Legendrian Intersections in the 1-Jet Bundle

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To Selma

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Declaration

The material in Chapter 1 is of an expository nature; the material in Chapters 2 through 5 is original, to the best of my knowledge, except where otherwise stated.

Summary

In this thesis we construct a family of generating functions for a Legendrian embedding, into the 1-jet bundle of a closed manifold, that can be obtained from the zero section through Legendrian embeddings, by discretising the action functional. We compute the second variation of a generating function obtained as above at a nondegenerate critical point and prove a formula relating the signature of the second variation to the Maslov index as the mesh goes to zero. We use this to prove a generalisation of the Morse inequalities thus refining a theorem of Chekanov. We also compute the spectral flow of the operator obtained by linearising the gradient equation of the action functional along a path connecting two nondegenerate critical points. We end by making a conjecture about the relation between the Floer connecting orbits and the gradient flow lines of the discrete action functional.

Introduction

In [1] Arnold calls a Legendrian embedding $f\colon L \to J^1M$, into the 1-jet bundle, that can be obtained from the zero section by a smooth deformation through Legendrian embeddings a quasi-function. Intersections of f with $M_0 \times \mathbb{R}$, where M_0 denotes the zero section of T^*M, are called critical points of f and these are said to be nondegenerate if the intersection is transverse. It can be shown that every such quasi-function is of the form $f = \psi|_{M_0\times\{0\}}$, where $\psi = \psi^1$ for some contact isotopy ψ^t of J^1M. Under the assumption that M is closed (that is, compact and boundaryless) we show that to each nondegenerate critical point $c \in L$ of f we may naturally assign a number

$$\mu(c, \psi^t) \in \frac{n}{2} + \mathbb{Z}$$

as the Maslov index of the path $\gamma\colon t \mapsto \psi^t((\psi^1)^{-1}f(c))$. Assuming that all critical points are nondegenerate we denote by p_k the number of critical points $c \in L$ such that $\mu(c, \psi^t) = k$ and by b_k the kth Betti number of M (with coefficients in any field). We prove the following:

Theorem A *Let $f\colon L \to J^1M$ be a quasi-function on a closed manifold M of dimension n having only nondegenerate critical points. Then*

$$p_{\frac{n}{2}-k} - p_{\frac{n}{2}-k+1} + \cdots \geq b_k - b_{k-1} + \cdots + (-1)^k b_0$$

for every $k \in \mathbb{Z}$.

Theorem A is a refinement of a theorem by Chekanov announced by Arnold in [1] (see also Chekanov [6]). This bounds the number of critical points, in the nondegenerate case, below by the sum of the Betti numbers. A proof of Chekanov's Theorem is also given by Chaperon [5] and is based on a result by Théret [25] concerning the existence of a generating function. Our proof is independent of their work.

A method of generating contact isotopies is by using a time-dependent Hamiltonian $H\colon [0,1] \times J^1M \to \mathbb{R}$. Indeed all contact isotopies may be obtained in this way. A special instance is given by the following: Given a Morse function $f\colon M \to \mathbb{R}$, let $H(x,y,z) = -f(x)$ define a time-independent Hamiltonian. The critical points of f are in 1-1 correspondence with the critical points of the associated quasi-function. In particular the Morse index $\mathrm{ind}_H(x_0)$ of a critical point $x_0 \in M$ of f is related to the Maslov index by

$$\mu(x_0, \psi_H^t) = \frac{n}{2} - \mathrm{ind}_H(x_0)$$

and thus in this special case Theorem A reduces to the classical Morse inequalities.

Our methods also refine a related theorem in the symplectic setting proved by Hofer [13] and Laudenbach and Sikorav [15]. This pertains to finding a lower bound for the number of points of intersection in the cotangent bundle of the zero section with its image under a time-dependent Hamiltonian symplectomorphism. An outline of our approach follows.

Let ψ^t be a contact isotopy with Hamiltonian $H\colon [0,1]\times J^1M \to \mathbb{R}$. When $M = \mathbb{R}^n$ we may associate to H an action functional as follows. Consider the path space

$$\mathcal{P} = \left\{c = (x,y)\colon [0,1] \to T^*\mathbb{R}^n \,\middle|\, y(0) = 0\right\},$$

and, given $c \in \mathcal{P}$, let $z = z_c\colon [0,1] \to \mathbb{R}$ be the unique solution of

$$\dot{z} = \langle y, \dot{x} \rangle - H(t,x,y,z) \qquad z(0) = 0.$$

Define the action functional $\Phi_H\colon \mathcal{P} \to \mathbb{R}$ by

$$\Phi_H(c) = \int_0^1 \Big(\langle y, \dot{x} \rangle - H(t,x,y,z) \Big)\, dt.$$

The fibre critical points of Φ_H with respect to the fibration $\mathcal{P} \to \mathbb{R}^n : (x,y) \mapsto x(1)$ generate the Legendrian submanifold $\psi^1(L_0)$.

We consider discretisations of the form $\Phi_H^N\colon \mathcal{P}^N \to \mathbb{R}$ for $N \in \mathbb{N}$ which also have this property. Here

$$\mathcal{P}^N = \left\{ c^N = (x_0, \dots, x_N, y_1, \dots, y_N) \in \mathbb{R}^{(2N+1)n} \,\middle|\, x_i,\, y_j \in \mathbb{R}^n \right\}$$

is the space of discrete paths which is fibred over $\mathbb{R}^n$ by $\mathcal{P}^N \to \mathbb{R}^n : c^N \mapsto x_N$ and Φ_H^N is the discrete action functional which is given by

$$\Phi_H^N(c^N) = \sum_{j=1}^N \Big(\langle y_j, x_j - x_{j-1} \rangle - V_{j-1}(x_{j-1}, y_j, z_{j-1}) \Big).$$

The functions $V_j\colon \mathbb{R}^{2n+1} \to \mathbb{R}$ are constructed by using the contact isotopy ψ^t, and the z_j are given by the iterative rule

$$z_0 = 0, \qquad z_j = \langle y_j, x_j - x_{j-1} \rangle - V_{j-1}(x_{j-1}, y_j, z_{j-1}) + z_{j-1}, \quad j > 0.$$

Such discretisations always exist for N sufficiently large and H having compact support. These are examples of generating functions which, in general, have been studied by, among others, Viterbo [27].

A path $c \in \mathcal{P}$ is a critical point of Φ_H if and only if (c, z_c) is a trajectory of ψ^t starting in $L_0 \times \{0\}$, where L_0 denotes the zero section of $T^*\mathbb{R}^n$, and ending in $L_0 \times \mathbb{R}$. Similarly, a discrete path $c^N \in \mathcal{P}^N$ is a critical point of Φ^N if and only if $(c^N, \{z_j\})$ corresponds to a "broken trajectory" of ψ^t with

$(x_0, y_0, z_0) \in L_0 \times \{0\}$ and $(x_N, y_N, z_N) \in L_0 \times \mathbb{R}$. Furthermore given such a path c^N, critical for Φ^N, and such that $\psi^1(L_0 \times \{0\})$ intersects $L_0 \times \mathbb{R}$ transversally at (x_N, y_N, z_N), $d^2\Phi^N_H(c^N)$ *will* be nondegenerate. We show that, given a critical point $c \in \mathcal{P}$ with the same property, for sufficiently large N the signature of the second variation of the discretisation is given by:

Propostion B

$$\operatorname{sign} d^2\Phi^N_H(c^N) = 2\mu(c^N, \psi^t).$$

This generalises a theorem of Robbin and Salamon in [20] for quadratic Hamiltonians. Also, it strengthens the theorem of Viterbo [26] which states that, in the symplectic case, the difference of the signature of the second variation of Φ^N at two critical points is independent of N, but at the cost of having to take N sufficiently large.

The action functional Φ_H and its discretisation are extended to the 1-jet bundle of a closed manifold M as follows. First, embed M in $\mathbb{R}^k$ for a suitable k. Second, lift H to any function $\widetilde{H}\colon J^1\mathbb{R}^k \to \mathbb{R}$ such that $\widetilde{H}|_{J^1\mathbb{R}^k|_M} = H \circ \rho$ where $\rho\colon J^1\mathbb{R}^k|_M \to J^1M$ is the natural projection map. Finally, define $\Phi_{H,M}$ to be $\Phi_{\widetilde{H}}$ restricted to the space of paths having $x(1) \in M$. Theorem A follows from generalising Proposition B and appealing to stabilised Morse theory (the Conley index).

Also in Part I we consider the linear first-order differential operator obtained by linearising the gradient equation of $\Phi_{H,M}$ along a path connecting two nondegenerate critical points. We show that this operator is Fredholm between the appropriate Banach spaces and, by considering the spectral flow, show that the Fredholm index is given as the difference of two Maslov indices.

In Part II, for ease of expression, we leave the contact setting and consider only the symplectic case. We consider the relation between bounded solutions of the gradient equation of $\Phi_{H,M}$, the Floer connecting orbits, and bounded gradient flow lines of the discrete action functional $\Phi^N_{H,M}$ as the parameter N tends to infinity. This part is mainly conjectural.

Part I

THE GENERALISED MORSE INEQUALITIES

Chapter 1

Preliminaries

The material in this chapter is of an expository nature. Standard references are McDuff and Salamon [17] and Arnold and Givental [2]. The key ideas discussed are the variational formulation of Hamilton's equations (Lemma 1.4) and the notion of restricted variational families given in Section 1.3.

1.1 Contact structures

A *contact structure* on a $2n+1$-dimensional manifold M is a maximally non-integrable field of hyperplanes $\xi \subset TM$. If we assume that ξ is transversally orientable, that is ξ is given by the kernel of some 1-form α, then the contact condition can be stated as

$$\alpha \wedge (d\alpha)^n \neq 0. \tag{1.1}$$

In this case α is said to be a *contact form* for the contact structure ξ. An important equivalent formulation of the contact condition (1.1) is that $d\alpha$ is a nondegenerate 2-form on ξ.

Note that, if a transverse orientation for ξ is specified, a contact form is unique only up to multiplication by a positive function and in general only up to multiplication by a nonzero function. Each choice of a contact form α is characterised by a vector field $Y = Y_\alpha \colon M \to TM$, called the *Reeb vector field*, defined by the relations

$$\iota(Y)\, d\alpha = 0, \qquad \alpha(Y) = 1.$$

Given a contact manifold (M, ξ) and an integral submanifold $L \subset M$ note that, for each $p \in M$, the tangent space T_pL is an isotropic subspace of the symplectic vector space $(\xi_p, d\alpha_p)$, that is, $d\alpha_p$ vanishes on T_pL. This implies, in particular, that the dimension of L can be at most n. In such a case L is said to be *Legendrian.*

A diffeomorphism $\psi \colon M \to M$ which preserves the oriented field of hyperplanes ξ is called a *contactomorphism*, that is, equivalently, if

$$\psi^* \alpha = e^g \alpha$$

for some function $g \colon M \to \mathbb{R}$.

Note that if $\psi \colon M \to M$ is a contactomorphism, then, by a direct calculation, its derivative restricts to give a linear conformal symplectomorphism

$$d\psi(p)|_{\xi_p} \colon (\xi_p, d\alpha_p) \to (\xi_{\psi(p)}, d\alpha_{\psi(p)}),$$

for every $p \in M$.

A smooth family $(\psi^t)_{0 \le t \le 1}$ of diffeomorphisms $\psi^t \colon J^1M \to J^1M$ with $\psi^0 = \mathrm{id}$ is called an *isotopy* of J^1M. When each ψ^t is a contactomorphism the family ψ^t is called a *contact isotopy.*

A *contact vector field* is defined to be a vector field $X \colon M \to TM$ which satisfies

$$\mathcal{L}_X \alpha = h\alpha$$

for some function $h \colon M \to \mathbb{R}$.

Contact Hamiltonians

Recall that for a symplectic manifold not all symplectic isotopies are generated by a time-dependent Hamiltonian function H. In contrast, the next lemma shows that the analogous statement in the case of a contact manifold with regard to contact isotopies *does* hold.

Lemma 1.1 ([17, Lemma 3.48]) *Let (M, α) be a contact manifold with Reeb field Y. Then*

(i) *a vector field $X: M \to TM$ is a contact vector field if and only if*

$$\iota(X)\alpha = -H, \qquad \iota(X)\, d\alpha = dH - (\iota(Y)\, dH)\, \alpha \tag{1.2}$$

for some function $H: M \to \mathbb{R}$;

(ii) *given any function $H: M \to \mathbb{R}$ there exists a unique vector field $X = X_H: M \to TM$ satisfying* (1.2).

1.2 1-Jet bundles

The model example of a contact manifold is provided by the 1-jet bundle $J^1M = T^*M \times \mathbb{R}$ with contact form

$$\alpha = dz - \lambda_{\mathrm{can}}.$$

Here z denotes the standard coordinate on $\mathbb{R}$ and λ_{can} the canonical 1-form on T^*M which is given by the defining property $\sigma^*\lambda_{\mathrm{can}} = \sigma$ for every 1-form $\sigma: M \to T^*M$. The Reeb field of α is given by $Y = \partial/\partial z$.

About any point $(c, z) \in J^1M$ one can construct a distinguished set of local coordinates $x_1, \dots, x_n, y_1, \dots, y_n, z$ with the property that α in these coordinates is given by

$$\alpha = dz - \sum_i y_i \, dx_i.$$

The following lemma gives the expression for a contact vector field in such a system of local coordinates.

Lemma 1.2 *An isotopy ψ^t of J^1M is a contact isotopy if and only if, in local coordinates as above, it is generated by the vector field*

$$\dot{x}_i = \frac{\partial H}{\partial y_i} \tag{1.3a}$$

$$\dot{y}_i = -\frac{\partial H}{\partial x_i} - y_i \frac{\partial H}{\partial z} \tag{1.3b}$$

$$\dot{z} = \langle y, \dot{x} \rangle - H \tag{1.3c}$$

for some smooth function $H \colon [0,1] \times J^1M \to \mathbb{R}$.

Proof. It is sufficient to consider the case $M = \mathbb{R}^n$. Given a contact isotopy ψ^t of $J^1\mathbb{R}^n$ define $X \colon [0,1] \times J^1\mathbb{R}^n \to TJ^1\mathbb{R}^n$ to be the time-dependent vector field satisfying

$$\frac{d}{dt}\psi^t(p) = X(t, \psi^t(p)).$$

This vector field is of contact type and thus, by a parametrised version of Lemma 1.1, is generated by a unique time-dependent Hamiltonian $H \colon [0,1] \times J^1\mathbb{R}^n \to \mathbb{R}$. Now we use the equations in Lemma 1.1. The first equation gives (1.3c) and, using the summation convention, the second equation unravels to give

$$\begin{aligned} \dot{x}_i \, dy_i - \dot{y}_i \, dx_i &= \iota(X) \, d\alpha \\ &= dH - (\iota(Y) \, dH) \, \alpha \\ &= \partial_{x_i} H \, dx_i + \partial_{y_i} H \, dy_i + \partial_z H \, dz - \partial_z H (dz - y_i \, dx_i) \\ &= (\partial_{x_i} H + y_i \partial_z H) \, dx_i + \partial_{y_i} H \, dy_i \end{aligned}$$

which is equivalent to (1.3a, b). This proves one implication of the lemma. The converse is given by reversing the above argument. □

The Hamilton–Jacobi equation

Given a function $H\colon [0,1] \times J^1M \to \mathbb{R}$, a function $S\colon [0,t_1] \times M \to \mathbb{R}$ is said to satisfy the *Hamilton–Jacobi equation* if

$$\partial_t S(t,x) + H(t,x,\partial_x S, S) = 0$$

and $S(0,x) = S_0(x)$ for some specified function $S_0\colon M \to \mathbb{R}$. It turns out that the characteristics of the Hamilton–Jacobi equation are those solutions of Hamilton's equations which begin in the 1-graph of S_0. In the following proposition, we formulate this more explicitly in local coordinates under the assumption that H has compact support.

Proposition 1.3 *Suppose that $S\colon [0,t_1] \times M \to \mathbb{R}$ is a solution of the Hamilton–Jacobi equation. Then given any solution of the equation*

$$\dot{x} = \partial_y H(t,x,\partial_x S, S),$$

on the interval $[0,t_1]$ the triple $(x,y,z)\colon [0,t_1] \to J^1M$, where $y(t) = \partial_x S(t,x(t))$ and $z(t) = S(t,x(t))$, solves Hamilton's equations.

Conversely, for t_1 sufficiently small, defining $S\colon [0,t_1] \times M \to \mathbb{R}$ by setting $S(t,x) = z(t)$, where $(x,y,z)\colon [0,t] \to J^1M$ is the unique solution of Hamilton's equations with boundary conditions

$$x(t) = x, \qquad y(0) = \partial_x S_0(x(0)), \qquad z(0) = S_0(x(0)),$$

gives a solution of the Hamilton–Jacobi equation.

Proof. We assume that $M = \mathbb{R}^n$. The first part of the proposition is straightforward to check. For the converse we first show that the function $S\colon [0,t_1] \times M \to \mathbb{R}$ constructed by solving Hamilton's equations satisfies

$$y(t) = \partial_x S(t,x)$$

for all $t \in [0,t_1]$. For this we fix $t \in [0,t_1]$ and consider the function $(-\varepsilon,\varepsilon) \to \mathbb{R}\colon s \to S(t, x + s\xi_1)$, where $\xi_1 \in \mathbb{R}^n$ is fixed, and denote by $(x_s, y_s, z_s)\colon [0,t] \to J^1M$ the unique solution of Hamilton's equation with $x_s(t) = x + s\xi_1$, $y_s(0) = \partial_x S_0(x_s(0))$ and $z_s(0) = S_0(x_s(0))$. Now

$$S(t, x + s\xi_1) = z_s(t) = \int_0^t \Big(\langle y_s, \dot{x}_s \rangle - H(\tau, x_s, y_s, z_s) \Big)\, d\tau$$

and it follows from Proposition 1.4 and Remark 1.5 below that differentiating this with respect to s and evaluating at $s = 0$ gives

$$\langle \partial_x S(t,x), \xi_1 \rangle = \langle y(t), \xi_1 \rangle.$$

Thus $\partial_x S(t,x) = y(t)$. It remains to show that S satisfies the Hamilton–Jacobi equation. This follows from

$$\langle y, \dot{x} \rangle - H = \dot{z} = \frac{d}{dt} S = \partial_t S + \partial_x S \dot{x} = \partial_t S + \langle y, \dot{x} \rangle$$

and the proof is complete. □

The above proposition shows that any contact isotopy with the property that the image of the zero section remains a 1-graph is characterised by a single time-dependent function solving the Hamilton–Jacobi equation. We consider the general case next.

The variational principle

Let $H\colon [0,1] \times J^1M \to \mathbb{R}$ be a time-dependent Hamiltonian. To ensure that (1.4) below has a solution $z\colon [0,1] \to \mathbb{R}$ for every smooth path $c\colon [0,1] \to$

T^*M assume that $\partial_z H$ is bounded. Let $\pi\colon \mathcal{P}_M \to M$ be the fibre bundle where

$$\mathcal{P}_M = \{c\colon [0,1] \to T^*M \mid c(0) \in M_0\}$$

is a space of smooth paths and π is the projection $c \mapsto \pi_{T^*M} c(1) \in M$. Here M_0 denotes the zero section of T^*M. To each path $c \in \mathcal{P}_M$ associate a function $z = z_c\colon [0,1] \to \mathbb{R}$ given as the unique solution of

$$\dot{z}(t) = \lambda_{\text{can}}(\dot{c}(t)) - H(t, c(t), z(t)), \qquad z(0) = 0. \tag{1.4}$$

Now define the *action functional* $\Phi_H\colon \mathcal{P}_M \to \mathbb{R}$ by

$$\Phi_H(c) = z(1) = \int_0^1 \Big(\lambda_{\text{can}}(\dot{c}) - H(t, c, z)\Big)\, dt.$$

A point $c \in \mathcal{P}_M$ is called a *fiber critical point* of Φ_H if the differential of Φ_H disappears along $T_c(\mathcal{P}_M)_{\pi(c)}$, the tangent space at c of the fibre over $\pi(c)$.

Proposition 1.4 *There is a 1-1 correspondence between the set of fibre critical points of Φ_H and the solutions of Hamilton's equations subject to $c(0) \in M_0$, $z(0) = 0$. Furthermore, $c \in \mathcal{P}_M$ is fibre critical if and only if*

$$\iota(\dot{c})d\lambda_{\text{can}} + dH_{t,z}(c) + \partial_z H(t, c, z)\lambda_{\text{can}} = 0 \tag{1.5}$$

Proof. Let $(-\varepsilon, \varepsilon) \to \mathcal{P}_M\colon s \mapsto c_s$ be a parametrised family of paths and abbreviate $c_0 = c$. Denote by $\gamma\colon [0,1] \to c^*T(T^*M)$ the associated variational field:

$$\gamma(t) = \frac{\partial}{\partial s}\bigg|_0 c_s(t).$$

Define $z_s\colon [0,1] \to \mathbb{R}$ for each s by (1.4) and denote the derivative with respect to s at $s = 0$ by ζ. Now ζ evaluated at $t \in [0,1]$ is given by the expression

$$
\begin{aligned}
\zeta(t) &= \frac{\partial}{\partial s}\bigg|_0 \int_0^t c_s^*(\lambda_{\mathrm{can}} - H_{z_s}\, d\tau) \\
&= \int_0^t c^*\mathcal{L}_\gamma(\lambda_{\mathrm{can}} - H_z\, d\tau) - \partial_z H\zeta\, d\tau \\
&= \int_0^t \Big(\iota(\gamma)(-\iota(\dot c)d\lambda_{\mathrm{can}} - dH_{\tau,z}) - \partial_z H\zeta\Big)\, d\tau + \lambda_{\mathrm{can}}(\gamma(t))
\end{aligned}
$$

where the third equality follows from Cartan's identity for the Lie derivative. Thus the 1-form α applied to the vector field (γ,ζ) satisfies

$$
\alpha(\gamma(t),\zeta(t)) = \int_0^t \Big(\iota(\gamma)(-\iota(\dot c)d\lambda_{\mathrm{can}} - dH_{\tau,z} - \partial_z H\lambda_{\mathrm{can}}) - \partial_z H\alpha(\gamma,\zeta)\Big)\, d\tau
$$
$$
\alpha(\gamma(0),\zeta(0)) = 0.
$$

This we can solve explicitly. Evaluating at $t = 1$ we find

$$
\alpha(\gamma(1),\zeta(1)) = \int_0^1 e^{\int_t^1 -\partial_z H\, d\tau}\iota(\gamma)(-\iota(\dot c)d\lambda_{\mathrm{can}} - dH_{t,z} - \partial_z H\lambda_{\mathrm{can}})\, dt.
$$

Thus it follows that $c \in \mathcal{P}_M$ is fibre critical if and only if (1.5) holds which, in local coordinates, is just equations (1.3a, b) or equivalently the second equation in (1.2). □

Remark 1.5 At a fibre critical point c the differential of Φ_H is given by

$$
d\Phi_H(c)(\gamma) = \lambda_{\mathrm{can}}(\gamma(1)).
$$

It thus follows that c is a critical point only if $c(1) \in M_0$.

Remark 1.6 The above variational principle is related to the Hamilton–Jacobi equation as follows. As before, define

$$
\mathcal{P}_t = \mathcal{P}_{M,t} = \{c\colon [0,t]\to T^*M \mid c(0) \in M_0\}
$$

and consider the functional $\Phi_t = \Phi_{H,t}\colon \mathcal{P}_t \to \mathbb{R}$ given by

$$\Phi_t(c) = z(t) = \int_0^t \Big(\lambda_{\mathrm{can}}(\dot{c}) - H(\tau, c, z)\Big)\, d\tau$$

where $z\colon [0,t] \to \mathbb{R}$ is given by (1.4). Next, define the *Green's function* $G_t\colon M \to \mathcal{P}_t$ by setting $G_t(x_1) = c$ where $c \in \mathcal{P}_t$ is the unique fibre critical point of Φ_t with $\pi_{T^*M}c(t) = x_1$. There exists such a fibre critical point if t is sufficiently small and H has compact support. Now define $S_t\colon M \to \mathbb{R}$, for t sufficiently small, by

$$S_t = \Phi_t \circ G_t.$$

This function is a solution of the Hamilton–Jacobi equation with $S_0 \equiv 0$. To see this note that by Proposition 1.4 $S(t, x_1) = z(t)$ where (c, z) is the unique solution of Hamilton's equations with $x(t) = x_1$, $c(0) \in M_0$ and $z(0) = 0$. The assertion now follows from the second part of Proposition 1.3.

1.3 Variational families

We start by briefly recalling some ideas from the theory of symplectic reduction. Suppose that (P, ω) is a symplectic manifold and that $Q \subset P$ is a coisotropic submanifold. This means that T_pQ^ω is contained in T_pQ for each $p \in Q$ where T_pQ^ω denotes the symplectic complement of T_pQ. Then the subspaces T_pQ^ω determine an isotropic distribution TQ^ω in TP which by the closedness of ω is integrable. It follows from Frobenius' theorem that Q is foliated by isotropic leaves. If we assume that the quotient space $P' = Q/{\sim}$ is a manifold, where the equivalence relation is given by: $p_0 \sim p_1$ if p_0 and p_1 lie in the same leaf, then we note that the P' is naturally a symplectic manifold.

We next recall that if L is a Lagrangian submanifold of M which intersects

Q cleanly then the image of L in P' is also Lagrangian, but may only be immersed.

To pass from Lagrangian immersions to Legendrian immersions we use the process of contactisation. This applies to all symplectic manifolds (P, ω) where the symplectic form is given as the differential of a 1-form $-\lambda$. Such manifolds are said to be *exact*. A *contactisation* of an exact symplectic manifold (P, ω) is a contact manifold $(P \times \mathbb{R}, dz - \lambda)$ where λ is a 1-form satisfying the relation $\omega = -d\lambda$. Here z denotes the coordinate on $\mathbb{R}$. Given an exact symplectic manifold (P, ω), a *contactisation* of a Lagrangian immersion $f: L \to P$ is a Legendrian immersion $\tilde{f}: L \to P \times \mathbb{R}$ lifting f. If the Lagrangian immersion $f: L \to P$ is *exact*, that is, $f^*\lambda = dS$ for some function $S: L \to \mathbb{R}$, then the lift $L \to P \times \mathbb{R}$: $c \mapsto (f(c), S(c))$ is Legendrian. Conversely it is easy to see that if f admits a lift then it must be exact. Note however that the contactisation of a Lagrangian immersion is not unique since the function S is defined only up to the addition of a locally constant function; this is the only nonuniqueness that can occur.

We now give the definition of a *variational family* [14]. This is a pair consisting of a fibre bundle $\pi: E \to X$ and a smooth function $\Phi: E \to \mathbb{R}$. The variational is called *transversal* if the graph of $d\Phi$ in T^*E intersects the conormal bundle of the fibres

$$N_E = \left\{(c, \eta) \in T^*E \,\middle|\, \eta \in (\ker d\pi(c))^\perp\right\}$$

transversally. For a transversal variational family (E, Φ) the set of fibre critical points is given by

$$\mathcal{C}_{E,\Phi} = \left\{c \in E \,\middle|\, d\Phi(c) \in (\ker d\pi(c))^\perp\right\}$$

which by transversality is a manifold. To each fibre critical point we can uniquely associate a covector $v^* \in T^*_{\pi(c)}X$ by the relation $v^* \circ d\pi(c) = d\Phi(c)$.

This gives rise to a map $\iota_\Phi\colon \mathcal{C}_{E,\Phi} \to J^1X$ given by mapping $c \in \mathcal{C}_{E,\Phi}$ to the 1-jet $(\pi(c), v^*, \Phi(c))$.

Proposition 1.7 *Given a transversal variational family (E, Φ), the map $\iota_\Phi\colon \mathcal{C}_{E,\Phi} \to J^1M$ is a Legendrian immersion.*

Proof. Note that the graph of $d\Phi$ is a Lagrangian submanifold of T^*E and N_E is a coisotropic submanifold. Note also that the quotient $N_E/\sim$ can be identified with T^*X. It follows from the above discussion that the map $\iota'_\Phi\colon \mathcal{C}_{E,\Phi} \to T^*X$, given by composing ι_Φ with the projection $J^1X \to T^*X$, is a Lagrangian immersion. Moreover the pullback of λ_{can} by ι'_Φ is given by

$$\iota'^*_\Phi \lambda_{\text{can}} = v^* \circ d(\pi|_{\mathcal{C}_{E,\Phi}}) = d(\Phi|_{\mathcal{C}_{E,\Phi}})$$

and so ι'_Φ is an exact immersion. Contactising, it follows that ι_Φ is a Legendrian immersion. □

Denote by L_Φ the image of ι_Φ and by X_0 the zero section of T^*X. Clearly, a point $c \in \mathcal{C}_{E,\Phi}$ is a critical point of Φ if and only if $\iota_\Phi(c)$ is in $L_\Phi \cap X_0 \times \mathbb{R}$. Moreover, when E is a finite dimensional vector bundle we have:

Lemma 1.8 *A point $c \in \operatorname{crit}\Phi$ is a nondegenerate critical point of Φ if and only if L_Φ and $X_0 \times \mathbb{R}$ intersect transversally at $\iota_\Phi(c)$.*

The proof of this uses the following elementary fact:

Lemma 1.9 *Let (V, ω) be a symplectic vector space and $N \subset V$ a coisotropic subspace. Then given two Lagrangian subspaces Λ_0 and Λ_1 satisfying*

$$\Lambda_0 \subset N, \qquad \Lambda_1 \cap N^\omega = \{0\},$$

Λ_0 is transverse to Λ_1 if and only if, in the quotient, the reduced spaces $\overline{\Lambda}_0$ and $\overline{\Lambda}_1$ are transversal.

Proof of Lemma 1.8. Apply Lemma 1.9 in case $V = T_p(T^*E)$, $N = T_pN_E$, $\Lambda_0 = T_pE_0$ and $\Lambda_1 = T_p(\mathrm{graph}(d\Phi))$ where $p = (c, d\Phi(c))$ and E_0 denotes the zero section of T^*E. $\square$

Example 1.10 (Action) Let $H\colon [0,1] \times J^1M \to \mathbb{R}$ be a time-dependent Hamiltonian and $\Phi_H\colon \mathcal{P}_M \to \mathbb{R}$ the action functional given in Section 1.2. By Proposition 1.4 and Remark 1.5 the pair $(\mathcal{P}_M, \Phi_H)$ is a variational family for the Legendrian submanifold $\psi^1(M_0 \times \{0\})$, where $\psi^t = \psi^t_H$ denotes the contact isotopy generated by H. Note that we do not say anything about transversality.

Restricted variational families

Let M be a submanifold of X and let $L \to J^1X\colon c \mapsto (f(c), S(c))$ be a Legendrian immersion transversal to $J^1X|_M$. We extend the notion of symplectic reduction to Legendrian immersions as follows.

Evidently $f\colon L \to T^*X$ is an exact Lagrangian immersion. Define $f'\colon L' \to T^*M$, where $L' = f^{-1}(T^*X|_M)$, by $f' = r \circ f \circ i$. Here $i\colon L' \hookrightarrow L$ is the inclusion and $r\colon T^*X|_M \to T^*M$ is the natural projection. By transversality L' is a manifold of the same dimension as M. Using $r^*\lambda_{\mathrm{can}} = j^*\lambda_{\mathrm{can}}$ where $j\colon T^*X|_M \hookrightarrow T^*X$ is the inclusion, it is now easy to see that f' is again an exact Lagrangian immersion with $f'^*\lambda_{\mathrm{can}} = d(S \circ i)$. The lift $L' \to J^1M\colon c \mapsto (f'(c), S \circ i(c))$ we refer to as the *reduction* of (f, S).

Suppose that $(\pi\colon E \to X,\ \Phi\colon E \to \mathbb{R})$ is a transversal variational family and denote by

$$(\pi_M = \pi|_{E_M}\colon E_M \to M,\ \Phi_M = \Phi|_{E_M}\colon E_M \to \mathbb{R}),$$

where $E_M = E|_M$, the variational family obtained by restricting to M. In the case of a finite dimensional vector bundle these are related by:

Lemma 1.11 *Given a transversal variational family* (E, Φ), *the variational family* (E_M, Φ_M) *is transversal if and only if* L_Φ *is transverse to* $J^1X|_M$. *Moreover, if both variational families are transversal then* L_{Φ_M} *is the reduction of* L_Φ.

Proof. The first part of the lemma follows easily by choosing local coordinates on X in which M is linear and the second part follows by a direct calculation.
□

Corollary 1.12 *If both* (E, Φ) *and* (E_M, Φ_M) *are transversal then* $c \in E_M$ *is a nondegenerate critical point of* Φ_M *if and only if* L_Φ *and* $TM^\perp \times \mathbb{R}$, *where* $TM^\perp \subset T^*X$ *denotes the conormal bundle of* M, *intersect transversally at* $\iota_\Phi(c)$.

Proof. This follows immediately from Lemmas 1.8, 1.9 and 1.11. □

Chapter 2

Continuous-time theory

In the second part of Section 1.3 we saw how by restricting a variational family to a submanifold we could obtain a variational family that generated the reduction of the original Legendrian immersion. We now consider the inverse of this process in a specific case, namely if M is a closed manifold embedded in $\mathbb{R}^k$, and $L = \psi(M_0 \times \{0\}) \subset J^1M$ is a Legendrian submanifold, where $\psi = \psi^1$ for some contact isotopy ψ^t of J^1M, we construct a variational family $(\mathcal{P}_{\mathbb{R}^k}, \Phi_H)$ whose restriction to M, $(\mathcal{P}, \Phi)$, generates L. This is carried out in Section 2.1. In Section 2.2 we consider the second variation of the function Φ thus obtained.

2.1 Construction of variational family

Let M^n be a closed manifold embedded in $\mathbb{R}^k$, and suppose that $H\colon [0,1] \times J^1\mathbb{R}^k \to \mathbb{R}$ is a time-dependent Hamiltonian with the property that for every $x \in M$

$$H(t,x,y,z) = H(t,x,y',z) \quad \text{whenever} \quad y - y' \perp T_xM.$$

Denoting by $\rho\colon J^1\mathbb{R}^k|_M \to J^1M$ the natural projection $(x,y,z) \mapsto (x, y|_{T_xM}, z)$, it follows that H descends to a time-dependent Hamiltonian $\overline{H}\colon [0,1] \times J^1M \to \mathbb{R}$ given by

$$H|_{J^1\mathbb{R}^k|_M} = \overline{H} \circ \rho. \tag{2.1}$$

We insist that H_t is C^1-bounded for every $t \in [0,1]$. This ensures that the Hamiltonian flows of H and $\overline{H}$ define global diffeomorphisms for all $t \in [0,1]$.

The next lemma shows how the Hamiltonian flow of H is related to that of $\overline{H}$ (*cf.* [3]).

Lemma 2.1 *Let H be a time-dependent Hamiltonian on $J^1\mathbb{R}^k$ and denote by φ^t the associated contact isotopy. Then the following are equivalent:*

(i) *H satisfies* (2.1) *for some function $\overline{H}\colon [0,1] \times J^1M \to \mathbb{R}$,*

(ii) *φ^t leaves $J^1\mathbb{R}^k|_M$ invariant,*

(iii) *φ^t descends to a contact isotopy on J^1M.*

Proof. (i) $\Rightarrow$ (ii). Let $X = (\dot{x}, \dot{y}, \dot{z})$ denote the vector field generated by H. It is sufficient to show that $X_t(p) \in T_p(J^1\mathbb{R}^k|_M)$ for all $p \in J^1\mathbb{R}^k|_M$. To see this note that, for any $p = (x,y,z) \in J^1\mathbb{R}^k|_M$,

$$\begin{aligned}\dot{x}_t(p) &= \partial_y H_t(x,y,z)\\ &= \partial_y H_t(x, \Pi(x)y, z)\\ &= \Pi(x)\partial_y H_t(x,y,z).\end{aligned}$$

Here T^*M has been identified with TM, using the induced inner product, and $\Pi(x) \in \mathbb{R}^{k\times k}$ denotes the orthogonal projection onto T_xM. It follows $\dot{x}_t(p) \in T_xM$ and hence $X_t(p) \in T_p(J^1\mathbb{R}^k|_M)$ as required.

(ii) $\Rightarrow$ (iii). We first claim that if a contactomorphism φ preserves $J^1\mathbb{R}^k|_M$ then it has the property that

$$\rho(\varphi(p)) = \rho(\varphi(q)) \quad \text{whenever} \quad \rho(p) = \rho(q). \tag{2.2}$$

To see this note that the subspace $N_p = T_p(J^1\mathbb{R}^k|_M) \cap \xi_p$ is a coisotropic subspace of the symplectic vector space $(\xi_p, d\alpha_p)$, for each $p \in J^1\mathbb{R}^k|_M$, and that the symplectic complement is given simply by $N_p^\omega = T_p(\rho^{-1}(\rho(p)))$. Also note that $d\varphi(p)$ restricts to a map from N_p to $N_{\varphi(p)}$. Property (2.2) now the fact that the fibers of ρ are connected.

Consequently, we may define $\psi^t : J^1M \to J^1M$ by $\rho \circ \varphi^t \circ j = \psi^t \circ \rho$, where j denotes the inclusion $J^1\mathbb{R}^k|_M \hookrightarrow J^1\mathbb{R}^k$. Clearly ψ^t is a diffeomorphism. That ψ^t is a contactomorphism follows from a direct calculation using the identity $j^*\alpha = \rho^*\alpha$.

(iii) $\Rightarrow$ (i). Since φ^t descends to a contact isotopy ψ^t on J^1M it is clear that φ^t preserves $J^1\mathbb{R}^k|_M$. Also it follows that the contact vector fields X, $\overline{X}$ generated by the isotopies φ^t, ψ^t respectively are related by $\rho_* X = \overline{X}$. By Lemma 1.1 these satisfy $\iota(X)\alpha = -H$, $\iota(\overline{X})\alpha = -\overline{H}$, where $\overline{H}$ is the Hamiltonian function generating ψ^t. A direct calculation now shows that H and $\overline{H}$ are related by (2.1). This completes the proof of the lemma. $\square$

Remark 2.2 It follows from Lemma 2.1 that, when H satisfies (2.1), there is a commutative diagram

$$\begin{tikzcd} J^1\mathbb{R}^k|_M \arrow[r, "\varphi^t"] \arrow[d, "\rho"] & J^1\mathbb{R}^k|_M \arrow[d, "\rho"] \\ J^1M \arrow[r, "\psi^t"] & J^1M \end{tikzcd}$$

but note that φ^t is not uniquely determined by the restriction of H to $J^1\mathbb{R}^k|_M$.

Now define the space

$$\mathcal{P} = \mathcal{P}_{\mathbb{R}^k,M} = \{c = (x,y)\colon [0,1] \to T^*\mathbb{R}^k \mid y(0) = 0,\ x(1) \in M\}$$

of paths in $T^*\mathbb{R}^k$, and let π be the projection $\mathcal{P} \to M$ given by $c = (x,y) \mapsto x(1)$. The action functional $\Phi = \Phi_{H,M}\colon \mathcal{P} \to \mathbb{R}$ is defined by

$$\Phi(c) = \int_0^1 \Big(\langle y(t), \dot{x}(t)\rangle - H(t, x(t), y(t), z(t))\Big)\, dt$$

where $z\colon [0,1] \to \mathbb{R}$ is the unique solution of the initial value problem

$$\dot{z}(t) = \langle y(t), \dot{x}(t)\rangle - H(t, x(t), y(t), z(t)), \qquad z(0) = 0. \tag{2.3}$$

Lemma 2.1 now implies the following:

Proposition 2.3 $(\mathcal{P}, \Phi)$ *is a variational family for the Legendrian submanifold* $\psi^1(M_0 \times \{0\}) \subset J^1M$.

In particular, $c \in \mathcal{P}$ is a critical point of Φ if and only if c solves equations (1.3a, b) and $c(1)$ is in $TM^\perp$, the conormal bundle of M. (Compare Corollary 1.12)

Remark 2.4 The variational families $(\mathcal{P}_M, \Phi_{\overline{H}})$, of Example 1.10, and $(\mathcal{P}, \Phi)$, considered above, both generate the Legendrian submanifold $\psi^1(M_0 \times \{0\}) \subset J^1M$. These are related as follows: let $c \in \mathcal{P}$ be a fibre critical point of Φ then $\bar{c} \in \mathcal{P}_M$, given by composing c with the projection $T^*\mathbb{R}^k|_M \to T^*M$, is a fibre critical point of $\Phi_{\overline{H}}$. Conversely any fibre critical point $\bar{c}$ of $\Phi_{\overline{H}}$ lifts uniquely to a fibre critical point c of Φ by using the contact isotopy φ^t.

2.2 The second variation

From the proof of Proposition 1.4 the differential of Φ is given by

$$d\Phi(c)(\gamma) = \int_0^1 e^{\int_t^1 -\partial_z H\, d\tau}\Big(\langle \eta, \dot{x} - \partial_y H\rangle + \langle \dot{y} + \partial_x H + y\partial_z H, \xi\rangle\Big)\, dt$$
$$+ \langle y(1), \xi(1)\rangle$$

where $\gamma \in T_c\mathcal{P} = \{\gamma = (\xi, \eta) \colon [0,1] \to \mathbb{R}^{2k} \,|\, \eta(0) = 0,\ \xi(1) \in T_{x(1)}M\}$. In order to deal with the boundary term $\langle y(1), \xi(1)\rangle$ we now restrict Φ to the space

$$\mathcal{E} = \mathcal{E}_{\mathbb{R}^k, M} = \{c \in \mathcal{P} \,|\, c(1) \in TM^\perp\}$$

of paths satisfying Lagrangian boundary conditions. Before giving a formula for the second variation of Φ at a critical point we introduce some notation.

Let $c \in \mathcal{P}$ be a critical point of Φ and denote by $z \colon [0,1] \to \mathbb{R}$ the corresponding solution of (2.3). Define, for reasons that will become apparent later, the matrix valued functions $G_{\xi\xi}, G_{\xi\eta}, G_{\eta\xi}, G_{\eta\eta} \colon [0,1] \to \mathbb{R}^{k\times k}$ and $G_\zeta \colon [0,1] \to \mathbb{R}$ by

$$\begin{aligned}
G_{\xi\xi} &= \partial^2_{xx}H + \partial^2_{xz}H y^T + y\partial^2_{zx}H + \partial^2_{zz}H yy^T \\
G_{\xi\eta} &= \partial^2_{xy}H + y\partial^2_{zy}H \\
G_{\eta\xi} &= \partial^2_{yx}H + \partial^2_{yz}H y^T \\
G_{\eta\eta} &= \partial^2_{yy}H \\
G_\zeta &= \partial_z H.
\end{aligned} \tag{2.4}$$

Here the derivatives of H are evaluated at $(t, c(t), z(t)) \in [0,1] \times J^1\mathbb{R}^k$. Note that $G_{\xi\xi}(t)$ and $G_{\eta\eta}(t)$ are symmetric and that $G_{\xi\eta}(t) = G_{\eta\xi}(t)^T$. Construct from these the symmetric matrix valued function $S = S_G \colon [0,1] \to \mathbb{R}^{2k\times 2k}$ by

$$S(t) = \begin{pmatrix} G_{\xi\xi}(t) & G_{\xi\eta}(t) + \frac{1}{2}G_\zeta(t)\mathbb{1} \\ G_{\eta\xi}(t) + \frac{1}{2}G_\zeta(t)\mathbb{1} & G_{\eta\eta}(t) \end{pmatrix}. \tag{2.5}$$

Also, denote by J_0 the $2k \times 2k$ matrix

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

and by $\mathcal{C}_c = \mathcal{C}_{c,M}$ the tangent space of $\mathcal{E}$ at c:

$$\mathcal{C}_c = \left\{\gamma\colon [0,1] \to \mathbb{R}^{2k} \,\middle|\, \gamma(0) \in \Lambda_0,\ \gamma(1) \in \Lambda_1\right\}.$$

Here $\Lambda_0 = \mathbb{R}^k \times \{0\}$ and $\Lambda_1 = T_{c(1)}(TM^\perp)$ are Lagrangian subspaces of $\mathbb{R}^{2k}$.

Proposition 2.5 *The second differential of Φ at a critical point c is of the form*

$$d^2\Phi(c)(\gamma_1, \gamma_2) = \langle A(c)\gamma_1, \gamma_2\rangle_{L^2}$$

where $\gamma_1, \gamma_2 \in \mathcal{C}_c$ and the second variation, $A(c)$, is given by

$$A(c)\gamma = e^{\int_t^1 -G_\zeta\, d\tau}\left(J_0\dot\gamma - S\gamma + \tfrac{1}{2}G_\zeta J_0\gamma\right). \tag{2.6}$$

Proof. Fix a two parameter family of curves

$$c_{s_1,s_2} = (x_{s_1,s_2}, y_{s_1,s_2}) \in \mathcal{E}, \qquad s_1,\ s_2 \in (-\varepsilon, \varepsilon)$$

with $c_{0,0} = c$ and denote the derivatives by

$$\xi_{i,s_1,s_2} = \frac{\partial}{\partial s_i}x_{s_1,s_2}, \quad \eta_{i,s_1,s_2} = \frac{\partial}{\partial s_i}y_{s_1,s_2}, \quad \zeta_{i,s_1,s_2} = \frac{\partial}{\partial s_i}z_{s_1,s_2}.$$

Here, of course, z_{s_1,s_2} is given by solving (2.3) with x, y, z replaced by x_{s_1,s_2}, y_{s_1,s_2}, z_{s_1,s_2}. Abbreviate $\xi_i = \xi_{i,0,0}$, $\eta_i = \eta_{i,0,0}$, $\zeta_i = \zeta_{i,0,0}$ and set $\gamma_i = (\xi_i, \eta_i)$, $\mu_i = (\xi_i, \eta_i, \zeta_i)$. We now proceed with the proof of the proposition.

Differentiate the identity

$$\dot z_{s_1,s_2} = \left\langle y_{s_1,s_2}, \dot x_{s_1,s_2}\right\rangle - H(\cdot, x_{s_1,s_2}, y_{s_1,s_2}, z_{s_1,s_2})$$

with respect to s_2 and s_1 successively and then evaluate at $s_1 = s_2 = 0$ to obtain

$$\begin{aligned}\partial_{s_1}\dot{\zeta}_2 &= \langle \partial_{s_1}\eta_2, \dot{x}\rangle + \langle \eta_2, \dot{\xi}_1\rangle + \langle \eta_1, \dot{\xi}_2\rangle + \langle y, \partial_{s_1}\dot{\xi}_2\rangle \\ &\quad - d^2H_t(\mu_1,\mu_2) - \langle \partial_x H, \partial_{s_1}\xi_2\rangle - \langle \partial_y H, \partial_{s_1}\eta_2\rangle - \partial_z H \partial_{s_1}\zeta_2.\end{aligned}$$

Using the fact that c is critical rewrite this as

$$\dot{\zeta} = \langle \eta_2, \dot{\xi}_1\rangle + \langle \eta_1, \dot{\xi}_2\rangle + \frac{d}{dt}\langle y, \partial_{s_1}\xi_2\rangle + \langle y\partial_z H, \partial_{s_1}\xi_2\rangle - \partial_z H\zeta - d^2H_t(\mu_1,\mu_2)$$

where $\zeta = \partial_{s_1}\zeta_2$ or more concisely, by using the substitution $\lambda = \zeta - \langle y, \partial_{s_1}\xi_2\rangle$, as

$$\frac{d}{dt}\lambda = \langle \eta_2, \dot{\xi}_1\rangle + \langle \eta_1, \dot{\xi}_2\rangle - \partial_z H\lambda - d^2H_t(\mu_1,\mu_2).$$

Since $\lambda(0) = 0$ the solution to this ODE evaluated at $t = 1$ is

$$\lambda(1) = \int_0^1 e^{\int_t^1 -G_\zeta\, d\tau}\Big(\langle \eta_2, \dot{\xi}_1\rangle + \langle \eta_1, \dot{\xi}_2\rangle - d^2H_t(\mu_1,\mu_2)\Big)\, dt.$$

Also since $0 = \partial_{s_1}\langle y(1), \xi_2(1)\rangle = \langle \eta_1(1), \xi_2(1)\rangle + \langle y(1), \partial_{s_1}\xi_2(1)\rangle$ it follows that $\lambda(1) = \zeta(1) + \langle \eta_1(1), \xi_2(1)\rangle$. Now partially integrate the second summand on the right hand side to obtain

$$\zeta(1) = \int_0^1 e^{\int_t^1 -G_\zeta\, d\tau}\Big(\langle \eta_2, \dot{\xi}_1\rangle - \langle \dot{\eta}_1, \xi_2\rangle - G_\zeta\langle \eta_1, \xi_2\rangle - d^2H_t(\mu_1,\mu_2)\Big)\, dt.$$

Since c is critical observe, by Proposition 1.4, that

$$\alpha(\mu_i(t)) = \zeta_i(t) - \langle y(t), \xi_i(t)\rangle = 0$$

for all $t \in [0,1]$ thus, in particular,

$$d^2H_t(\mu_1,\mu_2) = \langle G_{\xi\xi}\xi_1, \xi_2\rangle + \langle G_{\xi\eta}\eta_1, \xi_2\rangle + \langle G_{\eta\xi}\xi_1, \eta_2\rangle + \langle G_{\eta\eta}\eta_1, \eta_2\rangle.$$

The proposition now follows. $\square$

Remark 2.6 $A(c)$ extends to a self-adjoint operator on $L^2([0,1],\mathbb{R}^{2k})$ with dense domain

$$W^{1,2}_{c,M} = \left\{\gamma \in W^{1,2}([0,1];\mathbb{R}^{2k}) \,\middle|\, \gamma(0) \in \Lambda_0,\ \gamma(1) \in \Lambda_1\right\}.$$

In addition, $A(c)$ is injective if and only if $\Psi(1)\Lambda_0$ is transverse to Λ_1 where $\Psi(t) = \Psi_{c,H}(t) \in \mathrm{Sp}(2k)$ is given by

$$\dot{\Psi} = -J_0 S \Psi, \qquad \Psi(0) = \mathbb{1}.$$

In such a case we will say that c is a *nondegenerate* critical point of Φ.

Remark 2.7 The contact isotopy φ^t, generated by H, linearised along the path $a = (c,z)\colon [0,1] \to J^1\mathbb{R}^k$ and restricted to contact planes, gives rise to the path of linear conformal symplectomorphisms

$$d\varphi^t(a(0))|_{\xi_{a(0)}}\colon (\xi_{a(0)}, d\alpha_{a(0)}) \to (\xi_{a(t)}, d\alpha_{a(t)}).$$

This path may be naturally identified with the path $e^{\int_0^t -\frac{1}{2}G_\zeta\, d\tau}\Psi_{c,H}$, where $\Psi_{c,H}$ is given in Remark 2.6, via the canonical symplectomorphisms

$$(\xi_p, d\alpha_p) \to (\mathbb{R}^{2k}, \omega_0)\colon (\xi, \eta, \langle y, \xi\rangle) \mapsto (\xi, \eta)$$

where $p = (x, y, z)$.

Remark 2.8 Identify T^*M with TM using the induced inner product. The tangent space to $TM^\perp$ at $p = (x, y)$ is given by

$$T_p(TM^\perp) = \left\{(\xi, \eta) \in \mathbb{R}^{2k} : \xi \in T_xM,\ (d\Pi(x)\xi)y + \Pi(x)\eta = 0\right\}$$

where $\Pi(x) \in \mathbb{R}^{k\times k}$ denotes the orthogonal projection onto T_xM. Note that defining $I\!I_p\colon T_xM \to \mathbb{R}$ by $I\!I_p(\xi) = \frac{1}{2}\langle (d\Pi(x)\xi)y, \xi\rangle$ gives a quadratic form on T_xM, the *second fundamental form* of M at x along the normal vector y

Remark 2.9 Let $G\colon [0,1] \times J^1\mathbb{R}^k \to \mathbb{R}$ be the time-dependent Hamiltonian given by

$$G(t,\xi,\eta,\zeta) = \tfrac{1}{2}\langle G_{\xi\xi}(t)\xi,\xi\rangle + \langle G_{\eta\xi}(t)\xi,\eta\rangle + \tfrac{1}{2}\langle G_{\eta\eta}(t)\eta,\eta\rangle + G_\zeta(t)\zeta.$$

We call such Hamiltonians, where $G_{\xi\xi}(t), G_{\eta\xi}(t), G_{\eta\eta}(t) \in \mathbb{R}^{k\times k}$ with $G_{\xi\xi}(t)$, $G_{\eta\eta}(t)$ symmetric and $G_\zeta(t) \in \mathbb{R}$, *quadratic*, and define $G_{\xi\eta}(t) = G_{\eta\xi}(t)^T$. Define the functional $\Phi_c = \Phi_{c,H,M}\colon \mathcal{C}_c \to \mathbb{R}$ by

$$\Phi_c(\gamma) = \int_0^1 \Big(\langle \eta, \dot{\xi}\rangle - G(t,\xi,\eta,\zeta)\Big)\,dt + I\!I_{c(1)}(\xi(1)),$$

where $\zeta\colon [0,1] \to \mathbb{R}$ is the unique solution of the initial value problem

$$\dot{\zeta}(t) = \langle \eta(t), \dot{\xi}(t)\rangle - G(t,\xi(t),\eta(t),\zeta(t)), \qquad \zeta(0) = 0. \tag{2.7}$$

Notice that if c is a nondegenerate critical point of Φ, then 0 is the unique critical point of Φ_c, also nondegenerate, and in this case the above proposition shows that the second variation of Φ at c agrees with the second variation of Φ_c.

Remark 2.10 A quadratic Hamiltonian $G\colon [0,1] \times J^1\mathbb{R}^k \to \mathbb{R}$ as above generates the contact isotopy

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = e^{\int_0^t -\frac{1}{2}G_\zeta\,d\tau}\,\Psi_G(t)\begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix}$$
$$\zeta(t) = e^{\int_0^t -G_\zeta\,d\tau}\zeta_0 + \tfrac{1}{2}\int_0^t e^{\int_s^t -G_\zeta\,d\tau}\big(\langle G_{\eta\eta}\eta,\eta\rangle - \langle G_{\xi\xi}\xi,\xi\rangle\big)\,ds$$

where $\Psi_G\colon [0,1] \to \mathrm{Sp}(2k)$ is given by

$$\dot{\Psi}_G = -J_0 S_G \Psi_G, \qquad \Psi_G(0) = \mathbb{1}. \tag{2.8}$$

Conversely, given any smooth family of symplectic matrices $\Psi\colon [0,1] \to \mathrm{Sp}(2k)$ and any smooth positive function $a\colon [0,1] \to \mathbb{R}_{>0}$ there exists a contact isotopy of the form

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = a(t)\Psi(t) \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix}$$

$$\zeta(t) = a(t)^2\zeta_0 + Q(t, \xi_0, \eta_0)$$

where $\Psi = \Psi_G$ for some quadratic Hamiltonian G and $Q(t, \cdot, \cdot)$ is quadratic. We call such contact isotopies *pseudo-linear.*

Chapter 3

Discrete variational families

In this chapter we construct a family of finite dimensional variational families $(\mathcal{P}^N, \Phi^N)$ for the Legendrian submanifold $L = \psi(M_0 \times \{0\}) \subset J^1M$, where $\psi = \psi^1$ for some contact isotopy ψ^t of J^1M. This is done in two stages. The first stage, carried out in Section 3.1, is to consider the case of a contactomorphism C^1-close to the identity. The second stage, Section 3.2, is the general case. Alternative constructions are given by Chekanov [6] and Théret [25]. The corresponding construction in the case of cotangent bundles was first carried out by Laudenbach and Sikorav [15].

3.1 Generating functions of type V

We generalise the notion of a generating function of type V, as defined in [17], to the contact setting. Let $\varphi\colon J^1\mathbb{R}^k \to J^1\mathbb{R}^k$ be a contactomorphism and denote it as follows:

$$(x_0, y_0, z_0) \mapsto (x_1, y_1, z_1) = (u(x_0, y_0, z_0), v(x_0, y_0, z_0), w(x_0, y_0, z_0)).$$

A *generating function of type V* for φ is a function $V\colon \mathbb{R}^{2k+1} \to \mathbb{R}$ such that

$$\begin{aligned} x_1 - x_0 &= \frac{\partial V}{\partial y}(x_0, y_1, z_0) \\ y_1 - y_0 &= -\frac{\partial V}{\partial x}(x_0, y_1, z_0) - y_0 \frac{\partial V}{\partial z}(x_0, y_1, z_0) \\ z_1 - z_0 &= \langle y_1, x_1 - x_0 \rangle - V(x_0, y_1, z_0) \end{aligned} \tag{3.1}$$

if and only if $(x_1, y_1, z_1) = \varphi(x_0, y_0, z_0)$. In this case, observe that φ has compact support (that is, φ is equal to the identity outside a compact set) if and only if V has.

Proposition 3.1 (i) *Every contactomorphism $\varphi\colon J^1\mathbb{R}^k \to J^1\mathbb{R}^k$ which is sufficiently close to the identity in the C^1-topology admits a unique generating function of type V.*

(ii) *For each smooth function $V\colon \mathbb{R}^{2k+1} \to \mathbb{R}$ having compact support and sufficiently small first and second derivatives there exists a unique contactomorphism $\varphi\colon J^1\mathbb{R}^k \to J^1\mathbb{R}^k$ such that V is a generating function of type V for φ*

Proof. For (i), the assumption on φ ensures that the map

$$\mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1} \colon (x_0, y_0, z_0) \mapsto (x_0, v(x_0, y_0, z_0), z_0)$$

has a global inverse.[1] Thus there exists a map $f\colon \mathbb{R}^{2k+1} \to \mathbb{R}^k$ such that

$$y_0 = f(x_0, v(x_0, y_0, z_0), z_0).$$

[1] A continuously differentiable map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ has a continuously differentiable inverse if $\|\mathbb{1} - dF(x)\| \leq 1/2$ for all $x \in \mathbb{R}^n$. To see this note that, by the inverse function theorem, it is sufficient to show that F is bijective. The latter is seen as follows. The inequality $2 \cdot |F(x_1) - F(x_2)| \geq |x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}^n$ implies that F is injective. Surjectivity of F follows by considering the map $x \mapsto x - F(x) + y$, which is a contraction of the closed ball of radius R where $|y - F(0)| = R/2$.

Also, since $\varphi\colon (x_0, y_0, z_0) \mapsto (x_1, y_1, z_1)$ is a contactomorphism, we have

$$dz_1 - \sum y_{1i}\, dx_{1i} = e^g \left(dz_0 - \sum y_{0i}\, dx_{0i}\right)$$

for some function $g\colon \mathbb{R}^{2k+1} \to \mathbb{R}$. Thus, using the notation set up earlier, the chain rule gives

$$\begin{aligned} dz_1 - \sum y_{1i}\, dx_{1i} = &\sum \left(\partial_{x_{0i}} w - \langle y_1, \partial_{x_{0i}} u\rangle\right) dx_{0i} \\ &+ \sum \left(\partial_{y_{0i}} w - \langle y_1, \partial_{y_{0i}} u\rangle\right) dy_{0i} + \left(\partial_{z_0} w - \langle y_1, \partial_{z_0} u\rangle\right) dz_0. \end{aligned}$$

The contactomorphism condition now implies

$$\partial_{y_0} w - (\partial_{y_0} u)^T y_1 = 0, \tag{3.2}$$

and

$$\partial_{x_0} w - (\partial_{x_0} u)^T y_1 = -\left(\partial_{z_0} w - \langle y_1, \partial_{z_0} u\rangle\right) y_0. \tag{3.3}$$

We claim that $V\colon \mathbb{R}^{2n+1} \to \mathbb{R}$ defined by

$$\begin{aligned} V(x_0, y_1, z_0) &:= \langle y_1, x_1 - x_0\rangle - z_1 + z_0 \\ &= \langle y_1, u(x_0, f(x_0, y_1, z_0), z_0) - x_0\rangle - w(x_0, f(x_0, y_1, z_0), z_0) + z_0 \end{aligned}$$

satisfies (3.1).

This claim is proved by differentiating and appealing to the chain rule. The computation proceeds as follows. First we differentiate V with respect to z_0 to obtain

$$\begin{aligned} \partial_{z_0} V &= \langle y_1, \partial_{z_0} u + \partial_{y_0} u\, \partial_{z_0} f\rangle - \partial_{z_0} w - (\partial_{y_0} w)^T \partial_{z_0} f + 1 \\ &= -\left((\partial_{y_0} w)^T - y_1^T \partial_{y_0} u\right) \partial_{z_0} f - \partial_{z_0} w + \langle y_1, \partial_{z_0} u\rangle + 1 \\ &= -\partial_{z_0} w + \langle y_1, \partial_{z_0} u\rangle + 1. \end{aligned}$$

Here the last line is a consequence of (3.2), and thus

$$\partial_{z_0} V - 1 = -\left(\partial_{z_0} w - \langle y_1, \partial_{z_0} u\rangle\right). \tag{3.4}$$

Next we differentiate V with respect to x_0 and use identities (3.2), (3.3) and (3.4) to obtain

$$\begin{aligned}
\partial_{x_0} V &= \left((\partial_{x_0} u)^T + (\partial_{x_0} f)^T (\partial_{y_0} u)^T - \mathbb{1}\right) y_1 - \partial_{x_0} w - (\partial_{x_0} f)^T \partial_{y_0} w \\
&= -(\partial_{x_0} f)^T \left(\partial_{y_0} w - (\partial_{y_0} u)^T y_1\right) - \left(\partial_{x_0} w - (\partial_{x_0} u)^T y_1\right) - y_1 \\
&= -\partial_{z_0} V y_0 + y_0 - y_1.
\end{aligned}$$

This gives the formula for $y_1 - y_0$. To obtain the formula for $x_1 - x_0$ one proceeds similarly by differentiating V with respect to y_1. The uniqueness of generating functions of type V is clear.

For (ii), let $V\colon \mathbb{R}^{2k+1} \to \mathbb{R}$ be a smooth function satisfying the hypotheses. We construct φ as follows. Let $\mu\colon \mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}$ be the map defined by

$$(x_0, y_1, z_0) \mapsto \left(x_0, (1 - \partial_{z_0} V(x_0, y_1, z_0))^{-1} (y_1 + \partial_{x_0} V(x_0, y_1, z_0)), z_0\right),$$

and $\nu\colon \mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}$ the map defined by

$$(x_0, y_1, z_0) \mapsto (x_0 + \partial_{y_1} V, y_1, z_0 + \langle y_1, \partial_{y_1} V\rangle - V).$$

The hypotheses on V ensure that μ and ν are diffeomorphisms. We define φ to be the composition $\nu \circ \mu^{-1}$. That φ is a contactomorphism is seen directly as follows. Expand the formula for the contact 1-form $\alpha = dz_1 - \sum y_{1i}\, dx_{1i}$. After cancellations this becomes

$$dz_1 - \sum y_{1i}\, dx_{1i} = (1 - \partial_{z_0} V)\, dz_0 - \sum (\partial_{x_{0i}} V + y_{1i})\, dx_{0i}.$$

From the definition of the function y_0 we have the equality

$$\partial_{x_0} V + y_1 = (1 - \partial_{z_0} V)\, y_0$$

hence φ is a contactomorphism as required. □

The pseudo-linear case

When φ is the time-t_1 map of a contact isotopy generated by a quadratic Hamiltonian G the pair (ξ_1, η_1) is given in terms of (ξ_0, η_0) by

$$\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = a\Psi \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} \tag{3.5}$$

where $a > 0$ and

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2k)$$

are as given in Remark 2.10. In this case the condition of φ being C^1-close to the identity is replaced by requirement that

$$\det(D) \neq 0 \tag{3.6}$$

which holds for $t_1 > 0$ sufficiently small. Again, we say a generating function $\mathbb{R}^{2k+1} \to \mathbb{R}$: $(\xi, \eta, \zeta) \mapsto W(\xi, \eta, \zeta)$ is *quadratic* if it is quadratic in ξ and η and linear in ζ. A simple modification of the proof of Proposition 3.1 now gives:

Proposition 3.2 *Every contactomorphism φ: $J^1\mathbb{R}^k \to J^1\mathbb{R}^k$ generated by a quadratic Hamiltonian with* (3.6) *holding admits a unique quadratic generating function of type V.*

Conversely, for each function $W = W(\xi, \eta, \zeta)$ on $\mathbb{R}^{2k+1}$, quadratic in ξ and η and linear in ζ, satisfying

$$1 - \partial_\zeta W \neq 0, \qquad \det(\mathbb{1} + \partial^2_{\xi\eta} W) \neq 0, \tag{3.7}$$

there exists a unique pseudo-linear contactomorphism φ: $J^1\mathbb{R}^k \to J^1\mathbb{R}^k$ such that W is a generating function of type V for φ.

We now give an explicit formula for the quadratic generating function of the pseudo-linear contactomorphism φ above. Observe from (3.5) that when condition (3.6) holds

$$\begin{aligned}\xi_1 - \xi_0 &= (aA - \mathbb{1})\xi_0 + aB\eta_0 \\ &= (aA - \mathbb{1})\xi_0 + BD^{-1}\eta_1 - aBD^{-1}C\xi_0 \\ &= (aA - aBD^{-1}C - \mathbb{1})\xi_0 + BD^{-1}\eta_1 \\ &= \big(a(D^{-1})^T - \mathbb{1}\big)\xi_0 + BD^{-1}\eta_1.\end{aligned}$$

Here the last equality follows from the fact that $\Psi \in \mathrm{Sp}(2k)$. Also note that

$$\begin{aligned}\eta_1 - \eta_0 &= \eta_1 - a^{-1}D^{-1}\eta_1 + D^{-1}C\xi_0 \\ &= D^{-1}C\xi_0 + (\mathbb{1} - aD^{-1})\eta_1 + (a - a^{-1})D^{-1}\eta_1 \\ &= D^{-1}C\xi_0 + (\mathbb{1} - aD^{-1})\eta_1 + (a - a^{-1})D^{-1}(aC\xi_0 + aD\eta_0) \\ &= a^2D^{-1}C\xi_0 + (\mathbb{1} - aD^{-1})\eta_1 + (a^2 - 1)\eta_0.\end{aligned}$$

It follows that the generating function of type V for φ is given by

$$\begin{aligned}W(\xi_0, \eta_1, \zeta_0) = &-\tfrac{1}{2}\langle a^2D^{-1}C\xi_0, \xi_0\rangle + \big\langle\big(a(D^{-1})^T - \mathbb{1}\big)\xi_0, \eta_1\big\rangle \\ &+ \tfrac{1}{2}\langle BD^{-1}\eta_1, \eta_1\rangle + (1 - a^2)\zeta_0.\end{aligned} \tag{3.8}$$

3.2 Discrete-time variational theory

We return to the situation of Chapter 2, that is, M^n is a closed manifold embedded $\mathbb{R}^k$, and $H\colon [0,1] \times J^1\mathbb{R}^k \to \mathbb{R}$ satisfies (2.1) for some function $\overline{H}\colon [0,1] \times J^1M \to \mathbb{R}$. It follows that H will *not* have compact support. Hence we multiply by a smooth cutoff function and assume, by abuse of notation, that H is equal to zero outside a large compact set containing $\bigcup_{t\in[0,1]} \varphi^t(M)$, where φ^t denotes the contact isotopy generated by H, and M the set $M \times \{0\} \times \{0\} \subset J^1\mathbb{R}^k$.

We now construct a family finite dimensional variational families generating $\psi^1(M_0 \times \{0\}) \subset J^1M$, where ψ^t denotes the contact isotopy generated by $\overline{H}$, by discretising time. Pick an integer N and define

$$\varphi_j^{j+1} = \varphi^{\frac{j+1}{N}} \circ \left(\varphi^{\frac{j}{N}}\right)^{-1}$$

for $j = 0, \dots, N-1$. Then

$$\varphi^1 = \varphi_{N-1}^N \circ \varphi_{N-2}^{N-1} \circ \cdots \circ \varphi_0^1$$

and for N sufficiently large each φ_j^{j+1} satisfies the hypotheses of Proposition 3.1 (i). Hence for each j there exists a function $V_j \colon \mathbb{R}^{2k+1} \to \mathbb{R}$ such that

$$\begin{aligned} x_{j+1} - x_j &= \frac{\partial V_j}{\partial y} \\ y_{j+1} - y_j &= -\frac{\partial V_j}{\partial x} - y_j \frac{\partial V_j}{\partial z} \\ z_{j+1} - z_j &= \langle y_{j+1}, x_{j+1} - x_j \rangle - V_j \end{aligned} \tag{3.9}$$

if and only if $(x_{j+1}, y_{j+1}, z_{j+1}) = \varphi_j^{j+1}(x_j, y_j, z_j)$.

Now define, by analogy with the continuous-time case, the space

$$\mathcal{P}^N = \mathcal{P}^N_{\mathbb{R}^k, M} = \left\{ c = (x_0, \dots, x_N, y_1, \dots, y_N) \in \mathbb{R}^{(2N+1)k} \,\middle|\, x_N \in M \right\}$$

of discrete paths in $\mathbb{R}^{2k}$, and to each $c \in \mathcal{P}^N$ associate a sequence $(z_0, \dots, z_N)$ given by the iterative rule

$$z_0 = 0, \qquad z_j = \langle y_j, x_j - x_{j-1} \rangle - V_{j-1}(x_{j-1}, y_j, z_{j-1}) + z_{j-1}, \quad j > 0. \tag{3.10}$$

Let $\pi \colon \mathcal{P}^N \to M$ denote the projection $c \mapsto x_N$. The *discrete action functional* $\Phi^N = \Phi^N_{H,M} \colon \mathcal{P}^N \to \mathbb{R}$ can now be defined by

$$\Phi^N(c) = z_N = \sum_{j=1}^{N} \Big(\langle y_j, x_j - x_{j-1} \rangle - V_{j-1}(x_{j-1}, y_j, z_{j-1}) \Big).$$

Proposition 3.3 *$(\mathcal{P}^N, \Phi^N)$ is a variational family for the Legendrian submanifold $\psi^1(M_0 \times \{0\}) \subset J^1M$.*

This follows from:

Lemma 3.4 *There is a 1-1 correspondence between the set of fibre critical points of Φ^N and solutions of (3.9) (for $j = 0, \ldots, N-1$) with $y_0 = z_0 = 0$, $x_N \in M$.*

Proof. The partial derivatives of Φ^N are

$$\frac{\partial \Phi^N}{\partial x_j} = (1 - \partial_z V_{N-1}) \cdots (1 - \partial_z V_{j+1}) (y_j - y_{j+1} - \partial_x V_j - y_j \partial_z V_j)$$
$$\frac{\partial \Phi^N}{\partial y_{j+1}} = (1 - \partial_z V_{N-1}) \cdots (1 - \partial_z V_{j+1}) (x_{j+1} - x_j - \partial_y V_j)$$

for $j = 0, \ldots, N-1$ (on defining $y_0 = 0$). Since $1 - \partial_z V_l$ is nonzero for each l, this proves the lemma. □

As before, $c \in \mathcal{P}^N$ is a critical point of Φ^N if and only if it is a solution of Hamilton's discrete equations (3.9) and $c_N = (x_N, y_N) \in TM^\perp$. In particular, c is a critical point of Φ^N if and only if it is obtained by sampling a path $c \in \operatorname{crit} \Phi$.

Remark 3.5 In applications it will be useful to note the following. Let $x = x_N$ and $\xi = (x_0, \ldots, x_{N-1}, y_1, \ldots, y_N)$ denote the base and fibre coordinates respectively of $\mathcal{P}^N = M \times \mathbb{R}^{2m}$ where $m = Nk$. Then Φ^N can be written in the form

$$\Phi^N(x, \xi) = \tfrac{1}{2}\langle P\xi, \xi\rangle + W(x, \xi).$$

where $P \in \mathbb{R}^{2m \times 2m}$ is a nondegenerate symmetric matrix of signature zero and W is given by

$$W(x, \xi) = \langle x, y_N \rangle - \sum_{j=1}^{N} V_{j-1}(x_{j-1}, y_j, z_{j-1})$$

Also, the gradient of W with respect to ξ, $\partial_\xi W$, is bounded.

Remark 3.6 Suppose $L \subset J^1M$ is a Legendrian submanifold given by a variational family $(E = M \times \mathbb{R}^l, S)$. Then $\psi^1(L)$ is also given by a variational family. This is seen as follows.

Choose any function $\widetilde{S}\colon \widetilde{E} \to \mathbb{R}$, where $\widetilde{E} = \mathbb{R}^k \times \mathbb{R}^l$, satisfying $\widetilde{S}|_E = S$, and define the space

$$\widetilde{\mathcal{P}}^N = \left\{(\xi, c) \in \mathbb{R}^l \times \mathbb{R}^{(2N+1)k} \,\middle|\, c \in \mathcal{P}^N\right\}$$

of augmented discrete paths. Now define the generalised discrete action functional $\widetilde{\Phi}^N\colon \widetilde{\mathcal{P}}^N \to \mathbb{R}$ by

$$\widetilde{\Phi}^N(\xi, c) = z_N = \sum_{j=1}^{N} \Big(\langle y_j, x_j - x_{j-1}\rangle - V_{j-1}(x_{j-1}, y_j, z_{j-1})\Big) + z_0$$

where

$$z_0 = \widetilde{S}(x_0, \xi), \qquad z_j = \langle y_j, x_j - x_{j-1}\rangle - V_{j-1}(x_{j-1}, y_j, z_{j-1}) + z_{j-1}, \quad j > 0.$$

It is now easy to check that $(\widetilde{\mathcal{P}}^N, \widetilde{\Phi}^N)$ is a variational family for $\psi^1(L)$.

Remark 3.7 An alternative and more explicit way of constructing a discrete action functional is by setting

$$V_j(x_j, y_{j+1}, z_j) = H(t_j, x_j, y_{j+1}, z_j)(t_{j+1} - t_j)$$

where $t_j = j/N$ and N is sufficiently large so that the hypotheses of Proposition 3.1 (ii) hold for each V_j. A disadvantage of this method is that it does not generate the original Legendrian submanifold but one that is close to it. This is the approach used by Robbin and Salamon in the symplectic setting [20].

The second variation

We start with some notation. Let $c \in \mathcal{P}^N$ be a critical point of Φ^N and $\{z_j\}$ the associated sequence given by the iterative rule (3.10). For $j = 0, \ldots, N-1$ define $W_{j,\xi\xi}, W_{j,\xi\eta}, W_{j,\eta\xi}, W_{j,\eta\eta} \in \mathbb{R}^{k\times k}$ and $W_{j,\zeta} \in \mathbb{R}$ by

$$\begin{aligned}
W_{j,\xi\xi} &= \partial^2_{xx}V_j + \partial^2_{xz}V_j y_j^T + y_j\partial^2_{zx}V_j + \partial^2_{zz}V_j y_j y_j^T \\
W_{j,\xi\eta} &= \partial^2_{xy}V_j + y_j\partial^2_{zy}V_j \\
W_{j,\eta\xi} &= \partial^2_{yx}V_j + \partial^2_{yz}V_j y_j^T \\
W_{j,\eta\eta} &= \partial^2_{yy}V_j \\
W_{j,\zeta} &= \partial_z V_j
\end{aligned}$$

where V_j is evaluated at (x_j, y_{j+1}, z_j). Note that $W_{j,\xi\xi}, W_{j,\eta\eta}$ are symmetric and $W_{j,\xi\eta} = W_{j,\eta\xi}^T$. Abbreviate $\alpha_j^l = (1-\partial_z V_l)\cdots(1-\partial_z V_j)$ for $l \geq j$, setting it to 1 otherwise. Denote by $W_c^N = W_{c,M}^N$ the tangent space of $\mathcal{P}^N$ at c:

$$W_c^N = \left\{\gamma = (\xi_0, \ldots, \xi_N, \eta_1, \ldots, \eta_N) \in \mathbb{R}^{(2N+1)k} \,\middle|\, \xi_N \in T_{x_N}M\right\}.$$

We now give a formula for the second variation of Φ^N at c. As before, $\Pi(x) \in \mathbb{R}^{k\times k}$ denotes the orthogonal projection onto T_xM.

Proposition 3.8 *The second derivative of Φ^N at a critical point c is of the form*

$$d^2\Phi^N(c)(\gamma_1, \gamma_2) = \langle A^N(c)\gamma_1, \gamma_2\rangle$$

where $\gamma_1, \gamma_2 \in W_c^N$ and the second variation, $A^N(c)\colon W_c^N \to W_c^N\colon (\xi, \eta) \mapsto$

(u,v), *is given by the expressions*[2]

$$\begin{aligned} u_j &= \alpha_{j+1}^{N-1}(\eta_j - \eta_{j+1} - W_{j,\xi\xi}\xi_j - W_{j,\xi\eta}\eta_{j+1} - W_{j,\zeta}\eta_j) \\ v_{j+1} &= \alpha_{j+1}^{N-1}(\xi_{j+1} - \xi_j - W_{j,\eta\xi}\xi_j - W_{j,\eta\eta}\eta_{j+1}) \\ u_N &= (d\Pi(x_N)\xi_N)y_N + \Pi(x_N)\eta_N. \end{aligned}$$

Proof. Differentiate the formulae in the proof of Lemma 3.4. □

Remark 3.9 For $j = 0, \dots, N-1$ let $W_j \colon \mathbb{R}^{2k+1} \to \mathbb{R}$ be the function defined by

$$W_j(\xi_j, \eta_{j+1}, \zeta_j) = \tfrac{1}{2}\langle W_{j,\xi\xi}\xi_j, \xi_j\rangle + \langle W_{j,\eta\xi}\xi_j, \eta_{j+1}\rangle + \tfrac{1}{2}\langle W_{j,\eta\eta}\eta_{j+1}, \eta_{j+1}\rangle + W_{j,\zeta}\zeta_j.$$

Using these we may associate to the critical point c the function $\Phi_c^N = \Phi_{c,H,M}^N \colon W_c^N \to \mathbb{R}$ defined by

$$\Phi_c^N(\gamma) = \sum_{j=1}^{N} \Big(\langle \eta_j, \xi_j - \xi_{j-1}\rangle - W_{j-1}(\xi_{j-1}, \eta_j, \zeta_{j-1}) \Big) + \mathit{II}_{c_N}(\xi_N)$$

where ζ_j is given by the iterative rule

$$\zeta_0 = 0, \qquad \zeta_j = \langle \eta_j, \xi_j - \xi_{j-1}\rangle - W_{j-1}(\xi_{j-1}, \eta_j, \zeta_{j-1}) + \zeta_{j-1}, \quad j > 0 \tag{3.11}$$

and where II_{c_N} denotes the second fundamental form. As in the continuous-time case, if c is a nondegenerate critical point of Φ^N, then 0 is the unique critical point of Φ_c^N, also nondegenerate, and in this case Proposition 3.8 implies that the second variation of Φ^N at c agrees with the second variation of Φ_c^N.

Recall that in the continuous-time case given a critical point c of Φ we can construct a functional Φ_c whose second variation agrees with the second

[2] Define $\eta_0 = 0$.

variation of Φ at c (see Remark 2.9). The following proposition shows how Φ_c^N and Φ_c are related.

Proposition 3.10 *Φ_c^N is the discretisation of Φ_c.*

Proof. Let $(c, z)\colon [0,1] \to J^1M$ be a solution of Hamilton's equations beginning in $L_0 \times \{0\}$ and $(c^N, \{z_j\})$ the corresponding solution of Hamilton's discrete equations. Note that, by Remark 2.7, linearising Hamilton's equations for H along (c, z) and restricting to contact planes, which are canonically identified with $\mathbb{R}^{2k}$, yields the equations

$$\begin{aligned}\dot{\xi} &= G_{\eta\xi}\xi + G_{\eta\eta}\eta \\ \dot{\eta} &= -G_{\xi\xi}\xi - G_{\xi\eta}\eta - G_{\zeta}\eta.\end{aligned}$$

Similarly, linearising Hamilton's discrete equations along $(c^N, \{z_j\})$ and restricting to contact planes yields the equations

$$\begin{aligned}\xi_{j+1} - \xi_j &= W_{j,\eta\xi}\xi_j + W_{j,\eta\eta}\eta_{j+1} \\ \eta_{j+1} - \eta_j &= -W_{j,\xi\xi}\xi_j - W_{j,\xi\eta}\eta_{j+1} - W_{j,\zeta}\eta_j.\end{aligned}$$

It follows that the W_j define a family of generating functions of type V for the pseudo-linear contact isotopy generated by G and the result now follows. □

Corollary 3.11 *$(\mathcal{P}^N, \Phi^N)$ is always transversal as a variational family.*

Proof. We need to show that, given $\gamma = (\xi_0, \dots, \xi_N, \eta_1, \dots, \eta_N) \in W_c^N$ satisfying $\xi_N = 0$, if $d^2\Phi^N(c)(\gamma, \gamma')$ is 0 for all $\gamma' \in W_c^N$ then γ is identically 0. This is seen as follows. Suppose γ satisfies the hypotheses of the statement we are seeking to show, then

$$\begin{aligned}\xi_{j+1} - \xi_j &= W_{j,\eta\xi}\xi_j + W_{j,\eta\eta}\eta_{j+1} \\ \eta_{j+1} - \eta_j &= -W_{j,\xi\xi}\xi_j - W_{j,\xi\eta}\eta_{j+1} - W_{j,\zeta}\eta_j\end{aligned}$$

for $j = 1, \dots, N-1$, and $\Pi(x_N)\eta_N = 0$. Denoting by A_j the conformal symplectic matrix associated to W_j, it follows that

$$\begin{pmatrix} \xi_{j+1} \\ \eta_{j+1} \end{pmatrix} = A_j \begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix}$$

for $j = 1, \dots, N-1$, and that (ξ_N, η_N) is in the symplectic complement of the coisotropic subspace $N_{c_N} = T_{c_N}(T^*\mathbb{R}^k|_M)$. Since φ^t preserves $J^1\mathbb{R}^k|_M$, we have $A_j N_{c_j} = N_{c_{j+1}}$. It follows that $(\xi_0, 0)$ is in the symplectic complement of N_{c_0} which implies that $\xi_0 = 0$ and hence γ is identically 0, as required. $\square$

Corollary 3.12 *The second variation of Φ^N at a critical point c is nondegenerate if and only if $\Psi_{c,H}\Lambda_0$ is transverse to $\Lambda_1 = T_{c_N}(TM^\perp)$.*

Proof. This follows immediately from Corollaries 3.11 and 1.12 and Remark 2.7. $\square$

Remark 3.13 If the variational family (E, S), given in Remark 3.6, is transversal, then the variational family $(\widetilde{\mathcal{P}}^N, \widetilde{\Phi}^N)$ will also be transversal. This follows by arguing as in the proof of Corollary 3.11 and using Lemma 1.11.

Chapter 4

The signature

By considering the spectral flow we show that the Fredholm index of the operator obtained by linearising the gradient equation of Φ along a path connecting two nondegenerate critical points is given as the difference of two Maslov indices (Section 4.2). This, together with a theorem of Robbin and Salamon [20] in the symplectic setting for quadratic Hamiltonians, suggests that the signature of the second variation of the discrete action functional at a nondegenerate critical point should be related to the Maslov index. This is indeed shown to be the case (Section 4.3). In Section 4.4 stabilised Morse theory is used to prove the generalised Morse inequalities. We begin with a brief summary of the Maslov index.

4.1 The Maslov index

In this section we collect together the properties of the Maslov index for various spaces of paths that will be needed later. Proofs may be found in [21].

Let $\Lambda\colon [a,b] \to \mathcal{L}(k)$ be a path of Lagrangian subspaces and let $Z =$

$(X, Y)\colon [a, b] \to \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{2k})$ denote a choice of a frame for Λ. This means that $Z(t)\colon \mathbb{R}^k \to \mathbb{R}^{2k}$ is an injective linear map such that

$$\Lambda(t) = \operatorname{Im} Z(t), \qquad X(t)^T Y(t) = Y(t)^T X(t)$$

for all t. Also, let $V \in \mathcal{L}(k)$ be a fixed Lagrangian subspace. A *crossing* for the path Λ is a number $t \in [a, b]$ such that $\Lambda(t)$ intersects V nontrivially. At a crossing t the *crossing form* is defined to be the quadratic form

$$\Gamma(\Lambda, V, t)\colon \Lambda(t) \cap V \to \mathbb{R}$$

given by

$$v \mapsto \langle X(t)u, \dot{Y}(t)u \rangle - \langle Y(t)u, \dot{X}(t)u \rangle$$

where $v = Z(t)u$. A crossing t is said to be *regular* if $\Gamma(\Lambda, V, t)$ is nonsingular. When the path Λ has only regular crossings the *Maslov index*, $\mu(\Lambda, V)$, is defined by

$$\mu(\Lambda, V) = \tfrac{1}{2} \operatorname{sign} \Gamma(\Lambda, V, a) + \sum_{a<t<b} \operatorname{sign} \Gamma(\Lambda, V, t) + \tfrac{1}{2} \operatorname{sign} \Gamma(\Lambda, V, b)$$

where the sum is taken over all crossings t. By perturbing, keeping endpoints fixed, this definition extends to give a well defined Maslov index for all paths. The integers k_a and k_b are defined by

$$k_a = \dim \Lambda(a) \cap V, \qquad k_b = \dim \Lambda(b) \cap V.$$

We now state some properties of the Maslov index.

(Integrality) The integers μ, k_a and k_b are related by

$$\mu + \tfrac{1}{2}(k_a - k_b) \in \mathbb{Z}.$$

(Product) Under the natural identification of $\mathcal{L}(k) \times \mathcal{L}(k')$ as a submanifold of $\mathcal{L}(k + k')$

$$\mu(\Lambda \oplus \Lambda', V \oplus V') = \mu(\Lambda, V) + \mu(\Lambda', V').$$

(Localisation) For the path Λ given by the frame $t \mapsto (\mathbb{1}, A(t))$ and V the subspace $\mathbb{R}^k \times \{0\}$ the Maslov index is given by

$$\mu(\Lambda, V) = \tfrac{1}{2} \operatorname{sign} A(b) - \tfrac{1}{2} \operatorname{sign} A(a).$$

(Zero) If $\Lambda(t) \cap V$ has constant dimension for all t then $\mu(\Lambda, V) = 0$.

For a pair of Lagrangian paths $\Lambda, \Lambda' \colon [a, b] \to \mathcal{L}(k)$ the *relative crossing form* on $\Lambda \cap \Lambda'$ is defined by

$$\Gamma(\Lambda, \Lambda', t) = \Gamma(\Lambda, \Lambda'(t), t) - \Gamma(\Lambda(t), \Lambda', t).$$

When the pair Λ, Λ' have only regular crossings the *relative Maslov index*, $\mu(\Lambda, \Lambda')$, is defined by

$$\mu(\Lambda, \Lambda') = \tfrac{1}{2} \operatorname{sign} \Gamma(\Lambda, \Lambda', a) + \sum_{a<t<b} \operatorname{sign} \Gamma(\Lambda, \Lambda', t) + \tfrac{1}{2} \operatorname{sign} \Gamma(\Lambda, \Lambda', b). \tag{4.1}$$

The relative Maslov index has the following property.

(Naturality) For a Lagrangian pair Λ, Λ' and a symplectic path Ψ

$$\mu(\Psi\Lambda, \Psi\Lambda') = \mu(\Lambda, \Lambda').$$

Finally, for a path of symplectic matrices $\Psi \colon [a, b] \to \mathrm{Sp}(2k)$ the *Maslov index*, $\mu(\Psi)$, is defined by[1]

$$\mu(\Psi) = \mu(\Psi V, V)$$

where $V = \mathbb{R}^k \times \{0\}$. When Ψ is expressed in the form

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.2}$$

[1] Note that this definition differs from the one given in [21] since here the Maslov index is defined with respect to the horizontal and there with respect to the vertical.

note that the crossing form $\Gamma(\Psi, t)\colon \ker C(t) \to \mathbb{R}$ can be written as

$$\Gamma(\Psi, t)(y) = \langle A(t)y, \dot{C}(t)y\rangle.$$

The following properties of the Maslov index for symplectic paths will be used.

(Multiplication) Let Ψ, Ψ' be symplectic paths then

$$\mu(\Psi\Psi'V, V) = \mu(\Psi(1)\Psi'V, V) + \mu(\Psi\Psi'(0)V, V).$$

(Homotopy) Two symplectic paths Ψ, Ψ' with $\Psi(a) = \Psi'(a)$ and $\Psi(b), \Psi'(b) \in \mathrm{Sp}_0(2k)$ are homotopic within this class if and only if they have the same Maslov index, where $\mathrm{Sp}_0(2k)$ denote the set of matrices $\Psi \in \mathrm{Sp}(2k)$ with $\det C \neq 0$ in the block decomposition (4.2).

4.2 Fredholm theory

Let $c^-, c^+ \in \mathcal{P}$ be two nondegenerate critical points of Φ and $w\colon \mathbb{R} \to \mathcal{E}$ be any smooth family of paths such that

$$\lim_{s\to\pm\infty} w(s,t) = c^{\pm}(t), \qquad \lim_{s\to\pm\infty} \partial_s w(s,t) = 0, \tag{4.3}$$

where the convergence is uniform in t. From the proof of Proposition 1.4 the gradient of $\Phi\colon \mathcal{E} \to \mathbb{R}$ with respect to the standard L^2-metric on paths is given by

$$\operatorname{grad}\Phi(c)(t) = e^{\int_t^1 -\partial_z H(\tau,c,z)\,d\tau}\big(J_0\dot{c}(t) - \nabla H_{t,z}(c) - \partial_z H(t,c,z)L(c)\big),$$

where L denotes the vector field $(y, 0)$. Linearising the equation $\overline{\partial}_H(w) = \partial w/\partial s + \operatorname{grad}\Phi(w)$ in the direction of a vector field ζ along w now leads to the first-order differential operator given by

$$\mathcal{D}_w\zeta = \frac{\partial\zeta}{\partial s} + e^{\int_t^1 -G_\zeta\,d\tau}\left(J_0\frac{\partial\zeta}{\partial t} - S\zeta + \tfrac{1}{2}G_\zeta J_0\zeta\right) + C\zeta \tag{4.4}$$

where C is a multiplication operator arising from the linearisation of the exponential term and from the implicit nature of the contact equations. Here the symmetric matrix valued function $S\colon \mathbb{R}\times[0,1] \to \mathbb{R}^{2k\times 2k}$ and the function $G_\zeta\colon \mathbb{R}\times[0,1] \to \mathbb{R}$ are constructed using (2.4), (2.5) and the family of paths w. We will show that $\mathcal{D}_w$ is Fredholm between the Banach spaces $\mathcal{W}$ and $\mathcal{H}$ where

$$\begin{aligned} \mathcal{H} &= L^2(\mathbb{R}\times[0,1],\mathbb{R}^{2k}) \\ \mathcal{W}_w &= \left\{\zeta \in W^{1,2}(\mathbb{R}\times[0,1],\mathbb{R}^{2k}) \,\middle|\, \zeta(s,0) \in \Lambda_0,\, \zeta(s,1) \in \Lambda(s)\right\} \end{aligned}$$

and where $\Lambda_0 = \mathbb{R}^k \times \{0\}$ and $\Lambda(s) = T_{w(s,1)}(TM^\perp)$. Postponing the definition of the Maslov index, $\mu(\overline{c},\overline{H}) \in n/2 + \mathbb{Z}$, for critical points of $\Phi_{\overline{H}}\colon \mathcal{P}_M \to \mathbb{R}$, we now state the main theorem of this section.

Theorem 4.1 *Let $w\colon \mathbb{R} \to \mathcal{E}$ be a smooth curve satisfying* (4.3) *for nondegenerate critical points $c^-, c^+ \in \mathcal{P}$ of Φ. Then, denoting by $\overline{c}^\pm$ the projections of $c^\pm$ to $\mathcal{P}_M$, the operator $\mathcal{D}_w\colon \mathcal{W}_w \to \mathcal{H}$ is Fredholm and*

$$\operatorname{index} \mathcal{D}_w = \mu(\overline{c}^+,\overline{H}) - \mu(\overline{c}^-,\overline{H}).$$

The proof of Theorem 4.1 will be given later in this section. Next we define the relevant Maslov index.

Let $\overline{c} \in \mathcal{P}_M$ be a critical point of $\Phi_{\overline{H}}$ and denote by $c = (x,y) \in \mathcal{P}$ its unique lift to a critical point of Φ (see Remark 2.4). Induced by c is the path of symplectic matrices

$$\Psi_{c,H}\colon [0,1] \to \mathrm{Sp}(2k)$$

and this reduces, via symplectic reduction of the coisotropic distribution $T(T^*\mathbb{R}^k|_M)$, to the path of symplectic linear transformations

$$\Psi_{\overline{c},\overline{H}}(t)\colon (T_{\overline{c}(0)}(T^*M), -d\lambda_{\mathrm{can}\,\overline{c}(0)}) \longrightarrow (T_{\overline{c}(t)}(T^*M), -d\lambda_{\mathrm{can}\,\overline{c}(t)}).$$

Now define the *Maslov index* by

$$\mu(\bar{c}, \overline{H}) = \mu(\Psi_{\bar{c},\overline{H}}\overline{\Lambda}_0, \overline{\Lambda}_1) \tag{4.5}$$

where $\overline{\Lambda}_0 = T_{x(0)}M$ and $\overline{\Lambda}_1$ is the path obtained by reducing

$$\Lambda_1(t) = \left\{(\xi, \eta) \in \mathbb{R}^{2k} : \xi \in T_{x(t)}M,\ \Pi(x(t))(d\Pi(x(t))\xi)y(t) + \Pi(x(t))\eta = 0\right\}. \tag{4.6}$$

Here the right hand side of (4.5) is defined by choosing a unitary trivialisation of the vector bundle $\bar{c}^*T(T^*M)$. By the naturality property this is independent of the trivialisation chosen. By (4.1)

$$\mu(\bar{c}, \overline{H}) \in \frac{n}{2} + \mathbb{Z}.$$

Remark 4.2 The induced Riemannian structure on T^*M, via the embedding $M \hookrightarrow \mathbb{R}^k$, induces a splitting of $T_p(T^*M)$ into a horizontal and a vertical space:

$$T_p(T^*M) = H_p \oplus V_p,$$

given by

$$H_p = \left\{(\xi, (d\Pi(x)\xi)y) \in \mathbb{R}^{2k} : \xi \in T_xM\right\}$$
$$V_p = \left\{(0, \eta) \in \mathbb{R}^{2k} : \eta \in T_xM\right\}$$

where $p = (x, y)$. In this notation $\overline{\Lambda}_1(t)$ is just the horizontal subspace $H_{\bar{c}(t)}$.

Lemma 4.3 *Let $\bar{c} \in \mathcal{P}_M$ and $c \in \mathcal{P}$ be critical point of $\Phi_{\overline{H}}$ and Φ respectively and related as above. Then there is an equality*

$$\mu(\bar{c}, \overline{H}) = \mu(\Psi_{c,H}\Lambda_0, \Lambda_1(1)) + \tfrac{1}{2}\operatorname{sign} \mathit{\Pi}_{c(1)}$$

where $\Lambda_1\colon [0,1] \to \mathcal{L}(k)$ is given by (4.6).

The proof of this lemma uses the following fact about symplectic reductions.

Proposition 4.4 *Let $\Lambda_0, \Lambda_1\colon [0,1] \to \mathcal{L}(k)$ be a pair of Lagrangian paths and $N \subset \mathbb{R}^{2k}$ a coisotropic subspace such that*

$$\Lambda_0(t) \cap N^\omega = \{0\}, \qquad \Lambda_1(t) \subset N.$$

Then the Maslov index of the pair (Λ_0, Λ_1) agrees with the Maslov index of the reduced pair $(\overline{\Lambda}_0, \overline{\Lambda}_1)$:

$$\mu(\Lambda_0, \Lambda_1) = \mu(\overline{\Lambda}_0, \overline{\Lambda}_1).$$

Proof. By choosing a basis for the isotropic subspace N^ω and then extending it to a basis of the Lagrangian subspace $\Lambda_1(t)$ we can assume that

$$N = \mathbb{R}^n \times \{0\} \times \mathbb{R}^n \times \mathbb{R}^{k-n}$$
$$\Lambda_1 = \mathbb{R}^n \times \{0\} \times \{0\} \times \mathbb{R}^{k-n}$$

for all t. Hence it is sufficient to check that the assertion of the proposition holds when only Λ_0 is allowed to vary and Λ_1 and N are as above. Since $\Lambda_0(t) \cap N^\omega = \{0\}$ it follows that we may choose a Lagrangian frame for Λ_0 of the form $Z = (X, Y)\colon [0,1] \to \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{2k})$ where

$$X = \begin{pmatrix} \overline{X} & A \\ 0 & \mathbb{1} \end{pmatrix}, \qquad Y = \begin{pmatrix} \overline{Y} & B \\ C & D \end{pmatrix}$$

and thus a Lagrangian frame for $\overline{\Lambda}_0$ is given by $\overline{Z} = (\overline{X}, \overline{Y})\colon [0,1] \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{2n})$. We compute the crossing form for the pair (Λ_0, Λ_1).

A crossing for (Λ_0, Λ_1) is a number $t \in [0,1]$ such that there is a $u = (u', 0) \in \mathbb{R}^k$ such that $\overline{Y}(t)u' = 0$, that is, if and only if t is a crossing for

$(\overline{\Lambda}_0, \overline{\Lambda}_1)$. At a crossing t the crossing form is given by

$$\begin{aligned}\Gamma(\Lambda_0, \Lambda_1, t)(v) &= \langle X(t)u, \dot{Y}(t)u \rangle - \langle Y(t)u, \dot{X}(t)u \rangle \\ &= \left\langle \overline{X}(t)u', \dot{\overline{Y}}(t)u' \right\rangle \\ &= \Gamma(\overline{\Lambda}_0, \overline{\Lambda}_1, t)(v')\end{aligned}$$

where $v = Z(t)u$, $v' = \overline{Z}(t)u'$. This proves the proposition. □

Proof of Lemma 4.3. In view of Proposition 4.4 it is sufficient to show that

$$\mu(\Psi_{c,H}\Lambda_0, \Lambda_1) = \mu(\Psi_{c,H}\Lambda_0, \Lambda_1(1)) + \tfrac{1}{2}\operatorname{sign} I\!I_{c(1)}.$$

Let $\Phi\colon [0,1] \to \mathrm{Sp}(2k)$ be a path of symplectic matrices satisfying

$$\Lambda_1(t) = \Phi(t)\Lambda_1(1)$$

and abbreviate $\Psi = \Psi_{c,H}$. Then

$$\begin{aligned}\mu(\Psi\Lambda_0, \Lambda_1) &= \mu(\Psi\Lambda_0, \Phi\Lambda_1(1)) \\ &= \mu(\Phi^{-1}\Psi\Lambda_0, \Lambda_1(1)) \\ &= \mu(\Phi(1)^{-1}\Psi\Lambda_0, \Lambda_1(1)) + \mu(\Phi^{-1}\Psi(0)\Lambda_0, \Lambda_1(1)) \\ &= \mu(\Psi\Lambda_0, \Lambda_1(1)) + \mu(\Lambda_0, \Lambda_1)\end{aligned}$$

where the third equality is by the multiplication property of the Maslov index. By multiplying by a path of unitary matrices if necessary we may assume that

$$\Lambda_1(t) = \left\{(\xi, 0, -B(t)\xi, \eta) \in \mathbb{R}^{2k} : \xi \in \mathbb{R}^n,\ \eta \in \mathbb{R}^{k-n}\right\}$$

where $B\colon [0,1] \to \mathbb{R}^{n\times n}$ is a path of symmetric matrices with $B(0) = 0$. Now by the product and localisation properties of the Maslov index

$$\mu(\Lambda_0, \Lambda_1) = \tfrac{1}{2}\operatorname{sign} B(1) = \tfrac{1}{2}\operatorname{sign} I\!I_{c(1)}.$$

This proves the lemma. □

The proof of Theorem 4.1 is based on a modified version of Theorem 7.42 in [22] by Robbin and Salamon. We explain this next.

Denote by $A_S(s)\colon C^\infty([0,1],\mathbb{R}^{2k}) \to C^\infty([0,1],\mathbb{R}^{2k})$ the family of operators given by

$$(A_S(s))(\gamma)(t) = e^{\int_t^1 -G_\zeta(s,\tau)\,d\tau}\left(J_0\dot\gamma(t) - S(s,t)\gamma(t) + \tfrac{1}{2}G_\zeta(s,t)J_0\gamma(t)\right)$$

where $S\colon \mathbb{R}\times[0,1] \to \mathbb{R}^{2k\times 2k}$ is a family of matrices and $G_\zeta\colon \mathbb{R}\times[0,1]\to\mathbb{R}$ is fixed. Using this define the first-order differential operator $\mathcal{D}_S\colon \mathcal{W}\to\mathcal{H}$ by

$$\mathcal{D}_S\zeta = \frac{\partial\zeta}{\partial s} + e^{\int_t^1 -G_\zeta\,d\tau}\left(J_0\frac{\partial\zeta}{\partial t} - S\zeta + \tfrac{1}{2}G_\zeta J_0\zeta\right)$$

where

$$\mathcal{W} = \left\{\zeta\in W^{1,2}(\mathbb{R}\times[0,1],\mathbb{R}^{2k})\,\middle|\,\zeta(s,0),\,\zeta(s,1)\in\Lambda_0\right\}.$$

We make the following assumptions (compare conditions (CR-2) and (CR-3) in [22]):

(A) There exist symmetric matrices $S^\pm\colon [0,1]\to\mathbb{R}^{2k\times 2k}$ such that

$$\lim_{s\to\pm\infty}\sup_{0\le t\le 1}\|S(s,t)-S^\pm(t)\| = 0.$$

(B) Denote by $\Psi^\pm\colon [0,1]\to \mathrm{Sp}(2k)$ the path given by

$$\dot\Psi^\pm = -J_0S^\pm\Psi^\pm, \qquad \Psi^\pm(0) = \mathbb{1}.$$

The Lagrangian subspace $\Psi^\pm(1)\Lambda_0$ is transverse to Λ_0.

Theorem 4.5 *Assume the above. Then $\mathcal{D}_S\colon\mathcal{W}\to\mathcal{H}$ is Fredholm and*

$$\operatorname{index}\mathcal{D}_S = \mu(\Psi^+) - \mu(\Psi^-).$$

We first prove Theorem 4.5 in a special case.

Theorem 4.6 *Assume that* $S\colon \mathbb{R}\times[0,1]\to\mathbb{R}^{2k\times 2k}$ *is a family of symmetric matrices and that* (A) *and* (B) *hold. Assume further that* $s\mapsto\Psi(s,1)$ *has only regular crossings where* $\Psi\colon\mathbb{R}\times[0,1]\to\mathrm{Sp}(2k)$ *is given by* (4.7) *below. Then* $\mathcal{D}_S\colon\mathcal{W}\to\mathcal{H}$ *is Fredholm and*

$$\operatorname{index}\mathcal{D}_S=\mu(\Psi^+)-\mu(\Psi^-).$$

Theorem 4.6 follows from a spectral flow argument which we describe next.

Denote by $W^{1,2}_{\Lambda_0}$ the Hilbert space

$$W^{1,2}_{\Lambda_0}=\left\{\gamma\in W^{1,2}([0,1],\mathbb{R}^{2k})\,\middle|\,\gamma(0),\,\gamma(1)\in\Lambda_0\right\}.$$

Since the inclusion $W^{1,2}_{\Lambda_0}\subset L^2([0,1];\mathbb{R}^{2k})$ is compact the self-adjoint operator $A_S(s)$ with dense domain $W^{1,2}_{\Lambda_0}$ has compact resolvent and thus has a discrete spectrum consisting of real eigenvalues of finite multiplicity. These eigenvalues occur in continuous families. This means there exist functions

$$\lambda_j\colon\mathbb{R}\to\mathbb{R}$$

such that $\lambda_j(s)$ is an eigenvalue of $A_S(s)$ for every $s\in\mathbb{R}$. For an operator family of this form the spectral flow simply counts the number of eigenvalue families λ_j that cross from negative to positive as s goes from $-\infty$ to $+\infty$ minus the number of eigenvalue families that cross from positive to negative. It can be defined abstractly using the crossing operator.

Let $W\hookrightarrow H$ be a compact dense inclusion of real Hilbert spaces. Then a crossing for an operator family $A\colon\mathbb{R}\to\mathcal{L}(W,H)$ is a number $s\in\mathbb{R}$ for which $A(s)$ fails to be injective. At a crossing s the crossing operator is defined to be

$$\Gamma(A,s)=P\dot{A}(s)P\big|_{\ker A(s)}\colon\ \ker A(s)\to\ker A(s)$$

where P denotes the orthogonal projection onto the kernel of $A(s)$. As for the Maslov index, a crossing s is said to be regular if $\Gamma(A,s)$ is nonsingular. For an operator family with only regular crossings the spectral flow is defined as

$$\mu(A) = \sum_s \operatorname{sign}\Gamma(A,s)$$

where the sum is taken over all crossings s. An operator family having only regular crossings is a generic condition. Further details are given in [22].

For our given operator family, $A_S(s)$, injectivity holds precisely when

$$\Psi(s,1)\Lambda_0 \cap \Lambda_0 = \{0\}.$$

Here $\Psi(s,t) \in \mathrm{Sp}(2k)$ is given by

$$\frac{\partial\Psi}{\partial t}(s,t) = -J_0S(s,t)\Psi(s,t), \qquad \Psi(s,0) = \mathbb{1}. \tag{4.7}$$

When this operator is not injective elements in the kernel are of the form

$$\zeta(t) = e^{\int_0^t -\frac{1}{2}G_\zeta(s,\tau)\,d\tau}\Psi(s,t)v, \qquad v = (\xi,0), \qquad C(s,1)\xi = 0, \tag{4.8}$$

where we denote

$$\Psi(s,t) = \begin{pmatrix} A(s,t) & B(s,t) \\ C(s,t) & D(s,t) \end{pmatrix},$$

and the crossing operator, when thought of as a quadratic form on the kernel of A_S, can be written as

$$\begin{aligned}\Gamma(A_S,s)(\zeta) &= \int_0^1 \langle \zeta, \dot{A}_S(s)\zeta\rangle\,dt \\ &= \int_0^1 -e^{\int_t^1 -G_\zeta\,d\tau}\langle \zeta, (\partial_s S - \tfrac{1}{2}\partial_s G_\zeta J_0)\zeta\rangle\,dt. \end{aligned} \tag{4.9}$$

Proof of Theorem 4.6. Since the domain of $A_S(s)$ does not depend on s we can apply Theorem 3.12 in [22] to conclude that $\mathcal{D}_S$ is Fredholm and moreover by Theorem 4.21, also in [22], that the Fredholm index of $\mathcal{D}_S$ is given by the spectral flow of A_S. Thus[2]

$$\begin{aligned}\operatorname{index}\mathcal{D}_S &= \mu(A_S)\\ &= \mu(\Psi(\cdot,1))\\ &= \mu(\Psi^+) - \mu(\Psi^-).\end{aligned}$$

Here the second equality follows from Lemma 4.7 below and the last by considering the contractible loop of Lagrangian subspaces $\Psi\Lambda_0$ around the boundary of $[-T,T]\times[0,1]$ for T sufficiently large. □

Lemma 4.7 *The crossing operator of the spectral flow for A_S is related to the crossing operator of the Maslov index for $\Psi(\cdot,1)$ by*

$$\Gamma(A_S,s)(\zeta) = f(s)\Gamma(\Psi(\cdot,1),s)(\xi)$$

where γ and ξ are related by (4.8) *and $f(s) = e^{\int_0^1 -G_\zeta(s,\tau)\,d\tau}$.*

Proof. Differentiate the identity

$$S\Psi = J_0\partial_t\Psi$$

with respect to s and multiply on the left by Ψ^T to obtain

$$\Psi^T\partial_s S\Psi + \Psi^T S\partial_s\Psi = \Psi^T J_0\partial_s\partial_t\Psi.$$

Now integrate by parts to obtain

$$\begin{aligned}\int_0^1 \Psi^T\partial_s S\Psi\,dt &= \int_0^1 \Big(\Psi^T J_0\partial_t\partial_s\Psi - \Psi^T S\partial_s\Psi\Big)\,dt\\ &= \int_0^1 -\Big((\partial_t\Psi)^T J_0\partial_s\Psi + \Psi^T S\partial_s\Psi\Big)\,dt + \Psi(s,1)^T J_0\partial_s\Psi(s,1)\\ &= \Psi(s,1)^T J_0\partial_s\Psi(s,1).\end{aligned}$$

[2]Notice that our signs disagree with those in [22]. This is because we consider the operator $D_A = d/ds + A$ and they the operator $D_A = d/ds - A$.

From (4.9) and (4.8) note that the crossing operator $\Gamma(A_S, s)$ can be written as

$$\begin{aligned}\Gamma(A_S,s)(\zeta) &= \int_0^1 -f(s)\Big\langle \Psi(s,t)v, \Big(\partial_s S(s,t) - \tfrac{1}{2}\partial_s G_\zeta(s,t)J_0\Big)\Psi(s,t)v\Big\rangle\, dt \\ &= \int_0^1 -f(s)\big\langle \Psi(s,t)v, \partial_s S(s,t)\Psi(s,t)v\big\rangle\, dt.\end{aligned}$$

Thus

$$\begin{aligned}\Gamma(A_S,s)(\zeta) &= -f(s)\langle \Psi(s,1)v, J_0\partial_s\Psi(s,1)v\rangle \\ &= f(s)\langle A(s,1)\xi, \partial_s C(s,1)\xi\rangle \\ &= f(s)\Gamma(\Psi(\cdot,1),s)(\xi)\end{aligned}$$

where $v = (\xi, 0)$. □

Proof of Theorem 4.5. Let $\beta\colon \mathbb{R} \to [0,1]$ be a smooth cutoff function such that $\beta(s) = 1$ for $s \geq T$ and $\beta(s) = 0$ for $s \leq -T$. Define $S'\colon \mathbb{R}\times[0,1] \to \mathbb{R}^{2k\times 2k}$ by

$$S'(s,t) = \beta(s)S^+(t) + (1-\beta(s))S^-(t).$$

Now $\mathcal{D}_S$ and $\mathcal{D}_{S'}$ differ by a zeroth order operator which, by Lemma 3.18 in [22], is compact thus it follows that $\mathcal{D}_S$ is Fredholm and has the same index as $\mathcal{D}_{S'}$. The proof is completed by perturbing S' and appealing to Theorem 4.6. □

Proof of Theorem 4.1. Choose a family of unitary matrices $\Phi\colon \mathbb{R}\times[0,1] \to U(k)$ such that

$$\Phi(s,1)^{-1}\Lambda(s) = \Lambda_0, \qquad \Phi(s,0) = \mathbb{1}$$

for all $s \in \mathbb{R}$ and such that $\partial\Phi/\partial s$ tends to zero as $|s|$ tends to infinity. Now use the family Φ to transform $\mathcal{D}_w$ into $\mathcal{D}_{S'} + C' = \Phi^{-1}\circ\mathcal{D}_w\circ\Phi$ where S' is given by

$$S' = -e^{\int_t^1 G_\zeta\, d\tau}\Phi^{-1}\frac{\partial\Phi}{\partial s} - \Phi^{-1}J_0\frac{\partial\Phi}{\partial t} + \Phi^{-1}S\Phi$$

and where, by Lemma 3.18 in [22], $C' = \Phi^{-1}C\Phi$ is a compact perturbation. Note that the limit matrices, $S'^{\pm}(t)$, are symmetric and thus, by Theorem 4.5, that the operator $\mathcal{D}_w$ is Fredholm with the same index as $\mathcal{D}_{S'}$. Also note that the symplectic matrices $\Psi'^{\pm} \in \mathrm{Sp}(2k)$ associated to $S'^{\pm}$ satisfy

$$\Psi'^{\pm}(t) = \Phi^{\pm}(t)^{-1}\Psi^{\pm}(t)$$

where $\Phi^{\pm}(t) = \lim_{s\to\pm\infty}\Phi(s,t)$, $\Psi^{\pm} = \Psi_{c^{\pm},H}$. Thus

$$\begin{aligned}
\mu(\Psi'^{\pm}) &= \mu((\Phi^{\pm})^{-1}\Psi^{\pm}\Lambda_0, \Lambda_0) \\
&= \mu(\Phi^{\pm}(1)^{-1}\Psi^{\pm}\Lambda_0, \Lambda_0) + \mu((\Phi^{\pm})^{-1}\Psi^{\pm}(0)\Lambda_0, \Lambda_0) \\
&= \mu(\Psi^{\pm}\Lambda_0, \Lambda^{\pm}) - \mu(\Phi^{\pm}).
\end{aligned}$$

Hence

$$\begin{aligned}
\operatorname{index}\mathcal{D}_w &= \operatorname{index}\mathcal{D}_{S'} \\
&= \mu(\Psi'^{+}) - \mu(\Psi'^{-}) \\
&= \mu(\Psi^{+}\Lambda_0, \Lambda^{+}) - \mu(\Psi^{-}\Lambda_0, \Lambda^{-}) - \mu(\Phi^{+}) + \mu(\Phi^{-}) \\
&= \mu(\Psi^{+}\Lambda_0, \Lambda^{+}) - \mu(\Psi^{-}\Lambda_0, \Lambda^{-}) - \mu(\Lambda, \Lambda_0) \\
&= \mu(\Psi^{+}\Lambda_0, \Lambda^{+}) - \mu(\Psi^{-}\Lambda_0, \Lambda^{-}) + \tfrac{1}{2}\operatorname{sign} \Pi_{c^{+}(1)} - \tfrac{1}{2}\operatorname{sign} \Pi_{c^{-}(1)} \\
&= \mu(\overline{c}^{+}, \overline{H}) - \mu(\overline{c}^{-}, \overline{H}).
\end{aligned}$$

Here the last equality follows from Lemma 4.3 and the last but one from the localisation property of the Maslov index. □

4.3 The signature identity

In this section we generalise the signature theorem of Robbin and Salamon in [20] to the case of nondegenerate critical points of the action functional on the 1-jet bundle of a closed manifold.

Theorem 4.8 *Let $c \in \mathcal{P}$ be a nondegenerate critical point of Φ, $\overline{c}$ its projection to $\mathcal{P}_M$ and $c^N \in \mathcal{P}^N$ the corresponding critical point of Φ^N. Then, for sufficiently large N, the signature of the second variation of Φ^N at c^N is given by*

$$\operatorname{sign} A^N(c^N) = 2\mu(\overline{c}, \overline{H}).$$

This strengthens the theorem of Viterbo [26], which states that, in the symplectic case, the difference of the signature of the second variation of Φ^N at two critical points is independent of N, but at the cost of having to take N sufficiently large.

Below we prove a result which implies this theorem.

Assume that $G\colon [0,1] \times J^1\mathbb{R}^k \to \mathbb{R}$ is a quadratic Hamiltonian and denote by $\Psi_G\colon [0,1] \to \mathrm{Sp}(2k)$ the associated path of symplectic matrices given by (2.8). Let $F \subset \mathbb{R}^k$ be a linear subspace and $B\colon F \to F$ a linear transformation such that $\langle B\xi, \xi' \rangle = \langle \xi, B\xi' \rangle$ for all ξ, $\xi' \in F$. Define the Lagrangian subspace

$$\Lambda_{F,B} = \left\{ (\xi, \eta) \in \mathbb{R}^{2k} : \xi \in F,\ B\xi + \Pi_F \eta \right\}$$

Where $\Pi_F \in \mathbb{R}^{k\times k}$ denotes the orthogonal projection onto F. Now define the function $\Phi^N_{G,F,B}\colon W^N_F \to \mathbb{R}$ by

$$\Phi^N_{G,F,B}(\gamma) = \sum_{j=1}^{N} \Big(\langle \eta_j, \xi_j - \xi_{j-1} \rangle - W_{j-1}(\xi_{j-1}, \eta_j, \zeta_{j-1}) \Big) + \tfrac{1}{2}\langle B\xi_N, \xi_N \rangle$$

where

$$W^N_F = \left\{ \gamma = (\xi_0, \ldots, \xi_N, \eta_1, \ldots, \eta_N) \in \mathbb{R}^{(2N+1)k} \,\middle|\, \xi_N \in F \right\}$$

and where the W_j are computed using (3.8), and the ζ_j using (3.11). Denote by $A^N_{G,F,B}$ the second variation of $\Phi^N_{G,F,B}$.

In view of Proposition 3.10 and Lemma 4.3, Theorem 4.8 now follows from:

Theorem 4.9 *Let G, F and B be as above and assume that $\Psi_G(1)\Lambda_0$ is transverse to $\Lambda_{F,B}$. Then, for N sufficiently large, the signature of the second variation of $\Phi^N_{G,F,B}$ is given by*

$$\operatorname{sign} A^N_{G,F,B} = 2\mu(\Psi_G\Lambda_0, \Lambda_{F,B}) + \operatorname{sign} B.$$

The proof we give is based on that given by Robbin and Salamon [20] in the case where $F = \mathbb{R}^k$, $B = 0$ and G is independent of z.

Proof. Without loss of generality assume that

$$\Lambda_{F,B} = \left\{(\xi', 0, -B\xi', \eta'') \in \mathbb{R}^{2k} : \xi' \in \mathbb{R}^n,\ \eta'' \in \mathbb{R}^{k-n}\right\}$$

where $\dim F = n$ and where $B \in \mathbb{R}^{n\times n}$ is symmetric.

The proof of the theorem now proceeds in three steps.

Step 1. The theorem holds in the case

$$G(t, \xi', \xi'', \eta', \eta'', \zeta) = \tfrac{1}{2}\langle G_{\xi'\xi'}(t)\xi', \xi'\rangle.$$

In this case the path of symplectic matrices associated to G is given by

$$\Psi_G(t) = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ C(t) & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}$$

where

$$C(t) = \int_0^t -G_{\xi'\xi'}(\tau)\, d\tau.$$

Thus, from the product and the localisation properties of the Maslov index,

$$\mu(\Psi_G\Lambda_0, \Lambda_{F,B}) = \tfrac{1}{2}\operatorname{sign}(C(1) + B) - \tfrac{1}{2}\operatorname{sign} B.$$

Also, from (3.8),

$$W_j(\xi_j, \eta_{j+1}, \zeta_j) = \tfrac{1}{2}\langle W_{j,\xi'\xi'}\xi'_j, \xi'_j\rangle,$$

and defining

$$C^j = \sum_{i=0}^{j-1} -W_{i,\xi'\xi'}$$

for $j = 1, \ldots, N$, it is easy to see that $C^N = C(1)$. Hence the second variation is given by

$$\langle A^N_{G,F,B}\gamma, \gamma\rangle = 2\sum_{j=1}^{N}\langle \eta_j, \xi_j - \xi_{j-1}\rangle - \sum_{j=0}^{N-1}\langle W_{j,\xi'\xi'}\xi'_j, \xi'_j\rangle + \langle B\xi'_N, \xi'_N\rangle.$$

Now define the following system of coordinates on $W^N_F = \mathbb{R}^{2Nk+n}$:

$$\begin{aligned}
u'_j &= \xi'_j - \xi'_{j-1} \\
u''_j &= \xi''_j - \xi''_{j-1} \\
v'_j &= \eta'_j - \tfrac{1}{2}C^j(\xi'_j + \xi'_{j-1}) \\
v''_j &= \eta''_j \\
w &= \xi'_N.
\end{aligned}$$

In these coordinates the L^2-inner product of $u = (u'_1, u''_1, \ldots, u'_N, u''_N)$ and $v = (v'_1, v''_1, \ldots, v'_N, v''_N)$ is given by

$$\begin{aligned}
\langle u, v\rangle &= \sum_{j=1}^{N}\langle u'_j, v'_j\rangle + \sum_{j=1}^{N}\langle u''_j, v''_j\rangle \\
&= \sum_{j=1}^{N}\langle \xi'_j - \xi'_{j-1}, \eta'_j\rangle - \tfrac{1}{2}\sum_{j=1}^{N}\langle \xi'_j - \xi'_{j-1}, C^j(\xi'_j + \xi'_{j-1})\rangle \\
&\quad + \sum_{j=1}^{N}\langle \xi''_j - \xi''_{j-1}, \eta''_j\rangle \\
&= \sum_{j=1}^{N}\langle \xi_j - \xi_{j-1}, \eta_j\rangle + \tfrac{1}{2}\sum_{j=1}^{N}\langle C^j\xi'_{j-1}, \xi'_{j-1}\rangle - \tfrac{1}{2}\sum_{j=1}^{N}\langle C^j\xi'_j, \xi'_j\rangle \\
&= \sum_{j=1}^{N}\langle \xi_j - \xi_{j-1}, \eta_j\rangle + \tfrac{1}{2}\sum_{j=0}^{N-1}\langle C^{j+1}\xi'_j, \xi'_j\rangle - \tfrac{1}{2}\sum_{j=0}^{N}\langle C^j\xi'_j, \xi'_j\rangle \\
&= \sum_{j=1}^{N}\langle \xi_j - \xi_{j-1}, \eta_j\rangle + \tfrac{1}{2}\sum_{j=0}^{N-1}\langle (C^{j+1} - C^j)\xi'_j, \xi'_j\rangle - \tfrac{1}{2}\langle C^N\xi'_N, \xi'_N\rangle
\end{aligned}$$

$$= \sum_{j=1}^{N} \langle \xi_j - \xi_{j-1}, \eta_j \rangle - \tfrac{1}{2} \sum_{j=0}^{N-1} \langle W_{j,\xi'\xi'} \xi'_j, \xi'_j \rangle - \tfrac{1}{2} \langle C^N \xi'_N, \xi'_N \rangle$$
$$= \tfrac{1}{2} \langle A^N_{G,F} \gamma, \gamma \rangle - \tfrac{1}{2} \langle (C^N + B) w, w \rangle.$$

Thus the second variation satisfies

$$\tfrac{1}{2} \langle A^N_{G,F,B} \gamma, \gamma \rangle = \langle u, v \rangle + \tfrac{1}{2} \langle (C^N + B) w, w \rangle$$

and hence

$$\begin{aligned} \operatorname{sign} A^N_{G,F,B} &= \operatorname{sign} \left(C^N + B \right) \\ &= \operatorname{sign}(C(1) + B) \\ &= 2\mu(\Psi_G \Lambda_0, \Lambda_{F,B}) + \operatorname{sign} B. \end{aligned}$$

Step 2. The theorem holds for quadratic Hamiltonians independent of ζ.

Let G be a ζ-independent Hamiltonian with $\Psi_G(1)\Lambda_0$ transverse to $\Lambda_{F,B}$ and suppose, for a given value of k', that $G'\colon \mathbb{R}^{2k'} \to \mathbb{R}$ is any symplectic shear of the form

$$G'(\xi', \eta') = \tfrac{1}{2} \langle G'_{\xi'\xi'} \xi', \xi' \rangle$$

where $G'_{\xi'\xi'}$ is a nondegenerate symmetric matrix of signature zero. (It follows that k' is necessarily even.) From Step 1 it follows that

$$\operatorname{sign} A^N_{G',\mathbb{R}^{k'},0} = 2\mu(\Psi_{G'} \Lambda'_0, \Lambda'_0) = 0$$

where $\Lambda'_0 = \mathbb{R}^{k'} \times \{0\}$. Now define $G_0 = G \oplus G'\colon [0,1] \times J^1\mathbb{R}^{k+k'} \to \mathbb{R}$ by

$$G_0(t, \xi, \xi', \eta, \eta', \zeta) = G(t, \xi, \eta, \zeta) + G'(\xi', \eta').$$

By additivity of the signature

$$\operatorname{sign} A^N_{G_0, F \oplus \mathbb{R}^{k'}, B \oplus 0} = \operatorname{sign} A^N_{G,F,B}$$

and by the product property of the Maslov index

$$\mu(\Psi_{G_0}\Lambda_0 \oplus \Lambda_0', \Lambda_{F,B} \oplus \Lambda_0') = \mu(\Psi_G\Lambda_0, \Lambda_{F,B}).$$

Hence it suffices to prove the theorem for G_0 with F replaced by $\widetilde{F} = F \oplus \mathbb{R}^{k'}$ and B by $\widetilde{B} = B \oplus 0$.

Now, for k' sufficiently large, let G_1 be a symplectic shear of the form considered in Step 1, with F replaced by $\widetilde{F}$, satisfying

$$\mu(\Psi_{G_1}\widetilde{\Lambda}_0, \Lambda_{\widetilde{F},\widetilde{B}}) = \mu(\Psi_{G_0}\widetilde{\Lambda}_0, \Lambda_{\widetilde{F},\widetilde{B}})$$

where $\widetilde{\Lambda}_0 = \Lambda_0 \oplus \Lambda_0'$. This exists by the localisation and integrality properties of the Maslov index. Also, choose a symplectic matrix $\Phi \in \mathrm{Sp}(2k + 2k')$ such that $\Lambda_{\widetilde{F},\widetilde{B}} = \Phi\widetilde{\Lambda}_0$. Then

$$\mu(\Phi^{-1}\Psi_{G_1}\widetilde{\Lambda}_0, \widetilde{\Lambda}_0) = \mu(\Phi^{-1}\Psi_{G_0}\widetilde{\Lambda}_0, \widetilde{\Lambda}_0)$$

and the symplectic path $\Phi^{-1}\Psi_{G_i}$ satisfies

$$\Phi^{-1}\Psi_{G_i}(0) = \Phi^{-1}, \qquad \Phi^{-1}\Psi_{G_i}(1)\widetilde{\Lambda}_0 \cap \widetilde{\Lambda}_0 = \{0\}.$$

Thus by the homotopy property of the Maslov index there exists a homotopy of symplectic paths $\Phi^{-1}\Psi_\lambda\colon [0,1] \to \mathrm{Sp}(2k + 2k')$ between $\Phi^{-1}\Psi_{G_0}$ and $\Phi^{-1}\Psi_{G_1}$ and within the same class. From this we can construct a quadratic Hamiltonian G_λ such that $\Psi_\lambda = \Psi_{G_\lambda}$.

Now choose N sufficiently large such that for each λ the function $\Phi^N_{G_\lambda,\widetilde{F},\widetilde{B}}$ is defined. Then

$$\begin{aligned} \operatorname{sign} A^N_{G_0,\widetilde{F},\widetilde{B}} &= \operatorname{sign} A^N_{G_1,\widetilde{F},\widetilde{B}} \\ &= 2\mu(\Psi_{G_1}\widetilde{\Lambda}_0, \Lambda_{\widetilde{F},\widetilde{B}}) + \operatorname{sign}\widetilde{B} \\ &= 2\mu(\Psi_{G_0}\widetilde{\Lambda}_0, \Lambda_{\widetilde{F},\widetilde{B}}) + \operatorname{sign}\widetilde{B}. \end{aligned}$$

Here the first equality follows from Corollary 3.12.

Step 3. The general case.

To the quadratic Hamiltonian

$$G(t,\xi,\eta,\zeta) = \tfrac{1}{2}\langle G_{\xi\xi}(t)\xi,\xi\rangle + \langle G_{\eta\xi}(t)\xi,\eta\rangle + \tfrac{1}{2}\langle G_{\eta\eta}(t)\eta,\eta\rangle + G_\zeta(t)\zeta$$

associate the quadratic ζ-independent Hamiltonian

$$G'(t,\xi,\eta,\zeta) = \tfrac{1}{2}\langle G_{\xi\xi}(t)\xi,\xi\rangle + \langle (G_{\eta\xi}(t) + \tfrac{1}{2}G_\zeta(t)\mathbb{1})\xi,\eta\rangle + \tfrac{1}{2}\langle G_{\eta\eta}(t)\eta,\eta\rangle.$$

Now consider the homotopy $G_\lambda = \lambda G' + (1-\lambda)G$ between G and G'. Notice that $\Psi_{G_\lambda} = \Psi_G$ as symplectic paths for all λ. Step 3 now follows from Corollary 3.12. This completes the proof of the theorem. □

4.4 Proof of the generalised Morse inequalities

From now on we assume that the Hamiltonian $\overline{H}$ is such that $\psi^1(M_0)$ is transverse to $M_0 \times \mathbb{R}$, or equivalently, that the discrete action functional is Morse. Recall from Remark 3.5 that the discrete action functional $\Phi^N\colon M \times \mathbb{R}^{2m} \to \mathbb{R}$ can be written in the form

$$\Phi^N(x,\xi) = \tfrac{1}{2}\langle P\xi,\xi\rangle + W(x,\xi)$$

where $P \in \mathbb{R}^{2m\times 2m}$ is a nondegenerate symmetric matrix of signature zero and where the gradient of W in the direction of the fibre, $\partial_\xi W$, is bounded. We prove the generalised Morse inequalities by studying the critical points of Φ^N. As $M \times \mathbb{R}^{2m}$ is noncompact stabilised Morse theory will be used; see Conley [7].

In order to study the critical points of Φ^N we consider the (negative) gradient flow. This is defined by the equations

$$\frac{d\xi}{ds} = -P\xi - \frac{\partial W}{\partial \xi}(x,\xi), \qquad \frac{dx}{ds} = -\frac{\partial W}{\partial x}(x,\xi).$$

Now induced by P there is a splitting

$$\mathbb{R}^{2m} = E^- \oplus E^+$$

into the negative and positive eigenspaces. It follows that, for some $\delta > 0$,

$$\langle P\xi^-, \xi^- \rangle \leq -\delta|\xi^-|^2$$
$$\langle P\xi^+, \xi^+ \rangle \geq \delta|\xi^+|^2$$

for $\xi^- \in E^-$, $\xi^+ \in E^+$. Thus we can find constants $\varepsilon > 0$ and $R > 0$ such that

$$\frac{d}{ds}|\xi^-|^2 \geq \varepsilon \quad \text{if} \quad |\xi^-| \geq R$$
$$\frac{d}{ds}|\xi^+|^2 \leq -\varepsilon \quad \text{if} \quad |\xi^+| \geq R.$$

Indeed, for $|\xi^-| \geq R$ and R sufficiently large,

$$\frac{d}{ds}\frac{|\xi^-|^2}{2} = \left\langle \xi^-, -P\xi^- - \partial_\xi W \right\rangle \geq \delta|\xi^-|^2 - \sup|\partial_\xi W| \cdot |\xi^-| \geq \varepsilon,$$

and similarly for $|\xi^+|^2 \geq R$. Thus an isolating block, in the sense of Conley, for the compact invariant set Λ of all bounded orbits of the gradient flow, is given by

$$N = \left\{(x, \xi^- + \xi^+) : \xi^\pm \in E^\pm, \, |\xi^\pm| \leq R\right\}$$

with exit set

$$L = \left\{(x, \xi^- + \xi^+) \in N : |\xi^-| = R\right\}$$

for R sufficiently large.

We use the following notation

$$\begin{aligned}
b_k(\Lambda) &= \dim H_k(N, L) \\
b_k(M) &= \dim H_k(M) \\
c_k(\Lambda) &= \#\left\{c \in \Lambda \,|\, d\Phi^N(c) = 0,\ \mathrm{ind}_{\Phi^N}(c) = k\right\} \\
p_k(\Lambda) &= \#\left\{c \in \Lambda \,|\, d\Phi^N(c) = 0,\ \mu(c, \overline{H}) = k\right\}
\end{aligned}$$

where $\mathrm{ind}_{\Phi^N}(c)$ denotes the *Morse index* of Φ^N at c and is defined to be the number of negative eigenvalues of the Hessian $d^2\Phi^N(c)$. Here the numbers $b_k(\Lambda)$ are known as the *Conley-Betti numbers.* These are related to the $c_k(\Lambda)$ by the Morse inequalities.

Theorem 4.10 (Morse inequalities) *For $k = 0, \dots, n + 2m$*

$$c_k(\Lambda) - c_{k-1}(\Lambda) + \cdots \pm c_0(\Lambda) \geq b_k(\Lambda) - b_{k-1}(\Lambda) + \cdots \pm b_0(\Lambda)$$

with equality holding for $k = n + 2m$.

These inequalities are proved in [17].

The Conley–Betti numbers are related to the Betti numbers of M by the Thom isomorphism. Specifically, denote by E^m m-dimensional Euclidean space, by B^m and S^{m-1} the closed unit ball and unit sphere respectively in E^m, and abbreviate $E_0^m = E^m \setminus \{0\}$. Then by homotopy

$$(M\times B^m\times B^m, M\times\partial B^m\times B^m) \simeq (M\times B^m, M\times S^{m-1}) \simeq (M\times E^m, M\times E_0^m).$$

Thus

$$H_k(N, L) = H_k(M \times E^m, M \times E_0^m) = H_{k-m}(M)$$

where the last equality is the Thom isomorphism. In other words

$$b_k(\Lambda) = b_{k-m}(M). \tag{4.10}$$

Proof of Theorem A. By Theorem 4.8

$$c_k = p_{\frac{n}{2}+m-k}.$$

Now combine this and (4.10) with the Morse inequalities. This proves Theorem A, the generalised Morse inequalities. □

Part II

THE CONVERGENCE CONJECTURE

Chapter 5

The gradient flow

In this chapter we restrict our attention to the symplectic case. We begin by defining the gradient flow for the action functional and considering its discrete-time analogue (Sections 5.1 and 5.2). We briefly mention compactness in the space of connecting orbits for the action functional. Through formal considerations we are led to conjecture the existence of compactness in the space of gradient flow lines of the discrete action functional and also the possibility of approximating connecting orbits by gradient flow lines of the discrete action functional (Section 5.3). In the final section, Section 5.4, we discuss a possible approach towards solving the second part of this conjecture.

5.1 The continuous-time case

We begin by recalling some notation from Part I, now specialised to the symplectic case. Let M^n be a closed manifold embedded in $\mathbb{R}^k$, and let $\overline{H}\colon [0,1]\times T^*M \to \mathbb{R}$ be a time-dependent Hamiltonian. Fix any lift $H\colon [0,1]\times T^*\mathbb{R}^k \to$

$\mathbb{R}$ of $\overline{H}$ and let $\Phi\colon \mathcal{E} \to \mathbb{R}$ be the corresponding action functional given by

$$\Phi(c) = \int_0^1 \Big(\langle y, \dot{x} \rangle - H(t,x,y) \Big)\, dt,$$

where $\mathcal{E} = \big\{ c = (x,y)\colon [0,1] \to T^*\mathbb{R}^k \,\big|\, c(0) \in L_0,\ c(1) \in L_1 \big\}$. Here $L_0 = \mathbb{R}^k \times \{0\}$ and $L_1 = TM^\perp$. Denote by φ^t the Hamiltonian isotopy generated by H and assume that $\varphi^1(L_0)$ is transverse to L_1. It follows that all critical points of Φ will be nondegenerate in the sense of Remark 2.6.

On the space of paths consider the L^2-metric given by

$$\langle \gamma, \gamma' \rangle_{L^2} = \int_0^1 \langle \gamma(t), \gamma'(t) \rangle \, dt$$

where $\gamma, \gamma' \in T_c\mathcal{E} = \big\{ \gamma\colon [0,1] \to \mathbb{R}^{2k} \,\big|\, \gamma(0) \in \Lambda_0,\ \gamma(1) \in \Lambda_1 \big\}$. Here $\Lambda_0 = \mathbb{R}^k \times \{0\}$ and $\Lambda_1 = T_{c(1)}L_1$. A gradient flow line of Φ is by definition a family of paths $\mathbb{R} \to \mathcal{E}\colon s \mapsto w(s,\cdot)$ such that $\partial w/\partial s + \operatorname{grad}\Phi(w) = 0$. With the above choice of metric this is equivalent to the partial differential equation

$$\overline{\partial}_H(w) := \frac{\partial w}{\partial s} + J_0 \frac{\partial w}{\partial t} - \nabla H_t(w) = 0 \tag{5.1}$$

for smooth maps $w\colon \mathbb{R} \times [0,1] \to T^*\mathbb{R}^k$ satisfying the Lagrangian boundary conditions $w(s,0) \in L_0$, $w(s,1) \in L_1$. We will only consider solutions of (5.1) with finite energy where the energy of a solution is defined to be

$$E(w) = \int_{-\infty}^{\infty} \int_0^1 \left| \frac{\partial w}{\partial s} \right|^2 .$$

It turns out that the solutions with finite energy are exactly those which satisfy the limit conditions

$$\lim_{s \to \pm\infty} w(s,t) = c^\pm(t), \qquad \lim_{s \to \pm\infty} \partial_s w(s,t) = 0,$$

uniformly in t, for paths $c^\pm \in \operatorname{crit}\Phi$. Another equivalent condition is the existence of constants $\delta > 0$ and $c > 0$ such that

$$|\partial_s w(s,t)| \leq c e^{-\delta |s|}$$

for all $s \in \mathbb{R}$, $t \in [0,1]$; see Salamon [23] for details.

Now denote the moduli space of bounded solutions of (5.1) by

$$\mathcal{M} = \{w\colon \mathbb{R} \to \mathcal{E} \mid w \text{ satisfies (5.1)}, \; E(w) < \infty\}$$

and write

$$\mathcal{M}(c^-, c^+) = \left\{ w \in \mathcal{M} \,\middle|\, \lim_{s \to \pm\infty} w(s, \cdot) = c^{\pm} \right\}$$

for the space of connecting orbits between c^- and c^+. That the moduli spaces have certain compactness properties will follow from Floer's arguments in [10, 11] once we have established an *a priori* C^0-bound for the bounded solutions of (5.1). We do the latter next (*cf.* [18], [19]).

Denote by $B^k(R)$ the closed ball of radius R in $\mathbb{R}^k$ and let $S^{k-1}(R) = \partial B^k(R)$. Since $\mathbb{R}^{2k}$ is foliated by the J_0-convex hypersurfaces $S^{k-1}(R) \times \mathbb{R}^k$ and $\mathbb{R}^k \times S^{k-1}(R)$, and we assume that H has support inside $B^k(R) \times B^k(R)$ for some R, every connecting orbit $w \in \mathcal{M}$ must lie inside $B^k(R) \times B^k(R)$. Indeed, suppose not, then for some $w = (u, v) \in \mathcal{M}$ either u or v leaves $B^k(R)$. Suppose u leaves $B^k(R)$ and suppose $|u|$ achieves its maximum value of $R' > R$ at $(s_0, t_0) \in \mathbb{R} \times [0,1]$. Clearly s_0 must be finite and $t_0 \neq 1$. Also, by Lemma 2.4 in [16], (s_0, t_0) cannot be interior thus we can assume that $t_0 = 0$. Now $L_0 \cap (S^{k-1}(R') \times \mathbb{R}^k)$ is Legendrian in $S^{k-1}(R') \times \mathbb{R}^k$ hence the curve $s \mapsto w(s, 0)$ is tangent to the contact distribution at $(s_0, 0)$. By the Cauchy–Riemann equations w is also tangent to the contact distribution at $(s_0, 0)$. Hence the normal derivative of the subharmonic function $(s,t) \mapsto |u(s,t)|^2$ vanishes at $(s_0, 0)$, contradicting the strong maximum principle. A similar argument shows that v also cannot leave $B^k(R)$.

Denote by $C^\infty(T^*\mathbb{R}^k, H_0)$ the set of smooth Hamiltonians $H\colon [0,1] \times T^*\mathbb{R}^k \to \mathbb{R}$ which agree with H_0 up to second order on the union of trajectories $c([0,1])$ over all $c \in \operatorname{crit} \Phi$. In [12] Floer, Hofer and Salamon prove that there exists a subset $\mathcal{H}_{\text{reg}} \subset C^\infty(T^*\mathbb{R}^k, H_0)$, of second category, such that for every $H \in \mathcal{H}_{\text{reg}}$ the moduli space $\mathcal{M}(c^-, c^+, H)$ is a smooth manifold of finite dimension for all $c^\pm \in \operatorname{crit}\Phi_H$.

We remark that it is not clear whether the above statement continues to hold when one replaces $C^\infty(T^*\mathbb{R}^k, H_0)$ by the set of Hamiltonians which are lifts of time-dependent Hamiltonians on T^*M.

5.2 The discrete-time case

So that statements of convergence are easier to make, we adopt here a slightly different notation to that used in Part I.

Given an integer N denote by Σ^N the set

$$\Sigma^N = \{0, 1/N, 2/N \dots, 1\},$$

and abbreviate $\Sigma_i^N = \Sigma^N \setminus \{i\}$. For $t \in \Sigma^N$ use the shorthand $t^+ = t + 1/N$, $t^- = t - 1/N$. For each $t \in \Sigma^N$ assume that the symplectomorphism $\varphi^{t^+} \circ (\varphi^t)^{-1}$ admits a generating function (of type V) V_t^N. Given a function $f\colon \Sigma^N \to \mathbb{R}$ denote by $\delta f = \delta^N f\colon \Sigma_1^N \to \mathbb{R}$ the difference quotient of f given by

$$\delta f_t = N(f_{t^+} - f_t).$$

In the above notation the discrete path space may be written as

$$\mathcal{P}^N = \left\{c = (x, y) \,\middle|\, x\colon \Sigma^N \to \mathbb{R}^k,\ y\colon \Sigma_0^N \to \mathbb{R}^k,\ x_1 \in M\right\}$$

and the discrete action functional $\Phi^N\colon \mathcal{P}^N \to \mathbb{R}$ as

$$\Phi^N(c) = \sum_{t \in \Sigma_1^N} \Big(\langle y_{t^+}, \delta x_t\rangle - N V_t^N(x_t, y_{t^+})\Big) N^{-1}.$$

On the space of discrete paths we define the discrete L^2-metric by

$$\langle \gamma, \gamma' \rangle_{L^2} = \sum_{t \in \Sigma_1^N} \Big(\langle \xi_t, \xi_t' \rangle + \langle \eta_{t+}, \eta_{t+}' \rangle \Big) N^{-1} + \langle \xi_1, \xi_1' \rangle N^{-1}$$

where $\gamma = (\xi, \eta)$, $\gamma' = (\xi', \eta') \in T_c\mathcal{P}^N$, and where

$$T_c\mathcal{P}^N = \left\{ \gamma = (\xi, \eta) \,\middle|\, \xi \colon \Sigma^N \to \mathbb{R}^k,\ \eta \colon \Sigma_0^N \to \mathbb{R}^k,\ \xi_1 \in T_{x_1}M \right\}.$$

With respect to this matrix a family of discrete paths $w = (u, v) \colon \mathbb{R} \to \mathcal{P}^N$ is a gradient flow line if it solves the system of differential equations[1]

$$\begin{pmatrix} \dot{u}_t \\ \dot{v}_{t+} \end{pmatrix} + J_0 \begin{pmatrix} \delta u_t \\ \delta v_t \end{pmatrix} - N \nabla V_t^N(u_t, v_{t+}) = 0 \qquad (5.2^N)$$

$$\dot{u}_1 + N\Pi(u_1) v_1 = 0$$

where $\Pi(x) \in \mathbb{R}^{k \times k}$ denotes the orthogonal projection onto T_xM. We write this briefly as $\overline{\partial}_H^N(w) = 0$.

In the following lemma, by linear interpolation, we assume that V_t^N is defined for every $t \in [0, 1]$.

Lemma 5.1 *For each $t \in [0,1]$, $NV_t^N \to H_t$ as $N \to \infty$, in the C^∞-topology.*

Proof. Comparing the Hamiltonian difference and differential equations one easily gets C^0-convergence of $N\nabla V_t^N$ to ∇H_t, and thus, since each V_t^N is assumed to vanish outside a compact set, the same is true for NV_t^N and H_t. For higher derivatives use the differentiability theorem of ordinary differential equations. □

5.3 A conjecture

By replacing difference quotients by derivatives, we may think of the gradient equation (5.1) as a limit, in some sense, of the sequence of systems of

[1] Define $v_0 \colon \mathbb{R} \to \mathbb{R}^k$ to be the zero function.

differential equations (5.2^N). In addition, the boundary conditions in the discrete-time case lead naturally in the limit to those in the continuous-time case. Indeed, given any pair of paths $c^{\pm} \in \operatorname{crit}\Phi$ we may naturally associate, for all N sufficiently large, discrete paths, which we also denote by $c^{\pm}$, which are critical for Φ^N, by sampling the continuous ones. In a manner analogous to the continuous-time case we may define the spaces $\mathcal{M}^N(c^-, c^+)$ of gradient flow lines of the discrete action functional between the discrete paths c^- and c^+. Given any gradient flow line $w \in \mathcal{M}^N(c^-, c^+)$, by linear interpolation, we may extend it to a map $\tilde{w} = (\tilde{u}, \tilde{v})\colon \mathbb{R} \times [0,1] \to \mathbb{R}^{2k}$ satisfying (5.2^N) at mesh points and satisfying the boundary conditions $\tilde{v}(s,0) = 0$, $\tilde{u}(s,1) \in M$ for all $s \in \mathbb{R}$. Given a family of discrete paths $w^N = \left(u^N, v^N\right) \in \mathcal{M}^N(c^-, c^+)$, such that w^N converges uniformly to some map $w\colon \mathbb{R} \times [0,1] \to T^*\mathbb{R}^k$ and such that $\dot{w}^N$ also converges uniformly, it follows from (5.2^N) that $\Pi(u_1^N)v_1^N$ tends to zero as $N \to \infty$, giving the required limiting boundary condition.

The above observations, together with the fact that, by Theorems 4.1 and 4.8, for N sufficiently large the dimensions of the moduli spaces $\mathcal{M}(c^-, c^+)$ and $\mathcal{M}^N(c^-, c^+)$ agree, provide motivation for the following.

Conjecture 5.2 (I) *Let $w^N \in \mathcal{M}^N(c^-, c^+)$ be a sequence of gradient flow lines of the discrete action functional. Then there exists a subsequence, still denoted w^N, and sequences $s_j^N \in \mathbb{R}$ such that $w^N(s+s_j^N)$ converges to $w_j \in \mathcal{M}(c_{j-1}, c_j)$, uniformly on compact subsets of $\mathbb{R} \times [0,1]$ where $c_j \in \operatorname{crit}\Phi$ for $j = 0, \dots, m$ and $c_0 = c^-$, $c_m = c^+$.*

(II) *Given any connecting orbit $w \in \mathcal{M}(c^-, c^+)$ there exists a sequence of gradient flow lines $w^N \in \mathcal{M}^N(c^-, c^+)$, for all N sufficiently large, such that w^N converges to w.*

At present we have been unable to generalise the elliptic techniques crucially used in the continuous-time case. Consequently, we can say nothing further on the first part of this conjecture.

A possible way of approaching the second part of the conjecture is by using the implicit function theorem. We consider this in the next section.

5.4 An approach

Roughly speaking, the implicit function theorem says that given an approximate solution of the gradient flow equation of Φ^N there exists a true solution nearby provided that one can show that the linearised operator has a uniformly bounded right inverse independent of N.

A first step in this direction is the following.

An approximate solution

Let $w \in \mathcal{M}(c^-, c^+)$ be a connecting orbit. We construct an approximate solution $w^N \colon \mathbb{R} \to \mathcal{P}^N$ of (5.2) by defining

$$w_t^N(s) = w(s,t), \qquad s \in \mathbb{R},\ t \in \Sigma^N.$$

Lemma 5.3 *For any $\varepsilon > 0$*

$$\left\| \overline{\partial}_H^N(w^N) \right\|_{L^2} \leq \varepsilon$$

for all N sufficiently large.

Sketch of proof. Given $\varepsilon > 0$ choose $T > 0$ sufficiently large so that the L^2-norm of $\overline{\partial}_H^N(w^N)$ restricted to $(-\infty, -T] \times \Sigma^N \cup [T, \infty) \times \Sigma^N$ is at most $\varepsilon/2$ whenever the discretisation is defined. This is essentially a consequence of the exponential decay estimate for w and Lemma 5.1. Then choose N

sufficiently large so that the L^2-norm of $\overline{\partial}_H^N(w^N)$ can also be estimated on $[-T,T] \times \Sigma^N$ by $\varepsilon/2$. □

The linearised operators

Let $w\colon \mathbb{R} \to \mathcal{E}$ be a curve of paths. Linearising (5.1) in the direction of a vector field along w leads to the linear first-order differential operator given by

$$\mathcal{D}_w\zeta = \frac{\partial\zeta}{\partial s} + J_0\frac{\partial\zeta}{\partial t} - S\zeta,$$

where $S\colon \mathbb{R} \times [0,1] \to \mathbb{R}^{2k\times 2k}$ is the symmetric matrix valued function given by

$$S(s,t) = \operatorname{Hess} H_t(w(s,t)).$$

Now, let $w^N = (u,v)\colon \mathbb{R} \to \mathcal{P}^N$ be a curve of discrete paths. Linearising (5.2) in the direction of a vector field along w^N gives rise to the linear first-order differential operator $\zeta \mapsto \mathcal{D}^N_{w^N}\zeta = (f,g)$ given by

$$\begin{pmatrix} f_t \\ g_{t+} \end{pmatrix} = \begin{pmatrix} \dot{\xi}_t \\ \dot{\eta}_{t+} \end{pmatrix} + J_0 \begin{pmatrix} \delta\xi_t \\ \delta\eta_t \end{pmatrix} - S_t^N \begin{pmatrix} \xi_t \\ \eta_{t+} \end{pmatrix}$$

$$f_1 = \dot{\xi}_1 + N(d\Pi(u_1)\xi_1)v_1 + N\Pi(u_1)\eta_1$$

where $S_t^N\colon \mathbb{R} \to \mathbb{R}^{2k\times 2k}$ is the symmetric matrix valued function given by

$$S_t^N(s) = \operatorname{Hess} NV_t^N(u_t(s), v_{t^+}(s))$$

and $\eta_0(s)$ is defined to be zero.

Now let $w \in \mathcal{M}(c^-,c^+)$ be a connecting orbit and $w^N\colon \mathbb{R} \to \mathcal{P}^N$ be a family of curves of discrete paths approximating it. At present we have not been able to show that, after defining appropriate Banach spaces, $D^N = D^N_{w^N}$ has a uniformly bounded right inverse independent of N.

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