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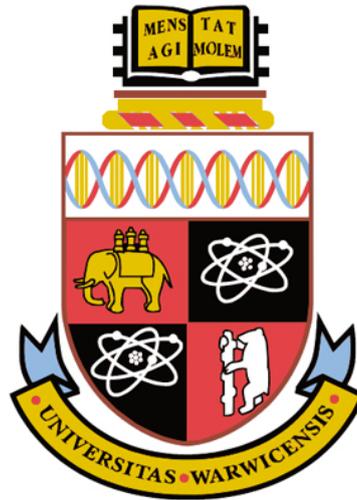
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Stationarity of asymptotically flat non-radiating electrovacuum spacetimes

by

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Thesis

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Declarations

I declare that, to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated, cited or commonly known.

The material in this thesis is submitted to the University of Warwick for the degree of Doctor of Philosophy, and has not been submitted to any other university or for any other degree.

Abstract

It is proven that a solution to the Einstein-Maxwell equations whose gravitational and electromagnetic radiation fields vanish at infinity is in fact stationary in a neighbourhood of spatial infinity. That is, if in adapted coordinates the Weyl and Faraday tensors decay suitably fast and there is an asymptotically-to-all-orders Killing vector field, then this is indeed a Killing vector field in the region outside the bifurcate horizon of a sphere of sufficiently large radius.

In particular, electrovacuum time-periodic spacetimes, which are truly dynamical, do not exist. This can be interpreted as a mild form of the statement: “Gravitational waves carry energy away from an isolated system”.

This is an extension of earlier work by Alexakis and Schlue, [5], and Bičák, Scholtz and Tod, [12], to include matter/energy models, in this case electromagnetism. It is also shown that the same result holds when the Einstein’s equations are coupled to a massless Klein-Gordon field.

Notation

$(\mathcal{M}, \mathbf{g})$ 4-dimensional spacetime. Definition 1.1.3.

∇ Levi-Civita connection of $(\mathcal{M}, \mathbf{g})$.

$\mathbf{R}, \mathbf{C}, \mathbf{S}, \mathbf{F}$ Riemann, Weyl, Schouten and Faraday tensors, respectively. Section 1.1.

$(t, r, \vartheta^2, \vartheta^3)$ space+time coordinates. Section 1.3.1.

$(u, s, \theta^2, \theta^3)$ outgoing null coordinates. Section 1.3.2

$(v, \underline{v}, y^2, y^3)$ asymptotically double null coordinates. Section 1.3.4.

(Σ, \mathbf{h}, K) Initial data set. Section 1.1.1.

$(S_{s,u}, \gamma)$ 2-dimensionial submanifold of $(\mathcal{M}, \mathbf{g})$. Section 1.3.2.

$\check{\nabla}, \check{\text{div}}, \dots$ connection, divergence, ..., operators on $S_{s,u}$.

$\check{\check{\nabla}}, \check{\check{\text{div}}}, \dots$ connection, divergence, ..., operators on the round sphere \mathbb{S}^2 .

$(\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ orthonormal frame adapted to Σ . Section 1.2.

(e_0, e_1, e_2, e_3) null frame adapted to $S_{s,u}$. Section 1.3.2

$(\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ asymptotically null frame adapted to $(v, \underline{v}, y^2, y^3)$. Proposition 3.3.1.

$\alpha, \beta, \mu, \nu, \dots$ spacetime indices, will run from 0 to 3.

a, b, c, \dots indices on Σ , will run from 1 to 3.

i, j, k, \dots indices on $S_{s,u}$, will run from 2 to 3.

$x := y$ definition of x .

$x \lesssim y$ means $x < cy$ for some positive constant c .

$x \simeq y$ means $c_1y < x < c_2y$ for some positive constants c_1, c_2 .

$V_{(\alpha_1 \dots \alpha_n)}$ is the symmetrisation of a tensor $V_{\alpha_1 \dots \alpha_n}$.

$V_{[\alpha_1 \dots \alpha_n]}$ is the anti-symmetrisation of a tensor $V_{\alpha_1 \dots \alpha_n}$.

\mathcal{L}_T is the Lie derivative in the direction of T .

$\hat{\mathcal{L}}_T$ is the modified Lie derivative in the direction of T . Section 3.2.

Chapter 0

Introduction

The most successful description of gravity is given by Einstein's Theory of General Relativity. His equations encode the evolution of the gravitational field, described by a Lorentzian metric, coupled to the matter/energy content present in the universe (cf. Section 1.1 and definition 1.1.3). The beauty of the Einstein's equations, but also one major obstruction to perform a PDE-analysis, lies in its coordinate independence; this is due to its geometric formulation. However, they adopt a more familiar quasi-linear second order hyperbolic structure when cast in suitable coordinates, e.g., wave coordinates. Then a formulation as an evolution problem given some initial data at a space-like slice is possible. The fundamental result of Choquet-Bruhat and Geroch, [17], [18], states that the resulting development is unique and maximal. We briefly review this approach in the Preliminaries chapter; we also recommend the chapter on Initial Value Formulation in Wald's book [65] for more details. This initial value formulation opened the path for a geometrical analysis of Einstein's equations; however, the result fails to provide any understanding about the geometrical and physical properties of the resulting spacetime. Here we cite just a few topics of interest in a very vague form: Geodesic completeness, presence of gravitational waves and radiation, stability of solutions.

In this thesis we focus on gravitational waves and radiation; or more precisely, on the consequences of the lack of them. Firstly, we give a brief account of their mathematical meaning. It is easier to start at the linear level following the steps of Einstein and Rosen [32]: Consider a small perturbation of the Minkowski metric. The perturbation then is required to satisfy the linearised Einstein's equations around the trivial solution. After imposing suitable gauge conditions, it is found that the perturbation obeys a wave equation and therefore it behaves as a

wave propagating causally on the background. Moreover, if we restrict ourselves to solutions decaying at infinity then the leading order term can be interpreted as power radiated to infinity. These perturbations are the so-called *gravitational waves*. The space of solutions, of both linear Einstein's and gauge equations, consists of the linear combination of two scalar waves, that is, gravitational waves have 2 degrees of freedom or polarisation modes. We refer the reader to [52] for more details.

It is worth mentioning at this point the remarkable observational achievement made recently by the LIGO collaboration [1]. They have detected for the first time a gravitational wave. Their observations indicate a wave-front compatible with the general relativistic description of the inspiral and merger of two black holes of ~ 30 and ~ 35 solar masses into a ~ 60 solar masses black hole. In addition, the settling behaviour of the resulting black hole is observed to be consistent with the description given by the stationary Kerr metric.

Now, a description of gravitational waves in the non-linear case is more delicate. To the author's knowledge there is no way of splitting the gravity field into "stationary" and "dynamic" parts; this is due to the non-linear nature of Einstein's equations. Whence the lack of meaning of the expression: Here is a gravitational wave (to be attributed to the dynamical part) propagating in a background (the stationary part). Nevertheless considerable efforts have been made to understand this statement and it is possible to make sense of it at infinity by imposing *asymptotically flat* boundary conditions¹. In Section 1.3 we present the Christodoulou-Klainerman stability and asymptotic analysis and conclusions, [22], which help us clarify the notion of radiation emitted to infinity.

The work presented here, roughly speaking, aims at proving a version of the folklore statement: Gravitational waves carry energy away from an isolated system. More precisely, a system with no radiation emitted to infinity must in fact be stationary. On physical grounds it is expected that truly time-periodic gravitational systems do not exist. This is due to the hyperbolic structure of the Einstein's equations; any dynamical solution loses energy through outward radiation. However as remarked by Alexakis-Schlue [5], this is a subtle mathematical question. It is true at the linearised level when we consider, for example, the free wave equation $\square\phi = 0$ on a Minkowski background: Formally, an outgoing wave with vanishing radiation

¹Alternatively, one could impose de Sitter or anti-de Sitter boundary conditions. In this thesis we focus on asymptotically flat spacetimes.

field has to be stationary; that is, if the function ϕ decays faster than r^{-1} in the null-outgoing direction then it must be time-independent [33]. This is no longer the case for a suitably perturbed wave operator $L = \square + V$ as remarked by Alexakis, Schlue and Shao in [7]. This problem can also be phrased as a question of uniqueness of solutions for differential operators given boundary conditions. Hörmander, [43], provided general conditions for the uniqueness property to hold across a hypersurface. In the context of Lorentzian geometry, these conditions reduce to requiring the hypersurface to be *pseudo-convex* (cf. Definition 3.1.1). Complementary to this is the work of Alinhac, [8], he showed that (generically) if one of Hörmander's conditions is violated then non-uniqueness of solutions across a hypersurface ensue.

To the author's knowledge, it was Papapetrou, [55],[56], who initiated the study of the relation radiation-stationarity for the full non-linear Einstein's equations. Although his result is not conclusive, it provided strong evidence for the validity of the theorem presented here. More precisely, Papapetrou showed the incompatibility of the following two conditions: a) A spacetime is stationary to the past of a characteristic hypersurface and non-radiative above it; b) There is a shock wave along that characteristic hypersurface or above it.

Later on, Bičák, Scholtz and Tod, [12], proved that a weakly-asymptotically simple², vacuum or electrovacuum, time-periodic spacetime which is analytic in a neighbourhood of (past) null infinity necessarily has a time-like Killing field in the interior. They followed ideas from Gibbons and Stewart, [40], but work in a different coordinate system and less restrictive gauge. Bičák-Scholtz-Tod used the (undesired) hypothesis of analyticity all the way up to infinity to split Einstein's equations order by order at infinity and then use the time-periodic condition to conclude time-independence inductively. Then the desired stationarity conclusion in the interior follows by analytic continuation.

The question was later considered by Alexakis and Schlue in [5]. They showed that a time-periodic or non-radiating spacetime which is suitably regular at infinity is stationary in a neighbourhood of spatial infinity. They remark that the regularity assumptions can be deduced from regularity of the initial data in the time-periodic case. However, it seems that their smoothness assumptions at infinity are rather strong (and difficult to unravel) in the non-radiating case. In this thesis, we adopt slight modifications of Alexakis' and Schlue's regularity assumptions aiming at fill-

²Admitting a smooth conformal compactification of (past in their case) null infinity.

ing some gaps concerning compatibility of the different coordinate systems used in [5], and also to clarify the strength and role of their smoothness and non-radiating condition at both spacelike and null infinity. We do this in Section 1.3.

Alexakis' and Schlue's approach relies again on first proving stationarity to all orders at infinity as in [12]. Then, in order to extend this condition to the interior, they use unique continuation from infinity techniques based on Carleman estimates in the spirit of [6]. For the first part of the proof, the exact null structure of the equations plays an important role in order to compute the metric, connection coefficients and curvature components to all orders at infinity from just the 'radiation field'. For the second part a wave equation satisfied by the deformation tensor of the Weyl curvature, $\mathcal{L}_T \mathbf{C}$, is derived and exploited to conclude stationarity in a neighbourhood of infinity by means of energy estimates.

The purpose of this thesis is to show that the Alexakis' and Schlue's result holds as well when gravity is coupled to electromagnetism. That is, gravity and electromagnetism cannot balance each other to produce a time-periodic solution. The key point is that the coupled Einstein-Maxwell system can be treated in a similar way as in [5]. The two deformation tensors, $\mathcal{L}_T \mathbf{C}$ and $\mathcal{L}_T \mathbf{F}$, also obey wave equations; however the coupling terms do not decay fast enough for the Alexakis-Schlue argument to work. Hence we are forced to revise and adapt their proof at the level of Carleman estimates to conclude the vanishing of the deformation tensors a little bit into the interior of the spacetime. An informal version of the main result of this thesis is

Theorem 1.4.2 *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be an asymptotically flat non-radiating solution of the Einstein-Maxwell equations. Then there exists a time-like vector field T in a neighbourhood of spatial infinity such that*

$$\mathcal{L}_T \mathbf{g} = 0 = \mathcal{L}_T \mathbf{F}.$$

In Section 3.4 we present a more precise statement together with a summary of the main steps of the proof. Now we comment briefly about the assumptions. In this thesis we will consider the class of asymptotically flat spacetimes admitting coordinates $(t, r, \vartheta^2, \vartheta^3)$ which are suitably close (to order 3 in r^{-1}) to Kerr-Newman in Boyer-Linquist coordinates, Definition 1.3.2. We also require the existence of coordinates $(u, s, \theta^2, \theta^3)$ adapted to future null infinity such that the metric with respect

to these coordinates admit an infinite asymptotic expansion in inverse powers of s which is well-behaved with respect to derivatives, Definition 1.3.8. We remark that this latter condition is morally equivalent to smoothness at future null infinity in the Penrose conformal picture, the advantage of this formulation lies in its adaptability to more general asymptotic conditions, e.g., polyhomogeneous expansions [66], [61]. We concentrate in this thesis in the smooth case and leave the possible generalisations to cope with expansions including logarithmic terms for future research.

By non-radiating we mean that that so-called radiation fields at future null infinity Ξ_{ij} and $\underline{A}(\mathbf{F})_i$ vanishes, where Ξ_{ij} is the traceless part of the incoming shear χ_{ij} and $\underline{A}(\mathbf{F})_i$ is the leading order of the Faraday tensor (cf. Section 1.2.1 for notation and definitions). This condition encodes the stationarity of the spacetime asymptotically; we will see this in Proposition 2.1.4. We also need control towards the past; to do so we impose a decay estimate of the form $\frac{1}{(1+|u|)^{1+\eta}}$, $\eta > 0$, for the deformation tensors. See Definition 1.4.1 for more details.

The conclusion of Theorem 1.4.2 can be seen from two point of views. Firstly, as a rigidity result for metrics with fast decaying curvature. Secondly, it is also a inheritance of symmetry result since the gravity and electromagnetic field both turned out to be time-independent. As we will see shortly, this is not the case for other matter-energy models coupled to gravity.

We remark that the above result can be generalised to include a massless Klein-Gordon scalar field coupled with gravity. A more precise statement, Theorem ??, is presented in Chapter 4 . We stress that the smoothness conditions at future null infinity are also assumed; however they are compatible with the linear analysis performed by Winicour, [66], to leading order.

Bičák-Scholtz-Tod also tackled this Einstein-Klein-Gordon problem in [13] (a continuation of [12]). They again go around the fall-off condition by requiring analyticity of the fields all the way up to infinity and conclude that time-periodic asymptotically flat Einstein-massless-Klein-Gordon systems are in fact stationary.

The results presented in this thesis are local around spatial infinity and hence are applicable to any system whose matter source is spatially compact. It is of considerable interest to mention here what is known about the case when the matter content extends to infinity. The simplest model to consider is that of a massive

Klein-Gordon field coupled to Einstein's equations. As mentioned above the techniques used in this thesis carry on to the Einstein-massless-Klein-Gordon system; however as we will see, the massive case is different.

In [16], Chodosh and Shlapentokh-Rothman constructed a 1-parameter family of solutions to the Einstein-Klein-Gordon equations bifurcating off the Kerr solution. The underlying family of spacetimes are asymptotically flat, stationary and axisymmetric. However, the corresponding scalar fields are non-zero and time-periodic. This is in contrast with our result where the symmetry is inherited by all the fields. The crucial difference in the assumptions is the presence of a non-zero mass for the scalar field, which acts as a non-decaying potential.

Related to this result is the existence of countably many time-periodic, spherically symmetric asymptotically flat *boson stars* given by Bizoń and Wasserman in [14]. These are solutions of the Einstein-Klein-Gordon equations where the underlying spacetime is static while the complex scalar field has a positive mass and takes the form of a standing wave $\phi(r)e^{i\omega t}$. Therefore the matter field does not inherit the time-like symmetry.

Also we mention here the unique continuation results for Klein-Gordon type equations on fixed backgrounds which are the cornerstone of the rigidity results proved in [5] and in this thesis. In [6], Alexakis-Schlue-Shao proved a unique continuation from infinity result for Klein-Gordon type equations,

$$\square_{\mathbf{g}}\phi + a^\alpha \nabla_\alpha \phi + V\phi = 0,$$

on asymptotically flat spacetimes, with a^α and V decaying suitably. They require infinite-order vanishing at infinity of ϕ and its first derivatives in order to conclude vanishing of ϕ locally. In the case of a merely bounded potential V (e.g. a massive Klein-Gordon equation), a stronger, exponential rate of decay at infinity is necessary to obtain the same result. It is worth mentioning that the infinite-order vanishing condition can be replaced by global regularity assumptions as in [7].

One can also consider different boundary conditions, such as de Sitter or anti-de Sitter. Holzegel and Shao, [42], proved a similar unique continuation result for the equation

$$\square_{\mathbf{g}}\phi + \sigma\phi = a^\alpha \nabla_\alpha \phi + V\phi, \quad \sigma \in \mathbb{R},$$

on asymptotically anti-de Sitter spacetimes, with a^α and V decaying suitably. Their method of proof follows closely that of [6] in that they assume vanishing conditions at infinity and together with novel Carleman estimates derived therein they conclude local vanishing in the interior. In contrast with the asymptotically flat case, they found that just by requiring the vanishing of a weighted rate of decay (with weight going to infinity as $\sigma \rightarrow -\infty$) of ϕ and its first derivatives on a sufficiently large portion of infinity they can prove unique continuation. The case for merely bounded potentials V is also treated and found that an infinite-order vanishing condition on ϕ is sufficient to conclude unique continuation.

0.1 Outline of the thesis

In **Chapter 1** of this thesis we present a brief review of the basic notions of Differential Geometry and General Relativity. Notions such as tensors, causality, curvature and Einstein's equations are defined. Illustrative and important examples of vacuum and electrovacuum spacetimes such as Minkowski, Schwarzschild, Kerr-Newman are presented; while more detailed asymptotic properties of these are included in the appendix. We also give a brief sketch of the Initial value formulation of Einstein's equations.

We devote Section 1.2 to review Christodoulou and Klainerman analysis based on the initial value formulation [22]. Their point of view is that all the asymptotic assumptions have to be made at the level of initial data and the decaying properties of the spacetime metric are then to be deduced from the former and the evolution equations. In their book, *The global non-linear stability of the Minkowski space* [22], they carried out this program for small perturbations of trivial initial data and proved that the resulting global solutions exist for all time and remain close to the trivial spacetime. Moreover they found precise rate of decay for the components of the Weyl tensor, cf Section 1.2.1. We also include the generalisation made by Zipser, [67], to include electromagnetism.

In this thesis we will be working with two main coordinates systems: Outgoing null coordinates, $(u, s, \theta^2, \theta^3)$, are used to compute asymptotic quantities to all orders at infinity with respect to the s -asymptotic expansion along outgoing null hypersurfaces C_u . On the other hand, asymptotically double-null coordinates $(v, \underline{v}, y^2, y^3)$ capture the decaying conditions in a neighbourhood of spatial infinity and are used to express the Carleman estimates necessary for the unique continua-

tion technique used by Alexakis and Schlue. We introduce them in Section 1.3. More precisely, we consider spacetimes suitably close to Kerr-Newman in Boyer-Linquist coordinates, $(t, r, \vartheta^2, \vartheta^3)$, and then construct coordinates $(u, s, \theta^2, \theta^3)$ based on [5] and [19]; it is a definite hypothesis that the two coordinate systems are compatible and that the spacetime metric is smooth at future null infinity, cf. Definition 1.3.8. We also show that such spacetimes close to Kerr-Newman admit the required Alexakis-Schlué coordinates, $(v, \underline{v}, y^2, y^3)$, in Proposition 1.3.11.

We conclude this chapter by discussing the notion of non-radiating spacetimes and we present our main result, Theorem 1.4.2.

In **Chapter 2** we use the structure equations together with the smoothness assumption to achieve the first step in the proof of the main theorem. That is, it is proven that a non-radiating spacetime is stationary to all orders at infinity with respect to the s -asymptotic expansion along outgoing null hypersurfaces.

This is done by recursively computing the connection and curvature coefficients to all orders at infinity (here is where the smoothness assumption of asymptotic flatness in Definition 1.3.8 plays an important role). The hierarchy found by BMS, also interpreted as signature levels in the language of Christodoulou-Klainerman, helps to understand the limiting structure of the Einstein's equations at future null infinity. Then the aforementioned hierarchy helps us identify levels where the equations become linear for the quantities belonging to that level. Moreover, the radiation fields can be regarded as the necessary initial data to run an induction argument and find recurrence relations. In particular, in the absence of radiation fields, all the asymptotic quantities are found to be time-independent, Proposition 2.1.4. Also, the procedure sheds light on the way the different terms in the s -expansion of the metric and connection coefficients couple to each other via the Einstein's equations, Proposition 2.1.1.

The above results together with an analytical condition already imply the stationarity of non-radiating electrovacuum spacetimes in a neighbourhood of infinity as in [12]. However, one of the goals of this thesis is to dispense with the analyticity assumption. We are able to remain in the smooth class by using the methods explained in Chapter 3 and conclude the stationarity of non-radiating spacetimes in the class of smooth metrics.

In **Chapter 3** we finish the proof of the main theorem. We explain how the previous asymptotic values at infinity can be translated to a local result in the spacetime. That is, the condition of stationarity to all orders at infinity is extended to a neighbourhood of spatial infinity.

The main technical tool is that of Carleman estimates, Theorem 3.1.2. These are estimates for functions decaying faster than any polynomial towards infinity. They can also be thought as energy estimates adapted to time-like boundary conditions where now we aim at controlling the bulk term while making the boundary terms vanish.

At this point we make a parenthesis to look for wave equations satisfied by the components of the deformation tensors $\mathcal{L}_T\mathbf{C}$ and $\mathcal{L}_T\mathbf{F}$. These are the Ionescu-Klainerman tensorial equations, [44]. We need to revise them in order to cope with a non-vacuum spacetime. These wave equations, written with respect to a suitable frame adapted to coordinates $(v, \underline{v}, y^2, y^3)$, in conjunction with the Carleman estimates are used to find L^2 -bounds for the components of the deformation tensors and its first derivatives. These bounds depend on an arbitrarily large parameter which can be taken to infinity to conclude the vanishing of the functions in the interior, cf. Proposition 3.3.1.

We conclude this chapter by putting all the pieces together and proving the main theorem, 1.4.2. We also present the notion of time-periodic spacetimes and we prove the corresponding stationarity result.

In **Chapter 4** we present the main conclusions of this thesis. We start by digressing the main construction and assumptions of our main result.

We also state a variation of Theorem 1.4.2 to include a Klein-Gordon field, as well as the corresponding recurrence relations. In addition, we propose a unique continuation conjecture for the Einstein's equations. Roughly speaking, it states that a spacetime is determined by its radiation field along null infinity and stationary data (to all orders) at spatial infinity.

Finally, an **Appendix** is included to review exact solutions relevant to the discussion of asymptotic flatness and gravitational radiation. The classical Kerr-Newman family (which includes Minkowski and Schwarzschild spacetimes) is pre-

sented in order to gain intuition about the leading order terms appearing in the asymptotic expansions of the connection coefficient and curvature components. Also a family of radiating solutions, the Robinson-Trautman metric, is reviewed. We pay special attention to their decaying properties and global structure.

Chapter 1

Preliminaries

1.1 Differential geometry and Einstein's equations

We start by setting notation. Let \mathcal{M} be a 4-dimensional smooth manifold. Recall that with this structure it is possible to define the tangent bundle $\mathcal{T}\mathcal{M}$, as well as its dual $\mathcal{T}^*\mathcal{M}$, and all the finite tensor products of these two. Recall that a *tensor* of type (r, s) is a smooth section of the vector bundle

$$\underbrace{\mathcal{T}\mathcal{M} \otimes \dots \otimes \mathcal{T}\mathcal{M}}_r \otimes \underbrace{\mathcal{T}^*\mathcal{M} \otimes \dots \otimes \mathcal{T}^*\mathcal{M}}_s \longrightarrow \mathcal{M}.$$

In particular a vector field is a $(1, 0)$ -tensor and a one-form is a $(0, 1)$ -tensor. We denote by $\mathfrak{X}(\mathcal{M})$ the space of smooth vector fields over \mathcal{M} .

Definition 1.1.1. A Lorentzian metric \mathbf{g} , also denoted as $\langle \cdot, \cdot \rangle$, on \mathcal{M} is a symmetric non-degenerate smooth section of $\mathcal{T}^*\mathcal{M} \otimes \mathcal{T}^*\mathcal{M}$ of signature $(-, +, +, +)$. This means that for each $p \in \mathcal{M}$ there is a basis of $\mathcal{T}_p^*\mathcal{M}$ with respect to which the components of \mathbf{g} form the matrix $\text{diag}(-1, 1, 1, 1)$.

It is necessary to work with components with respect to a basis. In this section $\{e_0, e_1, e_2, e_3\}$ will be an arbitrary local basis of vector fields on some open set $\mathcal{U} \subset \mathcal{M}$. We denote by $\{e^0, e^1, e^2, e^3\}$ the basis of 1-forms dual to $\{e_0, e_1, e_2, e_3\}$, that is, e^μ is the 1-form defined by $e^\mu(e_\nu) = \delta^\mu_\nu$. Using this notation the components of a tensor T of type (r, s) with respect to this basis are written as

$$T_{\mu\dots\nu}{}^{\rho\dots\sigma} := T(e_\mu, \dots, e_\nu, e^\rho, \dots, e^\sigma).$$

Then the tensor can be recovered from its components as follows,

$$T = T_{\mu\dots\nu\rho\dots\sigma} e^\mu \otimes \dots \otimes e^\nu \otimes e_\rho \otimes \dots \otimes e_\sigma.$$

Remark. Einstein's summation convention is used throughout: repeated Greek indices are to be understood as summed over the range 0, 1, 2, 3. In the coming sections we will work with submanifolds of dimension 2 and 3; we will use Latin indices a, b, \dots , to cover the values 1, 2, and 3, and Latin indices i, j, \dots , will run from 2 to 3.

Recall that the metric \mathbf{g} is non-degenerate, this has two implications. Firstly, the components of the metric, $\mathbf{g}_{\mu\nu}$, form a 4×4 invertible matrix. Secondly, \mathbf{g} induces an isomorphism $\mathcal{T}_p\mathcal{M} \rightarrow \mathcal{T}_p^*\mathcal{M}$ given by

$$X \mapsto X^\flat(\cdot) := \langle X, \cdot \rangle.$$

In components with respect to a basis, the previous translates to,

$$X_\mu^\flat = \mathbf{g}_{\mu\nu} X^\nu,$$

The inverse of this isomorphism is denoted by $\omega \mapsto \omega^\sharp$. In components it reads,

$$(\omega^\sharp)^\mu = \mathbf{g}^{\mu\nu} \omega_\nu,$$

where $\mathbf{g}^{\mu\nu}$ is the inverse matrix of $\mathbf{g}_{\mu\nu}$.

This procedure is known as *lowering* and *raising* indices, and generalises to tensors. We use it to identify all spaces of (p, q) -tensors with the same value of $p + q$.

Another important concept is causality:

Definition 1.1.2. *i) A vector X is said to be time-like, null or space-like if $\langle X, X \rangle < 0$, $\langle X, X \rangle = 0$ or $\langle X, X \rangle > 0$, respectively. A subspace, $V \subset \mathcal{T}_p\mathcal{M}$, is said to be time-like, null or space-like if the induced metric is Lorentzian, degenerate or positive definitive, respectively. This nomenclature carries on to curves, and in general to submanifolds, according to the character of the tangent space.*

ii) We say that $(\mathcal{M}, \mathbf{g})$ is time-orientable if it admits a globally defined smooth time-like vector field T . With respect to this choice, a time-like or null vector X is said to be future directed or past directed if $\langle T, X \rangle < 0$ or $\langle T, X \rangle > 0$,

respectively.

iii) A causal curve is a differentiable curve whose tangent at every point is a future directed time-like or null vector.

Derivative operator and curvature

In the presence of a metric there is a unique derivative operator, called the Levi-Civita connection, ∇ , which is a map from $\mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \rightarrow \mathfrak{X}(\mathcal{M})$ satisfying for any $X, Y, Z \in \mathfrak{X}(\mathcal{M})$:

i) Linearity,

$$\begin{aligned}\nabla_X(aY + bZ) &= a\nabla_XY + b\nabla_XZ, \\ \nabla_{aX+bZ}Y &= a\nabla_XY + b\nabla_ZY\end{aligned}$$

for any $a, b \in \mathbb{R}$.

ii) It is tensorial for the first entry, that is,

$$\nabla_{fX}Y = f\nabla_XY,$$

for any $f: \mathcal{M} \rightarrow \mathbb{R}$ smooth.

iii) Satisfies the Leibniz rule for the second entry, that is,

$$\nabla_X(fY) = X(f)Y + f\nabla_XY,$$

for any $f: \mathcal{M} \rightarrow \mathbb{R}$ smooth.

iv) It is torsion free,

$$[X, Y] = \nabla_XY - \nabla_YX.$$

v) It is compatible with the metric in the sense that,

$$X\langle Y, Z \rangle = \langle \nabla_XY, Z \rangle + \langle Y, \nabla_XZ \rangle.$$

The vector field ∇_XY is called the covariant derivative of Y in the direction of X . Now we note that this operation can be extended to one-forms. Given a one-form ω , we define $\nabla_X\omega$ to be the one-form satisfying:

$$\nabla_X\omega(Y) = X(\omega(Y)) - \omega(\nabla_XY),$$

for any vector field Y . Moreover, for any vector field X the operator ∇_X can be extended to tensor spaces by requiring the Leibniz rule to hold, for example

$$\nabla_X(Y \otimes \omega) := (\nabla_X Y) \otimes \omega + Y \otimes \nabla_X \omega.$$

Associated to a connection is its *Riemann curvature*, \mathbf{R} , this is a (1,3)-tensor defined as

$$\mathbf{R}(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z.$$

Consider the components with respect to an arbitrary local basis, $\mathbf{R}_{\alpha\beta\mu\nu}$, of the associated (0,4)-tensor $\langle \mathbf{R}(\cdot, \cdot), \cdot, \cdot \rangle$; they have the following properties:

- i) $\mathbf{R}_{\alpha\beta\mu\nu} = -\mathbf{R}_{\beta\alpha\mu\nu} = -\mathbf{R}_{\alpha\beta\nu\mu} = \mathbf{R}_{\mu\nu\alpha\beta}$.
- ii) $\mathbf{R}_{\alpha\beta\mu\nu} + \mathbf{R}_{\beta\mu\alpha\nu} + \mathbf{R}_{\mu\alpha\beta\nu} = 0$. This is known as the *first Bianchi identity*.
- iii) $\nabla_\alpha \mathbf{R}_{\beta\gamma\mu\nu} + \nabla_\beta \mathbf{R}_{\gamma\alpha\mu\nu} + \nabla_\gamma \mathbf{R}_{\alpha\beta\mu\nu} = 0$. Also known as the *second Bianchi identity* or just the Bianchi identity.

Due to these symmetries, the Riemann tensor has 20 algebraically independent components. By taking the trace over β and ν we get the *Ricci tensor*,

$$\mathbf{Ric}_{\alpha\mu} := \mathbf{g}^{\beta\nu} \mathbf{R}_{\alpha\beta\mu\nu}.$$

Finally, the *scalar curvature* is the trace of the Ricci tensor

$$R := \mathbf{g}^{\alpha\mu} \mathbf{Ric}_{\alpha\mu}.$$

The so-called *Einstein tensor* can be defined in terms of these,

$$\mathbf{G}_{\alpha\beta} := \mathbf{Ric}_{\alpha\beta} - \frac{1}{2} R \mathbf{g}_{\alpha\beta}.$$

It is worth remarking that the second Bianchi identity implies $\nabla^\alpha \mathbf{G}_{\alpha\beta} = 0$, this is known as the contracted Bianchi identities. The combination

$$\mathbf{S}_{\alpha\beta} := \mathbf{Ric}_{\alpha\beta} - \frac{1}{6} R \mathbf{g}_{\alpha\beta}$$

is called the *Schouten tensor*. It contains exactly the same information as the Ricci or Einstein tensors but has a simpler behaviour under conformal transformations.

It also allows to express the *Weyl tensor*, $\mathbf{C}_{\alpha\beta\mu\nu}$, in a neat formula:

$$\mathbf{C}_{\alpha\beta\mu\nu} := \mathbf{R}_{\alpha\beta\mu\nu} - \frac{1}{2}(\mathbf{g}_{\alpha\mu}\mathbf{S}_{\beta\nu} - \mathbf{g}_{\beta\mu}\mathbf{S}_{\alpha\nu} + \mathbf{g}_{\beta\nu}\mathbf{S}_{\alpha\mu} - \mathbf{g}_{\alpha\nu}\mathbf{S}_{\beta\mu}).$$

The above expression accomplishes the goal of splitting the Riemann tensor into trace and traceless components. That is, the Weyl tensor shares the symmetries i), ii) and iii) of the Riemann tensor; in addition, the contraction of any two of its indices vanishes. The Schouten and Weyl tensors have 10 algebraically independent components each.

We are now ready to state the principal postulate of General Relativity.

Definition 1.1.3. *A spacetime $(\mathcal{M}, \mathbf{g})$ is an orientable, time-orientable, Lorentzian 4-manifold satisfying*

$$\mathbf{G}_{\alpha\beta} + \Lambda\mathbf{g}_{\alpha\beta} = 8\pi\mathbf{T}_{\alpha\beta}.$$

These are known as the Einstein's equations. They encode the dynamics of both the spacetime and the matter within; as John Wheeler put it: "Space tells matter how to move; matter tells space how to curve". The parameter Λ is known as the *cosmological constant*; in this thesis only the case $\Lambda = 0$ will be considered. On the left-hand side of the equation we have only geometrical invariants assigned to the metric while on the right-hand side we have the *stress-energy tensor* \mathbf{T} ; this is a symmetric (0,2)-tensor containing all the physical information about the matter/energy fields present in the spacetime. We note that it must satisfy the conservation of energy condition, $\nabla^\alpha\mathbf{T}_{\alpha\beta} = 0$, to ensure compatibility of the Einstein's equations with the contracted Bianchi identity.

Examples.

- i) The stress-energy tensor for a *vacuum spacetime* is simply $\mathbf{T} = 0$. This gives a system of 10 second order PDEs for the 10 components of the metric tensor. Now we present some important solutions:

The *Minkowski metric* on \mathbb{R}^4 is given by

$$\mathbf{g}_M = -dt^2 + dx^2 + dy^2 + dz^2,$$

which in spherical coordinates reads,

$$\mathbf{g}_M = -dt^2 + dr^2 + r^2\check{\gamma},$$

where $\check{\gamma}$ is the standard round metric on \mathbb{S}^2 .

The *Schwarzschild metric* on $\mathbb{R} \times (2M, \infty) \times \mathbb{S}^2$ is

$$\mathbf{g}_{Sch} = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\check{\gamma},$$

with M a constant regarded as the mass of the system.

The *Kerr metric* in Boyer-Lindquist coordinates, $(t, r, \theta^2, \theta^3)$, is defined by,

$$\begin{aligned} \mathbf{g}_K = & - \left(1 - \frac{2Mr}{R^2}\right) dt^2 + \frac{2Mra \sin^2(\theta^2)}{R^2} dt d\theta^3 + \frac{R^2}{\Delta} dr^2 \\ & + R^2 (d\theta^2)^2 + \left(r^2 + a^2 + \frac{2Mra^2 \sin^2(\theta^2)}{R^2}\right) \sin^2(\theta^2) (d\theta^3)^2, \end{aligned}$$

where $R^2 := r^2 + a^2 \cos^2(\theta^2)$, $\Delta := r^2 + a^2 - 2Mr$, M and a are constants which can be interpreted as mass and angular momentum. The domain of definition of these coordinates is taken to be $\mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2$, where r_+ is the largest solution of $\Delta(r) = 0$ and we assume $M > |a|$.

Properties of these spacetimes are discussed in the Appendix.

- ii) A *Klein-Gordon* field φ is a complex-valued function coupled to the Einstein's equations via the following stress-energy tensor,

$$\mathbf{T}_{\alpha\beta} = \frac{1}{4\pi} \left(2\nabla_{(\alpha}\bar{\varphi}\nabla_{\beta)}\varphi - \mathbf{g}_{\alpha\beta}\nabla^\mu\bar{\varphi}\nabla_\mu\varphi - \mathbf{g}_{\alpha\beta}\kappa^2\bar{\varphi}\varphi \right).$$

Moreover, it is required to satisfy its own field equation:

$$\square_{\mathbf{g}}\varphi - \kappa^2\varphi = 0.$$

The constant κ is interpreted as the mass of the Klein-Gordon field and $\square_{\mathbf{g}} := \mathbf{g}^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the wave operator.

iii) An *electromagnetic field* or *Maxwell field* is represented by

$$\mathbf{T}_{\alpha\beta} = \frac{1}{4\pi} \left(\mathbf{F}_{\alpha\nu} \mathbf{F}^{\nu}_{\beta} - \frac{1}{4} \mathbf{g}_{\alpha\beta} \mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu} \right),$$

where $\mathbf{F}_{\alpha\beta}$ is a Faraday tensor; that is, $\mathbf{F}_{\alpha\beta}$ is an anti-symmetric (0,2)-tensor satisfying Maxwell equations,

$$\nabla^{\alpha} \mathbf{F}_{\alpha\beta} = 0,$$

$$\nabla_{[\alpha} \mathbf{F}_{\beta\nu]} = 0.$$

It is worth remarking that Maxwell equations imply the conservation of energy condition $\nabla^{\alpha} \mathbf{T}_{\alpha\beta} = 0$. The Maxwell equations are 8 additional PDEs coupled to the Einstein's equations. The whole system will be called *Einstein-Maxwell* equations.

One can generalise the Schwarzschild and Kerr solutions to obtain the corresponding charged versions, known as *Reissner-Nordström* and *Kerr-Newman* solutions, respectively:

$$\begin{aligned} \mathbf{g}_{RN} &= - \left(1 - \frac{2M}{r} + \frac{e^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2} \right)^{-1} dr^2 + r^2 \tilde{\gamma}, \\ \mathbf{g}_{KN} &= - \left(1 - \frac{2Mr - e^2}{R^2} \right) dt^2 + \frac{(2Mr - e^2)a \sin^2(\theta^2)}{R^2} dt(d\theta^3) + \frac{R^2}{\Delta} dr^2 \\ &\quad + R^2 (d\theta^2)^2 + \left(r^2 + a^2 + \frac{(2Mr - e^2)a^2 \sin^2 \theta}{R^2} \right) \sin^2(\theta^2) (d\theta^3)^2, \end{aligned}$$

where $R^2 := r^2 + a^2 \cos^2(\theta^2)$ and $\Delta := r^2 + a^2 - 2Mr + e^2$. Here the constant e is interpreted as the electromagnetic charge.

iv) A *perfect fluid*¹ in thermodynamic equilibrium with 4-velocity u_a , density ρ and pressure p is modelled by

$$\mathbf{T}_{ab} = (\rho + p)u_a u_b + p \mathbf{g}_{ab}.$$

The conservation of energy condition, $\nabla^a T_{ab} = 0$, has to be imposed in this case as the evolution equation for the system. In addition an equation of state, $p = p(\rho)$, is required to get a closed system. Of particular importance are the cases: dust, $p = 0$, and radiation fluid, $p = 3\rho$.

¹A fluid with vanishing viscosity and heat flux.

In this thesis the main focus will be on solutions to the Einstein-Maxwell equations, also referred to as *electrovacuum spacetimes*. We will also extend our results to Einstein-Klein-Gordon systems in Chapter 5. We will always work in (3+1)-dimensions.

1.1.1 Initial value formulation

From a physical point of view, it is desirable to cast the Einstein's equations as an evolutionary system, where the initial conditions determine completely the solution. As we will see this is indeed the case. The celebrated well-posedness result of Choquet-Bruhat and Geroch, [17], [18] establishes existence and uniqueness of solutions of the Einstein's equations given initial data, Theorem 1.1.4 below. The main idea is that the Einstein's equations can be reduced to a hyperbolic system where known techniques can be applied to obtain the desired result. Here we present the version as stated in Wald's book [65], we refer the reader to that reference for more details.

It is worth remarking that the Einstein's equations impose constraint equations on the initial data. In order to understand this feature we briefly review relevant notions of hypersurfaces in Lorentzian manifolds.

As before, consider a Lorentzian 4-manifold $(\mathcal{M}, \mathbf{g})$. In order to avoid undesired pathologies we require the existence of a Cauchy surface² $\Sigma \subset \mathcal{M}$. In this case $(\mathcal{M}, \mathbf{g})$ is said to be *globally hyperbolic*. It is a non-trivial result that global hyperbolicity implies the existence of a smooth global time function; this is a scalar function t whose gradient, ∇t , is time-like everywhere. Moreover, in such a case \mathcal{M} is diffeomorphic to $\mathbb{R} \times \Sigma$ and we denote by Σ_t the foliation induced by the level sets of t .

Let \mathbf{h} be the induced metric on Σ and \mathbf{n} be the future-directed unit normal field. Recall that Σ is space-like, so \mathbf{n} is time-like and \mathbf{h} is a Riemannian metric.

The second fundamental form of $\Sigma \subset \mathcal{M}$ is a (0,2)-tensor on Σ defined by

$$K(Y, Z) := \langle \nabla_Z Y, \mathbf{n} \rangle,$$

²A hypersurface with the property that any causal curve intersects it at precisely one point. In particular, a Cauchy surface must be space-like.

where Y, Z are vector fields tangent to Σ . We remark that, even though the vector fields Y, Z are defined only on Σ , the left-hand side of the above equation is unambiguous as $\nabla_Z Y$ depends only on the values of Y along the integral curves of Z ; the result however may fail to remain tangent to Σ . The second fundamental form measures precisely this failure. It can be checked that K is symmetric and $K(Y, Z) = -\langle Y, \nabla_Z \mathbf{n} \rangle$.

In the following we will choose a local basis $\{\hat{e}_0 = \mathbf{n}, \hat{e}_1, \hat{e}_2, \hat{e}_3\}$ adapted to Σ , that is, $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ are tangent to it. We remind the reader that Latin indices, a, b, \dots , run from 1 to 3.

Denote by $D, {}^3R_{abcd}, {}^3R_{ab}$ and 3R the Levi-Civita connection, the Riemann tensor, Ricci tensor and scalar curvature, respectively, associated to \mathbf{h} . The Gauss and Codazzi-Mainardi formulae relate the above invariants of $\Sigma \subset \mathcal{M}$ with the tangent-tangent and tangent-normal projections of the curvature of \mathcal{M} :

$$\begin{aligned} {}^3R_{abcd} + K_{ac}K_{bd} - K_{bc}K_{ad} &= \mathbf{R}_{abcd}, \\ D_b K^a{}_a - D_a K^a{}_b &= \mathbf{Ric}_{b0}. \end{aligned}$$

In a vacuum spacetime, $\mathbf{Ric} = 0$, the Gauss and Codazzi-Mainardi equations imply the so-called *vacuum constraint equations*,

$${}^3R + (K^a{}_a)^2 - K_{ab}K^{ab} = 0, \tag{1.1}$$

$$D_b K^a{}_a - D_b K^b{}_a = 0. \tag{1.2}$$

The previous discussion tells us that they are necessary conditions on data (Σ, \mathbf{h}, K) induced by a vacuum spacetime $(\mathcal{M}, \mathbf{g})$. The next theorem is a converse of this statement:

Theorem 1.1.4. *Let (Σ, \mathbf{h}, K) be an initial data set. That is, (Σ, \mathbf{h}) is a Riemannian 3-manifold and K is a symmetric 2-tensor satisfying the vacuum constraint equations (1.1) and (1.2). Then there exists a unique spacetime $(\mathcal{M}, \mathbf{g})$, called the maximal Cauchy development of (Σ, \mathbf{h}, K) , satisfying the following properties:*

1. $(\mathcal{M}, \mathbf{g})$ is a solution of the vacuum Einstein's equations.
2. There is an embedding $i : \Sigma \longrightarrow \mathcal{M}$ such that $i(\Sigma) \subset \mathcal{M}$ is a Cauchy surface.
3. The induced metric and second fundamental of $i(\Sigma) \subset \mathcal{M}$ are equal to $i^*(\mathbf{h})$ and $i^*(K)$, the push-forwards of \mathbf{h} and K , respectively.

4. Every other spacetime satisfying 1-3 can be mapped isometrically into $(\mathcal{M}, \mathbf{g})$.

We comment briefly on the non-vacuum case. The previous result is based on the existence and uniqueness theorem for quasilinear, second order hyperbolic systems of PDEs and thus it can be applied to include matter sources only when the dynamical equations satisfied by the matter fields are of this form and if \mathbf{T} depends only on the fields, the metric and the first derivatives of the fields and metric. In particular, the Klein-Gordon and Maxwell fields admit an initial value formulation, as well as a perfect fluid for appropriate choices of equation of state $P = P(\rho)$.

In the case of the Einstein-Maxwell equations one has to prescribe, in addition to (Σ, \mathbf{h}, K) , the *electric* and *magnetic fields* on Σ as initial data. These are defined as

$$E_a := \mathbf{F}_{a0}, \quad B_a := \star \mathbf{F}_{a0},$$

where $\star \mathbf{F}_{ab}$ is the Hodge dual³ of \mathbf{F}_{ab} . They also have to satisfy constraint equations on the initial slice which, in the case of no sources, read

$$D^a E_a = 0, \quad D^a B_a = 0.$$

Examples.

- i) Trivial initial data ($\Sigma \cong \mathbb{R}^3, \mathbf{h}_M = dx^2 + dy^2 + dz^2, K = 0$) develops into Minkowski spacetime.
- ii) Initial data for Schwarzschild spacetime is given on $\Sigma \cong (2M, \infty) \times \mathbb{S}^2$ with

$$\mathbf{h}_{Sch} = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \check{\gamma}, \quad K = 0.$$

- iii) Initial data for ReissnerNordström is given by

$$\begin{aligned} \mathbf{h}_{RN} &= \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 \check{\gamma}, \quad K = 0, \\ E &= -\frac{e}{r^2} dr, \quad B = 0. \end{aligned}$$

Now we proceed to a brief sketch of the main ideas of the proof. In order to understand the mathematical structure of the Einstein's equations as PDEs one

³ $\star \mathbf{F}_{ab} := \epsilon_{abcd} \mathbf{F}^{cd}$, where ϵ_{abcd} is the volume element of $(\mathcal{M}, \mathbf{g})$.

needs to write them explicitly in coordinates. However, in Definition 1.1.3, the Einstein's equations were presented from a geometrical point view. At this point their covariant form makes them particularly difficult to analyse. Specifically, relative to arbitrary coordinates (x^0, x^1, x^2, x^3) , the Ricci tensor takes the form

$$\mathbf{Ric}_{\mu\nu} = -\frac{1}{2} \sum_{\alpha,\beta=0}^3 \mathbf{g}^{\alpha\beta} (-2\partial_\beta \partial_{(\nu} \mathbf{g}_{\mu)\alpha} + \partial_\alpha \partial_\beta \mathbf{g}_{\mu\nu} + \partial_\mu \partial_\nu \mathbf{g}_{\alpha\beta}) + F_{\mu\nu}(\mathbf{g}, \partial\mathbf{g}),$$

where $F_{\mu\nu}(\mathbf{g}, \partial\mathbf{g})$ is a non-linear function of the metric and its first derivatives.

In particular, the vacuum Einstein's equations, $\mathbf{Ric}_{\mu\nu} = 0$, are not of a definite type. To remedy this, one has the freedom to impose gauge conditions aiming at fixing an appropriate coordinate system. One common choice of gauge conditions are given by the so-called wave coordinates; these are coordinates satisfying

$$H^\mu := \square_{\mathbf{g}} x^\mu = 0, \quad \mu = 0, 1, 2, 3.$$

In terms of the metric components, this expression is equivalent to

$$H^\mu = \sum_{\alpha=0}^3 \left(\partial_\alpha \mathbf{g}^{\alpha\mu} + \frac{1}{2} \mathbf{g}^{\alpha\mu} \sum_{\rho,\sigma=0}^3 g^{\rho\sigma} \partial_\alpha \mathbf{g}_{\rho\sigma} \right) = 0.$$

The advantage of this choice is more clearly seen by expressing the undesired 2nd order terms in $\mathbf{Ric}_{\mu\nu}$ in terms of H^μ . Explicitly,

$$\begin{aligned} \mathbf{Ric}_{\mu\nu} &= -\frac{1}{2} \sum_{\alpha,\beta=0}^3 \mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{g}_{\mu\nu} - \sum_{\alpha=0}^3 (\mathbf{g}_{\alpha(\mu} \partial_\nu) H^\alpha) + \hat{F}_{\mu\nu}(\mathbf{g}, \partial\mathbf{g}), \\ &=: R_{\mu\nu}^H - \sum_{\alpha=0}^3 (\mathbf{g}_{\alpha(\mu} \partial_\nu) H^\alpha). \end{aligned}$$

Therefore, the vacuum equations are equivalent to the system

$$R_{\mu\nu}^H := -\frac{1}{2} \sum_{\alpha,\beta=0}^3 \mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{g}_{\mu\nu} + \hat{F}_{\mu\nu}(\mathbf{g}, \partial\mathbf{g}) = 0, \quad (1.3)$$

$$H^\mu = 0. \quad (1.4)$$

Now, we recognise (1.3) as a quasi-linear, second order hyperbolic system of PDEs where known techniques based on energy estimates can be applied, [41], [51]. One

still has to solve the wave gauge equation, (1.4). The fundamental breakthrough of Choquet-Bruhat was to notice that they are preserved by the evolution. More precisely, the contracted Bianchi identities can be regarded as homogeneous evolution equations for H^μ ; therefore H^μ vanishes throughout the Cauchy development provided it vanishes on the initial Cauchy surface Σ .

1.2 Christodoulou-Klainerman estimates

In this section we revise relevant concepts and results from the Christodoulou and Klainerman non-linear asymptotic analysis of the initial value problem for initial data suitably close to trivial data. The construction of the canonical null foliations and an approximate time-like Killing vector field will be of particular importance for us; while we do not use them directly, we will assume that our coordinates and candidate Killing field agree with them to leading order.

In their book, *The Global Non-linear Stability of the Minkowski Space* [22], Christodoulou and Klainerman (CK) proved that any strongly asymptotically flat initial data set that satisfies a global smallness assumption, leads to a unique globally hyperbolic and geodesically complete solution of the Einstein's vacuum equations. Moreover the development is globally asymptotically flat in the sense that the curvature vanishes at infinity in all directions. We give a more precise statement in Theorem 1.2.2 below.

Let $(\mathcal{M}, \mathbf{g})$ be a globally hyperbolic Lorentzian 4-manifold with Cauchy surface Σ and induced data \mathbf{h}, K . Let t be a time function on \mathcal{M} with $\Sigma = \{t = 0\}$; then we can define local coordinates (t, x^1, x^2, x^3) in a neighbourhood of Σ by flowing coordinates (x^1, x^2, x^3) on Σ along integral lines of ∇t . The metric \mathbf{g} then takes the form,

$$\mathbf{g} = \phi^2(t, x) dt^2 + \mathbf{h},$$

where the function $\phi(t, x) := -\langle \nabla t, \nabla t \rangle^{-\frac{1}{2}}$ is called the *lapse function* of the foliation Σ_t .

The unit normal to Σ_t is given by

$$\mathbf{n} = \phi \nabla t = \frac{1}{\phi} \partial_t,$$

where $(\partial_t, \partial_{x^1}, \partial_{x^2}, \partial_{x^3})$ is the basis induced by the coordinates (t, x^1, x^2, x^3) . A

computation shows that with respect to these coordinates the second fundamental form is given by

$$K_{ab} = -\frac{1}{2\phi}\partial_t\mathbf{h}_{ab}$$

Moreover, it satisfies the so-called second-variation formula

$$\partial_t K_{ab} = -D_a D_b \phi + \phi(\mathbf{R}_{anbn} - K_{ac} K^c{}_b). \quad (1.5)$$

Taking the trace gives,

$$\partial_t K^a{}_a = -\Delta\phi + \phi(\mathbf{Ric}_{\mathbf{m}} + K_{ac} K^{ca}), \quad (1.6)$$

where $\Delta := D^a D_a$ is the Laplace operator associated to \mathbf{h} .

In addition to the constraint equations, (1.1) and (1.2), CK require the foliation to be maximal, that is,

$$K^a{}_a = 0.$$

This has the effect of making the following equations (implied by the constraints equations, second-variation formula and definition of K_{ab}) a determined system.

Constraint equations for a maximal foliation:

$$K^a{}_a = 0, \quad (1.7)$$

$$D^a K_{ab} = 0, \quad (1.8)$$

$${}^3R = K_{ab} K^{ab}. \quad (1.9)$$

Evolution equations for a maximal foliation:

$$\partial_t \mathbf{h}_{ab} = -2\phi K_{ab}, \quad (1.10)$$

$$\partial_t K_{ab} = -D_a D_b \phi + \phi({}^3R_{ab} - 2K_{ac} K^c{}_b). \quad (1.11)$$

Lapse equation of a maximal foliation

$$\Delta\phi = (K_{ab} K^{ab})\phi. \quad (1.12)$$

Now we present the class of asymptotically flat initial data sets considered by CK.

Definition 1.2.1. *We call an initial data set (Σ, \mathbf{h}, K) strongly asymptotic if there*

are coordinates (x^1, x^2, x^3) on $\Sigma \setminus K$, K compact, such that as $r_* := \left(\sum_{a=1}^3 (x^a)^2\right)^{\frac{1}{2}} \rightarrow \infty$,

$$\mathbf{h} = \left(1 + \frac{2M}{r_*}\right) ((dx^1)^2 + (dx^2)^2 + (dx^3)^2) + o_4(r_*^{-\frac{3}{2}}),$$

$$K_{ab} = o_3(-r_*^{\frac{5}{2}}).$$

In this case, we say that $\Sigma \setminus K$ is a *neighbourhood of spatial infinity* in Σ and that (x^1, x^2, x^3) , and the associated polar coordinates $(r_*, \vartheta^2, \vartheta^3)$, are *asymptotically flat coordinates*.

We proceed to state the main theorem of [22]. We omit the details of the smallness assumptions and the bounds on the curvature and focus just on the conclusion about the existence of an optical function; this is a function solving the eikonal equation, $\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$. Equivalently, it is a function whose level sets are null hypersurfaces.

Theorem 1.2.2. *(CK, [22]) Any complete, strongly asymptotically flat, maximal initial data set that satisfies a global smallness assumption, leads to a unique, globally hyperbolic, smooth and geodesically complete solution of the Einstein-vacuum equations, which is foliated by a normal maximal time function t , defined for all $t \geq -1$. Moreover, there exists a global smooth, exterior optical function, namely a solution of the eikonal equation defined everywhere in the exterior region $r_* \geq r_0/2$, with $r_0(t)$ representing the radius of the 2-surfaces of intersection between the hypersurfaces Σ_t and a fixed null cone C_0 with vertex at a point in Σ_{-1} .*

In particular, CK's theorem provides the long-time existence of solutions of the system (1.7)-(1.12) for small initial data⁴. They also obtained precise asymptotic conditions along the level sets of u , which we present in the following section.

Remark. We refer to the solutions obtained by Theorem 1.2.2 as *CK-spacetimes*. It is also important to remark that Zipser, Theorem 1 in [67], proved that the CK Theorem can be generalised to obtain solutions of the Einstein-Maxwell equations (provided the initial data of the Faraday tensor, \mathbf{F} , also satisfies a global smallness assumption). We call such solutions *CK-Zipser electrovacuum spacetimes*.

⁴Their methods also apply for incomplete initial data $\Sigma \setminus K$, the conclusions then hold only in the "future-outgoing" and "past-incoming" directions.

1.2.1 CK's conclusions

Now we list some conclusions of Christodoulou's and Klainerman's work regarding the asymptotic behaviour of solutions suitably close to Minkowski spacetime. Of particular interest are the decay rates obtained for the connection coefficients and the components of the Weyl tensor.

In their setting, Christodoulou-Klainerman used the foliation induced by t and u . That is, let $S_{t,u}$ be the space-like spheres defined as the intersection of the level sets of the maximal time function t and the canonical optical function u . The components of the connection and of the Weyl curvature are computed on each $S_{t,u}$ with respect to a frame $\{e_0, e_1, e_2, e_3\}$, where $e_0 = -\nabla u$ is the gradient of u , e_1 is the null conjugate of e_0 with respect to $S_{t,u}$, that is,

$$\langle e_0, e_1 \rangle = -2, \quad \langle e_1, e_1 \rangle = \langle e_1, X \rangle = 0, \quad X \in \mathcal{T}_p S_{t,u},$$

and $\{e_2, e_3\}$ is an orthonormal basis on $S_{t,u}$.

Let γ be the induced metric on $S_{t,u}$, and $\chi_{ij} := \langle \nabla_{e_i} e_0, e_j \rangle$, $\underline{\chi}_{ij} := \langle \nabla_{e_i} e_1, e_j \rangle$ be the second fundamental forms of $S_{t,u}$ with respect to e_0 and e_1 , respectively. CK showed that, for fixed u , $S_{t,u}$ converges to the standard round sphere embedded in Minkowski. That is, the following limits hold:

$$\begin{aligned} \lim_{C_u; r \rightarrow \infty} r^{-2} \gamma &= \check{\gamma}, & \lim_{C_u; r \rightarrow \infty} K[r^{-2} \gamma] &= 1, \\ \lim_{C_u; r \rightarrow \infty} r \operatorname{tr} \chi &= 2, & \lim_{C_u; r \rightarrow \infty} r \operatorname{tr} \underline{\chi} &= -2, \end{aligned}$$

here the limit $\lim_{C_u; r \rightarrow \infty}$ is taken along C_u , a level set of u , while letting the area function⁵ r tend to infinity; $\check{\gamma}$ is the standard round metric on \mathbb{S}^2 and K is the Gauss curvature.

Denote by $\hat{\chi}$ the trace-free part of χ , then to next order they obtained the existence of the following limits:

$$\begin{aligned} \lim_{C_u; r \rightarrow \infty} r^2 \hat{\chi} &= \Xi, & \lim_{C_u; r \rightarrow \infty} r(r \operatorname{tr} \chi - 2) &= H, \\ \lim_{C_u; r \rightarrow \infty} r \hat{\underline{\chi}} &= \underline{\Xi}, & \lim_{C_u; r \rightarrow \infty} r(r \operatorname{tr} \underline{\chi} - 2) &= \underline{H}, \end{aligned}$$

⁵I.e., $r = r(t, u) > 0$ is such that $\operatorname{Area}(S_{t,u}) = 4\pi r^2$.

where $\Xi, \underline{\Xi}$ are symmetric trace-less u -dependent 2-covariant tensors defined on \mathbb{S}^2 and H, \underline{H} are u -independent functions on \mathbb{S}^2 of vanishing mean. Moreover,

$$|\underline{\Xi}(u, \cdot)|_{\dot{\gamma}} \leq c(1 + |u|)^{-3/2}.$$

The Weyl tensor is decomposed into its so-called null components with respect to the frame $\{e_0, e_1, e_2, e_3\}$,

$$\alpha_{ij} = \mathbf{C}_{i0j0}, \quad \underline{\alpha}_{ij} = \mathbf{C}_{i1j1}, \quad (1.13)$$

$$2\beta_i = \mathbf{C}_{i010}, \quad 2\underline{\beta}_i = \mathbf{C}_{i110}, \quad (1.14)$$

$$4\rho = \mathbf{C}_{1010}, \quad \sigma = \mathbf{C}_{0123}. \quad (1.15)$$

These Weyl null components also decay. Explicitly, there exist a symmetric trace-less 2-tensor \underline{A} , a 1-form \underline{B} and functions P, Q defined on \mathbb{S}^2 and u -dependent satisfying

$$\lim_{C_u; r \rightarrow \infty} r \underline{\alpha} = \underline{A}, \quad \lim_{C_u; r \rightarrow \infty} r^2 \underline{\beta} = \underline{B}, \quad (1.16)$$

$$\lim_{C_u; r \rightarrow \infty} r^3 \rho = P, \quad \lim_{C_u; r \rightarrow \infty} r^3 \sigma = Q. \quad (1.17)$$

Moreover, these limits decay in u as follows,

$$|\underline{A}(u, \cdot)| \leq c(1 + |u|)^{-5/2}, \quad |\underline{B}(u, \cdot)| \leq c(1 + |u|)^{-3/2}, \quad (1.18)$$

$$|P(u, \cdot) - \overline{P}(u)| \leq c(1 + |u|)^{-1/2}, \quad |Q(u, \cdot) - \overline{Q}(u)| \leq c(1 + |u|)^{-1/2}, \quad (1.19)$$

and $\lim_{u \rightarrow -\infty} \overline{P}(u) = 0, \lim_{u \rightarrow -\infty} \overline{Q}(u) = 0$.

This is all consistent with the presence of peeling (cf. [53]). In contrast, they could only deduce a weaker fall-off for the remaining curvature components, namely,

$$|r^{\frac{7}{2}} \alpha_{ij}| \leq c, \quad |r^{\frac{7}{2}} \beta_i| \leq c, \quad \text{along } C_u.$$

For completeness we mention that the presence of the full peeling property would be achieved by $r^5 |\alpha_{ij}| \leq c$ and $r^4 |\beta_i| \leq c$, along C_u .

1.2.2 Mass

The *Hawking mass* enclosed by a 2-sphere $S_{t,u}$ is defined to be,

$$m(t, u) = \frac{r(t, u)}{2} \left(1 + \frac{1}{16\pi} \int_{S_{t,u}} \text{tr } \chi \text{tr } \underline{\chi} \right), \quad (1.20)$$

Christodoulou and Klainerman also deduced from their analysis the decay rate of the Hawking mass as well as its evolution equation. Explicitly, there exists a function $M(u)$ such that along a level set of u we have

$$|m(t, u) - M(u)| \leq c \frac{1}{r}.$$

Furthermore, in this context the Bondi mass formula for $M(u)$ takes the form

$$\partial_u M = -\frac{1}{32\pi} \int_{\mathbb{S}^2} |\Xi(u, \cdot)|^2 d\mu_{\dot{\gamma}}.$$

1.2.3 Gravity coupled with electromagnetism

The previous analysis was done for vacuum solutions suitably close to Minkowski spacetime. The generalisation to an Einstein-Maxwell system was carried out by Zipser in her PhD thesis [67]. She confirmed the stability of data close to Minkowski and found that the null components of the Faraday tensor decay as expected from the linear analysis (cf. [21]).

Recall the null decomposition of the Faraday tensor.

$$\alpha(\mathbf{F})_i = \mathbf{F}_{i0}, \quad \underline{\alpha}(\mathbf{F})_i = \mathbf{F}_{i1}, \quad (1.21)$$

$$\rho(\mathbf{F}) = \frac{1}{2} \mathbf{F}_{10}, \quad \sigma(\mathbf{F}) = \mathbf{F}_{23}. \quad (1.22)$$

As a consequence of Zipser's analysis we have that the following limits exist,

$$\lim_{C_u; r \rightarrow \infty} r \underline{\alpha}(\mathbf{F}) = \underline{A}(\mathbf{F}), \quad (1.23)$$

$$\lim_{C_u; r \rightarrow \infty} r^2 \rho(\mathbf{F}) = P(\mathbf{F}), \quad (1.24)$$

$$\lim_{C_u; r \rightarrow \infty} r^2 \sigma(\mathbf{F}) = Q(\mathbf{F}). \quad (1.25)$$

However, full peeling is not available for the remaining component. Instead

we have the weaker estimate along C_u ,

$$r^{\frac{5}{2}}|\alpha(\mathbf{F})_i| \leq c. \quad (1.26)$$

The generalisation made by Zipser for electrovacuum spacetimes has as a consequence that the same Hawking mass (1.20) also has a limit along C_u . We call this limit Bondi mass as well. The Bondi mass formula now takes the form,

$$\partial_u M = -\frac{1}{32\pi} \int_{\mathbb{S}^2} |\underline{\Xi}(u, \cdot)|^2 + |\underline{A}(\mathbf{F})(u, \cdot)|^2 d\mu_{\check{\gamma}}. \quad (1.27)$$

For this reason we call $\underline{\Xi}_{ij}$ and $\underline{A}(\mathbf{F})_i$ the *radiation fields* at future null infinity.

The above estimates together with the structure equations imply the following relations:

$$\begin{aligned} \partial_u \underline{\Xi} &= -\frac{1}{2} \underline{A}, & \check{\mathfrak{d}} \underline{\Xi} &= \underline{B}, \\ \partial_u \underline{\Xi} &= -\frac{1}{2} \underline{\Xi}, & \partial_u \underline{H} &= -\frac{1}{2} |\underline{\Xi}|^2, \end{aligned}$$

where $\check{\nabla}$ and $\check{\mathfrak{d}}$ stand for the connection and divergence operators on the round sphere, respectively. They tell us how the radiation field couple to the leading order terms of the Weyl tensor. In particular, $\underline{\Xi} = 0$ implies a stronger decay for $\underline{\alpha}$ and $\underline{\beta}$.

1.3 Coordinate systems

In this section we present the main technical assumptions required to ‘push’ the Einstein’s equations to infinity at all orders. Namely we state the class of asymptotically flat spacetimes that we will consider.

We focus on spacetimes which are sufficiently regular at spatial infinity and future null infinity. These concepts will be attached to the existence of coordinate systems suitably close to trivial ones. We explain them in the following subsections. It is important to remark that the issue of compatibility with the Einstein’s equation is left entirely open.

In this thesis we will work with spacetimes admitting coordinates $(t, r, \vartheta^2, \vartheta^3)$ such that the metric remains close to Kerr-Newman for r large enough and for all t (Definition 1.3.3 below).

Then, for this class of spacetimes, we construct in Section 1.3.2 coordinates $(u, s, \theta^2, \theta^3)$ with u a solution of the eikonal equation whose level sets are null hypersurfaces regarded as outgoing and intersecting null infinity as $s \rightarrow \infty$. We need to assume a smoothness condition with respect to these coordinates, Definition 1.3.8, in order to analyse Einstein's equations order by order at future null infinity.

Finally, we also need to consider approximately double null coordinates $(v, \underline{v}, y^2, y^3)$ which are relevant for the unique continuation from infinity techniques of Alexakis, Schlue and Shao, [6], and Alexakis and Schlue, [5]. We give, in Proposition 1.3.11, an explicit coordinate transformation $(t, r, \vartheta^2, \vartheta^3) \mapsto (v, \underline{v}, y^2, y^3)$ such that a spacetime close to Kerr-Newman in the sense of Definition 1.3.3 satisfies the required assumptions of [6].

Associated to all these coordinate systems are three radius parameters adapted to each one, r , s and \bar{r} , respectively. We also show compatibility of these functions in Lemma 1.3.9 and in Proposition 1.3.11 (see also the remark following its proof).

1.3.1 Space+time coordinates

Consider the open set $\mathcal{M} := \mathbb{R} \times (r_0, \infty) \times \mathbb{S}^2$ with coordinates $(t, r, \vartheta^2, \vartheta^3)$. We start by setting some notation about the decaying properties of functions with respect to these coordinates.

Notation. We use the symbol $x \lesssim y$ to mean $x \leq cy$ for some positive constant c .

Definition 1.3.1. *O*-notation with respect to coordinates $(t, r, \vartheta^2, \vartheta^3)$. We write $\phi = O(r^{-q})$, $q \in \mathbb{Z}$, if

$$|\phi| \lesssim \frac{1}{r^q}.$$

Similarly, we say that $\phi = O_k(r^{-q})$ if ϕ is C^k and

$$|(\partial_t)^{\alpha_t} (\partial_r)^{\alpha_r} (\partial_{\vartheta^2})^{\alpha_2} (\partial_{\vartheta^3})^{\alpha_3} \phi| \lesssim \frac{1}{r^{\alpha_r + q}}, \quad \alpha_t + \alpha_r + \alpha_2 + \alpha_3 \leq k.$$

Definition 1.3.2. Regularity at spatial infinity. Throughout this thesis we will call a Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ asymptotically flat and regular at spatial infinity if it admits coordinates $(t, r, \vartheta^2, \vartheta^3)$ such that for large r , the metric admits an expansion

of the form

$$\mathbf{g} = -dt^2 + dr^2 + (r^2\check{\gamma}_{ij} + O_2(1))d\vartheta^i d\vartheta^j + \mathbf{g}^\infty, \quad (1.28)$$

where the components $\mathbf{g}_{\alpha\beta}^\infty$ of \mathbf{g}^∞ belong to the class $O_2(r^{-1})$ and $\check{\gamma}_{ij} = \text{diag}(1, \sin^2(\vartheta^2))$ is the round metric on \mathbb{S}^2 .

Remark. In this thesis we will work with this class of spacetimes. This condition states that the initial regularity assumptions at spatial infinity are propagated throughout the evolution. This property may not hold for general electrovacuum spacetimes since logarithmic terms are expected to appear when $t \rightarrow \pm\infty$, [37], [62]. Hence, it would be desirable to exclude the existence of such terms from the Einstein's equations together with a non-radiating condition. However, this seems a difficult task closely related to the problem of regularity properties inherited at null infinity given initial data.

In addition to requiring regularity at spatial infinity we also specialised to spacetimes close to Kerr-Newman in the following sense.

Definition 1.3.3. We say that $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ is regular at spatial infinity and close to Kerr-Newman if there exist coordinates $(t, r, \vartheta^2, \vartheta^3)$ in an open set $\mathcal{U} \subset \mathcal{M}$ such that the following expansion holds,

$$\begin{aligned} \mathbf{g} &= \mathbf{g}_{KN} + \mathbf{g}^\infty \\ &= - \left(1 - \frac{2M}{r} + \frac{e^2}{r^2} + O_2(r^{-3}) \right) dt^2 + O_2(r^{-3}) dt dr \\ &+ \left(1 + \frac{2M}{r} - \frac{e^2}{r^2} - \frac{a^2 \sin^2 \vartheta^2}{r^2} + \frac{4M^2}{r^2} + O_2(r^{-3}) \right) dr^2 \\ &+ (r^2 \check{\gamma}_{ij} + O_2(1)) d\vartheta^i d\vartheta^j + \sum_{i=2}^3 (O_2(r^{-1}) dt d\vartheta^i + O_2(r^{-3}) dr d\vartheta^i), \end{aligned} \quad (1.29)$$

for all $t \in \mathbb{R}$, $r \geq r_0$ and $(\vartheta^2, \vartheta^3) \in \mathbb{S}^2$.

These spacetimes are important to us because they belong to the class where unique continuation from infinity results can be applied, [5], [6]. We will check this in Section 1.3.4 (Proposition 1.3.11) when we discuss the relevant coordinate systems.

1.3.2 Outgoing null coordinates and adapted frame

Here we explain the construction of one of the coordinate systems used in this thesis. Namely, we take from Alexakis and Schlue the construction of an optical function u on the Cauchy development of initial data satisfying the CK assumptions of Theorem 1.2.2. This function is complemented with coordinates along the leaves of the foliation to be described below. This procedure will allow us to define an approximate Killing field, which is roughly speaking, a time-like symmetry to first order.

We follow Alexakis-Schlue, [5], throughout. The initial conditions for the construction are given at a finite region of the spacetime and it is a definite hypothesis of this work that the resulting u share the same leading-order asymptotic properties with the CK canonical optical function. This is implicit in the smoothness assumption at future null infinity made in Definition 1.3.8 below.

Since we are not interested in constructing the spacetime itself, here we take the point of view that the CK-spacetime of Theorem 1.2.2 has already been constructed and just describe the construction of our optical function u .

Consider the initial Cauchy surface Σ and fix a sphere

$$S_0 = \{(x^1, x^2, x^3) | \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} = d_0\} \subset \Sigma.$$

At S_0 we can define the future-directed null normal vectors L and \underline{L} by the conditions

$$\langle L, \underline{L} \rangle = -2, \quad \langle L, L \rangle = \langle \underline{L}, \underline{L} \rangle = \langle L, X \rangle = \langle \underline{L}, X \rangle = 0, \quad X \in \mathcal{T}_p S_0.$$

These conditions fix L, \underline{L} up to rescaling by a function b on S_0 ,

$$L \mapsto bL, \quad \underline{L} \mapsto \frac{1}{b}\underline{L}.$$

Since Σ is asymptotically flat we can choose b so that the variation of area of S_0 along the direction of $L + \underline{L}$ vanishes. More precisely, let χ_{ij} and $\underline{\chi}_{ij}$ be the second fundamental forms of S_0 with respect to L and \underline{L} ; these are defined as,

$$\chi_{ij} = \langle \nabla_{e_i} L, e_j \rangle, \quad \underline{\chi}_{ij} = \langle \nabla_{e_i} \underline{L}, e_j \rangle.$$

where $\{e_2, e_3\}$ is a local basis of vector fields on S_0 . Then the traces, $\text{tr } \chi$ and $\text{tr } \underline{\chi}$,

measure the rate of change of area along the directions L and \underline{L} , respectively⁶. Also, due to asymptotic flatness one has $\text{tr } \chi \text{tr } \underline{\chi} < 0$. Thus, we can choose L and \underline{L} so that $\text{tr } \chi > 0$ and $\text{tr } \underline{\chi} < 0$; in this case we say that L is *outgoing* and \underline{L} is *incoming*. Also, we can choose b on S_0 such that

$$\text{tr } \chi + \text{tr } \underline{\chi} = 0 \quad \text{on } S_0.$$

Let C_0 and \underline{C}_0 be the null hypersurfaces consisting of null geodesics emanating from S_0 in the direction of L and \underline{L} , respectively. Thanks to Christodoulou-Klainerman [22], we can choose d_0 large enough so that the initial data on the region defined by $r_* > d_0$ satisfies the required smallness assumption required for Theorem 1.2.2. Then the generators of C_0 have no future end points; analogously the generators of \underline{C}_0 will not have past end points. We denote by $\mathcal{U} \subset \mathcal{M}$ the Cauchy development of the region $r_* > d_0$.

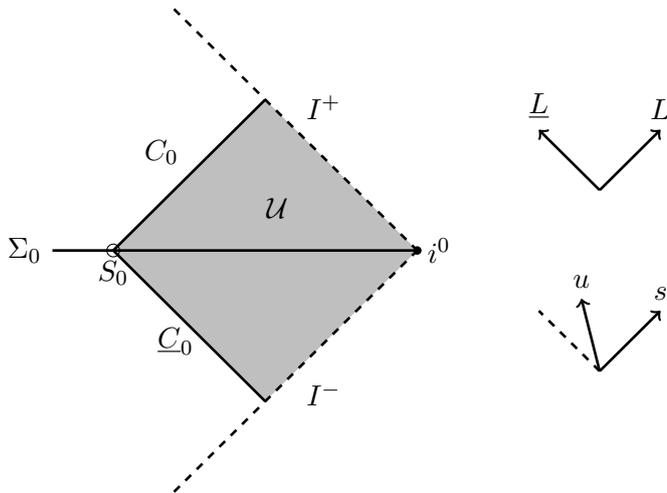


Figure 1.1: Coordinates in a neighbourhood of spatial infinity. The level sets of u are the outgoing null hypersurfaces C_u ruled by L . The level sets of s are time-like.

Next, L and \underline{L} are extended to \underline{C}_0 as follows: Let \underline{L} be extended by geodesics, that is,

$$\nabla_{\underline{L}} \underline{L} = 0, \quad \text{along } \underline{C}_0.$$

With the help of its affine function \underline{s} on \underline{C}_0 (that is, $\underline{L}(\underline{s}) = 1$ on \underline{C}_0 and $\underline{s} = 0$ on S_0) we define the *retarded time* function u on \underline{C}_0 to be just the following rescaling

⁶Indeed, if Φ_s is the flow associated to a local extension of L , then $\frac{d}{ds}|_{s=0} \text{Area}(\Phi_s(S_0)) = \int_{S_0} \text{tr } \chi$.

of \underline{s} on \underline{C}_0 ,

$$u := 2\underline{s},$$

Since u and \underline{s} define the same level sets on \underline{C}_0 , in the following we will denote by $S_{0,u}$ such level sets.

Next, extend L to \underline{C}_0 by defining it to be the conjugate null normal to \underline{L} on $S_{0,u}$, i.e., L is required to satisfy:

$$\langle L, \underline{L} \rangle = -2, \quad \langle L, L \rangle = \langle L, X \rangle = 0, \quad X \in \mathcal{T}_p S_{0,u}.$$

Finally we define u on \mathcal{U} as the solution of the eikonal equation

$$\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0,$$

with the already prescribed value of u along \underline{C}_0 . Alternatively, the level sets of u , denoted by C_u , can be defined as those generated by geodesics emanating from $S_{0,u}$ in the direction of L . Next, we extend L by geodesics,

$$\nabla_L L = 0 \quad \text{on } \mathcal{U},$$

Let s be the affine parameter of L with initial condition $-\underline{s}$ on \underline{C}_0 :

$$L \cdot s = 1 \quad \text{and} \quad s|_{S_{0,u}} = -\underline{s}.$$

The above procedure allows us to define coordinates $(u, s, \theta^2, \theta^3)$ on \mathcal{U} as follows. Firstly, choose coordinates (θ^2, θ^3) on S_0 and then u -flow them along \underline{L} on \underline{C}_0 , this gives coordinates (u, θ^2, θ^3) on \underline{C}_0 , where (θ^2, θ^3) are constant along \underline{L} -lines. Similarly, define the coordinates $(u, s, \theta^2, \theta^3)$ by s -flowing (u, θ^2, θ^3) along L to cover \mathcal{U} .

We define $S_{s,u}$ to be these spheres of intersection of the level sets of s and u . Then, let \underline{L} be extended to all of \mathcal{U} by taking it to be the null conjugate of L with respect to $S_{s,u}$. That is, \underline{L} is the unique vector field satisfying

$$\langle \underline{L}, \underline{L} \rangle = 0, \quad \langle L, \underline{L} \rangle = -2, \quad \langle \underline{L}, X \rangle = 0, \quad X \in \mathcal{T}_p S_{s,u}. \quad (1.30)$$

The candidate Killing vector field will be given by

$$T := \partial_u.$$

Lemma 1.3.4. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be a spacetime regular at spatial infinity and close to Kerr-Newman. Then, with respect to the coordinates $(u, s, \theta^2, \theta^3)$ on \mathcal{U} , the metric takes the form*

$$\mathbf{g} = -2duds + f^0 du^2 + \gamma_{ij} \left(d\theta^i - \frac{1}{2} f^i du \right) \left(d\theta^j - \frac{1}{2} f^j du \right), \quad (1.31)$$

for some smooth functions γ_{ij}, f^0, f^i . Moreover, we have

$$L = \partial_s = -\nabla u, \quad (1.32)$$

$$\underline{L} = f^0 \partial_s + 2\partial_u + f^i \partial_{\theta^i} \quad (1.33)$$

Proof. First we note that by the definition of the coordinates $L = \partial_s$ is a null vector field ruling the null hypersurfaces C_u , this justifies the absence of ds^2 and $dsd\theta^i$ terms in the above expression. Moreover, $L = \partial_s$ is geodesic, then $\langle \partial_u, \partial_s \rangle$ is constant along L -lines, indeed,

$$\begin{aligned} \partial_s \langle \partial_u, \partial_s \rangle &= \langle \nabla_{\partial_s} \partial_u, \partial_s \rangle + \langle \partial_u, \nabla_{\partial_s} \partial_s \rangle \\ &= \langle \nabla_{\partial_u} \partial_s, \partial_s \rangle = \frac{1}{2} \partial_u \langle \partial_s, \partial_s \rangle = 0. \end{aligned}$$

Hence $\langle \partial_u, \partial_s \rangle$ depends only on its value at \underline{C}_0 . Now we prove that $\langle \partial_u, \partial_s \rangle = -1$ at \underline{C}_0 . Let p be a point in \underline{C}_0 with coordinates $(u_0, -\frac{1}{2}u_0, \theta_0^2, \theta_0^3)$ and ϕ be a function in a neighbourhood of p , then we have,

$$\begin{aligned} \partial_u|_p(\phi) &= \lim_{u \rightarrow u_0} \frac{\phi(u, -\frac{1}{2}u_0, \theta_0^2, \theta_0^3) - \phi(u_0, -\frac{1}{2}u_0, \theta_0^2, \theta_0^3)}{u - u_0}, \\ &= \lim_{u \rightarrow u_0} \frac{\phi(u, -\frac{1}{2}u_0, \theta_0^2, \theta_0^3) - \phi(u, -\frac{1}{2}u, \theta_0^2, \theta_0^3)}{u - u_0} \\ &\quad + \lim_{u \rightarrow u_0} \frac{\phi(u, -\frac{1}{2}u, \theta_0^2, \theta_0^3) - \phi(u_0, -\frac{1}{2}u_0, \theta_0^2, \theta_0^3)}{u - u_0}, \\ &= \frac{1}{2} \partial_s|_p(\phi) + \frac{1}{2} \underline{L}|_p(\phi). \end{aligned}$$

Thus $\partial_u = \frac{1}{2}(L + \underline{L})$ at \underline{C}_0 , in particular $\langle \partial_u, \partial_s \rangle = -1$.

Now we focus on the expression for the null vector fields L and \underline{L} . Note, that ∇u is also a null vector field ruling C_u , therefore ∇u is parallel to L . Also, it satisfies $\langle \nabla u, \partial_u \rangle = 1$ and we already know $\langle L, \partial_u \rangle = -1$, so $L = -\nabla u$ as required.

Finally, a direct computation shows that, given the metric expression (1.31),

the vector field $\underline{L} := f^0 \partial_s + 2\partial_u + f^i \partial_{\theta^i}$ satisfies the conditions (1.30) and so it is the null conjugate of L with respect to $S_{s,u}$. ■

Choice of frame and gauge conditions

Now we introduce a framework in which to cast the connection, the curvature components and the equations relating them in a convenient way for our purposes.

Assume $\{e_\mu\} = \{e_0, e_1, e_2, e_3\}$ is a basis of vector fields satisfying

$$\langle e_\mu, e_\nu \rangle = \eta_{\mu\nu},$$

where

$$\eta_{\mu\nu} := \begin{pmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.34)$$

We refer to $\{e_\mu\} = \{e_0, e_1, e_2, e_3\}$ as a *null orthonormal frame*.

The associated connection coefficients⁷ are the components of the derivative operator with respect to this frame,

$$\omega_{\lambda\mu\nu} = \langle \nabla_{e_\lambda} e_\nu, e_\mu \rangle.$$

They satisfy $\omega_{\lambda\mu\nu} = -\omega_{\lambda\nu\mu}$, this is a consequence of $\eta_{\mu\nu}$ being a constant matrix. The Riemann curvature tensor components can be computed in terms of these (see [65] for details),

$$\mathbf{R}_{\rho\sigma\mu\nu} = e_\rho(\omega_{\sigma\mu\nu}) - e_\sigma(\omega_{\rho\mu\nu}) - \omega_{\rho\mu}{}^\alpha \omega_{\sigma\nu\alpha} + \omega_{\sigma\mu}{}^\alpha \omega_{\rho\nu\alpha} - \omega_{\sigma\rho}{}^\alpha \omega_{\alpha\mu\nu} + \omega_{\rho\sigma}{}^\alpha \omega_{\alpha\mu\nu}.$$

The splitting into trace and traceless part is the usual

$$\mathbf{R}_{\rho\sigma\mu\nu} = \mathbf{C}_{\rho\sigma\mu\nu} + \frac{1}{2}(\eta_{\rho\mu}\mathbf{S}_{\sigma\nu} - \eta_{\sigma\mu}\mathbf{S}_{\rho\nu} + \eta_{\sigma\nu}\mathbf{S}_{\rho\mu} - \eta_{\rho\nu}\mathbf{S}_{\sigma\mu}),$$

where $\mathbf{C}_{\rho\sigma\mu\nu}$ and $\mathbf{S}_{\mu\nu}$ are the components of the Weyl and Schouten tensors, respectively.

⁷Also known as Ricci or spin coefficients. Here we do not use that name to avoid confusion with the components of the Ricci tensor.

The *structure equations* are the combination of these two preceding equations, we write them schematically as

$$\mathbf{C} + \eta \wedge \mathbf{S} = e \wedge \omega + \omega \wedge \omega. \quad (1.35)$$

Gauge Conditions

Given arbitrary coordinates (x^0, x^1, x^2, x^3) let $\{\partial_{x^0}, \partial_{x^1}, \partial_{x^2}, \partial_{x^3}\}$ be the associated basis. We denote by h_μ^a the *orthonormalisation matrix*, that is, it is the change of basis defined by

$$e_\mu = h_\mu^a \partial_{x^a}.$$

We have made emphasis on the fact that a refers to an enumeration of the coordinate basis as opposed to the frame basis. In particular we have to be careful when contracting h_μ^a with the components of a tensor; the upstairs index can only be contracted with components of tensors with respect to the coordinate basis whereas the downstairs index can only be contracted with components of tensors with respect to the null-orthonormal frame. We will be using lower-case Latin indices, a, b, \dots , to enumerate the coordinate basis only in this section.

We can recover the metric components by the formula:

$$\mathbf{g}^{ab} = h_\mu^a \eta^{\mu\nu} h_\nu^b, \quad (1.36)$$

where \mathbf{g}^{ab} is the inverse matrix of $\mathbf{g}_{ab} := \langle \partial_{x^a}, \partial_{x^b} \rangle$.

The *frame equations* are PDEs relating the orthonormalisation matrix components with the connection coefficients:

Lemma 1.3.5. The frame equations. *The following equations hold*

$$e_\mu(h_\nu^a) - e_\nu(h_\mu^a) = (\omega_\mu^\rho{}_\nu - \omega_\nu^\rho{}_\mu) h_\rho^a. \quad (1.37)$$

Proof. This is precisely the torsion-free property of the connection,

$$[e_\mu, e_\nu] = \nabla_{e_\mu} e_\nu - \nabla_{e_\nu} e_\mu.$$

Applying this to the coordinate function x^a we get,

$$\begin{aligned} [e_\mu, e_\nu](x^a) &= 2e_{[\mu}(e_{\nu]}(x^a)), \\ &= 2e_{[\mu}(h_{\nu]}^\rho \partial_{x^\rho}(x^a)), \\ &= 2e_{[\mu}(h_{\nu]}^a). \end{aligned}$$

On the other hand,

$$\begin{aligned} [e_\mu, e_\nu](x^a) &= (\nabla_{e_\mu} e_\nu - \nabla_{e_\nu} e_\mu)(x^a), \\ &= (\omega_\mu^\rho{}_\nu - \omega_\nu^\rho{}_\mu) e_\rho(x^a), \\ &= (\omega_\mu^\rho{}_\nu - \omega_\nu^\rho{}_\mu) h_\rho{}^b \partial_{x^b}(x^a), \\ &= (\omega_\mu^\rho{}_\nu - \omega_\nu^\rho{}_\mu) h_\rho{}^b \delta_b{}^a, \\ &= (\omega_\mu^\rho{}_\nu - \omega_\nu^\rho{}_\mu) h_\rho{}^a. \quad \blacksquare \end{aligned}$$

In particular for the coordinates $(u, s, \theta^2, \theta^3)$ constructed in Section 1.3.2 the orthonormalisation matrix can be chosen to be (see Lemma 1.3.4)

$$h_\mu{}^a = \begin{pmatrix} 1 & f^0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & f^2 & & \\ 0 & f^3 & & h_i{}^k \end{pmatrix}.$$

This corresponds to the following choice of null orthonormal frame,

$$\begin{aligned} e_0 &:= L = \partial_s, \\ e_1 &:= \underline{L} = f^0 \partial_s + 2\partial_u + f^k \partial_{\theta^k}, \\ e_i &:= h_i{}^k \partial_{\theta^k}. \end{aligned}$$

Recall that the metric takes the form

$$\mathbf{g} = -2duds + f^0 du^2 + \gamma_{ij} \left(d\theta^i - \frac{1}{2} f^i du \right) \left(d\theta^j - \frac{1}{2} f^j du \right).$$

We call all these choices the *null gauge conditions*. Now we will see that with a judicious choice of $h_i{}^j$ we can achieve $\omega_{023} = 0$.

Lemma 1.3.6. *The orthonormal basis $\{e_i : i = 2, 3\}$ on the surfaces $S_{s,u}$ can be chosen such that $\omega_{023} = 0$.*

Proof. The basis $\{e_i : i = 2, 3\}$ is determined up to a rotation $\Theta_i{}^j$ in $SO(2)$. Under

the change of basis

$$e_i \mapsto \hat{e}_i = \Theta_i^j e_j,$$

the quantity ω_{023} transforms to $\hat{\omega}_{023}$ where

$$\begin{aligned}\hat{\omega}_{023} &= \langle \nabla_{e_0} \hat{e}_3, \hat{e}_2 \rangle, \\ &= \langle \nabla_{e_0} \Theta_3^i e_i, \Theta_2^j e_j \rangle, \\ &= \Theta_2^i (\Theta_3^j \omega_{0ij} + e_0(\Theta_3^j) \delta_{ij}).\end{aligned}$$

Thus, by solving the ODE, $\Theta_2^i (\Theta_3^j \omega_{0ij} + e_0(\Theta_3^j) \delta_{ij}) = 0$, we can set $\hat{\omega}_{023} = 0$ and omit the hat hereafter. ■

We will always be working with these choices.

Null components of the connection and curvature

We define the Christodoulou-Klainerman null components of the connection⁸ by,

$$\chi_{ij} := \langle \nabla_{e_i} e_0, e_j \rangle = \omega_{ij0}, \quad \underline{\chi}_{ij} := \langle \nabla_{e_i} e_1, e_j \rangle = \omega_{ij1}, \quad (1.38)$$

$$2\xi_i := \langle \nabla_{e_0} e_0, e_i \rangle = \omega_{0i0}, \quad 2\underline{\xi}_i := \langle \nabla_{e_1} e_1, e_i \rangle = \omega_{1i1}, \quad (1.39)$$

$$2\zeta_i := \langle \nabla_{e_1} e_0, e_i \rangle = \omega_{1i0}, \quad 2\underline{\zeta}_i := \langle \nabla_{e_0} e_1, e_i \rangle = \omega_{0i1}, \quad (1.40)$$

$$4\omega := \langle \nabla_{e_0} e_0, e_1 \rangle = \omega_{010}, \quad 4\underline{\omega} := \langle \nabla_{e_1} e_1, e_0 \rangle = \omega_{101}, \quad (1.41)$$

$$V_i := \langle \nabla_{e_i} e_0, e_1 \rangle = \omega_{i10}. \quad (1.42)$$

Note that, within the null gauge conditions, the frame equations tell us that $\chi_{ij}, \underline{\chi}_{ij}$ are symmetric 2-tensors, $2\zeta_i = V_i = -2\underline{\zeta}_i$, $\xi_i = 0$ and $\omega = 0$. The evolution of the orthonormalisation matrix is then given by

$$e_0(f^0) = 2\underline{\omega}, \quad e_i(f^0) = \underline{\xi}_i, \quad (1.43)$$

$$e_0(f^i) = -\zeta^k h_k^i, \quad e_i(h_j^l) - e_j(h_i^l) = (\omega_i^k{}_j - \omega_j^k{}_i) h_k^l, \quad (1.44)$$

$$e_0(h_i^j) = -\chi_i^k h_k^j, \quad e_1(h_i^j) - e_i(f^j) = (\omega_1^l{}_i - \omega_i^l{}_1) h_l^j. \quad (1.45)$$

⁸They use the last indices to refer to the null part of the frame, that is, their null pair (e_3, e_4) corresponds to our (e_1, e_0) .

The Weyl curvature null components are given by,

$$\begin{aligned}\alpha_{ij} &:= \mathbf{C}_{i0j0}, & \underline{\alpha}_{ij} &:= \mathbf{C}_{i1j1}, \\ 2\beta_i &:= \mathbf{C}_{i010}, & 2\underline{\beta}_i &:= \mathbf{C}_{i110}, \\ 4\rho &:= \mathbf{C}_{1010}, & 2\sigma &:= \mathbf{C}_{1023}.\end{aligned}$$

Due to the symmetries of the Weyl tensor we have that α_{ij} and $\underline{\alpha}_{ij}$ are trace-less and symmetric. Moreover, in $(3 + 1)$ -dimensions the above components determine completely the Weyl tensor, in particular we have

$$\begin{aligned}2\beta_i = \mathbf{C}_{i010} = 2\mathbf{C}_{0jji}, \quad i \neq j, & & 2\underline{\beta}_i = \mathbf{C}_{i110} = -2\mathbf{C}_{1jji}, \quad i \neq j, \\ 4\rho = \mathbf{C}_{1010} = -\mathbf{C}_{0212} & & 2\sigma = \mathbf{C}_{1023} = 2\mathbf{C}_{1203} \\ = -\mathbf{C}_{0313} = -\mathbf{C}_{2323}, & & = 2\mathbf{C}_{0312} = -2\mathbf{C}_{0213}.\end{aligned}$$

Finally, given a Faraday tensor, its null components are defined as:

$$\begin{aligned}\alpha(\mathbf{F})_i &= \mathbf{F}_{i0}, & \underline{\alpha}(\mathbf{F})_i &= \mathbf{F}_{i1}, \\ \rho(\mathbf{F}) &= \frac{1}{2}\mathbf{F}_{10}, & \sigma(\mathbf{F}) &= \mathbf{F}_{23}.\end{aligned}$$

These components determine completely the Faraday tensor.

1.3.3 Smoothness at future null infinity

Here we state the smoothness condition at future null infinity that we will require for our asymptotically flat spacetimes. This is a hypothesis tied to the coordinates $(u, s, \theta^2, \theta^3)$ and frame $\{e_0, e_1, e_2, e_3\}$ constructed previously, 1.3.2.

Definition 1.3.7. \mathcal{O} -notation with respect to coordinates $(u, s, \theta^2, \theta^3)$. We say that a function $f: \mathcal{U} \rightarrow \mathbb{R}$ is in $\mathcal{O}_k^\infty(s^{-q})$, $q \in \mathbb{Z}$, if it admits an infinite asymptotic expansion in s^{-n} , $n = q, q + 1, \dots$, and this expansion is well-behaved with respect to derivatives up to order k . Formally, we write $f = \mathcal{O}_k^\infty(s^{-q})$ if there are functions $f^{(n)}(u, \theta^2, \theta^3) \in C^k(\mathbb{R} \times \mathbb{S}^2)$ such that

$$\begin{aligned}f(u, s, \theta^2, \theta^3) &\sim \sum_{n=0}^{\infty} f^{(n+q)}(u, \theta^2, \theta^3) \frac{1}{s^{n+q}}, \\ \partial_u^l \partial_\theta^\alpha f(u, s, \theta^2, \theta^3) &\sim \sum_{n=0}^{\infty} \partial_u^l \partial_\theta^\alpha f^{(n+q)}(u, \theta^2, \theta^3) \frac{1}{s^{n+q}},\end{aligned}$$

where $\partial_\theta^\alpha = (\partial_{\theta^1})^{\alpha_1} (\partial_{\theta^2})^{\alpha_2}$, $l + \alpha_1 + \alpha_2 \leq k$.

$$\partial_s^l f(u, s, \theta^2, \theta^3) \sim \sum_{n=0}^{\infty} \left(\prod_{i=0}^{l-1} (-n - q - i) \right) f^{(n+q)}(u, \theta^2, \theta^3) \frac{1}{s^{n+q+l}}, \quad 1 \leq l \leq k.$$

Here $f \sim \sum_{n=0}^{\infty} f^{(n+q)} \frac{1}{s^{n+q}}$ means that for all $N \in \mathbb{N} \cup \{0\}$ there is a $C_N > 0$ so that

$$\left| f - \sum_{n=0}^N f^{(n+q)} \frac{1}{s^{n+q}} \right| \leq \frac{C_N}{s^{N+q+1}}.$$

Definition 1.3.8. Smoothness at future null infinity. A CK-Zipser electrovacuum spacetime $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ is called smooth at future null infinity if

$$f^0 - 1, f^i, h_i^j = \mathcal{O}_3^\infty(s^{-1}), \quad \omega_{\mu\nu\lambda}, \mathbf{F}_{\mu\nu} = \mathcal{O}_2^\infty(s^{-1}), \quad \mathbf{C}_{\mu\nu\alpha\beta} = \mathcal{O}_1^\infty(s^{-1}) \quad (1.46)$$

Moreover, we require compatibility with the coordinates $(t, r, \vartheta^2, \vartheta^3)$ in the sense that the Jacobian, $D\Phi$, of the change of coordinates $(u, s, \theta^2, \theta^3) \rightarrow (t, r, \vartheta^2, \vartheta^3)$ satisfies,

$$D\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}_2^\infty(s^{-1}). \quad (1.47)$$

I.e.,

$$\begin{aligned} \partial_u &= (1 + \mathcal{O}_2^\infty(s^{-1}))\partial_t + \mathcal{O}_2^\infty(s^{-1})\partial_r + \mathcal{O}_2^\infty(s^{-1})\partial_{\vartheta^2} + \mathcal{O}_2^\infty(s^{-1})\partial_{\vartheta^3}, \\ \partial_s &= (1 + \mathcal{O}_2^\infty(s^{-1}))\partial_t + (1 + \mathcal{O}_2^\infty(s^{-1}))\partial_r + \mathcal{O}_2^\infty(s^{-1})\partial_{\vartheta^2} + \mathcal{O}_2^\infty(s^{-1})\partial_{\vartheta^3}, \\ \partial_{\theta^2} &= \mathcal{O}_2^\infty(s^{-1})\partial_t + \mathcal{O}_2^\infty(s^{-1})\partial_r + (1 + \mathcal{O}_2^\infty(s^{-1}))\partial_{\vartheta^2} + \mathcal{O}_2^\infty(s^{-1})\partial_{\vartheta^3}, \\ \partial_{\theta^3} &= \mathcal{O}_2^\infty(s^{-1})\partial_t + \mathcal{O}_2^\infty(s^{-1})\partial_r + \mathcal{O}_2^\infty(s^{-1})\partial_{\vartheta^2} + (1 + \mathcal{O}_2^\infty(s^{-1}))\partial_{\vartheta^3}. \end{aligned}$$

We also assume a finiteness of poles condition. That is, we require all the asymptotic coefficients of curvature components, $\kappa = \alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}$, have finite limits at ‘spatial infinity’, i.e.,

$$\lim_{u \rightarrow -\infty} \left| \binom{n}{\kappa} \right| < \infty, \quad \forall n \in \mathbb{N}.$$

These are all technical conditions which are sufficient to allow us to deal with the Einstein’s equations one order at a time at infinity.

Remark. The purpose of this condition is to rule out *logarithmic singularities*.

For example, the function $\frac{\log s}{s}$ decays at infinity but it is not $\mathcal{O}(s^{-1})$. It is known that these kind of singularities are present in Cauchy developments of asymptotically flat initial data, [66], and actually are expected to be ubiquitous in dynamical spacetimes, [20]. However, as remarked by Christodoulou, [20], the coefficients accompanying the logarithmic singularities are time-independent for relevant systems, e.g., an N -body system with no incoming radiation. We believe that the arguments presented here can be extended to include time-independent logarithmic terms. Such terms would have the effect of changing the “limiting structure equations at infinity”, hence affecting the way the asymptotic terms couple to each other⁹. However by taking an extra time-derivative we should be able to get rid of the troublesome logarithmic terms and recover a system with the same properties as the one that we deal with in the smooth case. Then the finiteness of pole condition, which prohibits polynomial growth would allow us to conclude time independence. We do not attempt such approach here and we stick to the hypothesis of smoothness as above.

Now we show that the functions r and s appearing at the different coordinates systems are compatible in a suitable sense. Also, that the metric on the spheres $S_{u,s}$ approaches the round metric $s^2\check{\gamma}$ in the spacetimes under consideration.

Lemma 1.3.9. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be a CK-Zipser spacetime close to Kerr-Newman and smooth at future null infinity. Then,*

$$\lim_{C_u; s \rightarrow \infty} \frac{r}{s} = 1, \quad \text{and} \quad \gamma_{\theta^i \theta^j}(u, \theta^2, \theta^3) = s^2 \check{\gamma}_{ij} + \mathcal{O}_2(s).$$

In particular $c_1 r < s < c_2 r$ for some constants $c_1, c_2 > 0$. Moreover, h_i^j can be chosen to be $\text{diag}(1, \frac{1}{\sin(\theta^2)})$. (1)

Proof. Firstly, from

$$\partial_s = (1 + \mathcal{O}_2^\infty(s^{-1}))\partial_t + (1 + \mathcal{O}_2^\infty(s^{-1}))\partial_r + \mathcal{O}_2^\infty(s^{-1})\partial_{\theta^2} + \mathcal{O}_2^\infty(s^{-1})\partial_{\theta^3},$$

we get $\partial_s r = 1 + \mathcal{O}^\infty(s^{-1})$. Therefore $|r - s| \leq c \log s$ and $\lim_{s \rightarrow \infty} \frac{r}{s} = 1$.

Next, for metrics close to Kerr-Newman we have $\gamma_{\theta^i \theta^j} = r^2 \check{\gamma}_{ij} + \mathcal{O}(r^0)$, this together with the estimate for the Jacobian $D\Phi$ implies that

$$\gamma_{\theta^i \theta^j} = (D\Phi)_i^k \gamma_{\theta^k \theta^l} (D\Phi)_j^l = r^2 \check{\gamma}_{ij} + \mathcal{O}(r).$$

⁹In Proposition 2.1.1 we give a systematic way to understand the coupling to all orders in the smooth case.

Hence by smoothness assumption and $\lim_{s \rightarrow \infty} \frac{r}{s} = 1$ we get

$$\gamma_{\theta^i \theta^j} = s^2 \check{\gamma}_{ij} + \mathcal{O}_2^\infty(s).$$

Finally, recall that in Lemma 1.3.6 we solved an ODE to choose the orthonormal frame $\{e_2, e_3\}$ such that $\omega_{023} = 0$. This leaves the freedom of choosing initial conditions for the frame $\{e_1, e_2\}$. Since the metric $\frac{1}{s^2} \gamma_{\theta^i \theta^j}$ approaches the round metric at null infinity then we can choose h_i^j to be the standard orthonormalisation matrix of the round sphere. ■

1.3.4 Asymptotically double null coordinates

Now we introduce the class of spacetimes where the Carleman-type estimates from Alexakis-Schlue-Shao, [6], can be applied to obtain a unique continuation from infinity property. The Carleman estimates themselves will be discussed in Section 3.1.

The class of spacetimes considered in [6] are incomplete 4-manifolds of the form $\mathcal{M} = (-\infty, 0) \times (0, \infty) \times \mathbb{S}^2$ with coordinates $(v, \underline{v}, y^2, y^3)$. In order to discuss the asymptotic conditions we introduce notation first.

Definition 1.3.10. *O' -notation for $(v, \underline{v}, y^2, y^3)$ -coordinates. Consider smooth functions G and ϕ on \mathcal{M} . We say that $\phi = O'(G)$ if*

$$|\phi| \lesssim G.$$

Similarly we say that $\phi = O'_k(G)$ if $\phi = O'(G)$ and

$$|(\partial_v)^{\alpha_0} (\partial_{\underline{v}})^{\alpha_1} (\partial_{y^2})^{\alpha_2} (\partial_{y^3})^{\alpha_3} \phi| \lesssim |(\partial_v)^{\alpha_0} (\partial_{\underline{v}})^{\alpha_1} G|, \quad \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \leq k.$$

Alexakis-Schlue-Shao require the existence of a smooth positive function $\bar{r} = \bar{r}(v, \underline{v}, y^2, y^3)$ such that:

- i) The function \bar{r} is bounded on a level set of $v - u$, that is, there exist constants C_1, C_2 such that

$$\underline{v}(p) - v(p) = C_1 \quad \text{implies} \quad 1 \ll |\bar{r}(p)| < C_2.$$

ii) The differential of \bar{r} satisfies the following estimate

$$\left(1 + \frac{2m}{\bar{r}}\right) d\bar{r} = (1 + O'(\bar{r}^{-2}))d\underline{v} - (1 + O'(\bar{r}^{-2}))dv + \sum_{i=2}^3 O'(\bar{r}^{-1})dy^i.$$

Note that these two conditions imply that \bar{r} and $\underline{v} - v$ are comparable, that is, there exists a constant C such that $\bar{r} \leq C|\underline{v} - v|$.

With respect to the above, the class of metrics considered in [6] are of the form

$$\begin{aligned} \mathbf{g} = & \mathbf{g}_{vv}dv^2 - 4\mathbf{g}_{v\underline{v}}dv d\underline{v} + \mathbf{g}_{\underline{v}\underline{v}}d\underline{v}^2 + \bar{r}^2\gamma_{ij}dy^i dy^j \\ & \mathbf{g}_{iv}dy^i dv + \mathbf{g}_{i\underline{v}}dy^i d\underline{v}, \end{aligned} \quad (1.48)$$

where the metric components satisfy the following conditions:

$$\begin{aligned} \mathbf{g}_{v\underline{v}} = 1 - \frac{2m}{\bar{r}}, & \quad \mathbf{g}_{vv}, \mathbf{g}_{\underline{v}\underline{v}} = O_1'(\bar{r}^{-3}), \\ \mathbf{g}_{iv}, \mathbf{g}_{i\underline{v}} = O_1'(\bar{r}^{-1}), & \quad \gamma_{ij} = \check{\gamma}_{ij} + O_1'(\bar{r}^{-1}). \end{aligned}$$

Here, m is a positive function uniformly bounded away from 0, $m \geq m_{min} > 0$, and for some $\eta > 0$, the differential of m satisfies

$$|\partial_{y^i} m| = O'(\bar{r}(-v\underline{v})^{-\eta}), \quad |\partial_v m|, |\partial_{\underline{v}} m| = O'(\bar{r}^{-2}).$$

And to second order it is required that

$$|\square_{\mathbf{g}} \left(\frac{m}{\bar{r}}\right)| = O'((-v\underline{v})^{-1-\eta}).$$

We have to check that the CK-Zipser spacetimes satisfy the above conditions. While this may not be true in general we prove that it holds for spacetimes close to Kerr-Newman and regular at spatial infinity.

Proposition 1.3.11. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be a spacetime regular at spatial infinity and close to Kerr-Newman with coordinates $(t, r, \vartheta^2, \vartheta^3)$ (Definition 1.3.3). Then we can change to asymptotically double-null coordinates $(v, \underline{v}, y^2, y^3)$ in a domain $\mathcal{U} = (-\infty, 0) \times (0, \infty) \times \mathbb{S}^2 \subset \mathcal{M}$ such that the metric components verify condition (1.48) and the corresponding bounds.*

Proof. Recall the expansion of the metric \mathbf{g} for spacetimes close to Kerr-Newman:

$$\begin{aligned}\mathbf{g} = & - \left(1 - \frac{2M}{r} + \frac{e^2}{r^2} + O_2(r^{-3}) \right) dt^2 + O_2(r^{-3}) dt dr \\ & + \left(1 + \frac{2M}{r} - \frac{e^2}{r^2} - \frac{a^2 \sin^2(\vartheta^2)}{r^2} + \frac{4M^2}{r^2} + O_2(r^{-3}) \right) dr^2 \\ & + (r^2 \check{\gamma}_{ij} + O_2(r^0)) d\vartheta^i d\vartheta^j + \sum_{i=2}^3 (O_2(r^{-1}) dt d\vartheta^i + O_2(r^{-3}) dr d\vartheta^i),\end{aligned}$$

We note first that the change to double-null coordinates of the underlying Reissner-Nordström metric,

$$v = t - r_*, \quad \underline{v} = t + r_*, \quad \text{with} \quad \frac{dr_*}{dr} = \left(1 - \frac{2M}{r^2} + \frac{e^2}{r^2} \right)^{-1},$$

does not work. The troublesome term being $\frac{a^2 \sin^2(\vartheta^2)}{r^2}$ in \mathbf{g}_{rr} which will appear in \mathbf{g}_{vv} and $\mathbf{g}_{\underline{v}\underline{v}}$. Therefore we need to change first to “comoving coordinates” $(\bar{t}, \bar{r}, y^2, y^3)$ so that $\mathbf{g}_{\bar{r}\bar{r}}$ approaches the corresponding component of the Reissner-Nordström metric one order faster (cf. [6]). We achieve this by considering

$$\bar{t} = t, \quad y^3 = \vartheta^3, \quad \bar{r}^2 = r^2 + a^2 \sin^2(\vartheta^2), \quad \bar{r} \cos(y^2) = r \cos(\vartheta^2). \quad (1.49)$$

Notation. For the remaining of this proof we will use $\vartheta := \vartheta^2$ and $y := y^2$ to keep the notation simple.

Lemma 1.3.12. *Given the change of coordinates (1.49), the following estimates hold for some positive constant c ,*

$$\begin{aligned}|\cos \vartheta - \cos y| &\lesssim \frac{1}{\bar{r}^2}, & |\sin^2 \vartheta - \sin^2 y| &\lesssim \frac{1}{\bar{r}^2}, & \left| \bar{r} - r - \frac{a^2 \sin^2 y}{2\bar{r}} \right| &\lesssim \frac{1}{\bar{r}^3}, \\ \left| \frac{\partial r}{\partial \bar{r}} - 1 - \frac{a^2 \sin^2 y}{2\bar{r}^2} \right| &\lesssim \frac{1}{\bar{r}^3}, & \left| \frac{\partial r}{\partial y} \right| &\lesssim \frac{1}{\bar{r}}, \\ \left| \frac{\partial \vartheta}{\partial \bar{r}} \right| &\lesssim \frac{1}{\bar{r}^3}, & \left| \frac{\partial \vartheta}{\partial y} - 1 \right| &\lesssim \frac{1}{\bar{r}},\end{aligned}$$

Proof of Lemma. Note that $\bar{r}^2 = r^2 + a^2 \sin^2 \vartheta$ implies $r^2 - 1 \leq \bar{r}^2 \leq r^2 + 1$, so

$$\frac{1}{2}r < \bar{r} < 2r.$$

Hence $|\bar{r} - r| = \left| \frac{a^2 \sin^2 \vartheta}{\bar{r} + r} \right| < \frac{c}{\bar{r}}$. Next, we have

$$|\cos y - \cos \vartheta| = \left| \frac{r - \bar{r}}{r} \cos y \right| < \frac{c}{\bar{r}^2}$$

Similarly, $|\sin^2 y - \sin^2 \vartheta| = |\cos^2 y - \cos^2 \vartheta| = \left| \frac{r^2 - \bar{r}^2}{r^2} \cos^2 y \right| < \frac{c}{\bar{r}^2}$. Now we obtain an estimate for $\bar{r} - r$ to next order,

$$\begin{aligned} \left| \bar{r} - r - \frac{a^2 \sin^2 y}{2\bar{r}} \right| &= \left| \frac{a^2 \sin^2 \vartheta}{\bar{r} + r} - \frac{a^2 \sin^2 y}{2\bar{r}} \right| \\ &= a^2 \left| \frac{\bar{r}(\sin^2 \vartheta - \sin^2 y) + (\bar{r} - r) \sin^2 \vartheta + r(\sin^2 \vartheta - \sin^2 y)}{2\bar{r}(\bar{r} + r)} \right| \\ &< \frac{c}{\bar{r}^3}, \end{aligned}$$

where in the last line we used the previous estimates for $\bar{r} - r$ and $\sin^2 y - \sin^2 \vartheta$.

Now we estimate the Jacobian of $(\bar{r}, y) \mapsto (r, \vartheta)$. Differentiating (1.49) we obtain,

$$\begin{pmatrix} r & a^2 \cos \vartheta \sin \vartheta \\ \cos \vartheta & -\sin \vartheta \end{pmatrix} \begin{pmatrix} dr \\ d\vartheta \end{pmatrix} = \begin{pmatrix} \bar{r} & 0 \\ \cos y & -\bar{r} \sin y \end{pmatrix} \begin{pmatrix} d\bar{r} \\ dy \end{pmatrix}$$

So,

$$\begin{aligned} \begin{pmatrix} dr \\ d\vartheta \end{pmatrix} &= \begin{pmatrix} \frac{r\bar{r} + a^2 \cos \vartheta \cos y}{r^2 + a^2 \cos \vartheta} & -\frac{a^2 \bar{r} \sin y \cos \vartheta \sin \vartheta}{r^2 + a^2 \cos \vartheta} \\ \frac{\bar{r} \cos \vartheta - r \cos y}{(r^2 + a^2 \cos \vartheta) \sin \vartheta} & \frac{r\bar{r} \sin y}{(r^2 + a \cos \vartheta) \sin \vartheta} \end{pmatrix} \begin{pmatrix} d\bar{r} \\ dy \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial r}{\partial \bar{r}} & \frac{\partial r}{\partial y} \\ \frac{\partial \vartheta}{\partial \bar{r}} & \frac{\partial \vartheta}{\partial y} \end{pmatrix} \begin{pmatrix} d\bar{r} \\ dy \end{pmatrix} \end{aligned}$$

Based on this last expression and using the estimates already found for (r, ϑ) in

terms of (\bar{r}, y) we can deduce the remaining estimates.

$$\begin{aligned}
\left| \frac{\partial r}{\partial \bar{r}} - 1 - \frac{a^2 \sin^2 y}{2\bar{r}^2} \right| &= \left| \frac{r\bar{r} + a^2 \cos \vartheta \cos y}{r^2 + a^2 \cos^2 \vartheta} - 1 - \frac{a^2 \sin^2 y}{2\bar{r}^2} \right| \\
&= \left| \frac{2\bar{r}(r\bar{r} + a^2 \cos \vartheta \cos y) - 2\bar{r}(r^2 + a^2 \cos^2 \vartheta) - a^2 \sin^2 y(r^2 + a^2 \cos^2 \vartheta)}{2\bar{r}^2(r^2 + a^2 \cos^2 \vartheta)} \right| \\
&= \left| \frac{(2\bar{r}^2 r(\bar{r} - r) - a^2 r^2 \sin^2 \vartheta) + 2a^2 \bar{r}^2 \cos \vartheta (\cos y - \cos \vartheta) - a^4 \sin^2 \cos^2 \vartheta}{2\bar{r}^2(r^2 + a^2 \cos^2 \vartheta)} \right| \\
&< c \left| \frac{(2\bar{r}^2 r \frac{\sin^2 \vartheta}{r+\bar{r}} - r^2 \sin^2 \vartheta)}{\bar{r}^4} \right| + c \left| \frac{\bar{r}^2 (\cos y - \cos \vartheta)}{\bar{r}^4} \right| + \frac{c}{\bar{r}^4} \\
&< \frac{c}{\bar{r}^4}.
\end{aligned}$$

Similarly for the remaining partial derivatives. This finishes the proof of the Lemma.

We will also make use of the following estimate to change between $\frac{1}{r}$ and $\frac{1}{\bar{r}}$ up to $O(\bar{r}^{-3})$,

$$\left| \frac{1}{r} - \frac{1}{\bar{r}} \right| = \left| \frac{\bar{r} - r}{r\bar{r}} \right| < \frac{c}{\bar{r}^3}.$$

Now we proceed to find the asymptotic expansion of the metric with respect to the coordinates $(\bar{t}, \bar{r}, \vartheta^2, \vartheta^3)$. Clearly we have $\mathbf{g}_{\bar{t}\bar{t}} = \mathbf{g}_{tt}$, $\mathbf{g}_{\bar{t}y^3} = \mathbf{g}_{t\vartheta^3}$, $\mathbf{g}_{y^3 y^3} = \mathbf{g}_{\vartheta^3 \vartheta^3}$. Next, we prove the claim that $\left| \mathbf{g}_{\bar{r}\bar{r}} - 1 - \frac{2M}{\bar{r}} + \frac{e^2}{\bar{r}^2} - \frac{4M^2}{\bar{r}^2} \right| < \frac{c}{\bar{r}^3}$. Indeed,

$$\begin{aligned}
\mathbf{g}_{\bar{r}\bar{r}} &= \mathbf{g}_{rr} \left(\frac{\partial r}{\partial \bar{r}} \right)^2 + 2\mathbf{g}_{r\vartheta} \frac{\partial r}{\partial \bar{r}} \frac{\partial \vartheta}{\partial \bar{r}} + \mathbf{g}_{\vartheta\vartheta} \left(\frac{\partial \vartheta}{\partial \bar{r}} \right)^2 \\
&= \left(1 + \frac{2M}{\bar{r}} - \frac{e^2}{\bar{r}^2} - \frac{a^2 \sin^2(\vartheta^2)}{\bar{r}^2} + \frac{4M^2}{\bar{r}^2} + O(\bar{r}^{-3}) \right) \left(1 - \frac{a^2 \sin^2 y}{\bar{r}^2} + O(\bar{r}^{-3}) \right) \\
&\quad + O(\bar{r}^{-3}) + O(\bar{r}^{-4}) \\
&= 1 + \frac{2M}{\bar{r}} - \frac{e^2}{\bar{r}^2} + O(\bar{r}^{-3})
\end{aligned}$$

In the same way we can also estimate the following

$$\mathbf{g}_{\bar{t}\bar{r}} = O(\bar{r}^{-3}), \quad \mathbf{g}_{\bar{t}y^i}, \mathbf{g}_{\bar{r}y^i} = O(\bar{r}^{-1}), \quad \gamma_{yy} = \bar{r}^2 \gamma_{\vartheta\vartheta} + O(\bar{r}^0), \quad \gamma_{yy^3} = O(\bar{r}^0).$$

Finally we change to approximate double-null coordinates, that is,

$$v = \frac{\bar{t} - r_*}{2}, \quad \underline{v} = \frac{\bar{t} + r_*}{2}, \quad \text{with} \quad \frac{d\bar{r}}{dr_*} = 1 - \frac{2M}{\bar{r}^2} + \frac{e^2}{\bar{r}^2} + O(\bar{r}^{-3}),$$

We estimate the Jacobian,

$$\begin{aligned}\frac{\partial \bar{t}}{\partial v} &= 1, & \frac{\partial \bar{r}}{\partial v} &= -1 + \frac{2M}{\bar{r}} - \frac{e^2}{\bar{r}^2} + O(\bar{r}^{-3}), \\ \frac{\partial \bar{t}}{\partial \underline{v}} &= 1, & \frac{\partial \bar{r}}{\partial \underline{v}} &= 1 - \frac{2M}{\bar{r}} + \frac{e^2}{\bar{r}^2} + O(\bar{r}^{-3})\end{aligned}$$

Then, the metric takes the form,

$$\begin{aligned}\mathbf{g}_{vv} &= \mathbf{g}_{\bar{r}\bar{r}} \left(\frac{\partial \bar{r}}{\partial v} \right)^2 + 2\mathbf{g}_{\bar{r}\bar{t}} \frac{\partial \bar{r}}{\partial v} \frac{\partial \bar{t}}{\partial v} + \mathbf{g}_{\bar{t}\bar{t}} \left(\frac{\partial \bar{t}}{\partial v} \right)^2 \\ &= \left(1 + \frac{2M}{\bar{r}} - \frac{e^2}{\bar{r}^2} + \frac{4M^2}{\bar{r}^2} \right) \left(1 - \frac{4M}{\bar{r}} + \frac{2e^2}{\bar{r}^2} + \frac{4M^2}{\bar{r}^2} \right) + \left(-1 + \frac{2M}{\bar{r}} - \frac{e^2}{\bar{r}^2} \right) + O(\bar{r}^{-3}) \\ &= O(\bar{r}^{-3})\end{aligned}$$

Similarly

$$\mathbf{g}_{\underline{v}\underline{v}} = O(\bar{r}^{-3}), \quad \mathbf{g}_{\underline{v}\bar{v}} = -4 \left(1 - \frac{2M}{\bar{r}^2} + \frac{e^2}{\bar{r}^2} \right) + O(\bar{r}^{-3}), \quad \mathbf{g}_{vy^i}, \mathbf{g}_{\underline{v}y^i} = O(\bar{r}^{-1}).$$

Therefore, a metric \mathbf{g} close to Kerr-Newman can be brought to the Alexakis-Schlue-Shao form (1.48), with \bar{r} as constructed above and $m = M - \frac{e^2}{2\bar{r}} + O(\bar{r}^{-2})$. It remains to verify the estimates for \bar{r} and m , firstly,

$$\begin{aligned}\left(1 + \frac{2m}{\bar{r}} \right) d\bar{r} &= \left(1 + \frac{2M}{\bar{r}} + O(\bar{r}^{-2}) \right) \left(1 - \frac{2M}{\bar{r}} + O(\bar{r}^{-2}) \right) dr_* \\ &= d\underline{v} - dv + O(\bar{r}^{-2}).\end{aligned}$$

Also $\partial_v m, \partial_{\underline{v}} m, \partial_{y^i} m, \square_{\mathbf{g}} \frac{m}{\bar{r}} = O(\bar{r}^{-2})$. So we are only left to check that $\frac{1}{\bar{r}} < \frac{c}{(-v\underline{v})^\eta}$ for some $\eta > 0$. Indeed, for $\eta = \frac{1}{2}$ we have,

$$\sqrt{-v\underline{v}} \leq \frac{v + (-v)}{2} = \frac{r_*}{2} < c\bar{r}.$$

So m and \bar{r} satisfies the assumptions of Alexakis-Schlue-Shao. This finishes the proof of the proposition. ■

Remark. Now we just summarise the main relations between the asymptotic conditions adapted to the different coordinate systems. We use the symbol $x \simeq y$ to mean $c_1 x \leq y \leq c_2 x$ for c_1, c_2 positive constants. In this section we have found that

under the conditions of Proposition 1.3.11,

$$r \simeq s \simeq \bar{r} \simeq r_*. \quad (1.50)$$

In particular $\phi = O(r^{-q})$ if and only if $\phi = O'(\bar{r}^{-q})$. Also, if $\phi = \mathcal{O}(s^{-q})$ then $\phi = O'(\bar{r}^{-q})$ and $\phi = O(r^{-q})$. Finally we also have

$$v \simeq u \simeq t - r, \quad \underline{v} \simeq u + 2s \simeq t + r.$$

1.4 Main theorem and non-radiating condition

Now we are in condition to a precise version of our main theorem. We start by defining the non-radiating condition.

Based on the CK-Zipser analysis we know that radiation towards future null infinity can be quantify in terms of Ξ_{ij} and $\underline{A}(F)_i$. We will show this is the case with respect to our outgoing null coordinate system, frame and gauge choices in Proposition 2.1.3. However, we also need control towards the past, we achieve this by requiring a finer decay condition in terms of the optical function u .

Definition 1.4.1. Non-radiating spacetimes. *A smooth at future null infinity space-time $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ is called non-radiating if with respect to the frame $\{e_0, e_1, e_2, e_3\}$ the following hold:*

i) $\Xi_{ij} = 0$ and $\underline{A}(F)_i = 0$.

ii) Let ϕ be a component of $\mathcal{L}_T \mathbf{g}$, $\nabla \mathcal{L}_T \mathbf{g}$, $\mathcal{L}_T \mathbf{C}$, $\nabla \mathcal{L}_T \mathbf{C}$, $\mathcal{L}_T \mathbf{F}$ or $\nabla \mathcal{L}_T \mathbf{F}$, then for some $\eta > 0$,

$$|\phi| \lesssim \frac{1}{(1 + |u|)^{1+\eta}}.$$

Remarks.

1. We will show in Proposition 2.1.4 that condition *i)* implies that T is a Killing field to all orders in the sense that $\phi^{(n)} = 0$ for all $n \in \mathbb{N}$, with ϕ as in the definition above. In particular, this implies

$$|\phi| \lesssim \frac{1}{s^N}, \quad \text{for all } N \in \mathbb{N},$$

Therefore combining *i)* and *ii)* we get

$$|\phi|^2 \lesssim \frac{1}{s^N(1+|u|)^{1+\eta}}, \quad \text{for all } N \in \mathbb{N}. \quad (1.51)$$

2. Recall the CK and Zipser conclusions from [22] and [67]. Let¹⁰ $\tau_- = (1+u^2)^{\frac{1}{2}}$, $\tau_+ = (1+v^2)^{\frac{1}{2}}$ with $v = 2r - u$ and $r = r(t, u)$ the area parameter of $S_{t,u}$, then on the exterior region¹¹,

$$\begin{aligned} \sup_{\Sigma_t} \tau_+ \tau_-^{\frac{5}{2}} |\underline{\alpha}| &\leq c, & \sup_{\Sigma_t} \tau_+^2 \tau_-^{\frac{3}{2}} |\underline{\beta}| &\leq c, \\ \sup_{\Sigma_t} \tau_+^3 |\rho| &\leq c, & \sup_{\Sigma_t} \tau_+^3 \tau_-^{\frac{1}{2}} |\sigma| &\leq c, \\ \sup_{\Sigma_t} \tau_+^{\frac{7}{2}} |\beta| &\leq c, & \sup_{\Sigma_t} \tau_+^{\frac{7}{2}} |\alpha| &\leq c. \end{aligned}$$

And,

$$\begin{aligned} \sup_{\Sigma_t} \tau_+ \tau_-^{\frac{3}{2}} |\underline{\alpha}(\mathbf{F})| &\leq c, & \sup_{\Sigma_t} \tau_+^{\frac{5}{2}} |\alpha(\mathbf{F})| &\leq c, \\ \sup_{\Sigma_t} \tau_+^2 \tau_-^{\frac{1}{2}} |\rho(\mathbf{F})| &\leq c, & \sup_{\Sigma_t} \tau_+^2 \tau_-^{\frac{1}{2}} |\sigma(\mathbf{F})| &\leq c, \end{aligned}$$

This suggests that condition *ii)* is only relevant for the components α , β of the Weyl tensor and $\alpha(\mathbf{F})$ of the Faraday tensor.

Now we state our principal result. The neighbourhood of spatial infinity where stationarity is guaranteed is defined in terms of the coordinates $(v, \underline{v}, y^2, y^3)$,

$$\mathcal{D}_\epsilon := \{(v, \underline{v}, y^2, y^3) : 0 < \frac{1}{(-v)\underline{v}} < \epsilon\}.$$

Theorem 1.4.2. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be an electrovacuum spacetime regular at spatial infinity, close to Kerr-Newman and smooth at future null infinity in the sense of Definitions 1.3.3 and 1.3.8. Assume it is non-radiating, then there exists a time-like vector field T such that*

$$\mathcal{L}_T \mathbf{g} = 0 = \mathcal{L}_T \mathbf{F}, \quad \text{on } \mathcal{D}_\epsilon,$$

for $\epsilon > 0$ small enough.

¹⁰We abuse notation. Only in this Remark are t and u to be regarded as the CK time and optical functions, respectively; v and r are to be considered accordingly.

¹¹Where the exterior CK optical function is defined, see Theorem 1.2.2.

In Chapter 2 we prove that the non-radiating condition $i)$ does imply that T is killing to all orders at future null infinity with respect to coordinates $(u, s, \theta^2, \theta^3)$, Proposition 2.1.4. In Chapter 3 we extend this result into the neighbourhood of spatial infinity \mathcal{D}_ϵ . Firstly, we use the unique continuation techniques from Alexakis and Schlue [5] to prove that a spacetime with fast-decaying curvature and admitting a Killing field to all order, in a sense adapted to coordinates $(v, \underline{v}, y^2, y^3)$, is indeed stationary in \mathcal{D}_ϵ , Proposition 3.3.1. Finally we close the gap between the different frames and notions associated to the coordinates $(u, s, \theta^2, \theta^3)$ and $(v, \underline{v}, y^2, y^3)$ in Section 3.4.

Chapter 2

Asymptotic behaviour of the fields

In this chapter we compute asymptotic quantities at infinity. We follow Bičák-Scholtz-Tod [12] and Alexakis-Schlue [5] in their asymptotic analysis and extend their results to show that the radiation fields determine the evolution of the metric to all orders at infinity. Indeed, by assuming an asymptotic expansion we can take the limit for each order of the structure equations and find algebraic and u -transport relations at future null infinity to any order. These relations are well-suited for an induction process provided that we know the radiation fields, $\underline{\Xi}_{ij}$ and $\underline{A}(\mathbf{F})_i$ along future null infinity and initial conditions at some $u = u_0$. The main results of this chapter state that these data determines the solution to all orders at infinity, Propositions 2.1.1, 2.1.3 and 2.1.4.

The technique to compute the asymptotic quantities to all orders at infinity is so basic that we feel compelled to give an account of the main idea first with a toy model: An outgoing wave on Minkowski spacetime. The reader will find that the procedure is straightforward and the difficulty to apply it to the Einstein's equations lies only in the intricacy of the equations themselves.

Toy model. Consider the Minkowski metric in spherical coordinates:

$$\mathbf{g}_M = -dt^2 + dr^2 + r^2\check{\gamma},$$

where $\check{\gamma}$ is the round metric on the sphere \mathbb{S}^2 . We change to outgoing null coordinates which are better suited for the problem at hand. That is, let $u = t - r$, then the

metric can be written as

$$\mathbf{g}_M = -du^2 - 2dudr + r^2\check{\gamma}.$$

Note that the level sets of u are null hypersurfaces ruled by ∂_r . Now, let ϕ be a solution of the free wave equation,

$$\square_{\mathbf{g}_M}\phi = 0.$$

Assume moreover that ϕ admits an expansion of the form¹

$$\phi(u, r, \theta^2, \theta^3) = \sum_{n=1}^{\infty} \binom{(n)}{\phi}(u, \theta^2, \theta^3) \frac{1}{r^n},$$

which is well-behaved with respect to derivatives.

We wish to find $\binom{(n)}{\phi}$ in terms of $\binom{(1)}{\phi}$ and $\binom{(n)}{\phi_0} := \lim_{u \rightarrow -\infty} \binom{(n)}{\phi}$. To do so we consider $X_\mu := \nabla_\mu \phi$ and rewrite the wave equation as the 1st order system,

$$\begin{aligned} \nabla_{[\mu} X_{\nu]} &= 0, \\ \nabla^\mu X_\mu &= 0. \end{aligned}$$

For easy comparison with the CK notation we define the null components of X as

$$x := X_r = \partial_r \phi, \quad \underline{x} := X_u = \partial_u \phi, \quad \mathcal{X}_i := \frac{1}{r} X_i = \frac{1}{r} \partial_i \phi$$

In the following, $\check{\nabla}$ and $\check{\Delta}$ will denote the Levi-Civita connection and Laplace operator on the unit round sphere, respectively. In coordinates, the previous equations read

$$\begin{aligned} \partial_r \mathcal{X}_i &= \frac{1}{r} \partial_i x - \frac{1}{r} \mathcal{X}_i, & \partial_u \mathcal{X}_i &= \frac{1}{r} \partial_i \underline{x}, \\ \partial_r \underline{x} &= \partial_u x, & \check{\nabla}_{[i} \mathcal{X}_{j]} &= 0, \\ \partial_u x &= -\partial_r \underline{x} - \frac{2}{r} \underline{x} + \frac{1}{r^2} \partial_r(r^2 x) + \frac{1}{r} \check{\nabla}^j \mathcal{X}_j. \end{aligned}$$

These equations (the first column) imply the following recurrence relations for the

¹Such an expansion is morally equivalent to analyticity all the way to infinity; an assumption arguably incompatible with the wave equation. Here we assume it for the sake of simplicity. The role of the wave equation in this argument will be merely computational as opposed to evolutionary.

asymptotic quantities,

$$\begin{aligned} -(n-1)\cancel{X}^{(n)} &= \cancel{\text{grad}}^{(n)} x, \\ -n\cancel{x}^{(n)} &= \partial_u^{(n+1)} x, \\ \partial_u^{(n+1)} x &= (n-2)\cancel{x}^{(n)} - (n-2)x^{(n)} + \cancel{X}^i{}^j{}^{(n)} X_i. \end{aligned}$$

From these we can deduce an evolution equation for each $x^{(n+1)}$ in terms of lower order data,

$$\left(2 - \frac{2}{n}\right) \partial_u^{(n+1)} x = -\frac{1}{n-1} \cancel{\Delta}^{(n)} x - (n-2)x^{(n)},$$

which can be solved given initial data $x_0^{(n+1)}$ at $u = u_0$. Note that the asymptotic expansion for ϕ implies $x^{(1)} = 0$ and $\cancel{x}^{(1)} = -\phi$, in particular, we can compute $x^{(n+1)}$ inductively for $n \geq 2$ assuming that we know the radiation field $\phi^{(1)}$.

Therefore, we can compute the asymptotic quantities $x^{(n)}$, $\cancel{x}^{(n)}$ and $\cancel{X}^{(n)}$ for all $n \in \mathbb{N}$ along future null infinity in terms of the radiation field $\phi^{(1)}$ and initial conditions, $x_0^{(n)}$, $n \in \mathbb{N}$, at a “sphere at infinity” $u = u_0$. ■

This toy model reflects perfectly well the asymptotic relations when the equations can be extended smoothly up to infinity. The analysis for the Einstein-Maxwell equations is completely analogous albeit more technical. The previous relations will be used in Chapter 5 when we discuss the extension of the main theorem to include a (massless) Klein-Gordon scalar field.

2.1 Quantities to all orders at infinity

Throughout this chapter we will work with the coordinates $(u, s, \theta^2, \theta^3)$ and frame $\{e_0 = L, e_1 = \underline{L}, e_2, e_3\}$ as constructed in subsection 1.3.2. Also, the spacetimes considered will be assumed to be smooth at future null infinity in the sense of Definition 1.3.8. We will be using generically the word *quantities* to refer to the components of either the orthonormalisation matrix, the connection coefficients, the Weyl curvature or the Faraday tensor with respect to these coordinates and frame.

Now we proceed to show that the *radiation fields*,

$$\underline{\Xi}_{ij} := \hat{\chi}_{ij}^{(1)} = \underline{\chi}_{ij}^{(1)} - \text{tr} \underline{\chi} \eta_{ij} \quad \text{and} \quad \underline{A}(\mathbf{F})_i := \underline{\alpha}(\mathbf{F})_i^{(1)},$$

characterise the metric at infinity to all orders (up to stationary data to be described below). To do so we rely on the hierarchy discovered by BMS which is also tied to the splitting of the Weyl tensor into its null components and can be interpreted as signature levels.

Formally, in Proposition 2.1.1 we state recurrence relations satisfied by the different orders of the physical quantities. This is done by translating the structure equations to infinity, hence obtaining a non-linear algebraic system of equations for the asymptotic coefficients. Then the aforementioned hierarchy helps us to identify levels where the equations become linear for the quantities belonging to that level. Moreover, the structure of that hierarchy leads to identifying the radiation fields as the necessary initial data to run an induction argument.

Then we further specialise to the case when the radiation fields vanishes. Then by running the hierarchy-induction procedure we prove in Proposition 2.1.4 that ∂_u is a Killing symmetry to all orders at infinity. More precisely, if $\underline{\Xi}_{ij} = 0$ and $\underline{A}(\mathbf{F})_i = 0$ then all the asymptotic quantities are u -independent.

Before stating the main results we set some notation. Recall that $f^{(n)}$ denotes the best s^{-n} -approximation of f (see discussion preceding Definition 1.3.8). We refer to the corresponding functions associated to the orthonormalisation matrix, connection coefficients, etc., as *asymptotic quantities*.

The symbol $\{\phi_1, \dots, \phi_n\}$ will denote any expression involving the functions ϕ_1, \dots, ϕ_n . The symbol $[n]$ denotes an expression involving the connection coefficients up to order n and the orthonormalisation matrix up to order $n - 1$. That is,

$$[n] = \{\omega_{\mu\nu\lambda}^{(0)}, \dots, \omega_{\mu\nu\lambda}^{(n)}, h_{\mu}^{\nu(0)}, \dots, h_{\mu}^{\nu(n-1)}\},$$

with the convention that $[0] = 0$. The symbol $Q(\phi, \dots; \varphi, \dots)$ stands for a quadratic expression containing terms of the form $\phi\varphi$.

Proposition 2.1.1. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be a CK-Zipser electrovacuum spacetime smooth*

at future null infinity such that $\partial_s = e_0$ is a null geodesic vector field and ${}^2\omega_{023} = 0$. Then the asymptotic quantities satisfy the following recurrence relations for any $n \in \mathbb{N}$,

$$\alpha_{ij}^{(n+1)} = (n-2)\chi_{ij}^{(n)} - 2\alpha(\mathbf{F})_k \alpha(\mathbf{F})^k \eta_{ij}^{(1)} + \lfloor n-1 \rfloor, \quad (2.1a)$$

$$h_i^j = \chi_i^k h_k^j + \lfloor n-1 \rfloor, \quad (2.1b)$$

$$\omega_{jji}^{(n)} = \{\check{e}_k h_i^j, \lfloor n-1 \rfloor\} \quad (2.2a)$$

$$\beta_i^{(n+1)}, \rho(\mathbf{F}), \sigma(\mathbf{F}), \zeta_i, f^i = \{\chi_{ij}^{(n)}, \alpha(\mathbf{F})_i, \lfloor n-1 \rfloor\}, \quad n > 3 \quad (2.2b)$$

$$\rho^{(n+1)}, \sigma^{(n+1)}, \underline{\alpha}(\mathbf{F}), \underline{\omega}, \omega_{123}, \underline{\chi}_{ij}, \underline{\xi}_i, f^0 = \{\chi_{ij}^{(n)}, \alpha(\mathbf{F})_i, \lfloor n-1 \rfloor\}, \quad n > 2. \quad (2.3a)$$

Moreover,

$$2\partial_u \chi_{ij}^{(n+1)} = -n \underline{\chi}_{ij}^{(n)} + \{\chi_{ij}^{(n)}, \alpha(\mathbf{F})_i, \lfloor n-1 \rfloor\}, \quad (2.4a)$$

$$2\partial_u \alpha(\mathbf{F})_i = \{\alpha(\mathbf{F})_i, \rho(\mathbf{F}), \sigma(\mathbf{F}), \lfloor n \rfloor\}, \quad (2.4b)$$

$$\underline{\beta}_i^{(n+1)} = \partial_u \zeta_i^{(n+1)} + \lfloor n \rfloor, \quad (2.5a)$$

$$\underline{\alpha}_{ij}^{(n+1)} = \partial_u \underline{\chi}_{ij}^{(n+1)} + \lfloor n \rfloor. \quad (2.5b)$$

Before proving this proposition we remark that it can be interpreted as saying that the asymptotic quantities can be computed recursively starting from the radiation fields. The cases $n = 1, 2, 3$ are special. Roughly speaking $n = 1$ corresponds to the choice of gauges at infinity, with the exception of the radiation fields. On the other hand, for $n = 2$ we have to specify mass and EM-charge aspect functions as data. Finally, for $n = 3$ an angular momentum aspect vector is required. These aspect functions are not freely specifiable along all future null infinity; they obey

²Recall that in Section 1.3.2 we have constructed $\partial_s = e_0$ in such a way and by Lemma 1.3.6 we can make $\omega_{023} = 0$.

evolution equations given by the Einstein-Maxwell equations of signature -1 and -2 . Hence we really only need to specify aspect functions at some sphere at infinity $u = u_0$. We deal with these cases during the proof of Proposition 2.1.1. Here we state the main relations found there in:

Lemma 2.1.2. *Under the same hypothesis of Proposition 2.1.1 the following relations hold,*

$$\partial_u \operatorname{tr} \underline{\chi}^{(2)} = Q(\underline{\chi}^{(1)}; \underline{\chi}^{(1)}) + Q(\underline{\alpha}(\mathbf{F}); \underline{\alpha}(\mathbf{F})), \quad (2.6a)$$

$$\rho^{(3)}, \sigma^{(3)}, \underline{\alpha}(\mathbf{F})^{(2)}, \underline{\omega}, \omega_{123}^{(2)}, \hat{\chi}_{ij}^{(2)}, \underline{\xi}_i^{(2)}, f^0 = \{\chi_{ij}^{(2)}, \alpha(\mathbf{F})_i, \operatorname{tr} \underline{\chi}^{(2)}, [1]\}, \quad (2.6b)$$

$$\omega_{jji}^{(3)} = h^{-1} \cdot (\check{e}_k h_i^j + [2]), \quad (2.6c)$$

$$\partial_u \zeta_i^{(3)} = -\partial_u \omega_{jji}^{(3)} + Q(\underline{\alpha}(\mathbf{F}); \rho(\mathbf{F}), \sigma(\mathbf{F})) + [2], \quad (2.6d)$$

$$2\partial_u \rho(\mathbf{F}) = \check{\operatorname{div}} \underline{A}(\mathbf{F}), \quad (2.6e)$$

$$2\partial_u \sigma(\mathbf{F}) = \check{\operatorname{curl}} \underline{A}(\mathbf{F}). \quad (2.6f)$$

Here \check{e}_k is the standard orthonormal basis on the round sphere and $\check{\operatorname{div}}$ and $\check{\operatorname{curl}}$ are the corresponding operators. Equations (2.6a), (2.6d), (2.6e) and (2.6f) correspond to the aforementioned evolution formulas for the mass, angular momentum and EM-charges. From the point of view of a characteristic initial value formulation, these are constraint equations at future null infinity. All these recurrence relations allow us to compute all the asymptotic quantities in terms of the radiation fields and asymptotic initial data along a fixed C_{u_0} :

Proposition 2.1.3. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be CK-Zipser electrovacuum spacetime smooth at future null infinity. Then all the asymptotic quantities depend solely on the radiation fields $\underline{\Xi}_{ij}$, $\underline{A}(\mathbf{F})_i$ along future null infinity and the initial values at some $u = u_0$ of $\operatorname{tr} \underline{\chi}^{(2)}$, $\zeta_i^{(3)}$, $\rho(\mathbf{F})^{(2)}$, $\sigma(\mathbf{F})^{(2)}$ and $\chi_{ij}^{(n)}$ and $\alpha(\mathbf{F})_i^{(n)}$, $n \geq 2$.*

Along the same lines we can prove stationarity to all orders if the radiation fields vanish:

Proposition 2.1.4. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be a CK-Zipser electrovacuum spacetime smooth at future null infinity. Assume that $\underline{\Xi}_{ij} = 0 = \underline{A}(\mathbf{F})_i$. Then all the asymptotic quantities are u -independent.*

The proof of Proposition 2.1.1 relies on analysing the structure equations order by order with respect to the s -expansion assumed in the smoothness hypothesis

at null infinity. To do so it is convenient to understand the algebraic structure of the quadratic terms appearing in the equations. This was done by CK by introducing the concept of signature of a component of a tensor. We review this notion before proceeding to the proof of Proposition 2.1.1.

2.2 Structure equations and signature of null components

In order to understand the structure equations, (1.35), in the null gauge we introduce the concept of signature of a null component of a tensor, as discussed in [22] p 148. We just quote their clear exposition:

“Given any covariant tensor U at a point of spacetime, we define a null component of it to be any tensor tangent to the sphere $S_{u,s}$ at a point, which is derived from U by contractions with either e_0 or e_1 and projections to $S_{u,s}$. To any such component we assign a signature that it is defined as the difference between the total number of contractions with e_0 and the total number of contractions with e_1 . We are now ready to state the following heuristic principle.

Principle of Conservation of signature: *Consider an arbitrary covariant tensor U that can be expressed as a multilinear form in an arbitrary number of covariant tensors $U_1 \dots U_p$, with coefficients depending only on the spacetime metric and its volume form. Then the signature of any null term of U , expressed in terms of the null components of $U_1 \dots U_p$, is equal to the sum of the signatures of each constituent in the decomposition.”*

In particular the signature of a null component of tensor does not change by lowering/raising indices. Table 3.1 presents the signature of the CK null components. It also serves as dictionary between the slight variations of notation between CK and this thesis. For the convenience of the reader familiar with the Newman-Penrose (NP) notation, [53], we have also included their notation here.

The above principle will allow us to guess the quadratic terms appearing in the structure equations without performing the calculations. Indeed, we already know that in the null-geodesic gauge the only connection coefficient with signature equal 2 vanishes; this fact is the one responsible for the success of the hierarchy-induction argument used in the proof of Proposition 2.1.1 to obtain the asymptotic

Table 2.1: Signature of the components of the connection, Weyl curvature and Faraday tensor.

CK	NP	This thesis	Signature
L	l	e_0	1
\underline{L}	n	e_1	-1
e_i	m	e_i	0
ξ_i	κ	ω_{0i0}	2
$\hat{\chi}_{ij}, \text{tr } \chi$	σ, ρ	$\chi_{ij} = \omega_{ij0}$	1
$\omega, \not\chi_{Le_i}$	$\text{Re } \epsilon, \text{Im } \epsilon$	$\omega_{010}, \omega_{023}$	1
ζ_i	τ	ω_{1i0}	0
$\underline{\zeta}_i$	π	ω_{01i}	0
V_i	$\bar{\alpha} + \beta$	ω_{i10}	0
$\not\chi_{e_i e_j}$	$\bar{\alpha} - \beta$	$\omega_{223}, \omega_{332}$	0
$\hat{\underline{\chi}}_{ij}, \text{tr } \underline{\chi}$	λ, μ	$\underline{\chi}_{ij} = \omega_{ij1}$	-1
$\underline{\omega}, \not\chi_{\underline{L}e_i}$	$\text{Re } \gamma, \text{Im } \gamma$	$\omega_{101}, \omega_{123}$	-1
$\underline{\xi}_i$	ν	$\underline{\xi}_i = \omega_{1i1}$	-2
α_{ij}	Ψ_0	C_{i0j0}	2
β_i	Ψ_1	C_{i010}	1
ρ, σ	Ψ_2	C_{1010}, C_{1023}	0
$\underline{\beta}_i$	Ψ_3	C_{i110}	-1
$\underline{\alpha}_{ij}$	Ψ_4	C_{i1j1}	-2
-	Φ_{00}	S_{00}	2
-	Φ_{01}	S_{0i}	1
-	$\Phi_{11}, \Lambda, \Phi_{02}$	$S_{01}, S_{22}, S_{23}, S_{33}$	0
-	Φ_{12}	S_{1i}	-1
-	Φ_{22}	S_{11}	-2
$\alpha_i(F)$	ϕ_0	F_{0i}	1
$\rho(F), \sigma(F)$	ϕ_1	F_{01}, F_{23}	0
$\underline{\alpha}_i(F)$	ϕ_2	F_{1i}	-1

quantities to all orders at infinity.

Now we illustrate how to apply this principle to the structure equations. We will denote by $U(l)$ a null component of U with signature l . The signature 2 case yields schematically:

$$\begin{aligned}\mathbf{C}(2) + \eta \wedge \mathbf{S}(2) &= e(1)\omega(1) + e(0)\omega(2) + \omega(1) \wedge \omega(1) + \omega(0) \wedge \omega(2), \\ &= e(1)\omega(1) + \omega(1) \wedge \omega(1).\end{aligned}$$

where in the second line we have used the fact that the only connection coefficient with signature 2, $\omega_{0\alpha 0}$, vanishes in the null-geodesic gauge. More generally we have the following:

Hierarchy of structure equations: *When working in the null-geodesic gauge there is no connection coefficient of signature $l - 2$ on the right-hand side of the structure equation of signature l , $l = 2, 1, 0$.*

This is precisely the hierarchy found by Bondi, van der Burg and Metzner, and Sachs, [15], [59]. The exact form of the structure equations is presented below; we have grouped them in signature levels. Note that (2.7b), (2.8e), (2.8f), (2.9i) and (2.9j) are frame equations while (2.8g), (2.9k) are 1st Bianchi identities. All of these are satisfied on any Lorentzian manifold. In contrast, Einstein's equations appear implicitly when we regard $\mathbf{S}_{\mu\nu}$ as a quadratic expression in $\mathbf{F}_{\mu\nu}$. Finally, (2.8c), (2.8d), (2.9g), (2.9h), (2.10c), (2.10d) are Maxwell's equations.

a) The α_{ij} or signature 2 level. This includes the 2nd fundamental form χ_{ij} and h_i^j :

$$\alpha_{ij} + \frac{1}{2}\eta_{ij}\mathbf{S}_{00} = -e_0(\chi_{ij}) - \chi_{ik}\chi^k_j, \quad (2.7a)$$

$$e_0(h_i^j) = -\chi_i^k h_k^j. \quad (2.7b)$$

b) The β_i or signature 1 level. This includes the torsion ζ_i and the coefficients of

the induced connection on $S_{s,u}$, that is, ω_{ij} , and f^i :

$$2\beta_i + \mathbf{S}_{0i} = -2e_0(\zeta_i) - 4\zeta^k \chi_{ki}, \quad (2.8a)$$

$$\beta_i - \frac{1}{2}\mathbf{S}_{0i} = e_0(\omega_{jji}) - \zeta_j \chi_{ij} + \zeta_i \chi_{jj} + \chi_j^k \omega_{kji}, \quad i \neq j, \quad (2.8b)$$

$$e_0(\rho(\mathbf{F})) = -\mathcal{D}\text{iv} \alpha(\mathbf{F}) - \text{tr} \chi \rho(\mathbf{F}) - \zeta_i \alpha(\mathbf{F})^i, \quad (2.8c)$$

$$e_0(\sigma(\mathbf{F})) = -\mathcal{C}\text{ur}\ell \alpha(\mathbf{F}) - \text{tr} \chi \sigma(\mathbf{F}) + \epsilon^{ik} \zeta_i \alpha(\mathbf{F})_k, \quad (2.8d)$$

$$e_i(h_j^k) - e_j(h_i^k) = (\omega_i^n{}_j - \omega_j^n{}_i) h_n^k, \quad (2.8e)$$

$$e_0(f^i) = -\zeta^k h_k^i, \quad (2.8f)$$

$$\begin{aligned} e_1(\omega_{jji}) - e_j(\omega_{1ji}) &= -e_j(\underline{\chi}_{ji}) + e_i(\underline{\chi}_{jj}) \\ &\quad - (\omega \wedge \omega)_{ji1j} + (\omega \wedge \omega)_{1jji}, \quad i \neq j. \end{aligned} \quad (2.8g)$$

with no summation on repeated j 's. Here $\mathcal{D}\text{iv}$, $\mathcal{C}\text{ur}\ell$ and ϵ_{ik} are the divergence and curl operators and volume element on the spheres $S_{u,s}$, respectively.

c) The (ρ, σ) or signature 0 level. This includes $\underline{\omega}$, ω_{123} , $\underline{\chi}_{ij}$, $\underline{\xi}_i$ and f^0 :

$$\begin{aligned} -\rho + \frac{1}{2}(\mathbf{S}_{22} + \mathbf{S}_{33}) &= e_2(\omega_{323}) - e_3(\omega_{223}) + \frac{1}{2}\chi_{22}\underline{\chi}_{33} + \frac{1}{2}\chi_{33}\underline{\chi}_{22} \\ &\quad - \chi_{23}\underline{\chi}_{23} - (\omega_{232})^2 - (\omega_{323})^2, \end{aligned} \quad (2.9a)$$

$$\begin{aligned} -\sigma &= -e_2(\zeta_3) + e_3(\zeta_2) + \omega_{323}\zeta_3 - \omega_{232}\zeta_2 \\ &\quad - \chi_2^k \underline{\chi}_{3k} + \chi_3^k \underline{\chi}_{2k}, \end{aligned} \quad (2.9b)$$

$$\rho + \frac{1}{2}\mathbf{S}_{01} = e_0(\underline{\omega}) - 3\zeta^k \zeta_k, \quad (2.9c)$$

$$-2\sigma = e_0(\omega_{123}) + 4\zeta^k \omega_{k23}, \quad (2.9d)$$

$$-4\rho - \mathbf{S}_{jj} + \frac{1}{2}\mathbf{S}_{01} = -e_0(\underline{\chi}_{jj}) - 2e_j(\zeta_j) + 2\zeta^k \omega_{jkj} - \chi_j^k \underline{\chi}_{kj} - 2\zeta_j \zeta_j, \quad (2.9e)$$

$$-\sigma - \mathbf{S}_{23} = -e_0(\underline{\chi}_{23}) - 2e_2(\zeta_3) + 2\zeta^k \omega_{2k3} - 2\zeta_2 \zeta_3 - \chi_2^k \underline{\chi}_{k3}, \quad (2.9f)$$

$$e_0(\underline{\alpha}(\mathbf{F})_i) = -e_i(\rho(\mathbf{F})) + \epsilon_i^k e_k(\sigma(\mathbf{F})) - (\mathbf{F} \wedge \omega)_i, \quad (2.9g)$$

$$e_1(\alpha(\mathbf{F})_i) = e_i(\rho(\mathbf{F})) + \epsilon_i^k e_k(\sigma(\mathbf{F})) - (\star \mathbf{F} \wedge \omega)_i, \quad (2.9h)$$

$$e_0(f^0) = \underline{\omega}, \quad (2.9i)$$

$$e_i(f^0) = \underline{\xi}_i, \quad (2.9j)$$

$$e_1(\chi_{ij}) - e_i(\zeta_j) = e_0(\underline{\chi}_{ij}) - e_j(\zeta_i) - (\omega \wedge \omega)_{1i0j} + (\omega \wedge \omega)_{0j1i}. \quad (2.9k)$$

With no summation on repeated j 's.

d) The $\underline{\beta}_i$ or signature -1 level:

$$2\underline{\beta}_i - \mathbf{S}_{1i} = e_0(\underline{\xi}_i) + 2e_1(\zeta_i) - (\omega \wedge \omega)_{01i1} \quad (2.10a)$$

$$-\underline{\beta}_i - \frac{1}{2}\mathbf{S}_{1i} = e_1(\omega_{jji}) - e_j(\omega_{1ji}) - (\omega \wedge \omega)_{1jji}. \quad (2.10b)$$

$$e_1(\rho(\mathbf{F})) = \underline{\text{div}} \underline{\alpha}(\mathbf{F}) + \text{tr} \underline{\chi} \rho(\mathbf{F}) + \zeta_i \alpha(\mathbf{F})^i, \quad (2.10c)$$

$$e_1(\sigma(\mathbf{F})) = -\underline{\text{curl}} \underline{\alpha}(\mathbf{F}) - \text{tr} \underline{\chi} \sigma(\mathbf{F}) + \epsilon^{ik} \zeta_i \alpha(\mathbf{F})_k, \quad (2.10d)$$

With no summation on repeated j 's.

e) Finally, the $\underline{\alpha}_{ij}$ or signature -2 level:

$$\underline{\alpha}_{ij} + \frac{1}{2}\eta_{ij}\mathbf{S}_{11} = -e_i(\underline{\xi}_j) - e_1(\underline{\chi}_{ij}) - (\omega \wedge \omega)_{i1j1}. \quad (2.11)$$

2.3 Proofs of Propositions 2.1.1, 2.1.4, 2.1.3

Now we proceed to prove the results stated above. It is convenient to think of the structure-Einstein-Maxwell equations as a 1st order system schematically of the form:

$$\begin{aligned} \nabla h + \omega \wedge h &= 0, \\ \nabla \omega + \omega \wedge \omega &= \mathbf{C} + \eta \wedge \mathbf{S}, \\ \mathbf{S} &= \mathbf{F}^2, \\ \nabla \mathbf{F} + \omega \wedge \mathbf{F} &= 0. \end{aligned}$$

The procedure will be analogous to the one described at the beginning of the chapter for a wave equation.

Recall firstly the Einstein's equations:

$$\mathbf{S}_{\mu\nu} = 2\mathbf{F}_{\mu\sigma}\mathbf{F}^\sigma{}_\nu - \frac{1}{2}\eta_{\mu\nu}\mathbf{F}_{\alpha\beta}\mathbf{F}^{\alpha\beta}.$$

They are quadratic in $\mathbf{F}_{\mu\nu}$, in particular then $\mathbf{S}_{\mu\nu}^{(n+1)} = \{\mathbf{F}_{\mu\nu}^{(1)}, \dots, \mathbf{F}_{\mu\nu}^{(n)}\}$. A more

detailed null decomposition is given by

$$\begin{aligned}
\mathbf{S}_{00} &= 2\alpha(\mathbf{F})_i \alpha(\mathbf{F})^i, \\
\mathbf{S}_{01} &= \rho(\mathbf{F})^2 + \sigma(\mathbf{F})^2, \\
\mathbf{S}_{ij} &= Q(\rho(\mathbf{F}), \sigma(\mathbf{F}); \rho(\mathbf{F}), \sigma(\mathbf{F})) + Q(\alpha(\mathbf{F}); \underline{\alpha}(\mathbf{F})), \\
\mathbf{S}_{0i} &= Q(\alpha(\mathbf{F})_i; \rho(\mathbf{F}), \sigma(\mathbf{F})), \\
\mathbf{S}_{1i} &= Q(\underline{\alpha}(\mathbf{F})_i; \rho(\mathbf{F}), \sigma(\mathbf{F})), \\
\mathbf{S}_{11} &= 2\underline{\alpha}(\mathbf{F})_i \underline{\alpha}(\mathbf{F})^i.
\end{aligned}$$

Firstly, we state the leading order values of the relevant coefficients that we will be using to obtain the recurrence relations and also in the induction argument. They are a consequence of the smoothness assumption.

Lemma 2.3.1. *The connection coefficients vanish to order 1 with the exception of*

$$\underline{\chi}_{ij}^{(1)} = \underline{\Xi}_{ij} - \eta_{ij}, \quad \chi_{ij}^{(1)} = \eta_{ij}, \quad \omega_{332}^{(1)} = \frac{\cos(\theta^2)}{\sin(\theta^2)}.$$

Moreover,

$$\begin{aligned}
\alpha_{ij} &= \mathcal{O}(s^{-4}), & \beta_i &= \mathcal{O}(s^{-3}), & \rho, \sigma &= \mathcal{O}(s^{-3}), \\
\alpha(\mathbf{F})_i &= \mathcal{O}(s^{-3}), & \rho(\mathbf{F}), \sigma(\mathbf{F}) &= \mathcal{O}(s^{-2}).
\end{aligned}$$

Proof. We start with the α_{ij} -level. Recall that $h_i^j = \text{diag}(1, 1/\sin\theta^2)$, the frame equation (2.7b) to order 2 gives,

$$-h_i^j = -\chi_i^k h_k^j.$$

Hence $\chi_{ij}^{(1)} = \eta_{ij}$. Next, the structure equation (2.7a) to order 1 gives $\alpha_{ij}^{(1)} = 0$; to order 2 implies,

$$\alpha_{ij}^{(2)} + \frac{1}{2}\eta_{ij}\mathbf{S}_{00}^{(2)} = \chi_{ij}^{(1)} - \chi_{ik}^{(1)}\chi_j^k = 0.$$

Taking the trace we obtain $\mathbf{S}_{00}^{(2)} = 0$, so $\alpha(\mathbf{F})_i = 0$ and $\alpha_{ij}^{(2)} = 0$. Note that then

$\mathbf{S}_{00}^{(3)} = 2\alpha(\mathbf{F})_i \alpha(\mathbf{F})^i = 0$. Using the values already found, the same structure equation to order 3 gives,

$$\alpha_{ij}^{(3)} = 2\chi_{ij}^{(2)} - 2\chi_{ik}^{(1)}\chi_j^k = 0.$$

Therefore $\alpha_{ij} = \mathcal{O}(s^{-4})$.

Now note that $\mathbf{S}_{0i} = \mathcal{O}(s^{-3})$ since $\alpha^{(1)}(\mathbf{F})_i = 0$. Equation (2.8a) to order 1 gives $\beta_i^{(1)} = 0$. Equations (2.8a), (2.8b), (2.8e) and (2.8f) to order 2 are,

$$\begin{aligned} 2\beta_i^{(2)} &= 4\zeta_i^{(1)} - 4\zeta_k^{(1)}\chi_j^k = 0, \\ \beta_i^{(2)} &= -\omega_{jji}^{(1)} - \zeta_i^{(1)}\eta_{jj} + \omega_{jji}^{(1)} = \zeta_i^{(1)}, \quad i \neq j, \\ \check{e}_i(h_j^k) - \check{e}_j(h_i^k) &= (\omega_i^{n_j} - \omega_j^{n_i})h_n^k, \\ -f^i &= -\zeta^k h_k^i. \end{aligned}$$

So $\zeta_i^{(1)} = \beta_i^{(2)} = f^i = 0$ and $\omega_{jji}^{(1)}$ are the connection coefficients of the standard round sphere, explicitly, $\omega_{223}^{(1)} = 0$ and $\omega_{332}^{(1)} = \frac{\cos(\theta^2)}{\sin(\theta^2)}$.

The Maxwell equations (2.8c) and (2.8d) to order 2 read,

$$\begin{aligned} -\rho(\mathbf{F}) &= -\check{\text{div}}\alpha(\mathbf{F}) - 2\rho(\mathbf{F}) - \zeta^i \alpha(\mathbf{F})_i, \\ -\sigma(\mathbf{F}) &= -\check{\text{curl}}\alpha(\mathbf{F}) - 2\sigma(\mathbf{F}) - \check{\epsilon}^{ik} \zeta_i \alpha(\mathbf{F})_k. \end{aligned}$$

So $\rho(\mathbf{F}) = \sigma(\mathbf{F}) = 0$. To order 3 we get

$$\begin{aligned} -2\rho(\mathbf{F}) &= -\check{\text{div}}\alpha(\mathbf{F}) - 2\rho(\mathbf{F}), \\ -2\sigma(\mathbf{F}) &= -\check{\text{curl}}\alpha(\mathbf{F}) - 2\sigma(\mathbf{F}). \end{aligned}$$

So $\alpha(\mathbf{F})_i$ obeys the Hodge system

$$\begin{aligned} \check{\text{div}}\alpha(\mathbf{F}) &= 0, \\ \check{\text{curl}}\alpha(\mathbf{F}) &= 0, \end{aligned}$$

on the round sphere. Hence $\alpha(\mathbf{F})_i = 0$.

Moving on to the (ρ, σ) -level. Equations (2.9c), (2.9d), (2.9i) and (2.9j) to order 1 give $\overset{(1)}{\rho} = \overset{(1)}{\sigma} = 0 = \overset{(1)}{\underline{\omega}} = \overset{(1)}{\underline{\xi}_i}$. Also we know that $\mathbf{S}_{01} = \mathcal{O}(s^{-4})$ and $\mathbf{S}_{ij} = \mathcal{O}(s^{-3})$ since $\overset{(1)}{\rho}(\mathbf{F}) = \overset{(1)}{\sigma}(\mathbf{F}) = \overset{(1)}{\alpha}(\mathbf{F})_i = \overset{(2)}{\alpha}(\mathbf{F})_i = 0$. Therefore, equations (2.9a)-(2.9d) to order 2 read

$$\begin{aligned} -\overset{(2)}{\rho} &= \check{e}_2(\overset{(1)}{\omega_{323}}) - \check{e}_3(\overset{(1)}{\omega_{232}}) - (\overset{(1)}{\omega_{232}})^2 - (\overset{(1)}{\omega_{323}})^2 + \frac{1}{2}\text{tr}\overset{(1)}{\underline{\chi}}, \\ -\overset{(2)}{\sigma} &= -\check{e}_2(\overset{(1)}{\zeta_3}) + \check{e}_3(\overset{(1)}{\zeta_2}) + \overset{(1)}{\omega_{323}}\overset{(1)}{\zeta_3} - \overset{(1)}{\omega_{232}}\overset{(1)}{\zeta_2} - \overset{(1)}{\underline{\chi}}_{32} + \overset{(1)}{\underline{\chi}}_{23} = 0, \\ \overset{(2)}{\rho} &= -\overset{(1)}{\underline{\omega}} - 6\overset{(1)}{\zeta^k}\overset{(1)}{\zeta_k} = 0, \\ -2\overset{(2)}{\sigma} &= -\overset{(1)}{\omega_{123}} + 4\overset{(1)}{\zeta^k}\overset{(1)}{\omega_{k23}}. \end{aligned}$$

These imply $\overset{(2)}{\rho} = \overset{(2)}{\sigma} = \overset{(1)}{\omega_{123}} = 0$ and $\text{tr}\overset{(1)}{\underline{\chi}} = -2$. This finishes the proof of the Lemma. ■

Proof of Proposition 2.1.1. Now we compute the recurrence relations.

a) It can be seen that the $(n+1)$ -order of the α -level equations together with

$$\overset{(n+1)}{\mathbf{S}}_{00} = \{\overset{(3)}{\alpha}(\mathbf{F})_i, \dots, \overset{(n-2)}{\alpha}(\mathbf{F})_i\}$$

imply the recurrence relations (2.1) for $\overset{(n+1)}{\alpha_{ij}}$ and $\overset{(n)}{h_i^j}$.

b) Now we look at the $(n+1)$ -order of the β -level equations. From this level onwards the equations become more intricate. The important thing to remember is that we only need to keep track of the coefficients accompanying the variables that we want to find at a given order and level.

We start with the $(n+1)$ -order of the frame equation:

$$e_i(h_j^k) - e_j(h_i^k) = (\omega_i^n{}_j - \omega_j^n{}_i)h_n^k.$$

Note that the left-hand-side does not contain terms with $\overset{(n+1)}{h_j^k}$ since $e_i = h_i^j\partial_j$ and $h_i^j = \mathcal{O}(s^{-1})$. Writing down the equations explicitly for $k = 2, 3$ we get the

system

$$\begin{pmatrix} \binom{(1)}{h_2^2} & \binom{(1)}{-h_2^3} \\ \binom{(1)}{-h_3^2} & \binom{(1)}{h_3^3} \end{pmatrix} \begin{pmatrix} \binom{(n)}{\omega_{223}} \\ \binom{(n)}{\omega_{332}} \end{pmatrix} = \check{e}_k h_i^j + \lfloor n - 1 \rfloor.$$

This system tells us that $\binom{(n)}{\omega_{jji}} = \{\check{e}_k h_i^j, \lfloor n - 1 \rfloor\} = \{\check{e}_k \chi_{ij}, \lfloor n - 1 \rfloor\}$, where we have used the recurrence relation for $\binom{(n)}{h_i^j}$. In particular for $n = 3$ we get equation (2.6c).

We are left with the variables β_i , $\rho(\mathbf{F})$, $\sigma(\mathbf{F})$ and ζ_i . The $(n+1)$ -order of equations (2.8a)-(2.8d) give the following linear system valid for $n \geq 3$,

$$\begin{pmatrix} 2 & 0 & 0 & -(2n-4) \\ 1 & 0 & 0 & -1 \\ 0 & -(n-2) & 0 & 0 \\ 0 & 0 & -(n-2) & 0 \end{pmatrix} \begin{pmatrix} \binom{(n+1)}{\beta_i} \\ \binom{(n)}{\rho(\mathbf{F})} \\ \binom{(n)}{\sigma(\mathbf{F})} \\ \binom{(n)}{\zeta_i} \end{pmatrix} = \{\binom{(n)}{\chi_{ij}}, \alpha(\mathbf{F})_i, \lfloor n - 1 \rfloor\},$$

where the zeros on the first two rows correspond to the fact that $\alpha(\mathbf{F}) = 0 = \alpha(\mathbf{F})$ and the quadratic character of \mathbf{S}_{0i} stated before. Similarly the zeros on the last column come also from $\alpha(\mathbf{F}) = 0 = \alpha(\mathbf{F})$. The above system can be solved (note that the degenerate cases correspond to $n = 2, 3$) to obtain the desired recurrence relations.

Remark. The cases $n = 2, 3$ are special. For $n = 2$ the Maxwell equations degenerate; this translate to having to specify $\binom{(2)}{\rho(\mathbf{F})}$ and $\binom{(2)}{\sigma(\mathbf{F})}$ at some $u = u_0$ as initial data (these can be regarded as electromagnetic charges) and then evolve them using (2.6e) and (2.6f). For $n = 3$, the degeneracy is telling us that we have to prescribe $\binom{(3)}{\zeta_i}$ at some $u = u_0$ (angular momentum aspect vector) and evolve it using (2.6d). We will find the same situation on the (ρ, σ) -level where the corresponding initial data can be regarded as a mass aspect function.

- c) This is the 0-signature case; a signature count gives us that ξ_i does not appear on the $(\omega \wedge \omega)$ -term. Thus, the $(n + 1)$ -order of equations (2.9a)-(2.9g) give us

9 equations for 9 variables, namely, $\rho^{(n+1)}$, $\sigma^{(n+1)}$, $\alpha(\mathbf{F})_i^{(n)}$, $\underline{\omega}^{(n)}$, $\omega_{123}^{(n)}$ and $\underline{\chi}_{ij}^{(n)}$. The linear system can be solved provided it is non-degenerate, which can be checked by direct computation for $n > 2$. This gives the desired recurrence relation. Here we state the linear system obtained by considering the $(n+1)$ -order of equations (2.9a)-(2.9g):

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & n & 0 & * \\ 0 & 1 & 0 & 0 & 0 & 0 & -n & 0 \\ 1 & 0 & n-1 & 0 & 0 & 0 & 0 & * \\ 1 & 0 & 0 & n-1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & n-1 & 0 & 0 & * \\ n-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho^{(n+1)} \\ \sigma^{(n+1)} \\ \underline{\chi}_{22}^{(n)} \\ \underline{\chi}_{33}^{(n)} \\ \underline{\chi}_{23}^{(n)} \\ \underline{\omega}^{(n)} \\ \omega_{123}^{(n)} \\ \alpha(\mathbf{F})_i^{(n)} \end{pmatrix} = \begin{matrix} \text{Known} \\ \text{data} \end{matrix} \quad (2.12)$$

where ‘Known data’ can be described more precisely as a term of the form

$$\{\chi_{ij}^{(n)}, \zeta_i^{(n)}, \omega_{jji}^{(n)}, \alpha(\mathbf{F})_i^{(n)}, \rho(\mathbf{F})^{(n)}, \sigma(\mathbf{F})^{(n)}, [n-1]\}.$$

Once again the case $n = 2$ is special. The 1st, 5th and 6th rows become linearly dependent. Due to this degeneracy we need to know $\text{tr} \underline{\chi}^{(2)}$ along future null infinity; the Einstein’s equation at the $\underline{\alpha}$ -level provides us with an evolution equation, (2.6a). Hence we only have to specify $\text{tr} \underline{\chi}^{(2)}$ at a sphere at future null infinity $u = u_0$.

- d) Finally, equations (2.4) are obtained by taking the $(n+1)$ -order of equations (2.9k) and (2.9h), respectively.
- e) Now we focus on Lemma 2.1.2. The evolution equations for the EM-charges, (2.6e) and (2.6f), follow from Maxwell’s equations (2.10c) and (2.10d) to order 2, respectively.
- f) As explained before we can obtain an evolution equation for $\zeta_i^{(3)}$ by considering the system (2.10a) and (2.10b). The equation obtained after getting rid of the

Weyl terms is of the form:

$$\partial_u \zeta_i^{(3)} = -\partial_u \omega_{jji}^{(3)} + Q(\underline{\chi}^{(1)}; \zeta^{(2)}) + Q(\underline{\alpha}(\mathbf{F}); \rho^{(2)}, \sigma^{(2)}) + [2].$$

This proves the identity (2.6d).

g) Again a signature analysis as above helps us to find the structure of the $\underline{\alpha}$ -equation. We have that to 2nd order the trace of (2.11) reads

$$\partial_u \text{tr } \underline{\chi}^{(2)} = Q(\underline{\chi}^{(1)}; \underline{\chi}^{(1)}) + Q(\underline{\alpha}(\mathbf{F}); \underline{\alpha}(\mathbf{F})),$$

where we have used $\zeta_i^{(1)} = 0$. The remaining system for $\rho^{(3)}$, $\sigma^{(3)}$, etc, is non-degenerate and can be solved.

This finishes the proof of Proposition 2.1.1 and Lemma 2.1.2. ■

Proof of Propositions 2.1.4 and 2.1.3. All the hard work was done in the proof of Proposition 2.1.1. Now we show how to use the recurrence relations to find the asymptotic quantities to all orders.

We already know all the quantities to first order with the exception of $\Xi_{ij} := \hat{\chi}_{ij}^{(1)}$ and $\underline{A}(\mathbf{F})_i := \underline{\alpha}(\mathbf{F})_i$ which we assume given (for Proposition 2.1.3) or zero (for Proposition 2.1.4). Now, we procedure inductively for $n \geq 2$: Consider the asymptotic quantities,

$$\begin{aligned} & \chi_{ij}^{(k)}, \alpha(\mathbf{F})_i, \alpha_{ij}^{(k+1)}, h_i^j, \\ & \omega_{jji}^{(k)}, \beta_i^{(k+1)}, \rho(\mathbf{F}), \sigma(\mathbf{F}), \zeta_i, f^i, \\ & \rho^{(k+1)}, \sigma^{(k+1)}, \underline{\alpha}(\mathbf{F}), \underline{\omega}, \omega_{123}, \underline{\chi}_{ij}^{(k)}, \underline{\xi}_i^{(k)}, f^0. \end{aligned}$$

for $1 \leq k \leq n$.

- i) Suppose that they depend on $\Xi_{ij}, \underline{A}(\mathbf{F})_i, \text{tr } \underline{\chi}|_{u=u_0}, \rho(\mathbf{F})|_{u=u_0}, \sigma(\mathbf{F})|_{u=u_0}, \zeta_i|_{u=u_0}$ and $\chi_{ij}|_{u=u_0}$ and $\alpha(\mathbf{F})_i|_{u=u_0}$, $1 \leq k \leq n$, for Proposition 2.1.3.
- ii) or suppose that they are u -independent for Proposition 2.1.4.

The evolution equations (2.4a) and (2.4b) ensure that the induction hypothesis also holds for the quantities $\overset{(n+1)}{\chi}_{ij}$ and $\overset{(n+1)}{\alpha}(\mathbf{F})$; this is clear for case i). For case ii) we need to take an extra ∂_u derivative, which gives $\partial_u^2 \overset{(n+1)}{\chi}_{ij} = 0 = \partial_u^2 \overset{(n+1)}{\alpha}(\mathbf{F})$. Then the finiteness of pole condition, which prohibits polynomial growth in u , implies that they are also u -independent.

Now, the α -level recurrence equations (2.1a) and (2.1b) tell us that $\overset{(n+2)}{\alpha}_{ij}$ and $\overset{(n+1)}{h}_i{}^j$ depend on $\overset{(n+1)}{\chi}_{ij}$, $\overset{(n+1)}{\alpha}(\mathbf{F})$ and $[n]$ and so they also satisfy the induction hypothesis for $k = n + 1$.

We continue in this way. The β -level recurrence relations (2.2a) and (2.2b) do the work for $\overset{(n+1)}{\omega}_{jji}$, $\overset{(n+2)}{\beta}_i$, $\overset{(n+1)}{\rho}(\mathbf{F})$, $\overset{(n+1)}{\sigma}(\mathbf{F})$, $\overset{(n+1)}{\zeta}_i$, $\overset{(n+1)}{f}^i$. As discussed previously, we encounter a problem in the degeneracy of the Maxwell equations to order 3 and we have to specify $\overset{(2)}{\rho}(\mathbf{F})$ and $\overset{(2)}{\sigma}(\mathbf{F})$ at some $u = u_0$ and then evolve them using the ‘‘conjugate Maxwell equations’’ (2.6e) and (2.6f). The same issue arises for the β -structure equations to order 4 and we have to specify $\overset{(3)}{\zeta}_i$ at $u = u_0$ and evolve it using (2.6d); note that $\overset{(3)}{\omega}_{jji}$ is known from (2.2a). Therefore all the quantities at this level also satisfy the induction hypothesis i) for $k = n + 1$. Again for case ii) we need to take an extra ∂_u derivative for the degenerate case to show u -independence of $\overset{(2)}{\rho}(\mathbf{F})$, $\overset{(2)}{\sigma}(\mathbf{F})$ and $\overset{(3)}{\zeta}_i$.

The (ρ, σ) -level recurrence relation is analogous. Equation (2.4a) show that all the quantities at this level satisfy the induction hypothesis for $k = n + 1$. Again, we need to consider separately the case when the system (2.12) degenerates. The rank of the matrix on the left decreases by 1 for $n = 2$, so it is necessary to prescribe $\text{tr} \overset{(2)}{\chi}|_{u=u_0}$ and then evolve it using (2.6a); in particular, $\text{tr} \overset{(2)}{\chi}$ is u -independent if $\Xi_{ij} = \underline{A}(\mathbf{F})_i = 0$. Therefore the quantities of this level also satisfy the induction hypothesis.

This finishes the proof of Propositions 2.1.4 and 2.1.3. ■

Finally, we just check that the vanishing of the radiating fields is related to a stronger decay of the Weyl and Faraday tensors.

Lemma 2.3.2. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be a CK-Zipser electrovacuum spacetime smooth at*

future null infinity with $\Xi_{ij} = \underline{A}(\mathbf{F})_i = 0$, then

$$\mathbf{C}_{\rho\sigma\mu\nu} = \mathcal{O}(s^{-3}) \quad \text{and} \quad \mathbf{F}_{\mu\nu} = \mathcal{O}(s^{-2}).$$

Proof. We already know $\rho(\mathbf{F}) = \sigma(\mathbf{F}) = \alpha^{(1)}(\mathbf{F})_i = \alpha^{(2)}(\mathbf{F})_i = 0$ and by hypothesis $\underline{\alpha}^{(1)}(\mathbf{F})_i = 0$, hence $\mathbf{F}_{\mu\nu} = \mathcal{O}(s^{-2})$.

Then the Einstein equations imply $\mathbf{S}_{11}, \mathbf{S}_{1i} = \mathcal{O}(s^{-4})$. Next, the $\underline{\beta}$ and $\underline{\alpha}$ -structure equations, (2.10a) and (2.11) to order 1 give,

$$\begin{aligned} 2\underline{\beta}_i^{(1)} &= 4\partial_u \zeta_i^{(1)} = 0, \\ \underline{\alpha}_{ij}^{(1)} &= -2\partial_u \underline{\chi}_{ij}^{(1)} = 0. \end{aligned}$$

The same equations to order 2 read

$$\begin{aligned} 2\underline{\beta}_i^{(2)} &= 4\partial_u \zeta_i^{(2)} = 0, \\ \underline{\alpha}_{ij}^{(2)} &= f^0 \underline{\chi}_{ij}^{(1)} - 2\partial_u \underline{\chi}_{ij}^{(2)} + \underline{\chi}_i^k \underline{\chi}_{kj}^{(1)} = 0, \end{aligned}$$

where we have use the values of the connection coefficients to order 1 and the u -independence of Proposition 2.1.4.

Finally, recall that we already knew $\alpha_{ij} = \mathcal{O}(s^{-4})$ and $\beta_i, \rho, \sigma = \mathcal{O}(s^{-3})$. Therefore $\mathbf{C}_{\rho\sigma\mu\nu} = \mathcal{O}(s^{-3})$. ■

Chapter 3

Unique continuation from infinity

In this chapter we complete the proof of our main result, Theorem 1.4.2. In the previous chapter we showed that all the asymptotic quantities depend on the radiating fields Ξ_{ij} and $\underline{A}(\mathbf{F})_i$ and some stationary data at $u = u_0$, Proposition 2.1.3. In particular, if Ξ_{ij} and $\underline{A}(\mathbf{F})_i$ vanish then all the asymptotic quantities are u -independent, Proposition 2.1.4; that is, $T := \partial_u$ is a Killing field to all orders at infinity.

The goal now is to show that T is a genuine symmetry of the spacetime; we do this in Proposition 3.3.1 below. We start now by presenting the necessary techniques to prove this result. The motivation comes from successful applications of the so-called Carleman estimates to prove uniqueness of solutions of hyperbolic equations with *pseudo-convex* boundary conditions (see Definition 3.1.1 below). For example, Ionescu-Klainerman used this approach to prove local unique extension of Killing vector fields across a zero-pseudo-convex hypersurface in [44]. Also, in [4], Alexakis-Ionescu-Klainerman showed uniqueness of smooth stationary black holes for small perturbations of Kerr by proving the unique continuation property for the Simon-Mars tensor. Here we present the main results when we consider boundary conditions “at infinity” for a certain class of asymptotically flat spacetimes. Alexakis-Schlue-Shao showed in [6] that linear waves satisfy the unique continuation from infinity property provided they decay faster than any polynomial, see Theorem 3.1.3. The main technical tool is the new Carleman estimates they derived in the context of asymptotically flat spacetimes, we include them here as Theorem 3.1.2.

Also, we need to obtain wave equations for $\mathcal{L}_T \mathbf{g}$ and $\mathcal{L}_T \mathbf{F}$. It turns out it

is more convenient to work with wave equations for (modified versions of) $\mathcal{L}_T C$ and $\mathcal{L}_T \mathbf{F}$ and a transport equation for $\mathcal{L}_T \mathbf{g}$. We do this in the second section of this chapter. We revise and extend to the non-vacuum case, the tensorial equations satisfied by the deformation tensors $\mathcal{L}_T \mathbf{g}$, $\mathcal{L}_T \mathbf{C}$ and $\mathcal{L}_T \mathbf{F}$ obtained by Ionescu and Klainerman in [44]. These are a consequence of the usual wave equations satisfied by \mathbf{C} and \mathbf{F} , which in turn are implied by the Bianchi and Maxwell equations.

Then, in the third section we use Cartesian coordinates to cast the equations in a suitable form (with “fast decaying coefficients”) such that the Carleman estimates from Theorem 3.1.2 can be applied. Finally, a standard argument is used to bound the weighted L^2 -norms of the deformation tensors and hence conclude its vanishing in a neighbourhood of infinity. Special care has to be taken regarding the wave equation for $\mathcal{L}_T \mathbf{F}$ whose coefficients do not decay fast enough. To deal with this problem, we “borrow” some decay from the coupling coefficient appearing in the wave equation for $\mathcal{L}_T \mathbf{C}$.

Finally we wrap up all the results obtained so far to prove Theorem 1.4.2 in Section 3.4.

3.1 Carleman estimates

In [5], Alexakis and Schlue proved Theorem 1.4.2 for a vacuum spacetime. In order to generalise the proof to include a Maxwell field we need to adapt their argument at the level of Carleman estimates. These are inequalities for functions decaying faster than any polynomial at infinity. In conjunction with a wave equation satisfied by the function, this method can be used to prove the vanishing of the function in a neighbourhood of infinity.

An important ingredient in the deduction of the Carleman estimates is a *pseudo-convex* function. Geometrically, in the Lorentzian context, these are functions whose level sets are convex with respect to null geodesics; that is, any null geodesic tangent to a level set locally remains on one side of that level set. This is equivalent to the following quantitative condition, [43],

Definition 3.1.1. *A function f on $(\mathcal{M}, \mathbf{g})$ is pseudo-convex if there exists a function h such that*

$$h\mathbf{g} - \nabla^2 f$$

is positive definite when restricted to the tangent space of the level sets of f . Similarly, an hypersurface is said to be pseudo-convex if it is the level set of a pseudo-convex function when $v < 0$ and $\underline{v} > 0$.

Examples.

- In order to understand the geometry of pseudo-convex time-like hypersurfaces we start by analysing them in Minkowski spacetime. Consider double null coordinates $(v, \underline{v}, \theta^2, \theta^3)$, where

$$v = \frac{t - r}{2}, \quad \underline{v} = \frac{t + r}{2}.$$

The pseudo-convex time-like hypersurfaces considered by Alexakis-Schlue-Shao [6] are given by the positive level sets of the function $f_\varepsilon = \frac{1}{(-u+\varepsilon)(v+\varepsilon)}$, $\varepsilon > 0$. The ε -perturbation is necessary to accomplish the pseudo-convexity condition in the absence of a mass. See Figure 3.1.

- On positive-mass spacetimes the situation is qualitatively different. Consider for example Schwarzschild spacetime in double null coordinates, $(v, \underline{v}, \theta^2, \theta^3)$, recall that these are defined by

$$v = \frac{t - r_*}{2}, \quad \underline{v} = \frac{t + r_*}{2},$$

where

$$r_*(r) = \int_{r_0}^r \left(1 - \frac{2M}{s}\right)^{-1} ds, \quad r_0 > 2M.$$

Due to the presence of a mass now the function $f = \frac{1}{(-v)\underline{v}}$ is pseudo-convex.

Recall the class of spacetimes considered in Section 1.3.4. That is, let $\mathcal{M} = (-\infty, 0) \times (0, \infty) \times \mathbb{S}^2$ be endowed with coordinates $(v, \underline{v}, y^2, y^3)$ such that the metric satisfy the following asymptotic conditions:

$$\begin{aligned} \mathbf{g} &= O'_1(\bar{r}^{-3})dv^2 - 4 \left(1 - \frac{2m}{\bar{r}}\right) dv d\underline{v} + O'_1(\bar{r}^{-3})d\underline{v}^2 \\ &+ \bar{r}^2 (\check{\gamma}_{ij} + O'_1(\bar{r}^{-1})) dy^i dy^j + \sum_{i=2}^3 O'_1(\bar{r}^{-1}) dy^i dv + O'_1(\bar{r}^{-1}) dy^i d\underline{v}, \end{aligned}$$

Alexakis, Schlue and Shao, [6], proved that the function

$$f := \frac{1}{(-v)\underline{v}},$$

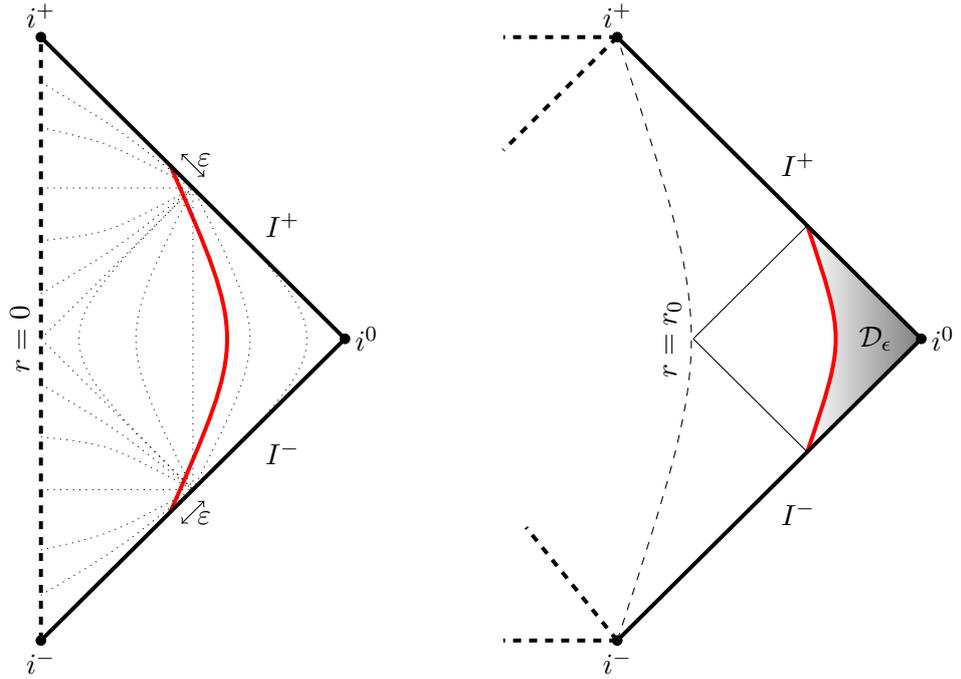


Figure 3.1: *Left.* Dotted lines represent the level sets of $\frac{1}{(-v)u}$ in Minkowski spacetime. One level set of $f_\epsilon = \frac{1}{(-v+\epsilon)(u+\epsilon)}$ is also shown (red line). In order to ensure the unique continuation property for a wave in Minkowski spacetime one has to prescribe initial data (to all orders) on more than half of null infinity, Theorem 2.3 in [6]. *Right.* One level set of $f = \frac{1}{(-v)u}$ is shown in Schwarzschild spacetime as well as the corresponding neighbourhood of infinity \mathcal{D}_ϵ (see 3.1). The pseudo-convex function depends now on a parameter $r_0 > 2M$; different choices of r_0 give rise to ‘parallel’ foliations. This behaviour is responsible for the localised result around spatial infinity for positive-mass spacetimes: Data required for unique continuation from infinity can be provided on small portions of null infinity, Theorem 2.5 in [6].

is pseudo-convex for this class of spacetimes. Therefore we expect uniqueness of solutions of the wave equation on an open set of the form

$$\{(v, \underline{v}, y^2, y^3) : \epsilon_1 < f(v, \underline{v}) < \epsilon\},$$

given boundary conditions on $\{(v, \underline{v}, y^2, y^3) : f(v, \underline{v}) = \epsilon_1\}$. One technicality arises in that this pseudo-convexity degenerates towards infinity, that is, when $\epsilon_1 \rightarrow 0$. To cope with this degeneracy, which takes the form of vanishing/blowing up weights towards infinity, Alexakis-Schlue-Shao rely on a reparametrisation of f ,

$$F(f) := \log f - f^{2\delta},$$

for some $\delta > 0$ to be chosen later. Then they are able to conclude the unique continuation from infinity property, Theorem 3.1.3 below, in a neighbourhood of the form

$$\mathcal{D}_\epsilon := \{(v, \underline{v}, y^2, y^3) : 0 < f(v, \underline{v}) < \epsilon\}. \quad (3.1)$$

We are now in position to state the main technical tools of this chapter. These are the Carleman-type estimates obtained by Alexakis, Schlue and Shao in [6] for spacetimes of positive mass in the sense of the expansion 1.48, with the corresponding asymptotic conditions (cf. Section 1.3.4). For convenience we introduce the weight function \mathcal{W} and associated weighted norms. For any $\lambda > 0$ and domain $\mathcal{D} = \mathcal{D}_\epsilon$, $\epsilon > 0$, we set

$$\mathcal{W} := e^{-\lambda F} f^{\frac{1}{2}}, \quad \|\cdot\|_{\mathcal{W}} := \|\mathcal{W} \cdot\|_2, \quad \|\phi\|_2 := \left(\int_{\mathcal{D}} |\phi|^2 d\mu_{\mathbf{g}} \right)^{\frac{1}{2}}.$$

Also, let

$$\Psi := \frac{m_{min} \log r}{r}.$$

Notation. Recall that we use the symbol $x \lesssim y$ to mean $x \leq cy$ for some positive constant c . Also, $x \simeq y$ means $x \lesssim y$ and $y \lesssim x$.

Theorem 3.1.2. (Carleman estimate near infinity for linear waves [5].) *Let $(\mathcal{M}, \mathbf{g})$ be an asymptotically flat spacetime with positive mass $m \geq m_{min} > 0$ in the sense of (1.48) and \mathcal{D}_ϵ a neighbourhood of infinity for some $\epsilon > 0$. Let $\delta > 0$ and let ϕ be a smooth function on \mathcal{D}_ϵ that vanishes to all orders at infinity in the sense that for*

each $N \in \mathbb{N}$ there exists an exhaustion¹ (\mathcal{U}_k) of \mathcal{D}_ϵ such that

$$\lim_{k \rightarrow \infty} \int_{\partial \mathcal{U}_k} \bar{r}^N (\phi^2 + |\partial \phi|^2) = 0. \quad (3.2)$$

Then, for $\epsilon > 0$ sufficiently small and $\lambda > 0$ sufficiently large,

$$\lambda^3 \|f^\delta \phi\|_{\mathcal{W}} + \lambda \|f^{-\frac{1}{2}} \Psi^{\frac{1}{2}} \nabla \phi\|_{\mathcal{W}} \lesssim \|f^{-1} \square \phi\|_{\mathcal{W}}. \quad (3.3)$$

Sketch of proof. Consider $\varphi = e^{-\lambda F(f)} \phi$. The idea is to obtain an energy estimate for φ , but here we wish for the bulk terms of the integral to be positive and for the boundary terms to vanish. That is, consider the modified energy current

$$J_\beta^w[\varphi] = Q_{\alpha\beta}[\varphi] \nabla^\alpha f + \left(\frac{1}{2} \partial_\beta w + \frac{1}{2} \lambda^2 (\nabla^\alpha f)(\nabla_\alpha f)(F')^2 \partial_\beta f \right) \varphi^2 - \frac{1}{2} w \partial_\beta (\varphi^2),$$

where $Q_{\alpha\beta}[\varphi] = (\nabla_\alpha \varphi)(\nabla_\beta \varphi) - \frac{1}{2} g_{\alpha\beta} (\nabla_\nu \varphi)(\nabla^\nu \varphi)$ is the standard energy-momentum tensor for $\square \varphi = 0$.

The function w is to be chosen appropriately so that $\text{div } J_\beta^w$ produces contractions with the tensor $hg - \nabla^2 f$, hence capturing the pseudo-convexity of f . Specifically, the choice

$$w = h - \frac{1}{2} \square f - \frac{1}{2},$$

produces positive bulk terms that are quadratic in $\partial_X \varphi$ for X tangent to the level sets of f ; in order to obtain positivity in the normal direction one relies on the choice of reparametrisation $F(f)$.

The above procedure ultimately results in an inequality of the form

$$\begin{aligned} \int_{\mathcal{D}_\epsilon} \mathcal{W}_L |\mathcal{P}\varphi|^2 &\geq C\lambda \int_{\mathcal{D}_\epsilon} \left(\mathcal{W}_N |\nabla_N \varphi|^2 + \mathcal{W}_T |\nabla_T \varphi|^2 + \mathcal{W}_T \sum |\nabla_{e_i} \varphi|^2 \right) \\ &\quad + C\lambda^3 \int_{\mathcal{D}_\epsilon} \mathcal{W}_0 |\varphi|^2 + \int_{\mathcal{D}_\epsilon} \mathcal{E}, \end{aligned}$$

where $\mathcal{P}(\varphi) = e^{-\lambda F(f)} \square (e^{\lambda F(f)} \varphi)$ and $\mathcal{W}_L, \mathcal{W}_N, \mathcal{W}_T, \mathcal{W}_0$ are positive weights. The only non-positive term is the error

$$\mathcal{E} = 2\lambda^2 (2F'h + F'' \nabla^\alpha f \nabla_\alpha f) \left(\nabla^\alpha f \nabla_\alpha \varphi - \left(w - \frac{1}{2} \right) \varphi \right) \varphi,$$

¹A nested family of subsets, with piece-wise smooth time-like boundaries, whose union is all of \mathcal{D}_ϵ .

which can be bounded appropriately and hence absorbed into the zero-order term, $\mathcal{W}_0|\varphi|^2$, of the above inequality. ■

Alexakis-Schlue-Shao used the above Carleman estimate to extend the infinite-order vanishing at infinity of a function ϕ into a neighbourhood of the spacetime. We state one of their main theorems here to illustrate the kind of result that we can get with this technique.

Theorem 3.1.3. (Unique continuation from infinity for linear waves, Theorem 2.5 in [6].) *Consider a metric \mathbf{g} on \mathcal{D}_ϵ of the form (1.48) satisfying all the bounds stated in Section 1.3.4. Let $P_{\mathbf{g}} = \square_{\mathbf{g}} + a^\alpha \partial_\alpha + V$ be a wave operator whose coefficients obey*

$$\begin{aligned} a^v &= O'(\underline{v}^{-1} \bar{r}^{-\frac{1}{2}}), & a^v &= O'((-v)^{-1} \bar{r}^{-\frac{1}{2}}), \\ a^i &= O'(f^{\frac{1}{2}} \bar{r}^{-\frac{3}{2}}), & V &= O'(f^{1+\eta}), \end{aligned}$$

for some $\eta > 0$. Let $\epsilon > 0$ and consider any \mathcal{C}^2 -solution ϕ on \mathcal{D}_ϵ of the equation $P_{\mathbf{g}}\phi = 0$ which in addition satisfies

$$\lim_{k \rightarrow \infty} \int_{\partial \mathcal{U}_k} \bar{r}^N (\phi^2 + |\partial \phi|^2) = 0.$$

for an exhaustion (\mathcal{U}_k) of \mathcal{D}_ϵ and all $N \in \mathbb{N}$. Then there exists $0 < \epsilon' < \epsilon$ so that $\phi = 0$ on $\mathcal{D}_{\epsilon'}$.

The proof is based on standard arguments based on the estimates of Theorem 3.1.2. The wave equation is used to substitute $\square \phi$ by lower order derivatives which can be absorbed on the left-hand side of (3.3) due to the decaying conditions. This allows to obtain weighted L^2 -bounds of ϕ and its first derivatives. Finally the result follows by taking λ to infinity. We omit a detailed proof since we will follow the same program in the proof of Proposition 3.3.1.

We will also be needing Carleman estimates for transport equations involving $\mathcal{L}_T \mathbf{g}$ and its first derivatives. These are covered by the following lemma, cf. Lemma 4.3 in [5].

Lemma 3.1.4. *Consider a metric \mathbf{g} on \mathcal{D}_ϵ of the form (1.48) satisfying all the bounds stated in Section 1.3.4 and a vector field*

$$L = \partial_{\underline{v}} + \sum_{\mu=0}^3 O'_2(\bar{r}^{-1}) \partial_\mu.$$

Let ϕ be a smooth function on \mathcal{D}_ϵ vanishing to all orders at infinity, in the sense that for any $N \in \mathbb{N}$ there is an exhaustion (\mathcal{U}_k) of \mathcal{D}_ϵ such that

$$\lim_{k \rightarrow \infty} \int_{\partial \mathcal{U}_k} r^N \phi^2 = 0.$$

Then for any $q \geq 1$ and $\lambda > 0$ sufficiently large,

$$\lambda \left\| \frac{1}{r} f^{-1} r^{-q} \phi \right\|_{\mathcal{W}} \lesssim \left\| f^{-1} \bar{r}^{-q} \nabla_L \phi \right\|_{\mathcal{W}}.$$

Remark. Consider a spacetime regular at spatial infinity and close to Kerr-Newman, in view of Proposition 1.3.11 we can change to approximately double-null coordinates. If the spacetime is also smooth at null infinity then $L := \partial_s$, satisfies $L = \partial_{\underline{v}} + \sum_{\mu=0}^3 \mathcal{O}'_2(\bar{r}^{-1}) \partial_\mu$. This can be seen from the compatibility assumption $\partial_u = \partial_t + \partial_r + \sum_{\mu=0}^3 \mathcal{O}(s^{-1}) \partial_\mu$ and from the estimates

$$\lim_{s \rightarrow \infty} \frac{r}{s} = 1, \quad |r - \bar{r}| < \frac{c}{\bar{r}}.$$

3.2 Ionescu-Klainerman tensorial equations

In this section we recall the equations satisfied by the deformation tensors deduced by Ionescu and Klainerman in [44]. We remind the reader that their work assumes a vacuum spacetime, that is, the Riemann and Weyl tensors are equal, $\mathbf{R}_{abcd} = \mathbf{C}_{abcd}$. This is in contrast with our current approach where the Riemann tensor is coupled to the Faraday tensor via the Einstein's equations. Hence we need to be careful now to distinguish between \mathbf{R} and \mathbf{C} .

Notation. Through out this chapter we will denote schematically by

$$U \cdot V$$

any linear combination of the product of two tensors U and V , and contractions thereof. For example the relation

$$\mathbf{R}_{\alpha\beta\mu\nu} = \mathbf{C}_{\alpha\beta\mu\nu} + \frac{1}{2} (\mathbf{g}_{\alpha\mu} \mathbf{S}_{\beta\nu} - \mathbf{g}_{\beta\mu} \mathbf{S}_{\alpha\nu} + \mathbf{g}_{\beta\nu} \mathbf{S}_{\alpha\mu} - \mathbf{g}_{\alpha\nu} \mathbf{S}_{\beta\mu}),$$

will be abbreviated to

$$\mathbf{R} = \mathbf{C} + \mathbf{g} \cdot \mathbf{S}.$$

Now we proceed to compute the Ionescu-Klainerman tensorial wave equa-

tions for $\mathcal{L}_T \mathbf{C}$ and $\mathcal{L}_T \mathbf{F}$. The idea is to use the wave equations satisfied by \mathbf{C} and \mathbf{F} and then commute the covariant and Lie derivatives with the help of Lemma 3.2.2. The relevant equations are naturally coupled to $\pi_{\alpha\beta} := \mathcal{L}_T \mathbf{g}_{\alpha\beta}$ and $\nabla_\alpha \pi_{\beta\mu}$, hence the necessity of finding (transport) equations for them. In order to get a closed system of equations we need to perform some algebraic tricks (related to the symmetries of the Weyl tensor) and work instead with auxiliary variables, $\mathbf{B}_{\alpha\beta}$ and $\mathbf{P}_{\alpha\beta\mu}$; see Proposition 3.2.3. This section is entirely based on [44].

We begin by noticing that $\mathcal{L}_T \mathbf{C}$ is not trace-less. Indeed,

$$\mathbf{g}^{\alpha\mu} \mathcal{L}_T \mathbf{C}_{\alpha\beta\mu\nu} = (-\mathcal{L}_T \mathbf{g}^{\alpha\mu}) \mathbf{C}_{\alpha\beta\mu\nu} = \pi^{\alpha\mu} \mathbf{C}_{\alpha\beta\mu\nu}.$$

To remedy this we define the modified Lie derivative

$$\hat{\mathcal{L}}_T \mathbf{C} := \mathcal{L}_T \mathbf{C} - \mathbf{B} \odot \mathbf{C}, \quad (3.4)$$

where \mathbf{B} is a 2-covariant tensor and $(\mathbf{B} \odot V)_{\alpha_1 \dots \alpha_n} := \sum_{i=1}^n \mathbf{B}_{\alpha_i}{}^\delta V_{\alpha_1 \dots \delta \dots \alpha_n}$ for any n -covariant tensor $V_{\alpha_1 \dots \alpha_n}$.

If we take

$$\mathbf{B} = \frac{1}{2}(\pi + \varpi),$$

with ϖ any antisymmetric 2-form, a simple calculation leads to

Lemma 3.2.1. *The modified Lie derivative of the Weyl tensor, $\mathbf{W} := \hat{\mathcal{L}}_T \mathbf{C}$, is a Weyl field, i.e.,*

- i) $\mathbf{W}_{\alpha\beta\mu\nu} = -\mathbf{W}_{\beta\alpha\mu\nu} = -\mathbf{W}_{\alpha\beta\nu\mu} = \mathbf{W}_{\mu\nu\alpha\beta}$.
- ii) $\mathbf{W}_{\alpha\beta\mu\nu} + \mathbf{W}_{\alpha\mu\nu\beta} + \mathbf{W}_{\alpha\nu\beta\mu} = 0$.
- iii) $\mathbf{g}^{\alpha\mu} \mathbf{W}_{\alpha\beta\mu\nu} = 0$.

Next, we need to compute the commutator of the Lie and covariant derivatives. This is given by the following:

Lemma 3.2.2. *For an arbitrary k -covariant tensor V and vector field T we have,*

$$\nabla_\beta \mathcal{L}_T V_{\alpha_1 \dots \alpha_k} - \mathcal{L}_T \nabla_\beta V_{\alpha_1 \dots \alpha_k} = \sum_{j=1}^k \Pi_{\alpha_j \beta \rho} V_{\alpha_1 \dots \rho \dots \alpha_k}, \quad (3.5)$$

$$\text{with } \Pi_{\alpha\beta\mu} := \frac{1}{2}(\nabla_\alpha \pi_{\beta\mu} + \nabla_\beta \pi_{\mu\alpha} - \nabla_\mu \pi_{\alpha\beta}). \quad (3.6)$$

Schematically we write, $[\nabla_\beta, \mathcal{L}_T]V = \Pi_\beta \odot V$.

Proof. We compute

$$\begin{aligned}\mathcal{L}_T V_{\alpha_1 \dots \alpha_n} &= T^\mu \nabla_\mu V_{\alpha_1 \dots \alpha_n} + \sum_i (\nabla_{\alpha_i} T^\mu) V_{\alpha_1 \dots \mu \dots \alpha_n}, \\ \nabla_\beta \mathcal{L}_T V_{\alpha_1 \dots \alpha_n} &= (\nabla_\beta T^\mu) \nabla_\mu V_{\alpha_1 \dots \alpha_n} + T^\mu \nabla_\beta \nabla_\mu V_{\alpha_1 \dots \alpha_n} \\ &\quad + \sum_i (\nabla_\beta \nabla_{\alpha_i} T^\mu) V_{\alpha_1 \dots \mu \dots \alpha_n} + (\nabla_{\alpha_i} T^\mu) \nabla_\beta V_{\alpha_1 \dots \mu \dots \alpha_n}, \\ \mathcal{L}_T \nabla_\beta V_{\alpha_1 \dots \alpha_n} &= T^\mu \nabla_\mu \nabla_\beta V_{\alpha_1 \dots \alpha_n} + (\nabla_\beta T^\mu) \nabla_\mu V_{\alpha_1 \dots \alpha_n} + \sum_i (\nabla_{\alpha_i} T^\mu) \nabla_\beta V_{\alpha_1 \dots \mu \dots \alpha_n}.\end{aligned}$$

Then

$$[\mathcal{L}_T, \nabla_\beta] V_{\alpha_1 \dots \alpha_n} = \sum_i (T^\mu \mathbf{R}_{\mu\beta\alpha_i}{}^\rho V_{\alpha_1 \dots \rho \dots \alpha_n} + (\nabla_\beta \nabla_{\alpha_i} T^\mu) V_{\alpha_1 \dots \mu \dots \alpha_n}).$$

And the results follows from the identity

$$\nabla_\beta \nabla_\alpha T_\mu = \mathbf{R}_{\mu\alpha\beta\nu} T^\nu + \Pi_{\alpha\beta\mu}.$$

To prove this last equality we just evaluate, commute derivatives and make use of the 1st Bianchi identity:

$$\begin{aligned}\mathbf{R}_{\mu\alpha\beta\nu} T^\nu + \Pi_{\alpha\beta\mu} &= \mathbf{R}_{\mu\alpha\beta\nu} T^\nu + \frac{1}{2} (\nabla_\alpha (\nabla_\beta T_\mu + \nabla_\mu T_\beta) + \nabla_\beta (\nabla_\alpha T_\mu + \nabla_\mu T_\alpha) \\ &\quad - \nabla_\mu (\nabla_\alpha T_\beta + \nabla_\beta T_\alpha)), \\ &= \mathbf{R}_{\mu\alpha\beta\nu} T^\nu + \frac{1}{2} (\mathbf{R}_{\alpha\mu\beta\nu} T^\nu + \mathbf{R}_{\beta\mu\alpha\nu} T^\nu + 2\nabla_\alpha \nabla_\beta T_\mu - \mathbf{R}_{\beta\alpha\mu\nu} T^\nu), \\ &= \mathbf{R}_{\mu\alpha\beta\nu} T^\nu + \frac{1}{2} (2R_{\alpha\mu\beta\nu} T^\nu + 2\nabla_\alpha \nabla_\beta T_\mu), \\ &= \nabla_\alpha \nabla_\beta T_\mu. \quad \blacksquare\end{aligned}$$

Now we present the variables and equations which play the crucial role in the unique continuation analysis. The variables are minor modifications of π , $\nabla\pi$, $\mathcal{L}_T \mathbf{C}$ and $\mathcal{L}_t \mathbf{F}$ which give us a closed system of equations.

Proposition 3.2.3. *Consider $\pi_{\alpha\beta} := \mathcal{L}_T \mathbf{g}_{\alpha\beta}$ and L^ν a future directed null vector field with affine parameter s . Assume $[T, L] = 0$, $L^\mu \nabla_\mu L_\alpha = 0$ and $\lim_{s \rightarrow \infty} \pi_{\alpha\beta} = 0$ along L -lines. Let $\varpi_{\alpha\beta}$ a 2-form solution of the transport equation*

$$\nabla_L \varpi_{\alpha\beta} = \pi_{\alpha\mu} \nabla_\beta L^\mu - \pi_{\beta\mu} \nabla_\alpha L^\mu, \quad (3.7)$$

with initial condition $\lim_{s \rightarrow \infty} \varpi_{\alpha\beta} = 0$ along L -lines. Define the tensors \mathbf{B} , \mathbf{P} and \mathbf{W} as follows,

$$\mathbf{B}_{\alpha\beta} := \frac{1}{2}(\pi_{\alpha\beta} + \varpi_{\alpha\beta}), \quad (3.8)$$

$$\mathbf{P}_{\alpha\beta\mu} := \frac{1}{2}(\nabla_{\alpha}\pi_{\beta\mu} - \nabla_{\beta}\pi_{\alpha\mu} - \nabla_{\mu}\varpi_{\alpha\beta}) = \Pi_{\alpha\mu\beta} - \nabla_{\mu}\mathbf{B}_{\alpha\beta}, \quad (3.9)$$

$$\mathbf{W}_{\alpha\beta\mu\nu} := \mathcal{L}_T \mathbf{C}_{\alpha\beta\mu\nu} - (\mathbf{B} \odot \mathbf{C})_{\alpha\beta\mu\nu}, \quad (3.10)$$

Then the following equations hold

$$\nabla_L \mathbf{B}_{\alpha\beta} = L^{\mu} \mathbf{P}_{\mu\beta\alpha} - \mathbf{B}_{\mu\beta} \nabla_{\alpha} L^{\mu}, \quad (3.11)$$

$$\nabla_L \mathbf{P}_{\alpha\beta\mu} = L^{\nu} (\mathcal{L}_T \mathbf{R}_{\alpha\beta\mu\nu} - \mathbf{B}_{\alpha}^{\delta} \mathbf{R}_{\delta\beta\mu\nu} - \mathbf{B}_{\beta}^{\delta} \mathbf{R}_{\alpha\delta\mu\nu}) + \mathbf{P}_{\alpha\beta\nu} \nabla_{\mu} L^{\nu}, \quad (3.12)$$

$$\square \mathbf{W} = \mathcal{L}_T (\square \mathbf{C}) + \nabla \mathbf{P} \cdot \mathbf{C} + \mathbf{B} \cdot \nabla \mathbf{C} + \nabla \mathbf{B} \cdot \nabla \mathbf{C} + \mathbf{B} \cdot \square \mathbf{C}. \quad (3.13)$$

During the proof we will make use of the following identities:

Lemma 3.2.4. *Under the same hypothesis of Proposition 3.2.3, the following hold*

$$L^{\beta} \pi_{\alpha\beta} = 0, \quad \mathbf{P}_{\alpha\beta\mu} L^{\mu} = 0, \quad L^{\beta} \varpi_{\alpha\beta} = 0.$$

Proof of Lemma. We start by showing $L^{\alpha} L^{\beta} \pi_{\alpha\beta} = 0$. Indeed,

$$\begin{aligned} L^{\alpha} L^{\beta} \pi_{\alpha\beta} &= L^{\alpha} L^{\beta} (\nabla_{\alpha} T_{\beta} + \nabla_{\beta} T_{\alpha}) = L^{\beta} T^{\alpha} \nabla_{\alpha} L_{\beta} + L^{\alpha} T^{\beta} \nabla_{\beta} L_{\alpha} \\ &= T^{\alpha} \nabla_{\alpha} (L_{\beta} L^{\beta}) = 0. \end{aligned}$$

Now, by commuting derivatives we can find a transport equation for $L^{\beta} \pi_{\alpha\beta}$.

$$\begin{aligned} L^{\mu} \nabla_{\mu} (L^{\beta} \pi_{\alpha\beta}) &= L^{\mu} \nabla_{\mu} (L^{\beta} (\nabla_{\alpha} T_{\beta} + \nabla_{\beta} T_{\alpha})), \\ &= L^{\mu} L^{\beta} (\nabla_{\alpha} \nabla_{\mu} T_{\beta} + \mathbf{R}_{\mu\alpha\beta}^{\delta} T_{\delta}) + L^{\mu} \nabla_{\mu} (T^{\beta} \nabla_{\beta} L_{\alpha}), \\ &= \nabla_{\alpha} (L^{\mu} L^{\beta}) \nabla_{\mu} T_{\beta} + \mathbf{R}_{\mu\alpha\beta}^{\delta} L^{\mu} L^{\beta} T_{\delta} + \mathbf{R}_{\mu\beta\alpha}^{\delta} L^{\mu} T^{\beta} L_{\delta} + T^{\mu} \nabla_{\mu} (L^{\beta} \nabla_{\beta} L_{\alpha}), \\ &= (\nabla_{\alpha} L^{\mu}) (L^{\beta} \pi_{\mu\beta}), \end{aligned}$$

which is a homogeneous equation for $L^{\beta} \pi_{\alpha\beta}$. In our context, the choice of T implies that the deformation tensor π vanishes to first order at infinity. In particular $L^{\beta} \pi_{\alpha\beta}$ also vanishes to first order at infinity; this together with the above transport equation implies that $L^{\beta} \pi_{\alpha\beta} \equiv 0$ as desired.

Next, a straightforward computation yields,

$$\begin{aligned} 2\mathbf{P}_{\alpha\beta\mu}L^\mu &= L^\mu(\nabla_\alpha\pi_{\beta\mu} - \nabla_\beta\pi_{\alpha\mu} - \nabla_\mu\varpi_{\alpha\beta}), \\ &= -\pi_{\beta\mu}\nabla_\alpha L^\mu + \pi_{\mu\alpha}\nabla_\beta L^\mu - \pi_{\alpha\mu}\nabla_\beta L^\mu + \pi_{\beta\mu}\nabla_\alpha L^\mu = 0, \end{aligned}$$

where we have used $L^\mu\pi_{\alpha\mu} = 0$ and the definition of $\varpi_{\alpha\beta}$.

Finally, for the last equality we deduce a transport equation for $L^\beta\varpi_{\alpha\beta}$ using its definition,

$$\begin{aligned} L^\mu\nabla_\mu(L^\beta\varpi_{\alpha\beta}) &= L^\beta L^\mu\nabla_\mu\varpi_{\alpha\beta}, \\ &= L^\beta(\pi_{\alpha\mu}\nabla_b L^\mu - \pi_{\beta\mu}\nabla_\alpha L^\mu) = 0, \end{aligned}$$

Hence $L^\beta\varpi_{\alpha\beta}$ is constant and has to vanish given the initial conditions for $\varpi_{\alpha\beta}$. This finishes the proof of the Lemma. ■

Proof of Proposition 3.2.3. For the transport equation for $\mathbf{B}_{\alpha\beta}$ we compute:

$$\begin{aligned} 2(L^\mu\mathbf{P}_{\mu\beta\alpha} - \mathbf{B}_{\mu\beta}\nabla_\alpha L^\mu) &= L^\mu(\nabla_\mu\pi_{\beta\alpha} - \nabla_\beta\pi_{\alpha\mu} - \nabla_\alpha\varpi_{\mu\beta}) - (\pi_{\mu\beta} + \varpi_{\mu\beta})\nabla_\alpha L^\mu \\ &= \nabla_L\pi_{\alpha\beta} + \pi_{\alpha\mu}\nabla_\beta L^\mu + \varpi_{\mu\beta}\nabla_\alpha L^\mu - (\pi_{\mu\beta} + \varpi_{\mu\beta})\nabla_\alpha L^\mu \\ &= 2\nabla_L\mathbf{B}_{\alpha\beta}, \end{aligned}$$

where we have used $L^\mu\pi_{\mu\beta} = L^\mu\varpi_{\mu\beta} = 0$ and the transport equation defining $\varpi_{\alpha\beta}$.

Next we deduce the transport equation for $\mathbf{P}_{\alpha\beta\mu}$. Recall the following identity² proved in [44] for $\tilde{\mathbf{P}}_{\alpha\beta\mu} := \frac{1}{2}(\nabla_\alpha\pi_{\beta\mu} - \nabla_\beta\pi_{\alpha\mu})$,

$$\nabla_\nu\tilde{\mathbf{P}}_{\alpha\beta\mu} - \nabla_\mu\tilde{\mathbf{P}}_{\alpha\beta\nu} = \mathcal{L}_T\mathbf{R}_{\alpha\beta\mu\nu} - \frac{1}{2}\pi_\alpha^\rho\mathbf{R}_{\rho\beta\mu\nu} - \frac{1}{2}\pi_\beta^\rho\mathbf{R}_{\alpha\rho\mu\nu}.$$

²Which is basically an antisymmetrised identity for $\nabla_\nu\nabla_\alpha(\nabla_{(\beta}T_{\mu)})$ necessary to cope with the symmetries of $\mathbf{R}_{\alpha\beta\mu\nu}$. The idea to prove it is to commute derivatives.

Note that $\mathbf{P}_{\alpha\beta\mu} = \tilde{\mathbf{P}}_{\alpha\beta\mu} - \frac{1}{2}\nabla_\mu\varpi_{\alpha\beta}$, hence

$$\begin{aligned} L^\nu(\nabla_\nu\mathbf{P}_{\alpha\beta\mu} - \nabla_\mu\mathbf{P}_{\alpha\beta\nu}) &= L^\nu\left(\mathcal{L}_T\mathbf{R}_{\alpha\beta\mu\nu} - \frac{1}{2}\pi_\alpha{}^\rho\mathbf{R}_{\rho\beta\mu\nu} - \frac{1}{2}\pi_\beta{}^\rho\mathbf{R}_{\alpha\rho\mu\nu}\right) \\ &\quad + \frac{1}{2}L^\nu(\nabla_\mu\nabla_\nu\varpi_{\alpha\beta} - \nabla_\nu\nabla_\mu\varpi_{\alpha\beta}), \\ &= L^\nu\left(\mathcal{L}_T\mathbf{R}_{\alpha\beta\mu\nu} - \frac{1}{2}\pi_\alpha{}^\rho\mathbf{R}_{\rho\beta\mu\nu} - \frac{1}{2}\pi_\beta{}^\rho\mathbf{R}_{\alpha\rho\mu\nu}\right) \\ &\quad - \frac{1}{2}L^\nu\left(\varpi_\alpha{}^\rho\mathbf{R}_{\rho\beta\mu\nu} - \frac{1}{2}\varpi_\beta{}^\rho\mathbf{R}_{\alpha\rho\mu\nu}\right), \\ &= L^\nu(\mathcal{L}_T\mathbf{R}_{\alpha\beta\mu\nu} - \mathbf{B}_\alpha{}^\rho\mathbf{R}_{\rho\beta\mu\nu} - \mathbf{B}_\beta{}^\rho\mathbf{R}_{\alpha\rho\mu\nu}), \end{aligned}$$

the final result follows by noticing that $L^\nu\nabla_\mu\mathbf{P}_{\alpha\beta\nu} = -\mathbf{P}_{\alpha\beta\nu}\nabla_\mu L^\nu$ since $L^\nu\mathbf{P}_{\alpha\beta\nu} = 0$.

Finally, we deduce the wave equation for \mathbf{W} by commuting Lie and covariant derivatives. We are interested only in the general structure of the equations, in particular, in the coefficients accompanying \mathbf{W} and $\nabla\mathbf{W}$. Hence we do not any longer keep track of the different contractions but just on the bilinear structure of products and the different terms involving our variables \mathbf{P} , \mathbf{B} and \mathbf{W} .

Notation. During the following computations we will substitute freely \mathbf{B} instead of π since $\pi_{\alpha\beta} = \mathbf{B}_{\alpha\beta} + \mathbf{B}_{\beta\alpha}$.

We start now with a divergence equation for \mathbf{W} .

Lemma 3.2.5. *The following holds:*

$$\nabla^\alpha\mathbf{W}_{\alpha\beta\mu\nu} = \mathbf{B}^{\alpha\delta}\nabla_\delta\mathbf{C}_{\alpha\beta\mu\nu} + \mathbf{g}^{\alpha\rho}(\mathbf{P}_\rho{}^\delta{}_\alpha\mathbf{C}_{\delta\beta\mu\nu} + \mathbf{P}_\beta{}^\delta{}_\rho\mathbf{C}_{\alpha\delta\mu\nu} + \mathbf{P}_\mu{}^\delta{}_\rho\mathbf{C}_{\alpha\beta\delta\nu} + \mathbf{P}_\nu{}^\delta{}_\rho\mathbf{C}_{\alpha\beta\mu\delta}).$$

Proof. We will prove the schematic version:

$$\operatorname{div}\mathbf{W} = \mathbf{B} \cdot \nabla\mathbf{C} + \mathbf{P} \cdot \mathbf{C}.$$

We have that

$$\nabla_\delta\mathbf{W} = \mathcal{L}_T(\nabla_\delta\mathbf{C}) + \Pi_\delta \odot \mathbf{C} - \nabla_\delta(\mathbf{B} \odot \mathbf{C}),$$

hence,

$$\begin{aligned}\operatorname{div} \mathbf{W} &= (\mathcal{L}_T \mathbf{g}) \cdot \nabla \mathbf{C} + \mathcal{L}_T \operatorname{div} \mathbf{C} + (\Pi_\delta - \nabla_\delta \mathbf{B}) \cdot \mathbf{C} - \mathbf{B} \cdot \nabla \mathbf{C}, \\ &= \mathbf{B} \cdot \nabla \mathbf{C} + \mathbf{P} \cdot \mathbf{C},\end{aligned}$$

where we used $\pi_{\alpha\beta} = \mathbf{B}_{\alpha\beta} + \mathbf{B}_{\beta\alpha}$, $\Pi_{\alpha\delta\beta} - \nabla_\delta \mathbf{B}_{\alpha\beta} = \mathbf{P}_{\alpha\beta\delta}$ and $\operatorname{div} \mathbf{C} = 0$. ■

We proceed similarly to obtain the wave equation for \mathbf{W} ,

$$\begin{aligned}\square \mathbf{W} &= \nabla^\delta (\mathcal{L}_T \nabla_\delta \mathbf{C} + \Pi_\delta \odot \mathbf{C} - \nabla_\delta (\mathbf{B} \odot \mathbf{C})), \\ &= \mathcal{L}_T \square \mathbf{C} + \Pi^\delta \cdot \nabla_\delta \mathbf{C} + \nabla^\delta (\Pi_\delta \odot \mathbf{C} - \nabla_\delta (\mathbf{B} \odot \mathbf{C})), \\ &= \mathcal{L}_T \square \mathbf{C} + \nabla \mathbf{B} \cdot \nabla \mathbf{C} + \nabla^\delta (\Pi_\delta - \nabla_\delta \mathbf{B}) \odot \mathbf{C} + \mathbf{B} \cdot \square \mathbf{C}, \\ &= \mathcal{L}_T \square \mathbf{C} + \nabla \mathbf{B} \cdot \nabla \mathbf{C} + \nabla \mathbf{P} \cdot \mathbf{C} + \mathbf{B} \cdot \square \mathbf{C},\end{aligned}$$

where we have used once more $\Pi_{\alpha\delta\beta} - \nabla_\delta \mathbf{B}_{\alpha\beta} = \mathbf{P}_{\alpha\beta\delta}$ and $\Pi = \nabla \pi = \nabla \mathbf{B}$ (schematically). This finishes the proof of Proposition 3.2.3. ■

Using the same argument as above we can prove the following general statement:

Lemma 3.2.6. *Let \mathbf{F} be a k -covariant tensor, then $\mathbf{E} := \mathcal{L}_T \mathbf{F} - \mathbf{B} \odot \mathbf{F}$ obeys*

$$\square \mathbf{E} = \mathcal{L}_T \square \mathbf{F} + \nabla \mathbf{B} \cdot \nabla \mathbf{F} + \nabla \mathbf{P} \cdot \mathbf{F} + \mathbf{B} \cdot \square \mathbf{F}. \quad (3.14)$$

Proof. We compute

$$\begin{aligned}\square \mathbf{E} &= \nabla^\delta (\mathcal{L}_T \nabla_\delta \mathbf{F} + \Pi_\delta \odot \mathbf{F} - \nabla_\delta (\mathbf{B} \odot \mathbf{F})), \\ &= \mathcal{L}_T \square \mathbf{F} + \Pi^\delta \cdot \nabla_\delta \mathbf{F} + \nabla^\delta (\Pi_\delta \odot \mathbf{F} - \nabla_\delta (\mathbf{B} \odot \mathbf{F})), \\ &= \mathcal{L}_T \square \mathbf{F} + \nabla \mathbf{B} \cdot \nabla \mathbf{F} + \nabla^\delta (\Pi_\delta - \nabla_\delta \mathbf{B}) \odot \mathbf{F} + \mathbf{B} \cdot \square \mathbf{F}, \\ &= \mathcal{L}_T \square \mathbf{F} + \nabla \mathbf{B} \cdot \nabla \mathbf{F} + \nabla \mathbf{P} \cdot \mathbf{F} + \mathbf{B} \cdot \square \mathbf{F}, \quad \blacksquare\end{aligned}$$

This last lemma can be applied to the Faraday tensor. To conclude this section we state the full system of equations relevant for the Carleman estimates. We start by recalling the wave equations satisfied by the Weyl and Faraday tensors,

as well as the Einstein's equations in schematic form:

$$\square \mathbf{C} = \mathbf{R} \cdot \mathbf{C} = (\mathbf{C} + \mathbf{g} \cdot \mathbf{S}) \cdot \mathbf{C}, \quad (3.15)$$

$$\square \mathbf{F} = \mathbf{R} \cdot \mathbf{F} = (\mathbf{C} + \mathbf{g} \cdot \mathbf{S}) \cdot \mathbf{F}, \quad (3.16)$$

$$\mathbf{S} = \mathbf{F} \cdot \mathbf{F} \quad (3.17)$$

Hence

$$\begin{aligned} \mathcal{L}_T \square \mathbf{C} &= (\mathcal{L}_T \mathbf{C} + \pi \cdot \mathbf{F}^2 + \mathcal{L}_T \mathbf{F} \cdot \mathbf{F}) \cdot \mathbf{C} + \mathbf{R} \cdot \mathcal{L}_T \mathbf{C}, \\ &= (\mathbf{W} + \mathbf{B} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{F}^2 + \mathbf{E} \cdot \mathbf{F} + \mathbf{B} \cdot \mathbf{F}^2) \cdot \mathbf{C} + \mathbf{R} \cdot \mathbf{W} + \mathbf{R} \cdot \mathbf{B} \cdot \mathbf{W}. \end{aligned}$$

Similarly, for the modified Lie derivative of the Faraday tensor we obtain

$$\mathcal{L}_T \square \mathbf{E} = (\mathbf{W} + \mathbf{B} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{F}^2 + \mathbf{E} \cdot \mathbf{F} + \mathbf{B} \cdot \mathbf{F}^2) \cdot \mathbf{F} + \mathbf{R} \cdot \mathbf{E} + \mathbf{R} \cdot \mathbf{B} \cdot \mathbf{F}.$$

Therefore, we have proved:

Lemma 3.2.7. *The deformation tensors $\mathbf{W} = \mathcal{L}_T \mathbf{C} - \mathbf{B} \odot \mathbf{C}$ and $\mathbf{E} = \mathcal{L}_T \mathbf{F} - \mathbf{B} \odot \mathbf{F}$ satisfy the following wave equations*

$$\begin{aligned} \square \mathbf{W} &= (\mathbf{R} + \mathbf{C}) \cdot \mathbf{W} + (\mathbf{C}^2 + \mathbf{F}^2 \cdot \mathbf{C} + \mathbf{R} \cdot \mathbf{C}) \cdot \mathbf{B} \\ &\quad + (\mathbf{F} \cdot \mathbf{C}) \cdot \mathbf{E} + \nabla \mathbf{C} \cdot \nabla \mathbf{B} + \mathbf{C} \cdot \nabla \mathbf{P}, \end{aligned} \quad (3.18)$$

$$\begin{aligned} \square \mathbf{E} &= \mathbf{F} \cdot \mathbf{W} + (\mathbf{F}^3 + \mathbf{F} \cdot \mathbf{C} + \mathbf{R} \cdot \mathbf{F}) \cdot \mathbf{B} \\ &\quad + (\mathbf{F}^2 + \mathbf{R}) \cdot \mathbf{E} + \nabla \mathbf{F} \cdot \nabla \mathbf{B} + \mathbf{F} \cdot \nabla \mathbf{P}. \end{aligned} \quad (3.19)$$

Remark. These wave equations together with the transport equations for \mathbf{B} and \mathbf{P} ,

$$\begin{aligned} \nabla_L \mathbf{B} &= L \cdot \mathbf{P} - \mathbf{B} \cdot \nabla L, \\ \nabla_L \mathbf{P} &= L \cdot (\mathcal{L}_T \mathbf{R} - \mathbf{B} \cdot \mathbf{R}) + \mathbf{P} \cdot \nabla L, \end{aligned}$$

form the system of equations for the components of \mathbf{W} , \mathbf{E} , \mathbf{P} and \mathbf{B} for which the Carleman estimates imply uniqueness of solutions given infinite-order data at infinity. We also remark that the modified versions of $\mathcal{L}_T \mathbf{C}$ and $\mathcal{L}_T \mathbf{F}$ are used to obtain a closed system of equations. Indeed, we do not have a nice L -transport equation for $\nabla_\mu \pi_{\alpha\beta}$ naturally coupled to \mathbf{R} , that is why we need to consider \mathbf{P} . Then the correction terms $\mathbf{B} \odot \mathbf{C}$ and $\mathbf{B} \odot \mathbf{F}$ have the effect that the wave equations for \mathbf{W} and \mathbf{E} are coupled to \mathbf{P} instead of \mathbf{R} .

3.3 Stationarity in a neighbourhood of infinity

Now we proceed to the last part of the proof of Theorem 1.4.2. Namely, we show that a vector field T is a symmetry of the gravitational and electromagnetic fields provided it is a symmetry to all orders at infinity. More precisely we have the following:

Proposition 3.3.1. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be an electrovacuum spacetime admitting asymptotically double-null coordinates $(v, \underline{v}, y^2, y^3)$ such that \mathbf{g} takes the form (1.48). Assume that, with respect to the frame*

$$\{\tilde{e}_0 = \partial_v, \tilde{e}_1 = \partial_{\underline{v}}, \tilde{e}_2 = \frac{1}{\bar{r}}\partial_{y^2}, \tilde{e}_3 = \frac{1}{\bar{r}}\partial_{y^3}\},$$

the Weyl and Faraday components satisfy:

$$|\mathbf{C}_{\alpha\beta\mu\nu}| = O'_1(\bar{r}^{-3}), \quad |\mathbf{F}_{\alpha\beta}| = O'_1(\bar{r}^{-2}). \quad (3.20)$$

Consider L and T vector fields with $L = \partial_{\underline{v}} + \sum_{\mu=0}^3 O'_2(\bar{r}^{-1})\tilde{e}_\mu$ null and satisfying $\nabla_L L = 0$, $[L, T] = 0$. Assume furthermore that T is a symmetry to all orders at infinity in the sense that,

$$\lim_{k \rightarrow \infty} \int_{\Omega_k} \bar{r}^N |\phi|^2 = 0, \quad (3.21)$$

for all the components, $\phi = \mathbf{W}_{\alpha\beta\mu\nu}$, $\nabla_\rho \mathbf{W}_{\alpha\beta\mu\nu}$, $\mathbf{E}_{\alpha\beta}$, $\nabla_\rho \mathbf{E}_{\alpha\beta}$, $\mathbf{B}_{\alpha\beta}$, $\nabla_\rho \mathbf{B}_{\alpha\beta}$, $\mathbf{P}_{\alpha\beta\mu}$, $\nabla_\rho \mathbf{P}_{\alpha\beta\mu}$, with respect to the frame $\{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$. And where

$$\Omega_k := \{(v, \underline{v}, y^2, y^3) : \underline{v} - v = k\} \cap \mathcal{D}_\epsilon, \quad \epsilon > 0.$$

Then T is in fact locally a genuine symmetry for $(\mathcal{M}, \mathbf{g}, \mathbf{F})$, namely

$$\mathcal{L}_T \mathbf{g} \equiv 0, \quad \mathcal{L}_T \mathbf{C} \equiv 0, \quad \mathcal{L}_T \mathbf{F} \equiv 0, \quad \text{on } \mathcal{D}_{\epsilon'}$$

for some $0 < \epsilon' < \epsilon$.

Remark. Note that conditions (3.20) roughly corresponds to the fast-decaying conclusion of Lemma 2.3.2 when the radiation fields at future null infinity vanish. Also, conditions (3.21) corresponds to those deduced from Proposition 2.1.4. We still have to deal with two issues to make these claims formal: i) We are using two different frames to compute components, ii) We are using different coordinates and asymptotic conditions. We deal with these problems in the next section, 3.4.

The strategy to prove Proposition 3.3.1 is analogous to the one used by Ionescu-Klainerman in [44] and Alexakis-Schlue in [5]. We use the wave equations for $\mathcal{L}_T \mathbf{C}$ and $\mathcal{L}_T \mathbf{F}$. When written with respect to the frame $\{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ those equations fit into the framework provided by the Carleman estimates of Theorem 3.1.2. Once we get those differential estimates a standard argument to deduce unique continuation follows through.

We emphasise here that the full system of equations does not fit into the Alexakis-Schlue argument since one of the coupling terms does not decay fast enough. Specifically, the term $\mathbf{F} \cdot \mathbf{W}$ on the wave equation for \mathbf{E} , (3.19), has a slow decaying coefficient since $\mathbf{F} = O'(\bar{r}^{-2})$ and we need a power strictly greater than 2 in order to run the Alexakis-Schlue argument. We remedy this problem by borrowing some decay from the other coupling term $(\mathbf{F} \cdot \mathbf{C}) \cdot \mathbf{E}$ in the wave equation for \mathbf{W} , (3.18), by using different λ -weights for each Carleman estimate. See equations (3.36) and (3.37), and proof below for more details.

3.3.1 Proof of Proposition 3.3.1

Now we use the Carleman estimates of Theorem 3.1.2 together with the wave equations for \mathbf{W} and \mathbf{E} to prove Proposition 3.3.1. We will follow the next ideas: 1. We start by writing everything with respect to the frame $\{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$. This is mainly for convenience since then all the associated connection coefficients decay at the same rate. 2. We also have to check that the fields \mathbf{W} and \mathbf{E} satisfy the vanishing condition (3.2). This latter condition is fulfilled on the “exterior” part thanks to the hypothesis of Proposition 3.3.1; however, in order to cope with the “interior” decay also included in the vanishing condition (3.2) a cut-off function needs to be introduced. 3. Finally, we apply the Carleman estimates and aim at absorbing the variables and their first derivatives into the LHS of the Carleman inequalities to obtain bounds of the L^2 -norms of \mathbf{W} , \mathbf{E} , \mathbf{B} and \mathbf{P} in a neighbourhood $\mathcal{D}_{\epsilon'}$.

1. Equations with respect to the frame $\{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$:

Firstly, we note that the metric satisfies

$$\tilde{\mathbf{g}}_{\alpha\beta} := \langle \tilde{e}_\alpha, \tilde{e}_\beta \rangle = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + O'_2(\bar{r}^{-1}).$$

Therefore the associated connection coefficients³, $\Gamma_{\alpha\beta}^\mu$, satisfy

$$\Gamma_{\alpha\beta}^\mu = O'_1(\bar{r}^{-2}) \quad \text{and} \quad \tilde{e}_\nu(\Gamma_{\alpha\beta}^\mu) = O'(\bar{r}^{-3}).$$

Similarly if the components of a tensor satisfy $V_{\alpha_1 \dots \alpha_n} = O'_1(\bar{r}^{-q})$ then $\nabla_\beta V_{\alpha_1 \dots \alpha_n} = \tilde{e}_\beta(V_{\alpha_1 \dots \alpha_n}) + \sum_i \Gamma_{\beta\alpha_i}^\nu V_{\alpha_1 \dots \nu \dots \alpha_n} = O'(\bar{r}^{-q-1})$.

In what follows, for brevity, we will denote by (V) the components of a tensor V with respect to the frame $\{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$. In particular, we have schematically

$$\begin{aligned} (\nabla V) &= \tilde{e}(V) + \Gamma \cdot (V), \\ (\square V) &= \square(V) + \Gamma \cdot \tilde{e}(V) + \tilde{e}(\Gamma) \cdot (V). \end{aligned}$$

Hence, the wave equations for the components of \mathbf{W} and \mathbf{E} are

$$\begin{aligned} \square(\mathbf{W}) &= [(\mathbf{R}) + \tilde{e}(\Gamma)] \cdot (\mathbf{W}) + \Gamma \cdot \tilde{e}(\mathbf{W}) + [(\mathbf{F}) \cdot (\mathbf{C})] \cdot (\mathbf{E}) + [(\mathbf{R}) \cdot (\mathbf{C}) + \tilde{e}(\mathbf{C}) \cdot \Gamma] \cdot (\mathbf{B}) \\ &\quad + \tilde{e}(\mathbf{C}) \cdot \tilde{e}(\mathbf{B}) + (\mathbf{C}) \cdot \tilde{e}(\mathbf{P}) + (\mathbf{C}) \cdot \Gamma \cdot (\mathbf{P}), \\ \square(\mathbf{E}) &= [(\mathbf{F}) + \tilde{e}(\Gamma)] \cdot (\mathbf{W}) + \Gamma \cdot \tilde{e}(\mathbf{E}) + [(\mathbf{F})^2 + \tilde{e}(\Gamma)] \cdot (\mathbf{E}) + [(\mathbf{R}) \cdot (\mathbf{F}) + \tilde{e}(\mathbf{F}) \cdot \Gamma] \cdot (\mathbf{B}) \\ &\quad + \tilde{e}(\mathbf{F}) \cdot \tilde{e}(\mathbf{B}) + (\mathbf{F}) \cdot \tilde{e}(\mathbf{P}) + (\mathbf{F}) \cdot \Gamma \cdot (\mathbf{P}), \end{aligned}$$

where we have just kept the leading order terms multiplying \mathbf{W} , \mathbf{E} , etc. In view of the asymptotic behaviour assumed for the Weyl and Faraday tensors we have the following estimates

$$\begin{aligned} \square(\mathbf{W}) &= O'(\bar{r}^{-3})(\mathbf{W}) + O'(\bar{r}^{-2})\tilde{e}(\mathbf{W}) + O'(\bar{r}^{-5})(\mathbf{E}) + O'(\bar{r}^{-6})(\mathbf{B}) \\ &\quad + O'(\bar{r}^{-4})\tilde{e}(\mathbf{B}) + O'(\bar{r}^{-5})(\mathbf{P}) + O'(\bar{r}^{-3})\tilde{e}(\mathbf{P}), \\ \square(\mathbf{E}) &= O'(\bar{r}^{-2})(\mathbf{W}) + O'(\bar{r}^{-3})(\mathbf{E}) + O'(\bar{r}^{-2})\tilde{e}(\mathbf{E}) + O'(\bar{r}^{-5})(\mathbf{B}) \\ &\quad + O'(\bar{r}^{-3})\tilde{e}(\mathbf{B}) + O'(\bar{r}^{-4})(\mathbf{P}) + O'(\bar{r}^{-2})\tilde{e}(\mathbf{P}). \end{aligned} \tag{3.22}$$

While these are morally the reason for the unique continuation we still need to

³Defined by the formula $\nabla_{\tilde{e}_\alpha} \tilde{e}_\beta = \Gamma_{\alpha\beta}^\mu \tilde{e}_\mu$.

compensate for the fact that the coefficient accompanying (\mathbf{W}) in the second equation does not decay fast enough. However the coupling term in the first equation, $O'(\bar{r}^{-5})(\mathbf{E})$, allows us to borrow some decay by modifying the Carleman weight.

We also need to write down the transport equations for \mathbf{B} and \mathbf{P} . Firstly, we need estimates for L and its derivatives. Recall $(L) = 1 + O'_2(\bar{r}^{-1})$, then

$$\begin{aligned}(\nabla L) &= \tilde{e}(L) + \Gamma \cdot (L) = O'_1(\bar{r}^{-2}). \\ (\nabla_L \tilde{e}_\alpha) &= (L) \cdot \Gamma = O'_1(\bar{r}^{-2}).\end{aligned}$$

Hence the transport equations for \mathbf{B} and \mathbf{P} read

$$\begin{aligned}\nabla_L(\mathbf{B}) &= (\mathbf{P}) + (\mathbf{B}) \cdot (\nabla L) + (\nabla_L \tilde{e}) \cdot (\mathbf{B}), \\ &= (\mathbf{P}) + O'(\bar{r}^{-1})(\mathbf{B}),\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}\nabla_L(\mathbf{P}) &= L \cdot ((\mathbf{W}) + (\mathbf{C}) \cdot (\mathbf{B}) + (\mathbf{F})^2 \cdot (\mathbf{B}) + (\mathbf{F}) \cdot (\mathbf{E}) + (\mathbf{B}) \cdot (\mathbf{C})) \\ &\quad + (\mathbf{P}) \cdot (\nabla L) + (\nabla_L \tilde{e}) \cdot (\mathbf{P}), \\ &= (\mathbf{W}) + O'(\bar{r}^{-2})(\mathbf{E}) + O'(\bar{r}^{-3})(\mathbf{B}) + O'(\bar{r}^{-1})(\mathbf{P}).\end{aligned}\tag{3.24}$$

We also have the following estimates

$$\begin{aligned}[\tilde{e}_\mu, L] &= O'(\bar{r}^{-2}), \\ (\nabla \nabla L) &= \tilde{e}(\nabla L) + \Gamma \cdot (\nabla L), \\ &= \tilde{e}(\tilde{e}(L)) + \tilde{e}(\Gamma) \cdot (L) + \Gamma \cdot \tilde{e}(L) + \Gamma \cdot \Gamma \cdot L, \\ &= O'(\bar{r}^{-2}).\end{aligned}$$

We use them to deduce transport equations for $\tilde{e}(\mathbf{B})$ and $\tilde{e}(\mathbf{P})$. Indeed, by \tilde{e} -differentiating (3.23) and (3.24), commuting derivatives and using the previous estimates we deduce,

$$\begin{aligned}\nabla_L \tilde{e}(\mathbf{B}) &= [\tilde{e}, L] \cdot (\mathbf{B}) + \tilde{e}(\mathbf{P}) + O'(\bar{r}^{-2})(\mathbf{B}) + O'(\bar{r}^{-1})\tilde{e}(\mathbf{B}), \\ &= O'(\bar{r}^{-1})\tilde{e}(\mathbf{B}) + O'(\bar{r}^{-2})(\mathbf{B}) + \tilde{e}(\mathbf{P}).\end{aligned}\tag{3.25}$$

And,

$$\begin{aligned}
\nabla_L \tilde{e}(\mathbf{P}) &= -[\tilde{e}, L] \cdot (\mathbf{P}) + \tilde{e}(\mathbf{W}) + O'(\bar{r}^{-3})(\mathbf{E}) + O'(\bar{r}^{-2})\tilde{e}(\mathbf{E}) \\
&\quad + O'(\bar{r}^{-4})(\mathbf{B}) + O'(\bar{r}^{-2})\tilde{e}(\mathbf{B}) + O'(\bar{r}^{-2})(\mathbf{P}) + O'(\bar{r}^{-1})\tilde{e}(\mathbf{P}), \\
&= \tilde{e}(\mathbf{W}) + O'(\bar{r}^{-4})(\mathbf{B}) + O'(\bar{r}^{-2})\tilde{e}(\mathbf{B}) + O'(\bar{r}^{-2})(\mathbf{P}) + O'(\bar{r}^{-1})\tilde{e}(\mathbf{P}) \\
&\quad + O'(\bar{r}^{-3})(\mathbf{E}) + O'(\bar{r}^{-2})\tilde{e}(\mathbf{E}). \tag{3.26}
\end{aligned}$$

2. Vanishing condition:

Before applying the Carleman estimates of Theorem 3.1.2 we have to guarantee that all the quantities vanish to all orders in the sense of (3.2), that is,

$$\lim_{k \rightarrow \infty} \int_{\partial \mathcal{U}_k} \bar{r}^N (\phi^2 + |\partial \phi|^2) = 0 \quad \text{for all } N \in \mathbb{N}.$$

We choose the exhaustion $\mathcal{U}_k := \{(v, \underline{v}, y^2, y^3) : \underline{v} - v < k\} \cap \mathcal{D}_\epsilon$, $k \in \mathbb{N}$. We deal with the “interior” and “exterior” parts of the the boundary of \mathcal{U}_k separately.

The “exterior” boundary of \mathcal{U}_k is precisely Ω_k , so the vanishing condition 3.21 of Proposition 3.3.1 already ensures the vanishing of this part. To deal with the “interior” part of $\partial \mathcal{U}_k$ a cut-off function is used. This technique is standard for unique continuation problems.

Let χ be a function on \mathcal{D}_ϵ defined by

$$\chi(f) = \frac{\exp\left(\frac{1}{\epsilon_1 - \epsilon_0} - \frac{1}{\epsilon_1 - f}\right)}{1 + \exp\left(-\frac{1}{f - \epsilon_0}\right)}.$$

This is a cut-off function whose level sets coincide with those of $F \circ f$ and

$$\chi = 1 \text{ on } \mathcal{D}_{\epsilon_0}, \quad \chi = 0 \text{ on } \mathcal{D}_\epsilon \setminus \mathcal{D}_{\epsilon_1}, \quad \epsilon_0 < \epsilon_1 < \epsilon.$$

Then the functions $\chi \cdot (\mathbf{W})$, $\chi \cdot (\mathbf{E})$, etc., satisfy the vanishing condition (3.2). The price to pay is that we have introduced extra terms in the wave equations, however these are easy to treat since they are supported only in the cut-off region

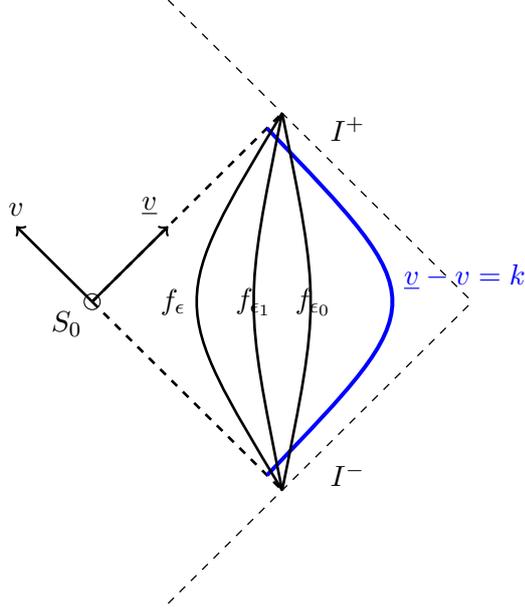


Figure 3.2: Schematic conformal picture of the level sets of $f = \epsilon, \epsilon_1, \epsilon_0$ and $\underline{u} - v = k$.

(we only need them to remain L^2 -bounded, see Lemma 3.3.2 below). Indeed,

$$\begin{aligned}
\Box(\chi \cdot (\mathbf{W})) &= (\Box\chi) \cdot (\mathbf{W}) + \tilde{e}(\chi) \cdot \tilde{e}(\mathbf{W}) + \chi \cdot \Box(\mathbf{W}), \\
&= (\Box\chi) \cdot (\mathbf{W}) + \tilde{e}(\chi) \cdot \tilde{e}(\mathbf{W}) + \tilde{e}(\chi) \cdot \{(\mathbf{W}), \tilde{e}(\mathbf{W}), \dots\} \\
&\quad + \{\chi \cdot (\mathbf{W}), \tilde{e}(\chi \cdot (\mathbf{W})), \dots\}, \\
&= \nabla\chi\mathbf{M} + \{\chi \cdot (\mathbf{W}), \tilde{e}(\chi \cdot (\mathbf{W})), \dots\}.
\end{aligned}$$

Hereafter we will use the symbol $\nabla\chi\mathbf{M}$ to denote multiples of $(\mathbf{W}), \tilde{e}(\mathbf{W})$ ($\mathbf{E}), \tilde{e}(\mathbf{E})$, etc., which are only supported in the cut-off region $\mathcal{D}_{\epsilon_1} \setminus \mathcal{D}_{\epsilon_0}$. Recall also that we have used the notation $\{(\mathbf{W}), (\mathbf{E}), \dots\}$ to denote a function involving $(\mathbf{W}), (\mathbf{E})$, etc. Hence, after applying the Carleman estimates we can focus only on the terms supported on \mathcal{D}_{ϵ_0} .

3. Applying the Carleman estimates.

We are now ready to apply Theorem 3.1.2 to the functions $\chi \cdot (\mathbf{W}), \chi \cdot (\mathbf{E})$, etc. To keep the notation simple and readable we will omit in the next argument the cut-off function and the parenthesis denoting components with respect to the frame $\{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$. We follow the standard procedure to bound the L^2 -norms of $\mathbf{W}, \mathbf{B}, \mathbf{P}$ and its first derivatives: The Carleman estimate for \mathbf{W} combined with its

wave equation reads

$$\begin{aligned}
\lambda^3 \|f^\delta \mathbf{W}\|_{\mathcal{W}} + \lambda \|f^{-\frac{1}{2}} \Psi^{\frac{1}{2}} \tilde{e}(\mathbf{W})\|_{\mathcal{W}} &\lesssim \|f^{-1} \square \mathbf{W}\|_{\mathcal{W}}, \\
&\lesssim \|f^{-1} \bar{r}^{-3} \mathbf{W}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-2} \tilde{e}(\mathbf{W})\|_{\mathcal{W}} \\
&\quad + \|f^{-1} \bar{r}^{-6} \mathbf{B}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-4} \tilde{e}(\mathbf{B})\|_{\mathcal{W}} \\
&\quad + \|f^{-1} \bar{r}^{-5} \mathbf{P}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-3} \tilde{e}(\mathbf{P})\|_{\mathcal{W}} \\
&\quad + \|f^{-1} \bar{r}^{-5} \mathbf{E}\|_{\mathcal{W}} + \|\nabla \chi \mathbf{M}\|_{\mathcal{W}}. \tag{3.27}
\end{aligned}$$

The following estimates will be used throughout:

$$f = \frac{1}{(-v)\underline{v}} \gtrsim \frac{1}{\bar{r}^2}, \quad f\Psi \gtrsim \frac{1}{\bar{r}^3}. \tag{3.28}$$

The first one is a consequence of AM-GM inequality, indeed, $\sqrt{(-v)\underline{v}} \leq \frac{v-\underline{v}}{2} \lesssim \bar{r}$. And since $\Psi = \frac{m_{\min} \log \bar{r}}{\bar{r}} \gtrsim \frac{1}{\bar{r}}$, the second estimate also follow.

In particular we have,

$$f^{-\frac{1}{2}} \Psi^{\frac{1}{2}} = f^{-1} (f\Psi)^{\frac{1}{2}} \gtrsim f^{-1} \bar{r}^{-\frac{3}{2}}, \tag{3.29}$$

$$f^\delta \gtrsim \frac{1}{\bar{r}^{2\delta}} > \frac{1}{\bar{r}}, \quad \text{for } 0 < 2\delta < 1. \tag{3.30}$$

These last inequalities tell us that the first two terms on the right-hand side of (3.27), $f^{-1} \bar{r}^{-3} \mathbf{W}$ and $f^{-1} \bar{r}^{-2} \tilde{e}(\mathbf{W})$, can be absorbed into the corresponding terms on the left-hand side since $f^{-1} \bar{r}^{-3} \lesssim f^\delta$ and $f^{-1} \bar{r}^{-2} \lesssim f^{-\frac{1}{2}} \Psi^{\frac{1}{2}}$ (λ can be arbitrarily large and we can always rescale the left-hand side by a small constant). This absorbing technique will be the main trick during the proof. We have thus obtained,

$$\begin{aligned}
\lambda^3 \|f^\delta \mathbf{W}\|_{\mathcal{W}} + \lambda \|f^{-\frac{1}{2}} \Psi^{\frac{1}{2}} \tilde{e}(\mathbf{W})\|_{\mathcal{W}} &\lesssim \|f^{-1} \bar{r}^{-6} \mathbf{B}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-4} \tilde{e}(\mathbf{B})\|_{\mathcal{W}} \\
&\quad + \|f^{-1} \bar{r}^{-5} \mathbf{P}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-3} \tilde{e}(\mathbf{P})\|_{\mathcal{W}} \\
&\quad + \|f^{-1} \bar{r}^{-5} \mathbf{E}\|_{\mathcal{W}} + \|\nabla \chi \mathbf{M}\|_{\mathcal{W}}. \tag{3.31}
\end{aligned}$$

Next, we aim at controlling the \mathbf{B} term. The Carleman estimate from Lemma (3.1.4) with $q = 4$, together with the transport equation for \mathbf{B} read

$$\begin{aligned}
\lambda \left\| \frac{1}{r} f^{-1} \bar{r}^{-4} \mathbf{B} \right\|_{\mathcal{W}} &\lesssim \|f^{-1} \bar{r}^{-4} \nabla_L \mathbf{B}\|_{\mathcal{W}}, \\
&\lesssim \|f^{-1} \bar{r}^{-4} \mathbf{P}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-5} \mathbf{B}\|_{\mathcal{W}} + \|\nabla \chi \mathbf{M}\|_{\mathcal{W}}.
\end{aligned}$$

We add this inequality to (3.31),

$$\begin{aligned} \lambda^3 \|f^\delta \mathbf{W}\|_{\mathcal{W}} + \lambda \|f^{-\frac{1}{2}} \Psi^{\frac{1}{2}} \tilde{e}(\mathbf{W})\|_{\mathcal{W}} + \lambda \|f^{-1} \bar{r}^{-5} \mathbf{B}\|_{\mathcal{W}} &\lesssim \|f^{-1} \bar{r}^{-5} \mathbf{B}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-4} \mathbf{P}\|_{\mathcal{W}} \\ &\quad + \|f^{-1} \bar{r}^{-6} \mathbf{B}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-4} \tilde{e}(\mathbf{B})\|_{\mathcal{W}} \\ &\quad + \|f^{-1} \bar{r}^{-5} \mathbf{P}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-3} \tilde{e}(\mathbf{P})\|_{\mathcal{W}} \\ &\quad + \|f^{-1} \bar{r}^{-5} \mathbf{E}\|_{\mathcal{W}} + \|\nabla \chi \mathbf{M}\|_{\mathcal{W}}. \end{aligned}$$

Observe that the terms $\|f^{-1} \bar{r}^{-5} \mathbf{B}\|_{\mathcal{W}}$ and $\|f^{-1} \bar{r}^{-6} \mathbf{B}\|_{\mathcal{W}}$ can be absorbed into the left-hand side. We get,

$$\begin{aligned} \lambda^3 \|f^\delta \mathbf{W}\|_{\mathcal{W}} + \lambda \|f^{-\frac{1}{2}} \Psi^{\frac{1}{2}} \tilde{e}(\mathbf{W})\|_{\mathcal{W}} + \lambda \|f^{-1} \bar{r}^{-5} \mathbf{B}\|_{\mathcal{W}} &\lesssim \|f^{-1} \bar{r}^{-5} \mathbf{E}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-4} \tilde{e}(\mathbf{B})\|_{\mathcal{W}} \\ &\quad + \|f^{-1} \bar{r}^{-4} \mathbf{P}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-3} \tilde{e}(\mathbf{P})\|_{\mathcal{W}} \\ &\quad + \|\nabla \chi \mathbf{M}\|_{\mathcal{W}}. \end{aligned} \quad (3.32)$$

Now we proceed similarly to absorb the terms \mathbf{P} , $\tilde{e}(\mathbf{B})$ and $\tilde{e}(\mathbf{P})$ into the left-hand side. Applying Lemma 3.1.4 to \mathbf{P} , $\tilde{e}(\mathbf{B})$ and $\tilde{e}(\mathbf{P})$ together with its transport equations (3.24), (3.25) and (3.26), we obtain,

$$\begin{aligned} \lambda \left\| \frac{1}{r} f^{-1} \bar{r}^{-3} \mathbf{P} \right\|_{\mathcal{W}} &\lesssim \|f^{-1} \bar{r}^{-3} \nabla_L \mathbf{P}\|_{\mathcal{W}} \\ &\lesssim \|f^{-1} \bar{r}^{-3} \mathbf{W}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-6} \mathbf{B}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-4} \mathbf{P}\|_{\mathcal{W}} \\ &\quad + \|f^{-1} \bar{r}^{-5} \mathbf{E}\|_{\mathcal{W}} + \|\nabla \chi \mathbf{M}\|_{\mathcal{W}}, \end{aligned} \quad (3.33)$$

$$\begin{aligned} \lambda \left\| \frac{1}{r} f^{-1} \bar{r}^{-3} \tilde{e}(\mathbf{B}) \right\|_{\mathcal{W}} &\lesssim \|f^{-1} \bar{r}^{-3} \nabla_L \tilde{e}(\mathbf{B})\|_{\mathcal{W}} \\ &\lesssim \|f^{-1} \bar{r}^{-5} \mathbf{B}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-4} \tilde{e}(\mathbf{B})\|_{\mathcal{W}} \\ &\quad + \|f^{-1} \bar{r}^{-4} \mathbf{P}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-3} \tilde{e}(\mathbf{P})\|_{\mathcal{W}} + \|\nabla \chi \mathbf{M}\|_{\mathcal{W}}, \end{aligned} \quad (3.34)$$

$$\begin{aligned} \lambda \left\| \frac{1}{r} f^{-1} \bar{r}^{-2} \tilde{e}(\mathbf{P}) \right\|_{\mathcal{W}} &\lesssim \|f^{-1} \bar{r}^{-2} \nabla_L \tilde{e}(\mathbf{P})\|_{\mathcal{W}} \\ &\lesssim \|f^{-1} \bar{r}^{-3} \mathbf{W}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-2} \tilde{e}(\mathbf{W})\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-6} \mathbf{B}\|_{\mathcal{W}} \\ &\quad + \|f^{-1} \bar{r}^{-5} \tilde{e}(\mathbf{B})\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-4} \mathbf{P}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-3} \tilde{e}(\mathbf{P})\|_{\mathcal{W}} \\ &\quad + \|f^{-1} \bar{r}^{-5} \mathbf{E}\|_{\mathcal{W}} + \|f^{-1} \bar{r}^{-4} \tilde{e}(\mathbf{E})\|_{\mathcal{W}} + \|\nabla \chi \mathbf{M}\|_{\mathcal{W}}. \end{aligned} \quad (3.35)$$

We add these inequalities to (3.32) and observe that the \mathbf{W} , \mathbf{B} , $\tilde{e}(\mathbf{B})$, \mathbf{P} and $\tilde{e}(\mathbf{P})$

terms can be absorbed into the left-hand side, thus obtaining,

$$\begin{aligned}
& \lambda^3 \|f^\delta \mathbf{W}\|_{\mathcal{W}} + \lambda \|f^{-\frac{1}{2}} \Psi^{\frac{1}{2}} \tilde{e}(\mathbf{W})\|_{\mathcal{W}} \\
& + \lambda \|f^{-1} \bar{r}^{-5} \mathbf{B}\|_{\mathcal{W}} + \lambda \|f^{-1} \bar{r}^{-4} \tilde{e}(\mathbf{B})\|_{\mathcal{W}} \\
& + \lambda \|f^{-1} \bar{r}^{-4} \mathbf{P}\|_{\mathcal{W}} + \lambda \|f^{-1} \bar{r}^{-3} \tilde{e}(\mathbf{P})\|_{\mathcal{W}} \lesssim \|f^{-1} \bar{r}^{-5} \mathbf{E}\|_{\mathcal{W}} + \|\nabla \chi \mathbf{M}\|_{\mathcal{W}}. \quad (3.36)
\end{aligned}$$

for sufficiently large λ and $0 < \delta < \frac{1}{2}$. An analogous argument gives (note the different Carleman parameter),

$$\begin{aligned}
& \lambda'^3 \|f^\delta \mathbf{E}\|_{\mathcal{W}'} + \lambda' \|f^{-\frac{1}{2}} \Psi^{\frac{1}{2}} \tilde{e}(\mathbf{E})\|_{\mathcal{W}'} \\
& + \lambda' \|f^{-1} \bar{r}^{-5} \mathbf{B}\|_{\mathcal{W}'} + \lambda' \|f^{-1} \bar{r}^{-4} \tilde{e}(\mathbf{B})\|_{\mathcal{W}'} \\
& + \lambda' \|f^{-1} \bar{r}^{-3} \mathbf{P}\|_{\mathcal{W}'} + \lambda' \|f^{-1} \bar{r}^{-2} \tilde{e}(\mathbf{P})\|_{\mathcal{W}'} \lesssim \|f^{-1} \bar{r}^{-2} \mathbf{W}\|_{\mathcal{W}'} + \|\nabla \chi \mathbf{M}\|_{\mathcal{W}'}. \quad (3.37)
\end{aligned}$$

We would like to add these last two inequalities and absorb the \mathbf{W} term on the left-hand side to obtain the desired bound. However, for $\lambda = \lambda'$, this is not possible as $f^{-1} \bar{r}^{-2} = O(1)$ does not decay fast enough. To remedy this, we make the observation that the norms depend on λ and by taking slightly different weights we can perform the procedure described previously. More precisely, we want to find λ' such that

$$\begin{aligned}
(e^{-\lambda F} f^{\frac{1}{2}}) f^{-1} \bar{r}^{-5} & \lesssim e^{-\lambda' F} f^{\frac{1}{2}} f^\delta, \\
(e^{-\lambda' F} f^{\frac{1}{2}}) f^{-1} \bar{r}^{-2} & \lesssim e^{-\lambda F} f^{\frac{1}{2}} f^\delta. \quad (3.38)
\end{aligned}$$

Indeed, we will show that the choice $\lambda' := \lambda - \delta$ achieves the previous inequalities. Firstly, note that

$$\begin{aligned}
\bar{r}^{-5} & \lesssim f^{1+2\delta}, \\
\bar{r}^{-2} & \lesssim f,
\end{aligned}$$

these are a consequence of estimates (3.28) and (3.30). They imply that

$$\begin{aligned}
f^{-1} \bar{r}^{-5} & \lesssim e^{\delta F} f^\delta, \\
e^{\delta F} f^{-1} \bar{r}^{-2} & \lesssim f^\delta,
\end{aligned}$$

since $\log f \geq F = \log f - f^{2\delta} \geq \log f - c$, with $c > 0$ constant and \bar{r} large. Finally, it is easy to see that these last inequalities are equivalent to (3.38), with $\lambda' := \lambda - \delta$.

Remark. It is worth noticing that the previous argument did not make any special

use of the power \bar{r}^{-5} accompanying \mathbf{E} . The procedure will work for any \bar{r}^{-q} with $q > 2$ by choosing $\delta > 0$ small enough.

Now we are in position to close the argument. We add inequalities (3.36) and (3.37), with $\lambda' = \lambda - \delta$. Inequalities (3.38) ensure that the terms $\|f^{-1}\bar{r}^{-5}\mathbf{E}\|_{\mathcal{W}}$ and $\|f^{-1}\bar{r}^{-2}\mathbf{W}\|_{\mathcal{W}'}$ can be absorbed into the left-hand side. So,

$$\begin{aligned}
& \lambda^3 \|f^\delta \mathbf{W}\|_{\mathcal{W}} + \lambda \|f^{-\frac{1}{2}} \Psi^{\frac{1}{2}} \tilde{e}(\mathbf{W})\|_{\mathcal{W}} \\
& + \lambda \|f^{-1}\bar{r}^{-5}\mathbf{B}\|_{\mathcal{W}} + \lambda \|f^{-1}\bar{r}^{-4}\tilde{e}(\mathbf{B})\|_{\mathcal{W}} \\
& + \lambda \|f^{-1}\bar{r}^{-4}\mathbf{P}\|_{\mathcal{W}} + \lambda \|f^{-1}\bar{r}^{-3}\tilde{e}(\mathbf{P})\|_{\mathcal{W}} \\
& + \lambda'^3 \|f^\delta \mathbf{E}\|_{\mathcal{W}'} + \lambda' \|f^{-\frac{1}{2}} \Psi^{\frac{1}{2}} \tilde{e}(\mathbf{E})\|_{\mathcal{W}'} \\
& + \lambda' \|f^{-1}\bar{r}^{-5}\mathbf{B}\|_{\mathcal{W}'} + \lambda' \|f^{-1}\bar{r}^{-4}\tilde{e}(\mathbf{B})\|_{\mathcal{W}'} \\
& + \lambda' \|f^{-1}\bar{r}^{-3}\mathbf{P}\|_{\mathcal{W}'} + \lambda' \|f^{-1}\bar{r}^{-2}\tilde{e}(\mathbf{P})\|_{\mathcal{W}'} \lesssim \|\nabla\chi\mathbf{M}\|_{\mathcal{W}} + \|\nabla\chi\mathbf{M}\|_{\mathcal{W}'}.
\end{aligned}$$

Now, the Carleman weights $e^{-\lambda F}$ and $e^{-\lambda' F}$ are monotonic functions, so on left-hand side we can substitute its minimum value attained at $f = \epsilon$. On the other hand, the terms on the right-hand side are only supported in the cut-off region where f is bounded, thus we can substitute the Carleman weights by the maximum value attained at $f = \epsilon_0$. Thus, after dropping the weight factors from the inequality we obtain the desired L^2 -bound,

$$\begin{aligned}
& \lambda^3 \|f^{\frac{1}{2}} f^\delta \mathbf{W}\|_2 + \lambda \|\Psi^{\frac{1}{2}} \tilde{e}(\mathbf{W})\|_2 \\
& + \lambda \|f^{\frac{1}{2}} f^{-1}\bar{r}^{-5}\mathbf{B}\|_2 + \lambda \|f^{\frac{1}{2}} f^{-1}\bar{r}^{-4}\tilde{e}(\mathbf{B})\|_2 \\
& + \lambda \|f^{\frac{1}{2}} f^{-1}\bar{r}^{-4}\mathbf{P}\|_2 + \lambda \|f^{\frac{1}{2}} f^{-1}\bar{r}^{-3}\tilde{e}(\mathbf{P})\|_2 \\
& + \lambda'^3 \|f^{\frac{1}{2}} f^\delta \mathbf{E}\|_2 + \lambda' \|\Psi^{\frac{1}{2}} \tilde{e}(\mathbf{E})\|_2 \lesssim \|\nabla\chi\mathbf{M}\|_2.
\end{aligned} \tag{3.39}$$

We wish to take $\lambda \rightarrow \infty$ to conclude vanishing of the L^2 -norms on the left-hand side. For this reason we need to make sure that the right-hand side is finite. Recall that $\nabla\chi\mathbf{M}$ consists of terms of the form $\tilde{e}(\chi) \cdot \phi$ and $\tilde{e}^2(\chi) \cdot \phi$ with ϕ vanishing to all orders at infinity. Thus the finiteness of $\|\nabla\chi\mathbf{M}\|_2$ is guaranteed by the following:

Lemma 3.3.2. *Let ϕ satisfy*

$$\lim_{k \rightarrow \infty} \int_{\Omega_k} \bar{r}^N |\phi|^2 = 0, \quad N \in \mathbb{N}$$

then

$$\int_{\mathcal{D}_\epsilon \setminus \mathcal{D}_{\epsilon_0}} |\tilde{e}(\chi) \cdot \phi|^2 + |\tilde{e}^2(\chi) \cdot \phi|^2 < +\infty.$$

Proof. We need bounds for $\tilde{e}(\chi)$ and $\tilde{e}^2(\chi)$. Note that $\tilde{e}(\chi) = \frac{d\chi}{df} \tilde{e}(f)$, so we can focus on ∂_v and $\partial_{\underline{v}}$ derivatives. Moreover it is easy to check that, $\frac{d\chi}{df}$ and $\frac{d^2\chi}{df^2}$ are bounded. Then, on $\mathcal{D}_\epsilon \setminus \mathcal{D}_{\epsilon_0}$ the following bounds hold:

$$|\tilde{e}_0(\chi)| \lesssim |\tilde{e}_0(f)| = \frac{1}{v^2 \underline{v}} = \frac{1}{-v} f < \epsilon^2 (\underline{v} - v),$$

where we have used $\frac{1}{-v} < \epsilon \underline{v} \leq \epsilon(\underline{v} - v)$ when $f = \frac{1}{-v \underline{v}} < \epsilon$. Similarly $|\tilde{e}_1(\chi)| \lesssim (\underline{v} - v)$. Also, for second derivatives we have

$$\begin{aligned} |\tilde{e}_0^2(\chi)| &\lesssim |\tilde{e}_0^2(f)| = -\frac{2}{v^3 \underline{v}} = \frac{1}{v^2} f \lesssim (\underline{v} - v)^2, \\ |\tilde{e}_0 \tilde{e}_1(\chi)| &\lesssim |\tilde{e}_0^2(f)| = \frac{1}{v^2 \underline{v}^2} = f^2 < \epsilon^2, \\ |\tilde{e}_1^2(\chi)| &\lesssim |\tilde{e}_0^2(f)| = -\frac{2}{v^3 \underline{v}} = \frac{1}{v^2} f \lesssim (\underline{v} - v)^2. \end{aligned}$$

Now we take k large enough so that $\int_{\Omega_{r_*}} \bar{r}^8 |\phi|^2 \leq 1$ for all $r_* \geq k$. Then

$$\int_{\Omega_{r_*}} (\underline{v} - v)^8 |\phi|^2 \lesssim \int_{\Omega_{r_*}} \bar{r}^8 |\phi|^2 \leq 1.$$

Next, we split the domain $\mathcal{D}_\epsilon \setminus \mathcal{D}_{\epsilon_0} = \mathcal{U}_0 \cup \mathcal{U}_\infty$ into bounded and unbounded parts:

$$\begin{aligned} \mathcal{U}_0 &:= (\mathcal{D}_\epsilon \setminus \mathcal{D}_{\epsilon_0}) \cap \{(v, \underline{v}, y^2, y^3) : \underline{v} - v < k\}, \\ \mathcal{U}_\infty &:= (\mathcal{D}_\epsilon \setminus \mathcal{D}_{\epsilon_0}) \cap \{(v, \underline{v}, y^2, y^3) : \underline{v} - v \geq k\}. \end{aligned}$$

Note that \mathcal{U}_0 is bounded so we are left to check

$$\int_{\mathcal{U}_\infty} |\tilde{e}(\chi) \cdot \phi|^2 + |\tilde{e}^2(\chi) \cdot \phi|^2 < +\infty.$$

In order to bound this integral, we compute it with respect to coordinates, (t, r_*, y^2, y^3) , adapted to the foliation of \mathcal{U}_∞ induced by Ω_{r_*} , the level sets of $r_* = \underline{v} - v$,

$$\begin{aligned} \int_{\mathcal{U}_\infty} |\tilde{e}(\chi) \cdot \phi|^2 &\lesssim \int_{\mathcal{U}_\infty} (\underline{v} - v)^2 |\phi|^2, \\ &= \int_k^\infty \left(\int_{\Omega_{r_*} \cap \mathcal{U}_\infty} (\underline{v} - v)^2 |\phi|^2 d\Omega_{r_*} \right) dr_*, \\ &\lesssim \int_k^\infty r_*^4 \frac{1}{r_*^8} dr_* < +\infty. \end{aligned}$$

Where, we have used that the volume element Ω_{r_*} satisfies $d\Omega_{r_*} = O'(\bar{r}^2) \lesssim r_*^2$.

In a similarly way we deduce , $\int_{\mathcal{U}_\infty} |\tilde{e}^2(\chi) \cdot \phi|^2 \lesssim \int_k^\infty r_*^6 \frac{1}{r_*^8} dr_* < +\infty$. Therefore the required L^2 -norm on $\mathcal{D}_\epsilon \setminus \mathcal{D}_{\epsilon_0}$ is bounded. ■

Finally, we restrict the left-hand side of (3.39) to be integrated over the smaller domain \mathcal{D}_{ϵ_0} where $\chi = 1$ and by taking $\lambda \rightarrow \infty$ we conclude that $\mathbf{B} \equiv 0$, $\mathbf{P} \equiv 0$, $\mathbf{W} \equiv 0$ and $\mathbf{E} \equiv 0$ on \mathcal{D}_{ϵ_0} . In particular

$$\mathcal{L}_T \mathbf{g} \equiv 0 \quad \text{and} \quad \mathcal{L}_T \mathbf{F} \equiv 0 \quad \text{on } \mathcal{D}_{\epsilon_0}.$$

This finishes the proof of Proposition 3.3.1 and Theorem 1.4.2. ■

3.4 Proof of Theorem 1.4.2

Here we include the last steps to put together Proposition 3.3.1 and Proposition 2.1.4 in order to prove our main result.

By hypothesis the radiation fields (with respect to frame $\{e_0, e_1, e_2, e_3\}$, cf. Section 1.3.2) Ξ_{ij} and $\underline{A}(F)_i$ vanish, hence we can apply Lemma 2.3.2 to conclude

$$\mathbf{C}_{\alpha\beta\mu\nu} = \mathcal{O}_1^\infty(s^{-3}), \quad \mathbf{F}_{\alpha\beta} = \mathcal{O}_1^\infty(s^{-2}).$$

So $\mathbf{C}_{\alpha\beta\mu\nu} = O'_1(\bar{r}^{-3})$ and $\mathbf{F}_{\alpha\beta} = O'_1(\bar{r}^{-2})$, since $s \lesssim \bar{r}$.

Notation. During the proof $V_{\alpha_1 \dots \alpha_n}$ will denote the components of a tensor V with respect to the basis $\{e_\mu\}$ (cf. Section 1.3.2) while $\tilde{V}_{\alpha_1 \dots \alpha_n}$ will be the components with respect to the basis $\{\tilde{e}_\mu\}$ (cf. Proposition 3.3.1).

Now we estimate the matrix for the change of basis $\{e_\mu\} \mapsto \{\tilde{e}_\nu\}$ defined by $\tilde{e}_\nu = \Theta_\nu^\mu e_\mu$. Recall the change of coordinates $(t, r, \vartheta^2, \vartheta^3) \mapsto (v, \underline{v}, y^2, y^3)$ considered in Proposition 1.3.11 and the compatibility condition with coordinates $(u, s, \theta^2, \theta^3)$ in Definition 1.3.8, then

$$\begin{aligned} \tilde{e}_0 &= \tilde{\partial}_v = \partial_t - (1 + O'(\bar{r}^{-1}))\partial_{\bar{r}} = 2\partial_u - \partial_s + O'(\bar{r}^{-1}) = e_1 + O'(\bar{r}^{-1}), \\ \tilde{e}_1 &= \tilde{\partial}_{\underline{v}} = \partial_t + (1 + O'(\bar{r}^{-1}))\partial_{\bar{r}} = \partial_s + O'(\bar{r}^{-1}) = e_0 + O'(\bar{r}^{-1}), \\ \tilde{e}_2 &= \frac{1}{\bar{r}}\tilde{\partial}_{y^2} = \frac{1}{\bar{r}}(\partial_{\vartheta^2} + O'(\bar{r}^{-1})) = e_2 + O'(\bar{r}^{-1}), \\ \tilde{e}_3 &= \frac{1}{\bar{r}}\tilde{\partial}_{y^3} = \frac{1}{\bar{r}}\partial_{\vartheta^3} = \sin(\theta^2)e_3 + O'(\bar{r}^{-1}), \end{aligned}$$

i.e., $|\Theta_\nu^\mu| \lesssim 1$. Therefore, $|\tilde{V}_{\alpha_1 \dots \alpha_n}| = |\Theta_{\alpha_1}^{\beta_1} \dots \Theta_{\alpha_n}^{\beta_n} V_{\beta_1 \dots \beta_n}| \lesssim \sum_{\beta_1 \dots \beta_n} |V_{\beta_1 \dots \beta_n}|$; so all the uniform estimates for V are inherited by \tilde{V} . In particular, $\tilde{V}_{\beta_1 \dots \beta_n} = O'(r^{-q})$ if $V_{\beta_1 \dots \beta_n} = O'(r^{-q})$. Thus, $\tilde{\mathbf{C}}_{\alpha\beta\mu\nu} = O'_1(\bar{r}^{-3})$ and $\tilde{\mathbf{F}}_{\alpha\beta} = O'_1(\bar{r}^{-2})$.

Now we need to check that the infinite-order vanishing conditions of Proposition 3.3.1 are satisfied.

Lemma 3.4.1. *Consider ϕ satisfying*

$$|\phi|^2 \lesssim \frac{1}{s^N(1+|u|)^{1+\eta}}, \quad \text{for all } N \in \mathbb{N}. \quad (3.40)$$

Then

$$\lim_{k \rightarrow \infty} \int_{\Omega_k} \bar{r}^n |\phi|^2 = 0, \quad \text{for all } n \in \mathbb{N}. \quad (3.41)$$

Proof of Lemma. Recall from Proposition 1.3.11 that $\bar{r} \lesssim r_* = \underline{v} - v$ and $v = t - r_* \lesssim u$. We will compute the integral over Ω_k with respect to coordinates $(t = \underline{v} + v, y^2, y^3)$; let $S_{t,k} := \{(v, \underline{v}, y^2, y^3) : \underline{v} + v = t, \underline{v} - v = k\}$, then

$$\begin{aligned} \int_{\Omega_k} \bar{r}^n |\phi|^2 d\Omega_k &\lesssim \int_{-\infty}^{\infty} \int_{S_{t,k}} r_*^n |\phi|^2 dS_{t,k} dt \\ &\lesssim \int_{-\infty}^{\infty} \int_{S_{t,k}} r_*^n \frac{1}{r_*^N (1+|u|)^{1+\eta}} dS_{t,k} dt \\ &\lesssim \int_{-\infty}^{\infty} \int_{S_{t,k}} \frac{dS_{t,k}}{r_*^{N-n} (1+|t-r_*|)^{1+\eta}} dt \\ &\lesssim \int_{-\infty}^{\infty} \frac{1}{k^{N-n-2} (1+|t-k|)^{1+\eta}} dt, \end{aligned}$$

where we have used $\text{Area}(S_{t,k}) \lesssim k^2$ in the last line. Taking $N = n + 3$ we get then

$$\int_{\Omega_k} \bar{r}^n |\phi|^2 d\Omega_k \lesssim \frac{1}{k} \int_{-\infty}^{\infty} \frac{1}{(1+|t|)^{1+\eta}} dt \lesssim \frac{1}{k} \rightarrow 0 \text{ as } k \rightarrow \infty,$$

for any $n \in \mathbb{N}$ as desired. ■

Then this lemma covers functions ϕ as in Definition 1.4.1, now we check the modified versions \mathbf{W} , \mathbf{E} , etc. We start by bounding $\varpi_{\alpha\beta}$ and $\nabla_\nu \varpi_{\alpha\beta}$. Recall that ϖ was defined as the solution of a transport equation of the form

$$\nabla_L \varpi = \mathcal{L}_T \mathbf{g} \cdot \nabla L.$$

This implies the following transport equation for $\nabla\varpi$,

$$\nabla_L \nabla\varpi = \nabla \mathcal{L}_T \mathbf{g} \cdot \nabla L + \mathcal{L}_T \mathbf{g} \cdot \nabla \nabla L + \nabla \mathcal{L}_T \mathbf{g} \cdot \varpi.$$

We will use the following Lemma to deduce the estimate (3.40) from these transport equations.

Lemma 3.4.2. *Let V be a tensor satisfying the tensorial transport equation*

$$\nabla_L V = W$$

Assume the components of W with respect to the frame $\{e_\mu\}$ satisfy (3.40). Then the components of V also satisfy (3.40).

Proof of Lemma. We need to analyse the extra terms arising from considering the components of the covariant derivative, we have

$$\nabla_\nu V_{\alpha_1 \dots \alpha_n} = e_\nu(V_{\alpha_1 \dots \alpha_n}) + \sum_i \omega_\nu^\beta{}_{\alpha_i} V_{\alpha_1 \dots \beta \dots \alpha_n}.$$

A signature consideration leads to

$$\nabla_0 V(\mathbf{s}) = \nabla_L V(\mathbf{s}) = L(V(\mathbf{s})) + \omega(0)V(\mathbf{s} + 1),$$

where $V(\mathbf{s})$ denotes a null component of the tensor V of signature \mathbf{s} . This is due to the fact that the only non-vanishing connection coefficient of the form $\omega_{0\alpha\beta}$ is $\omega_{0i1} = -2\zeta_i$ which has signature 0.

Hence the transport equation for V splits hierarchically into

$$\begin{aligned} \partial_s V(n) &= W, \\ \partial_s V(n-1) &= W + \zeta \cdot V(n), \\ &\vdots \\ \partial_s V(-n) &= W + \zeta \cdot V(-n+1). \end{aligned}$$

Now, simple integration and the fact that the components of W satisfy (3.40) implies

$$|V(n)| \lesssim \frac{1}{s^N (1 + |u|)^{1+\eta}}, \quad N \in \mathbb{N}.$$

We use these bounds on $V(n)$ and W together with the boundedness of ζ_i and the

transport equation for $V(n-1)$ to also conclude

$$|V(n-1)| \lesssim \frac{1}{s^N(1+|u|)^{1+\eta}}, \quad N \in \mathbb{N}.$$

Proceeding inductively we get that all the components of V satisfy (3.40). ■

These two Lemmas tell us that $\varpi_{\alpha\beta}$ and $\nabla_\nu \varpi_{\alpha\beta}$ vanish to all orders at infinity in the sense of (3.41). Then the bound $|\tilde{V}_{\alpha_1 \dots \alpha_n}| \lesssim \sum |V_{\beta_1 \dots \beta_n}|$ ensures that the same condition holds for the components with respect to the basis $\{\tilde{e}_\mu\}$.

Finally, let (V) denote the components of V with respect to the frame $\{\tilde{e}_\mu\}$. We know,

$$\begin{aligned} (\mathbf{B}) &= (\mathcal{L}_T \mathbf{g}) + (\varpi), \\ (\nabla \mathbf{B}) &= (\nabla \mathcal{L}_T \mathbf{g}) + (\nabla \varpi) + (\mathbf{B}) \cdot (\Gamma). \end{aligned}$$

Thus (\mathbf{B}) and $(\nabla \mathbf{B})$ also satisfy (3.41). We proceed similarly with the remaining tensors:

$$\begin{aligned} (\mathbf{W}) &= (\mathcal{L}_T \mathbf{C}) + (\mathbf{B}) \cdot (\mathbf{C}), \\ (\nabla \mathbf{W}) &= (\nabla \mathcal{L}_T \mathbf{C}) + (\nabla \mathbf{B}) \cdot (\mathbf{C}) + (\mathbf{B}) \cdot (\nabla \mathbf{C}) + (\mathbf{W}) \cdot (\Gamma), \\ (\mathbf{E}) &= (\mathcal{L}_T \mathbf{F}) + (\mathbf{B}) \cdot (\mathbf{F}), \\ (\nabla \mathbf{E}) &= (\nabla \mathcal{L}_T \mathbf{F}) + (\nabla \mathbf{B}) \cdot (\mathbf{F}) + (\mathbf{B}) \cdot (\nabla \mathbf{F}) + (\mathbf{E}) \cdot (\Gamma). \end{aligned}$$

We get that they also satisfy (3.41). Therefore condition (3.21) is fulfilled and we can apply Proposition 3.3.1 to the spacetimes under consideration and conclude stationarity in a neighbourhood of spatial infinity. ■

3.5 Time-periodic spacetimes

We finish this chapter by proving that time-periodic spacetimes are indeed stationary. The main focus will be on periodic spacetimes which are regular at spatial infinity and smooth at future null infinity, cf. [12] and [6].

Definition 3.5.1. Time-periodic spacetimes. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be a regular at spatial infinity and smooth at null infinity spacetime. We say that it is time-periodic if there exists a discrete smooth isometry Φ with time-like orbits. That is, $\Phi^* \mathbf{g} = \mathbf{g}$ and for any $p \in \mathcal{M}$ there is a future directed time-like curve from p to $\Phi(p)$. In*

addition we require

- There is a constant $T > 0$ such that, with respect to the coordinates $(t, r, \vartheta^2, \vartheta^3)$,

$$|t(\Phi(p)) - t(p)| < T \quad \text{for all } p \in \mathcal{M}.$$

- There is a constant $t_0 > 0$ such that, with respect to the coordinates $(u, s, \theta^2, \theta^3)$,

$$\lim_{C_{u;s \rightarrow \infty}} \Phi(u, s, \theta^2, \theta^3) = (u + t_0, \theta^2, \theta^3).$$

Remark. Note that \mathbf{F} is not required to satisfy the time-periodic symmetry.

In the presence of a discrete isometry, Φ , is useful to choose a set containing exactly one representative for each equivalence class defined by the isometry Φ ; we call such a set a *fundamental domain*. Formally, a connected set $\Omega \subset \mathcal{M}$ is a fundamental domain for Φ if

- The boundary of Ω consists of two smooth hypersurfaces Σ_1, Σ_2 such that $\Phi(\Sigma_1) = \Sigma_2$.
- For each point $p \in \Omega$ we have $\Phi(p) \notin \Omega$, and for each $q \notin \Omega$ there is an $n \in \mathbb{Z}$ such that $\phi^{(n)}(q) \in \Omega$.

As a consequence of Proposition 3.3.1 we have:

Corollary 3.5.2. Stationarity of time-periodic spacetimes. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ be an electrovacuum spacetime, regular at spatial infinity, close to Kerr-Newman, smooth at null infinity and time-periodic. Then there exists a time-like vector field T in a neighbourhood of spatial infinity such that*

$$\mathcal{L}_T \mathbf{g} = 0 = \mathcal{L}_T \mathbf{F}.$$

Proof. The time-periodicity condition at null infinity imply in particular that $\underline{\chi}^{(2)}(u, \theta^2, \theta^3) = (u + t_0, \theta^2, \theta^3)$. But $\underline{\chi}^{(2)}$ is non-decreasing since

$$2\partial_u \text{tr} \underline{\chi}^{(2)} = |\underline{\Xi}|^2 + |\underline{A}(F)|^2 \geq 0.$$

Hence $\underline{\Xi}_{ij}$ and $\underline{A}(F)_i$ vanish and the non-radiating condition *i*) of Definition 1.4.1 is satisfied. Let ϕ be as in Definition 1.4.1. Then we know that the non-radiating condition implies

$$|\phi| \lesssim \frac{1}{s^N} \quad N \in \mathbb{N}.$$

Now we check condition *ii*) of Definition 1.4.1. We can choose $\Sigma_1 = \{p \in \mathcal{M} : t(p) = 0\}$, then we know that $t(p) \in [0, T]$ for all $p \in \Omega$. Hence for $(u, s, \theta^2, \theta^3) \in \Omega$ we have

$$1 + |u|^2 \lesssim |t - r|^2 \lesssim |r|^2 \lesssim |s|^2.$$

Therefore, on Ω we have

$$|\phi| \lesssim \frac{1}{|s|^2} \lesssim \frac{1}{1 + |u|^2}.$$

But ϕ depends only on its values on Ω by time-periodicity, therefore this bound holds everywhere and we can apply Theorem 1.4.2 to conclude stationarity. ■

Chapter 4

Final remarks

In this chapter we review the main assumptions of Theorem 1.4.2 and explain possible ways to relax them. We focus on the construction of the candidate Killing field and the regularity assumptions at spatial and null infinity.

We also present conjectures regarding generalisations of the techniques presented here in two directions: Inheritance of symmetries for other matter models, e.g., Klein-Gordon; and unique continuation from infinity for the Einstein equations.

4.1 Assumptions of the main theorem revisited

4.1.1 Candidate Killing field

In Section 1.3 we constructed a coordinate system adapted to future null infinity $(u, s, \theta^2, \theta^3)$, the candidate Killing field was defined as $T = \partial_u$. The construction is motivated by Christodoulou [19] and Christodoulou-Klainerman, [22], however there is a crucial difference in that their construction is based on a limiting procedure while ours takes place at a finite region of the spacetime. More precisely, they construct on (an exterior region¹ of) each slab $\Sigma \times [-1, t_*]$ an optical function u_{t_*} then, their canonical optical function u is the limit as $t_* \rightarrow \infty$. Our construction, Section 1.3, on the other hand takes place at a sphere of finite radius S_0 . The main reason to do so is to exploit the gauge freedom associated to the parametrisation of the null geodesics ruling the incoming null hypersurface $\underline{\mathcal{C}}_0$. This gauge freedom was used to set $\langle \partial_u, \partial_s \rangle = -1$ (cf. Lemma 1.3.4). In the CK setting this latter equality just holds as $s \rightarrow \infty$.

¹c.f. CK Theorem 1.2.2

We also remark that the CK conclusions regarding the asymptotic behaviour of the fields with respect to the coordinates $(u, s, \theta^2, \theta^3)$ are also assumed implicitly in the the definition of smoothness at null infinity, 1.3.8. These considerations make the construction of the present candidate Killing field T unappealing.

We believe that a more appropriate construction can be achieved “from infinity” by tracing carefully the recurrence relations of Proposition 2.1.1 with less restrictive gauge choices. More precisely, CK already provide us with an approximate time-like vector field, ∂_t ; this can be taken as a first order approximation for T . Then, in the case of vanishing radiation fields, an analysis of the recurrence relations should hint at the higher-order correction terms for T by requiring it to be a Killing field to all orders at infinity. This program is left for future research.

4.1.2 Regularity of null infinity

Thanks to the work of Friedrich [34], [35], [36], Christodoulou-Klainerman [22], Klainerman-Nicoló [49], Valiente-Kroon [62] and many others, now it is widely accepted that smoothness at null infinity is not entirely compatible with the Einstein’s equations from an evolution point of view. That is, initial Cauchy data smooth up to infinity does not necessarily evolve into a spacetime having a smooth null infinity, [62]. Nevertheless, there are radiating spacetimes admitting a conformal compactification smooth at both past and future null infinities, see Cutler and Wald [28] and Chruściel and Delay [25].

In Section 1.3.3 we required that our spacetimes satisfy Definition 1.3.8, this encodes smoothness for \mathbf{g} at future null infinity in the sense of conformal compactification. To see this consider $\tilde{s} := s^{-1}$ and let $\tilde{\mathbf{g}} = \tilde{s}^2 \mathbf{g}$ be a conformal rescaling, then $\tilde{\mathbf{g}}$ extends continuously to $\tilde{s} = 0$ and admits infinitely many $\partial_{\tilde{s}}$ -derivatives but only derivatives up to third order in the (u, θ^2, θ^3) -directions. In this sense, this definition is still too strong. A less restrictive assumption requires however an understanding of the non-smooth terms appearing at null infinity arising from general asymptotically flat initial data, which we lack at present. Some progress in this direction was made by Klainerman and Nicoló in [48].

The non-smooth behaviour mentioned above carries on also to the change of coordinates $(t, r, \vartheta^2, \vartheta^3) \rightarrow (u, s, \theta^2, \theta^3)$. Therefore the assumption regarding this change of coordinates will require revision as well once we know more about the general asymptotic expansions at future null infinity.

4.1.3 Regularity of spatial infinity

Another important assumption of Theorem 1.4.2 was that of regularity at spatial infinity, Definition 1.3.2. As mentioned in the ensuing Remark, this is a “preservation of regularity” hypothesis.

Given suitably regular initial strongly asymptotically flat Cauchy data (Σ, h, k) , it is desirable to obtain a detailed asymptotic expansion, $\mathbf{g} = \eta + \mathbf{g}^\infty$, of the corresponding solution. Such expansion, most likely, will contain logarithmic terms owing to the non-smoothability at infinity. In addition, it will be of utmost importance to deduce asymptotic expansions for the change of coordinates corresponding to the main three choices: space+time, outgoing null and double-null coordinates. They are important to understand the failure of smoothness at infinity as well. Moreover, they play a crucial role when relating initial Cauchy data to null infinity data. The program just described is a formidable task closely related to the local (around infinity but global in time) stability of suitably regular and small initial data.

We stress here Friedrich’s analysis of his conformal equations, [37], [38]. Based on a suitably gauge he manage to write the Einstein equations as a part of a system extending smoothly up to infinity. His conformal gauge condition is such that spatial infinity is now depicted as a cylinder $\mathcal{I} \cong \mathbb{S}^2 \times (-1, 1)$ where each \mathbb{S}^2 -slice serves as a compactification of the time-like hypersurfaces Σ_t . More importantly, his conformal field equations form a 1st order system of PDEs which is hyperbolic up to spatial infinity² \mathcal{I} .

This analysis provides strong evidence about the preservation of regularity at spatial infinity for arbitrarily large but finite time, since the hyperbolic character of the equations allows the use of energy methods. In such a case, the regularity at spatial infinity assumption for time-periodic spacetimes can be dropped. However for non-radiating spacetimes this is still a strong regularity assumption.

4.2 Einstein-Maxwell-Klein-Gordon system

In this section we discuss the conclusion of Theorem 1.4.2 when other matter/energy models are considered. We present the result when a massless Klein-Gordon field is included. As remarked in the Introduction, this conclusion fails for massive matter

²It crucially loses its hyperbolic character on the spheres where spatial infinity “touches” null infinity, $\mathbb{S}^2 \times \{\pm 1\}$.

fields, [14], [16].

We start by putting the result into context. In [29], Dafermos establishes a similar rigidity theorem for spherically symmetric Einstein-matter systems which are time-periodic. This corresponds, roughly speaking, to a “no-hair” result for spherically symmetric time-periodic black holes, thus generalising the work of Bekenstein [11]. More precisely, he concludes that solutions to asymptotically flat spherically symmetric time-periodic Einstein-matter systems are either Schwarzschild or Reissner-Nordström spacetimes with vanishing matter fields. He assumes certain structure for the matter fields which includes, as examples, a wave map and a massive charged scalar field interacting with electromagnetism. Also, another important assumption on the underlying spacetime is that of the existence a bifurcate horizon. Indeed, Dafermos’ analysis (unlike ours) takes place at the event horizon, where he shows vanishing of initial conditions for a 2D-characteristic problem which implies the vanishing of the fields in the domain of outer communications³.

Here we adopt the “far-away” point of view and propose the following:

Conjecture 4.2.1. *Consider a spacetime solution of the Einstein-Maxwell equations coupled to a massless Klein-Gordon field, $(\mathcal{M}, \mathbf{g}, \mathbf{F}, \varphi)$, which is regular at spatial infinity, close to Kerr-Newman and smooth at future null infinity. Assume it is non-radiating, then there exists a time-like vector field T such that*

$$\mathcal{L}_T \mathbf{g} = 0 = \mathcal{L}_T \mathbf{F} = \mathcal{L}_T \varphi, \quad \text{on } \mathcal{D}_\epsilon,$$

for $\epsilon > 0$ small enough.

Remark. Here, smoothness at future null infinity is exactly as in Definition 1.3.8 with now $\varphi = \mathcal{O}_3^\infty(s^{-1})$. Also, non-radiating is as in Definition 1.4.1 with now

$$\overset{(1)}{\varphi} = 0 \quad \text{and} \quad |\mathcal{L}_T \varphi|, |e_\mu(\mathcal{L}_T \varphi)| \lesssim \frac{1}{(1 + |u|)^{1+\eta}}, \quad \eta > 0.$$

The strategy to prove the result would be the same as the one used for Theorem 1.4.2.

Firstly, we state the expected recurrence relations below. We use the notation

³In (1+1)-spacetime dimensions one has the nice property that an initial value problem set at a bifurcate horizon (or time-like hypersurface) is locally well-posed. This is seen by redefining the metric to be its negative.

of the Toy model at the beginning of Chapter 2; that is,

$$\begin{aligned} X_\mu &:= \nabla_\mu \varphi, \\ x &:= X_0 = e_0(\varphi), \quad \underline{x} := X_1 = e_1(\varphi), \quad \mathcal{X}_i := X_i = e_i(\varphi). \end{aligned}$$

Conjecture 4.2.2. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F}, \varphi)$ be a solution of the Einstein-Maxwell equations coupled to a massless Klein-Gordon field. Assume it is smooth at future null infinity with $\partial_s = e_0$ a null geodesic vector field and $\omega_{023} = 0$. Then the asymptotic quantities (with respect to coordinates $(u, s, \theta^2, \theta^3)$ and frame $\{e_0, e_1, e_2, e_3\}$ as defined in 1.3.2) satisfy the following recurrence relations for any $n \in \mathbb{N}$,*

$$\alpha_{ij}^{(n+1)} = (n-1)\chi_{ij}^{(n)} - 2 \left(\alpha(\mathbf{F})_k \alpha(\mathbf{F})^k + 2\text{Re} \bar{x} x \right) \eta_{ij} + \lfloor n-1 \rfloor, \quad (4.1a)$$

$$h_i^j = \chi_i^k h_k^j + \lfloor n-1 \rfloor, \quad (4.1b)$$

$$\beta_i^{(n+1)}, \rho(\mathbf{F})^{(n)}, \sigma(\mathbf{F})^{(n)}, \mathcal{X}_i^{(n)}, \omega_{jji}^{(n)}, \zeta_i^{(n)}, f^i = \{\chi_{ij}^{(n)}, \alpha(\mathbf{F})_i^{(n)}, \bar{x}^{(n)}, \lfloor n-1 \rfloor\}, \quad n > 3 \quad (4.2a)$$

$$\rho^{(n+1)}, \sigma^{(n+1)}, \underline{\alpha}(\mathbf{F})^{(n)}, \underline{x}^{(n)}, \underline{\omega}, \omega_{123}^{(n)}, \underline{\chi}_{ij}^{(n)}, \underline{\xi}_i^{(n)}, f^0 = \{\chi_{ij}^{(n)}, \alpha(\mathbf{F})_i^{(n)}, \bar{x}^{(n)}, \lfloor n-1 \rfloor\}, \quad n > 2. \quad (4.3a)$$

Moreover,

$$2\partial_u \chi_{ij}^{(n+1)} = -n \underline{\chi}_{ij}^{(n)} + \{\chi_{ij}^{(n)}, \alpha(\mathbf{F})_i^{(n)}, \bar{x}^{(n)}, \lfloor n-1 \rfloor\}, \quad (4.4a)$$

$$2\partial_u \alpha(\mathbf{F})_i^{(n+1)} = \{\alpha(\mathbf{F})_i^{(n)}, \rho(\mathbf{F})^{(n)}, \sigma(\mathbf{F})^{(n)}, \lfloor n \rfloor\}, \quad (4.4b)$$

$$2\partial_u \bar{x}^{(n+1)} = \{\bar{x}^{(n)}, \lfloor n \rfloor\}, \quad (4.4c)$$

and,

$$\underline{\beta}_i^{(n+1)} = \partial_u \zeta_i^{(n+1)} + \lfloor n \rfloor, \quad (4.5a)$$

$$\underline{\alpha}_{ij}^{(n+1)} = \partial_u \underline{\chi}_{ij}^{(n+1)} + \lfloor n \rfloor. \quad (4.5b)$$

$$(4.5c)$$

Next, let $Y_\mu := \mathcal{L}_T X_\mu - \mathbf{B}_\mu{}^\nu X_\nu$ be the deformation tensor associated to φ . Then the unique continuation from infinity would read,

Conjecture 4.2.3. *Let $(\mathcal{M}, \mathbf{g}, \mathbf{F}, \varphi)$ be a solution of the Einstein-Maxwell equations coupled to a massless Klein-Gordon field admitting asymptotically double-null coordinates $(v, \underline{v}, y^2, y^3)$ such that \mathbf{g} takes the form (1.48). Assume that, with respect to the frame*

$$\{\tilde{e}_0 = \partial_v, \tilde{e}_1 = \partial_{\underline{v}}, \tilde{e}_2 = \frac{1}{\bar{r}}\partial_{y^2}, \tilde{e}_3 = \frac{1}{\bar{r}}\partial_{y^3}\},$$

the Weyl, Faraday and Klein-Gordon components satisfy:

$$|\mathbf{C}_{\alpha\beta\mu\nu}| = O'_1(\bar{r}^{-3}), \quad |\mathbf{F}_{\alpha\beta}| = O'_1(\bar{r}^{-2}), \quad |X_\alpha| = O'_1(\bar{r}^{-2}). \quad (4.6)$$

Consider L and T vector fields with $L = \partial_{\underline{v}} + \sum_{\mu=0}^3 O'_2(\bar{r}^{-1})\tilde{e}_\mu$ null and satisfying $\nabla_L L = 0$, $[L, T] = 0$. Assume furthermore that T is a symmetry to all orders at infinity in the sense that,

$$\lim_{k \rightarrow \infty} \int_{\Omega_k} \bar{r}^N |\phi|^2 = 0, \quad (4.7)$$

for all the components, $\phi = \mathbf{W}_{\alpha\beta\mu\nu}, \nabla_\rho \mathbf{W}_{\alpha\beta\mu\nu}, \mathbf{E}_{\alpha\beta}, \nabla_\rho \mathbf{E}_{\alpha\beta}, Y_\alpha, \nabla_\rho Y_\alpha, \mathbf{B}_{\alpha\beta}, \nabla_\rho \mathbf{B}_{\alpha\beta}, \mathbf{P}_{\alpha\beta\mu}, \nabla_\rho \mathbf{P}_{\alpha\beta\mu}$, with respect to the frame $\{\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$. And where

$$\Omega_k := \{(v, \underline{v}, y^2, y^3) : \underline{v} - v = k\} \cap \mathcal{D}_\epsilon, \quad \epsilon > 0.$$

Then T is in fact locally a genuine symmetry for $(\mathcal{M}, \mathbf{g}, \mathbf{F}, \varphi)$, namely

$$\mathcal{L}_T \mathbf{g} \equiv 0, \quad \mathcal{L}_T \mathbf{C} \equiv 0, \quad \mathcal{L}_T \mathbf{F} \equiv 0, \quad \mathcal{L}_T \varphi \equiv 0, \quad \text{on } \mathcal{D}_{\epsilon'}$$

for some $0 < \epsilon' < \epsilon$.

4.3 Unique continuation for the Einstein's equations

The motivation for this section comes from the study of the well-posedness of the Einstein's equations when the initial data is given at a time-like or characteristic hypersurface. The regularity assumptions here play an important role. In the class of analytic functions, existence and uniqueness ensue via the Cauchy-Kowalevski theorem. In the class of smooth metrics the problem is more subtle and the tool of Carleman estimates has proven to be extremely useful. Roughly speaking, these are generalised energy estimates adapted to any initial hypersurface, we need however,

to require extra geometric assumptions on the hypersurface (the so-called pseudo-convexity condition, which for time-like hypersurfaces takes the form explained in Definition 3.1.1) in order to achieve local uniqueness for quasi-linear hyperbolic systems. Therefore, when the hypersurface is time-like pseudo-convex it is possible to prove uniqueness of solutions in the “exterior” region. The same result holds for characteristic hypersurfaces which are close to being pseudo-convex. As an example, here we quote a unique continuation theorem across a bifurcate horizon for the Einstein’s vacuum equations due to Alexakis, [3]. This result serves as motivation for Conjecture 4.3.2.

Consider a Cauchy surface Σ_0 with boundary $\partial\Sigma_0 = S$ and let $(\mathcal{M}, \mathbf{g})$ be its maximal Cauchy development under Einstein’s equations. It is known that in a neighbourhood of S the boundary of \mathcal{M} consists of a union of two null hypersurfaces I and J . Then $S = I \cap J$ is known as the bifurcate sphere and $I \cup J$ is known as the bifurcate horizon. The main theorem in [3] is:

Theorem 4.3.1. *Let $(\mathcal{M}, \mathbf{g})$, $(\hat{\mathcal{M}}, \hat{\mathbf{g}})$ be two vacuum spacetimes as described above. Denote by S, \hat{S} their bifurcate spheres and by $I \cup J, \hat{I} \cup \hat{J}$ their bifurcate horizons. Assume there exist points $P \in S, \hat{P} \in \hat{S}$ and relative open sets $V \subset \mathcal{M}, \hat{V} \subset \hat{\mathcal{M}}$ containing S, \hat{S} , respectively, and a diffeomorphism $\Phi : V \rightarrow \hat{V}$ so that $\mathbf{g} - \Phi^*\hat{\mathbf{g}}$ vanishes to third order on $(I \cup J) \cap V$. Then the metrics $\mathbf{g}, \hat{\mathbf{g}}$ are isometric in some relative open neighbourhoods of P, \hat{P} in $\mathcal{M}, \hat{\mathcal{M}}$.*

We would like to analyse the same problem from the point of view of data given at infinity. Examples of the free wave equation in a fixed background and the family of Weyl metrics lead to the conclusion that it is necessary to provide data to all orders at infinity. However, the recurrence relations of Proposition 2.1.1 suggest that these infinitely-many data are stationary, that is, they are given at spatial infinity and hence can be computed from Cauchy data. On the other hand the evolution of the system is determined solely by the radiation fields along null infinity, cf. Proposition 2.1.3.

Conjecture 4.3.2. (Unique continuation from infinity for Einstein’s equations.) *Consider two vacuum asymptotically flat spacetimes. Suppose that, in outgoing null adapted coordinates,*

- *Their radiation fields agree at null infinity.*
- *Their pole moments agree at spatial infinity. That is, the limits as $u \rightarrow -\infty$ of the mass aspect function, the angular momentum aspect vector and $\alpha_{ij}^{(n)}$, $n \in \mathbb{N}$, are equal.*

Similarly for past null infinity. Then, they are isometric in a neighbourhood of spatial infinity.

Remark. We stress once more that data is necessary to all orders at infinity. The family of Weyl spacetimes, Section A.4, provides a counter-example to the statement: *An asymptotically flat spacetime is determined in a neighbourhood of infinity by its Bondi mass, angular momentum and radiated power.* Indeed, a Weyl solution is static and hence it is a non-radiating spacetime. Moreover, the pole moments, given in this case by the coefficients a_l can vanish to any order at infinity (that is, $a_k = 0$ for $k < K$ for K arbitrary) and nevertheless the metric is non-flat, as one would naively expect from the vanishing of the leading terms: radiation, mass and angular momentum.

4.4 Conclusions

The goal of this thesis was to investigate the inheritance of symmetry property for the Einstein's equations when matter/energy models are included. Indeed, we showed the validity of this property in the context of asymptotically flat electrovacuum spacetimes: An asymptotic time-like symmetry to all orders at infinity is indeed a (local) symmetry of both gravity and electromagnetism, Proposition 3.3.1. We also provided weaker conditions for the first condition to hold, namely, an asymptotic time-like symmetry to first order in a non-radiating spacetime must be an asymptotic time-like symmetry to all orders, Proposition 2.1.4. We also sketched the proof for the same results when a massless Klein-Gordon field is also present, but we stressed that the conclusion no longer holds for the positive-mass case.

The assumed regularity assumptions are still artificial and probably restrict too much the class of spacetimes satisfying them. Nevertheless, we expect to extend the techniques employed here to asymptotic expansions including time-independent logarithmic terms. However, the precise class of regularity conditions compatible with physical systems are still not well understood and a generalisation in this direction seems to need a different approach. Also, it is important to remark that the regularity assumptions used in this thesis can be deduced from regular initial data if the resulting development is time-periodic, provided the preservation of regularity property at spatial infinity holds, see Section 4.1.3.

We expect also the same unique continuation property to hold for other matter model such as the massless Klein-Gordon field, Conjecture 4.2.1. In addition we

also propose a unique continuation from infinity result for the Einstein's equation themselves in Conjecture 4.3.2. The regularity assumptions for such a result will also have to be assumed at this point.

Appendix A

Exact solutions

A.1 Minkowski spacetime

Consider \mathbb{R}^4 with the standard Lorentzian metric,

$$\mathbf{g}_M = -dt^2 + dx^2 + dy^2 + dz^2.$$

We are interested in the causal structure of this spacetime and its behaviour at infinity. It is possible to gain a deep insight of the global structure by conformally embedding Minkowski spacetime into a bounded region of some other Lorentzian manifold. We start by writing the metric in spherical coordinates,

$$\mathbf{g}_M = -dt^2 + dr^2 + r^2\check{\gamma},$$

where $\check{\gamma} = d\theta^2 + \sin^2(\theta)d\phi^2$ is the standard round metric on \mathbb{S}^2 . Then define the null coordinates

$$\begin{aligned} u &= t - r, & v &= t + r, \\ q &= \arctan(u), & p &= \arctan(v), \end{aligned}$$

the latter being just a re-parametrization to bring infinity to a bounded region. Note that working with null coordinates guarantees the conformality of the coordinate transformation.

Finally we go back to time-radial coordinates,

$$t' = \frac{1}{2}(p + q), \quad r' = \frac{1}{2}(p - q).$$

It can be checked that the metric in these new coordinates is

$$\mathbf{g}_M = \sec^2(t' + r') \sec^2(t' - r') \underbrace{(-dt'^2 + dr'^2 + \sin^2(r')\check{\gamma})}_{\bar{\mathbf{g}}}.$$

So, except for the \sec^2 factors, we recognise it as the canonical metric for the space $\mathbb{R} \times S^3$.

Summarising, we have constructed an embedding $\Phi : (\mathbb{R}^4, \eta) \rightarrow (\mathbb{R} \times S^3, \bar{g})$ and a function $\Omega : \mathbb{R}^4 \rightarrow \mathbb{R}$ such that:

$$\bar{\mathbf{g}} = \Omega^2 \Phi_*(\mathbf{g}_M).$$

This means that all the causal structure is preserved by the embedding, that is, a vector is time-like (null or space-like) with respect to g if and only if it is time-like (null or space-like) with respect to \bar{g} .

Now, we observe that the image of \mathbb{R}^4 under the embedding is the region where $-\frac{\pi}{2} < t' + r' < \frac{\pi}{2}$ and $-\frac{\pi}{2} < t' - r' < \frac{\pi}{2}$. The boundary of this region is precisely where the conformal factor, Ω , vanishes and it consists of three points and two null surfaces, see Figure A.1. Note that i^-, i^+ can be regarded as past and future infinities, since any future directed time-like geodesic must start and end at these points; in the same manner i^0 is called spatial infinity. Similarly, the null hypersurfaces J^- and J^+ are called past and future null infinities, respectively.

Remark. It is worth noticing that the metric \bar{g} extends smoothly to infinity. This will not be the case for the following constructions involving mass (see remark following the description of the Schwarzschild spacetime). To the author's knowledge it remains an open question whether this condition characterises completely flat spacetime.

It is illustrative to compute the connection coefficients of the Minkowski spacetime in outgoing null coordinates. These values correspond to the leading orders of the connection coefficients of any asymptotically flat spacetime in this gauge.

Consider the Minkowski metric in outgoing null coordinates,

$$\mathbf{g}_M = -du^2 - 2dudr + r^2\check{\gamma},$$

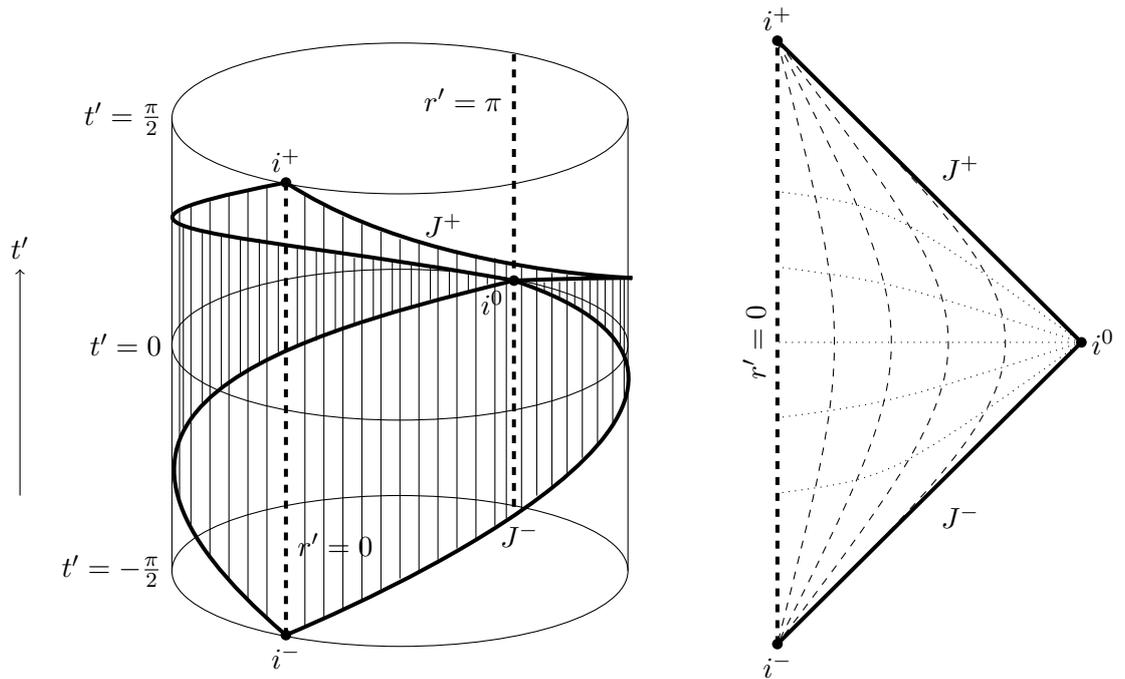


Figure A.1: Left: Embedding of Minkowski space into $\mathbb{R} \times S^3$. Coordinates (θ, ϕ) have been suppressed and each point represents one half of a 2-sphere of area $4\pi \sin^2 r'$. Note that points on the dashed lines are truly points, due to the singularity of polar coordinates. Right: The diagram in the (t', r') -plane is known as a *Penrose-Carter diagram*. Each point (with the exception of the line $r' = 0$ and i^0) represents a sphere. Dashed lines correspond to $\{r = \text{constant}\}$ and dotted to $\{t = \text{constant}\}$. Radial null curves are at 45° .

and choose the null frame,

$$\begin{aligned} e_0 &= L = \partial_r, & e_1 &= \underline{L} = 2\partial_u - \partial_r, \\ e_2 &= \frac{1}{r}\partial_\theta, & e_3 &= \frac{1}{r\sin\theta}\partial_\phi. \end{aligned}$$

That is, the orthonormalisation matrix is given by

$$h_\mu{}^\nu = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}.$$

The null second fundamental forms are then,

$$\chi_{ij} = -\underline{\chi}_{ij} = \frac{1}{r} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The other non-vanishing connection coefficients are ω_{223} and ω_{332} , that is, those corresponding to the induced connection on the sphere of radius r .

A.2 Schwarzschild spacetimes

It can be checked that the Schwarzschild metric

$$g_{Sch} = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \check{\gamma},$$

is a vacuum solution of the Einstein's equations. The constant M can be interpreted as the total mass of the system; this solution is regarded as modelling the vacuum space surrounding a massive spherical (non-rotating) body.

The coordinates (t, r, θ, ϕ) are valid only in the region $U = \mathbb{R} \times (2M, \infty) \times \mathbb{S}^2$, since the metric becomes degenerate at $r = 2M$. However this is a coordinate artefact and it is possible to extend the Schwarzschild solution by defining $r_* = \int^r \left(1 - \frac{2M}{\tilde{r}}\right)^{-1} d\tilde{r} = r + 2M \log(r - 2M)$ and null coordinates:

$$\begin{aligned} u &= t - r_*, & v &= t + r_*, \\ u' &= \exp(u/4M), & v' &= \exp(v/4M), \end{aligned}$$

And finally, $t' = \frac{1}{2}(v' + u')$, $x' = \frac{1}{2}(v' - u')$. In the coordinates (t', x', θ, ϕ) the

metric takes the form

$$\mathbf{g}_{Sch} = \exp(-r/2m) \frac{16m^2}{r} (-dt'^2 + dx'^2) + r(t', x')^2 \check{\gamma},$$

where r is defined implicitly by $t'^2 - x'^2 = -(r - 2M) \exp(r/2M)$. In this form it can be observed that the image of U corresponds only to $I = \{|x'| < t'\}$, whereas the metric is non-degenerate in the region $t'^2 - x'^2 < 2M$, hence this new metric, known as the Kruskal-Szekeres extension, smoothly extends the original. This is maximal, since the degeneracy at $r = 0$ is a true singularity, as can be checked by computing the Kretschmann scalar $R^{abcd}R_{abcd}$.

Once again it is possible to perform a similar process as before to embed the extended Schwarzschild solution into another manifold, this is done by defining the null coordinates $v'' = \arctan(v'/\sqrt{2M})$, $u'' = \arctan(u'/\sqrt{2M})$ and then $t'' = (v'' + u'')/2$, $x'' = (v'' - u'')/2$. The resulting structure is summarised in Figure A.2.

Remark. It can be checked that the resulting conformal metric extends continuously to infinity but it is not differentiable there; for example, $\frac{\partial r}{\partial u''}$ cannot be extended continuously. On the other hand, a smooth compactification of future null infinity only is considerably easy to achieve by working in outgoing null coordinates, (A.1). In order to achieve a regular infinity one needs to perform a more careful analysis if the rate of decay towards infinity in double-null coordinates. Indeed, Schmidt and Stewart, [60], provided a conformal compactification which is analytical at I^\pm and C^0 at i^0 .

Now we compute the connection coefficients and curvature components of the Schwarzschild spacetime, or more precisely, the charged version, known as the Reissner-Nordström spacetime. In outgoing null coordinates the metric is given by,

$$\mathbf{g}_{RN} = - \left(1 - \frac{2M}{r} + \frac{e^2}{r^2} \right) du^2 - 2dudr + r^2 \check{\gamma}, \quad (\text{A.1})$$

For brevity we will denote $f(r) := 1 - \frac{2M}{r} + \frac{e^2}{r^2}$. Consider the adapted null frame

$$\begin{aligned} e_0 &= L = \partial_r, & e_1 &= \underline{L} = 2\partial_u - f\partial_r, \\ e_2 &= \frac{1}{r}\partial_\theta, & e_3 &= \frac{1}{r \sin \theta}\partial_\phi. \end{aligned}$$

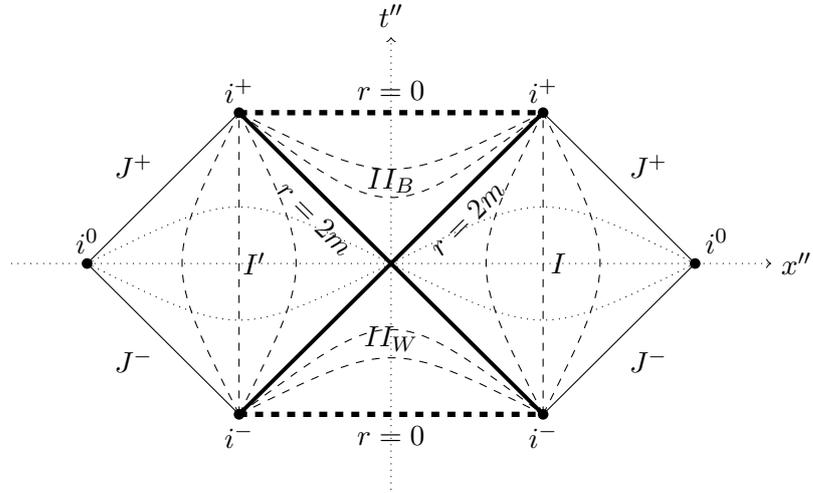


Figure A.2: Penrose-Carter diagram for Schwarzschild space in (t'', x'') coordinates. Dashed lines correspond to $\{r = \text{constant}\}$ and dotted to $\{t = \text{constant}\}$. Region I is the domain of the original coordinates. Region I' is isometric to region I , so is regarded as “another universe”. Region II_B is called a Black Hole, since it is impossible for any point in its interior to send signals to the exterior. Indeed, any future-directed time-like or null curve will inevitably fall into the singularity $r = 0$ within a finite proper time. Region II_W is called a White Hole since it has the opposite behaviour, any event in its interior is eventually seen by an observer in I (or in I').

The only non-vanishing connection coefficients are

$$\begin{aligned}\underline{\omega} &= -\frac{1}{2}\partial_r f = -\frac{M}{r^2} + \frac{e^2}{r^3}, \\ \chi_{ij} &= \frac{1}{r} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \underline{\chi}_{ij} &= \frac{f(r)}{r} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\end{aligned}$$

Together with ω_{223} and ω_{332} which again are the connection coefficients corresponding to a round sphere of radius r . Finally the only non-vanishing components of the

Weyl and Faraday tensors are

$$\rho = \frac{2M}{r^3},$$

$$\rho(\mathbf{F}) = -\frac{e}{r^2}.$$

Good intuition about the relation between the different choices of gauge conditions can be gained by comparing these values with those obtained using double-null coordinates. That is, consider the Reissner-Nordström metric in double null coordinates,

$$\mathbf{g}_{RN} = -2 \left(1 - \frac{2M}{r} + \frac{e^2}{r^2} \right) dudv + r^2 \check{\gamma},$$

where now r is to be interpreted as a function defined implicitly by $r_*(r) = \frac{v-u}{2}$, $dr_* = f^{-1}dr$. In particular

$$\frac{\partial r}{\partial v} = -\frac{\partial r}{\partial u} = \frac{1}{2}f(r).$$

The adapted null frame is now given by

$$e_0 = L = \partial_v, \quad e_1 = \underline{L} = \partial_u,$$

$$e_2 = \frac{1}{r}\partial_\theta, \quad e_3 = \frac{1}{r \sin \theta}\partial_\phi.$$

It is worth remarking that L does not longer define a geodesic field. This is easily seen from the values of the connection coefficients:

$$\omega = -\underline{\omega} = \partial_v f = \left(-\frac{2M}{r^2} + \frac{2e^2}{r^3} \right) \left(1 - \frac{2M}{r} + \frac{e^2}{r^2} \right),$$

$$\chi_{ij} = -\underline{\chi}_{ij} = \frac{f(r)}{2r} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The only non-vanishing components of the Weyl and Faraday tensors are

$$\rho = \frac{M}{r^3}f(r),$$

$$\rho(\mathbf{F}) = -\frac{e}{2r^2}f(r).$$

A.3 Kerr-Newman spacetimes

Now we study the asymptotic behaviour of a 3-parameter family of spacetimes representing the surroundings of a axisymmetric, stationary and charged massive object. These are given by the Kerr-Newman metrics depending on the constants M ,

a and e representing the mass, angular momentum and electric charge, respectively. In Boyer-Lindquist they take the form

$$\begin{aligned} \mathbf{g}_{KN} = & - \left(1 - \frac{2Mr - e^2}{R^2} \right) dt^2 + \frac{(2Mr - e^2)a \sin^2 \theta}{R^2} dt d\phi + \frac{R^2}{\Delta} dr^2 \\ & + R^2 d\theta^2 + \left(r^2 + a^2 + \frac{(2Mr - e^2)a^2 \sin^2 \theta}{R^2} \right) \sin^2 \theta d\phi^2, \end{aligned}$$

where $R^2 = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 + a^2 - 2Mr + e^2$.

The metric degenerates when $\Delta = 0$. This singularity however is again a coordinate artefact and the metric can be extended. See [41] for more details on how to obtain a maximal extension and the corresponding conformal compactification.

Here we are only interested in the asymptotic values towards future null infinity. Firstly, we change to outgoing null coordinates:

$$\begin{aligned} du &= dt - \frac{r^2 + a^2}{\Delta} dr, \\ d\phi &= d\phi - \frac{a}{\Delta} dr. \end{aligned}$$

The metric then takes the form,

$$\begin{aligned} \mathbf{g}_{KN} = & - \left(1 - \frac{2Mr - e^2}{R^2} \right) du^2 - 2dudr - \frac{2a \sin^2 \theta}{R^2} (2Mr - e^2) dud\phi + 2a \sin^2 \theta dr d\phi \\ & + R^2 d\theta^2 + \frac{(2Mr - e^2)a \sin^2 \theta}{R^2} dt d\phi. \end{aligned}$$

The connection coefficients and curvature components are more easily analysed using the NP formalism, [2]. That is, consider the null NP-frame:

$$\begin{aligned} l &= \partial_r, \\ n &= \frac{r^2 + a^2}{R^2} \partial_u - \frac{\Delta}{2R^2} \partial_\theta + \frac{a}{R^2} \partial_\phi, \\ m &= \frac{1}{\sqrt{2}(r + ia \cos \theta)} \left(ia \sin \theta \partial_u + \partial_\theta + i \frac{1}{\sin \theta} \partial_\phi \right). \end{aligned}$$

Then, the connection coefficients satisfy:

$$\begin{aligned}\kappa &= \sigma = \lambda = \nu = \epsilon = 0, \\ \rho &= -\frac{1}{r + ia \cos \theta}, & \mu &= -\frac{\Delta(r + ia \cos \theta)}{2R^4}, \\ \bar{\alpha} + \beta &= -\tau = \frac{ia \sin \theta}{2R^2}, & \pi &= \frac{ia \sin \theta}{\sqrt{2}(r + ia \cos \theta)^2}, \\ \gamma &= -\frac{\Delta(r + ia \cos \theta)}{2R^4} + \frac{r - M}{2R^2}.\end{aligned}$$

The Weyl curvature components are:

$$\begin{aligned}\Psi_0 &= \Psi_1 = \Psi_3 = \Psi_4 = 0, \\ \Psi_2 &= -\frac{M}{(r - ia \cos \theta)^3} + \frac{e^2}{(r + ia \cos \theta)(r - ia \cos \theta)^3}.\end{aligned}$$

Finally, the Faraday tensor components are:

$$\phi_0 = \phi_2 = 0, \quad \phi_1 = \frac{e}{\sqrt{2}(r - ia \cos \theta)^2}.$$

A.4 Weyl spacetimes

These are a family of static axisymmetric spacetimes. The *ansatz* for the metric is,

$$\mathbf{g}_W = -e^{2U} dt^2 + e^{2w-2U} dr^2 + r^2 e^{2w-2U} d\theta^2 + r^2 \sin^2 \theta e^{-2U} d\phi^2.$$

Here U and w are functions depending only on (r, θ) . It can be checked that it is a solution to the vacuum Einstein's equations if and only if

$$\begin{aligned}\Delta U &= 0, \\ \partial_r w &= \sin^2 \theta \left(r(\partial_r U)^2 - \frac{1}{r}(\partial_\theta U)^2 + \frac{2 \cos \theta}{\sin \theta} \partial_r U \partial_\theta U \right),\end{aligned}$$

where Δ is the 3D-flat Laplace operator in spherical coordinates. Therefore we can write U as an expansion in axi-symmetric spherical harmonics. Of particular interest are those which decay for large r ,

$$\begin{aligned}U(r, \theta) &= \sum_{k=1}^{\infty} \frac{a_k}{r^k} P_{k-1}(\cos \theta), \\ &= \frac{a_1}{r} + \frac{a_2 \cos \theta}{r^2} + \mathcal{O}(r^{-3}).\end{aligned}$$

It can be checked that w decays one order faster than U . Using the NP-frame

$$\begin{aligned} l &= \frac{1}{\sqrt{2}} (-e^{-U} \partial_t + e^{U-w} \partial_r), \\ n &= \frac{1}{\sqrt{2}} (-e^{-U} \partial_t - e^{U-w} \partial_r), \\ m &= \frac{1}{\sqrt{2}r} \left(e^{U-w} \partial_\theta + i \frac{e^U}{\sin \theta} \partial_\phi \right), \end{aligned}$$

we compute

$$\begin{aligned} \Psi_0 = \Psi_4 &= \frac{a_1^3 \sin^2 \theta e^{2U-2w}}{2r^5} + \mathcal{O}(r^{-6}), \\ \Psi_1 = \Psi_3 &= \frac{-e^{2U-2w} (a_1^3 \sin \theta \cos \theta - 3a_1 a_2 \sin \theta)}{2r^5} + \mathcal{O}(r^{-6}), \\ \Psi_2 &= -\frac{a_1 e^{2U-2w}}{r^3} + \mathcal{O}(r^{-5}). \end{aligned}$$

A.5 Robinson-Trautman spacetimes

Another important family of solutions is defined by the Robinson-Trautman metrics. These are vacuum spacetimes admitting a geodesic, twist-free, shear-free but diverging null congruence. Explicitly, there exists a vector field L^α satisfying: $L^\alpha \nabla_\alpha L^\beta = 0$, $\nabla_{[\alpha} L_{\beta]} = 0$ with associated shear $\hat{\chi}_{ij} = 0$ and expansion $\text{tr} \chi = \nabla_\alpha L^\alpha \neq 0$.

It can be shown, cf. [50], that under these assumptions there are coordinates (u, r, x^2, x^3) in which the metric takes the form

$$\mathbf{g}_{RT} = -H du^2 - 2dudr + r^2 \gamma_{ij} dx^i dx^j, \quad (\text{A.2})$$

with $H = \frac{1}{2}R + \frac{r}{12M} \Delta_\gamma R - \frac{2M}{r}$; Δ_γ and $R = R(\gamma)$ are the Laplace operator and Ricci scalar of γ , respectively. Here, $\gamma_{ij} = e^{2\tau} \bar{\gamma}_{ij}$ where $\tau = \tau(u, x^2, x^3)$ and $\bar{\gamma}_{ij} = \bar{\gamma}_{ij}(x^2, x^3)$ is a fixed metric on \mathbb{S}^2 . The constant M is related to the total Bondi mass of the system. The preferred null direction in these coordinates is given by $L = \partial_r$.

The metric (A.2) is a solution of the vacuum field equations if and only if

$$\Delta_\gamma R(\gamma) = \frac{1}{24M} \partial_u \tau.$$

This is a 4th-order parabolic equation¹. Chruściel, [23], proved global existence and nice asymptotic properties for this equation given initial conditions

$$\tau(u_0, x^i) = \tau_0(x^i) \quad \text{at } u = u_0.$$

In the context of the initial value formulation of the Einstein's equations, this corresponds to providing the metric g along the characteristic hypersurface $u = u_0$.

The resulting spacetime is $\mathcal{M} = [u_0, \infty) \times (0, \infty) \times \mathbb{S}^2$ and the metric is future asymptotically simple; it should be noted that the above coordinates are not asymptotically flat in the Bondi sense, though. The global structure is summarised in the following Penrose diagram.

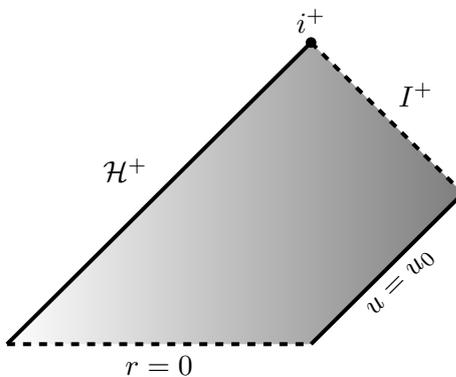


Figure A.3: Global structure of the Robinson-Trautman spacetimes obtained by solving a parabolic equation for $M > 0$. There is a curvature singularity at $r = 0$; the scalar $R^{abcd}R_{abcd}$ diverges as r^{-6} . In general (analytical initial data being the exception), it is not possible to extend the solution below the hypersurface $u = u_0$ within the Robinson-Trautman class as it would require solving a backwards heat equation. The issue of extending $(\mathcal{M}, \mathbf{g}_{RT})$ through \mathcal{H}^+ is quite subtle; the metric \mathbf{g}_{RT} can be \mathcal{C}^{117} -extended, one such extension is obtained by gluing a 180-rotated version of the above diagram to itself. And actually, it is possible to obtain infinitely many \mathcal{C}^5 -extensions by performing this gluing procedure with *any other* positive-mass Robinson-Trautman metric. We refer the reader to Chruściel [23] for more details.

¹Which is equivalent to $\gamma_{ij}\Delta_\gamma R(\gamma) = \frac{1}{24M}\partial_u\gamma_{ij}$, known as Calabi equation.

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