Representability in Supergeometry

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Abstract

[1] In this paper we use the notion of Grothendieck topology to present a unified way to approach representability in supergeometry, which applies to both the differential and algebraic settings.

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1 Introduction

Supergeometry is the mathematical tool originally developed to study supersymmetry. It was discovered, in the early 1970s, by the physicists Wess and Zumino ([WZ74]) and Salam and Strathdee ([SS74]) among others. Supergeometry grew out of the works of Berezin ([Be87]), Kostant([Kos77]) and Leites ([Lei80]), then, later on, by Manin ([Man88]), Bernstein ([Del98]) and others. These authors introduced an algebraic point of view on differential geometry, with emphasis on the methods that were originally developed in algebraic geometry by Grothendieck to handle schemes. In particular, the functor of points approach invented by Grothendieck turned out to be very useful to formalize the physical "anticommuting variables" of supersymmetry and provided a very useful tool to link algebra and geometry in a categorical way. The language developed by Grothendieck is, in fact, powerful enough to reveal the geometric nature not just of superschemes, but also of supermanifolds and superspaces in general.

In this paper we want to examine representability in the supergeometric context, with the use of Grothendieck topologies. In particular, we are able to prove a representability criterion (see Theorem 4.8), that can be applied to functors $\mathcal{C}^{\mathrm{op}} \longrightarrow (\mathrm{Set})$, where $\mathcal{C}$ is a *superspace site*, that is a full subcategory of the category of superspaces (SSpaces), with some additional very natural properties (see Def. 4.1). This broadens the range of application of the criterion, first published in [CCF11], to include, not only superschemes or supermanifolds, but also some less trivial categories like Leites regular supermanifolds and locally finitely generated superspaces, introduced by Alldridge et al. in [All13]. Our hope is that more general objects can be studied using this criterion, which formalizes the ideas of Grothendieck, adapting them to the supergeometric context.

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2 Preliminaries on Grothendieck topologies

We start with the notion of *Grothendieck topology*. For more details we refer the reader to [SGA4], Exposé ii, [2] [Vis05], [StPr].

Definition 2.1. Let $\mathcal{C}$ be a category. A *Grothendieck topology* $\mathcal{T}$ on $\mathcal{C}$ assigns to each object $U \in \mathrm{Ob}(\mathcal{C})$ a collection $\mathrm{Cov}(U)$ whose elements are families of morphisms with fixed target U, with the following properties.

(1) If $V \to U$ is an isomorphism, $\{V \to U\} \in \mathrm{Cov}(U)$.

(2) If $V \to U$ is an arrow and $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$, the fibered products $\{U_i \times_U V\}$ exist and the collection of projections $\{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}(V)$.

(3) If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ and for each i we have that $\{V_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(U_i)$, then $\{V_{ij} \to U_i \to U\}_{i \in I, j \in J_i} \in \mathrm{Cov}(U)$.

The pair $(\mathcal{C}, \mathcal{T})$ is called a *site*. The elements of $\mathrm{Cov}(U)$ are called *coverings*.

We may abuse the notation and write $\mathcal{U} \in \mathcal{T}$ or $\mathcal{U} \in \mathrm{Cov}(\mathcal{C})$ to indicate that $\mathcal{U} = \{U_i \to U\}_{i \in I}$ is a covering for the topology $\mathcal{T}$.

Now we discuss some key examples, which will be fundamental for our treatment.

Example 2.2.

1. Let us consider a topological space X and set $\mathcal{X}_{cl}$ to be the category with open sets as objects and inclusions as arrows. We say $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ if and only if $\bigcup_{i \in I} U_i = U$. We obtain a site $(X_{cl}, \mathcal{X}_{cl})$.

2. Let us consider the category (Sch) of schemes and define coverings of U to be collections of open embeddings whose images cover U. This is a topology, because of the existence and the properties of the fibered product in (Sch). This is called the *Zariski topology*.

Now we want to compare different topologies on the same category.

[2] In [SGA4] what we call "Grothendieck topology" is called "pretopology", we adhere to the terminology in [Vis05].

Definition 2.3. Let $\mathcal{C}$ be a category and let $\mathcal{U} = \{U_i \to U\}_{i\in I}$, $\mathcal{V} = \{V_j \to U\}_{j\in J}$ be two families of arrows with fixed target. We say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if for every $j \in J$ there exists $i \in I$ such that $V_j \to U$ factors through $U_i \to U$.

We say that the topology $\mathcal{T}$ is *subordinate* to the topology $\mathcal{T}'$ if every covering in $\mathcal{T}$ has a refinement that is a covering in $\mathcal{T}'$ and we write $\mathcal{T} \prec \mathcal{T}'$. If $\mathcal{T} \prec \mathcal{T}'$ and $\mathcal{T}' \prec \mathcal{T}$ we say that $\mathcal{T}$ and $\mathcal{T}'$ are *equivalent* i. e. $\mathcal{T} \equiv \mathcal{T}'$.

We are now in the position to define what a sheaf is in this framework.

Let $(\mathcal{C}, \mathcal{T})$ be a site, $\mathcal{U} = \{U_i \to U\}_{i\in I}$ a covering. Consider $F : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ and the fibered product $U_i \times_U U_j$. We denote the projection morphisms as

$$\mathrm{pr}_1 : U_i \times_U U_j \to U_i \quad \text{and} \quad \mathrm{pr}_2 : U_i \times_U U_j \to U_j$$

and the pullback morphisms $F(\mathrm{pr}_1)$, $F(\mathrm{pr}_2)$ as pr_1^* and pr_2^*.

Definition 2.4. Let $(C, \mathcal{T})$ be a site and consider $F : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$.

(1) F is *separated* if, given $\mathcal{U} = \{U_i \to U\}_{i\in I} \in \mathcal{T}$ and $x, y \in F(U)$ so that their pullback to $F(U_i)$ coincide for every i, we have $x = y$.

(2) F is a *sheaf* if, for every covering $\mathcal{U} = \{U_i \to U\}_{i\in I} \in \mathcal{T}$, the diagram

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} F(U_i \times_U U_j)$$

is the diagram of an equalizer, in other words the first arrow is injective with image

$$\{(\xi_i) \in \prod_i F(U_i) \mid \mathrm{pr}_1^*(\xi_i) = \mathrm{pr}_2^*(\xi_i) \text{ on } \prod_{i,j} F(U_i \times_U U_j)\}$$

where $\mathrm{pr}_i^* = F(\mathrm{pr}_i)$. If $\mathcal{U} = \{ f_i : U_i \to U \} \in \mathcal{T}$, we call the morphisms $F(f_i) : F(U) \to F(U_i)$ *restrictions*. Notice that the morphism $F(U) \to \prod_{i\in I} F(U_i)$ is induced by restrictions.

We have the following proposition (see [Vis05]).

Proposition 2.5. *Let $\mathcal{C}$ be a category and $\mathcal{T}$, $\mathcal{T}'$ two topologies on $\mathcal{C}$. If $\mathcal{T} \prec \mathcal{T}'$, then a sheaf in $\mathcal{T}'$ is also a sheaf in $\mathcal{T}$. Furthermore if $\mathcal{T} \equiv \mathcal{T}'$, then the two topologies have the same sheaves.*

We now introduce the *slice category*, which will be instrumental to discuss superspaces over a base superspace.

Definition 2.6. Let $\mathcal{C}$ be a category and let $S \in \mathrm{Ob}(\mathcal{C})$. We call $\mathcal{C}/S$ the *slice category of* $\mathcal{C}$ *over* S. This is the category with objects the pairs consisting of an object X in $\mathcal{C}$ and an arrow $f : X \to S$. Given two objects $f : X \to S$ and $h : Y \to S$, a morphism between them is an arrow $g : X \to Y$ of $\mathcal{C}$ such that $h \circ g = f$.

We now want to lift a Grothendieck topology from a site $(\mathcal{C}, \mathcal{T})$ to a slice category $\mathcal{C}/S$.

Definition 2.7. Let $(\mathcal{C}, \mathcal{T})$ be a site and let $\mathcal{C}/S$ be the slice category of $\mathcal{C}$ over S. The *slice topology* $\mathcal{T}/S$ on $\mathcal{C}/S$ is the topology which has as coverings of an object X/S families of morphisms $\mathcal{U}/S = \{ \varphi_i : U_i \to X \}$, where the φ_i are morphisms in $\mathcal{C}/S$, and $\mathcal{U}/S \in \mathcal{T}$.

One can readily check that $\mathcal{T}/S$ is a topology over $\mathcal{C}/S$.

Definition 2.8. Let us consider a category $\mathcal{C}$ and let $\mathcal{T}$ be a Grothendieck topology on it. $\mathcal{T}$ is said to be *subcanonical*, if all the representable functors are sheaves with respect to the topology $\mathcal{T}$. Recall that $F : \mathcal{C}^{\mathrm{op}} \longrightarrow (\mathrm{Set})$ is *representable* if there exists an object $X \in \mathrm{Ob}(\mathcal{C})$ such that $F \cong h_X$, where $h_X(Y) = \mathrm{Hom}(Y, X)$ and $h_X(f)\phi = f \circ \phi$. $\mathcal{T}$ is called *canonical* if is subcanonical and every subcanonical topology is subordinate to it.

One can show that if $\mathcal{T}$ is subcanonical on $\mathcal{C}$, then $\mathcal{T}/S$ is subcanonical on $\mathcal{C}/S$ (see [Vis05], Proposition 2.59).

3 Superspaces and functor of points

We introduce the category of superspaces and some important subcategories. For all of the terminology and notation we refer the reader to [Del98], [Man88], [Var04], [CCF11].

Definition 3.1. A *superspace* X is a topological space $|X|$ endowed with a sheaf (in the usual sense) of commutative super rings $\mathcal{O}_X$, which is called the *structure sheaf* of X, so that for every $p \in |X|$, the stalk $\mathcal{O}_{X,p}$ is a local super ring.

Superspaces forms a category denoted by (SSpaces) whose morphisms are given as follows. Given two superspaces $(|X|, \mathcal{O}_X)$ and $(|Y|, \mathcal{O}_Y)$, an arrow between them is a pair $f = (|f|, f^*)$, so that:

(1) $|f| : |X| \to |Y|$ is a continuous map.

(2) $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of super rings.

(3) The map of local super rings $f_p^* : \mathcal{O}_{Y,|f|(p)} \to \mathcal{O}_{X,p}$ is a local morphism for all p.

If X is a superspace, we can define its *functor of points*:

$$h_X : (\text{SSpaces})^{\text{op}} \longrightarrow (\text{Set}), \quad h_X(T) = \text{Hom}(T, X), \quad h_X(\phi)(f) = f \circ \phi.$$

One of the main purposes of this note is to establish when a functor $F : \mathcal{C}^{\text{op}} \longrightarrow (\text{Set})$, where $\mathcal{C}$ is a full subcategory of (SSpaces), is the functor of points of a superspace or more precisely of an object in the category $\mathcal{C}$.

Following Def. 2.6, we can define the category of *superspaces over a base superspace* S. The objects of this category are pairs consisting of a superspace X together with a morphism $X \longrightarrow S$. Morphisms are defined accordingly, as the morphisms of superspaces that commute with the given morphisms to S. A special case of particular interest to us is when $S = (|S|, k)$, where $|S|$ is a point and the structural sheaf is just a field k. We call the superspaces over such an S, *k-superspaces* and we denote them as $(\text{SSpaces})_k$. We shall see more on this later.

Remark 3.2. We observe that k-superspaces are simply superspaces whose structure sheaf is a sheaf of k-superalgebras. Moreover the morphisms of $(\text{SSpaces})_k$ are the superspace morphisms which preserve the k-superalgebra structure, i.e. which are k-linear.

Consider a superspace X. If $|U|$ is open in $|X|$, we can define the superspace $X_U = (|U|, \mathcal{O}_{X|U})$, together with the canonical morphism $j_{X|U} : X_U \to X$. We say that a morphism of superspaces $\varphi : Y \to X$ is an *open embedding*, if there exist an isomorphism $\phi : Y \to X_U$ and $\varphi = j_{X|U} \circ \phi$. We denote by $\varphi(Y)$ the superspace X_U. It makes sense to speak about union and intersections of open subsuperspaces U and V and we shall write, with an abuse of notation, $U \cup V$ and $U \cap V$ to denote them.

A morphism of superspaces $f : X \to Y$ is said to be a *closed embedding* if $|f|$ is a closed embedding and the morphism $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, i.e. it induces an isomorphism

$$\mathcal{O}_Y/\mathcal{I}_X \to f_*\mathcal{O}_X, \qquad \mathcal{I}_X := ker(f^*).$$

We call $\mathcal{I}_X$ the *vanishing ideal* of f or of X.

We say that a morphism $f : X \to Y$ between superspaces is an *embedding* if $f = g \circ h$ where h is a closed embedding and g is an open embedding. See [All13] for more details.

Open embeddings are particularly important because of the following fact, which is stated in the case of superspaces in [All13] and for ringed spaces in [EGA], Chapitre 0, 4.5.2.

Proposition 3.3. *Let $i : U \to X$ be an open embedding of superspaces. Then for every morphism $f : Y \to X$ of superspaces, the fibered product $U \times_X Y$ exists and:*

$$U \times_X Y = (|f|^{-1}(|X_U|), \mathcal{O}_{Y||f|^{-1}(|X_U|)}) = Y_{|f|^{-1}(U)}$$

Furthermore, $\mathrm{pr}_2 : U \times_X Y \to Y$ *is an open embedding.*

We now define a Grothendieck topology on (SSpaces).

Definition 3.4. Consider the category (SSpaces) and for $X \in \mathrm{Ob}((\mathrm{SSpaces}))$ define $\mathrm{Cov}(X)$ as the collections

$$\mathcal{U} = \{\varphi_i : U_i \to X\}_{i \in I}$$

where U_i are superspaces, the arrows φ_i are open embeddings and $\bigcup_{i \in I} |U_i| = |X|$, where, from now on, with an abuse of notation we write $|U_i|$ in place of $|\varphi_i|(|U_i|)$.

As one can readily check, we obtain a Grothendieck topology $\mathcal{S}$, that we call the *global super topology*. So we have a site $((\mathrm{SSpaces}), \mathcal{S})$.

Superspaces can be built by local data, that are suitably patched together.

Definition 3.5. A *gluing datum* $((U_i)_{i \in I}, (U_{ij})_{i,j \in I}, (\phi_{ij})_{i,j \in I})$ of superspaces consists of:

- a collection of superspaces $(U_i)_{i \in I} = (|U_i|, \mathcal{O}_{U_i})_{i \in I}$;

- a collection of open super subspaces $(U_{ij} \subseteq U_i)_{i,j\in I}$ so that for each $i \in I$, $U_{ii} = U_i$.
- a collection of isomorphisms $(\phi_{ji} : U_{ij} \to U_{ji})_{i,j\in I}$ so that the cocycle condition holds:

$$\phi_{ki} = \phi_{kj} \circ \phi_{ji} \quad \text{on} \quad U_{ij} \cap U_{ik}$$

If $((U_i)_{i\in I}, (U_{ij})_{i,j\in I}, (\phi_{ij})_{i,j\in I})$ is a gluing datum, there exists a superspace $X = (|X|, \mathcal{O}_X)$ and a family of open embeddings $\{\phi_i : U_i \to X\}_{i\in I}$ so that $X = \bigcup_{i\in I} \phi_i(U_i)$, $\phi_i = \phi_j \circ \phi_{ji}$ on U_{ij} for all $i, j \in I$ and finally $\phi_i(U_{ij}) = \phi_j(U_{ji}) = \phi_i(U_i) \cap \phi_j(U_j)$ for all $i, j \in I$. X and $\{\phi_i : U_i \to X\}_{i\in I}$ are uniquely determined up to isomorphism.

There are some full subcategories of (SSpaces) that are particularly interesting. We start by examining the differential setting.

Definition 3.6. A *real (resp. complex) superdomain* of dimension $(p|q)$ is a super ringed space $U^{p|q} = (U, \mathcal{O}_{U^{p|q}})$ where U is an open subset of $\mathbb{R}^p$ (resp. $\mathbb{C}^p$) and

$$\mathcal{O}_{U^{p|q}}(V) = \mathcal{O}_U \otimes \bigwedge(\theta^1, \dots, \theta^q)$$

for V open in U. $\mathcal{O}_U = C^\infty_U$ is the sheaf of C^∞ functions on U (resp. $\mathcal{O}_U = \mathcal{H}^\infty_U$ is the sheaf of holomorphic functions on U), $\bigwedge(\theta^1, \dots, \theta^q)$ is the exterior algebra in q odd coordinates $\theta^1, \dots, \theta^q$.

Let be $M = (|M|, \mathcal{O}_M)$ a k-superspace ($k = \mathbb{R}$ or $\mathbb{C}$). We say that M is a *supermanifold* if M is locally isomorphic, as k-superspace, to a real or complex superdomain.[3] Real supermanifolds are called *smooth supermanifolds*, while complex supermanifolds are called *holomorphic supermanifolds*; their categories are denoted as $(\mathrm{SMan})_\mathbb{R}$ and $(\mathrm{SMan})_\mathbb{C}$ or simply as (SMan), when we do not want to mark the difference between the smooth and holomorphic treatment.

As for superspaces we can easily define superschemes (or supermanifolds) on a base superscheme (or supermanifold) S. We leave to the reader the details.

Two further interesting examples of superspaces are the *Leites superspaces* and the *locally finitely generated superspaces* (see [All13]).

[3] In [All13] these are called *presupermanifolds*. In many references the definition of supermanifold requires more hypotheses (e.g. Hausdorff) on the topological space.

Definition 3.7. Let X be an object of $(\mathrm{SSpaces})_k$ ($k = \mathbb{R}$ or $\mathbb{C}$). We say that X is a *Leites regular superspace* if for every open subsuperspace $U \subseteq X$, and every integer p, the following map is a bijection

$$\Psi : \mathrm{Hom}(U, k^p) \to (\mathcal{O}_X(U)_0)^p, \quad \varphi \mapsto (\varphi^*(t_1), \dots, \varphi^*(t_p))$$

Here k^p is a superdomain, $t_1, ..., t_p$ are the canonical coordinates of k^p. We denote their category with $(\mathrm{SSpaces}^{\mathrm{L}})_k$.

Leites superspaces are a full subcategory of $(\mathrm{SSpaces})_k$ and open subsuperspaces of Leites regular superspaces are still Leites regular. Moreover, we have immediately that supermanifolds are Leites regular superspaces, because of Chart Theorem (see [Man88] Ch. IV).

Remark 3.8. Leites superspaces essentially describe the superspaces for which the Chart Theorem holds. Furthermore, it is possible to give a definition of Leites superspaces which comprehends not only differentiable or holomorphic supermanifolds, but also the real analytic ones. We do not introduce them here, referring the reader to [All13] for more details; their treatment is similar to the other ones.

We can now define the category of locally finitely generated k-superspaces. First of all, we need the definition of tidy embedding.

Definition 3.9. Consider an embedding $\varphi : Y \to X$ of superspaces with vanishing ideal $\mathcal{I}$. We say that φ is *tidy* if for every $x \in \varphi(|Y|)$, every open neighbourhood $U \subseteq |X|$ of x and every $f \in \mathcal{O}_X(U)$, we have:

$$(\forall y \in U \cap \varphi(|Y|), r \in \mathbb{N} : f_y \in \mathcal{I}_y + \mathfrak{m}^r_{X,y}) \Rightarrow f_x \in \mathcal{I}_x$$

(f_x is the image of f in $\mathcal{O}_{X,x}$). We say that a superspace X is *tidy* if id_X is a tidy embedding.

For more about tidiness, we refer the reader to [All13], Section 3.4. We note that open subsuperspaces of tidy superspaces are tidy.

We can now recall the definition of locally finitely generated superspace.

Definition 3.10. Let X be an object of $(\mathrm{SSpaces})_k$. We say that X is *finitely generated* if there exists a tidy embedding $\varphi : X \to k^{p|q}$. We say that X is *locally finitely generated* if it admits a cover by open subsuperspaces which are finitely generated. We denote the full subcategory of $(\mathrm{SSpaces})_k$ of locally finitely generated superspaces as $(\mathrm{SSpaces})_k^{\mathrm{lfg}}$.

Note that open subsuperspaces of locally finitely generated superspaces are still in $(\mathrm{SSpaces})_k^{\mathrm{lfg}}$. The following proposition is one of the main results in [All13] (see Sec. 5).

Proposition 3.11. $(\mathrm{SSpaces})_k^{\mathrm{lfg}}$ *is a full subcategory of the category of Leites regular superspaces; it is finitely complete (i.e. all the finite limits exist in it) and* $(\mathrm{SMan})_k$ *is a full subcategory of* $(\mathrm{SSpaces})_k^{\mathrm{lfg}}$*. Moreover finite limits are preserved by the inclusion.*

Observation 3.12. Notice that the following categories are all full subcategories of $(\mathrm{SSpaces})_k$: $(\mathrm{SMan})_k$, $(\mathrm{SSpaces}^{\mathrm{L}})_k$, $(\mathrm{SSpaces})_k^{\mathrm{lfg}}$. Similarly, the categories $(\mathrm{SMan})_k/S$, $(\mathrm{SSpaces}^{\mathrm{L}})_k/S$, $(\mathrm{SSpaces})_k^{\mathrm{lfg}}/S$ are full subcategories of $(\mathrm{SSpaces})_k/S$.

We now turn to examine the algebraic category.

Definition 3.13. A *superscheme* X is a superspace $(|X|, \mathcal{O}_X)$ where $(|X|, \mathcal{O}_{X,0})$ is an ordinary scheme and $\mathcal{O}_{X,1}$ is a quasi coherent sheaf of $\mathcal{O}_{X,0}$-modules. Morphisms of superschemes are the morphisms of the corresponding superspaces. We denote the category of superschemes by (SSch).

Notice that $(\mathrm{SSch})/S$ is a full subcategory of $(\mathrm{SSpaces})/S$.

4 Representability

In this section we examine a unified way to write representability theorems in supergeometry. Given a full subcategory $\mathcal{C}$ of (SSpaces), the question we want to study is how to characterize among all functors $F : \mathcal{C}^{\mathrm{op}} \longrightarrow (\mathrm{Set})$ those which are the functor of points of an object in $\mathcal{C}$.

We start with the notion of superspace site, which will be instrumental for the general result on representability we want to give.

Definition 4.1. We say that a site $(\mathcal{C}, \mathcal{T})$ is a *superspace site* if:

- $\mathcal{C}$ is a full subcategory of (SSpaces).
- $\mathcal{T}$ is a topology such that, for $\mathcal{U} = \{f_i : U_i \to X\}_{i \in I} \in \mathcal{T}$, we have that the arrows f_i are open embeddings and $\bigcup_{i \in I} |U_i| = |X|$.
- Given $X \in \mathrm{Ob}(\mathcal{C})$ and an open subset $|U| \subseteq |X|$, $X_U \in \mathrm{Ob}(\mathcal{C})$.

- Given a gluing datum $((U_i)_{i\in I}, (U_{ij})_{i,j\in I}, (\phi_{ij})_{i,j\in I})$ in $\mathcal{C}$, and X the corresponding superspace, with $\{\phi_i : U_i \to X\}_{i\in I}$ family of open embeddings, we have $X \in \mathrm{Ob}(\mathcal{C})$.

Notice that the last two conditions say that superspace site $\mathcal{C}$ is closed under the operation of gluing and under the operation of restriction to open subsets. It is immediate to extend the previous definition to $(\mathrm{SSpaces})/S$.

Observation 4.2. Proposition 3.11 and Observation 3.12, together with Prop. 3.3, give us that $(\mathrm{SMan})_k$, $(\mathrm{SSpaces}^{\mathrm{L}})_k$, $(\mathrm{SSpaces})_k^{\mathrm{lfg}}$, (SSch) are all examples of superspace sites, where the topology is given by the open embeddings i.e. it is the global super topology.

We have the following proposition.

Proposition 4.3. *Let be $(\mathcal{C}, \mathcal{T})$ a superspace site. Then $\mathcal{T}$ is subcanonical. Hence every representable functor $F : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is a sheaf for $\mathcal{T}$.*

Now we would like to establish the converse, that is, we want to understand, when given a superspace site $(\mathcal{C}, \mathcal{T})$, a sheaf is representable, in other words it is the functor of points of an object in $\mathcal{C}$. This is a most relevant question in supergeometry, since very often the only way to get hold of a supergeometric object is through its functor of points.

Definition 4.4. Given $F, G : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$, a natural transformation $f : F \to G$ between them is called *representable* or a *representable morphism* if for every object X in $\mathcal{C}$ and every natural transformation $g : h_X \to G$, the functor $F \times_G h_X$ is representable.

We now give the notion of open subfunctor.

Definition 4.5. Let us consider the functors $U, G : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ and a natural transformation $f : U \to G$. We say that U is an *open subfunctor* of G, if f is a monomorphism, it is representable and for every natural transformation $g : h_X \to G$, $X \in \mathcal{C}$, the second projection $\mathrm{pr}_2 : U \times_G h_X \to h_X$, corresponds to an open embedding $\mathrm{pr}_2^{\mathrm{Yon}} : Z \longrightarrow X$, with $h_Z \cong U \times_G h_X$, ($\mathrm{pr}_2^{\mathrm{Yon}} : Z \longrightarrow X$ corresponds to the second projection of the fibered product via the Yoneda's Lemma).

Definition 4.6. Let us consider a superspace site $(\mathcal{C}, \mathcal{T})$ and take a functor $F : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$. We say that a family $\{f_i : U_i \to F\}_{i \in I}$ of open subfunctors is an *open covering* of F, if for every $X \in \mathrm{Ob}(\mathcal{C})$ and every natural transformation $g : h_X \to F$, the family $\{(\mathrm{pr}_2)_i^{\mathrm{Yon}} : X_i \to X\}_{i \in I}$ is a covering of X, where the X_i are the superspaces such that $h_{X_i} \simeq U_i \times_F h_X$. If the U_i are representable functors, we will say that F has an *open covering by representable functors*.

Observation 4.7. Observe that, in a category $\mathcal{C}$, given $f : h_X \to h_Z$ and $g : h_Y \to h_Z$, then the fibered product $h_X \times_{h_Z} h_Y$ exists and it is representable if and only if $X \times_Z Y$ exists in $\mathcal{C}$ and in this case, we have $h_X \times_{h_Z} h_Y \simeq h_{X \times_Z Y}$. Then a morphism $f : h_X \to h_Z$ is representable if and only if, for every $g : h_Y \to h_Z$, $X \times_Z Y$ exists. So we have that in every superspace site, a morphism $f : h_X \to h_Z$ such that $f^{\mathrm{Yon}} : X \to Z$ is an open embedding is always representable (f^{Yon} denotes the morphism between superspaces corresponding to f via the Yoneda's Lemma).

On the other hand in the categories (SSch) and $(\mathrm{SSpaces})_k^{\mathrm{lfg}}$, fibered products always exist, hence the morphism $f : h_X \to h_Z$ is always representable, even in the case in which it is not an open embedding. We shall not need this fact in the sequel, see [CCF11] and [All13] for more details.

We are now ready to state our representability criterion, which is essentially a rewriting of the work by Grothendieck, in our context. Our proof generalizes the classical treatment in [GW10] (Theorem 8.9) and furthermore provides a unified perspective on the corresponding results in [CCF11] (Theorems 10.3.7 and 9.4.3, see also the footnote after Def. 3.6), besides including the categories $(\mathrm{SSpaces}^{\mathrm{L}})$, $(\mathrm{SSpaces})_k^{\mathrm{lfg}}$ (see Prop. 3.11) whose representability issues, to our knowledge, are not dealt with elsewhere. The reader is also invited to read the statement and the proof of Proposition 4.5.4, Chapitre 0 of [EGA], making a comparison.

Theorem 4.8. *Let us consider a superspace site $(\mathcal{C}, \mathcal{T})$ and a functor $F : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$. Then F is representable if and only if both of the following hold:*

(1) F is a sheaf for $(\mathcal{C}, \mathcal{T})$.

(2) F has a open covering by representable functors $\{f_i : U_i \to F\}_{i \in I}$.

Proof. Let us suppose that $F \simeq h_X$, i.e. F is representable. Then by Proposition 3.1.5, F is a sheaf and (1) holds. Moreover, we observe that the identity natural transformation $\mathrm{id}_F : F \to F$ is an open covering by representable functors and so also (2) holds.
Conversely, suppose that both (1) and (2) hold. Denote the open covering by representable functors of F by $\{f_i : U_i \simeq h_{X_i} \to F\}_{i\in I}$. We can assume without loss of generality that $\{f_i : h_{X_i} \to F\}_{i\in I}$ is our covering, i.e. that $U_i = h_{X_i}$. We want to build an $X \in \mathrm{Ob}(\mathcal{C})$ such that $h_X \simeq F$. We do this by constructing an appropriate gluing datum $((V_i)_{i\in I}, (V_{ij})_{i,j\in I}, (\phi_{ij})_{i,j\in I})$. We set $V_i = X_i$. By the definition of open subfunctor, we have that $h_{X_i} \times_F h_{X_j}$ is representable, and accordingly there exists an element $X_{ij} \in \mathrm{Ob}(\mathcal{C})$ such that

$$h_{X_i} \times_F h_{X_j} \simeq h_{X_{ij}}$$

with $\varphi_{ji} := (\mathrm{pr}_2)_i^{\mathrm{Yon}} : X_{ij} \to X_j$ an open embedding of superspaces. We set $V_{ji} = (\mathrm{pr}_2)_i^{\mathrm{Yon}}(X_{ij})$. The morphisms f_i are open subfunctors, so they are injective hence $(f_i)_Q : h_{X_i}(Q) \to F(Q)$ are also injective. So we can identify $(h_{X_i} \times_F h_{X_j})(Q)$ and $(h_{X_j} \times_F h_{X_i})(Q)$ with $h_{X_i}(Q) \cap h_{X_j}(Q) \subseteq F(Q)$ for all Q. Then $h_{X_j} \times_F h_{X_i} = h_{X_i} \times_F h_{X_j}$ and we have $X_{ij} \simeq X_{ji}$. We define $\phi_{ij} = \varphi_{ij} \circ \varphi_{ji}^{-1} : V_{ji} \simeq V_{ij}$ To show that $((V_i)_{i\in I}, (V_{ij})_{i,j\in I}, ((\phi_{ij})_{i,j\in I})$ is a gluing datum, we have to check the cocycle condition. Let X_{ijk} be the superspace whose functor of points is $h_{X_{ijk}} := h_{X_i} \times_F h_{X_j} \times_F h_{X_k}$. The X_{ijk}'s are all isomorphic for any permutation of the indices i, j, k, via the appropriate restrictions of ϕ_{ij}. We leave the easy details to the reader. Consequently it is almost immediate to verify the cocycle condition

$$\phi_{ki} = \phi_{kj} \circ \phi_{ji} \quad \text{on} \quad V_{ij} \cap V_{ik}.$$

Since $((V_i)_{i\in I}, (V_{ij})_{i,j\in I}, ((\phi_{ij})_{i,j\in I})$ is a gluing datum we get a superspace X, which, by definition of superspace site, is an object of $\mathcal{C}$. Notice that we have a covering $\mathcal{U} = \{X_i \to X\}_{i\in I} \in \mathcal{T}$, hence $\{h_{X_i} \to h_X\}_{i\in I}$ is an open covering by representable functors. We are left to prove that $F \simeq h_X$. We construct a natural transformation $\eta : F \to h_X$. For each superspace $T \in \mathrm{Ob}(\mathcal{C})$, we need to give a morphism $\eta_T : F(T) \to h_X(T)$. By Yoneda's Lemma, $F(T) \simeq \mathrm{Hom}(h_T, F)$, so we take $g \in \mathrm{Hom}(h_T, F)$. Consider the diagram:

$$\begin{tikzcd}
h_{Y_i} \simeq h_{X_i} \times_F h_T \arrow[r] \arrow[d, "(pr_1)_i"'] & h_T \arrow[d, "g"] \\
h_{X_i} \arrow[r, "f_i"'] & F
\end{tikzcd}$$

We have $(Y_i \to T)_{i\in I} \in \mathrm{Cov}(T)$. By Yoneda's lemma, we obtain a family of morphisms $(\mathrm{pr}_1)_i^{\mathrm{Yon}} : Y_i \to X_i$, that glue together, to give a morphism $t : T \to X$. So we set $\eta_T(g) = t$.
Now we have to build a natural transformation $\delta : h_X \to F$, which is the inverse of η. We define for each each superspace $T \in \mathrm{Ob}(\mathcal{C})$, a morphism $\delta_T : h_X(T) \to F(T)$. Take $t \in h_X(T)$ and set $Y_i = t^{-1}(X_i) = X_i \times_X T$. This is the preimage of the open superspace X_i in X via the morphism t. Then $(l_i : Y_i \to T)_{i\in I} \in \mathrm{Cov}(T)$. We obtain morphisms $g_i : Y_i \to X_i$, which correspond via Yoneda's Lemma to natural transformations $g_i' : h_{Y_i} \to h_{X_i} = U_i$. The morphisms $f_i \circ g_i'$ glue together, because F is a sheaf, to a morphism $g' : h_T \to F$, which corresponds by Yoneda's Lemma to an element $g \in F(T)$. We define $\delta_T(t) = g$. One can readily check that η and δ are natural transformations one inverse of the other. □

Our result has a straightforward generalization to $\mathcal{C}/S$, we leave the details to the reader.

Now we can state a corollary of the previous theorem, that can be useful in applications.

Corollary 4.9. *Let be $(\mathcal{C}, \mathcal{T})$ a superspace site and let be $\mathcal{T}'$ a topology on $\mathcal{C}$ so that $\mathcal{T} \prec \mathcal{T}'$. If a functor $F : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ satisfies*

(1) F is a sheaf for $(\mathcal{C}, \mathcal{T}')$,

(2) F has an open covering by representable functors $\{f_i : U_i \to F\}_{i\in I}$,

then it is representable. Moreover this condition is also necessary for the representability of F if $\mathcal{T}'$ is subcanonical.

Proof. If (1) and (2) hold, the hypothesis $\mathcal{T} \prec \mathcal{T}'$ implies that F is a sheaf for $(\mathcal{C}, \mathcal{T})$ and then F is representable by the previous theorem. Moreover, if $\mathcal{T}'$ is subcanonical, then a representable functor is by definition a sheaf for the topology $\mathcal{T}'$, and $\mathrm{id}_F : F \to F$ is a open covering by representable functors, then (1) and (2) are satisfied. □

The following observation is important for the applications of our main result 4.8.

Observation 4.10. The representability criterion 4.8 holds for the sites detailed in Obs. 4.2. In other words, to prove that a functor $F : \mathcal{C}^{\mathrm{op}} \longrightarrow (\mathrm{Set})$ is the functor of points of an object in $\mathcal{C}$, where $(\mathcal{C}, \mathcal{T})$ is a superspace site, it is enough to verify the properties (1) and (2) of 4.8.

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