GEOMETRIC MODELS FOR HIGHER GROTHENDIECK-WITT GROUPS IN $\mathbb{A}^1$-HOMOTOPY THEORY

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ABSTRACT. We show that the higher Grothendieck-Witt groups, a.k.a. algebraic hermitian K-groups, are represented by an infinite orthogonal Grassmannian in the $\mathbb{A}^1$-homotopy category of smooth schemes over a regular base for which 2 is a unit in the ring of regular functions. We also give geometric models for various $\mathbb{P}^1$- and S^1-loop spaces of hermitian K-theory.

CONTENTS

1. INTRODUCTION

For a regular noetherian separated scheme S of finite Krull dimension, denote by $\mathscr{H}_\bullet(S)$ the pointed unstable $\mathbb{A}^1$-homotopy category of smooth S-schemes, and by $[\ \ ,\ \ ]$ or $[\ \ ,\ \ ]_{\mathscr{H}_\bullet(S)}$ maps in that category [MV99]. A theorem of Morel and Voevodsky says that Quillen's algebraic K-theory is represented in $\mathscr{H}_\bullet(S)$ by $\mathbb{Z}\times BGL \sim \mathbb{Z}\times Gr_\bullet$ where for a vector bundle V on S, the scheme $Gr_d(V)$ denotes the Grassmannian scheme of d-planes in V, and $Gr_\bullet = \operatorname{colim}_n Gr_n(O_S^n \oplus O_S^n)$ is the infinite Grassmannian over S. More precisely [MV99, Theorem 3.13, p. 140], for any smooth S-scheme X there are natural isomorphisms for all $i \geq 0$

$$K_i(X) \cong [X_+ \wedge S^i, \mathbb{Z}\times Gr_\bullet] \cong [X_+ \wedge S^i, \mathbb{Z}\times BGL] \qquad (1.1)$$

where $S^i = \Delta^i/\partial\Delta^i$ is the simplicial i-sphere and $X_+ = X \sqcup +$ is X with a disjoint basepoint $+$ added. This is analogous to the fact that complex K-theory is represented in topology by $\mathbb{Z}\times BU$ and the infinite complex Grassmannian.

The purpose of this article is to prove a result analogous to (1.1) for the theory of higher Grothendieck-Witt groups, a.k.a. algebraic hermitian K-theory [Kar73],

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extended to schemes in [Sch10b]. Our result has already been used in the work of [AF12] and [Zib11] and opens the door to a classification of unstable operations in Grothendieck-Witt theory as done in [Rio10] for K-theory.

To state our main theorem, let $V = (V, \varphi)$ be an inner product space over S, that is, a vector bundle V over S equipped with a non-degenerate symmetric bilinear form $\varphi : V \otimes_S V \to O_S$, and let $GrO_d(V) \subset Gr_d(V)$ be the open subscheme, of the usual Grassmannian $Gr_d(V)$ of d-planes in V, of those subbundles E of V for which the from φ restricts to a non-degenerate form $\varphi_{|E}$ on E. Let H_S be the hyperbolic plane over S, that is, the rank 2 vector bundle O_S^2 equipped with the inner product $(x, y) \cdot (x', y') = xx' - yy'$. We define the *infinite orthogonal Grassmannian* (over S) as the colimit of schemes

$$GrO_\bullet = \operatorname{colim}_n GrO_{2n}(H^n \perp H^n)$$

where the colimit is taken over the maps

$$GrO_{2n}(H^n \perp H^n) \to GrO_{2n+2}(H^{n+1} \perp H^{n+1}) : E \mapsto H \perp E.$$

Moreover, let $O = \operatorname{colim}_n O(H^n)$ be the infinite orthogonal group over S where $O(V)$ denotes the group of isometries of an inner product space V. Let $B_{et}O = \operatorname{colim}_n B_{et}O(H^n)$ be the etale classifying space of O [MV99, p. 130]. Finally, for a scheme X with $\frac{1}{2} \in \Gamma(X, O_X)$ let $GW_i(X) = \pi_i GW(X)$ be the i-th higher Grothendieck-Witt group of X ([Sch10a, Definition 4.6] with $\mathcal{L} = O_X$ and $\varepsilon = 1$). For an affine scheme $X = \operatorname{Spec} A$ (with $\frac{1}{2} \in A$), these groups are Karoubi's hermitian K-groups of A [Sch10a, Remark 4.13]. Here is our main result.

Theorem 1.1. *Let S be a regular noetherian separated scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, O_S)$, and let X be a smooth S-scheme. Then there are natural isomorphisms*

$$GW_i(X) \cong [X_+ \wedge S^i, \mathbb{Z} \times GrO_\bullet]_{\mathscr{H}_\bullet(S)} \cong [X_+ \wedge S^i, \mathbb{Z} \times B_{et}O]_{\mathscr{H}_\bullet(S)}.$$

The proof of the K-theory analog of Theorem 1.1 has two steps. The first consists in showing that the K-theory presheaf K is homotopy invariant and satisfies the Nisnevich Brown-Gersten property. Both statements follow from Quillen's work [Qui73] and they imply $K_i(X) \cong [X_+ \wedge S^i, K]$. In the second step, one constructs explicit $\mathbb{A}^1$-weak equivalences $\mathbb{Z} \times Gr_\bullet \sim_{\mathbb{A}^1} \mathbb{Z} \times BGL \sim_{\mathbb{A}^1} K$. This was done in [MV99]; see also Remark 8.5.

For higher Grothendieck-Witt theory, the first step was proved by Hornbostel for affine schemes in [Hor05]. The extension to non-affine schemes follows from [Sch10b] and is also proved in [Sch12, Theorems 9.6, 9.8]. Thus, $[X_+ \wedge S^i, GW] \cong GW_i(X)$. Also, it is known from [PW10] that $B_{et}O \cong GrO_\bullet$ in $\mathscr{H}_\bullet(S)$; we give an alternative proof of a more precise statement in Proposition 8.1.

Denote by $\underline{\mathbb{Z}}$ the constant sheaf $\mathbb{Z}$. For a ring R, denote by ΔR the standard simplicial ring $n \mapsto \Delta^n R = R[T_0, ..., T_n]/\langle T_0 + \cdots + T_n - 1\rangle$. Theorem 1.1 is a consequence of the following which is proved in Theorem 8.2 and Proposition 8.1.

Theorem 1.2. *Let S be a regular noetherian separated scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, O_S)$. Then there are maps of simplicial presheaves on smooth S-schemes*

$$\underline{\mathbb{Z}} \times GrO_\bullet \to \underline{\mathbb{Z}} \times B_{et}O \to GW$$

which are weak equivalences of simplicial sets when evaluated at ΔR for any smooth affine S-scheme $\operatorname{Spec} R$*. In particular, these maps are isomorphisms in $\mathscr{H}_\bullet(S)$.*

We also give models for the n-th $\mathbb{P}^1$-loop space of GW and their S^1-loop spaces. Denote by $GW^n(X)$ the n-th shifted Grothendieck-Witt space of X ([Sch10b, Definition 7] with $\varepsilon = 1$, $Z = X$, $L = O_X$, $\mathcal{A}_X = O_X$, or [Sch12, Definition 9.1]), that is, the Grothendieck-Witt space of the category of bounded chain complexes of vector bundles on X with duality in $O_X[n]$, the line bundle O_X placed in degree $-n$. Let $GW^n : X \mapsto GW^n(X)$ be the corresponding simplicial presheaf made functorial as in [Sch12, Remark 9.4]. Then $GW^0 = GW$, and the presheaves GW^n are homotopy invariant [Sch12, Theorem 9.8] and satisfy the Nisnevich Brown-Gersten property [Sch12, Theorem 9.6]. Therefore,

$$GW_i^n(X) \cong [X_+ \wedge S^i, GW^n]$$

for all smooth S-schemes X. The motivic spaces GW^n are related by $\mathbb{A}^1$-weak equivalences $GW^n \sim \Omega_{\mathbb{P}^1} GW^{n+1}$ (a consequence of the $\mathbb{P}^1$-bundle theorem [Sch12, Theorem 9.10]) and isomorphisms $GW^n \cong GW^{n+4}$ [Sch10b, §8 Corollary 1], [Sch12, Remark 5.9]. The following is therefore a complete list of geometric models for the n-th $\mathbb{P}^1$-loop space $\Omega^n_{\mathbb{P}^1} GW \cong GW^{-n}$ of $\mathbb{Z} \times GrO_\bullet$ and their S^1-loop spaces, $n \in \mathbb{Z}$. Note that upon complex realization we obtain the 8 spaces of real Bott-periodicity.

Theorem 1.3. *Let S be a regular noetherian separated scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, O_S)$. Then there are isomorphisms in $\mathscr{H}_\bullet(S)$*

$$GW^n \cong \begin{cases} \mathbb{Z} \times GrO_\bullet & n = 0 \\ Sp/GL & n = 1 \\ \mathbb{Z} \times BSp & n = 2 \\ O/GL & n = 3 \end{cases} \qquad \Omega_{S^1} GW^n \cong \begin{cases} O & n = 0 \\ (GL/O)_{et} & n = 1 \\ Sp & n = 2 \\ GL/Sp & n = 3 \end{cases}$$

where Sp denotes the infinite symplectic group and $(GL/O)_{et}$ denotes the etale or scheme theoretic quotient.

More precise versions are proved in Theorems 8.2 and 8.4.

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2. Orthogonal Grassmannians

For a quasi-compact, separated scheme S, denote by Sch_S and Sm_S the categories of separated, finite type S-schemes and its full subcategory of smooth S-schemes, respectively.

Let $\mathscr{F}$ be a quasi-coherent sheaf on a scheme X. A *symmetric bilinear form* on $\mathscr{F}$ is a map $\varphi : \mathscr{F} \otimes_X \mathscr{F} \to O_X$ of O_X-modules such that $\varphi\tau = \varphi$ where $\tau : \mathscr{F} \otimes \mathscr{G} \cong \mathscr{G} \otimes \mathscr{F}$ is the switch map. The form φ is called *non-degenerate* and the pair $(\mathscr{F}, \varphi)$ is called an *inner product space* if $\mathscr{F}$ is a vector bundle and the adjoint $\hat{\varphi} : \mathscr{F} \to \mathscr{F}^* = Hom_{O_X}(\mathscr{F}, O_X) : \xi \mapsto \varphi(\ \ \otimes \xi)$ is an isomorphism. If $g : \mathscr{G} \to \mathscr{F}$ is a map of O_X-modules, then the restriction $\varphi_{|\mathscr{G}}$ of φ to $\mathscr{G}$ has as adjoint the map $g^*\hat{\varphi}g$. If $\mathscr{F}$ is a sheaf on S and $p : X \to S$ is an S-scheme, we may write $\mathscr{F}_X$ for the sheaf $p^*\mathscr{F}$.

Definition 2.1 (Orthogonal Grassmannians)**.** Let $\mathscr{F} = (\mathscr{F}, \varphi)$ be a quasi-coherent sheaf over S together with a symmetric bilinear form $\varphi : \mathscr{F} \otimes_S \mathscr{F} \to O_S$ which may be degenerate. The *Grassmannian of non-degenerate subspaces of $\mathscr{F}$* is the presheaf

$$GrO(\mathscr{F}) : (\mathrm{Sch}_S)^{op} \to \mathrm{Sets}$$

whose value at an S-scheme $p : X \to S$ is the set $GrO(\mathscr{F}_X)$ of finite rank locally free O_X-submodules $E \subset \mathscr{F}_X$ of $\mathscr{F}_X = p^*\mathscr{F}$ for which the restriction $\varphi_{|E}$ of the form φ to E is non-degenerate. For a map $f : X \to Y$ of S-schemes, the map $GrO(\mathscr{F}_Y) \to GrO(\mathscr{F}_X)$ is induced by the pullback f^* of quasi-coherent sheaves. For an integer $d \geq 0$ we let

$$GrO_d(\mathscr{F}) \subset GrO(\mathscr{F})$$

be the subpresheaf of those non-degenerate subspaces $E \subset \mathscr{F}$ which have constant rank d. If $X = \operatorname{Spec} R$ is affine, we may write $GrO_d(\mathscr{F}_R)$ and $GrO(\mathscr{F}_R)$ in place of $GrO_d(\mathscr{F}_X)$ and $GrO(\mathscr{F}_X)$.

Lemma 2.2. *Let $V = (V, \varphi)$ be an inner product space of rank n over S, and $0 \leq d \leq n$ be an integer. Then the presheaf $GrO_d(V)$ is represented by a scheme which is smooth and affine over S.*

Proof. To see that $GrO_d(V) \to S$ is smooth, we note that it is an open subscheme of the usual Grassmannian $Gr_d(V)$ of d-planes in V. More precisely, if we denote by ξ the universal rank d subbundle of V on $Gr_d(V)$, then the form on V restricts to a (degenerate) form $\varphi_{|\xi}$ on ξ, and $GrO_d(V)$ is the open subscheme of $Gr_d(V)$ where $\varphi_{|\xi}$ is non-degenerate, that is, $GrO_d(V)$ is the non-vanishing locus of the global section $\Lambda^d\hat{\varphi}$ of the line bundle $\underline{\mathrm{Hom}}_{O_X}(\Lambda^d\xi, \Lambda^d\xi^*)$ on $X = Gr_d(V)$. Since $Gr_d(V) \to S$ is smooth, so is $GrO_d(V) \to S$.

To see that $GrO_d(V) \to S$ is an affine morphism, note that for any S-scheme X, we have a natural bijection of sets

$$GrO(V_X) \cong \{p \in \mathrm{Hom}_{O_X}(V_X, V_X) |\ p = p^2,\ p^*\varphi = \varphi p\}.$$

The map is defined by $(i : M \subset V_X) \mapsto i(\varphi_{|M}^{-1})i^*\varphi$ and has inverse $p \mapsto \mathrm{Im}(p) \subset V_X$. This shows that $GrO(V)$ is a closed subscheme of the vector bundle $\underline{\mathrm{Hom}}_{O_S}(V, V)$ over S defined by two equations. In particular, $GrO(V) \to \underline{\mathrm{Hom}}_{O_S}(V, V) \to S$ are affine morphisms. As a closed subscheme of $GrO(V)$, the scheme $GrO_d(V)$ is also affine over S. $\square$

For an S-scheme X, let H_X be the hyperbolic plane over X, that is, the rank 2 vector bundle O_X^2 equipped with the inner product $(x, y) \cdot (x', y') = xy - x'y'$. Let H_X^n be its n-fold orthogonal sum (an inner product space over X) and let $H_X^\infty = \mathrm{colim}_n H_X^n$ be the infinite hyperbolic space (a quasi-coherent O_X-module with symmetric bilinear form). Order non-degenerate subspaces of H_X^∞ by inclusion. This defines a filtered category $\mathcal{H}$. Its objects are non-degenerate subspaces $V \subset H^\infty$ (which are inner product spaces), and maps are inclusions of subspaces. For a non-degenerate subspace $V \subset V'$ of an inner product space V', denote by $V' - V$ the orthogonal complement of V in V'.

Definition 2.3 (Infinite orthogonal Grassmannian)**.** For a vector bundle V of constant rank, write $|V|$ for its rank. The *infinite orthogonal Grassmannian* over S is the presheaf

$$GrO_\bullet = \mathrm{colim}_{V \subset H_S^\infty} GrO_{|V|}(V \perp H^\infty).$$

The colimit is taken over the non-degenerate subbundles of H_S^∞ of constant rank ordered by inclusion, and the transition maps are

$$GrO_{|V|}(V \perp H^\infty) \to GrO_{|V'|}(V' \perp H^\infty) : E \mapsto (V' - V) \perp E$$

whenever $V \subset V'$. Of course, it suffices to take the colimit over a cofinal subset such as the set $\{H_S^n |\ n \in \mathbb{N}\}$.

3. The etale classifying space

Let S be a scheme and $\mathscr{F} = (\mathscr{F}, \varphi)$ a quasi-coherent sheaf over S together with a symmetric bilinear form $\varphi : \mathscr{F} \otimes_S \mathscr{F} \to O_S$ which may be degenerate. For an S-scheme X, denote by

$$\mathcal{S}(\mathscr{F}_X)$$

the category of inner product spaces embedded in $\mathscr{F}_X$, that is, the category whose objects are the locally free O_X-submodules $E \subset \mathscr{F}_X$ of $\mathscr{F}_X = p^*\mathscr{F}$ for which the restriction $\varphi_{|E}$ of the form φ to E is non-degenerate. A map from $E_0 \subset \mathscr{F}_X$ to $E_1 \subset \mathscr{F}_X$ is an isometry $(E_0, \varphi_{|E_0}) \to (E_1, \varphi_{|E_1})$ which does not need to be compatible with the embeddings $E_0, E_1 \subset \mathscr{F}_X$. For a map $f : X \to Y$ of S-schemes, pull-back f^* of quasi-coherent modules defines a map $\mathcal{S}(\mathscr{F}_Y) \to \mathcal{S}(\mathscr{F}_X)$, and we obtain a presheaf of categories $X \mapsto \mathcal{S}(\mathscr{F}_X)$. Note that the set of objects of $\mathcal{S}(\mathscr{F}_X)$ is precisely $GrO(\mathscr{F}_X)$.

For an integer $d \geq 0$, we denote by

$$\mathcal{S}_d(\mathscr{F}_X) \subset \mathcal{S}(\mathscr{F}_X)$$

the full subcategory of those inner product spaces $E \subset \mathscr{F}_X$ which have constant rank d. Then $\mathcal{S}_d(\mathscr{F})$ is a presheaf of groupoids with presheaf of objects $GrO_d(\mathscr{F})$.

In the category $\mathcal{S}_{|V|}(V \perp H^\infty)$, the group of automorphisms of the object $V \subset V \perp H^\infty : v \mapsto (v, 0)$ is the group $O(V)$ of isometries of V. Thus we have a full inclusion $O(V) \to \mathcal{S}_{|V|}(V \perp H^\infty)$ of presheaves of categories. After etale sheafification, this inclusion becomes an equivalence of categories. This is because in a strictly henselian ring R with $\frac{1}{2} \in R$, every unit is a square, and thus, any two innner product spaces over R are isometric if and only if they have the same rank. It follows that the inclusion of categories induces a map of simplicial presheaves $BO(V) \to B\mathcal{S}_{|V|}(V \perp H^\infty)$ which is a weak equivalence at all strictly henselian R with $\frac{1}{2} \in R$. In other words, this map is a weak equivalence in the etale topology. In particluar, a globally fibrant model of $B\mathcal{S}_{|V|}(V \perp H^\infty)$ for the etale topology is also a globally fibrant model, denoted $B_{et}O(V)$, of $BO(V)$. Therefore, we obtain a sequence of maps

$$BO(V) \to B\mathcal{S}_{|V|}(V \perp H^\infty) \to B_{et}O(V) \tag{3.1}$$

which are weak equivalences in the etale topology, and the last presheaf is fibrant (in the etale topology).

Lemma 3.1. *Let V be an inner product space over a scheme S with $\frac{1}{2} \in \Gamma(S, O_S)$. Then for any affine S-scheme* $\operatorname{Spec} R$*, the map*

$$B\mathcal{S}_{|V|}(V \perp H^\infty)(R) \to B_{et}O(V)(R)$$

is a weak equivalence of simplicial sets. In particular, the following map is an $\mathbb{A}^1$-weak equivalence

$$B\mathcal{S}_{|V|}(V \perp H^\infty) \to B_{et}O(V).$$

Proof. This follows from [Jar01]. Let St be the stack associated with the sheaf of groupoids $\mathcal{S}_{|V|}(V \perp H^\infty)$, then St is a sheaf of groupoids satisfying the effective descent condition for the etale topology. So, $X \mapsto St(X)$ is a sheaf version of the category of $O(V)$-torsors over X. Since for affine X, the category $\mathcal{S}_{|V|}(V \perp H^\infty_X)$ is already the category of all $O(V)$-torsors, the map $\mathcal{S}_{|V|}(V \perp H^\infty)(X) \to St(X)$ is an equivalence of categories for X affine. Therefore, in the string of maps

$$B\mathcal{S}_{|V|}(V \perp H^\infty)(X) \to BSt(X) \to B_{et}O(V)(X),$$

the first map is a weak equivalence for every affine S-scheme X. The second map $BSt(X) \to B_{et}O(V)(X)$ is a weak equivalence of simplicial sets for all S-schemes X [Jar01, Theorem 6]. □

Definition 3.2. Set

$$\mathcal{S}_\bullet = \operatorname{colim}_{V \subset H^\infty_S} \mathcal{S}_{|V|}(V \perp H^\infty)$$

where for $V \subset V'$, the transition map $\mathcal{S}_{|V|}(V \perp H^\infty) \to \mathcal{S}_{|V'|}(V' \perp H^\infty)$ is defined by $E \mapsto (V' - V) \perp E$ on objects and by $g \mapsto 1_{V'-V} \perp g$ on maps.

Inclusion of zero-simplices and the second map in (3.1) define the string of maps of simplicial presheaves

$$GrO_{|V|}(V \perp H^\infty) \to \mathcal{S}_{|V|}(V \perp H^\infty) \to B_{et}O(V)$$

in which the second map is section-wise a weak equivalence on affine schemes, by Lemma 3.1. Passing to the colimit over the index category $\mathcal{H}$ defines the string of maps

$$GrO_\bullet \to \mathcal{S}_\bullet \to B_{et}O \tag{3.2}$$

in which the second map is a weak equivalence when evaluated at affine schemes.

4. The Grothendieck-Witt space

Let R be a commutative ring. Let $\mathscr{S}_R$ denote the category of inner product spaces over R with isometries as morphisms. This category is symmetric monoidal with respect to orthogonal sum $\perp$. In particular, we have the category $\mathscr{S}_R^{-1}\mathscr{S}_R$ as constructed in [Gra76] whose classifying space $B\mathscr{S}_R^{-1}\mathscr{S}_R$ is naturally weakly equivalent to $GW(R)$ [Sch04], [Sch12, Appendix A] (at least when $\frac{1}{2} \in R$ though this is also true without this hypothesis). Recall that the objects of $\mathscr{S}_R^{-1}\mathscr{S}_R$ are pairs of inner product spaces and a map $(A_0, A_1) \to (B_0, B_1)$ in that category is an equivalence class of data $[C, a_0, a_1]$ where C is an inner product space and $a_i : A_i \perp C \to B_i$ is an isometry for $i = 0, 1$. We have $[C, a_0, a_1] = [C', a_0', a_1']$ if and only if there is an isometry $f : C \cong C'$ such that $a_i'(1_{A_i} \perp f) = a_i$ for $i = 0, 1$.

The category $\mathscr{S}_R^{-1}\mathscr{S}_R$ is not convenient for our purposes as it is, a priori, not a small category, and it is not really functorial in R. In particular, the assignment $X \mapsto \mathscr{S}_R^{-1}\mathscr{S}_R$ with $R = \Gamma(X, O_X)$ does not define a presheaf. We remedy this as follows.

Definition 4.1 (The presheaf of Grothendieck-Witt spaces)**.** Let

$$\mathscr{G}W(R) \subset \mathscr{S}_R^{-1}\mathscr{S}_R$$

be the full subcategory whose objects are pairs (A, B) where $A \subset H^\infty_R \perp H^\infty_R$ and $B \subset (H^\infty)^{\perp 3}_R$ are finitely generated non-degenerate subspaces of $(H^\infty_R)^{\perp 2}$ and $(H^\infty_R)^{\perp 3}$, respectively. The ambient bilinear form spaces $(H^\infty_R)^{\perp 2}$ and $(H^\infty_R)^{\perp 3}$ are chosen so that we can construct certain maps below. The explicit ambient spaces don't matter as long as they are functorial in R and contain a copy of each inner product space over R.

From our definition, the category $\mathscr{G}W(R)$ is small, it is equivalent to $\mathscr{S}_R^{-1}\mathscr{S}_R$, and it is functorial in R. In particular, the assignment $X \mapsto \mathscr{G}W(R)$ with $R = \Gamma(X, O_X)$ does define a presheaf (of categories and hence of simplicial sets after application of the nerve functor).

From [Sch04], [Sch12, Appendix A], there is a map of presheaves $\mathscr{G}W \to GW$ which is a weak equivalence (of simplicial sets) for all affine schemes. We record a special case in the following Lemma.

Lemma 4.2. *Let S be a regular separated noetherian scheme of finite Krull dimension with $\frac{1}{2} \in S$. Then map of presheaves $\mathscr{G}W \to GW$ in $\Delta^{op}\,\mathrm{PSh}(\mathrm{Sm}_S)$ is a weak equivalence of simplicial sets at all* $\operatorname{Spec} R \to S$. *In particular the map of presheaves is a Nisnevich simplicial weak equivalence, and hence an $\mathbb{A}^1$-weak equivalence.* □

Definition 4.3. We define the presheaf of *reduced Grothendieck-Witt spaces* $\widetilde{\mathscr{G}W}$ as the presheaf of categories which for a ring $R = \Gamma(X, O_X)$ is the full subcategory

$$\widetilde{\mathscr{G}W}(R) \subset \mathscr{G}W(R)$$

of objects $(A, B) \in \mathscr{G}W(R)$ such that $A \subset H_R^\infty = 0 \perp H_R^\infty \subset (H_R^\infty)^{\perp 2}$, and $B \subset A \perp H_R^\infty \subset 0 \perp (H_R^\infty)^{\perp 2} \subset (H_R^\infty)^{\perp 3}$ and A, B have the same constant rank. For an integer i, we set $\widetilde{GW}_i(R) = \pi_i(\widetilde{\mathscr{G}W}(R))$ where the homotopy groups are taken with respect to the base point $(0, 0)$.

Consider the integers $\mathbb{Z}$ as a (symmetric monoidal) category with one object for each integer and only identity morphisms. Let $\mathbb{N} \subset \mathbb{Z}$ be the (full) subcategory of non-negative integers viewed as a symmetric monoidal category where the monoidal product is given by addition.

The functor

$$\mathbb{N}^{-1}\mathbb{N} \to \mathbb{Z} : (n, m) \mapsto n - m \tag{4.1}$$

induces a weak equivalence of simplicial sets (after application of the nerve functor) since all fibres are filtered categories and hence contractible.

Consider the ring R as an inner product space with bilinear form $R \otimes R \to R : x \otimes y \mapsto xy$. Then we have a map of presheaves of categories

$$\mathbb{N}^{-1}\mathbb{N} \to \mathscr{G}W : (n, m) \mapsto (R^n, R^m) \tag{4.2}$$

where the first factor R^n is considered as being in $H^n \perp 0 \subset H^\infty \perp H^\infty$ and the second factor R^m as being in $H^m \perp 0 \perp 0 \subset H^\infty \perp H^\infty \perp H^\infty$. Together with the inclusion $\widetilde{\mathscr{G}W} \subset \mathscr{G}W$ this defines a map of presheaves of categories

$$\mathbb{N}^{-1}\mathbb{N} \times \widetilde{\mathscr{G}W} \to \mathscr{G}W : (n, m), (A, B) \mapsto (R^n \perp A, R^m \perp B) \tag{4.3}$$

Lemma 4.4. *Let R be a connected ring with $\frac{1}{2} \in R$. Then the map (4.3) is a weak equivalence of simplicial sets. In particular, the maps (4.3) and (4.1) induce $\mathbb{A}^1$-weak equivalences*

$$\mathbb{Z} \times \widetilde{\mathscr{G}W} \xleftarrow{\sim} \mathbb{N}^{-1}\mathbb{N} \times \widetilde{\mathscr{G}W} \xrightarrow{\sim} \mathscr{G}W$$

Proof. For a connected ring R, the functor of categories $\mathscr{G}W(R) \to \mathbb{Z} : (A, B) \mapsto \operatorname{rk} A - \operatorname{rk} B$ is well defined and has $\widetilde{\mathscr{G}W}$ has homotopy fibre, by cofinality. Now, the functor (4.2) provides a splitting. Hence the result. □

Remark 4.5 (The Grothendieck-Witt space as a homotopy colimit). Let $\mathcal{I}$ be the category whose objects are the finitely generated non-degenerated subspaces $V \subset H^\infty$ and whose maps are all isometric embeddings, that is, a map from $V \subset H^\infty$ to $W \subset H^\infty$ is a map of O_X-modules $f : V \to W$ such that the form on W restricts to the form on V but f does not need to commute with the embeddings $V, W \subset H^\infty$. Composition is composition of O_X-module maps. In the notation of [Gra76], the category $\mathcal{I}$ is the category $\langle \mathcal{S}(H^\infty), \mathcal{S}(H^\infty)\rangle$. Note that our index category $\mathcal{H}$ is naturally a subcategory of $\mathcal{I}$. It is the subcategory with the same objects as $\mathcal{I}$ and with maps those isometric embeddings $f : V \to W$ which do commute with the embedding $V, W \subset H^\infty$.

We define a functor from $\mathcal{I}$ to the category of small categories which on objects is

$$V \mapsto \mathcal{S}_{|V|}(V \perp H^\infty)$$

and which sends an isometric embedding $g : V \to W$ to the functor

$$\begin{aligned} \mathcal{S}_{|V|}(V \perp H^\infty) \to \mathcal{S}_{|W|}(W \perp H^\infty) : \quad & E \mapsto (W - g(V))^- \perp \tilde{g}(E) \\ & e \mapsto 1_{(W-g(V))^-} \perp \tilde{g} e \tilde{g}^{-1} \end{aligned}$$

where $\tilde{g} = g \perp 1_{H^\infty} : V \perp H^\infty \to W \perp H^\infty$. Then we have an equality of categories

$$\widetilde{\mathscr{G}W}(R) = \mathrm{hocolim}_{V \in \mathcal{I}}\, \mathcal{S}_{|V|}(V \perp H^\infty)$$

where the right hand side is the homotopy colimit of categories as in [Tho79] whose construction is recalled in Appendix A.8.

Replacing $\mathcal{S}_{|V|}(V \perp H^\infty)$ with the full groupoid $\mathcal{S}(V \perp H^\infty)$ of all inner product spaces in $V \perp H^\infty$ and taking the homotopy colimit as above yields a model for the Grothendieck-Witt space $GW(R)$ of R.

5. The maps $GrO_\bullet \to B_{et}O \to \widetilde{\mathscr{G}W}$

Definition 5.1. By Remark 4.5, the (reduced) Grothendieck-Witt space is a homotopy colimit. In order to construct maps between various models for Grothendieck-Witt theory, we will need to express the presheaves $GrO_\bullet$ and $\mathcal{S}_\bullet \simeq B_{et}O$ as homotopy colimits as well. Recall that the presheaves $GrO_\bullet$ and $\mathcal{S}_\bullet$ are obtained as the colimits of sets $GrO_{|V|}(V \perp H^\infty)$ and of categories $\mathcal{S}_{|V|}(V \perp H^\infty)$ over the index category $\mathcal{H}$ of non-degenerate subspaces $V \subset H^\infty$. As usual, we consider sets as (discrete) categories and categories as simplicial sets (via the nerve functor) and thus sets as (constant) simplicial sets. Replacing the colimit over the (filtering) index category $\mathcal{H}$ with the corrensponding homotopy colimit as in Appendix A.8 yields the definition of the presheaves of categories $\mathscr{G}rO_\bullet$ and $\mathscr{S}_\bullet$. For $R = \Gamma(X, O_X)$, they are

$$\begin{aligned} \mathscr{G}rO_\bullet(R) &= \mathrm{hocolim}_{V \subset H^\infty}\, GrO_{|V|}(V_R \perp H_R^\infty) \\ \mathscr{S}_\bullet(R) &= \mathrm{hocolim}_{V \subset H^\infty}\, \mathcal{S}_{|V|}(V_R \perp H_R^\infty). \end{aligned}$$

By Lemma A.9, the homotopy colimit to colimit maps are weak equivalences of presheaves of simplicial sets

$$\mathscr{G}rO_\bullet \xrightarrow{\sim} GrO_\bullet, \quad \mathscr{S}_\bullet \xrightarrow{\sim} \mathcal{S}_\bullet. \tag{5.1}$$

The natural transformation of functors $\mathcal{H} \to \mathrm{Cat}$ which at $V \in \mathcal{H}$ is the inclusion of zero-simplices $GrO_{|V|}(V \perp H^\infty) \to \mathcal{S}_{|V|}(V \perp H^\infty)$ defines a map of presheaves of categories

$$\mathscr{G}rO_\bullet \to \mathscr{S}_\bullet \tag{5.2}$$

Furthermore, the inclusion $\mathcal{H} \subset \mathcal{I}$ defines a functor

$$\mathrm{hocolim}_{V \in \mathcal{H}} S_{|V|}(V \perp H^\infty) \longrightarrow \mathrm{hocolim}_{V \in \mathcal{I}} S_{|V|}(V \perp H^\infty),$$

that is, a map of presheaves of categories

$$\mathscr{S}_\bullet \to \widetilde{\mathscr{G}W}. \tag{5.3}$$

Write ΔR for the simplicial ring with $n \mapsto \Delta^n R = R[T_0, ..., T_n]/\langle T_0 + \cdots + T_n - 1\rangle$.

Theorem 5.2. *Let R be a commutative connected regular noetherian ring with $\frac{1}{2} \in R$. Then the maps (5.2) and (5.3) induce weak equivalences of simplicial sets*

$$\mathscr{G}rO_\bullet(\Delta R) \xrightarrow{\sim} \mathscr{S}_\bullet(\Delta R) \xrightarrow{\sim} \widetilde{\mathscr{G}W}(\Delta R).$$

The proof is in Corollary 7.3 and Proposition 7.4 in view of the weak equivalences (5.1).

6. Setting up the proof of Theorem 5.2

Let R be a commutative ring, V an inner product space of constant rank over R, and U an R-module equipped with a symmetric bilinear form. Denote by

$$GrO_V(U) \subset GrO_{|V|}(U)$$

the subset of those non-degenerate subspaces $W \subset U$ which are isometric to V. Scalar extension makes $GrO_V(U)$ into a presheaf on affine R-schemes. Similarly, denote by

$$\mathcal{S}_V(U) \subset \mathcal{S}_{|V|}(U)$$

the presheaf of full subcategories of those non-degenerate subspaces $W \subset U$ which are isometric to V. The presheaf of objects of $\mathcal{S}_V(U)$ is $GrO_V(U)$.

Let $\mathrm{Iso}_d(R)$ denote the set of isometry classes of inner product spaces over R of constant rank d. We define a map of sets

$$GrO_d(V \perp H^\infty_R) \to \mathrm{Iso}_d(R) : E \mapsto [E]$$

by sending a finitely generated non-degenerate subspace E of $V \perp H^\infty_R$ to its isometry class $[E] \in Iso_d(R)$. Similarly, we define a map of categories

$$\mathcal{S}_d(V \perp H^\infty_R) \to \mathrm{Iso}_d(R) : E \mapsto [E].$$

For an inner product space V over R of constant rank d, denote by $V : * \to \mathrm{Iso}_d(R)$ the map sending the point $*$ to the class $[V]$ of V in $\mathrm{Iso}_d(R)$. By definition, we have a cartesian diagram of sets

$$\begin{tikzcd} GrO_V(V \perp H^\infty_R) \arrow[r] \arrow[d] & GrO_{|V|}(V \perp H^\infty_R) \arrow[d] \\ * \arrow[r, "V"] & \mathrm{Iso}_{|V|}(R) \end{tikzcd} \tag{6.1}$$

and of categories

$$\begin{tikzcd} \mathcal{S}_V(V \perp H_R^\infty) \arrow[r] \arrow[d] & \mathcal{S}_{|V|}(V \perp H_R^\infty) \arrow[d] \\ * \arrow[r, "V"] & \mathrm{Iso}_{|V|}(R). \end{tikzcd} \tag{6.2}$$

Taking the colimit over the non-degenerate subspaces $V \subset H_R^\infty$ with transition maps as in Definitions 2.3 and 3.2, we obtain the cartesian diagrams of simplicial sets

$$\begin{tikzcd} GrO_{[0]}(R) \arrow[r] \arrow[d] & GrO_\bullet(R) \arrow[d] \\ * \arrow[r] & \widetilde{GW}_0(R) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}_{[0]}(R) \arrow[r] \arrow[d] & \mathcal{S}_\bullet(R) \arrow[d] \\ * \arrow[r] & \widetilde{GW}_0(R) \end{tikzcd}$$

where the upper left corners are $GrO_{[0]}(R) = \mathrm{colim}_{V \subset H_R^\infty} GrO_V(V \perp H_R^\infty)$ and $\mathcal{S}_{[0]}(R) = \mathrm{colim}_{V \subset H_R^\infty} \mathcal{S}_V(V \perp H_R^\infty)$.

Lemma 6.1. *Let R be a connected regular ring with $\frac{1}{2} \in R$. Then the cartesian diagrams of simplicial sets*

$$\begin{tikzcd} GrO_{[0]}(\Delta R) \arrow[r] \arrow[d] & GrO_\bullet(\Delta R) \arrow[d] \\ * \arrow[r] & \widetilde{GW}_0(\Delta R) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}_{[0]}(\Delta R) \arrow[r] \arrow[d] & \mathcal{S}_\bullet(\Delta R) \arrow[d] \\ * \arrow[r] & \widetilde{GW}_0(\Delta R) \end{tikzcd}$$

are homotopy cartesian, and the lower right corners are constant simplicial sets.

Proof. The Grothendieck-Witt group GW_0 is homotopy invariant for regular rings (with 2 a unit). For connected rings, the kernel $\widetilde{GW}_0$ of the rank map $GW_0 \to \mathbb{Z}$ is therefore also homotopy invariant. It follows that the lower right corner of the diagram is a constant simplicial set. Hence, the lower horizontal map is a fibration of (constant) simplicial sets. □

Diagram (6.1) maps to diagram (6.2) via the inclusion of zero simplices. By Lemma 6.1, we have a map of homotopy fibrations

$$\begin{tikzcd} GrO_{[0]}(\Delta R) \arrow[r] \arrow[d] & GrO_\bullet(\Delta R) \arrow[r] \arrow[d] & \widetilde{GW}_0(\Delta R) \arrow[d, "1"] \\ \mathcal{S}_{[0]}(\Delta R) \arrow[r] & \mathcal{S}_\bullet(\Delta R) \arrow[r] & \widetilde{GW}_0(\Delta R). \end{tikzcd} \tag{6.3}$$

The rest of this section is devoted to the proof of the following.

Proposition 6.2. *Let R be a commutative ring with $\frac{1}{2} \in R$ and V an inner product space over R. Then we have weak equivalences of simplicial sets*

$$GrO_V(V \perp H_{\Delta R}^\infty) \stackrel{\sim}{\to} B\mathcal{S}_V(V \perp H_{\Delta R}^\infty) \stackrel{\sim}{\leftarrow} BO(V_{\Delta R}),$$

where the first map is inclusion of zero-simplices and the second map is the inclusion of the endomorphism category of the object V into $\mathcal{S}_V(V \perp H^\infty)$. In particular, we have weak equivalences of simplicial sets

$$GrO_{[0]}(\Delta R) \stackrel{\sim}{\to} B\mathcal{S}_{[0]}(\Delta R) \stackrel{\sim}{\leftarrow} BO(\Delta R).$$

Let
$$O(V \perp H^\infty) = \mathrm{colim}_{W \subset V \perp H^\infty} O(W)$$
be the infinite orthogonal group based on $V \perp H^\infty$. It is the filtered colimit over the poset of finitely generated non-degenerate subspaces W of $V \perp H^\infty$ of the isometry groups $O(W)$ of W where for an inclusion $W \subset W'$, we embed $O(W)$ into $O(W')$ via $a \mapsto a \perp id_{W'-W}$. Our next aim is to identify the simplicial set $GrO_V(V \perp H^\infty_{\Delta R})$ with the simplicial set $BO(V_{\Delta R})$, up to homotopy. We will need the following lemma.

Lemma 6.3. *Let V be an inner product space over a commutative ring R with $\frac{1}{2} \in R$. Then the inclusion $H^\infty \subset V \perp H^\infty$ induces a homotopy equivalence of simplicial groups*
$$O(H^\infty_{\Delta R}) \to O(V \perp H^\infty_{\Delta_R}) : A \mapsto 1_V \perp A.$$

Proof. We first prove the claim when $V = H$. So, we need to show that $j : O(H^\infty_{\Delta_R}) \to O(H^\infty_{\Delta_R}) : A \mapsto 1_H \perp A$ is a homotopy equivalence. The point is that the two inclusions $j : O(H^n) \to O(H^{2n+2}) : A \mapsto 1_H \perp A \perp 1_{H^{n+1}}$ and $i : O(H^n) \to O(H^{2n+2}) : A \mapsto A \perp 1_{H^{n+2}}$ are naively $\mathbb{A}^1$-homotopic (see Appendix A.10 for a definition). This is because $i = c_g \circ j$ where $g = H(h \oplus h^{-1})$, $H : GL_{2n+2}(R) \to O(H^{2n+2})$ is the hyperbolic map and $h = \left(\begin{smallmatrix} 0 & 1 \\ I_n & 0 \end{smallmatrix}\right) \in GL_{n+1}(R)$ with $I_n \in GL_n(R)$ the identity matrix and $c_g : O(H^{2n+2}) \to O(H^{2n+2}) : x \mapsto gxg^{-1}$ denotes conjugation by g. Now, by the well-known formula
$$\left(\begin{smallmatrix} h & 0 \\ 0 & h^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -h^{-1} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right),$$
the element $h \oplus h^{-1} \in GL_{2n+2}(R)$ is a product of elementary matrices each of which is naively $\mathbb{A}^1$-homotopic to the identity by an elementary $\mathbb{A}^1$-homotopy. Therefore, g is naively $\mathbb{A}^1$-homotopic to the identity and the inclusions $j = c_g \circ i : O(H^n_{\Delta R}) \to O(H^{2n+2}_{\Delta R})$ and $i : O(H^n_{\Delta R}) \to O(H^{2n+2}_{\Delta R})$ are simplicially homotopic via a base-point preserving homotopy, by Lemma A.11. It follows that $j : \pi_k O(H^\infty_{\Delta R}) = \mathrm{colim}_n \pi_k O(H^n_{\Delta R}) \to \pi_k O(H^\infty_{\Delta R})$ is the identity map, hence an isomorphism for all $k \geq 0$. Since $O(H^\infty_{\Delta R})$ is an H-group, this implies the claim for $V = H$.

By induction, the claim is true for $V = H^n$. For general V, choose an embedding $V \subset H^n$. Then the composition of the first two and the composition of the last two maps in the following diagram are homotopy equivalences
$$O(H^\infty_{\Delta R}) \to O(V \perp H^\infty_{\Delta_R}) \to O(H^n \perp H^\infty_{\Delta R}) \to O(H^n \perp V \perp H^\infty_{\Delta_R})$$
since $V \perp H^\infty \cong H^\infty$. This finishes the proof of the Lemma. □

Let $V = (V, \phi_V)$ be an inner product space over a commutative ring R, and let $U = (U, \phi_U)$ be an R-module equipped with a symmetric bilinear form. For a commutative R-algebra A, let
$$\mathrm{St}(V, U)(A)$$
be the set of isometric embeddings $f : V_A \to U_A$ over A, that is, the set of those A-linear maps $f : V_A \to U_A$ such that $\phi_V = f^*\phi_U f$. Then $\mathrm{St}(V, U)$ is a presheaf on affine R-schemes.

For every commutative ring R with $\frac{1}{2} \in R$, the group $O(V \perp H^\infty_R)$ acts transitively from the left on the set $\mathrm{St}(V, V \perp H^\infty_R)$ via $(f, g) \mapsto f \circ g$. The action is transitive because any isometry between non-degenerate subspaces M, N of $V \perp H^n$ can be extended to an isometry of $V \perp H^{n+m}$ for some m as $(V \perp H^n) - M$ and $(V \perp H^n) - N$ are stably isometric. For the action above, the stabilizer of the element $i_V : V \to V \perp H^\infty : v \mapsto (v,0)$ of $\mathrm{St}(V, V \perp H^\infty)$ is the subgroup $O(H^\infty) \subset O(V \perp H^\infty) : A \mapsto 1_V \perp A$. Therefore, we obtain an isomorphism of presheaves of sets

$$O(H^\infty)\backslash O(V \perp H^\infty) \cong \mathrm{St}(V, V \perp H^\infty) : f \mapsto f \circ i_V. \tag{6.4}$$

Proposition 6.4. *Let V be an inner product space over a commutative ring R with $\frac{1}{2} \in R$. Then the simplicial set*

$$\mathrm{St}(V, V \perp H^\infty_{\Delta R})$$

is a contractible Kan set.

Proof. Contractibility follows from Proposition A.6 applied to the $O(H^\infty_{\Delta^\bullet R})$ equivariant homotopy equivalence $O(H^\infty_{\Delta^\bullet R}) \subset O(V \perp H^\infty_{\Delta^\bullet R})$ of Lemma 6.3 together with the isomorphism (6.4). The simplicial set is fibrant, by Proposition A.5. □

The group $O(V)$ of isometries of V acts from the right on $\mathrm{St}(V,U)$ via $(fg) \mapsto fg$ for $f \in \mathrm{St}(V,U)$ and $g \in O(V)$. The map $\mathrm{St}(V,U) \to GrO_V(U) : f \mapsto \mathrm{Im}(f)$ factors through the quotient map $\mathrm{St}(V,U) \to \mathrm{St}(V,U)/O(V)$ and yields an isomorphism of (presheaves of) sets

$$\mathrm{St}(V,U)/O(V) \cong GrO_V(U) : f \mapsto \mathrm{Im}(f). \tag{6.5}$$

For an inner product space V over R and a symmetric bilinear form R-module U, let $\mathcal{E}_V(U)$ be the category whose objects are the R-module maps $V \to U$ respecting forms and where a map from $a : V \to U$ to $b : V \to U$ is a map $c : \mathrm{Im}(a) \to \mathrm{Im}(b)$ of inner product spaces such that the diagram

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[dr, "b"'] & \mathrm{Im}(a) \arrow[d, "c"] \\ & \mathrm{Im}(b) \end{tikzcd}$$

commutes. Note that the set of objects of $\mathcal{E}_V(U)$ is the set $\mathrm{St}(V,U)$. The group $O(V)$ acts freely from the right on $\mathcal{E}_V(U)$ via

$$\mathcal{E}_V(U) \times O(V) \to \mathcal{E}_V(U) : (a,g) \mapsto ag,$$

the inclusion of zero simplices $\mathrm{St}(V,U) \to \mathcal{E}_V(U)$ is $O(V)$-equivariant, and the functor $\mathcal{E}_V(U) \to \mathcal{S}_V(U) : a \mapsto \mathrm{Im}(a)$ induces an isomorphism of simplicial sets

$$(B\mathcal{E}_V(U))/O(V) \cong B\mathcal{S}_V(U).$$

Lemma 6.5. *The category $\mathcal{E}_V(V \perp H^\infty)$ is contractible.*

Proof. The category $\mathcal{E}_V(V \perp H^\infty)$ is non-empty as it has $V \to V \perp H^\infty : v \mapsto (v,0)$ as object. Every object in $\mathcal{E}_V(V \perp H^\infty)$ is an initial object. Hence, this category is contractible. □

Proof of Proposition 6.2. The map of simplicial sets

$$\mathrm{St}(V, V \perp H^\infty)(\Delta R) \to \mathcal{E}_V(V \perp H^\infty)(\Delta R)$$

is $O(V_{\Delta R})$-equivariant, the simplicial group $O(V_{\Delta R})$ acts freely on both sides, and the map is a non-equivariant weak equivalence (of contractible simplicial sets), by Proposition 6.4 and Lemma 6.5. By Lemma A.6, the map on quotient simplicial sets $GrO_V(V \perp H^\infty_{\Delta R}) \to \mathcal{S}_V(V \perp H^\infty_{\Delta R})$ is also a weak equivalence. Finally, the inclusion $BO(V) \subset B\mathcal{S}_V(V \perp H^\infty)$ is a weak equivalence as it is the nerve of an equivalence of categories since $\mathcal{S}_V(V \perp H^\infty)$ is a connected groupoid and $O(V)$ is the set of automorphisms of the object $V \subset V \perp H^\infty$. □

7. E_∞-spaces and the end of the proof of Theorem 5.2

Even though diagram (6.3) is a map of homotopy fibrations which is a weak equivalence of simplicial sets on base and fibres, we can't conclude yet that the map on total spaces is a weak equivalence as well. This will be done by establishing that all maps in diagram (6.3) are maps of E_∞-spaces; see Proposition 7.2.

Informally, the "linear isometries" operad $\mathscr{E}$ has n-th space the space of isometric embeddings of $(H^\infty)^n$ into H^∞ with Σ_n permuting the n factors in $(H^\infty)^n$. More precisely, for a commutative ring R, let $\mathscr{E}(n)(R)$ be the set

$$\mathscr{E}(n)(R) = \lim_{V \subset H_R^\infty} \mathrm{St}(V^n, H_R^\infty).$$

The inverse limit ranges over (a cofinal subset of) the category $\mathcal{H}$ of all finitely generated non degenerate $V \subset H_R^\infty$, and $V^n = V \perp ... \perp V$ denotes n-fold orthogonal sum. The permutation group Σ_n on n-letters acts on $\mathscr{E}(n)$ by permuting the factors of V^n. This action is free. By Proposition 6.4, the simplicial sets $\mathrm{St}(V^n, H_{\Delta R}^\infty)$ are contractible Kan sets. By Proposition A.5, for $W \subset V$, the transition maps $\mathrm{St}(V^n, H_{\Delta R}^\infty) \to \mathrm{St}(W^n, H_{\Delta R}^\infty)$ are Kan fibrations in view of the identification (6.4). It follows that

$$\mathscr{E}(n)(\Delta R) = \lim_k \mathrm{St}(H^k \perp ... \perp H^k, H^\infty)(\Delta R)$$

is a contractible Kan set with a free Σ_n-action. We define the structure maps of the operad $\mathscr{E}$ by

$$\mathscr{E}(k) \times \mathscr{E}(j_1) \times \cdots \times \mathscr{E}(j_k) \to \mathscr{E}(j_1 + \cdots + j_k) : f, g_1, ..., g_k \mapsto f \circ (g_1 \perp ... \perp g_k)$$

Thus, we have proved the following lemma.

Lemma 7.1. *Let R be a commutative ring with $\frac{1}{2} \in R$. Then the operad $\mathscr{E}(\Delta R)$ defined above is an E_∞-operad.* □

Proposition 7.2. *For any commutative ring R with $\frac{1}{2} \in R$, the map*

$$GrO_\bullet(\Delta R) \to \mathcal{S}_\bullet(\Delta R)$$

is a map of group complete E_∞-spaces.

Proof. We make $\mathcal{S}_\bullet(R)$ into a module over the operad $\mathscr{E}$. The inclusion of zero-simplices $GrO_\bullet(R) \to \mathcal{S}_\bullet(R)$ will respect this action. So, the proposition will follow from Lemma 7.1.

To define the action of the operad $\mathscr{E}$, write $\mathcal{S}_\bullet$ as

$$\mathcal{S}_\bullet = \mathrm{colim}_{V \subset H^\infty} \mathcal{S}_{|V|}(V^- \perp V^+)$$

where V^- and V^+ are two copies of V and for $V \subset W$ the transition map is defined by

$$\mathcal{S}_{|V|}(V^- \perp V^+) \to \mathcal{S}_{|W|}(W^- \perp W^+) : E \mapsto (W - V)^- \perp E,\ g \mapsto 1_{(W-V)^-} \perp g.$$

Now, the action of $\mathscr{E}$ on $\mathcal{S}_\bullet$ is defined by

$$\mathrm{St}(V_1 \perp ... \perp V_k, W) \times \mathcal{S}_{|V_1|}(V_1^- \perp V_1^+) \times \cdots \times \mathcal{S}_{|V_k|}(V_k^- \perp V_k^+) \longrightarrow \mathcal{S}_{|W|}(W^- \perp W^+)$$

where for $g \in \mathrm{St}(V_1 \perp ... \perp V_k, W)$, the functor

$$\mathcal{S}_{|V_1|}(V_1^- \perp V_1^+) \times \cdots \times \mathcal{S}_{|V_k|}(V_k^- \perp V_k^+) \longrightarrow \mathcal{S}_{|W|}(W^- \perp W^+)$$

sends the object $(E_1, ..., E_k)$ to

$$(W - g(V_1 \perp ... \perp V_k))^- \perp g(E_1 \perp ... \perp E_k)$$

and the map $(e_1, ..., e_k) : (E_1, ..., E_k) \to (E'_1, ..., E'_k)$ to

$$1_{(W-g(V_1\perp...\perp V_k))^-} \perp g_{|E'_1} \circ e_1 \circ g^{-1}_{|E_1} \perp \cdots \perp g_{|E'_k} \circ e_k \circ g^{-1}_{|E_k}.$$

□

Corollary 7.3. *Let R be a connected regular ring with $\frac{1}{2} \in R$. Then the map*

$$GrO_\bullet(\Delta R) \to \mathcal{S}_\bullet(\Delta R)$$

is a weak equivalence of simplicial sets.

Proof. This follows from the map of homotopy fibrations (6.3) in view of Propositions 6.2 and 7.2. □

Proposition 7.4. *Let R be a connected regular noetherian ring with $\frac{1}{2} \in R$. Then the map (5.3) induces a weak equivalence of simplicial sets*

$$\mathscr{S}_\bullet(\Delta R) \xrightarrow{\sim} \widetilde{\mathscr{G}W}(\Delta R).$$

Proof. By the Group Completion Theorem [Gra76, Theorem, p. 221], the map

$$\mathscr{S}_\bullet(R) \to \widetilde{\mathscr{G}W}(R)$$

induces an isomorphism on integral homology groups. It follows that the map in the proposition is an isomorphism on integral homology groups as well. It is well-known that $GW(\Delta R) \simeq GW(R)$ and hence $\widetilde{\mathscr{G}W}(\Delta R) \simeq \widetilde{\mathscr{G}W}(R)$ are group complete H-spaces. By Proposition 7.2, the same is true for $\mathscr{S}_\bullet(\Delta R)$. Therefore, the map in the proposition is indeed a weak equivalence of simplicial sets. □

8. Geometric models for GW^n

In this section we will prove Theorem 1.3.

Proposition 8.1. *Let R be a regular noetherian ring with $\frac{1}{2} \in R$. Then the map (3.2) induces a weak equivalence of simplicial sets*

$$GrO_\bullet(\Delta R) \xrightarrow{\sim} (B_{et}O)(\Delta R).$$

In particular, for any regular noetherian scheme S with $\frac{1}{2} \in \Gamma(S, O_S)$, the canonical map $GrO_\bullet \to B_{et}O$ is isomorphism in $\mathscr{H}_\bullet(S)$.

Proof. For connected R, this follows from Corollary 7.3 in view of Lemma 3.1. Since both sides convert finite disjoint unions into cartesian products, we are done. □

Write $\underline{\mathbb{Z}}$ for the constant sheaf associated with the constant presheaf $\mathbb{Z}$. Recall that the presheaf $\pi_0 B_{et}O$ is homotopy invariant on regular noetherian rings R with $\frac{1}{2} \in R$ since on affine schemes it is the kernel of the rank map $GW \to \underline{\mathbb{Z}}$. Similarly, the presheaves $\pi_0 B_{et}GL$ and $\pi_0 B_{et}Sp$ are also homotopy invariant on affine schemes.

Note that in the next theorem, we have $B_{et}G = B_{Nis}G = B_{Zar}G$ for $G = GL$ and Sp but not for O since GL_n and Sp_n-torsors are Zariski-locally trivial whereas this is not the case for orthogonal groups.

Theorem 8.2. *The canonical maps of presheaves of simplicial sets*

$$\begin{array}{llllll} \underline{\mathbb{Z}} \times B_{et}O & \to & GW, & O & \to & \Omega_{S^1}GW, \\ \underline{\mathbb{Z}} \times B_{et}GL & \to & K, & GL & \to & \Omega_{S^1}K, \\ \underline{\mathbb{Z}} \times B_{et}Sp & \to & GW^2, & Sp & \to & \Omega_{S^1}GW^2 \end{array}$$

are weak equivalences of simplicial sets when evaluated at ΔR for any regular noetherian ring R (with $\frac{1}{2} \in R$ in case of O and Sp). In particular, all these maps are $\mathbb{A}^1$-weak equivalences.

Proof. The first statement for the orthogonal group was proved for connected rings in Theorem 5.2 (see also Lemma 4.4, diagram (5.1) and Proposition 8.1). Source and target of the map convert finite disjoint unions into cartesian products. So, the case of non-connected rings follows. For the second statement, consider the sequence

$$BO \to B_{et}O \to \pi_0 B_{et}O$$

which is section-wise a homotopy fibration. Since the base of the fibration is homotopy invariant on affine schemes, the sequence of simplicial sets

$$(BO)(\Delta R) \to (B_{et}O)(\Delta R) \to (\pi_0 B_{et}O))(\Delta R)$$

is a homotopy fibration with discrete base; see Proposition A.7. It follows that the spaces $(BO)(\Delta R) = B(O(\Delta R))$, $(B_{et}O)(\Delta R)$ and $(\underline{\mathbb{Z}} \times B_{et}O)(\Delta R) \simeq GW(\Delta R)$ all have equivalent S^1-loop spaces. But $\Omega_{S^1}B(O(\Delta R)) \simeq O(\Delta R)$ as is the case for any simplicial group in place of $O(\Delta R)$.

The case of the symplectic groups is *mutatis mutandis* the same as the orthogonal case replacing symmetric forms with alternating forms through-out.

The case of the general linear group is also *mutatis mutandis* the same provided one uses the correct dictionary. "Inner product spaces" should be replaced by "finitely generated projective modules". "Maps respecting forms" $(V, \varphi) \to (V', \varphi')$ are replaced by *direct maps* $(i, q) : P \to P'$, that is, pairs of maps $i : P \to P'$, $q : P' \to P$ such that $qi = 1_P$. Composition of direct maps are composition of the i's and q's. A *direct submodule* of a projective module Q therefore is a submodule $i : P \subset Q$ together with a retract $q : Q \to P$ such that $qi = 1$. The *direct complement* of a direct submodule $(i, q) : P \subset Q$ is the direct submodule $Q - P = \mathrm{Im}(1_Q - iq) \subset Q$ equipped with the retraction $q - iq : Q \to (Q - P)$. Note that $P \oplus (Q - P) = Q$ (as submodules of Q). The index category $\mathcal{H} = \{V \subset H^\infty\}$ in the definition of $GrO_\bullet$ and $\mathcal{S}_\bullet$ gets replaced by the category $\mathcal{H}'$ of finitely generated direct submodules of $R^\infty = \bigoplus_{\mathbb{N}} R$. Direct inclusions, that is inclusions together with retracts, make $\mathcal{H}'$ into a filtered category. With these definitions, the details of the proof for GL are left as an exercise. □

The following lemma applies to groups such as GL, O, Sp and the various forgetful and hyperbolic maps between them. Note that $(B_{et}G)(\Delta R)$ is an E_∞-space for $G = GL$, O, Sp, by Theorem 8.2, or Propositions 7.2 and 8.1 and their analogs for Sp and GL.

Lemma 8.3. *Let G be a presheaf of groups on Sm_S, and let $H \leq G$ be a presheaf of subgroups. Assume that for $\mathrm{Spec}\, R \in \mathrm{Sm}_S$ the map $(B_{et}H)(\Delta R) \to (B_{et}G)(\Delta R)$ is a map of group complete E_∞-spaces. Assume further that the presheaves $\pi_0 B_{et}G$ and $\pi_0 B_{et}H$ are homotopy invariant on affines. Then the canonical sequence*

$$(G/H)_{et} \to B_{et}H \to B_{et}G$$

is a homotopy fibration of simplicial sets when evaluated at ΔR *for any affine* $\operatorname{Spec} R \in \mathrm{Sm}_S$*.*

Proof. Write $\tilde{B}H$ for $(EG)/H$ and recall that the map $BH = (EH)/H \to (EG)/H = \tilde{B}H$ is a weak equivalence on all sections; see Proposition A.6. The sequence of presheaves $G/H \to \tilde{B}H \to BG$ is a fibration sequence of simplicial sets on all sections; see Proposition A.5. Taking fibrant replacements in the etale topology (or any other topology, say, with enough points) preserves section-wise homotopy fibrations. Therefore, the sequence $(G/H)_{et} \to \tilde{B}_{et}H \to B_{et}G$ is a homotopy fibration on all sections. Consider the commutative diagram of simplicial presheaves

$$\begin{tikzcd}
G/H \arrow[r] \arrow[d] & \tilde{B}H \arrow[r] \arrow[d] & BG \arrow[d] \\
(G/H)_{et} \arrow[r] \arrow[d] & \tilde{B}_{et}H \arrow[r] \arrow[d] & B_{et}G \arrow[d] \\
X \arrow[r] & \pi_0 \tilde{B}_{et}H \arrow[r] & \pi_0 B_{et}G
\end{tikzcd}$$

where X is the homotopy fibre (in this case, the kernel) of $\pi_0 \tilde{B}_{et}H \to \pi_0 B_{et}G$. In this diagram, all rows and columns are homotopy fibrations, and the bottom row is homotopy invariant on affines. Moreover, the lower vertical maps are surjective on π_0 (the left one because of the long exact sequence of homotopy groups associated with the middle row). For $\operatorname{Spec} R \in \mathrm{Sm}_S$, we therefore obtain a commutative diagram of simplicial sets

$$\begin{tikzcd}
(G/H)(\Delta R) \arrow[r] \arrow[d] & (\tilde{B}H)(\Delta R) \arrow[r] \arrow[d] & (BG)(\Delta R) \arrow[d] \\
(G/H)_{et}(\Delta R) \arrow[r] \arrow[d] & (\tilde{B}_{et}H)(\Delta R) \arrow[r] \arrow[d] & (B_{et}G)(\Delta R) \arrow[d] \\
X(\Delta R) \arrow[r] & (\pi_0 \tilde{B}_{et}H)(\Delta R) \arrow[r] & (\pi_0 B_{et}G)(\Delta R)
\end{tikzcd}$$

in which the columns are homotopy fibrations, by Proposition A.7. The bottom row is a homotopy fibration since it is the same as the bottom row of the previous diagram. The top row is a homotopy fibration, by Proposition A.5, since $(BN)(\Delta R) = B(N(\Delta R))$ for any presheaf of groups N. Furthermore, the lower vertical maps are surjective on π_0 since this was also the case in the previous diagram. The left column homotopy fibration maps to the homotopy fibration obtained by taking the homotopy fibres of the right horizontal maps. By the five lemma applied to the long exact sequence of homotopy groups (in which all homotopy groups and sets are abelian groups as all spaces involved are group complete E_∞-spaces, and the last non-trivial maps in the long exact sequences of homotopy groups are surjective) these two homotopy fibrations are weakly equivalent. It follows that the middle row is also a homotopy fibration. Since $BH \to \tilde{B}H$ is a section wise weak equivalence, the same is true for $B_{et}H \to \tilde{B}_{et}H$ and $\operatorname{Sing}_\bullet^{\mathbb{A}^1} B_{et}H \to \operatorname{Sing}_\bullet^{\mathbb{A}^1} \tilde{B}_{et}H$. This proves the claim. $\square$

In the following theorem, the sheafification map $Sp/GL \to (Sp/GL)_{Zar} = (Sp/GL)_{Nis} = (Sp/GL)_{et}$ is a weak equivalence in the Zariski-topology and hence an $\mathbb{A}^1$-weak equivalence; similarly for O/GL and GL/Sp. Thus, the following theorem together with Theorem 8.2 implies Theorem 1.3 from the Introduction.

Theorem 8.4. *There are canonical maps of simplicial presheaves*

$$\begin{array}{lllllll} (Sp/GL)_{et} & \to & GW^1, & \qquad & (GL/O)_{et} & \to & \Omega_{S^1}GW^1, \\ (O/GL)_{et} & \to & GW^3, & \qquad & (GL/Sp)_{et} & \to & \Omega_{S^1}GW^3 \end{array}$$

which are weak equivalences of simplicial sets when evaluated at ΔR where R is any regular noetherian ring with $\frac{1}{2} \in R$. In particular, all these maps are $\mathbb{A}^1$-weak equivalences.

Proof. For $n \in \mathbb{Z}$ there are homotopy fibrations $GW^n \xrightarrow{F} K \xrightarrow{H} GW^{n+1}$ where F and H denote forgetful and hyperbolic functor, respectively [Sch12, Theorem 6.1]. Since GW^n and K are homotopy invariant on regular rings [Sch12, Theorem 9.8], we have homotopy fibrations

$$GW^n(\Delta R) \xrightarrow{F} K(\Delta R) \xrightarrow{H} GW^{n+1}(\Delta R)$$

for any regular noetherian R with $\frac{1}{2} \in R$. The results now follow from Theorem 8.2 and Lemma 8.3. □

Remark 8.5. The proof given in [MV99] that $\mathbb{Z} \times Gr_\bullet \cong \mathbb{Z} \times BGL \cong K$ in $\mathscr{H}_\bullet(S)$ formally rests on [MV99, Proposition 1.9, p.126]. This proposition, however, is false as the following example shows.

Let T be the one-point-site, so that $\mathscr{H}_s(T)$ is the homotopy category of simplicial sets. Let R be a non-zero ring and $M = \bigsqcup_{n\in\mathbb{N}} BGL_n(R)$ be the monoid defined by $BGL_m \times BGL_n \to BGL_{m+n} : (A, B) \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. This monoid is commutative in $\mathscr{H}_s(T)$ because it is the classifying space of the symmetric monoidal category of finite rank free R-modules with isomorphisms as morphisms. Alternatively, the monoid multiplication is commutative because $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} B & 0 \\ 0 & A \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^{-1}$ and conjugation $c_g : G \to G : h \mapsto ghg^{-1}$ induces a map $c_g : BG \to BG$ on classifying spaces which is homotopic to the identity map. If we believe the conclusion of [MV99, Proposition 1.9, p.126], then we would have a weak equivalence of simplicial sets $\mathbb{Z} \times BGL(R) \sim \Omega B(M)$ which cannot exist since π_1 of the left hand side is non-abelian whereas π_1 of the right hand side is abelian.

Appendix A. Simplicial sets

We collect a few well-known facts about simplicial sets which are used throughout the paper. The standard reference nowadays is [GJ99].

Lemma A.1. *Given a sequence $X \to Y \to Z$ of simplicial sets in which $X \to Y$ is a surjective fibration and the composition $X \to Z$ is a fibration. Then the map $Y \to Z$ is a fibration.*

Proof. One checks that $Y \to Z$ has the right lifting property with respect to the maps $\Lambda^k_n \subset \Delta_n$. Any map $\Lambda^k_n \to Y$ lifts to a map $\Lambda^k_n \to X$. This is because we can lift the image of a zero simplex in Λ^k_n (as $X \to Y$ is surjective) and extend this lift to all of Λ^k_n since $X \to Y$ is a fibration and the inclusion of a point into Λ^k_n is an acyclic cofibration. Then the map $\Delta_n \to Z$ lifts to X since $X \to Z$ is a fibration. Composing this lift with the map $X \to Y$ yields the required map. □

Lemma A.2. *Given a cartesian square of simplicial sets*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

in which the right (and hence the left) vertical map is a surjective fibration. Then the upper horizontal map is a weak equivalence if and only if the lower horizontal map is a weak equivalence.

Proof. By properness of the model category of simplicial sets, if $Z \to W$ is a weak equivalence then so is $X \to Y$.

Assume that $X \to Y$ is a weak equivalence. Factoring $Z \to W$ into an acyclic cofibration and a fibration and pulling $Y \to W$ along that fibration, we can reduce to showing the claim in case $Z \to W$ is a fibration (and $X \to Y$ an acyclic fibration). Then we need to show that $Z \to W$ has the right lifting property with respect to all inclusions $\partial\Delta_n \subset \Delta_n$.

Given a map from $\partial\Delta_n \subset \Delta_n$ to $Z \to W$. Choose a lift of $\Delta_n \to W$ to Y which exists since $Y \to W$ is surjective. The universal property of Y as a pull-back yields a lift of $\partial\Delta_n \to Z$ to X making all diagrams commute. Since $X \to Y$ is an acyclic fibration, the map $\Delta_n \to Y$ lifts to X. Composing this map with $X \to Z$ yields the required lift. □

Lemma A.3. [GJ99, Lemma I.3.4] *Let G be a simplicial group. Then G is fibrant as simplicial set.*

Proposition A.4. *Let G be a simplicial group acting freely from the right on a simplicial set X. Then the quotient map $X \to X/G$ is a Kan fibration.*

Proof. We need to show that the map $X \to X/G$ has the right lifting property with respect to the standard generating acyclic cofibrations $\Lambda^n_k \subset \Delta^n$. Take a commutative square for which we have to find a lift. Pulling back along the map $\Delta^n \to E/G$, we see that it suffices to show the claim of the proposition in case $X/G = \Delta^n$. In this case, choose a section $s : \Delta^n \to X$ and define the map $\Delta^n \times G \to X : (x, g) \mapsto s(x)g$. Since G acts freely on X, this map is an isomorphism. Finally, the projection map $\Delta^n \times G \cong X \to \Delta^n$ is a fibration because G is fibrant. □

Proposition A.5. *Let G be a simplicial group and let $H \leq G$ be a simplicial subgroup. Let X be a simplicial set with a free G-action from the right. Then the map $X/H \to X/G$ is a Kan fibration. In particular, the map $G \to G/H$ is a Kan fibration, and the simplicial set G/H is fibrant.*

Proof. We apply Lemma A.1 to the sequence $X \to X/H \to X/G$ using Proposition A.4. So, $X/H \to X/G$ is a Kan fibration. Applied to $X = G$ and the inclusion of subgroups $\{e\} \subset H$, we obtain the Kan fibration $G \to G/H$. Applied to $X = G$ and the inclusion of groups $H \subset G$, we obtain the Kan fibration $G/H \to G/G = *$. □

Proposition A.6. *Let G be a simplicial group acting freely on the right on the simplicial sets X and Y. Let $X \to Y$ be a G-equivariant map which is a non-equivariant weak equivalence (that is, a weak equivalence forgetting the action). Then $X/G \to Y/G$ is a weak equivalence.*

Proof. Apply Lemma A.2 with $Z \to W$ the map $X/G \to Y/G$ and vertical maps the quotient maps. The diagram is cartesian because G acts freely on X and Y, and the right vertical map is a surjective fibration, by Proposition A.4. □

For a bisimplicial set X, denote by $\operatorname{diag} X$ the diagonal simplicial set $(\operatorname{diag} X)_n = X_{n,n}$. The following proposition follows from the Bousfield-Friedlander theorem [GJ99, Theorem IV.4.9] or from Mather's Cube Theorem [Mat76].

Proposition A.7. *Let $X_{\bullet\bullet} \to Y_{\bullet\bullet} \to Z_{\bullet\bullet}$ be a sequence of bisimplicial sets such that for all $p \in \mathbb{N}$, the sequence of simplicial sets $X_{p\bullet} \to Y_{p\bullet} \to Z_{p\bullet}$ is a homotopy fibration, and $Z_{\bullet\bullet}$ is constant in the p-direction, that is, $Z_{p_1,\bullet} \to Z_{p_2,\bullet}$ is the identity for all simplicial operators $[p_2] \to [p_1]$. Then the sequence of diagonal simplicial sets*

$$\operatorname{diag} X \to \operatorname{diag} Y \to \operatorname{diag} Z$$

is a homotopy fibration.

In order to construct certain maps in the body of our paper we will have to use homotopy colimits. The reason is that the K-theory and hermitian K-theory spaces are homotopy colimits themselves; see Remark 4.5. Below we recall the construction within the category of small categories, and in Lemma A.9 we recall a well-known basic fact that we will need.

Definition A.8 (Homotopy colimits)**.** Let $\mathscr{C}$ be a small category and $\mathscr{F} : \mathscr{C} \to \mathrm{Cat}$ a functor from $\mathscr{C}$ to the category Cat of small categories. The homotopy colimit

$$\operatorname{hocolim}_{\mathscr{C}} \mathscr{F}$$

is the category whose objects are pairs (X, A) with X and object of $\mathscr{C}$ and A an object of $\mathscr{F}(X)$. A map from (X, A) to (Y, B) is a pair (x, a) where $x : X \to Y$ is a map in $\mathscr{C}$ and $a : \mathscr{F}(x)A \to B$ is a map in $\mathscr{F}(Y)$. Composition $(y, b) \circ (x, a)$ of $(y, b) : (Y, B) \to (Z, C)$ and $(x, a) : (X, A) \to (Y, B)$ is the map $(y \circ x, b \circ F(y)a)$.

By a result of Thomason [Tho79], the nerve simplicial set $N_* \operatorname{hocolim}_{\mathscr{C}} \mathscr{F}$ is naturally homotopy equivalent to the Bousfield-Kan homotopy colimit of the diagram $N_*\mathscr{F} : \mathscr{C} \to \Delta^{op}\,\mathrm{Sets}$ of simplicial sets. We won't need this fact, but we will need the following special case. For that, recall that a poset $(\mathscr{P}, \leq)$ is considered a category with objects the elements of the poset and a unique map from $P \in \mathscr{P}$ to $Q \in \mathscr{P}$ if $P \leq Q$. The poset $(\mathscr{P}, \leq)$ is filtering if for every $P, Q \in \mathscr{P}$ there is a $R \in \mathscr{P}$ with $P, Q \leq R$.

Lemma A.9. *Let $(\mathscr{P}, \leq)$ be a filtering poset and let $\mathscr{F} : \mathscr{P} \to \mathrm{Cat}$ be a functor from $\mathscr{P}$ into the category* Cat *of small categories. Then the functor of categories*

$$\phi : \operatorname{hocolim}_{\mathscr{P}} \mathscr{F} \to \operatorname{colim}_{\mathscr{P}} \mathscr{F} : (P, A) \mapsto [P, A]$$

is a homotopy equivalence of simplicial sets.

Proof. By Quillen's theorem A [Qui73], it suffices to show that for every object $[P, A]$ of the category $\operatorname{colim}_{\mathscr{P}} \mathscr{F}$, the comma category $(\phi \downarrow [P, A])$ is contractible. For $A \in \mathscr{F}(P)$ and $P \leq Q$ write A_Q for the object $\mathscr{F}(P \leq Q)A$ in $\mathscr{F}(Q)$ which is the image of A under the functor $\mathscr{F}(P \leq Q) : \mathscr{F}(P) \to \mathscr{F}(Q)$. Contractibility of the comma category now follows from the equivalence of categories

$$\operatorname{colim}_{P \leq Q \in \mathscr{P}} (id \downarrow (Q, A_Q)) \cong (\phi \downarrow [P, A])$$

where for $Q \leq R$, the functor $(id \downarrow (Q, A_Q) \to (id \downarrow (R, A_R)$ sends $t : (T, B) \to (Q, A_Q)$ to $c \circ t : (T, B) \to (R, A_R)$ with $c : (Q, A_Q) \to (R, A_R)$ the map given by $id : A_R = \mathscr{F}(Q \leq R)A_Q \to A_R$. The left-hand category is a filtered colimit over categories with initial objects, hence a filtered colimit over contractible categories. Therefore, the left-hand category is contractible, and so is the right-hand category. □

Definition A.10. Let k be a commutative ring and F, G be simplicial presheaves on smooth affine k-schemes. An *elementary* $\mathbb{A}^1$*-homotopy* between two maps $h_0, h_1 : F \to G$ of presheaves is a map of presheaves $h : \mathbb{A}^1 \times F \to G$ such that $h_i = h \circ j_i$, $i = 0, 1$ where $j_i : \operatorname{Spec}(k) \to \mathbb{A}^1$ corresponds to the evaluation $k[t] \to k : t \mapsto i$. Elementary homotopy generates an equivalence relation called *naive* $\mathbb{A}^1$*-homotopy*. The following is a well-known fundamental fact from $\mathbb{A}^1$-homotopy theory.

Lemma A.11. *If $h_0, h_1 : F \to G$ are naively $\mathbb{A}^1$-homotopic then for every k-algebra R, the maps $h_0, h_1 : F(\Delta R) \to G(\Delta R)$ are simplicially homotopic.*

Proof. It suffices to prove the claim for elementary $\mathbb{A}^1$-homotopy. Let $h : \mathbb{A}^1 \times F \to G$ be an elementary homotopy between h_0 and h_1. The 1-simplex $id \in \mathbb{A}^1(\Delta^1) \cong \mathbb{A}^1(\mathbb{A}^1)$ of the simplicial set $\mathbb{A}^1(\Delta)$ defines a map of simplicial sets $\Delta^1 \to \mathbb{A}^1(\Delta_k) \to \mathbb{A}^1(\Delta_R)$ which induces the required homotopy $H : \Delta^1 \times F(\Delta_R) \to \mathbb{A}^1(\Delta_R) \times F(\Delta_R) \xrightarrow{h} G(\Delta_R)$. □

References

[AF12] Aravind Asok and Jean Fasel. A cohomological classification of vector bundles on smooth affine threefolds. *arXiv:1204.0770*, 2012.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[Gra76] Daniel Grayson. Higher algebraic K-theory. II (after Daniel Quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 217–240. Lecture Notes in Math., Vol. 551. Springer, Berlin, 1976.

[Hor05] Jens Hornbostel. A^1-representability of Hermitian K-theory and Witt groups. *Topology*, 44(3):661–687, 2005.

[Jar01] J. F. Jardine. Stacks and the homotopy theory of simplicial sheaves. *Homology Homotopy Appl.*, 3(2):361–384, 2001. Equivariant stable homotopy theory and related areas (Stanford, CA, 2000).

[Kar73] Max Karoubi. Périodicité de la K-théorie hermitienne. In *Algebraic K-theory, III: Hermitian K-theory and geometric applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 301–411. Lecture Notes in Math., Vol. 343. Springer, Berlin, 1973.

[Mat76] Michael Mather. Pull-backs in homotopy theory. *Canad. J. Math.*, 28(2):225–263, 1976.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[PW10] Ivan Panin and Charles Walter. On the motivic commutative ring spectrum *BO*. *arXiv:1011.0650*, 2010.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[Rio10] Joël Riou. Algebraic K-theory, $\mathbf{A}^1$-homotopy and Riemann-Roch theorems. *J. Topol.*, 3(2):229–264, 2010.

[Sch04] Marco Schlichting. Hermitian K-theory on a theorem of Giffen. *K-Theory*, 32(3):253–267, 2004.

[Sch10a] Marco Schlichting. Hermitian K-theory of exact categories. *J. K-Theory*, 5(1):105–165, 2010.

[Sch10b] Marco Schlichting. The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes. *Invent. Math.*, 179(2):349–433, 2010.

[Sch12] Marco Schlichting. Hermitian K-theory, derived equivalences, and Karoubi's Fundamental Theorem. *arXiv:1209.0848*, 2012.

[Tho79] R. W. Thomason. Homotopy colimits in the category of small categories. *Math. Proc. Cambridge Philos. Soc.*, 85(1):91–109, 1979.

[Zib11] Marcus Zibrowius. Witt groups of complex cellular varieties. *Doc. Math.*, 16:465–511, 2011.