Pairs of Closed Geodesics in Metric Graphs

by

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Contents

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Declarations

I hereby declare that the work contained in this thesis is the original work of the author, except where otherwise indicated, cited, or commonly known. The work was undertaken at the department of Mathematics, University of Warwick between October 2013 and July 2017 and has not been submitted, in whole or in part, for any other degree.

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Abstract

In this thesis we are interested in the problem of counting pairs of closed geodesics in metric graphs. We start with counting pairs of closed geodesics ordered by their word length on the metric graph and such that the difference of their geometric lengths is in a prescribed interval. Then we study a similar problem but where the interval is now allowed to shrink at a specific rate as the word length tends to infinity.

Next we study a variant on this problem where we fix a set of generators for the fundamental group of the graph and then order the closed geodesics by the word length of the corresponding conjugacy class with respect to these generators. Again we may also allow the interval to shrink at an appropriate rate.

We also study a restricted version of our first problem where we only count null homologous closed geodesics.

The final counting problem we study differs from the previous ones by counting pairs of geodesic paths instead of pairs of closed geodesics. These geodesic paths are also ordered by their word lengths in the metric graph and again the difference between the geometric lengths of these geodesic paths lies in a fixed interval.

The techniques we use in our study include coding metric graphs by subshifts of finite type and the concepts from the thermodynamic formalism that appear in the ergodic theory of these systems. In particular, we use the spectral properties of transfer operators and their relationship to pressure and entropy.

Chapter 1

Introduction

The problem of counting closed geodesics in negatively curved surfaces has been studied for different constraints on the closed geodesics. A breakthrough result was proved by Huber in 1961 [7] for compact surfaces with constant negative curvature. He gave an asymptotic formula for the number of closed geodesics, with respect to a bound on their geometric length. Denoting the geometric length of any closed geodesic γ by $\ell(\gamma)$, the result states that there exists $h > 0$ such that

$$\#\{\gamma : \ell(\gamma) \leq T\} \sim \frac{e^{hT}}{hT}, \text{ as } T \to +\infty \tag{1.0.1}$$

where $A(T) \sim B(T)$ means that $\lim_{T\to+\infty} A(T)/B(T) = 1$. This result was then generalised by Margulis in 1969 to compact manifolds of variable negative curvature [12].

In the early 80's, more general results were proved by Parry and Pollicott for Axiom A flows. Parry and Pollicott [15], [16] used new methods which involved dynamical zeta functions, subshifts of finite type and transfer operators introduced by Ruelle [23].

An analogue of the asymptotic in (1.0.1) when closed geodesics are counted with a bound on their word length is given by

$$\#\{\gamma : |\gamma| \leq N\} \sim \frac{e^{hT}}{e^h - 1}\frac{e^{hN}}{N}, \text{ as } N \to +\infty \tag{1.0.2}$$

where $|\gamma|$ denotes the word length of γ and is defined to be the shortest word length of elements in the conjugacy class (in the fundamental group of compact surface of negative curvature).

The two asymptotics (1.0.1) and (1.0.2) can be applied to another type of space, namely metric graphs, which are finite connected graphs with positive length on their edges. A closed geodesic γ in a metric graph is defined to be a closed path (with its cyclic permutation) with no backtracking. The geometric length of γ is the sum of the lengths of the edges that the closed path contains and the word length of γ is the number of edges in this closed path.

Another natural problem to study is the distribution of closed geodesics when their geometric lengths are compared, i.e., studying the distribution of the elements in a set

$$\{\ell(\gamma) - \ell(\gamma') : \gamma, \gamma' \text{ closed geodesics}\}.$$

However, this set is infinite in $\mathbb{R}$, a finite subset of this set that one can study can be formed by adding an upper bound on the word length of the closed geodesics. So, we can study the distribution of the elements in the set

$$\{\ell(\gamma) - \ell(\gamma') : \gamma, \gamma' \text{ closed geodesics}, |\gamma|, |\gamma'| \leq N\}.$$

For compact surfaces of negative curvature, Pollicott and Sharp [21], [22] studied pairs of closed geodesics, the difference of whose length lie in a prescribed interval $[a, b] \subset \mathbb{R}$ and ordered by word length. This is described by the set

$$\{(\gamma, \gamma') : \gamma, \gamma' \text{ closed geodesics}, |\gamma|, |\gamma'| \leq N, \ell(\gamma) - \ell(\gamma') \in [a, b]\}. \quad (1.0.3)$$

Moreover, in the same papers the authors proved an even stronger result in that the difference in the geometric lengths could be more localised by allowing the interval $[a, b]$ to shrink at some rate, say $\epsilon_N > 0$. Additionally, this result allows the interval $[a, b]$ to be positioned arbitrarily in the real line and this gives a uniform result not only an asymptotic. So for $z \in \mathbb{R}$, a uniform result has been obtained for the number of elements in the following set:

$$\{(\gamma, \gamma') : \gamma, \gamma' \text{ closed geodesics}, |\gamma|, |\gamma'| \leq N, \ell(\gamma) - \ell(\gamma') \in [a\epsilon_N + z, b\epsilon_N + z]\},$$

where $a < b$ are fixed.

In this thesis, we are going to study similar problems in the setting of metric graphs. For metric graphs, under certain conditions, we obtain asymptotic estimates similar to those for surfaces of negative curvature. The approach we use to prove these results begin with coding metric graphs by subshifts of finite type and the calculations involve using a transfer operator in order to analyse a pressure function which describes the spectrum of this operator. The counting process involves the Fourier transform of a counting function for pairs of closed geodesics. We are able to mimic the same techniques and calculations used to obtain asymptotics for pairs of closed geodesics in surfaces of negative curvature due to the fact that the coding of metric graphs we use results in a subshift of finite type, just as in the analysis of pairs of closed geodesics in surfaces of negative curvature. The subshift of finite type we use codes the metric graph by its oriented edges and the lengths on the edges are encoded by a function defined on the shift space.

We shall point out a difference in the the problem of counting with shrinking intervals, related to the rate that ϵ_N converges to zero. The results of [21], for surfaces of negative curvature (or, more generally, manifolds with 1/4-pinched negative curvature [22]), are valid when ϵ_N converges to zero subexponentially, i.e. $\limsup_{N\to+\infty} |\log \epsilon_N|/N = 0$. On the other hand, for metric graphs, our results are valid only when ϵ_N converges to zero at polynomial rate, i.e. slower than N^{-r}, for some $r > 0$. This difference occurs in our use of a result derived from the work of Dolgopoyat on the rate of mixing of hyperbolic flows [3] to give an estimate on the iterates of transfer operators. This gives an effective bound on their iterates, depending on the imaginary part of the parameter in the potential, but this bound is weaker than that held for the transfer operators that are associated to geodesic flows over negatively curved surfaces [2]. In this latter case, the transfer operators satisfy an exponential bound provided the number of iterates is sufficiently large compared to the logarithm of the imaginary part of the parameter in the potential. It is this stronger bound that allowed Pollicott and Sharp to obtain results with subexponentially shrinking intervals.

Further counting problems we shall study in this thesis include counting pairs of closed geodesics ordered by the word length of the corresponding conjugacy class in the fundamental group of the metric graph. The difference of their geometric lengths can lie in a fixed interval in $\mathbb{R}$ or in a family of shrinking intervals. We also find an asymptotic for pairs of closed geodesics in (1.0.3) with their homology classes fixed. In addition, we derive an asymptotic for the number of pairs of geodesic paths starting and ending at the same vertex in the metric graph. These geodesic paths are ordered by their word lengths and the difference of their geometric lengths lie in a fixed interval $[a, b] \subset \mathbb{R}$.

In all the counting problems we study in this thesis we require a condition on the geometric length of the closed geodesics we count. The condition is stated as follows

$$\{\ell(\gamma) - c|\gamma| : \gamma \text{ closed geodesic}\} \not\subset d\mathbb{Z},$$

for any $c, d \in \mathbb{R}$. This condition is translated naturally to a function defined on the shift space used to encode the geometric length of the closed geodesics. Furthermore, these functions are locally constant functions, i.e. depend on a finite number of coordinates. These functions are Hölder continuous functions with any positive exponent. For the analysis, we use a transfer operator defined on a complex Banach space of this class of functions. The spectrum of these operators has a nice structure described by a theorem due to David Ruelle that is similar to the very well-known Perron-Frobenius theorem. Using this, we obtain the dominating factor in the estimation process we perform to obtain our asymptotic.

Similar pair correlation results were studied in other areas of mathematics. For example, in number theory, in 1972 Hugh Montgomery introduced a new method for studying the zeros of the Riemann zeta function [14]. He investigated the distribution of the difference between two non-trivial zeros. For this he studied the Fourier transform of the distribution function of the difference of pairs of non-trivial zeros. Also, a correlation result for pairs of sequences, with their values given by quadratic forms, was proved by Jens Marklof [13]. He studied a pair correlation function that represents the counting of pairs of these sequences, where the difference between their values lies in an interval $[a, b] \subset \mathbb{R}$ and ordered by an upper bound on their values. His techniques also involved Fourier transform of the pair correlation function.

Now we discuss the structure of the thesis and give an outline of its content. In chapter 2, we give some preliminary material on the main ingredients in this thesis. We start with the main object of our study: metric graphs. Then, we review subshifts of finite type and define transfer operators on the Banach space of complex Hölder continuous functions defined on the shift space. After that, we discuss some concepts from thermodynamic formalism, including pressure and entropy. We then introduce the key tool in our analysis, the complex Ruelle-Perron-Frobenius theorem [19]. Using this theorem and with aid of perturbation theory we analyse the spectrum of the transfer operator which will be used in the estimation that lead to the asymptotics in the subsequent chapters.

In chapter 3, we study two problems for non-bipartite and bipartite metric graphs. The first problem is counting pairs of closed geodesic, ordered by word length, where the difference between their geometric lengths lie in an interval $[a, b] \subset \mathbb{R}$ (definitions of word length and geometric length will be introduced in section 2.1). In the second problem, we allow the interval $[a, b]$ to shrink at a specific rate and position the interval arbitrarily in the real line. As we mentioned earlier, the counting is done by coding the metric graphs by subshifts of finite type, where we have used the oriented edges of the graph for this coding. This coding results in a mixing subshift of finite type. However, in the case of bipartite graphs this coding does not give a mixing shift space. So we have to adjust the coding of bipartite graphs slightly to obtain a new subshift of finite type that is mixing. In this chapter, we accomplish all the estimations needed to obtain asymptotics in the case of fixed intervals and also a uniform result in the case of shrinking intervals. We will apply versions of these calculations to obtain the results in other chapters.

In chapter 4, compared to the counting problems we studied in chapter 3, we change the quantity we use to order the closed geodesics to word length of the corresponding conjugacy class in the fundamental group of the metric graph (with respect to a given set of generators). However, we are still looking at the difference in their geometric lengths to lie in a fixed interval in $\mathbb{R}$ and also the case when the interval is shrinking. In this chapter, we use the fact that the fundamental group of the metric graph, with the condition that the vertices have valency at least 3, is a free group with $k \geq 2$ generators. To obtain the results of this chapter we introduced a subshift of finite that coded the fundamental group of the metric graph, where the elements of the shift space are the infinite reduced words in the generators and their inverses. Combining some algebraic topological facts about the equivalence classes in the fundamental group of the metric graph, we reach a correspondence similar to that used in chapter 3. So we are able to apply the proofs of the results in chapter 3 to the results of this chapter.

Chapter 5 discuss a counting result similar to that in chapter 3, but with one more constraint on the closed geodesics. So we consider a subset of the set described in (1.0.3) by fixing the homology classes that the closed geodesics belong to. We denote the homology class of a closed geodesic γ by $[\gamma]$. The homology class $[\gamma]$ can be identified with an element in $\mathbb{Z}^k$, where k is the number of generators in the fundamental group of the metric graph. This follows from the fact that the first homology group of a metric graph is the abelianization of its fundamental group and this is isomorphic to $\mathbb{Z}^k$. The counting in this chapter is also done by using the same subshift of finite type we used in chapter 3, but with one more encoding function (defined on the shift space) that represents the homology class of a closed geodesic γ. This enables us to use results and techniques from the work of Pollicott and Sharp on $\mathbb{Z}^q$

and $\mathbb{R}^q$ extensions of subshifts of finite type, $q \geq 1$ [20].

Finally, in chapter 6 we derive an asymptotic formula for the number of pairs of geodesic paths in metric graphs that start and end at prescribed vertices on the graph such that the difference between their geometric lengths lies in a fixed interval $[a, b] \subset \mathbb{R}$ and ordered by word length. The geometric and word length of geodesic paths are defined in same way as in closed geodesics. Hence, we are looking at geodesic paths satisfying the same constraints as the closed geodesics in (1.0.3). For this we also used the subshift of finite type we introduced in chapter 3, which codes the metric graph by its oriented edges. As we are counting pairs of geodesic paths in this chapter not pairs of closed geodesics, we encode the geodesic paths by corresponding elements in the shift space in a slightly different way than in the case of closed geodesic. Nonetheless, we end up with a correspondence that allows us to follow the same techniques and analysis we have used to obtain the asymptotic for pairs of closed geodesics described in (1.0.3).

Chapter 2

Preliminaries

2.1 Metric graphs

In this chapter we introduce the main object of this research: metric graphs. We shall also define the different type of graphs that can be considered.

We define an *arc* A to be a space homeomorphic to the interval $[0, 1]$; the end points of A are the points p and q such that $A - \{p\}$ and $A - \{q\}$ are connected. A *finite linear graph* is a Hausdorff space that is written as the union of finitely many arcs, each pair of which intersect at common end points. The arcs in the collection are called the edges of the graph, and the end points of the arcs are called the vertices of the graph.

Definition 2.1.1. *A graph G is a finite linear graph identified as an ordered pair $G = (V, E)$, where V is the set of vertices and E a set of edges. Each pair of vertices $u, v \in V$ may be joined by one or more edges in E. We always assume graphs are connected, i.e. any two vertices are joined by a path along the edges.*

The graph we defined above is not a directed graph, i.e. there are no arrows on the edges and one may travel in either direction. So for each edge there are two possible orientations (directions). An edge with choice of orientation is called an oriented edge. We denote the set of oriented edges by E^o. The number of oriented edges is twice the number of edges, i.e. $|E^o| = 2|E|$.

If $e \in E^o$ goes from vertex v to vertex v' then $\overline{e}$ goes from v' to v. We say that v is the origin of the edge e, and v' is the terminus of e. In notation we write this as $\mathbf{o}(e) = v$ and $\mathbf{t}(e) = v'$. If e is a *loop* starting and ending at the same vertex then $\mathbf{o}(e) = \mathbf{t}(e)$, but there are still two orientations.

For a vertex v the degree of v is the number of edges originating in v, which we denote by $\deg(v)$. If an edge joins v to v, then it contributes 2 to $\deg(v)$.

Definition 2.1.2. *A geodesic path in a graph is a sequence of edges* $\boldsymbol{e} = e_0, e_1, ..., e_{n-1}$ *such that*

$$\boldsymbol{t}(e_i) = \boldsymbol{o}(e_{i+1}), \quad i = 0, ..., n-2.$$

and

$$e_{i+1} \neq \bar{e}_i, \quad i = 0, ..., n-2.$$

The last condition in the definition above means we do not allow back-tracking.

A geodesic path $\mathbf{e} = e_0, e_1, ..., e_{n-1}$ is closed if $\mathbf{t}(e_{n-1}) = \mathbf{o}(e_0)$. A closed geodesic path is said to be *prime* if it does not follow the same route more than once. For example if e_1, e_2 is a closed geodesic path then e_1, e_2, e_1, e_2 is a closed geodesic path but it is not prime.

Definition 2.1.3. *A closed geodesic is the set of cyclic permutations of a closed geodesic path, i.e. if* $\boldsymbol{e} = e_0, e_1, ..., e_{n-1}$ *is a closed geodesic path then*

$$\gamma = \{(e_0, e_1, ..., e_{n-1}), (e_1, e_2, ..., e_{n-1}, e_0), ..., (e_{n-1}, e_0, ..., e_{n-2})\}$$

is the corresponding closed geodesic.

A closed geodesic is *prime* if the paths it contains are prime.

Definition 2.1.4. *Let* $|\gamma|$ *be the word length of a closed geodesic* γ*, such that for* γ *equal to the permutations of* $(e_0, e_1, ..., e_{n-1})$*,* $|\gamma| = n$.

Definition 2.1.5. *A graph* $G = (V, E)$ *is said to be bipartite if we can split* V

into two disjoint sets V_1 and V_2, such that if vertices v and v' are joined by an edge then one of $\{v, v'\}$ is in V_1 and the other one is in V_2.

If a graph $G = (V, E)$ does not satisfy this definition,, then we call it a *non-bipartite* graph.

Definition 2.1.6. *A metric graph is a graph together with a positive length assigned to each edge.*

On a graph $G = (V, E)$ define a function $\ell : E \to \mathbb{R}^+$ that associates to each edge in G the length of the edge. A closed geodesic γ, consisting of the closed geodesic path $\mathbf{e} = e_0, e_1, ..., e_{n-1}$, has a *geometric length*

$$\ell(\gamma) = \ell(e_0) + \ell(e_1) + ... + \ell(e_{n-1}).$$

We shall denote metric graphs by the pair (G, ℓ).

2.2 Symbolic dynamics

2.2.1 Subshifts of finite type

In this section we review a class of symbolic dynamical systems called subshifts of finite type, which are used to code many dynamical systems. The simplest shift space is called the full shift, where the space contains all possible infinite sequences of elements taken from a finite set $\mathcal{A} = \{1, \ldots, k\}$ called the alphabet or state set. We define a shift transformation which moves each sequence one step to the left and delete the first element.

We are interested in using a subset of the full shift, defined by imposing a condition on which elements of $\mathcal{A}$ are allowed to follow each other. To identify which sequences are allowed and which are not we define a $0-1$ square matrix A, with its rows and columns indexed by $\mathcal{A} = \{1, \ldots, k\}$, such that for $i, j \in \mathcal{A}$, if j can follow i then $A(i, j) = 1$ if not then $A(i, j) = 0$. Using the matrix A this space of sequences can be represented in the following way

$$X_A^+ = \{x = (x_i)_{i=0}^{\infty} \in \mathcal{A}^{\mathbb{Z}^+} : A(x_i, x_{i+1}) = 1, \, \forall i \in \mathbb{Z}^+\}.$$

We denote the associated shift transformation by σ, where $\sigma(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$, i.e. the sequence is shifted one place to the left and the first element is deleted. The space (X_A^+, σ) is called a subshift of finite type or topological Markov chain. We define a metric on $\mathcal{A}$ by the formula $d(i,j) = 1 - \delta_{ij}$, where δ_{ij} is 0 if $i \neq j$ and 1 if $i = j$. In the topology defined by this metric $\mathcal{A}$ is compact and has the discrete topology. Naturally we give $\mathcal{A}^{\mathbb{Z}^+}$ the product topology and X_A^+ the subspace topology. A compatible metric for X_A^+ is the product metric given by

$$d((x_i)_{i=0}^{\infty}, (y_i)_{i=0}^{\infty}) = \sum_{k=0}^{\infty} \theta^k (1 - \delta_{x_i, y_i}),$$

for $0 < \theta < 1$. An equivalent and simpler metric, which is commonly used, is $d(x, y) = \theta^k$, where $k = \max\{m : x_n = y_n \text{ for } n < m\}$, i.e. k is the first place in which the sequences x, y disagree. These metrics make X_A^+ a compact space and σ a continuous map. We define the cylinder set in the shift space by fixing a finite set of co-ordinates. More precisely, in X_A^+ we define

$$[y_0, y_1, \dots, y_{n-1}]_{0,n} = \{x \in X_A : x_j = y_j,\, 0 \leq j \leq n-1\}.$$

These cylinder sets are both open and closed and they are non-empty if and only if $A(y_j, y_{j+1}) = 1$ for $j = 0, \dots, n-1$. Also, they form a countable basis for the topology on X_A^+ and so every open set in X_A^+ is a countable union of cylinder sets. The collection of all cylinders forms an algebra which generates the Borel σ-algebra. This fact will be useful to define a measure on subsets of the shift space X_A^+.

The matrix A is called *aperiodic* if there exists $n \geq 1$ such that for every pair of indices i and j, $A^n(i,j) > 0$. We call A *irreducible* if for each pair (i,j), there is an $n > 0$ such that $A^n(i,j) > 0$. The matrix A can also be categorised by its *period d*. We define the period of an index i to be $p(i) = \gcd\{n : A^n(i,i) > 0\}$. The period of the matrix A is defined by $d = \gcd\{p(i) : i \text{ index of } A\}$. The matrix A is aperiodic if it has period $d = 1$, when A is irreducible its period can be $d > 1$.

Another way to represent the shift space is by associating with the matrix A a directed graph Γ_A (which has arrows on the edges). The vertex set of Γ_A is $\mathcal{A} = \{1, 2, \ldots, k\}$. We have an edge from i to j if and only if $A(i, j) = 1$. In this way we can think of the sequences in X_A^+ as an infinite paths on Γ_A.

The properties of A discussed above can be interpreted in terms of the directed graph Γ_A. For example, the matrix A is irreducible if and only if, for any two vertices i, j in Γ_A, there exists a path along directed edges from i to j. It is aperiodic if and only if, there exists $n \geq 1$ such that, for any two vertices i and j in Γ_A, there exists a path of length n from i to j.

Dynamical systems is the study of the long-term behaviour of evolving systems. In the shift space this can be studied by the shift map iterated on given infinite sequences. The orbit of $x \in X_A^+$ is the set of points $\{x, \sigma x, \sigma^2 x, \ldots\}$. A point $x \in X_A$ is called a *periodic point* if there exists $n \in \mathbb{N}$ such that $\sigma^n x = x$ and n is called the period of x. The orbit of a periodic point x is the set $\{x, \sigma x, \sigma^2 x, \ldots \sigma^{n-1} x\}$ and is called a *periodic orbit*. If n is the smallest value such that $\sigma^n x = x$, we call the orbit of x a *prime periodic orbit*.

2.2.2 Product shift spaces

We are interested in studying pairs of closed geodesics in metric graphs. A useful symbolic dynamics model for this study will be a product subshift of finite type.

We shall use the subshift of finite type X_A^+ we defined earlier, to define a product shift space $\widetilde{X} = X_A^+ \times X_A^+$. To see that $\widetilde{X}$ is also a subshift of finite type, let $\widetilde{X}$ consists of pairs $(x, y) = ((x_i)_{i=0}^{\infty}, (y_i)_{i=0}^{\infty})$, with $x_i, y_i \in \mathcal{A}$, $i \in \mathbb{Z}^+$. If we identify a pair (x, y) of sequence with the sequence $((x_0, y_0), (x_1, y_1), \ldots)$ of pairs, we can regard $X_A^+ \times X_A^+$ as a subset of of $(\mathcal{A} \times \mathcal{A})^{\mathbb{Z}^+}$. Now to show how a pair can follow the other we use the matrix A used to define the subshift of finite type X_A^+ in the following way: a pair (x_i, y_i) can be followed by (x_j, y_j) if and only if both $A(x_i, x_j) = 1$ and $A(y_i, y_j) = 1$. So we can define a 0-1 square matrix $\widetilde{A}$ where the rows and columns are indexed by $\mathcal{A} \times \mathcal{A}$ such that for $(i, j), (i', j') \in \mathcal{A} \times \mathcal{A}$, (i', j') can follow (i, j) if and only if $\widetilde{A}((i, j), (i', j')) = 1$.

Note that $\widetilde{A}((i,j),(i',j')) = 1$ if and only if $A(i,i') = 1$ and $A(j,j') = 1$. From this $\widetilde{X}$ can be written as

$$\widetilde{X} = \{(x_i, y_i)_{i=0}^{\infty} \in (\mathcal{A} \times \mathcal{A})^{\mathbb{Z}^+} : \widetilde{A}((x_i, y_i), (x_{i+1}, y_{i+1})) = 1, \forall i \in \mathbb{Z}^+\}.$$

We see that this set defines a subshift of finite type as the matrix $\widetilde{A}$ determines which pairs are allowed to follow each other.

Similarly we can use the shift transformation $\sigma : X_A^+ \to X_A^+$ to define a product shift transformation $\widetilde{\sigma} : \widetilde{X} \to \widetilde{X}$. This is defined by $\widetilde{\sigma}(x,y) = (\sigma x, \sigma y)$ and we write $\widetilde{\sigma} = \sigma \times \sigma$. We give $\widetilde{X}$ the product topology, then $\widetilde{X}$ is compact and $\widetilde{\sigma}$ is continuous.

2.2.3 Hölder continuous functions and transfer operators

We are going to work with certain spaces of functions defined on subshifts of finite type X_A^+. Let $C(X_A^+, \mathbb{R})$ be the space of all continuous real valued functions and $C(X_A^+, \mathbb{C})$ be the corresponding complex space. We say that a function $f : X_A \to \mathbb{C}$ is Hölder continuous with exponent α, if there exist constants C and $\alpha > 0$ such that for all $x, y \in X_A$, $|f(x) - f(y)| \leq Cd(x,y)^{\alpha}$, α is called the Hölder exponent of f.

Given $\alpha > 0$, we let $C^{\alpha}(X_A^+, \mathbb{C})$ be the complex Banach space of α-Hölder continuous functions $f : X_A^+ \to \mathbb{C}$ with norm $\|f\| = |f|_{\alpha} + \|f\|_{\infty}$, where

$$|f|_{\alpha} = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)^{\alpha}} : x, y \in X_A^+, x \neq y\right\}$$

and $\|f\|_{\infty}$ is the supremum norm.

An equivalence relation on $C^{\alpha}(X_A^+, \mathbb{C})$ that we shall use in this thesis is cohomology. Two functions $f, g \in C^{\alpha}(X_A^+, \mathbb{C})$ are said to be *cohomologous* if there exists a continuous function $h \in C(X_A^+, \mathbb{C})$ such that $f = g + h \circ \sigma - h$.

Furthermore, we need to define a transfer operator which is defined on the Banach space of Hölder continuous function $C^{\alpha}(X_A^+, \mathbb{C})$ and induced by the shift transformation σ. Given a function $w \in C^{\alpha}(X_A^+, \mathbb{R})$, we let L_f :

$C^\alpha(X_A^+, \mathbb{C}) \to C^\alpha(X_A^+, \mathbb{C})$ be the transfer operator defined by

$$L_f w(x) = \sum_{\sigma y = x} e^{f(y)} w(y). \tag{2.2.1}$$

One can easily check that this operator is bounded and linear. The iterates of L_f have the form

$$L_f^n w(x) = \sum_{\sigma^n y = x} e^{f^n(y)} w(y),$$

where $f^n(y) = f(y) + f(\sigma y) + \cdots + f(\sigma^{n-1} y)$.

Similarly we can define the set of Hölder continuous functions of exponent α on the product shift space $\widetilde{X}$, denoted by $C^\alpha(\widetilde{X}, \mathbb{C})$. Let $F \in C^\alpha(\widetilde{X}, \mathbb{C})$, we define a transfer operator acting on $C^\alpha(\widetilde{X}, \mathbb{C})$ by

$$L_F w(x, y) = \sum_{\widetilde{\sigma}(x', y') = (x, y)} e^{F(x', y')} w(x', y').$$

The spectral properties of the transfer operator studied in relation to some thermodynamic formalism concepts such as pressure and entropy, will be one of the main tools in our work. For this we shall discuss the spectral radius of the transfer operator and the spectrum of the operator in later sections. We recall here their definitions and introduce the notation we shall use. The spectrum of L_f is defined by

$$\mathrm{spec}(L_f) = \{\lambda \in \mathbb{C} : (\lambda I - L_f) : C^\alpha(X_A^+, \mathbb{C}) \to C^\alpha(X_A^+, \mathbb{C}) \text{ is not invertible}\}$$

and the spectral radius is

$$\rho(L_f) = \sup\{|\lambda| : \lambda \in \mathrm{spec}(L_f)\}.$$

2.3 Thermodynamic formalism

In this section we going to discuss some concepts which originated in statistical mechanics such as the entropy and pressure. The material of this section is mainly taking from Walters' book on ergodic theory [30].

In the following settings and definitions we shall always assume that the space X is compact. Given a continuous transformation $T : X \to X$, we shall use the following notions related to ergodic theory: A measure μ is said to be T-invariant if $\mu(B) = \mu(T^{-1}B)$ for every $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra or, equivalently, we say that T is a measure preserving transformation (m.p.t).

Let $M(X,T)$ be the set of all Borel T-invariant probability measures endowed with the weak* topology (the smallest topology making each of the maps $\mu \mapsto \int_X f\, d\mu$, $f \in C(X,\mathbb{R})$, the set of all real-valued continuous functions defined on X). So the notion of convergence in $M(X,T)$ is going to be the weak* convergence, i.e, a sequence of measures μ_n weak* converges to μ as $n \to +\infty$ if, for every $f \in C(X,\mathbb{R})$, $\lim_{n\to+\infty} \int f\, d\mu_n = \int f\, d\mu$. The set $M(X,T)$ is non-empty, convex and compact with respect to the weak* topology.

2.3.1 Measure theoretic entropy and topological entropy

We shall discuss two concepts here: the entropy of a measure-preserving transformation T on a probability space which we refer to as measure theoretic entropy $h_\mu(T)$ and the entropy in the context of continuous transformation T on a compact space which we refer to as the topological entropy $h_{\text{top}}(T)$. Generally in dynamics the entropy is used to measure the complexity of the system under a transformation. The topological entropy also plays an important role as an invariant for the classification of continuous transformations up to (topological) conjugacy. The measure theoretic entropy plays a role in the classification of measure preserving transformations up to measure theoretic isomorphisms.

We are going to define the measure theoretic entropy in terms of countable partitions $\alpha = \{A_i\}_{i\in\mathcal{I}}$ of the probability space $(X,\mathcal{B},\mu)$ (i.e., $X = \bigcup_i A_i$ and $A_i \bigcap A_j = \emptyset\,, i \neq j$). First, define an information function $I(\alpha) : X \to \mathbb{R}$, which tells us how much information we receive about a point x if we know which element of α it lies in,

$$I(\alpha)(x) = -\sum_i \log \mu(A_i)\chi_{A_i}(x).$$

Then, we define the entropy of a partition α to be the average amount of information received knowing which element of α we are in:

$$H(\alpha) = \int I(\alpha)(x)\, d\mu(x) = -\sum_{A\in\alpha} \mu(A) \log \mu(A).$$

If α and β are two partitions of X, then we let $\alpha \bigvee \beta$, the join of α and β, denote the partition $\{A \cap B : A \in \alpha, B \in \beta\}$. Let $T^{-1}\alpha = \{T^{-1}A : A \in \alpha\}$. We define the entropy of a m.p.t $T : X \to X$ relative to a partition α (with $H(\alpha) < +\infty$) by

$$h(T, \alpha) = \lim_{n\to+\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} T^{-j}\alpha\right).$$

To remove the dependence on α in the definition, we take the supremum. So, we have that

$$h_\mu(T) = \sup\{h(T, \alpha) : \alpha \text{ countable and } H(\alpha) < +\infty\}.$$

In practice, this definition is not easy to be used to calculate the measure theoretic entropy of a m.p.t T, as we have to take the supremum over all finite entropy partitions. Hence we present Sinai's theorem, which is true under the condition that α is a generator or a strong generator. We say that a partition α with $H(\alpha) < +\infty$ is a generator for the probability space $(X, \mathcal{B}, \mu)$, if the sequence $\bigvee_{j=-(n-1)}^{n-1} T^{-j}\alpha \to \mathcal{B}$, as $n \to +\infty$. We call α with $H(\alpha) < +\infty$ a strong generator if $\bigvee_{j=0}^{n-1} T^{-j}\alpha \to \mathcal{B}$, as $n \to +\infty$.

Theorem 2.3.1 (Sinai's Theorem)**.** *If α is a strong generator or is a generator, then $h_\mu(T) = h(T, \alpha)$.*

Analogously to the measure theoretic entropy we are going to define the topological entropy of a continuous transformation T on a compact metric space X using open covers. Let α be an open cover of X, since X is compact, α has a finite subcover. Let $N(\alpha)$ be the cardinality of the smallest finite subcover of α. Then define the topological entropy of α to be $H_{\text{top}}(\alpha) = \log N(\alpha)$. Then we define the topological entropy of T relative to α as follows

$$h_{\text{top}}(T, \alpha) = \lim_{n \to +\infty} \frac{1}{n} H_{\text{top}} \left(\bigvee_{j=0}^{n-1} T^{-j} \alpha \right).$$

We can now define the topological entropy of T by

$$h_{\text{top}}(T) = \sup\{h_{\text{top}}(T, \alpha) : \alpha \text{ is an open cover of } X\}.$$

Analogous to Sinai's theorem, we have the following result for topological entropy, which provides a method to calculate the topological entropy of some continuous transformation $T : X \to X$. The following theorem relies on the condition that α is a generating cover or a strong generating cover for a continuous transformation $T : X \to X$. We call a finite cover α a generator for the homeomorphism $T : X \to X$ if for all $\epsilon > 0$, there exists $n > 0$ such that the cover $\bigvee_{j=-(n-1)}^{n-1} T^{-j}\alpha = \{B_1, \ldots, B_m\}$ consists of open sets each of which has diameter at most ϵ, i.e. $\sup_i\{\text{diam}(B_i)\} < \epsilon$. We call α a strong generating cover for a continuous map $T : X \to X$ if for all $\epsilon > 0$, there exists $n > 0$ such that the cover $\bigvee_{j=0}^{n-1} T^{-j}\alpha = \{B_1, \ldots, B_m\}$ consists of open sets each of which has diameter at most ϵ, i.e. $\sup_i\{\text{diam}(B_i)\} < \epsilon$.

Theorem 2.3.2. *If α is a strong generating cover for a continuous transformation $T : X \to X$ (or a generating cover for a homeomorphism $T : X \to X$), then*

$$h_{top}(T) = h_{top}(T, \alpha).$$

Topological entropy can also be define using separated or spanning sets which is known as Bowen's definition. Both definitions, using separated or spanning sets coincide with the definition using open covers (see chapter 7, [30]).

2.3.2 Variational principle and pressure

We now present the relation between the measure theoretic entropy and the topological entropy through a variational principle in $M(X,T)$, which is analogous to a well known variational principle in statistical mechanics.

Theorem 2.3.3 (The variational principle). *Let $T : X \to X$ be a continuous transformation of a compact metric space X. Then*

$$h_{top}(T) = \sup_{\mu \in M(X,T)} h_\mu(T).$$

The variational principle gives a natural way to pick out some members of $M(X,T)$. Among the measures $\mu \in M(X,T)$, if $h_{\text{top}}(T) = h_\mu(T)$, we call μ *a measure of maximal entropy* for T. To investigate this situation more, let $M_{\max}(X,T)$ be the collection of all measures with maximal entropy for T, i.e., $M_{\max}(X,T) = \{\mu \in M(X,T) : h_\mu(T) = h_{\text{top}}(T)\}$. To ensure that $M_{\max}(X,T)$ is non-empty we shall impose the condition that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous (i.e., $\mu_n \to \mu \Rightarrow h_\mu \geq \limsup_{n \to +\infty} h_{\mu_n}$). Then we have $M_{\max}(X,T)$ is non-empty due to the fact that an upper semi-continuous function on a compact metric space attains its supremum. The interesting case is when there is only one invariant measure that maximises entropy. A continuous transformation $T : X \to X$ is said to have *unique measure of maximal entropy* if $M_{\max}(X,T)$ consists of exactly one member. In the next section we are going to see that shifts of finite type have a unique measure with maximal entropy.

The topological entropy definition can be generalised to a weighted version to introduce another concept called pressure. The definition of pressure can be given using open covers or spanning sets or separated sets. Here we are going to give the definition of pressure using open covers. For $f \in C(X,\mathbb{R})$ and $n \geq 1$ we denote $\sum_{i=0}^{n-1} f(T^i x)$ by $f^n(x)$. Let α be an open cover of X, define

$$U_n(f,T,\alpha) = \inf \left\{ \sum_{B \in \beta} \sup_{x \in B} e^{f^n(x)} : \beta \text{ is a finite subcover of } \bigvee_{i=0}^{n-1} T^{-i}\alpha \right\}.$$

Then, if α is a strong generating cover of T, the topological pressure of T with a weight function f, is given by

$$P(f,T) = \lim_{n\to\infty} \frac{1}{n} \log U_n(T,f,\alpha).$$

Remark 2.3.1. *Taking $f = 0$, the zero function in $C(X,\mathbb{R})$, we have the topological pressure $P(0,T) = h_{top}(T)$.*

This remark brings us to the extension of the variational principle in Theorem 2.3.3. The following result was first proved by D. Ruelle for some transformations, in particular he proved it for subshifts of finite type as we are going to see in the next section. The result was generalised to all continuous transformations by P. Walters.

Theorem 2.3.4 (The variational principle)**.** *Let $T : X \to X$ be a continuous map of a compact metric space and let $f \in C(X,\mathbb{R})$. Then*

$$P(f,T) = \sup\left\{ h_\mu(T) + \int f \, d\mu : \mu \in M(X,T) \right\}.$$

As with the variational principle we presented earlier, this variational principle also gives a natural way of selecting members of $M(X,T)$. In a way this extends the idea of the measure of maximal entropy. Here we call the measure μ which gives $P(f,T) = h_\mu(T) + \int f \, d\mu$, *an equilibrium state* for f. Let $M_f(X,T)$ be the collection of all equilibrium states for f. A measure with maximal entropy is precisely an equilibrium state for 0 (the zero function) and hence $M_{\max}(X,T)$, the collection of all measures with maximal entropy for T, is the same as $M_0(X,T)$, the set of all equilibrium states for the zero function. For some functions f, the equilibrium state can be unique, i.e., $M_f(X,T)$ has just one member. The following result explains when this can happen.

Lemma 2.3.1. *Let $T : X \to X$ be a continuous map of a compact metric space and suppose the entropy map of T is upper semi-continuous at each point of $M(X,T)$. Then there is a dense subset of $C(X,\mathbb{R})$ such that each member f of this subset has a unique equilibrium state (i.e., $M_f(X,T)$ has just one member).*

2.3.3 Thermodynamic formalism of subshifts of finite type

We shall now apply the general results of thermodynamic formalism in the previous two subsection to subshifts of finite type. The material and results in this section are mostly taken from the monograph of Parry and Pollicott [17].

Let $M(X_A^+, \sigma)$ be the set of all σ-invariant probability measures on X_A^+. We shall need the following well known theorem, the Perron-Frobenius theorem to construct invariant measures on (X_A^+, σ).

Theorem 2.3.5 (Perron-Frobenius)**.** *Let B be a non-negative aperiodic $k \times k$ matrix. Then*

(i) there exists a positive eigenvalue $\lambda > 0$ such that all other eigenvalues $\lambda_i \in \mathbb{C}$ satisfy $|\lambda_i| < \lambda$;

(ii) the eigenvalue λ is simple (i.e., the corresponding eigenspace is one-dimensional);

(iii) there is a unique right eigenvector $v = (v_1, \ldots, v_k)^T$ such that $v_j > 0$, $\sum_{j=1}^{n} |v_j| = 1$ and $Bv = \lambda v$;

(iv) there is a unique left eigenvector $u = (u_1, \ldots, u_k)$ such that $u_j > 0$, $\sum_{j=1}^{n} |u_j| = 1$ and $Bu = \lambda u$;

(v) eigenvectors corresponding to eigenvalues other than λ are not positive: i.e., at least one co-ordinate is positive and at least one co-ordinate is negative.

Invariant measures can be constructed on (X_A^+, σ) in the following way: Let P be a $k \times k$ stochastic matrix (i.e., $P(i,j) \geq 0$, $i,j = 1, \ldots, k$ and $\sum_{i=1}^{k} P(i,j) = 1, i = 1, \ldots, k$) which is compatible with the matrix A, i.e., $P(i,j) > 0 \Leftrightarrow A(i,j) = 1$. Recall that the collection of all cylinder sets of X_A^+ form an algebra which generates the Borel σ-algebra. So we going to define a measure μ_P on cylinder sets, then by the Hahn-Kolomogrov Extension theorem this will uniquely define a measure on the whole Borel σ-algebra. Since A is aperiodic, P is aperiodic. By the Perron-Frobenius theorem, P has a unique maximal eigenvalue λ and since P is stochastic, $\lambda = 1$ and the right eigenvector is $(1, \dots, 1)$. Let $p = (p_1, \dots, p_k)$ be the corresponding left eigenvector, then define the probability measure μ_P by

$$\mu_P[y_0, y_1, \dots, y_n] = p_{y_0} P(y_0, y_1) \dots P(y_{n-1}, y_n), \tag{2.3.1}$$

this measure is called a Markov measure. Clearly this measure is σ-invariant. To study more properties of this measure and also to understand the dynamics of the shift space, we state three lemmas. The first two lemmas calculate the topological entropy and the measure theoretic entropy of the shift map $\sigma : X_A^+ \to X_A^+$. The third lemma shows that there is a unique Markov measure which maximises the entropy of $\sigma : X_A^+ \to X_A^+$ over all invariant probability measures, i.e., the shift map σ does have a unique measure with maximal entropy.

Lemma 2.3.2. *(Theorem 4.27, [30]) Let (X_A^+, σ) be a subshift of finite type. Then the measure theoretic entropy of σ with respect to the Markov measure μ_P is given by*

$$h_{\mu_P}(\sigma) = -\sum_{i,j} p_i P(i,j) \log P(i,j).$$

Lemma 2.3.3. *(Theorem 7.13, [30]) The topological entropy of the subshift of finite type (X_A^+, σ), with A aperiodic is given by $h_{top}(\sigma) = \log \lambda$, where λ is the largest positive eigenvalue of A.*

The variational principle in Theorem 2.3.3 applies to subshifts of finite type and hence we have $h_{\text{top}}(\sigma) = \sup_{\mu \in M(X_A^+, \sigma)} h_{\mu_P}(\sigma)$. In fact, subshifts of finite type have a unique measure with maximal entropy, i.e. there exists $\mu \in M(X_A^+, \sigma)$ such that

$$h_\mu(\sigma) = h_{\text{top}}(\sigma) = \log \lambda.$$

The measure that maximises the entropy is defined in the following way: By the Perron-Frobenius theorem there exists a unique maximal eigenvalue λ for A with corresponding left and right eigenvalue $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$, respectively. Define

$$P(i,j) = \frac{A(i,j)v_j}{\lambda v_i}, \; p_i = \frac{u_i v_i}{\sum_{i=1}^{k} u_i v_i}.$$

Then P is stochastic matrix and $pP = p$ and therefore P defines naturally a Markov measure μ on X_A^+. This measure is called the Parry measure, as it was constructed by William Parry.

Lemma 2.3.4. *(Theorem 8.10, [30]) The Parry measure is the unique measure with maximal entropy for the shift transformation $\sigma : X_A^+ \to X_A^+$ where A is aperiodic.*

Another object associated to the shift transformation σ is the transfer operator $L_f : C^\alpha(X_A, \mathbb{C}) \to C^\alpha(X_A^+, \mathbb{C})$ defined in (2.2.1). This transfer operator is going to be an important tool to carry out the analysis in the symbolic dynamic system that we are going to set up. We shall need the following theorem to obtain some information about the spectral properties of the transfer operator L_f.

Theorem 2.3.6 (Ruelle-Perron-Frobenius, RPF)**.** *[17] Let $f \in C^\alpha(X_A^+, \mathbb{R})$ be real valued and suppose A is aperiodic.*

(i) There is a simple maximal positive eigenvalue β of $L_f : C(X_A^+, \mathbb{C}) \to C(X_A^+, \mathbb{C})$ with a corresponding strictly positive eigenfunction $h \in C^\alpha(X_A^+, \mathbb{R})$.

(ii) The reminder of the spectrum of $L_f : C^\alpha(X_A^+, \mathbb{C}) \to C^\alpha(X_A^+, \mathbb{C})$ is contained in a disc of radius strictly smaller than β.

(iii) There is a unique probability measure μ such that $L_f^* \mu = \beta \mu$ (i.e. $\int L_f v \, d\mu = \beta \int v \, d\mu$, for all $v \in C(X_A^+, \mathbb{R})$).

(iv) $\frac{1}{\beta^n} L_f^n v \to h \int v \, d\mu$ uniformly for all $v \in C(X_A^+, \mathbb{R})$ where h is as above and $\int h \, d\mu = 1$.

As the transfer operator L_f is acting on $C^\alpha(X_A^+, \mathbb{R})$ the space of real Hölder continuous functions with exponent α, we want to understand something about the interaction in the shift space with the function f as a weight function. To achieve this we are going to study the pressure of f and one way to do that is to look at the variational principle that defines the pressure function.

As we can see that the variational principle in Theorem 2.3.4 applies to $f \in C^\alpha(X_A^+, \mathbb{R})$, we have that

$$P(f, \sigma) = \sup\left\{h_\mu(\sigma) + \int f\, d\mu : \mu \in M(X_A^+, \sigma)\right\}. \tag{2.3.2}$$

The next thing to think about when the variational principle is mentioned is when the supremum is attained. We have seen in the previous subsection that measures where the supremum is attained are called the *equilibrium states*. In the special case when $f = 0$, the zero function, the variational principle is reduced to the case that $P(0, \sigma)$. For abbreviation we write $P(0)$ instead of $P(0, \sigma)$. So, $P(0) = \sup\{h_\mu(\sigma),\, \mu \in M(X_A^+, \sigma)\}$. We also have by Remark 2.3.1 that $P(0) = h_{\text{top}}(\sigma)$. So we get the variational principle we discussed earlier, where we found that the supremum is attained at a unique measure which is the Parry measure (Lemma 2.3.4) and we have $h_\mu(\sigma) = h_{\text{top}}(\sigma) = \log \lambda$. Therefore, we have $P(0) = \log \lambda$, where λ is the maximal eigenvalue of the matrix A. In this case the measure of maximal entropy (Parry measure) is the equilibrium state of the function $f = 0$.

Now in the general case when $f \neq 0$, $f \in C^\alpha(X_A^+, \mathbb{R})$ the equilibrium state also exists and it is unique, i.e. there exists a measure $m \in M(X_A^+, \sigma)$ such that $P(f) = h_m(\sigma) + \int f\, dm$.

To find the equilibrium states $m \in M(X_A^+, \sigma)$, we shall be looking for measures that have a property called the Gibbs property. A measure μ on X_A^+ is called a *Gibbs measure* if there exits $g \in C(X_A^+, \mathbb{R})$ such that

$$A \leq \frac{\mu[x_0, \ldots, x_n]}{e^{g^n(x) + nC}} \leq B,$$

where $n \geq 0$ and A, B and C are constants and the measure μ is not necessarily σ-invariant.

We say that $f \in C^\alpha(X_A^+, \mathbb{R})$ or L_f is normalised when $L_f 1 = 1$. Proposition 3.2 and Corollary 3.2.1 of [17] state that for $f \in C^\alpha(X_A^+, \mathbb{R})$ normalised and $L_f^* m = m$ (m is the eigenmeasure given by the RPF theorem and m is σ-invariant), m satisfies the Gibbs property with $C = 0$. Hence, we can deduce that Gibbs measure exists for $f \in C^\alpha(X_A^+, \mathbb{R})$ when we normalise f, i.e., taking a function $g = f - \log h \circ \sigma + \log h - \log \beta$, where h and β are the positive eigenfunction and eigenvalue guaranteed by the RPF theorem, in particular there exists constants A', B' such that:

$$A' \leq \frac{m[x_0 \ldots x_n]}{e^{f^n(x) - n \log \beta}} \leq B',$$

for all $x \in X_A^+$. We can see that these inequalities satisfy the Gibbs property with $C = \log \beta$. Now we are going to see that m is actually the equilibrium state for $f \in C^\alpha(X_A^+, \mathbb{R})$ and it is unique. For $f \in C^\alpha(X_A^+, \mathbb{R})$ normalised and $L_f^* m = m$, Proposition 3.4 of [17] states that for any σ-invariant probability measure μ we have

$$h_\mu(\sigma) + \int f \, d\mu \leq 0,$$

with equality if and only if $\mu = m$. Now, for $f \in C^\alpha(X_A^+, \mathbb{R})$ which is not normalised, we take h and β as in the RPF theorem, the function $g = f - \log h \circ \sigma + \log h - \log \beta$ is normalised. Applying g to the inequality above gives that for any σ-invariant probability measure

$$h_\mu(\sigma) + \int g \, d\mu = h_\mu(\sigma) + \int f \, d\mu + \int (-\log h \circ \sigma + \log h) d\mu - \log \beta \leq 0,$$

with equality if and only if μ is the eigenmeasure for L_g, i.e., $d\mu = hdm$ where $L_f^* m = \beta m$. Hence, we have the following lemma:

Lemma 2.3.5. *(Theorem 3.5,[17]) Let $f \in C^\alpha(X_A^+, \mathbb{R})$, with A aperiodic and suppose μ_f is a σ-invariant Gibbs measure of f given by $d\mu_f = hdm$ where h and m are eigenfunction and eigenmeaure for L_f corresponding to the maximal eigenvalue $e^{P(f)}$. Then μ_f is the unique measure in $M(X_A^+, \sigma)$ for which*

$$P(f) = h_{\mu_f}(\sigma) + \int f \, d\mu_f,$$

where $P(f) = \log \beta$*,* β *is the maximal eigenvalue of* L_f *guaranteed by the RPF theorem.*

In the following lemma, we give more explicit information about the spectrum of L_f. We denote the spectral radius of L_f by $\rho(L_f)$.

Lemma 2.3.6. *Let* $f \in C^\alpha(X_A^+, \mathbb{R})$*, then*

(i) The unique maximal eigenvalue of L_f *is* $\beta = e^{P(f)}$*,* P *is the pressure function and* $L_f h = e^{P(f)} h$ *(*h *is the corresponding eigenfunction of* β*)(Theorem 3.5,[17]).*

(ii) $\rho(L_f) = e^{P(f)}$*, while the spectral radius of of the rest of the spectrum of* L_f *apart from the eigenvalue* $e^{P(f)}$ *is strictly less than* $e^{P(f)}$*, so in this case we say that* L_f *has a spectral gap.*

(iii) The equilibrium state m *(or the Gibbs measure) can be written as* $dm = hd\mu$*, where* h *and* μ *are the eigenfunction and the eigenmeasure corresponding to the maximal eigenvalue* $e^{P(f)}$ *(i.e.* $L_f^* \mu = e^{P(f)} \mu$*).*

As we can see that the maximal eigenvalue of the transfer operator can be written as an exponential of the pressure function. Analysis related to the pressure function is essential to analyse the transfer operator. First we shall we give a lemma to list some basic properties of the pressure as a functional. Then, we are going to give a lemma to calculate the first and the second derivatives of the pressure functions.

Lemma 2.3.7. *Let* $f \in C^\alpha(X_A^+, \mathbb{R})$*, then the pressure of* f *defined by* $P(f) = \sup\{h_\mu(\sigma) + \int f \, d\mu : \mu \in M(X_A^+, \sigma)\}$ *has the following properties:*

(i) P *is monotone increasing, i.e. if* $f, g \in C^\alpha(X_A^+, \mathbb{R})$ *and* $f \leq g$*, then* $P(f) \leq P(g)$*.*

(ii) P *is convex, i.e. for* $0 \leq \alpha \leq 1$*,* $P(\alpha f + (1-\alpha) g) \leq \alpha P(f) + (1-\alpha) P(g)$*.*

(iii) If f *is cohomologous to* $g + c$*, for some constant* c*, then* $P(f) = P(g) + c$*.*

(iv) P *is Lipschitz continuous, i.e.* $|P(f) - P(g)| \leq \|f - g\|_\infty$*.*

Lemma 2.3.8. *([17], Propositions 4.10, 4.11 and 4.12) If $f, g \in C^\alpha(X_A^+, \mathbb{R})$ and f is not cohomologous to a constant, then*

(i)

$$\left.\frac{dP(tf+g)}{dt}\right|_{t=0} = \int f\, dm_g,$$

(ii)

$$\left.\frac{d^2P(tf+g)}{dt^2}\right|_{t=0} = \lim_{n\to+\infty} \frac{1}{n}\int \left(f^n(x) - n\int f\, dm_g\right)^2 dm_g > 0,$$

where m_g is the equilibrium state of the function g.

A proof of this lemma will be given in Chapter 3 (Lemma 3.1.9) in the special case when $g = 0$.

2.3.4 Complex transfer operators

In this section we extend the definition of the transfer operators L_f to complex valued functions and also present the complex version of the RPF theorem, which was proved by M. Pollicott [19] (this can also be viewed as an extension of Wielandt's theorem for matrices). We shall also extend the pressure definition to complex valued functions.

Let $f \in C^\alpha(X_A^+, \mathbb{C})$, then we can define a complex transfer operator $L_f : C^\alpha(X_A^+, \mathbb{C}) \to C^\alpha(X_A^+, \mathbb{C})$ in similar way to when $f \in C^\alpha(X_A^+, \mathbb{R})$. Hence we have the following theorem which gives information about the spectrum of the transfer operators when f is complex valued.

Theorem 2.3.7. *(Complex RPF theorem)[17] For $f \in C^\alpha(X_A^+, \mathbb{C})$, we have $\rho(L_f) \le e^{P(\Re(f))}$. Moreover, we have one of the following cases.*

(i) If L_f has an eigenvalue β such that $|\beta| = e^{P(\Re(f))}$, then it is simple and unique and $L_f = \alpha M L_{\Re(f)} M^{-1}$, where M is a multiplication operator and $\alpha \in \mathbb{C}$, $|\alpha| = 1$ and so $\rho(L_f) = e^{P(\Re(f))}$. The rest of the spectrum is contained in a disc of radius strictly smaller than $e^{P(\Re(f))}$.

(ii) If L_f has no eigenvalue β with $|\beta| = e^{P(\Re(f))}$, then $\rho(L_f) < e^{P(\Re(f))}$.

Remark 2.3.2. *If we assume in the above theorem that f is locally constant depending on two coordinates for example, say $f = f(x_0, x_1)$, we can deduce the statement of Wielandt's theorem for matrices $M(i,j) = A(i,j)e^{f(i,j)}, 1 \leq i,j \leq k$, stated in the next theorem.*

Theorem 2.3.8. *(Wielandt's theorem)([4], p.57) Let N be a positive aperiodic matrix with $N(i,j) = |M(i,j)| \geq 0$ and let $\lambda > 0$ be the maximal positive eigenvalue for N. Then,*

(i) every eigenvalue of M in modulus is less than or equal to the maximal eigenvalue λ of N and

(ii) the equality happens if and only if M has the form $M = e^{ia}DND^{-1}$, where $0 \leq a \leq 2\pi$ and D is a diagonal matrix with diagonal entries are of unit modulus. In this case M has an eigenvalue equal to λe^{ia}.

Another way to formulate the Complex RPF theorem and so Wielandt's theorem is by a property related to $\Im(f)$.

Lemma 2.3.9. *[19] For $f \in C^{\alpha}(X_A^+, \mathbb{C})$*

(i) $\rho(L_f) = e^{P(\Re(f))}$ iff $\Im(f)$ is cohomologous to a function of the form $a + \psi$, where $\psi \in C(X_A^+, 2\pi\mathbb{Z})$ and $a \in \mathbb{R}$. In fact L_f has a simple eigenvalue $\beta = e^{ia+P(\Re(f))}$ and $\rho(L_f) = |\beta|$.

(ii) If $\Im(f)$ is not cohomologous to a function of the form $a + \psi$, where $\psi \in C(X_A^+, 2\pi\mathbb{Z})$ and $a \in \mathbb{R}$, then $\rho(L_f) < e^{P(\Re(f))}$.

Remark 2.3.3. *This lemma will be useful in analysing the spectrum of the transfer operator L_f. The condition on $\Im(f)$ will be connected to a condition we require on a function, related to $\Im(f)$, to be not cohomologous to $a + \phi$, where $\phi \in C(X_A^+, d\mathbb{Z}), a \in \mathbb{R}$. As we are going to see in chapter 3, we call this function a non-lattice function.*

The pressure $P(f)$ was defined in Lemma 2.3.6 in the case of real valued function f, where it is characterised by the unique maximal eigenvalue of the transfer operator L_f. Using perturbation theory this definition can be extended to complex valued functions in a neighbourhood of the real valued functions. To see this, we recall the definition of analytic functions in complex Banach spaces.

Definition 2.3.1. *Let B be a complex Banach space and let $D \subset \mathbb{C}$ be some open domain. A map $f : D \to B$ is said to be analytic if, for every bounded linear functional $u : B \to \mathbb{C}$, the map $u \circ f : D \to \mathbb{C}$ is analytic in the usual sense. If B_1 and B_2 are two complex Banach spaces and $D' \subset B_1$ is some open domain then a map $g : D' \to B_2$ is said to be analytic if the composition $g \circ f : D \to B_2$ is analytic for every open domain $D \subset \mathbb{C}$ and any analytic map $f : D \to B_1$ with $f(D) \subset D'$.*

We also have the following perturbation theory lemma from Kato's book ([8], VII.3).

Lemma 2.3.10. *Let $B(V)$ denote the Banach algebra of bounded linear operators on a complex Banach space V. If $L_g \in B(V)$ has a simple isolated eigenvalue λ_g with the corresponding eigenvector u_g then for any $\epsilon > 0$ there exists $\delta > 0$ such that if $L_f \in B(V)$ with $\|L_f - L_g\|| < \delta$ then*

(i) L_f has a simple isolated eigenvalue $\lambda(L_f)$ and corresponding eigenvector $u(L_f)$ with $\lambda(L_g) = \lambda_g, u(L_g) = u_g$ and such that

(ii) $L_f \mapsto \lambda(L_f), L_f \mapsto u(L_f)$ are analytic for $\|L_f - L_g\| < \delta$ and

(iii) for $\|L_f - L_g\| < \delta$, we have that $|\lambda(L_f) - \lambda_g| < \epsilon$ and $\mathrm{spec}(L_f) \setminus \{\lambda(L_f)\} \subset \{z \in \mathbb{C} : |z - \lambda_g| > \epsilon\}$.

Moreover, if $\Sigma_g = \mathrm{spec}\,(L_g) \setminus \{\lambda_g\}$ is contained in the interior of a circle C centred at $0 \in \mathbb{C}$ then provided $\delta > 0$ is sufficiently small, $\Sigma_f = \mathrm{spec}\,(L_f) \setminus \{\lambda(L_f)\}$ will also be contained in the interior of C.

The hypothesis of this lemma is satisfied when the transfer operator L_g where $g \in C^\alpha(X_A^+, \mathbb{R})$, as L_g has a simple isolated eigenvalue $\lambda_g = e^{P(g)}$ by the RPF theorem and lemma 2.3.6. Therefore, this lemma implies that for $f \in$

$C^\alpha(X_A^+, \mathbb{C})$ in a sufficiently small neighbourhood of $g \in C^\alpha(X_A^+, \mathbb{R})$ the transfer operator L_f has a simple eigenvalue at $e^{P(f)}$. This extends the definition of the pressure to a neighbourhood of $f \in C^\alpha(X_A^+, \mathbb{R})$. So this means that the pressure can only be defined in the case when L_f has a simple 'maximal' eigenvalue β and the rest of its spectrum is restricted to a disc of radius smaller then $|\beta|$ (part (i) of Theorem 2.3.7). For such functions f, the definition of pressure is given by $P(f) = \text{Log}\beta$, where Log is the principal branch of the complex function log. Moreover, by Lemma 2.3.10 the map $f \mapsto P(f)$ is analytic. Hence, we have the following lemma.

Lemma 2.3.11. *([17], Proposition 4.7) The domain of the pressure function P in $C^\alpha(X_A^+, \mathbb{C})$ is open and the function $f \mapsto P(f)$ is analytic from this domain into $\mathbb{C}$.*

2.3.5 Entropy and pressure of product shifts

We have $\sigma : X_A^+ \to X_A^+$ is a measure preserving transformation of the probability space $(X_A^+, \mathcal{B}, \nu)$, where ν is Borel $\sigma-$invariant probability measure. The direct product $\widetilde{\sigma} = \sigma \times \sigma$ is the measure preserving transformation of $(X_A^+ \times X_A^+, \mathcal{B} \times \mathcal{B}, \nu \times \nu) = (\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\nu})$ (see definition 1.2,[30]). Now, let $M(\widetilde{X}, \widetilde{\sigma})$ be the set of all $\widetilde{\sigma}$-invariant probability measures on $\widetilde{X}$.

To calculate the measure theoretic entropy of $\widetilde{\sigma}$ we use Theorem 4.21 in Walters [30]. This theorem states that if $(X_1, \mathcal{B}_1, \nu_1)$, $(X_2, \mathcal{B}_2, \nu_2)$ are two probability spaces and $T_1 : X_1 \to X_1$, $T_2 : X_2 \to X_2$ are measure-preserving transformations then the measure theoretic entropy $h_{\nu_1 \times \nu_2}(T_1 \times T_2) = h_{\nu_1}(T_1) + h_{\nu_2}(T_2)$. Applying this theorem to $\widetilde{\sigma}$ on $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\nu})$, we have that

$$h_{\widetilde{\nu}}(\widetilde{\sigma}) = 2h_\nu(\sigma). \tag{2.3.3}$$

Similarly we can calculate the topological entropy of $\widetilde{\sigma}$. We are going to use Theorem 7.10 in Walters [30] which states that if $(X_1, d_1), (X_2, d_2)$ are two compact metric spaces and T_i are continuous functions in $(X_i, d_i), i = 1, 2$ and a metric on $X_1 \times X_2$ defined by $d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$, then $T_1 \times T_2$ are continuous in $(T_1 \times T_2, d)$ and the topological entropy $h_{\text{top}}(T_1 \times T_2) = h_{\text{top}}(T_1) + h_{\text{top}}(T_2)$. Hence we have

$$h_{\text{top}}(\widetilde{\sigma}) = 2h_{\text{top}}(\sigma) = 2\log\lambda, \tag{2.3.4}$$

where λ is the maximal eigenvalue of the matrix A (see Lemma 2.3.3). Now we want to show that $\widetilde{\mu} = \mu \times \mu$ is the unique measure of maximal entropy for $\widetilde{\sigma}$. First, note that by (2.3.3) and (2.3.4) we have $h_{\mu\times\mu}(\widetilde{\sigma}) = 2h_\mu(\sigma) = 2h_{\text{top}}(\sigma) = h_{\text{top}}(\widetilde{\sigma})$. Then, by the variational principle that relate $h_{\widetilde{\nu}}(\widetilde{\sigma})$ and $h_{\text{top}}(\widetilde{\sigma})$ given as in Theorem 2.3.3, $h_{\text{top}}(\widetilde{\sigma}) = \sup\{h_{\widetilde{\nu}}(\widetilde{\sigma}) | \widetilde{\nu} \in M(\widetilde{X}, \widetilde{\sigma})\}$, $\widetilde{\mu} = \mu\times\mu$ is the unique measure of maximal entropy for $\widetilde{\sigma}$. Consequently, $h_{\widetilde{\mu}}(\widetilde{\sigma}) = h_{\text{top}}(\widetilde{\sigma}) = 2\log\lambda$.

The variational principle in (2.3.2) also holds for $F \in C^\alpha(\widetilde{X}, \mathbb{R})$

$$P(F, \widetilde{\sigma}) = \sup\left\{h_{\widetilde{\nu}}(\widetilde{\sigma}) + \int F\, d\widetilde{\nu} : \widetilde{\nu} \in M(\widetilde{X}, \widetilde{\sigma})\right\}. \tag{2.3.5}$$

The equilibrum state for F exists and it is unique as F is Hölder continuous function defined on a subshift of finite type space $\widetilde{X}$, we also have that

$$P(F, \widetilde{\sigma}) = \log\widetilde{\beta}, \tag{2.3.6}$$

where $\widetilde{\beta}$ is the maximal eigenvalue of L_F, see Lemma 2.3.5. For simplicity of notation we are going to use $P(F)$ where we mean $P(F, \widetilde{\sigma})$. This gives us that $e^{P(F)}$ is the maximal eigenvalue of the transfer operator L_F.

Remark 2.3.4. *Taking $F = 0$ in the pressure definition in 2.3.6, we get $P(0, \widetilde{\sigma}) = P(0) = \log\widetilde{\beta}$, where $\widetilde{\beta}$ is the maximal eigenvalue L_0. We also have that $P(0, \widetilde{\sigma}) = \sup\{h_{\widetilde{\nu}}(\widetilde{\sigma}) : \widetilde{\nu} \in M(\widetilde{X}, \widetilde{\sigma})\} = h_{\widetilde{\mu}}(\widetilde{\sigma})$. We have seen that there is a unique measure of maximal entropy $\widetilde{\mu}$ where $h_{\widetilde{\mu}}(\widetilde{\sigma}) = h_{top}(\widetilde{\sigma}) = 2\log\lambda$, λ the maximal eigenvalue of the matrix A. Hence at $F = 0$, the maximal eigenvalue of L_0, $\widetilde{\beta} = e^{P(0,\widetilde{\sigma})} = \lambda^2 = e^{2P(0,\sigma)}$.*

As a complex transfer operator can also be defined for $F \in C^\alpha(\widetilde{X}, \mathbb{C})$, the pressure function of a complex valued function $F \in C^\alpha(\widetilde{X}, \mathbb{C})$ can also be defined in a small neighbourhood of $F \in C^\alpha(\widetilde{X}, \mathbb{R})$ by analytic extension.

Chapter 3

Pairs of closed geodesics ordered by word length on (G, ℓ)

In this chapter we study pairs of closed geodesics on non-bipartite graphs and bipartite graphs. We shall obtain an asymptotic for the number of pairs of closed geodesics ordered by their word lengths on (G, ℓ), such that the difference of their geometric lengths lie in a fixed interval $[a, b] \subset \mathbb{R}$. Similarly, we are going to get an asymptotic for the case when the difference between the geometric lengths of these pairs lie in an interval that is allowed to shrink at a specific rate. In fact, we shall let the interval to be positioned arbitrarily in the real line to obtain a uniform result.

3.1 Non-bipartite graphs

We start to study the counting problem in non-bipartite graphs and then we are going to use the result in this section to get a result for bipartite graphs in the next section. The approach we choose to solve the counting problem is to use symbolic dynamics. We are going to use a subshift of finite type to model metric graphs.

3.1.1 Coding metric graphs by subshifts of finite type

We define a subshift of finite type which consists of infinite sequences of oriented edges $e_i \in E^o$, the set of oriented edges of the metric graph (G, ℓ). So, infinite paths in the graph G represented by infinite sequences of oriented edges in a shift space. Recall that edge $e \in E$ corresponds to two oriented edges, say $e, \overline{e}$. Then $e'' \in E^o$ follows $e' \in E^o$ if e'' begins at the terminus of e', i.e. $\mathbf{o}(e'') = \mathbf{t}(e')$. To ensure that we do not allow backtracking, we require that $e'' \neq \overline{e'}$. We define a $|E^o| \times |E^o|$ matrix A with rows and columns indexed E^o by

$$A(e', e'') = \begin{cases} 1 & \text{if } e'' \text{ follows } e' \text{ and } e'' \neq \overline{e'} \\ 0 & \text{otherwise .} \end{cases}$$

Hence we define the subshift of finite type:

$$X_A^+ = \{(e_j)_{j=0}^{\infty} : e_j \in E^o, A(e_j, e_{j+1}) = 1 \ \forall j \geq 0\},$$

with the associated shift map $\sigma : X_A^+ \to X_A^+ : \sigma(e_0, e_1, ...) = (e_1, e_2, ...)$. We can define the same metric on the subshifs of finite type that we defined in subsection 2.2.1. Given $0 < \theta < 1$, we define a metric on X_A^+ by $d_\theta(x, y) = \theta^N$, where N is the largest non-negative integer such that $x_i = y_i$, $i = 0, \ldots, N$. The topological entropy of σ is given $h_{\text{top}}(\sigma) = \log \lambda$, where λ is the maximal eigenvalue of the matrix A (see Lemma 2.3.3). We also have that A is aperiodic by the following lemma, proved by Kotani and Sunada in [9], Proposition 3.2.

Lemma 3.1.1. *Suppose G has* $\deg(v) \geq 3$ *for each vertex v, then*

(i) If G is not bipartite then A has period 1, i.e. A is aperiodic.

(ii) If G is bipartite then A is irreducible with period 2*.*

The length function $\ell : E \to \mathbb{R}^+$ which assigns length to the edges of G, can be coded by a locally constant function $r : X_A^+ \to \mathbb{R}^+$ defined by

$$r(e_0, e_1, ...) = \ell(e_0),$$

which is a Hölder continuous function for any exponent $\alpha > 0$. This function will be used as a weight function for the shift space X_A^+. The following lemma shows that periodic orbits in the shift space (X_A^+, σ) can be used to represent the closed geodesics in the metric graph (G, ℓ).

Lemma 3.1.2. *There is an exact correspondence between prime periodic orbits $\{x, \sigma x, ..., \sigma^{n-1}x\}$ (where $x \in X_A^+$) and prime closed geodesics γ in (G, ℓ) of word length $|\gamma| = n$. Moreover the geometric length of γ is given by $\ell(\gamma) = r^n(x)$.*

Proof. To show the exact correspondence, we define a map ϕ from the set of closed geodesics γ with $|\gamma| = n$ and the set of periodic orbits of period n in X_A^+. We shall see that this map is a bijection. Let $e = (e_0, \dots, e_{n-1})$ be a closed path in (G, ℓ) and let $x = (e_0, \dots, e_{n-1}, e_0, \dots, e_{n-1}, \dots)$ be a periodic point of period n, then we define ϕ by the following correspondence:

$$\begin{aligned} x &\rightarrow (e_0, e_1, ..., e_{n-1}) \\ \sigma x &\rightarrow (e_1, e_2, ..., e_{n-1}, e_0) \\ &\vdots \\ \sigma^{n-1}x &\rightarrow (e_{n-1}, e_0, ..., e_{n-2}). \end{aligned}$$

We can clearly see that ϕ is a bijection. Furthermore, by summing over each periodic point in X_A^+, we get the relation between the functions ℓ and r. Take a periodic point $\sigma^n x = x \in X_A^+$. Then its periodic orbit corresponds to a closed geodesic γ with $|\gamma| = n$. We have that

$$\begin{aligned} r^n(x) &= r(x) + r(\sigma x) + ... + r(\sigma^{n-1}x) \\ &= \ell(e_0) + \ell(e_1) + ... + \ell(e_{n-1}) = \ell(\gamma). \end{aligned}$$

□

We have mentioned at the beginning of this chapter that we are going to study pairs of closed geodesics. To do this we shall introduce a new subshift of finite type, the product shift defined as in subsection 2.2.2. Let $\widetilde{X} = X_A^+ \times X_A^+$ and consider the product transformation $\widetilde{\sigma} = \sigma \times \sigma$ on $\widetilde{X}$ defined by $\widetilde{\sigma}(x,y) = (\sigma x, \sigma y)$. We associate $\widetilde{X}$ with a 0-1 matrix $\widetilde{A}$. A pair (e_i, e_i') can be followed by another one (e_{i+1}, e'_{i+1}) if only if $A(e_i, e_{i+1}) = 1$ and $A(e'_i, e'_{i+1}) = 1$, i.e. $\widetilde{A}((e_i, e'_i), (e_{i+1}, e'_{i+1})) = 1$. Thus we have

$$\widetilde{X} = \{(e_i, e'_i)_{i=0}^{\infty} \in (E^o \times E^o) : \widetilde{A}((e_i, e'_i), (e_{i+1}, e'_{i+1})) = 1 \, \forall i \geq 0\}.$$

We have seen that $\widetilde{X}$ is a subshift of finite type. We can associate to $\widetilde{X}$ a continuous function $R : \widetilde{X} \to \mathbb{R}$ defined by $R(x,y) = r(x) - r(y)$. To see how this coding process comes together, we have the following lemma analogous to Lemma 3.1.2.

Lemma 3.1.3. *Let the periodic points $\widetilde{\sigma}^n(x,y) = (x,y) \in \widetilde{X}$, where $\sigma^n x = x, \sigma^n y = y \in X_A^+$ corresponds to closed geodesics γ, γ' with $|\gamma| = |\gamma'| = n$. Then we have that $R^n(x,y) = r^n(x) - r^n(y)$.*

Proof. Summing over periodic orbits $\widetilde{\sigma}^n(x,y) = (x,y) \in \widetilde{X}$ and using the definition of R and Lemma 3.1.2, we have that

$$\begin{aligned} R^n(x,y) &= R(x,y) + R(\widetilde{\sigma}(x,y) + ... + R(\widetilde{\sigma}^{n-1}(x,y)) \\ &= (r(x) - r(y)) + (r(\sigma x) - r(\sigma y)) + ... + (r(\sigma^{n-1}x) - r(\sigma^{n-1}y)) \\ &= r^n(x) - r^n(y) = \ell(\gamma) - \ell(\gamma'). \end{aligned}$$

□

3.1.2 Non-lattice condition on weight functions

In this section we discuss an important condition on the function $r : X_A^+ \to \mathbb{R}^+$ that we require in order to prove the main results in this thesis. Recall the definition of cohomologous functions from subsection 2.2.3, then we have the following definition of what we call non-lattice functions.

Definition 3.1.1. *We say that a function $f : X_A^+ \to \mathbb{R}$ is non-lattice if f is not cohomologous to a constant plus a function valued in a discrete subgroup of*

$\mathbb{R}$.

In other words, the function $r : X_A^+ \to \mathbb{R}$ is non-lattice if there are no continuous functions $\psi : X_A^+ \to \mathbb{R}$, $M : X_A^+ \to d\mathbb{Z}$ and $c \in \mathbb{R}$ such that $r = c + M + \psi\circ\sigma - \psi$. The importance of this condition for the function r is that we are going to use transfer operators (see subsection 2.2.3 for definition) L_{sr}, $s \in \mathbb{C}$, to carry out the analysis for studying pairs of closed geodesics with the constraints in the word length and geometric lengths. The fact that r is non-lattice is going to control the spectral properties of the transfer operator L_{sr}, $s \in \mathbb{C}$ (see Remark 2.3.3). Furthermore, as we are going to see in the next subsection, this condition on r will imply that the function R is not cohomologous to a constant. This property of R will be useful to analyse a pressure function $P(sR)$, $s \in \mathbb{R}$.

To guarantee that r is non-lattice we need the following assumption on the geometric lengths of closed geodesics:

The non-lattice condition: $\{\ell(\gamma) - c|\gamma| : \gamma \text{ closed geodesics}\} \not\subset d\mathbb{Z}$, for any $c, d \in \mathbb{R}$.

We can easily see that the non-lattice condition implies that r is non-lattice.

Lemma 3.1.4. *If the non-lattice condition holds, then r is non-lattice.*

Proof. Suppose that r is not non-lattice. Then, by definition, there are $c, d \in \mathbb{R}$ and a continuous function $M : X_A^+ \to d\mathbb{Z}$ such that, whenever $\sigma^n x = x$, we have $r^n(x) - nc = M^n(x)$. Hence

$$\{r^n(x) - nc : \sigma^n x = x, n \geq 1\} \subset d\mathbb{Z}.$$

Since $r^n(x) = \ell(\gamma)$, where γ is the closed geodesic corresponding to the periodic orbit of x, this is equivalent to $\{\ell(\gamma) - c|\gamma| : \gamma \text{ is a closed geodesic}\} \subset d\mathbb{Z}$, so the non-lattice condition fails to hold. □

Recall that considering $\mathbb{R}$ as vector space over $\mathbb{Q}$, we let L be the smallest subspace containing all the lengths of closed geodesics, i.e. $L = \text{span}_{\mathbb{Q}}\{\ell(\gamma) : \gamma \text{ is closed geodesic}\}$. The following lemma shows that the non-lattice condition is implied by requiring the dimension of $\dim_{\mathbb{Q}}(L)$ to be at least 3.

Lemma 3.1.5. *If* $\dim_{\mathbb{Q}}(L) \geq 3$*, then the non-lattice condition holds.*

Proof. Suppose that the non-lattice condition does not hold. Then, this means there are $c, d \in \mathbb{R}$ such that

$$\{\ell(\gamma) - c|\gamma| : \gamma \text{ closed geodesics}\} \subset d\mathbb{Z}.$$

Another way to write this is that there are $c, d \in \mathbb{R}$ such that

$$\ell(\gamma) = c|\gamma| + dM,$$

where $M \in \mathbb{Z}$. This implies that $l(\gamma) \in c\mathbb{Z} \oplus d\mathbb{Z} \subset c\mathbb{Q} \oplus d\mathbb{Q}$. This implies that $\dim_{\mathbb{Q}}(L) \leq 2$

□

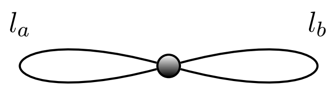
We are going to give three examples here. The first one shows a case when the non-lattice condition cannot be true for any choice of geometric lengths on the edges and so any geometric length of closed geodesic. While the second one shows that if the non-lattice condition holds, then it is not necessarily true that $\dim_{\mathbb{Q}}(L) \geq 3$. The third example shows a metric graph when the non-lattice condition is satisfied.

Example 3.1.1. *Consider a graph of one vertex and two edges (which are loops in this graph). To see why the non-lattice condition cannot be satisfied here, let us denote the edges (or the loops) a and b with lengths l_a and l_b, respectively (see Figure 3.1). Consider a closed geodesics γ with $|\gamma| = N$. Then $\ell(\gamma) = nl_a + (N-n)l_b$ for some $0 \leq n \leq N$. Therefore,*

$$\ell(\gamma) = n(l_a - l_b) \in (l_a - l_b)\mathbb{Z} + l_b\mathbb{Z}.$$

Hence, the non-lattice condition cannot be satisfied in this case.

Figure 3.1



Example 3.1.2. *Consider a graph of one vertex and three loops with edge lengths* $1, \sqrt{2}, 1+\sqrt{2}$ *as shown in Figure 3.2, so have that* $\dim_{\mathbb{Q}}(L) = 2$*. But in this metric graph the non-lattice condition holds. To see this, suppose that the non-lattice condition fails to hold, then there are* $c, d \in \mathbb{R}$ *such that* $\ell(\gamma) - c|\gamma| \in d\mathbb{Z}$*, for every closed geodesic* γ*. Just considering three loops (where* $|\gamma| = 1$*), this gives us the equations:*

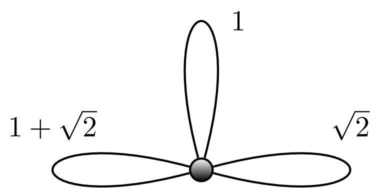
$$1 - c \in d\mathbb{Z}, \tag{3.1.1}$$

$$\sqrt{2} - c \in d\mathbb{Z}, \tag{3.1.2}$$

$$1 + \sqrt{2} - c \in d\mathbb{Z}. \tag{3.1.3}$$

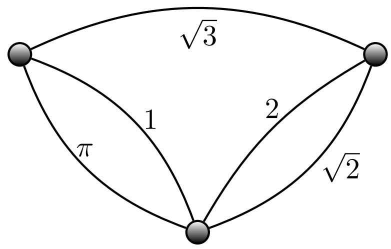
Subtracting (3.1.2) from (3.1.3) we get $1 \in d\mathbb{Z}$*. So* $d \in \mathbb{Q}$ *and hence* $d\mathbb{Z} \subset \mathbb{Q}$*. Also, subtracting (3.1.1) from (3.1.3) we get* $\sqrt{2} \in d\mathbb{Z} \subset \mathbb{Q}$*, i.e.* $\sqrt{2}$ *is rational, which is a contradiction. Therefore, the non-lattice condition holds.*

Figure 3.2



Example 3.1.3. *For the third example, to make sure that the non-lattice is satisfied we are going to draw a metric graph where we make sure that* $\dim_{\mathbb{Q}}(L) \geq 3$ *and hence by Lemma 3.1.5 the non-lattice condition holds. Consider a graph that has three vertices and five edges and assign lengths to the edges as shown in Figure 3.3. For the values of the edge lengths we have in this metric graph, clearly* $\dim(span_{\mathbb{Q}}\{\ell(\gamma) : \gamma \text{ is closed geodesic}\}) \geq 3$.

Figure 3.3



3.1.3 The pressure function and transfer operators

Recall the thermodynamic formalism for subshifts of finite type we introduced earlier in subsection 2.3.3. In this subsection we are going to study a pressure functions defined in terms of the functions $r : X_A^+ \to \mathbb{R}^+$ and $R : \widetilde{X} \to \mathbb{R}$. A family of transfer operators associated to the functions r and R will be used as a tool to analyse these pressure functions. In addition, the spectral properties of these transfer operators will play an important role in our analysis and calculations.

Let $M(X_A^+, \sigma)$ be the set of all σ-invariant probability measures on X_A^+. Similarly, let $M(\widetilde{X}, \widetilde{\sigma})$ be the set of all $\widetilde{\sigma}$-invariant probability measures on $\widetilde{X}$. By the variational principle (see Theorem 2.3.4), the pressure of a Hölder continuous function $f : X_A^+ \to \mathbb{R}$ is defined by

$$P(f) = \sup \left\{ h(\nu) + \int f \, d\nu : \nu \in M(X_A^+, \sigma) \right\}.$$

The supremum is attained at a unique equilibrium state in $M(X_A^+, \sigma)$ by Lemma 2.3.5. In a similar way by the variational principle of Hölder continuous functions $F \in C^\alpha(\widetilde{X}, \mathbb{R})$ (see formula 2.3.5), we can define the pressure of a continuous function $F : \widetilde{X} \to \mathbb{R}$ by

$$P(F) = \sup \left\{ h(\widetilde{\nu}) + \int F \, d\widetilde{\nu} : \widetilde{\nu} \in M(\widetilde{X}, \widetilde{\sigma}) \right\}.$$

The supremum is attained at a unique equilibrium state in $M(\widetilde{X}, \widetilde{\sigma})$. Pressures of the functions f and F also satisfies the identities

$$P(f) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} e^{f^n(x)}, \tag{3.1.4}$$

$$P(F) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{\widetilde{\sigma}^n (x,y) = (x,y)} e^{F^n(x,y)}. \tag{3.1.5}$$

We start our analysis by studying the spectrum and the spectral radius of the transfer operators L_{itr} and L_{-itr}

Lemma 3.1.6. *Let $L_{\pm itr} : C^{\alpha}(X_A^+, \mathbb{C}) \to C^{\alpha}(X_A^+, \mathbb{C})$ and suppose that r is non-lattice. Then,*

(i) for $t = 0$, L_0 has a simple maximal eigenvalue equal to $e^{P(0)}$ and so $\rho(L_0) = e^{P(0)}$. The rest of the spectrum is contained in a disc of radius strictly less than $e^{P(0)}$

(ii) For t in a sufficiently small neighbourhood of $t = 0$, $L_{\pm itr}$ has a simple isolated eigenvalue equal to $e^{P(\pm itr)}$ and the rest of the spectrum is contained in a disc of radius strictly less than $e^{P(0)}$.

(iii) For $t \neq 0$, $\rho(L_{\pm itr}) < e^{P(0)}$.

Proof. Part (i) follows from the RPF theorem and Lemma 2.3.6. Part (ii) follows from perturbation theory, Lemma 2.3.10. For part (iii), we need to use the assumption that r is non-lattice, note that by Lemma 2.3.9 part (i), $\rho(L_{itr}) = e^{P(0)}$ if and only if $\Im(itr) = tr$ is cohomologous to $a + \psi$, where $a \in \mathbb{R}$ and $\psi \in C(X_A^+, 2\pi\mathbb{Z})$. Then for all periodic points $x \in X_A^+$ such that $\sigma^n x = x$, $tr^n(x) - na \in 2\pi\mathbb{Z}$ and so for $t \neq 0$, $r^n(x) - n\frac{a}{t} \in \frac{2\pi}{t}\mathbb{Z}$. This means that r is cohomologous to $c + d\psi$, where $c = a/t$, $d = 2\pi/t$ and ψ is valued in $\mathbb{Z}$, i.e. is not non-lattice. Equivalently, if r is non-lattice, then for $t \neq 0$ $\rho(L_{itr}) < e^{P(0)}$. □

Next, we have the following lemma that relates $P(itR)$ and $P(\pm itr)$.

Lemma 3.1.7. *Suppose that $|t|$ is sufficiently small that $P(itR), P(itr)$ and $P(-itr)$ are defined. Then $P(itR)$ is real valued and*

$$e^{P(itR)} = e^{P(itr)+P(-itr)}.$$

Proof. Using the two identities (3.1.4) and (3.1.5) and the relation $R^n(x,y) = r^n(x) - r^n(y)$, we have that for some $s \in \mathbb{R}$

$$\begin{aligned} e^{P(sR)} &= \lim_{n\to+\infty} \left(\sum_{\tilde{\sigma}^n(x,y)=(x,y)} e^{sR^n(x,y)} \right)^{\frac{1}{n}} \\ &= \lim_{n\to+\infty} \left(\sum_{\sigma^n(x)=x} \sum_{\sigma^n(y)=y} e^{s(r^n(x)-r^n(y))} \right)^{\frac{1}{n}} \\ &= \lim_{n\to+\infty} \left(\sum_{\sigma^n(x)=x} e^{sr^n(x)} \right)^{\frac{1}{n}} \left(\sum_{\sigma^n(y)=y} e^{-sr^n(y)} \right)^{\frac{1}{n}} \\ &= e^{P(sr)+P(-sr)}. \end{aligned}$$

This implies that $e^{P(itR)} = e^{P(itr)+P(-itr)}$ by uniqueness of analytic extension. Now to prove that $P(itR)$ is real notice again from (3.1.5) and using analytic extension, we have

$$P(itR) = \lim_{n\to+\infty} \frac{1}{n} \log \sum_{\tilde{\sigma}^n(x,y)=(x,y)} e^{itR^n(x,y)}.$$

Since $R^n(x,y) = -R^n(y,x)$, we can group terms in the sum $\sum_{\tilde{\sigma}^n(x,y)=(x,y)} e^{itR^n(x,y)}$ as $e^{itR^n(x,y)} + e^{-itR^n(x,y)}$ which is in $\mathbb{R}$, and so $P(itR)$ is real. □

An important property of the function R, that will be useful in our subsequent analysis, is guaranteed by the following lemma.

Lemma 3.1.8. *If r is non-lattice, then R is not cohomologous to a constant.*

Proof. Let $t \neq 0$ be fixed and sufficiently small such that $e^{P(itR)}, e^{P(itr)}, e^{P(-itr)}$ are defined and have absolute value equal to the spectral radii of the corresponding transfer operator. Suppose R is cohomologous to a constant a. Let μ be the measure of maximal entropy for σ and $\widetilde{\mu} = \mu \times \mu$ is the measure of maximal entropy of $\widetilde{\sigma}$. Then

$$a = \int a \, d\widetilde{\mu} = \int R \, d\widetilde{\mu} = \int r \, d\mu - \int r \, d\mu = 0.$$

So itR is cohomologous to 0 and hence $e^{P(itR)} = e^{P(0,\widetilde{\sigma})} = e^{2P(0,\sigma)}$, where $P(0,\widetilde{\sigma})$ is the pressure with respect to the product shift $\widetilde{\sigma}$ and $P(0,\sigma)$ is the pressure with respect to the shift map σ (see Remark 2.3.4). Then with the relation from Lemma 3.1.7 we have that

$$e^{P(itR)} = e^{P(itr)+P(-itr)} = e^{2P(0,\sigma)}. \tag{3.1.6}$$

At the same time we know that $|e^{P(itr)}| \leq e^{P(0,\sigma)}$ and $|e^{P(-itr)}| \leq e^{P(0,\sigma)}$, which implies that

$$|e^{P(itr)+P(-itr)}| \leq e^{2P(0,\sigma)}.$$

This inequality with the equality in (3.1.6) implies that $\rho(L_{\pm itr}) = |e^{P(\pm itr)}| = e^{P(0,\sigma)}$. But if r is non-lattice then for $t \neq 0$, $\rho(L_{\pm itr}) < e^{P(0,\sigma)}$ by Lemma 3.1.6 part(iii) and Lemma 2.3.9 part(ii). This contradiction implies that r is not non-lattice. $\square$

Recall the definitions of the transfer operators L_f and L_F, where $f \in C^\alpha(X_A^+, \mathbb{R})$ and $F \in C^\alpha(\widetilde{X}, \mathbb{R})$, respectively. We are going to use the latter as a tool to study the pressure function $s \mapsto P(sR), s \in \mathbb{R}$, in particular to calculate the first and the second derivative of $P(sR)$ at $s = 0$. We shall use the following lemma later to find the first and second derivative of the pressure function $P(itR)$ at $t = 0$.

Lemma 3.1.9. *The first and second derivatives of the function $s \mapsto P(sR)$ at $s = 0$ are given by:*

(i)

$$\frac{d}{ds}P(sR)\Big|_{s=0} \quad = \quad \int R\, d\widetilde{\mu} = 0,$$

(ii)

$$P''(sR)\Big|_{s=0} \quad = \lim_{n\to+\infty} \frac{1}{n} \int (R^n)^2\, d\widetilde{\mu} > 0.$$

Proof. By the definition of the transfer operator we have

$$L_{sR}w(s)(x,y) = \sum_{\widetilde{\sigma}(x',y')=(x,y)} e^{sR(x',y')} w(s)(x',y').$$

We also have by (2.3.6) and the RPF theorem, that $e^{P(F)}$ is the unique maximal eigenvalue of L_F and hence

$$L_{sR}w(s)(x,y) = e^{P(sR)} w(s)(x,y).$$

Therefore

$$\sum_{\widetilde{\sigma}(x',y')=(x,y)} e^{sR(x',y')} w(s)(x',y') = e^{P(sR)} w(s)(x,y).$$

Differentiating both sides with respect to s we get

$$\begin{aligned}&\sum_{\widetilde{\sigma}(x',y')=(x,y)} e^{sR} \left(Rw(s)(x',y') + w'(s)(x',y') \right) \\ &= e^{P(sR)} \left(P'(sR)w(s)(x,y) + w'(s)(x,y) \right),\end{aligned}$$

and so,

$$L_{sR}\left(Rw(s) + w'(s) \right) = e^{P(sR)} \left(P'(sR)w(s) + w'(s) \right).$$

At $s = 0$ we have

$$L_0 \left(Rw(0) + w'(0) \right) = e^{P(0,\sigma)} \left(P'(sR) \mid_{s=0} w(0) + w'(0) \right).$$

Integrating both sides with respect to $\widetilde{\mu}$, gives

$$\int R\, d\widetilde{\mu} = P'(sR)\Big|_{s=0}.$$

Hence,

$$\begin{aligned}\frac{d}{ds}P(sR)\Big|_{s=0} &= \int R(x,y)d\widetilde{\mu} \\ &= \int r(x)\, d\widetilde{\mu}(x,y) - \int r(y)\, d\widetilde{\mu}(x,y) = 0.\end{aligned}$$

For part(ii) to get the second derivative, consider

$$L^n_{sR}w(s)(x,y) = \sum_{\widetilde{\sigma}(x',y')=(x,y)} e^{sR(x',y')}w(s)(x',y')$$

and

$$L^n_{sR}w(s)(x,y) = e^{nP(sR)}w(s)(x,y).$$

This implies that

$$\sum_{\widetilde{\sigma}(x',y')=(x,y)} e^{sR^n(x',y')}w(s)(x',y') = e^{nP(sR)}w(s)(x,y).$$

Differentiating both sides with respect to s we get

$$\begin{aligned}&\sum_{\widetilde{\sigma}(x',y')=(x,y)} e^{sR^n}\left(R^n w(s)(x',y') + w'(s)(x',y')\right) \\ &= e^{nP(sR)}\left(nP'(sR)w(s)(x,y) + w'(s)(x,y)\right).\end{aligned}$$

Differentiating a second time gives

$$\begin{aligned}&\sum_{\widetilde{\sigma}(x',y')=(x,y)} e^{sR^n}\left((R^n)^2 w(s)(x',y') + 2R^n w'(s)(x',y') + w''(s)(x',y')\right) \\ = \;& e^{nP(sR)}((nP''(sR)w(s)(x,y) + nP'(sR)w'(s)(x,y) \\ + \;& w''(s)(x,y) + (nP'(sR))^2 w(s)(x,y)).\end{aligned}$$

Letting $s = 0$, and using part(i) gives

$$\begin{aligned} L_0^n\left((R^n)^2 + 2R^n w'(0) + w''(0)\right) &= e^{nP(0,\widetilde{\sigma})}\left(nP''(sR)\Big|_{s=0} + 2nP'(sR)\Big|_{s=0} w'(0)\right. \\ &+ \left. w''(0) + \left(nP'(sR)\Big|_{s=0}\right)^2\right) \\ &= e^{nP(0,\widetilde{\sigma})}(nP''(sR)\Big|_{s=0} + w''(0)). \end{aligned}$$

Now, integrating the above equation with respect to $\widetilde{\mu}$, we have

$$\begin{aligned} \int L_0^n\left((R^n)^2 + 2R^n w'(0) + w''(0)\right) d\widetilde{\mu} &= e^{nP(0,\widetilde{\sigma})}\left(\int \left((R^n)^2 + 2R^n w'(0) + w''(0)\right) d\widetilde{\mu}\right) \\ &= e^{nP(0,\widetilde{\sigma})}(nP''(sR)\Big|_{s=0} + \int w''(0)\, d\widetilde{\mu}). \end{aligned}$$

Therefore,

$$P''(sR)\Big|_{s=0} = \frac{1}{n}\int (R^n)^2 d\widetilde{\mu} + 2\int \frac{R^n}{n} w'(0) d\widetilde{\mu}.$$

By Birkhoff's ergodic theorem we have that $\frac{R^n}{n}$ converges to $\int R\, d\widetilde{\mu}$, as $n \to +\infty$, $\widetilde{\mu}$ a.e. By part (i) of this lemma we have that $\int R\, d\widetilde{\mu} = 0$. This implies that $\lim_{n\to+\infty} \frac{R^n}{n} w'(0) = 0$ $\widetilde{\mu}$ a.e. Since $|\frac{R^n}{n} w'(0)| \le \|R\|_\infty \|w'(0)\|_\infty \in L^1$, then by the Dominated Convergence Theorem $\lim_{n\to+\infty} \int \frac{R^n}{n} w'(0) d\widetilde{\mu} = 0$. We can conclude that

$$P''(sR)\Big|_{s=0} = \lim_{n\to+\infty} \frac{1}{n}\int (R^n)^2\, d\widetilde{\mu}.$$

By Lemma 3.1.8 we have that R is not cohomologouse to a constant, hence as was shown by Parry and Pollicott in [17] (Proposition 4.12) this implies that

$$P''(sR)\Big|_{s=0} > 0.$$

□

The following lemma is technical. We shall use it to make a change of coordinates in small neighbourhood of $t = 0$ when writing the Taylor expansion of $e^{P(itR)}$.

Lemma 3.1.10 (Morse Lemma)**.** *Let $f : (-a, a) \to \mathbb{R}$ be a smooth function such that $f'(0) = 0$ and $f''(0) \neq 0$. Then there exists $0 < \epsilon < a$ and a smooth change of coordinates on $(-\epsilon, \epsilon)$, $u \to v(u)$ such that $f(u(v)) = f(0) \pm v^2$, where the sign will depend on that of $f''(0)$.*

The following lemma gives the Taylor expansion of $e^{P(itR)}$ in a neighbourhood of $t = 0$.

Lemma 3.1.11. *There exists $\sigma > 0$ such that the function $t \mapsto e^{P(itR)}$ has a Taylor expansion*

$$e^{P(itR)} = e^{2P(0,\sigma)} \left(1 - \frac{\sigma^2 t^2}{2} + O(|t|^3)\right),$$

where the order term is uniform on any bounded interval. Moreover, there exists a change of coordinates $v = v(t)$ such that for $t \in (-\epsilon, \epsilon)$, we have $e^{P(itR)} = e^{2P(0,\sigma)}(1 - v^2)$.

Proof. Since the function $s \mapsto P(sR)$ is real analytic and has an extension to a neighbourhood of the real line. We can find the Taylor expansion for the function $t \mapsto e^{P(itR)}$ for t sufficiently small. We have

$$\begin{aligned} e^{P(itR)} &= e^{2P(0,\sigma)} + \frac{d}{dt}P(itR)\bigg|_{t=0} e^{2P(0,\sigma)}t + \frac{1}{2}\frac{d^2}{dt^2}P(itR)\bigg|_{t=0} e^{2P(0,\sigma)}t^2 \\ &+ \frac{1}{3!}\frac{d^3}{dt^3}P(itR)|_{t=0}e^{2P(0,\sigma)}t^3 + ... \\ &= e^{2P(0,\sigma)} + \frac{d}{dt}P(itR)\bigg|_{t=0} e^{2P(0,\sigma)}t + \frac{1}{2}\frac{d^2}{dt^2}P(itR)\bigg|_{t=0} e^{2P(0,\sigma)}t^2 + O(|t|^3). \end{aligned}$$

Following the same differentiation we applied to get the first and second derivates of $P(sR)$ at $s = 0$ in Lemma 3.1.9, we deduce that

$$\frac{d}{dt}P(itR)|_{t=0} = i\int R\, d\tilde{\mu} = 0,$$

and

$$\frac{d^2}{dt^2}P(itR)\bigg|_{t=0} = i^2 \lim_{n\to+\infty} \frac{1}{n}\int (R^n)^2\, d\tilde{\mu} < 0.$$

Hence, the Taylor expansion of $e^{P(itR)}$ around $t = 0$ is obtained with $\sigma^2 = -\frac{d^2}{dt^2}P(itR)\Big|_{t=0}$. Let $f(t) = \frac{e^{P(itR)}}{e^{2P(0)}}$, notice that $f(0) = 1$, $f'(0) = \frac{1}{e^{2P(0)}} i \int R\, d\tilde{\mu} = 0$, and $f''(0) < 0$. The function $f(t)$ satisfies the hypothesis of the Morse Lemma, so we can make the required change of coordinates $f(v(t)) = \frac{e^{P(itR)}}{e^{2P(0)}} = 1 - v^2$ □

3.1.4 Pairs of closed geodesics in a fixed interval

We have the following asymptotic result for counting pairs of closed geodesics ordered by word length such that the difference of their lengths lies in an interval $[a, b]$.

Theorem 3.1.1. *Let (G, ℓ) be a non-bipartite metric graph such that for each vertex v, $\deg(v) \geq 3$. Suppose that the non-lattice condition holds. Then there exists $\beta > 1$ and $\sigma > 0$ such that for all $a < b$*

$$\begin{aligned} \pi(N, [a, b]) &= \#\{(\gamma, \gamma') \textit{ closed geodesics} : |\gamma|, |\gamma'| \leq N, \ell(\gamma) - \ell(\gamma') \in [a, b]\} \\ &\sim \frac{(b-a)}{\sqrt{2\pi}\sigma} \frac{\beta^2}{(\beta-1)^2} \frac{\beta^{2N}}{N^{5/2}}, \quad \textit{as } N \to +\infty. \end{aligned}$$

Now we introduce a counting function to help us approach the asymptotic formula we are looking for. We define

$$\phi_N(\chi) = \sum_{n,m=1}^{N} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi(r^n(x) - r^m(y)), \tag{3.1.7}$$

where χ is smooth function. By way of motivation, we should think of χ as an approximation to the indicator function of the interval $[a, b]$ but to carry out the analysis we will need to consider more general χ.

In order to find an asymptotic for the counting function, we shall need the following estimates due to D. Ruelle which connects the transfer operator $L_{\pm itr}$ and the sums over periodic points.

Lemma 3.1.12. *[24] There exists $0 < \theta < 1$ such that for any $x_0 \in X_A^+$*

$$\sum_{\sigma^n x = x} e^{\pm itr^n(x)} = \left(L_{\pm itr}^n 1(x_0)\right)\left(1 + O(\max\{1, |t| n\theta^n\})\right).$$

Consequently, we have the following two estimates by using the spectral properties of the transfer operator $L_{\pm itr}$.

Lemma 3.1.13. *[20] Let $K \subset \mathbb{R}$ be a compact set. There exists $\epsilon > 0, 0 < \theta < 1$ and a constant $C > 0$ such that*

(i) for $t \in (-\epsilon, \epsilon)$,

$$\sum_{\sigma^n x = x} e^{\pm itr^n(x)} = e^{nP(\pm itr)} + O(e^{P(0)n}\theta^{\frac{n}{2}}),$$

(ii) for $t \in K \setminus (-\epsilon, \epsilon)$, $|\sum_{\sigma^n x = x} e^{\pm itr^n(x)}| \leq Ce^{P(0)n}\theta^{\frac{n}{2}}$.

Proof. The two estimates can be obtained by combining Lemma 3.1.12 and Lemma 3.1.6 and setting $\theta^{1/2}e^{P(0)} = \sup_{t \in I} \rho(L_{\pm itr})$, where $I = (-\epsilon, \epsilon)$ or $I = K \setminus (-\epsilon, \epsilon)$. □

We also have the following useful relation when we sum over different periods of periodic points

Lemma 3.1.14.

$$\sum_{n,m=1}^{N} e^{nP(itr)} e^{mP(-itr)} = \frac{e^{(N+1)P(itR)}}{(e^{P(itr)} - 1)(e^{P(-itr)} - 1)}(1 + O(\delta^N)),$$

for some $0 < \delta < 1$.

Proof. We use the formula for the sum of a geometric progression to get

$$\begin{aligned}
\sum_{n,m=1}^{N} e^{nP(itr)mP(-itr)} &= \sum_{n=1}^{N}\sum_{m=1}^{N} e^{nP(itr)mP(-itr)} \\
&= \left(\frac{e^{(N+1)P(itr)}-1}{e^{P(itr)}-1}\right)\left(\frac{e^{(N+1)P(-itr)}-1}{e^{P(-itr)}-1}\right) \\
&= \frac{e^{(N+1)[P(itr)+P(-itr)]}}{(e^{P(itr)}-1)(e^{P(-itr)}-1)}(1+O(\delta^N)) \\
&= \frac{e^{(N+1)P(itR)}}{(e^{P(itr)}-1)(e^{P(-itr)}-1)}(1+O(\delta^N)),
\end{aligned}$$

for some $0 < \delta < 1$. □

We shall require that the Fourier transform of χ is compactly supported and such that for $|t| < \epsilon$ we have $\widehat{\chi}(t) = \widehat{\chi}(0) + O(|t|)$. In particular, we are going to assume that the support is contained in $[-M, M]$, for some $M > 0$. By the Fourier inversion formula we can write

$$\phi_N(\chi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n,m=1}^{N}\sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} e^{it(r^n(x)-r^m(y))}\widehat{\chi}(t)\,dt.$$

Since the support of $\widehat{\chi}(t)$ in contained in an interval $[-M, M]$, we can split the integral into two integrals. The first is on the interval $(-\epsilon, \epsilon)$, where we can use Lemma 3.1.13(i) to obtain a bound on the sum. The second integral is on the intervals $[-M, -\epsilon]$ and $[\epsilon, M]$ and here we can use Lemma 3.1.13 (ii) to bound the sum . Since $\widehat{\chi}$ is compactly supported, $\|\widehat{\chi}(t)\|_\infty = \sup_{t\in\mathbb{R}} |\widehat{\chi}(t)|$ is finite. So we get the following:

$$
\begin{aligned}
\phi_N(\chi) &= \frac{1}{2\pi}\int_{-M}^{M}\sum_{n,m=1}^{N}\sum_{\substack{\sigma^n x=x\\ \sigma^m y=y}} e^{it(r^n(x)-r^m(y))}\widehat{\chi}(t)\,dt \\
&= \frac{1}{2\pi}\int_{-M}^{M}\sum_{n,m=1}^{N}\sum_{\sigma^n x=x} e^{itr^n(x)} \sum_{\sigma^m y=y} e^{-itr^m(y)}\widehat{\chi}(t)\,dt \\
&= \frac{1}{2\pi}\int_{|t|<\epsilon}\left(\sum_{n,m=1}^{N}(e^{nP(itr)}+O(e^{P(0)n}\theta^{\frac{n}{2}}))(e^{mP(itr)}\right. \\
&\qquad \left. +O(e^{P(0)m}\theta^{\frac{m}{2}}))\right)\widehat{\chi}(t)\,dt \\
&\quad +\frac{2}{2\pi}\int_{\epsilon\le|t|\le M}\left(\sum_{n,m=1}^{N}O(e^{P(0)n}\theta^{\frac{n}{2}})O(e^{P(0)m}\theta^{\frac{m}{2}})\right)\widehat{\chi}(t)\,dt \\
&= \frac{1}{2\pi}\int_{|t|<\epsilon}\left(\sum_{n,m=1}^{N}e^{nP(itr)}e^{mP(itr)}(1+O(\theta^{n+m}))\right)\widehat{\chi}(t)\,dt \\
&\qquad +O(e^{2P(0)N}\theta^{N}) \\
&= \frac{1}{2\pi}\int_{|t|<\epsilon}\left(\frac{e^{(N+1)P(itR)}}{(e^{P(itr)}-1)(e^{P(-itr)}-1)}\right)\widehat{\chi}(t)\,dt \\
&\qquad +O(e^{2P(0)N}\max\{\theta^N,\delta^N\}),
\end{aligned}
$$

where in the last equation we used Lemma 3.1.14. Now we use some estimates around $t=0$ for the factors appearing in the last equation above. The estimates we are going to use are the following:

(1) $\widehat{\chi}(t)=\widehat{\chi}(0)+O(|t|)=\widehat{\chi}(0)(1+O(|t|))$,

(2) Lemma 3.1.11 to estimate $e^{P(itR)}$,

(3) $dt=\dfrac{1}{v'(t)}dv=\dfrac{1}{v'(0)}(1+O(|t|))dv$, where $v'(0)=\dfrac{\sigma}{\sqrt{2}}$,

(4) $\dfrac{1}{(e^{P(itr)}-1)(e^{P(-itr)}-1)}=\dfrac{1}{(e^{P(0)}-1)^2}+O(|t|)$.

Hence we can write $\phi_N(\chi)$ as follows,

$$\begin{aligned}
\phi_N(\chi) &= \frac{1}{2\pi}\int_{|t|<\epsilon}(e^{2P(0)}(1-v^2))^{N+1}\left(\frac{1}{(e^{P(0)}-1)^2}+O(|v|)\right) \\
&\quad \times\widehat{\chi}(0)(1+O(|v|))\frac{1}{v'(0)}(1+O(|v|))dv + O(e^{2P(0)N}\max\{\theta^N,\delta^N\}) \\
&= \frac{\widehat{\chi}(0)\sqrt{2}}{2\pi\sigma}\frac{e^{2P(0)(N+1)}}{(e^{P(0)}-1)^2}\int_{-\epsilon}^{\epsilon}(1-v^2)^{N+1}(1+R(v))dv \\
&\qquad + O(e^{2P(0)N}\max\{\theta^N,\delta^N\}) \\
&= \frac{\widehat{\chi}(0)\sqrt{2}}{2\pi\sigma}\frac{e^{2P(0)(N+1)}}{(e^{P(0)}-1)^2}\left(\int_{-\epsilon}^{\epsilon}(1-v^2)^{N+1}dv + \int_{-\epsilon}^{\epsilon}(1-v^2)^{N+1}R(v))dv\right) \\
&\qquad + O(e^{2e^{P(0)}N}\max\{\theta^N,\delta^N\}), \qquad (3.1.8)
\end{aligned}$$

where $R(v)$ is a smooth function with $R(0)=0$ which contains all terms of the form $O(|t|)$. Using the simple substitution, $w=v^2$ the leading term in (3.1.8) can be written as

$$\begin{aligned}
\int_{-\epsilon}^{\epsilon}(1-v^2)^{N+1}dv &= 2\int_0^{\epsilon}(1-v^2)^{N+1}dv \\
&= \int_0^{\epsilon^2}(1-w)^{N+1}w^{-1/2}dw \\
&= \int_0^1(1-w)^{N+1}w^{-1/2}dw - \int_{\epsilon^2}^1(1-w)^{N+1}w^{-1/2}dw \\
&= \frac{\Gamma(N+2)\sqrt{\pi}}{\Gamma(N+2+1/2)} + O((1-\epsilon^2)^N). \qquad (3.1.9)
\end{aligned}$$

The second part in the last line follows from a direct estimate of the integral above it. Whereas, the first part follows by using a relationship between two special functions known as the gamma function and the beta function. We recall their definitions here and how they are related. The gamma function has two definitions. The first definition of the gamma function is given by an infinite limit named after Euler defined as

$$\Gamma(z) = \lim_{n\to+\infty}\frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)}n^z, \qquad z\neq 0,-1,-2,-3,\ldots$$

Using this definition we have that $\Gamma(z+1)=z\Gamma(z)$ and that $\Gamma(1)=1$. We can then calculate $\Gamma(2) = 1$, $\Gamma(3) = 2\Gamma(2) = 2, \ldots$ and so we have

$$\Gamma(n) = 1 \cdot 2 \cdot 3 \cdots (n-1) = (n-1)!. \qquad (3.1.10)$$

The second definition of the gamma function, also called the Euler integral, is

$$\Gamma(z) = \int_0^{+\infty} e^{-x} x^{z-1}\, dx, \qquad \Re(z) > 0.$$

By this definition we can calculate $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. On the other hand, the beta function is defined by

$$B(m+1, n+1) = 2\int_0^{\frac{\pi}{2}} \cos^{2m+1}\theta \sin^{2m+1}\theta\, d\theta = \frac{m!n!}{(m+n+1)!}. \qquad (3.1.11)$$

Hence, equivalenty we can see from the identity in (3.1.10) that

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Combining this with the substitution of $x = \cos^2\theta$ into (3.1.11), result in the following relation between the beta function and the gamma function (see [28], p.236).

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

We apply this identity to obtain the first part in (3.1.9), where we let $n = N+2$, $m = 1/2$ and used $\Gamma(1/2) = \sqrt{\pi}$.

We are going to use the following Stirling formula, which gives an asymptotic for the Gamma function, in our subsequent calculations.

$$\Gamma(x) \sim \frac{x^x}{\sqrt{x}} e^{-x}\sqrt{2\pi}, \quad \text{as } x \to +\infty. \qquad (3.1.12)$$

We can use this to approximate the Gamma function in (3.1.9). Notice that

$$\lim_{N\to+\infty} \frac{\Gamma(N+2)\sqrt{N}}{\Gamma(N+2+1/2)} = \lim_{N\to+\infty} \sqrt{e}\frac{\sqrt{N}}{\sqrt{N+2}}\left(\frac{N+2}{N+2+1/2}\right)^{N+2} = 1,$$

since $\lim_{N\to+\infty}\left(\frac{N+2}{N+2+1/2}\right)^{N+2} = \frac{1}{\sqrt{e}}$. Therefore, we have that

$$\frac{\Gamma(N+2)}{\Gamma(N+2+1/2)} \sim \frac{1}{\sqrt{N}}. \tag{3.1.13}$$

For the second part of the integral in (3.1.8), since $R(v) = O(|t|)$ (i.e. $|R(v)| \leq C_0|v(t)|$) by simple substitution and integration we have the following estimate,

$$\begin{aligned} \left|\int_{-\epsilon}^{\epsilon}(1-v^2)^{N+1}R(v))dv\right| &\leq \int_{-\epsilon}^{\epsilon}|(1-v^2)^{N+1}||v|dv \\ &\leq C_0\left(\frac{(1-\epsilon^2)^{N+2}}{N+2} - \frac{1}{N+2}\right) \\ &= O\left(\frac{1}{N}\right) \end{aligned} \tag{3.1.14}$$

So now combining the estimates in (3.1.9),(3.1.13) and (3.1.14) with (3.1.8) we get the following asymptotic for $\phi_N(\chi)$, where $\widehat{\chi}(0) = \int \chi(x)\,dx$,

$$\phi_N(\chi) \sim \frac{\int \chi(x)\,dx}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\frac{e^{2P(0)(N+1)}}{\sqrt{N}}, \quad \text{as } N \to +\infty. \tag{3.1.15}$$

By the following standard result from probability theory, we can conclude that the asymptotic in (3.1.15) will still be true with $\chi : \mathbb{R} \to \mathbb{R}$ continuous, non-negative function with compact support.

Lemma 3.1.15. *([1], p.218) Let μ_n and μ be locally finite Borel measures on $\mathbb{R}$. Suppose there exists a function f_0 which is strictly positive such that $\hat{f}_0$ is compactly supported, and*

$$\int f\,d\mu_n \to \int f\,d\mu$$

for all f of the form $f(y) = e^{ivy}f_0(y)$, $v \in \mathbb{R}$. Then $\mu_n \to \mu$ weak (i.e. $\int g\,d\mu_n \to \int g\,d\mu$ for all continuous compactly supported $g : \mathbb{R} \to \mathbb{R}$).*

Lemma 3.1.16. *Let $\chi : \mathbb{R} \to \mathbb{R}$ be continuous, non-negative function with compact support. Then*

$$\phi_N(\chi) \sim \frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\int \chi(x)dx\frac{e^{2P(0)N}}{\sqrt{N}}, \quad \textit{as } N \to +\infty. \tag{3.1.16}$$

Proof. Equation (3.1.16) is equivalent to

$$\phi_N(\chi)\frac{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}{e^{2P(0)}}\frac{\sqrt{N}}{e^{2P(0)N}} \to \int \chi(x)dx, \quad \text{as } N \to +\infty.$$

Define a sequence of locally finite Borel measures m_N on $\mathbb{R}$ by

$$\int_{\mathbb{R}} \chi \, dm_N := \phi_N(\chi)\frac{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}{e^{2P(0)}}\frac{\sqrt{N}}{e^{2P(0)N}}.$$

By equation (3.1.15) we have that for $\chi : \mathbb{R} \to \mathbb{C}$ such that its Fourier transform $\hat{\chi}$ is compactly supported and $\widehat{\chi}(t) = \widehat{\chi}(0) + O(|t|)$,

$$\int_{\mathbb{R}} \chi(x) \, dm_N \to \int \chi(x)dx, \quad \text{as } N \to +\infty. \tag{3.1.17}$$

In particular, (3.1.17) holds for all functions of the form $\chi(x) = e^{ivx}\chi_0(x)$, where $v \in \mathbb{R}$, $\chi_0(x) > 0$. We can take $\chi_0(x) = \frac{\sin^2 x}{x^2} + \frac{\sin^2(\sqrt{2}x)}{2x^2}$. Hence, the result follows immediately by Lemma 3.1.15. □

3.1.5 Proof of Theorem 3.1.1

Here we prove a proposition which will lead to the proof of Theorem 3.1.1. First, we define a new counting function with $\chi : \mathbb{R} \to \mathbb{R}$ a continuous compactly supported non-negative function,

$$\omega_N(\chi) = \sum_{\substack{(\gamma,\gamma') \\ |\gamma|,|\gamma'| \le N}} \chi(\ell(\gamma) - \ell(\gamma')),$$

where γ, and γ' are prime closed geodesics.

Proposition 3.1.2. *We have that*

$$\omega_N(\chi) \sim \frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\int \chi(x)dx\frac{e^{2P(0)N}}{N^{5/2}}, \quad \textit{as } N \to +\infty$$

Proof. Note that the counting function $\omega_N(\chi)$ is for counting pairs of closed geodesics. To derive an asymptotic for $\omega_N(\chi)$ we are going to use Lemma 3.1.16, which describes an asymptotic for the counting function $\phi_N(\chi)$. As we have seen, the counting function $\phi_N(\chi)$ is defined for counting pairs of periodic points. Therefore, to use Lemma 3.1.16 we need to describe $\omega_N(\chi)$ in terms of periodic points. However, we know that there is an exact correspondence between periodic orbits and closed geodesics described by Lemma 3.1.2. We also know that every point in a prime periodic orbit $\{x, \sigma x, ..., \sigma^{n-1}x\}$ is a periodic point, and so corresponds to n terms in the sum $\sum_{\sigma^n x = x}$. Consequently, $\omega_N(\chi)$ can be written in the following form considering also that we are interested in counting prime pairs of closed geodesics.

$$\begin{aligned}\omega_N(\chi) = \quad & \sum_{n,m=1}^{N} \frac{1}{nm} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi(r^n(x) - r^m(y)) \\ & - \sum_{n,m=1}^{N} \frac{1}{nm} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y \\ \text{not both prime}}} \chi(r^n(x) - r^m(y)).\end{aligned}$$

We can estimate the second part of $\omega_N(\chi)$ using the fact that

$$\frac{1}{n} \sum_{\substack{\sigma^n x = x \\ x \text{ not prime}}} 1 = O(e^{nP(0)/2}),$$

and letting

$$\Theta_N(\chi) = \sum_{n,m=1}^{N} \frac{1}{nm} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi(r^n(x) - r^m(x)),$$

we can write $\omega_N(\chi)$ as

$$\omega_N(\chi) = \Theta_N(\chi) + O(e^{3NP(0)/2}).$$

So, we only need to find an asymptotic for $\Theta_N(\chi)$. We shall prove that

$$\lim_{N \to +\infty} \frac{N^{5/2}}{e^{2P(0)N}} \Theta_N(\chi) = \frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)} - 1)^2} \int \chi(x) dx. \tag{3.1.18}$$

We have that $\Theta_N(\chi) \geq \frac{1}{N^2}\phi_N(\chi)$, and by Lemma 3.1.16 we get,

$$\begin{aligned}
\liminf_{N\to+\infty} \frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(\chi) &\geq \liminf_{N\to+\infty} \frac{\sqrt{N}}{e^{2P(0)N}}\phi_N(\chi) \\
&= \frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\int \chi(x)\,dx. \qquad (3.1.19)
\end{aligned}$$

Considering the terms of large period and estimating the rest, we can write $\Theta_N(\chi)$ as

$$\begin{aligned}
\Theta_N(\chi) &= \sum_{n,m=[\alpha N]+1}^{N} \frac{1}{nm} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi(r^n(x) - r^m(x)) + O(N^2 e^{(1+\alpha)P(0)N}) \\
&\leq \frac{1}{(\alpha N)^2}\phi_N(\chi) + O(N^2 e^{(1+\alpha)P(0)N}),
\end{aligned}$$

where $\frac{1}{2} < \alpha < 1$. Hence,

$$\begin{aligned}
\limsup_{N\to+\infty} \frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(\chi) &\leq \frac{1}{\alpha^2}\limsup_{N\to+\infty} \frac{\sqrt{N}}{e^{2P(0)N}}\phi_N(\chi) \\
&\quad + \limsup_{N\to+\infty} \frac{N^{5/2}}{e^{2P(0)N}} O(N^2 e^{(1+\alpha)P(0)N}) \\
&= \frac{1}{\alpha^2}\frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\int \chi(x)dx,
\end{aligned}$$

where the second limit vanishes as $N \to +\infty$ since $\frac{1}{2} < \alpha < 1$ and for the first limit we used Lemma 3.1.16. Now, since we can take α arbitrarily close to 1, we have

$$\limsup_{N\to+\infty} \frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(\chi) \leq \frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\int \chi(x)\,dx \qquad (3.1.20)$$

Finally, combining (3.1.19) and (3.1.20) equation (3.1.18) follows and therefore the result holds. □

Now we can give the proof of Theorem 3.1.1.

Proof. First we approximate the indicator function $\chi_{[a,b]}$ by compactly supported continuous functions. Given $\epsilon > 0$, we can choose compactly supported continuous functions $f_1 \le \chi_{[a,b]} \le f_2$, and $\int f_2(x) - f_1(x)\,dx \le \epsilon$ such that

$$\int \chi_{[a,b]}(x)\,dx - \epsilon \le \int f_1(x)dx \le \int f_2(x)\,dx \le \int \chi_{[a,b]}(x)\,dx + \epsilon. \quad (3.1.21)$$

Applying Proposition 3.1.2 to f_1, and f_2 we have

$$\lim_{N\to+\infty} \frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(f_1) = \frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\int f_1(x)dx$$

and

$$\lim_{N\to+\infty} \frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(f_2) = \frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\int f_2(x)\,dx.$$

Hence, we have that

$$\frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(f_1) \le \frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(\chi_{[a,b]}) \le \frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(f_2).$$

Now as $N \to +\infty$, and using (3.1.21), we have the following

$$\begin{aligned}
\limsup_{N\to+\infty} \frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(\chi_{[a,b]})(x) &\le \frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\int f_2(x)\,dx \\
&< \frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\left(\int \chi_{[a,b]}(x)\,dx + \epsilon\right)
\end{aligned}$$

and

$$\begin{aligned}
\liminf_{N\to+\infty} \frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(\chi_{[a,b]}) &\ge \frac{e^{2h_0}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\int f_1(x)\,dx \\
&> \frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\left(\int \chi_{[a,b]}(x)\,dx - \epsilon\right).
\end{aligned}$$

Since we can take ϵ arbitrarily small, then we get

$$\begin{aligned}
\limsup_{N\to+\infty} \frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(\chi_{[a,b]}) &= \liminf_{N\to+\infty} \frac{N^{5/2}}{e^{2P(0)N}}\Theta_N(\chi_{[a,b]}) \\
&= \frac{e^{2P(0)}}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\int \chi_{[a,b]}(x)\,dx \\
&= \frac{b-a}{\sqrt{2\pi}\sigma}\frac{e^{2P(0)}}{(e^{P(0)}-1)^2}
\end{aligned}$$

and that completes the proof of the theorem where we let $\beta = e^{P(0)}$. $\square$

3.1.6 Pairs of closed geodesics in shrinking intervals

In this section, we are interested in counting pairs of closed geodesics in metric graphs ordered by word length and such that the difference between their geometric lengths fall in an interval that can be positioned arbitrarily in the real line and shrinks as the word length tends to infinity. Consider a sequence of intervals of the form $I_N(z) := [a\epsilon_N + z, b\epsilon_N + z]$, for some choice of $z \in \mathbb{R}$, where $\epsilon_N > 0$ tend to zero. We define the set of pairs with these properties as follows,

$$\pi(N, I_N(z)) = \#\{(\gamma, \gamma') : |\gamma|, |\gamma'| \leq N, \ell(\gamma) - \ell(\gamma') \in [a\epsilon_N + z, b\epsilon_N + z]\}.$$

A number theoretic property between the lengths of two closed geodesics will be one of the hypothesis of our main theorem in this section. This property is explained in the following definition.

Definition 3.1.2. *We say that $\alpha \in \mathbb{R}$ is Diophantine if there exist constants $c > 0$ and $\nu > 2$ such that*

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^\nu}$$

for all $\frac{p}{q} \in \mathbb{Q}$.

The following theorem, which we shall prove here, estimates $\pi(N, I_N(z))$.

Theorem 3.1.3. *Let (G, ℓ) be a non-bipartite metric graph such that for each vertex v, $\deg(v) \geq 3$. Suppose that the non-lattice condition holds and that there exists closed geodesics γ, γ' in G such that $\ell(\gamma)/\ell(\gamma')$ is Diophantine. Then there exists $\sigma > 0$ and $r > 0$ such that for any $a < b$ and a sequence $\epsilon_N > 0$ which tends to zero slower than the polynomial rate N^{-r},i.e., $\epsilon_N^{-1} = O(N^r)$, we have that*

$$\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} \left| \frac{\sigma N^{5/2}}{\epsilon_N e^{2P(0)N}} \pi(N, I_N(z)) - \frac{(b-a)e^{2P(0)}}{\sqrt{2\pi}(e^{P(0)} - 1)^2} e^{-z^2/2\sigma^2 N} \right| = 0.$$

In particular,

$$\pi(N, I_N(z)) \sim \frac{(b-a)e^{2P(0)}\epsilon_N}{\sqrt{2\pi}\sigma(e^{P(0)}-1)^2}\frac{e^{2P(0)N}}{N^{5/2}}, \; as\, N \to +\infty.$$

3.1.7 Proof of Threorem 3.1.3

Recall the the counting function

$$\phi_N(\chi) = \sum_{n,m=1}^{N} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi(r^n(x) - r^m(y))$$

where now we will take $\chi : \mathbb{R} \to \mathbb{R}$ to be a compactly supported C^k function (χ will be used to approximation the indicator function of the interval $[a, b]$). Additionally, we define the function

$$\mathcal{E}_N(u) = \sum_{n,m=1}^{N} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} e^{iu(r^n(x) - r^m(y))}.$$

We shall use these functions here, but in a rescaled manner. Since we are interested in obtaining a result for shrinking intervals $[\epsilon_N a + z, \epsilon_N b + z]$, we need to consider a sequence of rescaled functions $\chi_N^{(z)}(x) = \chi(\frac{x-z}{\epsilon_N})$. Using the Fourier transform formula we can write,

$$\widehat{\chi}_N^z(u) = e^{izu}\epsilon_N \widehat{\chi}(\epsilon_N u) \tag{3.1.22}$$

Then define the function

$$\phi_N(\chi_N^{(z)}) = \sum_{n,m=1}^{N} \sum_{\substack{\sigma^n x = x \\ \sigma^m y = y}} \chi_N^{(z)}(r^n(x) - r^m(y)).$$

We are going to prove Theorem 3.1.3 in three steps. First we define the quantity $A(N, z)$ in terms of the counting function $\phi_N(\chi_N^{(z)})$ (note here that we are counting all pairs of periodic points not only the prime ones):

$$A(N,z) := \left| \frac{\sigma\sqrt{N}}{\epsilon_N e^{2P(0)N}} \phi_N(\chi_N^{(z)}) - \frac{e^{2P(0)} \int \chi(x)dx}{(e^{P(0)}-1)^2} e^{-z^2/2\sigma^2 N} \right|.$$

In the second step we replace this counting function by another one which counts only prime periodic orbits. Therefore by the exact correspondence between closed orbits in the shift space and closed geodesic in the metric graph, described by Lemma 3.1.2, this new function counts prime closed geodesics. Then in the last step we replace the smooth function χ by the indicator function $\chi_{[a,b]}$.

Step 1: The first step to prove Theorem 3.1.3 is to prove the following lemma.

Lemma 3.1.17.

$$\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} A(N,z) = 0.$$

In order to prove this lemma first note that using Fourier inversion and (3.1.22) we can write the following

$$\begin{aligned}
\frac{\sigma\sqrt{N}}{\epsilon_N e^{2P(0)N}} \phi(\chi_N^{(z)}) &= \frac{1}{2\pi} \frac{\sigma\sqrt{N}}{\epsilon_N e^{2P(0)N}} \int_{-\infty}^{+\infty} \mathcal{E}_N(u) \widehat{\chi}_N^{(z)}(u)\, du \\
&= \frac{1}{2\pi} \frac{\sigma\sqrt{N}}{e^{2P(0)N}} \int_{-\infty}^{+\infty} \mathcal{E}_N(u) e^{izu} \widehat{\chi}(\epsilon_N u)\, du \\
&= \frac{e^{-2P(0)N}}{2\pi} \int_{-\infty}^{+\infty} \mathcal{E}_N\left(\frac{t}{\sigma\sqrt{N}}\right) e^{izt/\sigma\sqrt{N}} \widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right)\, dt,
\end{aligned}$$

where we obtained the last line above by substituting $t = u\sigma\sqrt{N}$. Combining this with the standard identity

$$e^{-z^2/2\sigma^2 N} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iuz/\sigma\sqrt{N}} e^{-u^2/2} du,$$

gives a new expression for $A(N,z)$

$$2\pi A(N,z) = \left| \int_{-\infty}^{+\infty} e^{izt/\sigma\sqrt{N}} \left\{ e^{-2h_0 N} \mathcal{E}_N \left(\frac{t}{\sigma\sqrt{N}} \right) \widehat{\chi} \left(\epsilon_N \frac{t}{\sigma\sqrt{N}} \right) \right.\right.$$
$$\left.\left. - \frac{e^{2h_0} \int \chi(x)dx}{(e^{P(0)}-1)^2} e^{-t^2/2} \right\} \right| .$$

Then we can split and bound $A(N,z)$ in the following form

$$2\pi A(N,z) \leq A_1(N,z) + A_2(N,z) + A_3(N,z),$$

where, given $\delta > 0$,

$$\begin{aligned} A_1(N,z) &= \left| \int_{-\delta\sigma\sqrt{N}}^{\delta\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \{ \, e^{-2h_0 N} \mathcal{E}_N \left(\frac{t}{\sigma\sqrt{N}} \right) \widehat{\chi} \left(\epsilon_N \frac{t}{\sigma\sqrt{N}} \right) \right. \\ &\quad \left. - \frac{e^{2P(0)} \int \chi(x)dx}{(e^{P(0)}-1)^2} e^{-t^2/2} \, \} \, dt \right|, \\ A_2(N,z) &= \left| \int_{|t|\geq\delta\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \left\{ e^{-2P(0)N} \mathcal{E}_N \left(\frac{t}{\sigma\sqrt{N}} \right) \widehat{\chi} \left(\epsilon_N \frac{t}{\sigma\sqrt{N}} \right) \right\} dt \right|, \\ A_3(N,z) &= \left| \int_{|t|\geq\delta\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \left\{ \frac{e^{2P(0)} \int \chi(x)dx}{(e^{P(0)}-1)^2} e^{-t^2/2} \right\} dt \right| . \end{aligned}$$

If we show that $\sup_{z\in\mathbb{R}} A_i(N,z) \to 0$ as $N \to +\infty$, , $i = 1,2,3$, then Lemma 3.1.17 follows.

First note that for $A_3(N,z)$ we can write

$$A_3(N,z) \leq \int_{|t|\geq\delta\sigma\sqrt{N}} \frac{e^{2P(0)} \int \chi(x)dx}{(e^{P(0)}-1)^2} e^{-t^2/2} dt = \frac{2e^{2P(0)} \int \chi(x)dx}{(e^{P(0)}-1)^2} \int_{\delta\sigma\sqrt{N}}^{+\infty} e^{-t^2/2}.$$

By simply noting that $e^{-t^2/2} \leq e^{-t}$ for $t > 2$, then for sufficiently large N, we have

$$\int_{\delta\sigma\sqrt{N}}^{+\infty} e^{-t^2/2} dt \leq \int_{\delta\sigma\sqrt{N}}^{+\infty} e^{-t} dt \to 0.$$

This implies that $\sup_{z\in\mathbb{R}} A_3(N,z) \to 0$ as $N \to +\infty$.

For $A_1(N,z)$, we use Lemma 3.1.14. Hence, for $0<\rho<1$ we get

$$\begin{aligned} A_1(N,z) &= \Bigg| \int_{-\delta\sigma\sqrt{N}}^{\delta\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \Bigg\{ e^{-2P(0)N} \frac{e^{(N+1)P(itR/\sigma\sqrt{N})}}{(e^{P(itr/\sigma\sqrt{N})}-1)(e^{P(-itr/\sigma\sqrt{N})}-1)} \\ &\qquad (1+O(\rho^N))\, \widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right) - \frac{e^{2P(0)}\int \chi(x)dx}{(e^{P(0)}-1)^2} e^{-t^2/2} \Bigg\} dt \Bigg|. \end{aligned}$$

We are going to use the Dominated Convergence Theorem (DCT) to show that $A_1(N,z)$ converges to zero uniformly in z. First, we note that in the domain of integration $[-\delta\sigma\sqrt{N}, \delta\sigma\sqrt{N}]$ the integrand converges to zero as $N\to+\infty$ since:

(i) $\widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right) \to \widehat{\chi}(0) = \int \chi(x)dx$;

(ii) $\dfrac{e^{P(itR/\sigma\sqrt{N})}}{(e^{P(itr/\sigma\sqrt{N})}-1)(e^{P(-itr/\sigma\sqrt{N})}-1)} \to \dfrac{e^{2P(0)}}{(e^{P(0)}-1)^2}$;

(iii) $e^{N(P(itR/\sigma\sqrt{N})-2h_0)} \to e^{-t^2/2}$. This follows from the Taylor series expansion $P(itR/\sigma\sqrt{N}) = 2P(0) - t^2/2N + O(|t/\sqrt{N}|^3)$.

Second, using Lemma 3.1.11, we can get the bound $e^{N(P(itR/\sigma\sqrt{N})-2P(0))} \le e^{-t^2/4}$ then since $e^{-t^2/2} \le e^{t^2/4}$ we also have the bound

$$|e^{N(P(itR/\sigma\sqrt{N})-2P(0))} - e^{-t^2/2}| \le e^{-t^2/4}.$$

Hence the DCT can be applied here to conclude that $\lim_{N\to+\infty}\sup_{z\in\mathbb{R}} A_1(N,z) = 0$.

To complete the proof of Lemma 3.1.17 we need to show that $\sup_{z\in\mathbb{R}} A_2(N,z) \to 0$ as $N\to\infty$. For that purpose we need the following lemma due to Dolgopyat which estimates the iterates of the transfer operator.

Lemma 3.1.18. *[3] There exists $\gamma>0$, $D>0$, $C>0$ and $c>0$ such that, for $|t|\ge a$ (for some $a>0$), we have*

$$\|L_{itr}^{2Km}1\|_\infty \le Ce^{2Kmh_0}\left(1-\frac{c}{|t|^\gamma}\right)^m,$$

where $K=[D\log|t|]$.

First using Lemma 3.1.12 we can bound $\mathcal{E}_N\left(t/\sigma\sqrt{N}\right)$ for t large in the following way:

$$\begin{aligned}
\left|\mathcal{E}_N\left(t/\sigma\sqrt{N}\right)\right| &= \left|\sum_{n,m=1}^{N}\sum_{\substack{\sigma^n x=x\\ \sigma^m y=y}} e^{(it(r^n(x)-r^m(x))/\sigma\sqrt{N}}\right| \\
&\leq \sum_{n,m=1}^{N}\left|\sum_{\sigma^n x=x} e^{itr^n(x)/\sigma\sqrt{N}}\sum_{\sigma^m y=y} e^{-itr^m(y)/\sigma\sqrt{N}}\right| \\
&= \sum_{n,m=1}^{N}\left|L^n_{itr/\sigma\sqrt{N}}1(x_0)L^m_{-itr/\sigma\sqrt{N}}1(y_0)\right| \\
&\quad \left(1+O(\max\{1,|t/\sigma\sqrt{N}|n\theta^n\})\right)\left(1+O(\max\{1,|t/\sigma\sqrt{N}|m\theta^m\})\right) \\
&\leq \sum_{n,m=1}^{N}\left|L^n_{itr/\sigma\sqrt{N}}1(x_0)L^m_{-itr/\sigma\sqrt{N}}1(y_0)\right|C_0|t| \qquad (3.1.23) \\
&\leq C_1\sum_{n,m=1}^{N}\|L^n_{itr/\sigma\sqrt{N}}1\|_\infty\|L^m_{-itr/\sigma\sqrt{N}}1\|_\infty|t|.
\end{aligned}$$

We obtained (3.1.23) by taking the maximum that both m and n can take, i.e. when $m=n=N$, then we consider only the dominating factors as N gets very large.

Now in order to use Lemma 3.1.18 to get a bound for the uniform norm of the iterates of the transfer operator, we shall write the iterates m and n as $m=2Km_1+l_1$ and $n=2Km_2+l_2$, where m_1,m_2 are integers, $0\leq l_1,l_2\leq 2K-1$ and $K=[D\log|t|]$. Therefore, we get that

$$\begin{aligned}
\left|\mathcal{E}_N\left(t/\sigma\sqrt{N}\right)\right| &\leq C_1\sum_{n,m=1}^{N}\|L^{2Km_1+l_1}_{itr/\sigma\sqrt{N}}1\|_\infty\|L^{2Km_2+l_2}_{-itr/\sigma\sqrt{N}}1\|_\infty|t| \\
&= C_1\sum_{n,m=1}^{N}\|L^{l_1}_{itr/\sigma\sqrt{N}}(L^{2Km_1}_{itr/\sigma\sqrt{N}}1)\|_\infty\|L^{l_2}_{-itr/\sigma\sqrt{N}}(L^{2Km_2}_{-itr/\sigma\sqrt{N}}1)\|_\infty|t| \\
&\leq C_1\sum_{n,m=1}^{N}\|L^{l_1}_{itr/\sigma\sqrt{N}}\|\|\|L^{l_2}_{-itr/\sigma\sqrt{N}}\|\|\|L^{2Km_1}_{itr/\sigma\sqrt{N}}1\|_\infty\|L^{2Km_2}_{-itr/\sigma\sqrt{N}}1\|_\infty|t| \\
&\leq C_2\sum_{n,m=1}^{N}|t|^{1+4D\log\|L_0\|}e^{2[D\log t](m_1+m_2)h_0}\left(1-\frac{c}{|\frac{t}{\sigma\sqrt{N}}|^\gamma}\right)^{m_1+m_2} \qquad (3.1.24)
\end{aligned}$$

The calculations in (3.1.24) has been done as follows: for $L^{l_j}_{itr/\sigma\sqrt{N}}$, $0 \le l_j \le 2[D\log t] - 1$, where $j = 1$ or 2

$$\|L^{l_j}_{itr/\sigma\sqrt{N}}\| \le \|L_{itr/\sigma\sqrt{N}}\|^{l_j} \le \|L_0\|^{2[D\log t]-1} \le |t|^{2D\log\|L_0\|},$$

and for the bound on $\|L^{2Km_j}_{-itr/\sigma\sqrt{N}}1\|_\infty$, $j = 1, 2$, we used Lemma 3.1.18.

Before we proceed to prove that $\sup_{z\in\mathbb{R}} A_2(N,z) \to 0$, as $N \to +\infty$, we would like to rewrite (3.1.24) by taking the dominating term in the sum when $m = n = N$, letting $\alpha = 1 + 4D\log\|L_0\|$ and since $l_1, l_2 \ge 0$, then $e^{-(l_1+l_2)h_0} \le 1$. Hence,

$$\begin{aligned} \left|\mathcal{E}_N\left(t/\sigma\sqrt{N}\right)\right| &\le C_2|t|^\alpha \sum_{n,m=1}^{N} e^{(n+m)h_0-(l_1+l_2)h_0}\left(1 - \frac{c}{|\frac{t}{\sigma\sqrt{N}}|^\gamma}\right)^{\frac{n+m-(l_1+l_2)}{2[D\log|t|]}} \\ &\le C_3|t|^\alpha e^{2Nh_0}\left(1 - \frac{c}{|\frac{t}{\sigma\sqrt{N}}|^\gamma}\right)^{\frac{2N-(l_1+l_2)}{2[D\log|t|]}}. \qquad (3.1.25) \end{aligned}$$

We shall also need the following lemma to bound $A_2(N,z)$.

Lemma 3.1.19. *If $\chi : \mathbb{R} \to \mathbb{R}$ is C^k and compactly supported then $\widehat{\chi}(u) = O(|u|^{-k})$, as $|u| \to +\infty$.*

Proof. Since χ is compactly supported, then there exists $M > 0$ such that $|\chi(t)| = 0$, for all $|t| \ge M$. Then using the Fourier transform we can write

$$\widehat{\chi}(u) = \int_{-M}^{M} e^{-itu}\chi(t)\,dt.$$

Integrating by parts will give

$$\widehat{\chi}(u) = \frac{-1}{iu}\chi(t)e^{-itu}\Big|_{-M}^{M} + \frac{1}{iu}\int_{-M}^{M} e^{-itu}\chi'(t)\,dt. \qquad (3.1.26)$$

The first term in (3.1.26) will vanish since $\chi(\pm M) = 0$. Then inductively since χ is C^k we have

$$\widehat{\chi}(u) = \left(\frac{-1}{iu}\right)^k \chi^{(k-1)}(t)e^{-itu}\Big|_{-M}^{M} + \left(\frac{1}{iu}\right)^k \int_{-M}^{M} e^{-itu}\chi^{(k)}(t)\,dt. \quad (3.1.27)$$

Again the first term will vanish for the same reason as above. Then by estimating the second term we can get a bound as $\chi^{(k)}$ is continuous in the compact interval $[-M, M]$

$$|\widehat{\chi}(u)| \leq \frac{1}{u^k} 2M \|\chi^k\|_\infty,$$

i.e., $|\widehat{\chi}(u)| = O(|u|^{-k})$. □

Now we shall start estimating A_2. First, using the bound on S_N in (3.1.25) we get that

$$\begin{aligned} A_2(N,z) &= \left| \int_{|t|\geq \delta\sigma\sqrt{N}} e^{izt/\sigma\sqrt{N}} \left\{ e^{-2h_0 N} \mathcal{E}_N\left(\frac{t}{\sigma\sqrt{N}}\right) \widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right)\right\} dt \right| \\ &\leq e^{-2h_0 N} \int_{|t|\geq \delta\sigma\sqrt{N}} \left| \mathcal{E}_N\left(\frac{t}{\sigma\sqrt{N}}\right)\right| \left|\widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right)\right| dt \\ &\leq C e^{-2h_0 N} \int_{\delta\sigma\sqrt{N}}^{+\infty} t^\alpha e^{2h_0 N} \left(1 - \frac{c}{(\frac{t}{\sigma\sqrt{N}})^\gamma}\right)^{\frac{2N-(l_1+l_2)}{2[D\log t]}} \left|\widehat{\chi}\left(\epsilon_N \frac{t}{\sigma\sqrt{N}}\right)\right| dt. \end{aligned}$$

We notice that if we split the above integral at $t = N^\beta$ such that $\frac{1}{2} < \beta < \frac{1}{2} + \frac{1}{\gamma}$, then A_2 can be bounded by a sum of two quantities that go to zero as N goes to infinity. So after splitting we have that

$$\begin{aligned} A_2(N,z) &\leq C\|\widehat{\chi}\|_\infty \int_{\delta\sigma\sqrt{N}}^{N^\beta} t^\alpha \left(1 - \frac{c}{(\frac{t}{\sigma\sqrt{N}})^\gamma}\right)^{\frac{2N-(l_1+l_2)}{2[D\log t]}} dt \\ &\quad + C' \frac{N^{k/2}}{\epsilon_N^k} \int_{N^\beta}^{\infty} t^{\alpha-k} \left(1 - \frac{c}{(\frac{t}{\sigma\sqrt{N}})^\gamma}\right)^{\frac{2N-(l_1+l_2)}{2[D\log t]}} dt. \quad (3.1.28) \end{aligned}$$

We have used Lemma 3.1.19 to estimate the Fourier transform in the second term in (3.1.28). Now in the first domain of integration $[\delta\sigma\sqrt{N}, N^\beta]$, note that we have $c\sigma^\gamma N^{\gamma/2-\beta\gamma+1} < N$ since $\beta > \frac{1}{2}$ and $\gamma > 0$. Therefore, we can use the standard estimate $(1-\frac{x}{n})^n < e^{-x}$, for $|x|<n$ to obtain the following bound

$$\begin{aligned}
\left(1-\frac{c}{(\frac{t}{\sigma\sqrt{N}})^\gamma}\right)^{\frac{N}{[D\log t]}} &\le \left(1-\frac{c\sigma^\gamma N^{\gamma/2}}{t^\gamma}\right)^{\frac{N}{D\log t}}\\
&\le \left(1-\frac{c\sigma^\gamma N^{\gamma/2}}{N^{\beta\gamma}}\right)^{\frac{N}{D\beta\log N}}\\
&= \left(1-\frac{c\sigma^\gamma N^{\gamma/2-\beta\gamma+1}}{N}\right)^{\frac{N}{D\beta\log N}}\\
&< e^{\frac{-c\sigma^\gamma N^{\gamma/2-\beta\gamma+1}}{D\beta\log N}}. \qquad (3.1.29)
\end{aligned}$$

Furthermore with the choice of $\frac{1}{2}<\beta<\frac{1}{2}+\frac{1}{\gamma}$, the exponential in (3.1.29) will go to zero superpolynomialy, i.e will go to zero faster than the reciprocal of any polynomial as $N\to+\infty$. This fact will be used to bound A_2.

We can also see that in the same domain of integration since $0\le l_1,l_2\le 2[D\log t]-1$, we get that

$$\begin{aligned}
\left(1-\frac{c}{(\frac{t}{\sigma\sqrt{N}})^\gamma}\right)^{\frac{-(l_1+l_2)}{2[D\log t]}} &\le \left(1-\frac{c}{(\frac{t}{\sigma\sqrt{N}})^\gamma}\right)^{\frac{-4[D\log t]+2}{2[D\log t]}}\\
&= \left(1-\frac{c}{(\frac{t}{\sigma\sqrt{N}})^\gamma}\right)^{-2}\left(1-\frac{c}{(\frac{t}{\sigma\sqrt{N}})^\gamma}\right)^{\frac{1}{[D\log t]}} \qquad (3.1.30)\\
&\le C'\left(1-\frac{c\sigma^\gamma}{N^{\gamma\beta-\gamma/2}}\right)^{\frac{1}{\beta D\log N}}. \qquad (3.1.31)
\end{aligned}$$

To get the bound in (3.1.31) we take smallest value of t in the domain of integration $[\delta\sigma\sqrt{N},N^\beta]$ for the first factor in (3.1.30) and the largest value of t for the second factor.

Now in the second domain of integration $[N^\beta,+\infty]$ we have the following bounds:

$$\left(1-\frac{c}{(\frac{t}{\sigma\sqrt{N}})^\gamma}\right)^{\frac{N}{2[D\log t]}}\left(1-\frac{c}{(\frac{t}{\sigma\sqrt{N}})^\gamma}\right)^{-2}\left(1-\frac{c}{(\frac{t}{\sigma\sqrt{N}})^\gamma}\right)^{\frac{1}{[D\log t]}}\le 1, \qquad (3.1.32)$$

as t gets very large and $\beta>\frac{1}{2}$.

Hence, we conclude from (3.1.29), (3.1.31) and (3.1.32) into (3.1.28) that

$$\begin{aligned} A_2(N,z) \quad &\leq \quad O\left(N^{\beta\alpha}\left(1-\frac{c\sigma^\gamma}{N^{\gamma\beta-\gamma/2}}\right)^{\frac{1}{\beta D\log N}} e^{-c\sigma^\gamma N^{\gamma/2-\beta\gamma+1}}\right) \\ &\quad + O\left(\frac{N^{k/2}}{\epsilon_N^k}\int_{N^\beta}^\infty t^{\alpha-k}\,dt\right). \end{aligned}$$

As $N\to\infty$ we can clearly see that the first term in the estimate for A_2 converges to zero since $\frac{1}{2}<\beta<\frac{1}{2}+\frac{1}{\gamma}$ and so $e^{-c\sigma^\gamma N^{\gamma/2-\beta\gamma+1}}$ converges to zero super polynomially as we indicated earlier. The second term will tend to zero provided we assume $k>\alpha+1$ and choose it to be sufficiently large since by assumption the sequence ϵ_N goes to zero slower than the rate N^{-r}, i.e. $\epsilon_N^{-1}=O(N^r)$, for some $r>0$.

This shows that $\sup_{z\in\mathbb{R}} A_2(N,z)\to 0$. By this we have proved that for $i=1,2,3$, $A_i(N,z)$ converges to zero uniformly in z. Therefore, Lemma 3.1.17 follows.

Step 2: The second step to prove our main theorem is to prove a uniform result for prime closed geodesics in the next proposition. To prove this we shall use the correspondence between closed geodesics and periodic orbits, this follows by Lemma 3.1.2. For this we shall replace $\phi_N(\chi_N^{(z)})$ (where we are counting periodic points) by

$$\begin{aligned} \omega_N(\chi_N^{(z)}) \quad &= \quad \sum_{\substack{(\gamma,\gamma')\\|\gamma|,|\gamma'|\leq N}} \chi_N^{(z)}(\ell(\gamma)-\ell(\gamma')) \\ &= \quad \sum_{n,m=1}^{N}\sum_{|\gamma|=n}\sum_{|\gamma'|=m}\chi_N^{(z)}(\ell(\gamma)-\ell(\gamma')), \end{aligned}$$

where γ,γ' are prime closed geodesics. Using the exact correspondence between periodic orbits in the shift space and closed geodesics in metric graphs described by Lemma 3.1.2, we can write $\omega_N(\chi_N^{(z)})$ in the following way

$$\begin{aligned}\omega_N(\chi_N^{(z)}) &= \sum_{n,m=1}^{N} \frac{1}{nm} \sum_{\substack{\sigma^n x = x\\ \sigma^m y = y}} \chi_N^{(z)}(r^n(x) - r^m(y)) \\ &\quad - \sum_{n,m=1}^{N} \frac{1}{nm} \sum_{\substack{\sigma^n x = x\\ \sigma^m y = y\\ \text{not both prime}}} \chi_N^{(z)}(r^n(x) - r^m(y)),\end{aligned}$$

where the function

$$\Theta_N(\chi_N^{(z)}) = \sum_{n,m=1}^{N} \frac{1}{nm} \sum_{\substack{\sigma^n x = x\\ \sigma^m y = y}} \chi_N^{(z)}(r^n(x) - r^m(y)),$$

is a summation over periodic orbits where pairs of periodic points might be both prime or not both prime. By estimating those that are not both prime using the fact that $\frac{1}{n} \sum_{\substack{\sigma^n x = x\\ x \text{ not prime}}} 1 = O(e^{nh_0/2})$, we can write $\omega_N(\chi_N^{(z)})$ as

$$\omega_N(\chi_N^{(z)}) = \Theta_N(\chi_N^{(z)}) + O(\|\chi\|_\infty e^{3h_0 N/2}). \tag{3.1.33}$$

Proposition 3.1.4.

$$\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} \left| \frac{N^{5/2}}{\epsilon_N e^{2h_0 N}} \omega_N(\chi_N^{(z)}) - \frac{e^{2h_0} \int \chi(x)dx}{\sigma\sqrt{2\pi}(e^{h_0}-1)^2} e^{-z^2/2\sigma^2 N} \right| = 0.$$

Proof. We shall show that this result is true for $\Theta_N(\chi_N^{(z)})$, then by (3.1.33) the same will be true for $\omega_N(\chi_N^{(z)})$. A relation that we can clearly see by considering $\Theta_N(\chi_N^{(z)}) \geq \frac{1}{N^2}\phi_N(\chi_N^{(z)})$ is that

$$\frac{N^{5/2}}{\epsilon_N e^{2h_0 N}} \Theta_N(\chi_N^{(z)}) \geq \frac{N^{1/2}}{\epsilon_N e^{2h_0 N}} \phi_N(\chi_N^{(z)}). \tag{3.1.34}$$

By considering large periods of periodic orbits, then for $0<\eta<1$ we get

$$\begin{aligned}\frac{N^{5/2}}{\epsilon_N e^{2h_0N}}\Theta_N(\chi_N^{(z)}) &= \frac{N^{5/2}}{\epsilon_N e^{2h_0N}}\sum_{n,m=[\eta N]+1}^{N}\frac{1}{nm}\sum_{\substack{\sigma^n x=x\\ \sigma^m y=y}}\chi_N^{(z)}(r^n(x)-r^m(x)) \\ &\quad +O\left(\frac{1}{\epsilon_N}\|\chi\|_\infty N^{5/2}e^{(\eta-1)h_0N}\right) \\ &\le \frac{N^{1/2}}{\eta^2\epsilon_N e^{2h_0N}}\phi_N(\chi_N^{(z)}) \\ &\quad +O\left(\frac{1}{\epsilon_N}\|\chi\|_\infty N^{5/2}e^{(\eta-1)h_0N}\right). \qquad (3.1.35)\end{aligned}$$

From Lemma 3.1.17 we have that

$$\lim_{N\to+\infty}\sup_{z\in\mathbb{R}}\frac{N^{1/2}}{\epsilon_N e^{2h_0N}}\phi_N(\chi_N^{(z)}) = \frac{e^{2h_0}\int\chi(x)dx}{\sigma\sqrt{2\pi}(e^{h_0}-1)^2}. \qquad (3.1.36)$$

Combining (3.1.34) and (3.1.35), then using (3.1.36) we obtain

$$\begin{aligned}0 &\le \lim_{N\to+\infty}\sup_{z\in\mathbb{R}}\left(\frac{N^{5/2}}{\epsilon_N e^{2h_0N}}\Theta_N(\chi_N^{(z)}) - \frac{N^{1/2}}{\epsilon_N e^{2h_0N}}\phi_N(\chi_N^{(z)})\right) \\ &\le \left(\frac{1}{\eta^2}-1\right)\frac{e^{2h_0}\int\chi(x)dx}{\sigma\sqrt{2\pi}(e^{h_0}-1)^2}.\end{aligned}$$

Since $0<\eta<1$, then we can take η arbitrarily close to 1, hence the desired result follows. □

Step 3: The last step in proving Theorem 3.1.3 is to replace the continuous function χ in Proposition 3.1.4 by the indicator function $\chi_{[a,b]}$ of the interval $[a,b]$. In order to show this we are going to approximate $\chi_{[a,b]}$ by two continuous compactly supported functions χ_- and χ_+, for a given $\epsilon>0$ such that $\chi_-\le\chi_{[a,b]}\le\chi_+$ and $\displaystyle\int\chi_+(x)-\chi_-(x)\,dx\le\epsilon$. Then, in particular, we have that

$$\int\chi_{[a,b]}(x)\,dx-\epsilon\le\int\chi_-(x)\,dx\le\int\chi_+(x)\,dx\le\int\chi_{[a,b]}(x)\,dx+\epsilon.$$

Using these relations and the fact that Proposition 3.1.4 applies to the two functions χ_- and χ_+, we deduce that

$$
\begin{aligned}
& -\frac{e^{2h_0}}{\sqrt{2\pi}\sigma(e^{h_0}-1)^2}\epsilon \\
& \leq \liminf_{N\to+\infty} \sup_{z\in\mathbb{R}} \left(\frac{N^{5/2}}{\epsilon_N e^{2h_0 N}} \pi(N, I_N(z)) - \frac{e^{2h_0}(b-a)}{\sigma\sqrt{2\pi}(e^{h_0}-1)^2} e^{-\sigma^2 z^2/2N} \right) \\
& \leq \limsup_{N\to+\infty} \sup_{z\in\mathbb{R}} \left(\frac{N^{5/2}}{\epsilon_N e^{2h_0 N}} \pi(N, I_N(z)) - \frac{e^{2h_0}(b-a)}{\sigma\sqrt{2\pi}(e^{h_0}-1)^2} e^{-\sigma^2 z^2/2N} \right) \\
& \leq \frac{e^{2h_0}}{\sqrt{2\pi}\sigma(e^{h_0}-1)^2}\epsilon.
\end{aligned}
$$

The proof of Theorem 3.1.3 is complete as we can take $\epsilon > 0$ arbitrarily small.

3.2 Bipartite graphs

In this section we give an asymptotic formula for the number of pairs of closed geodesics in bipartite graphs. These pairs are ordered by word length and such that the difference of their geometric length fall in a prescribed interval $[a, b] \subset \mathbb{R}$. We have the following result.

Theorem 3.2.1. *Let (G, ℓ) be a bipartite metric graph such that for each vertex v, $\deg(v) \geq 3$. Suppose that the non-lattice condition holds. Then, there exists $\beta > 1$ and $\sigma > 0$ such that for any $a < b$*

$$\begin{aligned} \pi(N, [a, b]) &:= \#\{(\gamma, \gamma') \text{ closed geodesics} : |\gamma|, |\gamma'| \leq N, \ell(\gamma) - \ell(\gamma') \in [a, b]\} \\ &\sim \frac{(b-a)}{\sqrt{2\pi}\sigma} \frac{\beta^2}{(\beta - 1)^2} \frac{\beta^{2[\frac{N}{2}]}}{[\frac{N}{2}]^{5/2}}, \quad \text{as } N \to +\infty. \end{aligned}$$

The proof of this theorem will follow the same steps as the proof of the same theorem when (G, ℓ) is a non-bipartite metric graph. We shall point out here the main differences.

First note that for non-bipartite graphs, we introduced the matrix A in section 3.1 to define the subshift of finite type X_A^+ and so by Lemma 3.1.1 A is aperiodic. But if we consider a bipartite graph then by the same lemma A is irreducible but not aperiodic and so the shift space is no longer mixing. For this reason we need to adjust the coding of the metric graph when it is bipartite so that we get a mixing subshift of finite type. Furthermore, Lemma 3.1.1 specifies that the matrix A has period 2 when the graph is bipartite. Recall the definition of the period of an index in A (see subsection 2.2.1) to be the greatest common divisor of $n \geq 1$ such that nth iterate of A from the index to itself is strictly positive. We have the following lemma from the book of Lind and Marcus on symbolic dynamics (Lemma 4.5.3, [10]).

Lemma 3.2.1. *If A is irreducible, then all indices have the same period and so the period of A is the period of any of its indices.*

Therefore, the period of every index in A is equal to 2. Also, recall that a matrix is aperiodic if it has period equal to 1, so we are going to define a new matrix B that is aperiodic. Let V_1 and V_2 be the two disjoint sets such that $V = V_1 \cup V_2$ and such that every edge joins a vertex in V_1 to a vertex in V_2. We define a matrix B indexed by the following set

$$\{(e, e') \in E^0 \times E^0 : e' \text{ follows } e \text{ with no backtracking}, e \text{ starts in } V_1\}, \tag{3.2.1}$$

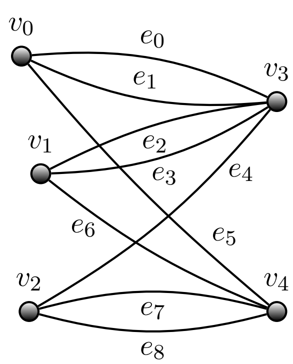
such that

$$B((e, e'), (e'', e''')) = \begin{cases} 1 & \text{if } e'' \text{ follows } e' \\ 0 & \text{otherwise .} \end{cases}$$

By construction, the period of every index is 1 and hence the period of the matrix B is 1. In the following example we illustrate this idea. For this, note that since B is indexed by the oriented edges of the metric graph, correspondingly we can think of the period as the greatest common divisor of the word length of the closed paths in the graph.

Example 3.2.1. *Consider the bipartite graph shown in Figure 3.4. Let* $V_1 = \{v_0, v_1, v_2\}$ *and* $V_2 = \{v_3, v_4\}$*. Following the way the matrix B is indexed, take* (e_0, e_1) *as an index. Then, to describe the period of this index let us take some possible closed paths from and to this index. We can take the closed paths* $((e_0, e_1))$, $((e_0, e_1), (e_0, e_4), (e_8, e_5))$*. These have word lengths* $1, 2, 3$ *respectively and their greatest common divisor is* 1.

Figure 3.4: A bipartite graph



Now we can define a subshift of finite type by

$$X_B^+ = \{(e_j, e_j')_{j=0}^{\infty} : \, B((e_j, e_j'), (e_{j+1}, e_{j+1}')) = 1, \text{ for all } j \geq 0\},$$

with a shift map

$$\sigma : X_B^+ \to X_B^+ : \sigma((e_0, e_0'), (e_1, e_1'), \ldots) = ((e_1, e_1'), (e_2, e_2'), \ldots).$$

To complete the coding of the metric graph (G, ℓ) by the subshift of finite type X_B^+, we need to define the function $r : X_B^+ \to \mathbb{R}^+$ in the following form:

$$r((e_0, e_0'), (e_1, e_1'), \ldots) = \ell(e_0) + \ell(e_0').$$

We can see that there is a bijection between the set of closed geodesics γ with $|\gamma| = 2n$ and the set of periodic orbits of period n for the shift of finite type X_B^+. Let $e = (e_0, e_1, \ldots, e_{2n-1})$ be a closed path in (G, ℓ) where e_0 starts in V_1 and $x = ((e_0, e_1), (e_2, e_3) \ldots, (e_{2n-2}, e_{2n-1}))$ a periodic point of period n in X_B^+, then we have

$$\begin{aligned}
x &\to (e_0, e_1, ..., e_{2n-1}) \\
\sigma x &\to (e_2, e_3, e_4, e_5, ..., e_0, e_1) \\
&\vdots \\
\sigma^{(n-1)} x &\to (e_{2n-2}, e_{2n-1}, e_0, e_1 ..., e_{2n-4}, e_{2n-3}).
\end{aligned}$$

Since we are counting pairs of closed geodesics, correspondingly we will look at pairs of periodic orbits. To do this, we shall consider a product space of subshifts of finite type. Let $\widetilde{X}_B = X_B^+ \times X_B^+$ and consider the product transformation $\widetilde{\sigma} = \sigma \times \sigma$ on $\widetilde{X}_B$ defined by $\widetilde{\sigma}(x, y) = (\sigma x, \sigma y)$. The states for $\widetilde{X}_B$ are pairs (i, j), where, $i, j \in X_B^+$ and the associated matrix $\widetilde{B}$ is given by $\widetilde{B}((i, j), (i', j')) = B(i, i')B(j, j')$. We can also associate to $\widetilde{X}_B$ a continuous function $R : \widetilde{X}_B \to \mathbb{R}$ defined by $R(x, y) = r(x) - r(y)$. Summing over periodic points in X_B^+, gives some useful relations between the three functions ℓ, r, and R. So for a periodic point $\widetilde{\sigma}^n(x,y) = (x,y) \in \widetilde{X}_B$, (i.e. $\sigma^n x = x$ and $\sigma^n y = y$) which correspond to closed geodesics γ and γ' with $|\gamma| = |\gamma'| = 2n$, we have that

$$\begin{aligned} r^n(x) &= r(x) + r(\sigma x) + ... + r(\sigma^{n-1}x) \\ &= (\ell(e_0) + \ell(e_1)) + (\ell(e_2) + \ell(e_3)) + ... + (\ell(e_{2n-1}) + \ell(e_{2n-1})) \\ &= \ell(\gamma) \end{aligned} \tag{3.2.2}$$

and

$$R^n(x,y) = r^n(x) - r^n(y) = \ell(\gamma) - \ell(\gamma').$$

For the analysis needed to prove Theorem 3.2.1 we need to check that the non-lattice condition on a bipartite metric graph implies that r is non-lattice. In terms of periodic points, r will be non-lattice if for any $a, b \in \mathbb{R}$ we have

$$\{r^n(x) - an : \sigma^n x = x, n \geq 1\} \not\subset b\mathbb{Z}. \tag{3.2.3}$$

From (3.2.2) we can see that $r^n(x)$ and the period n depend only on the periodic point x. Furthermore, it gives a relation between closed geodesics of word length $2n$ and the associated periodic points x. Then (3.2.3) can be written in the following form

$$\{\ell(\gamma) - (a/2)|\gamma| : \gamma \,\text{closed geodesic}\} \not\subset b\mathbb{Z}.$$

But this is exactly the non-lattice condition. Hence, r is non-lattice.

Example 3.2.2. *We can see the non-lattice condition is satisfied in the bipartite metric graph in Figure 3.4 by assigning lengths to the edges such that* $\dim_{\mathbb{Q}}(L)$ *cannot be less than 3, in the same way we had in Example 3.1.3 for the case of non-bipartite metric graphs.*

To obtain the asymptotic in Theorem 3.2.1, first note that from (3.2.2) and the bijection between the set of closed geodesics γ with $|\gamma| = 2n$ and the set of periodic orbits of period n for the shift of finite type X_B^+, we see that

$$\begin{aligned}\pi(2N,[a,b]) &= \#\{(\gamma,\gamma') \text{ closed geodesics} : |\gamma|,|\gamma'| \le 2N, \ell(\gamma)-\ell(\gamma') \in [a,b]\} \\ &= \#\{\text{pairs of periodic orbits in } X_B^+ : \text{periods } n \le N, \\ &\qquad\qquad r^n(x)-r^n(y) \in [a,b]\}.\end{aligned}$$

Secondly, we have the following proposition for counting pairs of periodic points (x,y) of period n such that $r^n(x)-r^n(y) \subset \mathbb{R}$ in $\widetilde{X}_B$. This proposition can be deduced from the analysis and calculations we have done in Chapter 3 to prove Theorem 3.1.1.

Proposition 3.2.2. *Let $\widetilde{X}_B$ be a mixing subshift of finite type such that*
(1) r is non-lattice,
(2) $\int R\, d\widetilde{\mu} = 0$, where $\widetilde{\mu}$ is the measure of maximal entropy for the shift map $\widetilde{\sigma}$ associated to $\widetilde{X}_B$. Then there exists $\beta > 1$and $\sigma > 0$ such that for all $a < b$

$$\begin{aligned}&\#\{\textit{pairs of periodic orbits in } X_B^+ : \textit{periods } n \le N, r^n(x)-r^n(y) \in [a,b]\} \\ &\sim \frac{(b-a)}{\sqrt{2\pi}\sigma}\frac{\beta^2}{(\beta-1)^2}\frac{\beta^{2N}}{N^{5/2}}, \quad \textit{as } N \to +\infty.\end{aligned}$$

Note that $\beta^2 = e^{h_{\text{top}}(\widetilde{\sigma})}$, where $h_{\text{top}}(\widetilde{\sigma})$ is the topological entropy of the shift map $\widetilde{\sigma}$ and $\sigma^2 = \sigma^2(R) = \lim_{n\to\infty} \frac{1}{n}\int (R^n)^2\, d\widetilde{\mu}$.

Finally, we also need to consider counting elements in the set $\pi(2N+1,[a,b])$. But in this set we are still counting closed geodesics in the bipartite graph, then counting the number of elements in the two sets $\pi(2N,[a,b])$ and $\pi(2N+1,[a,b])$ will correspond to the same set of elements in shift of finite type. Using Proposition 3.2.2 we deduce that

$$\pi(2N,[a,b]) = \pi(2N+1,[a,b]) \sim \frac{(b-a)}{\sqrt{2\pi}\sigma}\frac{\beta^2}{(\beta-1)^2}\frac{\beta^{2N}}{N^{5/2}}, \quad \text{as } N \to +\infty.$$

Combining the two cases $|\gamma|,|\gamma'| \le 2N$ and $|\gamma|,|\gamma'| \le 2N+1$, for $M \in \mathbb{N}$ we have

$$\pi(M,[a,b]) \sim \frac{(b-a)}{\sqrt{2\pi}\sigma}\frac{\beta^2}{(\beta-1)^2}\frac{\beta^{2[\frac{M}{2}]}}{[\frac{M}{2}]^{5/2}}, \quad \text{as } M \to +\infty,$$

which is the asymptotic needed to be proved in Theorem 3.2.1.

Chapter 4

Pairs of closed geodesics ordered by word length of conjugacy classes in $\pi_1(G)$

In this chapter we would like to present two results that are similar to Theorem 3.1.1 and Theorem 3.1.3. Although here, we are going to study metric graphs with respect to their fundamental groups. More specifically in the previous two theorems we studied pairs of closed geodesics ordered by their word lengths on the metric graph. In this chapter, the closed geodesics are going to be ordered by the word length of the corresponding conjugacy classes in the fundamental group of G, $\pi_1(G)$. For a closed geodesic γ in G, we denote the word length of its corresponding conjugacy class by $\|\gamma\|$. As we are going to see in this chapter, this is defined to be the shortest word length of elements in the conjugacy class which corresponds to γ. So we will be looking for an asymptotic for the number of element in the set

$$\{(\gamma, \gamma') : \|\gamma\|, \|\gamma'\| \leq N, \ell(\gamma) - \ell(\gamma') \in [a, b]\},$$

and in the case of shrinking intervals, the number of elements in the set

$$\{(\gamma, \gamma') : \|\gamma\|, \|\gamma'\| \leq N, \ell(\gamma) - \ell(\gamma') \in [a\epsilon_N + z, b\epsilon_N + z]\},$$

for some $z, a, b \in \mathbb{R}$, $a < b$ and $\epsilon_N > 0$.

We shall start by introducing some results that show the relation between metric graphs and their fundamental groups.

4.1 Graphs and free groups

An important result from algebraic topology that we rely on is that which describes the nature of the fundamental group of a finite connected graph. For this recall that a tree is a graph with no closed paths and that a tree in a finite connected graph G is maximal if it contains all the vertices of G.

Lemma 4.1.1. *(Proposition 1A.2, [6]) For a connected graph G with maximal tree T, $\pi_1(G)$ is a free group with basis the classes $[\iota_\alpha]$ corresponding to the edges e_α of $G - T$.*

In particular for the metric graph we considered earlier, a finite connected graph with $\deg(v) \geq 3$ for each vertex v, we have that the fundamental group F is a free group with a finite number of generators. More precisely, the number of generators is at least 2. The following lemma shows how the condition on the vertices implies that the number of generators is at least 2.

Lemma 4.1.2. *If G is a finite connected graph with $\deg(v) \geq 3$ for each vertex v, then the number of generators in $\pi_1(G) = F$ is greater then or equal to 2.*

Proof. Let T be a maximal tree for G. Then T contains at least two vertices which have degree 1 as a vertex in T, unless T consists of a single vertex. The only case where T has a single vertex is when G has one vertex and some number k loops. If the vertex in G has degree at least 3 then $k \geq 2$ and so the fundamental group has at least two generators. Otherwise, when T has at least two vertices of degree 1 in T, then since their degree in G is least 3, each will have at least one other edge in G. Hence, the fundamental group has at least two generators. □

Let F be the free group on k generators and let $\mathcal{A} = \{a_1^{\pm 1}, \ldots, a_k^{\pm 1}\}$ be a set of generators for F. Using these generators we can write finite words in unique reduced forms, i.e. we can write

$$x = x_0 x_1 \dots x_{n-1},$$

where $x_i \in \mathcal{A}$, $i = 0, 1, \dots, n-1$ and $x_{i+1} \neq x_i^{-1}$, $i = 0, 1, \dots, n-2$. This gives a unique shortest representation of any element $x \in F$ in term of the generators. We define the word length $|\cdot|$ on F by $|x| = n$. We are going to illustrate these definitions and concepts in an example of a metric graph.

Example 4.1.1. *In Figure 4.1 we give an example of a metric graph (except that we have put directions on edges for the purpose of finding the generators of the fundamental group of G). G contains 4 vertices and 8 edges, we are going to follow Lemma 4.1.1 to find the generators of the fundamental group of G.*

First we fixed a base point say v_0, then we choose a maximal tree T, an example is shown in Figure 4.2. We are going to have five generators corresponding to five edges in $G - T$. We list them as follows: $a_1 = e_0$, $a_2 = e_1^{-1}e_7e_1$, $a_3 = e_1^{-1}e_4^{-1}e_5^{-1}$, $a_4 = e_1^{-1}e_3e_6^{-1}e_5^{-1}$, $a_5 = e_5e_6e_2e_1$. So the fundamental group of G is generated by $\mathcal{A} = \{a_1^{\pm 1}, a_2^{\pm 1}, a_3^{\pm 1}, a_4^{\pm 1}, a_5^{\pm 1}\}$. Since any element in $\pi(G, v_0)$ can be represented in terms of these generators, then for example the closed path based at v_0, $e_0e_5e_6e_2e_1$ can be represented in terms of the generators as a_1a_5. The word length of a_1a_5 is $|a_1a_5| = 2$.

Figure 4.1: A graph G

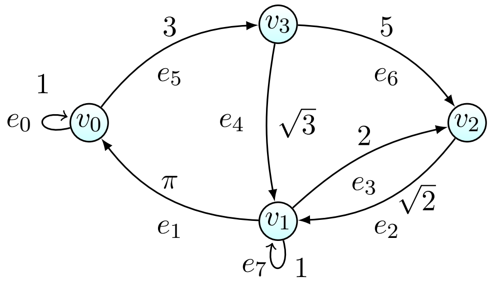
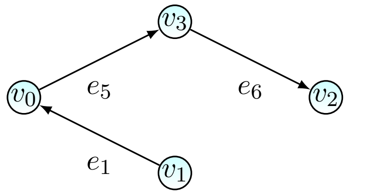


Figure 4.2: A maximal tree of the graph G



Let $\mathcal{C}(F)$ be the set of non-trivial conjugacy classes of F. A conjugacy class $w \in \mathcal{C}(F)$ might contain two possible forms of reduced words. Reduced words are of the form $x = x_0x_1 \dots x_{n-1}$ such that $x_{n-1} \neq x_0^{-1}$ and we call these words cyclically reduced words. The other form is when we have reduced words with $x_{n-1} = x_0^{-1}$ and we call these non-cyclically reduced words. Also, w contains the cyclic permutation of the cyclically reduced words. To define word length for w, note that the cyclically reduced word x and its cyclic permutation has word length n, while the non-cyclically reduced words will have word length greater than n. Hence, we define the word length $\|w\|$ of w (with respect to $\mathcal{A}$) as

$$\|w\| = n = \min_{x \in w} |x|.$$

We can consider the conjugacy class w^m formed by m-fold concatenation of elements of w, i.e. $w^m = \{x^m : x \in w\}$. Then for $m \geq 1$

$$\|w^m\| = m\|w\|.$$

Now we see how conjugacy classes in $\pi_1(G)$ relate to closed geodesics in G. First, recall that the fundamental group $\pi_1(G)$ contains all homotopy classes that are based at the same point. Then, if we consider the set of all homotopy classes when the base point is ignored (i.e. homotopy classes of closed paths based at different points) we obtain what we call free homotopy classes. So a free homotopy class contains closed paths that are based at different points but they are homotopic, i.e. can be continuously deformed to the same closed path. It is a well know result from algebraic topology that there exists a one-to-one correspondence between free homotopy classes and conjugacy classes in $\pi_1(G)$. In fact this holds for any path connected space. We also have that in G there is exactly one closed geodesic in each non-trivial free homotopy class ([29], p.14-15). Consequently, closed geodesics in a metric graph G correspond to non-trivial conjugacy classes in $\pi_1(G)$.

The following example shows how the closed geodesics, free homotopy classes and conjugacy classes can be related.

Example 4.1.2. *Consider the same metric graph in Figure 4.1 we used in Example 4.1.1. Let us take a free homotopy class where all closed path can be continuously deformed to the closed path* $e = e_4e_7e_3e_6^{-1}$. *Elements that belong to this free homotopy class include all permutations of the closed path* e *and also closed paths of the form* $e_5e_4e_7e_3e_6^{-1}e_5^{-1}$, $e_0e_5e_4e_7e_3e_6^{-1}e_5^{-1}e_0^{-1}$, $e_1^{-1}e_7e_3e_6^{-1}e_4e_1$, $e_0e_1^{-1}e_7e_3e_6^{-1}e_4e_1e_0^{-1}$*,etc. We see that* $e = e_4e_7e_3e_6^{-1}$ *and its cyclic permutations are the unique closed geodesic in this free homotopy class.*

To see the relation between this free homotopy class and conjugacy classes of F *we shall find a path that connects* e *to the base point* v_0*. An obvious choice is* e_1 *or* e_5*. Then we have* $e_5e_4e_7e_3e_6^{-1}e_5^{-1} = a_3^{-1}a_2a_4$*. So* $a_3^{-1}a_2a_4$ *belongs to a conjugacy class* $w \in \mathcal{C}(F)$*, where* w *corresponds to the free homotopy class represented by* e*. Hence in this case the word length of the conjugacy class corresponding to the closed geodesic formed by* $e = e_4e_7e_3e_6^{-1}$ *is equal to* $\|a_3^{-1}a_2a_4\| = 3$ *as this is a cyclically reduced element of* w *it would have the shortest word length among all other element* $x \in w$*. Note that the word length of the closed geodesic* $e = e_4e_7e_3e_6^{-1}$ *is* 4*.*

We shall define geometric length functions on F and on the conjugacy classes in $\mathcal{C}(F)$, then we are going to relate this to the geometric length of closed geodesics. To do that we consider the fact that the universal cover of G is an infinite tree $\mathcal{T}$, with the metric on G lifted to a metric $d_{\mathcal{T}}$ on $\mathcal{T}$. Recall that the metric on G is defined by a function $\ell : E \to \mathbb{R}^+$ that assigned a positive real number to each edge in G. For a fixed based point $o \in \mathcal{T}$ we define a based length function $L : F \to \mathbb{R}$ by

$$L(x) = d_{\mathcal{T}}(o, ox).$$

We also define the following length function that does not depend on base point,

$$\ell(x) = \inf_{o \in \mathcal{T}} d_{\mathcal{T}}(o, ox).$$

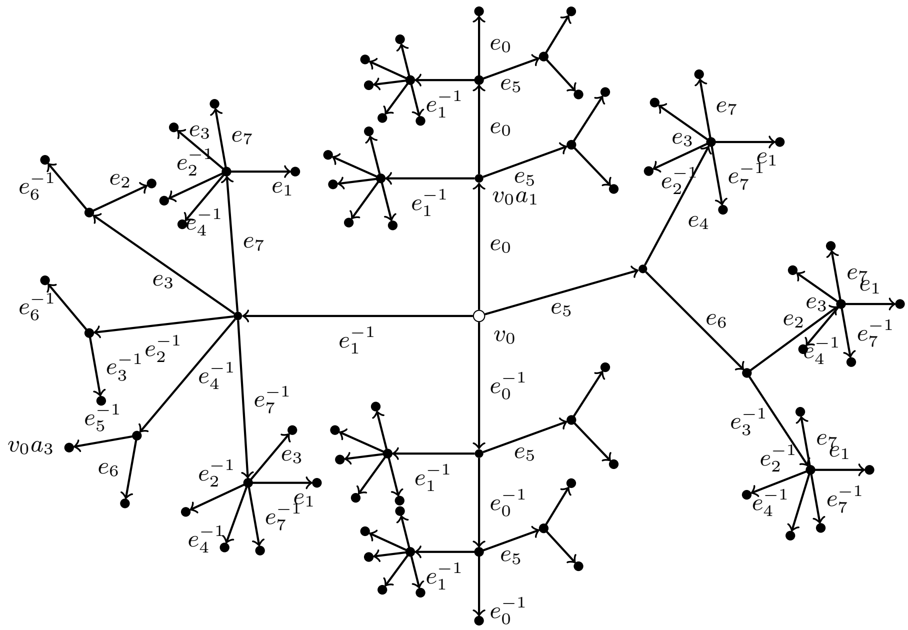
Note that if $x, z \in F$ are conjugate, i.e., there exists $y \in F$ such that $z = yxy^{-1}$, then $\inf_{o\in\mathcal{T}} d_{\mathcal{T}}(o, oz) = \inf_{o\in\mathcal{T}} d_{\mathcal{T}}(o, oyxy^{-1}) = \inf_{oy\in\mathcal{T}} d_{\mathcal{T}}(oy, oyx)$. Thus, ℓ does not depend on the base point but it depends on the conjugacy class of x only. So we can define ℓ as a function $\ell : \mathcal{C}(F) \to \mathbb{R}$. Now to see how this will relate to the geometric length of a closed geodesic γ in G, consider the fact that γ lifts to a geodesic path in $\mathcal{T}$ from a vertex $q \in \mathcal{T}$ to qy, where y belongs to a conjugacy class $w \in \mathcal{C}(F)$. Then $\ell(\gamma)$ will be equal to the length of the geodesic path from q to qy which we can find using the above definition. Furthermore, it is clear that $d_{\mathcal{T}}(q, qy)$ minimises $d_{\mathcal{T}}(o, oy)$ for $o \in \mathcal{T}$. Thus

$$\ell(\gamma) = \inf_{o\in\mathcal{T}} d_{\mathcal{T}}(o, oy) = \ell(y) = \ell(w).$$

Hence, $\ell(\gamma) = \ell(w)$. The following example gives an idea on how the geometric lengths relate to each other.

Example 4.1.3. *Consider the infinite tree $\mathcal{T}$ in Figure 4.3 which represents a universal cover of the metric graph G in Figure 4.1. Take a closed geodesic γ in G, represented by the cyclic permutation of the closed path $e = e_1^{-1}e_4^{-1}e_5^{-1}$. This closed path also represents an element in F because $e = a_3$. Using the infinite tree $\mathcal{T}$, the geometric length of a_3 is given by $L(a_3) = d_{\mathcal{T}}(v_0, v_0a_3) = \ell(e^{-1}) + \ell(e_4^{-1}) + \ell(e_5^{-1}) = \pi + \sqrt{3} + 3$. Note that a_3 is conjugate to $a_1a_3a_1^{-1}$, the length of the conjugacy class w that these belong to is $\ell(w) = \inf_{\substack{q\in\mathcal{T}\\ y\in w}} d_{\mathcal{T}}(q, qy) = \ell(a_3) = \ell(\gamma)$.*

Figure 4.3: Universal cover of G in Figure 4.1, infinite tree $\mathcal{T}$



We finish this section by listing some properties of the length function L which we are going to use to prove lemmas that come later. Two natural properties of L are: $L(x) = 0$ if and only if $x = 1$ and $L(x^{-1}) = L(x)$. Other properties are derived from Lyndon's work on length functions in groups [11]. For any $x, y, z \in F$, we have

A_1. $(x, y)_L \geq 0$, where $(x, y)_L = (L(x) + L(y) - L(x^{-1}y))/2$

A_2. $(x, y)_L < (x, z)_L$ implies that $(y, z)_L = (x, y)_L$; and

A_3. $(x, y)_L + (x^{-1}, y^{-1})_L > L(x) = L(y)$ implies that $x = y$.

We can also define the following Gromov product. For $x, y \in F$

$$(x, y) = (|x| + |y| - |x^{-1}y|)/2. \tag{4.1.1}$$

Since the map $f : F \to \mathcal{T}$ given by $f(x) = ox$ is a quasi-isometry between F equipped with the word metric $d_{\text{word}}(x, y) = |x^{-1}y|$ and $\mathcal{T}$, then there exists constants $D_1, D_2, K > 0$ such that for all $x, y \in F$,

$$D_1\,(x, y) - K \leq (x, y)_L \leq D_2\,(x, y) + K. \tag{4.1.2}$$

This inequality follows from Proposition 15 in chapter 5 of [5].

Now we state the two main results of this chapter.

4.2 Results

The first result is given by the following theorem which describes an asymptotic for counting pairs of closed geodesics the difference of whose geometric length is in a prescribed interval and the closed geodesics are ordered by the word length of the corresponding conjugacy classes in the fundamental group.

Theorem 4.2.1. *Let (G, ℓ) be a metric graph such that for each vertex v, $\deg(v) \geq 3$. Suppose that the non-lattice condition holds. Then there exists $\sigma > 0$ such that for all $a < b$,*

$$\begin{aligned}\pi(N, [a, b]) &= \#\{(\gamma, \gamma') : \|\gamma\|, \|\gamma'\| \leq N, \ell(\gamma) - \ell(\gamma') \in [a, b]\} \\ &\sim \frac{(b-a)(2k-1)^2}{4\sqrt{2\pi}\sigma(k-1)^2} \frac{(2k-1)^{2N}}{N^{5/2}}, \text{ as } N \to +\infty,\end{aligned}$$

where $k \geq 2$ is the number of generators of the fundamental group of G.

The second result differs from the first by letting the interval containing the differences of geometric lengths be arbitrarily positioned in the real line and allowed to shrink at a specific rate as the word length tends to infinity. This is represented in the following set

$$\pi(N, I_N(z)) = \#\{(\gamma, \gamma') : \|\gamma\|, \|\gamma'\| \leq N, \ell(\gamma) - \ell(\gamma') \in [a\epsilon_N + z, b\epsilon_N + z]\},$$

for some $z, a, b \in \mathbb{R}$, $a < b$ and $\epsilon_N > 0$. The following theorem gives an estimation of the growth of the number of elements in this set.

Theorem 4.2.2. *Let (G, ℓ) be a metric graph such that for each vertex v, $\deg(v) \geq 3$. Suppose that the non-lattice condition holds and that there exists γ, γ' closed geodesics in G such that $\ell(\gamma)/\ell(\gamma')$ is Diophantine. Then there exists $\sigma > 0$ and $r > 0$ such that for any $a < b$ and a sequence $\epsilon_N > 0$ which* *tends to zero slower than the polynomial rate* N^{-r}*,i.e.,* $\epsilon_N^{-1} = O(N^r)$*, we have that*

$$\lim_{N\to+\infty} \sup_{z\in\mathbb{R}} \left| \frac{\sigma N^{5/2}}{\epsilon_N (2k-1)^{2N}} \pi(N, I_N(z)) - \frac{(b-a)(2k-1)^2}{4\sqrt{2\pi}((k-1)^2} e^{-z^2/2\sigma^2 N} \right| = 0.$$

In particular,

$$\pi(N, I_N(z)) \sim \frac{(b-a)(2k-1)^2 \epsilon_N}{4\sqrt{2\pi}\sigma(k-1)^2} \frac{(2k-1)^{2N}}{N^{5/2}}, \ \ as\, N \to +\infty.$$

where $k \geq 2$ *is the number of generators of the fundamental group of* G*.*

In order to prove Theorem 4.2.1 and Theorem 4.2.2, we shall start by coding the fundamental group of G using a subshift of finite type.

4.3 Coding the fundamental group of G by a shift space

Recall that the fundamental group of G is a free group F with free basis $\mathcal{A} = \{a_1^{\pm 1}, \ldots, a_k^{\pm 1}\}$. Here we follow the coding described by Sharp in [26] and [27]. The shift space that we shall define will contain two different forms of infinite sequences. The first one is the infinite reduced words in $\mathcal{A}$ (infinite concatenation of elements in $\mathcal{A}$ in reduced form). The second is the finite reduced words in F followed by a string of 0s. We write these sequences together in the following set

$$X_C^+ = \{x = (x_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty} \mathcal{A} \cup \{0\} : C(x_i, x_{i+1}) = 1, \forall i \in \mathbb{N}\},$$

where C is $(2k+1) \times (2k+1)$ 0-1 matrix, indexed by $\mathcal{A} \cup \{0\}$, defined as follows

$$C(i,j) = \begin{cases} 1 & \text{if} \quad j \neq i^{-1} \text{ or } (i = \mathcal{A} \cup \{0\} \text{ and } j = 0) \\ 0 & \text{if} \quad j = i^{-1} \text{ or } (i = 0 \text{ and } j \in \mathcal{A}). \end{cases}$$

For the shift space we define the shift map $\sigma : X_C^+ \to X_C^+$ by $(\sigma x)_n = x_{n+1}$ (i.e. σ will delete the first element and shift the sequence to the left).

Since we will study periodic points in X_C^+, which are infinite reduced words, we will also consider the set of infinite reduced words

$$X_A^+ = \{x = (x_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty} \mathcal{A} : A(x_i, x_{i+1}) = 1, \forall i \in \mathbb{N}\},$$

where A is $(2k \times 2k)$ 0-1 matrix, indexed by $\mathcal{A}$, defined by $A(i,j) = 1$ if $j \neq i^{-1}$ and zero otherwise. This also represents a shift space with $\sigma : X_A^+ \to X_A^+$. The matrix A is a non-negative aperiodic matrix and hence by the Perron-Frobenius theorem, A has a unique maximal eigenvalue λ. This maximal eigenvalue can be calculated and is equal to $2k-1$. Then the measure of maximal entropy μ has the following explicit form.

$$\mu([y_0, \ldots, y_{n-1}]) = \frac{1}{2k(2k-1)^{(n-1)}},$$

which extends to Borel sets by the Hahn-Kolomogrov extension theorem. The topological entropy of σ is given by $h_{\mathrm{top}}(\sigma) = \log \lambda = \log(2k-1)$.

Using this coding we want to find a one-to-one correspondence between the set of conjugacy classes (and so closed geodesics) and the set of periodic orbits in X_A^+. Now to see the relation with the periodic orbits, consider a conjugacy class $w \in \mathcal{C}(F)$ and let $x \in w$ such that $x = x_0 x_1 \ldots x_{n-1}$ is cyclically reduced, then $\|w\| = n$ and all cyclical permutation of x have word length n. So we have a natural bijection between the set of conjugacy classes $w \in \mathcal{C}(F)$ and the set of periodic orbits of points $x \in X_A^+$ such that $\sigma^n x = x$.

As we are counting pairs of closed geodesics that correspond to pairs of conjugacy classes, then correspondingly we will look at pairs of periodic orbits. For that purpose we introduce a product shift space $\widetilde{X} = X_C^+ \times X_C^+$, with product shift map $\widetilde{\sigma} = \sigma \times \sigma$ defined by $\widetilde{\sigma}(x,y) = (\sigma x, \sigma y)$. The element of $\widetilde{X}$ are pairs of infinite sequences belong to X_C^+ and the associated matrix is given by $\widetilde{C}((i,j),(i',j')) = C(i,i')C(j,j')$. So, if both product factors are equal to 1 then $\widetilde{C}((i,j),(i',j')) = 1$ otherwise, it is equal to zero.

In a similar way, we can also introduce the product shift space $\widetilde{X} = X_A^+ \times X_A^+$ associate to the same product shift map as before $\widetilde{\sigma}$ and for a pair of infinite sequences $\widetilde{A}((i,j),(i',j')) = A(i,i')A(j,j')$.

We also need to code the length functions $L : F \to \mathbb{R}$ and $\ell : \mathcal{C}(F) \to \mathbb{R}$ in terms of a function $r : X_C^+ \to \mathbb{R}$. As we are going to see this function turns out to be a locally constant function. The construction for the function r is taken from [26].

First we have the following lemma.

Lemma 4.3.1. *There exists an integer $N \geq 1$ such that if $n \geq N$ and $x_0x_1 \dots x_{n-1}$ is a reduced word in F then*

$$L(x_0x_1 \dots x_{n-1}) - L(x_1 \dots x_{n-1}) = L(x_0x_1 \dots x_{N-1}) - L(x_1 \dots x_{N-1}).$$

Proof. Let $x = x_0, x_1, \dots, x_{n-1}$ and $y = x_0, x_1, \dots, x_{N-1}$. So to prove the lemma we are going to show that

$$L(x) - L(x_0^{-1}x) = L(y) - L(x_0^{-1}y). \tag{4.3.1}$$

First note that the Gromov product of x and y, we defined in (4.1.1), can be calculated as follows

$$(x, y) = 1/2(n + N - (n - N)) = N.$$

Now by starting with writing (4.3.1) as

$$L(x_0) + L(y) - L(x_0^{-1}y) = L(x_0) + L(x) - L(x_0^{-1}x),$$

which can be written as

$$(x_0, y)_L = (x_0, x)_L.$$

From property A_2, this will hold if

$$(x_0, x)_L < (x, y)_L. \tag{4.3.2}$$

To see how (4.3.2) can be satisfied we will use an inequality (4.1.2) to find an upper bound and a lower bound for $(x_0, x)_L$ and for $(x, y)_L$, so we get the following inequalities

$$B_1(x, x_0) - K \leq (x, x_0)_L \leq B_2(x, x_0) + K \tag{4.3.3}$$

and

$$B_1(x, y) - K \leq (x, y)_L \leq B_2(x, y) + K, \tag{4.3.4}$$

where B_1, B_2 and K are positive constants. From (4.3.3) we have that

$$1 \leq B_1^{-1}(x, x_0)_L + B_1^{-1}K \leq B_1^{-1}(B_2 + 2K) \tag{4.3.5}$$

since $(x, x_0) = 1$, then from (4.3.4) we get

$$N \leq B_1^{-1}(x, y)_L + B_1^{-1}K \leq B_1^{-1}(B_2 N + 2K) \tag{4.3.6}$$

since $(x, y) = N$. Therefore, we can see from (4.3.5) and (4.3.6) that (4.3.2) will be satisfied provided that

$$N \geq B_1^{-1}(B_2 + 2K).$$

□

Following the result of this lemma we can now define a locally constant function on the shift space X_C^+. (Note that we will use the notation $\dot{0}$ to denote an infinite sequence of zeros).

Definition 4.3.1. *Define a locally constant function* $r : X_C^+ \to \mathbb{R}$ *by*

$$r((x_n)_{n=0}^{\infty}) = L(x_0 x_1 \dots x_{N-1}) - L(x_1 \dots x_{N-1})$$

for some $N \in \mathbb{N}$ *(chosen so that the conclusion of Lemma 4.3.1 holds) and if*

$x \in X_C^+$ is of the form $(x_0, x_1, \ldots, x_{m-1}, \dot{0})$ then

$$r((x_0, x_1, \ldots, x_{m-1}, \dot{0})) = L(x_0 x_1 \ldots x_{m-1}) - L(x_1 \ldots x_{m-1})$$

The next lemma shows the function L has a relation with r when summed over periodic points.

Lemma 4.3.2. *Suppose that $x_0 x_1 \ldots x_{n-1}$ is a reduced word, then*

$$L(x_0 x_1 \ldots x_{n-1}) = r^n(x_0 x_1 \ldots x_{n-1}, \dot{0}).$$

Proof. Since $r^n(x) = r(x) + r(\sigma x) + \cdots + r(\sigma^{n-1} x))$ and by the above definition of the function r in terms of the function L, we have that

$$\begin{aligned} r^n(x_0 x_1 \ldots x_{n-1}, \dot{0}) &= r(x_0 x_1 \ldots x_{n-1}, \dot{0}) + r(x_1 \ldots x_{n-1}, \dot{0}) + \cdots + r(x_{n-1}, \dot{0}) \\ &= (L(x_0 x_1 \ldots x_{n-1}) - L(x_1 \ldots x_{n-1}) \\ &+ (L(x_1 \ldots x_{n-1}) - L(x_2 \ldots x_{n-1}) \\ &+ \cdots + (L(x_{n-1}) - L(1)) = L(x_0 x_1 \ldots x_{n-1}). \end{aligned}$$

□

Finally, we relate $r^n(x)$ to the geometric length of $w \in \mathcal{C}(F)$ and so to the geometric length of a closed geodesic γ in G.

Lemma 4.3.3. *Let the periodic orbit $\{x, \sigma x, \ldots, \sigma^{n-1} x\}$ ($\sigma^n x = x$) correspond to the conjugacy class $w \in \mathcal{C}(F)$ containing the cyclically reduced word $x_0 x_1 \ldots x_{n-1}$. Then*

$$\ell(w) = r^n(x).$$

Proof. Let $x^{(m)}$ be the reduced word obtained from the m-fold concatenation of $x_0x_1\dots x_{n-1}$ in a conjugacy class w. We have that

$$\begin{aligned}|r^{nm}(x)-r^{nm}(x^{(m)},\dot{0})| &\leq |r(x)-r(x^{(m)},0)|+|r(\sigma x)-r(\sigma x^{(m)},0)| \\ &+ \cdots+|r(\sigma^{nm-1}x)-r(\sigma^{nm-1}x^{(mn)},\dot{0})| \\ &\leq 2N\|r\|_\infty,\end{aligned}$$

we obtain the bound using the fact r is locally constant then there exists $N \geq 1$ such that at most N non-zero terms are bounded by $2\|r\|_\infty$. Note that $r^{mn}(x) = mr^n(x)$ and by Lemma 4.3.2 $r^{mn}(x^{(m)},\dot{0}) = L(x^{(m)})$ and so

$$r^n(x) = \lim_{m\to+\infty}\frac{1}{m}L(x^{(m)}).$$

After this, we only need to show that

$$\ell(w) = \lim_{m\to+\infty}\frac{1}{m}L(x^{(m)}). \tag{4.3.7}$$

To see this, let γ be the closed geodesic corresponding to the conjugacy class w. Then

$$\ell(w) = \ell(\gamma) = d_{\mathcal{T}}(q, qy)$$

for some $q \in \mathcal{T}$ and some $y \in w$. Furthermore,

$$d_{\mathcal{T}}(q, qy^m) = md_{\mathcal{T}}(q, qy).$$

By the triangle inequality,

$$d_{\mathcal{T}}(q, qy^m) - 2d_{\mathcal{T}}(q, o) \leq d_{\mathcal{T}}(o, oy^m) \leq d_{\mathcal{T}}(q, qy^m) + 2d_{\mathcal{T}}(q, o),$$

which implies that

$$\lim_{m\to+\infty}\frac{1}{m}L(y^m) = \lim_{m\to+\infty}\frac{1}{m}d_{\mathcal{T}}(q, qy^m).$$

Since y is conjugate to $x_0x_1\cdots x_{n-1}$, we have

$$\lim_{m\to+\infty}\frac{1}{m}L(y^m)=\lim_{m\to+\infty}\frac{1}{m}L(x^{(m)}). \tag{4.3.8}$$

Now

$$\begin{aligned}\lim_{m\to+\infty}\frac{1}{m}L(y^m) &= \lim_{m\to+\infty}\frac{1}{m}d_{\mathscr{T}}(q,qy^m)\\ &= \lim_{m\to+\infty}\frac{m}{m}d_{\mathscr{T}}(q,qy)=d_{\mathscr{T}}(q,qy)=\ell(w). \qquad (4.3.9)\end{aligned}$$

Combining (4.3.8) and (4.3.9) gives the result. □

Recall as we discussed earlier that a closed geodesic γ in G corresponds to conjugacy classes $w\in\mathcal{C}(F)$ with $\|\gamma\|=\|w\|=\min_{x\in w}|x|=n$ and we have $\ell(w)=\ell(\gamma)$, then by Lemma 4.3.3 for $x\in X_C^+$, $\sigma^n x=x$

$$\ell(\gamma)=r^n(x).$$

Using the function $r:X_C^+\to\mathbb{R}$ we define the function $R:\widetilde{X}\to\mathbb{R}$ by $R(x,y)=r(x)-r(y)$. By summing over periodic points, x and y with $\sigma^n x=x$ and $\sigma^n y=y$, gives a relation between the function R and the difference between the length of two closed geodesics γ and γ' as follows

$$R^n(x,y)=r^n(x)-r^n(y)=\ell(\gamma)-\ell(\gamma').$$

4.4 Proofs of Theorem 4.2.1 and Theorem 4.2.2

Although the counting problem in this section gives a new ordering to pairs of closed geodesics in a graph G, the proof of Theorem 4.2.1 and Theorem 4.2.2 will follow the proofs of Theorem 3.1.1 and Theorem 3.1.3 respectively. This is possible due to the fact that the coding of conjugacy classes gives a correspondence between conjugacy classes and periodic orbits. Then, since there is a correspondence between conjugacy classes and closed geodesics this again gives a correspondence between closed geodesics in G and periodic orbits in the shift space and so this is similar to the coding we introduced in section 3.1.1. For this correspondence, we also have the word length of the corresponding conjugacy classes of closed geodesics coded by the period of the periodic orbits. This is exactly what has been done in the earlier problem in Chapter 3, where we ordered closed geodesics by their word length on the graph and this word length is equal to the period of the corresponding periodic orbit in the shift space.

Moreover, the coding of the length function in this chapter results in the same relation we had in Chapter 3. In this chapter, for a periodic point $x \in X_A^+$ of period n with the corresponding closed geodesic γ which follows from the correspondence with conjugacy class $w \in \mathcal{C}(F)$ with $\|w\| = n$ and $\ell(w) = \ell(\gamma)$, we have the relation $r^n(x) = \ell(\gamma)$. Therefore, the non-lattice condition that we require in order to obtain our results in Chapter 3 still apply in the same way here and in particular we also have the function r is non-lattice. Consequently, this means that all the analysis and calculations used in Chapter 3 to prove Theorem 3.1.1 and Theorem 3.1.3 can be carried out in exactly the same way to prove Theorem 4.2.1 and Theorem 4.2.2 respectively.

Chapter 5

Pairs of closed geodesics with fixed homology classes in $H_1(G, \mathbb{Z})$

In this chapter, we study a counting problem similar to that we studied earlier in Chapter 3. However, we shall impose one more constraint on the closed geodesics we count. We shall count pairs of closed geodesics ordered by word length and such that the difference between their geometric length lie in a fixed interval $[a, b]$ and their homology classes are fixed.

We consider a metric graph (G, ℓ) with fundamental group $\pi_1(G) = F_k$, $(k \geq 2)$, a free group with k generators. Then the first homology group of G is the abelianization of its fundamental group, i.e. $H_1(G, \mathbb{Z}) = \pi_1(G)/[\pi_1(G), \pi_1(G)] \cong \mathbb{Z}^k$. Let γ and γ' be two closed geodesics in (G, ℓ), and let $[\gamma], [\gamma']$ be their corresponding homology classes in $H_1(G, \mathbb{Z})$, respectively. For simplicity, we are going to consider the case where we fix $([\gamma], [\gamma']) = (0, 0) \in \mathbb{Z}^{2k}$ and the word length of both closed geodesics is the same, i.e. $|\gamma|, |\gamma'| = N$. So, we want to find an asymptotic formula for the number of elements in the following set

$$\{(\gamma, \gamma') : \gamma, \gamma' \text{ closed geodesic, } |\gamma|, |\gamma'| = N, \ell(\gamma) - \ell(\gamma') \in [a, b], [\gamma], [\gamma'] = 0\}.$$

To obtain an asymptotic we are going to use the same approach we used in

Chapter 3. We are going to use the subshift of finite type X_A^+ we defined in Chapter 3 with the two functions r and R defined on X_A^+ (see subsection 3.1.1). However, we define an additional function here to represent the homology class of a closed geodesic in terms of the shift space X_A^+, we introduce a locally constant function $f : X_A^+ \to \mathbb{Z}^k$ depending on one coordinate, with components $f_j : X_A^+ \to \mathbb{Z}$ for $j = 1, \dots, k$. Clearly each f_j is Hölder continuous for every positive exponent. We define f by $f(e_0, e_1, \dots) = [C_{o(e_0)} \circ e_0 \circ C_{t(e_0)}^{-1}]$, where for any $v \in V$, C_v is a chosen path from a fixed vertex, say v_0, to v. Summing f over a periodic orbit of x such that $\sigma^n x = x$ and considering γ as the corresponding closed geodesic to this periodic orbit, gives that

$$\begin{aligned} f^n(x) &= f(e_0, e_1, \dots, e_{n-1}) + f(e_1, , \dots, e_{n-2}) + \dots + f(e_{n-1}, e_0, \dots, e_{n-2}) \\ &= [C_{o(e_0)} \circ e_0 \circ C_{t(e_0)}^{-1}] + [C_{o(e_1)} \circ e_1 \circ C_{t(e_1)}^{-1}] + \dots \\ &\quad + [C_{o(e_{n-1})} \circ e_{n-1} \circ C_{t(e_{n-1})}^{-1}] \\ &= [C_{o(e_0)} \circ e_0 \circ e_1 \circ \dots \circ e_{e_{n-1}} \circ C_{t(e_{n-1})}^{-1}] = [\gamma] \in \mathbb{Z}^k. \end{aligned} \tag{5.0.1}$$

We see that there is a one-to-one correspondence between between closed geodesics in (G, ℓ), with $|\gamma| = N$ and closed orbits $\{x, \sigma x, \dots, \sigma^{n-1} x\}$ in X_A^+, where the length of the closed geodesic is given by $r^n(x) = \ell(\gamma)$ and its homology class is given by $f^n(x) = [\gamma]$.

The following two lemmas describe some of the important properties of the function f, which will be very useful in the subsequent calculations.

Lemma 5.0.1. *(i) The set $\{f^n(x) : \sigma^n x = x, n \in \mathbb{N}\}$ generates $\mathbb{Z}^k$.*

(ii) $\int f d\mu = 0$, where μ is the measure of maximal entropy of σ.

Proof. To prove (i), by the one-to-one correspondence between closed geodesics and periodic orbits, as we mentioned earlier, we have that the set $\{f^n(x) : \sigma^n x = x, n \in \mathbb{N}\} = \{[\gamma] : |\gamma| = n, n \in \mathbb{N}\} = H_1(G, \mathbb{Z})$. Then, since $H_1(G, \mathbb{Z}) \cong \mathbb{Z}^k$, we can say that $\{f^n(x) : \sigma^n x = x, n \in \mathbb{N}\}$ generates $\mathbb{Z}^k$.

For (ii), if μ is the measure of maximal entropy on X_A^+, the periodic points $x \in X_A^+$ with $\sigma^n x = x$ are equidistributed with respect to μ. More precisely, we have the following identity

$$\int f\,d\mu = \lim_{n\to+\infty} \frac{1}{\#\{x : \sigma^n x = x\}} \sum_{\sigma^n x = x} \frac{f^n(x)}{n},$$

(Theorem 8.17, [30]). By the one-to-one correspondence between periodic orbits in X_A^+ and closed geodesics γ in (G, ℓ), we have that $f^n(x) = [\gamma]$. Then since γ and $-\gamma$ have the same word length but reverse homology classes, i.e. $|\gamma| = |-\gamma|$ and $[\gamma] = -[\gamma]$,

$$\sum_{\sigma^n x = x} \frac{f^n(x)}{n} = \sum_{|\gamma| = n} \frac{[\gamma]}{n} = 0.$$

Hence,

$$\int f\,d\mu = 0.$$

□

Lemma 5.0.2. *$\langle f, v\rangle$ is not cohomologous to a constant unless $v = 0$.*

Proof. Suppose that $\langle f, v\rangle$ is cohomologous to a constant, say $c \in \mathbb{R}$. Integrating with respect to the maximal measure μ and applying part (ii) of Lemma 5.0.1, gives that

$$c = \int c\,d\mu = \int \langle f, v\rangle\,d\mu = 0.$$

Hence $\langle f, v\rangle$ is cohomologous to 0. Summing over periodic orbit of period n, $\langle f^n(x), v\rangle = 0$ for all $x \in X_A^+$ such that $\sigma^n x = x$. But, by Lemma 5.0.1 (i), $\{f^n(x) : \sigma^n x = x, n \in \mathbb{N}\} = \mathbb{Z}^k$, i.e, $f^n(x)$ spans $\mathbb{Z}^k$, so $v = 0$. □

Now we can start the counting process by defining the following counting function.

$$\psi_N(\chi) = \sum_{\substack{\sigma^N x = x, \sigma^N y = y \\ f^N(x), f^N(y) = 0}} \chi(r^N(x) - r^N(y)),$$

where χ is a smooth function such that its Fourier transform is compactly supported. We also require that for t close to 0, say $t \in (-\epsilon, \epsilon)$, we have $\widehat{\chi}(t) = \widehat{\chi}(0) + O(|t|)$. Let $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$. By considering an orthogonal identity of the form:

$$\int_{\mathbb{T}^k} e^{2\pi i \langle f^N(x), v\rangle}\, dv \quad = \quad \begin{cases} 1 & \text{if } f^N(x) = 0 \\ 0 & \text{otherwise} \end{cases}$$

and using the Fourier inversion formula, we can write $\psi_N(\chi)$ as follows.

$$\begin{aligned} \psi_N(\chi) &= \frac{1}{2\pi} \int_{\mathbb{T}^k} \int_{\mathbb{T}^k} \int_{-\infty}^{+\infty} \sum_{\substack{\sigma^N x = x \\ \sigma^N y = y}} e^{it(r^N(x) - r^N(y))} \\ &\qquad e^{2\pi i \langle f^N(x), v\rangle} e^{2\pi i \langle f^N(y), w\rangle} \widehat{\chi}(-t)\, dt\, dw\, dv \\ &= \frac{1}{2\pi} \int_{\mathbb{T}^k} \int_{\mathbb{T}^k} \int_{-\infty}^{+\infty} \sum_{\sigma^N x = x} e^{itr^N(x) + 2\pi i \langle f^N(x), v\rangle} \\ &\qquad \sum_{\sigma^N y = y} e^{-itr^N(y) + 2\pi i \langle f^N(x), w\rangle} \widehat{\chi}(-t)\, dt\, dv\, dw. \end{aligned}$$

Using Lemma 3.1.12, we have for any $x_0, y_0 \in X_A^+$ there exists $\theta \in (0,1)$ such that

$$\begin{aligned} \psi_N(\chi) &= \frac{1}{2\pi} \int_{\mathbb{T}^k} \int_{\mathbb{T}^k} \int_{-\infty}^{+\infty} L^N_{itr + 2\pi i \langle f, v\rangle} 1(x_0) L^N_{-itr + 2\pi i \langle f, w\rangle} 1(y_0) \\ &\quad \times (1 + O(\max\{1, |t| N\theta^N\}))(1 + O(\max\{1, |t| N\theta^N\})) \widehat{\chi}(-t)\, dt\, dv\, dw. \end{aligned}$$

To estimate $\psi_N(\chi)$ we shall study the spectrum of the transfer operators $L_{itr+2\pi i\langle f, v\rangle}$ and $L_{-itr+2\pi i\langle f, w\rangle}$.

We introduce the quotient subgroup $\mathbb{Z}^k/\Delta_f$ which has been proven to be cyclic generated by $\Delta_f + c_f$ in [18], where

$$\Delta_f = \{f^n(x) - f^n(y) : \sigma^n x = x, \sigma^n y = y, n \in \mathbb{N}\}$$

and

$$c_f = f^{n+1}(z) - f^n(x),$$

for a chosen x, z with $\sigma^n x = x, \sigma^{n+1} z = z$. We also introduce a character subgroup $\Delta_f^\perp$ of $\widehat{\mathbb{Z}}^k$, defined by

$$\Delta_f^\perp = \{\xi \in \widehat{\mathbb{Z}}^k : \xi = 1 \text{ on } \Delta_f\}$$

In [18] Parry and Schmidt showed that $\mathbb{Z}^k/\Delta_f$ has finite order. Let $d = |\mathbb{Z}^k/\Delta_f|$. Since $\mathbb{Z}^k/\Delta_f$ is cyclic, $\mathbb{Z}^k/\Delta_f \cong \widehat{\mathbb{Z}^k/\Delta_f}$. Also, $\Delta_f^\perp \cong \widehat{\mathbb{Z}^k/\Delta_f}$. Then, $|\Delta_f^\perp| = |\widehat{\mathbb{Z}^k/\Delta_f}| = |\mathbb{Z}^k/\Delta_f| = d$. In the same paper the following lemma was also proven.

Lemma 5.0.3. *(Proposition 5.1, [18]) The function $f - c_f$ is cohomologous to a function g taking values in Δ_f. Moreover, the function g can be chosen to be a function of two coordinates.*

We shall use this lemma in the proof of the next lemma about the spectral radius of the transfer operator $L_{itr+2\pi i\langle f,v\rangle}$ and its maximal eigenvalues.

Lemma 5.0.4. *(i) Let $r : X_A^+ \to \mathbb{R}^+$ be a non-lattice function. Then, $\rho(L_{itr+2\pi i\langle f,v\rangle}) = e^{P(0)}$ if and only if $(t,v) = (0, v^{(j)})$, $j = 0, \ldots, d-1$ where $v^{(j)}$ are characterised by the set $\{e^{2\pi i\langle \cdot, v^{(j)}\rangle} : j = 0, \ldots, d-1\} = \Delta_f^\perp$.*

(ii) At the values $(t,v) = (0, v^{(j)})$, $j = 0, \ldots, d-1$, each transfer operator $L_{itr+2\pi i\langle f,v^{(j)}\rangle}$ has a simple and unique maximal eigenvalue equal to $e^{P(0)}e^{2\pi ij/d}$.

Proof. For the proof of (i), by the Complex RPF theorem, if the transfer operator $L_{itr+2\pi i\langle f,v\rangle}$ has an eigenvalue of modulus $e^{P(0)}$, then it is simple and unique and $L_{itr+2\pi i\langle f,v\rangle} = e^{ia}ML_0M^{-1}$, where M is a multiplication operator and $e^{ia} \in \mathbb{C}$, $|e^{ia}| = 1$ and so $\rho(L_{itr+2\pi i\langle f,v\rangle}) = e^{P(0)}$. By Lemma 2.3.9 part(i) this is equivalent to saying that $\rho(L_{itr+2\pi i\langle f,v\rangle}) = e^{P(0)}$ if and only if $tr+2\pi\langle f,v\rangle$ is cohomologous to $a+\phi$,where $a \in \mathbb{R}, \phi \in C(X_A^+, 2\pi\mathbb{Z})$, i.e.

$$tr + 2\pi\langle f, v\rangle - a = \phi + h \circ \sigma - h,$$

where $h \in C(X_A^+, \mathbb{R})$. Summing over a periodic orbit of x with $\sigma^n x = x$ gives that

$$tr^n(x) - na + 2\pi\langle f^n(x), v\rangle \in 2\pi\mathbb{Z}. \tag{5.0.2}$$

Let γ be a closed geodesic with $|\gamma| = n$, corresponding to the periodic orbit with $\sigma^n x = x$. Then $r^n(x) = \ell(\gamma)$ and $f^n(x) = [\gamma]$. Thus, we have that for $t \neq 0$

$$\ell(\gamma) - \frac{a}{t}|\gamma| + \frac{2\pi}{t}\langle[\gamma], v\rangle \in \left(\frac{2\pi}{t}\right)\mathbb{Z}. \tag{5.0.3}$$

Now consider the closed geodesic $-\gamma$ which has the reverse direction of γ. Then, $\ell(-\gamma) = \ell(\gamma)$, $|-\gamma| = |\gamma|$ and $[-\gamma] = -[\gamma]$. Therefore,

$$\ell(\gamma) - \frac{a}{t}|\gamma| - \frac{2\pi}{t}\langle[\gamma], v\rangle \in \left(\frac{2\pi}{t}\right)\mathbb{Z}. \tag{5.0.4}$$

Adding (5.0.3) and (5.0.4) gives that

$$\ell(\gamma) - \frac{a}{t}|\gamma| \in \left(\frac{2\pi}{2t}\right)\mathbb{Z},$$

which contradicts the assumption that r is non-lattice. Hence, t must equal to 0. Now substituting back into our first assumption that $tr + 2\pi\langle f, v\rangle$ is cohomologous to a constant plus a function in $C(X_A^+, 2\pi\mathbb{Z})$, for $t = 0$, implies that we are now assuming that $2\pi\langle f, v\rangle$ is cohomologous to a constant plus a function in $C(X_A^+, 2\pi\mathbb{Z})$. As we can see from (5.0.2), this means $2\pi\langle f^n(x), v\rangle - na \in 2\pi\mathbb{Z}$. This is equivalent to saying that

$$e^{2\pi i\langle f^n(x), v\rangle} = e^{2\pi i na}. \tag{5.0.5}$$

We need to find for what values $v \in \mathbb{R}^k/\mathbb{Z}^k$ this equation is satisfied. For this we are going to define an element of the character group of $\mathbb{Z}^k$, here we are going to follow the argument used to prove Proposition 3 in [20]. Note that if we define a character $\xi \in \widehat{\mathbb{Z}}^k$ such that $\xi(\cdot) = e^{2\pi i\langle\cdot, v\rangle}$, v satisfies (5.0.5). Then, for x, y with $\sigma^n x = x$ and $\sigma^n y = y$,

$$\xi(f^n(x) - f^n(y)) = e^{2\pi i \langle f^n(x) - f^n(y), v\rangle} = e^{2\pi i n a} e^{-2\pi i n a} = 1,$$

i.e. $\xi = 1$ on Δ_f, and so $\xi \in \Delta_f^{\perp}$. Therefore,

$$\{e^{2\pi i \langle \cdot, v\rangle} : v \text{ satisfies } e^{2\pi i \langle f^n(x), v\rangle} = e^{2\pi i n a}, \sigma^n x = x, n \in \mathbb{N}\} \subset \Delta_f^{\perp}.$$

Now, if we assume $\xi \in \Delta_f^{\perp}$, then by Lemma 5.0.3, we have that $f^n(x) = g^n(x) + nc_f$, with $g(\sigma^j x) \in \Delta_f$, $j = 0, \ldots, n-1$. Hence,

$$\xi(f^n(x)) = \xi(g^n(x))\xi(nc_f) = \prod_{j=0}^{n-1} \xi(g(\sigma^j x))\xi^n(c_f) = \xi^n(c_f). \tag{5.0.6}$$

We also have $\xi^d(c_f) = \xi(dc_f) = 1$, since $dc_f \in \Delta_f$. So $\xi(c_f)$ is the dth root of unity. Let $\xi(c_f) = e^{2\pi i a}$, then combining the definition of ξ with (5.0.6) we have

$$e^{2\pi i \langle f^n(x), v\rangle} = \xi(f^n(x)) = e^{2\pi i n a},$$

so v satisfies (5.0.5). Hence,

$$\Delta_f^{\perp} \subset \{e^{2\pi i \langle \cdot, v\rangle} : v \text{ satisfies } e^{2\pi i \langle f^n(x), v\rangle} = e^{2\pi i n a}, \sigma^n x = x, n \in \mathbb{N}\}.$$

So we conclude that

$$\{e^{2\pi i \langle \cdot, v\rangle} : v \text{ satisfies } e^{2\pi i \langle f^n(x), v\rangle} = e^{2\pi i n a}, \sigma^n x = x, n \in \mathbb{N}\} = \Delta_f^{\perp}.$$

For the proof of (ii), as the cyclic group $\mathbb{Z}^k/\Delta_f$ is generated by $\Delta_f + c_f$ and has order d, there are d characters each determined by sending $\Delta_f + c_f$ to a different dth root of unity in $\mathbb{C}$. Since $\xi(\Delta_f + c_f) = \xi(c_f)$ and $\xi(c_f)$ is a dth root of unity, we have the following,

$$\{\xi(c_f) : \xi \in \Delta_f^{\perp}\} = \{e^{2\pi i j/d} : 0 \leq j \leq d-1\}. \tag{5.0.7}$$

From part (i), there are d values of $(t,v) = (0, v^{(j)}), 0 \le j \le d-1$, for which $\rho(L_{itr+2\pi i\langle f,v\rangle}) = e^{P(0)}$. Then, each transfer operator $L_{itr+2\pi i\langle f,v^{(j)}\rangle}$ has an eigenvalue $\lambda(t,v)$ such that $|\lambda(t,v)| = e^{P(0)}$. By the Complex RPF theorem this eigenvalue is simple, unique and maximal. Combining this with (5.0.7), the statement in part (ii) of the lemma follows. □

To complete our study of the spectrum of the transfer operator $L_{itr+2\pi i\langle f,v\rangle}$, we need to look at the rest of the spectrum apart from the maximal eigenvalue. As a consequence of Lemma 5.0.4 and the Complex RPF theorem we have the following.

(i) The rest of the spectrum of the transfer operator $L_{itr+2\pi i\langle f,v\rangle}$ apart from the maximal eigenvalues at $(t,v) = (0, v^{(j)}), j = 0, \ldots, d-1$, is contained in a disc of radius strictly less than $e^{P(0)}$.

(ii) For $(t,v) \neq (0, v^{(j)})$, $j = 0, \ldots, d-1$, the transfer operator $L_{itr+2\pi i\langle f,v\rangle}$ has a spectral radius strictly less than $e^{P(0)}$.

We also have the following lemma.

Lemma 5.0.5. *(Lemma 3, [20]) For (t,v) close to $(0, v^{(j)})$, $j = 0, \ldots, d-1$ we have*

$$e^{P(itr+2\pi i\langle f,v\rangle)} = e^{2\pi ij/d} e^{P(itr+2\pi i\langle f,v-v^{(j)}\rangle)}.$$

We shall follow the strategy we used in Section 3.1.4, where we start the estimation of $\psi_N(\chi)$ by splitting the integration into two different parts. One part is going to be in a small neighbourhood of the values $(0, v^{(j)})$, $j = 0, \ldots, d-1$, where at these values, by Lemma 5.0.4 part (ii) the transfer operator $L_{itr+2\pi\langle f,v^{(j)}\rangle}$ has a maximal eigenvalue $e^{P(0)}e^{2\pi ij/d}$. Since these are simple and isolated eigenvalues, perturbation theory gives that $(t,v) \mapsto e^{P(itr+2\pi i\langle f,v\rangle)}$ is analytic on a small neighbourhood of $(0, v^{(j)})$, $j = 0, \ldots d-1$, i.e. the transfer operator $L_{itr+2\pi i\langle f,v\rangle}$ has a unique simple eigenvalue at $e^{P(itr+2\pi i\langle f,v\rangle)}$ in a sufficiently small neighbourhood of $(t,v) = (0, v^{(j)})$, $j = 0, \ldots, d-1$ and by Lemma 5.0.5 $e^{P(itr+2\pi i\langle f,v\rangle)} = e^{2\pi ij/d} e^{P(itr+2\pi i\langle f,v-v^{(j)}\rangle)}$ in these neighbourhoods. For the rest of the spectrum we apply (i). The other part is away from these values, we apply (ii). The same applies to the transfer operator $L_{-itr+2\pi i\langle f,w\rangle}$. Assuming that the Fourier transform of χ is compactly supported where the support is contained in an interval $[-M,M]$, for some $M>0$ and such that for $|t|<\epsilon$, $\widehat{\chi}(-t)=\widehat{\chi}(0)+O(|t|)$. Then, there exists $\theta\in(0,1)$ and for sufficiently small $\epsilon,\delta_1,\delta_2$ let

$$\begin{aligned} B(\delta_1,\delta_2,\epsilon) &= \mathbb{T}^k\times\mathbb{T}^k\times\mathbb{R}\setminus\bigcup_{j=0}^{d-1}\bigcup_{j'=0}^{d-1}\{w:\|w-v^{(j')}\|_2<\delta_1\} \\ &\quad \times\{v:\|v-v^{(j)}\|_2<\delta_2\}\times\{t:|t|<\epsilon\}. \end{aligned}$$

Then we have

$$\begin{aligned} \psi_N(\chi) &= \sum_{j=0}^{d-1}\sum_{j'=0}^{d-1}\frac{1}{2\pi}\int_{\|w-v^{(j')}\|_2<\delta_1}\int_{\|v-v^{(j)}\|_2<\delta_2}\int_{|t|<\epsilon}(e^{NP(itr+2\pi i\langle f,v\rangle)} \\ &\qquad +O(\theta^{N/2}e^{NP(0)}))(e^{NP(-itr+2\pi i\langle f,w\rangle}+O(\theta^{N/2}e^{NP(0)}))\widehat{\chi}(-t)\,dtdvdw \\ &+ \int_{B(\delta_1,\delta_2,\epsilon)}O(\theta^{N/2}e^{NP(0)})\widehat{\chi}(-t)\,dtdvdw \\ &= \sum_{j=0}^{d-1}e^{2\pi iNj/d}\sum_{j'=0}^{d-1}e^{2\pi iNj'/d}\frac{1}{2\pi}\int_{\|w-v^{(j')}\|_2<\delta_1}\int_{\|v-v^{(j)}\|_2<\delta_2}\int_{|t|<\epsilon} \\ &\qquad \times(e^{NP(itr+2\pi i\langle f,v-v^{(j)}\rangle)}e^{NP(-itr+2\pi i\langle f,w-v^{(j')}\rangle)})(1+O(e^{2P(0)N}\theta^N)) \\ &\qquad \times\widehat{\chi}(-t)\,dtdvdw+O(e^{2P(0)N}\theta^N) \\ &= \left(\sum_{j=0}^{d-1}e^{2\pi iNj/d}\right)^2\frac{1}{2\pi}\int_{\|w\|_2<\delta_1}\int_{\|v\|_2<\delta_2}\int_{|t|<\epsilon}(e^{NP(itr+2\pi i\langle f,v\rangle)} \\ &\qquad e^{NP(-itr+2\pi i\langle f,w\rangle)})\widehat{\chi}(-t)\,dtdvdw+O(e^{2P(0)N}\theta^N) \end{aligned}$$

In the last step we have used a simple substitution to write the maximal eigenvalues for each $j=0,\ldots,d-1$ as $e^{2\pi ij/d}e^{P(itr+2\pi i\langle f,v\rangle)}$ where (t,v) is in a small neighbourhood of $(0,v^{(0)})=(0,0)\in\mathbb{R}\times\mathbb{T}^k$. Note that

$$\left(\sum_{j=0}^{d-1}e^{2\pi iNj/d}\right)^2 = \begin{cases} d^2 & \text{if } d\mid N \\ 0 & \text{otherwise.}\end{cases}$$

In fact, when d does not divide N there exists no periodic point $x\in X_A^+$ with

$\sigma^N x = x$ and $f^N(x) = 0$. Therefore, $\psi_N(\chi) = 0$ in this situation. To explain this we follow the argument given in [20], suppose $x \in X_A^+$ with $\sigma^N x = x$ and $f^N(x) = 0$. By Lemma 5.0.3 we have that $f - c_f$ is cohomologous to $g \in \Delta_f$ and so $f^N(x) - Nc_f = g^N(x) \in \Delta_f$. Then, since $f^N(x) = 0$, we get $-Nc_f \in \Delta_f$. This implies that $N(\Delta_f + c_f) = \Delta_f + Nc_f = \Delta_f$. We know that $\Delta_f + c_f$ is the generator of $\mathbb{Z}^k/\Delta_f$ which has the order d. Hence, d divides N.

Writing a function $F : \widetilde{X} \to \mathbb{Z}^{2k}$ such that $F(x, y) = (f(x), f(y))$, $u = (v, w)$, also as in Lemma 3.1.14 we can write $e^{P(itr+2\pi i\langle f,v\rangle)}e^{P(-itr+2\pi i\langle f,w\rangle)} = e^{P(itR+2i\pi\langle F,u\rangle)}$. Hence, we have

$$\psi_N(\chi) = \frac{d^2}{2\pi}\int_{\|u\|_2<\delta}\int_{|t|<\epsilon} e^{NP(itR+2\pi i\langle F,u\rangle)}\widehat{\chi}(-t)\,dtdu + O(\theta^N e^{2P(0)N}).$$

To help us analyse the function $(t, u) \mapsto e^{P(itR+2\pi i\langle F,u\rangle)}$, the following lemma studies the partial derivatives of the pressure function $P(itR + 2\pi i\langle F, u\rangle)$.

Lemma 5.0.6. *We have the following derivatives for the function* $(t, u) \mapsto P(itR + 2\pi i\langle F, u\rangle)$*:*

(i) For $j = 1, \ldots, 2k$

$$\left.\frac{\partial P(itR + 2\pi i\langle F, u\rangle)}{\partial u_j}\right|_{(t,u)=(0,0)} = 2\pi i\int F_j\,d\widetilde{\mu} = 0$$

and

$$\left.\frac{\partial P(itR + 2\pi i\langle F, u\rangle)}{\partial t}\right|_{(t,u)=(0,0)} = i\int R\,d\widetilde{\mu} = 0.$$

(ii) For $j = 1, \ldots, 2k$

$$\left.\frac{\partial^2 P(itR + 2\pi i\langle F, u\rangle)}{\partial^2 u_j}\right|_{(t,u)=(0,0)} = (2\pi i)^2\lim_{n\to\infty}\frac{1}{n}\int(F_j^n(x))^2 d\widetilde{\mu} < 0,$$

and

$$\left.\frac{\partial^2 P(itR + 2\pi i\langle F, u\rangle)}{\partial^2 t}\right|_{(t,u)=(0,0)} = i^2\lim_{n\to\infty}\frac{1}{n}\int(R^n(x))^2 d\widetilde{\mu} < 0.$$

(iii) $\nabla P(itR + 2\pi i\langle F, u\rangle)\Big|_{(t,u)=(0,0)} = 0,$

(iv) $\nabla^2 P(itR + 2\pi i\langle F, u\rangle)\Big|_{(t,u)=(0,0)}$ *is strictly negative definite.*

Proof. For part (i), since the two functions f and r are locally constant functions and so Hölder continuous functions for any positive exponent, the first partial derivatives follow from Lemma 2.3.8 (part (i)). By Lemma 5.0.1, we have $\int f\, d\mu = 0$, this implies that $\int F\, d\widetilde{\mu} = 0$. Also, by Lemma 3.1.9, $\int R\, d\widetilde{\mu} = 0$.

For part (ii), we apply Lemma 2.3.8 (part (ii)) to get the second partial derivatives. Then, since $\langle f, v\rangle$ is not cohomologous to a constant by Lemma 5.0.2 then $\langle F, u\rangle$ is not cohomologous to a constant. Also, by Lemma 3.1.8 R is not cohomologous to a constant. Therefore, we have the second partial derivatives are strictly negative. Part (iii) of the lemma follows from part (i). Also, part (iv) follows from part (ii).

$\square$

Lemma 5.0.7. *In a small neighbourhood of* $(t, u) = (0, 0) \in \mathbb{R} \times \mathbb{T}^{2k}$*, the function* $(t, u) \mapsto e^{P(itR+2\pi i\langle F,u\rangle)}$ *has a Taylor expansion*

$$\begin{aligned} e^{P(itR+2\pi i\langle F,u\rangle)} &= e^{P(0)}\big(1 - \langle (t,u), \nabla^2 P(itR + 2\pi i\langle F,u\rangle)|_{(0,0)}(t,u)\rangle \\ &\qquad + O(\max\{|t|^3, \|u\|^3\})\big)\,. \end{aligned}$$

Moreover, there exists a change of coordinates τ *such that for* (t, u) *close* $(0, 0)$*, we have* $e^{P(itR+2\pi i\langle F,u\rangle)} = e^{2P(0)}(1 - \|\tau\|_2^2)$*, where* $\tau = (\tau_1, \dots, \tau_{2k+1})$

Proof. The function $(t, u) \mapsto e^{P(itR+2\pi i\langle F,u\rangle)}$ is analytic on a small neighbourhood of $(t, u) = (0, 0)$, so we can write its Taylor expansion as

$$\begin{aligned} e^{P(itR+2\pi i\langle F,u\rangle)} = {} & e^{2P(0)} + e^{2P(0)}\langle (t,u), \nabla P(itR + 2\pi i\langle F,u\rangle)|_{(0,0)}\rangle \\ & + e^{2P(0)}\langle (t,u), \nabla^2 P(itR + 2\pi i\langle F,u\rangle)|_{(0,0)}(t,u)\rangle \\ & \qquad + O(\max\{|t|^3, \|u\|^3\}) \\ & = e^{2P(0)}\big(1 - \langle (t,u), \nabla^2 P(itR + 2\pi i\langle F,u\rangle)|_{(0,0)}(t,u)\rangle \\ & \qquad + O(\max\{|t|^3, \|u\|^3\})\big)\,. \end{aligned}$$

We applied Lemma 5.0.6 (iii) and (iv) in the last equation. We see that a function $g(t,u) = e^{P(itR+2\pi i\langle F,u\rangle)}/e^{2P(0)}$, has $g(0,0)=1$, $\nabla g(0,0)=0$ and $\nabla^2 g(0,0)<0$. Thereby, we can apply the Morse lemma to make a change of coordinates on a small neighbourhood of $(t,u)=(0,0)$ such that $g(t,u) = 1-\|\tau\|_2^2$, where τ is close to 0 in $\mathbb{R}^{2k+1}$. □

Following a similar type of calculations as in [20], we use this lemma and we estimate the other terms in a small neighbourhood of $(t,u)=(0,0)$, hence we have for small $a>0$

$$\begin{aligned}\psi_N(\chi) &= \frac{d^2}{2\pi}\int_{\|\tau\|_2<a}(e^{2P(0)}(1-\|\tau\|_2^2))^N\widehat{\chi}(0)(1+O(\|\tau\|_2))|J(\tau)|\,d\tau \\ &\quad +O(\theta^N e^{2P(0)N}) \\ &= \frac{d^2}{2\pi}|J(0)|\widehat{\chi}(0)e^{2P(0)N}\int_{\|\tau\|_2<a}(1-\|\tau\|_2^2)^N(1+Q(\tau))\,d\tau \\ &\quad +O(\theta^N e^{2P(0)N}),\end{aligned}$$

where

(1) $|J(\tau)|$ is the jacobian determinant of the change of coordinates.

(2) $Q(\tau)$ is defined by $|J(0)|(1+Q(\tau)) = |J(\tau)|$, such that $Q(\tau)$ contains all terms of the form $O(\|\tau\|_2)$.

(3) $|J(0)| = 2^{k+1/2}(\det D)^{-1/2}$, where $D = -\nabla^2 P(itR+2\pi i\langle F,u\rangle)|_{(t,u)=(0,0)}$.

To see how we obtained the formula in (3), we follow the argument given by Sharp in [25]. Define a $(2k+1)\times(2k+1)$ matrix by $M(i,j) = [\frac{\partial\tau_i(\vartheta)}{\partial\vartheta_j}]_{\vartheta=0}$, where $\vartheta = (t,u_1,\dots,u_{2k})$. Then we have

$$\begin{aligned}D(i,j) &= -(\nabla^2 P(itR+2\pi i\langle F,u\rangle)|_{(t,u)=(0,0)})(i,j) \\ &= 2\sum_{l=1}^{2k+1}\left[\frac{\partial\tau_l(\vartheta)}{\partial\vartheta_i}\right]_{\vartheta=0}\left[\frac{\partial\tau_l(\vartheta)}{\partial\vartheta_j}\right]_{\vartheta=0} = 2(M^TM)(i,j).\end{aligned}$$

Since $\det M = \det M^T$,

$$\det D = 2^{2k+1}(\det M)^2.$$

Also we have that $|\det M| = 1/|J(0)|$. Hence the formula for $|J(0)|$ follows.

Changing τ to spherical polar coordinates $(\rho, \Omega), 0 \leq \rho \leq a, \Omega \in S^{(2k+1)-1} = S^{2k}$, we get that

$$\begin{aligned}\psi_N(\chi) = \quad & \frac{d^2}{2\pi}|J(0)|\mathrm{vol}(S^{2k})\widehat{\chi}(0)e^{2P(0)N}\int_{S^{2k}}\int_0^a (1-\rho^2)^N \\ & \times(1+Q(\rho\Omega))\rho^{2k}\, d\rho d\Omega + O(e^{2P(0)N}\theta^N).\end{aligned}$$

We shall estimate the following integrals.

$$\begin{aligned}\int_{S^{2k}}\int_0^a (1-\rho^2)^N\rho^{2k}(1+Q(\rho\Omega))\, d\rho d\Omega = \int_0^a (1-\rho^2)^N\rho^{2k}\, d\rho \\ + \int_{S^{2k}}\int_0^a (1-\rho^2)^N\rho^{2k}Q(\rho\Omega)\, d\rho d\Omega. \qquad (5.0.8)\end{aligned}$$

To estimate the first part of the integral we use a simple substitution $\eta = \rho^2$ then we have

$$\begin{aligned}\int_0^a (1-\rho^2)^N\rho^{2k}\, d\rho &= \frac{1}{2}\int_0^{a^2}(1-\eta)^N\eta^{k-1/2}\, d\eta \\ &= \frac{1}{2}\int_0^1 (1-\eta)^N\eta^{k-1/2}\, d\eta - \frac{1}{2}\int_{a^2}^1 (1-\eta)^N\eta^{k-1/2}\, d\eta \\ &= \frac{\Gamma(k+1/2)\Gamma(N+1)}{\Gamma(N+1+k+1/2)} + O((1-a^2)^N).\end{aligned}$$

Using Stirling's formula we can see that

$$\frac{\Gamma(N+1)}{\Gamma(N+1+k+1/2)} \sim \frac{N^{N+1/2}e^{k+1/2}}{(N+k+1/2)^{N+k+1}}.$$

Then one can easily prove that $\lim_{N\to+\infty}\frac{N^{N+1/2}e^{k+1/2}N^{k+1/2}}{(N+k+1/2)^{N+k+1}} = 1$. Consequently,

$$\frac{\Gamma(N+1)}{\Gamma(N+1+k+1/2)} \sim \frac{1}{N^{k+1/2}}.$$

For the second part of the integral in (5.0.8), since $Q(\rho\Omega) = O(\rho)$ and following a similar type of calculations as in the first part of (5.0.8), we have

$$\begin{aligned}\left|\int_{S^{2k}}\int_0^a (1-\rho^2)^N \rho^{2k} Q(\rho\Omega)\, d\rho d\Omega\right| &\leq \text{Const.} \int_0^a (1-\rho^2)^N \rho^{2k+1}\, d\rho \\ &= \text{Const.} \int_0^{a^2} (1-\eta)^N \eta^k\, d\eta \\ &\leq \text{Const.} \int_0^1 (1-\eta)^N \eta^k\, d\eta \\ &= O\left(\frac{\Gamma(k+1)\Gamma(N+1)}{\Gamma(N+1+k+1)}\right)\end{aligned}$$

where $\frac{\Gamma(N+1)}{\Gamma(N+1+k+1)} \sim N^{-(k+1)}$.

Combining the estimations of the two parts of the integral in (5.0.8), we have the following asymptotic formula for $\psi_N(\chi)$ when d divides N

$$\psi_{dN}(\chi) \sim \frac{C\, d^2 \int \chi(x)dx}{2\pi} \frac{e^{2P(0)dN}}{(dN)^{k+1/2}}, \qquad \text{as } N \to +\infty$$

where $C = |J(0)|\text{vol}(S^{2k})\Gamma(k+\frac{1}{2})$ and $\int \chi(x)dx = \widehat{\chi}(0)$. In the case where d does not divide N, $\psi_N(\chi) = 0$ as we have explained earlier. One can compare this asymptotic to the asymptotic we obtained in (3.1.15). The constant C can be given by $C = \frac{\pi^{k+1}(2k)!}{2^{k-1/2}(k!)^2 \det(D)^{1/2}}$.

Applying the same techniques and calculations after (3.1.15), in particular Lemma 3.1.15, Lemma 3.1.16 and Proposition 3.1.2, we obtain the following theorem.

Theorem 5.0.1. *Let (G,ℓ) be a non-bipartite metric graph such that for each vertex v, $\deg(v) \geq 3$. Suppose that the non-lattice condition holds. Then for fixed homology classes $(\alpha,\beta) = (0,0) \in \mathbb{Z}^{2k}$ and for all $a < b$,*

$$\begin{aligned}&\pi(dN,[a,b],(0,0)) \\ &= \#\{(\gamma,\gamma') : \gamma,\gamma' \textit{ closed geodesic}, |\gamma|,|\gamma'| = dN, \ell(\gamma)-\ell(\gamma') \in [a,b], [\gamma],[\gamma'] = 0\} \\ &\sim \frac{C\, d^2(b-a)}{2\pi} \frac{e^{2P(0)dN}}{(dN)^{k+5/2}}, \qquad \textit{as } N \to +\infty\end{aligned}$$

and $\pi(N,[a,b],(0,0)) = 0$ when d does not divide N.

Chapter 6

Counting pairs of geodesic paths in metric graphs

In this chapter we study counting pairs of geodesic paths in non-bipartite metric graphs (G, ℓ). The set of pairs of geodesics paths we are studying here is described in the same way as the set of closed geodesics studied in Chapter 3. So pairs of geodesic paths are ordered by word length, such that the difference between their geometric lengths lie in a fixed interval $[a, b] \subset \mathbb{R}$. We shall obtain an asymptotic for this set of geodesic paths where we assume that the paths start and end at prescribed vertices of the graph.

To obtain the asymptotic for the geodesic paths, we are going to code the metric graph (G, ℓ) by a subshift of finite type. As we have mentioned earlier, the counting problem we are studying here has the same constraints on the word length and the geometric length as the counting problem of closed geodesics in Chapter 3. Therefore, as expected, we will use the same subshifts of finite type X_A^+ we have used in Chapter 3, where we code (G, ℓ) by its oriented edges. Also, we are going use the same counting techniques and analysis we have used in counting closed geodesics in Chapter 3.

We can write X_A^+ as

$$X_A^+ = \{(\eta_i)_{i=0}^{\infty} : \eta_i \in E^o, A(\eta_i, \eta_{i+1}) = 1 \ \forall i \geq 0\}$$

and $\widetilde{X} = X_A^+ \times X_A^+$ defined as in Chapter 3. To represent the length of geodesic paths and the difference between the lengths of two geodesic paths, we recall the two functions $r : X_A^+ \to \mathbb{R}^+$ and $R : \widetilde{X} \to \mathbb{R}$ defined by $r(\eta_0, \eta_1, \dots) = \ell(\eta_0)$ and $R(x, y) = r(x) - r(y)$, $x, y \in X_A^+$. Now to represent the geodesic paths that start at a vertex I and end at a vertex J, we let

$$\mathrm{Out}(I) = \{e \in E^o : o(e) = I\}, \, \mathrm{In}(J) = \{e \in E^o : t(e) = J\}.$$

So a geodesic path $\eta = \eta_0, \dots, \eta_{n-1}$ that starts at a vertex I and ends at a vertex J, will have an initial edge $\eta_0 \in \mathrm{Out}(I)$ and a final edge $\eta_{n-1} \in \mathrm{In}(J)$.

We shall show how geodesic paths in a metric graph (G, ℓ) correspond to elements in X_A^+. Also, how the length of a geodesic path can be represented in terms of the function r. Let $\eta = \eta_0, \dots, \eta_{n-1}$ be a geodesic path with $\eta_0 \in \mathrm{Out}(I)$ and $\eta_{n-1} \in \mathrm{In}(J)$. We write $[\eta] \subset X_A^+$ for the corresponding cylinder and choose a point $x_\eta \in [\eta]$. Let $|[\eta]|$ denote the word length of the cylinder $[\eta]$. We then have the following:

(i) $r^n(x_\eta) = r(x_\eta) + r(\sigma x_\eta) + \cdots + r(\sigma^{n-1} x_\eta) = \ell(\eta_0) + \ell(\eta_1) + \cdots + \ell(\eta_{n-1}) = \ell(\eta)$;

(ii) if $\eta' = \eta'_0, \dots, \eta'_{n-1}$ is another geodesic path with $\eta'_0 \in \mathrm{Out}(I)$ and $\eta'_{n-1} \in \mathrm{In}(J)$ then $R^n(x_\eta, x_{\eta'}) = r^n(x_\eta) - r^n(x_{\eta'}) = \ell(\eta) - \ell(\eta')$;

(iii) for any $[a, b] \subset \mathbb{R}$,

$$\begin{aligned} &\#\{(\eta, \eta') : \eta, \eta' \text{ geodesic paths, } |\eta|, |\eta'| \leq N, \ell(\eta) - \ell(\eta') \in [a, b]\} \\ &= \#\{(x_\eta, x_{\eta'}) : |[\eta]|, \, |[\eta']| \leq N, R^n(x_\eta, x_{\eta'}) \in [a, b]\}. \end{aligned}$$

It will be important that the function r is a non-lattice function. As we have seen earlier in subsection 3.1.2, the function r is non-lattice if the non-lattice condition is satisfied, i.e. for any $c, d \in \mathbb{R}$, $\{\ell(\gamma) - c|\gamma| : \gamma \text{ is a closed geodesic }\} \not\subset d\mathbb{Z}$ (see Lemma 3.1.4). So in this chapter we still assume that the non-lattice condition holds to guarantee that the function r is non-lattice.

6.1 Pairs of geodesic paths in a fixed interval

Let $\mathcal{P}_n = \mathcal{P}_n(I, J) = \{\eta : \eta \text{ is a geodesic path from } I \text{ to } J, |\eta| = n\}$. We are looking for an asymptotic for the following set:

$$\{(\eta, \eta') : \eta, \eta' \in \mathcal{P}_n, n \leq N, \ell(\eta) - \ell(\eta') \in [a, b]\}.$$

The following theorem gives an asymptotic on how the number of elements in this set grows as N goes to $+\infty$.

Theorem 6.1.1. *Let (G, ℓ) be a non-bipartite graph with $\deg(v) \geq 3$, for each vertex v. Suppose that η and η' are geodesic paths in (G, ℓ) where both start at vertex I and end at vertex J. If the non-lattice condition is satisfied, then there exists constants $\beta > 1$, σ, $\kappa_{IJ} > 0$ such that for any $a < b$,*

$$\begin{aligned} \pi(I, J, N, [a, b]) &= \#\{(\eta, \eta') : \eta, \eta' \in \mathcal{P}_n, n \leq N, \ell(\eta) - \ell(\eta') \in [a, b]\} \\ &\sim \kappa_{IJ} \frac{(b-a)}{\sqrt{2\pi}\sigma} \frac{1}{(\beta-1)^2} \frac{\beta^{2N}}{\sqrt{N}}, \quad \text{as } N \to +\infty. \end{aligned}$$

The constants appearing in the asymptotic are defined as follows:

$$\kappa_{IJ} = \sum_{\substack{i,i' \in \mathrm{Out}(I) \\ j,j' \in \mathrm{In}(J)}} C_j C_{j'} (e_i w)(e_{i'} w),$$

where e_i is the row vector with 1 in ith position and 0 elsewhere and w is the right eigenvector corresponding to the maximal eigenvalue β of the matrix of A. The eigenvector w is normalised such that $u \cdot w = 1$, where u is the left eigenvector corresponding to β, normalised to be a probability vector. The constant C_j is defined by $e_j^T = C_j w + \overline{w}$, where $\overline{w} \in L_2$, the span of the generalised eigenspaces of the other eigenvalues of A.

To count the number of geodesic paths of word length less than or equal to N such that the difference between their geometric length is in a fixed interval

$[a, b]$, we start with the following counting function:

$$\Phi_N(\chi) = \sum_{n,m=1}^{N} \sum_{\substack{\eta \in \mathcal{P}_n \\ \eta' \in \mathcal{P}_m}} \chi(\ell(\eta) - \ell(\eta')),$$

where χ is a smooth function such that its Fourier transform is compactly supported. We also require that for t close to 0, say $t \in (-\epsilon, \epsilon)$, we have $\widehat{\chi}(t) = \widehat{\chi}(0) + O(|t|)$. Using the Fourier inversion formula,

$$\Phi_N(\chi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{n,m=1}^{N} \sum_{\substack{\eta \in \mathcal{P}_n \\ \eta' \in \mathcal{P}_m}} e^{it(\ell(\eta) - \ell(\eta'))} \widehat{\chi}(t)\, dt.$$

We define weighted matrices $A_s(i,j) = e^{s\ell(i)} A(i,j)$, $s \in \mathbb{C}$. We have that $A^{(n-1)}(i,j)$ is the number of geodesic paths, of length n, which start with an edge $i \in \mathrm{Out}(I)$ and end with edge $j \in \mathrm{In}(J)$. Hence, the number of geodesic paths of length n starting at vertex I and ending at vertex J is $\sum_{\substack{i \in \mathrm{Out}(I) \\ j \in \mathrm{In}(J)}} A^{(n-1)}(i,j)$. Then $\Phi_N(\chi)$ can be written in the following form assuming that the support of $\widehat{\chi}$ is contained in an interval $[-M, M]$, for some $M > 0$.

$$\begin{aligned} \Phi_N(\chi) &= \frac{1}{2\pi} \int_{-M}^{M} \sum_{n,m=1}^{N} \sum_{\substack{i \in \mathrm{Out}(I) \\ j \in \mathrm{In}(J)}} A_{it}^{(n-1)}(i,j) \sum_{\substack{i \in \mathrm{Out}(I) \\ j \in \mathrm{In}(J)}} A_{-it}^{(m-1)}(i,j) \widehat{\chi}(t)\, dt. \\ &= \frac{1}{2\pi} \int_{M}^{M} \sum_{n,m=1}^{N} \sum_{\substack{i \in \mathrm{Out}(I) \\ j \in \mathrm{In}(J)}} (e_i A_{it}^{(n-1)} e_j^T) \sum_{\substack{i \in \mathrm{Out}(I) \\ j \in \mathrm{In}(J)}} (e_i A_{-it}^{(m-1)} e_j^T) \widehat{\chi}(t)\, dt, \end{aligned}$$

In a similar way as we have analysed the spectrum of the transfer operator $L_{\pm itr}$, using the Complex RPF theorem, which is described by Lemma 3.1.6. For the matrices $A_{\pm it}$ we have that:

(i) At $t = 0$, $A_{it} = A$ has a simple maximal eigenvalue $\beta = \beta(0)$ and all other eigenvalues $|\mu(0)| < \beta(0)$. Furthermore, β has unique left and right eigenvectors u and w normalised so that u is a probability vector and $u \cdot w = 1$. This follows from the Perron-Frobenius theorem for matrices

(Theorem 2.3.5).

(ii) For t sufficiently close to 0, say $t \in (-\epsilon, \epsilon)$, by perturbation theory A_{it} has a simple eigenvalue $\beta(it)$ depending analytically on t such that $\beta(0) = \beta$, the maximal eigenvalue of the matrix A. For the rest of the eigenvalues $\mu(it)$ of A_{it}, we have $|\mu(it)| < \beta(0)$. Also, by perturbation theory, the left and right eigenvectors $u(it)$ and $w(it)$ associated to $\beta(it)$ depend analytically on t such that $u(it)$ and $w(it)$ are chosen so that $u(0) = u$ and $w(0) = w$. We can choose to normalise $w(it)$ such that $u(it) \cdot w(it) = 1$.

Let $L_1 = \mathbb{C}w$ and L_2 be the span of the generalised eigenspaces of the other eigenvalues of A. Then $\mathbb{C}^{|E^o|} = L_1 \oplus L_2$, so we can write e_j^T uniquely as $e_j^T = C_j w + \overline{w}$, where $\overline{w} \in L_2$. Therefore, the contribution to $\Phi_N(\chi)$ of $t \in (-\epsilon, \epsilon)$ can be calculated in the following way.

$$\begin{aligned}
&\int_{-\epsilon}^{\epsilon} \sum_{n,m=1}^{N} \sum_{\substack{i \in \mathrm{Out}(I) \\ j \in \mathrm{In}(J)}} (e_i A_{it}^{(n-1)}(C_j w(it) + e_i A_{it}^{(n-1)} \overline{w}(it))) \\
&\qquad \times \sum_{\substack{i \in \mathrm{Out}(I) \\ j \in \mathrm{In}(J)}} (e_i A_{-it}^{(m-1)}(C_j w(-it) + e_i A_{it}^{(m-1)} \overline{w}(-it)) \widehat{\chi}(t)\, dt \\
= &\int_{-\epsilon}^{\epsilon} \sum_{n,m=1}^{N} \sum_{\substack{i \in \mathrm{Out}(I) \\ j \in \mathrm{In}(J)}} (C_j e_i w(it) \beta(it)^{(n-1)} n + e_i \overline{w}(-it) \mu(it)^{(n-1)}) \\
&\qquad \times \sum_{\substack{i \in \mathrm{Out}(I) \\ j \in \mathrm{In}(J)}} (C_j e_i w(-it) \beta(-it)^{(m-1)} + e_i \overline{w}(-it) \mu(-it)^{(m-1)}) \widehat{\chi}(t)\, dt.
\end{aligned}$$

For $t \in [-M, -\epsilon] \cup [\epsilon, M]$ we use the relation that $r(x_\eta) = \ell(\eta_0)$, $x_\eta \in [\eta]$, to write $A_{it}(i,j) = A(i,j) e^{itr(x_\eta)}$. We know that the function r is non-lattice then by Wielandt's theorem, the matrix A_{it} does not have an eigenvalue of modulus equal to $\beta(0)$ and so for $t \neq 0$, $|\beta(it)| < \beta(0)$. Hence, we have that for some constant $\theta \in (0,1)$,

$$\begin{aligned}
\Phi_N(\chi) = & \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\sum_{n,m=1}^{N}\sum_{\substack{i\in\mathrm{Out}(I)\\ j\in\mathrm{In}(J)}}(C_je_iw(it)\beta(it)^{(n-1)}+O(\theta^{\frac{n}{2}}\beta(0)^n))\\
& \times\sum_{\substack{i\in\mathrm{Out}(I)\\ j\in\mathrm{In}(J)}}(C_je_iw(-it)\beta(-it)^{m-1}+O(\theta^{\frac{m}{2}}\beta(0)^m))\widehat{\chi}(t)\,dt\\
& +\int_{M\le|t|\le\epsilon}\sum_{n,m=1}^{N}(O(\theta^{\frac{n}{2}}\beta(0)^n))(O(\theta^{\frac{m}{2}}\beta(0)^m))\widehat{\chi}(t)\,dt\\
= & \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\sum_{n,m=1}^{N}\sum_{\substack{i,i'\in\mathrm{Out}(I)\\ j,j'\in\mathrm{In}(J)}}(C_jC_{j'}(e_iw(it))(e_{i'}w(-it))\beta(it)^{n-1}\beta(-it)^{m-1})\\
& \times(1+O(\theta^{n+m}))\widehat{\chi}(t)\,dt+O(\theta^N\beta(0)^{2N}).
\end{aligned}$$

We can view the weighted matrices A_s as finite dimensional transfer operators. Then, by Lemma 2.3.6, $e^{P(f)}$ is the maximal eigenvalue of the transfer operator L_f, $f\in C^\alpha(X_A^+,\mathbb{R})$. Furthermore, extending the pressure definition for $f\in C^\alpha(X_A^+,\mathbb{C})$ in a small neighbourhood of $f\in C^\alpha(X_A^+,\mathbb{R})$, gives that the simple eigenvalue $\beta(it)$ of the weighted matrix A_{it} can be written as $\beta(it)=e^{P(itr)}$. Hence, the maximal eigenvalue of the matrix A, $\beta=\beta(0)=e^{P(0)}$. Using these facts and Lemma 3.1.14, we have

$$\begin{aligned}
\Phi_N(\chi) = & \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\sum_{n,m=1}^{N}\sum_{\substack{i,i'\in\mathrm{Out}(I)\\ j,j'\in\mathrm{In}(J)}}(C_jC_{j'}(e_iw(it))(e_{i'}w(-it))e^{(n-1)P(itr)}\\
& \times e^{(m-1)P(-itr)})(1+O(\theta^{n+m}))\widehat{\chi}(t)\,dt+O(\theta^Ne^{2NP(0)})\\
= & \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\sum_{\substack{i,i'\in\mathrm{Out}(I)\\ j,j'\in\mathrm{In}(J)}}(C_jC_{j'}(e_iw(it))(e_{i'}w(-it))\\
& \times\left(\frac{e^{NP(itR)}}{(e^{P(itr)}-1)(e^{P(-itr)}-1)}\right)\widehat{\chi}(t)\,dt+O(e^{2P(0)}\max\{\theta^N,\delta^N\}).
\end{aligned}$$

Now from this point we can just follow the estimations and calculations performed for counting pairs of closed geodesics in a fixed interval $[a,b]$ in section 3.1.4. Consequently, we obtain the following asymptotic, which is similar to Lemma 3.1.16 for closed geodesics.

Lemma 6.1.1. *Let $\chi : \mathbb{R} \to \mathbb{R}$ be continuous, non-negative function with compact support. Then*

$$\Phi_N(\chi) \sim \frac{\sum_{\substack{i,i' \in \mathrm{Out}(I) \\ j,j' \in \mathrm{In}(J)}} C_j C_{j'} (e_i w)(e_{i'} w)}{\sqrt{2\pi}\sigma (e^{P(0)} - 1)^2} \int \chi(x)dx \frac{e^{2NP(0)}}{\sqrt{N}}, \text{ as } N \to +\infty.$$

Then to obtain the asymptotic in Theorem 6.1.1, we approximate the indicator function $\chi_{[a,b]}$ from above and below by two continuous and compactly supported functions in a similar way as we have proven Theorem 3.1.1 (in page 55). To do this we use Lemma 6.1.1.

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