GEOMETRIC STRUCTURES, GROMOV NORM AND KODAIRA DIMENSIONS

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Abstract. We define the Kodaira dimension for 3-dimensional manifolds through Thurston's eight geometries, along with a classification in terms of this Kodaira dimension. We show this is compatible with other existing Kodaira dimensions and the partial order defined by non-zero degree maps. For higher dimensions, we explore the relations of geometric structures and mapping orders with various Kodaira dimensions and other invariants. Especially, we show that a closed geometric 4-manifold has nonvanishing Gromov norm if and only if it has geometry $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^2(\mathbb{C})$ or $\mathbb{H}^4$.

Contents

1. Introduction

Complex Kodaira dimension $\kappa^h(M,J)$ provides a very successful classification scheme for complex manifolds. This notion is generalized by several authors (*c.f.* [31, 39, 40, 26, 27]) to symplectic manifolds, especially of dimension two and four. In these two dimensions, this symplectic Kodaira dimension is independent of the choice of symplectic structures [31]. In other words, it is a smooth invariant of the manifold which is thus denoted by $\kappa^s(M)$. In dimension four, the smaller the symplectic Kodaira dimension, the more we know. Symplectic 4-manifolds with $\kappa^s=-\infty$ are diffeomorphic to rational or ruled surfaces [36]. When $\kappa^s=0$, all known examples are K3 surface, Enrique surface and T^2 bundles over T^2. Moreover, it is shown in [31] that a symplectic manifold with $\kappa^s=0$ has the same homological invariants as one of the manifolds listed above. When $\kappa^s=1$ or 2, no classification is possible since symplectic manifolds in both categories could admit arbitrary finitely presented group as their fundamental group [19].

In [9], the authors prove that complex and symplectic Kodaira dimensions are compatible with each other. More precisely, when a 4-manifold M admits at the same time both complex and symplectic structures (but the structures are not necessarily compatible with each other), then $\kappa^s(M)=\kappa^h(M,J)$. In [35], a general framework of "additivity of Kodaira dimension" is provided to further understand the compatibility of various Kodaira dimensions in possibly different dimensions. In particular, it is shown that the Kodaira dimensions are additive for fiber bundles, Lefschetz fibrations and coverings.

Higher dimensional generalizations of Kodaira dimension, *e.g.* symplectic Kodaira dimension in dimensions six or higher, are less understood except for a proposed definition in [33]. Like complex Kodaira dimension, it will no longer be a smooth invariant. Hence, the study of this notion in higher dimensions will be associated to the study of deformation classes of symplectic structures and symplectic birational geometry.

As suggested by the additivity framework, dimension three should also attach certain counterpart of Kodaira dimension. In this paper, we give a definition of Kodaira dimension $\kappa^t(M)$ in dimension three through Thurston's eight 3-dimensional geometries and the Geometrization Theorem. It takes value from $-\infty$, 0, 1 or $\frac{3}{2}$. The half integer $\frac{3}{2}$ is a new phenomenon, since complex or symplectic Kodaira dimension only takes value from integers. Certain classification with respect to $\kappa^t(M)$ is given. This notion is then discussed in the framework of "additivity of Kodaira dimension". In this sense, it is compatible with the complex Kodaira dimension and symplectic Kodaira dimension in dimension 4. Remarkably, we show that the 3-dimensional Kodaira dimension is compatible with the partial order defined by non-zero degree maps.

Theorem 1.1. *If $f: M^3 \longrightarrow N^3$ is a non-zero degree map, then $\kappa^t(M)\geq \kappa^t(N)$.*

Evidently, the theorem implies

Corollary 1.2. *If M^3 and N^3 are equivalent with respect to nonzero degree maps, then $\kappa^t(M) = \kappa^t(N)$.*

This result could also be viewed as the first step towards a relative version of 3-dimensional Kodaira dimension as what we did for 4-dimensional symplectic manifolds in [35].

The *Gromov norm* (or *simplicial volume*) is a homotopy invariant of oriented closed manifolds which is introduced by Gromov in [20]. It is defined by minimizing the sum of the absolute values of the coefficients over all singular chains representing the fundamental class. A remarkable fact in dimension three is that a closed geometric 3-manifold has nonzero Gromov norm if and only if it is hyperbolic.

There are nineteen 4-dimensional geometries, which were classified by Filipkiewicz [14]. As in dimension three, we divide the nineteen geometries into 4 categories: $-\infty, 0, 1$ and 2, corresponding to the 4 possible values of Kodaira dimensions of 4-manifolds. We call this number the Kodaira dimension κ^g of geometric 4-manifolds.

We give structural description of geometric manifolds with $\kappa^g = 1$ (Proposition 4.3 and Theorem 4.4). Especially, we show

Theorem 1.3. *Any closed geometric 4-manifold with geometry $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{SL_2} \times \mathbb{E}$ or $\mathbb{H}^3 \times \mathbb{E}$ admits a foliation by geodesic circles.*

Usually people expect, as suggested by Ricci flow, that 4-manifolds are glued together from Einstein manifolds and collapsed pieces. However, by the Hitchin-Thorpe theorem, complex surfaces or symplectic 4-manifolds with Kodaira dimension 1 do not admit any Einstein metric. Hence, Proposition 4.3 and Theorem 4.4 describe "collapsed pieces" in Kodaria dimension 1.

Theorem 1.3 implies the following fact which identifies closed geometric 4-manifolds with vanishing Gromov norm.

Corollary 1.4. *A closed geometric 4-manifold has nonzero Gromov norm if and only if it has geometry $\mathbb{H}^4$, $\mathbb{H}^2 \times \mathbb{H}^2$ or $\mathbb{H}^2(\mathbb{C})$.*

The non-vanishing part of the above corollary is due to [25, 5, 20]. After posting this paper, the author was kindly informed by Pablo Suárez-Serrato that the Corollary 1.4 has been obtained by him in [52]. Actually, [52] also establishes other equivalence conditions. Especially, it shows that a closed geometric 4-manifold not modeled on $\mathbb{H}^4$, $\mathbb{H}^2 \times \mathbb{H}^2$ or $\mathbb{H}^2(\mathbb{C})$ if and only if it admits an $\mathcal{F}$-structure in the sense of Cheeger-Gromov [8], which implies Gromov norm zero by [8, 44]. Theorem 4 in [52] claims the statement of Theorem 1.3 for $\widetilde{SL_2} \times \mathbb{E}$ and $\mathbb{H}^3 \times \mathbb{E}$. However, it does not give a proof of the essential fact that the natural foliation by lines descend to a foliation by circles when the quotient manifold is closed. Thus the argument in [52] is incomplete. In fact, the same issue for 3-manifolds is treated carefully in [51].

For dimensions greater than or equal to four, we also explore the mapping orders defined by maps respecting various structures, *e.g.* complex, symplectic or J-holomorphic. In the spirit of Theorem 1.1, we then discuss the relations of these mapping orders with Kodaira dimensions and other invariants associated to different structures, *e.g.* the Gromov norm, topological entropy, J-(anti)-invariant cohomology *etc.*. Several structural properties of non-zero degree maps and degree one maps are discussed along this line. Various questions are raised during the discussions.

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Unless it says specifically otherwise, our objects are connected closed oriented manifolds.

2. Preliminaries

In this section, we will recall the definitions and basic properties of several important notions used in the paper.

2.1. **Kodaira dimensions.** We mentioned in the introduction that for complex manifolds, sympletic 4-manifolds and Lefschetz fibrations for 4-manifolds, we also have suitable definitions of Kodaira dimensions. Let us first recall the definitions.

Definition 2.1. *Suppose (M, J) is a complex manifold of real dimension $2m$. The holomorphic Kodaira dimension $\kappa^h(M, J)$ is defined as follows:*

$$\kappa^h(M,J)=\begin{cases} -\infty & \text{if } P_l(M,J)=0 \text{ for all } l\geq 1,\\ 0 & \text{if } P_l(M,J)\in\{0,1\}, \text{ but } \not\equiv 0 \text{ for all } l\geq 1,\\ k & \text{if } P_l(M,J)\sim cl^k;\ c>0.\end{cases}$$

Here $P_l(M, J)$ is the l-th plurigenus of the complex manifold (M, J) defined by $P_l(M, J) = h^0(\mathcal{K}_J^{\otimes l})$, with $\mathcal{K}_J$ the canonical bundle of (M, J).

Definition 2.2. *For a minimal symplectic 4-manifold (M^4, ω) with symplectic canonical class K_ω, the Kodaira dimension of (M^4, ω) is defined in the following way:*

$$\kappa^s(M^4,\omega) = \begin{cases} -\infty & \textit{if } K_\omega\cdot[\omega]<0 \textit{ or } \quad K_\omega\cdot K_\omega<0,\\ 0 & \textit{if } K_\omega\cdot[\omega]=0 \textit{ and } \quad K_\omega\cdot K_\omega=0,\\ 1 & \textit{if } K_\omega\cdot[\omega]>0 \textit{ and } \quad K_\omega\cdot K_\omega=0,\\ 2 & \textit{if } K_\omega\cdot[\omega]>0 \textit{ and } \quad K_\omega\cdot K_\omega>0.\end{cases}$$

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its minimal models.

Here K_ω is defined as the first Chern class of the cotangent bundle for any almost complex structure compatible with ω.

LeBrun [26, 27, 28] has studied the relations between the Yamabe invariant and Kodaira dimensions. Especially, he proposed a definition of general type for arbitrary 4-manifolds in [28]: namely, when the Yamabe invariant is negative. Recall that the Yamabe invariant is defined as

$$Y(M) = \sup_{[\hat{g}]\in\mathcal{C}} \inf_{g\in[\hat{g}]} \int_M s_g dV_g,$$

where g is a Riemannian metric on M, s_g is the scalar curvature of g, and $\mathcal{C}$ is the set of conformal classes on M. When $Y(M)\le 0$, the invariant is simply the supremum of the scalar curvatures of unit-volume constant-scalar-curvature metrics on M. There is an interesting question of LeBrun: if M^4 admits a symplectic structure and $Y(M^4)<0$, is $\kappa^s(M^4)=2$? It is clear that, for minimal M^4, $\kappa^s(M^4)=2$ would imply $Y(M^4)<0$ since $Y(M)\le -4\pi\sqrt{2K_M^2}$ when the Seiberg-Witten invariant is nonzero and $K_M^2\ge 0$ (see *e.g.* [26]). On the other hand, the answer to LeBrun's question is positive for Kähler surfaces [27].

Finally, let us recall that the Kodaira dimension $\kappa^l(g,h,n)$ of Lefschetz fibrations defined in [9]. Here g and h denote the fiber and base genus of a Lefschetz fibration and n is the number of singular fibers.

Definition 2.3. *Given a relative minimal (g,h,n) Lefschetz fibration with $h\ge 1$, define the Kodaira dimension $\kappa^l(g,h,n)$ as follows:*

$$\kappa^l(g,h,n) = \begin{cases} -\infty & \textit{if } g=0,\\ 0 & \textit{if } (g,h,n)=(1,1,0),\\ 1 & \textit{if } (g,h)=(1,\ge 2) \textit{ or } (g,h,n)=(1,1,>0) \textit{ or } (\ge 2,1,0),\\ 2 & \textit{if } (g,h)\ge(2,2) \textit{ or } (g,h,n)=(\ge 2,1,\ge 1).\end{cases}$$

The Kodaira dimension of a non-minimal Lefschetz fibration with $h\ge 1$ is defined to be that of its minimal models.

Here, a Lefschetz fibration is called relative minimal if no fiber contains a sphere of self-intersection -1.

The three Kodaira dimensions $\kappa^h,\kappa^s,\kappa^l$ are compatible with each other.

2.2. **Geometric structures.** Let us recall the definition of a geometry structure *a la* Thurston. A *model geometry* is a simply connected smooth manifold X together with a transitive action of a Lie group G on X with compact stabilizers. A model geometry is called *maximal* if G is maximal among groups acting smoothly and transitively on X with compact stabilizers. Sometimes this condition is included in the definition of a model geometry. A *geometric structure* on a manifold M is a diffeomorphism from M to X/Γ for some model geometry X, where Γ is a discrete subgroup of G acting freely on X. If a given manifold admits a geometric structure, then it admits one whose model is maximal. In other words, different geometries are distinguished from their fundamental groups. In this paper, a *geometry* means a maximal model geometry such that at least one model X/Γ has finite volume.

There is a unique geometry in dimension one, that of the Euclidean line $\mathbb{E}^1$ (or sometimes denoted by $\mathbb{R}$). There are three geometries in dimension 2: the spherical geometry S^2, the Euclidean geometry $\mathbb{E}^2$ and the hyperbolic geometry $\mathbb{H}^2$.

In dimension three we have the following eight maximal geometric structures:

(1) Spherical geometry S^3;
(2) The geometry of $S^2 \times \mathbb{E}$;
(3) Euclidean geometry $\mathbb{E}^3$;
(4) Nil geometry Nil;
(5) Sol geometry Sol;
(6) The geometry of $\mathbb{H}^2 \times \mathbb{E}$;
(7) The geometry $\widetilde{SL_2(\mathbb{R})}$;
(8) Hyperbolic geometry $\mathbb{H}^3$.

Here $\widetilde{SL_2(\mathbb{R})}$ is the universal cover of $PSL_2(\mathbb{R})$, the unit tangent bundle of $\mathbb{H}^2$. The spherical geometry S^3 could also be viewed as the double cover of the unit tangent bundle of S^2. The Nilpotent group is the group of 3×3 upper triangular matrices of the form

$$B = \begin{pmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

The solvable group $Sol = \mathbb{R}^2 \rtimes_\phi \mathbb{R}$, where $\phi(t)(x, y) = (e^t x, e^{-t} y)$.

For the application in next section, we divide the eight Thurston geometries into four categories:

$$\begin{aligned} -\infty &: \quad S^3 \text{ and } S^2 \times \mathbb{E}; \\ 0 &: \quad \mathbb{E}^3, Nil \text{ and } Sol; \\ 1 &: \quad \mathbb{H}^2 \times \mathbb{E}, \widetilde{SL_2(\mathbb{R})}; \\ \tfrac{3}{2} &: \quad \mathbb{H}^3. \end{aligned}$$

On the other hand, we also have 19 geometries in dimension 4 (see [14]). As in dimension 3, we separate the 19 geometries into 4 categories:

$-\infty$: $\mathbb{P}^2(\mathbb{C})$, S^4, $S^3 \times \mathbb{E}$, $S^2 \times S^2$, $S^2 \times \mathbb{E}^2$, $S^2 \times \mathbb{H}^2$, Sol_0^4 and Sol_1^4 ;
0 : $\mathbb{E}^4$, Nil^4, $Nil^3 \times \mathbb{E}$ and $Sol^4_{m,n}$(including $Sol^3 \times \mathbb{E}$);
1 : $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{SL_2} \times \mathbb{E}$, $\mathbb{H}^3 \times \mathbb{E}$ and F^4;
2 : $\mathbb{H}^2(\mathbb{C})$, $\mathbb{H}^2 \times \mathbb{H}^2$ and $\mathbb{H}^4$.

Let us recall the definition of non-product geometries in the list. First, S^4, $\mathbb{H}^4$, $\mathbb{P}^2(\mathbb{C})$ and $\mathbb{H}^2(\mathbb{C}) = SU(2,1)/S(U(2)\times U(1))$ are Riemannian symmetric spaces.

Next, nilpotent Lie groups and solvable Lie groups are realized by semidirect product: $Nil^4 = \mathbb{R}^3 \rtimes_U \mathbb{R}$, $Sol^4_{m,n} = \mathbb{R}^3 \rtimes_{T_{m,n}} \mathbb{R}$. Here $U(t) = \exp(tB)$ and $T_{m,n}(t) = \exp(tC_{m,n})$ with

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_{m,n} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

where $e^a > e^b > e^c$ are roots of $\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$ with m, n positive integers. Especially, $a > b > c$ are real and $a + b + c = 0$. If $m = n$, then $b = 0$ and $Sol^4_{m,n} = Sol^3 \times \mathbb{E}$.

When there are two equal roots for $\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$, i.e. when $m^2n^2 + 18mn = 4(m^3 + n^3) + 27$, the geometry is denoted by Sol_0^4. There is another solvable group Sol_1^4 which is represented as a matrix group

$$B = \begin{pmatrix} 1 & \beta & \gamma \\ 0 & a & \alpha \\ 0 & 0 & 1 \end{pmatrix}, \quad a, \alpha, \beta, \gamma \in \mathbb{R}, a > 0$$

Finally we have the geometry F^4 with isometry group $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ with the natural action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$. The geometry F^4 is the only geometry in the list to admit no compact model (although, by definition of geometry, we have some models with finite volume). However, it does admit complex structures and even Kähler structures.

In [59], Wall studies the relations between complex structures and the geometries. In summary, the geometries

$$S^4, \mathbb{H}^4, \mathbb{H}^3 \times \mathbb{E}, Nil^4, Sol^4_{m,n}$$

do not admit a complex structure compatible with the geometric structure. In the remaining cases except Sol_1^4, the complex structure on the maximal relevant geometry is unique. For Sol_1^4 we have two complex structures, denoted by Sol_1^4 and $Sol_1'^4$. Among those admitting complex structures, the geometries

$$S^3 \times \mathbb{E}, Sol_0^4, Sol_1^4, Sol_1'^4, Nil^3 \times \mathbb{E}, \widetilde{SL_2} \times \mathbb{E}$$

admit no compatible Kähler structures. The first four are in Class VII of Kodaira's list of complex surfaces. The remaining two are in Class VI. A compact model for $Nil^3 \times \mathbb{E}$ is the so-called Kodaira-Thurston manifold.

All the remaining geometries admit compatible Kähler structures. Moreover, the Kodaira dimension of these Kähler structures is the same as the category number of the corresponding geometric structures.

2.3. **Gromov norm.** The *Gromov norm*, or sometimes called the *simplicial volume*, is a norm on the homology (with real coefficients) given by minimizing the sum of the absolute values of the coefficients over all singular chains representing a cycle. The Gromov norm $||M||$ of the manifold M is the Gromov norm of the fundamental class. More precisely, let $|\cdot|_1 : C_k(M;\mathbb{R}) \to \mathbb{R}$ be the l^1 norm on real singular chains: for $z = \sum c_i\sigma_i \in C_k(M;\mathbb{R})$,

$$|z|_1 := \sum |c_i|.$$

Then the Gromov norm is

$$||M|| := \inf\{|z|_1 | [z] = [M]\} \in \mathbb{R}_{\geq 0}.$$

Let us summarize several fundamental properties of Gromov norm that will be used in the paper. The first is the gluing result in [20], which says that the Gromov norm of 3-manifolds is additive for connected sums and gluing along incompressible tori. In particular, it implies a 3-manifold has vanishing Gromov norm if and only if this is a graph manifold.

The second follows directly from the definition. Let $f : M \to N$ be a map of oriented closed connected manifolds of the same dimension, then

$$||M|| \geq |\deg f| \cdot ||N||.$$

If f is a covering map, then the equality holds. Because homotopy equivalences of oriented closed connected manifolds have degree ± 1, it follows that Gromov norm is a homotopy invariant.

Next, we recall some vanishing properties. First, any manifolds admitting a self-map f of $|\deg f| > 1$ have vanishing Gromov norm. Second, any oriented closed connected smooth manifold that admits a non-trivial S^1-action has vanishing Gromov norm [60]. Third, the Gromov norm of oriented closed connected manifolds with amenable fundamental group is zero [20].

3. Kodaira dimension of 3-manifolds

This section focuses on dimension 3. There are three types of results, as divided by subsections, which also play as stereotype for results in later sections. The first subsection gives the definition of the Kodaira dimension for 3-manifolds in terms of geometric structures. It takes value from $-\infty$, 0, 1 and 1.5. Classification is given for manifolds with Kodaira dimension $-\infty$ and 0. In the second subsection, we show that the Kodaira dimension for 3-manifolds is compatible with the mapping order given by nonzero degree maps. The third section is on the compatibility of this new Kodaira dimension with complex and symplectic Kodaira dimensions, in the sense of "additivity" introduced in [35]. These results motivate questions in symplectic and complex geometry.

3.1. **Geometric structures and Kodaira dimension of 3-manifolds.** We start with the discussion of the geometrization theorem, which says that every closed 3-manifold can be decomposed uniquely into pieces that each has one of Thurston's eight geometric structures.

The first step of the decomposition is the prime decomposition, the existence and uniqueness is due to Kneser and Milnor respectively.

Theorem 3.1. *Every compact, orientable 3-manifold can be decomposed into the connected sum of a unique (finite) collection of prime 3-manifolds.*

This reduces the study further decomposition to prime manifolds. The geometrization theorem was conjectured by Thurston and finally proved by Perelman using Hamilton's Ricci flow [46, 47, 48] (see also [6, 24, 41]).

Theorem 3.2. *Every oriented prime closed 3-manifold can be cut along tori, so that the interior of each of the resulting manifolds has a geometric structure with finite volume.*

Notice that there is a unique minimal way of cutting an irreducible oriented 3-manifold along tori into pieces that are Seifert manifolds or atoroidal called the JSJ decomposition. It is not quite the same as the decomposition in the above theorem, because some of the pieces in the JSJ decomposition might not have finite volume geometric structures. Moreover, there are many inequivalent cuttings of Theorem 3.2 depending on the initial metric to start the Ricci flow.

Given a 3-manifold M^3, we first decompose it into prime pieces and then further exploit a toroidal decomposition for each prime summand, such that at the end each piece admits one of the eight geometric structures (recalled in Section 2.2) with finite volume. By Theorem 3.1, the decomposition is unique. We call this a T-decomposition. For example, $\mathbb{R}P^3\#\mathbb{R}P^3$ admits a geometric structure of type $S^2\times\mathbb{E}$. But in this paper, we should first decompose it into two $\mathbb{R}P^3$, then these two prime pieces admit spherical geometry.

We are ready to give the following definition of Kodaira dimension of 3-manifolds:

Definition 3.3. *For an oriented 3-dimensional manifold M^3, we define the Kodaira dimension $\kappa^t(M^3)$ as follows:*

1. $\kappa^t(M^3)=-\infty$ *if for any T-decomposition, each piece has geometric type in category $-\infty$;*
2. $\kappa^t(M^3)=0$ *if for any T-decomposition, we have at least a piece with geometry type in category 0, but no piece has type in category 1 or $\frac{3}{2}$;*
3. $\kappa^t(M^3)=1$ *if for any T-decomposition, we have at least one piece in category 1, but no piece has type in category $\frac{3}{2}$.*
4. $\kappa^t(M^3)=\frac{3}{2}$ *if for any T-decomposition, we have at least one hyperbolic piece.*

For non-orientable M, we define $\kappa^t(M)$ to be that of its oriented double cover $\widetilde{M}$.

It is worth noting that κ^t is not automatically well-defined since the decomposition in Theorem 3.2 might not be unique. In fact, depending on the choice of the initial metric, the Ricci flow will cut up a manifold into geometric pieces in many inequivalent ways.

Before showing the Kodaira dimension $\kappa^t(M^3)$ is well-defined, let us recall an important result of Thurston (see Theorem 4.7.10 of [56]), which is used several times throughout the paper.

Theorem 3.4 (Thurston). *Non-closed* 3*-dimensional geometric manifolds with finite volume exist only for geometries in category* 1 *or* $\frac{3}{2}$, *i.e.* $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}$ *and* $\widetilde{SL_2(\mathbb{R})}$.

Recall that the Gromov norm is additive when gluing along tori. This fact implies that the Gromov norm of a 3-manifold is proportional to the sum of the volume of the hyperbolic pieces under a geometric decomposition. In particular, a 3-manifold has zero Gromov norm if and only if this is a graph manifold. In other words, the manifold with $\kappa^t = \frac{3}{2}$ is characterized by the non-vanishing of the Gromov norm.

We are ready to show that Definition 3.3 is well-defined.

Theorem 3.5. *The* 3*-dimensional Kodaira dimension* κ^t *is well-defined.*

Proof. For a manifold M^3, if we have a decomposition with every piece from category $-\infty$, then by Theorem 3.4, there are no toroidal decompositions. Because the prime decomposition is unique, then we know that the case $\kappa^t = -\infty$ is well-defined.

Similarly, if we have a decomposition with at least one piece from category 0, but no piece from category 1, then by Theorem 3.4, there are no toroidal decompositions. Because the prime decomposition is unique, then we know that the case $\kappa^t = 0$ is well-defined.

Moreover, M^3 has a hyperbolic piece in a T-decomposition if and only if it has non-vanishing Gromov norm. Hence the the case $\kappa^t = \frac{3}{2}$ is also well-defined.

Finally, the case $\kappa^t = 1$ is just the complementary of cases $\kappa^t = \frac{3}{2}$, $\kappa^t = 0$ and $\kappa^t = -\infty$.

Hence, Definition 3.3 is well-defined. □

Furthermore, we can classify the manifolds with $\kappa^t = -\infty$ or 0.

Proposition 3.6. *Let* M^3 *be a* 3*-dimensional manifold with* $\kappa^t(M^3) = -\infty$ *or* 0*. Then* $M = (M_1 \# M_2 \# \cdots \# M_n)$*, where each* M_i *is prime and of the following types:*

(1) *spherical, i.e. it has a Riemannian metric of constant positive sectional curvature;*
(2) $S^2 \times S^1$*;*

(3) *nontrivial S^2 bundle over S^1;*
(4) *Seifert fibrations with zero orbifold Euler characteristic;*
(5) *the mapping torus of an Anosov map of the 2-torus or quotient of these by groups of order at most 8.*

Moreover, if $\kappa^t(M^3) = -\infty$, then each M_i is of type $(1)-(3)$. If $\kappa^t(M^3) = 0$, then at least one M_i is of type (4) or (5).

Proof. We first decompose M into prime manifolds. Then we decompose those prime ones to pieces of finite volume with geometries in category $-\infty$ and 0. By Theorem 3.4, the finite volume geometric pieces with structures in category $-\infty$ or 0 are compact without boundary. Hence we only have prime decompositions.

When $\kappa^t(M^3) = -\infty$, each M_i has If some geometric piece with finite volume has $\kappa^t = -\infty$. For spherical geometry S^3, by elliptization conjecture which is a corollary of Theorem 3.2, it has a Riemannian metric of constant positive sectional curvature. If the geometry is $S^2 \times \mathbb{E}$, then it is (2) or (3).

When $\kappa^t = 0$, we have three more geometries: $\mathbb{E}^3$, Nil, Sol. Euclidean and Nil are Seifert fiber spaces with orbifold Euler number 0. Compact manifolds with Sol geometry are either T^2 bundles over S^1 with monodromy of Anosov type or quotient of these by groups of order at most 8. These correspond to the last two types listed in the statement. □

Remark 3.7. *There are two more non-orientable prime 3-manifolds with the geometry in category $-\infty$ and 0: $\mathbb{R}P^2 \times S^1$ and the mapping torus of the antipode map of S^2, which is the non-orientable fiber bundle of S^2 over S^1.*

Remark 3.8. *We would like to thank an anonymous referee making the suggestion of distinguishing $\mathbb{H}^3$ by assigning $\frac{3}{2}$ to it in Definition 3.3. In the previous versions of the paper, we define the Kodaira dimension of a 3-manifold as the integer part of our current κ^t. For convenience, we denote it by $\kappa^{tt} := [\kappa^t]$ where $[k]$ means the greatest integer no greater than k. In other words, the manifolds with $\kappa^t(M) = 1$ or $\frac{3}{2}$ are all set to have $\kappa^{tt} = 1$. This version of Kodaira dimension is also well-defined and all the main properties, like Theorem 3.11, still hold.*

Moreover, as suggested by Tian-Jun Li, we could have the following numerical description of κ^{tt} for irreducible manifolds. Let $vb_1(M)$ be the supremum of $b_1(\tilde{M})$ among all finite covers $\tilde{M}$. Then at least for irreducible 3-manifolds, $\kappa^{tt} = -\infty$ when $vb_1 = 0$, and $\kappa^{tt} = 0$ when vb_1 is finite and positive, and $\kappa^{tt} = 1$ when vb_1 is infinite. This follows immediately from Agol's resolution of virtual Haken conjecture. By Theorem 3.4, It is easy to see that this description also works for a general 3-manifold if we assume there are no pieces of $S^2 \times \mathbb{E}$ geometry. There are 4 closed manifolds of $S^2 \times \mathbb{E}$ geometry: two prime but not irreducible, one not prime, one not orientable.

The main reason we take the current definition is the compatibility with higher dimensional geometric manifolds. See Remark 3.14.

3.2. **Nonzero degree maps.** In this section, we would like to discuss how κ^t changes under non-zero degree maps.

We start with showing $\mathbb{H}^3$ is the "largest" geometry among the eight under non-zero degree maps.

Lemma 3.9. *Suppose $f : M^3 \longrightarrow N^3$ is a non-zero degree map. If at least one of the geometric pieces for N has geometry $\mathbb{H}^3$, then at least one of the geometric pieces for M is hyperbolic, i.e. $\kappa^t(M) = \frac{3}{2}$.*

Proof. Since N has a hyperbolic piece, $||N|| > 0$. Then by the definition of Gromov norm, $||M|| \geq \deg(f) \cdot ||N|| > 0$. Hence M also has a hyperbolic piece. □

In general, we think that the Thurston norm and Gabai's result on taut foliation should be useful for a version of Lemma 3.9 for other geometries.

The next theorem is the first step towards a definition of the relative Kodaira dimension of 3-manifolds. We will need the following lemma (see for example Lemma 1.2 in [50]).

Lemma 3.10. *If there is a non-zero degree map $f : M \to N$, then $f_*\pi_1(M)$ has finite index in $\pi_1(N)$.*

Theorem 3.11. *If $f : M^3 \longrightarrow N^3$ is a non-zero degree map, then $\kappa^t(M) \geq \kappa^t(N)$.*

Proof. Since 3-manifolds are almost determined by their fundamental groups, let us first recall that how the fundamental groups determine geometric manifolds, see [51, 1]. In the following bullets, let L be a geometric manifold.

- $\pi_1(L)$ is finite if and only if the geometric structure on L is spherical.
- $\pi_1(L)$ is virtually cyclic but not finite if and only if the geometric structure on L is $S^2 \times \mathbb{E}$.
- $\pi_1(L)$ is virtually abelian but not virtually cyclic if and only if the geometric structure on L is Euclidean.
- $\pi_1(L)$ is virtually nilpotent but not virtually abelian if and only if the geometric structure on L is Nil.
- $\pi_1(L)$ is virtually solvable but not virtually nilpotent if and only if the geometric structure on L is Sol.
- $\pi_1(L)$ has an infinite index normal cyclic subgroup but is not virtually solvable if and only if the geometric structure on L is either $\mathbb{H}^2 \times \mathbb{E}$ or $\widetilde{SL_2(\mathbb{R})}$.

Here a group G is said to be virtually having some property P if there is a finite index subgroup H of G which has this property P. When G is the fundamental group of a 3-manifold, then it is virtually solvable if and only if it is solvable (*e.g.* Theorem 1.20 of [1]). We remark that when the fundamental group is a virtually abelian/nilpotent/solvable group, it has an infinite index normal cyclic subgroup (*e.g.* Lemma 3.2 and the proof of Theorem 4.17 in [51]).

Before continuing the proof, let us recall that a 3-manifold has Gromov norm zero if and only if it is a graph manifold. When a graph manifold has $\kappa^t \leq 0$, then each irreducible piece in the prime decomposition is (closed and) geometric. Especially, the fundamental group is the free product of several groups of types we listed above, because the fundamental group of the connected sum is the free product of the fundamental groups when dimension is greater than two. However, for a general 3-manifold, its fundamental group is the free product with amalgamation along torus or trivial group by Theorem 3.2.

Let us first prove that when $\kappa^t(M) = -\infty$, then $\kappa^t(N) = -\infty$ as well. In this case, $\pi_1(M)$ is the free product of several virtually cyclic groups G_i. We denote a cyclic subgroup in G_i as H_i. Especially, any subgroup of the free product of several virtually cyclic groups like $\pi_1(M)$ cannot contain an infinite index normal cyclic subgroup. We prove it by contradiction. If there is such a cyclic group C which is generated by an element from some G_i, then any element a satisfying the property $a^{-1}Ca \subset C$ is contained in G_i. This is because otherwise there will be a nontrivial relation involving elements of G_i and of at least another G_j, which contradicts to the fact that $\pi_1(M)$ is a free product of G_i. On the other hand, if C is generated by a "mixed" element that is not in a single G_i. Without loss, we could assume the generator of C cannot be written as a power of another element. Then any element a satisfying the property $a^{-1}Ca \subset C$ is contained in C. By Lemma 3.10, $f_*(\pi_1(M))$ is of finite index in $\pi_1(N)$. Thus $f_*(\pi_1(M))$ is the free product of cyclic groups and of finite index in $\pi_1(N)$. Because f_* is a group homomorphism, any subgroup of $f_*(\pi_1(M))$ also does not contain an infinite index normal cyclic subgroup. So if $\pi_1(N)$ contains a subgroup G with an infinite index normal cyclic subgroup C', then $C'' = f_*(\pi_1(M)) \cap C'$ is a normal subgroup of $H = f_*(\pi_1(M)) \cap G \leq f_*(\pi_1(M))$ with infinite index since H/C'' is of finite index in G/C'. Moreover, the group C'' is a nontrivial subgroup of $f_*(\pi_1(M))$ since otherwise $C' \cdot f_*(\pi_1(M))$ will be infinite distinct cosets. Hence $\pi_1(N)$ also does not contain a subgroup with an infinite index normal cyclic subgroup. In addition by Lemma 3.9, $||N|| = 0$ and hence N has to be a graph manifold. If a group has a subgroup with an infinite index normal cyclic subgroup, this subgroup will be preserved under free product or free product with amalgamation along tori. Thus $\kappa^t(N) = -\infty$.

Similarly when $\kappa^t(M) = 0$, $\pi_1(M)$ is the free product of several virtually solvable groups. By the similar reasoning as above, any subgroup of the free product of virtually solvable groups cannot contain an infinite index normal cyclic subgroup which is not virtually solvable. There are still two possibilities: when the cyclic group is contained in some (virtually solvable group) G_i and when it is not. In the first case, the elements a such that $a^{-1}Ca \subset C$ are contained in G_i. Hence any such subgroup containing an infinite index normal cyclic subgroup is virtually solvable. In the second case, these elements a are exactly the centralizers of C and constitute a cyclic group as well. Then by Lemma 3.10, $\pi_1(N)$ contains as a subgroup of finite index $f_*(\pi_1(M))$, which is the free product of several virtually solvable groups as this property is preserved under group homomorphism. In addition, any subgroup of $f_*(\pi_1(M))$ does not contain an infinite index normal cyclic subgroup which is not solvable. By the same argument as in the case $\kappa^t(M) = -\infty$, so is $\pi_1(N)$. And by Lemma 3.9, $||N|| = 0$ and hence N has to be a graph manifold. Thus $\kappa^t(N) \leq 0$.

Finally, when $\kappa^t(M) = 1$, then $||M|| = 0$. By Lemma 3.9, we have $||N|| = 0$ as well. Hence $\kappa^t(N) \leq 1$.

This completes our proof. □

Theorem 3.11 suggests a definition of relative Kodaira dimension of 3-manifolds. When f is a degree $k > 2$ map between M and N, it can be deformed to a branched covering whose branch locus is a link. Hence the possible definition of relative Kodaira dimension has its own interests for study, whenever the branched locus is non-empty.

By a result of Rong [50], we also know that a non-zero degree map between Seifert manifolds with infinite π_1 is homotopic to a fiber preserving pinch followed by a fiber preserving branched covering. In this case, we can reduce our situation to that of dimension 2, and get $\kappa^t(M) \geq \kappa^t(N)$ in turn.

3.3. **Comparing with other Kodaira dimensions.**

3.3.1. *Additivity.* We recall the definitions of Kodaira dimensions for complex manifolds, sympletic 4-manifolds and Lefschetz fibrations for 4-manifolds in Section 2.1. In this section, we will compare our Kodaira dimension κ^t with these ones.

Now we are ready to compare these Kodaira dimensions with our κ^t (in fact, with $\kappa^{tt} = [\kappa^t]$).

Proposition 3.12. $[\kappa^t(M)] = \kappa^h(M^3 \times S^1)$ *when* $M^3 \times S^1$ *admits a complex structure.* $[\kappa^t(M)] = \kappa^l(M^3 \times S^1)$ *when* $M^3 \times S^1$ *admits a Lefschetz fibration.* $[\kappa^t(M)] = \kappa^s(M^3 \times S^1)$ *when* $M^3 \times S^1$ *admits a symplectic structure. In all these cases, the manifold* M *is a surface bundle over* S^1*.*

Proof. In [13], Etgü proved that when $M \times S^1$ admits a complex structure or a Lefschetz fibration, M is a surface bundle over S^1. Then from the genus of the surfaces, we determine the Kodaira dimension: when the surface is S^2, T^2 or Σ_g ($g \geq 2$) respectively, $[\kappa^t] = -\infty$, 0 or 1 respectively by Definition 3.3. At the same time, $\kappa^l = -\infty$, 0 or 1 respectively by Definition 2.3. Furthermore, when M is a surface bundle over circle, $M^3 \times S^1$ is a surface bundle over torus. Thus by classification results on these manifolds in [9], we also have $[\kappa^t] = \kappa^h$ in this case.

When $M^3 \times S^1$ admits a sympletic structure, then $[\kappa^t] = \kappa^s$ is a consequence of the Taubes conjecture proved by Friedl and Vidussi [15]. Their theorem says that $M^3 \times S^1$ admits a sympletic structure if and only if M^3 is a surface bundle over circle. Then the same argument as above shows that $[\kappa^t] = \kappa^s$. □

Finally, let us discuss more on additivity in the sense of [35] up to dimension 4. Roughly speaking, we call the Kodaira dimension of a fibration is additive, if the Kodaira dimension of the total space is the sum of the Kodaira dimensions of the fiber and the base (might be in the relative sense if the fibration is not a bundle). For convenience of discussion, we could also define the topological Kodaira dimension κ^t for manifolds of dimension up to 2.

The 2-dimensional Kodaira dimension is defined in the normal sense by the sign of the Euler class. Namely, $\kappa^t(S^2) = -\infty$, $\kappa^t(T^2) = 0$ and $\kappa^t(\Sigma_g) = 1$ for $g \geq 2$.

The only closed connected 0-dimensional manifold is a point, and the only closed connected 1-dimensional manifold is diffeomorphic to a circle. We define κ^t of them to be 0.

When dimension n is three (or a larger odd number), our Kodaira dimension could take value the half integers $\frac{n}{2}$. Then we modify the additivity in the following manner: $[Kod(total)] = [Kod(fiber) + Kod(base)]$ where $[k]$ means the greatest integer no greater than k.

Bundles in dimension three contains three cases: covering spaces, circle bundles over surface and surface bundles over circle. The covering map preserves κ^t follows from the fundamental group description of geometric structures in Theorem 3.11. The additivity of other two cases are both straightforward to check by definition.

For dimension four, discussions in this section imply the additivity for the product $M^3 \times S^1$. The corresponding results of [15] for circle bundles and mapping tori are further discussed in [16] and [32] respectively. For a surface bundle over surface, when the base is a positive genus surface, the additivity is established in [9]. When the base is S^2, the bundle is either a ruled surface or a Hopf surface; the latter case occurs when the fiber is T^2 and homologically trivial. Hence the additivity holds.

3.3.2. *Symplectic 4-manifolds and complex manifolds.* Motivated by the discussions in Section 3.1 and the next, we have the following question.

Question 3.13.
(1) *Let M be a smooth $2n$-dimensional complex manifold with nonvanishing Gromov norm. Is $\kappa^h(M) = n$?*
(2) *Let M be a smooth 4-dimensional symplectic manifold with nonvanishing Gromov norm. Is $\kappa^s(M) = 2$?*

If M is a Kähler surface, the question is positively answered by the result of [45, 44]. Precisely, they showed that M admits an $\mathcal{F}$-structure if and only if the Kodaira dimension is different from 2 in [45]. On the other hand, the existence of $\mathcal{F}$-structure implies vanishing Gromov norm. Moreover, all known examples of compact complex surfaces which are not of Kähler type have $\mathcal{F}$-structure and thus vanishing Gromov norm. In other words, the complex part of Question 3.13 for complex surfaces is reduced to answer the following: Does every complex surface of Class VII have Gromov norm 0?

In general, if the answer is positive for M and N, so is the product manifold $M \times N$ since the Kodaira dimension is additive and

$$||M|| \cdot ||N|| \leq ||M \times N|| \leq \binom{\dim M + \dim N}{\dim M} \cdot ||M|| \cdot ||N||.$$

For the symplectic part, $\kappa^s(M) = -\infty$ implies vanishing Gromov norm since all these manifolds are rational or ruled surfaces which have amenable fundamental groups. It is most interesting to know whether $\kappa^s(M) = 0$ would imply $||M|| = 0$.

Remark 3.14. *The main reason we take the current definition of $\kappa^t(M^3)$ instead of $\kappa^{tt}(M^3)$ is the compatibility of Kodaira dimensions for geometric manifolds, which is suggested by an anonymous referee. As suggested by Question 3.13 and Theorem 4.7, one would like to expect the Kodaira dimension of a closed $2n$-manifold of non-zero Gromov norm to be n. Since the product of two hyperbolic 3-manifolds has non-trivial Gromov norm, and if we assume the additivity of the Kodaira dimensions, it would follow that a hyperbolic 3-manifold should have Kodaira dimension $\frac{3}{2}$.*

The following question is partially motivated by Theorem 3.11. We expect the same conclusion is valid for symplectic 4-manifolds and (J, J') pseudoholomorphic maps (or symplectic maps) between them with respect to symplectic Kodaira dimension.

Question 3.15. *Suppose that (M_1, ω_1) and (M_2, ω_2) are symplectic 4-manifolds and almost complex structures J_i are tamed by ω_i. If f is a (J_1, J_2)-pseudo-holomorphic map (i.e. $f \circ J_1 = J_2 \circ f$) of non-zero degree from (M_1, ω_1) to (M_2, ω_2), is $\kappa^s(M_1, \omega_1) \geq \kappa^s(M_2, \omega_2)$?*

Recall that an almost complex structure J is *tamed* by a symplectic form ω if $\omega(v, Jv) > 0$ for any $v \neq 0$.

We could answer this question positively when $\kappa^s(M_1, \omega_1) = -\infty$.

Proposition 3.16. *Under the assumptions of Question 3.15, if $\kappa^s(M_1, \omega_1) = -\infty$, then $\kappa^s(M_2, \omega_2) = -\infty$.*

Proof. In this situation, M_1 could be covered by J_1-holomorphic spheres in certain homology class A. As f is onto M_2, not all these spheres will be contracted under f. This implies the class f_*A is non-trivial. After composing $f : M_1 \to M_2$, any J_1-holomorphic sphere $S^2 \to M_1$ in class A will be a J_2-holomorphic sphere in class f_*A. All these J_2-holomorphic spheres will cover M_2 since f is surjective. This implies that the Kodaira dimension $\kappa^s(M_2) = -\infty$ as well. □

4. Kodaira dimensions, geometric structures, and the mapping order for 4-manifolds

This section generalizes ideas in Section 3 to dimension 4. First, we define Kodaira dimension for geometric 4-manifolds and establish structural results. Then we discuss the mapping order given by non-zero degree maps in topological, smooth, complex, or symplectic category.

4.1. Kodaira dimension κ^g of geometric 4-manifolds.

As we have seen, there are Kodaira dimensions available for complex and symplectic 4-manifolds. On the other hand, we also have 19 geometries in dimension 4. It is natural to ask whether we could define Kodaira dimension for 4-manifolds, at least for irreducible ones, through the 19 geometries. In dimension 4, we do not have the decomposition theorem as in dimension 3. Hence we will only focus on the 19 types of geometric manifolds.

We have divided 19 geometries into 4 categories in Section 2.2. This leads to the following

Definition 4.1. *Let M^4 be a 4-dimensional geometric manifold. The Kodaira dimension $\kappa^g(M)$ is defined to be the category number of M.*

Let us collect some useful information for the hyperbolic geometry $\mathbb{H}^3$ extracted from [51]. We use the upper half 3-space model

$$\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$$

with the metric

$$ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2).$$

The isometry group is generated by reflections and an isometry is determined by its restriction to the 2-sphere at infinity $\mathbb{C} \cup \{\infty\}$, where the xy-plane is identified with $\mathbb{C}$.

The group of orientation preserving isometries of $\mathbb{H}^3$ can be identified with the group of Möbius transformations $PSL(2, \mathbb{C})$ of $\mathbb{C} \cup \{\infty\}$. If we identify the point $(x, y, z) \in \mathbb{R}^3_+$ with the quaternion $x + yi + zj$, the 2×2 complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbb{R}^3_+$ by

$$w \mapsto (aw + b)(cw + d)^{-1},$$

where w is a quaternion of the form $x + yi + zj, z > 0$. This yields all orientation preserving isometries of $\mathbb{H}^3$. It follows that each orientation preserving isometry of $\mathbb{H}^3$ fixes one or two points of the sphere at infinity. There isometries are called *parabolic* and *hyperbolic* respectively. If α is an isometry of $\mathbb{H}^3$, let fix(α) denote the set of points on the sphere at infinity which are fixed by α.

Lemma 4.2 (Lemma 4.5 in [51]). (1) *If α and β are two non-trivial orientation preserving isometries of $\mathbb{H}^3$, then α and β commute if and only if fix(α) = fix(β).*

(2) *If α is a non-trivial orientation preserving isometry of $\mathbb{H}^3$, then the group $C(\alpha)$ of all orientation preserving isometries which commute with α is abelian and isomorphic to $\mathbb{R}^2$ or $S^1 \times \mathbb{R}$.*

We remark the orientation preserving isometry group of $\mathbb{H}^2$ can be identified with $PSL(2, \mathbb{R})$. There are three types of orientation preserving isometry: rotations, parabolics and hyperbolics. They are characterized by the number of points, *i.e.* 0, 1 or 2, left fixed on the circle at infinity. Thus similar results of Lemma 4.2 hold. For the second statement, $C(\alpha)$ is abelian and is isomorphic to S^1 if α is a rotation and isomorphic to $\mathbb{R}$ otherwise.

We have the following

Proposition 4.3. *Let G be a discrete group of isometries of $\mathbb{H}^3 \times \mathbb{E}$ which acts freely and has quotient M. Then one of the following three statements holds:*

(1) *the natural foliation of $\mathbb{H}^3 \times \mathbb{E}$ by lines descends to a foliation on M by circles;*
(2) *the natural foliation of $\mathbb{H}^3 \times \mathbb{E}$ by lines gives M the structure of a line bundle over some hyperbolic 3-manifold;*
(3) *the natural foliation of $\mathbb{H}^3 \times \mathbb{E}$ by lines descends to a foliation of M by lines in which each line has non-closed image in M. In this case, G must by isomorphic to $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group.*

Especially, in the last two cases M is not a closed manifold.

Proof. We identify the isometry group of $\mathbb{H}^3 \times \mathbb{E}$ with $\mathrm{Isom}(\mathbb{H}^3) \times \mathrm{Isom}(\mathbb{R})$. As G is discrete, $K = G \cap \mathrm{Isom}(\mathbb{R})$ is discrete and so must be 1, $\mathbb{Z}_2$, $\mathbb{Z}$ or $D(\infty)$. As G acts freely, K is 1 or $\mathbb{Z}$. Let Γ denote the image of the projection $G \to \mathrm{Isom}(\mathbb{H}^3)$. Then we have the exact sequence

$$0 \to K \to G \to \Gamma \to 0.$$

In the case when K is infinite cyclic, each line $\{x\} \times \mathbb{E}$ descends to a circle. Hence M is foliated by circles. When K is trivial, then $G \cong \Gamma$. If Γ is a discrete group of isometries of $\mathbb{H}^3$, then the quotient would be a line bundle over $\mathbb{H}^3/\Gamma$. For this case, the quotient is not closed.

When Γ is an indiscrete group of $\mathrm{Isom}(\mathbb{H}^3)$. Replacing G by a subgroup of index two if necessary, we can suppose that Γ is orientation preserving. We will now consider the projection $G \to \mathrm{Isom}(\mathbb{E})$. Let L be the image of the kernel under the isomorphism $G \to \Gamma$. L is a discrete group of isometries of $\mathbb{H}^3$.

Suppose L is non-trivial. Conjugation of L by each element of Γ induces an automorphism of L. As L is discrete, an element of Γ sufficiently close to the identity must commute with L. Since Γ is not discrete, there must be a non-trivial element of Γ which centralizes L. Now the centralizer of any non-trivial element in $\mathrm{Isom}^+(\mathbb{H}^3) = PSL(2, \mathbb{C})$ is always abelian, and actually $\mathbb{R} \times \mathbb{R}$ or $S^1 \times \mathbb{R}$, by Lemma 4.2. So it follows L is abelian. As L is discrete and torsion free, L must be $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. On the other hand, Γ is indiscrete, hence Γ has a subgroup Γ_1 of index at most two which centralizes L. Since L is abelian, each element has the same fixed points set by Lemma 4.2. Again by the same lemma, each element in Γ_1 has the same fixed points set and thus Γ_1 is abelian. If Γ_1 consists of hyperbolic isometries there is a unique geodesic l in $\mathbb{H}^3$ left invariant by Γ_1. Let G_1 be the subgroup of G corresponding to Γ_1. We see that G_1 leaves invariant the plane $l \times \mathbb{R}$. As this plane is isometric to Euclidean plane and G_1 must act discretely on it, we know G_1 is $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.

If Γ_1 consists of parabolic isometries, without loss we could assume the common fixed point at infinite S^2 is ∞. Hence Γ_1 leaves invariant each line $x = y =$ const. Taking l to be one of these lines, same argument as the above hyperbolic isometry case applies to complete the proof when L is non-trivial.

When L is trivial, since the orientation preserving part of $\mathrm{Isom}(\mathbb{R})$ is isomorphic to $\mathbb{R}$. Hence G has a subgroup of index at most two which is abelian. All the arguments above apply to show G_1 is actually $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. Especially, all these analysis imply that the quotient is not a closed manifold if not foliated by S^1. □

We could also give precise description of 4-manifolds of geometries $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{SL_2} \times \mathbb{E}$. Especially, we have the following result for closed geometric 4-manifolds with $\kappa^g = 1$.

Theorem 4.4. *A closed geometric 4-manifold M with $\kappa^g = 1$ is foliated by geodesic circles.*

Proof. For geometries in category 1, since F^4 does not admit any closed manifold model, we will focus on the remaining three. For $\mathbb{H}^3 \times \mathbb{E}$, the statement follows from Proposition 4.3. We have two more cases.

1. $\mathbb{H}^2 \times \mathbb{E}^2$**:**

For $\mathbb{H}^2 \times \mathbb{E}^2$, its isometry group is identified with $\mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{R}^2)$. Hence $K = G \cap \mathrm{Isom}(\mathbb{R}^2)$ is discrete and torsion free, so has to be 1, $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. If K is non-trivial, then at least one line in E^2 descends to a circle. Hence the quotient manifolds is foliated by geodesic circles.

If $K = 1$, we know again $G \cong \Gamma$ where Γ is the image of the projection $G \to \mathrm{Isom}(\mathbb{H}^2)$. If Γ is discrete, G cannot be cofinite since

$$\mathrm{vol}(\mathbb{H}^2 \times \mathbb{E}^2/G) = \mathrm{area}(\mathbb{H}^2/\Gamma) \cdot \mathrm{area}(\mathbb{E}^2/K).$$

In fact, M is a 2-dimensional vector bundle over the hyperbolic surface $\mathbb{H}^2/\Gamma$. If Γ is not discrete, we will show that G has a subgroup of finite index isomorphic to $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. The argument is similar to that of Proposition 4.3. Let L be the image of the kernel of the projection $G \to \mathrm{Isom}(\mathbb{E}^2)$ under the isomorphism $G \to \Gamma$. L is a discrete group of isometries of $\mathbb{H}^2$. Suppose L is non-trivial first. As Γ is indiscrete, there must be a non-trivial element of Γ which centralizes L. Now the centralizer of any non-trivial element in $PSL(2, \mathbb{R})$ is always abelian, hence L is $\mathbb{Z}$ since it is discrete and torsion free as well. Hence Γ has a subgroup Γ_1 of index at most two which centralizes L, and Γ_1 is abelian. By the remark after Lemma 4.2 for $\mathrm{Isom}(\mathbb{H}^2)$, Γ_1 consists of hyperbolic or parabolic isometries since $L \cong \mathbb{Z}$ prevents the case of rotations. Moreover, whenever Γ_1 consists of hyperbolic isometries or parabolic isometries, there will be an Euclidean space $l \times \mathbb{E}^2 \subset \mathbb{H}^2 \times \mathbb{E}^2$ left invariant by G_1 and G_1 must act discretely on it. It follows that G_1 is $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

If L is trivial, then G is identical to both Γ and its image under the projection $G \to \mathrm{Isom}(\mathbb{E}^2)$, say F. If F is discrete, then it is clear that M is an $\mathbb{H}^2$ bundle over the Euclidean surface $\mathbb{E}^2/F$.

Hence we could assume both F and Γ are not discrete. Recall the exact sequence

$$0 \to \mathbb{R}^2 \to \mathrm{Isom}(\mathbb{E}^2) \to O(2) \to 0.$$

Then $J = F \cap \mathbb{R}^2$ is a normal subgroup of F.

Since translations with same length are in the same conjugacy class, if J is not discrete, we could choose a discrete subgroup J' which is also normal in F. Since J' is discrete and F is indiscrete, F has a subgroup F_1 of index at most two which centralizes J'. Since the rotations and translations do not commute, $F_1 \cong F_1 \cap \mathbb{R}^2$. In particular, this implies F_1 and its corresponding group Γ_1 in Γ are abelian. There is an Euclidean plane (for rotations) or an Euclidean 3-space (for hyperbolic or parabolic isometries) fixed by G_1 which acts discretely on it. Hence G_1 is $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. This finishes the proof that closed $\mathbb{H}^2 \times \mathbb{E}^2$ manifolds are foliated by circles.

2. $\widetilde{SL_2} \times \mathbb{E}$:

For $\widetilde{SL_2} \times \mathbb{E}$ geometry, first notice $\mathrm{Isom}(\widetilde{SL_2} \times \mathbb{E}) = \mathrm{Isom}(\widetilde{SL_2}) \times \mathrm{Isom}(\mathbb{E})$. Let us look at the image of G under the projection $\mathrm{Isom}(\widetilde{SL_2} \times \mathbb{E}) \to \mathrm{Isom}(\mathbb{E})$. If the kernel $K = G \cap \mathrm{Isom}(\widetilde{SL_2})$ is trivial, then the image has to be indiscrete. In this case, G_1 is an abelian group. Then we look at the other projection $\mathrm{Isom}(\widetilde{SL_2} \times \mathbb{E}) \to \mathrm{Isom}(\widetilde{SL_2})$. If the kernel is nontrivial, the quotient manifold is foliated by geodesic circles. Then G is identified with its image under this projection. We further project it under $\mathrm{Isom}(\widetilde{SL_2}) \to \mathrm{Isom}(\mathbb{H}^2)$. Again, if the kernel is nontrivial, it is an S^1 manifold. Hence we could assume G is identified with its image under the composition of the above two projections. An abelian subgroup of $PSL(2, \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$ fixes a line l. Hence G_1 leaves invariant and acts discretely on the 3-space $l \times \mathbb{E}^2$. It follows that G_1 is $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

If the kernel K is nontrivial, it is discrete in $\mathrm{Isom}(\widetilde{SL_2})$. There are three cases by the classification of $\widetilde{SL_2}$ geometry. In the first case, the corresponding 3-manifold is a line bundle over a non-closed surface. In this scenario, the line bundle structure would be inherited by the 4-manifold. Hence, the quotient is a non-closed manifold. In the second case, the 3-manifold is a Seifert fibration. In this situation, the quotient 4-manifold would also be S^1-foliated.

We are left with the case when $K \cap \mathbb{E}$ is trivial and the image of K under $\mathrm{Isom}(\widetilde{SL_2}) \to \mathrm{Isom}(\mathbb{H}^2)$ is indiscrete. In this case, K is $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group. Especially, we notice that G has a subgroup G_1 of index at most two such that it has discrete abelian normal subgroup H which is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.

Then let us look at the image of G under the composition of the projections
$$\mathrm{Isom}(\widetilde{SL_2} \times \mathbb{E}) \to \mathrm{Isom}(\widetilde{SL_2}) \to \mathrm{Isom}(\mathbb{H}^2).$$
If the kernel is nontrivial, then the quotient 4-manifold is S^1-foliated since the kernel in each step has to be infinite cyclic. Hence we could assume G is identified with its image Γ in $PSL(2, \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$. Since Γ is indiscrete and it has nontrivial discrete abelian normal subgroup H, Γ has a subgroup Γ_1 of index at most two which centralizes H. Hence by Lemma 4.2, Γ_1 is abelian. Then as we argued in Proposition 4.3, G_1 leaves invariant a 3-space $l \times \mathbb{E}^2$. Since this 3-space is isometric to the Euclidean space and G_1 acts discretely on it, it follows G_1 is $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. In summary, either our quotient manifold is not closed or it is foliated by geodesic circles. □

Remark 4.5. *By* [58], *we know that if M admits a Riemannian metric with respect to which all the circles are geodesic, then a double cover of M admits a non trivial smooth S^1 action. In particular, it implies M has vanishing Gromov norm.*

It is interesting to compare Theorem 4.4 with complex surfaces with Kodaira dimension 1. It is well known that all these complex surfaces are elliptic surfaces. While, Theorem 4.4 implies closed geometric manifolds with $\kappa^g = 1$ also admit some fibration structures with Calabi-Yau fibers.

Theorem 4.4 also sits well in a broader setting. As suggested by the solution of the geometrization conjecture by Ricci flow, building blocks of 4-manifolds should consist of Einstein manifolds and collapsed pieces. Einstein 4-manifolds are far more complicated than Einstein 3-manifolds. However, we have the following Hitchin-Thorpe theorem [21, 54].

Theorem 4.6 (Hitchin-Thorpe)**.** *Any compact oriented Einstein 4-manifold (M, g) satisfies $2\chi + 3\sigma \geq 0$. The equality holds if and only if (M, g) is finitely covered by a Calabi-Yau K3 surface or by a 4-torus.*

Notice when M is almost complex with canonical class K, then the inequality reads as $K^2 \geq 0$. Hence it implies there are no (minimal or non-minimal) symplectic or complex Einstein 4-manifolds of Kodaira dimension 1. Hence very likely, geometric manifolds with $\kappa^g = 1$ are those "collapsed pieces" building blocks of Kodaira dimension 1. Our Proposition 4.3 and Theorem 4.4 describes these pieces.

Theorem 4.4 enables us to have the following characterization of closed geometric 4-manifolds of maximal Kodaira dimension by Gromov norm.

Theorem 4.7. *A closed geometric 4-manifold M has nonzero Gromov norm if and only if $\kappa^g(M) = 2$.*

Proof. First we show that for any geometric manifold with $\kappa^g = 2$, the Gromov norm is non-vanishing. This is because if M is a closed oriented locally symmetric space of non-compact type, then $||M|| > 0$ [25]. The examples include the three geometries with $\kappa^g = 2$. Especially, if M is hyperbolic

$$||M|| = \frac{1}{v_4}\mathrm{Vol}(M),$$

where v_4 is the maximal volume of ideal geodesic triangles. If M has geometry $\mathbb{H}^2 \times \mathbb{H}^2$ [5],

$$||M|| = \frac{3}{2\pi^2}\mathrm{Vol}(M).$$

We then want to show that the Gromov norm vanishes for geometric 4-manifolds in the other three categories. Closed manifolds of geometries $\mathbb{E}^4$, Nil^4, $Nil^3 \times \mathbb{E}$, Sol_0^4, Sol_1^4 and $Sol_{m,n}^4$ would have solvable fundamental group. The Gromov norm of such a closed manifold is zero since a solvable group is in particularly amenable. For the geometric closed manifolds with $\kappa^g = -\infty$, if the geometry is $\mathbb{P}^2(\mathbb{C})$, S^4 or $S^2 \times S^2$, then the fundamental group is finite and thus amenable. This implies the Gromov norm is zero. All the remaining with $\kappa^g = -\infty$ have a factor of S^2 or S^3. Let G be the discrete group of isometries of these geometries, then the original natural foliation by 2-spheres or 3-spheres of each geometry is preserved by the isometries. Hence any such geometric manifold would inherit a foliation by 2 or 3 dimensional spherical geometries. Since spherical geometries in dimension 2 or 3 admit nontrivial group actions, especially S^1 actions, this action would extend to the whole geometric manifold. Hence by [60], the Gromov norm is zero.

For geometries with $\kappa^g = 1$, Theorem 4.4 shows closed geometric manifolds are foliated by geodesic circles which in turn implies Gromov norm 0 by Remark 4.5 or Proposition 3 of [52]. $\square$

It is easy to see that κ^g is preserved under finite covering, since the geometry is preserved. Theorem 4.7 implies there is no non-zero degree map from M to N such that $\kappa^g(M) < 2$ and $\kappa^g(N) = 2$. Recently, [42] shows a similar result as Theorem 3.11 for closed geometric 4-manifold, *i.e.* κ^g is monotone with respect to the existence of maps of non-zero degree.

We end this subsection by discussing the relations with symplectic structures. First, symplectic 4-manifolds with $\kappa^s = -\infty$ are rational or ruled surfaces. Hence, $\mathbb{P}^2(\mathbb{C})$, $S^2 \times S^2$, $S^2 \times \mathbb{E}^2$ and $S^2 \times \mathbb{H}^2$ in $\kappa^g = -\infty$ admit symplectic models. Meanwhile, S^4, Sol_0^4 and Sol_1^4 and $S^3 \times \mathbb{E}$ does not admit any symplectic models. Geometries $\mathbb{E}^4$, Nil^4, $Nil^3 \times \mathbb{E}$ and $Sol^3 \times \mathbb{E}$ with $\kappa^g = 0$ admit symplectic models. They are realized by T^2 bundles over T^2 [18]. All geometries with $\kappa^g = 1$ except F^4 also admit symplectic structures. They are realized by surface bundles over torus. Finally, for geometries with $\kappa^g = 2$, product of surfaces $\Sigma_g \times \Sigma_h$ has geometry $\mathbb{H}^2 \times \mathbb{H}^2$, ball quotients have $\mathbb{H}^2(\mathbb{C})$. However, it is conjectured that any closed hyperbolic 4-manifold would have all Seiberg-Witten invariants vanish, which in particular implies it does not admit symplectic structures. Partial results towards this conjecture were obtained in [30].

4.2. **Remarks on mapping orders.**

4.2.1. *Partial orders by non-zero degree maps.* In dimension two, orientable surfaces are ordered by their genus, which is finer than Kodaira dimension. This order could also be viewed as introduced by maps between manifolds. More precisely, there is a non-zero degree map from Σ_g to Σ_h if and only if $g \geq h$. Thus, we could introduce a partial ordered set (in this case it is totally ordered). The elements are surfaces up to homotopy/homeomorphism/diffeomorphism. And we say M^2 is larger than N^2, or $M^2 \succ N^2$, if there is a non-zero degree map from M^2 to N^2. Notice that it does define an order because once $M^2 \succ N^2$, and $N^2 \succ M^2$, then $M^2 \cong N^2$.

This partial order could be generalized to higher dimensions. It is usually called the Gromov partial order. However, there are several issues. Let us first focus on closed orientable manifolds. First of all, it is very sensitive to the category of maps we choose. We are interested in continuous or differentiable maps, and sometimes may require to preserve the symplectic/complex structures. This is not a problem when the dimension is three. In dimension four, Duan and Wang [12] show that, when we are working on continuous maps and topological manifolds, the simply connected 4-manifolds are ordered by their intersection forms. There are topological 4-manifolds not admitting smooth structures. However, if we concentrate on smooth manifolds, then it does not really matter if we look at continuous maps or differentiable map. This is because a continuous map between smooth manifolds is homotopic to a differentiable one (*c.f.* [4] Proposition 17.8). At the same time, degree is a homotopy invariant. In other words, the smooth non-zero degree map cannot distinguish the exotic smooth structures. For example, the exotic 7-spheres and the standard 7-sphere (smoothly) 1-dominate each other, since we could find differentiable degree 1 maps (from either direction) which is homotopic to homeomorphisms.

The second issue is this mapping "order" is not necessarily a partial order. In other words, if we define that M and N are in the same equivalence class when we have a non-zero degree map from M to N and a non-zero degree map from N to M, then the equivalence classes are no longer manifolds up to homotopy/homeomorphism/diffeomorphism, even in dimension three. For example, S^3 and the lens spaces $L(p, q)$ are in the same equivalence class because there is a quotient map from S^3 to $L(p, q)$, and we know that any three manifold dominates S^3. We are interested in determining the manifolds in a given class. Lemma 3.10 is useful to deal with this issue.

The last but not the least, we have to restrict ourselves to irreducible manifolds at least when dimension is four. Let us take a look at an example of LeBrun [29]. Suppose M is a (non-spin) complete intersection surface of general type, and $N = S^2 \times S^2/\mathbb{Z}_2$. We know that $M\#N$ have a degree one map onto M by contracting the N portion in the direct sum to a point. While on the other hand, the double cover of $M\#N$ is $k\mathbb{C}P^2\#l\overline{\mathbb{C}P^2}$. Another more direct example is taking $M\#\mathbb{C}P^2 = (k-1)\mathbb{C}P^2\#l\overline{\mathbb{C}P^2}$ and looking at the degree one map from it to M. These examples imply that this order would not be interesting if we include the reducible ones into our objects. It only detects the size of the intersection form. This is not the right order in our mind since $k\mathbb{C}P^2\#l\overline{\mathbb{C}P^2}$ should be among the simplest ones by their topological types. Notice that any symplectic 4-manifolds and complex surfaces are irreducible.

There are three facts worth noting for the mapping order(s). The first one is that for any manifold M^n there is always a continuous non-zero degree map onto S^n. Thus S^n is always the minimal one in the order. A fact related to this is that any symplectic/complex 4-manifold could be realized as a symplectic/complex ramified cover of $\mathbb{C}P^2$. So $\mathbb{C}P^2$ is the minimal manifold in the symplectic/complex order. Another fact is the so called Gromov hyperbolization. It says that for any manifold N, one can find a hyperbolic manifold $H(N)$, such that it maps onto N through a non-zero degree map. In other words, hyperbolic manifold is large with respect to the order.

Now let us assume our manifolds are smooth. Although the mapping order defined using differentiable maps does not give more information than that given by continuous maps, it indeed gives us richer structures when the map is regarding some geometric structures. In a rougher scale, we expect various Kodaira dimensions are compatible with this order, in the sense that if $M \succ N$ in a suitable category, then $\kappa(M) \geq \kappa(N)$ for Kodaira dimension κ defined in the same category. The three facts mentioned in the previous paragraph are evidences. We have shown in Theorem 3.11 that it is indeed true for κ^t. It is also true for holomorphic Kodaira dimension (see Theorem 5.1). The corresponding statement for symplectic 4-manifolds is phrased in Question 3.15.

4.2.2. *Degree* 1 *maps.* The degree 1 maps are self interesting and are studied extensively in the literatures.

In dimension three, there is a good formulation of degree one maps using surgeries. First, a well known result says that any 3-manifold could be constructed from S^3 by a (± 1) surgery along a link, each of whose components are unknots (or equivalently we could do a sequence of surgeries along unknots). A result of Boileau and Wang shows that any surgery along homotopically trivial knot could be realized by a degree one map. Especially, this gives a proof of the result that S^3 is 1-dominated by any 3-manifolds.

Since surgeries on 3-manifolds correspond to attaching 2-handles on 4-manifolds with boundary, it is natural to have a version of Boileau-Wang's result in this situation as well. Notice the degree of a proper map between compact manifolds with boundary is defined by using the relative cohomology $H^n(M, \partial M; \mathbb{Z})$, which is isomorphic to $H_0(M; \mathbb{Z}) = \mathbb{Z}$ by Lefschetz duality.

Proposition 4.8. *Suppose M is a compact 4-manifold with boundary, and M' is the 4-manifold obtained by attaching an irreducible 2-handle H along a homotopically trivial knot k on ∂M, then there is a degree one map $f : M' \to M$. The same statement is true when we attach 2-handles along a link l with each component homotopically trivial.*

Proof. Let us first recall the settings of Proposition 3.1 in [3]. Since k is null-homotopic in ∂M, k can be obtained from a trivial know k' by finitely many self-crossing changes of k'. Let D' be an embedded disk in M bounded by k'. A singular disk $D \subset \partial M$ with $\partial D = k$ is obtained by by identification of pairs of arcs in D' following the self-crossing-changes from k' to k. The singular disk D obtained in ∂M with $\partial D = k$ has the homotopy type of a graph. Let $N(D)$ be a regular neighborhood of D in M. Then $N(D)$ is an irreducible handlebody. i.e. homeomorphic to $D^4 = D^2 \times D^2$. We could make a suitable choice of $N(D)$ such that the attaching region $N(k) \subset N(D)$.

Let us construct a degree one map $f : M' = M - N(k) \cup_\phi H \to M = \overline{M - N(k)} \cup N(k)$. where $H = D^2 \times D^2$.

First, the map f at part $M - N(k)$ in M' is defined to be identity. For $\partial M'$, it could be viewed as obtained from a surgery along k from ∂M defined by the map ϕ. Hence we could define f on this part as in [3]. Especially, we notice that the part of $\partial H = S^1 \times D^2 \cup D^2 \times S^1$ which is not attached to M is mapped to $N(D)$.

Combining what we said on $M - N(k)$ and on $\partial M'$, the whole boundary $\partial H \subset N(D)$. Since H is D^4, we can extend the map to whole M' by sending $H = D^4$ into $N(D)$.

Since M is a compact 4-manifold, $N(D)$ is a proper subset of M, we know the degree of f is one. □

The next step is to analyze the case of maps between closed 4-manifolds. The 3-handles and 4-handles attachings are uniquely determined by the 1-handles and 2-handles, especially we know that the union of 3-handles and 4-handles will be diffeomorphic to the boundary sum of m $S^1 \times D^3$. In particular, we know the 2-handlebody X_2 has boundary $\#_m S^1 \times S^2$. Back to M' and M in previous proposition, we have already established the map between the corresponding 2-handlebodies M' and M. If there are no 3-handles to be attached for both, i.e. $\partial M' = \partial M = S^3$, then the degree one map could be extended to the unique closed-ups of M' and M.

The relations of degree 1 maps with symplectic birational geometry will be discussed in Section 5.1.

5. Higher Dimensional Kodaira dimensions and equivalence classes of mapping order

This section is on higher dimension manifolds. We discuss the compatibility of Kodaira dimensions in the sense of mapping order and additivity.

5.1. **The mapping order for complex and almost complex manifolds.** For complex surfaces, the Kodaira dimension behaves just as we expect, *i.e.* it regards the (meromorphic) mapping order and preserved by covering. Namely, we have the following (see [57]).

Theorem 5.1.
- *Let $f : M \to N$ be a generically surjective meromorphic mapping of complex manifolds such that* $\dim M = \dim N$. *Then we have $\kappa^h(M) \geq \kappa^h(N)$.*
- *Let $f : M \to N$ be a finite unramified covering of complex manifolds. Then we have $\kappa^h(M) = \kappa^h(N)$.*

The complex projective space $\mathbb{C}P^n$ is smallest with respect to this mapping order, in the sense that when $M = \mathbb{C}P^n$, N has to be $\mathbb{C}P^n$ as well.

Example 5.2. *Let M_1 be the algebraic surface homeomorphic but not diffeomorphic to $\mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ constructed in* [43]*. We know that there are differentiable degree one maps from each direction because we can homotope the homeomorphism from both directions. Theorem 5.1 tells us that there is no non-trivial holomorphic map $f : \mathbb{C}P^2\#k\overline{\mathbb{C}P^2} \to M_1$. However, we have $f : M_1\#k\overline{\mathbb{C}P^2} \to \mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ with $k \geq 5$. There is no such map for $k < 5$ since they are simply connected which is ordered by their intersection forms by* [12].

Notice that Theorem 5.1 is related to (and could be viewed as 0-dimensional generalization of) the Iitaka conjecture, which states that a fiber space $f : X \to Z$ satisfies $\kappa^h(X) \geq \kappa^h(Z) + \kappa^h(F)$ where F is a general fiber of f. Here, an (analytic) fiber space is a proper surjective morphism with connected fibres. Actually, the Iitaka conjecture is one of the main motivations for our additivity principle of Kodaira dimensions.

Furthermore, the mapping order also regards other invariants. Recall that the algebraic dimension $a(M)$ of a complex manifold is defined as the transcendence degree over $\mathbb{C}$ of the field $\mathbb{C}^{Mer}(M)$ of meromorphic functions. When $f : M \to N$ is a surjective holomorphic map, the algebraic dimensions $a(M) = a(N)$ (see [57]).

In the almost complex setting, it is worth noting that a (J, J') holomorphic map makes the J-anti-invariant cohomology dimension h_J^-, which is introduced in [34, 10], non-decreasing. Let (M^{2n}, J) be an almost complex manifold. The almost complex structure acts on the bundle of real 2-forms Λ^2 as an involution, by $\alpha(\cdot, \cdot) \to \alpha(J\cdot, J\cdot)$. This involution induces the splitting into J-invariant, respectively, J-anti-invariant 2-forms

$$\Lambda^2 = \Lambda_J^+ \oplus \Lambda_J^-.$$

We denote by Ω^2 the space of 2-forms on M (C^∞-sections of the bundle Λ^2), Ω_J^+ the space of J-invariant 2-forms, *etc.* Let also $\mathcal{Z}^2$ denote the space of closed 2-forms on M and let $\mathcal{Z}_J^\pm = \mathcal{Z}^2 \cap \Omega_J^\pm$. Then

$$H_J^\pm(M) = \{\mathfrak{a} \in H^2(M;\mathbb{R}) | \exists\ \alpha \in \mathcal{Z}_J^\pm \text{ such that } [\alpha] = \mathfrak{a}\}.$$

The dimensions $\dim H_J^\pm(M)$ is denoted as $h_J^\pm(M)$.

The following result should be compared with a similar statement for Betti numbers (and $b_2^\pm$ for 4-manifolds).

Proposition 5.3. *If there is a surjective equi-dimensional (J, J') holomorphic map $f : M \to N$ for two almost complex manifolds (M, J) and (N, J'), then $h_J^\pm \geq h_{J'}^\pm$.*

Proof. We only show it for h_J^-. The argument for h_J^+ is similar. Recall when α is a J'-anti-invariant two form on N,

$$\alpha(J'X, J'Y) = -\alpha(X, Y).$$

Now $f^*\alpha$ is a two form on M, and

$$f^*\alpha(JX, JY) = \alpha(J'f_*X, J'f_*Y) = -\alpha(f_*X, f_*Y) = -f^*\alpha(X, Y).$$

Because f has non-zero degree, $f^*\alpha$ is non-trivial and

$$(f^*\alpha)^n = \deg(f) \cdot \alpha^n.$$

Finally, f^* commutes with the differential d. Hence $h_J^- \geq h_{J'}^-$. ☐

There is no such example with $h_J^- > h_{J'}^-$ coming into the author's mind when $f : (M, J) \to (N, J')$ is a surjective equi-dimensional (J, J') holomorphic map. Generalizations of Proposition 5.3 are discussed in [53].

Now, let us look at degree 1 maps. First, since all birational maps are of degree 1, it is of interest to understand the relation of it with symplectic birational geometry (see [33]). However, a plain differentiable degree 1 map will not preserve the birational class starting from dimension 4. Boileau and Wang [3] proves that any 3-manifold M is 1-dominated by a hyperbolic manifolds which is meanwhile a surface bundle $H(M)$. Thus $H(M) \times S^1$ again 1-dominates $M \times S^1$. Once M is a surface bundle, both could be endowed with a symplectic structure. Hence, a degree one map could change (complex/symplectic) Kodaira dimension of manifolds, thus the symplectic birational equivalence class. Symplectic fiber sum construction provides more such examples. Moreover, Example 5.2 provides a simply connected example. Hence, one has to impose more conditions on the map in addition to its degree in order to preserve the birational equivalence. A natural question (as mentioned to the author by Tian-Jun Li) is

Question 5.4. *Suppose $f : M_1 \to M_2$ is a (J_1, J_2) pseudo holomorphic map of degree 1, where J_1 and J_2 are almost complex structures tamed by symplectic structures ω_1 and ω_2. Is the map a composition of blow downs?*

Blow-downs compatible with J_i are very rigid objects. This excludes a lot of possibilities for the target (see [11]).

We could show that the map is birational if it is a holomorphic map. Let $f : M \to N$ be a degree one holomorphic map, then except possibly the set D where $\det(\frac{\partial z_i}{\partial w_j}) = 0$, other parts are $1 : 1$. Here z_i and w_j are holomorphic local coordinates of N and M respectively. This zero locus D is a complex subvariety of complex codimension one in M. Thus M and N are birational. When both M and N are projective varieties, the birational morphism is factored as several blowdowns.

For general almost complex structures, it is apparent that the Jacobi matrix $Df(x)$ of f in any point $x \in M_1$ is positive definite. Moreover, the degree is calculated locally as the sum of the signs of the determinant of Jacobian at each preimage. Hence if $f : M_1 \to M_2$ is a finite (*i.e.* the preimage of any point is a finite set) (J_1, J_2) pseudo holomorphic map of degree 1, then f is a diffeomorphism.

More generally, we would like to know whether it is true that any non-zero degree holomorphic map is homotopic to a composition of blow-downs and branched covering? One evidence is that every algebraic surface could be realized as a branched covering of $\mathbb{C}P^2$. Another notable fact is that for any equi-dimensional dominating morphism $f : M \to N$, we have the ramification formula $K_M = f^*K_N + R_f$, where the effective $\mathbb{Q}$-divisor $R_f \geq 0$ is called the ramification divisor. We expect the ramification formula would still hold for (J, J')-pseudoholomorphic maps. This would help us to understand Question 3.15.

Finally, it is amusing to look at the case when the degree is zero. Let us first suppose the complex dimension is one, then the Liouville's theorem tells us that any such map is a constant map, i.e. it maps onto a point in $\mathbb{C}P^1$. This statement could be generalized to any genus target. This is because any non-compact Riemannian surface is Stein and further we know any Stein manifold could be biholomorphically embedded into $\mathbb{C}^N$. For higher dimensions, if $f : M \to N$ is of degree zero, then f maps into a proper subvariety of N.

5.2. **Higher dimensional Kodaira dimensions and additivity.** For higher (even) dimensions, we still have the definition of Kodaira dimensions for complex manifolds. However, it is not known if there is a suitable generalization of 4-dimensional symplectic Kodaira dimension. One difference between dimensions no more than four and higher is that for dimension larger than 4 the holomorphic Kodaira dimension depends not only on the smooth structure but also on the complex structure.

The following example is due to Răsdeaconu, but may not be that well-known. Hence we reproduce it here for the convenience of readers. Let us take $M = \mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$ and N is the Barlow surface [2] which is a complex surface of general type homeomorphic to M. One can also take

$M = \mathbb{C}P^2\#5\overline{\mathbb{C}P^2}$ and N is the complex surface of general type homeomorphic to M constructed in [43]. Then $M \times \Sigma_g$ is diffeomorphic to $N \times \Sigma_g$. This is because that, at first, they are $h-$cobordant because M and N are so. Second, they are $s-$cobordant because the Whitehead group $Wh(M\times\Sigma_g) = Wh(N\times\Sigma_g) = Wh(\Sigma_g) = 0$. Then by the $s-$cobordism theorem proved independently by Mazur, Stallings, and Barden, they are diffeomorphic to each other. On the other hand, they are not the same as complex manifolds since they have distinct Kodaira dimensions. The Kodaira dimension $\kappa^h(M \times \Sigma_g) = -\infty$ as $\kappa^h(M) = -\infty$. However, $\kappa^h(N \times \Sigma_g) = 2$ when $g = 1$ and $\kappa^h(N \times \Sigma_g) = 3$ when $g > 1$. In [49], there are various examples of diffeomorphic manifolds with different Kodaira dimensions constructed. It is worth noting that none of them is simply connected.

In [33], Li and Ruan propose a possible way to define symplectic Kodaira dimension in dimension 6. First let us recall the following definition.

Definition 5.5. *A symplectic* 6*-manifold is minimal if it does not contain any rigid stable uniruled divisor.*

Here a (symplectic) uniruled divisor is nothing but a rational or ruled 4-manifold. A uniruled divisor is *stable* if one of its uniruled classes A has a nontrivial Gromov-Witten invariant of the ambient manifold with $K_\omega(A) \leq -1$. A uniruled divisor is *rigid* if none of its uniruled class is uniruled in the ambient manifold.

This definition only takes care about the divisorial contraction. However, for algebraic 3-folds, flip or small contraction cannot happen in smooth category.

Assume (M, ω) is a minimal symplectic manifold of dimension 6, then Li and Ruan propose the following definition of simplectic Kodaira dimension:

(1)
$$\kappa^s(M,\omega) = \begin{cases} -\infty & \text{if one of } K_\omega^i \cdot [\omega]^{3-i} < 0, \\ k & \text{if } K_\omega^i \cdot [\omega]^{3-i} = 0 \text{ for } i > k \text{ and } K_\omega^i \cdot [\omega]^{3-i} > 0 \text{ for } i \leq k. \end{cases}$$

There is an issue of well definedness. For example, one cannot yet exclude the possibility of a minimal symplectic 6-manifold (M, ω) with

$$K_\omega \cdot [\omega]^2 = 0, K_\omega^2 \cdot [\omega] = 0 \text{ but } K_\omega^3 > 0, \tag{2}$$

although there are no counterexample in the author's sight as well.

In dimension 4, a similar issue is resolved only with the help of Seiberg-Witten invariant which has no counterpart in higher dimensions.

For product symplectic manifolds of type $M^4 \times \Sigma_g$ with $g \geq 0$ and product symplectic forms, we can verify that the proposed definition is good in the sense that no bad cases like (2) would happen. It also satisfies the additivity and is compatible with complex Kodaira dimension.

Proposition 5.6. *Suppose* (M^4, ω_M) *is a symplectic* 4*-manifold and* $(M^4 \times \Sigma_g, \omega_M \times \omega_g)$ *is minimal, then the following additivity relation holds*

$$\kappa^s(M^4 \times \Sigma_g, \omega_M \times \omega_g) = \kappa^s(M^4) + \kappa^s(\Sigma_g).$$

Moreover, when M^4 *admits a complex structure* J*,*

$$\kappa^s(M^4 \times \Sigma_g, \omega_M \times \omega_g) = \kappa^h(M^4 \times \Sigma_g, J \times j).$$

Proof. First notice that minimality of a 4-dimensional symplectic manifold only depends on the diffeomorphism type. So we will say M^4 is minimal instead of saying (M^4, ω_M) is so.

If M^4 is not minimal with E as an exceptional curve, then $D = E \times \Sigma_g$ is a rigid stable uniruled divisor. Here $A = [E]$ is a uniruled class in D and $K_{\omega_M \times \omega_g}(A) = -1$.

Hence M^4 is minimal. Notice that the canonical class $K = K_{M \times \Sigma_g} = K_M + K_{\Sigma_g}$ and $[\omega] = [\omega_M] + [\omega_g]$. We calculate

$$K^3 = K_M^2 \cdot K_{\Sigma_g},$$

$$K^2 \cdot [\omega] = K_M^2 \cdot [\omega_g] + 2K_M \cdot [\omega_M] \cdot K_{\Sigma_g},$$
$$K \cdot [\omega]^2 = [\omega_M]^2 \cdot K_{\Sigma_g} + 2[\omega_M] \cdot K_M \cdot [\omega_g].$$

If $g = 0$, then we need to prove $\kappa^s(M^4 \times S^2, \omega_M \times \omega_0) = -\infty$, or one of the products $K \cdot [\omega]^2, K^2 \cdot [\omega], K^3$ is negative. Notice $K_{S^2} = -2 < 0$. Both $K^3 \geq 0, K^2 \cdot [\omega] \geq 0$ would imply $K_M^2 \leq 0, K_M \cdot [\omega_M] \leq 0$. However, this would imply $K \cdot [\omega]^2 < 0$.

If $g = 1$, then $K_{\Sigma_1} = 0$ and $K^3 = 0$. And the signs of $K_\omega^2 \cdot [\omega]$ and $K \cdot [\omega]^2$ are determined by that of $K^2(M)$ and $K_M \cdot [\omega_M]$ respectively. Hence $\kappa^s(M^4 \times \Sigma_g, \omega_M \times \omega_g) = \kappa^s(M^4)$.

If $g \geq 2$, then $K_{\Sigma_g} > 0$ and $\kappa^s(\Sigma_g) = 1$. If $\kappa^s(M) \geq 0$, then $K \cdot [\omega]^2 > 0$, $K^3 \geq 0, K^2 \cdot [\omega] \geq 0$. Furthermore, $K^3 = 0$ if and only if $K_M^2 = 0$, i.e. $\kappa^s(M) = 0$ or 1. And in addition $K^2 \cdot [\omega] = 0$ if and only if $\kappa^s(M) = 0$. This verifies $\kappa^s(M^4 \times \Sigma_g, \omega_M \times \omega_g) = \kappa^s(M^4) + 1$ when $\kappa^s(M) \geq 0$.

When $\kappa^s(M) = -\infty$, we want to show one of the product $K_\omega \cdot [\omega]^2, K_\omega^2 \cdot [\omega], K_\omega^3$ is negative. Actually, we will show it is always true for any (possibly non-minimal) rational or ruled 4-manifold. If $K_M^2 < 0$, then $K^3 < 0$. If $K_M^2 = 0$ and $K_M \cdot [\omega_M] < 0$, then $K^2 \cdot [\omega] < 0$. Hence we could assume $K_M^2 > 0$ and $K_M \cdot [\omega_M] < 0$. The only possibilities are $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ when $k < 9$ and $S^2 \times S^2$. For all these cases, if we let $K_M = 3H + \sum_i E_i$ for $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ and $K_M = -2S_1 - 2S_2$ for $S^2 \times S^2$, then ω can be written as $aH - \sum_i b_i E_i$ and $aS_1 + bS_2$ with all the coefficient positive. If both $K^2 \cdot [\omega]$ and $K \cdot [\omega]^2$ are non-negative, then we have

$$K_M^2 \cdot [\omega_M]^2 \geq 4(K_M \cdot [\omega_M])^2$$

from our formula for $K^2 \cdot [\omega]$ and $K \cdot [\omega]^2$. It is straightforward to check that the inequality is impossible for all the above cases, i.e. one of $K^2 \cdot [\omega]$ and $K \cdot [\omega]^2$ has to be negative. This completes the proof of our first statement.

The second statement follows from the facts that $\kappa^s(M^4) = \kappa^h(M^4)$ and that the complex Kodaira dimension is additive for the product complex structure, i.e. $\kappa^h(M \times \Sigma_g, J \times j) = \kappa^h(M, J) + \kappa^h(\Sigma_g, j)$. □

5.3. **Remarks on equivalence classes, Entropy and Gromov norm.** In the last section, we mention that the order defined through non-zero degree maps may not be a partial order. This raises a question that what is each equivalent class like. Precisely, for which manifolds M and N, we could have non-zero degree maps $f : M \to N$ and $g : N \to M$? Especially, can we find some invariants for each class?

There is another mapping order, defined in a similar manner but through degree one maps (instead of general non-zero degree maps). In dimension three, the degree one mapping order is indeed a partial order [50], *i.e.* each equivalence class contains exactly one manifold. This is because 3-manifolds are almost determined by their fundamental groups and the fundamental groups of 3-manifolds are residually finite and thus are Hopfian. In other words, it is more or less a group theory reasoning. However, when dimension is getting higher, it does not give rise a partial order when we identify two manifolds if they are homeomorphic. This is because there are examples of homotopy equivalent but not homeomorphic manifolds, *e.g.* certain lens spaces. But on the other hand, the homotopy equivalent manifolds 1-dominate each other by definition.

It is then natural to ask that what will happen if we identify two manifolds in the same equivalence class of degree one mapping order when they are merely homotopy equivalent. Let us consider aspherical closed oriented n-manifolds with Hopfian fundamental groups. By the same argument as Rong's for 3-manifolds, if such M and N 1-dominate each other, then they are homotopy equivalent. If Borel conjecture holds, they are homeomorphic to each other. Recall the Borel conjecture: Let M and N be closed and aspherical topological manifolds, if they are homotopy equivalent, then they are homeomorphic to each other.

In addition, Gromov norm is also an invariant for the equivalence classes for the degree one mapping order. This is because $||M|| \geq \deg(f) \cdot ||N||$ if there is a map $f : M \to N$. It is also amusing to note that if we use the original mapping order, then each equivalence class is either the same as the one given by degree one order, or each manifold in the equivalence class has Gromov norm 0. Especially, by Gromov's proof of Mostow rigidity, the equivalence class of degree one mapping order containing a hyperbolic manifold is exactly this manifold. This is because, by above discussion, any two manifolds in an equivalence class would have the same Gromov norm. Moreover, any degree 1 map between hyperbolic manifolds with same volume is a homotopy equivalence and thus an isometry.

One advantage to consider this degree one mapping order is that we may prevent the issue of reducibility as indicated in LeBrun's example. Another advantage of this mapping order is that the set of topological entropies would be an invariant for each equivalence class. The *Shub topological entropy* $S(f)$ of a map $f : N \to N$ is defined as $\log \lambda(f)$, where $\lambda(f)$ is the maximal spectral radius $f_* : H_l(N, \mathbb{R}) \to H_l(N, \mathbb{R})$ among all l.

Proposition 5.7. *Assume M and N are equivalent through degree one map, i.e. there are $g : M \to N$ and $h : N \to M$ both of degree one. Then, for any map $f_1 : M \to M$, the composed map $f_2 = g \circ f_1 \circ h : N \to N$ would have $S(f_2) = S(f_1)$.*

Proof. This follows from Nakayama's lemma. One corollary of it says the following (see Theorem 2.4 of [38]).

Lemma 5.8. *Suppose R is a commutative ring. If Z is a finitely generated R-module and $f : Z \to Z$ is a surjective endomorphism, then f is an isomorphism.*

In our situation, homology groups are finitely generated $\mathbb{Z}$-modules. Since degree one map $g : M \to N$ (resp. $h : N \to M$) would induce epimorphisms $g_* : H_k(M) \to H_k(N)$ (resp. $h_* : H_k(N) \to H_k(M)$). See for example Lemma 1.2 in [50]. Hence the compositions $g_* \circ h_*$, $h_* \circ g_*$ are epimorphisms, and thus isomorphisms by Lemma 5.8. This implies $H_k(N) \cong H_k(M)$ and g_*, h_* are isomorphisms. Hence the Shub topological entropy $S(f_2) = S(f_1)$. □

Notice that entropy invariant (and its variants) could make complementary use with Gromov norm. The latter detects hyperbolic pieces and the entropy sees the others because if one admits a self degree > 1 map then Gromov norm has to be 0.

Gromov norm has other interesting applications in the 4-manifolds theory. For example, we know that $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$ all have Gromov norm 0 (even for each homology class of them). This implies that the exotic differential structures of them will not admit any metric with negative sectional curvature. Another example is that any manifold with amenable fundamental group (*e.g.* trivial, nilpotent, solvable, abelian...) will not be greater than the ones with negative sectional curvature under the mapping order. Moreover, we have the following, which is surely known to the experts.

There does not exist two homotopic closed Riemannian manifolds such that one has negative sectional curvature and the other has

(1) *non-negative Ricci curvature; or*
(2) *almost non-negative sectional curvature.*

Proof. By Cheeger-Gromoll [7], if a manifold admits a metric with non-negative Ricci curvature, then its fundamental group is virtual abelian, *i.e.* there is an abelian subgroup of it with finite index. On the other hand, a group is amenable if it has a finite index amenable subgroup and an abelian group is amenable, so the fundamental group of a manifold with non-negative section curvature has to be amenable. Similarly, if a manifold admits almost non-negative sectional curvature (*i.e.* it admits a sequence of Riemannian metrics $\{g_n\}_{n\in\mathbb{N}}$ whose sectional curvatures and diameters satisfy $\sec(M, g_n) \geq -\frac{1}{n}$ and $\mathrm{diam}(M, g_n) \leq \frac{1}{n}$), then the fundamental groups of such manifolds are virtually nilpotent [17], which is also amenable. This implies the Gromov norm of such a manifold is zero. Therefore, it cannot admit any metric with negative sectional curvature because Gromov norm of such a manifold would have nonzero Gromov norm [22]. □

Notice that negative sectional curvature cannot be replaced by negative Ricci curvature because we know that any manifold of dimension greater than two could admit a metric with negative Ricci curvature [37].

For non-closed manifolds, the situation is different: there are homotopy types of manifolds, *e.g.* $\mathbb{R}^n$, which admit complete metrics of non-negative and negative sectional curvatures respectively. In general, Cheeger-Gromoll soul takes care of the complete non-negative metrics. On the other hand, the classical Hadarmard-Cartan theorem says that if M^n is a connected complete Riemannian manifold with non-positive sectional curvature, then its universal covering space is diffeomorphic to $\mathbb{R}^n$. It is probably true that if a closed manifold admits both non-positive and non-negative sectional curvature, then it has to be flat. All closed flat n-manifolds are finitely covered by T^n.

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