VARIATIONS ON A THEME OF SOLOMON

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ABSTRACT.

The main aim of this thesis is to examine the structure of the Hecke algebra $H_K(G,B)$ of a finite group $G$ with a split $(B,N)$ pair of rank $n$ and characteristic $p$, with Weyl group $W$, over a field $K$ of characteristic $p$. We then see how this relates to the structure of the $KG$-module $L$ induced from the principal $KB$-module.

Chapters 1, 2 and 3 are introductory. Chapter 1 deals with the properties of finite Coxeter groups, and gives Solomon's decomposition of the group algebra of a finite Coxeter group over the field of rational numbers. In Chapter 2 we discuss finite groups with split $(B,N)$ pairs and their irreducible modular representations. Chapter 3 deals with Hecke algebras and the generic ring and some of its specialisations.

In Chapter 4, we examine the structure of the $0$-Hecke algebra $H$ of type $(W,R)$ over any field, which is defined in Chapter 3; the algebra $H_K(G,B)$ is an example of one of these. $H$ has $2^n$ distinct irreducible representations where $n = |R|$, all of which are one-dimensional, and correspond in a natural way with subsets of $R$. $H$ can be written as a direct sum of $2^n$ indecomposable left (or right) ideals, in a similar manner to the Solomon decomposition of the underlying Weyl group $W$. 
In Chapter 5, we also obtain decompositions of the generic ring similar to Solomon's decomposition of the underlying Coxeter group. These decompositions carry over to some specialisations of the generic ring; in particular, we get Solomon's decomposition of the Coxeter group and decompositions of the Hecke algebra of a finite group with \((B,N)\) pair over a field of characteristic zero.

Certain homology modules which arise from the Tits building of a finite group with a \((B,N)\) pair, called relative Steinberg modules because of the character of the group they afford, are discussed in Chapter 6.

Finally, in Chapter 7, we see that \(L\) is a direct sum of \(2^n\) indecomposable \(KG\)-modules, each of which is a relative Steinberg module. We can deduce what \(|\mathcal{H}|\) of the composition factors of \(L\) are, but there are in general others.
**NOTATION.**

\( \mathbb{Z} \) The rational integers.

\( \mathbb{Z}^+ \) The non-negative rational integers.

\( \mathbb{Q} \) The rational number field.

\( \mathbb{R} \) The real number field.

DCC Descending chain condition.

ACC Ascending chain condition.

\( K^* \) If \( K \) is a field, \( K^* = K - \{0\} \).

Let \( S \) be a set and \( J \) a subset of \( S \). Then:

\( |J| \) The number of elements in \( J \).

\( \hat{J} \) The complement of \( J \) in \( S \).

\( \{i_1, \ldots, i_s, \ldots i_n\} \) The set \( \{i_1, \ldots, i_n\} - \{i_s\} \).

\( < J > \) The group or the algebra generated by \( J \).

\( W_1 \cdots \hat{W}_j \cdots W_n = W_1 \cdots W_{j-1} W_{j+1} \cdots W_n \)

\( \bigcup \) Set union.

\( \bigcap \) Set intersection.

\((w_i w_j w_i \ldots)_n \) and \((w_i w_j \ldots)_n \) The product of the first \( n \) terms of the sequence \( w_i, w_j, w_i, w_j, \ldots \)

\((\ldots w_i w_j w_i)_n \) and \((\ldots w_j w_i)_n \) The product of the first \( n \) terms of the alternating sequence \( w_i, w_j, w_i, w_j, \ldots \) written from right to left.
Chapter 1: **FINITE COXETER GROUPS.**

(1.1) **Finite Reflection Groups.**

Most of the results in this section are proved in either Bourbaki [1], Carter [3] or Steinberg [24].

Let \( V \) be an \( n \)-dimensional real Euclidean space, with a positive definite symmetric scalar product \((\cdot, \cdot)\). An orthogonal linear transformation \( w \) of \( V \) is called a reflection if \( w \neq 1 \) and \( w \) fixes some hyperplane \( H \) pointwise. If \( r \) is a vector perpendicular to \( H \), then \( w \) is given by

\[
(1.1.1) \quad w(x) = x - \frac{2(x, r)}{(r, r)} r \quad \text{for all } x \in V.
\]

We write \( w = w_r \). Note that \( w_r^2 = 1 \) and \( w_r(r) = -r \). \( w_r \) is an automorphism of \( V \), and if \( \sigma \) is an automorphism of \( V \), then

\[
\sigma w_r \sigma^{-1} = w_{\sigma(r)}.
\]

We call a group \( W \) a reflection group if \( W \) is generated by a set of reflections, and \( |W| < \infty \).

(1.1.2) **Definition:** A finite subset \( \Phi \) of \( V \) is called a system of roots if the following conditions are satisfied:

1. \( \Phi \) is a set of generators for \( V \).
2. \( 0 \notin \Phi \), and if \( r \in \Phi \) and \( cr \in \Phi \) for some \( c \in \mathbb{R} \), then \( c = \pm 1 \).
3. \( w_r(\Phi) = \Phi \) for each \( r \in \Phi \), where \( w_r \) is the reflection defined in (1.1.1).

The elements of \( \Phi \) are called roots, and from the definition it follows that \( \Phi = -\Phi \). Let \( W = W(\Phi) \) be the group generated by the reflections \( \{w_r : r \in \Phi\} \). Then the
restriction of $\Phi$ to $\phi$ is faithful, and hence $W$ is a finite group. $W$ is called the Weyl group of the root system $\Phi$.

**NOTE:** Bourbaki [1] and Carter [3] have the following additional condition to define a system of roots:

(4) $\frac{2(r,s)}{(r,r)} \in \mathbb{Z}$ for all $r$ and $s \in \phi$.

We do not include this condition in our definition as we wish to associate root systems with arbitrary finite reflection groups.

(1.1.3) **Lemma** (Curtis [9], Richen [18]): Every finite group $W$ generated by reflections of a real Euclidean space $V$ can be identified with the Weyl group of some root system.

**Proof:** First, by dividing out the space of invariant vectors under $W$, we may assume that $W$ is a finite group generated by reflections, which leaves no non-zero vectors fixed.

Let $\phi$ be the set of all unit vectors which are perpendicular to the hyperplane fixed by some reflection in $W$. Then

1. $\phi$ is a system of roots. For if $r,s \in \phi$, then $w_r(s) \in \phi$ as $w_r(s)$ is a unit vector orthogonal to the hyperplane fixed pointwise by the reflection $w_r w_s w_r^{-1} = w_{w_r(s)}$. So $\phi$ is finite and generates $V$, and $r$ and $cr \in \phi$ implies $c=\pm 1$, and $r \in \phi$ implies $w_r \in W$.

2. The $w_r$'s for $r \in \phi$ generate $W$. 

(1.1.4) **Definition:** A subset \( \Phi^+ \) of roots of \( \Phi \) is called a positive system if it consists of the roots which are positive relative to some ordering of \( V \).

(1.1.5) **Definition:** A subset \( \Pi \) of roots of \( \Phi \) is called a simple system if

1. the elements of \( \Pi \) are linearly independent, and
2. every root is a linear combination of the elements of \( \Pi \) in which the coefficients are either all non-negative or all non-positive.

(1.1.6) **Proposition:** Each simple system is contained in a unique positive system, and each positive system contains a unique simple system.

Fix a simple system \( \Pi \) of \( \Phi \). Then \( \Pi \) determines a partial order relation \( > \) on \( V \), for which the non-negative elements are \( \sum_{a \in \Pi} c_a a \), with all \( c_a \geq 0 \). Let

\[
\Phi^+ = \{ a \in \Phi : a > 0 \}, \quad \Phi^- = \{ a \in \Phi : a < 0 \},
\]

the positive and negative roots respectively.

(1.1.7) **Lemma:** Let \( a \in \Pi \). Then \( w_a(a) = -a \) and \( w_a(\Phi^+ - \{a\}) = \Phi^+ - \{a\} \).

(1.1.8) **Lemma:** Let \( W_0 \) be the subgroup of \( W(\Phi) \) generated by the reflections \( \{ w_a : a \in \Pi \} \). Then \( W_0(\Pi) = \phi \) and \( W_0 = W(\Phi) \).

(1.1.9) **Definition:** Let \( \Pi \) be a set of simple roots of \( \Phi \).
The corresponding set of reflections \( R = \{ w_a : a \in \Pi \} \) is called the set of fundamental reflections, and \( w_a (a \in \Pi) \) is called a fundamental reflection. We write \( \Pi = \{ r_1, r_2, \ldots, r_n \} \) and \( R = \{ w_1, w_2, \ldots, w_n \} \), where \( w_i = w_{r_i} \) and \( n = \dim V \).

(1.1.10) **Definition:** For \( w \in W \), let \( l(w) \) be the minimal length of all possible expressions of \( w \) as a product of the fundamental reflections: we say \( l(w) \) is the 'length of \( w \'.

An expression \( w = w_{i_1} w_{i_2} \ldots w_{i_t} \), with \( w_{i_j} \in R \) for all \( j, 1 \leq j \leq t \), is called reduced if \( l(w) = t \).

(1.1.11) **Definition:** For \( w \in W \), define
\[
n(w) = |\{ a \in \Phi^+ : w(a) \in \Phi^- \}|
\]
Let \( \Phi^+_w = \{ a \in \Phi^+ : w(a) \in \Phi^+ \} \), \( \Phi^-_w = \{ a \in \Phi^+ : w(a) \in \Phi^- \} \).

(1.1.12) **Proposition:**

(1) For all \( w \in W \), \( l(w) = l(w^{-1}) \), \( n(w) = n(w^{-1}) \), and \( l(w) = n(w) \).

(2) Let \( w \in W \), and \( w \neq 1 \). Then \( |\Phi^-_w| \geq 1 \).

(3) For all \( w \in W \), \( w_i \in R \),
\[
\begin{align*}
l(w_i w) &= l(w) + 1 \quad \text{if} \quad w^{-1}(r_i) \in \Phi^+ \quad \text{if} \quad w^{-1}(r_i) \in \Phi^- \quad \text{if} \quad w(r_i) \in \Phi^+ \quad \text{if} \quad w(r_i) \in \Phi^- \\
l(w_i w) &= l(w) - 1 \quad \text{if} \quad w^{-1}(r_i) \in \Phi^- \\
l(w w_i) &= l(w) + 1 \quad \text{if} \quad w(r_i) \in \Phi^+ \\
l(w w_i) &= l(w) - 1 \quad \text{if} \quad w(r_i) \in \Phi^- \\
\end{align*}
\]

(4) Let \( w = w_{i_1} \ldots w_{i_t} \), with \( w_{i_j} \in R \) for all \( j, 1 \leq j \leq t \). If \( l(w) < t \), then for some \( j \) and \( k, 1 \leq j < k < t-1 \), we have
\[
\begin{align*}
(a) \quad r_{i_j}^{i_{j+1}} \cdots r_{i_k}^{i_{k+1}} \\
(b) \quad w_{i_j}^{i_{j+1}} \cdots w_{i_k}^{i_{k+1}} = w_{i_j}^{i_{j+1}} \cdots w_{i_k}^{i_{k+1}} \\
(c) \quad w = w_{i_1}^{\hat{i}_{i_j}} \cdots w_{i_k}^{\hat{i}_{i_k}} w_{i_t}^{i_{i_t}}.
\end{align*}
\]
(5) For \( w \in W, r_i \in \Pi \),

(a) \( w(r_i) \in \phi^- \) if and only if there is a reduced expression for \( w \) ending with \( w_i \).

(b) \( w^{-1}(r_i) \in \phi^- \) if and only if there is a reduced expression for \( w \) beginning with \( w_i \).

(6) Suppose \( w_{i_1} \cdots w_{i_t} \) is a reduced expression but \( w_{i_0} w_{i_1} \cdots w_{i_t} \) is not reduced (where \( w_{i_j} \in R \) for all \( j, 0 \leq j \leq t \)). Then for some \( k, 1 \leq k \leq t \), \( w_{i_1} \cdots w_{i_k} = w_{i_0} w_{i_1} \cdots w_{i_k-1} \).

(7) We can pass between any two reduced expressions for \( w \in W \) by substitutions of the form

\[
(w_i w_j w_i \cdots)_{n_{ij}} = (w_j w_i w_j \cdots)_{n_{ij}}
\]

where \( w_i, w_j \in R, w_i \neq w_j \), and \( n_{ij} \) = order of \( w_i w_j \) in \( W \).

(8) For \( w \in W \), if \( w(\phi^+) = \phi^+ \) or \( w(\Pi) = \Pi \) or \( n(w) = 0 \), then \( w = 1 \).

(9) There is a unique element \( w_0 \in W \) of maximal length, with the properties that \( w_0^2 = 1 \), \( w_0(\phi^+) = \phi^- \), \( w_0(\Pi) = -\Pi \) and for all \( w \in R \), there exists \( w_j \in R \) such that \( w_0 w_i w_0 = w_j \).

(10) For any \( v \in V, w \in W \), \( w(v) = v \) if and only if \( w \) is a product of reflections corresponding to roots perpendicular to \( v \).
(1.2) Finite Coxeter Systems.

The results in this section are proved in either Bourbaki [1], Carter [3] or Steinberg [24].

(1.2.1) Definition: A Coxeter system \((W,R)\) is a group \(W\) generated by a finite set of involutions \(R\), such that \(W\) has a presentation

\[
W = \langle w_i \in R: (w_iw_j)^{n_{ij}} = 1 \text{ with } n_{ii} = 1, n_{ij} > 2 \text{ if } i \neq j \rangle
\]

\((W,R)\) is a finite Coxeter system if \(|W| < \infty\). In this case \(W\) is called a finite Coxeter group.

(1.2.2) Theorem: Let \(W\) be a finite group generated by a set of involutions \(R = \{w_i: 1 \leq i \leq n\}\) satisfying the following 'exchange condition':

(E) If \(w_{i_1}w_{i_2}...w_{i_k}\) is a reduced expression but \(w_{i_0}w_{i_1}...w_{i_k}\)

is not, where \(w_{i_j} \in R\) for \(0 \leq j < k\), then for some \(m,\)

\[1 \leq m < k, \quad w_{i_1}...w_{i_m} = w_{i_0}w_{i_1}...w_{i_m}^{-1}\]

(a) Then \((W,R)\) is a Coxeter system, i.e. \(W\) has a presentation as an abstract group

\[
W = \langle w_i \in R: (w_iw_j)^{n_{ij}} = 1, \text{ with } n_{ii} = 1, n_{ij} > 2 \text{ if } i \neq j \rangle.
\]

(b) Let \(W(\Phi)\) be the Weyl group of a system of roots, with \(R\) the set of fundamental reflections. Then \((W(\Phi),R)\) is a Coxeter system.

(c) To every Coxeter system \((W,R)\), where \(W\) is a finite group, there corresponds a root system \(\Phi\), and there is an isomorphism \(T: W \to W(\Phi)\) such that \(T(R)\) is the set of fundamental reflections of \(W(\Phi)\) for some set of simple roots.
Hence we can identify a Coxeter group $W$ with a Euclidean reflection group, and thus speak of the corresponding root system $\Phi$ and simple system $\Pi$.

(1.2.3) **Theorem:** $w_i, w_j \in R$ are conjugate in the finite Coxeter group $W$ if and only if there exists a sequence $w_{k_1} = w_i, w_{k_2}, \ldots, w_{k_s} = w_j$ with $w_{k_t} \in R$ for all $t, 1 \leq t \leq s$, such that $w_{k_t} w_{k_t+1}$ has odd order for all $t, 1 \leq t \leq s-1$.

(1.2.4) **Theorem:** Let $(W, R)$ be a Coxeter system. For $J \subseteq R$, let $W_J = \langle w \in J \rangle$, and let $\Pi_J$ be the set of simple roots corresponding to $J$. Let $V_J$ be the subspace of $V$ spanned by the elements of $\Pi_J$, and let $\Phi_J = \Phi \cap V_J$. Then:

1. $\Phi_J$ is a root system.
2. $\Pi_J$ is a simple system in $\Phi_J$.
3. $W_J$ is the Weyl group of $\Phi_J$.
4. $(W_J, J)$ is a Coxeter system.
5. If $l_J$ denotes the length function on $W_J$, then for all $w \in W_J$, $l_J(w) = l(w)$.

(1.2.5) **Theorem:** $J \mapsto W_J$ is a lattice isomorphism from the collection of subsets of $\Pi$ ($\cup, \cap$) to the collection of subgroups of $W$ ($\langle \rangle, \cap$), i.e. for all $J, K \subseteq R$,

$$W_J \cup K = \langle W_J, W_K \rangle$$

$$W_J \cap K = W_J \cap W_K.$$
a subset $J$ of $R$ such that for all $w_j \in J$ and for all $w_k \in K$, where $K = R - J$, $w_j w_k = w_k w_j$. In this case, $W = W_J \times W_K$ (a direct product of groups). Otherwise, $(W, R)$ is indecomposable.

The finite Coxeter systems have been classified - see Bourbaki [1]. In Appendix 1, we give the classification of the finite indecomposable Coxeter systems.

(1.3) **Distinguished Coset Representatives.**

Proofs of results in this section can be found in Solomon [20], if not given below.

Let $(W, R)$ be a finite Coxeter system.

(1.3.1) **Definition:** For all $J \subseteq R$, define the sets

$$X_J = \{ w \in W : w(\prod_J) \subseteq \Phi^+ \}$$

$$Y_J = \{ w \in W : w(\prod_J) \subseteq \Phi^+, w(\prod_J) \subseteq \Phi^- \}$$

where $\hat{J} = R - J$.

Note that $X_J = \bigcup_{J \subseteq K \subseteq R} Y_K$, and for distinct subsets $J_1$, $J_2$ of $R$, $Y_{J_1} \cap Y_{J_2} = \emptyset$, the empty set.

(1.3.2) **Lemma:** The set $X_J$ is a set of left coset representatives for $W$ mod $W_J$. If $w \in W$ and $w = xu$ with $x \in X_J$, $u \in W_J$, then $l(w) = l(x) + l(u)$.

Let $w_{oJ}$ be the unique element of maximal length in $W_J$, for all $J \subseteq R$.

(1.3.3) **Lemma:** Let $J \subseteq R$ and let $\hat{J}$ be its complement in $R$. If $y \in Y_J$, then $yw_{oJ} \in X_{\hat{J}}$ and $l(y) = l(yw_{oJ}) + l(w_{oJ}).$
Proof: $w_0\hat{J}(\prod J) = -\prod J$. Since $y \in Y_J$, $y(\prod J) \subset \phi^-$ and hence $yw_0\hat{J}(\prod J) \subset \phi^+$. So $yw_0\hat{J} \in X_J^\hat{J}$.

Then $y = (yw_0\hat{J})w_0\hat{J}$ with $yw_0\hat{J} \in X_J^\hat{J}$, $w_0\hat{J} \in W_J$. By lemma (1.3.2), $l(y) = l(yw_0\hat{J}) + l(w_0\hat{J})$.

(1.3.4) **COROLLARY:** If $y \in Y_J$, then $y = w_0\hat{J}$ with $l(y) = l(w) + l(w_0\hat{J})$ and $w \in X_J^\hat{J}$.

(1.3.5) **LEMMA:** $w_0\hat{J}$ is the unique element of minimal length in $Y_J$, and $w_0w_0\hat{J}$ is the unique element of maximal length in $Y_J$.

**Proof:** By corollary (1.3.4), $w_0\hat{J}$ is the unique element of minimal length in $Y_J$. Consider the map $f:Y_J \to Y_J^\hat{J}$ given by $f(y) = w_0y$ and the map $g:Y_J^\hat{J} \to Y_J$ given by $g(x) = w_0x$. Then $f$ and $g$ are mutually inverse isomorphisms of the sets $Y_J$ and $Y_J^\hat{J}$, so $|Y_J| = |Y_J^\hat{J}|$. Since $Y_J^\hat{J}$ has $w_0\hat{J}$ as its unique element of minimal length, it follows that $w_0w_0\hat{J}$ is the unique element of maximal length in $Y_J$.

(1.3.6) **COROLLARY:** $|Y_J| = |Y_J^\hat{J}|$ for all $J \subseteq R$.

(1.3.7) **PROPOSITION:** (1) Let $w \in W$ satisfy

(a) $w(\prod J) \subseteq \phi^-$

(b) $w(r_j) = r_k$ for all $r_j \in \prod J$, and some $r_k \in \prod$.

Then $w$ is the unique element of maximal length in $Y_J$, and conversely, the unique element of maximal length in $Y_J$ satisfies (a) and (b).

(2) Let $w \in W$ satisfy
(a) $w(\prod_j) \subseteq \phi^+$

(b) $w(r_j) = -r_k$ for all $r_j \in \prod_j$, and some $r_k \in \prod$

Then $w$ is the unique element of minimal length in $Y_J$, and conversely, the unique element of minimal length in $Y_J$ satisfies (a) and (b).

**Proof:** We have that $w_ow_oJ$ is the unique element of maximal length in $Y_J$, and satisfies (a) and (b) of (1), and that $w_oJ$ is the unique element of minimal length in $Y_J$, and satisfies (a) and (b) of (2).

(1) We prove that $w = w_ow_oJ$. Clearly $w_ow_oJ(\prod_j) = -w(\prod_j) \subseteq \phi^-$. Let $r_i \in \prod_J$. Then $w_ow_oJ(r_i) = w(r_i + s)$, where $s$ is a linear combination of the elements of $\prod_j$, with all coefficients non-negative. Let $\prod_{\ J} = \{r_{j_1}, \ldots, r_{j_t}\}$. Let $r_{j_k}^*$ be the element of $\prod$ such that $w(r_{j_k}) = r_{j_k}^*$. Let

$\prod_{\ J}^* = \{r_{j_k}^* : 1 < k < t\}$. Then $w(s) \in V_{\ J}^*$. So for $w_ow_oJ(r_i) \in \phi^+$, we must have that $w(r_i) \in V_{\ J}^*$. But each element of $V_{\ J}^*$ is the image under $w$ of an element of $V_J$. As $w$ is an automorphism of $V$, $w(r_i) \notin V_{\ J}^*$, and so $w_ow_oJ(\prod) \subseteq \phi^-$. Hence $w_ow_oJ = w_o$.

(2) Show $w_ow_oJ = 1$, using an argument similar to the above.

In Appendix 2, we give some examples of Coxeter groups, and calculate the subsets $Y_J$ for all $J \subseteq R$.

(1.3.8) **PROPOSITION:** Let $w \in W$, with $w = xu$, where $x \in X_J$, $u \in W_J$, and $l(w) = l(x) + l(u)$. Suppose $w(r_i) \in \phi^+$. Then

(1) if $w_i \in W_J$, $ww_i = xv$ where $v = uw_i \in W_J$ and

$l(ww_i) = l(w) + 1 = l(x) + l(v)$. 
(2) if \( w_i \not\in W_J \) then \( ww_i = yv \) where \( y \in X_J \), \( v \in W_J \),
\[ l(ww_i) = l(y) + l(v) \text{ and } l(y) > l(x). \]

**Proof:** (1) Clear since \( w(r_i) \in \phi^+ \) and \( w_i \in W_J \).

(2) \( ww_i = yv = xuw_i \). Now, \( l(ww_i) = l(w) + 1 = l(xu) + 1 = l(x) + l(u) + 1. \) Since \( yv(r_i) \in \phi^- \), let \( y = w_i \ldots w_i \), where \( l(v) = s-r \), and with \( w_i \in R \) for all \( j, 1 \leq j \leq s \). Then \( yv = w_i \ldots w_i \) is reduced but \( yvw_i \) is not, and so there exists \( j, 1 \leq j \leq s \) such that
\[ w_i \ldots w_i = w_i \ldots w_i. \]

There are now two cases to consider.

(a) \( r+1 \leq j \leq s \). Then \( yv = yw_i \ldots w_i \).

That is, \( w_i \ldots w_i = w_i \ldots w_i \). Then \( w_i \ldots w_i (r_i) \in \phi^- \).

But \( w_i \ldots w_i \in W_J \), and \( r_i \not\in \bigcap J \) — contradiction. So we must have:

(b) \( 1 \leq j \leq r \). Then \( yv = w_i \ldots \hat{w_i} \ldots w_i v = xu \).

Thus \( w_i \ldots \hat{w_i} \ldots w_i v = xu \). Since \( v, u \in W_J \), then by the uniqueness of the expression of \( w \) as a product of elements \( xu \), with \( x \in X_J \), \( u \in W_J \), we have \( l(v) < l(u) \), and thus
\[ l(x) < l(y). \]

(1.3.9) **LEMMA:** If \( K \subseteq R \), then \( |Y_K| > 1 \) unless \( K \) and \( \hat{K} \) are mutually commuting sets (i.e. unless for all \( w_i \in K \) and all \( w_j \in \hat{K} \) we have that \( w_i w_j = w_j w_i \)). If \( K \) and \( \hat{K} \) are mutually commuting sets, then \( |Y_K| = |Y_{\hat{K}}| = 1 \). In particular, \( \emptyset \) and \( R \) are mutually commuting sets.

**Proof:** Obviously \( |Y_K| \geq 1 \), as \( w_0 \hat{K} \in Y_K \).
(a) Suppose there exists \( w_k \in K \) such that \( w_0 K^+(r_k) \in \Phi^+ \), but \( w_0 K^+(r_k) \neq r_i \) for any \( r_i \in \prod K \). Then \( w_k w_0 K^+ \in Y_K \), for if \( r_j \in \prod K \), \( w_k w_0 K^+(r_j) = w_k(s) \in \Phi^+ \) where \( s \in \Phi^+ - \{r_k\} \), and if \( r_j \in \prod K \), then \( w_k w_0 K^+(r_j) = -w_k(s) \), where \( s \in \prod K \), and so \( w_k w_0 K^+(r_j) \in \Phi^- \). Hence \( |Y_K| > 1 \) and so \( |Y_K| > 1 \) by (1.3.6).

(b) Suppose that for all \( r_k \in \prod K \), \( w_0 K^+(r_k) = r_k \). Then \( w_0 K^+ \) is a product of reflections each of which fix \( r_k \), and \( w_0 K^+ w_k = w_k w_0 K^+ \) for all \( w_k \in K \). Further, \( w_j w_k = w_k w_j \) for all \( r_j \in \prod K \), \( r_k \in \prod K \), and so \( K \) and \( \hat{K} \) are mutually commuting sets. Since there is no \( w_k \in R \) for which \( w_k w_0 K^+ \in Y_K \), then \( |Y_K| = 1 \), and so also \( |Y_K| = 1 \).

(1.4) Solomon's Decomposition of the Group Algebra of a Finite Coxeter Group.

These results and their proofs can be found in Solomon [20].

Let \( A = Q[\hat{W}] \), the group algebra of a finite Coxeter group \( W \) over \( Q \). Let \( R \) be a set of fundamental reflections which generate \( W \). For each subgroup \( W_J \) of \( W \), let \( A_J = Q[\hat{W}_J] \), the group algebra of \( W_J \) over \( Q \). Define

\[
(1.4.1)\quad e_J = \frac{1}{|W_J|} \sum_{w \in W_J} w, \quad o_J = \frac{1}{|W_J|} \sum_{w \in W_J} \varepsilon(w)w
\]

where \( \varepsilon: W \to \{\pm 1\} \) is a homomorphism of \( W \) such that \( \varepsilon(w_i) = -1 \) for all \( w_i \in R \). (\( \varepsilon \) is called the alternating character of \( W \), and \( \varepsilon(w) = (-1)^l(w) \) for all \( w \in W \)). \( e_J \) and \( o_J \)
are idempotents in the centre of $A_J$.

(1.4.2) **Theorem:** Let $(W, R)$ be a finite Coxeter system, and let $A = Q[W]$. Then

$$A = \sum_{J \in R} A_{e_J o_J},$$

where $\dim A_{e_J o_J} = |Y_J|$, and $A_{e_J o_J}$ has basis $\{ye_J o_J : y \in Y_J\}$.

Moreover, for any $K \subseteq R$,

$$A_{e_K} = \sum_{J \subseteq K} A_{e_J o_J e_K}.$$

Further, if $\phi_K$ is the character of $W$ induced from the principal character of $W_K$, for $K \subseteq R$, and if $\epsilon$ is the alternating character of $W$, then for any subset $J$ of $R$ we have

$$\sum_{J \subseteq K \subseteq R} (-1)^{|K-J|} \phi_K = \sum_{J \subseteq K \subseteq R} (-1)^{|K-J|} \epsilon.$$

If $J = \emptyset$, this reduces to $\epsilon = \sum_{J \subseteq R} (-1)^{|J|} \phi_J$.

**Example:** $W = W(A_2) \cong S_3$, the symmetric group on 3 symbols.

$W = \langle w_1, w_2 : w_1^2 = w_2^2 = (w_1 w_2)^3 = 1 \rangle$.

$J = \emptyset, \ Y_\emptyset = \{w_1 w_2 w_1\}, \ e_\emptyset = 1, \ o_\{w_1, w_2\} = \frac{1}{6}(1-w_1-w_2+w_1 w_2+w_1 w_2+w_1)-w_1 w_2 w_1)$

$J = \{w_1\}, \ Y_{\{w_1\}} = \{w_2, w_1 w_2\}, \ e_{\{w_1\}} = \frac{1}{2}(1+w_1), \ o_{\{w_2\}} = \frac{1}{2}(1-w_2)$

$J = \{w_2\}, \ Y_{\{w_2\}} = \{w_1, w_2 w_1\}, \ e_{\{w_2\}} = \frac{1}{2}(1+w_2), \ o_{\{w_1\}} = \frac{1}{2}(1-w_1)$

$J = \{w_1,w_2\}, \ Y_{\{w_1,w_2\}} = \{1\}, \ e_{\{w_1,w_2\}} = \frac{1}{6}(1+w_1+w_2+w_1 w_2+w_1 w_2+w_1 w_2 w_1)$

Set $A = Q[W]$, and then we have:
A = \mathbb{A} e^0 \{w_1, w_2\} \ast \mathbb{A} e\{w_1\}^0 \{w_2\} \ast \mathbb{A} e\{w_2\}^0 \{w_1\} \ast \mathbb{A} e\{w_1, w_2\}^0 \emptyset,

where \mathbb{A} e\emptyset \{w_1, w_2\} has basis \{\frac{1}{6}(w_1 w_2 w_1 - w_1 w_2 - w_2 w_1 + w_1 + w_2 - 1)\},
\mathbb{A} e\{w_1\}^0 \{w_2\} has basis \{\frac{1}{4}(1 - w_2 - w_2 w_1 + w_1 w_2 w_1), \frac{1}{4}(w_1 - w_1 w_2 + w_2 w_1 - w_1 w_2 w_1)\}, \mathbb{A} e\{w_2\}^0 \{w_1\} has basis \{\frac{1}{4}(1 - w_1 - w_1 w_2 + w_1 w_2 w_1)\}, \mathbb{A} e\{w_2\} \emptyset \{w_1\} has basis \{\frac{1}{4}(w_2 - w_2 w_1 + w_1 w_2 - w_1 w_2 w_1)\}, and \mathbb{A} e\{w_1, w_2\}^0 \emptyset has basis
\{\frac{1}{6}(1 + w_1 + w_2 + w_1 w_2 + w_2 w_1 + w_1 w_2 w_1)\}.

(1.4.3) \textbf{Theorem}: For any J \subseteq R, \mathbb{A} e_J^0 \emptyset and \mathbb{A} e \emptyset_J J are isomorphic A-modules.

(1.4.4) \textbf{Inversion Formula}.

Let R be a finite set and let f be a function which has for domain the set of all subsets J of R and which takes values in some additive abelian group. If

\[ g(K) = \sum_{K \preceq J \subseteq R} f(J), \]

then

\[ f(J) = \sum_{J \preceq K \subseteq R} (-1)^{|K - J|} g(K) \]
Chapter 2: **GROUPS WITH (B,N) PAIRS.**

(2.1) **Groups with (B,N) Pairs.**

(2.1.1) **Definition:** A group $G$ is said to have a (B,N) pair $(G,B,N,R)$ if there exist subgroups $B$ and $N$ of $G$ satisfying the following axioms:

1. $G = \langle B,N \rangle$
2. $H = B \cap N \triangleleft N$
3. The group $W = \frac{N}{H}$ is finite, and is generated by a set of involutions $R = \{w_1, \ldots, w_n\}$.
4. For all $w_1 \in R$ and $w \in W$, $w_1 Bw \subseteq BwB \cup Bw_1WB$
5. For all $w_1 \in R$, $w_1 Bw_1 \neq B$.

$W$ is called the Weyl group of the (B,N) pair. The natural homomorphism from $N$ to $W$ will be denoted by $\theta$. Since elements $w \in W$ belong to $\frac{N}{H}$, and $H \triangleleft N$, $H \triangleleft B$, the sets $wH$, $wB$ or $Bw$, defined as $nH$, $nB$ or $Bn$ for any $n \in N$ with $\theta(n) = w$, are well-defined in $G$.

(2.1.2) **BRUHAT'S THEOREM** (Tits [25]): Let $G$ be a group with a (B,N) pair, with Weyl group $W$. Then

$$G = \bigcup_{w \in W} BwB$$

and $BwB = Bw'B$ if and only if $w = w'$ for $w, w' \in W$.

(2.1.3) **THEOREM** (Matsumoto [17]): Let $G$ be a group with a (B,N) pair $(G,B,N,R)$, with Weyl group $W$. Then $(W,R)$ is a Coxeter system.
Hence by (1.2.2c), $W$ is isomorphic to the Weyl group $W(\phi)$ of a system of roots $\phi$, in such a way that $R$ corresponds to the set of fundamental reflections for a simple system $\Pi$ of $\phi$. We now identify $W$ with $W(\phi)$, and $R$ with the set of fundamental reflections in $W(\phi)$ for the simple system $\Pi$ of $\phi$, and say $\phi$ is the root system corresponding to $W$.

**EXAMPLE:** Let $G = \text{Gl}_n(q)$, the group of non-singular $n \times n$ matrices over the field $\text{GF}(q)$. Let $B$ be the subgroup of upper triangular matrices, and $N$ the subgroup of monomial matrices. Then $B \cap N = H$, the subgroup of diagonal matrices. $W = \frac{N}{H} \cong S_n$, the symmetric group on $n$ symbols.

(2.1.4) **Definition:** A Borel subgroup of $G$ is a subgroup conjugate to $B$. A parabolic subgroup of $G$ is a subgroup conjugate to some subgroup $G_J = BW_JB$, where $J \subseteq R$.

(2.1.5) **Lemma** (Tits [25]): (1) The subgroups $G_J$, for $J \subseteq R$, are the only subgroups of $G$ which contain $B$.

(2) Two different parabolic subgroups which contain a common Borel subgroup are not conjugate.

(3) The normaliser of a parabolic subgroup is itself.

(2.1.6) **Lemma:** If $G$ has a $(B,N)$ pair $(G,B,N,R)$, then for all $J \subseteq R$, $G_J$ has a $(B,N)$ pair $(G_J,B,N_J,J)$, where $N_J$ is the
inverse image of \( W_J \) under the natural homomorphism \( \theta: N \rightarrow W \).

(2.1.7) **Lemma:** Let \( G \) be a finite group with a \((B,N)\) pair \((G,B,N,R)\) and Weyl group \( W \). Then for all \( w_i \in R \), for all \( w \in W \), \( B_{w_i} B w B = \begin{cases} B w_i w B & \text{if } l(w_i w) = l(w) + 1 \\ BwB U B w_i w B & \text{if } l(w_i w) = l(w) - 1. \end{cases} \)

(2.1.8) **Theorem** (Curtis [8]): Let \( G \) be a finite group with a \((B,N)\) pair \((G,B,N,R)\), and let \( W \) be the Weyl group of \( G \). Then there is a bijection between the family of parabolic subgroups \( G_J \) of \( G \) and the family of parabolic subgroups \( W_J \) of \( W \). The subgroup \( G_J = B w_j B \) corresponds to \( W_J \). Further, there exists a one-one correspondence between the \((G_J,G_K)\)-cosets of \( G \) and the \((W_J,W_K)\)-cosets of \( W \).

(2.1.9) **Theorem** (Feit-Higman [13]): If \( G \) is a group with a \((B,N)\) pair whose Weyl group is a dihedral group of order \( 2n \), then \( n = 2,3,4,6 \) or 8.

(2.1.10) **Definition:** A \((B,N)\) pair \((G,B,N,R)\) is saturated if \( B \cap N = \bigcap_{n \in N} n^{-1} B n \).

(2.1.11) **Lemma:** If \( G \) is a group with a \((B,N)\) pair \((G,B,N,R)\), then \( G \) has a saturated \((B,N)\) pair \((G,B,N',R)\), with the same subgroup \( B \) and Weyl group \( W \). (**Proof:** see Richen [18].)
(2.2) Groups with split \((B,N)\) pairs.

\((2.2.1)\) **Definition:** A group \(G\) has a split \((B,N)\) pair of rank \(n\) and characteristic \(p\) if

1. \(G\) has a \((B,N)\) pair \((G,B,N,R)\) of rank \(n\) (i.e. \(|R| = n\)).
2. \(B = UH\) where \(U\) is a normal subgroup of \(B\) and an \(n\)-group, and \(H\) is an abelian \(p'\)-subgroup of \(B\).
3. \(H = \bigcap_{n \in N} B^n\).

We write \((G,B,N,R,U)\) for the split \((B,N)\) pair of \(G\).

For each \(w \in W\), choose \(w_i \in N\) such that \(e(w) = w\).
Write \(n_i = n_{w_i}\) for \(i = 1, \ldots, n\), and \(n_0 = n_{w_0}\). Define

\[
\begin{align*}
V &= U^n_0 \\
X_i &= U \cap V^n_i \\
X_{-i} &= X_i^n = U^n_i \cap U^n_0 \\
U_i &= U \cap U^n_i \\
U^+_w &= U \cap U^n_w \\
U^-_w &= U \cap V^n_w \\
\end{align*}
\]

(2.2.2)

Since \(H\) normalises \(U\), these definitions are independent of the choice of coset representative. Proofs of the following results in this section can be found in either Carter-Lusztig [4] or Richen [18].

\((2.2.3)\) **Theorem:** Let \(G\) be a finite group with a split \((B,N)\) pair \((G,B,N,R,U)\), with Weyl group \(W\), root system \(\Phi\) and simple system \(\Pi\). Then:

1. \(r_i \in \Phi^+_w\) implies \(U^-_{ww_i} = X_i (U^-_w)^{n_i}\) and \(X_i \cap (U^-_w)^{n_i} = 1\).
(2) \( r_i \in \phi_w^- \) implies \( U_w^- = X_i(U_w^-)^{n_i} \) and \( X_i \cap (U_w^-)^{n_i} = 1 \).

(3) For all \( w \in W \), \( U = U_w^+ U_w^- = U_w^- U_w^+ \) with \( U_w^+ \cap U_w^- = 1 \).

In particular, \( U = X_i U_i = U_i X_i \), \( U_i \cap X_i = 1 \) for all \( i = 1, \ldots , n \).

(4) \( G = \bigcup_{w \in W} B_n U_w^- \).

(5) \( B \cap V = 1 \).

(6) Let \( n \in N \) and \( \Theta(n) = w \). Then \( X_i^n \subseteq U \) if \( l(w_i w) = l(w) + 1 \).

(7) Suppose \( r_i, r_j \in \prod \) with \( w(r_i) = r_j \). Let \( n \in N \) satisfy \( \Theta(n) = w \). Then \( X_i^{n-1} = X_j \).

(8) Let \( w = w_{i_1} \cdots w_{i_k} \) where \( w_j \in R \) for \( j = 1, \ldots , k \), and \( l(w) = k \). Then

\[
U_w^- = X_i^{n_{i_k}}(X_i^{n_{i_{k-1}}}) \cdots (X_i^{n_{i_1}}).
\]

(2.2.4) **COROLLARY:** Each element \( g \in G \) has a unique expression of the form \( g = b n_w u_w \), where \( b \in B \), \( w \in W \) and \( u_w \in U_w^- \).

(2.2.5) **COROLLARY:** \( U \) is a \( p \)-Sylow subgroup of \( G \).

(2.2.6) **Definition:** Let \( H_i = < X_i X_{i-1} > \cap H \).

(2.2.7) **LEMMA:** (1) The coset representative \( n_i \) can be chosen in \( < X_i X_{i-1} > \).

(2) If \( r_i, r_j \in \prod \) with \( w(r_i) = r_j \), let \( n \in N \) satisfy \( \Theta(n) = w \); then \( H_i^{n-1} = H_j \).

(3) \( < X_i X_{i-1} > = X_i H_i \cup X_i H_i n_i X_i \).

(4) Let \( x_i \in X_{i-1} \). Then \( n_i^{-1} x_i n_i \in X_i H_i n_i X_i \).

(2.2.8) **Definition** (Structure equation): For each \( x_i \in X_{i-1} \)
write \( n_1^{-1} x_1 n_1 = f_1(x_1) h_1(x_1) n_1 g_1(x_1) \) where \( f_1(x_1), g_1(x_1) \in X_1 \), and \( h_1(x_1) \in H_1 \).

(2.2.9) **Theorem:** Let \( \Sigma \) be the set of \( W \)-conjugates of the \( X_i \)'s, \( 1 \leq i \leq n \). Then \((W, \Sigma)\) is a permutation group under

\[
w : w'X_i w^{-1} \rightarrow ww'X_i w^{-1} w^{-1}.
\]

and \( X_i^{w^{-1}} \rightarrow w(r_1) \) defines an isomorphism \((W, \Sigma) \cong (W, \phi)\).

(2.3) **Irreducible Representations of Finite Groups with Split \((B, N)\) Pairs.**

The results and their proofs in this section can be found in Curtis [9], Richen [18] and Carter-Lusztig [4].

Let \( K \) be a splitting field for \( H \) of characteristic \( p \), and let \( G \) be a finite group with a split \((B, N)\) pair \((G, B, N, R, U)\) of rank \( n \) and characteristic \( p \).

(2.3.1) **Lemma:** Let \( M \) be an arbitrary left \( KG \)-module. Then \( M \) contains a one-dimensional \( B \)-invariant subspace.

(2.3.2) **Definition:** An element \( m \neq 0 \) in a left \( KG \)-module \( K \) is called a weight element of weight \((X; \mu_1, \ldots, \mu_n)\) where \( X : B \rightarrow K^* = K \setminus \{1\} \) is a homomorphism, and the \( \mu_i \in K \), provided that

\[
 bm = X(b)m \text{ for all } b \in B
\]

\[
 S_i m = \mu_i m \text{ for all } 1 \leq i \leq n
\]

where \( S_i = \sum_{u \in X_i} u n_1 \) for \( i = 1, \ldots, n \).
(2.3.3) **THEOREM:** Let $G$ be a group with a split $(B,N)$ pair of rank $n$ and characteristic $p$, and $K$ a field as above. Then:

(a) Every left $KG$-module contains a weight element.

(b) If $m \in M$ is a weight element, then $KGm = KVm$.

(c) Each irreducible left $KG$-module contains a unique line fixed by $U$, and hence a unique line fixed by $B$.

(d) If $M_1$ and $M_2$ are irreducible modules containing weight elements of the same weight, then $M_1 \cong M_2$. Conversely, each irreducible module determines a unique weight.

(2.3.4) **LEMMA:** Let $G$ and $K$ be as above. Let $(\chi; \mu_1, \ldots, \mu_n)$ be the weight of an irreducible $KG$-module. Then $\mu_1 = 0$ or $-1$, and $\mu_i \neq 0$ implies $\chi/H_i = 1$.

(2.3.5) **THEOREM:** Let $G$ be a group with a split $(B,N)$ pair $(G,B,N,R,U)$ of rank $n$ and characteristic $p$, and let $K$ be an algebraically closed field of characteristic $p$. Let $\chi: B \rightarrow K^*$ be a homomorphism, and let $\mu_1, \ldots, \mu_n$ be elements of $K$ such that $\mu_i = 0$ or $-1$. Then $(\chi; \mu_1, \ldots, \mu_n)$ is the weight of an irreducible $KG$-module if and only if $\chi/H_i = 1$ whenever $\mu_i \neq 0$.

(2.3.6) **LEMMA:** Every irreducible left $KG$-module has dimension less than $|U|$, except for the irreducible module of weight $(1_B; -1, -1, \ldots, -1)$, whose dimension is $|U|$. 
(2.4) Construction of the Irreducible Modules.

(2.4.1) THEOREM (Richen [18]): If G is a finite group, H any subgroup and K any field, then every irreducible KG-module is contained in a module induced from an irreducible KH-module.

Let G be a finite group with a split (B,N) pair of rank n and characteristic p. Let K be any field. By the above theorem, we can restrict our search for the irreducible KG-modules by considering KG-modules induced from irreducible representations of B. If M is the principal KU-module, KG\cdot KU\cdot M is the KG-module induced from M. It is convenient to describe this module as a space of functions on the right cosets of U in G: let \mathcal{I} be the set of functions f: G/U \to K.

\mathcal{I} can be made into a left KG-module by defining xf \in \mathcal{I}, given x \in G, f \in \mathcal{I}, by

\[(xf)(Ug) = f(Ugx) \text{ for all } Ug \in G/U.\]

(2.4.2) LEmA: \mathcal{I} is isomorphic, as left KG-module, to KG\cdot KU\cdot M.

The results given in the rest of this section are proved in Carter-Lusztig [4].

Let X be a one-dimensional representation of H with values in K. Define the subspace \mathcal{I}_X of \mathcal{I} by:

(2.4.3) \[\mathcal{I}_X = \{f \in \mathcal{I} : f(Uhg) = X(h)f(Ug) \text{ for all } h \in H, g \in G\}\]

\mathcal{I}_X is a KG-submodule of \mathcal{I}. Now the linear character X of H can be extended to a linear character of B with U in
the kernel. Let $M_X$ be the one-dimensional left $KB$-module affording the character $X$.

(2.4.4) **Lemma**: $\mathcal{F}_X$ is isomorphic, as left $KG$-module, to $KG\otimes_{KB} M_X$.

(2.4.5) **Lemma**: Suppose the characteristic of $K$ is prime to $|H|$ and that $K$ is a splitting field for $H$. Then

$\mathcal{F} = \bigoplus_{X} \mathcal{F}_X$, summed over all one-dimensional representations $X$ of $H$.

Let $\phi_X \in \mathcal{F}$ be the function taking values

$\phi_X(Uh) = X(h)$

$\phi_X(Ug) = 0$ if $g \notin B$.

Then $\phi_X \in \mathcal{F}_X$ and $\phi_X$ generates the left $KG$-module $\mathcal{F}_X$.

(2.4.7) **Definition**: Given a linear character $X$ of $H$ and an element $w \in W$, define $w(X)$ to be the linear character of $H$ given by $(w(X))(h) = X(h^w)$ for all $h \in H$.

(2.4.8) **Lemma**: If $w_i(X) \neq X$, then $\sum_{x_i \in X_i \setminus \{1\}} w_i(X)(h_i(x_i)) = 0$, where the $h_i$ is given in (2.2.8). Also, $\sum_{x_i \in X_i \setminus \{1\}} X(h_i(x_i)) = 0$.

(2.4.9) **Definition**: For each $f \in \mathcal{F}$, define $T_n f \in \mathcal{F}$ by:

$$(T_n f)(Ug) = \sum_{Ug' \subseteq Un^{-1} Ug} f(Ug').$$

(2.4.10) **Proposition**: (1) $T_n$ is an endomorphism of the $KG$-module $\mathcal{F}$, and transforms the submodule $\mathcal{F}_X$ into $\mathcal{F}_{w(X)}$, where $\theta(n) = w$.

(2) The $U$-invariant functions in $\mathcal{F}_X$
form a subspace of dimension \(|W|\). The functions \(T_w^w w^{-1}(X)\) for all \(w \in W\) form a basis for this subspace.

(3) Let \(n, n' \in \mathbb{N}\), with \(\theta(n) = w\), \(\theta(n') = w'\). Then \(T_{n}^{n} = T_{nn'}\) provided \(l(ww') = l(w) + l(w')\).

Write \(T_i = T_{n_i}^n\) for \(i = 1, \ldots, n\). The elements \(T_1, \ldots, T_n\) are called the fundamental endomorphisms of \(\mathcal{F}\).

Now assume \(K\) is a field of characteristic \(p\), which is a splitting field for \(H\).

(2.4.11) Lemma: Let \(X\) be a linear \(K\)-character of \(H\). If \(X/H_i = 1\), then \(v_i(X) = X\). If \(X/H_i \neq 1\), then \(\sum_{x_i \in X_i} \chi(h_i(x_i)) = 0\).

We thus have \(T_i^2 = \begin{cases} 0 & \text{if } X/H_i \neq 1 \text{ on } \mathcal{F}_X \\ -T_i & \text{if } X/H_i = 1 \text{ on } \mathcal{F}_X \end{cases}\).

(2.4.12) Definition: For each linear \(K\)-character \(X\) of \(H\), define \(J_0(X) = \{v_i \in R: X/H_i = 1\}\)

\[= \{v_i \in R: T_i^2 = -T_i \text{ on } \mathcal{F}_X\}.\]

For each subset \(J\) of \(J_0(X)\) and each element \(w \in W\), define a \(KG\)-homomorphism \(\theta^J_w : \mathcal{F}_X \to \mathcal{F}_w(X)\) as follows: write \(w = w_{j_1} \cdots w_{j_k}\) where \(l(w) = k\), and write \(\theta^J_w = \theta_{j_k} \cdots \theta_{j_1}\), where

\[\theta_{j_i} : \mathcal{F}_{w_{j_{i-1}}} \cdots w_{j_1}(X) \to \mathcal{F}_{w_{j_{i-1}}} \cdots w_{j_1}(X) \text{ is defined by}\]

\[\theta_{j_i} = \begin{cases} T_{j_i} & \text{if } w_{j_1} \cdots w_{j_{i-1}} (r_{j_i}) \notin \phi_j \\ 1 + T_{j_i} & \text{if } w_{j_1} \cdots w_{j_{i-1}} (r_{j_i}) \in \phi_j \end{cases}\]

In the latter case, \(T_{j_i}^2 = -T_{j_i} \) on \(\mathcal{F}_{w_{j_{i-1}}} \cdots w_{j_1}(X)\).

Now, \(\theta^J_w\) is a non-zero \(KG\)-homomorphism from \(\mathcal{F}_X\).
into $\mathcal{J}_{w}(\mathcal{X})$, and is determined to within a scalar multiple by $w$ and $J$. We now restrict attention to the homomorphism $\Theta_{w_0}^J : \mathcal{J}_X \to \mathcal{J}_{w_0}(\mathcal{X})$.

(2.4.13) **Definition:** For each subset $J \subseteq R$, define $\bar{J}$ by

$$\bar{J} = \{ w_i \in R : -w_0(r_i) \in \prod \}.$$  

$\bar{J}$ is the image of $J$ under the opposition involution $w_0$.

(2.4.14) **Proposition:** The subspace $\Theta_{w_0}^J \mathcal{J}_X$ of $\mathcal{J}_{w_0}(\mathcal{X})$ lies in an eigenspace of the map $T_i : \mathcal{J}_{w_0}(\mathcal{X}) \to \mathcal{J}_{w_0}(\mathcal{X})$ for each $i$. The eigenvalue of $T_i$ on $\Theta_{w_0}^J \mathcal{J}_X$ is given by:

$$0 \text{ if } w_i \in \bar{J}$$

$$-1 \text{ if } w_i \not\in \bar{J}, w_i \in \bar{J}_0(\mathcal{X})$$

$$0 \text{ if } w_i \not\in \bar{J}_0(\mathcal{X}).$$

(2.4.15) **Corollary:** $\Theta_{w_0}^J \varphi_X$ is a $U$-invariant vector in $\mathcal{J}_{w_0}(\mathcal{X})$ which is an eigenvector for each $T_i$.

(2.4.16) **Definition:** Define $f_X^J = \Theta_{w_0}^J \varphi_X$. $f_X^J$ is determined by $J$ and $\mathcal{X}$ to within a non-zero scalar multiple.

(2.4.17) **Proposition:** $S_i f_X^J = \mu_i f_X^J$ where $\mu_i = \begin{cases} 0 \text{ if } w_i \in J \\ -1 \text{ if } w_i \in J_0(\mathcal{X}) - J \\ 0 \text{ if } w_i \not\in J_0(\mathcal{X}) \end{cases}$

(2.4.18) **Theorem:** The module $\Theta_{w_0}^J \mathcal{J}_X$ is an irreducible $KG$-submodule of $\mathcal{J}_{w_0}(\mathcal{X})$. It has a unique one-dimensional $U$-invariant subspace, and this subspace is spanned by $f_X^J$. 
The stabiliser of this one-dimensional subspace is the parabolic subgroup \( G_J \) of \( G \). The one-dimensional representation of \( B \) on this subspace is \( \mathcal{X} \). The irreducible modules \( \theta^J_{w_0} \mathcal{X} \) for distinct pairs \((J, \mathcal{X})\) are not isomorphic.

(2.4.19) **Lemma:** Every irreducible KG-module is isomorphic to a submodule of \( \mathcal{J}_\mathcal{X} \) for some \( \mathcal{X} \).

(2.4.20) **Theorem:** The modules \( \theta^J_{w_0} \mathcal{X} \) for \( J \subseteq J_0(\mathcal{X}) \) are the only irreducible submodules of \( \mathcal{J}_w(\mathcal{X}) \). There is a natural bijection between the isomorphism classes of irreducible KG-modules and the pairs \((J, \mathcal{X})\) with \( J \subseteq J_0(\mathcal{X}) \).

Note that the irreducible KG-modules are absolutely irreducible since the construction and proof of irreducibility of the modules remains essentially the same when the field \( K \) is replaced by its algebraic closure.

Denote the irreducible module of weight \((\mathcal{X}; \mu_1, \ldots, \mu_n)\) by \( M(\mathcal{X}; \mu_1, \ldots, \mu_n) \).

(2.4.21) **Theorem:** Let \( M = KGm \), where \( m \in M \) is a weight element of weight \((\mathcal{X}; \mu_1, \ldots, \mu_n)\). Then \( M \) has a unique maximal submodule \( M' \), and \( M/M' \cong M(\mathcal{X}; \mu_1, \ldots, \mu_n) \).

**Proof:** \( M(\mathcal{X}; \mu_1, \ldots, \mu_n) \) is generated as KG-module by a weight element \( v \). Define a KG-homomorphism \( \sigma: M \to M(\mathcal{X}; \mu_1, \ldots, \mu_n) \) by \( \sigma(m) = v \). \( \sigma \) is a well-defined KG-homomorphism, as \( m \) is a weight element of the same weight as \( v \), and is clearly onto.
Let $M' = \ker \sigma$. Then $M/M' \cong M(\mathcal{X}; \mu_1', \ldots, \mu_n')$.

Let $M_1$ be a maximal submodule of $M$. Then there is a KG-homomorphism $\Psi : M \rightarrow M(\mathcal{X}' ; \mu_1', \ldots, \mu_n')$ with kernel $M_1$.

If $\Psi(m) = 0$, since $M = KGM$, then we must have $\Psi = 0$. But we have supposed $\Psi \neq 0$. So $\Psi(m) \neq 0$, and as $\Psi$ is a KG-homomorphism, $\Psi(m)$ is a weight element of $M(\mathcal{X}' ; \mu_1', \ldots, \mu_n')$ of weight $(\mathcal{X}; \mu_1', \ldots, \mu_n')$. By (2.4.18), we must have $\mathcal{X} = \mathcal{X}'$, and $\mu_1 = \mu_1'$ for each $i$. Then $\Psi$ is a scalar multiple of $\sigma$, so $\ker \sigma = \ker \Psi$, i.e. $M_1 = M'$.

**Remark:** Unfortunately, nothing seems to be known about the composition factors of $M'$.

(2.4.22) **Note:** Let $G(p)$ be a finite Chevalley group, and $H$ a maximal torus. Then the irreducible representations of $H$ over the field $GF(p)$ are the functions

$$X_{a_1, \ldots, a_n} : H \rightarrow GF(p)$$

given by $(h_1, \ldots, h_n) \mapsto h_1^{a_1} h_2^{a_2} \cdots h_n^{a_n}$, where $0 \leq a_i \leq p-1$ for all $i$. The representation theory of algebraic groups leads one to associate the point $(a_1, \ldots, a_n)$, $0 \leq a_i \leq p-1$, with the pair $(\mathcal{X}, J)$ where $\mathcal{X} = X_{a_1, \ldots, a_n}$ and $w_i \in J$ if and only if the $i$-th co-ordinate of $(a_1, \ldots, a_n)$ is 0. So $(0, \ldots, 0)$ is associated with the pair $(\mathcal{X}_0, \ldots, 0, \{w_1, \ldots, w_n\})$ giving the unit representation, and $(p-1, \ldots, p-1)$ is associated with the pair $(\mathcal{X}_0, \ldots, 0, \emptyset)$ giving the Steinberg representation.
Chapter 3: HECKE ALGEBRAS AND THE GENERIC RING.

(3.1) The Hecke Algebra.

We will define the Hecke algebra of a group $G$ with respect to a subgroup $B$ as in Bourbaki [1] (pages 54-5, exercises 22, 23, 24).

Let $B$ be a subgroup of a group $G$. Suppose each double coset $B g B$ is a finite union of right cosets of $B$ in $G$. Let $k$ be a commutative ring. Let $G/B = \{B g : g \in G\}$, and $B \backslash G/B = \{B g B : g \in G\}$. For $t \in G/B$, let $b_t$ denote the map from $G$ to $k$ defined by

$$b_t(g) = \begin{cases} 1 & \text{if } g \in t \\ 0 & \text{if } g \notin t \end{cases}$$

for all $g \in G$. For $t \in B \backslash G/B$, let $a_t$ denote the map from $G$ to $k$ defined by

$$a_t(g) = \begin{cases} 1 & \text{if } g \in t \\ 0 & \text{if } g \notin t \end{cases}$$

for all $g \in G$. Let $L$ be the $k$-module generated by the $b_t$ for $t \in G/B$, and $H$ the $k$-module generated by the $a_t$ for $t \in B \backslash G/B$. For $t$, $t' \in B \backslash G/B$, define

$$(3.1.1) \quad a_t \ast a_{t'} = \sum_{t''} m(t, t'; t'') a_{t''},$$

where $m(t, t'; t'')$ is the number of right cosets of $B$ contained in $t' \cap t^{-1} x$ for any $x \in t''$, and extend by linearity to $H$. This gives $H$ a $k$-algebra structure, admitting $a_B$ as unit element. $H$ is called the Hecke algebra.
of $G$ with respect to $B$, and written $H_k(G,B)$.

$L$ can be made into a left $H$-module, by defining for each $t \in B \backslash G / B$ and each $t' \in G / B$:

\[(3.1.2) \ a_t \ast b_{t'} = \sum_{t'' \subseteq tt'} b_{t''}\]

and extending by linearity to an action of $H$ on $L$.

$G$ operates on $L$ in the following way: for $g \in G$,

t \in G / B, define $(gb_t) \in L$ as

\[(3.1.3) \ gb_t = b_{tg^{-1}}\]

\[(3.1.4) \ \text{THEOREM: The action of } H \text{ on } L \text{ defines an isomorphism between } H \text{ and the ring of } kG\text{-endomorphisms } E_{kG}(L) \text{ of the left } kG\text{-module } L.\]

Remark: It is also true that

\[m(t,t';t'') = \text{the number of left cosets of } B \text{ in } t \cap x(t')^{-1}\]

for all $x \in t''$.

Now suppose $G$ is a finite group with a $(B,N)$ pair $(G,B,N,R)$. Then, for all $w_i \in R$, the double coset $Bw_iB$ is the union of a finite number of right cosets of $B$. Hence for all $w \in W$, where $W$ is the Weyl group of $G$, $BwB$ is the union of a finite number of right cosets of $B$. Thus we can form the Hecke algebra $H_k(G,B)$ for any commutative ring $k$.

Since each $t \in B \backslash G / B$ is of the form $BwB$, for some $w \in W$, we write $a_w$ for $a_{BwB}$. Then the map $a_w : G \rightarrow k$ satisfies

\[a_w(g) = \begin{cases} 1 & \text{if } g \in BwB \\ 0 & \text{if } g \notin BwB. \end{cases}\]
The $\{a_w : w \in W\}$ form a $k$-basis for $H_k(G, B)$, and multiplication is given by:

$$a_w \ast a_{w'} = \sum_{w'' \in W} m(w, w'; w'')a_{w''},$$

where $m(w, w'; w'')$ = the number of right cosets of $B$ in $Bw'B \cap Bw^{-1}Bw''b$ for any $b \in B$.

(3.1.5) **Definition:** For any $w \in W$, $w \neq 1$, define

$$q_w = |B : B \cap B^w|$$

= the number of right cosets of $B$ in $BwB$.

Define $q_1 = 1$.

(3.1.6) **Lemma:** For any $w \in W$, $w \neq 1$, let $w = w_1 \cdots w_s$ be a reduced expression for $w$ with each $w_i \in R$ for $1 \leq j \leq s$.

Then $q_w = q_{w_1} \cdots q_{w_s}$, and $q_w$ is independent of the reduced expression for $w$. In particular, if $w_i, w_j \in R$ are conjugate in $W$, then $q_{w_i} = q_{w_j}$.

**Proof:** We prove $q_w = q_{w_1} \cdots q_{w_s}$ by induction on $l(w)$. It is obviously true if $l(w) = 0$ or $1$. So suppose $l(w) > 1$.

Let $w' = w_{i_1} \cdots w_{i_s}$; then $w = w_1 w'$ and $l(w) = l(w') + 1$. By induction, $q_{w'} = q_{w_{i_1}} \cdots q_{w_{i_s}}$. Now let $b_1, \ldots, b_r, b'_1, \ldots, b'_t$ be elements of $B$ such that $\{Bw'_1b''_{i=1}^r \}$ is precisely the set of cosets of $B$ in $Bw'B$, and $\{Bw'_{i_1}b''_{j=1}^t \}$ is precisely the set of cosets of $B$ in $Bw'B$. We show that $\{Bw'_{i_1}b'_jw''b'_i : 1 \leq i \leq r, 1 \leq j \leq t \}$ is precisely the set of cosets of $B$ in $BwB$.

Let $BwB$ be a coset of $B$ in $BwB$. Then $Bw'B = Bw'_{i_1}b'_1$.
for some $i, 1 < i < r$. So there exists $b' \in B$ such that $w'b = b'w'b_i$. Then $Bwb = Bw_i w'b = Bw_i b'w'b_i$. Now for some $j, 1 < j < t$, we have $Bw_i b' = Bw_i b'_j$, and so $Bwb = Bw_i b'_j w'b_i$. Hence $q_w q_{w_i} = q_{w_i} q_w$. Conversely, since $l(w_i w') = l(w') + 1$, each coset $Bw_i b'_j w'b_i \subseteq Bw_i w'B = BwB$.

It remains to show that if $Bw_i b'_j w'b_i = Bw_i b'_j w'b_u$, for some $j, l \in \{1, \ldots, t\}, i, u \in \{1, \ldots, r\}$, then $b_i = b_u$ and $b'_j = b'_1$. Suppose $b_i = b_u$. Then $Bw_i b'_j = Bw_i b'_1$, so $j = 1$.

So suppose $i \neq u$; there exists $b \in B$ with $b w_i b'_j w'b_i = w_i b'_1 w'b_u$. Then $w_i b w_i b'_j w'b_i = b'_1 w'b_u$. But $w_i b w_i \in B \cup Bw_i B$, so $w_i b w_i = b'$ for some $b' \in B$, or $w_i b w_i = b' w_i b''$ for some $b', b'' \in B$. In the first case we have $b'_i b w_i b_i = b'_i w'B$ and so we must have $i = u$, contrary to assumption. In the second case, $b' w_i b'' b'_j w'b_i = b'_i w'B$, and as $l(w_i w') > l(w')$, $b' w_i b'' b'_j w'b_i \in Bw_i w'B = BwB$.

But $b'_i w'B \subseteq Bw'B$ - this is impossible, as if $w \neq w'$, then $BwB \cap Bw'B = \emptyset$. Thus, $q_w = q_{w_i} q_{w'} = q_{w_i} \cdots q_{w_i}$.

The rest of the lemma follows from (1.2.3), and by noting that if $w_i w_j$ has odd order $n_{ij}$, where $w_i, w_j \in R$, then $(q_{w_i} q_{w_j} q_{w_i} \cdots)_{n_{ij}} = (q_{w_j} q_{w_i} q_{w_j} \cdots)_{n_{ij}}$ as $(w_i w_j w_i \cdots)_{n_{ij}}$ and $(w_j w_i w_j \cdots)_{n_{ij}}$ are reduced expressions in $W$ which are equal.
Let $w_i \in R$, and $w \in W$. Evaluating various $m(w,w';w'')$'s we get the following equations:

\[(3.1.7) \quad a_{w_i} \ast a_{w_i} = q_{w_i} a_1 + (q_{w_i} - 1) a_{w_i}\]

\[(3.1.8) \quad a_{w_i} \ast a_w = \begin{cases} a_{w_i} w \text{ if } l(w_i w) = l(w) + 1 \\ (q_{w_i} - 1) a_w + q_{w_i} a_{w_i} w \text{ if } l(w_i w) = l(w) - 1 \end{cases}\]

We then get the following two theorems about the structure of $H_k(G,B)$:

\[(3.1.9) \quad \text{THEOREM: } H_k(G,B) \text{ is the associative } k\text{-algebra with } k\text{-basis } \{a_w : w \in W\} \text{ and multiplication given by }\]

\[a_{w_i} \ast a_w = \begin{cases} a_{w_i} w \text{ if } l(w_i w) = l(w) + 1 \\ (q_{w_i} - 1) a_w + q_{w_i} a_{w_i} w \text{ if } l(w_i w) = l(w) - 1 \end{cases}\]

for all $w_i \in R$ and all $w \in W$. $a_1$ is the identity element.

\[(3.1.10) \quad \text{THEOREM: } H_k(G,B) \text{ is the associative } k\text{-algebra with identity } a_1 \text{ generated by } \{a_{w_i} : w_i \in R\} \text{ subject to the relations: }\]

\[a_{w_i} \ast a_{w_i} = (q_{w_i} - 1) a_{w_i} + q_{w_i} a_1 \text{ for all } w_i \in R\]

\[a_{w_i} \ast a_{w_j} \ast a_{w_i} \ast \cdots \text{ for all } w_i, w_j \in R, i \neq j, \text{ where } n_{ij} = \text{order of } w_i w_j \text{ in } W.\]

Now we wish to examine the left $kG$-module $L$. Let $M$ be the principal $kB$-module, (i.e. $M$ affords the trivial representation of $B$ over $k$), and consider the $kG$-module $M^G = kG \otimes_{kB} M$ induced from $M$. Then the left
kG-module $L$ is isomorphic (as left $kG$-module) to $M^G$.

Hence $L$ affords the representation $(1_B)^G$ of $G$ over $k$.

We may regard $L$ as the set of functions from the right cosets of $B$ in $G$ to $k$; then $L$ has $k$-basis \{ $f_{Bg}$: $Bg \in G/B$ \}

where $f_{Bg}: G/B \to k$ is given by

$$f_{Bg}(Bg') = \begin{cases} 1 & \text{if } Bg' = Bg, \\ 0 & \text{if } Bg' \neq Bg. \end{cases}$$

Further, for all $g' \in G$, for all $Bg \in G/B$, we have

$$g'f_{Bg} = f_{B(g'g^{-1})}$$

More generally, if $f \in L$, then for all $g' \in G$, for all $Bg \in G/B$, we have $(g'f)(Bg) = f(Bgg')$.

By (3.1.4), $E_{kG}(L)$ is isomorphic to $H_k(G,B)$, which has $k$-basis \{ $a_w$: $w \in W$ \}, such that

(3.1.11) $a_w f_{Bg} = \sum_{Bg' \subseteq BwBg} f_{Bg'}$ for all $w \in W$ and all $Bg \in G/B$.

(3.1.12) **Lemma:** For any $f \in L$, and for any $w \in W$,

$$a_w f(Bg) = \sum_{Bg' \subseteq Bw^{-1}Bg} f(Bg')$$

**Proof:**

$$a_w f_{Bx}(Bg) = \sum_{Bx' \subseteq BwBx} f_{Bx'}(Bg) = \sum_{Bg' \subseteq Bw^{-1}Bg} f_{Bx}(Bg').$$

Result now follows as every element of $L$ is a $k$-linear combination of elements of the form $f_{Bx}$ with $Bx \in G/B$.

Now take $k = K$, a field of characteristic $p$, $p \neq 0$.

Suppose that for all $w \in R$, $q_w = 0$ in $K$. Let $L = \{ f: G/B \to K \}$, and then by (3.1.4) and (3.1.10) we have that

$$H_K(G,B) \cong E_{kG}(L)$$ is the associative $K$-algebra with
identity $a_1$ generated by $\{a_{w_i} : w_i \in R\}$ subject to the relations:

$$a_{w_i}^2 = -a_{w_i} \text{ for all } w_i \in R$$

$$(a_{w_i} a_{w_j} a_{w_i} \cdots)_{n_{ij}} = (a_{w_j} a_{w_i} a_{w_j} \cdots)_{n_{ij}} \text{ for all } w_i, w_j \in R, i \neq j, \text{ where } n_{ij} = \text{order of } w_i w_j \text{ in } \hat{W}.$$ 

Moreover, if $w \in \hat{W}$ has a reduced expression $w_{i_1} \cdots w_{i_s}$ with all $w_{i_j} \in R$, then $a_w = a_{w_{i_1}} \cdots a_{w_{i_s}}$, and for any $f \in L$,

$$a_w f(B_g) = \sum_{B_{g'} \subseteq Bw^{-1}B_g} f(B_{g'}).$$

Such an algebra is an example of a $0$-Hecke algebra, which we will define below. In Chapter 4, we will examine the structure of a $0$-Hecke algebra. When $k=K$ is a field of characteristic $p$, and all $q_{w_i} = 0$ in $K$, we will denote the Hecke algebra $H_K(G,B)$ by $H_K(0)$.

(3.1.13) **Definition:** The $0$-Hecke algebra $H_K$ over the field $K$ of type $(\hat{W},R)$, where $(\hat{W},R)$ is a finite Coxeter system, is the associative algebra over $K$ with identity $a_1$ generated by $\{a_{w_i} : w_i \in R\}$ subject to the relations:

(a) $a_{w_i}^2 = -a_{w_i} \text{ for all } w_i \in R$

(b) $(a_{w_i} a_{w_j} a_{w_i} \cdots)_{n_{ij}} = (a_{w_j} a_{w_i} a_{w_j} \cdots)_{n_{ij}} \text{ for all } w_i, w_j \in R, i \neq j, \text{ where } n_{ij} = \text{order of } w_i w_j \text{ in } \hat{W}.$
For all \( w \in W \), define \( a_w = a_{w_{i_1}} \cdots a_{w_{i_t}} \), where

\[ w = w_{i_1} \cdots w_{i_t} \]

is a reduced expression for \( w \) in terms of the elements of \( R \). \( a_w \) is clearly independent of the expression for \( w \), by (1.1.12(7)) and (3.1.13(b)).

(3.1.14) PROPOSITION: Every element of \( H_K \) is a \( K \)-linear combination of elements \( a_w \), for \( w \in W \).

Proof: Consider products of the form \( a_{w_{i_1}} \cdots a_{w_{i_t}} \). Consider the corresponding expression \( w_{i_1} \cdots w_{i_t} \) in \( W \). If it is reduced, then \( w_{i_1} \cdots w_{i_t} = w \) for some \( w \in W \), and

\[ a_w = a_{w_{i_1}} \cdots a_{w_{i_t}} \].

So suppose \( w_{i_1} \cdots w_{i_t} \) is not reduced; then there exists an \( r, 1 \leq r < t \) such that \( w' = w_{i_1} \cdots w_{i_r} \) is reduced, but \( w'w_{i_{r+1}} \) is not reduced. Hence by (1.1.12(5)) there is a reduced expression for \( w' \) ending with \( w_{i_{r+1}} \), say \( w' = w_{j_1} \cdots w_{j_r} \), where \( w_{j_r} = w_{i_{r+1}} \). Then

\[ a_{w_{i_1}} \cdots a_{w_{i_r}} a_{w_{i_{r+1}}} = a_{w_{j_1}} \cdots a_{w_{j_r}} a_{w_{i_{r+1}}} = - a_{w_{j_1}} \cdots a_{w_{j_r}} \]

as \( j_r = i_{r+1} \) and \( a_{w_{j_r}} = -a_{w_{j_r}} \). Continuing in this way, we show \( a_{w_{i_1}} \cdots a_{w_{i_t}} = \pm a_{w_{k_1}} \cdots a_{w_{k_s}} \) for some \( w = w_{k_1} \cdots w_{k_s} \) in \( W \).
with $l(w) = s$. Hence the result.

Now from Bourbaki [1], exercise 23, page 55, we have that \( \{a_w : w \in W\} \) are linearly independent over $K$ and so form a $K$-basis for $H_K$.

Example: Let $G = G(q)$ be a Chevalley group over the finite field $\mathbb{F} = \mathbb{F}_q(q)$ of $q$ elements, where $q = p^n$ for some prime $p$ and positive integer $n$. Then $G$ has a (3,$n$) pair $(\mathbb{F}, n, \phi, R)$ with Weyl group $W$ such that for each $w_i \in R$ there is a strictly positive integer $c_i$ such that $c_{w_i} = q^{c_i}$. Suppose $K$ is a field of characteristic $p$. Then for all $w_i \in R$, $q_{w_i} = 0$ in $K$, and the Hecke algebra $H_K(G, R)$ is an $O$-Hecke algebra.

In general, let $G$ be a finite group with a split (3,$n$) pair $(G, R, N, R, N)$ of rank $n$ and characteristic $p$ with Weyl group $W$, and let $K$ be a field of characteristic $p$. Then for all $w_i \in R$, $q_{w_i} = 0$ in $K$, and $H_K(G, R) \cong H_{\text{Hecke}}(G)$ is an $O$-Hecke algebra of type $(W, R)$ over $K$. 
(3.2) Systems of Groups with (B,N) Pairs.

(3.2.1) Definition (Curtis, Iwahori, Kilmoyer [10]):

A system $S$ of finite groups with (B,N) pairs of type $(W,R)$ consists of a Coxeter system $(W,R)$, an infinite set $\mathcal{Q}$ of prime powers $q$, a set of positive integers $\{c_i : w_i \in R\}$, and for each $q \in \mathcal{Q}$, a finite group $G(q)$ with a (B,N) pair $(G(q),B(q),N(q),R)$ having $(W,R)$ as its Coxeter system, such that the following conditions are satisfied:

1. $c_i = c_j$ for $w_i, w_j \in R$ if $w_i$ and $w_j$ are conjugate in $W$.
2. For each group $G = G(q) \in S$,
   \[
   \text{ind}_{B(q)}w_i = |B(q):B(q) \cap B(q)^{w_i}| = q^{c_i} \quad \text{for all } w_i \in R.
   \]

Examples: $S = \{G(q) = SL_n(q), \text{ the group of } n \times n \text{ matrices of determinant 1 over the field } GF(q) \text{ of } q \text{ elements}\}$

$S$ is a system of groups with (B,N) pairs of type $(W,R)$, where $W = W(A_{n-1})$. Ind$_{B(q)}w_i = q$ for all $w_i \in R$, so $c_i = 1$ for all $w_i \in R$.

More generally, each of the families of finite Chevalley groups, and twisted Chevalley groups (see Carter [3]) forms a system of (B,N) pairs.

(3.2.2) Definition: For all $w \in W$, $w \neq 1$, define

\[
\text{ind}_{B(q)}w = |B(q):B(q) \cap B(q)^w|
\]

Let ind$_{B(q)}1 = 1$. 


(3.2.3) **Lemma:** For any \( w \in W, \ w \neq 1, \)

\[
\text{ind}_{B(q)}w = q^{c(w)}
\]

where \( c(w) = c_{i_1} + \ldots + c_{i_s} \) for any reduced expression

\( w = w_{i_1} \ldots w_{i_s} \) of \( w \) with \( w_{i_j} \in R \) for all \( j \).

(3.2.4) **Definition:** Let \( u \) be an indeterminate. The characteristic function of \( S \) is the polynomial

\[
\psi(u) = \sum_{w \in W} \psi_w(u)
\]

where \( \psi_w(u) = u^{c(w)} \) for all \( w \in \mathcal{W}, \ w \neq 1, \) and \( \psi_1(u) = 1. \)

(3.3) **The 'Classical' Hecke Algebra.**

Let \( S \) be a system of finite groups with \((B,N)\) pairs of type \((W,R)\). Let \( K \) be a field of characteristic zero, and let \( q \in \mathcal{O} \). Define in the group algebra \( KG(q) \) the idempotent \( b(q) = \frac{1}{|B(q)|} \sum_{x} x, \) and the left ideal

\[
V(q) = KG(q) \cdot b(q).
\]

If we regard \( V(q) \) as left \( KG \)-module, we see that \( V(q) \) affords a representation of \( G(q) \) over \( K \) with character \( (1_{B(q)})_{G(q)} \). In particular,

\[
\dim_K V(q) = |G(q):B(q)|. \quad \text{(see, for example, Curtis and Reiner [11]).}
\]

(3.3.1) **Definition:** Define the Hecke algebra \( H_K(q) = H_K(G(q), B(q)) \)

as \( H_K(q) = b(q) \cdot KG(q) \cdot b(q). \)

(3.3.2) **Lemma:** \( H_K(q) \) acts by right multiplication on \( V(q) \), and if \( h \in H_K(q) \), the map \( \Theta(h) : v \rightarrow vh \) (for all \( v \in V(q) \))
defines a $K\mathbf{G}(q)$-endomorphism of $V(q)$. The map

$$\theta : H_K(q) \rightarrow \mathbb{E}n_{K\mathbf{G}(q)}(V(q))$$

is an isomorphism of $K$-algebras.

The structure of the Hecke algebra $H_K(q)$ has been determined by Iwahori [16] and Matsumoto [17], and is as follows:

(3.3.3) **Theorem:** $H_K(q)$ has $K$-basis $\{h_w : w \in W\}$ where

$$h_w = \text{ind}_{B(q)wB(q)}b(q)n(q)wB(q)$$

for any $n(q)_w \in B(q)wB(q)$ for all $w \in W$. $h_1 = b(q)$ is the identity element of $H_K(q)$. For any $w \in W$, $w \neq 1$, and for any reduced expression $w = w_{i_1} \ldots w_{i_s}$ for $w$, with the $w_{i_j} \in R$,

$$h_w = h_{w_{i_1}} \ldots h_{w_{i_s}}.$$ 

$H_K(q)$ is generated as $K$-algebra with identity element $h_1$ by $\{h_{w_{i_j}} : w_{i_j} \in R\}$, subject to the relations:

$$h_{w_{i_j}}^2 = q^{c_{i_j}}h_1 + (q^{c_{i_j} - 1})h_{w_{i_j}}$$

$$(h_{w_{i_j}}h_{w_{j}}h_{w_{i_j}}h_{w_{j}} \ldots)_{n_{ij}} = (h_{w_{j}}h_{w_{i_j}}h_{w_{j}}h_{w_{i_j}} \ldots)_{n_{ij}}$$

for all $w_{i_j}, w_j \in R$, $i \neq j$, where $n_{ij}$ is order of $w_{i_j}w_j$ in $W$.

(3.3.4) **Corollary:** For all $t, v, w \in W$ there exist polynomials

$$z_{t, v, w}(u) \in \mathbb{Z}[u]$$

such that

(1) for any $q \in \mathcal{P}$,

$$h_t h_v = \sum_{w \in W} z_{t, v, w}(q)h_w.$$

(2) $\sum_{w \in W} z_{t, v, w}(u)z_{w, x, y}(u) = \sum_{s \in W} z_{t, s, y}(u)z_{v, x, s}(u)$
for all \( t, v, x, y \in W \).

**Proof:** (1) Follows from (3.3.3).

(2) Replacing \( u \) by \( q \) for any \( q \in \mathcal{P} \), this equation is true by the associativity of \( H_K(q) \). Since \( \mathcal{P} \) is an infinite set, the equation must be an identity in \( u \).

(3.4) The **Generic Ring**.

The following definition is due to J. Tits:

(3.4.1) **Definition:** The generic ring \( A_{G,S}(u) = A_{G}(u) \) of the system \( S \) of finite groups with \( (B,N) \) pairs of type \((W,R)\) is the associative algebra over \( \mathcal{O} = Q[u] \), where \( u \) is an indeterminate, with identity \( a_1 \) and basis \( \{a_w : w \in W\} \) satisfying:

\[
    a_wa_w = \begin{cases} 
        a_w & \text{if } l(w_1w) > l(w) \\
        u^{c_i}a_w + (u^{c_i-1})a_w & \text{if } l(w_1w) < l(w)
    \end{cases}
\]

for all \( w \in W \), and \( w_1 \in R \).

Alternatively, \( A_{G}(u) \) is the algebra over \( \mathcal{O} \) with identity \( a_1 \) and basis \( \{a_w : w \in W\} \), and multiplication given by:

\[
    a_xa_y = \sum_{w \in W} z_{x,y,w}(u)a_w \quad \text{for all } x, y \in W
\]

where \( z_{x,y,w}(u) \in Z[u] \) is the polynomial determined in (3.3.4)

Let \( K \) be any extension ring of \( \mathcal{O} \).

(3.4.2) **Definition:** The generic ring \( A_K(u) \) of the system
$S$ of finite groups with $(B,N)$ pairs of type $(W,R)$ is the algebra over $K$ with identity $a_1$ and basis $\{a_w : w \in W\}$, and with multiplication given by

$$a_x a_y = \sum_{w \in W} z_{x,y,w}(u) a_w$$ for all $x,y \in W$.

Note: By (3.3.3) and (3.3.4(2)), $A_\sigma(u)$ and $A_K(u)$, for any extension ring $K$ of $\sigma$, are the associative algebras over $\sigma$ and $K$ respectively generated by $\{a_w : w \in R\}$ subject to the relations:

$$a_w^2 = u^c_i a_1 + (u^c_i - 1) a_w$$ for all $w \in R$, where $a_1$ is the identity element.

$$(a_w a_j a_w \ldots)_{n_{ij}} = (a_w a_j a_w \ldots)_{n_{ij}}$$ for all $w, w_j \in R$, $i \neq j$, where $n_{ij}$ = order of $w_i w_j$ in $W$. Further, for all $w \in W$,

$$a_w = a_{w_1} \ldots a_{w_s}$$ where $w_1 \ldots w_s$ is a reduced expression for $w$ in terms of the elements of $R$.

(3.4.3) Definition: Let $f : \sigma \to Q$ be a homomorphism of commutative rings. Then define

$$A_f,\sigma(f(u)) = Q \otimes_\sigma A_\sigma(u)$$

where $Q$ is viewed as a $(Q, \sigma)$ bimodule by way of $f$:

$$axh = axf(h)$$ for all $a, x \in Q$, $h \in \sigma$.

$A_f,\sigma(f(u))$ is an associative algebra over $Q$, called a specialised algebra of $A_\sigma(u)$. It has $Q$-basis $\{1 \otimes a_w : w \in W\}$, and multiplication is given by:

$$a_x f a_y, f = \sum_{w \in W} f(z_{x,y,w}(u)) a_w, f$$ for all $x, y \in W$. 


Note that $f$ induces a ring epimorphism $f': A_\mathcal{O}(u) \to A_{f', \mathcal{O}}(f(u))$.

**Examples:**

(1) Let $f_q$ be the $\mathbb{Q}$-linear map $f_q: \mathcal{O} \to \mathbb{Q}$ given by $f_q(u) = q$, where $q \in \mathbb{Q}$. Then $f_q$ induces a ring epimorphism $f'_q: A_\mathcal{O}(u) \to A_{f_q', \mathcal{O}}(q) \cong H_{\mathbb{Q}}(q)$.

(2) Let $f_1$ be the $\mathbb{Q}$-linear map $f_1: \mathcal{O} \to \mathbb{Q}$ given by $f_1(u) = 1$. Then $f_1$ induces a ring epimorphism $f'_1: A_\mathcal{O}(u) \to A_{f_1', \mathcal{O}}(1) \cong \mathbb{Q}W$, the group algebra of $W$ over $\mathbb{Q}$.

(3) Let $f_0$ be the $\mathbb{Q}$-linear map $f_0: \mathcal{O} \to \mathbb{Q}$ given by $f_0(u) = 0$. Then $f_0$ induces a ring epimorphism $f'_0: A_\mathcal{O}(u) \to A_{f_0', \mathcal{O}}(0) \cong H_{\mathbb{Q}}$, the $0$-Hecke algebra over $\mathbb{Q}$.

**Definition (due to Green [14]):** Let $k$ be a subfield of $\mathbb{C}$, and let $q \in k$. Then $A_k(q)$ is defined as the algebra over $k$ with identity $a_1$ and $k$-basis $\{a_w : w \in W\}$, and with multiplication given by:

$$a_xa_y = \sum_{w \in W} z_{x,y,w}(q)a_w \text{ for all } x, y \in W,$$

where $z_{x,y,w}(u) \in \mathbb{Z}[u]$ is the polynomial determined in (3.3.4).

**Note:** By (3.3.3) and (3.3.4(2)) $A_k(q)$ is the associative algebra over $k$ with identity $a_1$, generated by $\{a_w : w \in W\}$ subject to the relations

$$a_{w_i}^2 = q^{c_i}a_1 + (q^{c_i-1})a_{w_i} \text{ for all } w_i \in R$$

$$(a_{w_i} a_{w_j} a_{w_i} \ldots)_n_{ij} = (a_{w_j} a_{w_i} a_{w_j} \ldots)_n_{ij} \text{ for all } w_i, w_j \in R,$$

if $i \neq j$, where $n_{ij}$ is the order of $w_i w_j$ in $W$. 
\[ a_w = a_{w_1} \ldots a_{w_s} \] where \( w_1 \ldots w_s \) is a reduced expression for \( w \) in terms of the elements of \( R \).

(3.4.5) **Lemma** (Green [14]): (1) For each \( q \in \mathcal{P} \), there is a \( k \)-algebra isomorphism \( s_{k,q}: A_k(q) \to H_k(q) \) which takes \( a_w \to h_w \) for all \( w \in W \).

(2) There is a \( k \)-algebra isomorphism \( s_{k,1}: A_k(1) \to kW \) which takes \( a_w \to w \) for all \( w \in W \).

**Notation:** For the remainder of this section, we will use the following notation:

\[ \sigma = Q[u] \]
\[ K_0 = Q(u) \]
\[ K = \text{a finite field extension of } K_0 \text{ which is a splitting field for } A_K(u). \]
\[ I = \text{integral closure of } \sigma \text{ in } K \]
\[ k = \text{subfield of } C. \]

(3.4.6) **Theorem** (Tits - see Green [14]): \( A_K(u) \) is semi-simple.

(3.4.7) **Corollary:** \( A_{K_0}(u) \) is semi-simple, hence separable as the characteristic of \( K_0 \) is zero.

(3.4.8) **Theorem** (Tits - see Green [14]): If \( q \) is any complex number such that \( q \psi(q) \neq 0 \), then \( A_C(q) \cong CW \). In particular this holds for any real positive \( q \). If \( q \) is any element of \( \mathcal{P} \), then \( H_C(q) \cong CW \).
We now discuss specialisations of $A_K(u)$, which are defined by Green [14] (section 4).

Definition: If $P$ is a prime ideal of $I$, let $K_P$ be the ring
$$K_P = \left\{ \frac{a}{b} : a \in I, b \in I-P \right\}$$
where $I-P = \{ i \in I : i \notin P \}$. Then a specialisation of $K$ with nucleus $P$ is a ring homomorphism $f : K_P \rightarrow C$ such that $f(1) = 1$ and $\ker f = PK_P$. If $a \in K$, say $f(a)$ is defined if and only if $a \in K_P$. The range $k = f(K_P)$ of $f$ is a subfield of $C$.

Let $f$ be a specialisation of $K$, with nucleus $P$ and range $k$. $f$ can be extended to a ring epimorphism
$$f : A_{K_P}(u) \rightarrow A_K(q)$$
where $f(u) = q$ and $A_{K_P}(u) = \{ \sum s_wa_w : s_w \in K_P \text{ for all } w \in W \}$. Then if $x = \sum s_wa_w \in A_K(u)$, where $s_w \in K$ for all $w \in W$, we say $f(x)$ is defined if and only if $x \in A_{K_P}(u)$; in this case, $f(x) = \sum f(s_w)a_w$.

(3.4.9) THEOREM (Green [14]): Given any specialisation $f_o$ of $K_o$, whose nucleus is the prime ideal $P_o$ of $O$, then there exists at least one specialisation $f$ of $K$ which extends $f_o$. If $P$ is the nucleus of $f$, then $P_o = P \cap O$.

(3.4.10) COROLLARY (Green [14]): Given any element $q$ of $C$ there exists a specialisation $f$ of $K$ such that $f(u) = q$. 
If \( q \in \mathcal{Q} \), and if \( P \) is the nucleus of \( f \), then
\[
(u-q)\mathcal{Q} = P \cap \mathcal{Q}.
\]

**Proof:** There is a unique specialisation \( f_0 \) of \( K_0 \) such that \( f_0(u) = q \), and so the existence follows from the theorem above. If \( q \in \mathcal{Q} \), the nucleus of \( f_0 \) is \((u-q)\mathcal{Q}\).

Suppose that for every \( q \in \mathcal{Q} \) we have a specialisation \( f_q \) of \( K \) such that \( f_q(u) = q \). Let \( P_q \) be the nucleus of \( f_q \). If \( q, q' \in \mathcal{Q} \), \( q \neq q' \), then by the corollary \( P_q \neq P_q' \).

Finally we look at some specialisations of \( A_K(u) \), where \( K \) is an extension ring of \( \mathcal{Q} \).

(3.4.11) Let \( B_0 = \{g(u)/h(u) : g(u), h(u) \in \mathcal{Q}, u|h(u)\} \).

Let \( f_0 : B_0 \to Q \) be the \( Q \)-linear homomorphism of rings defined by \( f_0(u) = 0 \). Then \( f_0 \) induces a ring epimorphism \( f_0' : A_{B_0}(u) \to A_{\mathcal{Q}}(0) \cong H_Q \), the \( 0 \)-Hecke algebra over \( Q \).

(3.4.12) For any \( q \in \mathcal{Q} \), let \( B_q = \{g(u)/h(u) : g(u), h(u) \in \mathcal{Q}, (u-q)x h(u)\} \).

Let \( f_q : B_q \to Q \) be the \( Q \)-linear ring homomorphism given by \( f_q(u) = q \). Then \( f_q \) induces a ring epimorphism \( f_q' : A_{B_q}(u) \to A_{\mathcal{Q}}(q) \cong H_Q(q) \).
(3.5) **The Algebra $H_o$.**

The algebra $H_o$ was defined by Starkey [22] (section 1.5). By (3.4.9) there exists a specialisation $f_o$ of $K$ with nucleus $P$ and range $k_o$ (where $k_o$ is a subfield of $C$) such that $f_o(u) = 0$. Also, $P \cap \mathcal{O} = u\mathcal{O} = (u)$. $f_o$ induces a ring epimorphism $f_o': A_{K_P}(u) \to A_{k_o}(0)$ given by

$$f_o'(\sum s_w a_w) = \sum f_o(s_w) a_w,$$

where $s_w \in K_P$ for all $w \in W$.

(3.5.1) **Definition:** Let $H_o$ be the $k_o$-algebra with identity $a_1$ and basis $\{a_w : w \in W\}$ and multiplication given by

$$a_x a_y = \sum_{w \in W} x_{x,y,w}^{(0)} a_w$$

for all $x, y \in W$.

From the above we see that $H_o = A_{k_o}(0)$.

By the Iwahori-Matsumoto theorem (3.3.3), we have that $H_o$ is generated as $k_o$-algebra with identity $a_1$ by

$\{a_w : w \in R\}$ subject to the relations:

$$a_{w_i}^2 = -a_{w_i}$$

for all $w_i \in R$

$$(a_{w_i} a_{w_j} a_{w_i} \ldots) a_{n_{ij}} = (a_{w_j} a_{w_i} a_{w_j} \ldots) a_{n_{ij}}$$

for all $w_i, w_j \in R$, $i \neq j$, where $n_{ij} =$ order of $w_i w_j$ in $W$.

Also,

$$a_w = a_{w_1} \ldots a_{w_s}$$

where $w_{1} \ldots w_{s}$ is a reduced expression for $w$ in terms of the elements of $R$.

Unlike the algebra $A_K(u)$, the algebra $H_o$ is not semi-simple. However, $H_o$ is the $O$-Hecke algebra $H_{k_o}$, whose
structure we will determine in Chapter 4.

Starkey [22] has defined the decomposition numbers for $\mathbb{H}_o$, and we will give the definition of these and relate them to the Cartan matrix of $\mathbb{H}_o$ in Chapter 5.
Chapter 4: **DECOMPOSITIONS OF THE O-HECKE ALGEBRAS.**

(4.1) **Introduction.**

Let $K$ be a field. Let $(W, R)$ be a finite Coxeter system, and let $H = H_K$ be the O-Hecke algebra of type $(\omega, R)$ over $K$, as defined in (3.1.13). By (3.1.14) $H$ has $K$-basis $\{a_w : w \in W\}$, where $a_w = a_{w_1} \ldots a_{w_s}$ if $w = w_1 \ldots w_s$ with $l(w) = s$. We write $a_1 = 1$.

(4.1.1) **LEMMA:** For all $w_i \in R$ and all $w \in W$, we have

$$a_{w_i} a_w = \begin{cases} a_{w_{i}w} & \text{if } l(w_{i}w) = l(w) + 1 \\ -a_w & \text{if } l(w_{i}w) = l(w) - 1 \end{cases}$$

$$a_w a_{w_i} = \begin{cases} a_{ww_{i}} & \text{if } l(ww_{i}) = l(w) + 1 \\ -a_w & \text{if } l(ww_{i}) = l(w) - 1 \end{cases}$$

**Proof:** If $l(w_{i}w) = l(w) + 1$, then $a_{w_{i}w} = a_{w_i} a_w$ by the definition of $a_{w_i}$. Similarly, if $l(ww_{i}) = l(w) + 1$ then $a_{ww_{i}} = a_w a_{w_i}$. Suppose $l(w_{i}w) = l(w) - 1$; then there is a reduced expression for $w$ beginning with $w_i$: say $w = w_i w'$ where $l(w) = l(w') + 1$. Then $a_w = a_{w_i} a_{w'}$, and so

$$a_{w_i} a_w = a_{w_i} a_{w_{i}} a_{w'} = -a_{w} a_{w'} = -a_w.$$  

Similarly, if $l(ww_{i}) = l(w) - 1$, there exists a reduced expression for $w$ ending with $w_i$, and in this case we get that $a_w a_{w_i} = -a_w$. 
(4.1.2) COROLLARY:

(1) For all \( w, w' \in W \), \( a_w a_{w'} = \pm a_{w''} \), for some \( w'' \in W \), with \( l(w'') \geq \max (l(w), l(w')) \).

(2) For all \( w, w' \in W \), \( a_w a_{w'} = a_{ww'} \), if and only if \( l(ww') = l(w) + l(w') \).

(3) For all \( w, w' \in W \), \( a_w a_{w'} = (-1)^{l(w')} a_w \), if and only if \( w(r_i) \in \Phi^- \) for each \( r_i \in \prod J \), where \( J = \{ w_i \in R : w_i \text{ occurs in some reduced expression for } w' \} \).

(4) For all \( w, w' \in W \), \( a_w a_{w'} = (-1)^{l(w)} a_{w'} \), if and only if \( (w')^{-1}(r_i) \in \Phi^- \) for each \( r_i \in \prod J \), where \( J = \{ w_j \in R : w_j \text{ occurs in some reduced expression for } w \} \).

(5) For all \( w, w' \in W \), \( a_w a_{w'} = \pm a_{w''} \), with \( l(w'') > l(w) \), where \( l(w) \geq l(w') \), if and only if there exists \( r_i \in \prod J \), where \( J = \{ w_j \in R : w_j \text{ occurs in some reduced expression for } w' \} \) such that \( w(r_i) \in \Phi^+ \).

(6) Let \( w_0 \) be the unique element of maximal length in \( W \). Then for all \( w \in W \), \( a_w a_{w_0} = (-1)^{l(w)} a_{w_0} \) and \( a_{w_0} a_w = (-1)^{l(w)} a_{w_0} \).

Example: Let \( H = H(A_2) \), the \( 0 \)-Hecke algebra with \( K \)-basis \( \{ a_w : w \in W(A_2) \} \), which is generated as \( K \)-algebra by \( \{ a_{w_1}, a_{w_2} \} \) subject to the relations \( a_{w_1}^2 = -a_{w_1} \), for \( i = 1, 2 \), and \( a_{w_1} a_{w_2} a_{w_1} = a_{w_2} a_{w_1} a_{w_2} \). The multiplication table is given below: set \( a_1 = 1 \), \( a_{w_1} = x \), \( a_{w_2} = y \).
(4.2) **The Nilpotent Radical of** \( H \).

**Definition:** (1) The nilpotent radical \( N \) of an algebra \( A \) is the sum of all nilpotent ideals of \( A \).

(2) The Jacobson radical \( J(A) \) of an algebra \( A \) is the intersection of all maximal ideals of \( A \).

(4.2.1) **PROPOSITION:** If \( A \) has the DCC (= descending chain condition) then \( N = J(A) \) and \( N \) is nilpotent.

**Proof:** See, for example, Curtis and Reiner [11].

Let \( N \) be the nilpotent radical of \( H \). Since \( H \) is a finite-dimensional algebra over the field \( K \), \( H \) has the DCC and the ACC, and so \( N \) is also the Jacobson radical of \( H \), and is the unique maximal nilpotent ideal of \( H \).

(4.2.2) **LEMMA:** Let \( L \) be an ideal of \( H \). Then either

(a) \( a \omega \in L \), or

(b) \( a \omega \cdot L = 0 \).

**Proof:** By (4.1.2(6)), for all \( x \in H \), \( x a \omega = a \omega \cdot x = x a \omega \).
for some $M \in K$. Hence, for all positive integers $n$,

$$x^n a_{w_0} = a_{w_0} x^n = M^n a_{w_0}$$

If $M \neq 0$, then $M^n \neq 0$ and so $x^n \neq 0$ for all positive integers $n$. So if there exists an $x \in L$ such that $x a_{w_0} = M a_{w_0}$ for some $M \neq 0$, $M \in K$, then $a_{w_0} \in L$. Otherwise $a_{w_0} \in L = 0$.

(4.2.3) **PROPOSITION:** $(-1)^{l(w_0)} a_{w_0}$ is an idempotent in $H$, and $H = H(-1)^{l(w_0)} a_{w_0} \oplus H(1 - (-1)^{l(w_0)}) a_{w_0}$, where

$$H(-1)^{l(w_0)} a_{w_0} \cong K.$$  

**Proof:** By (4.1.2(6)) we have that $(-1)^{l(w_0)} a_{w_0}$ is an idempotent in the centre of $H$, and so the direct sum decomposition follows by basic ring theory. The two summands are ideals of $H$, and since for all $x \in H$ we have $x a_{w_0} = M a_{w_0}$, $M \in K$, it follows that $H(-1)^{l(w_0)} a_{w_0} \cong K$.

There is a natural composition series for $H$, consisting of (two-sided) ideals of $H$ such that every factor is a one-dimensional $H$-module. This series arises as follows: list the basis elements $\{a_w : w \in \mathcal{W}\}$ in order of increasing length of $w$ - if $w, w' \in \mathcal{W}$ have the same length, it does not matter in which order they occur in the list. Rename these elements $h_1, h_2, \ldots, h_{|\mathcal{W}|}$ respectively; note that $h_1 = a_1 = 1$, $h_{|\mathcal{W}|} = a_{w_0}$. Let $H_j$ be the ideal
of \( H \) generated by \( \{ h_m : m \geq j \} \). \( H_j \) has \( K \)-basis \( \{ h_m : m \geq j \} \) and dimension \( |W| - j + 1 \). Then

\[ (4.2.4) \quad H = H_1 > H_2 > \ldots > H |W| = a_w H > 0 \]

is the natural composition series of \( H \) described above.

\( H_i / H_{i+1} \) is a one-dimensional \( H \)-module, with basis \( h_i + H_{i+1} \), where \( h_i = a_w \) for some \( w \in W \). Now either \( a_w^2 = (-1)^{l(w)} a_w \) or \( a_w^2 \in H_{i+1} \). In the first case, the factor ring \( H_i / H_{i+1} \) is generated by an idempotent, and in the second case it is nilpotent.

\[ (4.2.5) \quad \text{LEMMA: The number of factors which are generated by an idempotent is equal to } 2^n, \text{ where } n = |R|. \]

Proof: The factors which are generated by idempotents correspond to elements \( w \in W \) such that \( a_w^2 = (-1)^{l(w)} a_w \).

Let \( w \in W \) be such an element. Write \( w = w_{i_1} \ldots w_{i_s} \), where \( l(w) = s \), and let \( J = \{ w_{i_j} : 1 \leq j \leq s \} \). Then \( w \in W_J \), and by (4.1.2(3)), \( w(\bigcap J) \subseteq \phi^- \). Hence \( w = w_{0J} \). Conversely, for each subset \( J \) of \( R \), \( a_{w_{0J}}^2 = (-1)^{l(w_{0J})} a_{w_{0J}} \). Hence the number of factors which are generated by an idempotent is equal to the number of subsets of \( R \), i.e. \( 2^n \), where \( n = |R| \).

By Schreier's theorem, any series of ideals of \( H \) can be refined to a composition series, and all so obtained
have the same number of terms in them as the natural series, and with the factors in one-one correspondence with those of the natural series. In particular, consider

\[ H > N > 0. \]

This can be refined to a composition series

\[ H = H'_1 > H'_2 > \ldots > H'_r = N > \ldots > H'_{|w|} > 0. \]

Now each factor \( \frac{H_i}{H_{i+1}} \), \( i > r \), is nilpotent as \( H_i < N \), and each factor \( \frac{H'_i}{H'_{i+1}} \), \( i+1 < r \), must be generated by an idempotent as \( \frac{H_i}{N} \leq \frac{H}{N} \), a semi-simple ring. Hence the number of factors which are nilpotent is equal to the dimension of \( N \). Thus, \( \text{dim } N = |w|-2^n \), where \( n = |R| \).

We can, in fact, give a precise basis of \( N \).

(4.2.6) **Theorem:** Let \( w \in W \), and suppose \( w \neq w_{oJ} \) for any \( J \subseteq R \). Write \( w = w_{i_1} \ldots w_{i_s} \), \( l(w) = s \), and let \( J = J(w) = \{ w_{i_j} : 1 \leq j \leq s \} \).

Then \( E(w) = a_w + (-1)^{l(w_{oJ}(w)) + l(w) + 1} a_{w_{oJ}(w)} \) is nilpotent, and \( \{ E(w) : w \in W, w \neq w_{oJ} \text{ for any } J \subseteq R \} \) is a basis of \( N \).

**Proof:** We show that \( E(w) \) is nilpotent by induction on \( l(w_{oJ}(w)) - l(w) \). Note that if \( w = w_{oJ} \) for some \( J \subseteq R \), then \( E(w_{oJ}) = 0 \). Suppose \( l(w_{oJ}(w)) - l(w) = 1 \). Then since a reduced expression for \( w \) involves all \( w_i \in J(w) \), \( w \neq w_{oJ}(w) \), there exists \( r_j \in \bigcap J(w) \) such that \( w(r_j) \in \phi^+ \). So

\[ a_w^2 = (-1)^{l(w) - 1} a_{w_{oJ}(w)} \]

Thus,

\[ E(w)^2 = a_w^2 + a_w a_{w_{oJ}(w)} + a_{w_{oJ}(w)} a_w + a_{w_{oJ}(w)}^2 \]
So, \( E(w)^2 = a_w a_{\omega J(w)} (-1)^{l(w)} + 2(-1)^{l(w)} + (-1)^{l(\omega J(w))} a_{\omega J(w)} \)

\[ = 0, \text{ as } l(\omega J(w)) = l(w)+1. \]

Now suppose \( l(\omega J(w)) - l(w) > 1 \). Consider the product \( a_w a_w \). Since \( w \neq \omega J(w) \), there exists \( r_i \in \bigcap J(w) \) such that \( w(r_i) \in \mathfrak{p}^+ \). As any reduced expression for \( w \) involves all \( w_i \in J(w) \), we have \( a_w a_w = (-1)^{2l(w)} a_{\omega J(w)} \), with \( w' \in W_J(w) \) and \( l(w') > l(w) \). Further, \( J(w') = J(w) \).

Then \( E(w)^2 = a_w^2 + 2(-1)^{l(\omega J(w))} a_{\omega J(w)} \)

\[ + (-1)^{l(\omega J(w))} a_{\omega J(w)} \]

\[ = (-1)^{l(w')} a_{w'} + (-1)^{l(\omega J(w))} a_{\omega J(w)} \]

\[ = (-1)^{l(w')} (a_{w'} + (-1)^{l(\omega J(w'))} a_{\omega J(w')}) \]

\[ = (-1)^{l(w')} E(w'). \]

As \( l(w') > l(w) \), we have either \( w' = \omega J(w) \), and so \( E(w') = 0 \), or \( w' \neq \omega J(w) \), and then by induction on \( l(\omega J(w)) - l(w) \), we have that \( E(w') \) is nilpotent. So \( E(w) \) is nilpotent.

Finally note that we get a nilpotent element for each \( w \in W \), \( w \neq \omega J \) for any \( J \subseteq R \). The set of all \( E(w) \), \( w \neq \omega J \) for any \( J \subseteq R \), is obviously linearly independent, and there are \( |W| - 2^n \) elements in all, where \( n = |R| \). Hence they are a \( K \)-basis for \( N \).
(4.2.7) **COROLLARY:** \( \mathbb{H}/\mathbb{N} \) is commutative.

**Proof:** We show that \( a_{w_i}a_{w_j} - a_{w_j}a_{w_i} \in \mathbb{N} \) for all \( w_i, w_j \in \mathbb{R} \).

If \( a_{w_i}a_{w_j} = a_{w_j}a_{w_i} \), the result is obvious. So suppose

\[ a_{w_i}a_{w_j} \neq a_{w_j}a_{w_i} \].

Then we can form \( E(w_iw_j) \) and \( E(w_jw_i) \), and

\[ E(w_iw_j) - E(w_jw_i) = a_{w_i}a_{w_j} - a_{w_j}a_{w_i} \in \mathbb{N} \]

as each of \( E(w_iw_j) \) and \( E(w_jw_i) \) is.

We now give some examples of this basis of \( \mathbb{N} \).

1. \( H_K \) of type \( (W(A_2), \{w_1, w_2\}) \)

   \( \mathbb{N} \) has dimension 2 and K-basis \( \{a_{w_1w_2} + a_{w_1w_2w_1}, a_{w_2w_1w_2}, a_{w_1w_2w_1} - a_{w_0}\} \),

   where \( w_0 = w_1w_2w_1 \).

2. \( H_K \) of type \( (W(A_3), \{w_1, w_2, w_3\}) \)

   \( \mathbb{N} \) has dimension 16, and K-basis:

\[
\begin{align*}
   &a_{w_1w_2} + a_{w_1w_2w_1} \\
   &a_{w_1w_2} + a_{w_1w_2w_1} \\
   &a_{w_2w_3} + a_{w_2w_3w_2} \\
   &a_{w_3w_2} + a_{w_3w_2w_2} \\
   &a_{w_3w_2w_1} + a_{w_0} \\
   &a_{w_1w_3w_2} + a_{w_0} \\
   &a_{w_1w_3w_2} + a_{w_0} \\
   &a_{w_1w_3w_2} + a_{w_0} \\
   &a_{w_2w_1w_3} + a_{w_0} \\
   &a_{w_2w_1w_3} + a_{w_0} \\
   &a_{w_2w_1w_3} + a_{w_0} \\
   &a_{w_2w_1w_3} + a_{w_0}
\end{align*}
\]

where \( w_0 = w_1w_2w_1w_3w_2w_1 \).
(4.3) The Irreducible Representations of $H$.

We investigate the one-dimensional $H$-modules which arise from the natural composition series of $H$. Let the factor $H_i/H_{i+1}$ be generated as left $H$-module by $a_w + H_{i+1}$. The action of $H$ on this element is determined as follows: for each $w_i \in R$,

$$a_{w_i}(a_w + H_{i+1}) = \begin{cases} -(a_w + H_{i+1}) & \text{if } w^{-1}(r_i) \not\in \phi^- \\ 0 & \text{if } w^{-1}(r_i) \in \phi^+ \end{cases}$$

For any $w \in W$, let $J(w) = \{w_{i_j} : 1 \leq j \leq s\}$ where $w = w_{i_1} \cdots w_{i_s}$ is a reduced expression for $w$. Then

$$a_{w'}(a_w + H_{i+1}) = \begin{cases} (-1)^{l(w')}(a_w + H_{i+1}) & \text{if } w^{-1}(\prod J(w')) \subseteq \phi^- \\ 0 & \text{if there exists } r_i \in \prod J(w') \text{ such that } w^{-1}(r_i) \in \phi^+ \end{cases}$$

Hence the action of $H$ on $a_w + H_{i+1}$ depends on the element $w^{-1} \in W$.

(4.3.1) Definition: For each $J \subseteq R$, let $\lambda_J$ be the one-dimensional representation of $H$ defined by

$$\lambda_J(a_{w_i}) = \begin{cases} 0 & \text{if } w_i \in J \\ -1 & \text{if } w_i \notin J \end{cases}$$

For all $w \in W$, let $w = w_{i_1} \cdots w_{i_s}$ with $l(w) = s$. Then

$$\lambda_J(a_w) = \lambda_J(a_{w_{i_1}}) \cdots \lambda_J(a_{w_{i_s}}).$$

Extend $\lambda_J$ by linearity to $H$.

For each $J \subseteq R$, let $H_i(J)/H_{i+1}(J)$ be the factor of the natural series of $H$ which is generated by
Then the left $H$-module $H_i(J)/H_i(J)+1$ affords the representation $\lambda_J$ of $H$.

Since each composition factor of $H$ is one-dimensional, it follows that all irreducible representations of $H$ are one-dimensional. Let $\mu$ be an irreducible representation of $H$. Then $\mu$ is completely determined by the values $\mu(a_{w_1})$ for all $w_1 \in R$. Since $\mu$ is an algebra homomorphism, we must have that $\mu(a_{w_1})^2 = -\mu(a_{w_1})$ for all $w_1 \in R$. Let $\mu(a_{w_1})=u_1 \in K$ for all $w_1 \in R$. Then $u_1^2 = -u_1$ in $K$ implies that $u_1=0$ or $u_1=-1$. Thus we can describe each irreducible representation of $H$ by an $n$-tuple $(u_1, \ldots, u_n)$, where $n=|R|$, with $u_i=0$ or $-1$ for all $i$. In particular, $\lambda_J$ corresponds to the $n$-tuple $(u_1, \ldots, u_n)$ where $u_i=0$ if $w_i \in J$ and $u_i=-1$ if $w_i \notin J$. There are $2^n$ such irreducible representations, and they all occur in the natural series of $H$.

We determine $2^n$ maximal left ideals of $H$ as follows: for each $J \subseteq R$, form the $n$-tuple $(u_1, \ldots, u_n)$, where $u_i=0$ if $w_i \in J$ and is $-1$ otherwise. Let $M_J$ be the left ideal of $H$ generated by $\{a_{w_i}u_i^{-1}: w_i \in R\}$. Then $M_J = \ker \lambda_J$, and as $\lambda_J$ is irreducible, $M_J$ is a maximal left ideal of $H$.

Now, $H/N$ is semi-simple Artinian, so if we extend $K$ to its algebraic closure $\overline{K}$ and consider $H$ as an algebra over $\overline{K}$, we deduce that
\[ H/N \cong K \oplus K \oplus \ldots \oplus K, \text{ a direct sum of } 2^n \text{ fields.} \]

(Actually, we will show that
\[ H/N \cong K \oplus K \oplus \ldots \oplus K, \text{ } 2^n \text{ copies of } K, \]
regardless of which field \( K \) is.)

(4.4) **Some Decompositions of \( H \).**

In this section we will determine two decompositions of \( H \) as a direct sum of \( 2^n \) left ideals.

For each subset \( J \) of \( R \), let \( H_J \) be the subalgebra of \( H \) generated by \( \{ a_w : w \in J \} \). Define elements \( e_J \) and \( o_J \) in \( H_J \) as follows:

\[ e_J = \sum_{w \in W_J} a_w \]
\[ (4.4.1) \]
\[ o_J = (-1)^{l(w_J)} a_{w_J} \]

\( e_J \) and \( o_J \) are in the centre of \( H_J \) by inspection.

(4.4.2) **Lemma:** Let \( w_J = w_{i_1} \ldots w_{i_s} \), \( l(w_J) = s \). Then

\[ e_J = (1 + a_{w_{i_1}})(1 + a_{w_{i_2}}) \ldots (1 + a_{w_{i_s}}) \]

and is independent of the reduced expression for \( w_J \).

**Notation:** For all \( w \in W \), if \( w = w_{i_1} \ldots w_{i_t} \) with \( l(w) = t \), write \( [1 + a_w] = (1 + a_{w_{i_1}}) \ldots (1 + a_{w_{i_t}}) \). By the following proof, it follows that \([1 + a_w]\) is independent of the reduced expression for \( w \).

**Proof:** Firstly we show that \([1 + a_{w_{oJ}}]\) is independent of the reduced expression for \( w_{oJ} \). Since we can pass from
one reduced expression for \( w_{0j} \) to another by substitutions
of the form \((w_i w_j \ldots)_{n_{ij}} = (w_j w_i \ldots)_{n_{ij}}, i \neq j, \) where
\( n_{ij} \) is the order of \( w_i w_j \) in \( W \), we need to show that
\[
\left[ 1 + a(w_i w_j \ldots)_{n_{ij}} \right] = \left[ 1 + a(w_j w_i \ldots)_{n_{ij}} \right].
\]
To do this, we use induction on \( n, n \leq n_{ij} \), to show that
\[
\left[ 1 + a(w_i w_j \ldots)_{n} \right] = 1 + \sum_{m=1}^{n} a(w_i w_j \ldots)_{m} + \sum_{m=1}^{n-1} a(w_j w_i \ldots)_{m}.
\]
This is clearly true for \( n=1 \). Suppose it is true for all integers \( \leq k \), and suppose that \( k \) is odd. Then
\[
\left[ 1 + a(w_i w_j \ldots)_{k+1} \right] = \left[ 1 + a(w_i w_j \ldots)_{k} \right] (1 + a_{w_j})
\]
\[
= \left( 1 + \sum_{m=1}^{k} a(w_i w_j \ldots)_{m} + \sum_{m=1}^{k-1} a(w_j w_i \ldots)_{m} \right) (1 + a_{w_j})
\]
\[
= \left( 1 + \sum_{m=1}^{k} a(w_i w_j \ldots)_{m} + \sum_{m=1}^{k-1} a(w_j w_i \ldots)_{m} \right) + a_{w_j}
\]
\[
\quad + \left( \frac{1}{2} (k-1) \right) a_{w_j} + \sum_{m=0}^{k-1} a(w_i w_j \ldots)_{2m+1} a_{w_j} + \sum_{m=1}^{k-1} a(w_j w_i \ldots)_{2m} a_{w_j}
\]
\[
\quad + \left( \frac{1}{2} (k-1) \right) a_{w_j} + \sum_{m=1}^{k-1} a(w_i w_j \ldots)_{2m-1} a_{w_j} + \sum_{m=1}^{k-1} a(w_j w_i \ldots)_{2m-1} a_{w_j}.
\]
Now, \( a(w_i w_j \ldots)_{2m-1} a_{w_j} = -a(w_i w_j \ldots)_{2m} a_{w_j}, 1 \leq m \leq \frac{1}{2} (k-1), \) and
\( a(w_i w_j \ldots)_{2m} a_{w_j} = -a(w_j w_i \ldots)_{2m-2} a_{w_j}, 1 \leq m \leq \frac{1}{2} (k-1), \) where
\( a(w_j w_i \ldots)_{0} = 1. \) Then
\[
\left[ 1 + a(w_i w_j \ldots)_{k+1} \right] = 1 + \sum_{m=1}^{k} a(w_i w_j \ldots)_{m} + \sum_{m=1}^{k-1} a(w_j w_i \ldots)_{m}
\]
\[
\quad + a(w_i w_j \ldots)_{k} a_{w_j} + a(w_j w_i \ldots)_{k-1} a_{w_j}.
\]
Similarly, we get the same result if we had assumed that 

\[ k \] was even.

Similarly, we can show that for all \( n \), \( n \leq n_{ij} \),

\[
\left[ 1 + a(w_{i}w_{j}...)_{n} \right] = 1 + \sum_{m=1}^{n} a(w_{i}w_{j}...)_{m} + \sum_{m=1}^{n-1} a(w_{j}w_{i}...)_{m}
\]

So for all \( n \leq n_{ij} \),

\[
\left[ 1 + a(w_{i}w_{j}...)_{n} \right] - \left[ 1 + a(w_{j}w_{i}...)_{n} \right] = a(w_{i}w_{j}...)_{n} - a(w_{j}w_{i}...)_{n}
\]

When \( n = n_{ij} \), this difference is zero. So

\[
\left[ 1 + a(w_{i}w_{j}...)_{n_{ij}} \right] = \left[ 1 + a(w_{j}w_{i}...)_{n_{ij}} \right]
\]

and thus \( [1 + a_{W_{OJ}}] \) is independent of the reduced expression for \( W_{OJ} \) chosen.

Finally, \( [1 + a_{W_{OJ}}] \) is a linear combination of certain \( a_{w} \) with \( w \in W_{J} \). We show by induction on \( l(w) \) for all \( w \in W_{J} \) that \( a_{w} \) occurs in the expansion of \( [1 + a_{W_{OJ}}] \) with coefficient 1. If \( l(w) = 0 \), then \( w = 1 \) and obviously \( a_{1} = 1 \) occurs with coefficient 1. Suppose \( l(w) > 0 \). Let \( w = w'w_{j} \), \( w' \in W_{J} \), \( w_{j} \in J \), where \( l(w) = l(w')+1 \). By induction \( a_{w'} \) occurs in \( [1 + a_{W_{OJ}}] \) with coefficient 1. Choose an expression for \( W_{OJ} \) ending in \( w_{j} \), and then

\[
[1 + a_{W_{OJ}}] = [1 + a_{W_{OJ}w_{j}}] (1 + a_{w_{j}})
\]

Since \( l(w'w_{j}) > l(w') \), the only contribution to \( a_{w} \) from the last bracket is from the 1. If instead we take \( a_{w_{j}} \) from
the last bracket, we get \( a_w \), with coefficient 1. Now suppose \( a_w \) occurs in \( \left[ 1 + a_{woj} w_j \right] \) with coefficient \( m \). Then
\[
ma_w(1 + a_w) = ma_w + ma_w a_w
\]
\[
= ma_w - ma_w \text{ as } w(r_j) \in \phi^- 
\]
\[
= 0.
\]
Thus \( a_w \) occurs in the expansion of \( \left[ 1 + a_{woj} \right] \) with coefficient 1, and hence \( e_j = \left[ 1 + a_{woj} \right] \).

(4.4.3) **COROLLARY:** \( e_j \) and \( o_j \) are idempotents in the centre of \( H_j \).

**Proof:** For all \( w_i \in R \), \( (1 + a w_i)(1 + a w_i) = (1 + a w_i) \), and \( a w_i^2 = -a w_i \). The result follows as \( e_j = \left[ 1 + a_{woj} \right] \), and \( w_{oj} \) has a reduced expression ending in \( w_j \) for all \( w_j \in J \).

(4.4.4) **LEMMA:** (1) If \( J, L \subset R \), \( o_je_L = 0 \) if \( J \cap L \neq \emptyset \)
and \( e_j o_L = 0 \) if \( J \cap L \neq \emptyset \).

(2) If \( L \subset J \subset R \), \( e_L e_j = e_j = e_j e_L \)
and \( o_j o_L = o_j = o_j o_L \).

**Proof:** (1) If \( J \cap L \neq \emptyset \), choose \( w_i \in J \cap L \). Then we can choose reduced expressions for \( w_{oj} \) and \( w_{ol} \) ending and beginning respectively with \( w_i \). Then \( o_j e_L = a w_i (1+a w_i) \ldots \)
and so \( o_j e_L = 0 \). Similarly, \( e_j o_L = (1+a w_i) a w_i \ldots = 0 \).

(2) For each \( w_i \in J \), \( (1+a w_i) e_j = e_j + a w_i e_j \).
Since \( a w_i = -o_\{w_i\} \), \( (1+a w_i) e_j = e_j \) by (1). Also,
\( a w_i o_j = -o_j \) if \( w_i \in J \), and thus the results follow.
(4.4.5) **Lemma:** Let $y \in Y_J$ for some $J \subseteq R$. Then $a_{y_0} = a_y$ and $a_{y_0} e_J = \sum_{w \in W_J} a_{yw}$, with $l(yw) = l(y) + l(w)$ for all $w \in W_J$.

**Proof:** By (1.3.4), if $y \in Y_J$, then $y = w o_J$ for some $w \in W$ with $l(y) = l(w) + l(o_J)$. Hence $a_{y_0} = (-1)^{l(w)} a_w w o_J$ and so $a_{y_0} = a_y$. Now for all $w \in W_J$, as $y \in Y_J \subseteq X_J$, by (1.3.2) we have $l(yw) = l(y) + l(w)$. So for all $w \in W_J$ $a_y a_w = a_{yw}$. Thus $a_{y_0} e_J = a_y e_J = \sum_{w \in W_J} a_y a_w = \sum_{w \in W_J} a_{yw}$ and $l(yw) = l(y) + l(w)$ for all $w \in W_J$.

(4.4.6) **Lemma:** For $y \in Y_J$, $a_y$ occurs in the expansion of $a_y e_J o_J$ with coefficient 1, and if, for any $w \in W$, $a_w$ occurs in the expansion of $a_y e_J o_J$ with non-zero coefficient, then $w = y$ or $l(w) > l(y)$.

**Proof:** By (4.4.5), $a_y e_J = \sum_{w \in W_J} a_{yw}$ with $l(yw) = l(y) + l(w)$ for all $w \in W_J$. Then

$$a_y e_J o_J = \sum_{w \in W_J} a_{yw} o_J = a_y o_J + \sum_{w \in W_J} a_{yw} o_J.$$ 

By (4.4.5) $a_y o_J = a_y$, and for all $w \in W_J$, $w \neq 1$, $a_y w o_J = \pm a_w$, for some $w' \in W$, with $l(w') > l(yw) > l(y)$.

(4.4.7) **Theorem:** (1) The elements $\{a_{y_0} e_J : y \in Y_J, J \subseteq R\}$ are linearly independent and form a basis of $H$.

(2) The elements $\{a_y e_J o_J : y \in Y_J, J \subseteq R\}$ are linearly independent and form a basis of $H$. 
Proof: (1) Suppose that for each \( y \in Y_J \) and each \( J \subseteq R \) there is an element \( k_y \in K \) such that \( \sum_{J \subseteq \mathbb{R}} \sum_{y \in Y_J} k_y a_y o_j e_j = 0 \).

Let \( S_n = \sum_{J \subseteq \mathbb{R}} \sum_{y \in Y_J} k_y a_y o_j e_j \). We show that if \( S_n = 0 \), \( l(y) \neq n \)
then \( k_y = 0 \) whenever \( l(y) = n \) and hence \( S_{n+1} = 0 \).

Let \( y_1, \ldots, y_t \) be those elements of \( W \) for which \( l(y_i) = n \). Then by (4.4.5), if \( y_i \in Y_J(i) \) for some \( J(i) \subseteq R \),
\[
a_{y_i} o_j(i) e_j(i) = a_{y_i} + (a \text{ linear combination of certain } a_w \text{ where } l(w) > l(y_i))
\]
Hence, \( S_n = \sum_{i=1}^{t} k_{y_i} a_{y_i} + (a \text{ linear combination of certain } a_w \text{ with } l(w) > n) \).

If \( S_n = 0 \), then as \( \{a_w : w \in W\} \) are a basis of \( H \), we must have \( k_{y_i} = 0 \) for all \( i \), \( 1 \leq i \leq t \). So \( S_{n+1} = 0 \).

Since \( S_0 = 0 \), \( k_y = 0 \) for all \( y \) whenever \( l(y) = 0 \), and \( S_1 = 0 \). By induction, all \( k_y \) are zero, and so \( \{a_y o_j e_j : y \in Y_J, J \subseteq R\} \) is a set of linearly independent elements. As there are \( |W| \) of them, as \( \sum_{J \subseteq \mathbb{R}} |Y_J| = |W| \), they must form a basis of \( H \).

(2) A similar argument gives the result, since by (4.4.6) \( a_y e_j o_j = a_y + (a \text{ linear combination of certain } a_w \text{ where } l(w) > l(y)) \)
for \( y \in Y_J \).
(4.4.8) COROLLARY: (1) For any \( L \subseteq R \), the elements of the set \( \{ a\cdot y \cdot e\}_J : y \in Y_J, J \subseteq L \} \) are linearly independent.

(2) For any \( L \subseteq R \), the elements of the set \( \{ a\cdot e\}_J : y \in Y_J, L \subseteq J \} \) are linearly independent.

Proof: (1) For any \( y \in Y_J \), \( a\cdot y \cdot e \) = \( a\cdot y \) and so \( a\cdot y \cdot e \cdot o = a\cdot e \cdot o \cdot y \). Then \( a\cdot e \cdot o \cdot L = \sum_{w \in W_J} a\cdot y \cdot w \cdot o \cdot L \). As \( J \subseteq L \), \( L \subseteq J \) and so \( a\cdot w \cdot o \cdot L = a\cdot w \cdot o \).

Then \( a\cdot e \cdot L = a\cdot o \cdot e \cdot L = a\cdot y \cdot y \cdot w \cdot e \cdot L \) as \( y \in Y_J \).

\[
= a\cdot y + (\text{a linear combination of certain } a\cdot w \text{ with } 1(w) > 1(y))
\]

Now we may use a similar argument to that used in the proof of (4.4.7) to deduce that the given elements are linearly independent.

(2) For any \( y \in Y_J \), \( a\cdot e \cdot o \cdot L = a\cdot y \cdot e \cdot L \cdot o \cdot (\sum_{w \in W_J} k\cdot a\cdot w) \) where \( k\cdot w \in K \) and \( k\cdot w = 0 \) if \( 1(w) < 1(y) \). Then

\[
a\cdot e \cdot o \cdot L = a\cdot y \cdot e \cdot L + (\sum_{w \in W_J} k\cdot a\cdot w) \cdot e \cdot L, k\cdot w \in K \text{ given as above},
\]

\[
= a\cdot y + (\sum_{w \in W_J} k\cdot a\cdot w) \text{ for certain } k\cdot w \in K, \text{ with } k\cdot w = 0 \text{ if } 1(w) < 1(y).
\]

Once again we can use a similar argument to that in the proof of (4.4.7) to get the result.

(4.4.9) THEOREM: (1) For each \( a \in H \), and for any \( J \subseteq R \), there exist elements \( k\cdot y \in K \) such that

\[
a\cdot y \cdot e \cdot J = \sum_{y \in Y_J} k\cdot y \cdot a\cdot o \cdot e \cdot J.
\]
(2) For each $a \in H$, and for any $J \subseteq R$, there exist elements $k_y \in K$ such that

$$a e^J \cdot o^J = \sum_{y \in Y_J} k_y a^J \cdot o^J.$$  

**Proof:** We will prove the first part by a method which cannot be directly applied to the second part. However, the method we will use to prove the second part can be altered slightly to apply to the first.

(1) As $\{a_w : w \in W\}$ is a basis of $H$, we may write $a = \sum_{w \in W} u_w a_w$, with $u_w \in K$ for all $w \in W$. It is thus sufficient to express $a_w e^J$ as a linear combination of the elements $\{a_y o^J : y \in Y_J\}$ for all $w \in W$. Use induction on $l(w)$ to prove this.

If $l(w) = 0$, then $w = 1$, and $o^J = (-1)^{l(w)} a_w o^J$. Result is true for $w = 1$ as $a_w o^J \in Y_J$.

Suppose $l(w) > 0$. Let $w = w_1 w'$ for some $w_1 \in R$, $w' \in W$, $l(w) = l(w') + 1$. By induction,

$$a_{w_1} o^J e^J = \sum_{y \in Y_J} u_y a_y o^J e^J$$

for some $u_y \in K$. Then $a_w o^J e^J = a_w a_{w_1} o^J e^J = \sum_{y \in Y_J} u_y a_y a_{w_1} o^J e^J$. Hence for each $y \in Y_J$, we have to express $a_{w_1} a_y o^J e^J$ as a combination of $\{a_w o^J e^J : v \in Y_J\}$. Now for any $y \in Y_J$,
(4.4.10) \( a_{w_i y^0 J} = \begin{cases} -a_{y^0 J} & \text{if } y^{-1}(r_i) \in \phi^- \\ 0 & \text{if } y^{-1}(r_i) = r_j \text{ for some } r_j \in \prod J, \end{cases} \)

as then \( a_{w_i a y^0 J} = a_{y^0 J} a_{w_j} \)

\( a_{w_i y^0 J} \) where \( w_i y \in Y_J \) if \( y^{-1}(r_i) \in \phi^+ \)

\( y^{-1}(r_i) \neq r_j \) for any \( r_j \in \prod J \).

Hence the result follows.

(2) Since \( \{a_y e_L e_L^* : y \in Y_L, L \subseteq R \} \) is a basis of \( H \), there exist elements \( u_y \in K \) such that

\[ a e_J o^*_J = \sum_{L \subseteq R} \sum_{y \in Y_L} u_y a_y e_L e_L^* J. \]

Choose any \( M \subseteq R \) with \( M \cap \hat{J} \neq \emptyset \). Then \( a e_J o^*_M = 0 \), so

\[ \sum_{L \subseteq R} \sum_{y \in Y_L} u_y a_y e_L e_L^* M = 0. \]

But \( o^* e_M = 0 \) if \( \hat{L} \cap M \neq \emptyset \). So the only non-zero terms in the above equation involve those \( L \subseteq R \) for which \( \hat{L} \cap M = \emptyset \).

Thus \( \sum_{L \subseteq R} \sum_{y \in Y_L} u_y a_y e_L e_L^* M = 0. \) By (4.4.8(2)), \( u_y = 0 \)

for all \( y \in Y_L, M \subseteq L \subseteq R \). Hence we have that \( u_y = 0 \) for all \( y \in Y_L \), with \( L \cap \hat{J} \neq \emptyset \). So we now have

\[ a e_J o^*_J = \sum_{L \subseteq J} \sum_{y \in Y_L} u_y a_y e_L e_L^* J. \]

Let \( S_J = \{w \in W : u_w \neq 0, w \in Y_L \text{ for some } L \subseteq J \} \). Suppose \( S_J \neq \emptyset \). Choose an element \( y_o \in S_J \) of minimal length, and suppose \( y_o \in Y_{J_o} \) for some \( J_o \subseteq J \). Consider

\[ a e_J o^*_{J_o} = \sum_{L \subseteq J} \sum_{y \in Y_L} u_y a_y e_L e_L^* J_o. \]
As \( J_0 \subseteq J \), \( a_{eJ} j_{eJ} \cdot j_{eJ} = 0 \). Then

\[
\sum_{L \subseteq J} \sum_{y \in Y_L} u_y a_y e_L e_{J_0} j_{eJ} = 0
\]

(*)

Now if \( L \subseteq J \) and \( y \in Y_L \),

\[
a_y e_L e_{J_0} j_{eJ} = a_y j_{eJ} + \sum_{w \in W} k_w a_w \quad \text{where} \quad k_w \in K, \quad l(w) > l(y)
\]

and \( a_y j_{eJ} = \pm a_w \), for some \( w \in W \) with \( l(w) \geq l(y) \).

Since \( y_0 \) is of minimal length in \( S_J \), the coefficient of \( a_y j_{eJ} \) on the left side of (*) is \( u_{y_0} \). As \( \{ a_w : w \in W \} \) is a basis of \( H \), so \( u_{y_0} = 0 \) - contradiction. Hence \( S_J = \emptyset \)

and \( a_{eJ} j_{eJ} = \sum_{y \in Y_J} u_y a_y e_J e_{J_0} j_{eJ} \).

Remark: Let \( z \in Z \). We can regard \( z \) as an element of \( K \) in a natural way - it is the element \( z_1 K = 1_K + 1_K + \ldots + 1_K \) (\( z \) times)

where \( 1_K \) is the identity of \( K \).

(4.4.11) COROLLARY: (1) For each \( w \in W \), there exist rational integers \( u_y = u_y(w) \) such that \( a_w j_{eJ} e_J = \sum_{y \in Y_J} u_y a_y j_{eJ} e_J \).

(2) For each \( w \in W \), there exist rational integers \( u_y = u_y(w) \) such that \( a_w e_J j_{eJ} = \sum_{y \in Y_J} u_y a_y e_J j_{eJ} \).

Proof: (1) Follows from the proof of (4.4.9(1)).

(2) List the elements \( y_1, \ldots, y_m \) of \( Y_J \) in order of increasing length; if \( i < j \) then \( l(y_1) < l(y_j) \). Let \( c_{ij} \) be the coefficient of \( a_{y_i} \) in \( a_{y_j} e_J j_{eJ} \). Clearly \( c_{ij} \) is an integer as \( a_{y_j} j_{eJ} \) is an integral combination of certain elements \( a_{w'}, w' \in W \). Also, \( c_{ii} = 1 \) for all \( i, 1 \leq i \leq m \), and \( c_{ij} = 0 \).
if $i < j$ by (4.4.6). Let $h_i$ be the coefficient of $a_{\gamma_i}$ in $a_{\omega_j} o_{\omega_j}$. Clearly $h_i$ is an integer, and

$$h_i = \sum_{j=1}^{m} k_j c_{ij}$$

where $a_{\omega_j} o_{\omega_j} = \sum_{i=1}^{m} k_i a_{\gamma_i}$ for some $k_i \in K$. Hence,

$$h_i = \sum_{j=1}^{i-1} k_j c_{ij} + k_i$$

Let $i=1$. Then $h_1 = k_1$, an integer. Now use increasing induction on $i$ to show $k_i$ is an integer for all $i$, $1 \leq i \leq m$.

(4.4.12) **Theorem:** (1) $H_{\omega_j}$ is a left ideal of $H$ with $K$-basis $\{a_{\gamma_j} \omega_j : y \in Y_j\}$. Hence $\dim H_{\omega_j} = |Y_j|$. Moreover, $H = \bigoplus_{J \subseteq R} H_{\omega_j}$, a direct sum of $2^n$ left ideals, where $n = |R|$.

(2) $H_{e_j} o_j$ is a left ideal of $H$ with $K$-basis $\{a_{\gamma_j} e_j o_j : y \in Y_j\}$. Hence $\dim H_{e_j} o_j = |Y_j|$. Moreover, $H = \bigoplus_{J \subseteq R} H_{e_j} o_j$, a direct sum of $2^n$ left ideals, where $n = |R|$.

**Proof:** The results follow by (4.4.7), (4.4.9) and the fact that $\dim H = |Y| = \sum_{J \subseteq R} |Y_j|$.

(4.4.13) **Corollary:** $H_{\omega_j}$ and $H_{e_j} o_j$ are indecomposable left ideals of $H$, for all $J \subseteq R$, and they are isomorphic as left ideals of $H$.

**Proof:** From the theory of Artinian rings and the fact that $H/N$ is a direct sum of $2^n$ irreducible components (see remarks at the end of (4.3)), it follows that $H$ can be expressed as the direct sum of $2^n$ indecomposable left
ideals. Hence we must have that $H_{o_j}^e$ and $H_{o_j}^e$ are indecomposable left ideals of $H$ for all $J \subseteq R$.

To show they are isomorphic, we first show that $H_{o_j}^e = H_{o_j}^e$. Since each element of $H_{o_j}^e$ is of the form

$$\sum_{y \in Y_J} k_y a_y e_{o_j}$$

for some $k_y \in K$, and each $y \in Y_J$ is of the form $y = w w_{o_j}$ for some $w \in W$ with $l(y) = l(w) + l(w_{o_j})$, we have that $a_y = a_y (-1)$

$$l(w_{o_j}) a_w_{o_j} = a_y e_{o_j}.$$ Thus

$$\sum_{y \in Y_J} k_y a_y e_{o_j} = \sum_{y \in Y_J} k_y a_y o_{o_j} e_{o_j}$$

and so $H_{o_j}^e \subseteq H_{o_j}^e$. But obviously $H_{o_j}^e \subseteq H_{o_j}^e$, and so we have equality.

Now define the homomorphism $f_J$ of left ideals of $H$

$$f_J: H_{o_j}^e \rightarrow H_{o_j}^e$$

by $f_J(a o_{o_j}) = a o_{o_j} e_{o_j}$, for all $a o_{o_j} \in H_{o_j}^e$. As $f_J$ is given by right multiplication by $o_j$, it is well-defined and is a homomorphism of left ideals of $H$. $f_J$ is onto, since $H_{o_j}^e = H_{o_j}^e$ and an element $a o_{o_j} e_{o_j} \in H_{o_j}^e$ is the image under $f_J$ of $a o_{o_j}$. $f_J$ is one-one as $\dim H_{o_j}^e = \dim H_{o_j}^e$. Hence $f_J$ is an isomorphism of left ideals of $H$.

(4.4.14) **COROLLARY:** (1) For any $L \subseteq R$,

$$H_{o_j}^e \subseteq \bigoplus_{J \subseteq L} H_{o_j}^e_{L} \subseteq \bigoplus_{J \subseteq L} |Y_J| = |X_L|.$$ 

(2) For any $L \subseteq R$,

$$H_{o_j}^e \subseteq \bigoplus_{J \subseteq L} H_{o_j}^e_{L} \subseteq \bigoplus_{J \subseteq L} |Y_J| = |X_L|. $$
Proof: Use (4.4.12) and (4.4.8).

(4.4.15) **Theorem:** For any \( J \subseteq \mathbb{R} \),

\[
\text{He}_J = \{ a \in H : a_w w_i = 0 \text{ for all } w_i \in J \} \\
= \{ a \in H : a(1 + a_w) = a \text{ for all } w_i \in J \}.
\]

Further, \( \text{He}_J = \bigoplus_{J \subseteq L} \text{Ho}^*_L e_L \), and \( \text{He}_J \) has basis \( \{ a_w e_J : w \in X_J \} \)
and dimension \( |X_J| \). Consider the map \( \theta : \text{He}_J \to \text{Ho}^*_J e_J \) given
by projection. Then \( \theta : \text{He}_J / \bigoplus_{J \subseteq L} \text{He}_J \cong \text{Ho}^*_J e_J \). Finally,

\[
\text{Ho}^*_J e_J = \{ a \in H : a_w w_i = 0 \text{ for all } w_i \in J, a e_L = 0 \text{ for all } L \supset J \}
\]

\[
= \text{He}_J \cap \bigcap_{J \subseteq L} \ker e_L , \text{ where}
\]

\[
\ker e_L = \{ a \in H : a e_L = 0 \}.
\]

**Proof:** Clearly \( \text{He}_J \leq \{ a \in H : a_w w_i = 0 \text{ for all } w_i \in J \} \).

Conversely take \( a \in H \) and suppose \( a_w w_i = 0 \text{ for all } w_i \in J \).

Then \( a(1 + a_w) = a \text{ for all } w_i \in J \), and so \( a e_J = a \), and

\( a \in \text{He}_J \). Thus \( \text{He}_J = \{ a \in H : a_w w_i = 0 \text{ for all } w_i \in J \} \).

Similarly, \( \text{He}_J = \{ a \in H : a(1 + a_w) = a \text{ for all } w_i \in J \} \).

Then \( \text{Ho}^*_L e_L \leq \text{He}_J \) for all \( L \supset J \), and so \( \bigoplus_{J \subseteq L} \text{Ho}^*_L e_L \leq \text{He}_J \).

By (4.4.14), \( \dim \text{He}_J = |X_J| \), and as \( \dim \text{Ho}^*_L e_L = |Y_L| \),

we have \( \text{He}_J = \bigoplus_{J \subseteq L} \text{Ho}^*_L e_L \).

Let \( a = \sum_{w} u_w a_w \in \text{He}_J \), where \( u_w \in K \). Let \( w_i \in J \).

Then \( a_w w_i = 0 \), and so \( \sum_{w} u_w a_w w_i = 0 \).

Now \( \sum_{w} u_w a_w w_i = \sum_{w} u_w a_w w_i - \sum_{w} w(w_i) \in \phi^+ \), \( w(w_i) \in \phi^- \).
That is, \[ \sum_{w \in W} u_{ww_1} w - \sum_{w \in W} u_w w = 0. \]

Since \[ \{a_w : w \in W\} \text{ form a basis of } H, \]
we have \[ u_{ww_1} = u_w \]
for all \( w \in W \) with \( w(r_1) \in \phi^- \). Hence \[ u_w = u_{ww_1} \]
for all \( w \in W \),
with \( w(r_1) \in \phi^+ \). Now if \( w \in W \), \( w \) can be expressed uniquely in the form \( w = yw_j \), where \( y \in X_j \), \( w_j \in W_j \) and \( l(w) = l(y) + l(w_j) \).

Write \( w_j = w_{i_1} \ldots w_{i_t} \), \( w_j \in J \), \( l(w_j) = t \). By the above we have \[ u_y = u_{yw_{i_1}} = u_{yw_{i_1}w_{i_2}} = \ldots = u_{yw_j} = u_w. \]

Hence \( a = \sum_{y \in X_j} u_y a_y e_j \). Conversely, for each \( y \in X_j \),
\( a_y e_j \in H e_j \), and as \( \{a_y e_j : y \in X_j\} \) is linearly independent and \( \dim H e_j = |X_j| \), \( \{a_y e_j : y \in X_j\} \) is a basis of \( H e_j \).

Since \( H e_j = \sum_{J \subseteq L} H o_L e_L \), \( H e_j \) also has basis \( \{a_y o_L e_L : y \in Y_L, L \supseteq J\} \).

Consider \( \theta: \sum_{J \subseteq L} H o_L e_L \to H e_j \) given by \[ \theta(\sum_{J \subseteq L} H o_L e_L) = \sum_{J \subseteq L} H o_L e_L. \]

Since each \( H o_L e_L \)
is a left \( H \)-module, \( \theta \) is a left \( H \)-module homomorphism, and is onto. \( \ker \theta = \{a \in H e_j : a = \sum_{J \subseteq L} H o_L e_L \} \)
\[ = \{a \in H e_j : a \in \sum_{J \subseteq L} H o_L e_L \} \]
\[ = \sum_{J \subseteq L} H e_L. \]

and \( \theta: H e_j / \ker \theta \cong H o_j e_j \).

Finally, \( H o_j e_j \leq \{a \in H : a_{w_i} = 0 \text{ for all } w_i \in J, \}
a_{L} = 0 \text{ for all } L \supseteq J\}. \]

Let \( a = \sum_{J \subseteq L} H o_L e_L, u_y \in K, \)
\[ L y \in Y_L. \]
satisfy $aa_{w_1} = 0$ for all $w_1 \in J$ and $ae_L = 0$ for all $L \supset J$.

Since $a \in He_J$, $u_y = 0$ for all $y \in Y_L$ if $J \not\subseteq L$. So

$$a = \sum_{J \subseteq L} \sum_{y \in Y_L} u_y a y o y e_L.$$ 

Set $S_J = \{w \in W : u_w \neq 0, w \in Y_L, L \supset J\}$.

Suppose $S_J \neq \emptyset$. There exists an element $y_o$ of minimal length in $S_J$; suppose $y_o \in Y_M, M \supset J$. Then $ae_M = 0$.

Also, $o y e_M = 0$ as $M \supset J$. For other $L \supset J$, if $y \in Y_L$,

$$y o y e_L e_M = y e_L e_M = a y + (a \text{ combination of certain } a_w, w \in W,$$

with $1(w) > 1(y)$)

Then $ae_M = 0$ gives

$$\sum_{J \subseteq L} \sum_{y \in Y_L} u_y a y o y e_L = 0.$$ 

As $y_o$ is of minimal length in $S_J$, the coefficient of $a y_o$ in the left side of the last equation is $u_{y_o}$. By the linear independence of $\{a_w : w \in W\}$, we must have $u_{y_o} = 0$ — contradiction.

Hence $S_J = \emptyset$ and $a = \sum_{y \in Y_J} u_y a y o y e_J \in Ho^J e_J$. Thus

$$Ho^J e_J = \{a \in He_J : ae_L = 0 \text{ for all } L \supset J\}.$$

(4.4.16) **Theorem**: For any $J \subseteq R$,

$$Ho_J = \{a \in H : a(1 + a_{w_i}) = 0 \text{ for all } w_i \in J\}.$$

$Ho_J$ has basis $\{a_w : w \in Y_J, \hat{L} \subseteq J\}$, dimension $|X_J|$ and $Ho_J = \bigoplus_{J \subseteq L} \bigoplus_{L \supset J} He^o_{o,L}$. Further, the map $\Theta: Ho_J \to He^o_{o,J}$ given by projection defines an isomorphism $\Theta: Ho_J \cong He^o_{o,J}$.

Finally, $He^o_{o,J} = \{a \in Ho_J : ao_L = 0 \text{ for all } L \supset J\}$.

**Proof**: Clearly $Ho_J \subseteq \{a \in H : a(1 + a_{w_i}) = 0 \text{ for all } w_i \in J\}$.

Conversely, take $a \in H$ such that $a(1 + a_{w_i}) = 0 \text{ for all } w_i \in J$. 


Then $a a_{w_i} = -a$ for all $w_i \in J$; in particular,

$$aa_{w_{OJ}} = (-1)^{l(w_{OJ})} a,$$

and so $a = a_{OJ}$. Hence $a \in H_{OJ}$, and

$$H_{OJ} = \{ a \in H : a(1 + a_{w_i}) = 0 \text{ for all } w_i \in J \}.$$

Let $a = \sum_{w \in W} u_w a_w \in H_{OJ}$, $u_w \in K$. Then $a(1 + a_{w_i}) = 0$ for all $w_i \in J$; thus

$$\sum_{w \in W} u_w a_w + \sum_{w \in W} u_w a_{w_{w_1}} - \sum_{w \in W} u_w a_w = 0 \text{ for all } w_i \in J.$$

If $w(r_i) > 0$, the coefficient of $a_w$ on the left side is $u_w$, so $u_w = 0$ if $w(r_i) > 0$. If $w(r_i) < 0$, the coefficient of $a_w$ on the left side is $u_w + u_{w w_1} - u_w = u_{w w_1} = 0$ as $w_{w_1}(r_i) > 0$. Hence $a = \sum_{w \in W} u_w a_w = \sum_{L \subseteq J} \sum_{y \in Y_L} u_y a_y$, $w(\prod L) < 0$.

Conversely, if $y \in \{ w \in W : w \in Y_L, \hat{L} \subseteq \hat{J} \}$ then $a_y \in H_{OJ}$. Thus $\{ a_y : y \in Y_L, \hat{L} \subseteq \hat{J} \}$ is a basis of $H_{OJ}$. So

$$\dim H_{OJ} = \sum_{L \subseteq J} |Y_L| = |X_J|.$$ Obviously $\sum_{L \subseteq J} H_{L \cap OJ} \subseteq H_{OJ}$, and considering dimensions they must be equal. Hence

$\{ a_{y \in L \cap OJ} : y \in Y_L, L \supseteq J \}$ is also a basis of $H_{OJ}$.

The rest of the proof follows similar lines as (4.4.15), and uses the proof of (4.4.9(2)).

(4.4.17) Let $\psi_J$ be the character of the representation of $H$ on $H_{OJ} e_J$. Then $\psi_J$ takes values as follows: For each $w \in W$, let $w = w_{i_1} \cdots w_{i_t}$ be a reduced expression for $w$, and set $J(w) = \{ w_{i_j} : 1 \leq j \leq t \}$. Then
\[ \psi_J(a_w) = (-1)^{l(w)}N_J(w), \]

where \( N_J(w) = \) the number of elements \( y \in Y_J \) such that \( y^{-1}(\cap J(w)) \subseteq \phi^- \).

**Proof:** Use (4.4.10).

(4.4.18a) **LEMMA:** Let \( \phi_J \) be the character of the representation of \( H \) on \( H e_J \). Then \( \phi_J \) takes values as follows: for \( w \in W \) let \( w = w_{i_1} \ldots w_{i_t} \) be a reduced expression for \( w \). Set \( J(w) = \{ w_{i_j} : 1 \leq j \leq t \} \). Then

\[ \phi_J(a_w) = (-1)^{l(w)}M_J(w) \]

where \( M_J(w) = \) the number of elements \( x \in X_J \) such that \( x^{-1}(\cap J(w)) \subseteq \phi^- \). Also, \( M_J(w) = \sum_{J \subseteq L} N_L(w) \).

**Proof:** \( H e_J \) has basis \( \{ a_w e_J : w \in X_J \} \). For any \( w_i \in R \),

\[ a_{w_i} a_w e_J = \begin{cases} -a_{w_i} e_J & \text{if } w^{-1}(r_i) < 0 \\ a_{w_i} w_i e_J & \text{where } w_i w \in X_J \text{ if } w^{-1}(r_i) > 0, \\ & \text{and } w^{-1}(r_i) \neq r_j \text{ for any } r_j \in \prod, \\ 0 & \text{if } w^{-1}(r_i) = r_j \text{ for some } r_j \in \prod, \end{cases} \]

and \( a_{w_i} a_w = a_w a_{w_i} \), and \( a_{w_i} e_J = 0 \).

The result now follows.

(4.4.18b) **LEMMA:** Let \( \mu_J \) be the character of the representation of \( H \) on \( H o_J \). Then \( \mu_J \) takes values as follows: for each \( w \in W \), let \( w = w_{i_1} \ldots w_{i_t} \) be a reduced expression for \( w \), and set \( J(w) = \{ w_{i_j} : 1 \leq j \leq t \} \). Then

\[ \mu_J(a_w) = (-1)^{l(w)}L_J(w) \]
where \( L_J(w) = \) the number of elements \( z \in Z_J \) such that 
\[
z^{-1}(\prod_J(w)) \subseteq \phi^-, \text{ and } Z_J = \{w \in W: \prod_J(w) \subseteq \phi^-\}.
\]
Note that 
\[
Z_J = \bigcup_{L} \phi^L.
\]

**Proof:** \( H \) has basis \( \{a_w: w \in Z_J\} \). For all \( w_i \in R \),
\[
a_{w_i}a_w = \begin{cases} -a_w & \text{if } w^{-1}(r_i) < 0 \\ a_{w_i}w & \text{if } w^{-1}(r_i) > 0 \end{cases}
\]

If \( w \in Z_J, w_i \in R \) and \( w^{-1}(r_i) > 0 \), then \( w_iw \in Z_J \), for if \( r_j \in \prod_J, w(r_j) = -s \) for some \( s \in \phi^+ \), and \( w_i(s) < 0 \) if and only if \( s = r_i \). But if \( s = r_i \), \( w^{-1}(r_i) = -r_j \) is impossible.

The result now follows.

(4.4.19) **Corollary:**

1. \( \phi_J = \sum_{J \subseteq L} \Psi^L \) for all \( J \subseteq R \).
2. \( \mu_J = \sum_{J \subseteq L} \Psi^L \) for all \( J \subseteq R \).

A direct sum decomposition of \( H \) into indecomposable left ideals is equivalent to expressing the identity of \( H \) as a sum of mutually orthogonal primitive idempotents.

Let 
\[
1 = \sum_{J \subseteq R} q_J \quad \text{and} \quad 1 = \sum_{J \subseteq R} p_J
\]
be the decompositions of \( 1 \) corresponding to the decompositions
\[
H = \sum_{J \subseteq R} H \phi_J e_J \quad \text{and} \quad H = \sum_{J \subseteq R} H e_J \phi_J \] respectively, where
\[
H \phi_J = H e_J \phi_J \quad \text{and} \quad H p_J = H e_J \phi_J. \]

There does not appear to be a specific expression for the \( q_J \) or the \( p_J \) in terms of \( \{a_y \phi^J e_J: y \in Y_J\} \) or \( \{a_y \phi^J e_J: y \in Y_J\} \) respectively, but in Appendix 3 we give some examples of \( \{q_J\} \) and \( \{p_J\} \).
(4.4.20) **Theorem**: Let \( \{ q_J : J \subseteq R \} \) be a set of mutually orthogonal primitive idempotents with \( q_J \in H o_j e_j \) for all \( J \subseteq R \) such that \( 1 = \sum_{J \subseteq R} q_J \). Then \( H o_j e_j = H q_J \), and if \( N \) is the nilpotent radical of \( H \), \( N o_j e_j = N q_J \) is the unique maximal left ideal of \( H q_J \), and \( H q_J / N q_J \cong K \).

\( H q_J / N q_J \) affords the representation \( \lambda_J \) of \( H \) defined in (4.3.1). Finally, \( H / N = \sum_{J \subseteq R} H q_J / N q_J \cong K \otimes K \otimes \cdots \otimes K \), where \( 2^n \) summands, where \( n = |R| \).

**Proof**: By the theory of Artinian rings, \( N q_J \) is the unique maximal left ideal of \( H q_J \), and \( H / N = \sum_{J \subseteq R} H q_J / N q_J \).

Since \( q_J \in H o_j e_j \), \( H q_J \subseteq H o_j e_j \). As \( H = \sum_{J \subseteq R} H q_J = \sum_{J \subseteq R} H o_j e_j \), we must have \( H q_J = H o_j e_j \) for all \( J \subseteq R \). Then \( N q_J = MH q_J = NH o_j e_j = N o_j e_j \) is the unique maximal left ideal of \( H q_J \). But \( \{ \sum_{y \in Y_J} u_y o_j e_j : u_y \in K \} \) is a maximal left ideal of \( H o_j e_j \), by looking at (4.4.10), and so

\( N q_J = \{ \sum_{y \in Y_J} u_y o_j e_j : u_y \in K \} \). Then \( H q_J / N q_J \) is a one-dimensional \( H \)-module generated by \( a_{o_j e_j} + N q_J \) which affords the representation \( \lambda_J \) of \( H \), and since every element of \( H q_J / N q_J \) is of the form \( k a_{o_j e_j} + N q_J \) for some \( k \in K \), \( H q_J / N q_J \cong K \) for all \( J \subseteq R \). Hence the result.
(4.4.21) **THEOREM:** Let \( \{p_J : J \subseteq R \} \) be a set of mutually orthogonal primitive idempotents with \( p_J \in \text{He}_J \) for all \( J \subseteq R \) such that \( 1 = \sum_{J \subseteq R} p_J \). Then \( \text{He}_J = \text{Hp}_J \), and if \( N \) is the nilpotent radical of \( H \), \( \text{He}_J = \text{Hp}_J \) is the unique maximal left ideal of \( \text{Hp}_J \), and \( \text{Hp}_J / \text{Hp}_J \) affords the representation \( \lambda_J \) of \( H \) defined in (4.3.1). Finally, \( H / : = \bigoplus_{J \subseteq R} \text{Hp}_J / \text{Hp}_J \cong K \oplus K \oplus \ldots \oplus K \), \( 2^n \) summands, where \( n = |R| \).

**Proof:** We have similar relations to (4.4.10) for the elements \( a_y e_J \) with \( y \in Y_J \); they are as follows:

(4.4.10') Let \( y \in Y_J \) and let \( v \in K \). Then

\[
a_{v_j} a_y e_J = \begin{cases} -a_y e_J & \text{if } y^{-1}(v) < 0 \\
0 & \text{if } y^{-1}(v) = r \text{ for some } r \in \prod_J \\
a_{v_j} a_y e_J & \text{where } v_j y \in Y_J \text{ if } y^{-1}(v) > 0 \\
\text{and } y^{-1}(v) \neq r \text{ for any } r \in \prod_J.
\end{cases}
\]

The result now follows using the proof of (4.4.10)

since \( \{ \sum_{y \in Y_J} v_y a_y e_J : v_y \in K \} \) is a maximal left ideal \( \text{He}_J \neq \text{He}_J \) of \( \text{He}_J \).

(4.4.22) **LEMMA:** \( \{k a_w v_{OJ} e_J : k \in K \} \) and \( \{k a_w v_{OJ} e_J : k \in K \} \) are minimal submodules of \( \text{Ho}_J e_J \) and \( \text{He}_J e_J \) respectively, where \( w_{OJ} \) is the unique element of maximal length in \( Y_J \).

These minimal left ideals both afford the representation
\( \lambda_{\bar{J}} \) of \( \mathbb{H} \), where \( \bar{J} = \{ w_1 \in \mathbb{H} : \text{there exists } w_j \in J \text{ with } w_0 w_j = w_j w_0 \} \), or, alternatively, \( \prod_{\bar{J}} \) is defined by \( w_0 (\prod_{\bar{J}}) = -\prod_{\bar{J}} \).

**Proof:** The submodules given are clearly minimal by (1.3.5) and (1.3.7). Since \( w_0 w_0^{-1} = w_0 \bar{w}_0 \), these minimal left ideals both afford the representation \( \lambda_{\bar{J}} \) of \( \mathbb{H} \) as

\[
(w_0 \bar{w}_0^{-1})^{-1} = w_0 \bar{w}_0 \in \gamma_{\bar{J}}.
\]
Examples:

(1) \( H \) of type \((W(A_1), \{w_1\})\)

\[ H = \{ k_11 + k_2a_{w_1} : k_1, k_2 \in K \} \]

\[
\begin{array}{ccc}
\mathcal{J} & -e_J & o_J \\
\emptyset & 1 & 1 \\
\{w_1\} & 1 + a_{w_1} & -a_{w_1} \\
\end{array}
\]

Then \( H = H(1 + a_{w_1}) \oplus H(-a_{w_1}) \), where \( H(1 + a_{w_1}) \) has basis \((1 + a_{w_1})\}, and \( H(-a_{w_1}) \) has basis \( \{a_{w_1}\} \). Notice that in this case the two decompositions are identical, i.e. \( o_{\emptyset} e_{\{w_1\}} = e_{\{w_1\}} o_{\emptyset} \) and \( o_{\{w_1\}} e_{\emptyset} = e_{\emptyset} o_{\{w_1\}} \).

(2) \( H \) of type \((W(A_2), \{w_1, w_2\})\).

\( H \) has \( K \)-basis \( 1, a_{w_1}, a_{w_2}, a_{w_1}w_2, a_{w_2}w_1, a_{w_1}w_2w_1 \).

\[
\begin{array}{ccc}
\mathcal{J} & -e_J & o_J \\
\emptyset & 1 & 1 \\
\{w_1\} & 1 + a_{w_1} & -a_{w_1} \\
\{w_2\} & 1 + a_{w_2} & -a_{w_2} \\
\{w_1, w_2\} & (1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1}) & -a_{w_1}w_2w_1 \\
\end{array}
\]

Then \( H = H(-a_{w_1}w_2w_1) \oplus H(-a_{w_1})(1 + a_{w_2}) \oplus H(-a_{w_1})(1 + a_{w_2}) \oplus H(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1}) \)

and \( H = H(-a_{w_1}w_2w_1) \oplus H(1 + a_{w_2})(-a_{w_1}) \oplus H(1 + a_{w_1})(-a_{w_2}) \oplus H(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1}) \)

where \( H(-a_{w_1}w_2w_1) \) has \( K \)-basis \( \{a_{w_1}w_2w_2\}, H(-a_{w_1})(1 + a_{w_2}) \)
has K-basis \{a_{w_1}w_1(1 + a_{w_2}), a_{w_2}w_1(1 + a_{w_1})\},

\[ H(-a_{w_2})(1 + a_{w_1}) \text{ has K-basis } \{a_{w_1}(1 + a_{w_2}), a_{w_2}w_1(1 + a_{w_1})\}, \]

\[ H(1 + a_{w_2})(-a_{w_1}) \text{ has K-basis } \{a_{w_1} - a_{w_2}w_1, a_{w_2}w_1 + a_{w_1}w_2w_1\}, \]

\[ H(1 + a_{w_1})(-a_{w_2}) \text{ has K-basis } \{a_{w_2} - a_{w_1}w_2w_1, a_{w_1}w_2 + a_{w_1}w_2w_1\}, \]

and \[ H(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1}) \text{ has K-basis } \{(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})\}. \] Also,

\[ H(-a_{w_1})(1 + a_{w_2}) \cong H(1 + a_{w_1})(-a_{w_2}) \text{ as left ideals and } \]

\[ H(-a_{w_2})(1 + a_{w_1}) \cong H(1 + a_{w_1})(-a_{w_2}) \text{ as left ideals.} \]

(4.4.23) **Note:**

(1) By the same methods we also have that

\[ H = \sum_{J \subseteq R} e_J o_J H \quad \text{and} \quad H = \sum_{J \subseteq R} o_J e_J H, \]

both being direct sum decompositions of \( H \) into \( 2^n \) right ideals, where \( n = |R| \). Further, we have that \( e_J o_J H \) has K-basis \( \{e_J a_{w^{-1}}: w \in Y_J\} \), and that \( o_J e_J H \) has K-basis \( \{o_J a_{w^{-1}}: w \in Y_J\} \). All the results for the left ideals \( H e_J, H o_J, H e_J o_J \) and \( H o_J e_J \) have analogues for the right ideals \( e_J H, o_J H, o_J e_J H \) and \( e_J o_J H \) respectively.

(2) Suppose \( y \in Y_J \); then \( y = (y_J w_J) w_{o_J} \) with \( l(y) = l(y_J w_J) + l(w_{o_J}) \).

Then \( a_{y e_J o_J} = a_{y w_{o_J} w_J} e_J o_J = 0 \) unless \( J = \emptyset \). Suppose there exists an element \( w_1 \in J \) such that \( w_1w_J = w_Jw_1 \) for all \( w_J \in W_J \). Then \( a_{y o_J e_J} = 0 \). So we cannot get similar results using the elements \( a_{y e_J o_J} \) and \( a_{y o_J e_J} \) where \( y \in Y_J \).
(4.5) The Cartan Matrix of $H$.

We have that $H = \sum_{J \subseteq R} U_J$, where $U_J = H e^j$ is an indecomposable left $H$-module. Thus $\{U_J : J \subseteq R\}$ are the principal indecomposable $H$-modules. $\{U_J/\text{rad } U_J : J \subseteq R\}$, where rad $U_J$ is the unique maximal submodule of $U_J$, are irreducible $H$-modules, such that $M_J = U_J/\text{rad } U_J$ affords the representation $\lambda_J$ of $H$.

Definition: The Cartan matrix $C$ of $H$, where $H$ is of type $(\mathbb{W}, R)$, with $|R| = n$, is a $2^n \times 2^n$ matrix with rows and columns indexed by the subsets of $R$, and if we write $C = (c_{J,L})$, then

\[ c_{J,L} = \text{the number of times } M_L \text{ is a composition factor of } U_J. \]

(4.5.1) THEOREM: For all $J, L \subseteq R$,

\[ c_{J,L} = |Y_J \cap (Y_L)^{-1}| = |Y_L \cap (Y_J)^{-1}| = c_{L,J}. \]

Hence $C$ is a symmetric matrix.

Proof: $U_J$ has $K$-basis $\{a_{y} e^{J} : a_{y} e^{J} : y \in Y_J\}$. Let $y_1, \ldots, y_s$ be all the elements of $Y_J$ written in order of increasing length; if $i > j$ then $l(y_i) \geq l(y_j)$. Then set

\[ U_J(i) = \{ \sum_{j \geq i} k_{y} a_{y} e^{J} : k_{y} e^{J} : y \in K\} \]

of $H$ for all $i$, and $U_J(i) > U_J(i+1)$ for all $i$, $1 \leq i \leq s-1$. Then $U_J = U_J(1) > U_J(2) > \ldots > U_J(s) > 0$ is a composition series of $U_J$, with $U_J(i)/U_J(i+1)$ being an irreducible
H-module with basis $a_y e_j + U_j(i+1)$ and affording the irreducible representation $\lambda_L$, defined in (4.3.1), where we determine $L$ as follows: recall (4.4.10).

Let $w_j \in R$ and $y_1 \in Y_J$; then

$$a_{w_j} a_{y_1} e_J = \begin{cases} -a_y e_j & \text{if } y_1^{-1}(r_j) < 0 \\ 0 & \text{if } y_1^{-1}(r_j) = r_k \text{ for some } r_k \in \mathbb{N} \\ a_{w_j} y_1 e_j & \text{where } w_j y_1 = y_1 \text{ for some } y_1 \in Y_J \\ & \text{with } i < 1, \text{ if } y_1^{-1}(r_j) > 0 \text{ but } y_1^{-1}(r_j) \neq r_k \text{ for any } r_k \in \mathbb{N} \end{cases}$$

Hence $\lambda_L: a_{w_j} \rightarrow -1$ if $y_1^{-1}(r_j) < 0$

$$0 \text{ if } y_1^{-1}(r_j) > 0$$

That is, $y_1^{-1} \in Y_L$.

Hence $c_{jL}$ is the number of elements $y \in Y_J$ such that $y^{-1} \in Y_L$.

$$= |Y_J \cap (Y_L)^{-1}| = |Y_L \cap (Y_J)^{-1}|$$

since if $y \in Y_J \cap (Y_L)^{-1}$, then $y^{-1} \in Y_L \cap (Y_J)^{-1}$.

Some examples of the Cartan matrix for various types of $H$ are given in Appendix 4.

**Definition:** Let $E$ be a centrally primitive idempotent in $H$.

Then the block $B = B(E)$ is the class of all finitely generated $H$-modules $V$ satisfying $EV = V$.

Since $1 = \sum_{i=1}^{t} e_i$, a sum of pairwise orthogonal centrally primitive idempotents in $H$, then any $H$-module $V$ can be written $V = 1.V = \sum_{i=1}^{t} e_i V$. So if $V$ is indecomposable,
\[ V = e_1 V \text{ for some } e_1 \text{ and } V \in B(e_1). \] Thus every non-zero finitely generated indecomposable \( H \)-module is in a unique block. Furthermore, if \( V \in B = B(e) \), then for all \( v \in V \), \( ev = v \).

**Definition:** Let \( e_1 \) and \( e_2 \) be primitive idempotents in \( H \). Then we say \( e_1 \) and \( e_2 \) are linked if there is a sequence 
\[ e_1 = e_{i_1}, e_{i_2}, \ldots , e_{i_n} = e_2 \] of primitive idempotents such that for each \( j \), \( H e_{i_j} \) and \( H e_{i_{j+1}} \) have a common irreducible component.

(4.5.2) **Lemma:** The primitive idempotents \( e_1 \) and \( e_2 \) of \( H \) are linked if and only if \( H e_1 \) and \( H e_2 \) are in the same block.

**Proof:** See Dornhoff [12], theorem 46.2.

**Definition:** If \( e \) is a centrally primitive idempotent in \( H \) and \( B = B(e) \) is the corresponding block, then the Cartan matrix of the algebra \( H e = e H e \) is the Cartan matrix of the block \( B \).

(4.5.3) **Theorem** (Dornhoff [12], theorem 46.3): Let \( A \) be a finite-dimensional algebra over the field \( K \), and let \( B_1, \ldots , B_m \) be all of the distinct blocks of \( A \). Let \( C \) be the Cartan matrix of \( A \), \( C_1 \) the Cartan matrix of \( B_1 \). Then

1. with a suitable arrangement of rows and columns,

\[
C = \begin{bmatrix}
C_1 & C_2 & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{bmatrix}
\]

2. for any \( i, 1 \leq i \leq m \), it is impossible to arrange the
rows and columns of $C_1$ so that $C_1 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ for some matrices $A$ and $B$.

(4.5.4) **Theorem:** Let $H$ be the $0$-Hecke algebra over the field $K$ of type $(W,R)$, where $W$ is indecomposable. Then if $|R| > 1$, $H$ has three blocks. If $|R| = 1$, then $H$ has two blocks.

**Proof:** If $|R| = 1$, then $W = W(A_1)$ and $H = H_1 + H(-a_{w_1})$, where $R = \{w_1\}$. Both $(1 + a_{w_1})$ and $(-a_{w_1})$ are primitive idempotents as well as being central. Hence $H$ has only two blocks.

Now suppose that $|R| > 1$. $\begin{bmatrix} 1 + a_{w_0} \\ (-1)^{l(w_0)} a_{w_0} \end{bmatrix}$ are primitive and centrally primitive idempotents in $H$ and so correspond to two distinct blocks. The other primitive idempotents in $H$, i.e. $\{q_J: J \neq \emptyset, R\}$ as in (4.4.20), determine at least one other block. We will show that provided $W$ is indecomposable the Cartan matrix $C'$ corresponding to the indecomposables $U_J$ for $J \neq \emptyset, R$ and the irreducibles $M_L$ for $L \neq \emptyset, R$ cannot be expressed in the form $C' = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$.

Suppose that $C'$ can be put in the form above. Let $S_1 = \{J \subset R: U_J \text{ and } M_J \text{ index the rows and columns of } C_1\}$, $S_2 = \{J \subset R: U_J \text{ and } M_J \text{ index the rows and columns of } C_2\}$. Suppose for some $J \subset R$, $|J| = n-1$ (where $n = |R|$), that $J \in S_1$. Then we show
(1) for all \( L \subseteq R \) with \(|L| = n-1\), \( L \in S_1 \).

(2) by decreasing induction on \(|J|\) for all \( J \neq \emptyset, R \) that \( J \in S_1 \).

(a) Suppose \( J = \{w_1, \ldots, \hat{w}_j, \ldots, w_n\} \) and \( L = \{w_1, \ldots, \hat{w}_{j+1}, \ldots, w_n\} \), where the nodes corresponding to \( w_j \) and \( w_{j+1} \) in the graph of \( W \) are joined. Then the order of \( w_j w_{j+1} \) is greater than 2. Now \( w_{oJ} = w_j \in Y_J \) and \( w_{oL} = w_{j+1} \in Y_L \). Since the order of \( w_j w_{j+1} \) is greater than 2, we have that \( w_{j+1} w_j \in Y_J \) and \( w_j w_{j+1} \in Y_L \); that is, \( w_{j+1} w_j \in Y_J \cap (Y_L)^{-1} \). Hence \( J \in S_1 \) if and only if \( L \in S_1 \).

Hence if there is some \( J \in S_1 \), with \(|J| = n-1\), then all \( L \) with \(|L| = n-1\) are in \( S_1 \) by the above.

(b) Suppose that for all \( J \subseteq R \) with \(|J| > m\) that \( J \in S_1 \).

Choose \( L \subseteq R \) with \(|L| = m\). We show \( L \in S_1 \). Suppose \( L = \{w_{i_1}, \ldots, w_{i_m}\} \) with \( 1 \leq i_1 < i_2 < \ldots < i_m \leq n \). Since \( W \) is indecomposable, \( L \neq \emptyset, R \) then \(|Y_L| > 1\). Choose some \( w_{i_j} \in L \) and \( w_k \in \hat{L} \) such that \( w_{i_j} w_k \) has order \( r \), where \( r \geq 3 \). Then \( w_{i_j} w_{oL} \in Y_L \) (as \( w_{oL} (r_{i_j}) \neq r_i \) for any \( r_i \in \prod L \), since otherwise \( w_{oL} (r_{i_j}) = r_i \) for some \( r_i \in \prod L \) implies that \( r_{i_j} = r_i \) and \( w_{oL} \) is a product of reflections corresponding to roots orthogonal to \( r_{i_j} \), and so for all \( w_k \in \hat{L} \), \( w_{i_j} w_k = w_{w_{i_j}} \), contradiction). Now consider \( (w_{i_j} w_{oL})^{-1} = w_{oL} w_{i_j} \). Suppose \( w_{i_1} \in L \), \( w_{i_1} \neq w_{i_j} \). Then \( w_{oL} w_{i_j} (r_{i_1}) \in \phi^+ \). Also \( w_{oL} w_{i_j} (r_{i_j}) \in \phi^- \). Suppose \( w_k \in \hat{L} \).
Then $w_{\hat{L}}w_{i_j}(r_k) = w_\hat{L}(r_k + ur_{i_j})$ with $u > 0$

$$= w_\hat{L}(r_k) + uw_\hat{L}(r_{i_j}).$$

If $u = 0$, i.e. if $w_{i_j}w_k = w_kw_{i_j}$, then $w_{\hat{L}}w_{i_j}(r_k) \in \mathcal{P}^-.$

If $u > 0$, as $w_\hat{L}(r_k) = -r$ for some $r \in \prod_i^L$, and $w_\hat{L}(r_{i_j}) \in \mathcal{P}^+$, $w_\hat{L}(r_{i_j}) \neq r_{i_s}$ for any $r_{i_s} \in \prod_i^L$, we have

$$w_{\hat{L}}w_{i_j}(r_k) \in \mathcal{P}^+.$$ Hence $w_{\hat{L}}w_{i_j} \in Y_m$, where

$$M = \{L - \{w_{i_j}\} \cup \{w_k \in \hat{L} : w_{i_j}w_k \text{ has order} > 2\} \}

= \{L - \{w_{i_j}\} \cup \{w_k \in \hat{L} : \text{the node corresponding to} w_k \text{ in the graph of} \ L \text{ is joined to that corresponding to} w_{i_j}\}.$$

Now $|M| > |L|$ if the node corresponding to $w_{i_j}$ is joined to at least two nodes corresponding to elements of $\hat{L}$, and then $L \in S_i$ by induction.

Let $P_i$ be the node of the graph of $\mathcal{L}$ which corresponds to $w_i \in \mathcal{L}$, $1 \leq i \leq n$. Then suppose $P_{i_j}$ is joined to only one $P_k$ for all $w_k \in \hat{L}$. Then the above argument shows that $L = \{w_{i_1}, \ldots, w_{i_m}\}$ and $M = \{w_{i_1}, \ldots, \hat{w}_{i_j}, \ldots, w_{i_m}, w_k\}$ belong to the same $S_i$, where $i=1$ or $i=2$. Since $|L| < n-2$, $|\hat{L}| \geq 2$. Let $w_{k_1}$ and $w_{k_2}$ be any two elements of $\hat{L}$, such that there exists a sequence $P_{k_1} = P_{j_0}, P_{j_1}, \ldots, P_{j_r} = P_{k_2}$ of nodes such that $P_{j_i}$ and $P_{j_{i+1}}$ are joined for all $i$, $0 \leq i < r-1$, and $P_{j_1}$ corresponds to an element of $L$ for all $1 \leq i < r-1$.

If $r = 1$, then $P_{k_1}$ and $P_{k_2}$ are joined. Without loss of
generality, we may suppose there exists \( w_i \in L \) such that \( P_i \) is joined to \( P_{k_1} \). Then let \( M = \{ L - \{ w_i \} \} \cup \{ w_k \} \). M and L belong to the same \( S_1 \), and by the above, as M has an element \( w_{k_1} \) such that \( w_{k_1} w_i \) and \( w_{k_1} w_{k_2} \) both have order > 2, where \( w_i, w_{k_2} \in M \), \( w_i \neq w_{k_2} \), then \( M \in S_1 \).

If \( r = 2 \), then \( L \) and \( N \) are in the same \( S_1 \), where 
\[ L = \{ L - \{ w_j \} \} \cup \{ w_{k_1}, w_{k_2} \}, \]
and by induction \( M \in S_1 \).

If \( r > 2 \), define \( L_0 = L \)
\[ L_1 = \{ L - \{ w_j \} \} \cup \{ w_{j_0} \}, \]
\[ \ldots \ldots \]
\[ L_{r-2} = \{ L_{r-3} - \{ w_j \} \} \cup \{ w_{j_{r-3}} \} \]
Then \( L_0, L_1, \ldots L_{r-2} \) are all in the same \( S_1 \), and by the above, \( L_{r-2} \in S_1 \).

Hence \( L \in S_1 \). Then \( S_2 = \emptyset \), and so \( H \) has precisely three blocks.

(4.5.5) **Theorem**: Let \( H \) be a \( O \)-Hecke algebra of type \((W,R)\). Suppose \( W \) is decomposable, and let \( W = W_1 \times W_2 \times \cdots \times W_r \), where each \( W_i \) is an indecomposable Coxeter group, and the corresponding Coxeter system is \((W_i,R_i)\). Let \( H_i \) be the \( O \)-Hecke algebra of type \((W_i,R_i)\), and let \( m_i \) be the number of blocks of \( H_i \). Then \( H \) has \( m_1 m_2 \cdots m_r \) blocks.

**Proof**: Suppose that \( 1 = \sum_{i=1}^{t} e_i \) where the \( e_i \) are mutually orthogonal centrally primitive idempotents in \( H \). Then the number of blocks of \( H \) is equal to \( t \).
Now for all \( w \in W_i, w' \in W_j \), where \( 1 \leq i,j \leq r \) and \( i \neq j \), we have that \( a_w a_{w'} = a_{w'} a_w = a_{w'} a_w \), and so it follows that if \( f_i \) is a central primitive idempotent of \( H_i \), then \( f_1 \cdots f_r \) is a central primitive idempotent of \( H \). Suppose

\[
1_{H_i} = \sum_{j=1}^{t(i)} f_{ij}
\]

where for a fixed \( i \), \( \{f_{ij} : 1 \leq j \leq t(i)\} \) is a set of mutually orthogonal central primitive idempotents in \( H_i \).

Then

\[
1_H = \sum_{j_1=1}^{t(1)} \cdots \sum_{j_r=1}^{t(r)} f_1 j_1 \cdots f_r j_r,
\]

a sum of mutually orthogonal central primitive idempotents in \( H \), and so \( H \) has \( t(1)t(2) \cdots t(r) \) blocks, where \( t(i) = m_i \).
Chapter 5: **DECOMPOSITIONS OF THE GENERIC RING.**

Let $A = A_{B_0}(u)$ be the generic ring of the system $S$ of finite groups with $(B,N)$ pairs of type $(W,R)$, where $B_0 = \{g(u)/h(u): g(u), h(u) \in Q[u], u \not| h(u)\}$. $A$ is the associative algebra over $B_0$ with basis $\{a_w: w \in W\}$, and multiplication is given by the following: for all $w_i \in R$ and all $w \in W$, we have

$$a_{w_i}a_w = \begin{cases} a_{w_i} & \text{if } l(w_iw) > l(w) \\ u_i^\ell a_{w_i}w + (u_i - 1)a_w & \text{if } l(w_iw) < l(w) \end{cases}$$

Also, if $w = w_{i_1} \ldots w_{i_s}$ is a reduced expression for $w \in W$, then $a_w = a_{w_{i_1}} \ldots a_{w_{i_s}}$.

Now each group $G(q) \in S$ has a parabolic subgroup $G_J(q)$ for each subset $J \subseteq R$, and $G_J(q)$ is itself a finite group with a $(B,N)$ pair of type $(W_J,J)$.

**Definition:** Let $A_J = A_{J,B_0}(u)$ be the generic ring of the system $S_J$ of finite groups with $(B,N)$ pairs of type $(W_J,J)$ with $J$ the same as for $S$, and $\{c_i: w_i \in J\}$ are the same as in $S$. Each $G(q) \in S$ determines a $G_J(q) \in S_J$.

**Definition:** For all $w \in W$, $w \neq 1$, let $w = w_{i_1} \ldots w_{i_s}$ be a reduced expression for $w$, and define

$$c_w = c_{i_1} + \ldots + c_{i_s}$$

If $w = 1$, let $c_1 = 0$. (see also (3.2.3)).

**Lemma:** $c_w$ is independent of the reduced expression
for \( w \), and since all \( c_i \) are positive integers, \( c_w \) is a positive integer for all \( w \in W, w \neq 1 \).

**Proof:** Since we can get from one reduced expression for \( w \) to another by substitutions of the form

\[
(w_i w_j \cdots)_{n_{ij}} = (w_j w_i w_j \cdots)_{n_{ij}}
\]

where \( n_{ij} \) is the order of \( w_i w_j \) in \( W \), we need to show

\[
c_i + c_j + c_i + \cdots (n_{ij} \text{ terms}) = c_j + c_i + c_j + \cdots (n_{ij} \text{ terms}).
\]

If \( n_{ij} \) is even this is obvious. If \( n_{ij} \) is odd, then \( w_i \) and \( w_j \) are conjugate in \( W \{w_i, w_j\} \) and so \( c_i = c_j \), and again the result is obvious.

(5.4) **Definition:** The characteristic function of \( S_J \) for all \( J \subseteq R \) is the polynomial

\[
f(u)_J = \sum_{w \in W_J} c_w u^w \in \mathbb{Z}[u].
\]

(Compare (3.2.4)). \( f(u)_\emptyset = 1 \).

(5.5) **Definition:** For each \( J \subseteq R \), define

1. \( e_J = \frac{1}{f(u)_J} \sum_{w \in W_J} c_w a_w \)
2. \( o_J = \frac{1}{f(u)_J} \sum_{w \in W_J} (-1)^{l(w)} u^{c_{w_{OJ}} o_J} a_w \) where

\( w_{OJ} \) is the unique element of maximal length in \( W_J \).

\( e_J \) and \( o_J \) are elements of \( A_J \). We will now show that they are central idempotents in \( A_J \).
(5.6) \textbf{Lemna}: (1) For all $w_i \in J$, $a_{w_i} e_j = u^i_e e_j = e_j a_{w_i}$.

(2) For all $w_i \in J$, $a_{w_i} o_j = -o_j = o_j a_{w_i}$.

\textbf{Proof}: (1) $a_{w_i} e_j = \frac{1}{f(u)_J} \sum_{w \in W_j} a_{w_i} w$

$= \frac{1}{f(u)_J} \sum_{w \in W_j} a_{w_i} w \quad w^{-1}(r_i) > 0$

$+ \frac{1}{f(u)_J} \sum_{w \in W_j} \{u^i_a w_i + (u^i_{i-1}) a_w\} \quad w^{-1}(r_i) < 0$

$= \frac{1}{f(u)_J} \sum_{w \in W_j} \{a_{w_i} w + u^i_a w_i + (u^i_{i-1}) a_{w_i} w\} \quad w^{-1}(r_i) > 0$

$= u^i_e e_j$.

Similarly $e_j a_{w_i} = u^i_e e_j$.

(2) $a_{w_i} o_j = \frac{1}{f(u)_J} \sum_{w \in W_j} (-1)^{l(w)} u^{c_{WW oJ} a_{w_i} w}$

$= \frac{1}{f(u)_J} \sum_{w \in W_j} (-1)^{l(w)} u^{c_{WW oJ} [u^i_a w_i + (u^i_{i-1}) a_w]} \quad w^{-1}(r_i) > 0$

Suppose $WW_{oJ} = w_{i_1} \ldots w_{i_s}$, $l(WW_{oJ}) = s$, and $w^{-1}(r_i) < 0$.

Then $(WW_{oJ})^{-1}(r_i) > 0$ (as $r_i \in \prod J$), and so

$w_i WW_{oJ} = w_i w_{i_1} \ldots w_{i_s}$, $l(w_i WW_{oJ}) = s+1$. Hence

$c_{Wi WW_{oJ}} = c_i + c_{WW_{oJ}}$. So

$a_{w_i} o_j = \frac{1}{f(u)_J} \sum_{w \in W_j} (-1)^{l(w)} \{u^{c_{WW oJ} a_{w_i} w - u^{c_{WW oJ} a_w}}$

$= (u^{c_{WW oJ} - u^{c_{WW oJ} a_{w_i} w}}) a_{w_i} w\}$. 
Hence \( a_{w_1} o_J = -o_J \). Similarly \( o_J a_{w_1} = -o_J \).

(5.7) **COROLLARY:** \( e_J \) and \( o_J \) are idempotents in the centre of \( A_J \), for all \( J \subseteq R \).

**Proof:** Follows from (5.6).

(5.8) **LEMMA:** Let \( J \subseteq L \subseteq R \), and let \( X_J^L = \{y_1, \ldots, y_s\} \), where \( X_J^L = \{w \in W_L : w(\prod J) \in \Phi^+\} \). (By (1.3.2) the set \( X_J^L \) is a set of left coset representatives for \( W_L \mod W_J \), and each element of \( W_L \) can be expressed uniquely in the form \( y_i w_J \) for some \( i \) and some \( w_J \in W \) with \( l(y_i w_J) = l(y_i) + l(w_J) \)).

Then (1) \( e_L = \frac{f(u)_J}{f(u)_L} \sum_{i=1}^s a_{y_i} e_J = \frac{f(u)_J}{f(u)_L} \sum_{i=1}^s e_J a_{y_i} -1 \), and

(2) \( o_L = \frac{f(u)_J}{f(u)_L} \sum_{i=1}^s (-1)^{l(y_i)} w_{oJ} w_{oL}^{-1} c_{y_i} a_{y_i} o_J \)

\( = \frac{f(u)_J}{f(u)_L} \sum_{i=1}^s (-1)^{l(y_i)} w_{oJ} w_{oL}^{-1} c_{y_i} a_{y_i} o_J -1 \).

**Proof:** (1) \( \frac{1}{f(u)_L} \sum_{w \in W_L} a_w = \frac{1}{f(u)_L} (\sum a_{y_i} w \in W_J) \)

\( = \frac{f(u)_J}{f(u)_L} \sum_{i=1}^s a_{y_i} e_J \)

and similarly for the other equality, as \( \{y_1^{-1}, \ldots, y_s^{-1}\} \)

are a set of right coset representatives for \( W_L \mod W_J \).

(2) \( o_L = \frac{1}{f(u)_L} \sum_{i=1}^s (-1)^{l(y_i)} (-1)^{l(w)} c_{y_i w oL} a_{y_i} w \)

Now let \( y_i w_{oL} = w_{i_1} \ldots w_{i_t} \), \( l(y_i w_{oL}) = t \). Since \( w \in W_J \), \( l(y_i w) = l(y_i) + l(w) \) and so \( w_{oL} = (y_i)^{-1} w_{i_1} \ldots w_{i_t} \), with

\( l(w_{oL}) = l((y_i)^{-1}) + t \). Then
Now for all \( w \in W_J \), \( w w o_L = (w w o_J)(w o_J w o_L) \) and
\[ c_{w w o_L} = c_{w w o_J} + c_{w o_J w o_L} \]. Hence
\[ c_{w o_L} w w = c_{w o_L} - c_{y_i} = c_{w o_J w o_L} - c_{y_i} \]. So
\[ o_L = \frac{1}{f(u)_L} \sum_{i=1}^{s} \sum_{w \in W_J} l(y_i) (c_{w o_J w o_L} - c_{y_i}) a_{y_i} \]
\[ x (-1)^{l(w)} c_{w o_J a_w} \],
\[ = \frac{f(u)_J}{f(u)_L} \sum_{i=1}^{s} (-1)^{l(y_i)} (c_{w o_J w o_L} - c_{y_i}) a_{y_i} o_J. \]

For the other equality, note that for \( w \in W_J \),
\[ c_{w y_i w o_L} = c_{w w o_J} + c_{w o_J w o_L} - c_{y_i}. \]

(5.9) COROLLARY: For all \( J \subseteq L \subseteq R \),
\[ e_J e_L = e_L = e_L e_J \]
and \( o_J o_L = o_L = o_L o_J. \)

Proof: Use (5.8) and (5.7).

(5.10) LEMMA: If \( J, L \subseteq R \) and \( J \cap L \neq \emptyset \), then \( o_J e_L = 0 \) and \( e_J o_L = 0. \)

Proof: Let \( M = \{w_i\} \). Then
\[ o_M e_M = \frac{1}{f(u)_M^2} \left( u c_{i1} - a_w \right)(1 + a_w) \]
\[ = \frac{1}{f(u)_M^2} \left( u c_{i1} + u c_{i1 a_w} - a_w - a_w^2 \right) = 0 \]
Similarly $\epsilon_{n^m} = 0$. The result now follows using (5.8).

**NOTE:** From the definition of $A$, it is clear that for all $w, w' \in \mathcal{W}$, we have $a_w a_{w'} = \sum_{w'' \in \mathcal{W}} k_{w''} a_{w''}, k_{w''} \in \mathcal{O}$, summed over certain $w'' \in \mathcal{W}$ for which

$$|l(w) - l(w')| \leq l(w'') \leq l(w) + l(w').$$

(5.11) **LEITHE:** (1) For all $y \in Y_j$,

$$a_{y_{o^j}e_{j}} = \frac{1}{f(u)_{j}f(u)_{j}} a_{y_{w_{o^j}}} + (\text{a linear combination of terms } a_w \text{ with } l(w) < l(y_{w_{o^j}})).$$

(2) For all $y \in Y_j$,

$$a_{y_{o^j}o_{j}} = \frac{1}{f(u)_{j}f(u)_{j}} a_{y} + (\text{a linear combination of terms } a_w \text{ with } l(w) > l(y)).$$

**Proof:** (1) Since $y \in Y_j$, $y = (yw_{o^j})w_{o^j}$ with $yw_{o^j} \in X_j$ and $l(y) = l(yw_{o^j}) + l(w_{o^j})$. Then for all $w \in W_j$, $l(yw_{o^j}w) = l(yw_{o^j}) + l(w)$.

By (5.6(2)) we have

$$a_{y_{o^j}e_{j}} = (-1)^{l(w_{o^j})} a_{yw_{o^j}e_{j}} w_{o^j}$$

$$= \frac{1}{f(u)_{j}} \sum_{w \in W_j} (-1)^{l(w)} c_{w_{o^j}} a_{yw_{o^j}w_{o^j}}$$

$$= \frac{1}{f(u)_{j}} a_{y} + (\text{a combination of terms } a_w, l(w) < l(y)).$$

Thus $a_{y_{o^j}e_{j}} = \frac{1}{f(u)_{j}f(u)_{j}} \sum_{w \in W_j} \sum_{v \in V_j} (-1)^{l(w)} c_{w_{o^j}} a_{yw_{o^j}w_{o^j}a_{v}}$.

Since $l(yw_{o^j}) = l(y) + l(w_{o^j})$, $a_{yw_{o^j}w_{o^j}} = a_{yw_{o^j}}$. Now for all $w \in W_j, v \in V_j$, $a_{yw_{o^j}w_{o^j}} a_{v} = \sum k_{w''} a_{w''}$, summed over certain $w'' \in \mathcal{W}$ with $l(w'') \leq l(yw_{o^j}w) + l(v) \leq l(yw_{o^j}w) + l(v) + l(v)$.

$$\leq l(yw_{o^j}w) + l(w_{o^j}) + l(v_{o^j}).$$
Hence $a_{yw_{OJ}}$ can only occur with non-zero coefficient in a product $a_{yw_{OJ}}^\wedge w_v$ where $l(w)=l(w_{OJ})$ and $l(v)=l(w_{OJ})$, that is, where $w=w_{OJ}$ and $v=w_{OJ}$. Now if $w\neq w_{OJ}$ for all $v \in W_j$,

$a_{yw_{OJ}}^\wedge w_v = \sum k_w a_w^{'}$, summed over certain $w^{'} \in W$ with $l(w^{'}) = l(yw_{OJ})+l(w)+l(v) < l(yw_{OJ})$. Hence

$$a_{yjOJ}^\wedge = \frac{1}{f(u)_{j}^\wedge f(u)_{j}^\wedge} \sum_{w \in W_j} \sum_{v \in W_j^\wedge} (-1)^{l(v)} a_{yw}^\wedge a_v^\wedge$$

$$+ \frac{1}{f(u)_{j}^\wedge} a_{yjOJ}^\wedge.$$ 

Now if $w\neq 1$, $w \in W_j$, $a_{yw}^\wedge a_v^\wedge = \sum k_w a_w^{'}$, summed over certain $w^{'} \in W$ with $l(yw)-l(v) < l(w^{'})$ (since $v \in W_j^\wedge$, $l(y) \geq l(v)$).

As $l(yw) > l(y)$ and $l(v) < l(w_{OJ})$, $l(yw_{OJ}) = l(y)-l(w_{OJ})$,

$$l(yw_{OJ}) < l(yw)-l(v)$$ for any $v \in W_j^\wedge$. From (1) we have

$$a_{yjOJ}^\wedge = \frac{1}{f(u)_{j}^\wedge} a_{yw_{OJ}}^\wedge + (a \text{ linear combination of terms } a_w, l(w) > l(yw_{OJ})).$$

Hence $a_{yjOJ}^\wedge = \frac{1}{f(u)_{j}^\wedge f(u)_{j}^\wedge} a_{yw_{OJ}}^\wedge + (a \text{ linear combination of certain } a_w, l(w) > l(yw_{OJ})).$

(5.12) THEOREM: (1) The elements $\{a_{yjOJ}^\wedge : y \in Y_j\}$ are linearly independent, for all $J \subseteq R$.

(2) The elements $\{a_{yjOJ}^\wedge : y \in Y_j\}$ are linearly independent, for all $J \subseteq R$. 

Proof: (1) Suppose we have a relation $\sum_{y \in Y_J} b_y a_y \delta_J = 0$, where the $b_y \in B_0$. Set $S_n = \sum_{y \in Y_J} b_y a_y \delta_J$. Then if $S_n = 0$, $1(y) \geq n$ by (5.11(1)). $b_y = 0$ for all $y \in Y_J$ with $1(y) = n$, as $\{a_w : w \in W\}$ are a basis of $A$, and so $S_{n+1} = 0$.

Let $m$ be the length of the maximal element of $Y_J$. Then $S_m = 0$ is the given relation. By decreasing induction it follows that $b_y = 0$ for all $y \in Y_J$. Hence the given elements are linearly independent.

(2) Suppose we have a relation $\sum_{y \in Y_J} b_y a_y \delta_J = 0$, where the $b_y \in B_0$. Set $S_n = \sum_{y \in Y_J} b_y a_y \delta_J$. Then if $S_n = 0$, $1(y) \geq n$ by (5.11(2)). $b_y = 0$ for all $y \in Y_J$ with $1(y) = n$, as $\{a_w : w \in W\}$ are a basis of $A$, and so $S_{n+1} = 0$. The relation given above is $S_m = 0$, where $m$ is the length of the shortest element in $Y_J$. Thus by induction, all $b_y = 0$, and so the given elements are linearly independent.

(5.13) Theorem: (1) The elements $\{a_y \delta_J : y \in Y_J, J \subseteq R\}$ are linearly independent and hence form a basis of $A$.

(2) The elements $\{a_y \delta_J : y \in Y_J, J \subseteq R\}$ are linearly independent and hence form a basis of $A$.

Proof: (1) Suppose we have a relation $\sum_{J \subseteq R} \sum_{y \in Y_J} b_y a_y \delta_J = 0$, where the $b_y \in B_0$. We may suppose there exists
at least one non-zero coefficient $b_y$ such that $b_y = g(u)/h(u)$ with $u \not| g(u)$ and $u \not| h(u)$, $g(u), h(u) \in \mathbb{Q}[u]$.

Consider the specialisation $f_0$ of $B_0$ defined in (3.4.11). $f_0$ induces a ring epimorphism $f_0': A \rightarrow H_Q$, the $Q$-Hecke algebra of type $(W,R)$ over $Q$. In particular, writing the standard basis elements of $H_Q$ as $\{h_w : w \in W\}$ and the idempotents defined in (4.4.1) as $E_J$ and $O_J$ respectively, we see that $f_0'(a_y O_J E_J) = h_y O_J E_J$. Now apply $f_0'$ to the given relation:

$$\sum_{J \subseteq R} \sum_{y \in Y_J} f_0(b_y) h_y O_J E_J = 0.$$  

As $\{h_y O_J E_J : y \in Y_J, J \subseteq R\}$ are linearly independent, $f_0(b_y) = 0$ for all $y \in W$. But $\ker f_0 = uB_0$, and we have supposed that at least one of the non-zero coefficients $b_y$ was not in $uB_0$. Hence all $b_y$ are zero, and the given elements are linearly independent.

(Note: If all $b_y \in uB_0$, then the original relation becomes $u \sum_{J \subseteq R} \sum_{y \in Y_J} b_y ' a_y O_J E_J = 0$ for some $b_y ' \in B_0$, and as $B_0$ is an integral domain and $A$ an algebra, we have $\sum_{J \subseteq R} \sum_{y \in Y_J} b_y ' a_y O_J E_J = 0.$)

(2) Use a similar argument.

We would like to express $A$ as a direct sum of left ideals of the form $A O_J E_J$ or $A E_J O_J$. Notice that for all
$y \in Y_J$, $a \cdot y \in \text{span}(e_J)$, and so
\dim \text{span}(e_J) \geq |Y_J|$ and $\dim \text{span}(e_J) \geq |Y_J|$. If we can show
that $\dim \text{span}(e_J) = |Y_J|$ and $\dim \text{span}(e_J) = |Y_J|$ for all $J \subseteq R$,
then \{a \cdot y : y \in Y_J\} is a basis of $\text{span}(e_J)$ and
\{a \cdot e_J : y \in Y_J\} is a basis of $\text{span}(e_J)$; then by (5.13)
we have $A = \sum_{J \subseteq R} \text{span}(e_J)$ and $A = \sum_{J \subseteq R} \text{span}(e_J)$. We begin by
examining $\text{span}(e_J)$ and $\text{span}(e_J)$ for all $J \subseteq R$.

(5.14) THEOREM: $A = \{a \in A : a \cdot \{w_j\} = 0 \text{ for all } w_j \in J\}$
= \{a \in A : a \cdot \{w_j\} = a \text{ for all } w_j \in J\}.

$\text{span}(e_J)$ has basis \{a \cdot e_J : y \in X_J\} and dimension $|X_J|$. Further,
\{a \cdot e_L : y \in Y_L, J \subseteq L\} is a basis of $\text{span}(e_J)$ and
$\text{span}(e_J) = \sum_{J \subseteq L} \text{span}(e_L)$. 

Proof: Clearly, $\text{span}(e_J) \subseteq \{a \in A : a \cdot \{w_j\} = 0 \text{ for all } w_j \in J\}$.
Conversely, suppose $a \in A$ satisfies $a \cdot \{w_j\} = 0 \text{ for all } w_j \in J$.
Then $a \cdot w_j = u \cdot a \text{ for all } w_j \in J$, and so $a = \sum \frac{1}{f(u)} b(w_j)$.
But $a \cdot w_j = u \cdot a \text{ for all } w \in Y_J$, and so $a \cdot w_j = a$. Then
$a \in \text{span}(e_J)$, and $\text{span}(e_J) = \{a \in A : a \cdot \{w_j\} = 0 \text{ for all } w_j \in J\}$.

Clearly also, $\text{span}(e_J) \subseteq \{a \in A : a \cdot \{w_j\} = a \text{ for all } w_j \in J\}$.
$a \cdot \{w_j\} = a \text{ gives } a(1 + a \cdot w_j) = (u \cdot J + 1) a$, and thus $a \cdot w_j = u \cdot a$, so as before, $\text{span}(e_J) = \{a \in A : a \cdot \{w_j\} = a \text{ for all } w_j \in J\}$.

Let $a = \sum b(w_j)$ for all $w_j \in J$. Then $a \cdot \{w_j\} = 0$ gives
$\sum b(w_j) (u \cdot J - a \cdot w_j) = 0$. So
Since \{a_w : w \in W\} is a basis of A, the coefficient of each \(a_w\) is zero. Suppose \(w(r_j) > 0\); the coefficient of \(a_w\) is

\[ u^c J b_w = u^c J b_{ww_j} \]

As this is zero, \(b_w = b_{ww_j}\). Similarly if \(w(r_j) < 0\), the coefficient of \(a_w\) gives us that \(b_w = b_{ww_j}\).

Let \(x \in X_j\) and \(w \in W_j\). Then writing \(w = w_{i_1} \cdots w_{i_t}\),

\[ l(w) = t, w_{i_j} \in J \text{ for all } j \]

we see that

\[ b_x = b_{xw_{i_1}} = b_{xw_{i_1}w_{i_2}} = \cdots = b_{xw}. \]

Then \(a = \sum f(u)_j b_{a x e_j}\). Conversely, each \(a x e_j \in A e_j\), \(x \in X_j\)

and as \(\{a x e_j : x \in X_j\}\) is linearly independent, it is a basis of \(A e_j\). Thus \(\text{dim } A e_j = |X_j|\).

Finally, for all \(L \supseteq J\) we have \(A \cap e_L \subseteq A e_j\), and so \(\sum A \cap e_L \subseteq A e_j\). Moreover, \(\{a y e_L : y \in Y_L, L \supseteq J\}\) is a set of linearly independent elements in \(\sum A \cap e_L\), so also in \(A e_j\). Since \(\text{dim } A e_j = |X_j|\), this set is also a basis of \(A e_j\), and so we must have \(A e_j = \sum A \cap e_L\).

(5.15) \textbf{THEOREM:} \(A o_j = \{a \in A : a e\{w_j\} = 0 \text{ for all } w_j \in J\}\)

\[ = \{a \in A : a o\{w_j\} = a \text{ for all } w_j \in J\} \]

\(A o_j\) has basis \(\{a y o_j : y \in X_j\}\) and dimension \(|X_j|\). Further \(\{a y e L : y \in Y_L, L \supseteq J\}\) is a basis of \(A o_j\) and \(A o_j = \sum A e L\).
Proof: Clearly \( \text{Ao}_J \subseteq \{ a \in A : ae(w_j) = 0 \text{ for all } w_j \in J \} \).

Conversely, suppose \( a \in A \) satisfies \( ae(w_j) = 0 \) for all \( w_j \in J \).

Then \( a_{w_j} = -a \) for all \( w_j \in J \), and so \( a_{w} = (-1)^{l(w)}a \) for all \( w \in \mathcal{W}_J \). Thus \( \text{ao}_J = a \), and so \( a \in \text{Ao}_J \). Thus \( \text{Ao}_J = \{ a \in A : ae(w_j) = 0 \text{ for all } w_j \in J \} \).

Similarly we can show that \( \text{Ao}_J = \{ a \in A : ao(w_j) = a \text{ for all } w_j \in J \} \).

Now let \( a = \sum_{w \in \mathcal{W}_J} b_w a_w \in \text{Ao}_J \). Let \( w_j \in J \); then \( ae(w_j) = 0 \) gives \( \sum_{w \in \mathcal{W}_J} b_w a_w (1 + a_{w_j}) = 0 \). So

\[
\sum_{w \in \mathcal{W}_J} b_w a_w + \sum_{w \in \omega \mathcal{W}_J} b_w a_{w_w} j + \sum_{w \in \mathcal{W}_J} b_w (u^j a_{w_j} j + (u^j-1)a_w) = 0.
\]

Since \( \{a_w : w \in \mathcal{W}_J\} \) is a basis of \( A \), the coefficient of each \( a_w \) in the above expression is zero. Suppose \( w(r_j) > 0 \); the coefficient of \( a_w \) is \( b_w + u^j b_{w_{w_j}} = 0 \). Thus \( b_w = -u^j b_{w_{w_j}} \), \( \text{when } w(r_j) > 0 \). If \( w(r_j) < 0 \), the coefficient of \( a_w \) is \( b_w + b_{w_{w_j}} + b_w (u^j-1) = b_{w_{w_j}} + u^j b_w = 0 \).

Let \( x \in X_J \) and \( w \in W_J \), with \( w = w_{i_1} \ldots w_{i_t} \), \( l(w) = t \), \( w_{i_j} \in J \), and then:

\[
b_x = -u_{i_1} c_{i_1} b_{x_{w_{i_1}}} = u_{i_1} c_{i_2} b_{x_{w_{i_1} w_{i_2}}} = \ldots = (-1)^{l(w)} u^c_{w} b_{x_{w}}.
\]

Hence \( a = \sum_{x \in X_J} f(u)_x b_x a_x \in \text{ao}_J \), and conversely each \( a_x \in \text{ao}_J \) for all \( x \in X_J \). As \( \{ a_x \in \text{ao}_J : x \in X_J \} \) are linearly independent, they are a basis of \( \text{ao}_J \), and \( \dim \text{ao}_J = |X_J| \).

Finally, for all \( L \supseteq J \) we have \( \text{Ae}_L \subseteq \text{ao}_J \), and
so $\sum_{J \subseteq L} A_{\mathcal{L}, J} \subseteq A_{\mathcal{J}}$. Moreover, $\{a_y e_{\mathcal{L}, J} : y \in Y_L, L \supseteq J\}$ is a set of linearly independent elements in $\sum_{J \subseteq L} A_{\mathcal{L}, L}$, so also in $A_{\mathcal{J}}$. Since $\dim A_{\mathcal{J}} = |Y_J|$, this set must also be a basis of $A_{\mathcal{J}}$, and hence $A_{\mathcal{J}} = \sum_{J \subseteq L} A_{\mathcal{L}, L}$.

(5.16) Theorem: (1) For all $J \subseteq R$, $A_{\mathcal{J}} = \sum_{J \subseteq L} A_{\mathcal{L}, L}$.

(2) For all $J \subseteq R$, $A_{\mathcal{J}} = \sum_{J \subseteq L} A_{\mathcal{L}, L}$.

Proof: Let $a \in A$, then $a e_{\mathcal{J}, J} \subseteq A_{\mathcal{J}, J} \subseteq A_{\mathcal{J}}$. Since $\{a_y e_{\mathcal{L}, J} : y \in Y_L, L \supseteq J\}$ forms a basis of $A_{\mathcal{J}, J}$, there exist elements $b_y \in B_0$ such that

$$a e_{\mathcal{J}, J} = \sum_{J \subseteq L} \sum_{y \in Y_L} b_y a_y e_{\mathcal{L}, L}.$$ 

Then $a e_{\mathcal{J}, J} \cap = \sum_{J \subseteq L} \sum_{y \in Y_L} b_y a_y e_{\mathcal{L}, L} = \sum_{J \subseteq L} b_y a_y e_{\mathcal{L}, L}$ since $L \cap J \neq \emptyset$ if $J \subseteq L$. Hence $\dim A_{\mathcal{J}, J} \subseteq |Y_J|$. By a similar argument, $\dim A_{\mathcal{J}, J} \subseteq |Y_J|$.

Now $A_{\mathcal{J}} = \sum_{J \subseteq L} A_{\mathcal{L}, L}$, and so $A_{\mathcal{J}, J} = (\sum_{J \subseteq L} A_{\mathcal{L}, L}) e_{\mathcal{J}, J}$. Thus $A_{\mathcal{J}, J} \subseteq \sum_{J \subseteq L} A_{\mathcal{L}, L} e_{\mathcal{J}, J} = A_{\mathcal{J}, J}$. Since $B_0$ is a principal ideal domain,

$$\dim A_{\mathcal{J}, J} \subseteq \dim A_{\mathcal{J}, J} \subseteq |Y_J|.$$ 

But we have previously shown $\dim A_{\mathcal{J}, J} \supseteq |Y_J|$, and so $\dim A_{\mathcal{J}, J} = |Y_J|$. Hence $\{a_y e_{\mathcal{J}, J} : y \in Y_J\}$ is a basis of $A_{\mathcal{J}, J}$. Similarly, $\{a_y e_{\mathcal{J}, J} : y \in Y_J\}$ is a basis of $A_{\mathcal{J}, J}$. Thus

$A_{\mathcal{J}} = \sum_{J \subseteq L} A_{\mathcal{L}, J} e_{\mathcal{L}, L}$, and $A_{\mathcal{J}} = \sum_{J \subseteq L} A_{\mathcal{L}, L}$.
(5.17) COROLLARY: \( \text{Aeo}_{e_{J}} \cong \text{Ae}_{J}/ \bigoplus_{J \subseteq L} \text{Aeo}_{L} \) and

\( \text{Ae}_{J} \cong \text{Ae}_{J}/ \bigoplus_{J \subseteq L} \text{Aeo}_{L} \) as left \( \text{A} \)-modules for all \( J \subseteq R \).

Proof: As \( \text{Ae}_{J} = \bigoplus_{J \subseteq L} \text{Aeo}_{L} \), define the left \( \text{A} \)-module homomorphism \( f: \text{Ae}_{J} \to \text{Aeo}_{e_{J}} \) by projection. Clearly \( f \) is onto, and \( \ker f = \bigoplus_{J \subseteq L} \text{Aeo}_{L} = \bigoplus_{J \subseteq L} \text{Aeo}_{L} \). Also as \( \text{Aeo}_{J} = \bigoplus_{J \subseteq L} \text{Aeo}_{L} \), we can define a left \( \text{A} \)-module homomorphism \( g: \text{Aeo}_{J} \to \text{Aeo}_{e_{J}} \) by projection, and so the result follows.

(5.18) COROLLARY: (1) \( A = \bigoplus_{J \subseteq R} \text{Aeo}_{e_{J}} \) where \( \text{Aeo}_{e_{J}} \) has basis \( \{ a_{y}o_{J}^{e} : y \in Y_{J} \} \) and dimension \( |Y_{J}| \).

(2) \( A = \bigoplus_{J \subseteq R} \text{Aeo}_{J} \) where \( \text{Aeo}_{J} \) has basis \( \{ a_{y}e_{J}^{o} : y \in Y_{J} \} \) and dimension \( |Y_{J}| \).

Proof: Follows from (5.16) as in its proof we got that \( \dim \text{Aeo}_{e_{J}} = |Y_{J}| \) and \( \dim \text{Aeo}_{J} = |Y_{J}| \).

Note: Let \( K \) be any extension ring of \( B_{0} \), and let \( A_{K} = A_{K}(u) \) be the generic ring over \( K \). Then \( A_{K} \cong K \otimes_{B_{0}} A \), and we also have the decomposition of (5.18) for \( A_{K} \). In particular, if \( K = Q(u) \), then we have the decompositions of (5.18) of \( A_{K} \) as a direct sum of \( 2^{n} \) left ideals, \( n = |R| \).

Note also that the left ideals of \( A_{K} \) for any extension of \( B_{0} \) which occur in the decompositions given in (5.18) are not necessarily indecomposable left ideals.
(5.19) **COROLLARY:** For all $J \subseteq R$, $A_{\sigma J}e_J$ and $A_{\epsilon J}o_J$ are isomorphic $A$-modules.

**Proof:** By (5.16), $A_{\sigma J}e_Jo_J = A_{\epsilon J}o_J$ for all $J \subseteq R$ and $A_{\sigma J}e_Jo_J$ has basis $\{a_y o_{\sigma J}o_J : y \in Y_J\}$. Consider the $A$-module homomorphism $\Psi_J : A_{\sigma J}e_J \to A_{\sigma J}e_Jo_J$ given by right multiplication by $o_J$. If $\sum_{y \in Y_J} u_y a_y o_{\sigma J}o_J \in A_{\sigma J}e_Jo_J$, then $\Psi_J : \sum_{y \in Y_J} u_y a_y o_{\sigma J}o_J \to \sum_{y \in Y_J} u_y a_y o_{\sigma J}o_J$, and so $\Psi_J$ is onto. Moreover, as $\Psi_J(a_y o_{\sigma J}o_J) = a_y o_{\sigma J}o_J$ for all $y \in Y_J$, and the $\{a_y o_{\sigma J}o_J : y \in Y_J\}$ are a basis of $A_{\sigma J}e_Jo_J$, it follows that $\Psi_J$ is one-one. Hence $\Psi_J$ is an isomorphism of $A$-modules, and $A_{\sigma J}e_J \cong A_{\sigma J}e_Jo_J = A_{\epsilon J}o_J$ for all $J \subseteq R$.

(5.20) **REMARK:**

(1) Suppose $y \in Y_J$; then $y = (y^w o_J)w o_J$ with $y^w o_J \in X_J$ and $l(y) = l(y^w o_J) + l(w o_J)$. Now

$$a_{\sigma J}e_J = \frac{1}{f(u)_{\sigma J}} u c_{w o_J} a_{\sigma J} + \frac{1}{f(u)_{\epsilon J}} \sum_{w \in W_J, w \neq 1} (-1)^{l(w)} u c_{w o_J} a_{w o_J}.$$  

Then $a_{y J}e_J = a_{y w o_J} e_J u c_{w o_J}$ by (5.6)

$$= \frac{1}{f(u)_{y J}} \sum_{v \in W_J} u c_{w o_J} a_{y w o_J} v \quad \text{where for all}$$

$v \in W_J$, $l(y w o_J v) = l(y w o_J) + l(v)$.

Also, for any $w \in W_J$, $w \neq 1$, $a_{y w e_J} = \sum_{w', k_{w'}} a_{w}$, where $k_{w'} \in B_0$, summed over certain $w' \in W$ with

$$l(w') \geq l(y) + 1 - l(w o_J) > l(y w o_J).$$

Hence $a_{\sigma J}e_J = \frac{1}{f(u)_{\sigma J} f(u)_{\epsilon J}} u c_{w o_J} a_{y w o_J} + (\text{a sum of certain}$. 
Thus it follows that \( \{a_y e_J^y : y \in Y_J^y \} \) are a set of linearly independent elements in \( A_0 e_J^y \), and so are a basis of \( A_0 e_J^y \) since \( |Y_J^y| = |Y_J| \) for all \( J \subseteq R \).

(2) Similarly, if \( y \in Y_J^y \), then

\[
a_y e_J^y = a_y w_{oJ}^c e_J^y,
\]

and so

\[
a_y e_J^y = \frac{1}{f(u)_j f(u)_{\wedge}} w_{oJ}^c (-1)^{l(w_o^\wedge)} a_y w_{oJ}^\wedge + ( \text{a linear combination of certain terms } a_w^c \text{ with}
\]

\[
l(w) < l(y w_o^\wedge) = l(y) + l(w_o^\wedge).
\]

Thus \( \{a_y e_J^y : y \in Y_J^y \} \) are also a basis of \( A e_J^y \) for all \( J \subseteq R \).
(5A) Some Specialisations of $K = Q(u)$.

For any extension ring $K'$ of $\mathcal{O} = Q[u]$, let $A_{K'} = A_K(u)$ be the generic ring of the system $S$ of finite groups with $(B,N)$ pairs of type $(W,R)$ over $K'$, with identity $a_1 = 1$ and basis $\{a_w : w \in W\}$ as in (3.4.2). Let $K = Q(u)$. Define the idempotents $e_J$, $\omega_J$ for each $J \subseteq R$ as in (5.5), and then by (5.18) and the note after it, we have the following two decompositions of $A_K$ into direct sums of left ideals:

$$A_K = \bigoplus_{J \subseteq R} A_K e_J$$

and

$$A_K = \bigoplus_{J \subseteq R} A_K \omega_J$$

where $A_K e_J$ has dimension $|Y_J|$ and basis $\{a_y e_J : y \in Y_J\}$ for all $J \subseteq R$, and $A_K \omega_J$ has dimension $|Y_J|$ and basis $\{a_y \omega_J : y \in Y_J\}$ for all $J \subseteq R$.

Now for all $J \subseteq R$, and for all $y \in Y_J$, the element $a_y \omega_J$ has the form $\frac{1}{f(u)_J f(u)}$ (an element of $A_\mathcal{O}$) and the element $a_y e_J$ has the form $\frac{1}{f(u) \omega_J f(u)}$ (an element of $A_\mathcal{O}$).

We say $a_y \omega_J$ and $a_y e_J$ are defined in $A_{K'}$, for any extension ring $K'$ of $\mathcal{O}$ if $\frac{1}{f(u)_J}$ and $\frac{1}{f(u) \omega_J}$ are both elements of $K'$. If $K'$ is an extension ring of $\mathcal{O}$ which is contained in $K$ such that $\{a_y \omega_J : y \in Y_J, J \subseteq R\}$ are all defined in $A_{K'}$, then they are linearly independent over $K'$, and $A_{K'}$ has a decomposition $A_{K'} = \bigoplus_{J \subseteq R} A_{K'} e_J$, where $A_{K'} e_J$ has dimension $|Y_J|$ and basis $\{a_y e_J : y \in Y_J\}$. Note that $\{a_y \omega_J : y \in Y_J, J \subseteq R\}$ are all defined in $A_K$, if and only
if \( \{a_y e_J^J : y \in Y_J, J \subseteq R \} \) are all defined in \( A_K \). Hence \( A_K \) also has the decomposition \( A_K = \sum_{J \subseteq R} A_K e_J^J \) where \( A_K e_J^J \) has dimension \( |Y_J| \) and basis \( \{a_y e_J^J : y \in Y_J \} \) for all \( J \subseteq R \).

**Definition:** Suppose \( \mathcal{O} \triangleleft K' \subseteq K \), and \( \{a_y e_J^J : y \in Y_J, J \subseteq R \} \) are defined in \( A_K' \). Then we say \( A_K \) has the Solomon decomposition property, SDP.

Let \( P \) be a prime ideal of \( \mathcal{O} \), and let \( K_P = \{a/b : a \in \mathcal{O}, b \in \mathcal{O}-P\} \). Then \( K_P \) is a subring of \( K \) which contains \( \mathcal{O} \). A specialisation \( F \) of \( K \) with nucleus \( P \) is a ring homomorphism \( F:K_P \to \mathbb{C} \) with kernel \( PK_P \) and image \( k_o \), a subfield of \( \mathbb{C} \). \( F \) induces a ring epimorphism \( F:A_K \to A_{k_o}(F(u)) \). (see (3.4.4)).

Now let \( F \) be a specialisation of \( K \) with nucleus \( P \) and image \( k_o \). Suppose \( \frac{1}{f(u)} \in K_P \) for all \( J \subseteq R \). Then \( A_{K_P} \) has the SDP, and we have the decompositions

\[
A_{K_P} = \sum_{J \subseteq R} A_{K_P} e_J^J \quad \text{and} \quad A_{K_P} = \sum_{J \subseteq R} A_{K_P} e_J^J
\]

where \( A_{K_P} e_J^J \) has dimension \( |Y_J| \) and basis \( \{a_y e_J^J : y \in Y_J \} \)

and \( A_{K_P} e_J^J \) has dimension \( |Y_J| \) and basis \( \{a_y e_J^J : y \in Y_J \} \)

for all \( J \subseteq R \). Now \( F \) induces a ring epimorphism \( F:A_K \to A_{k_o} \) (where \( F(u) = c \in \mathbb{C} \). Let \( F(e_J) = E_J \) and \( F(o_J) = 0_J \). \( E_J \) and \( o_J \) are idempotents as \( F \) is a ring epimorphism. Further, \( F \) induces module epimorphisms \( F_J \) and
and $F_J'$ as follows:

$$F_J: A_k \circ e_J \to A_k (c) E_J$$

and

$$F_J': A_k \circ e_J o_J \to A_k (c) E_J o_J$$

where $a \in A_k$ for all $J \subseteq R$. Since $F$ is an epimorphism $F: A_k \to A_k (c)$, it follows that

$$A_k (c) = \sum_{J \subseteq R} A_k (c) E_J$$

and

$$A_k (c) = \sum_{J \subseteq R} A_k (c) E_J o_J$$

where $A_k (c) E_J$ and $A_k (c) E_J o_J$ are left ideals of $A_k (c)$ of dimension $|Y_J|$, with bases $\{a_y E_J : y \in Y_J\}$ and $\{a_y E_J o_J : y \in Y_J\}$ respectively.

**Examples:**

(1) Let $P = u \delta$ and let $F: K \to C$ be the $Q$-linear map defined by $F(u) = 0$. Then the image of $F$ is $Q$, and $F$ induces an epimorphism $F: A_k \to A_k (0) = H_Q$, the $0$-Hecke algebra of type $(W, R)$ over $Q$. If $E_J$ and $O_J$ are the idempotents defined in (4.4.1), then we see that $F(e_J) = E_J$ and $F(o_J) = O_J$ for all $J \subseteq R$, and the resulting decompositions of $H_Q$ which we obtain are the same as obtained in (4.4.12).

(2) Let $P = (u-1) \delta$ and let $F: K \to C$ be the $Q$-linear map defined by $F(u) = 1$. Then the image of $F$ is $Q$, and $F$ induces an epimorphism $F: A_k \to A_k (1) = QW$. For all $J \subseteq R$,

$$F(e_J) = \frac{1}{|\mathcal{W}_J|} \sum_{w \in \mathcal{W}_J} w$$

and

$$F(o_J) = \frac{1}{|\mathcal{W}_J|} \sum_{w \in \mathcal{W}_J} (-1)^{l(w)} w,$$

which are the idempotents defined in (1.4.1). Then, letting
\( A = QW, \) we have the decompositions

\[
A = \sum_{J \subseteq R} A_{0 \gamma} E_J \quad \text{and} \quad A = \sum_{J \subseteq R} A_{E_J} O_J
\]

where \( A_{0 \gamma} E_J \) and \( A_{E_J} O_J \) are left ideals of \( A \), of dimension \( |Y_J| \) and bases \( \{y_{0 \gamma} E_J : y \in Y_J\} \) and \( \{y_{E_J} O_J : y \in Y_J\} \) respectively. Note that the second decomposition is the one given in (1.4.2).

(3) For any \( q \in \mathcal{P} \), let \( P = (u-q)\sigma \). Let \( F:K_{p} \to C \) be the \( Q \)-linear map defined by \( F(u) = q \). Then the image of \( F \) is \( Q \), and \( F \) induces a ring epimorphism \( F:K_{p} \to A_{Q}(q) \cong H_{Q}(q) \).

The idempotents \( E_J \) and \( O_J \) in \( H_{Q}(q) \) are as follows:

\[
E_J = \frac{1}{|G_J(q):B(q)|} \sum_{w \in \mathcal{W}_J} h_w
\]

\[
O_J = \frac{1}{|G_J(q):B(q)|} \sum_{w \in \mathcal{W}_J} (-1)^{l(w)} q^{l(w_{OJ} w)} h_w
\]

We thus have the decompositions \( H_{Q}(q) = \sum_{J \subseteq R} H_{Q}(q)_{0 \gamma} E_J \) and \( H_{Q}(q) = \sum_{J \subseteq R} H_{Q}(q)_{E_J} O_J \), where for all \( J \subseteq R \), \( H_{Q}(q)_{0 \gamma} E_J \) and \( H_{Q}(q)_{E_J} O_J \) are left ideals of \( H_{Q}(q) \) of dimension \( |Y_J| \) and bases \( \{h_{0 \gamma} E_J : y \in Y_J\} \) and \( \{h_{E_J} O_J : y \in Y_J\} \) respectively.
(5B) **Decomposition Numbers of** $H_0$.

The algebra $H_0$ defined in (3.5) is not semi-simple, and \( \{ U_J = H_0^J e_J : J \subseteq R \} \) is a full set of the isomorphism classes of principal indecomposable $H_0$-modules. Also, \( \{ U_J^f = U_J / \text{rad} U_J : J \subseteq R \} \) is a full set of isomorphism classes of the irreducible $H_0$-modules. The Cartan matrix $C$ of $H_0$ is defined in (4.5), and if $C = (c_{JL})$, then $c_{JL}$ is the number of times $U_J^f$ occurs as a composition factor of $U_J$.

**Notation:**
- $\sigma = Q[u]
- \mathbb{K}_0 = Q(u)
- K = \text{a finite field extension of } \mathbb{K}_0 \text{ which is a splitting field for } A_K = A_K(u)
- I = \text{integral closure of } \sigma \text{ in } K
- \mathbb{k}_0 = \text{subfield of } C
- A_B = A_B(u) \text{ is the generic ring of the system } S \text{ of finite groups with } (B, N) \text{ pairs of type } (W, R) \text{ over } B, \text{ where } B \text{ is any extension ring of } \sigma.
- f_o = \text{a specialisation of } K \text{ with nucleus } P \text{ and range } \mathbb{k}_o \text{ such that } f_o(u) = 0; f_o \text{ induces a ring epimorphism } f_o': A_K^K \rightarrow A_{\mathbb{k}_o}(0) = H_0$

(5B.1) **THEOREM** (Dornhoff [12], no. 48.1(iv)): If $M$ is any finitely generated $A_K$-module, then $M \cong X_K$ for some finitely
generated $K_p$-free $A_{K_p}$-module $X$, where $X_K = K \otimes_{K_p} X$.

**Proof:** Since $I$ is a Dedekind domain, $K_p$ is a principal ideal domain. Let $M$ have $K$-basis $m_1, \ldots, m_n$ and let $\{a_w : w \in \mathcal{W}\}$ be a $K_p$-basis of $A_{K_p}$. Then $\{1_K \otimes a_w : w \in \mathcal{W}\}$ is a $K$-basis of $A_K = K \otimes_{K_p} A_{K_p}$. Let $X$ be the $A_K$-submodule of $M$ generated by all $(1_K \otimes a_w)m_i$. Then $K \otimes_{K_p} X = M$. Since $M$ is $K_p$-torsion free, so is $X$; $K_p$ is a principal ideal domain, and so $X$ is $K_p$-free.

(5B.2) **Definition:** Let $\{X_i\}_{i=1}^s$ be a set of $K_p$-free $A_{K_p}$-modules such that $\{(X_i)_K\}$ are a set of irreducible $A_K$-modules.

(5B.3) **Definition:** The decomposition matrix of $H_0$, $D = (d_{ij})$, is an $s \times 2^n$ matrix (where $n = |R|$) with entries $d_{ij}$, $1 \leq i \leq s$, $J \subseteq R$, defined by:

$$d_{ij} = \text{the number of times } M_J \text{ occurs as a composition factor of } f_0'(X_i).$$

Since $P \cap \mathcal{C} = u\mathcal{C}$, we can define idempotents $e_J$ and $o_J$ in $A_{K_p}$ as follows:

$$e_J = \frac{1}{f(u)_J} \sum_{w \in \mathcal{W}_J} a_w$$

(5B.4)

$$o_J = \frac{1}{f(u)_J} \sum_{w \in \mathcal{W}_J} (-1)^{l(w)} u^{c_{WN,0J}} a_w$$

with notation as in (5.5). For all $J \subseteq R$, define the $A_{K_p}$-module $(U_J)_{K_p} = A_{K_p} o_J e_J$
and the $A_K$-module $(U_J)_K = A_K e_J$. We see that

$$(U_J)_K = K \theta_{K_P}(U_J)_{K_P}.$$ 

(5B.5) **LEMMA** (Dornhoff [12], 46.1): Let $A$ be any ring, $e$ an idempotent in $A$, and $V$ an $A$-module. Define

$$f : eV \rightarrow \text{Hom}_A(Ae, V)$$

by $f(v)(ae) = ae v$ for all $v \in eV, a \in A$. Then

1. $f$ is an isomorphism of additive groups.
2. if $A$ is an $R$-algebra for the commutative ring $R$, then $f$ is an $R$-isomorphism.

**Proof:** Clearly $f$ is a group homomorphism. If $f(v) = 0$ for some $v \in V$, then $0 = f(v)e = ev = v$ so $f$ is one-one. If $h \in \text{Hom}_A(Ae, V)$, $h(e) = h(e^2) = e h(e)$, so $h(e) \in eV$. Then $f(h(e)) = h$ so $f$ is onto.

(5B.6) **THEOREM** (Dornhoff [12], 48.4): Let $U_J$ be a principal indecomposable $H_o$-module, and $V$ any finitely generated $K_p$-free $A_{K_P}$-module. Then

$$\dim_{K_P}\text{Hom}_{A_{K_P}}((U_J)_{K_P}, V) = \dim_{K_p}\text{Hom}_{H_o}(U_J, f_0(V)) = \dim_{K_p}\text{Hom}_{A_K}(U_J, V)_K$$

where $V_K = K \theta_{K_P} V$, and if $v_1, \ldots, v_m$ are a $K_p$-basis of $V$, then

$$f_0(V) = \{ \sum_{i=1}^m f_0(k_i)v_i : k_i \in K_p, \sum_{i=1}^m k_i v_i \in V \}.$$ 

**Proof:** There exist idempotents $E_J \in H_o, E_J, K, K_P \in A_{K_P}$ and $E_J, K \in A_K$ such that $U_J = H_o E_J$, $(U_J)_{K_P} = A_{K_P} E_J, K_P$ and

$$(U_J)_K = A_K E_J, K.$$ Then $\text{Hom}_{A_{K_P}}(A_{K_P} E_J, K_P, V) \cong E_J, K_P V$, a finitely
generated $K_P$-free $K_P$-module. Let $d = \dim_{K_P} \text{Hom}_{K_P}(A_{K_P} E_J, K_P^e, V)$.

Now $\text{Hom}_{K_P}(A_{K_P} E_J, K_P^e, V) \cong \text{Hom}_{K_P}(A_{K_P}(1 \otimes E_J, K_P^e), V_K) \\
\cong (1 \otimes E_J, K_P^e) V_K \cong K_P \otimes_{K_P} E_J, K_P^e V$.

Since $E_J, K_P^e V$ is $K_P$-free, this last has dimension $d$.

Finally, $\text{Hom}_{K_P}(H_O E_J, f_0(V)) \cong E_J f_0(V) \cong f_0(E_J, K_P^e V)$ and this last has dimension $d$ over $k_0$.

(5B.7) LEMMA (Dornhoff [12], 48.5): Let $A$ be any ring, $P$ a projective $A$-module, and assume that

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

is an exact sequence of $A$-modules. Define

$$a^*: \text{Hom}_A(P, L) \rightarrow \text{Hom}_A(P, M) \text{ by } a^*(f) = a \cdot f$$

$$b^*: \text{Hom}_A(P, M) \rightarrow \text{Hom}_A(P, N) \text{ by } b^*(g) = b \cdot g$$

Then $0 \rightarrow \text{Hom}_A(P, L) \xrightarrow{a^*} \text{Hom}_A(P, M) \xrightarrow{b^*} \text{Hom}_A(P, N) \rightarrow 0$ is an exact sequence of abelian groups.

(5B.8) THEOREM (Dornhoff [12], 48.6): Let $U_J = H_O E_J$ for some idempotent $E_J \in H_O$. Let $V$ be a finitely generated $H_O$-module, and assume $K_L$ occurs $n_L$ times as a composition factor of $V$. Then $n_L = \dim_{K_O} E_L V$.

Proof: Since $V$ is a finitely generated $H_O$-module, there exists a sequence of submodules

$$0 = V_{i_{n+1}} < V_{i_n} < \ldots < V_{i_1} < V_{i_0} = V$$

such that $V_{i_j}$ is a $H_O$-module for each $j, 0 \leq j \leq n+1$, and $V_{i_j}/V_{i_{j+1}} \cong M_{i_j}$, an irreducible $H_O$-module. Then we
have an exact sequence of $H_0$-modules

$$0 \to V_{i+1} \to V_i \to M_i \to 0.$$

Then for each $J \subseteq R$,

$$0 \to \text{Hom}_{H_0}(U_J, V_{i+1}) \to \text{Hom}_{H_0}(U_J, V_i) \to \text{Hom}_{H_0}(U_J, M_i) \to 0$$

is an exact sequence of abelian groups. In particular,

$$\dim_{k_0} \text{Hom}_{H_0}(U_J, V_i) - \dim_{k_0} \text{Hom}_{H_0}(U_J, V_{i+1}) = \dim_{k_0} \text{Hom}_{H_0}(U_J, M_i).$$

So

$$\dim_{k_0} \text{Hom}_{H_0}(U_J, V) = \sum_j \dim_{k_0} \text{Hom}_{H_0}(U_J, M_j) = \sum_i n_i \dim_{k_0} \text{Hom}_{H_0}(U_J, M_i).$$

Now if $J \neq L$, $\text{Hom}_{H_0}(U_J, M_L) = 0$, and if $J = L$, then

$$\dim_{k_0} \text{Hom}_{H_0}(U_J, M_J) = 1.$$ Hence

$$\dim_{k_0} \text{Hom}_{H_0}(U_J, V) = \dim_{k_0} E_J V = n_J.$$

(5B.9) **Lemma** (Dornhoff [12], 48.8(i)): $(U_J)_K = \sum_i a_{iJ}(X_i)_K$

**Proof:** Since $A_K$ is semi-simple,

$$(U_J)_K = \sum_{i=1}^n a_{iJ}(X_i)_K,$$

where the $a_{iJ} \in \mathbb{Z}$.

Since $K$ is a splitting field for $A_K$,

$$\text{Hom}_{A_K}((X_i)_K, (X_j)_K) = \begin{cases} K & \text{if } i = j \\ (0) & \text{if } i \neq j \end{cases}$$

and so

$$a_{iJ} = \dim_{k_0} \text{Hom}_{A_K}((U_J)_K, (X_i)_K) = \dim_{k_0} \text{Hom}_{H_0}(U_J, f_0(X_i)) \text{ by (5B.6)}$$

$$= \dim_{k_0} E_J f_0(X_i) = d_{iJ}. $$
THEOREM (Dornhoff [12], 48.8(ii)): \( C = D^t D \), where \( D^t \) is the transpose of \( D \).

**Proof:** By definition of \( c_{iJ} \), \( c_{iJ} = \dim_k k^i \) \( E_i U_J \). Thus

\[
c_{iJ} = \dim_k \text{Hom}_{H_o} (U_i U_J) \quad \text{by (5B.5)}
\]

\[
= \dim_k \text{Hom}_{A_K} ((U_i)_K, (U_J)_K) \quad \text{by (5B.6)}
\]

\[
= \dim_k \text{Hom}_{A_K} (\sum_i d_i E_i(X_i)_K, \sum_j d_j E_j(X_j)_K) \quad \text{by (5B.9)}
\]

\[
= \sum_i d_i d_j.
\]

**EXAMPLE:** Let \( H_o \) be of type \((\mathfrak{W}(A_3), \{w_1, w_2, w_3\})\). Then from Starkey [22], appendix 6, we have that the decomposition matrix \( D \) of \( H_o \) is as follows:

\[
D = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Then \( D^t D = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

which is the Cartan matrix of \( H_o \) given in Appendix 4, no. (3).
Chapter 6: THE RELATIVE STEINBERG MODULES OF A
FINITE GROUP WITH (B,N) PAIR.

Introduction:

Let $G(q)$ be a Chevalley group over the field $GF(q)$, where $q=p^n$ for some prime integer $p$, and $n > 0$. Steinberg [23] showed that $G(q)$ has a remarkable irreducible character of degree equal to a power of $q$. Later, Curtis [8] showed that any finite group $G$ with a (B,N) pair has an irreducible character $\mu$ which is equal to Steinberg's in the case $G = G(q)$, and $\mu$ may be written as an alternating linear combination of characters induced from characters of parabolic subgroups of $G$:

$$\mu = \sum_{J \subseteq R} (-1)^{|J|} (1^G_J)$$

where $1^G_J$ is the principal character of the parabolic subgroup $G_J$ which corresponds to the subset $J$ of $R$. Solomon [21] shows that this formula has a homological source when $|R| \geq 2$, i.e. that $\mu$ corresponds to the representation of $G$ on a 'homology module' for a particular simplicial complex.

(6.1) The Relative Steinberg Modules of a Finite Group
with a (B,N) Pair.

Definition: The relative Steinberg character of type $J$ of a finite group $G$ with a (B,N) pair, where $J \subseteq R$, is the
character
\[ \mu_J = \sum_{K \leq K \leq R} (-1)^{|K-J|} (1_{G_K})^G \]

where \( 1_{G_K} \) is the principal character of the parabolic subgroup \( G_K \) of \( G \).

When \( |J| \geq 2 \), we will show that \( \mu_J \) corresponds to the representation of \( G \) on a 'homology module' for a subcomplex of the complex considered by Solomon above.

The Tits Complex of \( G \) (refer to Tits [26])

Let \( G \) be a finite group with \((B,N)\) pair of rank \( n \), and let \( G^1, \ldots, G^n \) be the maximal parabolic subgroups of \( G \) containing \( B \). (If \( R = \{w_1, \ldots, w_n\} \), then for each \( i = 1, \ldots, n \), let \( R_i = \{w_1, \ldots, \hat{w_i}, \ldots, w_n\} \), and set \( G^i = G_{R_i}^1 \) for all \( i \).) Let \( V^i \) be the collection of cosets \( gG^i \), for \( g \in G \), and let \( V = V^1 \cup V^2 \cup \ldots \cup V^n \). Then the Tits complex of \( G \) is a simplicial complex \( \Delta \) of dimension \( n-1 \) which has \( V \) as its set of vertices.

A collection \( S \) of vertices is a simplex of \( \Delta \) if and only if \( \cap v \) is non-empty. If \( S \) and \( S' \) are collections of vertices \( v \in S \) which are simplexes, we say \( S \) is a face of \( S' \) if every vertex of \( S \) is a vertex of \( S' \).

The Tits Complex of \( G \) with respect to \( G_J^r \)

Let \( G^1, \ldots, G^n \) be the maximal parabolic subgroups of \( G \) containing \( B \), numbered so that \( G^1, \ldots, G^r \) contain \( G_J \), but none of \( G^{r+1}, \ldots, G^n \) contain \( G_J \). (In this case, we have that \( J = \{w_1, \ldots, w_r\} \). Let \( V_J = V^1 \cup V^2 \cup \ldots \cup V^r \). Then the Tits
complex of $G$ with respect to $G_J$ is a simplicial complex $\Delta_J$ of dimension $r-1$ which has $V_J$ as its set of vertices.

A collection $S$ of vertices of $V_J$ is a simplex of $\Delta_J$ if and only if $\cap S$ is non-empty. If $S$ and $S'$ are collections of vertices of $V_J$ which are simplexes, then $S$ is a face of $S'$ if every vertex of $S$ is a vertex of $S'$.

Note that if $J = \emptyset$, $\Delta_{\emptyset} = \emptyset$. Moreover, for each $J \subseteq R$, $\Delta_J$ is a subcomplex of $\Delta$. In particular, if $J, K \subseteq R$, and $J \subseteq K$, then $\Delta_K$ is a subcomplex of $\Delta_J$.

Let $S = \{g_{i_0}^i \ldots g_{i_p}^i \}$, $1 \leq i_0 < i_1 < \ldots < i_p \leq r$ be a collection of vertices of $V_J$ which form a $p$-simplex $\sigma$ of $\Delta_J$. Write

$$\sigma = (g_{i_0}^i \ldots g_{i_p}^i), 1 \leq i_0 < i_1 < \ldots < i_p \leq r.$$ 

There is a natural $G$-action on $\Delta_J$ defined as follows:

if $\sigma = (g_{i_0}^i \ldots g_{i_p}^i)$ is a $p$-simplex of $\Delta_J$, $g \in G$,

define the $p$-simplex $g\sigma = (gg_{i_0}^i \ldots gg_{i_p}^i)$ of $\Delta_J$. $g\sigma$ is a $p$-simplex of $\Delta_J$ as $g \cap gg_{i_0}^i \ldots gg_{i_p}^i$ is non-empty as $\sigma$ is a $p$-simplex of $\Delta_J$.

Note that we have defined an ordering on the simplexes of $\Delta_J$ by insisting that the vertices of a simplex be written in the order above.

For each subset $L = \{w_{i_0}^i \ldots w_{i_p}^i\}$ of $\hat{J}$, define the standard $p$-simplex $\sigma_L$ as follows:

$$\sigma_L = (g_{i_0}^i \ldots g_{i_p}^i), 1 \leq i_0 < i_1 < \ldots < i_p \leq r.$$
(6.1.1) **Lemma**: Each p-simplex $\sigma$ of $\Delta_J$ is conjugate under $G$ to precisely one $\sigma_L$, for some $L \subseteq J$, $|L| = p+1$.

**Proof**: Let $\sigma = (g_0^i \ldots g_p^i)$, $1 \leq i_0 < i_1 < \ldots < i_p < r$, be any p-simplex of $\Delta_J$. Then there exists $g \in G$, $g \neq 0$, such that $g \in \bigcap_{j=0}^p g_j^i$. Then $gG_j^i \cap g_j^i \neq \emptyset$ for each $j$, and so $gG_j^i = g_j^i$ for all $j$. So $\sigma = (gG_0^i \ldots gG_p^i) = g(G_0^i \ldots G_p^i)$, i.e. $\sigma = g\sigma_L$, where $L = \{w_{i_0}, \ldots, w_{i_p}\}$.

Now suppose $\sigma = g_1\sigma_{L_1} = g_2\sigma_{L_2}$, with $g_1, g_2 \in G$, and $L_1, L_2 \subseteq J$. Then $\sigma_{L_1} = g\sigma_{L_2}$, where $g = g_1^{-1}g_2 \in G$. Thus, for each $w_i \in L_1$, there exists $w_j(i) \in L_2$ such that $G_i = gG_j(i)$. So $g \in G_j(i)$ as $1 \in G_i$; continuing in this way we get that $g \in \bigcap G_j$, and hence $g\sigma_{L_2} = \sigma_{L_2} = \sigma_{L_1}$. Hence the result.

(6.1.2) **Lemma**: (1) $\{g \in G : g\sigma_L = \sigma_L\} = G_L$.

(2) If $\sigma = g\sigma_L$, then $\{x \in G : \sigma = x\sigma_L\} = gG_L$.

**Proof**: (1) $g\sigma_L = \sigma_L$ if and only if $g \in \bigcap_{w_i \in L} G_i = G_L$.

(2) Let $\sigma = g\sigma_L$. Then $g\sigma_L = x\sigma_L$ if and only if $g^{-1}x = G_L$. By (1), $G_L = g^{-1}x\sigma_L$ if and only if $g^{-1}x \in G_L$, i.e. $g\sigma_L = x\sigma_L$ if and only if $x \in gG_L$.

(6.1.3) **Proposition**: There is a one-one correspondence between simplexes $\sigma$ of $\Delta_J$ and cosets $gG_K$ of $G$ with $J \subseteq K \subseteq R$ and $g \in G$, given by

$\sigma = g(G_0^i \ldots G_p^i) \rightarrow g(\bigcap_{j=0}^p G_j^i) = gG_K$, where $\hat{K} = \{w_{i_0}, \ldots, w_{i_p}\}$.
Definition: A simplex of dimension \( r-1 \) in \( \Delta_J \) is called a chamber. The chamber \( \mathcal{O}_J = (G^1, \ldots, G^r) \) is called the fundamental chamber.

In the remainder of this section, we will prove the following two main theorems:

(6.1.4) **Theorem 1:** Let \( \Delta_J \) be the Tits complex of a finite group \( G \) with a \((B,N)\) pair with respect to a parabolic subgroup \( G_J \), with \( |\hat{J}| = r \geq 2 \). Then the homology groups of \( \Delta_J \) with integral coefficients are:

\[
\begin{align*}
H_0(\Delta_J) &= \mathbb{Z} \\
H_1(\Delta_J) &= 0 \text{ for any } i \text{ with } 1 \leq i \leq r-2, \text{ or } i > r. \\
H_{r-1}(\Delta_J) &= \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} (t \text{ summands})
\end{align*}
\]

where \( t = \sum_{w \in Y_J} |B : B \cap B^{w^{-1}}| \). If \( \xi : W \to \{ \pm 1 \} \) is the alternating character of \( W \) and \( \mathcal{O}_J \) is the fundamental chamber of \( \Delta_J \), then the \((r-1)\)-chains

\[
\begin{align*}
z_1 &= \sum_{w \in W^*_J} \xi(w) y_1 w \mathcal{O}_J^r \\
\vdots \\
z_1 &= \sum_{w \in W^*_J} \xi(w) y_1 w \mathcal{O}_J^r
\end{align*}
\]

are cycles. If \( b_{li_1}, \ldots, b_{li_1} \) are coset representatives for

\[
\begin{align*}
\vdots \\
b_{li_1}, \ldots, b_{li_1}
\end{align*}
\]

in \( B \), then \( b_{li_1} z_1, \ldots, b_{li_1} z_1 \) are cycles which

\[
\begin{align*}
\vdots \\
b_{li_1} z_1, \ldots, b_{li_1} z_1
\end{align*}
\]
form a basis for $H_{r-1}(\Delta_J)$.

Note: If $G = G(q)$ is a finite Chevalley group, $t = \sum_{w \in Y_J} q^{l(w)}$.

(6.1.5) **THEOREM 2:** Let $\Delta_J$ be the Tits complex of a finite group $G$ with a $(B,N)$ pair with respect to a parabolic subgroup $G_J$, with $|J| \geq 2$. Then the action of $G$ on $\Delta_J$ defines a $Q[G]$-module structure on $H_{r-1}(\Delta_J) \otimes Q$ which affords the character

\[
\mu_J = \sum_{K \leq R} (-1)^{|K-J|} (1_{G_K})^G.
\]

We call the $G$-module $H_{r-1}(\Delta_J) \otimes Q$ the relative Steinberg module of type $J$.

(6.1.6) **LEMMA:** For each $K \leq R$,

\[
G = \bigcup_{w \in X_K} BwG_K
\]

where $X_K$ is defined as in (1.3.1).

(6.1.7) **COROLLARY:** Each simplex $\sigma$ of $\Delta_J$ has the form $bw_{\sigma_L}$ for a unique subset $L$ of $\hat{J}$ and a unique $w \in X^\cap_{L}^\sigma$.

Let $D_J$ denote the set of chambers of $\Delta_J$. We say $c_1, c_2 \in D_J$ are adjacent if $c_1 \cap c_2$ is a simplex of dimension $r-2$.

**Definition:** If $\sigma \in \Delta_J$, define $D_J(\sigma) = \{c \in D_J : \sigma < c\}$, where '$<$' means 'is a face of'.

**Definition:** For $\sigma \in \Delta_J$, define $l(\sigma) = \min_{w} l(w)$, over all $w \in W$ for which $\sigma$ can be written $\sigma = bw_{\sigma_L}$, some $L \subseteq \hat{J}$. 

It is easy to see that \( l(\sigma) = l(w) \) where \( \sigma = bw\sigma_L \), with \( L \subseteq \hat{J} \) and \( w \in X_L^\hat{J} \).

(6.1.8) **PROPOSITION:** Let \( \sigma = bw\sigma_L \in \Delta_j \), with \( w \in X_L^\hat{J} \), \( b \in B \), \( L \subseteq \hat{J} \). Then \( D_j(\sigma) = \{ b'w'\sigma_j' : w' \in X_j^\hat{L} \subseteq bwBw'G_j \} \), where \( X_j^\hat{L} = \{ w \in W_L^\hat{J} : w(\cap_j) \subseteq \phi^+ \} \).

This follows because if \( J \subseteq K \), if \( w \in X_K \) and \( b \in B \),
then \( bwG_K = \bigcup_{w' \in X_j^\hat{L}} bwBw'G_j \).

(6.1.9) **LEMMA** (Solomon [21]): Let \( \sigma \in \Delta = \Delta_\emptyset \). Then

(1) there exists a unique \( c_0 \in D(\sigma) = D_\emptyset(\sigma) \) such that \( l(c_0) < l(c) \) for all \( c \in D(\sigma) \).

(2) if \( c \in D(\sigma) \) and \( l(c) = l(c_0) + m \), then there exist \( c_0, c_1, \ldots, c_m = c \in D(\sigma) \) such that \( c_i, c_{i+1} \) are adjacent and \( l(c_i) = l(c_0) + i \) for all \( i, 0 \leq i \leq m-1 \).

**Proof:** (1) Suppose \( \sigma = bw\sigma_K \), \( w \in X_K^\hat{\sigma} \), for some \( K \subseteq R \). Then \( D(\sigma) = \{ b'w'\sigma_R : w' \in W_K^\hat{\omega}, b'w'B \subseteq bwBw'B \} \) and if \( w' \in W_K^\hat{\omega} \), \( l(w') = l(w) + l(w') \). Then \( c_0 = bw\sigma_R \) has the required properties.

(2) Suppose \( c \in D(\sigma) \). Then \( c = b'w\sigma_R \), for some \( b' \in B \), \( w' \in W_K^\hat{\omega} \), where \( b'w'B \subseteq bwBw'B \). Let \( w' = W_1 \ldots W_s \), \( l(w') = s \), and \( w_j \in R \) for each \( j \). Since \( w \in X_K^\hat{\sigma} \), \( w \in W_K^\hat{\omega} \), we have that \( l(ww') = l(w) + l(w') \), and so \( l(c) = l(c_0) + l(w') = l(c_0) + s \).

For \( j = 1, 2, \ldots, s \), set \( c_j = b'w_1 \ldots w_j \sigma_R \). Then each \( c_j \in D(\sigma) \), and \( l(c_j) = l(c_0) + j \). It remains to show \( c_j, c_{j+1} \) are adjacent for \( j = 0, 1, \ldots, s-1 \). Since \( w_1 \ldots w_j (r_{i+1} \ldots r_i) \in \phi^+ \),
set $K = \mathbb{R} - \{w_{j+1}\}$, and then $b'ww_i \cdots w_i \sigma_K$ is a face of dimension $n-2$ of both $c_j$ and $c_{j+1}$. Hence the result.

(6.1.10) **PROPOSITION:** Let $s \in \Delta_J$. Then if $s = bwG_L$ with $w \in X_L$, 

1) there exists a unique $c_o \in D_J(s)$, $c_o = bwG_J$, $w \in X_J$, such that if $c \in D_J(s)$, $c = b'ww'G_J$, $w' \in X_J$, $w' \neq w$, then $l(c) < l(c_o)$.

2) if $c \in D_J(s)$, $c \neq c_o$, then there exists a sequence $c_o, c_1, \ldots, c_m = c \in D_J(s)$ such that $c_i, c_{i+1}$ are adjacent for each $i = 0, 1, \ldots, m-1$ and $l(c_i) < l(c_{i+1})$ for each $i = 0, 1, \ldots, m-1$.

**Proof:** (1) Since $s = bwG_L$, $w \in X_L$, $D_J(s) = \{b'ww'G_J: w' \in X_J', b'ww'G_J \subseteq bwG_J \}$. $c_o = bwG_J$ has the required properties, for if $w' \in X_J', l(ww') = l(w) + l(w')$, since $w \in X_L$.

(2) Regard $s$ as an element of $\Delta_J$. By (6.1.9) there exists a unique $\tilde{c}_o = bwG_R \in D_J(s)$ such that if $\tilde{c} \in D_J(s)$, $\tilde{c} \neq \tilde{c}_o$, then $l(\tilde{c}_o) < l(\tilde{c})$. For $c = b'ww'G_J \in D_J(s)$, where $b'ww'G_J \subseteq bwG_J$ and $w' \in X_J$, consider $\tilde{c} = b'ww'G_R \in D_J(s)$. By (6.1.9), if $l(w') = m$, there exists a sequence $\tilde{c}_o, \tilde{c}_1, \ldots, \tilde{c}_m = \tilde{c}$ of chambers in $D_J(s)$ such that $\tilde{c}_i, \tilde{c}_{i+1}$ are adjacent and $l(\tilde{c}_i) = l(\tilde{c}_o) + i$ for all $i$, $0 \leq i \leq m-1$. If $w' = w_i \cdots w_m$, $w_j \in R$, then $\tilde{c}_j = b'ww_i \cdots w_j \sigma_R$. Now define $c_o, c_1, \ldots, c_m = c$ as follows: $c_o = bwG_J$

$$c_j = b'ww_i \cdots w_j \sigma_R = b'ww(j)G_J, \text{ where } ww(j) \in X_J,$$

$l(ww(j)) = l(w) + l(w(j))$, and $l(w(j)) \leq l(w_i \cdots w_j)$.

We show by induction on $j$, for $j > 0$, that
(a) \(c_{j-1}, c_j\) are either equal or adjacent.

(b) \(c_j \in D_j(s)\)

(c) if \(c_{j-1}, c_j\) are adjacent, then \(l(c_{j-1}) < l(c_j)\).

Then by omitting repetitions we have the required sequence of chambers. Now \(c_0 = bw\sigma J^c \in D_j(s)\). Consider \(c_1 = b'ww_{i_1}^\circ \sigma J^c\). If \(w_{i_1} \in W_j\), then \(c_0 = c_1\). So suppose \(w_{i_1} \not\in W_j\).

Then \(w \in X_j \cup \{w_{i_1}\}\), so \(ww_{i_1} \in X_j^L\). Hence \(c_1 = b'ww_{i_1}^\circ \sigma J^c \in D_j(s)\), \(l(c_0) < l(c_1)\). Moreover, \(c_0\) and \(c_1\) are adjacent as they both contain \(bw\sigma J^c - \{w_{i_1}\}\).

Suppose that for all \(k < j\), we have that \(c_{k-1}, c_k\) are both in \(D_j(s)\) and are either equal or adjacent, and if they are adjacent, \(l(c_{k-1}) < l(c_k)\). Now \(c_{j-1} = b'ww(j-1)\sigma J^c\), for some \(w(j-1) \in X_j^L\), and \(c_j = b'ww_{i_1} \ldots w_{i_j}^\circ \sigma J^c\). If \(w_{i_j} \in W_j\), then \(c_j = c_{j-1}\). So suppose \(w_{i_j} \not\in W_j\). Then we can write

\[ww_{i_1} \ldots w_{i_j} = ww(j)w_{j}\], where \(w_j \in W_j\), \(w(j) \in X_j^L\), and

\(l(ww(j)w_{j}) = l(w) + l(w(j)) + l(w_{j})\). Then \(c_j = b'ww(j)\sigma J^c \in D_j(s)\) as \(w(j) \in X_j^L\), \(b'ww(j)\sigma J \subseteq bwBw(j)\sigma J^c\). \(c_{j-1}\) and \(c_j\) are adjacent as \(b'ww(j-1)\sigma J^c - \{w_{i_j}\} = b'ww_{i_1} \ldots w_{i_{j-1}}^\circ \sigma J^c - \{w_{i_j}\}\) is an \(r\)-2 dimensional face of both. \(l(c_{j-1}) < l(c_j)\) by (1.3.8).

Let \(\Delta(J)\) be the set of \(s \in \Delta J\) such that \(s < c\) for some \(c \in D_j\), \(c = bw\sigma J^c\), with \(w \in X_j\), \(w \not\in Y_j\).

(6.1.11) **Lemma:** \(\Delta(J) = \Delta_J - \{c_1, \ldots c_t\}\), where \(c_1, \ldots c_t\) are all the chambers of the form \(bw\sigma J^c\), \(b \in B\), \(w \in Y_j\). Further, \(\Delta(J)\) is a subcomplex of \(\Delta J\).
Proof: Clearly \( c_1 \ldots c_t \not\in \Delta(J) \). Suppose \( s \) is a proper face of, say, \( c_1 \). We may suppose \( s \) is of dimension \( r-2 \). Then 
\[
s = bw\mathcal{S}_j^J - \{w_i\}, \quad \text{for some } w_i \in \hat{J}, \quad \text{with } w \in X_J \cup \{w_i\}, \quad b \in B.
\]
So \( w \notin Y_J \). But 
\[
D_J(s) = \{b'ww'\mathcal{S}_i^J : w' \in X_J^J \cup \{w_i\}, \quad b'ww'G_j \subseteq bwBw'G_j \}.
\]
In particular, \( s < c = bw\mathcal{S}_J^J, \quad w \in X_J \cup \{w_i\}, \quad w \notin Y_J \). So \( s \in \Delta(J) \).

**Definition:** If \( s \in \Delta_J \), let \( \Phi(s) \) be the subcomplex consisting of \( s \) and its faces.

**Definition:** Let \( \Delta(J)_k \), for \( k \in \mathbb{Z}^+ \), be the subcomplex of \( \Delta(J) \) consisting of all \( s \in \Delta(J) \) such that there exists a chamber \( c = bw\mathcal{S}_J^J \in D_J, \ 1(w) < k, \ w \in X_J \) with \( s < c \).

(6.1.12) **Lemma:** \( \Delta(J)_0 = \Phi(\mathcal{S}_J^J) \).

**Proof:** Clearly \( \Phi(\mathcal{S}_J^J) \subset \Delta(J)_0 \). So suppose \( s \in \Delta(J)_0 \). Then 
\[
s = bw\mathcal{S}_L \quad \text{for some } L \subset \hat{J}, \ w \in X_L \hat{J} \ \text{. Now} \]
\[
D_J(s) = \{b'ww'\mathcal{S}_i^J : w' \in X_L J ^J, \quad b'ww'G_J \subseteq bwBw'G_J \}.
\]
Since \( s \in \Delta(J)_0 \), \( \mathcal{S}_J^J = b'ww'\mathcal{S}_i^J \) for some \( b' \in B \), \( w' \in X_L J ^J \). But 
\[
ww' \in X_J, \quad \text{and } l(ww') = l(w) + l(w').
\]
Hence \( w = 1, \ w' = 1 \), and \( s = \mathcal{S}_L \). Thus \( s \in \Phi(\mathcal{S}_J^J) \).

(6.1.13) **Lemma:** Let \( c \in D(J) = \) the set of chambers of \( \Delta(J) \), and suppose \( c = bw\mathcal{S}_J^J, \ w \in X_J, \ l(w) = k > 0 \). Let \( s_1, \ldots s_r \) be the \( r-2 \) dimensional faces of \( c \) numbered such that 
\[
s_1, \ldots s_p \in \Delta(J)_{k-1}, \ s_{p+1}, \ldots s_r \not\in \Delta(J)_{k-1}. \quad \text{Then } 1 \leq p < r-1, \quad \text{and } \Phi(c) \cap \Delta(J)_{k-1} = \bigcup_{i=1}^{p} \Phi(s_i).
\]

**Proof:** (a) Suppose \( l(ww_i) = l(w) + 1 \), with \( w_i \in \hat{J} \). Then
$w \in X \cup \{w_i\}$ and $bw_{J - \{w_i\}}$ is a face of maximal dimension in $c$. 

(b) Suppose $l(w_1) = l(w) - 1$, with $w_i \in \hat{J}$. Then $w = w_1(i)w_2(i)$, with $w_1(i) \in X \cup \{w_i\}$, $w_2(i) \in X \cup \{w_i\}$ and $l(w) = l(w_1(i)) + l(w_2(i))$. Then $bw_1(i)\Sigma_{J - \{w_i\}}$ is a face of maximal dimension in $bw_{J - \{w_i\}}$, and $l(w_1(i)) < l(w)$. 

Now suppose $w \in Y^*_L$, with $L \subseteq \hat{J}$. Assume that the $w_i \in \hat{J}$ are numbered such that $w_1, \ldots, w_p$ are the reflections in $L$, and $w_{p+1}, \ldots, w_r$ those in $\hat{J} - L$. Let $s_i = bw_1(i)\Sigma_{J - \{w_i\}}$ if $1 \leq i \leq p$, and $s_j = bw_{J - \{w_j\}}$ for $p + 1 \leq j \leq r$. Then $s_1, \ldots, s_p \in \Delta(J)_{k - 1}$ but $s_{p+1}, \ldots, s_r \notin \Delta(J)_{k - 1}$. As $k > 0$, $1 \leq p \leq r - 1$.

For any $1 \leq i \leq p$, $s_i \in \Delta(J)_{k - 1}$, so $\Phi(s_i) \subseteq \Delta(J)_{k - 1}$. Hence $\bigcup_{i=1}^{p} \Phi(s_i) \subseteq \Phi(c) \cap \Delta(J)_{k - 1}$. Conversely, take any $t \in \Phi(c) \cap \Delta(J)_{k - 1}$, such that $t \notin \bigcup_{i=1}^{p} \Phi(s_i)$. Obviously, $t \notin c$ as $t \in \Delta(J)_{k - 1}$, and $t$ cannot equal any of $s_{p+1}, \ldots, s_r$, as none of these are in $\Delta(J)_{k - 1}$. Since $\Phi(c) = c \cup \bigcup_{i=1}^{r} \Phi(s_i)$, we must have that $t < s_{q_1}, t < s_{q_2}, \ldots, t < s_{q_u}$ for some $s_{q_i}$, with $p + 1 \leq q_i \leq r$ for all $i$. As $c \in D_J(t)$, there exists a unique $c_0 \in D_J(t)$ such that $l(c_0) \leq l(c')$ for all $c' \in D_J(t)$.

Now $c_0 \neq c$ as $t \in \Delta(J)_{k - 1}$, and so $l(c_0) \leq k - 1$. By (6.1.10) there exists a sequence of chambers $c_0, c_1, \ldots, c_m = c$ in $D_J(t)$ such that $c_i, c_{i+1}$ are adjacent for all $i$, $0 \leq i \leq m - 1$, and $l(c_i) < l(c_{i+1})$. But then $c_{m-1} \cap c_m$ is an $r - 2$ dimensional simplex contained in $c_{m-1}$ and in $c_m$, and so must contain $t$. 
So for some \( j, 1 < j < u \), \( c_{m-1} \cap c_m = s_j \). Then as \( s_j < c_{m-1} \), and \( l(c_{m-1}) < l(c_m) = k \), \( s_j \in \Delta(J)_{k-1} \) — contradiction.

Hence \( \Phi(c) \cap \Delta(J)_{k-1} = \bigcup_{i=1}^{p} \Phi(s_i) \).

(6.1.14) **Lemma:** Let \( K \) be an abstract simplicial complex. If \( s_1, \ldots, s_m \) are simplexes of \( K \) with at least one common vertex, then \( \bigcup_{i=1}^{m} \Phi(s_i) \) has the homology of a point. (See Hilton and Wylie [15] for a proof).

(6.1.15) **Corollary:** \( \Phi(c) \cap \Delta(J)_{k-1} = \bigcup_{i=1}^{p} \Phi(s_i) \) has the homology of a point.

**Proof:** Since \( p < r \), the \( \{s_i\}_{i=1}^{p} \) have at least one common vertex.

(6.1.16) **Lemma:** Let \( K \) be a simplicial complex which is a union \( K = L \cup L_1 \cup L_2 \cup \ldots \cup L_n \) of subcomplexes. Suppose

(a) each \( L_i \) has the homology of a point
(b) each \( L \cap L_i \) has the homology of a point
(c) \( L_i \cap L_j \subseteq L \) when \( i \neq j \).

Then \( K \) and \( L \) have isomorphic homology groups.

**Proof:** Use induction on \( n \). True for \( n=0 \). Write \( K = K_1 \cup L_n \), where \( K_1 = L \cup L_1 \cup L_2 \cup \ldots \cup L_{n-1} \), and by induction \( K \) and \( K_1 \) have isomorphic homology groups. Also, \( K_1 \cap L_n \) has the homology of a point. Consider the Mayer-Vietoris exact sequence (see [15]):

\[
\ldots \rightarrow H_p(K_1 \cap L_n) \rightarrow H_p(K_1) \oplus H_p(L_n) \rightarrow H_p(K_1 \cup L_n) \rightarrow H_{p-1}(K_1 \cap L_n) \rightarrow \ldots
\]
For all \( p > 1 \), \( H_p(K_1 \cap L_n) = 0 \), \( H_p(L_n) = 0 \), and \( H_0(K_1 \cap L_n) \cong \mathbb{Z} \), \( H_0(L_n) \cong \mathbb{Z} \). Thus if \( p > 1 \) we have
\[
0 \rightarrow H_p(K_1) \rightarrow H_p(K_1 \cup L_n) \rightarrow 0
\]
i.e. \( H_p(K_1) \cong H_p(K_1 \cup L_n) \).

If \( p = 1 \), we have
\[
0 \rightarrow H_1(K_1) \rightarrow H_1(K_1 \cup L_n) \rightarrow \mathbb{Z} \rightarrow H_0(K_1) \cong \mathbb{Z} \rightarrow H_0(K_1 \cup L_n) \rightarrow 0.
\]
We must show: \( H_1(K_1) \cong H_1(K_1 \cup L_n) \)
and \( H_0(K_1) \cong H_0(K_1 \cup L_n) \).

But \( H_0(K_1 \cup L_n) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \), \( r \) copies, where \( r \) is equal to the number of connected components of \( K_1 \cup L_n \). Since
\( H_0(L_n) \cong \mathbb{Z} \), \( L_n \) is connected, and as \( H_0(K_1 \cap L_n) \cong \mathbb{Z} \), there is a vertex \( v \) of \( K_1 \) such that \( v \) is connected to every vertex of \( L_n \). Hence the number of connected components of \( K_1 \cup L_n \) equals the number of connected components of \( K_1 \).

That is, \( H_0(K_1 \cup L_n) \cong H_0(K_1) \). Hence consider
\[
0 \rightarrow H_1(K_1) \xleftarrow{\alpha} H_1(K_1 \cup L_n) \xrightarrow{\beta} \mathbb{Z} \xrightarrow{\gamma} H_0(K_1) \cong \mathbb{Z} \xrightarrow{\delta} H_0(K_1) \).
\]
\( \ker \alpha = 0 \), so \( \alpha \) is a monomorphism. \( \delta \) is a projection, and so \( \gamma \) is an injection. Thus \( \ker \gamma = 0 \), and \( \text{im} \beta = 0 \).

So \( \ker \beta = H_1(K_1 \cup L_n) \cong \text{im} \alpha \). \( \alpha \) is an isomorphism and so \( H_1(K_1) \cong H_1(K_1 \cup L_n) \).

Now fix an integer \( k > 0 \). Let \( a_1, \ldots, a_m \) be all the chambers of \( \Delta(J) \) for which \( l(a_i) = k \). Let \( K = \Delta(J)_k \), \( L = \Delta(J)_{k-1} \) and \( L_i = \Phi(a_i) \). Each \( \Phi(a_i) \) has the homology of a point, each \( \Phi(a_i) \cap \Delta(J)_{k-1} \) has the homology of
a point, and \( \Phi(a_i) \cap \Phi(a_j) \) for \( i \neq j \) is in \( \Delta(J)_{k-1} \). So by (6.1.16), \( \Delta(J)_k \) and \( \Delta(J)_{k-1} \) have isomorphic homology groups. Now \( \Delta(J)_0 = \Phi(C_j) \) has the homology of a point, and so it follows by induction that \( \Delta(J)_k \) has the homology of a point for all non-negative integers \( k \). But

\[
\Delta(J) = \Delta(J)_{k_0} = \Delta(J)_{k}
\]

for all integers \( k \geq k_0 \), for some integer \( k_0 \), and so \( \Delta(J) \) has the homology of a point.

**Definition:** Let \( K \) be an oriented simplicial complex. Let \( C_p(K) \) be the group of \( p \)-chains (with coefficients in \( \mathbb{Z} \)), \( \partial: C_p(K) \to C_{p-1}(K) \) the boundary operator for all \( p \), \( \mathbb{Z}_p(K) \) the group of \( p \)-cycles of \( K \), \( B_p(K) \) the group of \( p \)-boundaries of \( K \), and \( H_p(K) = \mathbb{Z}_p(K)/B_p(K) \) the \( p \)th-homology group of \( K \).

**Proof of Theorem 1:**

Recall that \( \Delta(J) = \Delta - \{c_1, \ldots, c_t\} \), where \( c_1, \ldots, c_t \) are all the chambers of the form \( bwC_j \), \( b \in B \), \( w \in Y_J \).

We have \( C_p(\Delta(J)) = C_p(\Delta) \) for all \( p = 0, 1, \ldots, r-2 \), and so \( H_p(\Delta_J) = H_p(\Delta(J)) \) for all \( p = 0, \ldots, r-2 \). Since \( \Delta(J) \) has the homology of a point, \( H_0(\Delta_J) = \mathbb{Z} \), and \( H_i(\Delta_J) = 0 \) for \( 1 \leq i \leq r-2 \).

Now \( C_{r-1}(\Delta_J) \cong C_{r-1}(\Delta(J)) \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \) (\( t \) copies of \( \mathbb{Z} \)).

Consider \( \partial: C_{r-1}(\Delta(J)) \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \to C_{r-2}(\Delta(J)) = C_{r-2}(\Delta_J) \)

and \( \partial': C_{r-1}(\Delta(J)) \to C_{r-2}(\Delta(J)) \). Since \( H_{r-2}(\Delta_J) = H_{r-2}(\Delta(J)) = 0 \), \( \mathrm{im} \partial = \mathrm{im} \partial' \cong Z_{r-2}(\Delta_J) = Z_{r-2}(\Delta(J)) \). But \( H_{r-1}(\Delta(J)) = 0 \), so \( \ker \partial' = 0 \).
Thus \( \ker d \cong \mathbb{Z} \ast \mathbb{Z} \ast \ldots \ast \mathbb{Z} \) (\( t \) copies). Hence

\[
H_{r-1}(\Delta_J) \cong \mathbb{Z} \ast \mathbb{Z} \ast \ldots \ast \mathbb{Z} \) (\( t \) copies). Further, we have

\[
H_i(\Delta_J) = 0 \text{ for all } i > r.
\]

Finally, let \( y_1 = w_0 \hat{\in} Y_J \). Then each \( y_1 \in Y_J \) has the form \( y_1 = w(i) y_1 \), with \( l(y_1) = l(w(i)) + l(y_1) \), and so \( z_1 = w(i) z_1 \). So it is sufficient to show that \( z_1 \) is an \((r-1)\)-cycle. Consider first \( d(\sigma^\Delta_J)\):

\[
d(\sigma^\Delta_J) = d((G^1, \ldots, G^r)) = \sum_{i=1}^{r} (-1)^{i+1} (G^1, \ldots, \hat{G}_i, \ldots, G^r) = \sum_{i=1}^{r} (-1)^{i+1} \sigma^\Delta_J-{\{w_1\}}.
\]

Now \( z_1 = \sum_{w \in W^\Delta_J} (w) w \sigma^\Delta_J = \sum_{w \in W^\Delta_J} (w) w \sigma^\Delta_J = \sum_{w \in W^\Delta_J} (w) z_1 \), where \( z = \Sigma_{w \in W^\Delta_J} (w) w \sigma^\Delta_J \). So it is sufficient to show that \( z \) is an \((r-1)\)-cycle. Now

\[
d(z) = \sum_{w \in W^\Delta_J} (w) \sum_{i=1}^{r} (-1)^{i+1} \sigma^\Delta_J-{\{w_1\}} = \sum_{w \in W^\Delta_J} (w) \sum_{i=1}^{r} (-1)^{i+1} \sigma^\Delta_J-{\{w_1\}}.
\]

For each \( w \in W^\Delta_J \), either \( l(w w_1) > l(w) \), and \( w \in x_{\{w_1\}} \), or \( l(w w_1) < l(w) \), and then \( w \) has the form \( w = w' w_1 \) with \( l(w) = l(w') + 1 \). So

\[
\sum_{w \in W^\Delta_J} (w) w \sigma^\Delta_J-{\{w_1\}} = \sum_{w \in W^\Delta_J} (w) w \sigma^\Delta_J-{\{w_1\}} + \sum_{w \in W^\Delta_J} (w) w \sigma^\Delta_J-{\{w_1\}}
\]

\[
= \sum_{w \in W^\Delta_J} (w) w \sigma^\Delta_J-{\{w_1\}} - \sum_{w \in W^\Delta_J} (w) w \sigma^\Delta_J-{\{w_1\}} = 0.
\]

Hence \( z \) is an \((r-1)\)-cycle, and thus \( z_1, \ldots, z_1 \) are all \((r-1)\)-cycles. Each \( z_1 \) has an expression in which there is a unique
simplex, \( y_i \triangle \), whose length is strictly greater than the length of any other simplex which occurs with non-zero coefficient in the expression for \( z_1 \), and hence it follows that the \( t \) distinct \((r-1)\)-cycles given are linearly independent and so form a basis for \( H_{r-1}(\Delta_j) \). Thus we have proved THEOREM 1.

\((6.1.17)\) **Conditions for an \((r-1)\)-chain to be a cycle.**

We have already that

\[
d(\triangle_j) = d((G^1, \ldots, G^r)) = \sum_{i=1}^{r} (-1)^{i+1} \triangle_{j}^{i} - \{w_i\}.
\]

Consider an arbitrary \((r-1)\)-chain \( c = \sum_{w \in X_j} a_{bw} \cdot \triangle_{j}^{i} - \{w_i\} \), where \( a_{bw} \in \mathbb{Z} \). Then

\[
d(c) = \sum_{w \in X_j} a_{bw} \cdot \sum_{i=1}^{r} (-1)^{i+1} \triangle_{j}^{i} - \{w_i\}.
\]

Now, each \( w \in X_j \) can be written uniquely in the form \( w = w_1(i)w_2(1) \), with \( w_1(i) \in X_j \cup \{w_i\} \), \( w_2(1) \in X_j \cup \{w_i\} \), and

\[
l(w) = l(w_1(i)) + l(w_2(1)).
\]

Then

\[
d(c) = \sum_{w \in X_j} a_{bw} \cdot \sum_{i=1}^{r} (-1)^{i+1} \triangle_{j}^{i} - \{w_i\},
\]

and so \( d(c) = 0 \) if and only if the coefficient of each \((r-2)\)-simplex \( bw \triangle_{j}^{i} - \{w_i\} \), \( w \in X_j \cup \{w_i\} \), is zero. Hence \( d(c) = 0 \) if and only if for each \((r-2)\)-simplex \( bw \triangle_{j}^{i} - \{w_i\} \), \( w \in X_j \cup \{w_i\} \), \( 1 \leq i \leq r \), we have

\[
\sum_{w' \in X_j \cup \{w_i\}} a_{bw'w} = 0.
\]

\( b'w'w \triangle_j \subseteq bwbw' \triangle_j \)
Rewrite this as $a_{bw} + \sum_{1 \neq w' \in X_J \cup \{w_1\}} a_{b'w'w'} = 0$. So if $b'w'wG_J \subseteq bwBw'G_J$

$w \in Y_K$ for some $K \supseteq J$, then $a_{bw}$, for any $b \in B$, can be expressed as a sum of $a_{b'w'}$'s, with $l(w') > l(w)$. Hence we conclude that each $a_{bw}$ can be expressed as a linear combination of $\{a_{b'y}: y \in Y_J, b' \in B\}$.

**Proof of THEOREM 2:**

Each $g \in G$ induces a non-singular linear transformation of $C_p(\Delta_J) \otimes \mathbb{Q}$, $H_{r-1}(\Delta_J) \otimes \mathbb{Q}$, and $H_0(\Delta_J) \otimes \mathbb{Q}$, giving rise to characters $\phi_p$, $\theta_{r-1}$ and $\theta_0$ respectively. By the Hopf trace formula, we have

$$(-1)^{r-1} \theta_{r-1} + \theta_0 = \sum_{p=0}^{r-1} (-1)^p \phi_p.$$  

$G$ acts as a permutation representation on the oriented simplexes of $C_p(\Delta_J)$, and the orbits are subsets $O_L = \{gO_L: g \in G, L \subseteq \hat{J}\}$.

So $\phi_p$ is the character of the permutation representation of $G$ on $C_p(\Delta_J) \otimes \mathbb{Q}$. Let $x_L$ be the character of the permutation representation of $G$ on $O_L$. Then $\phi_p = \sum_L x_L$, summed over all $L \subseteq \hat{J}$ with $|L| = p+1$. Hence

$$\sum_{p=0}^{r-1} (-1)^p \phi_p = \sum_{\emptyset \neq L \subseteq \hat{J}} (-1)^{|L|} x_L.$$  

Now $\theta_0$ is the character of the representation of $G$ on $H_0(\Delta_J) \otimes \mathbb{Q}$. As $\Delta_J$ is connected, $\theta_0$ is the principal character of $G$, that is, $\theta_0 = \mathbf{1}_G$. 


G acts transitively on $O_L$. Let $U_L = \{ u \in G : uO_L = O_L \}$. Then $u \in U_L$ if and only if $u \in G^*_L$. So $U_L = G^*_L$. $x_L$ is the character of the permutation representation of $G$ on $O_L$, and so corresponds to the character of the permutation representation of $G$ on $G^*_L$: that is, $x_L = (1_{G^*_L})^G$. Thus

$$- \sum_{\phi \neq L \subseteq \hat{J}} (-1)^{|L|} |x_L| = - \sum_{\phi \neq L \subseteq \hat{J}} (-1)^{|L|} (1_{G^*_L})^G$$

$$= - \sum_{K \leq K \leq R} (-1)^{|K-J|} (1_{G^*_K})^G (-1)^x$$

Hence $(-1)^{r-1} \Theta_{r-1} = \sum_{K \leq K \leq R} (-1)^{r-1} (1_{G^*_K})^G - 1_G$

That is, $\Theta_{r-1} = \sum_{K \leq K \leq R} (-1)^{|K-J|} (1_{G^*_K})^G$, and Theorem 2 has been proved.

(6.1.18) **COROLLARY**: The restriction of $\mu_J$ to $Q[B]$ is given by $\mu_{J/B} = \sum_{w \in Y_J} (1_{B \cap B^{-1}})^B$

**Proof**: $B$ permutes the basis elements of $H_{r-1}(\Delta_J) \otimes Q$ in the same way as $B$ permutes the cosets of $B \cap B^{-1}$ in $B$, for $w \in Y_J$.

(6.1.19) **COROLLARY**: The space of $B$-invariant vectors in $H_{r-1}(\Delta_J) \otimes Q$ is of dimension $|Y_J|$.

(6.1.20) **COROLLARY**: Let $K$ be any field. Let $\Delta_J$ be the Tits complex of a finite group $G$ with $(B,N)$ pair with
respect to a parabolic subgroup $G_J$ of $G$, with $|J| = r \geq 2$.

Then regarding $Z$ as a subring of $K$ in a natural way, we have that $H_0(\Delta_J) \otimes_Z K \cong K$

$H_1(\Delta_J) \otimes_Z K = 0$ for all $i$, $1 < i < r-2$, and $i > r$

$H_{r-1}(\Delta_J) \otimes_Z K \cong K \oplus K \oplus \ldots \oplus K$ ($t$ copies)

where $t = \sum_{w \in Y_J} |B_w B \cap B^w|$. Further, the action of $G$ on $\Delta_J$

defines a $K[G]$-module structure on $H_{r-1}(\Delta_J) \otimes_Z K$ which

affords the character

$$\mu_J = \sum_{L \leq J} (-1)^{|L-J|} (1_{G_L})^G,$$

where $1_{G_L}$ is the principal character of $G_L$.

**Note:** We call any $K[G]$-module $M$ which affords the character $\mu_J$ above, where $K$ and $G$ are as in (6.1.20),
a relative Steinberg module of type $J$. Corollary (6.1.20)
gives us one such module $M = H_{r-1}(\Delta_J) \otimes_Z K$ whenever $|J| = r \geq 2$. 
A Degenerate Form of the Relative Steinberg Modules.

The Coxeter complex $C$ of a finite Coxeter system $(W,R)$ can be viewed as a degenerate form of the Tits building of a finite group $G$ with a $(B,N)$ pair $(G,B,N,R)$ whose Weyl group is $W$. The maximal parabolic subgroups $w_1, \ldots, w_n$ of $W$ (where $n = |R|$) are used to define the simplicial complex $C$ in the same way that the maximal parabolic subgroups $G_1, \ldots, G_n$ of $G$ are used to define $\Delta$. Similarly, we can define the Coxeter complex $C_J$ of $W$ with respect to $W_J$, for $J \subseteq R$.

Using similar arguments to those for the relative Tits complex $\Delta_J$ of $G$ with respect to $G_J$, we have the following results:

(6.2.1) Theorem 1: Let $C_J$ be the simplicial Coxeter complex of a finite Coxeter system $(W,R)$ with respect to a parabolic subgroup $W_J$ of $W$, with $|J| = r \geq 2$. Then the homology groups of $C_J$ with integral coefficients are as follows:

\[ H_0(C_J) \cong \mathbb{Z} \]
\[ H_i(C_J) = 0, \text{ if } 1 \leq i < r - 2 \text{ or } i > r. \]
\[ H_{r-1}(C_J) \cong \mathbb{Z} \otimes \mathbb{Z} \otimes \cdots \otimes \mathbb{Z}, \text{ } t \text{ summands,} \]

where $t = |Y_J|$. If $\varepsilon : W \to \{\pm 1\}$ is the alternating character of $W$, and $\Omega_J$ is the fundamental chamber of $C_J$, then the $(r-1)$-chains $\zeta_i = \sum_{w \in W_J} (w) y_i w \Omega_J$, for $1 \leq i \leq t$, where $Y_J = \{y_1, \ldots, y_t\}$, are cycles which form a basis for $H_{r-1}(C_J)$. 

(6.2.2) **THEOREM 2**: Let $C_J$ be the Coxeter complex of a finite Coxeter system $(W,R)$ with respect to a parabolic subgroup $W_J$ of $W$, with $|J|=r\geq 2$. Then the action of $W$ on $C_J$ defines a $Q[W]$-module structure on $H_{r-1}(C_J) \otimes Q$ which affords the character

$$\xi_J = \sum_{K<K<R} (-1)^{|K-J|} (1_{W_K})^W$$

where $1_{W_K}$ is the principal character of $W_K$.

(6.2.3) **COROLLARY**: Let $A = Q[W]$, and let $e_J$ and $o_J$ be the idempotents defined in (1.4.1). Then there is an isomorphism of the $A$-modules $A \circ e_J$ and $H_{r-1}(C_J) \otimes Q$, where $|J| = r \geq 2$, given by mapping the basis elements $\{y_{i}o_{j}e_{j} : 1 \leq i \leq t\}$ of $A \circ e_J$ to the basis elements $\{z_{i} : 1 \leq i \leq t\}$ of $H_{r-1}(C_J) \otimes Q$, where $y_{i}o_{j}e_{j} \rightarrow z_{i}$ for all $i$, $1 \leq i \leq t$. 


Chapter 7: ON THE DECOMPOSITION OF THE MODULE L.

(7.1) General Decompositions of L.

Let G be a finite group with a (B, N) pair (G, B, N, R) of rank n with Weyl group W. Let K be a field. Let M be the principal KB-module (that is, M affords the representation $1_B$ of B); then KG $\otimes_{KB} M$ is the KG-module $M^G$ induced from M, and affords the representation $(1_B)^G$ of G. Let $G/B$ denote the set of right cosets of B in G and let L be the set of functions $f: G/B \rightarrow K$, as in (3.1), after (3.1.10).

L becomes a left KG-module by defining for all $x \in G$ the function $xf \in L$, given by $(xf)(Bg) = f(Bgx)$ for all $Bg \in G/B$.

(7.1.1) Theorem: L is isomorphic, as left KG-module, to KG $\otimes_{KB} M$.

Let $H = E_{KG}(L)$. Then by (3.1.10) we have that $H$ is generated as K-algebra with identity $a_1$ by $\{a_{w_i}: w_i \in R\}$ subject to the relations:

$$a_{w_i}^2 = q_i a_1 + (q_i-1)a_{w_i}$$

for all $w_i \in R$

where $q_i = |B:B \cap B^{w_i}|$ for all $w_i \in R$.

$$(a_{w_i} a_{w_j} a_{w_i} \ldots)^{n_{ij}} = (a_{w_j} a_{w_i} a_{w_j} \ldots)^{n_{ij}}$$

for all $w_i, w_j \in R$, $i \neq j$, where $n_{ij} = \text{order of } w_i w_j \text{ in } W$.

For all $w \in W$, define $a_w = a_{w_1} a_{w_2} \ldots a_{w_s}$, where $w = w_1 \ldots w_s$ is a reduced expression for $w$. Then $H$ is the K-algebra with identity $a_1$ and K-basis $\{a_w: w \in W\}$, with an associative multiplication given by
\[ a_w a_w = \begin{cases} a_{w^{-1}} & \text{if } l(w) = l(w) + 1 \\ q a_{w^{-1}} + (q-1)a_w & \text{if } l(w) = l(w) - 1. \end{cases} \]

for all \( w \in W, w_i \in R. \)

Now let \( S \) be a system of finite groups with \((B,N)\) pairs of type \((W,R)\). Then for each \( G = G(q) \in S \), where \( q \in \mathbb{Z} \), we have \( q_i = |B(q):B(q) \cap B(q)_{w_i}| = q_i \) for all \( w_i \in R. \)

There are two cases to consider:

1. Suppose that the characteristic of \( K \) is not equal to \( p \), where \( q = p^s \) for some positive integer \( s \). We will assume that \( K \) is a field of characteristic \( 0 \) in this case, and then \( H \cong H_K(q) \), as defined in (3.3).

2. Suppose that the characteristic of \( K \) is equal to \( p \). Then \( H \cong H_K(0) \), which is defined in (3.1).

In chapters 4 and 5 we have given some decompositions of \( H_K(q) \) and \( H_K(0) \), which are as follows:

1. \( H_K(q) = \sum_{J \subseteq R} E_J 0_J H_K(q) \) and \( H_K(q) = \sum_{J \subseteq R} 0_J E_J H_K(q), \)

where for all \( J \subseteq R, E_J \) and \( 0_J \) are defined as follows:

\[ E_J = \frac{1}{f(q)_J} \sum_{w \in W_J} a_w \]

\[ 0_J = \frac{1}{f(q)_J} \sum_{w \in W_J} (-1)^{l(w)} q^{c_{w_{wJ}}0_J} a_w \]

where \( f(q)_J = \sum_{w \in W_J} q^{c_w}. \)

2. \( H_K(0) = \sum_{J \subseteq R} e_J 0_J H_K(0) \) and \( H_K(0) = \sum_{J \subseteq R} 0_J e_J H_K(0), \)

where for all \( J \subseteq R, e_J \) and \( 0_J \) are defined by:
\[ e_J = \sum_{v \in W_J} a_w \left[ 1 + a_w o_J \right], \text{ with notation as in (4.4)}, \]
\[ o_J = (-1)^{l(w o_J)} a_w o_J. \]

Further, for all \( J \subseteq R \), \( e_J o_J H_K(0) \) and \( o_J e_J H_K(0) \) are indecomposable right ideals of \( H_K(0) \).

(7.1.2) **Theorem:** Let \( V \) be a module over a ring \( R \). Then a direct sum decomposition of \( V \), \( V = \sum_{i \in I} V_i \), is equivalent to writing \( 1_V \) as a sum of mutually orthogonal idempotents \( e_i \), \( 1_V = \sum_{i \in I} e_i \), in \( \text{End}_R(V) \).

**Proof:** Suppose \( V = \sum_{i \in I} V_i \), a direct sum of \( R \)-submodules of \( V \). Then for each \( v \in V \), \( v = \sum_{i \in I} v_i \) uniquely for some \( v_i \in V_i \). Consider the map \( e_i : V \to V \) given by \( e_i(\sum_{j \in I} v_j) = v_i \). \( e_i \) is an \( R \)-endomorphism of \( V \), and \( e_i^2 = e_i \neq 0 \) if \( V_i \neq 0 \). If \( i \neq j \), then \( e_i e_j = 0 \). Finally, \( 1_V = \sum_{i \in I} e_i \).

Conversely, suppose that \( 1_V = \sum_{i \in I} e_i \), where the \( e_i \) are mutually orthogonal idempotents. Define \( V_i = e_i V \) for each \( i \in I \). Then for all \( v \in V \), \( v = 1_V v = \sum_{i \in I} e_i v \). Hence \( V = \sum_{i \in I} V_i \). To show that this sum is direct, we have to show that the expression \( v = \sum_{i \in I} e_i v \) is unique. Suppose \( v = \sum_{i \in I} e_i v = \sum_{i \in I} v_i' \) where \( v_i' \in V_i \) for all \( i \). Then each \( v_i' \) is of the form \( e_i v(i) \) for some \( v(i) \in V_i \). Now for all \( j \in I \), \( e_j v = e_j(\sum_{i \in I} e_i v) = e_j(\sum_{i \in I} e_i v(i)) = e_j v(j) = v_j' \).
Hence \( V = \sum_{i \in I} V_i \ast \)

Denote by \( L(q) \) and \( L(0) \) the KG-module \( L \) when \( K \) is a field of characteristic 0 or characteristic \( p \) respectively. Then we have

\[ H_K(q) \cong \mathbb{E}_K(L(q)) \]

and \( H_K(0) \cong \mathbb{E}_K(L(0)) \).

In the following work, if we do not wish to consider the characteristic of \( K \), we will just write \( L \).

(7.1.3) \text{COROLLARY:} (1) \( L(q) = \sum_{J \subseteq R} e_J o_J L(q) \) and

\[ L(q) = \sum_{J \subseteq R} o_J e_J L(q) \]

are decompositions of \( L(q) \) as direct sums of KG-submodules.

(2) \( L(0) = \sum_{J \subseteq R} e_J o_J L(0) \) and

\[ L(0) = \sum_{J \subseteq R} o_J e_J L(0) \]

are decompositions of \( L(0) \) as direct sums of indecomposable KG-submodules.

\text{Proof:} The decompositions of \( H_K(q) \) and \( H_K(0) \) correspond to expressing the identity of \( H_K(q) \) and the identity of \( H_K(0) \) as sums of mutually orthogonal idempotents: if, say, in \( H_K(0) \) we have \( 1 = \sum J \subseteq R p_J \), where \( p_J \in o_J e_J H_K(0) \), and the elements \( p_J \) for all \( J \subseteq R \) are a set of mutually orthogonal primitive idempotents with \( p_J H_K(0) = o_J e_J H_K(0) \), then by (7.1.2) we have that \( L(0) = \sum_{J \subseteq R} p_J L(0) \). Since for all \( J \subseteq R \) we have that \( p_J L(0) = p_J H_K(0) L(0) = o_J e_J H_K(0) L(0) = o_J e_J L(0) \), the required decomposition follows. Similarly
in the other cases.

(7.1.4) **Definition:** Let \( f_{Bg} \) be the function of \( L \) defined by:

\[
 f_{Bg}(Bg') = \begin{cases} 
 0 & \text{if } Bg' \neq Bg \\
 1 & \text{if } Bg' = Bg 
\end{cases}
\]

Then \( \{ f_{Bg} : Bg \in G/B \} \) is a \( K \)-basis of \( L \), and is called the set of characteristic functions of \( L \).

(7.1.5) **Lemma:** \( f_B \) generates the \( KG \)-module \( L \).

**Proof:** Any element of \( L \) is a \( K \)-linear combination of elements \( f_{Bg} \), for some \( Bg \in G/B \). Since for all \( g \in G \) we have \( f_{Bg} = g^{-1}f_B \), \( f_B \) generates \( L \) as \( KG \)-module.

(7.1.6) **Corollary:**

1. \( E_J O_J f_B \) and \( O_J E_J f_B \) generate the \( KG \)-modules \( E_J O_J L(q) \) and \( O_J E_J L(q) \) respectively.
2. \( e_J O_J f_B \) and \( O_J e_J f_B \) generate the \( KG \)-modules \( e_J O_J L(0) \) and \( O_J e_J L(0) \) respectively.

(7.1.7) **Proposition:**

1. \( E_J L(q) = \{ f \in L(q) : O_{\{w_i\}} f = 0 \text{ for all } w_i \in J \} \) = \{ \( f \in L(q) : E_{\{w_i\}} f = f \text{ for all } w_i \in J \} \}. \( E_J L(q) \) has dimension \( \sum_{w \in X_J} q^c_w = |G(q) : G_J(q)| \) and basis \( \{ E_J f_{Bw} : w^{-1} \in X_J, Bw \subseteq BwB \} \). Further, let \( K^G_J \) be the principal \( KG_J(q) \)-module. Then \( E_J L(q) \) and the \( KG \)-module \( K^G_J \) are isomorphic \( KG \)-modules.

2. \( e_J L(0) = \{ f \in L(0) : O_{\{w_i\}} f = 0 \text{ for all } w_i \in J \} \) = \{ \( f \in L(0) : e_{\{w_i\}} f = f \text{ for all } w_i \in J \} \).

\( e_J L(0) \) has dimension \( \sum_{w \in X_J} q^c_w = |G(q) : G_J(q)| \) and basis
\( \{ e_j f_{Bwb} : w^{-1} \in X_J, Bwb \subseteq BwB \} \). Further, let \( H_J \) be the principal \( KG_J(q) \)-module. Then \( e_J L(0) \) and the \( KG \)-module \( H_J^G \) are isomorphic \( KG \)-modules.

**Proof:** (1) Clearly \( E_J L(q) = \{ f \in L(q) : O \{ w_i \} f = 0 \) for all \( w_i \in J \} \). Conversely, let \( f \in L(q) \) satisfy \( O \{ w_i \} f = 0 \) for all \( w_i \in J \). Then \( (q^{c_i} - a_i) f = 0 \) for all \( w_i \in J \); that is, \( a_i f = q^{c_i} f \) for all \( w_i \in J \). Hence for all \( w \in W_J \), \( a_i f = q^{c_i} f \). Then \( E_J f = \frac{1}{f(q)} \sum_{w \in W_J} a_i f = \sum_{w \in W_J} q^{c_i} f = f \), and so \( f \in E_J L(q) \).

Similarly, if \( E \{ w_i \} f = f \), then \( (1 + a_i f = (q^{c_i} + 1) f \), and hence \( a_i f = q^{c_i} f \), and it follows that

\[ E_J L(q) = \{ f \in L(q) : E \{ w_i \} f = f \) for all \( w_i \in J \} \].

Since \( \{ f_{Bwb} : w \in W, Bwb \subseteq BwB \} \) form a \( K \)-basis of \( L(q) \), let \( f = \sum_{w \in W} \sum_{Bwb} k_{Bwb} f_{Bwb} \in E_J L(q) \), where the \( k_{Bwb} \in K \).

Let \( w_i \in J \); then \( O \{ w_i \} f = 0 \) gives

\[ * q^{c_i} \sum_{w \in W} \sum_{Bwb} k_{Bwb} f_{Bwb} - \sum_{w \in W} \sum_{Bwb} k_{Bwb} a_i f_{Bwb} = 0. \]

Now for all \( w \in W \), \( a_i f_{Bwb} = \sum_{Bg \subseteq BwB} f_{Bg} \). Suppose \( w^{-1}(r_i) > 0 \). Then \( BwBwB \subseteq BwB \), and \( \{ Bg \subseteq BwBwB \} = \{ BwBwB \} \). Suppose that \( w^{-1}(r_i) < 0 \). Then \( w = w_i w' \) for some \( w' \in W \) with \( l(w) = l(w') + 1 \). Then \( BwBwB \subseteq BwB \cup BwBwB \), and \( \{ Bg \subseteq BwBwB \} = BwBwB \cup \{ Bwb' \subseteq BwBwB \} \).

Now consider the coefficients of the \( f_{Bwb} \) on the
left hand side of equation *. Suppose \( w^{-1}(r_i) > 0 \). Then the
coefficient of \( f_{Bw_b} \) is \( \sum_{Bw_i w'} \sum_{Bw_i w''} k_{Bw_i w'}^{Bw_i w''} k_{Bw_i w'} \).

Since \( \{f_{Bv_b} : v \in W, Bv_b \subseteq BvB\} \) is a basis of \( L(q) \), the
coefficients of the \( f_{Bv_b} \) occurring on the left hand side
of * must all be zero. Thus, if \( w^{-1}(r_i) > 0 \),
\[
q^c_i k_{Bw_b} - \sum_{Bw_i w'} \sum_{Bw_i w''} k_{Bw_i w'}^{Bw_i w''} = 0. \tag{a}
\]

Now choose any coset \( Bw_i w' \subseteq Bw_i Bw_b \). Since \( w^{-1}(r_i) > 0 \),
\( (w_i w)^{-1}(r_i) < 0 \), and so the coefficient of \( f_{Bw_i w'} \) on the
left hand side of * is
\[
q^c_i k_{Bw_i w'} - k_{Bw'} - \sum_{Bw_i w''} k_{Bw_i w''} = 0. \tag{b}
\]

So
\[
(q^c_i + 1) k_{Bw_i w'} - k_{Bw'} - \sum_{Bw_i w''} k_{Bw_i w''} = 0. \tag{c}
\]

Since \( Bw_i w' \subseteq Bw_i Bw_b, Bw' = Bw_b \), and \( \{Bw_i w'' \subseteq Bw_i Bw_b\} = \{Bw_i w'' \subseteq Bw_i Bw_b\} \).
Then (b) can be written:
\[
(q^c_i + 1) k_{Bw_i w'} - k_{Bw'} - \sum_{Bw_i w''} k_{Bw_i w''} = 0 \tag{c}
\]

Subtract (c) from (a):
\[
(q^c_i + 1) k_{Bw_b} - (q^c_i + 1) k_{Bw_i w'} = 0.
\]

Since \( q^c_i + 1 \neq 0 \), \( k_{Bw_b} = k_{Bw_i w'} \). Hence for all \( w^{-1} \in X_{w_i} \),
for any \( Bw_b \subseteq BwB \), \( k_{Bw_b} = k_{Bw_i w'} \), for all \( Bw_i w' \subseteq Bw_i Bw_b \).

Now let \( w^{-1} \in X_{j} \). Then for any \( w' \in W_{j} \), we have
\( l(w' w) = l(w') + l(w) \). Let \( y \in W \). Then \( y = w_j x \) for some \( w_j \in \mathcal{L}_{X_{j}} \)
and some \( x^{-1} \in X_{j} \) with \( l(y) = l(w_j) + l(x) \). Let \( w_j = v_{i_1} \cdots v_{i_s} \)
be a reduced expression for \( w_j \); for \( 1 \leq j \leq s, v_{i_j} \in \mathcal{L}_{j} \). Choose
any coset $B w J w b' \subseteq B w J B w b$: then $B w b' = B w b$ and by the above
\[ k_{B w J w b'} = k_{B w i_1 \ldots w_i w b'} = k_{B w i_3 \ldots w_i w b'} = \ldots = k_{B w b'} = k_{B w b}. \]
Now $E_J f_{B w b} = \frac{1}{f(q)_J} \sum_{w_j \in W_J} \sum_{B g \subseteq B w J B w b} f_{B g}$, and so
\[ f = \sum_{w^{-1} \in X_J} \sum_{B w b \subseteq B w B} k_{B w b} E_J f(q)_J f_{B w b}. \]
Conversely for all $w^{-1} \in X_J$ and all $B w b \subseteq B w B$, $E_J f_{B w b} \in E_J L(q)$. Now for any $w^{-1} \in X_J$, $f_{B w b}$ occurs with non-zero coefficient in $E_J f_{B w b}$ but in no other $E_J f_{B w'b'}$, where $(w')^{-1} \in X_J$, with $B w'b' \neq B w b$. (See (1.3.2) for this.) Hence $\{E_J f_{B w b}: w^{-1} \in X_J, B w b \subseteq B w B\}$ is a set of linearly independent elements which generate $E_J L(q)$, and so is a basis of $E_J L(q)$. Thus
\[ \dim E_J L(q) = \sum_{w \in X_J} c_w \text{ as } c_w = c_{w^{-1}} \text{ for all } w \in W. \]

Finally, let $G / G_J$ denote the set of right cosets of $G_J$ in $G$ and let $L_J(q)$ be the set of functions $f: G / G_J \to K$.
$L_J(q)$ becomes a left $K G_J$-module by defining for all $x \in G$ the function $x f \in L_J(q)$ given by
\[ (x f)(G_J g) = f(G_J g x) \text{ for all } G_J g \in G / G_J. \]
Let $M_J$ be the principal $K G_J$-module. Then as in (7.1.1), $L_J(q)$ is isomorphic as $K G_J$-module to $M_J^G$.

Now $L_J(q)$ has $K$-basis $\{f_{G_J w b}: w^{-1} \in X_J, G_J w b \subseteq G_J w B\}$ where $f_{G_J w b}$ is given by
\[ f_{G_J w b}(G_J g) = \begin{cases} 1 & \text{if } G_J g = G_J w b \\ 0 & \text{if } G_J g \neq G_J w b \end{cases} \text{ for all } G_J g \in G / G_J. \]
Then the map $\phi: E_J L(q) \to L_J(q)$ given by $\phi(E_J f_{B w b}) = f_{G_J w b}$.
for all $w^{-1} \in X$, $BwB \subseteq BwB$, extended by linearity to $E_{j}L(q)$, is clearly an isomorphism of KG-modules.

(2) Similarly, by noting that for all $w \in W$, $w \neq 1$, $q^{c}w = 0$ in $K$.

(7.1.8) **PROPOSITION:** (1) $O_{j}L(q) = \{ f \in L(q): E_{\{ w_{i} \}} f = 0 \}

for all $w_{i} \in J \} = \{ f \in L(q): O_{\{ w_{i} \}} f = f \}

for all $w_{i} \in J \}$. Similarly, $O_{j}L(0) = \{ f \in L(0): O_{\{ w_{i} \}} f = 0 \}

for all $w_{i} \in J \}$. o

**Proof:** (1) Clearly $O_{j}L(q) = \{ f \in L(q): E_{\{ w_{i} \}} f = 0 \}

for all $w_{i} \in J \}$. Conversely, let $f \in L(q)$ satisfy $E_{\{ w_{i} \}} f = 0

for all $w_{i} \in J \}$. Then $(1 + a_{w_{i}})f = 0$ for all $w_{i} \in J \}; that

is, $a_{w_{i}} f = -f$ for all $w_{i} \in J \}. Hence if $w \in W_{j}$, $a_{w} f =\ (-1)^{l(w)} f,$

so $O_{j}f = f$, and $f \in O_{j}L(q)$. Then

$O_{j}L(q) = \{ f \in L(q): E_{\{ w_{i} \}} f = 0 \}

for all $w_{i} \in J \}$.

Similarly, if $O_{\{ w_{i} \}} f = f$, then $(q^{c_{1}} - a_{w_{i}}) f = (1 + q^{c_{1}}) f,$

and so $a_{w_{i}} f = -f$, and so also

$O_{j}L(q) = \{ f \in L(q): O_{\{ w_{i} \}} f = f \}

for all $w_{i} \in J \}$. Suppose $f = \sum_{w \in W} \sum_{Bwb \subseteq BwB} k_{Bwb} f_{Bwb} \in O_{j}L(q),

where the $k_{Bwb} \in K$. Then $E_{\{ w_{i} \}} f = 0$ for all $w_{i} \in J \} and so

$f + \sum_{w \in W} \sum_{Bwb \subseteq BwB} k_{Bwb} a_{w_{i}} f_{Bwb} = 0$ for all $w_{i} \in J \}. That is,

$f + \sum_{w \in W} \sum_{Bwb \subseteq BwB} \sum_{B_{i}B_{w}} k_{Bwb} f_{B_{i}B_{w}} = 0$ for all $w_{i} \in J \}.
Now if \( w^{-1}(r_i) > 0 \), then \( B_{w_1}wB \subseteq B_{w_1}wB \), and \( (w_1w)^{-1}(r_i) < 0 \).

If \( w^{-1}(r_i) < 0 \), then \( w = w_1w' \) for some \( w' \in W \) with
\[
l(w) = l(w') + 1, \text{ and } (w')^{-1}(r_i) > 0.
\]
Moreover,
\[
\{B_{w_1}wB' \subseteq B_{w_1}wB\} = B_{w'B} \cup \{B_{w'} \subseteq B_{w_1}B_{w'B}, B_{w'} \neq B_{wB}\}.
\]

Suppose \( w^{-1}(r_i) > 0 \). Then \( f_{B_{wB}} \) occurs on the right side of the last equation with coefficient
\[
k_{B_{wB}} + \sum_{B_{w_1}wB' \subseteq B_{w_1}wB} k_{B_{w_1}wB'} = 0.
\]
Since \( \{f_{B_{wB}} : w \in W, B_{wB} \subseteq B_{wB}\} \) are a basis of \( L(q) \), this coefficient is zero.

Thus
\[
k_{B_{wB}} + \sum_{B_{w_1}wB' \subseteq B_{w_1}wB} k_{B_{w_1}wB'} = 0 \quad \text{for all } w \in W,
\]
\( w^{-1}(r_i) > 0, B_{wB} \subseteq B_{wB}, w_i \in J \).

Suppose \( w^{-1}(r_i) < 0 \). Then as the coefficient of \( f_{B_{wB}} \) is zero, we have
\[
k_{B_{wB}} + \sum_{B_{w_1}wB' \subseteq B_{w_1}wB} k_{B_{w_1}wB'} = 0.
\]
i.e.
\[
k_{B_{w_1}wB} + \sum_{B_{w_1}wB' \subseteq B_{w_1}wB} k_{B_{w_1}wB'} = 0.
\]
But this is the same type of equation as above, as
\[
(w_1w)^{-1}(r_i) > 0.
\]
Hence for all \( w_i \in J \), for all \( w^{-1} \in X \{w_i\} \), for all
\( B_{wB} \subseteq B_{wB}, k_{B_{wB}} + \sum_{B_{w_1}wB' \subseteq B_{w_1}wB} k_{B_{w_1}wB'} = 0.\)

Let \( x^{-1} \in X_J \) and \( w \in W_J \), \( w \neq w_{oJ} \). There exists \( w_j \in J \) such that \( (wx)^{-1}(r_j) > 0 \), and so for any \( B_{wxB} \subseteq B_{wxB} \),
\[
k_{B_{wxB}} + \sum_{B_{w_1}wxB' \subseteq B_{w_1}wxB} k_{B_{w_1}wxB'} = 0.
\]
Do the same for each \( B_{w_1wxB} \subseteq B_{w_1wxB} \), provided \( w_j \neq w_{oJ} \).
This process terminates, and if \( w_{oJ} = w'w \), where
\[
l(w_{oJ}) = l(w') + l(w),\]
we have
\[
k_{BwxB} + (-1)^{l(w')} + 1 \sum_{Bw_{oJ}x} k_{Bw_{oJ}x'} = 0.
\]
Thus if \( w \in W \), \( w = w_Jx \) with \( x^{-1} \in X_J, w_J \in W_J \), and
\[
l(w) = l(w_J) + l(x),\]
then for any \( BwB \subseteq Bw_{oJ}x \), \( k_{BwB} \) can be expressed as a linear combination of \( \{ k_{Bw_{oJ}x} \} \):
\[
Bw_{oJ}x \subseteq Bw_{oJ}w_JBw_{oJ}x_B.\]
Now for each \( x^{-1} \in X_J \), for each
\[
Bw_{oJ}x \subseteq Bw_{oJ}xB, \text{ define } F_{Bw_{oJ}x} = \sum_{w \in W_J} (-1)^{l(w)}f_{Bw_{oJ}x}.
\]
Then
\[
f = \sum_{x^{-1} \in X_J} \sum_{Bw_{oJ}x \subseteq Bw_{oJ}xB} (-1)^{l(w)}k_{Bw_{oJ}x}F_{Bw_{oJ}x}.
\]
Conversely, for all \( w_i \in J, E_{\{ w_i \}}F_{Bw_{oJ}x} = 0 \), for all
\( x^{-1} \in X_J \) and all \( Bw_{oJ}x \subseteq Bw_{oJ}xB \). Hence \( \{ F_{Bw_{oJ}x}: x^{-1} \in X_J, Bw_{oJ}x \subseteq Bw_{oJ}xB \} \) is a basis of \( O_JL(q) \), and so
\[
dim O_JL(q) = \sum_{w \in Z_J} c_{w_{oJ}x}.\]
Now clearly if \( x^{-1} \in X_J \) then
\[
(w_{oJ}x)^{-1}((\prod J)) \subseteq \phi^-.\]
Conversely, suppose that \( w \in W \) and
\[
w^{-1}((\prod J)) \subseteq \phi^- \). Then \( (w_{oJ}w)^{-1} \in X_J \), and so \( w = w_{oJ}x^{-1} \) for
some \( x \in X_J \). Thus \( \dim O_JL(q) = \sum_{w \in Z_J} c_w \).

(2) Done similarly.

(7.1.9) **PROPOSITION:** (1) \( E_JL(q) = \sum_{L \supseteq J} E_{L \supseteq J}O_JL(q) \) and
\[
O_JL(q) = \sum_{L \supseteq J} O_{L \supseteq J}E_{L \supseteq J}L(q) \]
for all \( J \subseteq R \).

(2) \( e_JL(0) = \sum_{L \supseteq J} e_{L \supseteq J}O_JL(0) \) and
\[
o_JL(0) = \sum_{L \supseteq J} o_{L \supseteq J}E_{L \supseteq J}L(0) \]
for all \( J \subseteq R \).
Proof: (1) For each $L \supseteq J$, $E_L O^J L(q) \subseteq E_J L(q)$. So
\[ \sum_{L \supseteq J} E_L O^J L(q) \subseteq E_J L(q). \] But $E_J L(q) = E_J H_K(q) L(q)$ and so
\[ E_J L(q) = \left( \sum_{L \supseteq J} E_L O^J H_K(q) \right) L(q) \subseteq \sum_{L \supseteq J} E_L O^J H_K(q) L(q) \]
\[ = \sum_{L \supseteq J} E_L O^J L(q). \]
Hence $E_J L(q) = \sum_{L \supseteq J} E_L O^J L(q)$. Similarly, as
$O_J H_K(q) = \sum_{L \supseteq J} O_L E^J H_K(q)$, we get $O_J L(q) = \sum_{L \supseteq J} O_L E^J L(q)$.

(2) Done similarly as $e_J H_K(0) = \sum_{L \supseteq J} e_L O^J H_K(0)$ and
$O_J H_K(0) = \sum_{L \supseteq J} O_L e^J H_K(0)$.

(7.1.10) Corollary: (1) For all $J \subseteq R$,
\[ \dim E_J O^J L(q) = \sum_{w \in Y_J} q^w \] and \[ \dim e_J O^J L(0) = \sum_{w \in Y_J} q^w. \]
(2) For all $J \subseteq R$,
\[ \dim O_J E_J L(q) = \sum_{w \in Y_J} q^w \] and \[ \dim o_J e_J L(0) = \sum_{w \in Y_J} q^w. \]

Proof: (1) $\dim E_J L(q) = \sum_{w \in X_J} q^w$. We show by decreasing
\[ \dim E_J O^J L(q) = \sum_{w \in Y_J} q^w. \]
induction on $|J|$ that $\dim E_J O^J L(q) = \sum_{w \in Y_J} q^w$.

Suppose $J = R$. Then $E_R L(q) = E_R O^R L(q)$ has dimension
$\sum_{w \in X_R} q^w$. But $X_R = Y_R$ and so $\dim E_R O^R L(q) = \sum_{w \in Y_R} q^w$.

Suppose $|J| < |R|$. $E_J L(q) = \sum_{L \supseteq J} E_L O^J L(q)$. By
induction, $\dim E_L O^J L(q) = \sum_{w \in Y_L} q^w$ for all $L \supseteq J$. Hence
\[ \dim E_J O^J L(q) = \dim E_J L(q) - \sum_{L \supseteq J} \dim E_L O^J L(q) \]
So \( \dim E_J O_J L(q) = \sum_{w \in X_J} q^c w - \sum_{y \in Y_J} q^c y \)
\( \leq \sum_{w \in Y_J} q^c w \), as \( X_J = \bigcup_{L \supseteq J} Y_L \), a disjoint union. Similarly, \( \dim e_J O_J L(0) = \sum_{w \in Y_J} q^c w \).

(2) As \( \dim O_J L(q) = \sum_{w \in Z_J} q^c w \), and
\( O_J L(q) = \sum_{L \supseteq J} O_J L^E L(q) \), we show by decreasing induction on \(|J|\) that \( \dim O_J E_J L(q) = \sum_{w \in Y_J} q^c w \).

If \( J = R \), \( Z_R = Y_R \) and \( O_R L(q) = O_R E_R L(q) \) and so \( \dim O_R L(q) = \sum_{w \in Y_R} q^c w \). Suppose that \( J \subset R \). Then
\( Z_J = \bigcup_{L \supseteq J} Y_L = \bigcup_{L \supseteq J} Y_L \), both being disjoint unions.

So \( \dim O_J E_J L(q) = \dim O_J L(q) - \sum_{L \supseteq J} \dim O_L E_L L(q) \)
\( = \sum_{w \in Y_J} q^c w \), by the above comments on \( Z_J \).

(7.1.11) **COROLLARY:** (1) \( E_J O_J L(q) \cong E_J L(q) / \sum_{L \supseteq J} E_L L(q) \) and
\( e_J O_J L(0) \cong e_J L(0) / \sum_{L \supseteq J} E_L L(0) \) for all \( J \subseteq R \).

(2) \( O_J E_J L(q) \cong O_J L(q) / \sum_{L \supseteq J} O_L L(q) \) and
\( o_J e_J L(0) \cong o_J L(0) / \sum_{L \supseteq J} O_L L(0) \) for all \( J \subseteq R \).

(7.1.12) **COROLLARY:** (1) \( E_J O_J L(q) \) affords the character
\( \chi_J = \sum_{L \supseteq J} (-1)^{|L-J|} (1_{G_L})^G \)
of \( G = G(q) \) over \( K \).
(2) \( e_J o \Lambda L(0) \) affords the character \( \mu_J = \sum_{L \supset J} (-1)^{|L-J|} (1_G)_L^G \) of \( G = G(q) \) over \( K \).

**Proof:** (1) Use decreasing induction on \( |J| \).

Suppose \( J = R \). Then \( E_R L(q) = \{ f \in L(q) : f(Bg) = f(Bg') \} \) for all \( Bg, Bg' \in G/B \) = \{constant functions of \( L(q) \)\}. So \( E_R L(q) \) affords the character \( 1_G \) of \( G \); i.e., \( u_R = 1_G = (1_G)^G \).

Suppose \( |J| < |R| \). Then

\[
E_J L(q) = E_J o \Lambda L(q) \oplus \bigoplus_{L \supset J} E_L o \Lambda L(q).
\]

Now \( E_J L(q) \) affords the character \( (1_G)_J^G \) of \( G \) by (7.1.7), and by induction \( E_L o \Lambda L(q) \) affords the character \( \mu_L \) of \( G \).

Hence \( E_J o \Lambda L(q) \) affords the character \( \mu_J' \) of \( G \), where

\[
\mu_J' = (1_G)_J^G - \sum_{L \supset J} \mu_L
= (1_G)_J^G - \sum_{L \supset J} \sum_{L \supset L} (-1)^{|L-L'|} (1_G)_L^{L'}.
\]

Now if \( K \supset J \), the coefficient of \( (1_G)_K^G \) in this is

\[\sum_{L \subseteq K \subseteq M} (-1)^{|K-L|}.\]

Suppose \( M = J \cup \{ w_1, \ldots, w_r \} \), with \( w_j \in J \) for all \( j, 1 \leq j \leq r \).

Then

\[
\sum_{L \subseteq K \subseteq M} (-1)^{|K-L|} = 1 - r + \frac{r(r-1)}{2} - \ldots + (-1)^{r-1} \frac{r!}{(r-1)!} \frac{r!}{1!} \quad \text{for any positive integer } k \leq r,
\]

where \( r = |M-J| \) and for any positive integer \( k \leq r \),

\[
r_{C_k} = \frac{(r)!}{(r-k)!k!} = \frac{1}{r} \binom{r}{k}.
\]

Then \( \mu_J' = u_J = \sum_{L \supset J} (-1)^{|L-J|} (1_G)_L^G \).

(2) Done similarly.
(7.1.13) **COROLLARY:** If \( \mu_J \) is the character of \( G \) given in (1) or (2) of (7.1.12), then for all \( J \subseteq R \),
\[
(1_G)^G_J = \sum_{L \supseteq J} \mu_L.
\]
In particular, if \( J = \emptyset \), then \( (1_B)^G = \sum_{J \subseteq R} \mu_J \).

**Proof:** Use the inversion formula, (1.4.4).

(7.1.14) **THEOREM:** (1) \( E_J \circ J L(q) \cong 0_J E_J L(q) \) as left KG-modules, for all \( J \subseteq R \).

(2) \( e_J \circ J L(0) \cong o_J e_J L(0) \) as left KG-modules, for all \( J \subseteq R \).

**Proof:** (1) From (5.19), the map \( \Psi_J : E_J \circ J H_K(q) \rightarrow 0_J E_J H_K(q) \) given by left multiplication by \( O_J \) is an isomorphism of right \( H_K(q) \)-modules, and \( 0_J E_J H_K(q) = 0_J E_J L(q) \).

Define \( \Psi_J' : E_J \circ J L(q) \rightarrow 0_J E_J L(q) \) by left multiplication by \( O_J \). \( \Psi_J' \) is well-defined and is a homomorphism of left KG-modules. \( \Psi_J' \) is onto, as
\[
0_J E_J O_J L(q) = 0_J E_J O_J H_K(q)L(q) = 0_J E_J H_K(q)L(q) = 0_J E_J L(q).
\]
As \( \dim E_J O_J L(q) = \dim 0_J E_J L(q) \), \( \Psi_J' \) is one-one. Hence for all \( J \subseteq R \), \( E_J O_J L(q) \cong 0_J E_J L(q) \).

(2) Done similarly, using (4.4.13).

(7.1.15) **PROPOSITION:** If \( J = R \) or \( R - \{ w_i \} \) for any \( w_i \in R \),

(1) \( E_J O_J L(q) = \{ f \in E_J L(q) : E_M f = 0 \text{ for all } M \supseteq J \} \), and

(2) \( e_J O_J L(0) = \{ f \in e_J L(0) : e_M f = 0 \text{ for all } M \supseteq J \} \).

**Proof:** Clearly, for any \( J \subseteq R \), we have that
E_J \Omega J \gamma L(q) \leq \{ f \in E_J L(q) : E_M f = 0 \text{ for all } M \supset J \}, \text{ and}

e_J \Omega J \gamma L(0) \leq \{ f \in e_J L(0) : e_M f = 0 \text{ for all } M \supset J \}.

The result is obviously true if J = R. So suppose
J = R - \{ w_i \} for some w_i \in R. Let f \in E_J L(q) = E_R L(q) \oplus E_J \Omega J \gamma L(q)
by (7.1.9(1)). Suppose that f = f_R + f_J , where f_R \in E_R L(q) and f_J \in E_J \Omega J \gamma L(q); assume E_R f = 0. Then clearly E_R f_J = 0, and so E_R f_R = f_R = 0. Hence f = f_J and
E_J \Omega J \gamma L(q) = \{ f \in E_J L(q) : E_R f = 0 \}. Similarly, if J = R - \{ w_i \},
e_J \Omega J \gamma L(0) = \{ f \in e_J L(0) : e_R f = 0 \}.

For the rest of this section, we will consider
both L(q) and L(0) together, writing them as L, and writing
the corresponding idempotents E_J, O_J and e_J, o_J respectively
as e_J, o_J, for all J \subseteq R. Then for all J \subseteq R, we have that
e_J \Omega J \gamma L \leq \{ f \in e_J L : E_M f = 0 \text{ for all } M \supset J \}.

(7.1.16) Lemma: If f \in L, J \subseteq R, then for all g \in G_J,
e_J f(B g) = e_J f(B).

Proof: e_J f \in e_J L, and the result follows by (7.1.7), as
we can consider e_J f as a function on the right cosets of
G_J in G.

Hence any element of e_J L is determined by its values
on the cosets B w_b, where w^{-1} \in X_J and B w_b \subseteq B W. Suppose
f \in e_J L; then e_J f = f. Let M \supset J and suppose that e_M f = 0.
Since e_M f \in e_M L, e_M f is determined by its values on the
cosets B w_B, where w^{-1} \in X_M and B w_B \subseteq B W. Now let w^{-1} \in X_M.
and $Bwb \subseteq BwB$. Then $e_Mf(Bwb) = 0$ gives

$$
\sum_{v \in W_M} \sum_{Bg \subseteq BvBwb} f(Bg) = 0.
$$

Now each $v \in W_M$ has the form $v = v(1)v(2)$, where $v(1) \in W_J$, $(v(2))^{-1} \in X^M_J$, and $l(v) = l(v(1)) + l(v(2))$. So the above equation becomes

$$
\sum_{v^{-1} \in X^M_J} \sum_{w_J \in W_J} \sum_{Bg \subseteq BwJvBwb} f(Bg) = 0; \text{ that is,}
$$

$$
\sum_{v^{-1} \in X^M_J} \sum_{Bg \subseteq BvBwb} \sum_{w_J \in W_J} \sum_{Bg' \subseteq BwJbg} f(Bg') = 0
$$

Since $e_Jf = f$, $f(Bg) = f(Bg')$ if $gg'^{-1} \in G_J$, we have that

$$
\sum_{w_J \in W_J} \sum_{Bg' \subseteq BwJbg} f(Bg') = f(q)\sum_{w_J \in W_J} \sum_{Bg' \subseteq BwJbg} f(Bg), \text{ and so}
$$

$$
e_Mf(Bwb) = f(q)\sum_{v^{-1} \in X^M_J} \sum_{Bg \subseteq BvBwb} f(Bg) = 0
$$

Now if $K$ is of characteristic $0$, $f(q)_J \neq 0$ for all $J \subseteq R$, and if the characteristic of $K$ is $p$, where $q = p^s$ for some positive integer $s$, then $f(q)_J = 1$ in $K$ for all $J \subseteq R$. Hence for all $w^{-1} \in X^M_J$ and all $Bwb \subseteq BwB$,

$$
\sum_{v^{-1} \in X^M_J} \sum_{Bg \subseteq BvBwb} f(Bg) = 0 \quad (*)
$$

Now let $L_J = \{f \in e_JL : e_Mf = 0 \text{ for all } M \supset J\}$. Then if $f \in L_J$, $w^{-1} \in X^M_J$ for some $M \supset J$ and $Bwb \subseteq BwB$, then

$$
\sum_{v^{-1} \in X^M_J} \sum_{Bg \subseteq BvBwb} f(Bg) = 0
$$

Suppose $\hat{J} = \{w_1, \ldots, w_r\}$, and let $M_i = J \cup \{w_i\}$, $1 < i < r$. Then for each $M_i$ and for each coset $Bwb \subseteq BwB$ where $w^{-1} \in X^M_{M_i}$,
\[ \sum_{v \in X_J}^{M_i} f(Bg) = 0 \quad \text{[A]} \]

Since \( L_J \leq e_J \), we may consider any \( f \in L_J \) as a function on the cosets of \( G_J \) in \( G \), and then [A] becomes:

\[ \sum_{v \in X_J}^{M_i} f(G_Jg) = 0, \text{ for all } BwB \subseteq BwBwB \]

where \( w^{-1} \in X_{M_i} \) \[ \text{[B]} \]

Consider the Tits complex \( \Delta_J \) of \( G \) with respect to \( G_J \), as in (6.1), and in particular, consider the homology module \( H_{r-1}(\Delta_J) \otimes K \). Suppose \( |J| \geq 2 \). By (6.1.17)

\[ c = \sum_{w \in X_J}^{M_i} \sum_{bwB \subseteq BwB} k^c_{bwBwJ}, \text{ where the } k^c_{bwBwJ} \in K, \text{ is an} \]

\((r-1)\)-cycle of \( \Delta_J \) (with coefficients in \( K \)), if and only if for each \( M_i \), for each \( w \in X_{M_i} \) and for each \( bwB \subseteq BwB \),

\[ \sum_{v \in X_J}^{M_i} \sum_{gG_J \subseteq bwBwG_J} k^c_g = 0 \quad \text{[C]} \]

Comparing [B] and [C] for each \( v \in X_{M_i} \) and each \( w \in X_{M_i} \), we have that \( f(G_Jv^{-1}w^{-1}b^{-1}) \) satisfies the same equations as \( k^c_{bwBw} \). Hence for each \((r-1)\)-cycle

\[ c = \sum_{w \in X_J}^{M_i} \sum_{bwB \subseteq BwB} k^c_{bwBwJ}, \text{ define the function} \]

\( f^c \in L_J \) by \( f^c(G_Jv^{-1}w^{-1}b^{-1}) = k^c_{bwBw} \) for all \( w \in X_{M_i}, v \in X_{J} \), and \( bwB \subseteq BwB \). Similarly, given any function \( f \in L_J \), we can define an \((r-1)\)-cycle \( c(f) \in \Delta_J \) by

\[ c(f) = \sum_{w \in X_J}^{M_i} \sum_{bwB \subseteq BwB} k^c_{bwBwJ}, \text{ where for all } w \in X_J \text{ and} \]

\[ \text{[D]} \]
all \(\text{bwB} \subseteq \text{BwB}, k_{\text{bw}}^c = f(G_j w^{-1} b^{-1})\). Then if \(c_1, \ldots, c_t\) are a basis of \(H_{r-1}(\Delta J) \otimes K\), \(f^{c_1}, \ldots, f^{c_t}\) must be linearly independent elements of \(L_j\), and if \(f_1, \ldots, f_s\) are a basis of \(L_j\), then \(c(f_1), \ldots, c(f_s)\) are linearly independent \((r-1)\)-cycles. Thus \(\dim L_J = \dim H_{r-1}(\Delta J) \otimes K = \dim e_J o_J L\), where \(|\hat{J}| \geq 2\). Hence if \(|\hat{J}| \geq 2\),

\(e_J o_J L = \{f \in e_J L : e_{M} f = 0 \text{ for all } M \supset J\}\). So:

(7.1.17) \text{THEOREM: For all } J \subseteq R,

\(e_J o_J L = \{f \in e_J L : e_{M} f = 0 \text{ for all } M \supset J\}\)

(7.1.18) \text{THEOREM: For all } J \subseteq R, \text{ where } |\hat{J}| \geq 2, |\hat{J}| = r,

\(e_J o_J L\) and \(H_{r-1}(\Delta J) \otimes K\) are isomorphic KG-modules.

\text{Proof:} We have a K-space isomorphism

\[ T_J : H_{r-1}(\Delta J) \otimes K \rightarrow e_J o_J L \]

given as follows: if \(c = \sum_{w \in X_J} \sum_{\text{bwB} \subseteq \text{BwB}} k_{\text{bw}}^c \text{bw}_{\text{J}}^f\),

where the \(k_{\text{bw}}^c \in K\), and \(c \in H_{r-1}(\Delta J) \otimes K\), define

\(T_J(c) = f^c\), where for all cosets \(G_j w^{-1} b^{-1}\) of \(G_j\) in \(G\),

\(f^c(G_j w^{-1} b^{-1}) = k_{\text{bw}}^c \), \(w \in X_J\). Then

\(f^c = \sum_{w \in X_J} \sum_{\text{bwB} \subseteq \text{BwB}} k_{\text{bw}}^c \text{bw}_{\text{J}}^f G_j w^{-1} b^{-1}\), a linear combination

of the characteristic functions of \(e_J L\).

Further, \(T_J\) is a KG-module isomorphism, for if \(g \in G\),

\(c \in H_{r-1}(\Delta J) \otimes K, c = \sum_{w \in X_J} \sum_{\text{bwB} \subseteq \text{BwB}} k_{\text{bw}}^c \text{bw}_{\text{J}}^f\), then

\(gc = \sum_{w \in X_J} \sum_{\text{bwB} \subseteq \text{BwB}} k_{\text{bw}}^c \text{gbw}_{\text{J}}^f\), that is,
\[ gc = \sum_{w \in X_J} \sum_{b \in B} k_{bw} g_{b^{-1}w} \text{. Then} \]

\[ g_{f c} = \sum_{w \in X_J} \sum_{b \in B} k_{bw} f_{g_{b^{-1}w}^{-1}} \]

\[ = \sum_{w \in X_J} \sum_{b \in B} k_{bw} f_{G_J w^{-1}b^{-1}g^{-1}} \]

\[ = \sum_{w \in X_J} \sum_{b \in B} k_{bw} f_{G_J w^{-1}b^{-1}} = f_{gc} \cdot \]

(7.1.19) **COROLLARY:** If \( K \) is a field of characteristic \( p \) and \( G \) is a finite group with a split \((B,N)\) pair of characteristic \( p \), then for all \( J \subseteq R \) with \( |J| = r \geq 2 \), \( H_{r-1}(\Delta_J) \otimes K \) is an indecomposable \( KG \)-module.

Let \( G_J \) be a parabolic subgroup of \( G \). Then

\[ (1_B^{G_J}) = \sum_{L \subseteq J} \psi_L^{(J)}, \text{ where } \psi_L^{(J)} = \sum_{M \subseteq L} (-1)^{|M-L|} \cdot (1_{G_M}^{G_J}) \cdot L \subseteq M \subseteq J \]

where the \( \psi_L^{(J)} \) are ordinary characters of \( G_J \) if the characteristic of \( K \) is 0, and Brauer characters if the characteristic of \( K \) divides \( q \), where \( G_J = G_J(q) \).

Let \( w_1 \in \hat{J} \), and let \( L_1 = J \cup \{w_1\} \). Then

\[ G_{L_1}^{G_{L_1}} \cong ((1_B^{G_J})^{G_{L_1}})^{G_{L_1}} \text{ as } KG_{L_1} \text{-modules, and} \]

\[ (1_B^{G_{L_1}}) = \sum_{M \subseteq L_1} \phi_M^{(L_1)} \text{, where} \]

\[ \phi_M^{(L_1)} = \sum_{N \subseteq M \subseteq L_1} (-1)^{|N-M|} \cdot (1_{G_N}^{G_{L_1}}) \cdot M \subseteq N \subseteq L_1 \]
(7.1.20) **PROPOSITION:** With notation as above,

\[
(\psi_{j}^{M})_{G_{L_{1}}} = \phi_{M}^{(L_{1})} + \phi_{M \cup \{w_{i}\}}^{(L_{1})}
\]

**Proof:**

\[
\begin{align*}
\left(\psi_{j}^{M}\right)_{G_{L_{1}}} &= \sum_{M \subseteq N \subseteq J} (-1)^{|N-M|} \left((1_{G_{N}})^{G_{L_{1}}}ight)_{G_{L_{1}}} \\
&= \sum_{M \subseteq N \subseteq J} (-1)^{|N-M|} \left((1_{G_{N}})^{G_{L_{1}}}ight)_{G_{L_{1}}} \\
&= \sum_{M \subseteq N \subseteq L_{1}} (-1)^{|N-M|} \left((1_{G_{N}})^{G_{L_{1}}}ight)_{G_{L_{1}}} \\
&= \sum_{M \subseteq N \subseteq L_{1}} (-1)^{|N-M|} \left((1_{G_{N}})^{G_{L_{1}}}ight)_{G_{L_{1}}} \\
&\quad + \sum_{M \cup \{w_{i}\} \subseteq N \subseteq L_{1}} (-1)^{|N-(M \cup \{w_{i}\})|} \left((1_{G_{N}})^{G_{L_{1}}}ight)_{G_{L_{1}}} \\
&= \phi_{M}^{(L_{1})} + \phi_{M \cup \{w_{i}\}}^{(L_{1})}
\end{align*}
\]
(7.2) On the Composition Factors of \( L(\Omega) \).

We restrict attention in this section to the case where \( G \) is a finite group with a split \((B,N)\) pair \((G,B,N,R,U)\) of rank \( n \) and characteristic \( p \), and with Weyl group \( W \). Let \( K \) be a field of characteristic \( p \). Then

\[
L = L(\Omega) = \sum_{J \subseteq R} e_{\Omega}^{J} L,
\]

a direct sum of indecomposable \( KG \)-modules. This decomposition arises from the direct sum decomposition of \( A = \mathbb{E}_{KG}(L) \) as \( A = \sum_{J \subseteq R} e_{\Omega}^{J} A \), where \( e_{\Omega}^{J} A \) is a right \( A \)-module of dimension \( |Y_{J}| \) over \( K \) and basis \( \{ e_{\Omega}^{y} a_{J}^{-1} : y \in Y_{J} \} \). List the elements \( y_{1}, \ldots y_{s} \) of \( Y_{J} \) in order of increasing length; if \( i < j \) then \( l(y_{i}) \leq l(y_{j}) \). Let \( A_{J,i} = \{ \sum_{j=i}^{s} k_{j} e_{\Omega}^{y} a_{J}^{-1} : k_{j} \in K \} \).

Then \( A_{J,1} = e_{\Omega}^{J} A > A_{J,2} > \ldots > A_{J,s} > 0 \) is a composition series for \( e_{\Omega}^{J} A \) of right \( A \)-modules.

Consider the corresponding series of \( KG \)-submodules of \( L \): \( e_{\Omega}^{J} L = A_{J,1} L > A_{J,2} L > \ldots > A_{J,s} L > 0 \) - (7.2.1)

(7.2.2) \textbf{Lemma}: \( A_{J,i} L / A_{J,i+1} L \) is generated as \( KG \)-module by \( e_{\Omega}^{y} a_{J}^{-1} f_{B} + A_{J,i+1} L \) for all \( i, 1 \leq i \leq s \), where \( A_{J,s+1} L = 0 \).

\textbf{Proof}: Result follows as \( f_{B} \) generates \( L \) as \( KG \)-module.

(7.2.3) \textbf{Lemma}: For all \( i, 1 \leq i \leq n \), \( a_{wi} f_{B} = S_{i} f_{B} \), where \( S_{i} \) is defined in (2.3.2).

\textbf{Proof}: We show both \( a_{wi} f_{B} \) and \( S_{i} f_{B} \) are functions in \( L \).
which take value 1 on $Bw_1B$ and 0 outside, and hence are equal.

$$a_w f_B(Bg) = \sum_{B' \subset Bw_1B} f_B(Bg')$$

$$= \begin{cases} 1 & \text{if } B \subset Bw_1B \\ 0 & \text{otherwise.} \end{cases}$$

Now $B \subset Bw_1B$ if and only if $Bg \subset Bw_1B$. Suppose $Bg \subset Bw_1B$; then for some $b \in B$, $Bg = Bw_1b$, and

$$Bw_1Bw_1b \cap B = (Bw_1Bw_1 \cap B)b = (Bw_1B \cap Bw_1)w_1b = Bw_1w_1b = B.$$

Thus $a_w f_B(Bg) = 1$ if and only if $Bg \subset Bw_1B$.

$$S_i f_B(Bg) = \sum_{u \in X_i} u f_B(Bg)$$

and so $S_i f_B(Bg) = \sum_{u \in X_i} u f_B(Bg)$

$$= \sum_{u \in X_i} f_B(Bgun_i).$$

Now $f_B(Bgun_i) = 0$ unless $Bgun_i = B$, i.e. unless $Bg = Bn_i^{-1}u^{-1}$, and then $g \in Bn_i^{-1}B = Bw_1B$. Conversely, suppose $g \in Bn_i^{-1}B$, and let $g \in Bn_i^{-1}u^{-1}$ for some $u \in X_i$. Then

$$S_i f_B(Bg) = \sum_{u \in X_i} f_B(Bn_i^{-1}u^{-1}u'n_i) = \sum_{u \in X_i} f_B(Bn_i^{-1}un_i).$$

$f_B(Bn_i^{-1}un_i) = 0$ unless $n_i^{-1}un_i \in B$. But $n_i^{-1}un_i \in X_i$, and $X_i^{n_i} = U^{n_i} \cap V \subset V$. So $f_B(Bn_i^{-1}un_i) = 0$ unless $n_i^{-1}un_i \in B \cap V = 1$. So $S_i f_B(Bg) = f_B(B) = 1$. Hence $S_i f_B$ takes value 1 on $Bn_i^{-1}B = Bw_1B$ and 0 on $Bn_w^{-1}B$ if $w \neq w_1$.

The result now follows.

(7.2.4) THEOREM: $A_{j,i}^L/A_{j,i+1}^L$ is generated as KG-module by a weight element of weight $(1_B, [i_1, \ldots, i_n])$, where
$i_j = 0$ if $y_i^{-1}(r_j) > 0$ and $i_j = -1$ if $y_i^{-1}(r_j) < 0$: that is, if $A_{J,i}/A_{J,i+1}$ is isomorphic to the $A$-module affording the representation $\lambda_L$, where $y_i^{-1} \in Y_L$, then $A_{J,i}/A_{J,i+1}$ is generated as $K\Gamma$-module by a weight element of weight

$(1_B, \{i_1, \ldots, i_n\})$, where $i_j = \begin{cases} 0 & \text{if } w_j \in L \\ -1 & \text{if } w_j \in \hat{L}. \end{cases}$

**Proof:** $A_{J,i}/A_{J,i+1}$ is generated as $K\Gamma$-module by

$$e_j \circ a_{y_i}^{-1} f_B + A_{J,i+1}.$$

For all $b \in B$,

$$b(e_j \circ a_{y_i}^{-1} f_B + A_{J,i+1}) = b(e_j \circ a_{y_i}^{-1} f_B) + A_{J,i+1} = e_j \circ a_{y_i}^{-1} f_B + A_{J,i+1}.$$

Hence $e_j \circ a_{y_i}^{-1} f_B + A_{J,i+1}$ is $U$-invariant and affords the character $1_B$ of $B$.

Now for each $w_j \in R$, consider

$$S_j (e_j \circ a_{y_i}^{-1} f_B + A_{J,i+1}) = S_j e_j \circ a_{y_i}^{-1} f_B + A_{J,i+1} = e_j \circ a_{y_i}^{-1} S_j f_B + A_{J,i+1} \text{ as } S_j \in K\Gamma$$

$$= e_j \circ a_{y_i}^{-1} a_{w_j} f_B + A_{J,i+1}$$

$$= \begin{cases} -(e_j \circ a_{y_i}^{-1} f_B + A_{J,i+1}) & \text{if } y_i^{-1}(r_j) < 0 \\ 0 & \text{if } y_i^{-1}(r_j) > 0. \end{cases}$$

From now on, we assume that $K$ is a splitting field for the subgroup $H$ of $G$.

*(7.2.5) Corollary:* $A_{J,i}/A_{J,i+1}$ has a unique
maximal submodule $A^L_j/A_{j+1}$ such that

$$( A^L_j/A_{j+1} ) / ( A^L_j/A_{j+1} ) \cong M(1_{B};i_1,...,i_n),$$

the irreducible KG-module of weight $(1_{B};i_1,...,i_n)$.

Proof: Use (2.4.21).

(7.2.6) Theorem: Let $G$ be a finite group with a split $(B,N)$ pair $(G,B,N,R,U)$ of rank $n$ and characteristic $p$, and let $W$ be the Weyl group of $G$. Let $K$ be a field of characteristic $p$ which is a splitting field for the subgroup $H$ of $G$. Let $W$ be the principal $K$-$B$-module, and $M^G$ the KG-module induced from $M$. Then the irreducible KG-module $M(1_{B};J) = M(1_{B};\mu_1,...,\mu_n)$, where $\mu_i=0$ if $w_i \in J$ and $\mu_i=-1$ if $w_i \in \hat{J}$, occurs as a composition factor of $M^G$ with multiplicity at least $|Y_J|$.

Proof: Consider the composition series for $A$ obtained by taking a composition series for each $e_J o_{J} A$, for all $J \subseteq R$. This gives us a corresponding series of KG-submodules. Let $A = A_1 > A_2 > A_3 > ... > A_{|\lambda|} > 0$ be the composition series for $A$. Then each $A_i/A_{i+1}$ is an irreducible right $A$-module, generated by $e_J o_{J} A y + A_{i+1}$ for some $J \subseteq R$ and some $y^{-1} \in Y_J$. This affords the representation $\lambda_N$ of $A$, where $y \in Y_N$. The corresponding factor in the series of KG-modules, $A_i^L/A_{i+1}^L$, (as $L \cong K^G$ as left KG-module), is generated as KG-module by a weight vector.
of weight \((1_B; \mu_1, \ldots, \mu_n)\), where \(\mu_i = 0\) if \(w_i \in N\), and \(\mu_i = -1\) if \(w_i \in \hat{N}\), by (7.2.4). By (7.2.5), \(A_i^L/A_{i+1}^L\) has as a composition factor the irreducible KG-module \(M(1_B; \mu_1, \ldots, \mu_n)\). Each such element \(y \in Y_N\) gives rise to \(M(1_B; \mu_1, \ldots, \mu_n)\) in this way, and so \(M(1_B; \mu_1, \ldots, \mu_n) = M(1_B, N)\) occurs in the composition series of \(M^G\) with multiplicity at least \(|Y_N|\).

(7.2.7) Theorem: With \(G\) and \(K\) as in (7.2.6), \(e_j^{o_j a_{yF}}(w_o w_{oJ})^{-1}L\) is an irreducible submodule of \(L\), where \(w_o w_{oJ}\) is the unique element of maximal length in \(Y_J\).

Proof: The KG-module \(\mathcal{F}_{1_B}^J\) of (2.4.3) is the same as the KG-module \(L\) which we have discussed in this section, and the KG-endomorphism \(T_n^J\) of \(\mathcal{F}_{1_B}^J\) is the same as the KG-endomorphism \(a_w\) of \(L\), where \(\Theta(n) = w\). Hence for each \(J \subseteq R\), the submodule \(\Theta_{w_o}^J L\) is an irreducible KG-submodule of \(L\), where \(\Theta_{w_o}^J\) is defined as in (2.4.12) in terms of the elements \(a_{w_i} \in A = E_n KG(L)\) in place of the elements \(T_i\) for all \(w_i \in R\).

We will prove the theorem by showing that for all \(J \subseteq R\), \(\Theta_{w_o}^J L = e_j^{o_j a_{yF}}(w_o w_{oJ})^{-1}L\), where \(J\) is defined in (2.4.13).

Write \(N = \hat{J}\). Then \(\Theta_{w_o}^N L\) is an irreducible submodule of \(L\), and by (2.4.14) and (2.4.17) we have

\[
\Theta_{w_o}^N L \leq \{ f \in L : a_{w_i} f = 0 \text{ for all } w_i \in J, \quad (1 + a_{w_i}) f = 0 \text{ for all } w_i \in \hat{J} \}.
\]
So $\Theta^N_{w_0} \leq \{ f \in e_{J}: e_{w_0} f = 0 \text{ for all } M \supseteq J \} = e_{\Theta^J} L$ by (7.1.17). Now $\Theta^N_{w_0} \in A$, and if $a \in A$ satisfies $a f_B = 0$, then we must have $a = 0$. So $\Theta^N_{w_0} \in e_{\Theta^J} A$. Let

$$\Theta^N_{w_0} = \sum_{y \in Y_J} k_y e_{\Theta^J} y^{-1}, \quad k_y \in K.$$ 

By (2.4.17), $S^N_{1} e_{\Theta^N_{w_0}} f_B = \begin{cases} \Theta^N_{w_0} f_B & \text{if } w_i \in \hat{N} \\ 0 & \text{if } w_i \in N. \end{cases}$

Thus since $S^N_{1} e_{\Theta^N_{w_0}} f_B = e_{\Theta^N_{w_0}} S^N_{1} f_B = e_{\Theta^N_{w_0}} a f_B$ for all $w_i \in R$, we have $e_{\Theta^N_{w_0}} aw_i = \begin{cases} -e_{\Theta^N_{w_0}} & \text{if } w_i \in \hat{N} \\ 0 & \text{if } w_i \in N. \end{cases}$

Now, for all $y \in Y_J$,

$$e_{\Theta^J} y^{-1} a w_i = \begin{cases} -e_{\Theta^J} y^{-1} & \text{if } y^{-1}(r_i) < 0 \\ 0 & \text{if } y^{-1}(r_i) = r_j \text{ for some } r_j \in \prod_J \\ e_{\Theta^J} (w_i y)^{-1} & \text{if } y^{-1}(r_i) > 0 \text{ and } y^{-1}(r_i) \neq r_j \end{cases}$$

for any $r_j \in \prod_J$, with $w_i y \in Y_J$.

So $e_{\Theta^N_{w_0}} aw_i = \sum_{y \in Y_J} k_y e_{\Theta^J} y^{-1} + \sum_{y \in Y_J} k_y e_{\Theta^J} (w_i y)^{-1}$. (2)

If $w_i \in \hat{N}$, then we must have, comparing (1) and (2), that $k_y = 0$ for all $y \in Y_J$ with $y^{-1}(r_i) > 0$. Hence

$$e_{\Theta^N_{w_0}} = \sum_{y \in Y_J} k_y e_{\Theta^J} y^{-1}.$$ 

$$y^{-1}(\prod_N) < 0$$

Further, if $w_i \in N$, then comparing (1) and (2) we get

(a) $k_y = 0$ if $y^{-1}(r_i) < 0$, and $y^{-1}(r_i) = -r_j$ for some $r_j \in \prod_J$. 

(b) If \( y^{-1}(r_i) < 0 \) and \( y^{-1}(r_i) \neq -r_j \) for any \( r_j \in \Pi_j \), then \( k_y = k_{w_i}y \).

(c) If \( y^{-1}(r_i) > 0 \) and \( y^{-1}(r_i) \neq r_j \) for any \( r_j \in \Pi_j \), then \( k_y = k_{w_i}y \).

Choose \( y \in Y_j \) such that \( k_y \neq 0 \). Then \( y^{-1}(\Pi_N^c) < 0 \), and \( y^{-1}(r_i) \neq -r_j \) for any \( r_i \in \Pi_N \) with \( r_j \in \Pi \). Hence for all \( r_i \in \Pi_N \), either \( y^{-1}(r_i) < 0 \) but \( y^{-1}(r_i) \neq -r_j \) for any \( r_j \in \Pi_j \) or \( y^{-1}(r_i) > 0 \).

If \( y^{-1}(r_i) < 0 \) and \( y^{-1}(r_i) \neq -r_j \), some \( r_j \in \Pi_j \), for all \( r_i \in \Pi_N \), then \( y^{-1}(\Pi) < 0 \) and so \( y^{-1} = w_o \). So \( y = w_o \) and \( J = \emptyset \). In this case \( \emptyset_{w_o} = a_{w_o} \) and the result is true.

Hence suppose there exists \( r_i \in \Pi_N \) such that \( y^{-1}(r_i) > 0 \). If \( y^{-1}(r_i) \neq r_j \) for any \( r_j \in \Pi_j \), then \( k_y = k_{w_i}y \). Continuing in this way, we end up with an element \( y_o \in Y_J \), \( k_{y_o} \neq 0 \), such that \( y_o^{-1}(\Pi_L^c) < 0 \) for some \( L \supseteq \hat{N} \), and if \( r_i \in \Pi_L \), then \( y_o^{-1}(r_i) = r_j \) for some \( r_j \in \Pi_j \). Then by (1.3.7(1)), \( y_o^{-1} \) is the unique element of maximal length in \( Y_L \), so \( y_o^{-1} = w_o w_o \hat{L} \). Then \( y_o = w_o w_o \hat{L} \). But \( y_o \in Y_J \), and so \( J = \hat{L} \).

That is, \( N = \hat{J} = \hat{L} \). So \( y_o^{-1}(\Pi_{\hat{N}}^c) < 0 \) and if \( r_i \in \Pi_{\hat{N}} \), then \( y_o^{-1}(r_i) = r_j \) for some \( r_j \in \Pi_J \). In particular, there is no \( r_i \in \Pi_{\hat{N}} \) with \( y_o^{-1}(r_i) < 0 \). Hence \( k_{y_o} \) is the only non-zero coefficient and

\[
\theta_{\hat{J}}^J = \theta_{\hat{N}}^N = e_{j_0} e_{j_0}^{-1} (w_o w_o)^{-1} k, \quad \text{for some} \ k \in K.
\]
Hence \( e_j Q a \left( w_o w_o j \right)^{-1} L \) is irreducible.

(7.2.8) **COROLLARY:** For all \( J \subseteq R \), \( e'_J \) is a scalar multiple of \( e_j Q a \left( w_o w_o j \right)^{-1} \).

We now know what \(|W|\) of the composition factors of \( L \) are, but there may be others. We will give some examples later.

Consider the natural composition series of \( A \), given by (4.2.4). Let

\[
A = I_1 > I_2 > \ldots > I_{|W|} > 0
\]

be this natural composition series. Then for all \( j, 1 \leq j \leq |W| \), \( I_j \) is a two-sided ideal of \( A \) and \( I_j / I_{j+1} \) is a left and a right \( A \)-module, where \( I_{|W|+1} = 0 \).

This gives us a series of KG-submodules of \( L \) which are also \( A \)-submodules of \( L \):

\[
L = I_1 L > I_2 L > \ldots > I_{|W|} L > 0.
\]

If \( I_j / I_{j+1} \) is generated as \( A \)-module by \( a_v + I_{j+1} \) for some \( v \in W \), then \( I_j L / I_{j+1} L \) is generated as \( A \)-module and as KG-module by \( \sum w \neq B + I_{j+1} L \).

(7.2.9) **LEMMA:** \( \sum w \neq B + I_{j+1} L \) is a weight vector of weight \((1_B; \mu_1, \ldots, \mu_n)\) where \( \mu_i = 0 \) if \( w(r_i) > 0 \) and \( \mu_i = -1 \) if \( w(r_i) < 0 \). Hence \( I_j L / I_{j+1} L \) has a unique maximal KG-submodule \( I_{j+1} L / I_{j+1} L \) such that
\[
\left( \frac{I^j L}{I^j L} \right) / \left( \frac{I^j L}{I^j L} \right) \cong M(1_B; \mu_1, \ldots, \mu_n) \]
as KG-modules.

**Proof:** \(a_w f_B + I_{j+1} L\) is clearly \(U\)-invariant and affords the character \(1_B\) of \(B\).

\[
S_i (a_w f_B + I_{j+1} L) = S_i a_w f_B + I_{j+1} L = a_w S_i f_B + I_{j+1} L = a_w a_{w_i} f_B + I_{j+1} L = \begin{cases} -(a_w f_B + I_{j+1} L) & \text{if } w(r_i) < 0 \\ 0 & \text{if } w(r_i) > 0. \end{cases}
\]

Hence \(a_w f_B + I_{j+1} L\) is a weight element of weight \((1_B; \mu_1, \ldots, \mu_n)\) where \(\mu_i = 0\) if \(w(r_i) > 0\) and \(\mu_i = -1\) if \(w(r_i) < 0\).

The rest follows by (2.4.21).

Hence using the natural composition series of \(A\), we also get that \(M(1_B; \mu_1, \ldots, \mu_n)\) occurs as a composition factor of the KG-module \(M^G\) with multiplicity at least \(|Y_J|\), where \(J = \{w_1 \in R; \mu_1 = 0\}\).

**EXAMPLES:**

(1) Let \(G = SL_2(p)\), the group of 2x2 matrices of determinant 1 over the field \(GF(p)\) of \(p\) elements; \(G\) has a split \((B,N)\) pair with \(B(p)\) the subgroup of upper triangular matrices, \(N(p)\) the subgroup of monomial matrices, and \(H(p)\) the subgroup of diagonal matrices. \(W = \frac{N(p)}{H(p)}\) and \(W\) is the Weyl group of type \(A_1\); also \(W \cong S_2\).
Let $K$ be a field of characteristic $p$ which is a splitting field for \( R(p) \). Let $L$ be the KG-module induced from the trivial $KB(p)$-module. Then

\[
L = (1 + a_{w_1})L \oplus a_{w_1}L
\]

where \((1 + a_{w_1})L\) is an irreducible KG-module of dimension 1 and weight \((1_B;0,0)\), and \(a_{w_1}L\) is an irreducible KG-module of dimension $p$ and weight \((1_B;-1)\).

(2) Let $G = SL_3(p)$, the group of $3 \times 3$ matrices of determinant one over the field $GF(p)$ of $p$ elements. Then $G$ has a split $(B,N)$ pair, where $B$ is the subgroup of upper triangular matrices, $N$ is the subgroup of monomial matrices, $H$ is the subgroup of diagonal matrices and the Weyl group $W$ of $G$ is of type $A_2$. Write $R = \{w_1, w_2\}$.

Let $K$ be a field of characteristic $p$ which is a splitting field for $H$, and let $L$ be the KG-module induced from the principal $KB$-module. Then:

\[
L = (1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})L \oplus (1 + a_{w_1})a_{w_2}L
\]

\[\oplus (1 + a_{w_2})a_{w_1}L \oplus a_{w_1}w_2w_1L.\]

Now, \((1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})L\) is an irreducible KG-module of dimension 1 and weight \((1_B;0,0)\), and \(a_{w_1}w_2w_1L\) is an irreducible KG-module of dimension $p^3$ and weight \((1_B;-1,-1)\).

\((1 + a_{w_1})a_{w_2}L\) is an indecomposable KG-module
of dimension $p^2 + p$. It has an irreducible submodule $(1 + a_{w_1})a_{w_2}^L$ of weight $(1_B; -1, 0)$. By unpublished work of Braden, $\dim M(1_B; -1, 0) = \frac{p^2 + p}{2}$ and $\dim M(1_B; 0, -1) = \frac{p^2 + p}{2}$. The factor module $$(1 + a_{w_1})a_{w_2}^L/(1 + a_{w_1})a_{w_2}w_1^L$$ contains an isomorphic copy of $M(1_B; 0, -1)$, and considering dimensions, $$(1 + a_{w_1})a_{w_2}^L/(1 + a_{w_1})a_{w_2}w_1^L \cong M(1_B; 0, -1)$$. So we have the following situation:

There is a similar situation for the indecomposable KG-module $(1 + a_{w_2})a_{w_1}^L$, as follows:

In this example, there are only $|W|$ composition factors of $L$.

(3) Let $G = Sp_4(p)$, the group of $4 \times 4$ matrices $T$ over the field $GF(p)$ which satisfy $T^*ET = E$, where $E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$. $G$ has a split $(B, N)$ pair, and the Weyl group $\mathcal{W}$ of $G$ is of
Let \( K \) be a field of characteristic \( p \) which is a splitting field for the subgroup \( H \) of \( G \), and let \( L \) be the \( KG \)-module induced from the principal \( KB \)-module. Then:

\[
L = \left[ 1 + a_{w_1}w_2w_1w_2 \right] L \oplus (1 + a_{w_1})a_{w_2}L \oplus (1 + a_{w_2})a_{w_1}L \\
\oplus a_{w_1}w_2w_1w_2L.
\]

The irreducible \( KG \)-submodules of \( L \) are (for \( p > 2 \))

(a) \( \left[ 1 + a_{w_1}w_2w_1w_2 \right] L \), of dimension 1 and weight \((1_B;0,0)\),
(b) \( (1 + a_{w_1})a_{w_2}w_1w_2L \), of weight \((1_B;0,-1)\) and dimension \( \frac{p(p+1)(2p+1)}{6} \)
(c) \( (1 + a_{w_2})a_{w_1}w_2w_1L \), of weight \((1_B;1,0)\) and dimension \( \frac{p(p+1)(p+2)}{6} \)
(d) \( a_{w_1}w_2w_1w_2L \), of dimension \( p^4 \) and weight \((1_B;1,1)\).

So we need only consider the indecomposable submodules \((1 + a_{w_1})a_{w_2}L\) and \((1 + a_{w_2})a_{w_1}L\). We have a series of \( KG \)-submodules of \((1 + a_{w_1})a_{w_2}L\) as follows:

\[
(1 + a_{w_1})a_{w_2}L > (1 + a_{w_1})a_{w_2}w_1L > (1 + a_{w_1})a_{w_2}w_1w_2L > 0.
\]

The top factor, \((1 + a_{w_1})a_{w_2}L/(1 + a_{w_1})a_{w_2}w_1L\), contains a copy of \( K(1_B;0,-1) \), the middle factor

\[
(1 + a_{w_1})a_{w_2}w_1L/(1 + a_{w_1})a_{w_2}w_1w_2L \]

contains a copy of \( K(1_B;1,0) \), and the bottom factor \((1 + a_{w_1})a_{w_2}w_1w_2L \approx K(1_B;0,0) \)
But \( \dim (1 + a_{w_1}) a_{w_2} L = p^3 + p^2 + p \), and

\[
2 \dim M(1_B; 0, -1) + \dim M(1_B; -1, 0) = \frac{5p^3 + 9p^2 + 4p}{6}; \text{ hence}
\]

there are composition factors of dimension \( \frac{p(p-1)(p-2)}{6} \) to be accounted for.

Similarly, \( (1 + a_{w_2}) a_{w_1} L \) contains \( M(1_B; -1, 0) \) with multiplicity at least 2 and \( M(1_B; 0, -1) \) with multiplicity at least 1. In this case there are composition factors of dimension \( \frac{p(p-1)(2p-1)}{6} \) to be accounted for.

Now, the irreducible KG-modules where \( G = \text{Sp}_4(p) \) and \( K = \text{GF}(p) \) are in one-one correspondence with points \((a, b)\) in the restricted fundamental region \( 0 \leq a \leq p-1, 0 \leq b \leq p-1 \). Let \( d(a, b) \) be the dimension of the irreducible representation of the algebraic group \( \text{Sp}_4(C) \) with dominant weight \((a, b)\). \( d(a, b) \) is given by Weyl's dimension formula, and is as follows:

\[
d(a, b) = \frac{(a+1)(b+1)(a+b+2)(a+2b+3)}{6}
\]

Let \( M(a, b) \) be the irreducible KG-module corresponding to the pair \((a, b), 0 \leq a \leq p-1, 0 \leq b \leq p-1\), and let \( m(a, b) = \dim M(a, b) \). If \( p > 2 \), then we have

\[
m(0, 0) = d(0, 0) = 1
\]
\[
m(p-1, p-1) = d(p-1, p-1) = p^4
\]
\[
m(p-1, 0) = d(p-1, 0) = \frac{p(p+1)(p+2)}{6}
\]
\[
m(0, p-1) = d(0, p-1) = \frac{p(p+1)(2p+1)}{6}
\]
\[ m(p-3,0) = d(p-3,0) = \frac{p(p-1)(p-2)}{6} \]

\[ m(0,p-2) = d(0,p-2) = \frac{p(p-1)(2p-1)}{6} \]

The corresponding points occur in the restricted fundamental region as follows:

From (2.4.22), \( M(0,0) \) corresponds to \( M(1,0,0) \),

\( M(p-1,p-1) \) corresponds to \( M(1,1,-1) \),

\( M(p-1,0) \) corresponds to \( M(1,1,0) \),

and \( M(0,p-1) \) corresponds to \( M(1,0,1) \).

Conjecture: \((1 + a_{w_1})a_{w_2}L\) contains \( M(p-3,0) \) with multiplicity 1, and \((1 + a_{w_2})a_{w_1}L\) contains \( M(0,p-2) \) with multiplicity 1.

Example: When \( p = 3 \), \( m(p-3,0) = m(0,0) = 1 \), and
\( m(0, p-2) = m(0,1) = 5. \) Clearly \( M(0,0) \) is the only possibility for an extra factor of \( (1 + a_{w_1})a_{w_2}L \).

(4) Let \( G = \text{SL}_4(p) \), the group of 4x4 matrices of determinant one over the field \( \text{GF}(p) \) of \( p \) elements. Then

\( G \) has a split \((B,N)\) pair, with \( B \) the subgroup of upper triangular matrices, \( N \) the subgroup of monomial matrices, \( H \) the subgroup of diagonal matrices and the Weyl group \( W \) of \( G \) is of type \( A_3 \). Write \( R = \{w_1, w_2, w_3\} \).

Let \( K \) be a field of characteristic \( p \) which is a splitting field for \( H \), and let \( L \) be the \( KG \)-module induced from the principal \( KB \)-module. Then:

\[
L = \left[ 1 + a_{w_1 w_2 w_3 w_1 w_2 w_1} \right] L \otimes \left[ 1 + a_{w_1 w_2 w_1} \right] a_{w_3} L
\]

\[
\otimes \left[ 1 + a_{w_1 w_3} \right] a_{w_2} L \otimes \left[ 1 + a_{w_2 w_3 w_2} \right] a_{w_1} L
\]

\[
\otimes (1 + a_{w_1}) a_{w_2 w_3 w_2} L \otimes (1 + a_{w_2}) a_{w_1 w_2} L
\]

\[
\otimes (1 + a_{w_3}) a_{w_1 w_2 w_1} L \otimes a_{w_1 w_2 w_3 w_1 w_2 w_1} L.
\]

Now \( \left[ 1 + a_{w_1 w_2 w_3 w_1 w_2 w_1} \right] L \) is irreducible, of dimension 1 and weight \((1_B; 0, 0, 0)\), and \( a_{w_1 w_2 w_3 w_1 w_2 w_1} L \) is irreducible, of dimension \( p^6 \) and weight \((1_B; -1, -1, -1)\).

\[
\left[ 1 + a_{w_1 w_2 w_1} \right] a_{w_3} L \text{ has dimension } p^3 + p^2 + p, \text{ and contains }
\]

\( N(1_B; 0, 0, -1), M(1_B; 0, -1, 0) \) and \( K(1_B; -1, 0, 0) \).

\[
\left[ 1 + a_{w_1 w_3} \right] a_{w_2} L \text{ has dimension } p^4 + p^3 + 2p^2 + p, \text{ and }
\]
contains $M(1_B;0,0,-1)$, $M(1_B;0,-1,0)$ twice, $M(1_B;-1,0,0)$

and $M(1_B;-1,0,-1)$.

$\left[1 + a_{w_2w_3w_2}\right] a_{w_1} L$ has dimension $p^3+p^2+p$, and

contains $M(1_B;0,0,-1)$, $M(1_B;0,-1,0)$ and $M(1_B;-1,0,0)$.

$(1 + a_{w_1}) a_{w_2w_3w_2} L$ has dimension $p^5+p^4+p^3$, and

contains $M(1_B;0,-1,-1)$, $M(1_B;-1,0,-1)$ and $M(1_B;-1,-1,0)$.

$(1 + a_{w_2}) a_{w_1w_3w_2} L$ has dimension $p^5+2p^4+p^3+p^2$, and

contains $M(1_B;0,-1,-1)$, $M(1_B;-1,0,-1)$ twice, $M(1_B;-1,-1,0)$

and $M(1_B;0,-1,0)$.

$(1 + a_{w_3}) a_{w_1w_2w_1} L$ has dimension $p^5+p^4+p^3$, and

contains $M(1_B;0,-1,-1)$, $M(1_B;-1,0,-1)$ and $M(1_B;-1,-1,0)$.

We look at the cases when $p=2$ and $p=3$.

<table>
<thead>
<tr>
<th></th>
<th>$p=2$</th>
<th>$p=3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim $M(1_B;0,0,-1)$</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>dim $M(1_B;0,-1,0)$</td>
<td>6</td>
<td>19</td>
</tr>
<tr>
<td>dim $M(1_B;-1,0,0)$</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>dim $M(1_B;0,-1,-1)$</td>
<td>20</td>
<td>126</td>
</tr>
<tr>
<td>dim $M(1_B;-1,-1,0)$</td>
<td>20</td>
<td>126</td>
</tr>
<tr>
<td>dim $M(1_B;-1,0,-1)$</td>
<td>15</td>
<td>69</td>
</tr>
</tbody>
</table>

The case $p=2$

We must have that $(1 + a_{w_1}) a_{w_2w_3w_2} L$, $(1 + a_{w_2}) a_{w_1w_3} L$

and $(1 + a_{w_3}) a_{w_1w_2w_1} L$ each contain two copies of
$M(1_B;0,0,0)$. We then have all the composition factors in this case.

The case $p=3$.

\[
\left[1 + a_{w_1 w_2 w_1}\right] a_{w_3} L \quad \text{and} \quad \left[1 + a_{w_2 w_3 w_2}\right] a_{w_1} L
\]

have no other factors other than those already given.

\[
\left[1 + a_{w_1 w_2}\right] a_{w_2} L \quad \text{must also contain two copies of} \quad M(1_B;0,0,0).
\]

Both \((1 + a_{w_1}) a_{w_2 w_3 w_2} L\) and \((1 + a_{w_3}) a_{w_1 w_2} L\) have other factors of total dimension 30, and \((1 + a_{w_2}) a_{w_1 w_3} L\) has other factors of dimension totalling 32.
Appendix 1: CLASSIFICATION OF FINITE COXETER SYSTEMS.

Let \((W,R)\) be a finite Coxeter system. The graph \(D\) of \((W,R)\) is a graph where the nodes are in one-one correspondence with the elements of \(R\), and the number of bonds connecting nodes \(P_i\) and \(P_j\) is equal to \(n_{ij}-2\), where \(n_{ij}\) is the order of \(w_i w_j\) in \(W\), for all \(w_i, w_j \in R\). \(P_i\) and \(P_j\) are not connected if and only if \(w_i w_j = w_j w_i\), \(i \neq j\).

If \((W,R)\) is indecomposable, then its graph \(D=D(W)\) is connected. If \((W,R)\) is not indecomposable, then its graph \(D\) is a disjoint union of connected components:

\[ D = D_1 \cup D_2 \cup \ldots \cup D_s \]

where each \(D_i\) is a non-zero connected graph. Each \(D_i\) determines a subset \(J_i\) of \(R\), and hence a parabolic subgroup \(W_{J_i}\) of \(W\). Then, \(W = W_{J_1} \times W_{J_2} \times \ldots \times W_{J_s}\), a direct product of subgroups.

THEOREM: Let \((W,R)\) be a finite indecomposable Coxeter system. Then the graph of \((W,R)\) must be one of the following:

- \( A_n, n \geq 1 \)
- \( B_n, n \geq 2 \)
- \( D_n, n \geq 4 \)
- \( E_6 \)
- \( E_7 \)
\( E_8 \)

\( P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_6 \quad P_7 \)

\( F_4 \)

\( P_1 \quad P_2 \quad P_3 \quad P_4 \)

\( G_2 \)

\( P_1 \quad P_2 \quad P_3 \)

\( H_3 \)

\( P_1 \quad P_2 \quad P_3 \quad P_4 \)

\( H_4 \)

\( o--o \quad (m-2) \text{ bonds} \)

\( I_2(m), \ m=3, \text{ or } m \geq 5. \)

\( P_1 \quad P_2 \)
Appendix 2: SOME EXAMPLES OF FINITE COXETER GROUPS.

We give some examples of finite Coxeter groups, calculating for each the sets $Y_J$, the orders of the elements in $Y_J$, and the set $K \subset R$ such that $w^{-1} \in Y_K$, where $w \in Y_J$. Finally we list the conjugacy classes of the group.

The groups we discuss are:

(1) $W(A_1) \cong S_2$, $|W(A_1)| = 2$.

(2) $W(A_2) \cong S_3$, $|W(A_2)| = 6$.

(3) $W(A_3) \cong S_4$, $|W(A_3)| = 24$.

(4) $W(A_4) \cong S_5$, $|W(A_4)| = 120$.

(5) $W(B_2) \cong D_8$, the dihedral group of order 8.

(6) $W(B_3)$, $|W(B_3)| = 48$.

(7) $W(G_2) \cong D_{12}$, the dihedral group of order 12.

(8) $W(I_2(8)) \cong D_{16}$, the dihedral group of order 16.

(9) $W(A_1) \times W(A_1) \cong D_4$, the dihedral group of order 4.

(10) $W(A_1) \times W(A_2)$, $|W(A_1) \times W(A_2)| = 12$.

(11) $W(A_2) \times W(A_2)$, $|W(A_2) \times W(A_2)| = 36$. 
\( W = W(A_1) = \langle w_1 : w_1^2 = 1 \rangle \) \((\cong S_2)\)

\[
\begin{array}{|c|c|c|c|}
\hline
J & \{w : w \in Y_f\} & |w| & K : w^{-1} \in Y_K \\
\hline
\emptyset & \{w_1\} & 2 & \emptyset \\
\{w_1\} & \{1\} & 1 & \{w_1\} \\
\hline
\end{array}
\]

Conjugacy classes:

\( C_1 = \{1\} \), \( C_2 = \{w_1\} \).

\[
\begin{array}{|c|c|c|c|}
\hline
J & \{w : w \in Y_f\} & |w| & K : w^{-1} \in Y_K \\
\hline
\emptyset & \{w_1w_2w_1\} & 2 & \emptyset \\
\{w_1\} & \left\{ \begin{array}{l}
\{w_2\} \\
\{w_1w_2\} \\
\end{array} \right. & 3 & \{w_2\} \\
\{w_2\} & \left\{ \begin{array}{l}
\{w_1\} \\
\{w_2w_1\} \\
\end{array} \right. & 3 & \{w_1\} \\
\{w_1, w_2\} & \{1\} & 1 & \{w_1, w_2\} \\
\hline
\end{array}
\]

Conjugacy classes:

\( C_1 = \{1\} \), \( C_2 = \{w_1, w_2, w_1w_2w_1\} \), \( C_3 = \{w_1w_2, w_2w_1\} \).
\begin{align*}
W = W(A_3) &= \langle w_1, w_2, w_3 : w_1^2 = w_2^2 = w_3^2 = (w_1 w_2)^2 = (w_2 w_3)^3 = 1 \rangle \tag{3} \equiv S_4 \end{align*}

\begin{array}{|c|c|c|}
\hline
J & \{ w : w \in Y_J \} & |w| \hline
\emptyset & \{ w_1 w_2, w_1 w_3 w_2, w_2 w_1 \} & 2 & \emptyset \\
\{ w_1 \} & \{ w_2 w_3, w_2 w_1 w_3, w_1 w_2 w_3 \} & 2 & \{ w_1 \} \\
& \{ w_1 w_2 w_3, w_2 w_1 w_2 w_3, w_1 w_2 w_1 \} & 3 & \{ w_2 \} \\
\{ w_2 \} & \{ w_1 w_3, w_2 w_1 w_3, w_1 w_2 w_3 \} & 2 & \{ w_1 \} \\
& \{ w_1 w_2 w_1, w_3 w_1 w_2 w_1, w_2 w_3 w_1 w_2 w_1 \} & 4 & \{ w_2 \} \\
\{ w_3 \} & \{ w_1 w_2 w_1, w_3 w_1 w_2 w_1, w_2 w_3 w_1 w_2 w_1 \} & 3 & \{ w_1 \} \\
\{ w_1, w_2 \} & \{ w_3, w_2 w_3, w_1 w_2 w_3 \} & 2 & \{ w_1, w_2 \} \\
& \{ w_1 w_2, w_3, w_1 w_2 w_3 \} & 3 & \{ w_1, w_3 \} \\
\{ w_1, w_3 \} & \{ w_2, w_1 w_3, w_3 w_1 w_2 w_3 \} & 2 & \{ w_1, w_3 \} \\
& \{ w_2 w_1 w_3, w_1 w_2 w_3 w_2, w_2 w_1 w_3 w_2 \} & 4 & \{ w_3 \} \\
\{ w_2, w_3 \} & \{ w_1 w_2, w_3, w_1 w_2 w_3 \} & 3 & \{ w_3 \} \\
& \{ w_2 w_1, w_3, w_3 w_1 w_2 w_3 \} & 4 & \{ w_3 \} \\
\{ w_1, w_2, w_3 \} & \{ 1 \} & 1 & \{ w_1, w_2, w_3 \} \\
\hline
\end{array}
Conjugacy classes:

\[ C_1 = \{1\}, \quad C_2 = \left\{ \begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array} \right\}, \quad C_3 = \left\{ \begin{array}{c} w_1 w_3 \\ w_2 w_1 w_3 w_2 \\ w_1 w_2 w_3 w_2 \end{array} \right\} \]

\[ C_4 = \left\{ \begin{array}{c} w_1 w_2 \\ w_2 w_1 \\ w_2 w_3 \\ w_3 w_2 \\ w_1 w_2 w_3 w_2 \\ w_2 w_3 w_2 w_1 \\ w_1 w_2 w_1 w_3 \\ w_1 w_3 w_2 w_1 \end{array} \right\}, \quad C_5 = \left\{ \begin{array}{c} w_1 w_2 w_3 \\ w_2 w_1 w_2 \\ w_2 w_3 w_1 \\ w_3 w_2 w_1 \end{array} \right\} \]
\( W = W(A_4) = \langle w_1, w_2, w_3, w_4 : w_i^2 = 1 \text{ for } 1 \leq i \leq 4, (w_1 w_2)^3 = 1, (w_1 w_3)^2 = (w_1 w_4)^2 = (w_2 w_4)^2 = 1, (w_2 w_3)^3 = (w_3 w_4)^3 = 1 \rangle \)

| \( J \) | \{ \( w : w \in Y_4 \) \} | \( |w| \) | \( K : w^{-1} \in Y_K \) |
|---|---|---|---|
| \( \emptyset \) | \{ \( w_1 w_2 w_3 w_4, w_1 w_2 w_3 w_1 w_2 w_4 \) \} | 2 | \( \emptyset \) |
| \( \{ w_1 \} \) | \{ \( w_2 w_3 w_2 w_4, w_1 w_2 w_3 w_4, w_1 w_2 w_3 w_4 w_1 w_2 w_3 w_2 \) \} | 6 | \( \{ w_2 \} \) |
| \( \{ w_2 \} \) | \{ \( w_1 w_2 w_3 w_4 w_3, w_2 w_1 w_3 w_4, w_1 w_2 w_3 w_4, w_1 w_2 w_3 w_4 w_1 w_2 w_3 w_2 \) \} | 4 | \( \{ w_3, w_4 \} \) |
| \( \{ w_3 \} \) | \{ \( w_1 w_2 w_1 w_4, w_1 w_2 w_3 w_4, w_2 w_1 w_3 w_4, w_1 w_2 w_3 w_4, w_1 w_2 w_3 w_4 w_1 w_2 w_3 w_2 \) \} | 6 | \( \{ w_3 \} \) |
\[ W(A_4) \text{ continued.} \]

| \( J \) | \( \{ w : w \in Y_J \} \) | \( |w| \) | \( K : w^{-1} \in Y_K \) |
|---|---|---|---|
| \( \{ w_4 \} \) | \[ w_1w_2w_1w_3w_2w_1 \]
| | \[ w_1w_2w_4w_3w_1w_2w_1 \]
| | \[ w_1w_3w_2w_1w_4w_3w_2w_1 \]
| | \[ w_2w_1w_3w_2w_4w_3w_1w_2w_1 \] | 2 | \( \{ w_4 \} \) |
| | \[ w_1w_2w_3w_4w_3 \] | 6 | \( \{ w_3 \} \) |
| | \[ w_1w_2w_3w_4w_3 \] | 5 | \( \{ w_2 \} \) |
| | \[ w_2w_3w_2w_4w_3w_1w_2w_1 \] | 4 | \( \{ w_1 \} \) |
| \( \{ w_1, w_2 \} \) | \[ w_2w_3w_4w_3 \] | 2 | \( \{ w_1, w_2 \} \) |
| | \[ w_1w_2w_3w_4w_3 \] | 3 | \( \{ w_1, w_3 \} \) |
| | \[ w_1w_2w_3w_4w_3 \] | 4 | \( \{ w_2, w_3 \} \) |
| | \[ w_2w_3w_2w_4w_3w_1w_2w_1 \] | 4 | \( \{ w_1, w_4 \} \) |
| | \[ w_1w_2w_3w_4w_1w_2w_3 \] | 5 | \( \{ w_2, w_4 \} \) |
| | \[ w_1w_2w_3w_4w_1w_2w_3 \] | 6 | \( \{ w_3, w_4 \} \) |
| \( \{ w_2, w_3 \} \) | \[ w_1w_4 \] | 2 | \( \{ w_2, w_3 \} \) |
| | \[ w_2w_1w_4 \] | 6 | \( \{ w_1, w_3 \} \) |
| | \[ w_1w_3w_4 \] | 6 | \( \{ w_2, w_4 \} \) |
| | \[ w_2w_1w_3w_4 \] | 5 | \( \{ w_1, w_3, w_4 \} \) |
| | \[ w_2w_1w_3w_4 \] | 5 | \( \{ w_1, w_2, w_4 \} \) |
| | \[ w_1w_2w_3w_4w_1w_2w_3 \] | 4 | \( \{ w_3, w_4 \} \) |
| | \[ w_2w_3w_2w_1w_4 \] | 4 | \( \{ w_1, w_4 \} \) |
| | \[ w_3w_4w_3w_2w_1 \] | 4 | \( \{ w_1, w_2 \} \) |
| | \[ w_1w_2w_3w_4w_2w_1 \] | 3 | \( \{ w_2, w_4 \} \) |
| | \[ w_2w_3w_4w_3w_2w_1 \] | 3 | \( \{ w_1, w_3 \} \) |
| | \[ w_1w_2w_3w_4w_2w_1 \] | 2 | \( \{ w_2, w_3 \} \) |
\[ W(A_4) \text{ continued.} \]

| \( J \) | \( \{ w : w \in Y_J \} \) | \( |w| \) | \( K : w^{-1} \in Y_K \) |
|---|---|---|---|
| \( \{ w_3, w_4 \} \) | \[ w_1 w_2 w_1 \] | 2 | \( \{ w_3, w_4 \} \) |
| | \[ w_1 w_2 w_1 \] | 3 | \( \{ w_2, w_4 \} \) |
| | \[ w_2 w_1 w_3 w_2 w_1 \] | 4 | \( \{ w_1, w_4 \} \) |
| | \[ w_1 w_4 w_3 w_2 w_1 \] | 4 | \( \{ w_2, w_3 \} \) |
| | \[ w_2 w_4 w_3 w_2 w_1 \] | 5 | \( \{ w_1, w_3 \} \) |
| | \[ w_3 w_4 w_2 w_1 w_3 w_2 w_1 \] | 6 | \( \{ w_1, w_2 \} \) |
| \( \{ w_1, w_3 \} \) | \[ w_2 w_4 \] | 2 | \( \{ w_1, w_3 \} \) |
| | \[ w_1 w_2 w_4 \] | 6 | \( \{ w_2, w_3 \} \) |
| | \[ w_3 w_4 w_2 \] | 4 | \( \{ w_1, w_2, w_4 \} \) |
| | \[ w_1 w_3 w_2 w_4 \] | 5 | \( \{ w_2, w_4 \} \) |
| | \[ w_2 w_3 w_2 w_4 \] | 3 | \( \{ w_1, w_4 \} \) |
| | \[ w_3 w_4 w_3 w_2 \] | 3 | \( \{ w_1, w_2 \} \) |
| | \[ w_1 w_2 w_3 w_4 w_2 \] | 4 | \( \{ w_2, w_4 \} \) |
| | \[ w_2 w_1 w_3 w_2 w_4 \] | 6 | \( \{ w_1, w_2, w_4 \} \) |
| | \[ w_1 w_3 w_4 w_3 w_2 \] | 4 | \( \{ w_2 \} \) |
| | \[ w_2 w_3 w_4 w_3 w_2 \] | 2 | \( \{ w_1, w_3 \} \) |
| | \[ w_1 w_2 w_1 w_3 w_2 w_4 \] | 5 | \( \{ w_3, w_4 \} \) |
| | \[ w_1 w_2 w_3 w_4 w_3 w_2 \] | 3 | \( \{ w_2, w_3 \} \) |
| | \[ w_2 w_1 w_3 w_4 w_3 w_2 \] | 2 | \( \{ w_1, w_3 \} \) |
| | \[ w_1 w_2 w_3 w_4 w_1 w_3 w_2 \] | 4 | \( \{ w_3 \} \) |
| | \[ w_2 w_3 w_2 w_1 w_4 w_3 w_2 \] | 6 | \( \{ w_1, w_4 \} \) |
| | \[ w_1 w_2 w_3 w_4 w_2 w_1 w_3 w_2 \] | 5 | \( \{ w_2, w_4 \} \) |
| $w$ | $\{w : w \in Y_J\}$ | $|w|$ | $k : w^{-1} \in Y_K$ |
|-----|----------------------|------|------------------|
| $w_2 \cdot w_4$ | \[
\begin{align*}
w_1w_3 & \\
w_2w_1w_3 & \\
w_1w_4w_3 & \\
w_1w_2w_1w_3 & \\
w_2w_1w_4w_3 & \\
w_2w_2w_1w_1 & \\
w_1w_2w_1w_4w_3 & \\
w_1w_2w_3w_2w_1 & \\
w_2w_4w_3w_2w_1 & \\
w_1w_4w_2w_1w_3 & \\
w_1w_2w_4w_3w_2w_1 & \\
w_1w_3w_2w_4w_3w_1 & \\
w_1w_3w_2w_4w_3w_2w_1 & \\
w_2w_1w_3w_2w_4w_3w_1 & \\
w_2w_1w_3w_2w_4w_3w_2w_1 &
\end{align*}
\] | 2 | \[w_2, w_4\] |
| 4 | \[w_1, w_3, w_4\] |
| 6 | \[w_2, w_3\] |
| 3 | \[w_3, w_4\] |
| 5 | \[w_1, w_3\] |
| 3 | \[w_1, w_4\] |
| 4 | \[w_1, w_4\] |
| 6 | \[w_1, w_2, w_4\] |
| $w_1 \cdot w_4$ | \[
\begin{align*}
w_2w_3w_2 & \\
w_1w_2w_3w_2 & \\
w_2w_4w_3w_2 & \\
w_1w_2w_4w_3w_2 & \\
w_3w_4w_3w_2w_1 & \\
w_1w_2w_1w_4w_3w_2 & \\
w_1w_3w_2w_4w_3w_1 & \\
w_1w_3w_2w_4w_3w_2w_1 & \\
w_2w_1w_3w_2w_4w_3w_1 & \\
w_2w_1w_3w_2w_4w_3w_2w_1 & \\
\end{align*}
\] | 2 | \[w_1, w_4\] |
| 3 | \[w_2, w_4\] |
| 3 | \[w_1, w_3\] |
| 4 | \[w_3, w_4\] |
| 4 | \[w_2, w_3\] |
| 4 | \[w_1, w_2\] |
| 4 | \[w_1, w_3\] |
| 5 | \[w_3\] |
| 5 | \[w_2\] |
| 6 | \[w_1, w_3\] |
| 6 | \[w_2, w_4\] |
| 2 | \[w_1, w_4\] |
$W(A_4)$ continued.

| $J$ | $\{w: w \in Y_j\}$ | $|w|$ | $K: w^{-1} \in Y_K$ |
|-----|-----------------|-----|-----------------|
| $\{w_1, w_2, w_3\}$ | \{ $w_4$, $w_3 w_4$, $w_2 w_3 w_4$, $w_1 w_2 w_3 w_4$ \} | 2 | $\{w_1, w_2, w_3\}$ |
| | | 3 | $\{w_1, w_2, w_4\}$ |
| | | 4 | $\{w_1, w_3, w_4\}$ |
| | | 5 | $\{w_2, w_3, w_4\}$ |
| $\{w_1, w_2, w_4\}$ | \{ $w_3$, $w_2 w_3$, $w_4 w_3$, $w_1 w_2 w_3$, $w_2 w_4 w_3$, $w_1 w_2 w_4 w_3$, $w_2 w_1 w_3 w_2 w_4 w_3$ \} | 2 | $\{w_1, w_2, w_4\}$ |
| | | 3 | $\{w_1, w_3, w_4\}$ |
| | | 4 | $\{w_2, w_3, w_4\}$ |
| | | 5 | $\{w_2, w_3\}$ |
| $\{w_1, w_3, w_4\}$ | \{ $w_2$, $w_1 w_2$, $w_3 w_2$, $w_1 w_3 w_2$, $w_4 w_3 w_2$, $w_2 w_1 w_3 w_2$, $w_1 w_4 w_3 w_2$, $w_2 w_1 w_4 w_3 w_2$, $w_3 w_2 w_1 w_4 w_3 w_2$ \} | 2 | $\{w_1, w_3, w_4\}$ |
| | | 3 | $\{w_2, w_3, w_4\}$ |
| | | 4 | $\{w_2, w_3\}$ |
| | | 5 | $\{w_1, w_3\}$ |
| $\{w_2, w_3, w_4\}$ | \{ $w_1$, $w_2 w_1$, $w_2 w_2 w_1$, $w_4 w_3 w_2 w_1$ \} | 2 | $\{w_2, w_3, w_4\}$ |
| | | 3 | $\{w_1, w_3, w_4\}$ |
| $\{w_1, w_2, w_3, w_4\}$ | \{ $1$ \} | 1 | $\{w_1, w_2, w_3, w_4\}$ |
$W(A_4)$ Conjugacy classes:

$C_1 = \{1\}$, $C_2 = \left\{ \begin{array}{c} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_1 w_2 w_1 \\ w_2 w_3 w_2 \\ w_3 w_4 w_3 \\ w_1 w_2 w_3 w_2 w_1 \\ w_2 w_3 w_4 w_3 w_2 \\ \end{array} \right\}$

$C_3 = \left\{ \begin{array}{c} w_1 w_2 \\ w_2 w_1 \\ w_2 w_3 \\ w_3 w_2 \\ w_3 w_4 w_3 \\ w_4 w_3 \\ w_2 w_3 w_2 w_1 \\ w_1 w_2 w_1 w_3 \\ w_3 w_4 w_3 w_2 \\ w_2 w_3 w_2 w_1 \\ \end{array} \right\}$

$C_4 = \left\{ \begin{array}{c} w_1 w_3 \\ w_1 w_4 \\ w_2 w_4 \\ w_2 w_1 w_3 w_2 \\ w_1 w_2 w_4 w_3 \\ w_3 w_4 w_2 w_3 \\ w_1 w_2 w_1 w_3 w_2 w_1 \\ w_2 w_1 w_3 w_4 w_3 w_2 \\ w_1 w_3 w_2 w_4 w_3 w_1 \\ w_2 w_3 w_2 w_4 w_3 w_2 \\ \end{array} \right\}$
$W(A_4)$ Conjugacy classes:

$C_5 = \{ w_1 w_2 w_3, w_2 w_1 w_3, w_1 w_3 w_2, w_2 w_3 w_4, w_3 w_2 w_1, w_2 w_4 w_3, w_3 w_4 w_2, w_4 w_3 w_2, \ldots\}$
\[ W(A_4) \text{ Conjugacy classes:} \]

\[ C_6 = \begin{cases} 
\{ w_1 w_2 w_4 \} \\
\{ w_2 w_1 w_4 \} \\
\{ w_1 w_3 w_4 \} \\
\{ w_1 w_4 w_3 \} \\
\{ w_2 w_1 w_3 w_2 \} \\
\{ w_1 w_3 w_2 w_4 \} \\
\{ w_2 w_3 w_2 w_1 w_3 w_2 \} \\
\{ w_1 w_2 w_4 w_3 w_2 \} \\
\{ w_1 w_2 w_3 w_4 w_2 \} \\
\{ w_2 w_1 w_3 w_2 \} \\
\{ w_2 w_3 w_2 w_1 \} \\
\{ w_1 w_2 w_3 w_1 w_2 w_4 \} \\
\{ w_1 w_2 w_3 w_2 w_4 \} \\
\end{cases} \]

\[ C_7 = \begin{cases} 
\{ w_1 w_2 w_3 w_4 \} \\
\{ w_1 w_2 w_4 w_3 \} \\
\{ w_2 w_1 w_3 w_4 \} \\
\{ w_2 w_1 w_4 w_3 \} \\
\{ w_1 w_3 w_2 w_4 \} \\
\{ w_1 w_4 w_3 w_2 \} \\
\{ w_3 w_2 w_4 w_1 \} \\
\{ w_4 w_3 w_2 w_1 \} \\
\{ w_1 w_2 w_3 w_4 \} \\
\{ w_1 w_2 w_4 w_3 \} \\
\{ w_2 w_1 w_3 w_4 \} \\
\{ w_2 w_1 w_4 w_3 \} \\
\{ w_1 w_3 w_2 w_4 \} \\
\{ w_1 w_4 w_3 w_2 \} \\
\{ w_3 w_2 w_4 w_1 \} \\
\{ w_4 w_3 w_2 w_1 \} \\
\end{cases} \]
(5) \( W = W(B_2) = \langle w_1, w_2 : w_1^2 = w_2^2 = (w_1 w_2)^4 = 1 \rangle \)

| \( J \) | \( \{ w : w \in Y_J \} \) | \( |w| \) | \( K : w^{-1} \in Y_K \) |
|---|---|---|---|
| \( \emptyset \) | \( \{ w_1 w_2 w_1 w_2 \} \) | 2 | \( \emptyset \) |
| \( \{ w_1 \} \) | \( \{ w_2 \} \) | 2 | \( \{ w_1 \} \) |
|  | \( \{ w_1 w_2 \} \) | 4 | \( \{ w_2 \} \) |
|  | \( \{ w_2 w_1 \} \) | 2 | \( \{ w_1 \} \) |
| \( \{ w_2 \} \) | \( \{ w_1 \} \) | 2 | \( \{ w_2 \} \) |
|  | \( \{ w_2 w_1 \} \) | 4 | \( \{ w_1 \} \) |
| \( \{ w_1 w_2 \} \) | \( \{ 1 \} \) | 1 | \( \{ w_1, w_2 \} \) |

Conjugacy classes:

\( C_1 = \{ 1 \} \), \( C_2 = \{ w_2 \} \), \( C_3 = \{ w_1 \} \), \( C_4 = \{ w_1 w_2 \} \), \( C_5 = \{ w_1 w_2 w_1 w_2 \} \)
\((6) \ W = W(B_3) = \langle w_1, w_2, w_3 : w_1^2 = w_2^2 = w_3^2 = 1, \ (w_1 w_2)^4 = (w_1 w_3)^2 = (w_2 w_3)^3 = 1 \rangle. \)

<p>| (J) | ({w : w \in \mathcal{Y}_J}) | (|w|) | (K : w^{-1} \in \mathcal{Y}_K) |
|---|---|---|---|
| (\emptyset) | ({w_1 w_2 w_1 w_2 w_3 w_2 w_1 w_2 w_3}) | 2 | (\emptyset) |
| ({w_1}) | ({w_2 w_3 w_2}) | 2 | ({w_1}) |
| ({w_1}) | ({w_1 w_2 w_3 w_2 w_1 w_2 w_3 w_2}) | 3 | ({w_3}) |
| ({w_1}) | ({w_2 w_3 w_1 w_2 w_3 w_2 w_1 w_2 w_3}) | 4 | ({w_1}) |
| ({w_1, w_2, w_3}) | ({w_1 w_2 w_3 w_2 w_1 w_2 w_3 w_2}) | 6 | ({w_1, w_3}) |
| ({w_2}) | ({w_1 w_3}) | 2 | ({w_2}) |
| ({w_2, w_3}) | ({w_2 w_3 w_1 w_2 w_3 w_2 w_1 w_2 w_3}) | 3 | ({w_1, w_3}) |
| ({w_2, w_3}) | ({w_2 w_3 w_1 w_2 w_3 w_2 w_1 w_2 w_3}) | 4 | ({w_2}) |
| ({w_3}) | ({w_3 w_1 w_2 w_1 w_2 w_3 w_2 w_1 w_2 w_3}) | 6 | ({w_3}) |
| ({w_3}) | ({w_3 w_1 w_2 w_1 w_2 w_3 w_2 w_1 w_2 w_3}) | 4 | ({w_1}) |
| ({w_3}) | ({w_3 w_1 w_2 w_1 w_2 w_3 w_2 w_1 w_2 w_3}) | 2 | ({w_2}) |</p>
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<th>$J$</th>
<th>${ w: w, Y_J }$</th>
<th>$w$</th>
<th>$K: w^{-1}$</th>
<th>$y_K$</th>
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</table>
$W(B_3)$ Conjugacy classes:

$c_1 = \{1\}$
$c_2 = \left\{ \begin{array}{c} w_1 \\ w_2w_1w_2 \\ w_3w_2w_1w_2w_3 \end{array} \right\}$
$c_3 = \left\{ \begin{array}{c} w_2 \\ w_3 \\ w_1w_2w_1w_2 \end{array} \right\}$
$c_4 = \left\{ \begin{array}{c} w_1w_3 \\ w_2w_3w_1w_2 \\ w_1w_2w_3w_1w_2w_1 \\ w_3w_2w_1w_2w_3w_2w_1w_2 \end{array} \right\}$
$c_5 = \left\{ \begin{array}{c} w_1w_2 \\ w_2w_1 \\ w_1w_2w_3w_2 \end{array} \right\}$
$c_6 = \left\{ \begin{array}{c} w_2w_3 \\ w_3w_2 \\ w_1w_2w_3w_1w_3 \\ w_3w_1w_2w_1 \\ w_2w_3w_1w_2w_1w_2w_3w_1w_2w_3w_1w_2w_3 \end{array} \right\}$
$c_7 = \left\{ \begin{array}{c} w_1w_2w_3 \\ w_3w_2w_1 \\ w_2w_1w_3 \\ w_1w_3w_2 \\ w_2w_1w_2w_3w_2w_1w_2w_3w_1w_2w_3w_1w_2w_3 \end{array} \right\}$
$c_8 = \left\{ \begin{array}{c} w_1w_2w_1w_2 \\ w_3w_2w_1w_2w_3w_1 \\ w_3w_2w_1w_2w_3w_2w_1w_2w_3w_1w_2w_3w_1w_2w_3 \end{array} \right\}$
$c_9 = \left\{ \begin{array}{c} w_1w_2w_3w_1w_2 \end{array} \right\}$
\[(7) \quad W = W(G_2) = \langle w_1, w_2 : w_1^2 = w_2^2 = (w_1 w_2)^6 = 1 \rangle\]

| \(J\) | \(\{ w : w \in I_J \}\) | \(|w|\) | \(K : w^{-1} \in I_K\) |
|---|---|---|---|
| \(\emptyset\) | \(\{ w_1 w_2 w_1 w_2 w_1 w_2 \}\) | 2 | \(\emptyset\) |
| \(\{w_1\}\) | \(\{ w_2 \}\) | 2 | \(\{ w_1 \}\) |
| \(\{w_2\}\) | \(\{ w_1 \}\) | 2 | \(\{ w_2 \}\) |
| \(\{w_1, w_2\}\) | \(\{ 1 \}\) | 1 | \(\{ w_1, w_2 \}\) |

Conjugacy classes:

\(C_1 = \{ 1 \}, \quad C_2 = \left\{ \begin{array}{c} w_1 \\ w_2 \\ w_1 w_2 \\ w_1 w_2 \end{array} \right\}, \quad C_3 = \left\{ \begin{array}{c} w_2 \\ w_1 w_2 \\ w_1 w_2 \\ w_1 w_2 \end{array} \right\}, \quad C_4 = \left\{ \begin{array}{c} w_1 w_2 \\ w_2 w_1 \\ w_1 w_2 \end{array} \right\}, \quad C_5 = \left\{ \begin{array}{c} w_1 w_2 w_1 w_2 \\ w_2 w_1 w_2 \end{array} \right\}, \quad C_6 = \{ w_1 w_2 w_1 w_2 w_1 w_2 \}
\[
(8) \ W = W(I_2(8)) = \langle w_1, w_2 : w_1^2 = w_2^2 = (w_1 w_2)^8 = 1 \rangle \ (\cong D_{16})
\]

| \( J \) | \( \{ w : w \in Y_J \} \) | \(| w | \) | \( K : w^{-1} \in Y_K \) |
|---|---|---|---|
| \( \emptyset \) | \{ \( w_1^2 w_2, w_1 w_2, w_1^2 w_1, w_2 w_1 w_2 \} \) | 2 | \( \emptyset \) |
| \( \{ w_1 \} \) | \{ \( w_1^2, w_1 w_2, w_1^2 w_1, w_2 w_1 w_2 \} \) | 2 | \( \{ w_1 \} \) |
| \( \{ w_2 \} \) | \{ \( w_1^2, w_1 w_2, w_1^2 w_1, w_2 w_1 w_2 \} \) | 2 | \( \{ w_2 \} \) |
| \( \{ w_1, w_2 \} \) | \{ \( 1 \) \} | 1 | \( \{ w_1, w_2 \} \) |

**Conjugacy classes:**

\( C_1 = \{ 1 \}, \ C_2 = \left\{ \frac{w_1}{w_2} \right\}, \ C_3 = \left\{ \frac{w_1}{w_2} \right\}, \ C_4 = \left\{ \frac{w_1 w_2}{w_2 w_1} \right\}, \ C_5 = \left\{ \frac{w_1 w_2}{w_2 w_1} \right\}, \ C_6 = \left\{ \frac{w_1 w_2}{w_2 w_1} \right\}, \ C_7 = \left\{ \frac{w_1 w_2}{w_2 w_1} \right\} \)
(9) \( W = W(A_1) \times W(A_1) = \langle w_1, w_2 : w_1^2 = w_2^2 = (w_1w_2)^2 = 1 \rangle \)

|                   | \{w : w \in Y_J\} | |        | \{w^{-1} \in Y_K\} |
|-------------------|--------------------|-----|-------------------|
| \( \emptyset \)   | \{w_1w_2\}        | 2   | \( \emptyset \)   |
| \{ w_1 \}         | \{ w_2 \}         | 2   | \{ w_1 \}         |
| \{ w_2 \}         | \{ w_1 \}         | 2   | \{ w_2 \}         |
| \{ w_1, w_2 \}    | \{ 1 \}           | 1   | \{ w_1, w_2 \}    |

**Conjugacy classes:**

\( C_1 = \{ 1 \} \), \( C_2 = \{ w_1 \} \), \( C_3 = \{ w_2 \} \), \( C_4 = \{ w_1w_2 \} \).
(10) \( W = W(A_1) \times W(A_2) = \langle w_1, w_2, w_3 : w_1^2 = w_2^2 = w_3^2 = 1, \\
(w_1w_2)^2 = (w_1w_3)^2 = (w_2w_3)^3 = 1 \rangle \)

\[
\begin{array}{|c|c|c|}
\hline
J & \{ w : w \in Y_J \} & |w| & \{ K : w^{-1} \in Y_K \} \\
\hline
\emptyset & \{ w_1w_2w_3w_2 \} & 2 & \emptyset \\
\{ w_1 \} & \{ w_2w_3w_2 \} & 2 & \{ w_1 \} \\
\{ w_2 \} & \{ w_1w_3 \} & 2 & \{ w_2 \} \\
& \{ w_1w_2w_3 \} & 6 & \{ w_3 \} \\
\{ w_3 \} & \{ w_1w_2 \} & 2 & \{ w_3 \} \\
& \{ w_1w_3w_2 \} & 6 & \{ w_2 \} \\
\{ w_1, w_2 \} & \{ w_3 \} & 2 & \{ w_1, w_2 \} \\
& \{ w_2w_3 \} & 3 & \{ w_1, w_3 \} \\
\{ w_1, w_3 \} & \{ w_2 \} & 2 & \{ w_1, w_3 \} \\
& \{ w_2w_3 \} & 3 & \{ w_1, w_2 \} \\
\{ w_2, w_3 \} & \{ w_1 \} & 2 & \{ w_2, w_3 \} \\
\{ w_1, w_2, w_3 \} & \{ 1 \} & 1 & \{ w_1, w_2, w_3 \} \\
\hline
\end{array}
\]

**Conjugacy classes:**

\( C_1 = \{ 1 \}, \ C_2 = \{ w_1 \}, \ C_3 = \left\{ \begin{array}{c} w_2 \\ w_3 \\ w_2w_3 \\
\end{array} \right\}, \ C_4 = \left\{ \begin{array}{c} w_1w_2 \\ w_1w_3 \\ w_1w_2w_3 \\
\end{array} \right\}, \ C_5 = \left\{ \begin{array}{c} w_2w_3 \\ w_3w_2 \\
\end{array} \right\}, \ C_6 = \left\{ \begin{array}{c} w_1w_2w_3 \\ w_1w_3w_2 \\
\end{array} \right\} \)
\[ W = W(A_2) \times W(A_2) = \langle w_1, w_2, w_3, w_4 : w_1^2 = w_2^2 = w_3^2 = w_4^2 = 1, \]

\[
(w_1w_2)^3 = (w_1w_3)^2 = (w_1w_4)^2 = 1, \\
(w_2w_3)^2 = (w_2w_4)^2 = (w_3w_4)^3 = 1 \]

| \( J \) | \( \{ w : w \in Y_J \} \) | \( |w| \) | \( K : w^{-1} \in Y_K \) |
|-------|-----------------|-----|-----------------|
| \( \emptyset \) | \{ \( w_1w_2w_3w_4w_3 \) \} | 2 | \( \emptyset \) |
| \( \{ w_1 \} \) | \{ w_2w_3w_4w_3 \} | 2 | \( \{ w_1 \} \) |
|  | \{ w_1w_2w_3w_4w_3 \} | 6 | \( \{ w_2 \} \) |
| \( \{ w_2 \} \) | \{ w_1w_3w_4w_3 \} | 2 | \( \{ w_2 \} \) |
|  | \{ w_2w_1w_3w_4w_3 \} | 6 | \( \{ w_1 \} \) |
| \( \{ w_3 \} \) | \{ w_1w_2w_4 \} | 2 | \( \{ w_3 \} \) |
|  | \{ w_3w_4w_1w_2w_1 \} | 6 | \( \{ w_4 \} \) |
| \( \{ w_4 \} \) | \{ w_3w_4w_1w_2w_1 \} | 6 | \( \{ w_3 \} \) |
| \( \{ w_1, w_2 \} \) | \{ w_3w_4w_3 \} | 2 | \( \{ w_1, w_2 \} \) |
| \( \{ w_1, w_3 \} \) | \{ w_2w_4 \} | 2 | \( \{ w_1, w_3 \} \) |
|  | \{ w_1w_2w_4 \} | 6 | \( \{ w_2, w_3 \} \) |
|  | \{ w_3w_2w_4 \} | 6 | \( \{ w_1, w_4 \} \) |
|  | \{ w_1w_3w_2w_4 \} | 3 | \( \{ w_2, w_4 \} \) |
| \( \{ w_1, w_4 \} \) | \{ w_2w_3 \} | 2 | \( \{ w_1, w_4 \} \) |
|  | \{ w_1w_2w_3 \} | 6 | \( \{ w_2, w_4 \} \) |
|  | \{ w_4w_2w_3 \} | 6 | \( \{ w_1, w_3 \} \) |
|  | \{ w_1w_4w_2w_3 \} | 3 | \( \{ w_2, w_3 \} \) |
| \( \{ w_2, w_3 \} \) | \{ w_1w_4 \} | 2 | \( \{ w_2, w_3 \} \) |
|  | \{ w_2w_1w_4 \} | 6 | \( \{ w_1, w_3 \} \) |
|  | \{ w_3w_1w_4 \} | 6 | \( \{ w_2, w_4 \} \) |
|  | \{ w_2w_3w_1w_4 \} | 3 | \( \{ w_1, w_4 \} \) |
\[ W(A_2) \times W(A_2) \] continued.

| \( J \) | \( \{ w : w \in J \} \) | \( |w| \) | \( K : w^{-1} \in Y_K \) |
|-------|-----------------|-----|------------------|
| \{ \( w_2, w_4 \) \} | \{ \( w_1 w_3 \), \( w_3 w_4 \), \( w_4 w_3 \), \( w_2 w_4 w_3 \) \} | 2 | \{ \( w_2, w_4 \) \} |
| \{ \( w_3, w_4 \) \} | \{ \( w_1 w_2 w_1 \) \} | 2 | \{ \( w_3, w_4 \) \} |
| \{ \( w_1, w_2, w_3 \) \} | \{ \( w_4 \), \( w_3 w_4 \) \} | 2 | \{ \( w_1, w_2, w_3 \) \} |
| \{ \( w_1, w_2, w_4 \) \} | \{ \( w_3 \), \( w_4 w_3 \) \} | 2 | \{ \( w_1, w_2, w_3 \) \} |
| \{ \( w_1, w_3, w_4 \) \} | \{ \( w_2 \), \( w_1 w_2 \) \} | 2 | \{ \( w_1, w_3, w_4 \) \} |
| \{ \( w_2, w_3, w_4 \) \} | \{ \( w_1 \), \( w_2 w_1 \) \} | 2 | \{ \( w_2, w_3, w_4 \) \} |
| \{ \( w_1, w_2, w_3, w_4 \) \} | \{ 1 \} | 1 | \{ \( w_1, w_2, w_3, w_4 \) \} |

Conjugacy classes:

\( C_1 = \{ 1 \} \),  \( C_2 = \{ w_1 \}, \)  \( C_3 = \{ w_3 \}, \)  \( C_4 = \{ w_1 w_2 \}, \)  \( C_5 = \{ w_3 w_4 \}, \)  \( C_6 = \{ w_1 w_3, w_1 w_4, w_2 w_3, w_2 w_4, w_1 w_2 w_1, w_1 w_3 w_4 w_3, w_2 w_3 w_4 w_3 \} \).
$W(A_2) \times W(A_2)$ continued.

\[ C_7 = \begin{array}{c}
  w_1 \ w_2 \ w_4 \\
  w_1 \ w_2 \ w_3 \\
  w_2 \ w_1 \ w_3 \\
  w_2 \ w_1 \ w_4 \\
  w_1 \ w_2 \ w_3 \ w_4 \ w_3 \\
  w_2 \ w_1 \ w_3 \ w_4 \ w_3
\end{array} \quad C_8 = \begin{array}{c}
  w_4 \ w_2 \ w_3 \\
  w_3 \ w_2 \ w_4 \\
  w_4 \ w_1 \ w_3 \\
  w_3 \ w_2 \ w_4 \\
  w_3 \ w_4 \ w_1 \ w_2 \ w_1 \\
  w_4 \ w_3 \ w_1 \ w_2 \ w_1
\end{array} \]
Appendix 3: IDEMPOTENTS IN THE DECOMPOSITIONS OF THE O-HECKE ALGEBRA.

A direct sum decomposition of the O-Hecke algebra \( H \) over the field \( K \) is equivalent to writing the identity element of \( H \) as a sum of mutually orthogonal primitive idempotents. Let \( 1 = \sum_{J \subseteq R} q_J \) and \( 1 = \sum_{J \subseteq R} p_J \) be the decompositions of 1 corresponding to the decompositions \( H = \sum_{J \subseteq R} H_{q_J} \) and \( H = \sum_{J \subseteq R} H_{p_J} \) respectively, where

\[
H_{q_J} = H_{q_J e_J} \quad \text{and} \quad H_{p_J} = H_{p_J o_J} \quad \text{for all} \quad J \subseteq R.
\]

Since \( q_J \in H_{q_J e_J} \) and \( p_J \in H_{p_J o_J} \), there exist elements \( b_y, u_y \in K \) such that \( q_J = \sum_{y \in Y_J} b_y a y o_J e_J \) and

\[
p_J = \sum_{y \in Y_J} u_y a y e_J o_J.
\]

(A3.1) Lemma: Suppose there exist elements \( q_J^1 \in H_{q_J e_J} \) and \( p_J^1 \in H_{p_J o_J} \) such that \( 1 = \sum_{J \subseteq R} q_J^1 \) and \( 1 = \sum_{J \subseteq R} p_J^1 \). Then \( \{q_J^1: J \subseteq R\} \) and \( \{p_J^1: J \subseteq R\} \) are both sets of mutually orthogonal primitive idempotents, and \( H_{q_J^1} = H_{q_J^1 e_J} \) and \( H_{p_J^1} = H_{p_J^1 o_J} \) for all \( J \subseteq R \).

Proof: \( q_J^1 = q_J^1 q_L = \sum_{L \subseteq R} q_J^1 q_L \). Then for all \( L \subseteq R \),

\[
q_J^1 q_L \in H_{o_J e_L}, \quad \text{and as} \quad \sum_{L \subseteq R} H_{o_J e_L} \quad \text{is direct, we have}
\]

\[
q_J^1 q_L = 0 \quad \text{if} \quad J \neq L, \quad \text{and} \quad q_J^1 q_J^1 = q_J^1. \quad \text{Hence} \quad \{q_J^1: J \subseteq R\} \quad \text{is a set of mutually orthogonal idempotents, and} \quad H_{q_J^1} = H_{q_J^1 e_J}
\]

for all \( J \subseteq R \). Since \( 1 = \sum_{J \subseteq R} q_J^1 \), we have

\[
H = \sum_{J \subseteq R} H_{q_J^1} \quad \subseteq \quad \sum_{J \subseteq R} H_{q_J e_J} = H. \quad \text{Hence} \quad H_{q_J^1} = H_{q_J e_J} \quad \text{for all}
\]

\( J \subseteq R \).
J \subseteq R, and as $H_0 e_J$ is an indecomposable $H$-module for all $J \subseteq R$, $q_J^1$ is primitive for all $J \subseteq R$.

Similarly for \{p_J^1: J \subseteq R\}.

(A3.2) \textbf{THEOREM:} Let $H$ be the 0-Hecke algebra of type $(W,R)$ where $W$ is the dihedral group of order $2n$. For each $J \subseteq R$, let

$$q_J = \sum_{y \in Y_J} (-1)^{l(y)} a_{y} e_J.$$

Then \{q_J: J \subseteq R\} is a set of mutually orthogonal primitive idempotents, with $H q_J = H_0 e_J$ for all $J \subseteq R$, and

$$H = \sum_{J \subseteq R} H q_J^1.$$

\textbf{Proof:} Since $q_J \in H_0 e_J$ for each $J \subseteq R$, it is sufficient to prove that $1 = \sum_{J \subseteq R} q_J^1$.

(a) Suppose $n = 2m$ is even. Then

$$q_{\emptyset} = a(w_1 w_2 w_1 \cdots)_{2m},$$

$$q_{\{w_1\}} = -(a_{w_2} + a_{w_2 w_1 w_2} + \cdots + a_{w_2 w_1 w_2 \cdots})_{2m-1}(1 + a_{w_1})$$

$$= 2m \sum_{k=1}^m a_{w_2 w_1 w_2 \cdots} k$$

$$q_{\{w_2\}} = -\sum_{k=1}^m a_{w_1 w_2 w_1 \cdots} k$$

$$q_{\{w_1, w_2\}} = \sum_{w \in W} a_w$$

Let $S = q_{\emptyset} + q_{\{w_1\}} + q_{\{w_2\}} + q_{\{w_1, w_2\}}$. Then by inspection $1$ occurs in $S$ with coefficient $1$, and for all $w \in W$, $w \neq 1$, $a_w$ occurs with zero coefficient. Hence $S = 1$, and the result is true.
(b) Suppose \( n = 2m+1 \) is odd. Then

\[
q_\emptyset = -a(w_1w_2w_1\cdots)_{2m+1},
\]

\[
q_{\{w_1\}} = -\sum_{k=1}^{2m} a(w_2w_1w_2\cdots)_{2m+1},
\]

\[
q_{\{w_2\}} = -\sum_{k=1}^{2m} a(w_1w_2w_1\cdots)_{2m+1},
\]

\[
q_{\{w_1, w_2\}} = \sum_{w \in W} a_w.
\]

Clearly, \( q_\emptyset + q_{\{w_1\}} + q_{\{w_2\}} + q_{\{w_1, w_2\}} = 1 \), and the result is true.

(A3.3) **Theorem:** Let \( H \) be the \( 0 \)-Hecke algebra of type \( (W, R) \) where \( W \) is a dihedral group of order \( 2n \). Let \( p_\emptyset = (-1)^{1(w_0)} a_{w_0} \),

\[
p_{\{w_1, w_2\}} = \sum_{w \in W} a_w, \text{ and for each of } J = \{w_1\}, \{w_2\} \text{ let}
\]

\[
p_J = (-1)^n \sum_{y \in Y_J} a_y e_{w_0 J}
\]

where for all \( y \in Y_J \), \( n_y = \left\lceil \frac{1(y)+1}{2} \right\rceil \), where if \( x \in Q \), we denote by \( [x] \) the largest rational integer which is less than or equal to \( x \). Then \( \{p_J : J \subseteq R\} \) is a set of mutually orthogonal primitive idempotents, with \( H p_J = H e_{w_0 J} \) for all \( J \subseteq R \), and

\[
H = \sum_{J \subseteq R} H p_J.
\]

**Proof:** Since \( p_J \in He_{w_0 J} \) for each \( J \subseteq R \), it is sufficient to prove that \( 1 = \sum_{J \subseteq R} p_J \).

(a) Suppose \( n = 2m \) is even; then

\[
p_\emptyset = a(w_1w_2w_1\cdots)_{2m},
\]
\[ p_{\{w_1\}} = (-1) \sum_{y \in Y_{\{w_1\}}} n_{yx} a_y (1 + a_{w_1} a_{w_2}) \]

\[ = \sum_{k=1}^{2m-1} \binom{k+1}{2} a_{\ldots w_2 w_1 w_2} a_{w_2} \]

\[ + \sum_{k=1}^{2m-1} \binom{k+1}{2} a_{\ldots w_2 w_1 w_2} a_{w_1} a_{w_2} \]

\[ = -\sum_{k=1}^{2m-1} \binom{k+1}{2} a_{\ldots w_2 w_1 w_2} + \sum_{k=3}^{2m-1} \binom{k-1}{2} a_{\ldots w_2 w_1 w_2} \]

Now if \( k = 1 \) or \( k = 2 \), \( a_{\ldots w_2 w_1 w_2} \) occurs in the expression for \( p_{\{w_1\}} \) with coefficient \((-1)\). Since \( \binom{k+1}{2} - \binom{k-1}{2} = 1 \) for all \( k \), \( 3 \leq k \leq 2m-1 \), the coefficient of \( a_{\ldots w_2 w_1 w_2} \) for all \( k \), \( 3 \leq k \leq 2m-1 \), in the expression for \( p_{\{w_1\}} \) is \((-1)\). Finally, we have that \( \binom{2m-1}{2} - \binom{2m}{2} = -1 \), and hence

\[ p_{\{w_1\}} = -\sum_{k=1}^{2m} a_{\ldots w_2 w_1 w_2} \]

and similarly

\[ p_{\{w_2\}} = -\sum_{k=1}^{2m} a_{\ldots w_2 w_1 w_2} \]

Now clearly we have that \( p_{\emptyset} + p_{\{w_1\}} + p_{\{w_2\}} + p_{\{w_1, w_2\}} = 1 \), and hence the result.

(b) Suppose \( n = 2m+1 \) is odd; then

\[ p_{\emptyset} = -a_{w_1 w_2 w_1 \ldots} a_{w_2 w_1 \ldots} \]

\[ p_{\{w_1\}} = \sum_{k=1}^{2m} \binom{k+1}{2} a_{\ldots w_2 w_1 w_2} a_{w_2} \]

\[ + \sum_{k=1}^{2m} \binom{k+1}{2} a_{\ldots w_2 w_1 w_2} a_{w_1} a_{w_2} \]
Thus, \( p\{w_1\} = - \sum_{k=1}^{2m} \left[ \frac{k+1}{2} \right] a(\cdots w_2 w_1 w_2) k + \frac{2m}{k=3} \left[ \frac{k-1}{2} \right] a(\cdots w_2 w_1 w_2) k \)

\[
+ \left[ \frac{2m}{2} \right] a(\cdots w_2 w_1 w_2) 2m+1 - \left[ \frac{2m+1}{2} \right] a(\cdots w_2 w_1 w_2) 2m+1
\]

\[
= - \sum_{k=1}^{2m} a(\cdots w_2 w_1 w_2) k
\]

since \( \left[ \frac{k+1}{2} \right] = 1 \) if \( k = 1 \) or \( 2 \), \( \left[ \frac{k+1}{2} \right] - \left[ \frac{k-1}{2} \right] = 1 \) for all \( k \), \( 3 \leq k \leq 2m \), and \( \left[ \frac{2m}{2} \right] - \left[ \frac{2m+1}{2} \right] = 0 \). Similarly,

\[
p\{w_2\} = - \sum_{k=1}^{2m} a(\cdots w_1 w_2 w_1) k'
\]

and then we have that \( p\emptyset + p\{w_1\} + p\{w_2\} + p\{w_1, w_2\} = 1 \),

and hence the result.

**EXAMPLE:** \( W = W(G_2) \), the dihedral group of order 12.

Let \( R = \{w_1, w_2\} \); then

\[
q\emptyset = a_{w_1 w_2 w_1 w_2 w_1 w_2}
\]

\[
q\{w_1\} = -a_{w_2} (1 + a_{w_1}) - a_{w_2 w_1 w_2} (1 + a_{w_1})
- a_{w_2 w_1 w_2} (1 + a_{w_1})
\]

\[
q\{w_2\} = -a_{w_1} (1 + a_{w_2}) - a_{w_1 w_2 w_1} (1 + a_{w_2})
- a_{w_1 w_2 w_1} (1 + a_{w_2})
\]

\[
q\{w_1, w_2\} = (1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1}) \times (1 + a_{w_2}).
\]

\[
p\emptyset = a_{w_1 w_2 w_1 w_2 w_1 w_2}
\]
\[
p_{w_1} = (a_{w_2} + a_{w_1}w_2 + 2a_{w_2}w_1w_2 + 2a_{w_1}w_2w_1w_2 + 33a_{w_2}w_1w_2w_1w_2) \\
\times (1 + a_{w_1})a_{w_2}.
\]

\[
p_{w_2} = (a_{w_1} + a_{w_2}w_1 + 2a_{w_1}w_2w_1 + 2a_{w_2}w_1w_2w_1 + 3a_{w_1}w_2w_1w_2w_1) \\
\times (1 + a_{w_2})a_{w_1}.
\]

\[
p_{w_1, w_2} = q_{w_1, w_2}.
\]

The idempotents \( \{q_j\} \) and \( \{p_j\} \) for the 0-Hecke algebra of type \((W,R)\), where \( W = W(A_3) \) are as follows:

\[
q_{w_1} = -a_{w_2}w_1w_2(1 + a_{w_1})
\]

\[
q_{w_2} = (a_{w_1}w_3 - a_{w_1}w_3w_2w_1w_3)(1 + a_{w_2})
\]

\[
q_{w_3} = -a_{w_1}w_2w_1(1 + a_{w_3})
\]

\[
q_{w_1, w_2} = -a_{w_3}(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1})
\]

\[
q_{w_1, w_3} = -(a_{w_2} + a_{w_2}w_1w_3w_2)(1 + a_{w_3})(1 + a_{w_1})
\]

\[
q_{w_2, w_3} = -a_{w_1}(1 + a_{w_2})(1 + a_{w_3})(1 + a_{w_2})
\]

\[
q_{w_1, w_2, w_3} = \sum_{w \in W(A_3)} a_{w}.
\]

\[
p_{\emptyset} = a_{w_1}w_2w_1w_3w_2w_1
\]

\[
p_{w_1} = (a_{w_2}w_3w_2 + a_{w_1}w_2w_3w_2 + a_{w_2}w_1w_2w_3w_2)(1 + a_{w_1})a_{w_2}w_3w_2.
\]
The idempotents \( \{ q_j \} \) for the 0-Hecke algebra of type \((W,R)\) where \( W = W(A_4) \) are as follows:

\[
p_{\{w_2\}} = (a_{w_1}w_3 + a_{w_2}w_1w_3 - 3a_{w_1}w_3w_2w_1w_3)(1 + a_{w_2})a_{w_1}w_3.
\]

\[
p_{\{w_3\}} = (a_{w_1}w_2w_1 + a_{w_2}w_1w_2w_1 + a_{w_2}w_3w_1w_2w_1)(1 + a_{w_3})
\]

\[
p_{\{w_1,w_2\}} = (a_{w_3} + a_{w_2}w_3 + a_{w_1}w_2w_3)[1 + a_{w_1}w_2w_1]a_{w_3}.
\]

\[
p_{\{w_1,w_3\}} = (a_{w_2} + a_{w_1}w_2 + a_{w_3}w_2 + a_{w_1}w_3w_2 + a_{w_2}w_1w_3w_2)
\]

\[
\times (1 + a_{w_1})(1 + a_{w_3})a_{w_2}.
\]

\[
p_{\{w_2,w_3\}} = (a_{w_1} + a_{w_2}w_1 + a_{w_3}w_2w_1)[1 + a_{w_2}w_3w_2]a_{w_1}.
\]

\[
p_{\{w_1,w_2,w_3\}} = \sum_{w \in \Pi(A_4)} a_w^*.
\]

The idempotents \( \{ q_j \} \) for the 0-Hecke algebra of type \((W,R)\) where \( W = W(A_4) \) are as follows:

\[
q_{\emptyset} = a_{w_1}w_2w_3w_4w_1w_2w_3w_1w_2w_1.
\]

\[
q_{\{w_1\}} = a_{w_2}w_3w_2w_4w_3w_2(1 + a_{w_1}).
\]

\[
q_{\{w_2\}} = (a_{w_1}w_3w_4w_3 + a_{w_1}w_4w_3w_2w_1w_3w_4w_3)(1 + a_{w_2}).
\]

\[
q_{\{w_3\}} = (a_{w_1}w_2w_1w_4 + a_{w_1}w_2w_3w_4w_1w_3w_2w_1)(1 + a_{w_3}).
\]

\[
q_{\{w_4\}} = a_{w_1}w_2w_1w_3w_2w_1(1 + a_{w_4}).
\]

\[
q_{\{w_1,w_2\}} = -a_{w_3}w_4w_3(1 + a_{w_1})(1 + a_{w_2})(1 + a_{w_1}).
\]

\[
q_{\{w_2,w_3\}} = (a_{w_1}w_4 - a_{w_1}w_2w_3w_4w_3w_2w_1)(1 + a_{w_2})(1 + a_{w_3})
\]

\[
\times (1 + a_{w_2}).
\]
\[ q_{\{w_3, w_4\}} = -a_{w_1 w_2 w_1} (1 + a_{w_3})(1 + a_{w_4})(1 + a_{w_3}). \]

\[ q_{\{w_1, w_3\}} = (a_{w_2 w_4} - a_{w_2 w_3 w_4} w_2 w_3 - a_{w_1 w_2 w_3 w_4} w_3 w_2 - a_{w_1 w_2 w_3} w_4 w_3 w_2 - a_{w_1 w_2 w_3} w_3 w_4 w_1 w_3 w_2) (1 + a_{w_1})(1 + a_{w_3}). \]

\[ q_{\{w_2, w_4\}} = (a_{w_1 w_3} - a_{w_1 w_2 w_3} w_2 w_3 - a_{w_1 w_2} w_4 w_3 w_2 - a_{w_1 w_3} w_2 w_4 w_3 w_2 - a_{w_1 w_3} w_2 w_4 w_3 w_2 - a_{w_1 w_2 w_3} w_4 w_3 w_2 - a_{w_1 w_2} w_4 w_3 w_2) (1 + a_{w_2})(1 + a_{w_4}). \]

\[ q_{\{w_1, w_4\}} = -(a_{w_2 w_3 w_2} + a_{w_2 w_1 w_3 w_4} w_2 w_3 - a_{w_2 w_1 w_3} w_4 w_3 w_2 - a_{w_1 w_3} w_2 w_4 w_3 w_2 - a_{w_1 w_3} w_2 w_4 w_3 w_2 - a_{w_1 w_2} w_4 w_3 w_2) (1 + a_{w_1})(1 + a_{w_4}). \]

\[ q_{\{w_1, w_2, w_3\}} = -a_{w_4} [1 + a_{w_1 w_2 w_3} w_1 w_2 w_1]. \]

\[ q_{\{w_2, w_3, w_4\}} = -a_{w_1} [1 + a_{w_2 w_3 w_4} w_2 w_3 w_2]. \]

\[ q_{\{w_1, w_2, w_4\}} = -(a_{w_3} + a_{w_3 w_4} w_2 w_3) [1 + a_{w_1 w_2} w_1 w_4]. \]

\[ q_{\{w_1, w_3, w_4\}} = -(a_{w_2} + a_{w_2 w_1} w_3 w_2) [1 + a_{w_1 w_3} w_4 w_3]. \]

\[ q_{\{w_1, w_2, w_3, w_4\}} = \sum_{w \in \mathcal{W}(A_4)} a_{w}. \]
Appendix 4: **The Cartan Matrix of the O-Hecke Algebra.**

Let \( C = (c_{jL})_{j,L \subseteq R} \) be the Cartan matrix of the O-Hecke algebra of type \((W,R)\) over a field \(K\). By (4.5.1),

\[
c_{jL} = |Y_j \cap (Y_L)^{-1}| = c_{LJ}.
\]

The Cartan matrix of the O-Hecke algebra of type \((W,R)\), where \(W\) is one of the finite Coxeter groups listed below are given in the next few pages.

1. \(W(A_1)\).
2. \(W(A_2)\).
3. \(W(A_3)\).
4. \(W(A_4)\).
5. \(W(B_2)\).
6. \(W(B_3)\).
7. \(W(G_2)\).
8. \(W(I_2(8))\).
9. \(W(A_1) \times W(A_1)\).
10. \(W(A_1) \times W(A_2)\).
11. \(W(A_2) \times W(A_2)\).

**Notation:** Let \(R = \{w_1, w_2, \ldots, w_n\}\), \(|R| = n\). The rows and columns of \(C\) will be indexed by sets \((i_1, \ldots, i_r)\), \(1 \leq i_1 < i_2 < \ldots < i_r \leq n\), with \(i_j \in \mathbb{Z}\) for all \(j\), where the element in the \((i_1, \ldots, i_r) \times (j_1, \ldots, j_s)\) position is the element \(c_{jL}\), where \(J = \{w_{i_1}, \ldots, w_{i_r}\}\) and \(L = \{w_{j_1}, \ldots, w_{j_s}\}\).
(1) $W(A_1)$

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(2) $W(A_2)$

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(5) \( W(B_2) \).

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(2) & 0 & 1 & 2 & 0 \\
(1,2) & 0 & 0 & 0 & 1 \\
\end{array} \]

(6) \( W(B_3) \).

\[ \begin{array}{cccccccc}
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(2) & 0 & 2 & 3 & 2 & 0 & 2 & 1 \\
(3) & 0 & 1 & 2 & 2 & 0 & 0 & 0 \\
(1,2) & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\
(1,3) & 0 & 1 & 2 & 0 & 2 & 4 & 2 \\
(2,3) & 0 & 0 & 1 & 0 & 1 & 2 & 3 \\
(1,2,3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array} \]

(7) \( W(G_2) \).

\[ \begin{array}{cccc}
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(1) & 0 & 3 & 2 & 0 \\
(2) & 0 & 2 & 3 & 0 \\
(1,2) & 0 & 0 & 0 & 1 \\
\end{array} \]
\begin{align*}
(8) & \mathcal{W}(I_2(8)) \\
\begin{array}{cccc}
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(1) & 0 & 4 & 3 & 0 \\
(2) & 0 & 3 & 4 & 0 \\
(1,2) & 0 & 0 & 0 & 1 \\
\end{array} \\
(9) & \mathcal{W}(A_1) \times \mathcal{W}(A_1) \\
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(2) & 0 & 0 & 1 & 0 \\
(1,2) & 0 & 0 & 0 & 1 \\
\end{array} \\
(10) & \mathcal{W}(A_1) \times \mathcal{W}(A_2). \\
\begin{array}{cccccccc}
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(2,3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
(1,2,3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\end{align*}
\[
\begin{array}{cccccccccccc}
\emptyset & (1) & (2) & (3) & (4) & (1,2) & (1,3) & (1,4) & (2,3) & (2,4) & (3,4) & (1,2,3) & (1,2,4) & (1,3,4) & (2,3,4) & (1,2,3,4) \\
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\end{array}
\]

\[W(A_2) \times W(A_2)\]
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