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Bernays and the Completeness Theorem

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1. Introduction

A well-known result in Reverse Mathematics is the equivalence of the formalized version of the Gödel Completeness Theorem [8] – i.e. every countable, consistent set of first-order sentences has a model – and Weak König’s Lemma [WKL] – i.e. every infinite tree of 0-1 sequences contains an infinite path – over the base theory RCA_0 .¹ It is less well known how the Completeness Theorem came to be studied in the setting of second-order arithmetic and computability theory. The first goal of this note will be to recount these developments against the backdrop of the latter phases of the Hilbert program, culminating in the publication of the second volume of Hilbert and Bernays’s [13] *Grundlagen der Mathematik* in 1939. This work contains a detailed formalization of the Completeness Theorem in a system similar to first-order Peano arithmetic [PA] – a result which has come to be known as the *Arithmetized Completeness Theorem*. Its second goal will be to illustrate how reflection on this result informed Bernays’s views about the philosophy of mathematics, in particular in regard to his engagement with the maxim “consistency implies existence”.

2. Origins and arithmetization

Hilbert posed the problem of the completeness of the first-order predicate calculus in his 1928 Bologna address [10] after having investigated (in conjunction with Bernays) a variety of completeness notions in his 1917/1918 and 1921/1922 Göttingen lectures [5].² These lectures also serve as the basis for the axiomatization of first- and second-order logic in Hilbert and Ackermann’s 1928 textbook *Grundzüge der theoretischen Logik* [11]. This was in turn Gödel’s source in his 1929 dissertation [8]. An important technique employed in Hilbert’s lectures and in the *Grundzüge* is the method of *arithmetical interpretations*. In the case of propositional logic, such an interpretation is an assignment of the values 0 (true) and 1 (false) to the propositional variables X, Y, Z, \dots , together with the interpretation of disjunction as multiplication and negation as $1 - X$. A formula is taken to be valid if it evaluates to 0 under all such interpretations. In his 1917/1918 lectures Hilbert used this definition to prove both the soundness of the propositional calculus and also its so-called *Post Completeness*

¹ This is one of several results originally announced by Friedman [6] whose proof later appeared in Simpson’s monograph [19] IV.3.

² See [21] discussion of this evolution.

– i.e. if an underivable formula were to be added to the calculus as a scheme, the system would become inconsistent. In his 1918 *Habilitationschrift*, Bernays would go on to adapt Hilbert’s presentation to prove the completeness of the propositional calculus in the now familiar form – i.e. “every valid formula is provable”.

In order to address the analogous questions about first-order logic, Hilbert and Ackermann [11] described a method for transforming first-order formulas into propositional ones whereby each predicate symbol is associated with a collection of numerical substitution instances derived by replacing universally quantified formulas with conjunctions and existentially quantified with disjunctions. This yields a method for constructing counter-models for formulas falsifiable in a finite domain which Gödel showed could be extended to arbitrary formulas.

Suppose that F is an irrefutable sentence which we wish to show is satisfied in a model \mathcal{M} . Gödel described how to construct a family of propositional formulas A_0, A_1, A_2, \dots composed of numerical substitution instances of the primitive predicates appearing in F in a manner similar to that originally described by Hilbert which mimics the dependencies between the bound variables of F . He also showed that if all of the A_i are irrefutable in the propositional calculus, then F is irrefutable in the predicate calculus. The completeness theorem for the propositional calculus can now be invoked to yield propositional valuations v_i satisfying each of the A_i . Relying on the manner in which these formula are constructed, it can be shown that the v_i assign compatible truth values to their propositional variables and that there are finitely many ways of extending v_i to v_{i+1} . König’s Lemma can thus be employed to obtain a valuation v_ω satisfying all of the A_i simultaneously.³ This valuation may in turn be used to construct a model \mathcal{M} satisfying F by taking the extension of a k -ary predicate $P(\vec{j})$ to be the set $\{\vec{j} \in \mathbb{N}^k : v_\omega(P(\vec{j})) = 0\}$.

On this basis Gödel observed that his proof yields a model with domain \mathbb{N} . In the second (1938) edition of [11], Hilbert and Ackermann went one step further in stressing that the construction of the valuations v_i also provides arithmetical *definitions* of the predicates appearing in F . Prior to the treatment of the Completeness Theorem in the second volume of the *Grundlagen* Hilbert and Bernays had also elaborated on the significance of arithmetical models in the first two chapters of the first volume. They begin this discussion by distinguishing between what they refer to as *contentual* and *formal axiomatics*. The former is typified by Euclidean geometry and physical theories whose fundamental notions draw on experience and which may thus be understood contentually. Formal axiomatic theories are typified by Hilbert’s axiomatizations of geometry and abstract away from intuitive content. They begin instead with the assumed existence of “a fixed system of things . . . which is delimited

³ Although Gödel does not mention König’s Lemma by name, its role in the proof of completeness appears to have first been noted by Kreisel [14] (p. 268).

from the outset and constitutes a domain of subjects for all predicates of the propositions of the theory” and whose existence is “an idealizing assumption that properly augments the assumptions formulated in the axioms” (p. 2). As such structures “transcend the realm of experience” (p. 3), Hilbert and Bernays arrive at the conclusion that the corresponding axiomatic theories can only be appropriately grounded by providing a proof of their consistency.

This provides the context for their introduction of arithmetical models:⁴

We are therefore forced to investigate the consistency of theoretical systems without considering actuality, and thus we find ourselves already at the standpoint of formal axiomatics. ¶ Now, one usually treats this problem . . . with the *method of arithmetization*. The objects of a theory are represented by numbers or systems of numbers and the basic relations by equations or inequations, such that, on the basis of this translation, the axioms of the theory turn out either as arithmetical identities or provable sentences . . . or as a system of conditions whose joint satisfiability can be demonstrated via arithmetical existence sentences . . . [12], p. 3

Much of the second volume of the *Grundlagen* is devoted to the exposition of techniques by which Hilbert and Bernays hoped to provide a finitist consistency proof for theories such as the systems of second-order arithmetic in which they ultimately proposed to formalize analysis (Supplement IV of [13]). But they also provide a treatment of Gödel’s incompleteness theorems (§5.1), before ultimately considering various ways in which proof theoretic results such as those of Ackermann and Gentzen may call for a “transgression” of the original *finiten Standpunkt* (§5.3). Preceding this is their presentation of the Completeness Theorem (§4.2). As a prelude to this, they remark (§3.5c) that this result simplifies the decision problem for first-order logic by showing that derivability may be identified with validity and irrefutability with satisfiability. But they also observe that since this result is not guaranteed to provide finite counter-models it must be initially understood as belonging to the set-theoretic understanding of logic as opposed to the finitist one.

In light of this, Hilbert and Bernays go on to introduce the notion of an *effectively satisfiable* [*effektiv Erfüllbar*] formula ([13], pp. 189-190) – i.e. one which upon being put into prenex normal form can be transformed by effectively replacing atomic formulas with truth values and formulas containing free variables with computable [*berechenbar*] number theoretic predicates so that each substitution instance with numerals is made true in an arithmetical model. For irrefutable formulas which have no finite models (e.g. of the sort discussed in §1 of [12]) they suggest that such an interpretation would constitute a “finitist sharpening” ([13], p. 191) of the Completeness Theorem. But upon introducing the definition of effective satisfiability, they go on to conjecture that completeness would *fail* if this notion were to be substituted

⁴ See also [12] pp. 18-19 and pp. 37-42.

for the traditional (non-effective) definition of satisfiability in its statement.⁵

These considerations set the stage for Hilbert and Bernays's formalization of the Completeness Theorem. Unlike modern presentations, this treatment begins with a careful exposition (§4.1) of Gödel's method of the arithmetization of syntax. This machinery is employed (§4.2a) to define a primitive recursive function $q(x)$ such that the truth of $q(\mathbf{n}) = 0$ for each natural number n (where \mathbf{n} is the numeral for n) implies the irrefutability of a given first-order formula F in the predicate calculus.⁶

The proof of the Completeness Theorem given in §4.2b proceeds by showing how these arithmetized definitions can be used to formalize the steps in Hilbert and Ackermann's rendition of Gödel's proof as summarized above. This leads to a method for transforming a formula F of an arbitrary first-order language into an arithmetical formula F^* by replacing its primitive predicate and function symbols by arithmetical formulas of appropriate arities extracted from Gödel's construction such that if $q(\mathbf{n})$ holds for all n (i.e. F is irrefutable) then F^* is a true arithmetical sentence. This proof yields one form of what is now called the *Arithmetized Completeness Theorem*: if F is a sentence which is consistent relative to the axioms of the the predicate calculus (and thus $q(\mathbf{n}) = 0$ holds for all n), then F has an arithmetically definable model – i.e. one whose domain is a set of natural numbers definable by a formula of first-order arithmetic and whose primitive relation and function symbols are interpreted by the extension of such formulas in the standard model $\mathcal{N} = \langle \mathbb{N}, 0, s, +, \times \rangle$.

Just prior to their presentation of Gödel's Incompleteness Theorem, Hilbert and Bernays describe the significance of this form of his Completeness Theorem as follows:

The transition from the formula F to the formula F^* through number-theoretic substitution has the significance of providing a number-theoretic model for the axiom system, since the formulas of the axioms by the replacement of the symbols for the basic predicates through number-theoretic expressions go over into provable number-theoretic expressions ... There thereby exists for a consistent axiom system of the kind considered a number-theoretic model in the deductive sense. ¶ This result has its significance as a completeness theorem, that is as a kind of deductive closure of the predicate calculus, only under the assumption of the consistency of the number-theoretic formalism ... ¶ This consideration indicates that the task of a proof of the consistency of the full number-theoretic formalism endures as an unfinished problem in our investigation. [13], p. 253

3. After the *Grundlagen*

As Hilbert acknowledged in his introduction to [12], both volumes of the *Grund-*

⁵ “It is rather to be assumed that in general it is not possible for a irrefutable prenex formula to be satisfiable by computable logical functions.” [13], p. 191.

⁶ As Hilbert and Bernays only introduce the (now) standard formalization of consistency as $\neg \text{Bew}(\ulcorner \perp \urcorner)$ in §5.2, the definition of $q(x)$ given in §4.1 more closely resembles the contemporary notion of *Herbrand consistency* (cf., e.g. [9], p. 179 ff).

lagen were composed by Bernays. And in fact one of the earliest applications of the Arithmetized Completeness Theorem grew out of Bernays's contemporaneous work on the system which is now known as Gödel-Bernays set theory [GB]. Bernays originally presented this system as a two-sorted theory of sets (x, y, z, \dots) and classes (X, Y, Z, \dots) with separate set and class membership relations.⁷ It is a consequence of Bernays's *Class Theorem* ([1], p. 72) that GB proves the existence of all *predicatively definable classes* – i.e. those definable without the use of bound class variables. From this it can be seen to follow that this system conservatively extends ZF. Nonetheless, Bernays originally provided a finite axiomatization which can be conjoined to yield a single sentence containing \in as its sole non-logical symbol. This in turn suggests GB as a natural theory to consider in regard to Hilbert and Bernays's conjecture that there exist irrefutable first-order formulas which are not effectively satisfiable.

In the course of attempting to demonstrate this, Kreisel [14] isolated a fact which is implicit in Hilbert and Bernays's proof of the Arithmetized Completeness Theorem – i.e. that if $q(x)$ and F^* are defined as above, then not only does the *truth* of $\forall x(q(x) = 0)$ in the standard model entail that of F^* , but also the conditional $\forall x(q(x) = 0) \rightarrow F^*$ is in fact *provable* in a first-order arithmetical system such as PA. He then used this fact in conjunction with the availability of an additional interpretation of PA into the language of GB to construct a sentence similar to that arising in Russell's paradox. But rather than leading to a contradiction, Kreisel showed that the relevant sentence is undecidable in GB, modulo the assumption of its ω -consistency.⁸

Kreisel thus showed how, upon formalization in a system like PA, the Arithmetized Completeness Theorem can be used to obtain a form of Gödel's First *Incompleteness* Theorem for a system S consisting of GB without the axiom of infinity.⁹ But in addition to this, he also initiated the study of *arithmetical models of set theory* – i.e. structures of the form $\mathfrak{A} = \langle \mathbb{N}, E \rangle$ satisfying the axioms of theories like S where $E \subseteq \mathbb{N} \times \mathbb{N}$ is an arithmetically definable set interpreting membership. In particular, Kreisel explicitly stated that no such model can be such that E is a recursive relation.¹⁰ Since S admits a finite axiomatization, this confirmed Hilbert and Bernays's

⁷ As Bernays observes in [1], his axiomatization is based on an earlier formulation of von Neumann which was first presented in his 1929-1930 Göttingen lectures.

⁸ The independent statement is explicitly constructed on p. 274 of [14] and roughly expresses that the arithmetical interpretation R^* of the Russell class does not bear the relation \in^* – i.e. the arithmetical interpretation of \in – to itself.

⁹ Kreisel's axiomatization of S is given on p. 48 of [15].

¹⁰ As Wang observed in his review of [14], the proof Kreisel sketched doesn't actually yield this result but (in effect) only the weaker statement that the definition of \in^* produced by applying the Arithmetized Completeness Theorem to GB can never be Δ_1^0 . This anticipates the later work (summarized in the introductory remarks to [8]) which collectively showed that any consistent statement has an arithmetical model in which all of its relations are Δ_2^0 -definable and also that this statement fails for $\Sigma_1^0 \cup \Pi_1^0$.

conjecture.

These results would be of isolated interest were they not also related to the early study of non-standard models of arithmetic and set theory. For as Rabin [17] observed, if $\mathfrak{A} \models \mathbf{S}$ and \mathcal{M} is a model of arithmetic obtained by interpreting arithmetical notions in the conventional manner in \mathfrak{A} , then \mathcal{M} cannot be isomorphic to the standard model \mathcal{N} . Thus although the Arithmetized Completeness Theorem can be employed to yield an arithmetical definition of a model of an apparently stronger theory of sets (modulo the assumption of its formal consistency statement), the models so obtained are non-standard with respect to how they interpret arithmetical notions themselves. In light of this, Wang [20] (p. 32) describes such models as ‘not very “natural”’ and Rabin [17] (p. 408) described them as ‘highly pathological’.

4. Bernays’s reappraisal

Although Bernays was not directly involved with obtaining any of the foregoing results, he made similar use of the Arithmetized Completeness Theorem in the last of his series of papers on GB [3] (pp. 87-93). After first describing how the Ackermann interpretation of Zermelo-Fraenkel set theory without the axiom of infinity can be extended to provide an interpretation of \mathbf{S} , he then observes that the methods of the *Grundlagen* can also be employed to reduce the consistency of this system to that of number theory. In particular, he states that arithmetizing the Henkin completeness construction for \mathbf{S} yields a “constructive” proof of the consistency of this system relative to the consistency of the “number theoretic frame” ([2], p. 93).

In order to appreciate the significance which Bernays appears to have invested in such a reduction, it is important to keep in mind that from the 1940s onwards, he expressed increasing confidence that the proof theoretic work of Ackermann, Gentzen, Schütte and others had provided constructive consistency proofs for arithmetic and subsystems of analysis (e.g., [2], p. 87). This in turn appears to have informed his appraisal of the foundational significance of the Completeness Theorem itself.

A useful waypoint here is provided by the controversy between Hilbert and Frege concerning whether the consistency of various sets of geometric axioms entails the existence of a model in which they are satisfied (the view conventionally attributed to Hilbert) or whether consistency should be *defined* in terms of existence of a model (the view conventionally attributed to Frege).¹¹ Gödel acknowledged his awareness of this controversy in the introduction to his dissertation as did Bernays in an unpublished note from the mid-1920s.¹² Therein he observes that while consistency proofs

¹¹ This was, of course, the subject of the 1899 exchange of letters between Frege and Hilbert (reprinted in [7]) which arose in light of Frege’s reaction to the publication of the first edition of Hilbert’s *Grundlagen der Geometrie*.

¹² The note in question is reprinted in [18], pp. 389-390. It dates from between 1925 and

have thus far proceeded by exhibiting models, the development of proof theory provides an alternative to Frege's position in the form of a deductive understanding of consistency.

This point is taken up in greater detail in Bernays's 1930 paper "The philosophy of mathematics and Hilbert's proof theory" [4]. This paper begins by reviewing the rationale for using formal logic and proof theory as tools for investigating the foundational questions which had arisen in mathematics in light of the paradoxes. But it is also notable in that it provides a summary of the status of Hilbert's finitist consistency program on the eve of the Incompleteness Theorems. After expressing optimism that the systems of formal arithmetic which were currently under investigation would be shown to be not only consistent but also deductively complete, he concedes that "a considerable field of problems remains open" (p. 59) in this area.

But Bernays also asserts that the unavailability of such results does not pose a problem for the use of proof theoretic methods to investigate formal axiomatics, inclusive of systems which posit infinite totalities. For as he writes

[T]he formalism of statements and proofs, which represent our idea-formation does not coincide with the formalism of that structure we intend in the concept-formation. The formalism is sufficient to formulate our ideas about infinite manifolds and to draw from these the logical consequences, but it is, in general, not capable of producing the manifold, as it were, combinatorially from within. [4], p. 59

Bernays's view at this point thus appears to have been that while the systems employed in formal axiomatics are sufficient for formalizing the practice of subjects like analysis, they do not themselves provide models of their axioms and also presumably that this would continue to be the case even if consistency proofs in the style then under consideration (e.g. as based on the ϵ -substitution method) became available.

Although Bernays does not mention the Completeness Theorem by name in his 1930 paper, its role comes to the surface in his 1950 essay "Mathematical existence and consistency" [4]. He begins with the observation that the equation of existence with consistency is salutary as it serves to dispel the impression that mathematics is concerned with a form of "ideal" existence which is unlike that of theoretical entities in the physical sciences. But he also remarks that there is still the problem of finding a general way of understanding consistency so that its equation with satisfiability in a model is more than a matter of definition. He goes on to reiterate the claim that development of proof theory has made it possible to delimit such a conception within number theory and analysis – i.e. "that of the impossibility of arriving deductively at an inconsistency".

1928 and is found in the Hilbert Nachlass, Cod. 685: 9,2. Thanks to an anonymous referee for drawing my attention to this.

In this case Bernays states that the Completeness Theorem does indeed show that “each requirement that does not lead deductively to an inconsistency can . . . be satisfied” and thus that the “identification of existence with consistency appears to receive exact confirmation”. But he also stresses that this coincidence is “far from obvious”. For as he notes, even if we define the *inconsistency* of a formula A as the non-existence of a model in which A is satisfied, the passage from the consistency of $\neg A$ to the existence of a satisfying model still requires that the validity of the principle that the failure of the negation of a universal proposition (i.e. not all \mathcal{M} , $\mathcal{M} \models A$) entails an existential one (i.e. there exists $\mathcal{M} \models A$). Noting that this form of inference is in general non-constructive, Bernays issues the following admonishment:

The common acceptance of the explanation of mathematical existence in terms of consistency is no doubt due in considerable part to the circumstance that on the basis of the simple cases one has in mind, one forms an unduly simplistic idea of what consistency (compatibility) of conditions is. One thinks of the compatibility of conditions as something which the complex speaks of directly such that one only needs to make clear the content of the conditions in order to see whether they are in agreement or not. In fact, however, the role of the conditions is that they affect each other in functional use and by combination. That which results in such a way is not contained as a constituent part of what is given through the conditions. [4], p. 98

These remarks were written at a time when the metamathematical ramifications of the Completeness Theorem were just beginning to be understood. Nonetheless, they are indicative that Bernays already realized that the methods of Gödel or of Henkin do not always yield a model which can reasonably be described as being “produced from within” or being a “constituent part” of a consistent set of axioms. The non-constructive aspects of the Completeness Theorem are, of course, now well known in virtue of its equivalence to WKL.¹³ However Bernays also appears to have appreciated this aspect in virtue of the relationship of the Completeness Theorem to results such as the Löwenheim-Skolem theorem which are now often thought of as means of demonstrating the *non-categoricity* of theories such as PA or GB.

He returned to comment on such connections in his last (1970) philosophical paper “Schematic Correspondence and Idealized Structures”. Here Bernays first observed that once we have adopted the axiomatic method, the Completeness Theorem provides a form of “harmony” between the semantic and deductive consequences of a set of first-order statements. But he then goes on to observe that the formalization of analysis and set theory naturally leads to the use of *second-order* axiomatizations – an approach which is already explored in Hilbert’s Göttingen lectures and is developed in detail in Supplement IV of the second volume of the *Grundlagen*.

¹³ See [19] XIII.2 for discussion of this in the context of Reverse Mathematics and computability theory and (e.g.) [16] in the constructive setting.

But as Bernays was also in a good position to appreciate by this time, it is also possible to effect a kind of reduction of second- to first-order logic by treating the membership (or predication) relation of a second-order theory as a “basic axiomatic relation, analogously to the incidence relation in geometry” ([4], p. 184). Once such a step is undertaken, he remarks that it becomes possible to construct not only non-standard models of arithmetic but also models of set theoretic systems like S or GB which are non-standard in virtue of possessing either a countable domain or a restricted range of second-order quantification.¹⁴

The question thus arises how we should regard such models – e.g. in comparison to their “intended” counterparts. Bernays appears to have remained agnostic on the significance of second-order logic for securing categorical descriptions of mathematical structure which have occupied many latter-day theorists. For as he had already made clear in [1], he viewed his own axiomatization of set theory in as a *two-sorted first-order system*. And he also stops short of endorsing a “non-axiomatic” understanding of second-order theories. For as he stressed earlier neither concept-formation nor proof in classical mathematics require us to “transgress” an axiomatic understanding of such theories ([4], p. 117) – e.g. by adopting a semantic understanding of second-order consequence.

Bernays did, however, use this opportunity to underscore a point which appears to apply specifically to the Arithmetized Completeness Theorem and which elaborates his remarks in [2]:

By following the idea of Hilbert to make the deductive structure the object of study, the theory is in a sense projected into number theory. The resulting number-theoretic structure is, in general, essentially different from the structure intended by the theory. But it can serve to recognize the consistency of the theory from a standpoint that is more elementary than the assumption of the intended structure. [4], p. 186

The context of this passage makes clear that Bernays was well aware that a model of set theory or analysis constructed in this manner must be non-standard with respect not only to its interpretation of membership and domain of quantification, but also its interpretation of “natural number” as well. This in turn raises the question of how his invocation of a ‘more elementary’ number theoretic standpoint can be reconciled at this stage with both his appreciation of the role of the Completeness Theorem in securing assertions of mathematical existence and also his recognition that this result can itself be formalized in first-order arithmetic. Although subsequent work in Reverse Mathematics has shed considerable light on related matters (see, e.g., [19], IX.2-3), a more thorough examination of this question must await another occasion.

¹⁴ Kreisel additionally applied the arithmetized completeness theorem to S in precisely this way in the course of his incompleteness proof in [14].

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