# Controlling the statistical properties of expanding maps

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**Abstract**

How can one change a system, in order to change its statistical properties in a prescribed way? In this note we consider a control problem related to the theory of linear response. Given an expanding map of the unit circle with an associated invariant density we can consider the inverse problem of finding which first order changes in the transformation can achieve a given first order perturbation in the density. We show the general mathematical structure of the problem, the existence of many solutions in the case of expanding maps of the circle and the existence of optimal ones. We investigate in depth the example of the doubling map, where we give a complete solution of the problem.

## 1 Introduction

An important idea, which has attracted much interest in recent years, is that of linear response. The basic principle is that in many cases a first order change in a system leads to a corresponding first order change in its equilibrium state. There has been a wealth of work on linear response theory for dynamical systems after the pioneering work of Ruelle, who developed a formula for the first derivative of the physical or SRB measure for hyperbolic systems [19]. In subsequent works the approach was simplified, applied to some system outside the uniform hyperbolic case and applied to other kinds of more or less chaotic or hyperbolic system (see [2] for a detailed overview, and [3] for very recent results on intermittent systems).

Many problems involving physical and social systems are modelled by chaotic dynamics and therefore many mathematical ideas have been developed and implemented in the context of the prediction and description of the statistical behaviour of a chaotic system. Linear response itself is used to understand the behaviour of complex systems out of equilibrium (see e.g. [17] for an application to models of the climate evolution).

Besides merely trying to understand the behaviour of a given system it is also important to attempt to understand the extent to which it can be controlled. For example, *can one determine which small changes in the system will change the statistical behaviour in a prescribed direction? Can these changes be chosen optimally, in an appropriate sense?*

Understanding the general behaviour in light of these questions will be of help in designing efficient strategies of intervention for the control and management of complex, chaotic systems.

We shall initiate the investigation of these specific questions in the context of particularly simple models of chaotic systems. In particular, we will concentrate on expanding maps of the circle. We shall consider the general mathematical structure of the problem and show that in the case of circle expanding maps, it has many solutions amongst which we can choose an optimal one, in a suitable sense. As an illustration, in this article we will also present a complete solution to the problem in the particular case of the linear doubling map, finding a solution which minimizes the $L^2$ norm (and other natural norms on the space).

To formulate the problem more precisely, we introduce some notation. Let $T_0 : X \to X$ be a $C^4$ expanding orientation preserving map of the circle $X = \mathbb{R}/\mathbb{Z}$ of degree $d \geq 2$. Let $T_\delta : X \to X$, where $\delta \in (-\eta, \eta)$ be a family of $C^3$ expanding maps. Let us suppose that the dependence of the family on $\delta$ is differentiable at $0$, hence can be written

$$T_\delta(x) = T_0(x) + \delta\epsilon(x) + o_{C^3}(\delta) \text{ for } x \in X.$$

where $\epsilon \in C^3(X, \mathbb{R})$, and $o_{C^3}(\delta)$ denotes a term whose $C^3$ norm tends to zero faster than $\delta$, as $\delta \to 0$.[1]

[1]More precisely we say that $T_\delta$ is a differentiable family of $C^3$ expanding maps if there exists $\epsilon \in C^3(X, \mathbb{R})$ such that $\|(T_\delta - T_0)/\delta - \epsilon\|_{C^3} \to 0$ as $\delta \to 0$, where

$$\|f(x)\|_{C^3} = \sup_{x\in X} |f(x)| + \sup_{x\in X} |f'(x)| + \sup_{x\in X} |f''(x)| + \sup_{x\in X} |f'''(x)|$$

is the usual norm on $C^3$ functions.

**Example 1.1.** *A simple example might be where* $d \geq 2$ *and* $T_0(x) = dx \pmod 1$ *and* $T_\delta(x) = dx + \delta \sin(2\pi x) \pmod 1$ *then trivially* $\epsilon(x) = \sin(2\pi x)$.

For each $T_\delta$ we can associate a unique invariant $C^2$ density and it is known that these vary in a differentiable way $\rho_\delta = \rho_0 + \delta\rho^{(1)} + \cdots$. It is a folklore result that the density of the unique physical invariant measure of such a family of maps varies in a differentiable way. For a more precise statement we have the following.

**Lemma 1.2.** *Let us assume that* $T_\delta$ *is a* $C^1$ *family of* $C^3$ *expanding maps. The density* $\rho_\delta \in C^1(X, \mathbb{R})$ *has a continuously differentiable dependence on* $\delta$.

Another reformulation of the response problem would be to consider the integrals of a fixed smooth function with respect to the varying natural measure (see e.g. [2]).

A more general statement of Lemma 1.2 appears as Theorem 20 in [16]. The proof is omitted, but it is a standard approach using the Implicit Function Theorem for Banach manifolds. With this notation we can now formulate the basic problem that we want to address:

**Problem 1.3.** *What first order change* $\epsilon(x)$ *in* $T_0$ *will result in a given first order change* $\rho^{(1)}$ *in the density?*

Before addressing this question, we recall some general results about linear response of systems under small perturbations. For hyperbolic systems (including Anosov diffeomorphisms and flows) we have the luxury that this class is open which ensures that under small perturbations the system remains hyperbolic. Moreover, such systems are structurally stable which makes it easier to apply ideas from thermodynamic formalism [8]. In this context Ruelle formulated an explicit expression for the derivative of the SRB measure [19] in terms of the perturbation of the system, which was subsequently reproved using a different method in [18] [11] and [7]. Linear response results nowadays have been proved for several classes of dynamical systems, even outside of these uniformly expanding and hyperbolic and structurally stable settings, see for example [9], [4],[3],[6],[12] and the survey [2].

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## 2 Linear response for expanding maps

A standard tool used in characterising the invariant densities for expanding maps is the transfer operator. In this section we recall the definition of the transfer operator and we present a theorem describing the linear response for operators satisfying certain assumptions. This tool will be directly applied to get the linear response formula for expanding maps.

**Definition 2.1.** *Let $\mathcal{L}_\delta : L^1(X,\mathbb{R}) \to L^1(X,\mathbb{R})$ be the transfer operators associated to an expanding map $T_\delta$, defined by*

$$\mathcal{L}_\delta w(x) = \sum_{i=1}^{d} \frac{w(y_i^\delta)}{T'_\delta(y_i^\delta)}$$

*where the summation is over the $d$ pre-images $\{y_i^\delta\}_{i=1}^d := T_\delta^{-1}(x)$ of $x \in X$.*

The following is a classical result but can be also taken as a definition of the invariant density.

**Definition 2.2.** *An invariant density $\rho_\delta$ for $T_\delta$ is a fixed point for the operators $\mathcal{L}_\delta$ acting on $L^1(X,\mathbb{R})$ (i.e., $\mathcal{L}_\delta \rho_\delta = \rho_\delta$).*

Now let us take a general point of view and describe a general result on linear response in terms of fixed points of operators. Let us consider the action of these operators $\mathcal{L}_\epsilon$ on different spaces. Let $B_w, B_s, B_{ss}$ denote abstract Banach spaces of Borel measures on $X$ equipped with norms $|| \, ||_w, || \, ||_s, || \, ||_{ss}$ respectively, such that $|| \, ||_w \leq || \, ||_s \leq || \, ||_{ss}$. We suppose that $\mathcal{L}_\delta$, $\delta \geq 0$, has a unique fixed point $h_\delta \in B_{ss}$. Let $\mathcal{L} := \mathcal{L}_0$ be the unperturbed operator and $h \in B_{ss}$ be its invariant measure. Let us consider the space of zero average measures

$$V_s^0 = \{v \in B_s, v(X) = 0\}.$$

Following an approach from [18], we present a general setting in which differentiable dependence and *a formula for the derivative of the physical measure* of a family of positive operators $\mathcal{L}_\delta$ can be obtained (see [5] for a proof of the statement in this form). We need to consider several norms for our operators, let us denote

$$||\mathcal{L}_\delta^k h||_{B_w \to B_w} = \sup_{\|h\|_w \leq 1} \|\mathcal{L}_\delta^k h\|_{B_w}$$

$$||\mathcal{L}_\delta^k h||_{B_s \to B_w} = \sup_{\|h\|_s \leq 1} \|\mathcal{L}_\delta^k h\|_{B_w} \text{ and } \quad ||\mathcal{L}_\delta^k h||_{V_s^0 \to B_s} = \sup_{\|h\|_{V_s^0} \leq 1} \|\mathcal{L}_\delta^k h\|_{B_w}.$$

The result we require is the following.

**Theorem 2.3.** *Suppose that the following assumptions hold:*

1. *The norms* $||\mathcal{L}_\delta^k||_{B_w \to B_w}$ *are uniformly bounded with respect to* $k$ *and* $\delta \geq 0$.

2. $\mathcal{L}_\delta$ *is a perturbation of* $\mathcal{L}$ *in the following sense: there is a constant* $C$ *independent of* $\delta$ *such that*
$$||\mathcal{L}_\delta - \mathcal{L}||_{B_s \to B_w} \leq C\delta. \tag{1}$$

3. *The operators* $\mathcal{L}_\delta$*, have uniform rate of contraction on* $V_s^0$*: there are* $C_1 > 0$*,* $0 < \rho < 1$*, such that* $\forall \delta \in [0,1]$
$$||\mathcal{L}_\delta^n||_{V_s^0 \to B_s} \leq C_1 \rho^n. \tag{2}$$

4. *There is an operator* $\mathbb{L} : B_{ss} \to B_s$ *such that* $\forall f \in B_{ss}$
$$\lim_{\delta \to 0} ||\delta^{-1}(\mathcal{L} - \mathcal{L}_\delta)f - \mathbb{L}f||_s = 0. \tag{3}$$
*Let*
$$\hat{h} = (Id - \mathcal{L})^{-1}\mathbb{L}h. \tag{4}$$
*Then*
$$\lim_{\delta \to 0} ||\delta^{-1}(h - h_\delta) - \hat{h}||_{B_w} = 0;$$
*i.e.* $\hat{h}$ *represents the derivative of* $h_\delta$ *for small increments of* $\delta$.

Let us see how to apply Theorem 2.3 to our setting. The assumptions of the above theorem are valid for the perturbations of circle expanding maps we are considering with the choices of spaces:

1. $B_w = L^1(X)$ with the norm $\|f\|_{L^1} = \int_X |f(x)|dx$;
2. $B_s = W^{1,1}$ with the norm $\|f\|_{W^{1,1}} = \int_X |f(x)|dx + \int_X |f'(x)|dx$;
3. $B_{ss} = C^2(X)$ with the norm $\|f\|_{C^2(X)} = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty$ where $\|f\|_\infty = \sup_{x \in X} |f(x)|$.

In this context, item $1)$ of Theorem 2.3 is trivial on $L^1$ as the transfer operators are weak contractions. Items $2)$ and $3)$ of Theorem 2.3 are proved for example in [10], Section 6. The existence of the operator $\mathbb{L}$, and an explicit formula for it, will be proved in the next section (see Proposition 3.1). From this follows the differentiability of the physical measure and the linear response formula (4) for our family of expanding maps.

# 3 The derivative operator for circle expanding maps

In this section we present a detailed description of the structure of the operator $\mathbb{L}$ in our case.

**Proposition 3.1.** *Let $T_\delta$ be a family of expanding maps as considered before. Let $w \in C^3(X, \mathbb{R})$. For each $x \in X$ we can write*

$$\begin{aligned} \mathbb{L}w(x) &= \lim_{\delta \to 0} \left( \frac{\mathcal{L}_\delta w(x) - \mathcal{L}_0 w(x)}{\delta} \right) \\ &= -\mathcal{L}_0 \left( \frac{w\epsilon'}{T_0'} \right)(x) - \mathcal{L}_0 \left( \frac{\epsilon w'}{T_0'} \right)(x) + \mathcal{L}_0 \left( \frac{\epsilon T_0''}{(T_0')^2} w \right)(x) \end{aligned} \tag{5}$$

*and the convergence is also in the $C^1$ topology.*

Before presenting the proof of Proposition 3.1 we state a simple lemma.

**Lemma 3.2.** *If $y_i^\delta \in T_\delta^{-1}(x)$ then we can expand*

$$y_i^\delta = y_i^0 + \delta \left( -\frac{\epsilon(y_i^0)}{T_0'(y_i^0)} \right) + o_{C^2}(\delta).$$

*Proof of Lemma 3.2.* We denote by $\{y_i^\delta\}_{i=1}^d := T_\delta^{-1}(x)$ and $\{y_i^0\}_{i=1}^d := T_0^{-1}(x)$ the $d$ preimages under $T_\delta$ and $T_0$, respectively, of a point $x \in X$. Let us write

$$y_i^\delta(x) = y_i^0(x) + \delta\epsilon_i(x) + F_i(\delta, x).$$

We will show that $F_i(\delta, x) = o_{C^2}(\delta)$. Substituting this into the identity $T_\delta(y_i^\delta(x)) = x$ and and using that $T_\delta(x) = T_0(x) + \delta\epsilon(x) + o_{C^3}(\delta)$ we can expand

$$\begin{aligned} x &= T_\delta(y_i^\delta(x)) \\ &= T_0(y_i^\delta(x)) + \delta\epsilon(y_i^\delta(x)) + E(\delta, y_i^\delta(x)) \\ &= T_0(y_i^0(x) + \delta\epsilon_i(x) + F_i(\delta, x)) \\ &\quad + \delta\epsilon(y_i^0(x) + \delta\epsilon_i(x) + F_i(\delta, x)) + E(\delta, y_i^\delta(x)). \end{aligned} \tag{6}$$

We can write the first term in the final line of (6) as

$$\begin{aligned} &T_0(y_i^0(x) + \delta\epsilon_i(x) + F_i(\delta, x)) \\ &= T_0(y_i^0(x)) + T_0'(y_i^0(x))(\delta\epsilon_i(x) + F_i(\delta, x)) + o_{C^2}(\delta) \end{aligned}$$

and the second term in the last line as

$$\delta\epsilon(y_i^0(x) + \delta\epsilon_i(x) + F_i(\delta, x)) = \delta\epsilon(y_i^0(x)) + \delta F_i(\delta, x) + o_{C^2}(\delta)$$

and use that $T_0(y_i^0(x)) = x$ to cancel terms on either side of (6) to get that

$$0 = T_0'(y_i^0(x))(\delta\epsilon_i(x) + F_i(\delta, x)) + \delta\epsilon(y_i^0(x)) + \delta F_i(\delta, x) + E(\delta, y_i^\delta(x)) + o_{C^2}(\delta).$$

Thus we can identify the first order terms as $\delta T_0'(y_i^0(x))\epsilon_i(x) + \delta\epsilon(y_i^0(x))$ and then what is left is

$$T_0'(y_i^0(x))F_i(\delta, x) + \delta F_i(\delta, x) = -E(\delta, y_i^\delta(x)) + o_{C^2}(\delta)$$

from which the result follows. □

We now return to the proof of Proposition 3.1.

*Proof of Proposition 3.1.* We again denote by $\{y_i^\delta\}_{i=1}^d := T_\delta^{-1}(x)$ and $\{y_i^0\}_{i=1}^d := T_0^{-1}(x)$ the $d$ preimages under $T_\delta$ and $T_0$, respectively, of a point $x \in X$. Futhermore, we assume that the indexing is chosen so that $y_i^\delta$ is a perturbation of $y_i^0$, for $1 \le i \le d$. We can write

$$\begin{aligned}
&\frac{\mathcal{L}_\delta w(x) - \mathcal{L}_0 w(x)}{\delta} \\
&= \frac{1}{\delta}\left(\sum_{i=1}^d \frac{w(y_i^\delta)}{T_\delta'(y_i^\delta)} - \sum_{i=1}^d \frac{w(y_i^0)}{T_0'(y_i^0)}\right) \\
&= \underbrace{\frac{1}{\delta}\left(\sum_{i=1}^d w(y_i^\delta)\left(\frac{1}{T_\delta'(y_i^\delta)} - \frac{1}{T_0'(y_i^\delta)}\right)\right)}_{=:(I)} + \underbrace{\frac{1}{\delta}\left(\sum_{i=1}^d \frac{w(y_i^\delta) - w(y_i^0)}{T_0'(y_i^\delta)}\right)}_{=:(II)} \\
&+ \underbrace{\frac{1}{\delta}\left(\sum_{i=1}^d w(y_i^0)\left(\frac{1}{T_0'(y_i^\delta)} - \frac{1}{T_0'(y_i^0)}\right)\right)}_{=:(III)}.
\end{aligned} \tag{7}$$

For the first term we first differentiate the expansion $T_\delta(x) = T_0(x) + \delta\epsilon(x) + o_{C^3}(\delta)$ in $x$ to get:

$$T_\delta'(x) = T_0'(x) + \delta\epsilon'(x) + o_{C^2}(\delta).$$

We can then write

$$
\begin{aligned}
(I) &= \frac{1}{\delta}\left(\sum_{i=1}^{d} w(y_i^\delta)\left(\frac{1}{T'_\delta(y_i^\delta)} - \frac{1}{T'_0(y_i^\delta)}\right)\right) \\
&= \frac{1}{\delta}\left(\sum_{i=1}^{d} \frac{w(y_i^\delta)}{T'_\delta(y_i^\delta)}\left(1 - \frac{T'_\delta(y_i^\delta)}{T'_0(y_i^\delta)}\right)\right) \\
&= \frac{1}{\delta}\left(\sum_{i=1}^{d} \frac{w(y_i^\delta)}{T'_\delta(y_i^\delta)}\left(1 - \left(\frac{T'_0(y_i^\delta) + \delta\epsilon'(y_i^\delta) + o_{C^2}(\delta)}{T'_0(y_i^\delta)}\right)\right)\right) \\
&= \left(-\sum_{i=1}^{d} \frac{w(y_i^\delta)\epsilon'(y_i^\delta)}{T'_\delta(y_i^\delta)T'_0(y_i^\delta)}\right) + o_{C^2}(1).
\end{aligned}
$$

Thus we have that

$$
\begin{aligned}
\lim_{\delta\to 0}\frac{1}{\delta}\left(\sum_{i=1}^{d} w(y_i^\delta)\left(\frac{1}{T'_\delta(y_i^\delta)} - \frac{1}{T'_0(y_i^\delta)}\right)\right) &= \lim_{\delta\to 0}\left(-\sum_{i=1}^{d}\frac{w(y_i^\delta)\epsilon'(y_i^\delta)}{T'_0(y_i^\delta)^2}\right) \\
&= -\mathcal{L}_0\left(\frac{w\epsilon'}{T'_0}\right)
\end{aligned}
$$

and the limit converges in $C^1$. For the second term of (7) we can use Lemma 3.2 to write

$$
\begin{aligned}
w(y_i^\delta) &= w(y_i^0) + w'(y_i^0)\left(\frac{dy_i^\delta}{d\delta}|_{\delta=0}\right)\delta + o_{C^1}(\delta) \\
&= w(y_i^0) + w'(y_i^0)\left(-\frac{\epsilon(y_i^0)}{T'_0(y_i^0)}\right)\delta + o_{C^1}(\delta).
\end{aligned}
$$

Thus

$$
\begin{aligned}
(II) = \frac{1}{\delta}\sum_{i=1}^{d}\frac{w(y_i^\delta) - w(y_i^0)}{T'_0(y_i^\delta)} &= \frac{1}{\delta}\sum_{i=1}^{d}\frac{w'(y_i^0)}{T'_0(y_i^\delta)}\left(-\frac{\epsilon(y_i^0)}{T'_0(y_i^0)}\right) + o_{C^1}(1) \\
&= -\sum_{i=1}^{d}\frac{\epsilon(y_i^0)w'(y_i^0)}{T'_0(y_i^0)T'_0(y_i^\delta)} + o_{C^1}(1)
\end{aligned}
$$

and therefore, both pointwise and in the $C^1$ topology

$$
\lim_{\delta\to 0}\frac{1}{\delta}\sum_{i=1}^{d}\frac{w(y_i^\delta) - w(y_i^0)}{T'_0(y_i^\delta)} = -\mathcal{L}_0\left(\frac{\epsilon w'}{T'_0}\right)(x).
$$

Finally, for the third term we can write

$$\begin{aligned} T_0'(y_i^\delta) &= T_0'(y_i^0) + T_0''(y_i^0)\left(\frac{dy_i^\delta}{d\delta}|_{\delta=0}\right)\delta + o_{C^1}(\delta) \\ &= T_0'(y_i^0) + T_0''(y_i^0)\left(-\frac{\epsilon(y_i^0)}{T_0'(y_i^0)}\right)\delta + o_{C^1}(\delta), \end{aligned}$$

again using the Lemma 3.2. Therefore

$$\begin{aligned} (III) =& \frac{1}{\delta}\left(\sum_{i=1}^d w(y_i^0)\left(\frac{1}{T_0'(y_i^\delta)} - \frac{1}{T_0'(y_i^0)}\right)\right) \\ =& \frac{1}{\delta}\left(\sum_{i=1}^d w(y_i^0)\left(\frac{T_0'(y_i^0) - T_0'(y_i^\delta)}{T_0'(y_i^\delta)T_0'(y_i^0)}\right)\right) \\ =& \frac{1}{\delta}\left(\sum_{i=1}^d w(y_i^0)\left(\frac{-\left(T_0'(y_i^0) + T_0''(y_i^0)\left(-\frac{\epsilon(y_i^0)}{T_0'(y_i^0)}\right)\delta\right) + T_0'(y_i^0)}{T_0'(y_i^\delta)T_0'(y_i^0)}\right)\right) + o_{C^1}(1) \\ =& \frac{1}{\delta}\left(\sum_{i=1}^d w(y_i^0)\left(\frac{\epsilon(y_i^0)T_0''(y_i^0)}{T_0'(y_i^0)^2 T_0'(y_i^\delta)}\right)\right) + o_{C^1}(1) \end{aligned}$$

and thus, finally,

$$\lim_{\delta\to 0}\frac{1}{\delta}\left(\sum_{i=1}^d w(y_i^0)\left(\frac{1}{T_0'(y_i^\delta)} - \frac{1}{T_0'(y_i^0)}\right)\right) = \mathcal{L}_0\left(\frac{\epsilon T_0''}{(T_0')^2}w\right)(x)$$

in $C^1$. □

# 4 The control problem

We can now use the preceding results to address Problem 1.3, finding first order perturbations to the expanding maps corresponding to a given first order perturbation in the density.

Putting together the information given by Theorem 2.3 and Proposition 3.1 in the particular case that $w = \rho$ is the invariant density for $T_0 : X \to X$ we arrive at an equation that allows us to address the question posed in Problem

1.3. More precisely, given the required direction of change $\rho^{(1)} \in C^1(X, \mathbb{R})$ we want to find $\epsilon(x)$ such that the associated operator $\mathbb{L}$ satisfies

$$(I - \mathcal{L}_0)\rho^{(1)} = \mathbb{L}\rho(x).$$

Thus by Proposition 3.1, given $\rho^{(1)}$ we need to solve for $\epsilon(x)$ such that

$$(I - \mathcal{L}_0)\rho^{(1)}(x) = \mathcal{L}_0 \left( -\frac{\rho\epsilon'}{T_0'} - \frac{\epsilon\rho'}{T_0'} + \frac{\epsilon T_0''}{(T_0')^2} \right)(x). \tag{8}$$

The problem hence involves the solution of a differential equation for $\epsilon : X \to \mathbb{R}$ in the space of functions representing infinitesimal changes of the system. We remark that given a function $f$, the equation $f = \mathcal{L}_0 g$ typically may have many solutions, as $T_0$ is not bijective.

Thus in the solution of Problem 1.3 there are two steps:

1. Firstly we have to choose a solution $g$ to $(I - \mathcal{L}_0)\rho^{(1)} = \mathcal{L}_0\,(g)$. Typically there will be an infinite dimensional family of solutions; and

2. Secondly, we can solve for $\epsilon$ such that

$$g = -\epsilon' \left( \frac{\rho}{T_0'} \right) - \epsilon \left( \frac{\rho'}{T_0'} \right) + \epsilon \left( \frac{T_0''}{(T_0')^2} \right).$$

   In particular, we have to solve a linear first order inhomogeneous differential equation.

Finally, amongst the possible solutions $\epsilon(x)$ of this problem, it is natural to look for one which is optimal, in a suitable sense. In particular, we will search for a solution of minimal size with respect to some natural norm to be described later. We will prove that under suitable assumption such an optimal solution exists.

## 4.1 Existence of a solution

Our first main conclusion about existence is the following.

**Theorem 4.1.** *Let $k \geq 4$. For a $C^k$ expanding map of the circle $T_0$, any $C^{k-2}$ first order perturbation $\rho^{(1)}$ in the density can be realised by a suitable first order $C^{k-1}$-perturbation $\epsilon$ in the transformation. Moreover there is an infinite dimensional space of perturbations achieving this.*

Before giving the proof of Theorem 4.1 we state a lemma regarding the solutions of $f = \mathcal{L}_0 g$. In the setting of expanding maps of the circle this takes a simple form. Let $C^{k-1}(X, \mathbb{R})$ be the Banach space of $C^{k-1}$ functions on the unit circle $X$, for $k \geq 1$.

Let $T : X \to X$ be a $C^k$ expanding map of the circle $X$. It is known that such maps have a unique $C^{k-1}$ invariant density. Let $h : X \to X$ be an orientation preserving $C^k$ diffeomorphism. We can define a new map $S : X \to X$ by conjugation $S = h \circ T \circ h^{-1}$.

We can define transfer operators $\mathcal{L}_S : C^{k-1}(X) \to C^{k-1}(X)$ and $\mathcal{L}_T : C^{k-1}(X) \to C^{k-1}(X)$ by

$$\mathcal{L}_T w(x) = \sum_{Ty=x} \frac{w(y)}{T'(y)} \text{ and } \mathcal{L}_S w(x) = \sum_{Sy=x} \frac{w(y)}{S'(y)}.$$

**Lemma 4.2.** *For each orientation preserving $C^k$ diffeomorphism $h$ we can write $(\mathcal{L}_S w) \circ h = \frac{1}{h'} \mathcal{L}_T \left( (w \circ h) h' \right)$.*

*Proof.* We can differentiate $S \circ h = h \circ T$ and use the chain rule to write

$$(S' \circ h) h' = (h' \circ T) T'.$$

Let us write $y = h(y')$ and $x' = h^{-1}(x)$ then we have

$$\begin{aligned} \mathcal{L}_S w(x) = \sum_{S(hy')=x} \frac{w(y)}{S' \circ h(y')} &= \sum_{T(y')=h^{-1}(x)} \frac{w(hy') h'(y')}{h'(Ty') T'(y')} \\ &= \sum_{T(y')=x'} \frac{(w \circ h)(y') h'(y')}{h'(Ty') T'(y')} \\ &= \frac{1}{h'(x')} \sum_{T(y')=x'} \frac{(w \circ h)(y') h'(y')}{T'(y')}. \end{aligned}$$

This corresponds to the required identity. □

If we assume that $T$ has an invariant density $\rho$ then we know that $\mathcal{L}_T \rho = \rho \in C^{k-1}(X)$. By a suitable choice of coordinates we can assume that $S(0) = 0 = T(0)$ are corresponding fixed points. We can then define $h : X \to X$ by $h(x) = \int_0^x \rho(u) du$. Then $h'(x) = \rho(x)$ and by the Lemma 4.2 with $w$ taking the constant value $1$ we can write

$$(\mathcal{L}_S 1) \circ h = \frac{1}{h'} \mathcal{L}_T (1 h') = \frac{1}{\rho} \mathcal{L}_T(\rho) = 1.$$

We then conclude that $(\mathcal{L}_S 1) = 1$. In particular, the transformation $S : X \to X$ has density $1$ (i.e., $S$ preserves the Haar measure on $X$).

**Lemma 4.3.** *Let $T : X \to X$ be a $C^k$ map with $k \geq 2$. For any $f \in C^{k-1}(X)$ we can find $g \in C^{k-1}(X)$ such that $f = \mathcal{L}_T(g)$. Moreover, we can find an infinite dimensional set of such solutions $g$.*

*Proof.* Given $f \in C^{k-1}(X)$ we can first look for a solution $g \in C^{k-1}(X)$ to $f = \mathcal{L}_S g$. However, since $\mathcal{L}_S 1 = 1$ we see that it merely suffices to choose $g = f \circ S$, since then

$$\begin{aligned}\mathcal{L}_S(g)(x) = \mathcal{L}_S(f \circ S)(x) &= \sum_{Sy=x} \frac{f(Sy)}{S'(y)} = f(x) \sum_{Sy=x} \frac{1}{S'(y)} \\ &= f(x) \underbrace{(\mathcal{L}_S 1)(x)}_{=1} = f(x)\end{aligned}$$

and the result follows. Furthermore, there are clearly uncountably many such solutions $g$.

In the general case, we can apply the identity in the previous lemma $\mathcal{L}_T\left((w \circ h)\rho\right) = \rho(\mathcal{L}_S w) \circ h$ with the choice $w = (f/\rho) \circ h^{-1} \circ S$. This gives that

$$\begin{aligned}\mathcal{L}_T\left(((f/\rho) \circ h^{-1} \circ S \circ h)\rho\right) &= \rho(\mathcal{L}_S((f/\rho) \circ h^{-1} \circ S)) \circ h \\ &= \rho((f/\rho) \circ h^{-1}) \circ h = f.\end{aligned}$$

Thus it suffices to let $g = ((f/\rho) \circ h^{-1} \circ S \circ h)\rho$ to get the required result.

Given a solution $g$, every point of the set $g + ker(\mathcal{L}_T)$ is again a solution. Moreover, $\mathcal{L}_T$ is an infinite dimensional space. Indeed consider the intervals $I_1, ..., I_n$ where the branches of $T_i$ of $T$ are defined, then given a $C^{k-1}$ density $\rho_1$ supported on the interior of $I_1$ it is easy to find a density $\rho_2$ supported in $I_2$ such that $\mathcal{L}_T(\rho_1) = -\mathcal{L}_T(\rho_2)$. □

*Proof of Theorem 4.1.* Recall that in equation (8) we have

$$(I - \mathcal{L}_0)\rho^{(1)}(x) = \mathcal{L}_0\left(-\frac{\rho\epsilon'}{T_0'} - \frac{\epsilon\rho'}{T_0'} + \frac{\epsilon T_0''}{(T_0')^2}\right)(x).$$

The left hand side is a $C^{k-2}$ function, by assumption. Since $T_0$ is $C^k$ then we have $\rho \in C^{k-1}(X)$. By Lemma 4.3 we can choose a $C^{k-1}$ solution $g_1$ for the

step 2 described above, such that $(I - \mathcal{L}_0)\rho^{(1)}(x) = \mathcal{L}_0\,(g)\,(x)$. Finally this means that the differential equation we need to solve is

$$g_1 = -\epsilon' \left(\frac{\rho}{T_0'}\right) - \epsilon \left(\frac{\rho'}{T_0'}\right) + \left(\frac{\epsilon T_0''}{(T_0')^2}\right),$$

which has at least $C^{k-2}$ coefficients, and hence a $C^{k-1}$ family of solutions, proving the statement. Each such solution is a solution to Problem (1.3) by Theorem 2.3. □

In particular, taking $k = 5$ gives the following corollary.

**Corollary 4.4.** *For a $C^5$ expanding map of the circle $T_0$, any first order perturbation $\rho^{(1)} \in C^3(X)$ in the density can be realised by a suitable first order perturbation $\epsilon \in C^4(X)$ in the transformation.*

## 4.2 Optimal solutions

We now consider the problem of finding an optimal solution amongst the many different possible solutions. It is natural to minimize the size of the perturbation in a suitable norm. Since we are considering smooth dynamics and perturbations we can choose the following natural Sobolev-type norm

$$||f||_{abcd} = ||f||_2 + a||f'||_2 + b||f''||_2 + c||f'''||_2 + d||f^{iv}||_2$$

for given $a, b, c, d \geq 0, d > 0$. With this norm, the associated space $H^4$ of functions having $L^2$ fourth derivatives is a Hilbert space.

**Proposition 4.5.** *If $\rho^{(1)} \in C^3$ and $T_0 \in C^5$ then in the space of $H^4$ solutions to (8) there is a unique minima with respect to the $||\cdot||_{abcd}$ norm.*

*Proof.* We begin by observing that the equation

$$(I - \mathcal{L}_0)\rho^{(1)} = \mathcal{L}_0 \left(-\frac{\rho\epsilon'}{T_0'} - \frac{\epsilon\rho'}{T_0'} + \frac{\epsilon T_0''}{(T_0')^2}\right) \tag{9}$$

is equivalent to $g = \tilde{L}(\epsilon)$, where $g = (I - \mathcal{L}_0)\rho^{(1)}$ and $\tilde{L}$ is the linear operator on the Right Hand Side of equation (9). We observe that $\tilde{L}$ is continuous as an operator from $H^4$ to $L^2(X)$ and thus $ker(\tilde{L})$ is a closed space in $H^4$. Moreover, the space of solutions of 9 in $H^4$ is not empty, because of Corollary 4.4. We can therefore deduce that the space of solutions to (8) is a closed

affine space on which we can search for an element of minimum norm. Finally, since we are in a Hilbert space there is a unique minima. Indeed, we can take a solution $v$ of (9) and then subtract its projection on $ker(\tilde{L})$, this is orthogonal to $ker(\tilde{L})$ and thus to the affine space of solutions $ker(\tilde{L}) + v$. This solution necessarily minimizes the norm. □

**Remark 4.6.** *The minimal solution found in Proposition 4.5 is an actual solution of the initial problem. Indeed let us call $\epsilon_0$ this solution. Then*

$$\mathcal{L}_0\left(-\frac{\rho\epsilon_0'}{T_0'} - \frac{\epsilon_0\rho'}{T_0'} + \frac{\epsilon_0 T_0''}{(T_0')^2}\right) \in W^{1,1}$$

*and then by Theorem 2.3 (which can be applied since the solution $\epsilon_0$ is in $C^3$) we get*

$$\rho^{(1)} = (I - \mathcal{L}_0)^{-1}\mathcal{L}_0\left(-\frac{\rho\epsilon_0'}{T_0'} - \frac{\epsilon_0\rho'}{T_0'} + \frac{\epsilon_0 T_0''}{(T_0')^2}\right) \tag{10}$$

*is the required linear response associated to the first order perturbation $\epsilon_0$.*

**Remark 4.7.** *The main point of the optimization procedure is an orthogonalization. It seems that this can be efficiently implemented by an algorithm to produce the optimal solution, once an effective characterization of $ker(\tilde{L})$ is provided.*

# 5 Example: the doubling map

In this section we consider a simple example which can be easily analysed using classical Fourier series.

Let $T : X \to X$ be the doubling map given by $T(x) = 2x \pmod 1$ then $T_0' = 2$, $T_0'' = 0$ and $\rho = 1$ and $\rho' = 0$. Let us write the prescribed perturbation $\rho^{(1)}(x)$ in the density as a Fourier series, i.e.,

$$\rho^{(1)}(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$$

and then observe that since

$$\mathcal{L}_0\left(e^{2\pi i n x}\right) = \begin{cases} e^{2\pi i (n/2) x} & \text{if n is even} \\ 0 & \text{if n is odd} \end{cases}$$

we have that the equation we need to solve becomes

$$(I - \mathcal{L}_0)\rho^{(1)}(x) = \mathcal{L}_0\left(-\frac{\epsilon'}{2}\right)$$

where

$$(I - \mathcal{L}_0)\rho^{(1)}(x) = \sum_{n\in\mathbb{Z}}(a_n - a_{2n})e^{2\pi inx}.$$

Moreover, given any function $f(x)$ written in the form

$$f(x) = \sum_{n\in\mathbb{Z}} b_n e^{2\pi inx}$$

we see that

$$(\mathcal{L}_0 f)(x) = \sum_{n\in\mathbb{Z}} b_{2n} e^{2\pi inx}.$$

Comparing coefficients, for $f(x)(= -\frac{\epsilon'(x)}{2})$ to be a solution to $(I - \mathcal{L}_0)\rho^{(1)} = \mathcal{L}_0(f)$ now corresponds to

1. $b_n = a_{n/2} - a_n$ if $n$ is even;
2. $b_n$ have no restrictions if $n$ is odd

and finally we see that the (infinitely many) solutions to the linear perturbation are solutions to

$$-\epsilon'(x) = 2f(x) = \sum_{n\in\mathbb{Z}}(2b_n)e^{2\pi inx}.$$

That is

$$\epsilon(x) = -\sum_{n\in\mathbb{Z}} \frac{b_n}{\pi in} e^{2\pi inx}$$

(where the constant of integration corresponding to $b_0$ is actually zero), and a solution is

$$\epsilon(x) = -\sum_{n\in 2\mathbb{Z}} \frac{a_{n/2} - a_n}{\pi in} e^{2\pi inx} + \sum_{n\in\mathbb{Z}-2\mathbb{Z}} \frac{c_n}{\pi in} e^{2\pi inx}.$$

For every $\{c_i\} \in l^2$.

$$\|\epsilon\|_2^2 = \sum_{n\in\mathbb{Z}} \frac{|a_{2n} - a_n|^2}{\pi^2(2n)^2} + \sum_{n\in\mathbb{Z}} \frac{|c_{2n+1}|^2}{\pi^2(2n+1)^2}.$$

In particular, this is minimised when $c_{2n+1} = 0$ for $n \in \mathbb{Z}$ and leaves us with a distinguished solution

$$\epsilon_0(x) = \sum_n \frac{(a_n - a_{2n})}{2\pi i n} e^{2\pi i 2nx}.$$

Similarly for the $|| \cdot ||_{abcd}$ norm we can reason as before, having the same distinguished solution $\epsilon_0$.

**Example 5.1.** *In the particular case of the doubling map and $\rho^{(1)} = \sin(2\pi x)$ we have that $a_1 = \frac{1}{2i}$ and $a_{-1} = -\frac{1}{2i}$ and all the other terms are zero. Thus $2b_2 = i$ and $2b_{-2} = -i$ and all the other terms are zero. Thus we can write*

$$\begin{aligned}\epsilon(x) &= \frac{e^{4i\pi x}}{4\pi} + \frac{e^{-4i\pi x}}{4\pi} + \sum_{n\in\mathbb{Z}} \frac{b_{2n+1}}{2\pi n} e^{2\pi i(2n+1)x} \\ &= \frac{1}{2\pi}\cos(4\pi x) + \sum_{n\in\mathbb{Z}} \frac{b_{2n+1}}{2\pi n} e^{2\pi i(2n+1)x}.\end{aligned}$$

*If we additionally want to choose the $u(x)$ so as to minimise the $L^2$-norm then*

$$\epsilon_0(x) = \frac{1}{2\pi}\cos(4\pi x)$$

*and*

$$\|\epsilon_0\|_2 = \sqrt{\int_0^1 \left(\frac{1}{2\pi}\cos(4\pi x)\right)^2 dx} = \frac{\sqrt{8}}{8\pi}.$$

**Remark 5.2.** *It is natural to consider in which way the previous results could be generalized to systems with more dimensions and with contracting directions. In this case, using suitable anisotropic norms, we can have a spectral gap, and probably the general structure of the problem remains similar, with Proposition 2.3 applying to a suitable space of distributions, in a way similar to that which we have seen for circle expanding maps, however the formula in 3.1 is quite specific to the expanding case, and in the general case a suitable generalization of the derivative operator should apply to measures and distributions.*

**Remark 5.3.** *After the completion of this manuscript, B. Kloeckner ([13]) investigated a control problem similar to the one considered here, although in that work, only restricted perturbations of the system coming from a smooth*

*conjugacy were considered. With this point of view, using ideas similar to those in [14], it was proved that there is always at least one solution of the control problem for a large class of systems preserving a smooth invariant measure. However, the class of admissible changes used in that paper is much smaller than in the present work, and the class of solutions found is correspondingly smaller. In particular, the optimal changes (arising from conjugacies) found in [13] for expanding maps may be very different from the ones we found in the present work, as it is shown in Section 2.4 of [13] for the doubling map.*

# References


[1] Baladi, V., On the susceptibility function of piecewise expanding interval maps, *Comm. Math. Phys.*, (2007) 839-859.

[2] V. Baladi, Linear response, or else, *ICM proceedings Seoul 2014* , vol III, pp525-545

[3] V. Baladi, M. Todd. Linear response for intermittent maps. arXiv:1508.02700 (to appear in Comm. Math. Phys.)

[4] V. Baladi and D. Smania, Linear response formula for piecewise expanding unimodal maps, *Nonlinearity*, (2008) 677–711.

[5] W. Bahsoun, S. Galatolo, A. Nisoli, A Rigorous Computational Approach to Linear Response, arXiv:1506.08661

[6] W. Bahsoun, B. Saussol Linear response in the intermittent family: differentiation in a weighted $C^0$-norm. arXiv:1512.01080

[7] O. Butterley and C. Liverani, Smooth Anosov Flows: Correlation Spectra and Stability, *J. Mod. Dyn.*, 1(2):301-322, 2007.

[8] G. Contreras, Regularity of topological and metric entropy of hyperbolic flows. *Math. Z.* 210 (1992) 97-111.

[9] D. Dolgopyat On differentiability of SRB states for partially hyperbolic systems. *Inv. Math.* 155 (2004) 389-449.

[10] S. Galatolo, Statistical properties of dynamics. Introduction to the functional analytic approach, arXiv:1510.02615

[11] S.Gouezel C. Liverani Banach spaces adapted to Anosov systems *Erg. Th. Dyn. Sys.* 26, 1, 189–217, (2006).

[12] A. Korepanov. Linear response for intermittent maps with summable and nonsummable decay of correlations. arXiv:1508.06571

[13] Kloeckner B. The linear request problem. arXiv:1606.02428

[14] Moser J. On the volume elements on a manifold. Trans. AMS, 120: 286-294, 1965

[15] M. Pollicott and O. Jenkinson, Computing invariant densities and metric entropy, *Comm. Math. Phys.* (2000), 211: 687-703.

[16] O. Jenkinson and M. Pollicott. Entropy, Exponents and invariant densities for hyperbolic systems: Dependence and computation, in *Modern Dynamical Systems and its Applications* (eds. M. Brin, B. Hasselblatt, Y. Pesin), C.U.P., Cambridge, 2004.

[17] V. Lucarini, F. Lunkeit, F. Ragone Predicting Climate Change using Response Theory: Global Averages and Spatial Patterns arXiv:1512.06542 [physics.ao-ph]

[18] C. Liverani Invariant measures and their properties. A functional analytic point of view, Dynamical Systems. Part II: in *Topological Geometrical and Ergodic Properties of Dynamics*. Proceedings, by Scuola Normale Superiore, Pisa (2004).

[19] Ruelle, D., Differentiation of SRB states, *Comm. Math. Phys.* 187 (1997) 227–241.

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