Correction to scaling analysis of diffusion-limited aggregation

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Abstract

Diffusion-limited aggregation is consistent with simple scaling. However, strong subdominant terms are present, and these can account for various earlier claims of anomalous scaling. We show this in detail for the case of multiscaling.

Key words: diffusion-limited aggregation, correction to scaling, finite size scaling
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1 Introduction

Since its introduction by Witten and Sander in 1981 [1], diffusion-limited aggregation (DLA) has been the fundamental stochastic model of quasistatic growth processes where the growth is limited by a diffusion process. The model can be described in simple terms: a rigid aggregate grows by the capture of a low density of Brownian particles, which attach to it on first contact. A highly ramified branching structure is produced, which—at least on first sight—appears to be fractal.

One of the most basic questions asked about DLA is whether the growing clusters obey simple scaling, i.e. are they indeed simple fractals? Based on numerical simulations, it has been suggested that the scaling is more complex:

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multiple divergent length scales might be present [2], the ensemble variance of cluster radii might have anomalous scaling [3], there could be more than one fractal dimension [4], or the clusters might obey multiscaling where the fractal dimension continuously depends on the position [5].

We claim that these anomalous scaling claims are *wrong*, they are misled by finite size transients. In particular, we will show in detail that subdominant terms in the scaling account for the apparent "multiscaling" observed in small to medium size simulations. For clarity we should mention that here *multi-scaling* refers to the space dependent fractal dimension (anomalous scaling). This should not be confused with the well established *multifractality* of the harmonic (growth) measure, which is consistent with the simple asymptotic scaling of the clusters.

2 Simple scaling

In this section we will look at the scaling of various characteristic lengths: the *deposition radius* $R_{\rm dep} = \langle r \rangle$ (the average distance of newly arriving particles from the center), the cluster's *gyration radius* $R_{\rm gyr} = \sqrt{\frac{1}{N}\sum_{N'=1}^{N}\langle r^2\rangle_{N'}}$, the *root-mean-square radius* $R_2 = \sqrt{\langle r^2\rangle}$, and the *penetration depth* ξ (the width of the active zone ring, where newly deposited particles land). According to our results, DLA obeys simple scaling; all length scales scale with the same fractal dimension. To illustrate this, we look at one of the anomalous scaling claims mentioned in the introduction: that the cluster radius $R_{\rm dep}$ does not scale in the same way as the penetration depth ξ [2] — although it is worth mentioning that this claim has been questioned very soon [6]. The ratio of the two, often called *relative penetration depth*, $\Xi = \xi/R_{\rm dep}$, in our measurements obeys the asymptotic form [7,8] for large N:

$$\Xi(N) \;\approx\; \Xi_\infty \,(1 + CN^{-\nu})\,. \tag{1}$$

On Figure 1 we plot Ξ against $N^{-\nu}$ with an appropriately chosen ν; the linear behavior at $N^{-\nu} \to 0$ clearly indicates the validity of Eq. (1). It is not easy to obtain numerically the exponent ν: systematic errors (fitting data far from the asymptotic point) have to be balanced with large statistical errors (fitting close to the asymptotic point). Nevertheless, all data presented in this paper is consistent with a single "universal" exponent $\nu = 0.33 \pm 0.06$.

When the ratios of various lengths defined on DLA obey Eq. (1), and the lengths have an asymptotic power-law dependence on N, then they can be

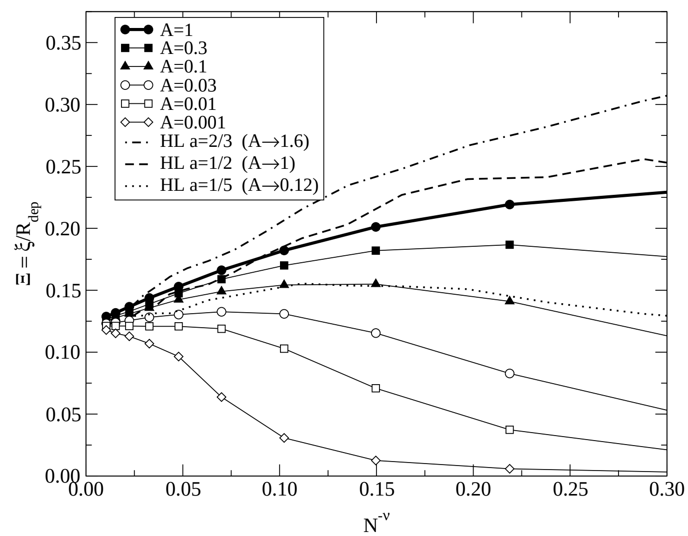

Fig. 1. Finite size scaling of the relative penetration depth $\Xi = \xi/R_{\rm dep}$, with correction-to-scaling exponent $\nu = 0.33$. The thick line connecting full circles corresponds to the standard random-walker-based DLA; it approaches a finite asymptotic value from above. The other curves with symbols show simulations with decreased shot noise (when a growth occurs, instead of a full particle, only a thin layer of width A is added; details of this off-lattice noise reduction technique can be found in Ref. [7]). Moderate noise reduction accelerates the convergence to the asymptotic value, while for strong noise reduction the approach is from below. The dashed lines with no symbols correspond to simulations based on iterative conformal maps of Hastings and Levitov [9]. In all cases the relative penetration depth approaches the *same* finite asymptotic value $\Xi_\infty = 0.121 \pm 0.003$.

written in a scaling form with a leading subdominant term [7]:

$$R(N) \;\approx\; \widehat{R} N^{1/D}(1 + \tilde{R} N^{-\nu}) \tag{2}$$

A numerical proof of this finite size scaling is plotted on Figure 2. For completeness, we collected the corresponding coefficients $\widehat{R}$ and $\tilde{R}$ for many characteristic lengths in Table 1.

The coefficients are not independent, it is easy to derive some relations between them by neglecting higher order corrections: $\widehat{R}_2 = \sqrt{\widehat{R}^2_{\rm dep} + \widehat{\xi}^2_0}$ and $\tilde{R}_2 = (\tilde{R}_{\rm dep}\widehat{R}^2_{\rm dep} + \tilde{\xi}_0\widehat{\xi}^2_0)/(\widehat{R}^2_{\rm dep} + \widehat{\xi}^2_0)$, or for the gyration radius $\widehat{R}_{\rm gyr} = \widehat{R}_2/\sqrt{1+2/D}$ and $\tilde{R}_{\rm gyr} = \tilde{R}_2(1+2/D)/(1-\nu+2/D)$. The measured coefficients satisfy these relations within error.

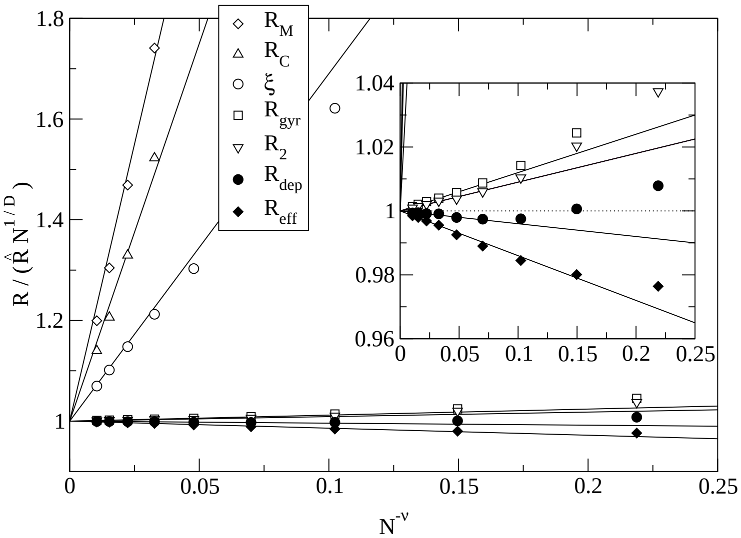

Fig. 2. Correction to scaling fits of various lengths, $D = 1.711$ and $\nu = 0.33$. Some of the quantities have positive correction (open symbols), others negative (filled symbols). The largest corrections is taken by lengths having low asymptotic values. **Inset**: y axis magnified around 1.

	definition	$\widehat{R}$	$\tilde{R}$
deposition radius	$R_{\text{dep}} = \langle r \rangle$	0.733(1)	-0.04(2)
root-mean-square radius	$R_2 = \sqrt{\langle r^2 \rangle}$	0.738(1)	0.09(2)
gyration radius	$R_{\text{gyr}} = \sqrt{\frac{1}{N} \sum_{N'=1}^{N} \langle r^2 \rangle_{N'}}$	0.501(1)	0.12(2)
effective (Laplacian) radius [8]	$R_{\text{eff}} = \langle \exp\left(\int dq \ln r\right) \rangle$	0.726(1)	-0.14(3)
effective radius variability	$\delta R_{\text{eff}} = \sqrt{\text{var}[\exp\left(\int dq \ln r\right)]}$	0.0086(10)	15
maximal radius	$R_{\max} = \langle \max_q r \rangle$	0.892(3)	1.0
maximal radius variability	$\delta R_{\max} = \sqrt{\text{var}[\max_q r]}$	0.034(2)	13.
seed to center-of-charge dist.	$R_{\text{C}} = \sqrt{\langle \lvert \int dq\, \mathbf{r} \rvert^2 \rangle}$	0.027(3)	15.(10)
seed to center-of-mass distance	$R_{\text{M}} = \sqrt{\langle \lvert \frac{1}{N} \sum_{N'=1}^{N} \mathbf{r}_{N'} \rvert^2 \rangle}$	0.016(1)	22.(6)
ensemble penetration depth	$\xi_0 = \sqrt{\langle r^2 \rangle - \langle r \rangle^2}$	0.091(1)	6.9(8)

Table 1
Coefficients of correction to scaling fits of form Eq. (2), with $D = 1.711$ and $\nu = 0.33$. In the definitions r denotes the distance of the N-th particle from the seed, $\langle \cdot \rangle$ is the average over the ensemble of clusters, and $\int dq$ is the average over the harmonic measure of a fixed cluster. The harmonic measure, or charge, is the probability measure of growth. The error in the last digit (when known) is indicated in parentheses.

3 Correction to scaling analysis of multiscaling

We start with the multiscaling assumption: it has been suggested [10] that the particle density of an N-particle cluster at distance $xR_{\rm gyr}$ away from the center scales with $R_{\rm gyr}$ with an x-dependent co-dimension:

$$g_N(xR_{\rm gyr}) = C(x) R_{\rm gyr}^{-d+D(x)} \tag{3}$$

From this scaling law $D(x)$ can be obtained as

$$-d + D(x) = \left.\frac{\partial \ln g_{N(R_{\rm gyr})}(xR_{\rm gyr})}{\partial \ln R_{\rm gyr}}\right|_x . \tag{4}$$

In direct numerical measurements [5] a non-trivial $D(x)$ was obtained, forming the basis of multiscaling claims.

Now we consider the distribution of r, the distance of the N-th particle from the seed. From the definitions in Table 1, the mean of r is $R_{\rm dep}$ and its standard deviation is ξ_0. The probability density of r can be written as

$$\frac{1}{\xi_0(N)}\, h\!\left(\frac{r - R_{\rm dep}(N)}{\xi_0(N)}\right) , \tag{5}$$

if we assume that the shape h of the distribution is independent of N. After replacing the sum over the particles with an integral, for the particle density we obtain

$$2\pi r\, g_N(r) = \int_0^N dN' \;\frac{1}{\xi_0(N')}\; h\!\left(\frac{r - R_{\rm dep}(N')}{\xi_0(N')}\right) \tag{6}$$

A formula similar to this has been suggested earlier [11]. At this point we can calculate [7] the function $D(x)$ from Eqs. (4) and (6), because we already know the correction to scaling approximation (2) of $R_{\rm gyr}(N)$, $R_{\rm dep}(N)$ and $\xi_0(N)$. The only extra ingredient needed is the functional form of h, which we measured directly. It is a normalized probability density of zero mean and unit variance, and as shown on Figure 3, it turns out to be well approximated by the standard normal distribution.

Now we compare $D(x)$ calculated with correction to scaling forms with that of earlier direct measurements on Figure 4a: the agreement is rather good. However, our method indicates (Figure 4b) that for larger size clusters $D(x)$ collapses to a constant: for $N \to \infty$ the radii approach pure scaling. From this

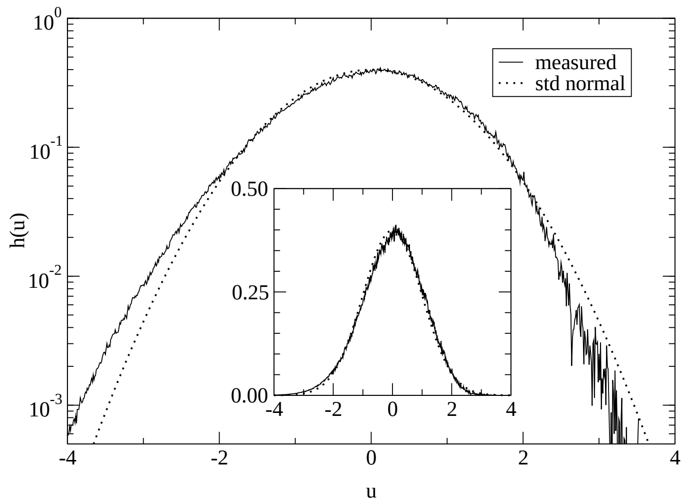

Fig. 3. The scaling function h, measured in random-walker-based simulation (the histogram bin width is $\Delta u = 0.01$), compared to standard normal distribution. h falls off faster than Gaussian at large positive u, compensated with slower fall off at negative u, but overall it is well approximated with the density of the standard normal distribution. **Inset**: the same quantities on linear scale.

we conclude that the observed "multiscaling" is a small size transient caused by the strong correction to scaling of the radii (mostly of ξ_0).

4 Summary

We have seen that current numerical measurements are consistent with the simple scaling picture of DLA. Subdominant terms, however, are strongly present: earlier anomalous scaling claims—including divergent length scales and multiscaling—were misled by them. The correction to scaling analysis, calculating the effect of the dominant correction, explains these earlier observations, even for a complicated quantities like $D(x)$. It remains a challenge for the future to predict theoretically the—so far only empirical—correction to scaling parameters.

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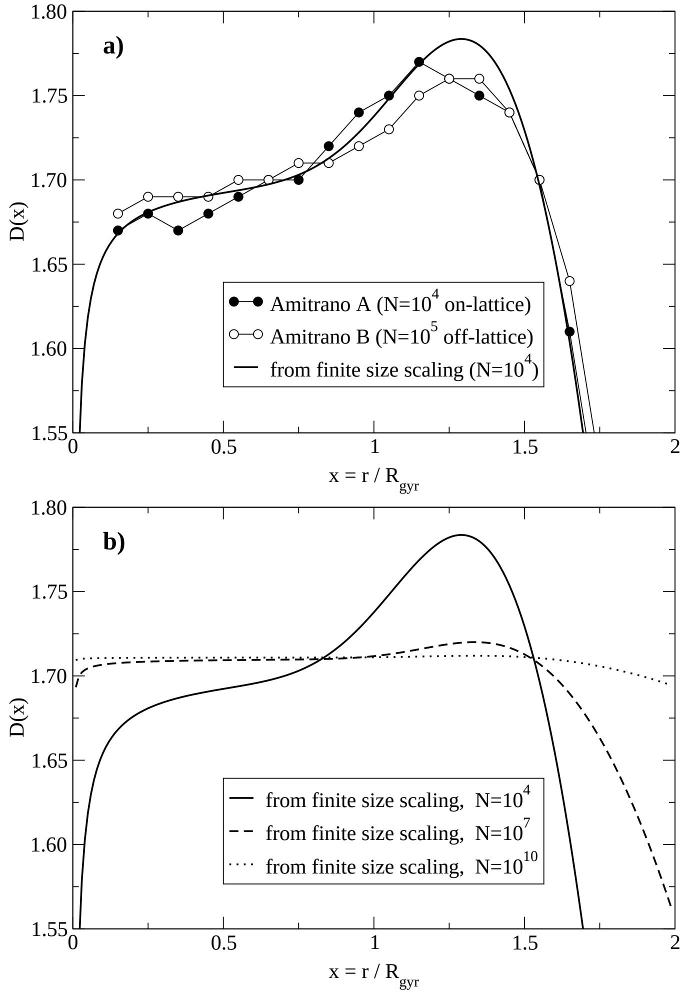

Fig. 4. Apparent "multiscaling dimensions". **a)** Comparison of the directly measured dimensions from Ref. [5] and the finite size scaling prediction at $N = 10^4$ (the size corresponding to the simulations). **b)** Finite size scaling predictions at sizes $N = 10^4$, 10^7 and 10^{10}. In the limit $N \to \infty$, the dimension approaches a constant: $D(x) \to D$.

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