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The One-way Fubini Property and Conditional Independence: An Equivalence Result

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Version dated 2016 April 25th

Abstract

A general parameter process defined by a continuum of random variables is not jointly measurable with respect to the usual product $\sigma$-algebra. For the case of independent random variables, a one-way Fubini extension of the product space was constructed in [11] to satisfy a limited form of joint measurability. For the general case we show that this extension exists if and only if there is a countably generated $\sigma$-algebra given which the random variables are essentially pairwise conditionally independent, while their joint conditional distribution also satisfies a suitable joint measurability condition. Applications include new characterizations of essential pairwise independence and essential pairwise exchangeability through regular conditional distributions with respect to the usual product $\sigma$-algebra in the framework of a one-way Fubini extension.

Keywords: Continuum of random variables, joint measurability problem, one-way Fubini property, conditional distributions, characterizations of conditional independence.
1 Introduction

In this paper, a continuous parameter random process (or simply a process) is formalized as a collection of random variables indexed by points in an atomless measure space. This process is then also called a continuum of random variables. When the index space is the time line, such a process is usually assumed to be jointly measurable with respect to time and the random state in the usual product $\sigma$-algebra of probability theory.

Following the work on oceanic games by Milnor and Shapley [21], as well as the other pioneering contributions by Aumann [2], [3] and Hildenbrand [14], economists and game theorists have long been interested in the “continuum limit” of an economic model or game as the number of agents or players tends to infinity. Agents or players in such a limit are indexed by points in an atomless probability space. Such “continuum” models of random processes involving many agents work well for systemic risks taking the form of common random shocks that influence a non-negligible set of agents. Reality suggests, however, that these systemic risks are supplemented by risks at the individual level in the form of idiosyncratic micro shocks that influence a negligible set of agents. As shown in Corollary 1 below, however, a process that generates a continuum of random shocks satisfies a standard joint measurability condition only if there is essentially no idiosyncratic risk at all.

The first references to this non-measurability issue occur are by Doob [6, Theorem 2.2, p. 113] and [7, p. 67]. Indeed, this failure of joint measurability led him to claim in [7, p. 102] that processes with mutually independent random variables are only useful in the discrete parameter case.\footnote{Here we note that studying a continuum of (conditionally) independent random variables within an appropriate analytic framework has allowed the discovery of several new connections between some basic concepts in probability theory. For example, Theorem 1 of [12] shows the essential equivalence of pairwise and mutual conditional independence, which also implies the essential equivalence of pairwise and multiple versions of exchangeability. For other results related to the exact law of large numbers and its converse, see [25].} For the case of a process with a continuum of independent random variables, however, we constructed in [11] an extension of the product space satisfying a limited form of joint measurability, which we associated with a “one-way Fubini” property of double integrals. We also pointed out that a two-way Fubini extension as in [25] may not be possible in general (see Remark 3.2 in [11]).
The main aim of this paper is to characterize completely all processes that satisfy the one-way Fubini property, without assuming independence. In particular, the main result, Theorem 1, shows that a process satisfies the one-way Fubini property if and only if there is a countably generated $\sigma$-algebra $C$ such that: (i) the random variables are essentially pairwise conditionally independent given $C$; and (ii) the joint conditional distribution of the random variables given $C$ satisfies a suitable joint measurability condition.

We also discuss several applications. In particular, using the one-way Fubini property along with regular conditional distributions with respect to the usual product $\sigma$-algebra, Proposition 4 provides new characterizations for the basic concepts of independent and exchangeable random variables. In a more general setting, these characterizations allow us to show the duality between independence and exchangeability through the random variables and sample functions generated by a process.

In the sequel, we introduce the basic concepts in Section 2. The main result is stated in Section 3 and proved in Sections 4 and 5. As a first application of the general results proved earlier, Section 6 shows that any function that is jointly measurable in the usual sense differs fundamentally from a process that includes non-trivial idiosyncratic micro shocks. The second application in Section 7 considers, in the framework of a one-way Fubini extension, regular conditional distributions of a process with respect to the usual product $\sigma$-algebra. As a corollary, we use the regular conditional distributions to give new characterizations of essential pairwise independence and essential pairwise exchangeability as well as to demonstrate their duality.

2 Basic Definitions

We first fix some notation. Let $(T, \mathcal{T}, \lambda)$ be a complete atomless probability space. Let $(\Omega, \mathcal{A}, P)$ be a complete, countably additive probability space. Let $X$ be a complete separable metric space with the Borel $\sigma$-algebra $\mathcal{B}$. A process $g$ is a mapping from $T \times \Omega$ to $X$ such that, for all $t \in T$, the mapping $g_t(\cdot) = g(t, \cdot)$ is $\mathcal{A}$-measurable (i.e., $g_t$ is a random variable defined on $(\Omega, \mathcal{A}, P)$). Thus, the probability spaces $(T, \mathcal{T}, \lambda)$ and $(\Omega, \mathcal{A}, P)$ are used as the parameter and sample spaces, respectively, for the process $g$.

In the following two subsections, we introduce the two main concepts in this paper: (i) the one-way Fubini extension; (ii) regular conditional independence.
2.1 One-way Fubini property

The following definition was introduced in [11].

**Definition 1.** (1) A probability space \((T \times \Omega, W, Q)\) extends the usual product probability space \((T \times \Omega, T \otimes \mathcal{A}, \lambda \times P)\) provided that \(W \supseteq T \otimes \mathcal{A}\), with \(Q(E) = (\lambda \times P)(E)\) for all \(E \in T \otimes \mathcal{A}\).

The extended space \((T \times \Omega, W, Q)\) is a one-way Fubini extension of the space \((T \times \Omega, T \otimes \mathcal{A}, \lambda \times P)\) provided that, given any real-valued \(W\)-integrable function \(f\):

(i) for \(\lambda\)-almost all \(t \in T\), the function \(\omega \mapsto f_t(\omega)\) is integrable on \((\Omega, \mathcal{A}, P)\);

(ii) the function \(t \mapsto \int_{\Omega} f_t dP\) is integrable on \((T, \mathcal{T}, \lambda)\), with \(\int_{T \times \Omega} f dQ = \int_{T} (\int_{\Omega} f_t dP) \, d\lambda\).

The space \((T \times \Omega, W, Q)\) is a (two-way) Fubini extension of the space \((T \times \Omega, T \otimes \mathcal{A}, \lambda \times P)\) provided that, given any real-valued \(W\)-integrable function \(f\), one has in addition:

(iii) for \(P\)-almost all \(\omega \in \Omega\), the function \(f_\omega\) is integrable on \((T, \mathcal{T}, \lambda)\);

(iv) its integral w.r.t. \(t\) satisfies \(\int_{T \times \Omega} f dQ = \int_{\Omega} (\int_{T} f_\omega d\lambda) \, dP\).

(2) A process \(g : T \times \Omega \to X\) is said to satisfy the one-way Fubini property if there is a one-way Fubini extension \((T \times \Omega, W, Q)\) such that \(g\) is \(W\)-measurable.

2.2 Regular conditional independence

Recall that a \(\sigma\)-algebra \(\mathcal{C}\) on \(\Omega\) is said to be countably generated if there exists a countable family \(\{C_n\}_{n=1}^{\infty}\) of subsets of \(\Omega\) such that \(\mathcal{C} = \sigma(\{C_n\}_{n=1}^{\infty})\), the smallest \(\sigma\)-algebra including the whole family — see, for example, [4] (Ex. 2.11, p. 34). As shown in [4] (Ex. 20.1, p. 270), the \(\sigma\)-algebra \(\mathcal{C}\) is countably generated if and only if there exists a Borel measurable mapping \(\theta : \Omega \to \mathbb{R}\) such that \(\mathcal{C} = \sigma(\{\theta\})\), the smallest \(\sigma\)-algebra that makes the function \(\theta\) Borel measurable.

Given the probability space \((\Omega, \mathcal{A}, P)\), a sub-\(\sigma\)-algebra \(\mathcal{C} \subset \mathcal{A}\) is said to be essentially countably generated if it is the strong completion of a countably generated \(\sigma\)-algebra \(\mathcal{C}'\), in the sense that

\[ \mathcal{C} = \{ A \in \mathcal{A} \mid \exists A' \in \mathcal{C}' : P(A \triangle A') = 0 \}. \]

See [25, Definition 2.2]. For a nontrivial example, see [25, Proposition 5.6].
For simplicity, from now on we describe a $\sigma$-algebra as countably generated even when it is only essentially countably generated. Of course, the extra sets in the essentially countably generated $\sigma$-algebra are all null.

Let $\mathcal{M}(X)$ be the space of Borel probability measures on $X$ endowed with the topology of weak convergence of measures.³

**Definition 2.** Let $g$ a process from $T \times \Omega$ to $X$, and $C$ be a countably generated sub-$\sigma$-algebra of $\mathcal{A}$.

1. Two random variables $\phi$ and $\psi$ from $(\Omega, \mathcal{A}, P)$ to $X$ are said to be conditionally independent given $C$ if, for any Borel sets $B_1, B_2 \in \mathcal{B}$, the conditional probabilities satisfy
   \[
   P(\phi^{-1}(B_1) \cap \psi^{-1}(B_2) | C) = P(\phi^{-1}(B_1) | C) P(\psi^{-1}(B_2) | C).
   \] (1)

2. The process $g$ is said to be essentially pairwise conditionally independent given $C$ if, for $\lambda$-a.e. $t_1 \in T$, the random variables $g_{t_1}$ and $g_{t_2}$ are conditionally independent given $C$ for $\lambda$-a.e. $t_2 \in T$.⁴

3. A $T \otimes C$-measurable mapping $\mu$ from $T \times \Omega$ to $\mathcal{M}(X)$ is said to be an essentially regular conditional distribution process of $g$ given $C$ if, for $\lambda$-a.e. $t \in T$, the $C$-measurable mapping $\omega \mapsto \mu_t^\omega$ is a regular conditional distribution $P(g_t^{-1} | C)$ of the random variable $g_t$.

4. The process $g$ is said to be regular conditionally independent if there exists a countably generated sub-$\sigma$-algebra $C$ of $\mathcal{A}$ such that $g$ is essentially pairwise conditionally independent given $C$, and also $g$ admits an essentially regular conditional distribution process given $C$.

An important special case of this definition that is discussed in [10] is when all the variables $g_t$ are conditionally independent and identically distributed, i.e., exchangeable with $\mu_{t, \omega} = \mu^\omega_t$, independent of $t$; see also Proposition 4 below. The conditional independence assumption implies in general that the probability space $(\Omega, \mathcal{A}, P)$ is rich, in the sense that any nontrivial submeasure space is not essentially countably generated. See [16], [17] and [25], for example, for recent applications of such measure spaces.

³Recall that a mapping $\phi$ from a measurable space $(I, \mathcal{I})$ to $\mathcal{M}(X)$ is measurable w.r.t. the Borel $\sigma$-algebra generated by the topology of weak convergence of measures on $\mathcal{M}(X)$ if and only if it is event-wise measurable — i.e., for every event $B \in \mathcal{B}$, the mapping $i \mapsto \phi(i)(B)$ is $\mathcal{I}$-measurable (see, for example, [10, p. 748]).

⁴Note that this condition is weaker than requiring the random variables $g_{t_1}$ and $g_{t_2}$ to be conditionally independent given $C$ for $\lambda \times \lambda$-a.e. pair $(t_1, t_2) \in T \times T$. In addition, this also implies that the process satisfies essential mutual conditional independence, as shown in Theorem 1 of [12].
The following theorem characterizes the one-way Fubini property.

**Theorem 1.** A process $g$ from $T \times \Omega$ to $X$ satisfies the one-way Fubini property if and only if it is regular conditionally independent.

The Loeb product framework introduced in [18] (see also [20]) provides a rich class of Fubini extensions, as shown in [1], [15], [19] and [23]. For these we note that Theorem 1 also implies that any process on a Loeb product probability space must be regular conditionally independent.

**Proof of Necessity**

Proof of the necessity part of Theorem 1: Suppose that the product probability space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ has a one-way Fubini extension $(T \times \Omega, \mathcal{W}, Q)$, where $g$ is $\mathcal{W}$-measurable. Then, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the function $(t, \omega) \mapsto 1_A(\omega)1_{g^{-1}(B)}(t, \omega)$ is $\mathcal{W}$-measurable. By the one-way Fubini property, the mapping

$$t \mapsto \int_{\Omega} 1_A(\omega)1_{g^{-1}(B)}(t, \omega)dP = P(A \cap g_t^{-1}(B))$$

is $\mathcal{T}$-measurable. That is, the process $g$ has event-wise measurable conditional probabilities, as in property (3) in the statement of Theorem 1 in [13]. So property (1) of that theorem follows: specifically, the process $g$ is regular conditionally independent with respect to a suitable conditioning $\sigma$-algebra $\mathcal{C}$. 

We remark that the appropriate conditioning $\sigma$-algebra in this result is the Monte Carlo $\sigma$-algebra $\mathcal{C}^g$ specified in Definition 3 of [13].

**Proof of Sufficiency**

Throughout this section, let $g$ be a regular conditionally independent process from $T \times \Omega$ to $X$. Thus, there exists a countably generated sub-$\sigma$-algebra $\mathcal{C}$ of $\mathcal{A}$ such that $g$ is essentially pairwise conditionally independent given $\mathcal{C}$, and also $g$ admits an essentially regular conditional distribution process given $\mathcal{C}$. 

5
Define the mapping $H : T \times \Omega \rightarrow T \times \Omega \times X$ by $H(t, \omega) := (t, \omega, g(t, \omega))$. Let $\mathcal{E} := \mathcal{T} \otimes \mathcal{A} \otimes \mathcal{B}$ denote the product $\sigma$-algebra on $T \times \Omega \times X$. Let $\mathcal{F} := \{H^{-1}(E) : E \in \mathcal{E}\}$. Then it is clear that $\mathcal{F}$ is a $\sigma$-algebra. Also, the first two components of $H(t, \omega)$ are given by the identity mapping $\text{id}_{T \times \Omega}$ on $T \times \Omega$, while the last component is $g(t, \omega)$. Hence, $\mathcal{F}$ is the smallest $\sigma$-algebra such that $\text{id}_{T \times \Omega}$ and $g$ are both measurable. This means that $\mathcal{F}$ is the smallest extension of the product $\sigma$-algebra $\mathcal{T} \otimes \mathcal{A}$ such that $g$ is measurable.

Given any event $E \in \mathcal{E} = \mathcal{T} \otimes \mathcal{A} \otimes \mathcal{B}$, along with any fixed $t \in T$ and $\omega \in \Omega$, let $E_t$ denote the section $\{\omega \in \Omega \times X \mid (t, \omega, x) \in E\}$ and $E_{t\omega}$ the section $\{x \in X \mid (t, \omega, x) \in E\}$.

The proof of the sufficiency part of Theorem 1 relies on the following:

**Proposition 1.** Given any event $E \in \mathcal{E} = \mathcal{T} \otimes \mathcal{A} \otimes \mathcal{B}$, for $\lambda$-a.e. $t \in T$ the following four properties hold:

(i) the set $H^{-1}_t(E_t)$ is $\mathcal{A}$-measurable;
(ii) the mapping $\omega \mapsto \mu_{t\omega}(E_{t\omega})$ is $\mathcal{A}$-measurable;
(iii) $P(H^{-1}_t(E_t)) = \int_{\Omega} \mu_{t\omega}(E_{t\omega})dP$;
(iv) the mapping $t \mapsto P(H^{-1}_t(E_t))$ is $\lambda$-integrable.

In order to prove the proposition, we need several lemmas.

**Lemma 1.** For all $s, t \in T$ and $B \in \mathcal{B}$, one has

$$\mathbb{E}[1_{g^{-1}_s(B)}|\mathbb{E}(1_{g^{-1}_t(B)}|C)] = \mathbb{E}[\mathbb{E}(1_{g^{-1}_s(B)}|C)\mathbb{E}(1_{g^{-1}_t(B)}|C)]$$

**Proof.** By the law of iterated expectations,

$$\mathbb{E}[1_{g^{-1}_s(B)}\mathbb{E}(1_{g^{-1}_t(B)}|C)] = \mathbb{E}[\mathbb{E}(1_{g^{-1}_s(B)}\mathbb{E}(1_{g^{-1}_t(B)}|C)|C)]$$

because the function $\omega \mapsto \mathbb{E}(1_{g^{-1}_t(B)}|C)(\omega)$ is already $\mathcal{C}$-measurable — see, for example, [8] (p. 266).

Fix any Borel set $B$ in $X$. For each $t \in T$, define the function $h_t := 1_{g^{-1}_t(B)} - \mathbb{E}(1_{g^{-1}_t(B)}|C)$, which is a random variable on $(\Omega, \mathcal{A}, P)$.

**Lemma 2.** If $g_s$ and $g_t$ are conditionally independent given $\mathcal{C}$, then $h_s$ and $h_t$ are uncorrelated random variables with zero mean.
Proof. By the law of iterated expectations,

\[ E h_t = E 1_{g_t^{-1}(B)} - E [E (1_{g_t^{-1}(B)} | c)] = E 1_{g_t^{-1}(B)} - E 1_{g_t^{-1}(B)} = 0 \]

and similarly \( E h_s = 0 \). Furthermore,

\[
E h_s h_t = E [1_{g_s^{-1}(B)} 1_{g_t^{-1}(B)}] - E [1_{g_s^{-1}(B)} E (1_{g_t^{-1}(B)} | c)]
- E [1_{g_t^{-1}(B)} E (1_{g_s^{-1}(B)} | c)] + E [E (1_{g_s^{-1}(B)} | c) E (1_{g_t^{-1}(B)} | c)]
= E [E (1_{g_s^{-1}(B)} 1_{g_t^{-1}(B)} | c)] - E [E (1_{g_s^{-1}(B)} | c) E (1_{g_t^{-1}(B)} | c)]
\]

by Lemma 1 and the law of iterated expectations. But

\[ E (1_{g_s^{-1}(B)} 1_{g_t^{-1}(B)} | c) = E (1_{g_s^{-1}(B)} | c) E (1_{g_t^{-1}(B)} | c) \]

because \( g_s \) and \( g_t \) are conditionally independent given \( C \). So \( E h_s h_t = 0 \), implying that the two zero-mean random variables are uncorrelated.

Lemma 3. Suppose that the component random variables \( f_t \) \((t \in T)\) are all square-integrable and are almost surely uncorrelated — i.e., suppose each \( f_t \in L^2(\Omega, \mathcal{A}, P) \) and, for a.e. \( t_1 \in T \), one has \( E(f_{t_1}, f_{t_2}) = E f_{t_1} \cdot E f_{t_2} \) for a.e. \( t_2 \in T \). Then, for every \( A \in \mathcal{A} \), one has \( \int_A f_t dP = P(A) E f_t \) for \( \lambda \)-a.e. \( t \in T \).

Proof. This is Lemma 1 of [11], which was proved by considering orthogonal projections in Hilbert space.

Lemma 4. Given any \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), for \( \lambda \)-a.e. \( t \in T \) one has

\[ \int_A 1_{g_t^{-1}(B)} dP = \int_A E (1_{g_t^{-1}(B)} | c) dP \]

Proof. Because of Lemma 2, we can apply Lemma 3 to the bounded and so square-integrable random variables \( h_t \) \((t \in T)\). Hence, for \( \lambda \)-a.e. \( t \in T \)

\[ \int_A h_t dP = P(A) E h_t = 0 \]

Then the definition of \( h_t \) implies the claimed result directly.

Let \( \mathcal{D} \) be the collection of all events \( E \in \mathcal{E} \) whose sections \( E_t \) and \( E_{t\omega} \) satisfy properties (i)–(iii) in the statement of Proposition 1, for \( \lambda \)-a.e. \( t \in T \).
Lemma 5. Each measurable triple product set \( E = S \times A \times B \in \mathcal{E} \) satisfies (i)–(iii) of Proposition 1, implying that \( E \in \mathcal{D} \).

Proof. First, if \( t \notin S \), then \( E_t = \emptyset \) and \( \mu_{t\omega}(E_{t\omega}) = 0 \) for all \( \omega \in \Omega \), so (i)–(iii) hold trivially.

On the other hand, suppose that \( t \in S \).

(i) Then \( E_t = A \times B \), so \( H_t^{-1}(E_t) = A \cap g_t^{-1}(B) \), which is the intersection of the two \( \mathcal{A} \)-measurable sets \( A \) and \( g_t^{-1}(B) \). Thus, \( H_t^{-1}(E_t) \) is \( \mathcal{A} \)-measurable.

(ii) Furthermore, \( E_{t\omega} = B \) if \( \omega \in A \), but \( E_{t\omega} = \emptyset \) if \( \omega \notin A \). So the mapping \( \omega \mapsto \mu_{t\omega}(E_{t\omega}) = 1_A(\omega) \mu_{t\omega}(B) \) is obviously \( \mathcal{A} \)-measurable for all \( t \in S \).

(iii) For all \( t \in S \), Lemma 4 implies that

\[ P(H_t^{-1}(E_t)) = P(A \cap g_t^{-1}(B)) = \mathbb{E}[1_A \mathbb{E}(1_{g_t^{-1}(B)}|c)]. \]

Using the definition of \( \mu_{t\omega} \) gives

\[ P(H_t^{-1}(E_t)) = \mathbb{E}[1_A \mathbb{E}(1_{g_t^{-1}(B)}|c)] = \mathbb{E}[1_A \mu_{t\omega}(B)] = \int_{\Omega} \mu_{t\omega}(E_{t\omega}) dP \]

as required. \( \Box \)

Lemma 6. The family \( \mathcal{D} \) is a Dynkin (or \( \lambda \)-) class in the sense that:

(a) \( T \times \Omega \times X \in \mathcal{D} \);

(b) if \( E, E' \in \mathcal{D} \) with \( E \supseteq E' \), then \( E \setminus E' \in \mathcal{D} \);

(c) if \( E_n \) is an increasing sequence of sets in \( \mathcal{D} \), then \( \cup_{n=1}^{\infty} E_n \in \mathcal{D} \).

Proof. (a) \( T \times \Omega \times X \in \mathcal{D} \) as a triple product of measurable sets.

(b) If \( E, E' \) satisfy properties (i)–(iii) with \( E \supseteq E' \), then \( E \setminus E' = E_t \setminus E'_t \) and \( (E \setminus E')_{t\omega} = E_{t\omega} \setminus E'_{t\omega} \). Hence:

(i) For \( \lambda \)-a.e. \( t \in T \), the set \( H^{-1}((E \setminus E')_t) = H_t^{-1}(E_t) \setminus H_t^{-1}(E'_t) \) is \( \mathcal{A} \)-measurable.

(ii) The mapping \( \omega \mapsto \mu_{t\omega}((E \setminus E')_{t\omega}) = \mu_{t\omega}(E_{t\omega}) - \mu_{t\omega}(E'_{t\omega}) \) is \( \mathcal{A} \)-measurable.

(iii) Also

\[ P(H^{-1}((E \setminus E')_t)) = P(H_t^{-1}(E_t)) - P(H_t^{-1}(E'_t)) \]

\[ = \int_{\Omega} [\mu_{t\omega}(E_{t\omega}) - \mu_{t\omega}(E'_{t\omega})] dP = \int_{\Omega} \mu_{t\omega}((E \setminus E')_{t\omega}) dP \]

Hence, \( E \setminus E' \in \mathcal{D} \).
(c) If $E^n$ is an increasing sequence in $D$, then:

(i) For $\lambda$-a.e. $t \in T$, the set $H^{-1}_t(\cup_{n=1}^{\infty} E^n_t) = \cup_{n=1}^{\infty} H^{-1}_t(E^n_t)$ is $\mathcal{A}$-measurable.

(ii) The mapping $\omega \mapsto \mu_t(\cup_{n=1}^{\infty} E^n_t) = \lim_{n \to \infty} \mu_t(E^n_t)$ is $\mathcal{A}$-measurable.

(iii) Also

$$P(H^{-1}_t(\cup_{n=1}^{\infty} E^n_t)) = \lim_{n \to \infty} P(H^{-1}_t(E^n_t)) = \lim_{n \to \infty} \int_{\Omega} \mu_t(E^n_t)dP$$

by the monotone convergence theorem for integrals.

Hence, $\cup_{n=1}^{\infty} E^n_t \in D$.

This completes the proof that $D$ is a Dynkin class.

Proof of Proposition 1: We can apply Dynkin’s $\pi$–$\lambda$ theorem to establish that $D = E = T \otimes A \otimes B$, because the set of products of measurable sets is a $\pi$-system — i.e., closed under finite intersections (see [5], p. 44 and [9], p. 404). This verifies parts (i)–(iii).

Also, applying the ordinary Fubini theorem to the integrand $(t, \omega) \mapsto \mu_{t\omega}(E^n_{t\omega})$ on the product space $(T \times \Omega, T \otimes \mathcal{C}, \lambda \times P)$ shows that the mapping $t \mapsto \int_{\Omega} \mu_{t\omega}(E^n_{t\omega})dP$ is $\lambda$-integrable. So (iii) implies (iv).

Proof of the sufficiency part of Theorem 1: Let $g$ be a regular conditionally independent process. Let $\mathcal{F}$ be the $\sigma$-algebra $H^{-1}(E)$ as defined at the beginning of this section. Hence, given any $F \in \mathcal{F}$, there exists at least one $E \in \mathcal{E}$ such that $F = H^{-1}(E)$. By Proposition 1, the section $F_t = H^{-1}_t(E_t) \in A$ for $\lambda$-a.e. $t \in T$. The same result implies that the mapping $F \mapsto \nu(F) := \int_T P(F_t)d\lambda$ defines a unique set function $\nu$ on the $\sigma$-algebra $\mathcal{F}$. Arguing as in the proof of Theorem 1 in [11], it follows that $\nu$ is a uniquely defined probability measure, whose restriction to the product $\sigma$-algebra $T \otimes A$ is $\lambda \times P$. Hence $(T \times \Omega, \mathcal{F}, \nu)$ extends $(T \times \Omega, T \otimes \mathcal{A}, \lambda \times P)$.

To show that $(T \times \Omega, \mathcal{F}, \nu)$ is a one-way Fubini extension, we use exactly the same argument as that used to prove Theorem 1 in [11], without any need even to change notation. Even that argument was a simple adaptation of the standard proof of the usual Fubini Theorem — see, for example, [22] (p. 308).
6 Some General Results

As above, assume that for some countably generated sub-\(\sigma\)-algebra \(C\) of \(A\), the process \(g\) is essentially pairwise conditionally independent given \(C\), and admits a \(T \otimes C\)-measurable, essentially regular conditional distribution process \(\mu\).

**Proposition 2.** Let \(h\) be any measurable function mapping the product space \((T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)\) to a Polish space \(Y\). Then, for \(\lambda\)-almost all \(t \in T\), the two random variables \(g_t\) and \(h_t\) are conditionally independent given \(C\).

**Proof.** Let \(\mathcal{D}\) denote the Borel \(\sigma\)-algebra on \(Y\). For any \(B \in \mathcal{B}, C \in \mathcal{C}\) and \(D \in \mathcal{D}\), consider the set \(E = (h^{-1}(D) \times B) \cap (T \times C \times X)\). Since \(h\) is \(T \otimes \mathcal{A}\)-measurable, the set \(E\) belongs to \(T \otimes \mathcal{A} \otimes \mathcal{B}\). For each \(t \in T\), it is clear that \(E_t = (h_t^{-1}(D) \times B) \cap (C \times X) = (C \cap h_t^{-1}(D)) \times B\), and so \(H_t^{-1}(E_t) = C \cap h_t^{-1}(D) \cap g_t^{-1}(B)\). It is also clear that \(E_{t\omega} = B\) when \(\omega \in C \cap h_t^{-1}(D)\), and \(E_{t\omega} = \emptyset\) when \(\omega \notin C \cap h_t^{-1}(D)\). By Proposition 1, for \(\lambda\)-a.e. \(t \in T\) we have

\[
P(H_t^{-1}(E_t)) = P(C \cap h_t^{-1}(D) \cap g_t^{-1}(B)) = \int_{\Omega} \mu_{t\omega}(E_{t\omega})dP = \int_{C \cap h_t^{-1}(D)} \mu_{t\omega}(B)dP.
\]

By the properties of conditional expectation, and the fact that \(\mu_{t\omega}(B)\) is \(C\)-measurable for \(\lambda\)-a.e. \(t \in T\), we obtain

\[
\int_{C} \mathbb{E} \left( 1_{h_t^{-1}(D)} 1_{g_t^{-1}(B)} | C \right) dP = \int_{C} 1_{h_t^{-1}(D)} 1_{g_t^{-1}(B)} dP = \int_{C} 1_{h_t^{-1}(D)} \mu_{t\omega}(B) dP = \int_{C} \mathbb{E} \left( 1_{h_t^{-1}(D)} | C \right) \mu_{t\omega}(B) dP = \int_{C} \mathbb{E} \left( 1_{h_t^{-1}(D)} | C \right) \mathbb{E} \left( 1_{g_t^{-1}(B)} | C \right) dP.
\]

Hence, for \(\lambda\)-a.e. \(t \in T\), one has the pairwise conditional independence condition

\[
P(h_t^{-1}(D) \cap g_t^{-1}(B) | C) = P(h_t^{-1}(D) | C) P(g_t^{-1}(B) | C) \quad (2)
\]

Let \(C^\pi = \{C_n\}_{n=1}^\infty, B^\pi = \{B_m^\pi\}_{m=1}^\infty, \) and \(D^\pi = \{D_k^\pi\}_{k=1}^\infty\) be countable \(\pi\)-systems generating \(C, B, \) and \(D\) respectively. For each triple \((k, m, n),\) there
exists a set $T_{kmn}$ with $\lambda(T_{kmn}) = 1$ such that for all $t \in T_{kmn}$, (2) holds with $C = C_n$, $B = B^n_m$, and $D = D^n_k$. Let $T^* := \cap_{k=1}^\infty \cap_{m=1}^\infty \cap_{n=1}^\infty T_{kmn}$. It is obvious that $\lambda(T^*) = 1$. Now, whenever $t \in T^*$, (2) with $C = C_n$, $B = B^n_m$, and $D = D^n_k$, must hold for all triples $(k, m, n)$ simultaneously.

Because $C^\pi$ is a $\pi$-system that generates $C$, Dynkin’s $\pi–\lambda$ theorem (see [9], p. 404) implies that (2) must hold whenever $t \in T^*$, for all $C \in C$, all $B \in B^\pi$, and all $D \in D^\pi$ simultaneously. Finally, because $B^\pi$ and $D^\pi$ are $\pi$-systems that generate $B$ and $D$ respectively, (2) must hold whenever $t \in T^*$, $C \in C$, $B \in B$ and $D \in D$. Therefore for all $t \in T^*$, the random variables $g_t$ and $h_t$ are conditionally independent given $C$. This completes the proof.

Suppose that a standard joint measurability condition is imposed on a process $g$ that is used to model many agents who face idiosyncratic micro shocks combined with macroeconomic risks that generate the conditioning $\sigma$-algebra $C$. Then following corollary, which is Proposition 4 of [13], shows that there is essentially no idiosyncratic risk at all. The corollary generalizes the type of non-measurability result shown for independent random variables in Proposition 2.1 of [25], and for exchangeable random variables in Proposition 2 of [10].

**Corollary 1.** If $g$ is measurable on $(T \times \Omega, T \otimes A, \lambda \times P)$, then for $\lambda$-almost all $t \in T$, the random variable $g_t$ is $C$-measurable.

**Proof.** Proposition 2 implies that for $\lambda$-a.e. $t \in T$, the random variable $g_t$ is conditionally independent of itself, given $C$. Thus, for any $B \in B$,

$$P(g_t^{-1}(B) \cap g_t^{-1}(B)|C) = P(g_t^{-1}(B)|C) = P(g_t^{-1}(B)|C) P(g_t^{-1}(B)|C)$$

This evidently implies that $P(g_t^{-1}(B)|C) \in \{0, 1\}$ for all $\omega \in \Omega$. Let $A \in C$ denote the subset of $\Omega$ on which $P(g_t^{-1}(B)|C) = 1$. Then $P(g_t^{-1}(B)|C)$ almost surely has the same value as the indicator function $1_A$. It follows that $P(g_t^{-1}(B) \cap C) = P(A \cap C)$ for all $C \in C$. In particular, $P(g_t^{-1}(B) \cap A) = P(A)$ and $P(g_t^{-1}(B) \cap (\Omega \setminus A)) = 0$, which implies that $P(g_t^{-1}(B)\Delta A) = 0$. Therefore $g_t^{-1}(B) \in C$ for each $B \in B$, which completes the proof. \qed

By Theorem 1, the product probability space $(T \times \Omega, T \otimes A, \lambda \times P)$ has a one-way Fubini extension $(T \times \Omega, F, \nu)$ such that $g$ is $F$-measurable. The following proposition shows that, in the framework of a one-way Fubini extension $(T \times \Omega, F, \nu)$ the essentially regular conditional distribution process $\mu$ generates regular conditional distributions of the process $g$ with respect to the usual product $\sigma$-algebra $T \otimes A$. 

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Proposition 3. The $\mathcal{T} \otimes \mathcal{C}$-measurable mapping $\mu : T \times \Omega \to \mathcal{M}(X)$ satisfies $\nu(g^{-1}|T \otimes A) = \mu_{t,\omega}$ for $(\lambda \times P)$-a.e. $(t,\omega) \in T \times \Omega$.

Proof. Take any $G \in \mathcal{T} \otimes \mathcal{A}$, and any $B \in \mathcal{B}$. Let $E = G \times B$. For each $t \in T$, it is clear that $E_t = G_t \times B$, and $H_t^{-1}(E_t) = G_t \cap g_t^{-1}(B)$. It is also clear that $E_{t,\omega} = B$ when $\omega \in G_t$, and $E_{t,\omega} = \emptyset$ when $\omega \notin G_t$. By Proposition 1, for $\lambda$-a.e. $t \in T$ we have

$$P(H_t^{-1}(E_t)) = P(G_t \cap g_t^{-1}(B)) = \int_\Omega \mu_{t,\omega}(E_{t,\omega})dP = \int_{G_t} \mu_{t,\omega}(B)dP.$$  

Taking the integral of each side w.r.t. the measure $\lambda$ on $T$ gives

$$\int_T P(G_t \cap g_t^{-1}(B))d\lambda = \int_T \int_{G_t} \mu_{t,\omega}(B)dPd\lambda. \quad (3)$$

But the one-way Fubini property implies that

$$\int_G 1_{g_t^{-1}(B)}d\nu = \int_T \left[ \int_\Omega 1_{G(t,\omega)} 1_{g_t^{-1}(B)}dP \right]d\lambda = \int_T P(G_t \cap g_t^{-1}(B))d\lambda \quad (4)$$

Also, the usual Fubini property for $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ implies that

$$\int_T \int_{G_t} \mu_{t,\omega}(B)dPd\lambda = \int_G \mu_{t,\omega}(B)d(\lambda \times P) = \int_G \mu_{t,\omega}(B)d\nu \quad (5)$$

because $(T \times \Omega, \mathcal{F}, \nu)$ is a one-way Fubini extension. Together (3), (4) and (5) imply that $\int_G 1_{g_t^{-1}(B)}d\nu = \int_G \mu_{t,\omega}(B)d\nu$. Because the choice of $G \in \mathcal{T} \otimes \mathcal{A}$ and $B \in \mathcal{B}$ were arbitrary, it follows that $\nu(g^{-1}|T \otimes A) = \mu_{t,\omega}$ for $(\lambda \times P)$-a.e. $(t,\omega) \in T \times \Omega$. \hfill $\square$

7 Independence and Exchangeability

The following is part of Definition 5 in [13].

Definition 3. A process $g$ from $T \times \Omega$ to $X$ is said to be essentially pairwise exchangeable if there exists a fixed joint probability measure $\pi$ on $(X \times X, \mathcal{B} \otimes \mathcal{B})$ such that for $\lambda$-a.e. $t_1 \in T$, the random variables $g_{t_1}$ and $g_{t_2}$ have the given joint distribution $\pi$ for $\lambda$-a.e. $t_2 \in T$.\footnote{As in footnote 4, note that this condition is weaker than requiring the random variables $g_{t_1}$ and $g_{t_2}$ to have the joint distribution $\pi$ for $\lambda \times \lambda$-a.e. pair $(t_1, t_2) \in T \times T$. We also note that essential pairwise exchangeability is equivalent to its finite or countably infinite multivariate versions; see [12, Corollary 3] and [24, Theorem 4 and Proposition 3.5].}
Given any process that is measurable in a one-way Fubini extension, the following proposition characterizes essential pairwise independence and essential pairwise exchangeability through regular conditional distributions with respect to the relatively smaller product σ-algebra $T \otimes A$.

**Proposition 4.** Let $(T \times \Omega, W, Q)$ be a one-way Fubini extension of the product probability space $(T \times \Omega, T \otimes A, \lambda \times P)$, and $f$ any $W$-measurable process from $(T \times \Omega, W, Q)$ to a Polish space $X$. Let the mapping $(t, \omega) \mapsto \mu'_t\omega = Q(f^{-1}|T \otimes A)$ be a regular conditional distribution of $f$ with respect to $T \otimes A$. Then, the random variables $\omega \mapsto f_t(\omega)$ are:

1. essentially pairwise independent if and only if $(t, \omega) \mapsto \mu'_t\omega$ is essentially a function only of $t$;
2. essentially pairwise exchangeable if and only if $(t, \omega) \mapsto \mu'_t\omega$ is essentially a function only of $\omega$.

Before giving the proof of Proposition 4, we state two lemmas. The following lemma is a special case of Lemma 2 in [12].

**Lemma 7.** Let $g$ be a process from $T \times \Omega$ to $X$. Let $C \subseteq A$ be a countably generated σ-algebra on $\Omega$ and $\mu$ a $T \otimes C$-measurable mapping from $T \times \Omega$ to $M(X)$. Assume that for each fixed $A \in A$ and $B \in B$,

$$P(A \cap g_t^{-1}(B)) = \int_A \mu_t\omega(B) \, dP$$

for $\lambda$-a.e. $t \in T$. Then the process $g$ is essentially pairwise independent conditional on $C$, with $P(g_t^{-1}|C) = \mu_t\omega$ for $\lambda$-a.e. $t \in T$.

The following lemma is part of Propositions 6 and 7 in [13].

**Lemma 8.** Let $g$ be a process from $T \times \Omega$ to $X$. The process $g$ is essentially pairwise exchangeable if and only if there exists a measurable mapping $\omega \mapsto \mu_\omega$ from $(\Omega, A)$ to $M(X)$ such that for each $A \in A$ and $B \in B$, one has

$$P(A \cap g_t^{-1}(B)) = \int_A \mu_\omega(B) \, dP$$

for $\lambda$-a.e. $t \in T$.

**Proof of Proposition 4:** Fix any $A \in A$ and $B \in B$. For any $S \in T$, the definition of $\mu'$ implies that $\int_{S \times A} 1_{f_t^{-1}(B)} \, dQ = \int_{S \times A} \mu'_t(B) \, dQ$. Because the mapping $(t, \omega) \mapsto \mu'_t\omega$ must be measurable w.r.t. $T \otimes A$, the usual Fubini property implies that $\int_S \int_A 1_{f_t^{-1}(B)} \, dP \, d\lambda = \int_S \int_A \mu'_t\omega(B) \, dP \, d\lambda$. But the choice of $S \in T$ was arbitrary, so

$$P(A \cap f_t^{-1}(B)) = \int_A \mu'_t\omega(B) \, dP$$

for $\lambda$-a.e. $t \in T$. 

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(1) Suppose that the random variables \( f_t \) are essentially pairwise independent. Take \( C = \{ \Omega, \emptyset \} \). Then \( f_t \) is essentially pairwise conditionally independent given \( C \), and admits a \( T \otimes C \)-measurable, essentially regular conditional distribution process \( \mu' = P f_t^{-1} \). Proposition 3 then implies that \( Q(f^{-1}|T \otimes A) = \mu' = P f_t^{-1} \), which is essentially a function only of \( t \).

Conversely, suppose that \( (t, \omega) \mapsto \mu'_{t \omega} \) is essentially a function only of \( t \). Then we can say that \( (t, \omega) \mapsto \mu'_{t \omega} \) is \( T \otimes \{ \Omega, \emptyset \} \)-measurable and satisfies Equation (7). By Lemma 7 with \( C = \{ \Omega, \emptyset \} \), the random variables \( \omega \mapsto f_t(\omega) \) are essentially pairwise independent.

(2) Suppose that the random variables \( \omega \mapsto f_t(\omega) \) are essentially pairwise exchangeable. By Lemma 8, there exists a measurable mapping \( \omega \mapsto \mu_\omega \) from \( (\Omega, A) \) to \( M(X) \) such that for each \( A \in A \) and \( B \in B \), one has \( P(A \cap f_t^{-1}(B)) = \int_A \mu_\omega(B) dP \) for \( \lambda \)-a.e. \( t \in T \). Let \( C \) be the \( \sigma \)-algebra generated by the mapping \( \omega \mapsto \mu_\omega \). By Lemma 7, the process \( f \) is essentially pairwise conditionally independent given \( C \), and admits a \( T \otimes C \)-measurable, essentially regular conditional distribution process \( \mu \). It then follows from Proposition 3 that \( \mu'_{t \omega} = \mu_\omega \) for \( \lambda \times P \)-almost all \( (t, \omega) \in T \times \Omega \).

Conversely, suppose that \( (t, \omega) \mapsto \mu'_{t \omega} \) is essentially a function only of \( \omega \). Then, Equation (7) and Lemma 8 imply that the random variables \( \omega \mapsto f_t(\omega) \) are essentially pairwise exchangeable.

Using the framework of Loeb product spaces, it is shown in [24, Theorem 5] that the basic notions of independence and exchangeability are in fact dual to each other, in the sense that essential pairwise independence of the random variables is equivalent to essential pairwise exchangeability of the sample functions generated by the relevant process. Proposition 4 makes this duality result transparent and allows it to be extended from the Loeb product spaces used in [24] to the more general setting of a two-way Fubini extension.

**Corollary 2.** Let \( (T \times \Omega, \mathcal{W}, Q) \) be a two-way Fubini extension of the product probability space \( (T \times \Omega, T \otimes A, \lambda \times P) \), and \( (t, \omega) \mapsto f(t, \omega) \) a \( \mathcal{W} \)-measurable

\[\text{regular conditional distribution of } f \text{ with respect to } T \otimes A \text{ as in Proposition 3 is stated for the minimal one-way Fubini extension of the product probability space in which } f \text{ is measurable. On the other hand, } (T \times \Omega, \mathcal{W}, Q) \text{ is a general one-way Fubini extension in which } f \text{ is measurable, and thus includes the minimal one-way Fubini extension. By the definition of conditional expectations, it is easy to see that the regular conditional distribution of } f \text{ with respect to } T \otimes A \text{, as viewed in an extended probability space, } (T \times \Omega, \mathcal{W}, Q) \text{ remains the same.} \]
process from \((T \times \Omega, W, Q)\) to a Polish space \(X\). Then the random variables \(\omega \mapsto f_t(\omega)\) are essentially pairwise independent if and only if the sample functions \(t \mapsto f_\omega(t)\), regarded as random variables on \((T, \mathcal{T}, \lambda)\), are essentially pairwise exchangeable.

Proof. Let \(\mu' = Q(f^{-1}|\mathcal{T} \otimes \mathcal{A})\) be a regular conditional distribution of \(f\) with respect to \(\mathcal{T} \otimes \mathcal{A}\). Then part (1) of Proposition 4 implies that the random variables \(\omega \mapsto f_t(\omega)\), are essentially pairwise independent if and only if \((t, \omega) \mapsto \mu'_{t\omega}\) is essentially a function only of \(t\).

By viewing \((\Omega, \mathcal{A}, P)\) as the parameter space and \((T, \mathcal{T}, \lambda)\) as the sample space, the two-way Fubini extension property on \((T \times \Omega, W, Q)\), together with Part (2) of Proposition 4, imply that \((t, \omega) \mapsto \mu'_{t\omega}\) is essentially a function only of \(t\) if and only if the functions \(t \mapsto f_\omega(t)\) are essentially pairwise exchangeable, when viewed as random variables on \((T, \mathcal{T}, \lambda)\).

The result follows immediately from the above two equivalence results.

\[\square\]

References


