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# Steady states in Leith's model of turbulence

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**Abstract**

We present a comprehensive study and full classification of the stationary solutions in Leith's model of turbulence with a generalised viscosity. Three typical types of boundary value problems are considered: Problems 1 and 2 with a finite positive value of the spectrum at the left (right) and zero at the right (left) boundaries of a wave number range, and Problem 3 with finite positive values of the spectrum at both boundaries. Settings of these problems and analysis of existence of their solutions are based on a phase–space analysis of orbits of the underlying dynamical system. One of the two fixed points of the underlying dynamical system is found to correspond to a 'sharp front' where the energy flux and the spectrum vanish at the same wave number. The other fixed point corresponds to the only *exact* power-law solution—the so-called dissipative scaling solution. The roles of the Kolmogorov, dissipative
Q1 and thermodynamic scaling, as well as of sharp front solutions, are discussed.

Keywords: Leith model of turbulence, stationary solutions, solvability of boundary value problems, bottleneck phenomenon

SQ1 (Some figures may appear in colour only in the online journal)

## 1. Introduction

Leith's model of turbulence is a nonlinear degenerate inhomogeneous parabolic equation with absorption of the form [1–3]:

$$\frac{\partial E}{\partial t} = \frac{1}{8}\frac{\partial}{\partial k}\left(k^{\frac{11}{2}}E^{\frac{1}{2}}\frac{\partial}{\partial k}(k^{-2}E)\right) - \nu k^z E, \tag{1}$$

where $E \equiv E(k, t)$ is the one-dimensional energy spectrum, $k$ is the absolute value of the wave number, $\nu = \text{const} > 0$ is a viscosity coefficient and $z = \text{const} > 0$ is the degree of the viscous dissipation. The usual kinematic viscosity corresponds to $z = 2$, friction dissipation —to $z = 0$, hyper-viscous dissipation (often used in numerics)—to $z > 2$. We will be interested in the stationary version of this equation:

$$\frac{1}{8}\frac{\partial}{\partial k}\left(k^{11/2}E^{1/2}\frac{\partial}{\partial k}(E/k^2)\right) = \nu k^z E. \tag{2}$$

Quantity

$$\epsilon(k) = -\frac{1}{8}k^{11/2}E^{1/2}\frac{\partial}{\partial k}(E/k^2) \tag{3}$$

has the meaning of the energy flux through $k$. Clearly $\epsilon(k)$ is always a monotonously decreasing function for $\nu > 0$ and constant for $\nu = 0$.

Transient solutions of the inviscid Leith model, i.e. equation (1) with $\nu = 0$, arising from an initial spectrum compactly supported at low $k$ were investigated in [2, 3]. These solutions precede the formation of a steady cascade in the full Leith model. It was shown that this regime becomes self-similar just before the breaking of the energy conservation (which occurs once the cascade has proceeded far enough to generate a finite flux of energy to $k = \infty$). This regime is interesting because it does not exhibit the scaling inherited from the Kolmogorov spectrum. Namely, the transient spectrum was found to have a power-law asymptotics with an exponent which is smaller than the Kolmogorov index. The self-similar solutions which were analyzed numerically in [2] has a 'sharp' nonlinear front which accelerates explosively reaching $k = \infty$ at a finite time $t = t_*$. In [7] we recovered this result analytically and established the existence of a self-similar solution with a power-law asymptotic on the low-wavenumber end and a sharp boundary on the high-wavenumber end which propagates to infinite wavenumbers in a finite-time $t_*$. It was shown that such a self-similar solution is realised by a heteroclinic orbit of the corresponding dynamical system. It was proven that this solution has a power-law asymptotic with an anomalous exponent $x^*$ which is greater than the Kolmogorov value, $x^* > 5/3$, and less than value $x_2 \approx 1.95$ corresponding to a Hopf bifurcation. The existence of weak solutions (the spectrum evolving from an arbitrary finitely supported initial data) of the initial-boundary value problem was proven and convergence to the self-similar solution as $t \to t_*$ was established. In paper [8], the symmetry analysis was applied to describe all essentially different invariant solutions of the Leith model with or without viscosity.

The present paper is devoted to the study and full classification of the stationary solutions of the Leith model (1). In absence of viscosity, the general stationary solution was found in [2, 3]:

$$E_{P,Q}(k) = ck^2(Pk^{-11/2} + Q)^{2/3}, \tag{4}$$

where $c = (24/11)^{2/3}$ and $P$ and $Q$ are arbitrary constants. For $Q = 0$, this gives the pure Kolmogorov cascade solution, whereas for $P = 0$ this is a pure thermodynamic spectrum. For

the general solution, both the constant flux energy $\epsilon(k) = P = \text{const} \neq 0$ and a thermodynamic part $Q \neq 0$ are present as a nonlinear combination which is a nonlinear mixture of the Kolmogorov cascade (dominating at small $k$) and a thermal Rayleigh–Jeans spectrum (dominating at large $k$). Respectively, solution $E_{P,Q}$ with $P$, $Q \neq 0$ was called a 'warm cascade' spectrum in [2, 3]. Such a warm-cascade solution describes a bottleneck phenomenon of spectrum stagnation near the dissipative scale. It is a prototype of the bottleneck phenomenon in the numerical simulations of the Euler (inviscid) turbulence using spectral methods, where the energy spectrum accumulates at high wavenumbers near the truncation wavenumber [4]. It also similar to a real physical bottleneck phenomenon in superfluid turbulence—an energy accumulation at the classical–quantum crossover scale, an effect predicted in [5].

The bottleneck effect was shown to exist, although in a much milder form, in viscous (Navier–Stokes) fluids too, even without a cut-off wave number [6]. It was explained in [6] by using the fact that the Navier–Stokes are nonlocal in the $k$-space.

In the present paper we will study the stationary solutions of the viscous Leith model, i.e. solutions of equation (1). We will see that the inviscid warm-cascade spectra (4) still play an important role in some relevant asymptotic regimes. However, we will see that in absence of a maximum (cut-off) wave number such spectra (or any milder bottleneck) require presence of an extra energy source at $k = \infty$. This results agrees with the view of [6], nonlocality of interaction in the $k$-space is important—the property absent in the Leith model. However, we will see that the warm-cascade described by the Leith model is still relevant to the situations where a maximum (cut-off) wave number is naturally present.

The present paper is structured as follows. In section 2, we perform the change of independent and dependent variables which transforms the stationary viscous Leith's type model into an autonomous nonlinear ordinary differential equation. The corresponding dynamical system is presented and its fixed points are found and classified. This is followed by an analysis of the inviscid asymptotics and the power-law scalings, and the behaviour of solutions near a sharp front, where both the stationary spectrum $E(k)$ and the flux of energy are vanishing. Section 3 is devoted to a qualitative analysis of the dynamical system based on phase portraits for different values of $z$ found numerically. The solutions are interpreted in terms of their physical meanings as low and high Reynolds number direct and inverse energy cascades. Section 4 is devoted to rigorous proofs of the assertions made in section 3, including the formulation of relevant types of Cauchy problems, studying their solvability and identifying several classes of qualitatively different solutions, their dependence on $z$ and on the initial conditions. The full classification is given in terms of the three sets of the qualitatively different orbits existing for any $z$. A summary and discussion of results is given in section 5.

## 2. Autonomous dynamical system and its basic solutions

### *2.1. The stationary model as an autonomous dynamical system*

To introduce into consideration the autonomous dynamical system, we change variables as

$$s = \ln k/k_0, \quad E = k^{2z-3} f^2, \tag{5}$$

where $k_0$ is the left or right (depending on the particular problem) boundary of the considered $k$-range. Then equation (2) is transformed into the following autonomous ODE

$$2f\frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + 4\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2 + (12z - 19)f\frac{\mathrm{d}f}{\mathrm{d}s} + Df^2 = 8\nu f, \tag{6}$$

where

$$D = (3z - 2)(2z - 5).$$

For the flux (3), we have in terms of $f$ and $s$:

$$\epsilon(s) = -\frac{1}{8}\mathrm{e}^{(3z-2)s}\left[(2z - 5)f^3 + f^2\frac{\mathrm{d}f}{\mathrm{d}s}\right]. \tag{7}$$

Since $\epsilon(k)$ is always a monotonously decreasing function for $\nu > 0$, $\epsilon(s)$ is also always a monotonously decreasing function for $\nu > 0$.

We can write equation (6) in the form of a dynamical system in $2D$ phase space by introducing a new variable $g$ via

$$\frac{\mathrm{d}f}{\mathrm{d}s} = (f + g). \tag{8}$$

Then from (8) and (6) we have

$$2f\frac{\mathrm{d}g}{\mathrm{d}s} + 2f(f + g) + 4(f + g)^2 + (12z - 19)f(f + g) + Df^2 = 8\nu f. \tag{9}$$

Notice that (9) is singular at $f = 0$. To remove the singularity, we introduce a new 'time' variable $\tau$ by

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = f\frac{\mathrm{d}}{\mathrm{d}s}$$

and as a result the dynamical system reads

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = (f + g)f, \tag{10}$$

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} + 2(f + g)^2 + \left(6z - \frac{17}{2}\right)f(f + g) + \frac{D}{2}f^2 = 4\nu f. \tag{11}$$

For equilibria we have either $f = 0$, $g = 0$ or

$$f + g = 0, \quad Df^2 = 8\nu f.$$

Therefore, we always have fixed point $P1 = (0, 0)$, and sometimes also fixed point $P2 = \frac{8\nu}{D}(1, -1)$. The latter exists only for $D > 0$ since from the physics $f$ must be a non-negative function. A linearised version of the dynamical system near the fixed point $P1 = (0, 0)$ reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} f \\ g \end{pmatrix} = 4\nu\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} \tag{12}$$

with degenerate eigenvalues $\lambda_1 = \lambda_2 = 0$ and a single eigenvector $(0, 1)$. Correspondingly, near $P2$ we have the following linearised system

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} f \\ g \end{pmatrix} = \frac{8\nu}{D}\begin{pmatrix} 1 & 1 \\ -\frac{1}{2}D - A & -A \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}, \tag{13}$$

where $A = 6z - \frac{17}{2}$. The eigenvalues are given by

$$\lambda_{\pm} = \frac{4\nu}{D}\left(\frac{19}{2} - 6z \pm \sqrt{24z^2 - 76z + 281/4}\right).$$

The expression under the square root is always positive. However, $P2$ exists only for $z < 2/3$ and $z > 5/2$. We have stability if $z > 19/12$ and instability otherwise. Thus, $P2$ is an unstable node for $z < 2/3$ and a stable node for $z > 5/2$.

### 2.2. Asymptotes to inviscid solutions

Let us assume that $z$ is of order one and not too close to $2/3$ or $5/2$. Then it is clear from (11) that the viscous term can be neglected if $f \gg \nu$ or/and $|g| \gg \nu$. Thus, the general stationary solution for regions $f \gg \nu$ or/and $|g| \gg \nu$ is given by the warm-cascade spectrum (4). If both $P$ and $Q$ are positive then such a spectrum grows unbounded at both small and large $k$. However, we will see later that solutions with either $P$ or $Q$ (but not both simultaneously) negative are also of interest. Cases $P < 0$, $Q > 0$ and $P > 0$, $Q < 0$ correspond to solutions that have a sharp front on the left and the right sides of the $k$-range respectively. Both types are majorating solutions for orbits in the case with $\nu > 0$ and same values of $P$ and $Q$ at large $f$.

### 2.3. Power law scalings

For reference, let us first find power-law solutions in the inviscid case ($\nu = 0$), i.e. $E(k) \sim k^{-x}$. We have

$$f = k^{\frac{3-2z}{2}} E^{1/2} = k^{\frac{3-2z-x}{2}} \tag{14}$$

and respectively

$$g = \partial_s f - f = \frac{1 - 2z - x}{2} k^{\frac{3-2z-x}{2}}. \tag{15}$$

This corresponds to $f(s)$ in the form $f \sim \mathrm{e}^{ps}$ with $p = (3 - x - 2z)/2$. Then from (6) with $\nu = 0$ we have: $6p^2 + (12z - 19)p + D = 0$, i.e. $p = (19 \pm 11)/12 - z$. Here, the plus sign corresponds to the thermodynamic and the minus to the Kolmogorov spectra. For Kolmogorov solution (corresponding to $Q = 0$ in the mixed solution (4)):

$$f_{P,0} = \mathrm{e}^{(2/3-z)s} = k^{2/3-z}. \tag{16}$$

In this case $f_{P,0}$ is an increasing function of $k$ if $z < 2/3$, constant for $z = 2/3$, and decreasing otherwise. For thermodynamic solution ($P = 0$):

$$f_{0,Q} = \mathrm{e}^{(5/2-z)s} = k^{5/2-z}. \tag{17}$$

In this case $f_{0,Q}$ is an increasing function of $k$ if $z < 5/2$, constant for $z = 5/2$, and decreasing otherwise.

Pure power-law Kolmogorov and thermodynamic spectra are not solutions when $\nu \neq 0$. However, in this case there also exists a power law solution—it corresponds the fixed point $P2$. For such a 'viscous scaling' we have $f$ constant so that $x = 3 - 2z$. If the viscous scaling is steeper than Kolmogorov, it will be observed on the low-$k$ side of the spectrum. This occurs for $3 - 2z > 5/3$ i.e. for $z < 2/3$.

For the ratio we have $g/f = (1 - 2z - x)/2$. For Kolmogorov solution:

$$g_{P,0}/f_{P,0} = -1/3 - z. \tag{18}$$

For thermodynamic solution:

$$g_{0,Q}/f_{0,Q} = 3/2 - z. \tag{19}$$

### *2.4. A sharp-front solution*

Let us analyse the behaviour of solutions of equation (2) under the assumption that there exists a finite point $k_*$ where the stationary spectrum $E(k)$ and the energy flux $\epsilon$, are vanishing, which implies

$$E(k_*) = \mathrm{d}E(k)/\mathrm{d}k|_{k=k_*} = 0. \tag{20}$$

Let us seek a solution of equation (2) for $k < k_*$ in the form

$$E(k) = A(k - k_*)^y. \tag{21}$$

Assuming that $k_* - k \ll k_*$, in the leading order in $(k_* - k)/k_*$ by direct calculation we have

$$y(3y/2 - 1)k_*^{11/2-2}A^{1/2}(k - k_*)^{3y/2-2} = 8\nu k_*^z(k - k_*)^y. \tag{22}$$

From here it follows that $y = 4$ and $A = 4\nu^2 k_*^{2z-7}/25$. Therefore the function

$$E(k) = \frac{4\nu^2 k_*^{2z-7}}{25}(k - k_*)^4 \tag{23}$$

satisfies equation (2) and the condition (20). For the phase variables $f$ and $g$ we have respectively

$$f = k_*^{\frac{3-2z}{2}}A^{1/2}(k - k_*)^2, \tag{24}$$

$$g \approx \partial_s f = 2A^{1/2}k_*^{\frac{5-2z}{2}}(k - k_*) = -(\nu f/5)^{1/2}. \tag{25}$$

This solution corresponds to the motion in a small vicinity of fixed point $P1$ on its slow manifold (both its stable and the unstable parts). It is not captured by the linear analysis near $P1$ because such motion is nonlinear due to the zero eigenvalue. Note that there is no $k_*$ dependence in this expression. Solutions with different $k_*$ (and therefore with different energy flux) correspond to the same orbit, namely the slow manifold of $P1$. The different energy flux corresponds to different choices of the initial wave number. For example, we fix the flux if we specify condition $f(k_0) = f_0$: then the value $f_0$ will correspond to a unique starting point on the slow manifold. Thus, the 'time' to the collapse is uniquely determined by the starting point on the slow manifold i.e. $s_* = \ln(k_*/k_0)$ will be a unique function of $f_0$.

Note that there is also another orbit connecting to the fixed point from the positive side. Asymptotic consideration similar to the one above give for this orbit

$$g \approx \partial_s f = 2A^{1/2}k_*^{\frac{5-2z}{2}}(k - k_*) = (\nu f/5)^{1/2}, \tag{26}$$

where now the cut off is on the left side, $k_* < k$.

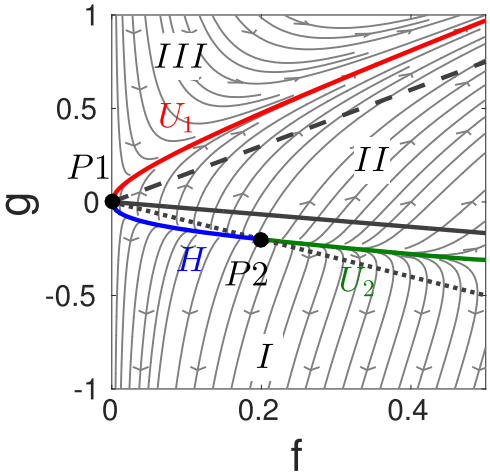


**Figure 1.** Case $z < 2/3$. Phase portrait for $z = 0$, $\nu = 1/4$. The solid straight line shows the Kolmogorov scaling, the dashed line—thermodynamic, and the dotted line is $f = -g$.

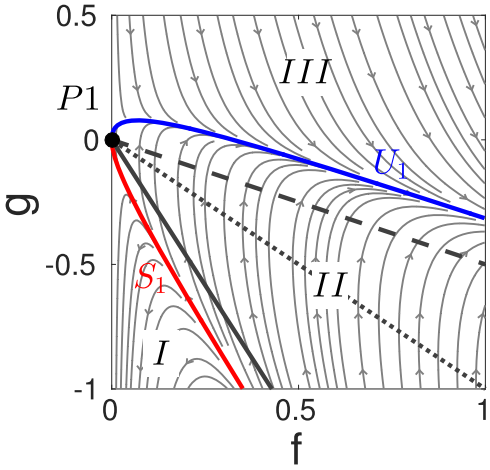


**Figure 2.** Case $2/3 < z < 5/2$. Phase portrait for $z = 2$, $\nu = 1/4$. The solid straight line shows the Kolmogorov scaling, the dashed line—thermodynamic, and the dotted line is $f = -g$.

## 3. Phase analysis of orbits: qualitative considerations and overview of results

The easiest way to understand the main features of the steady state solutions for different parameter values is to consider the phase space plots of the respective dynamical systems. There are three qualitatively different cases: $z < 2/3$, $2/3 < z < 5/2$ and $z > 5/2$; see figures 1, 2 and 3 respectively. As we said before, there are two fixed points for $z < 2/3$ and $z > 5/2$, ($P1 = (0.0)$ and $P2 = (8\nu/D, -8\nu/D)$, and only one for $2/3 < z < 5/2$, ($P1 = (0.0)$). The Kolmogorov and the thermodynamic scalings correspond to straight lines with slopes $-1/3 - z$ and $3/2 - z$ respectively. Since $z > 0$, the Kolmogorov slope is

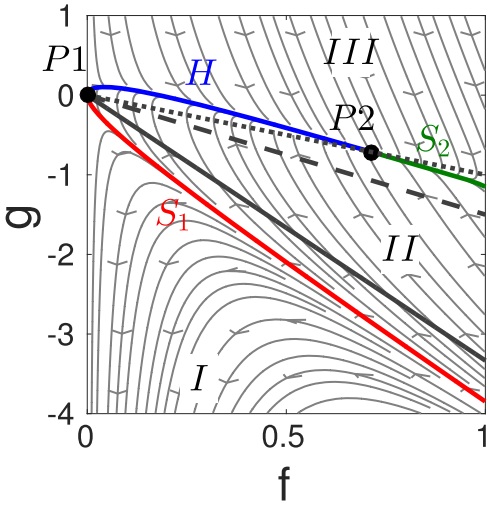


**Figure 3.** Case $z > 5/2$. Phase portrait for $z = 3$, $\nu = 5/8$. The solid straight line shows the Kolmogorov scaling, the dashed line—thermodynamic, and the dotted line is $f = -g$.

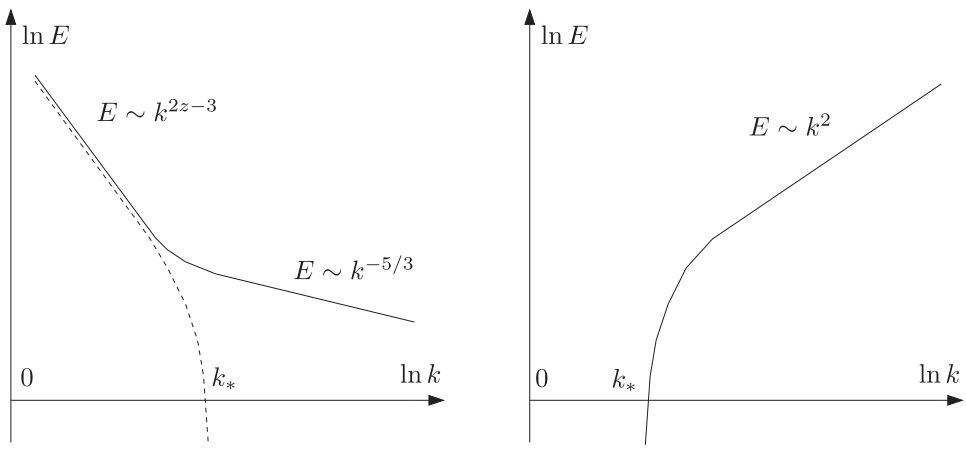


**Figure 4.** Sketches of spectra in case $z < 2/3$. Left panel corresponds to the direct cascade: solid line is the high-Reynolds-number spectrum described by orbit $U_2$; dashed line is the low-Reynolds-number spectrum described by orbit $H$. Right panel corresponds to the inverse cascade described by orbit $U_1$ (similar spectrum is associated to orbit $U_1$ for $2/3 < z < 5/2$).

always negative and below the thermodynamic one (the latter is positive for $z < 3/2$ and negative otherwise). It is also instructive to mark the line $g = -f$ where the orbits are vertical (have an infinite slope), i.e. $f(s)$ reaches a local maximum or minimum. This line passes through $P2$ when the latter exists.

### *3.1. Separatrices*

The most physically important orbits are represented by separatrices. These solutions are generic in the sense that they correspond to a single energy source at one of the ends of the $k$-range and no sinks (i.e. the energy is dissipated by the viscosity only).

There are two separatrices in the case $2/3 < z < 5/2$: the stable manifold of $P1$ asymptoting to the Kolmogorov line at infinity, $S_1$, and the unstable manifold of $P1$ asymptoting to the thermodynamic line, $U_1$. There are three separatrices in each of the $z < 2/3$ and $z > 5/2$ cases. One of them is a heteroclinic orbit connecting $P1$ and $P2$. For $z < 2/3$, the other two separatrices are represented by the unstable manifold of $P2$ asymptoting to the Kolmogorov line, $U_2$, and the unstable manifold of $P1$ asymptoting to the thermodynamic line, $U_1$. For $z > 5/2$, the other two separatrices are represented by the stable manifold of $P2$ asymptoting to the thermodynamic line, $S_2$, and the stable manifold of $P1$ asymptoting to the Kolmogorov line, $S_1$.

Let us analyse solutions corresponding to the separatrices.

*3.1.1. Case $z < 2/3$.* We start with the case $z < 2/3$. Orbit $U_2$ corresponds to a direct energy cascade from low to high $k$: it starts with the dissipative scaling near $P2$, $E \sim k^{2z-3}$, where the energy flux is gradually weakening, followed by transition to the Kolmogorov scaling, $E \sim k^{-5/3}$, at high $k$ where the energy flux saturates to a constant $k$-independent value; see figure 4, left. The $-5/3$ scaling will continue to infinite $k$ corresponding to the fact that the dissipation is negligible in this range. A solution of this kind was first found in [10]. Note that orbit $U_2$ corresponds to an energy source at the low $k$ boundary strong enough for $f$ to be greater than $8\nu/D$ (i.e. to the right of $P2$). We will refer to this solution as high-Reynolds-number direct cascade.

**Remark 3.1.** At large $k$ (corresponding to large $f$ in this case) the dissipation is negligible and the solution tends to one of the inviscid solutions (4). Importantly, the solution in this case is pure Kolmogorov, $Q = 0$. Indeed, any finite $Q$ would lead to deviation from the Kolmogorov line at $k \to \infty$, which is not the case here.

Orbit $H$ (a heteroclinic orbit connecting $P1$ and $P2$) corresponds to forcing with $f < 8\nu/D$—we will call it the low-Reynolds-number direct cascade; see figure 4, left. This solution also starts with the dissipative scaling near $P2$, $E \sim k^{3-2z}$, where the energy flux is gradually weakening. However, this scaling does not transition to the inviscid $-5/3$ scaling (as in of the high-Reynolds-number case) but drops to zero, together with the energy flux, at a finite wave number $k_*$. This corresponds to arrival of $H$ at fixed point $P1$ and described by solution (24) in the vicinity of $k_*$.

**Remark 3.2.** Importantly, viscous scaling will show up on the spectra for the direct or inverse energy cascade only for very special initial data corresponding to $f_0 = f|_{s=0}$ close to $8\nu/D$, i.e. such that $E(k_0)$ is close to $k_0^{2z-3}(8\nu/D)^2$. For a generic case this condition is not satisfied, i.e. the starting point on orbits $U_2$ or $H$ is far from $P2$ and there is no viscous scaling range on the spectrum.

Finally, orbit $U_1$ corresponds to an inverse energy cascade; see figure 4, right. Here, the energy forcing is at high (or infinite) wave number $k_0$ and the energy flux $\epsilon(k)$ is negative at $k < k_0$, gradually decreasing in magnitude toward lower $k$'s and turning into zero, together with the spectrum itself, at a finite $k_*$. At high $k$, the dissipation is negligible and the spectrum

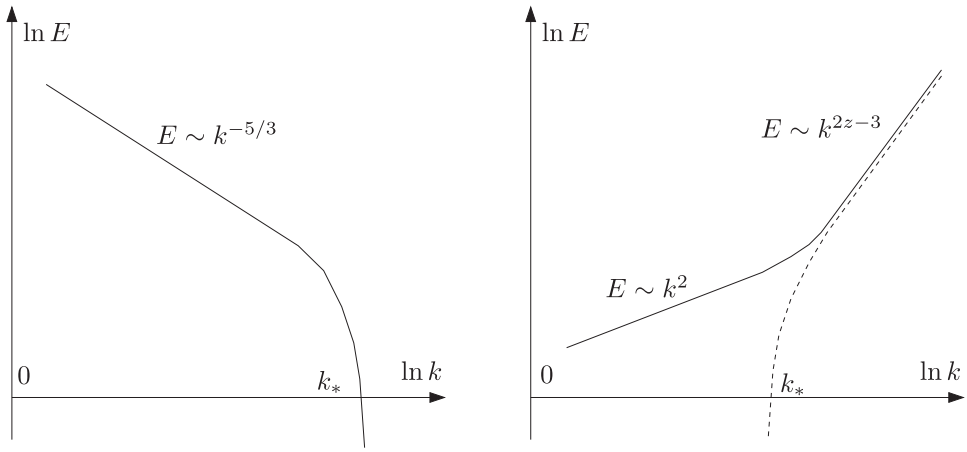


**Figure 5.** Sketches of spectra in case $z > 5/2$. Left panel corresponds to the direct cascade described by orbit $S_1$ (similar spectrum is associated to orbit $S_1$ for $2/3 < z < 5/2$). Left panel corresponds to the inverse cascade: solid line is the high-Reynolds-number spectrum described by orbit $S_2$; dashed line is the low-Reynolds-number spectrum described by orbit $H$.

tends to the warm-cascade solution (4), dominated by the thermodynamic part, but with a finite flux correction which is negative and almost $k$-independent in the high-$k$ range, $\epsilon(k) \to P = \text{const} < 0$. On the dimensional grounds we can estimate:

$$P \sim Q^{\frac{2-3z}{5-2z}} \nu^{\frac{11}{5-2z}}. \tag{27}$$

*3.1.2. Case* $2/3 < z < 5/2$. This case is the simplest because there is no $P2$ equilibrium. Orbit $S_1$ corresponds to a direct energy cascade. It starts with the Kolmogorov scaling at low $k$'s (there viscosity is negligible) and terminates, with zero flux, at a finite right boundary $k = k_*$ (corresponding to the arrival at $P1$); see figure 5, left. In the low-$k$ range the Kolmogorov scaling has a finite thermal correction with $Q < 0$.

Orbit $U_1$ corresponds to an inverse-cascade spectrum which terminates, with zero flux, at a finite left boundary $k = k_*$; see figure 4, right. At large $k$ the spectrum asymptotes to a warm-cascade spectrum dominated by the thermodynamic part, but with a finite flux $P$. The value of $P$ can be estimated as before by equation (27).

*3.1.3. Case* $z > 5/2$. For $z > 5/2$, the orbit $S_1$ behaves qualitatively similar to the behaviour of $S_1$ in the case $2/3 < z < 5/2$; see figure 5, left. Orbits $S_2$ and $H$ describe a high-Reynolds and a low-Reynolds number inverse cascades respectively; see figure 5, right. The energy source is located at the right boundary of the $k$-range, $k_0$, near which it has the dissipative scaling $E \sim k^{2z-3}$ if (and only if) $E(k_0)$ is close to $k_0^{2z-3}(8\nu/D)^2$ (see case $z < 2/3$). At small $k$ the dissipation is negligible and the solution is close to pure thermodynamic. We have $P = 0$ because otherwise the flux part would win at $k \to 0$ in the warm-cascade solution (4), which is not the case here. The low-Reynolds number inverse cascade does not transition to the thermodynamic scaling: it terminates, with zero spectrum and flux, at a finite left boundary $k = k_*$.

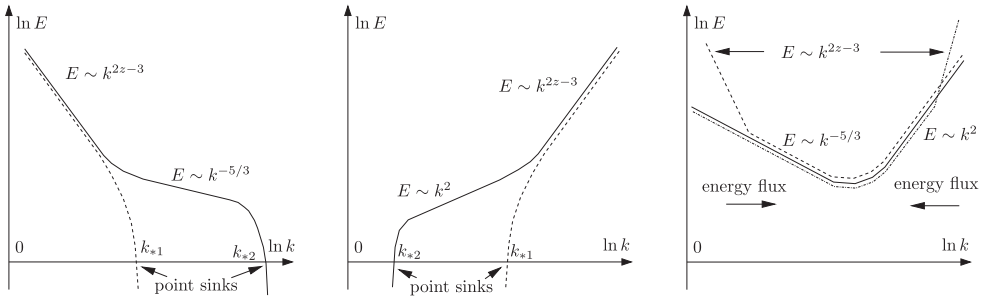


**Figure 6.** Sketches of spectra corresponding to orbits which are not separatrices. *Left*: high-Reynolds (solid line) and low-Reynolds number direct cascades (dashed line) described by the part I orbits in case $z < 2/3$. Spectra for $z > 2/3$ are similar except $E \sim k^{2z-3}$ part is absent. *Centre*: high-Reynolds (solid line) and low-Reynolds number inverse cascades (dashed line) described by the part III orbits in case $z > 5/2$. Spectra for $z < 5/2$ are similar except $E \sim k^{2z-3}$ part is absent. *Right*: dual (converging) cascade spectra described by the part II orbits in cases $2/3 < z < 5/2$ (solid line), $z < 2/3$ (dashed line) and $z > 5/2$ (dashed–dotted line).

### *3.2. Orbits other than separatrices*

The other orbits are less interesting because they correspond to more artificial boundary conditions with extra sources and sinks, but we will consider them too for completeness. Orbits from part I of the phase space (below the lower separatrix/separatrices) correspond to warm direct-cascade spectra that turn into zero at finite right boundary $k = k_*$ with a finite positive value of the energy flux, which implies presence of a *point sink* at $k_*$; see figure 6, left. Orbits from part III (above the upper separatrix/separatrices) correspond to warm inverse-cascade spectra that turn into zero at finite left boundary $k = k_*$ with a finite negative value of the energy flux, which, again, implies presence of a *point sink* at $k_*$; see figure 6, centre. Orbits from part II (in between of the lower and the upper separatrices) which have their two ends on the opposite sides of the thermodynamic line correspond to spectra with *two point sources* located at both ends of the $k$-range (energy fluxes converging toward the centre of this range are dissipated by the viscosity); see figure 6, right. If both ends of the type II orbit are below (above) the thermodynamic line then the we get a warm direct (inverse) cascade spectrum with a *point sink* and the right (left) end of the $k$-range. Obviously, if one of the orbit's ends is exactly on the thermodynamic line, there are no point sinks. This special case is relevant to some numerical simulations, as will be discussed in the Conclusions section.

## 4. Rigorous analysis of solutions

### *4.1. Boundary value problems*

For equation (2) we are interested in studying the following boundary value problems. Find solutions of equation (2) supplemented by the conditions

$$E(k_0) = E_0, \quad E(k_*) = 0, \quad k_* > k_0 \tag{28}$$

such that $E(k) > 0$ for $k \in [k_0, k_*)$ (which corresponds to $s = \ln k/k_0 < 0$) and

$$E(k_0) = E_0, \quad E(k_*) = 0, \quad 0 < k_* < k_0 \tag{29}$$

for $k \in [k_*, k_0)$ which corresponds to $s < 0$. We will refer to these as Problem 1 and Problem 2 respectively. In an addition, we also consider Problem 3:

$$E(k_0) = E_0, \quad E(k_1) = E_1. \tag{30}$$

We show that not for all combinations $E_0$, $E_1$, $k_*$ and $k_1$ Problems 1–3 are solvable.

It will be convenient to write equation (6) in an equivalent form:

$$\hat{F}(f) \equiv 2\frac{\mathrm{d}}{\mathrm{d}s}\left(f^2\frac{\mathrm{d}f}{\mathrm{d}s}\right) + (12z - 19)f^2\frac{\mathrm{d}f}{\mathrm{d}s} + Df^3 - 8\nu f^2 = 0. \tag{31}$$

Notice that equation (31) admits the translation group of transformations of independent variable $s \to s + b$, where $b$ is a constant with the infinitesimal operator $X = \frac{\partial}{\partial s}$. For the original equation (2) this symmetry is transformed into the scaling symmetry. Therefore we set $k_0 = 1$ for the first boundary condition in (28). In terms of $f(s)$ the boundary value problems are formulated as solving equation (31) with the following boundary conditions. Problem 1f:

$$f(0) = f_0 > 0, \quad f(s_*) = 0, \quad s_* > 0, \tag{32}$$

Problem 2f:

$$f(0) = f_0 > 0, \quad f(s_*) = 0, \quad s_* < 0 \tag{33}$$

and Problem 3f:

$$f(0) = f_0 > 0, \quad f(s_1) = f_1 > 0. \tag{34}$$

To study these boundary value problems, we apply the well-developed methods of the theory of nonlinear ODEs, see e.g. [9]. First of all, we notice that solutions of these problems are (if exist) unique. Also, it is easy to establish that the Kolmogorov solution denoted by $f_{P,0}$ is a super solution of equation (31), i.e. $\hat{F}(f_{P,0}) \leqslant 0$. The same is true for the thermodynamic spectrum $f_{0,Q}$ and for the general solution of the inviscid form of equation (31)

$$f_{P,Q}(s) = c^{1/2}\mathrm{e}^{(5-2z)s/2}(P\mathrm{e}^{-11s/2} + Q)^{1/3}. \tag{35}$$

### *4.2. Case $z < 2/3$*

Let us consider the case $z < 2/3$ which means that $D > 0$ and $12z - 19 < 0$. We will consider equation (31) and the phase portrait of the dynamical system (10) and (11) to establish which data guarantees solvability of the Problems 1 and 2. Instead of directly using the boundary conditions (32) and (33) let us employ a shooting method. Namely, let us supplement equation (31) by the initial conditions

$$f(0) = f_0 > 0, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}s}f\right|_{s=0} = 0. \tag{36}$$

These initial conditions mean that we study the orbits of the dynamical system which start on the line $f + g = 0$ of the phase plane $(f, g)$. Note that the direction of velocity $(\mathrm{d}f/\mathrm{d}\tau, \mathrm{d}g/\mathrm{d}\tau)$ on the line $f + g = 0$ is vertically down for $f > f_D$ and vertically up for $f < f_D$, where $f_D = 8\nu/D$. Recall that equation (31) has an exact positive solution $f(s) \equiv f_D$ which corresponds to the unstable node $P2$ on the phase plane.

We begin with a preliminarily analysis of the behaviour of orbits of the dynamical system (10) and (11).

**Lemma 4.1.** *The orbits of the dynamical system (10), (11) intersecting the line $f + g = 0$ with $f \geqslant 48\nu/5D$ do not approach the fixed point $P1$. Instead, these orbits approach, in finite time, the $g$-axis without intersecting it, so that $f \to 0$, $g \to -\infty$ as $s \to s_* > 0$.*

To prove this Lemma, we consider an integral identity obtained by multiplying equation (31) by $f^2\,\mathrm{d}f/\mathrm{d}s$, integrating over $[0, s]$ and setting $\mathrm{d}f/\mathrm{d}s = 0$ at $s = 0$:

$$2\left(f^2\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2(s) + (12z - 19)\int_0^s f^4\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^2\mathrm{d}\eta + \Phi(f) = \Phi(f_0), \tag{37}$$

where $\Phi(f) = \frac{D}{6}f^6 - \frac{8\nu}{5}f^5$. Function $\Phi(f)$ has zeros at $f = 0$ and $f = \frac{48\nu}{5D}$; $\Phi(f) < 0$ for $f \in \left(0, \frac{48\nu}{5D}\right)$, and $\Phi(f) > 0$ for $f > \frac{48\nu}{5D}$. The minimum of $\Phi(f)$ is achieved at $f = \frac{8\nu}{D}$. First, let us show that at least for $f_0 \geqslant 48\nu/5D$ the corresponding solutions are decreasing functions up to the intersection with the $s$-axis. Indeed, since $f_0 \geqslant 48\nu/5D > 8\nu/D$ the solutions are decreasing functions for small $s$ because $f(s)$ has a local maximum at $s = 0$: $f'(0) = 0, f''(0) = g'(0) < 0$. Now suppose that $f(s)$ also has a positive minimum at $s > 0$ (including the case $f(s) \to \mathrm{const} > 0$ as $s \to \infty$): then the first term in (37) is zero, the second term is negative and, therefore, $\Phi(f(s)) > \Phi(f_0)$. But this means that $f(s) > f_0$ which contradicts the assumption that this is a minimum, i.e. has to be less than the maximum. Therefore, there can be no positive minima, as required.

Now let us show that $f$ can vanish at finite point $s = s_*$ only. Assume that $s_* = \infty$: then $\mathrm{d}f/\mathrm{d}s \to 0$ as $s \to \infty$ and the first term of the left-hand side of (37) tends to zero. So do the third and the fourth terms on the left-hand side, whereas the second term is bounded from above by a negative number (in principle it could be $-\infty$ if the integral was divergent). But then we arrive at a contradiction as the right-hand side in (37) is positive. Therefore $s_* < \infty$.

By the same argument we see that at the point $s_* < \infty$, where $f(s_*) = 0$ we must have $\lim_{s\to s_*}(f^2\mathrm{d}f/\mathrm{d}s) = \mathrm{const} \neq 0$ and, therefore, $\lim_{s\to s_*}\mathrm{d}f(s)/\mathrm{d}s = -\infty$. In view of the formula $\mathrm{d}f/\mathrm{d}s = f + g$ we get that $g(s) \to -\infty$ as $s \to s_*$ ($\tau \to \tau^*$). Therefore the corresponding orbits of the dynamical system (10) and (11) cannot approach the equilibrium $P1$ (where $\mathrm{d}f(s)/\mathrm{d}s = 0$), but instead asymptote to the $g$-axis with $g \to -\infty$.

**Corollary 4.1.** *At least for $f_0 \geqslant 48\nu/5D$ there exits $s_* < \infty$ such that the boundary value problem (31), (32) is solvable. Flux $\epsilon$ defined by (3) and (7) tends to a positive constant as $s \to s_*$.*

**Corollary 4.2.** *The boundary value problem (31) and (32) with $s_* = \infty$ has no solutions at least for $f_0 \geqslant 48\nu/5D$.*

**Remark 4.1.** We show later that the orbits for all $f_0 > f_D$ asymptote to the $g$-axis, $(f, g) \to (0, -\infty)$.

On the orbits asymptoting to the $g$-axis the flux $\epsilon$ defined by (7) is positive at each $s$. We see that $\epsilon > 0$ for $\mathrm{d}f/\mathrm{d}s < 0$, which includes the line $g + f = 0$ and below. For the orbits that go to $P1$ (i.e. the heteroclinic orbit $H$) we have $\epsilon(s_*) = 0$ since at this point $f = \mathrm{d}f(s)/\mathrm{d}s = 0$. For $f_0 \geqslant 48\nu/5D$ we showed that $\lim_{s\to s_*}(f^2\mathrm{d}f/\mathrm{d}s) = \mathrm{const} < 0$ (actually, the same is true for all orbits with $f_0 > 8\nu/D$). Thus for small $f$ (and therefore large

negative $\mathrm{d}f/\mathrm{d}s \approx g$) we have from (7):

$$\lim_{s\to s_*} \epsilon(s) = A\mathrm{e}^{(3z-2)s_*}, \quad A = \mathrm{const} > 0. \tag{38}$$

Therefore, $\epsilon(s_*) > 0$, as required. Moreover, $\epsilon(s_*)$ is a decreasing function of $s_*$.

**Remark 4.2.** Each orbit of (10), (11) is invariant under shifts along the trajectory $\tau \to \tau + a$ or the translation symmetry (with respect to $s \to s + b$) of equation (31). According to formula $E(k) = k^{2z-3}f^2$, the translation symmetry generates a one-parametric family of solutions $E(k, b) = \mathrm{e}^{b(z-3/2)}E(bk)$ for each known solution $E(k)$. In the other words, each orbit on the $(f, g)$ plane corresponds to not just one, but to a one-parametric family of solutions.

To prove the existence of separatrices, we present a technical result concerning the direction of the vector field on the Kolmogorov line $f^{\mathrm{K}}: g + (1/3 + z)f = 0$ and thermodynamic line $f^{\mathrm{T}}: g + (z - 3/2)f = 0$. Notice that $f^{\mathrm{K}}$ is located above of the line $f + g = 0$ and below the thermodynamic line which is located in the first quadrant of the phase plane for $z < 2/3$. We write $g = g(f)$ where

$$\frac{\mathrm{d}g}{\mathrm{d}f} = -\frac{2(f+g)}{f} - (6z - 17/2) - \frac{D}{2(f+g)} + \frac{4\nu}{f+g} \equiv G(f, g).$$

**Lemma 4.2.** *$G(f, -(1/3 + z)f) > -(1/3 + z)$ for the Kolmogorov line and $G(f, -(z - 3/2)f) > -(z - 3/2)$ for the thermodynamic line.*

By simple calculations, we have $G(f, -(1/3 + z)f) = -(1/3 + z) + 4\nu/((2/3 - z)f)$ for the Kolmogorov line and $G(f, -(z - 3/2)f) = -(z - 3/2) + 4\nu/((5/2 - z)f)$ for the thermodynamic line. Geometrically lemma 4.2 states that along $f^{\mathrm{K}}$ and $f^{\mathrm{T}}$ the flow is directed into the domains $g + (1/3 + z)f > 0$ and $g + (z - 3/2)f > 0$ respectively.

Now we show the existence of a heteroclinic connection between the fixed points $P2$ with $P1$.

**Lemma 4.3.** *There exists an orbit $H$ (a heteroclinic connection) of (10), (11) which emerges out the unstable node $P2$ and goes to the equilibrium $P1$ with time.*

Consider a Cauchy problem for equation (31) supplemented with the initial data

$$f(0) = f_0 \equiv f_D - \varepsilon, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}s}f\right|_{s=0} = f_m < 0, \qquad m = 1, \ldots \tag{39}$$

for arbitrary small $\varepsilon > 0$. Notice that vertical line $f = f_D - \varepsilon$ intersects the Kolmogorov line on the phase plane and also the orbits obtained in lemma 4.1. First of all, we indicate that for sufficiently small values of $|f_m|$ the orbits, which correspond to solutions of (31) and (39), intersect the line $g + f = 0$ since the direction of the velocity field $(\mathrm{d}f/\mathrm{d}s, \mathrm{d}g/\mathrm{d}s)$ is directed into $g + f > 0$ for $f < f_D$. Therefore, these solutions achieve local minima. At the same time, there exists $f_m$ such that the corresponding solutions decrease monotonically with time vanishing at some finite points. Indeed, the line $f = f_D - \varepsilon$ always intersects the orbits obtained in lemma 4.1 and $f_m$, together with $f_0$, fix the coordinates of these intersections. For

completeness, we notice that if we take $f_m$ positive and sufficiently large such that we are located above of the Kolmogorov line then solutions of (31) and (39) are increasing functions, see lemma 4.2. Therefore, there exists families of orbits with different behaviours on the phase plane and it should be a separatrix which separates them. To prove it, let us consider the set $A = \{f_m,\ f_m < 0\}$ with the properties: the corresponding solution of the Cauchy problem has a positive minimum at $s = s_{\min}$. This set is not empty and not a single element set. Moreover, $A$ is bounded below, see the discussion above. It means that $\inf A$ exists, which we denote by $f_A$, and the solution we denote by $f(s; f_A)$. We denote the corresponding orbit by $H$. According to the definition of the infimum any vicinity of $f_A$ produces solutions both in and outside of $A$, i.e. $\mathrm{d}f/\mathrm{d}s = 0$ at the boundary $s = s_*$ which, by definition, is where $f = 0$. (We discard possibility for $f(s; f_A) \to \text{const} < \infty$ with $\mathrm{d}f/\mathrm{d}s \to 0$ at $s \to \infty$, as there are no fixed points in the system other than $P1$ and $P2$.) However, the point where $f = \mathrm{d}f/\mathrm{d}s = 0$ is $P1$ and, therefore, orbit $H$ goes to $P1$ and reaches it (as it follows from the asymptotics of solution near $P1$ found in section 2.4) in finite $s = s_*$.

If we now change $\tau$ to the inverse time $\hat{\tau} = -\tau$ then $H$ goes to the stable node $P2$. It follows from the Poincaré–Bendixon theorem. Indeed, let us consider a finite area of the ($f$, $g$)-plane: the $g$-axis, the Kolmogorov line, a horizontal line somewhere below $P2$, $g = g_b$ const $< -f_D$ and the vertical line $f = -g_b$. Easy calculation shows that on the boundary of this domain vector field $(\mathrm{d}f/\mathrm{d}\hat{\tau}, \mathrm{d}g/\mathrm{d}\hat{\tau})$ is either directed into this domain or along the boundary. Note that there is only one orbit passing through $P1$ within the specified domain—the orbit $H$. The other orbit passing $P1$ is describes motion toward this fixed point (in the reverse time) from outside of the specified domain. This follows from the local structure of this trajectory (and of $H$) $f \sim g^2$, see equations (26) and (25). Thus, $H$ cannot return to $P1$ as this would imply leaving the specified domain first, which is impossible by the construction—i.e. $H$ is not a homoclinic orbit. Therefore, $H$ approaches the stable (in the reverse time) node $P2$ and this is a heteroclinic connection.

**Corollary 4.3.** *There exists a solution of Problem 1f which vanishes together with the flux $\epsilon$ defined by (7) at a finite point $s = s_*$. This solution is represented by the orbit $H$.*

Indeed, since for $H$ we have $f = \mathrm{d}f/\mathrm{d}s = 0$ at $s = s_*$, the right-hand side of (7) is zero.

Existence of the orbit $H$ allows us to generalise lemma 4.1 to all $f_0 > f_D$ as stated in remark 4.1. Indeed, such orbits are bound by $H$ on one side and by the orbits with $f \geqslant 48\nu/5D$ which approach the $g$-axis. Thus they also asymptote to the $g$-axis.

Now let us prove that there exists an orbit $U_2$ which originates at the fixed point $P2$ and asymptotes to the Kolmogorov line $f^{\mathrm{K}}$ for large $\tau$, see figure 1.

**Lemma 4.4.** *There exists an orbit $U_2$ which emerges out the unstable node $P2$ and transitions to the Kolmogorov line as $\tau \to \infty$*

Consider a Cauchy problem for equation (31) supplemented with the initial data

$$f(0) = f_0 = f_D + \varepsilon, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}s} f\right|_{s=0} = f_m, \qquad m = 1, \ldots. \tag{40}$$

Here $\varepsilon > 0$ is arbitrarily small. Note that vertical line $f = f_D + \varepsilon$ intersects the Kolmogorov line on the phase plane. Let us consider the solution $f(s; f_m)$ of (31) and (40). Assume that $f(s; f_m)$ is large at sufficiently large $s$, namely that there exists $s_0 > 0$ such that $f(s; f_m) \gg f_D$ for $s > s_0$ so that the right-hand side of (31) is negligible compared to the

last term on the left-hand side, and in the leading order we have an inviscid equation:

$$2\frac{\mathrm{d}}{\mathrm{d}s}\left(f^2\frac{\mathrm{d}f}{\mathrm{d}s}\right) + (12z - 19)f^2\frac{\mathrm{d}f}{\mathrm{d}s} + Df^3 = 0. \tag{41}$$

Solutions of this equation are

$$f_{P,Q}(s) = c^{1/2}\mathrm{e}^{(5-2z)s/2}(P\mathrm{e}^{-11s/2} + Q)^{1/3}. \tag{42}$$

They correspond to the warm-cascade solutions (4). In view of the theorem of continuous dependence of solutions on the right-hand side applied to equation (31), we have that for $f(s_0)/f_D \to \infty$, $f(s)$ converges to $f_{P,Q}(s)$ for $s \geqslant s_0$ at least in the Hölder norm $C^{2+\alpha}$, $0 < \alpha < 1$ on each compact interval. Therefore $f(s; f_m)$ converges to $f_{P,Q}(s)$ for $s \geqslant s_0$.

Function $f_{P,Q}(s)$ is the two-parametric general solution of the inviscid form of the inviscid equation (41), where $c > 0$ and $P$, $Q$ are constant parameters. Parameter $P$ is the flux $\epsilon$ which is constant on $f_{P,Q}(s)$. Case $Q = 0$ gives the Kolmogorov solution $f_{P,0}$ parametrised by $P$, whereas for $P = 0$ we get the thermodynamic solution $f_{0,Q}$ parametrised by temperature $Q$. For different values of the parameters $P$ and $Q$ we have both increasing and decreasing behaviour of $f_{P,Q}(k)$. It follows from (42) that the Kolmogorov solution $f_{P,0}$ is positive everywhere and is a solution of the minimal growth among solutions $f_{P,Q}(s)$. If $P$ and $Q$ have different signs, $f_{P,Q}(s)$ vanishes at $s = (2/11)\ln(-P/Q)$. Therefore for the growing $f(s; f_m)$ the corresponding orbits either asymptote to the Kolmogorov line or pass above this line asymptoting to the thermodynamic line as $\tau \to \infty$, as the $Q$-term in (42) is always dominant for large $s$.

Consider the set $C = \{\mathrm{d}f(0)/\mathrm{d}s = f_m\}$ with the properties: corresponding solutions of the Cauchy problem (31) and (40) have one positive maximum at $s = s_{\max}$. On the phase plane $(f, g)$ the points of local maximum of $f(s; f_m)$ are located on line $f + g = 0$ for $f > f_D$ and go along this line as $f_m$ grows. Clearly, values of $f_m$ such that the corresponding orbits intersect line $g + f = 0$ exist: for example one can take an initial point very close and just above line $g + f = 0$. Thus, set $C$ is not empty. Also, set $C$ is bounded from above: at the very least it is bounded by the value $f_m$ corresponding to the Kolmogorov line. Indeed, the orbits cross the Kolmogorov line from the lower to the upper side, and, therefore, once crossing it will never cross back, see lemma 4.2. Therefore, $\sup C < \infty$ exists, let us denote it by $f_C$, and the solution $f(s; f_C)$ of (31), (40) is a positive increasing function defined for $s \geqslant 0$. Orbit $U_2$, corresponding to $f(s; f_C)$, asymptotes to the Kolmogorov line on the phase plane as $s \to \infty$. Indeed, $f(s; f_C)$ presents the minimally monotonically growing solution as $s \to \infty$ in the sense for $f_m = f_C - \delta$ with arbitrary small $\delta > 0$ the corresponding solution of the Cauchy problem is not a monotonically increasing function. For positive $\delta \to 0$ we have $s_{\max} \to \infty$ and $f(s_{\max}) \to \infty$ so that for $s \sim s_{\max}$ the solution converges to an inviscid solution $f_{P,Q}(s)$ with a finite positive $P$ and negative $Q \to 0$ (see remark 4.3 below). For negative $\delta \to 0$ we have no maximum, but for large $s$ the solution also converges to an inviscid solution $f_{P,Q}(s)$, now with a finite positive $P$ and positive $Q \to 0$. Therefore, $f(s; f_C) \to f_{P,0}(s)$ for $s \to \infty$. By construction, orbit $U_2$ is above of the line $g + f = 0$. Also, it remains below the Kolmogorov line, because crossing this line and then asymptoting back to it would contradict the monotonous decrease of $\epsilon(s)$ property.

Let us now show that orbit $U_2$ emerges out of the unstable node $P2$. Consider a finite domain bounded by the Kolmogorov line, line $g + f = 0$ for $f \geqslant f_D$, orbit $H$ for $f < f_D$ and a vertical line $f = f_D + \varepsilon$. For the inverse time $\hat{\tau} = -\tau$, the vector field $(\mathrm{d}f/\mathrm{d}\hat{\tau}, \mathrm{d}g/\mathrm{d}\hat{\tau})$ is directed either inwards or parallel to the boundaries of this domain (e.g. for the Kolmogorov line see lemma 4.2). Thus, by the Poincaré–Bendixon theorem the orbit $U_2$ must approach $P2$.

$U_2$ cannot approach $P1$ since the only orbit in the fourth quadrant that goes to $P1$ is $H$ (see section 2.4).

**Remark 4.3.** We established that there exist a monotonically growing solution $f(s; f_C)$ and converging to $f_{P,0}(s)$ in the Hölder norm $C^{2+\alpha}$ as $s \to \infty$. Respectively, the flux $\epsilon(s; f_C)$ calculated for $f(s; f_C)$ converges to the flux $\epsilon_{P,0}(s)$ for the Kolmogorov solution $f_{P,0}$ as $s \to \infty$. Direct calculation shows that $\epsilon_{P,0}(s) \equiv P$ is a positive constant for all $s$. Thus $\epsilon(s; f_C) \to P$ as $s \to \infty$. Let us now take $f_m \in C$ sufficiently close to $f_C$: there exists an interval $[a_m, b_m]$, $a_m > 0$, $b_m < \infty$ where $f(s; f_m) \gg f_D$ or/and $|g(s; f_m)| \gg f_D$. Hence the difference $|f(s; f_m) - f_{P,Q}(s)|$ is small in the norm of the Hölder space $C^{2+\alpha}[a_m, b_m]$ for some choice of the parameters $P$ and $Q$, where $Q$ is always a negative quantity. Therefore the flux $\epsilon(s)$ tends to constant $P$ and $f_{P,Q}(s)$ is an asymptotic of $f(s; f_m)$.

Let us denote by $O_{\mathrm{I}}$ the set of orbits which located below of $H \cup U_2 \cup P2$. By $O_{\mathrm{II}}$ we will denote the set of orbits which emerge out $P2$ and located above of $H \cup U_2 \cup P2$. As we will shortly show, the latter set is bounded from above on the $(f, g)$-plane by yet another separatrix, $U_1$. The set of orbits in the first quadrant of the $(f, g)$-plane above $U_1$ will be called $O_{\mathrm{III}}$.

The flux $\epsilon$ is positive on the $g$-axis and on the line $g + f = 0$ of the fourth quadrant. All orbits from $O_{\mathrm{II}}$ can be obtained by starting from different initial points on the Kolmogorov line. Moving backwards in time one can see, by the Poincaré–Bendixon theorem using the domain bounded by $H \cup U_2 \cup P2$ and the Kolmogorov line, that the orbits converge onto $P2$. Moving forwards in time we get solutions $f(s)$ that grow monotonously and, therefore, at large $s$ converge to $f_{P,Q}(s)$ with $Q > 0$. Note that, in spite of the two parameters in $f_{P,Q}(s)$, the respective family of orbits is one-parametric due to the translational symmetry mentioned in remark 4.2. But the $Q$-part always wins in $f_{P,Q}(s)$ over the $P$-part at large $s$, so all these orbits asymptote to the thermodynamic line. This applies to the limiting orbit starting at $P1$. In fact we already know, that when run backwards in time the trajectory starting at $P1$ also ends at starting at $P2$—like any orbit from the $O_{\mathrm{II}}$ set: this is orbit $H$. Thus we have proven the following lemma concerning the orbit starting at $P1$ and running forward in time.

**Lemma 4.5.** *There exists an orbit $U_1$ of the dynamical system (10) and (11) which emerges out the fixed point $P1$ and asymptotes to the thermodynamical line.*

**Remark 4.4.** Orbit $U_1$ starts with zero flux at $P1$, $\epsilon(0) = 0$, and asymptotes to $f_{P,Q}(s)$ with $Q > 0$ and $P < 0$. Thus, $U_1$ lies above the thermodynamic line (on which $\epsilon(s) = 0$).

Note that this remark does not contradict the preceding lemma because at large $s$ the $Q$-term is dominant over the $P$-term in $f_{P,Q}(s)$ for any finite $P$ and $Q$.

Thus, $U_1$ represents an inverse energy cascade solution vanishing, together with the flux, at a finite wave number. The fact that $\epsilon(0) = 0$ follows from (7) upon substitution $f = \mathrm{d}f/\mathrm{d}s = 0$. The fact that $\epsilon \to P < 0$ for $s \to \infty$ follows from the fact that $\epsilon(s)$ is always a monotonically decreasing function of $s$.

**Corollary 4.4.** *$U_1$ realises a solution of Problem 2f which vanishes, together with $f^2\mathrm{d}f/\mathrm{d}s$ and $\epsilon$, at $s = s^{**}$, $s^{**} < 0$.*

This statement follows from the shift invariance of the solutions. Changing $s \to s + s^{**}$, we shift the point where $s = 0$ to somewhere on $U_1$ away from $P1$. This will generate the required solution of Problem 2f which vanishes at $s = s^{**}$ with zero flux.

Now, let us consider set $O_{\mathrm{III}}$. It is clear that the $O_{\mathrm{III}}$-orbits represent monotonously increasing $f(s)$ which asymptotes to the inviscid $f_{P,Q}(s)$ solutions with $Q > 0$ and $P < 0$ for $s \to \infty$. However, $O_{\mathrm{III}}$ are very different from $O_{\mathrm{II}}$ near the left boundary of the $s$-interval.

Indeed, since $f(s)$ is monotonously growing, moving backwards in time the $O_{\mathrm{III}}$-orbits will reach small values of $f$ such that $f \ll f_D$ and $f \ll g$. Then the dynamical system (10) and (11) reduces to:

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = gf, \quad \frac{\mathrm{d}g}{\mathrm{d}\tau} = -2g^2,$$

solving which we have

$$g = \frac{1}{2(\tau - \tau^*)}, \quad f = C(\tau - \tau^*)^{1/2}, \quad \tau > \tau^*, \tag{43}$$

where $C$ is a positive constant.

Thus, the $O_{\mathrm{III}}$-orbits have a sharp left boundary $s_* < 0$, i.e. correspond to solutions of Problem 2f on $s_* < s < 0$ such that $f(s) \to 0$ as $s \to s_*$. In fact, $f(s)$ vanishes at a finite point that follows from (43). Indeed, in terms of $f(s)$ the obtained solution reads:

$$f(s) = (4C/9)|s - s_*|^{3/4}. \tag{44}$$

Note that at $s_*$ we have $f^2\mathrm{d}f/\mathrm{d}s = C^2/2$, which means that $\epsilon(s_*)$ is a finite negative number, see (7). Therefore, any $O_{\mathrm{III}}$-orbit also corresponds to an inverse energy cascade situation, but now with a finite amount of (negative) flux left at the point $s_*$ where the spectrum turns into zero. For realisability of such a solution one has to put a point sink of energy at the boundary $s = s_*$.

We will now put together the classification of the orbits.

**Theorem 4.1.** *$H \cup U_2 \cup P2$ and $U_1$ divide the phase plane $(f, g)(f > 0, -\infty < g < \infty)$ into the parts I, II and III with the different behaviours of orbits. In part I orbits emerge out of $P2$ and always asymptote to the negative part of the $g$-axis. In part II orbits emerge out of $P2$, the fourth quadrant intersecting the $f$-axis (never intersecting $g$-axis), and asymptote to the thermodynamic line as $\tau \to \infty$. In part III orbits emerge with $(f, g) \to (0, +\infty)$, go down along the $g$-axis, and then turn up asymptoting to the thermodynamic line as $\tau \to \infty$.*

Now we will consider how the orbits classified in theorem 4.1 could be linked to solutions of Problems 1, 2 and 3, or equivalently 1f, 2f and 3f. It is clear that there exist parameters of these problems for which solutions exist. Below we will show that not for all values of parameters there exist a solution.

**Theorem 4.2.** *There exist choices of parameters $(E_0, k_*)$ (of $(f_0, s_*)$) for which Problem 1 (Problem 1f) is not solvable.*

The Problem 1 is equivalent to the Problem 1f, so we will stick here to the Problem 1f. Initial condition $f(0) = f_0$ corresponds to the points on the vertical line $f = f_0$ on the $(f, g)$-

plane. Only orbits from the $O_{\mathrm{I}}$ family and the orbit $H$ are relevant to the Problem 1f, as only these orbits describe solutions vanishing at a finite right boundary, $s_* > 0$. It is clear that $s_*$ is a monotonously increasing function of $g_0$ for fixed $f_0$. Indeed, larger $g_0$ for fixed $f_0$ mean smaller negative values of $\mathrm{d}f/\mathrm{d}s(0)$. If such solutions had smaller $s_*$ than the values of $s_*$ corresponding to larger negative $\mathrm{d}f/\mathrm{d}s(0)$ then the two solutions would have to intersect at some $s > 0$, which is impossible due to uniqueness of solution of the Problem 1f (arising by choosing the intersection point at the right boundary). Secondly, for $g_0 \to -\infty$ at fixed $f_0$ we have $s_* \to 0$, which follows from the inviscid asymptotics of the solution valid at large $|g|$. Consider the case $f_0 < f_D$. From what we just said it follows that $s_*$ is bounded from above by a finite value corresponding to $s_*$ of the solution generated by the heteroclinic orbit $H$; let us call it $s_{*H}$. Thus we have proven that for any $f_0 < f_D$ there exist $0 < s_{*H} < +\infty$ such that the Problem 1f does not have solution with pair $(f_0, s_*)$ if $s_* > s_{*H}$ and have solution if $s_* \leqslant s_{*H}$. Now consider the case $f_0 \geqslant f_D$. In this case the value of possible $s_*$ is not bounded: it tends to infinity when $(f_0, g_0)$ approaches to $U_2$. Thus we have proven that for any $f_0 \geqslant f_D$ the Problem 1f has solution with pairs $(f_0, s_*)$ for any $s_* > 0$.

**Theorem 4.3.** *There exist choices of parameters* $(E_0, k_*)$ *(of* $(f_0, s_*)$*) for which Problem 2 (Problem 2f) is not solvable.*

The relevant orbits for this case are the ones from $O_{\mathrm{III}}$ and the separatrix $U_1$. In the same way as in the previous theorem, one can show that $s_*$ is a monotonously increasing function of $g_0$ for fixed $f_0$, and that $s_* \to 0$ for $g_0 \to +\infty$ at fixed $f_0$. But for any fixed $f_0$, the value of $s_*$ is bounded from below by some finite $s_* < 0$ corresponding to $U_1$; let us call it $s_{*U}$. Thus we have proven that for any $f_0$ there exist $-\infty < s_{*U} < 0$ such that the Problem 2f does not have solution with pair $(f_0, s_*)$ if $s_* < s_{*U}$ and has solution if $s_* \geqslant s_{*U}$.

**Remark 4.5.** If we are interested in solutions with finite $s_*$ at which both $f = \epsilon = 0$ then physically it makes sense to generalise the Problem 1f (2f) by postulating $f(s) \equiv 0$ in the range $s_* > s_{*H}$ $(s_* < s_{*U})$.

**Theorem 4.4.** *There exist choices of parameters* $(E_0, E_1, k_1)$ *(of* $(f_0, f_1, s_1)$*) for which Problem 3 (Problem 3f) is not solvable.*

This is the most general problem and relevant solutions may be given by orbits from all three parts of the phase plane. Clearly, in the limit $f_1 \to 0$ Problem 3f transforms into Problem 1f, and in the limit $f_0 \to 0$ and after shifting $s \to s - s_1$ it transforms into Problem 2f. Therefore by continuity we conclude from the previous two theorems that Problem 3f has no solutions for sufficiently large $s_1$ and small $f_0$ or/and $f_1$. However, for $f_D < f_0 < f_1$ Problem 3f is solvable for any $s_1$. This follows from the monotonously increasing (decreasing) and unbounded dependence of $s_1$ on $g_0$ when the latter is above (below) $U_2$ when $f_0$ and $f_1$ are fixed. Indeed, $s_1 \to 0$ for $g_0 \to \pm\infty$ and $s_1 \to +\infty$ for $(g_0, f_0) \to U_2$. In particular, for large $f$ or/and $g$ the solutions become $f_{P,Q}$ and, using equation (35), we find $P$ and $Q$ for any $(f_0, f_1, s_1)$:

$$P = c^{-3/2}[f_0^3 - f_1^3 \mathrm{e}^{3(z-5/2)s_1}]/(1 - \mathrm{e}^{-11s_1/2}), \tag{45}$$

$$Q = c^{-3/2}[f_0^3 - f_1^3 \mathrm{e}^{(3z-2)s_1}]/(1 - \mathrm{e}^{11s_1/2}). \tag{46}$$

### *4.3. Case $z > 5/2$*

In this case, $f(s)$ for the Kolmogorov and the thermodynamic spectra are decreasing. Since $z > 5/2$ then $D > 0$ and equation (31) again admits an exact positive solution $f(s) = f_D = 8\nu/D$.

The following technical lemma will be used later.

**Lemma 4.6.** *Solutions of the Cauchy problems for equation (31) are always bounded functions together with $f^2|\mathrm{d}f/\mathrm{d}s|$.*

It is easily established from the integral relation arising from (31):

$$2\left(f^2\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2(s) + (12z - 19)\int_0^s f^4\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^2\mathrm{d}\eta + \Phi(f(s)) = \Phi(f_0) + 2\left(f^2\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2(0). \tag{47}$$

Since now $12z - 19 > 0$, it follows from (47) that

$$\Phi(f(s)) < |\Phi(f_0)| + 2\left(f^2\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2(0). \tag{48}$$

Therefore $f(s) < K$ where the constant $K$ depends on $f(0)$, $\mathrm{d}f(0)/\mathrm{d}s$, $\nu$ and $z$. The same holds for $f^2|\mathrm{d}f/\mathrm{d}s|$ because $\Phi(f(s)) \geqslant \Phi(f_D)$.

**Lemma 4.7.** *The orbits of the dynamical system (10), (11) which intersect the line $g + f = 0$ cannot approach the $g$-axis for $g \leqslant 0$ as time $\tau$ evolves.*

Indeed, according to the formula (7), $\epsilon \leqslant 0$ above and on the thermodynamic line (this includes line $g + f = 0$) and $\epsilon > 0$ otherwise (including the $g$-axis for $g < 0$). But, according to equation (2), $\mathrm{d}\epsilon/\mathrm{d}k \leqslant 0$ (hence $\mathrm{d}\epsilon/\mathrm{d}s \leqslant 0$). Therefore the orbits starting on the line $g + f = 0$ cannot also approach the $g$-axis where $\epsilon > 0$ for $g \leqslant 0$.

**Remark 4.6.** The flux $\epsilon$ for orbits which go to the stable node $P2$ takes arbitrary negative values as $k \to \infty$.

**Lemma 4.8.** *There exist orbits (named by $O_1$-set) of the dynamical system (10), (11) which approach the $g$-axis for $g < 0$ of the phase plane $(f, g)$. These orbits are always below of the Kolmogorov line.*

The velocity field $(\mathrm{d}f/\mathrm{d}\tau, \mathrm{d}g/\mathrm{d}\tau)$ on the $g$-axis for $g < 0$ are directed down along this axis. By starting from different initial points near by the $g$-axis, we get orbits which asymptote to this axis with time as follows from the representation (43). If we change $\tau$ to the reverse time $\hat{\tau} = -\tau$ then these orbits go to infinity never intersecting the Kolmogorov line since the velocity field $G(f, -(1/3 + z)f) < 0$ i.e. is directed into the region $g + (z + 1/3)f < 0$.

**Lemma 4.9.** *There exists an orbit $S_1$ of the dynamical system (10), (11) which goes from infinity to the fixed point $P1$.*

Again, let us consider a Cauchy problem for the equation (31) with the initial data

$$f(0) = f_0 \equiv f_D - \varepsilon \left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{s=0} = f_m < 0 \tag{49}$$

and proceed as in the case with $z < 2/3$. Consider the set $B = \{f_m < 0\}$ with the properties: the corresponding solutions of the Cauchy problem (31) and (49) achieve a positive minimum at $s = s_{\min}$. It is clear that this set is not empty.

Consider an algebraic second-order curve $g = g(f)$ determined by the condition $\mathrm{d}g/\mathrm{d}\tau = 0$. From (11) we have:

$$g = g(f): \quad 2(f+g)^2 + \left(6z - \frac{17}{2}\right) f(f+g) + \frac{D}{2} f^2 - 4\nu f = 0. \tag{50}$$

This curve goes through the fixed point $P1$. There exists a branch $g = g_1(f)$ of $g = g(f)$ which is located below of the Kolmogorov line since $\mathrm{d}g/\mathrm{d}\tau > 0$ along $f^{\mathrm{K}}$. The velocity field $(\mathrm{d}f/\mathrm{d}\tau, \mathrm{d}g/\mathrm{d}\tau)$ restricted on $g_1(f)$ has the components: $\mathrm{d}f/\mathrm{d}\tau < 0$ and $\mathrm{d}g/\mathrm{d}\tau = 0$. Hence the orbits starting at $g \leqslant g_1(f)$ will always remain in the region $g \leqslant g_1(f)$, which implies $\mathrm{d}f/\mathrm{d}s = g + f < 0$, i.e. $f(s)$ is a monotonously decreasing function. Therefore $f_B = \inf B$ exists: we have $f_B \geqslant g_1(f_0) + f_0$.

Repeating the same arguments as the ones we used before for the case $z < 2/3$, we get that there exists a solution of (31) and (49) with the following property: the minimum of $f(s; f_B)$ is achieved at a point $s_*$ where again $f = \mathrm{d}f/\mathrm{d}s = 0$. Therefore there exists a solution of Problem 1f such that $f^2 \mathrm{d}f/\mathrm{d}s = 0$ at $s = s_*$. This solution on the phase plane corresponds to an orbit $S_1$ which goes to $P1$. If we change the time on the inverse time $\hat{\tau}$ then $S_1$ emerges out the fixed point $P1$ and goes to infinity never intersecting the Kolmogorov line (see lemma 4.8) and, therefore, never intersecting the line $g + f = 0$ (see lemma 4.7). This orbit corresponds to a monotonously increasing $f(\hat{\tau})$ which asymptotes to the pure Kolmogorov inviscid solution $f_{P,0}$. Here, $Q = 0$ follows from the fact that $S_1$ remains below the Kolmogorov line (implying $Q \leqslant 0$) and the fact that for $Q < 0$ the solution $f_{P,Q}$ would not be monotonously increasing.

Thus, the orbit $S_1$ corresponds to a direct energy cascade whose energy flux is gradually decreased by the viscous dissipation, so that both the spectrum and the energy flux turn into zero at a finite wave number $k_* = k_0 \mathrm{e}^s$.

**Lemma 4.10.** *There exists an orbit $H$ of the dynamical system (10) and (11) connecting $P1$ with $P2$ (a heteroclinic connection).*

Recall that there are always two (and only two) orbits connecting to $P1$, see section 2.4. One of them, entering into $P1$, is the orbit $S_1$ discussed above. Let us call the other orbit by $H$ —it is emerging out of $P1$ into the first quadrant. By lemma 4.6, $f(s)$ for such an orbit is bounded. By the same Lemma, $|g(s)|$ is bounded too provided that $f$ is bounded from below, which is indeed the case, if we start on $H$ stepping slightly away from $P1$. In this case $f$ cannot approach zero at $g < 0$ because this would mean achieving a positive flux $\epsilon$ which is impossible since the staring flux is negative and $\epsilon(s)$ cannot increase. Neither $f$ can approach zero at $g > 0$ because the vector field there is directed toward positive $f$. Thus, both $f(s)$ and $|g(s)|$ are bounded, and the orbit $H$ goes to the fixed point $P2$ by the Poincaré–Bendixon theorem.

**Lemma 4.11.** *The orbits starting at the line $g + f = 0$ approach the $g$-axis for $g > 0$ in the reverse time, $\hat{\tau} \to \infty$, for $f_0 \geqslant f_D$.*

In order to prove lemma 4.11, we consider the Cauchy problem for equation (31)

$$f(0) = f_0, \quad \left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{s=0} = 0, \quad s < 0. \tag{51}$$

The proof is similar to the case of $z < 2/3$ and is based on the same integral identity (37), but now for $s < 0$. In fact, all orbits $f_0 \geqslant f_D$ approach the $g$-axis as they approach the $g$-axis for $f_0 \geqslant 48\nu/5D$ and cannot intersect the orbit $H$. Asymptotic analysis of (10), (11) near the $g$-axis for $g > 0$ gives again $f \to (4C_1/9)|s - s_*|^{3/4}$ as $s \to s_*$ for a finite $s_*$ and constant $C_1$ i.e. $f(s)$ vanishes at a finite point.

**Lemma 4.12.** *There exists an orbit $S_2$ of the dynamical system (10) and (11) which goes from infinity to the fixed point $P2$. Orbit $S_2$ asymptotes to the thermodynamic line in the reverse time, so that $f \to f_{0,Q}$ as $\hat{\tau} \to \infty$.*

Any orbit starting on or above the thermodynamic line and with $f_0 > 0$ will end at $P2$. This can be shown in the same way as in the proof that $H$ goes to $P2$ in lemma 4.10. Reversed in time, all these orbits go to the regions where either $f \to \infty$ or/and $|g| \to \infty$, so that $f \to f_{P,Q}$ with $Q > 0$ and $P$ either positive, or negative, or zero. The orbit corresponding to $P = 0$ is $S_2$: it emanates from $P2$ and asymptotes to the thermodynamic line. Orbits with $P > 0$ asymptote to the Kolmogorov line, because for any finite positive $P$ and $Q$ the $P$-part wins in $f_{P,Q}(s)$ at $s \to -\infty$. We will call the set of these orbits $O_{\mathrm{II}}$. Orbits with $P < 0$ correspond to $f(s)$ that vanishes at finite $s = s_* < 0$. We will call the set of these orbits $O_{\mathrm{III}}$.

**Remark 4.7.** The orbits from $O_{\mathrm{III}}$ are located above the thermodynamic line and, therefore, characterised by the flux $\epsilon$ which is negative. The orbits which approach the $g$-axis for $g < 0$ (we will call then the $O_{\mathrm{I}}$-set) lie below the Kolmogorov line can be characterised, for large $f$ or/and $g$, by negative temperature $Q$. Note that for the same orbit, the value of $Q$ is usually different near the $g$-axis from its value for large $f$.

Summarising, we have the following classification of the orbits.

**Theorem 4.5.** *$S_1$ and $H \cup S_2 \cup P2$ divide the phase plane $(f, g)(f > 0, -\infty < g < \infty)$ into parts I, II and III with different behaviours of orbits. In part I (below $S_1$) the orbits always approach the $g$-axis for $g < 0$. Between $S_1$ and $H \cup S_2 \cup P2$, i.e. in II, the orbits go from infinity along $S_1$ to the stable node $P2$. Some of these orbits have an intermediate asymptotics—the $S_2$ orbit. Part III contains the orbits located above $H \cup S_2 \cup P2$. For the reverse time $\hat{\tau}$, the orbits emerge out the unstable node $P2$ and go to the first quadrant approaching the $g$-axis.*

Similarly to how it was done in the case $z < 2/3$, one can prove that the Problems 1, 2 and 3 (and respectively 1f, 2f and 3f) are not solvable for some sets of parameters.

### *4.4. Case* $2/3 < z < 5/2$

We have the following three possibilities: $12z - 19 > 0$, $12z - 19 < 0$ and $12z - 19 = 0$.

Consider the case of $12z - 19 \geqslant 0$ and give a qualitative analysis of the behaviour of orbits. If we start on the line $g + f = 0$ then we have from the integral identity (37):

$$\Phi(f(s)) \leqslant \Phi(f_0). \tag{52}$$

Since $\Phi(f(s))$ is a negative monotonously decreasing function, it follows from (52) that $f(s) > f_0 > 0$. The maximum principle guarantees that no local maximum of $f(s)$ can be achieved on the interval $(0, \infty)$. Indeed, since $D < 0$, according to (31) the second derivative of $f(s)$ cannot be negative at points where $\mathrm{d}f(s)/\mathrm{d}s = 0$. Therefore the solutions $f(s)$ monotonously increase. This can also be easily observed from the phase analysis of the dynamical system (10) and (11). Consider the angle $\Gamma : \{g = 0, g + f = 0\}$, $g < 0$ of the plane $(f, g)$. The direction of the velocity field $(\mathrm{d}f/\mathrm{d}\tau, \mathrm{d}g/\mathrm{d}\tau)$ restricted on the line $g + f = 0$ is directed into $\Gamma$. Therefore orbits cannot leave $\Gamma$ with time.

**Lemma 4.13.** *There exist orbits which approach the $g$-axis for $g < 0$. Also there exist an orbit $S_1$ which goes to the fixed point $P1$. This orbit is located below the Kolmogorov line.*

Consider the dynamical system (10), (11) and the algebraic curve (50). This curve goes through the fixed point $P1$. Consider the angle $\Lambda : \{f = 0, g + f = 0\}$ with $g < 0$. The vector field $(\mathrm{d}f/\mathrm{d}\tau, \mathrm{d}g/\mathrm{d}\tau)$ on the $g$-axis has components $(0, -g^2)$ and $\mathrm{d}g/\mathrm{d}\tau > 0$ on $g + f = 0$ except the fixed point $P1$. Therefore there exists a branch $g = g_1(f)$ of $g = g(f)$ which is located inside of $\Lambda$. The velocity field $(\mathrm{d}f/\mathrm{d}\tau, \mathrm{d}g/\mathrm{d}\tau)$ restricted on $g_1(f)$ has the components: $\mathrm{d}f/\mathrm{d}\tau < 0$ and $\mathrm{d}g/\mathrm{d}\tau = 0$. Hence $(\mathrm{d}f/\mathrm{d}\tau, \mathrm{d}g/\mathrm{d}\tau)$ is directed into $\Psi_1 : \{f = 0, g = g_1(f)\}$ and orbits do not leave $\Psi_1$. This means that there exists a set of initial data

$$f(0) = f_0, \quad \left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{s=0} = f_m, \quad f_m < 0 \tag{53}$$

such that solutions of (31) and (53) are monotonously decreasing functions (because region $\Psi_1$ is below the line $f + g = 0$). Local solvability of the problem (31), (53) follows from the theory of ODE. Thus, there exist two families of orbits. The first family $O_{\mathrm{I}}$ presents orbits which cross the curve $g_1(f) = 0$ approaching the $g$-axis. The second family $O_{\mathrm{II}}$ consists of the orbits which go into $\Gamma$. Therefore, there must be an orbit $S_1$ which splits $O_{\mathrm{I}}$ and $O_{\mathrm{II}}$. The existence of $S_1$ can be proven by using the same arguments as before. Consider a Cauchy problem for the equation (31) with the following condition:

$$f(0) = f_0, \quad \left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{s=0} = f_m < 0 \tag{54}$$

with arbitrary $f_0 > 0$. Define set $E = \{f_m\}$ with the properties: the corresponding solution of the Cauchy problem has a positive minimum at some $s = s_{\min}$. This set is not empty and bounded from below in view of the discussion above. Denote by $f_E = \inf E$ and let $f(s; f_E)$ be a solution of (31) and (54). As before (see the proof of lemma 4.3), we conclude that there exists a finite $s_*$ such that $f(s_*; f_E) = 0$ together with $f^2(s; f_E)\mathrm{d}f(s; f_E)/\mathrm{d}s|_{s=s_*}$. This solution corresponds to the orbit $S_1$ which goes to $P1$. The orbit $S_1$, and therefore all the orbits from $O_{\mathrm{I}}$, are always below the Kolmogorov line because the velocity field is crossing this line in the upward direction (i.e. when traced back in time $S_1$ will not cross the Kolmogorov line).

**Lemma 4.14.** *There exist orbits which approach the $g$-axis for $g > 0$ as the reverse time $\hat{\tau} \to \infty$. Also there exists an orbit $U_1$ which emerges out the fixed point $P1$. This orbit asymptotes to $f_{P,Q}$ with $Q > 0$ and $P < 0$, and lies above of the thermodynamic line.*

Consider orbits from $O_{\mathrm{II}}$. These orbits go to infinity and, after crossing the line $f + g = 0$, the corresponding $f(\tau)$ monotonously grow to infinity. The orbits cross the thermodynamic line and then asymptote towards it as $\tau \to \infty$. In reverse time $f(\hat{\tau})$ also grows monotonously to infinity after crossing the line $f + g = 0$. The orbits cross the Kolmogorov line and then asymptote towards it as $\hat{\tau} \to \infty$. Thus for each $O_{\mathrm{II}}$-orbit asymptotically the value of the energy flux tends to a positive constant at the right boundary of the $s$-interval and to a negative constant at the left boundary. Physically, this corresponds to a system with two energy sources at both ends of the wave number range which produce energy fluxes from the boundaries toward the middle of the wave number range, gradually decreased by the viscosity and turning into zero at some point within the wave number range.

The vector field $(\mathrm{d}f/\mathrm{d}\tau, \mathrm{d}g/\mathrm{d}\tau)$ restricted on the $g$-axis for $g > 0$ towards to the fixed point $P1$. Therefore there exist orbits which approach the $g$-axis with $g > 0$ as the reverse time $\hat{\tau} \to \infty$. We shall denote these orbits by $O_{\mathrm{III}}$. The orbits from $O_{\mathrm{III}}$ asymptote to the thermodynamic line as $\tau \to \infty$. The proof is the same as in the case of $z < 2/3$. Notice that $(\mathrm{d}f/\mathrm{d}\tau, \mathrm{d}g/\mathrm{d}\tau)$ restricted on the second branch $g = g_2(f)$ of the algebraic curve (50) is directed into $\Psi_2 : \{g = g_2(f), f = 0\}$ and $g = g_2(f)$ is above of the line $g + f = 0$. This means that orbits from $O_{\mathrm{III}}$ are always inside $\Psi_2$. $O_{\mathrm{II}}$ and $O_{\mathrm{III}}$ present orbits with different behaviours. The existence of an orbit which splits up $O_{\mathrm{II}}$ and $O_{\mathrm{III}}$ can be proven by using the same argument as in lemma 4.5. We denote this orbit by $U_1$ and in terms of the reverse time $\hat{\tau}$ the orbit $U_1$ realises a solution of Problem 2f such that $s_* < 0, |s_*| < \infty$ and $f^2(s)\mathrm{d}f(s)/\mathrm{d}s$ vanishes at $s = s_*$. The family $O_{\mathrm{III}}$ corresponds to solutions of Problem 2f which vanish at finite times with $f^2\mathrm{d}f/\mathrm{d}s \neq 0$ at these points. Here again we use the asymptotic solution $f(s) \approx (4C_1/9)|s - s_*|^{3/4}$ near the $g$-axis with $g > 0$.

The case $12z - 19 < 0$ is considered similar with the same classification theorem for orbits of the dynamical system. Summarising the results, we have the following classification of the orbits.

**Theorem 4.6.** *The orbits $S_1$ and $U_1$ divide the phase plane on $I$, $II$ and $III$ parts with different behaviours of orbits. The orbits from $I$ go along the separatrix $S_1$ approaching the $g$-axis at $g < 0$ with a finite flux, $\epsilon > 0$. Part $II$ consists of the orbits located between $S_1$ and $U_1$, the latter representing asymptotes for each of these orbits as $\hat{\tau} \to \infty$ and $\tau \to \infty$ respectively. For these orbits $\epsilon < 0$ as $\tau \to \infty$ and $\epsilon > 0$ as $\hat{\tau} \to \infty$. The set $III$ represents orbits emerging at the first quadrant and approaching $U_1$ with time. For the reverse time $\hat{\tau}$ these orbits approach the $g$-axis at $g > 0$ as $\hat{\tau} \to \infty$ with a finite flux, $\epsilon < 0$.*

Similarly to how it was done in the case $z < 2/3$, one can prove that the Problems 1, 2 and 3 (and respectively 1f, 2f and 3f) are not solvable for some sets of parameters.

**Remark 4.8.** Generalising remark 4.5 for any $z$: one can extend the Problem 1f (2f) in which $f(s_*) = \epsilon(s_*) = 0$ by postulating $f(s) \equiv 0$ for $s$ which is greater (less) than the maximal (minimal) allowed $s_*$.

## 5. Conclusions

In this paper we have presented an exhaustive study and full classification of all possible stationary solutions of the Leith model of turbulence with dissipation represented by equation (1) by the phase plane analysis of the corresponding dynamical system. Different solutions are realised depending on the degree of the dissipation $z$, the effective Reynolds number $f_D/8\nu$, position of the forcing (at the left and right boundaries for the direct and inverse cascades respectively), absence or presence of extra dissipation or/and forcing at the boundaries (dual cascades diverging to the centre of the $k$-range, point sinks instantly absorbing the remaining flux). Such solutions may or may not have a finite front, they may asymptotically tend to inviscid 'warm cascade' solutions at $k \to \infty$ with a finite constant flux $P$ and/or temperature $Q$, they may exhibit viscous scaling at high or low ends of the $k$-range. Many possible physical situations were linked to three types of the boundary value problems —Problems 1–3. In spite of the behavioural richness, the solutions may be divided into three distinct classes corresponding to the orbits the phase plane divided by separatrices which connect fixed points of the corresponding dynamical system with each other or with infinity.

The most physically relevant solutions are represented by the separatrices themselves. Let us mention another interesting solution corresponding to a warm direct cascade with $Q > 0$ such that the energy flux $\epsilon$ is zero at the right boundary. This is a typical solution in numerical simulations of turbulence by pseudo-spectral methods, implying there is a maximal wave number at which the energy flux is reflected. In our classification such solutions are to be found by solving Problem 3: e.g. one of such solutions could be obtained by first picking an arbitrary orbit from Part II of the phase plane, then picking its left end arbitrarily, and then placing its right end onto the thermodynamic line (see the discussion in section 3).

In section 3 we gave a qualitative description of the solutions, including the phase portraits and sketches of typical spectra. In section 4 we presented rigorous proofs of the statements made in section 3. The table below provides a brief summary of our solutions with emphasis on their physical meanings.

In future, it remains to be shown that the steady state solutions found in the present paper are attractors of the evolving system. It is also interesting to study scenarios of reaching the steady states. Based on a numerical evidence, the authors of papers [2, 3] suggested that the steady state in the direct cascade forms as a reflection wave propagating from high to low $k$'s in which the Kolmogorov scaling is gradually replacing an initially steeper transient power law. This seems to be the typical behaviour for finite capacity turbulent systems [11], and it is also observed in integral/kinetic equation closures [12]. However, such a scenario has not been explained analytically yet, and this would be an important subject for future work.

| Orbits | Physics | Problems |
|---|---|---|
| $U_1$, $z < \frac{5}{2}$ | Warm inverse cascade, zero flux at a sharp left end. | 2 |
| $S_1$, $z > \frac{2}{3}$ | Negative-$Q$ direct cascade, zero flux at a sharp right end. | 1 |
| $U_2$, $z < \frac{2}{3}$ | High-Re cold direct cascade; (possibly) the dissipative scaling near the left end and Kolmogorov near the right end. | 3 |
| $S_2$, $z > \frac{5}{2}$ | High-Re inverse cascade; (possibly) the dissipative scaling near the right end and thermodynamic near the left end with $\epsilon \to 0$ at $k \to 0$. | 3 |

(Continued.)

| Orbits | Physics | Problems |
|---|---|---|
| $H$, $z < \frac{2}{3}$ | Low-Re direct cascade, zero flux at a sharp right end. No scaling ranges. | 1 |
| $H$, $z > \frac{5}{2}$ | Low-Re inverse cascade, zero flux at a sharp left end. No scaling ranges. | 2 |
| $O_{\mathrm{I}}$, any $z$ | Direct cascade, point sink at a sharp right end. | 1 |
| $O_{\mathrm{III}}$, any $z$ | Inverse cascade, point sink at a sharp left end. | 2 |
| $O_{\mathrm{II}}$, any $z$ | Direct cascade, finite spectrum and point sink at right end, *or* inverse cascade, finite spectrum and point sink at left end, *or* converging direct and inverse cascades, point sources at both ends, *or* inverse cascade, finite spectrum and zero flux at the left end, *or* direct cascade, finite spectrum and zero flux at the right end. | 3 |

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