Self-similar evolution of Alfven wave turbulence

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Abstract. We study self-similar solutions of the kinetic equation for MHD wave turbulence derived in [1]. Motivated by finding the asymptotic behaviour of solutions for initial value problems, we formulate a nonlinear eigenvalue problem comprising in finding a number x^* such that the self-similar shape function $f(\eta)$ would have a power-law asymptotic η^{-x^*} at low values of the self-similar variable η and would be the fastest decaying positive solution at $\eta \to \infty$. We prove that the solution $f(\eta)$ of this problem has a tail decaying as a power-law, and not exponentially or super-exponentially. We present a relationship between the power-law exponents in the regions $\eta \to 0$ and $\eta \to \infty$, and an integral relation for $f(\eta)$ and x^*. We confirm these relationships by solving numerically the nonlinear eigenvalue problem, and find that $x^* \approx 3.80$.

Figure 1. The integration area Δ_k.

1. Introduction

3D MHD turbulence in incompressible conducting fluid embedded in a strong uniform magnetic field consists of a large set of weakly interacting Alfvén waves. In the leading order of nonlinearity, such waves do not transfer their energy among the modes with different values of the wave vector component parallel to the magnetic field, $k_\parallel$. This leads to the facts that the 3D wave action spectrum separates as $n^{(3D)}(k_\parallel, \mathbf{k}_\perp, t) = f(k_\parallel) n(\mathbf{k}_\perp, t)$ with a time-independent (set by initial conditions) parallel part $f(k_\parallel)$. The transverse spectrum $n(\mathbf{k}_\perp, t)$, assuming it to be isotropic in the $\mathbf{k}_\perp$-plane, is evolving according to the following kinetic equation [1, 2, 3],

$$\frac{\partial n(k,t)}{\partial t} = \iint_{\Delta_k} W(k, k_1, k_2)\, n(k_1)\left[n(k_2) - n(k)\right] dk_1 dk_2, \tag{1}$$

where $k = |\mathbf{k}|$, $k_1 = |\mathbf{k}_{1\perp}|$, $k_2 = |\mathbf{k}_{1\perp}|$. The integration area Δ_k is determined by the triangle inequalities,

$$\Delta_k = \left\{(k_1, k_2) \,:\, (k \le k_1 + k_2) \cap (k_1 \le k + k_2) \cap (k_2 \le k_1 + k)\right\}. \tag{2}$$

This is sketched in figure 1. The interaction coefficient is

$$W(k, k_1, k_2) = k k_2 \cos^2 \theta_2 \sin \theta_1, \tag{3}$$

where θ_2 is the angle between $\mathbf{k}$ and $\mathbf{k}_2$, and θ_1 is the angle between $\mathbf{k}$ and $\mathbf{k}_1$. Thus,

$$\cos^2 \theta_2 = \left(\frac{k^2 - k_1^2 + k_2^2}{2kk_2}\right)^2, \quad \sin \theta_1 = \frac{\sqrt{2(k^2 k_1^2 + k^2 k_2^2 + k_1^2 k_2^2) - k^4 - k_1^4 - k_2^4}}{2kk_1}.$$

Equation (1) has a two stationary power-law solutions: $n(k) \sim k^0$ corresponding to a thermodynamic equilibrium and $n(k) \sim k^{-3}$ corresponding to the Kolmogorov-Zakharov (KZ) state with constant flux of energy from small to large wave numbers k.

The aim of this paper is to study the transient self-similar behaviour discovered in numerical experiments of [1]. The self-similar solutions are similar to the ones analysed in [4, 5, 6, 7] for the nonlinear diffusion models of Leith type [8]: they have a propagating front which accelerates explosively, reaching $k = \infty$ in a finite time t_*. The main feature of such self-similarity is that the scaling exponents cannot be find by dimensional considerations and from the existence of conservation laws. In particular, the low-wavenumber asymptotic of the self-similar spectrum is a power law with an anomalous (non-Kolmogorov) exponent. Following the Zeldovich-Raizer terminology, this type of behaviour is usually called a self-similarity of the second kind [9, 10].

The high-wavenumber boundary condition for the self-similar solution in the nonlinear diffusion equation is a sharp front – with the solution being identically equal to zero beyond a finite support. We deal with an integro-differential equation for which there exist no finite support solutions. Thus, the question about the boundary conditions at the high wavenumbers has to be readdressed.

In the present paper we argue that the correct boundary conditions for the self-similar solution in the considered kinetic-equation model are: (i) power-law asymptotic at low wave numbers (with an exponent x^* determined by the nonlinear eigenvalue problem) and (ii) the fastest decay at large wave numbers within the class of positive functions. We show that the fastest decay at infinity also takes form of a power law, and we predict a simple relation between the power law exponents in the vicinities of zero and infinity. We further predict the following integral relationship between the shape of the self-similar function $f(\eta)$ (defined below in (5)) and exponent x^*

$$D = \int_0^\infty f(\eta)\eta^3 d\eta = \frac{8}{\pi(4 - x^*)\left[(x^* - 1) + \sqrt{(x^* - 1)^2 + 1}\right]}. \tag{4}$$

By numerical simulations, we find $f(\eta)$ and the exponent x^*, and confirm the above-mentioned integral relationship. With an accuracy of about one percent we find $x^* = 3.8$, which compares with $x^* \approx 3.33$ previously obtained by numerical simulation of the time-dependent kinetic equation (1)) in [1]. The reason for the discrepancy is not yet known, but we suspect that it is related to the logarithmic discretisation used in [1] that may not have resolved well the structure of propagating front of the spectrum.

2. Self-similar solutions and the nonlinear eigenvalue problem

We look for self-similar solutions of the second kind of equation (1) in the form

$$n(t, k) = \frac{1}{\tau^a}f(\eta), \quad \eta = \frac{k}{\tau^b}, \quad \tau = t^* - t. \tag{5}$$

To eliminate τ, we need to impose the condition $a = 1+4b$. Then the self-similar shapes $f(\eta)$ obey the following equation,

$$xf + \eta f' = \frac{1}{b} \iint_{\Delta_\eta} W(\eta, \eta_1, \eta_2) f(\eta_1)[f(\eta_2) - f(\eta)]d\eta_1\eta_2, \tag{6}$$

where

$$x = \frac{a}{b}, \quad b = \frac{1}{x - 4},$$

and Δ_η is given by (2) where k, k_1, k_2 are replaced by η, η_1, η_2 respectively.

2.1. Boundary conditions

Equation (6) has to be complimented with boundary conditions at $\eta \to 0$ and $\eta \to \infty$. Based on our experience with the self-similar solutions of the Leith-type nonlinear PDE models [4, 6, 7], we postulate the condition on the right boundary as

$$f(\eta) \to \eta^{-x} \quad \text{for} \quad \eta \to 0. \tag{7}$$

Three remarks are due here.

 (i) Our system is scale-invariant because the interaction coefficient is a homogeneous
    function

$$W(\lambda\eta, \lambda\eta_1, \lambda\eta_2) = \lambda^2 W(\eta, \eta_1, \eta_2). \tag{8}$$

Hence, if $f(\eta)$ is a solution of equation (6) satisfying condition (7) then $\tilde{f}(\eta) = \lambda^{-4} f(\lambda\eta)$ is also a solution – it satisfies condition $\tilde{f}(\eta) \to \lambda^{-4}\eta^{-x}$ for $\eta \to 0$. Thus it is enough to consider condition (7) without a pre-factor in front of the power law.

 (ii) The self-similar formulation with the boundary condition (7) is self-consistent only
    if x corresponds to convergence of the integral in (6) at $\eta_1 \to 0$ and $\eta_2 \to 0$ i.e. only
    if $x < 4$, see the Appendix.

 (iii) As $\eta \to 0$, the right-hand side of equation (6) becomes vanishingly small compared
    to each of the terms on the left-hand side for $x < 4$, which ensures that $f = \eta^{-x}$
    satisfies this equation.

The second boundary condition is the condition on the right boundary. In the Leith-type PDE models, this condition was that $f(\eta) \equiv 0$ for $\eta \geq \eta^*$ for some constant η^* and that the energy flux turns into zero at $\eta = \eta^*$ [4, 6]. There is only one value of x, $x = x^*$, for which such a boundary condition can be satisfied, and finding x^* constitutes the nonlinear eigenvalue problem to be solved. The self-similar solution corresponding to $x = x^*$ is the only one that forms asymptotically at large k in the initial value problem of the evolution equation under consideration with initial data in a finite support.

Notice that finite-support solutions are impossible for the integral equation (6). Indeed, suppose that there exists a finite-length support set A such that $f(\eta) > 0$ for $\eta \in A$ and $f(\eta) \equiv 0$ otherwise. Then for some η outside of the support A, at a finite but sufficiently close distance to its boundary, there exist finite intersection $A \cap \Delta_\eta$. For all η_1 and η_2 in $A \cap \Delta_\eta$ the integrand in the right hand side of (6) is finite and positive. Hence, equation (6) cannot be satisfied in this case.

It is however natural to think that the self-similar solution chosen by the evolution will correspond to $x = x^*$ for which $f(\eta)$ is positive everywhere and tends to zero at $\eta \to \infty$ in a fastest way among the solutions with different values of x. This is the second boundary condition which we postulate. This constitutes the nonlinear eigenvalue problem of finding $x = x^*$ for which this boundary condition is satisfied simultaneously with the condition at $\eta \to 0$.

The condition that the solution must remain positive for all η arises from the positivity of $n(k, t)$ which is preserved by the kinetic equation (1). Note that not for all x are the self-similar solutions positive. In fact, $x = x^*$ separates the values of x for which the solution is positive from the values for which it crosses zero at some η. Detecting when such zero-crossings disappear will be exploited by us for finding x^* numerically.

3. Large-η asymptotics

Finding the exponent x^* analytically is difficult and probably even impossible. The same is true even for the simplest Leith-type PDE models. However, important relations between this quantity and the other properties of function $f(\eta)$ can be established via considering the large-η asymptotics of this function.

First of all, let us consider a possibility that at $\eta \gg 1$ tail of $f(\eta)$ the interaction is local, i.e. that the leading order contribution to the integral of (6) comes from $\eta_1, \eta_2 \sim \eta$, and not from the regions $\eta_1 \ll \eta, \eta_2$, $\eta_2 \ll \eta, \eta_1$ or $\eta_1 \approx \eta_2 \gg \eta$. In this case $f(\eta)$ must be bound between two power laws: $f(\eta) > const/\eta^4$ (for convergence at $\eta_1 \ll \eta, \eta_2$ and $\eta_2 \ll \eta, \eta_1$) and $f(\eta) < const/\eta^2$ (for convergence at $\eta_1 \approx \eta_2 \gg \eta$); see the Appendix. So let us take $f(\eta) = C\eta^y$ with $-4 < y < -2$ and $C = $ const. Then equation (6) becomes

$$x + y = C\eta^{y+4} \frac{1}{b} \iint_{\Delta_1} W(1, \kappa_1, \kappa_2)\kappa_1^y[\kappa_2^y - 1]d\kappa_1\kappa_2 \quad \text{(where } \kappa_{1,2} = \eta_{1,2}/\eta). \text{(9)}$$

Because of the pre-factor η^{y+4} the right-hand side tends to infinity as $\eta \to \infty$. Thus this equation can only be satisfied when the integral is zero, i.e. when y corresponds to the stationary KZ solution. But this solution must be rejected because it does not conserve energy – the energy flux is constant on this solution, and the energy is lost at $\eta \to \infty$ at a constant rate. Therefore, the interaction at the $\eta \gg 1$ tail of $f(\eta)$ is nonlocal. Nonlocal interaction with region $\eta_1 \approx \eta_2 \gg \eta$ implies slowly decaying tails with $f(\eta) > const/\eta^2$. Such spectra contain infinite energy $2\pi \int_0^\infty kn(k, t)\, dk$ and, therefore, cannot develop out of a finite-energy initial data. Thus, the nonlocal interaction takes place with the large-scale regions $\eta_1 \ll \eta, \eta_2$ and $\eta_2 \ll \eta, \eta_1$.

3.1. Absence of exponential and super-exponential tails

Consider first the region $\eta_1 \ll \eta, \eta_2$ and suppose that the main contribution comes from the scales $\eta_1 \sim 1$. Here we have the second small parameter $\sigma = \eta_2 - \eta$ such that within Δ_η we have $|\sigma| \leq \eta_1 \sim 1 \ll \eta$. Taylor expanding the interaction coefficient, we have

$$W(\eta, \eta_1, \eta_2) = \eta^2\sqrt{1 - \frac{\sigma^2}{\eta_1^2}} + \eta^2 o(\eta_1^2) + \eta^2 o(\sigma^2), \tag{10}$$

and equation (6) becomes

$$xf + \eta f' \approx \frac{\eta^2}{b} \int_0^\infty f(\eta_1)d\eta_1 \int_{-\eta_1}^{\eta_1} \sqrt{1 - \frac{\sigma^2}{\eta_1^2}} \, [f(\eta + \sigma) - f(\eta)] \, d\sigma. \tag{11}$$

Suppose that the tail of $f(\eta)$ is decaying so rapidly that $|\eta_1 f'(\eta)|$ is not small compared to $|f(\eta)|$ so that one cannot Taylor expand the square bracket in the above equation. This is the case, e.g., for the exponential and super-exponential functions, $f(\eta) \sim e^{-\mu\eta^d}$ with $\mu = const \sim 1$ and $d \geq 1$. Then the first term on the left-hand side of (11) can be neglected,

$$f' \approx \frac{\eta}{b} \int_0^\infty f(\eta_1)d\eta_1 \int_{-\eta_1}^{\eta_1} \sqrt{1 - \frac{\sigma^2}{\eta_1^2}} \, [f(\eta + \sigma) - f(\eta)] \, d\sigma. \tag{12}$$

By the direct substitution, we see that the exponential function ($d = 1$) does not solve this equation. But for $d > 1$ the second term in the square bracket is sub-dominant and can be neglected. Since $b < 0$, we have in this case

$$bf' \approx \eta \int_0^\infty f(\eta_1)d\eta_1 \int_{-\eta_1}^{\eta_1} \sqrt{1 - \frac{\sigma^2}{\eta_1^2}} f(\eta + \sigma)d\sigma$$

$$> \eta \int_0^\infty f(\eta_1)d\eta_1 \int_{-\eta_1}^0 \sqrt{1 - \frac{\sigma^2}{\eta_1^2}} f(\eta + \sigma)d\sigma$$

$$> \eta f(\eta) \int_0^\infty f(\eta_1)d\eta_1 \int_{-\eta_1}^0 \sqrt{1 - \frac{\sigma^2}{\eta_1^2}} d\sigma = \frac{\pi}{4}\eta f(\eta) \int_0^\infty f(\eta_1)\eta_1 d\eta_1 = \infty \quad (13)$$

since $f(\eta_1) \to \eta_1^{-x}$ with $x > 2$ for $\eta_1 \to 0$. Thus, we arrive at an inequality which is false and, therefore, the exponential and super-exponential tails are not possible. Taking into account the region $\eta_2 \ll \eta, \eta_1$ would not change this conclusion because the respective contribution is strictly positive.

3.2. Power-law decay at $\eta \gg 1$

For supra-exponential tails, such that $|f'(\eta)| \ll |f(\eta)|$, e.g. when $f(\eta) \sim e^{-\mu\eta^d}$ with $\mu = const \sim 1$ and $0 < d < 1$ or $f(\eta) \sim \eta^{-y}$ with $y > 0$, the square brackets in equation (11) can be Taylor expanded in $\eta_1 \sim \sigma \sim 1 \ll \eta$. Similarly, the integrand of (6) can be expanded in $\eta_2 \sim \rho = \eta_1 - \eta \sim 1 \ll \eta$. The sum of the respective contributions from the regions $\eta_1 \ll \eta, \eta_2$ and $\eta_2 \ll \eta, \eta_1$ (expressions (35) and (38) respectively, see the Appendix) leads to the following ODE,

$$xf + \eta f' = N^{-1} \left[3\eta f' + \eta^2 f'' + 2f \right], \quad (14)$$

where

$$N = \frac{16b}{\pi D} < 0 \quad \text{and} \quad D = \int_0^\infty f(\eta_1)\eta_1^3 \, d\eta_1. \quad (15)$$

We should consider N to be a given constant which makes this ODE linear and easy to solve. Note that the integral D is convergent at $\eta_1 \to 0$ since $f(\eta_1) \to \eta_1^{-x}$ with $x < 4$. Convergence at $\eta_1 \to \infty$ is assumed for now but will be checked *a posteriori*. It requires that $f(\eta) < const \, \eta^{-4}$ at $\eta \to \infty$. Such a convergence on both ends of implies that the main contribution in the integral D comes from the region $\eta_1 \sim 1$, which is consistent with the Taylor expansion used.

Let us introduce

$$N^*(x) = -2 \left[(x - 1) + \sqrt{(x - 1)^2 + 1} \right] < 0. \quad (16)$$

For $N \neq N^*$, equation (14) has two fundamental power-law solutions:

$$f = \eta^{\lambda_1} \quad \text{and} \quad f = \eta^{\lambda_2}, \quad (17)$$

with $\lambda_1 \neq \lambda_2$:

$$\lambda_1 = -1 + \frac{N}{2} - \frac{1}{2}\sqrt{N^2 + 4N(x - 1) - 4}, \quad \text{and} \quad (18)$$

$$\lambda_2 = -1 + \frac{N}{2} + \frac{1}{2}\sqrt{N^2 + 4N(x - 1) - 4}. \quad (19)$$

For $N = N^*$, the exponents of the power laws degenerate, $\lambda_1 = \lambda_2 = \lambda^*$, and the fundamental solutions become

$$f = \eta^{\lambda^*} \quad \text{and} \quad f = \eta^{\lambda^*} \ln \eta. \tag{20}$$

The linear combination of these solutions can be written in the form

$$f = C\eta^{\lambda^*} \ln(\eta_0/\eta) \tag{21}$$

with some constants C and η_0 (the latter being positive) and

$$\lambda^* = -1 + \frac{N^*}{2}. \tag{22}$$

We can see that solution (21) crosses zero at $\eta = \eta_0$ excepting for the case $\eta_0 \to \infty$, in which case one simply has $f = C\eta^{\lambda^*}$.

3.3. The tail corresponding to the solution with $x = x^*$

The nonlinear eigenvalue problem we have formulated requires finding $x = x^*$ for which the tail of $f(\eta)$ decays in the fastest way while remaining positive.

For $N < N^*$, the exponents λ_1 and λ_2 are complex. The corresponding real-valued solutions have infinitely many zero crossings and, therefore, cannot correspond to the solution of the nonlinear eigenvalue problem.

The solutions that stay positive for all η are only possible for $N \geq N^*$. These are the power laws, the steepest among which corresponds to the lowest value of N, namely $N = N^*$, and this is the solution that corresponds to $x = x^*$. Choosing the solution that remains positive in this case, we finally have:

$$f(\eta) = \eta^{\lambda^*}, \quad \lambda^* = -1 + \frac{N^*(x^*)}{2} = -x^* - \sqrt{(x^* - 1)^2 + 1}. \tag{23}$$

Here we have used

$$N^*(x^*) = -2\left[(x^* - 1) + \sqrt{(x^* - 1)^2 + 1}\right]. \tag{24}$$

Now we can check the consistency of our approach based on the Taylor expansion which requires convergence at infinity of the integral (15) defining N. The condition for this is $\lambda^* < -4$ which means $x^* > 7/5$. Since $x^* > 2$, this consistency condition is satisfied.

Equation (24) combined with (15) leads to the prediction of the relationship (4) between x^* and $f(\eta)$. This prediction will be put to test via numerical simulations in the next Section.

4. Numerical simulations

Self-similar solutions have been found by numerical simulation of the governing equation. Equation (6) can be transformed via the substitution $g(\eta) = f(\eta)\eta^x$ in order to simplify the boundary condition and cancel one of the terms on the left hand side. In terms of the new function $g(\eta)$, the equation to be solved is now

$$g' = \frac{1}{8b} \iint_{\Delta_\eta} W(\eta, \eta_1, \eta_2)\eta^{x-1}\eta_1^{-x}g(\eta_1)[\eta_2^{-x}g(\eta_2) - \eta^{-x}g(\eta)]d\eta_1\eta_2, \tag{25}$$

with the interaction coefficient $W(\eta, \eta_1, \eta_2)$ defined as in (3). The left boundary condition is then

$$g(\eta) \to 1 \quad \text{for} \quad \eta \to 0. \tag{26}$$

Equation 25 is solved by an iterative scheme given by

$$\frac{g_{n+1}(\eta + \delta) - g_{n+1}(\eta)}{\delta} = \tag{27}$$

$$\frac{1}{8b} \iint_{\Delta_\eta} W(\eta, \eta_1, \eta_2)\eta^{x-1}\eta_1^{-x}g_n(\eta_1)[\eta_2^{-x}g_n(\eta_2) - \eta^{-x}g_n(\eta)]d\eta_1\eta_2.$$

The two-dimensional integral is calculated as in [11] apart from the discretisation δ, which is chosen to be linear rather than logarithmic. This also differs from the methodology in [1] which retained the logarithmic discretisation. The logarithmic discretisation allows for a wider range of wavenumbers to be computed, but we suspect that this comes at the cost of not well resolving the propagating front of the system.

The initial function $g_0(\eta)$ is chosen to be an indicator function on the set $\{0, 1\}$ and the computation is performed over the values of η in the range $\{\eta_{\min}, \eta_{\max}\}$. Along with the discretisation δ, this makes three parameters which should be taken to their respective limits. The iteration procedure converges to a solution for a given $\eta_{\min}$ and $\eta_{\max}$ providing the ratio $\delta/\eta_{\min}$ is sufficiently small. The sensitivity of these solutions to the specific choice of $\eta_{\min}$ and $\eta_{\max}$ should then be checked.

It should be noted that the iteration procedure does not converge to a solution on the whole domain but instead on a sub-domain containing the first two thirds of the η's. We attribute this to the truncation of the area over which we integrate illustrated in Figure 1. As η gets closer to $\eta_{\max}$, the rectangle becomes wider but more of the region with $\eta_1, \eta_2 > \eta_{\max}$ gets lost.

Providing that $\eta_{\max}$ is large enough to resolve the decay of the function, increasing it further has no effect on the shape of the solution. We have chosen the moderate value $\eta_{\max} = 15$ in order to reduce computational cost. There is a greater sensitivity on the solution to the value of $\eta_{\min}$ due to the singularity of the integrand at zero η_1 and η_2. Experimentally, the decay of $g(\eta)$ is steeper for smaller values of $\eta_{\min}$. Obviously, the smaller $\eta_{\min}$ is made, the smaller δ must be chosen, and thus the larger the computational cost. This constraint is more severe for the convergence of the solution at small η's. However, as we approach larger values of η where the function is closer to zero, we do find good convergence. For identifying the value of x^* and for comparing with the theory developed in the previous section, this is the region of our interest. The remaining parameters used for what follows are $\eta_{\min} = 0.00625$ and $\delta = 0.0015625$.

4.1. Identifying x^*

The solution which corresponds to $x = x^*$ is one which stays positive for all η and the fastest decaying. From our simulations we find
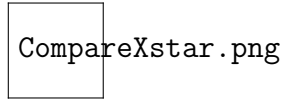
$$x^* = 3.80 \pm 0.01. \tag{28}$$

Figure 2. The solution for $x = x^*$ compared with the solution for $x = 3.33$, the previously reported value of x^*.
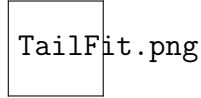

Figure 3. Power law fits compared to the solution for x^*. Two power laws are fitted as a guide, these seem to fit in two small regions. The theoretical prediction is in between these two guides.

This is plotted in Figure 4.1 along with the solution for $x = 3.33$, *i.e.* the exponent found by Galtier *et. al* in [1]. The solution for $x = 3.33$ can be seen to cross zero, thus violating one of the conditions for a valid solution.

The solution found for $x = x^* = 3.80$ can then be compared to the theory developed in Section 3. Substituting x^* into equation (24), we find

$$N^*(x^*) = -11.55. \tag{29}$$

This can be compared with the value of N found for the simulation with $x = 3.8$,

$$N_{x=3.8} = -11.23, \tag{30}$$

which was found using relations (15). This is accurate within under 3% and since N^* corresponds to a power law tail, this can be seen as some verification of this hypothesis.

From (22) we expect the tail to behave as

$$g(\eta) = C\eta^{-2.97}. \tag{31}$$

We have tried to fit this prediction with our numerical solution but only a small range for the tail is available and in this range the prediction does fits in a small region only. While this fit is not conclusive, it is at least consistent with the prediction. This power law fit is shown in Figure 4.1 along with two further power laws for an eye guide. An exponential and super-exponential fits were also attempted. They did not prove to be consistent with our numerics.

5. Conclusions

In this paper, we considered self-similar solutions of the integro-differential kinetic equation (1) describing the MHD wave turbulence. Such self-similar solutions are of the second kind, which, by definition, means that the self-similarity parameters can not be uniquely fixed by a dimensional analysis based on a conservation law. Namely, there remains a single parameter which depends on the shape of the self-similar solution globally. This parameter was to be found by solving a nonlinear eigenvalue problem, i.e. matching the solution to relevant boundary conditions at the two ends of the interval

of the self-similar variable: in our case $\eta \to 0$ and $\eta \to \infty$. The boundary conditions were to be chosen from the consideration that the respective solution is asymptotically approached as $t \to t^*$ (with some $t^* < \infty$) by the solution of the initial value problem with initial data in the finite range of wave numbers.

We postulated the following boundary conditions defining the nonlinear eigenvalue problem of the self-similar solutions. At $\eta \to 0$, the self-similar solution must tend to a power-law asymptotics, $f(\eta) \to \eta^{-x}$. The second boundary condition is that at $\eta \to \infty$ one must satisfy $f(\eta) \to 0$ where the decay to zero is the fastest among the solutions corresponding to different parameters x in the class of positive functions $f(\eta)$. The respective value $x = x^*$ and the respective function $f(\eta)$ comprise the solution of the nonlinear eigenvalue problem. Our conjecture (yet unproven) is that the postulated boundary value problem does yield a self-similar solution which is asymptotically approached as $t \to t^*$ (with some $t^* < \infty$) by the solution of the initial value problem with initial data in the finite range of wave numbers.

We also proved that the tail of $f(\eta)$ at $\eta \to \infty$ cannot be exponentially or super-exponentially decaying. Instead, the tail is shown to be a power law with the index λ^* related to x^* as in (23). This leads to the prediction of a relation (4) between x^* and the integral of the solution $f(\eta)$ which was confirmed by numerical simulations.

The value x^* depends on the global shape of $f(\eta)$ rather than its asymptotics at the boundaries $\eta \to 0$ and $\eta \to \infty$, and, perhaps, it cannot be found analytically. Our numerical solution of the stated nonlinear eigenvalue problem yields the value $x^* = 3.80 \pm 0.01$.

It is interested to compare our solution with self-similar solutions of the Smoluchowski kinetic equation for sticky particles describing creation of infinite-mass clusters in a finite time. This effect is called gelation. For example, in one of the integrable cases with the reaction rate equal to the product of the masses of the two colliding particles, m_1, m_2, there is a well-known analytical solution [12] which at $t \to t^*$ tends to the self-similar solution $n(m, t) = (t^* - t)^5 f(\eta)$ with $\eta = (t^* - t)^2 m$ and $f(\eta) = \eta^{-5/2} e^{-\eta/2}$; see Ref [13]. Like in our case, this is a self-similarity of the second kind. However, there are two striking differences with our MHD system: (i) the large η decay is exponential and (ii) the power law at small η coincides with the KZ scaling. One can relate the absence of the exponential decay in our system to the presence of the backscatter term which is absent in the Smoluchowski equation. Indeed, a single MHD wave can decay into two waves, whereas a single particle in the Smoluchowski system in not allowed to get split into two.

In future, it would be useful to continue analysing the large-η asymptotics of the self-similar solutions in other kinetic equations of Wave Turbulence as well as various Smoluchowski kinetic equations, and, in particular, establish the conditions of the exponential and the power-law behaviours.

6. Appendix: convergence of the integral

6.1. Region $\eta_1 \ll \eta, \eta_2$

Let us introduce new variables κ_1 and θ_1:

$$\eta_1 = \eta\,\kappa_1, \quad \eta_2 = \sqrt{\eta^2 + \eta_1^2 - 2\eta\eta_1 \cos\theta_1} = \eta\sqrt{1 + \kappa_1^2 - 2\kappa_1 \cos\theta_1} = \eta\,\kappa_2.$$

Then $\kappa_1 \in [0, \infty]$, $\theta_1 \in [0, \pi]$ and we have

$$St = \iint\limits_{\Delta_k} W(\eta, \eta_1, \eta_2)\, f(\eta_1)\, [f(\eta_2) - f(\eta)]\, d\eta_1 d\eta_2 =$$

$$\int\limits_0^\eta \int\limits_{\eta-\eta_1}^{\eta+\eta_1} W(\eta, \eta_1, \eta_2)\, f(\eta_1)\, [f(\eta_2) - f(\eta)]\, d\eta_1 d\eta_2 + \int\limits_\eta^\infty \int\limits_{\eta_1-\eta}^{\eta+\eta_1} W(\eta, \eta_1, \eta_2)\, f(\eta_1)\, [f(\eta_2) - f(\eta)]\, d\eta_1 d\eta_2$$

$$= \eta^4 \int\limits_0^1 \int\limits_0^\pi W(1, \kappa_1, \kappa_2)\, f(\eta\kappa_1) \left[f(\eta\sqrt{1 + \kappa_1^2 - 2\kappa_1 \cos\theta_1}) - f(\eta) \right] \frac{\partial\kappa_2}{\partial\theta_1} d\theta_1\, d\kappa_1 +$$

$$+ \eta^4 \int\limits_1^\infty \int\limits_0^\pi W(1, \kappa_1, \kappa_2)\, f(\eta\kappa_1) \left[f(\eta\sqrt{1 + \kappa_1^2 - 2\kappa_1 \cos\theta_1}) - f(\eta) \right] \frac{\partial\kappa_2}{\partial\theta_1} d\theta_1\, d\kappa_1 =$$

$$= \eta^4 \int\limits_0^\infty \int\limits_0^\pi V_{1\kappa_1\theta_1}\, f(\eta\kappa_1) \left[f(\eta\sqrt{1 + \kappa_1^2 - 2\kappa_1 \cos\theta_1}) - f(\eta) \right] d\theta_1\, d\kappa_1, \tag{32}$$

where

$$W(1, \kappa_1, \kappa_2) = \kappa_2 \cos^2\theta_2 \sin\theta_1 = \kappa_2 \frac{(1 - \kappa_1 \cos\theta_1)^2}{\kappa_2^2} \sin\theta_1, \quad \frac{\partial\kappa_2}{\partial\theta_1} = \frac{\kappa_1 \sin\theta_1}{\kappa_2},$$

and

$$V_{1\kappa_1\theta_1} = W(1, \kappa_1, \kappa_2) \frac{\partial\kappa_2}{\partial\theta_1} = \frac{\kappa_1 \sin^2\theta_1 (1 - \kappa_1 \cos\theta_1)^2}{1 + \kappa_1^2 - 2\kappa_1 \cos\theta_1}. \tag{33}$$

Let us consider a contribution to the integral St that comes from the region $\kappa_1 \ll 1$. We will call it St_1. We get the expansions

$$V_{1\kappa_1\theta_1} = \kappa_1 \sin^2\theta_1 + O(\kappa_1^3),$$

$$f\left(\eta\sqrt{1 + \kappa_1^2 - 2\kappa_1 \cos\theta_1}\right) - f(\eta) =$$

$$-\kappa_1\eta \frac{\partial f(\eta)}{\partial\eta} \cos\theta_1 + \frac{1}{2}\kappa_1^2 \left(\eta\frac{\partial f(\eta)}{\partial\eta} \sin^2\theta_1 + \eta^2 \frac{\partial^2 f(\eta)}{\partial\eta^2} \cos^2\theta_1 \right) + O(\kappa_1^3). \tag{34}$$

Note that for validity of the expansion (34) for values $\eta_1 \sim 1 \ll \eta$, function $f(\eta)$ must be supra-exponential, i.e. satisfying $|f'(\eta)| \ll |f(\eta)|$ (see the explanation in the main text).

The term with $\cos \theta_1$ in (34) is annihilated by integration over θ_1 as result of a symmetry with respect to the centre of $[0, \pi]$. After integrating over θ_1 we get

$$St_1 = \eta^4 \int_0^\infty \int_0^\pi V_{1\kappa_1\theta_1} \, f(\eta\kappa_1) \left[f\left(\eta\sqrt{1 + \kappa_1^2 - 2\kappa_1 \cos\theta_1}\right) - f(\eta) \right] d\theta_1 \, d\kappa_1$$

$$\approx \frac{\pi}{16} \left[3\eta \frac{\partial f(\eta)}{\partial \eta} + \eta^2 \frac{\partial^2 f(\eta)}{\partial \eta^2} \right] \int_0^\infty f(\eta\kappa_1) \eta^4 \kappa_1^3 \, d\kappa_1. \tag{35}$$

For solutions with asymptotics $f \to \eta^{-x}$ for $\eta \to 0$, we have the convergence condition

$$x < -4. \tag{36}$$

6.2. Region $\eta_2 \ll \eta, \eta_1$

Here, it is convenient to use the variables κ_2 and θ_2:

$$\eta_2 = \eta \, \kappa_2, \quad \eta_1 = \sqrt{\eta^2 + \eta_2^2 - 2\eta\eta_2 \cos\theta_2} = \eta\sqrt{1 + \kappa_2^2 - 2\kappa_2 \cos\theta_2} = \eta \, \kappa_1$$

with $\kappa_2 \in [0, \infty]$ and $\theta_2 \in [0, \pi]$. So

$$St = \iint_{\Delta_k} W(\eta, \eta_1, \eta_2) \, f(\eta_1) \left[f(\eta_2) - f(\eta) \right] d\eta_1 d\eta_2 =$$

$$\int_0^\eta \int_{\eta-\eta_1}^{\eta+\eta_1} W(\eta, \eta_1, \eta_2) \, f(\eta_1) \left[f(\eta_2) - f(\eta) \right] d\eta_1 d\eta_2 + \int_\eta^\infty \int_{\eta_1-\eta}^{\eta+\eta_1} W(\eta, \eta_1, \eta_2) \, f(\eta_1) \left[f(\eta_2) - f(\eta) \right] d\eta_1 d\eta_2$$

$$= \eta^4 \int_0^\infty \int_0^\pi V_{1\theta_2\kappa_2} \, f\left(k\sqrt{1 + \kappa_2^2 - 2\kappa_2 \cos\theta_2} \right) \left[f(\eta\kappa_2) - f(\eta) \right] d\theta_2 \, d\kappa_2, \tag{37}$$

where

$$\frac{\partial \kappa_1}{\partial \theta_2} = \frac{\kappa_2 \sin\theta_2}{\kappa_1},$$

and

$$V_{1\theta_2\kappa_2} = \frac{\kappa_2^2 \cos^2\theta_2 \sin\theta_2}{\sqrt{1 + \kappa_2^2 - 2\kappa_2 \cos\theta_2}} \sin\theta_1 = \frac{\kappa_2^3 \cos^2\theta_2 \sin^2\theta_2}{1 + \kappa_2^2 - 2\kappa_2 \cos\theta_2}.$$

Let us consider a contribution to the integral St that comes from the region $\kappa_2 \ll 1$. We will call it St_2. We get again the expansions:

$$V_{1\theta_2\kappa_2} = \kappa_2^3 \cos^2\theta_2 \sin^2\theta_2 + O(\kappa_2^4),$$

$$f\left(\eta\sqrt{1 + \kappa_2^2 - 2\kappa_2 \cos\theta_2} \right) = f(\eta) - \kappa_2\eta \frac{\partial f(\eta)}{\partial \eta} \cos\theta_2 + O(\kappa_2^2),$$

where, again, $f(\eta)$ is assumed to be supra-exponential. After integrating over θ_2 we have

$$St_2 = \eta^4 \int_0^\infty \int_0^\pi V_{1\theta_2\kappa_2} f\left(\eta\sqrt{1+\kappa_2^2 - 2\kappa_2\cos\theta_2}\right)[f(\eta\kappa_2) - f(\eta)]\, d\theta_2\, d\kappa_2$$

$$\approx \frac{\pi}{8} f(\eta) \int_0^\infty f(\eta\kappa_2)\, \eta^4 \kappa_2^3\, d\kappa_2. \tag{38}$$

The integral is the same as for the region $\eta_1 \ll \eta, \eta_2$, and therefore it is convergent again when $x < 4$.

6.3. Region $\eta_1, \eta_2 \gg \eta$.

For completeness, let us consider convergence of the integral at the upper end of the integration domain, $\eta_1, \eta_2 \gg \eta$. For the low-η asymptotics this would mean $\eta_1, \eta_2 \sim 1 \gg \eta$, whereas for the high-$\eta$ tail we have $\eta_1, \eta_2 \gg \eta \gg 1$. Then we assume in equations (32) and (33) that $\kappa_1 \gg 1$ and Taylor expands in $\kappa_1^{-1} \le \varepsilon \ll 1$. This gives for contributions with $\kappa_1 \ge \varepsilon^{-1}$:

$$St_3 \approx \eta^4 \int_{\varepsilon^{-1}}^\infty \int_0^\pi V_{1\kappa_1\theta_1} f(\eta\kappa_1)[f(\eta\kappa_1) - f(\eta)]\, d\theta_1\, d\kappa_1, \tag{39}$$

$$V_{1\kappa_1\theta_1} \approx \kappa_1 \sin^2\theta_1 \cos^2\theta_1 + O(1), \tag{40}$$

$$\sqrt{1 + \kappa_1^2 - 2\kappa_1\cos\theta_1} \approx \kappa_1 - \cos\theta_1 + O\left(\kappa_1^{-1}\right). \tag{41}$$

Integration over θ_1 gives $\pi/8$. For decaying $f(\eta)$ we have

$$St_3 \approx \frac{\pi}{8}\eta^4 \int_{\varepsilon^{-1}}^\infty \kappa_1 f(\eta\kappa_1)[f(\eta\kappa_1) - f(\eta)]\, d\kappa_1 \approx -\eta^2 f(\eta)\frac{\pi}{8} \int_{\eta\varepsilon^{-1}}^\infty \eta_1 f(\eta_1)\, d\eta_1.$$

For convergence of this integral $f(\eta)$ must decay faster than $1/\eta^2$.

This condition is satisfied at $\eta \ll 1$ end if $x > 2$ and at the tail $\eta \gg 1$ if $\lambda < -2$. Both inequalities are true for the solution of the nonlinear eigenvalue problem since $-\lambda^* > x^* \approx 3.8$.

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