Continuity of pullback and uniform attractors

Luan T. Hoang[a], Eric J. Olson[b], James C. Robinson[c]

[a]*Department of Mathematics and Statistics, Texas Tech University Box 41042, Lubbock, TX 79409-1042, USA.*
[b]*Department of Mathematics/084, University of Nevada Reno, NV 89557, USA.*
[c]*Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK.*

Abstract

We study the continuity of pullback and uniform attractors for non-autonomous dynamical systems with respect to perturbations of a parameter. Consider a family of dynamical systems parameterized by $\lambda \in \Lambda$, where Λ is a complete metric space, such that for each $\lambda \in \Lambda$ there exists a unique pullback attractor $\mathscr{A}_\lambda(t)$. Using the theory of Baire category we show under natural conditions that there exists a residual set $\Lambda_* \subseteq \Lambda$ such that for every $t \in \mathbb{R}$ the function $\lambda \mapsto \mathscr{A}_\lambda(t)$ is continuous at each $\lambda \in \Lambda_*$ with respect to the Hausdorff metric. Similarly, given a family of uniform attractors $\mathbb{A}_\lambda$, there is a residual set at which the map $\lambda \mapsto \mathbb{A}_\lambda$ is continuous. We also introduce notions of equi-attraction suitable for pullback and uniform attractors and then show when Λ is compact that the continuity of pullback attractors and uniform attractors with respect to λ is equivalent to pullback equi-attraction and, respectively, uniform equi-attraction. These abstract results are then illustrated in the context of the Lorenz equations and the two-dimensional Navier–Stokes equations.

Keywords: Pullback attractor, uniform attractor.

1. Introduction

The theory of attractors plays an important role in understanding the long time behavior of dynamical systems, see Babin and Vishik [1], Billotti and LaSalle [3], Chueshov [7], Hale [11], Ladyzhenskaya [17], Robinson [22] and Temam [27]. For the autonomous theory, we consider a family of dissipative dynamical systems parameterized by Λ such that for each $\lambda \in \Lambda$ the corresponding dynamical system possesses a unique compact global attractor $\mathscr{A}_\lambda \subseteq Y$, where Y is a complete metric space with metric d_Y. Under very mild assumptions (see for example [12] and the references therein) the map $\lambda \mapsto \mathscr{A}_\lambda$ is known to be upper semicontinuous. This means that

$$\rho_Y(\mathscr{A}_\lambda, \mathscr{A}_{\lambda_0}) \to 0 \qquad \text{as} \qquad \lambda \to \lambda_0$$

Email addresses: `luan.hoang@ttu.edu` (Luan T. Hoang), `ejolson@unr.edu` (Eric J. Olson), `J.C.Robinson@warwick.ac.uk` (James C. Robinson)

where $\rho_Y(A,C)$ denotes the Hausdorff semi-distance

$$\rho_Y(A,C) = \sup_{a\in A}\inf_{c\in C} d_Y(a,c). \tag{1.1}$$

However, lower semicontinuity

$$\rho_Y(\mathscr{A}_{\lambda_0}, \mathscr{A}_\lambda) \to 0 \qquad \text{as} \qquad \lambda \to \lambda_0,$$

and hence full continuity with respect to the Hausdorff metric, is much harder to prove.

For autonomous systems, general results on lower semicontinuity require strict conditions on the structure of the unperturbed global attractor, which are rarely satisfied for complicated systems (see Hale and Raugel [13] and Stuart and Humphries [25]). However, Babin and Pilyugin [2] and Luan et al. [14] showed, using the theory of Baire category, that continuity holds for λ_0 in a residual set $\Lambda_* \subseteq \Lambda$ under natural conditions when Λ is a complete metric space. We recall this result for autonomous systems as Theorem 1.1 below.

Let Λ and X be complete metric spaces. We will suppose that $S_\lambda(\cdot)$ is a parameterized family of semigroups on X for $\lambda \in \Lambda$ that satisfies the following properties:

(G1) $S_\lambda(\cdot)$ has a global attractor $\mathscr{A}_\lambda$ for every $\lambda \in \Lambda$;

(G2) there is a bounded subset D of X such that $\mathscr{A}_\lambda \subseteq D$ for every $\lambda \in \Lambda$; and

(G3) for $t > 0$, $S_\lambda(t)x$ is continuous in λ, uniformly for x in bounded subsets of X.

Note that condition (G2) can be strengthened and (G3) weakened by replacing *bounded* by *compact*. These modified conditions will be referred to as conditions (G2′) and (G3′).

Theorem 1.1. *Under assumptions* (G1–G3) *above—or under the assumptions* (G1), (G2′) *and* (G3′)*—$\mathscr{A}_\lambda$ is continuous in λ at all λ_0 in a residual subset of Λ. In particular the set of continuity points of $\mathscr{A}_\lambda$ is dense in Λ.*

The proof developed in [14] of the above theorem, which appears there as Theorem 5.1, is more direct than previous proofs (e.g. in [2]) and can be modified to establish analogous results for the pullback attractors and uniform attractors of non-autonomous systems. This is the main purpose of the present paper. After briefly introducing some definitions and notations concerning attractors and Baire category theory in Section 2, in Section 3 we prove Theorem 3.3, our main result concerning pullback attractors. Section 4 then contains Theorem 4.1, which provides similar results for uniform attractors. In addition, we investigate the continuity of pullback and uniform attractors on the entire parameter space Λ. It was proved by Li and Kloeden [19] (see also [14]) that when Λ is compact, the continuity of the global attractors on Λ is equivalent to equi-attraction of the semigroups. In Section 5, we extend this result and the notion of equi-attraction to non-autonomous and uniform attractors. Theorem 5.2 shows for pullback attractors that continuity is equivalent to pullback equi-attraction, while Theorem 5.3 shows for uniform attractors that continuity is equivalent to uniform equi-attraction.

We note that the continuity of pullback attractors is investigated by Carvalho et al. [5], who extend the autonomous results to non-autonomous systems, under strong conditions on the structure of the pullback attractors. Similarly, the notion of equi-attraction defined in Section 5 is a difficult property to discern for any concrete family of dynamical system. In contrast, the continuity results from Sections 3 and 4 only require standard conditions that are met in many applications. We demonstrate this in Section 6 with the Lorenz system of ODEs and the two-dimensional Navier–Stokes equations.

2. Preliminaries

We begin by setting our notation and recalling the definition of the Hausdorff metric. Given a metric space (Y, d_Y), denote by $B_Y(y, r)$ the ball of radius r centered at y,

$$B_Y(y, r) = \{y \in Y : \; d_Y(x, y) < r\}.$$

Write Δ_Y for the symmetric Hausdorff distance

$$\Delta_Y(A, C) = \max(\rho_Y(A, C), \rho_Y(C, A)) \tag{2.1}$$

where ρ_Y is the semi-distance between two subsets A and C of Y defined in (1.1). Denote by $CB(Y)$ the collection of all non-empty closed, bounded subsets of a metric space Y, which is itself a metric space with metric given by the symmetric Hausdorff distance Δ_Y.

In the same way that a continuous semigroup may be used to describe an autonomous dynamical system, the concept of a non-autonomous process may be used to describe a non-autonomous dynamical system.

Definition 2.1. *Let (X, d_X) be a complete metric space. A* process $S(\cdot, \cdot)$ *on X is a two-parameter family of maps $S(t, s)\colon X \mapsto X$, $s \in \mathbb{R}$, $t \geq s$, such that*

(P1) $S(t, t) = \mathrm{id}$*;*

(P2) $S(t, \tau)S(\tau, s) = S(t, s)$ *for all $t \geq \tau \geq s$; and*

(P3) $S(t, s)x$ *is continuous in x, t, and s.*

Given a non-autonomous process, there are two common ways to characterize its asymptotic behavior: roughly speaking, the limit of $S(t, s)$ for a fixed t as $s \to -\infty$ leads to the definition of the pullback attractor, while the limit of $S(t + s, s)$ as $t \to \infty$ leads to the uniform attractor (given sufficient uniformity in s). While both methods give rise to the same object for autonomous dynamics, they may be different in the non-autonomous case.

We begin with a formal definition of the pullback attractor, obtained by taking the limit as $s \to -\infty$.

Definition 2.2. *A family of compact sets $\mathscr{A}(\cdot) = \{\mathscr{A}(t)\colon t \in \mathbb{R}\}$ in X is the* pullback attractor *for the process $S(\cdot, \cdot)$ if*

(A1) *$\mathscr{A}(\cdot)$ is invariant: $S(t, s)\mathscr{A}(s) = \mathscr{A}(t)$ for all $t \geq s$;*

(A2) *$\mathscr{A}(\cdot)$ is pullback attracting: for any bounded set B in X and $t \in \mathbb{R}$*

$$\rho_X\big(S(t, s)B, \mathscr{A}(t)\big) \to 0 \qquad \text{as} \qquad s \to -\infty;$$

and

(A3) $\mathscr{A}(\cdot)$ *is minimal, in the sense that if* $C(\cdot)$ *is any other family of compact sets that satisfies (A1) and (A2) then* $\mathscr{A}(t) \subseteq C(t)$ *for all* $t \in \mathbb{R}$*.*

The uniform attractor is obtained by taking the limit as $t \to \infty$.

Definition 2.3. *A set* $\mathbb{A} \subseteq X$ *is the* uniform attractor *if it is the minimal compact set such that*

$$\lim_{t\to\infty} \sup_{s\in\mathbb{R}} \rho_X\big(S(t+s,s)B, \mathbb{A}\big) = 0 \tag{2.2}$$

for any bounded $B \subseteq X$*.*

We finish this section by stating a few basic facts from the theory of Baire category including an abstract residual continuity result. Recall that a set is *nowhere dense* if its closure contains no non-empty open sets, and a set is *residual* if its complement is the countable union of nowhere dense sets. It is a well-known fact that any residual subset of a complete metric space is dense.

The following result, an abstract version of Theorem 7.3 in Oxtoby [21], was proved as Theorem 5.1 in [14], and forms a key part of our proofs.

Theorem 2.4. *Let* $f_n \colon \Lambda \mapsto Y$ *be a continuous map for each* $n \in \mathbb{N}$*, where* Λ *is a complete metric space and* Y *is any metric space. If* f *is the pointwise limit of* f_n*, that is, if*

$$f(\lambda) = \lim_{n\to\infty} f_n(\lambda) \qquad \textit{for each} \qquad \lambda \in \Lambda$$

(implicit in this is the requirement that the limit exists) then the points of continuity of f *form a residual subset of* Λ*.*

3. Residual continuity of pullback attractors

In this section we consider the continuity of pullback attractors. Let Λ be a complete metric space and $S_\lambda(\cdot,\cdot)$ a parameterized family of processes on X with $\lambda \in \Lambda$. Suppose that

(L1) $S_\lambda(\cdot,\cdot)$ has a pullback attractor $\mathscr{A}_\lambda(\cdot)$ for every $\lambda \in \Lambda$;

(L2) there is a bounded subset D of X such that $\mathscr{A}_\lambda(t) \subseteq D$ for every $\lambda \in \Lambda$ and every $t \in \mathbb{R}$; and

(L3) for every $s \in \mathbb{R}$ and $t \geq s$, $S_\lambda(t,s)x$ is continuous in λ, uniformly for x in bounded subsets of X.

We denote by (L2′) and (L3′) the assumptions (L2) and (L3), respectively, with *bounded* replaced by *compact*.

The following result is proved for the autonomous case as Lemma 3.1 in [14]; we omit the proof for the non-autonomous case, which is identical.

Lemma 3.1. *Assume either* (L2) *and* (L3)*, or* (L2′) *and* (L3′)*. Then for any* $s \in \mathbb{R}$ *and* $t \geq s$*, the map* $\lambda \mapsto \overline{S_\lambda(t,s)D}$ *is continuous from* Λ *into* $CB(X)$*.*

We also need the following related continuity result for $S_\lambda(t,s)B$. Note that the result only treats sets $B \in CB(K)$ for some compact K, which is crucial to the proof.

Lemma 3.2. *Assume that* (L3′) *holds, and let K be any compact subset of X. Then for any $t \geq s$, the mapping $(\lambda, B) \mapsto S_\lambda(t,s)B$ is (jointly) continuous in $(\lambda, B) \in \Lambda \times CB(K)$.*

Proof. Since every $B \in CB(K)$ is compact and $S_\lambda(t,s)x$ is continuous in x, it follows that the image $S_\lambda(t,s)B$ is compact too. Now suppose that $s \in \mathbb{R}$, $t \geq s$, $\lambda_0 \in \Lambda$, $B_0 \in CB(K)$ and $\epsilon > 0$. Condition (L3′) ensures that there exists a $\delta_1 \in (0,1)$ such that

$$d_\Lambda(\lambda_0, \lambda) < \delta_1 \quad \text{implies that} \quad d_X\big(S_\lambda(t,s)x, S_{\lambda_0}(t,s)x\big) < \epsilon/2 \quad \text{for every} \quad x \in K.$$

Since K is compact, the map $x \mapsto S_{\lambda_0}(t,s)x$ is uniformly continuous on K; in particular, there is a $\delta_2 \in (0,1)$ such that

$$d_X(x,y) < \delta_2 \quad \text{with} \quad x, y \in K \quad \text{implies that} \quad d_X\big(S_{\lambda_0}(t,s)x, S_{\lambda_0}(t,s)y\big) < \epsilon/2.$$

Set $\delta = \min\{\delta_1, \delta_2\}$.

Take $\lambda \in \Lambda$ with $d_\Lambda(\lambda, \lambda_0) < \delta$ and $B \in CB(K)$ with $\Delta_X(B, B_0) < \delta$. For any $b \in B$ there is a $b_0 \in B_0$ such that $d_X(b, b_0) < \delta$. Therefore

$$\begin{aligned} d_X\big(S_\lambda(t,s)b, S_{\lambda_0}(t,s)b_0\big) &\leq d_X\big(S_\lambda(t,s)b, S_{\lambda_0}(t,s)b\big) \\ &\qquad + d_X\big(S_{\lambda_0}(t,s)b, S_{\lambda_0}(t,s)b_0\big) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence

$$\rho_X\big(S_\lambda(t,s)B, S_{\lambda_0}(t,s)B_0\big) \leq \epsilon. \tag{3.1}$$

The hypothesis $\Delta_X(B, B_0) < \delta$ also implies that for any $b_0 \in B_0$, there is $b \in B$ such that $d_X(b, b_0) < \delta$. Consequently

$$\rho_X\big(S_\lambda(t,s)B_0, S_{\lambda_0}(t,s)B\big) \leq \epsilon. \tag{3.2}$$

Combining (3.1) and (3.2) now yields $\Delta_X\big(S_\lambda(t,s)B, S_{\lambda_0}(t,s)B_0\big) \leq \epsilon$, which proves the joint continuity as claimed. □

As (L3) is a stronger hypothesis than (L3′) we note that Lemma 3.2 also holds under (L3). We now use Lemma 3.2 to prove the residual continuity of pullback attractors.

Theorem 3.3. *Let $S_\lambda(\cdot,\cdot)$ be a family of processes on (X,d) each satisfying* (P1–P3) *and suppose that* (L1) *holds along with either*

(i) (L2′) *and* (L3′), *or*

(ii) (L2), (L3), *and for any $\lambda_0 \in \Lambda$ and $t \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$\overline{\bigcup_{B_\Lambda(\lambda_0,\delta)} \mathscr{A}_\lambda(t)} \qquad \textit{is compact.} \tag{3.3}$$

Then, there exists a residual set Λ_* in Λ such that for every $t \in \mathbb{R}$ the function $\lambda \mapsto \mathscr{A}_\lambda(t)$ is continuous at each $\lambda \in \Lambda_*$.

Proof. From Lemma 3.1 it follows that for each $n \in \mathbb{Z}$ and $s < n$ the function $\lambda \mapsto \overline{S_\lambda(n,s)D}$ is continuous. Moreover, since by either (L2) or (L2′) we have $D \supseteq \mathscr{A}_\lambda(s)$, then from the invariance of the attractor (A1) it follows that

$$\overline{S_\lambda(n,s)D} \supseteq S_\lambda(n,s)D \supseteq S_\lambda(n,s)\mathscr{A}_\lambda(s) = \mathscr{A}_\lambda(n) \qquad \text{for every} \qquad s \leq n. \tag{3.4}$$

Therefore, the pullback attraction property (A2) yields

$$\mathscr{A}_\lambda(n) = \lim_{s \to -\infty} \overline{S_\lambda(n,s)D}, \tag{3.5}$$

where the convergence is with respect to the Hausdorff metric. It follows from Theorem 2.4 that there is a residual set Λ_n of Λ at which the map $\lambda \in \Lambda \mapsto \mathscr{A}_\lambda(n)$ is continuous. Since the countable intersection of residual sets is still residual, then $\Lambda_* = \bigcap_{n \in \mathbb{Z}} \Lambda_n$ is a residual set at which $\lambda \mapsto \mathscr{A}_\lambda(n)$ is continuous for every $n \in \mathbb{Z}$.

We now use the invariance of $\mathscr{A}_\lambda(\cdot)$ to obtain continuity for every $t \in \mathbb{R}$. For $t \notin \mathbb{Z}$ there is $n \in \mathbb{Z}$ such that $t \in (n, n+1)$. Moreover,

$$\mathscr{A}_\lambda(t) = S_\lambda(t,n)\mathscr{A}_\lambda(n). \tag{3.6}$$

In case (i) set $K = D$; in case (ii) define

$$K = \overline{\bigcup_{B_\Lambda(\lambda_0,\delta)} \mathscr{A}_\lambda(t)}$$

where $\delta > 0$ has been chosen by (3.3) such that K is compact. Since $\mathscr{A}_\lambda(n)$ is continuous at $\lambda \in \Lambda_*$ and $\mathscr{A}_\lambda(n) \subseteq K$ for all $\lambda \in B_\Lambda(\lambda_0, \delta)$, Lemma 3.2 guarantees that $S_\lambda(t,n)B$ is continuous in $(\lambda, B) \in \Lambda \times CB(K)$. Viewing (3.6) as a composition of continuous functions now yields the continuity of $\mathscr{A}_\lambda(t)$ at $\lambda \in \Lambda_*$. $\square$

4. Residual continuity of uniform attractors

This section develops the theory of residual continuity of uniform attractors with respect to a parameter. A key component of the proof is the expression for the uniform attractor as a union of the uniform ω-limit sets given by

$$\mathbb{A}_\lambda = \overline{\bigcup_{n \in \mathbb{N}} \Omega_\lambda(B_X(0,n))}, \tag{4.1}$$

where, as in Chapter VII of [6], we define

$$\Omega_\lambda(B) = \bigcap_{\tau \in \mathbb{R}} \overline{\bigcup_{s \in \mathbb{R}} \bigcup_{t \geq \tau} S_\lambda(t+s,s)B} \qquad \text{for any set} \qquad B \subseteq X.$$

We can now state our main result on uniform attractors.

Theorem 4.1. *Suppose that there exists a compact set $K \subseteq X$ such that*

(a) *for every bounded $B \subseteq X$ and each $\lambda \in \Lambda$ there exists a $t_{B,\lambda}$ such that*

$$S_\lambda(t+s,s)B \subseteq K \qquad \text{for all} \qquad t \geq t_{B,\lambda} \quad \text{and} \quad s \in \mathbb{R}; \tag{4.2}$$

and

(b) *for any $t > 0$ the mapping $S_\lambda(t+s,s)x$ is continuous in $\lambda \in \Lambda$ uniformly for $s \in \mathbb{R}$ and $x \in K$.*

Then the uniform attractor $\mathbb{A}_\lambda$ is continuous in λ at a residual subset of Λ.

In the preceding theorem, assumption (a) is sufficient for the existence of a uniform attractor given by the uniform ω-limit (4.1) with λ fixed, and assumption (b) provides some uniform continuity of the processes S_λ in a way that depends only on the elapsed time. More specifically, given $\lambda_0 \in \Lambda$, $t > 0$ and $\epsilon > 0$, there is $\delta > 0$ depending only on λ_0, t and ϵ such that for any $\lambda \in \Lambda$ with $d_\Lambda(\lambda, \lambda_0) < \delta$

$$d_X\big(S_\lambda(t+s,s)x, S_{\lambda_0}(t+s,s)x\big) < \epsilon \qquad \text{for all} \qquad s \in \mathbb{R} \quad \text{and} \quad x \in K.$$

Proof. Take $\lambda \in \Lambda$. Applying (2.2) to $S = S_\lambda$ gives

$$\lim_{t\to\infty} \sup_{s\in\mathbb{R}} \rho_X\big(S_\lambda(t+s,s)B, \mathbb{A}_\lambda\big) = 0$$

for any bounded $B \subseteq X$. Since

$$\rho_X\Bigg(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)B}, \mathbb{A}_\lambda\Bigg) = \rho_X\Bigg(\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)B, \mathbb{A}_\lambda\Bigg) \leq \sup_{s\in\mathbb{R}} \rho_X\big(S_\lambda(t+s,s)B, \mathbb{A}_\lambda\big),$$

we obtain

$$\lim_{t\to\infty} \rho_X\Bigg(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)B}, \mathbb{A}_\lambda\Bigg) = 0. \tag{4.3}$$

Let $T_n = \tau_{B,\lambda}$ for $n \in \mathbb{N}$ be given by (4.2) where $B = B_X(0,n)$. Then

$$\begin{aligned}
\Omega_\lambda(B_X(0,n)) &\subseteq \bigcap_{\tau\geq T_n} \overline{\bigcup_{s\in\mathbb{R}} \bigcup_{t\geq\tau} S_\lambda(t+s,s)B_X(0,n)} \\
&= \bigcap_{\tau\geq T_n} \overline{\bigcup_{s\in\mathbb{R}} \bigcup_{t\geq\tau} S_\lambda(t+s,T_n+s)S_\lambda(T_n+s,s)B_X(0,n)} \\
&\subseteq \bigcap_{\tau\geq T_n} \overline{\bigcup_{s\in\mathbb{R}} \bigcup_{t\geq\tau} S_\lambda(t+s,T_n+s)K} = \bigcap_{\tau\geq T_n} \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{t\geq\tau} S_\lambda(t-T_n+\eta,\eta)K} \\
&= \bigcap_{\tau\geq T_n} \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq\tau-T_n} S_\lambda(s+\eta,\eta)K} = \bigcap_{t\geq 0} \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq t} S_\lambda(s+\eta,\eta)K}.
\end{aligned}$$

It follows from (4.1) that

$$\mathbb{A}_\lambda \subseteq \bigcap_{t\geq 0} \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq t} S_\lambda(s+\eta,\eta)K}. \tag{4.4}$$

Let $T = \tau_{K,\lambda}$ in (4.2). By (4.4) we have for $t \geq 0$ that

$$\begin{aligned}\mathbb{A}_\lambda &\subseteq \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq t+T} S_\lambda(s+\eta,\eta)K} \\ &\subseteq \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq t+T} S_\lambda\big(t+(s-t+\eta), s-t+\eta\big) S_\lambda(s-t+\eta,\eta)K} \\ &\subseteq \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq t+T} S_\lambda\big(t+(s-t+\eta), s-t+\eta\big)K}.\end{aligned}$$

Thus

$$\mathbb{A}_\lambda \subseteq \overline{\bigcup_{z\in\mathbb{R}} S_\lambda(t+z,z)K}. \tag{4.5}$$

Taking $B = K$ in (4.3) yields

$$\Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K}, \mathbb{A}_\lambda\right) \to 0 \qquad \text{as} \qquad t\to\infty. \tag{4.6}$$

Define $K_\lambda(t) = \bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K$, fix $\lambda_0 \in \Lambda$ and let $t > 0$. Given $\epsilon > 0$ choose $\delta > 0$ as in (b). Then for any $x \in K$, we have

$$d_X\big(S_\lambda(t+s,s)x, S_{\lambda_0}(t+s,s)x\big) < \epsilon \qquad \text{for any} \qquad s\in\mathbb{R}.$$

Hence

$$\rho_X\big(K_\lambda(t), K_{\lambda_0}(t)\big) \leq \epsilon \qquad \text{and} \qquad \rho_X\big(K_{\lambda_0}(t), K_\lambda(t)\big) \leq \epsilon$$

imply that

$$\Delta_X\big(\overline{K_\lambda(t)}, \overline{K_{\lambda_0}(t)}\big) \leq \epsilon.$$

Consequently

$$\lambda \mapsto \overline{K_\lambda(t)} = \overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K}$$

is continuous from Λ into $BC(X)$. The result now follows from (4.6) and Theorem 2.4. □

5. Continuity everywhere and equi-attraction

In this section, we extend the notion of equi-attraction and the results on continuity of global attractors with respect to a parameter Λ in [19] to non-autonomous systems. We first show that that the continuity of pullback attractors with respect to a parameter $\lambda \in \Lambda$ is equivalent to pullback equi-attraction when Λ is compact. Next, we prove similar results for uniform equi-attraction and uniform attractors. Our methods are based on those used in Section 4 of [14].

Assume (L1) and (L2) throughout this section where D is the set specified in (L2). Now consider the following conditions:

(U1) Pullback equi-dissipativity at time $t \in \mathbb{R}$: there exists $s_0 \leq t$ and a bounded set B such that

$$S_\lambda(t,s)D \subseteq B \qquad \text{for every} \qquad s \leq s_0 \quad \text{and} \quad \lambda\in\Lambda.$$

(U2) Pullback equi-attraction at time $t \in \mathbb{R}$:

$$\lim_{s\to-\infty} \sup_{\lambda\in\Lambda} \rho_X\big(S_\lambda(t,s)D, \mathscr{A}_\lambda(t)\big) = 0.$$

(U3) There is a bounded set D_1 and a function $s_*(t)$ such that $s_*(t) \le t$ and

$$S_\lambda(t,s)D_1 \subseteq D_1 \qquad \text{for every} \qquad s \le s_*(t) \quad \text{and} \quad \lambda \in \Lambda. \tag{5.1}$$

We remark that (U3) is the uniform version of (U1) commonly obtained while proving the existence of pullback attractors. In the autonomous case (U3) is identical to condition (4.5) in [14]. Our analysis here relies on the following version of Dini's theorem, which also appears in [14].

Lemma 5.1 (Theorem 4.1 in [14])**.** *Let K be a compact metric space and Y be a metric space. For each $n \in \mathbb{N}$, let $f_n\colon K \mapsto Y$ be a continuous map. Assume f_n converges to a continuous function $f\colon K \mapsto Y$ as $n \to \infty$ in the following monotonic way*

$$d_Y\big(f_{n+1}(x), f(x)\big) \le d_Y\big(f_n(x), f(x)\big) \qquad \textit{for all } n \in \mathbb{N} \quad \textit{and for every } x \in K.$$

Then f_n converges to f uniformly on K as $s \to \infty$.

First, we deal with the pullback attractors.

Theorem 5.2. *Fix $t \in \mathbb{R}$. If* (U1) *and* (U2) *hold then $\mathscr{A}_\lambda(t)$ is continuous at every $\lambda \in \Lambda$. Conversely, suppose that Λ is a compact metric space and that* (U3) *holds; then if $\lambda \mapsto \mathscr{A}_\lambda(t)$ is continuous on Λ then* (U2) *holds with D replaced with D_1.*

Proof. Let $t \in \mathbb{R}$ be such that (U1) and (U2) hold. Following the same arguments used to obtain (3.4) and (3.5) in the proof of Theorem 3.3, except with t replacing n, we have for each $\lambda \in \Lambda$ and $s \le t$ that

$$\mathscr{A}_\lambda(t) \subseteq S_\lambda(t,s)D \tag{5.2}$$

and

$$\mathscr{A}_\lambda(t) = \lim_{s\to-\infty} \overline{S_\lambda(t,s)D}. \tag{5.3}$$

By (U2) the convergence in (5.3) is uniform in $\lambda \in \Lambda$ as $s \to -\infty$. Moreover, Lemma 3.1 implies for $s \le t$ that the function $\lambda \mapsto \overline{S_\lambda(t,s)D}$ is continuous in λ on Λ. Thus, the limit function $\lambda \mapsto \mathscr{A}_\lambda(t)$ is continuous on Λ.

We now prove the converse. Assume (U3) and that $\lambda \mapsto \mathscr{A}_\lambda(t)$ is continuous on Λ for some $t \in \mathbb{R}$. Let $s_0 = s_*(t) - 1$, $s_1 = s_*(s_0) - 1$ and $s_{n+1} = s_*(s_n) - 1$ for $n \ge 1$. Then the sequence $\{s_n\}_{n=0}^\infty$ is strictly decreasing and $s_n \to -\infty$ as $n \to \infty$. By (P2) and (U3) we have

$$S(t, s_{n+1})D_1 = S(t, s_n)S(s_n, s_{n+1})D_1 \subseteq S(t, s_n)D_1. \tag{5.4}$$

Replacing D by D_1 in (5.2) and (5.3) yields

$$\mathscr{A}_\lambda(t) \subseteq S_\lambda(t,s)D_1 \qquad \text{for all} \qquad s \le t \tag{5.5}$$

and

$$\mathscr{A}_\lambda(t) = \lim_{s\to-\infty} \overline{S_\lambda(t,s)D_1}. \tag{5.6}$$

We infer from (5.4) and (5.5) that

$$\Delta_X\big(\overline{S(t,s_{n+1})D_1}, \mathscr{A}_\lambda(t)\big) \le \Delta_X\big(\overline{S(t,s_n)D_1}, \mathscr{A}_\lambda(t)\big). \tag{5.7}$$

Therefore, the convergence given in (5.6) is monotonic along the sequence $s = s_n$, and consequently, Lemma 5.1 implies $\overline{S_\lambda(t,s_n)D_1}$ converges to $\mathscr{A}_\lambda(t)$ as $n\to\infty$ uniformly in $\lambda\in\Lambda$. Thus,

$$\lim_{n\to\infty}\sup_{\lambda\in\Lambda}\rho_X\big(\overline{S_\lambda(t,s_n)D_1}, \mathscr{A}_\lambda(t)\big) = 0. \tag{5.8}$$

To obtain (5.8) in the continuous limit as $s\to-\infty$ suppose $s\in(s_{n+2}, s_{n+1})$. Then by definition $s < s_*(s_n)$ so that by (U3) we obtain

$$\mathscr{A}_\lambda(t) \subseteq S(t,s)D_1 = S(t,s_n)S(s_n,s)D_1 \subseteq S(t,s_n)D_1.$$

Hence,

$$\rho_X\big(\overline{S_\lambda(t,s)D_1}, \mathscr{A}_\lambda(t)\big) \le \rho_X\big(\overline{S_\lambda(t,s_n)D_1}, \mathscr{A}_\lambda(t)\big) \quad \text{for every} \quad \lambda\in\Lambda.$$

This and (5.8) prove

$$\lim_{s\to-\infty}\sup_{\lambda\in\Lambda}\rho_X\big(\overline{S_\lambda(t,s)D_1}, \mathscr{A}_\lambda(t)\big) = 0,$$

which is exactly (U2) with D replaced by D_1. ☐

For uniform attractors we work under the standing assumption that there exists a set K such that (a) of Theorem 4.1 holds. We say that $\mathbb{A}_\lambda$ is uniformly equi-attracting if

$$\lim_{t\to\infty}\sup_{\lambda\in\Lambda, s\in\mathbb{R}}\rho_X\big(S_\lambda(t+s,s)K, \mathbb{A}_\lambda\big) = 0. \tag{5.9}$$

In our analysis we further consider the case where any trajectory starting in K uniformly re-enters K within a certain time T_0. This is characterized by the following condition.

(U4) Assume there exists $T_0 \ge 0$ such that

$$S_\lambda(t+s,s)K \subseteq K \qquad \text{for all} \qquad t\ge T_0, \quad \lambda\in\Lambda \quad \text{and} \quad s\in\mathbb{R}. \tag{5.10}$$

We are now ready to prove our main result on uniform equi-attraction.

Theorem 5.3. *If $\mathbb{A}_\lambda$ is uniformly equi-attracting, then $\mathbb{A}_\lambda$ is continuous on Λ. Conversely, if $\mathbb{A}_\lambda$ is continuous, Λ is compact and* (U4) *is satisfied, then $\mathbb{A}_\lambda$ is uniformly equi-attracting.*

Proof. Under our standing assumption about the existence of K, we have that (4.3), (4.5) and (4.6) hold. To prove that $\mathbb{A}_\lambda$ is continuous, recall from Theorem 4.1 that

$$\lambda \mapsto \overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K}$$

is continuous. By (4.5) and (5.9) the limit (4.6) is uniform in λ. In other words,

$$\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K} \to \mathbb{A}_\lambda \quad \text{as} \quad t\to\infty \quad \text{uniformly for} \quad \lambda\in\Lambda.$$

Therefore, $\mathbb{A}_\lambda$ is continuous at every $\lambda \in \Lambda$.

Conversely, let T_0 be in (U4) and set $T_* = T_0 + 1 \geq 1$. Let $t_n = nT_*$ for all $n \in \mathbb{N}$. Then $t_n \to \infty$ as $n \to \infty$. For $s \in \mathbb{R}$,

$$\begin{aligned} S_\lambda(t_{n+1}+s,s)K = S_\lambda(t_n+T_*+s,s)K &= S_\lambda(t_n+T_*+s,T_*+s)S_\lambda(T_*+s,s)K \\ &\subseteq S_\lambda(t_n+T_*+s,T_*+s)K \subseteq \bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K. \end{aligned}$$

Thus,

$$\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_{n+1}+s,s)K} \subseteq \overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K}. \tag{5.11}$$

The inclusion (4.5) then yields

$$\Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_{n+1}+s,s)K}, \mathbb{A}_\lambda\right) \leq \Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K}, \mathbb{A}_\lambda\right).$$

which is the monotonicity needed for Lemma 5.1. It follows that the convergence in (4.6) taken along the sequence $t = t_n$ is uniform in λ. In other words, that

$$\lim_{n\to\infty} \sup_{\lambda\in\Lambda} \Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K}, \mathbb{A}_\lambda\right) = 0. \tag{5.12}$$

To obtain uniformity in the continuous limit as $t \to \infty$ suppose $t \in (t_{n+1}, t_{n+2})$. Then $t - t_n > T_0$ and

$$\begin{aligned} S_\lambda(t+s,s)K &= S_\lambda(t+s,t-t_n+s)S_\lambda(t-t_n+s,s)K \\ &\subseteq S_\lambda(t+s,t-t_n+s)K = S_\lambda(t_n+z,z)K \end{aligned}$$

with $z = t - t_n + s$. Thus,

$$S_\lambda(t+s,s)K \subseteq \bigcup_{z\in\mathbb{R}} S_\lambda(t_n+z,z)K.$$

Together with (4.5) we obtain that

$$\mathbb{A}_\lambda \subseteq \overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K} \subseteq \overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K}$$

and therefore

$$\Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K}, \mathbb{A}_\lambda\right) \leq \Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K}, \mathbb{A}_\lambda\right). \tag{5.13}$$

Combining (5.13) with (5.12) yields

$$\lim_{t\to\infty}\sup_{\lambda\in\Lambda}\Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}}S_\lambda(t+s,s)K},\mathbb{A}_\lambda\right)=0$$

and consequently

$$\lim_{t\to\infty}\sup_{\lambda\in\Lambda}\rho_X\left(\bigcup_{s\in\mathbb{R}}S_\lambda(t+s,s)K,\mathbb{A}_\lambda\right)=0. \tag{5.14}$$

This implies that $\mathbb{A}_\lambda$ is uniformly equi-attracting. □

We make the following four remarks. First, that our results for uniform attractors in Theorem 5.3 have not been established before in literature. Second, our result in Theorem 5.2 on everywhere continuity implying the equi-attraction is simpler and less technical than the similar ones in [15, 4]. Third, our result requires certain boundedness, but not any extra compactness for the family of the processes $S_\lambda(\cdot,\cdot)$. Finally, there are other papers (e.g. [20]) that establish the equivalence of equi-attraction and everywhere continuity for a nonautonomous dynamical system (θ,ϕ) with a cocycle mapping ϕ on X driven by an autonomous dynamical system θ acting on a base or parameter space P. Such considerations are notably different from ours.

6. Applications

In this section we demonstrate the applicability of the abstract theory developed in Sections 3 and 4 to some well-known systems of ordinary and partial differential equations. We also present some natural examples that are relevant for the autonomous theory developed in [14] (see also [2]). We begin with the following simple observation about residual sets. The proof is elementary but included for the sake of clarity and completeness.

Lemma 6.1. *Let (X,d) be a metric space, and suppose that*

$$X=\bigcup_{j=1}^{\infty}X_j. \tag{6.1}$$

If $Y_j\subseteq X_j$ is residual in (X_j,d) for all $j\geq 1$, then the set

$$Y=\bigcup_{j=1}^{\infty}Y_j \tag{6.2}$$

is residual in X.

Proof. First, observe that if $X'\subseteq X$ and $Z\subseteq X'$ is nowhere dense in X' then Z is nowhere dense in X. Second, observe that if A is residual in X and $A\subseteq B$, then B is residual in X. By hypothesis $X_j\setminus Y_j=\bigcup_{i=1}^{\infty}A_{ji}$ where each A_{ji} is nowhere dense in X_j. It follows that

$$X\setminus Y=\bigcup_{j=1}^{\infty}(X_j\setminus Y)=\bigcup_{j=1}^{\infty}\bigcap_{k=1}^{\infty}(X_j\setminus Y_k)\subseteq\bigcup_{j=1}^{\infty}(X_j\setminus Y_j)=\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty}A_{ji}.$$

By the first observation each A_{ji} is nowhere dense in X. Since $X\setminus\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty}A_{ji}\subseteq Y$, the second observation implies that Y is residual in X. □

6.1. The Lorenz system

The first application of our theory concerns the system of three ordinary differential equations introduced by Lorenz in [18]. Namely, we consider

$$\begin{cases} x' = -\sigma x + \sigma y, \\ y' = rx - y - xz, \\ z' = -bz + xy, \end{cases} \tag{6.3}$$

where σ, b and r are positive constants. These equations have been widely studied as a model of deterministic nonperiodic flow. The standard bifurcation parameter of the Lorenz equations is r, see [24], but we will consider continuity of the global attractor of the autonomous system with respect to the full parameter set $\lambda = (\sigma, b, r)$. Since physical measurements and numerical computations in general employ only approximate values, then considering perturbations in all three parameters makes sense from a mathematical point of view. As an example, we point out that Tucker [28] considered an open neighborhood of the standard choice of parameters $\lambda = (10, 8/3, 28)$ in his work on the Lorenz equations.

As shown in Doering and Gibbon [10], see also Temam [27], for any $\lambda \in (0,\infty)^3$ the solutions to (6.3) generate a semigroup $S_\lambda(t)$ for which there exists a corresponding global attractor $\mathscr{A}_\lambda$. Therefore, the requirement (G1) of Theorem 1.1 is met. The estimates in [10, 27] also show that for any compact subset Π of $(0,\infty)^3$ that there is a a bounded set D such that given any bounded set $B \in \mathbb{R}^3$ there is $T > 0$ such that

$$S_\lambda(t)B \subseteq D \qquad \text{for all} \qquad t \geq T \quad \text{and} \quad \lambda \in \Pi.$$

This guarantees that (G2) holds. Assumption (G3), the continuity of $S_\lambda(t)$ with respect to λ, can be verified by considering the equation for the difference of two solutions with different values of the parameters and using a standard Gronwall-type argument. Thus, the conditions of Theorem 1.1 are satisfied, as a consequence of which we have the following result.

Theorem 6.2. *There is a residual and dense subset Λ_* in $(0,\infty)^3$ such that the function from $(0,\infty)^3 \to CB(\mathbb{R}^3)$ given by $\lambda \mapsto \mathscr{A}_\lambda$ is continuous at every $\lambda \in \Lambda_*$.*

Proof. For each $n \in \mathbb{N}$ let $\Lambda_n = [n^{-1}, n]^3$ and define $\Phi_n \colon \Lambda_n \mapsto CB(\mathbb{R}^3)$ by $\Phi_n(\lambda) = \mathscr{A}_\lambda$. Theorem 1.1 implies there is a residual set $\Lambda_{*,n}$ in $[n^{-1}, n]^3$ such that Φ_n is continuous at each point in $\Lambda_{*,n}$ with respect to the Hausdorff metric. Set

$$\Lambda_* = \bigcup_{n=2}^{\infty} \left(\Lambda_{*,n} \cap \Lambda_n^\circ\right) \qquad \text{where} \qquad \Lambda_n^\circ = (n^{-1}, n)^3.$$

Since, the function $\Phi \colon (0,\infty)^3 \to CB(\mathbb{R}^3)$ defined by $\Phi(\lambda) = \mathscr{A}_\lambda$ is continuous at each point in $\Lambda_{*,n} \cap \Lambda_n^\circ$, then Φ is continuous at each point in Λ_*. Since $\Lambda_{*,n} \cap \Lambda_n^\circ$ is residual and dense in Λ_n°, then Lemma 6.1 implies that Λ_* is a residual subset of $(0,\infty)^3$. Moreover, since each $\Lambda_{*,n} \cap \Lambda_n^\circ$ is dense in $(n^{-1}, n)^3$, then Λ_* is dense in $(0,\infty)^3$. $\square$

As a simple illustration of the non-autonomous theory, let $r(t)$ be a fixed C^1-function on $\mathbb{R}$ and R_0 a constant such that

$$|r(t)|, |r'(t)| \leq R_0 \qquad \text{for all} \qquad t \in \mathbb{R}. \tag{6.4}$$

Consider the family of systems of ordinary differential equations given by

$$\begin{cases} x' = -\sigma x + \sigma y, \\ y' = r(t)x - y - xz, \\ z' = -bz + xy \end{cases} \tag{6.5}$$

indexed by the parameter $\lambda = (\sigma, b)$.

Note that the model (6.5) and assumption (6.4) are relevant in some climate models, see for example [9]. In particular, the function $r(t)$ can be a finite sum of sinusoidal functions. Making the standard change of variable $w = z - \sigma - r(t)$ we rewrite (6.5) as

$$\begin{cases} x' = -\sigma x + \sigma y, \\ y' = -y - \sigma x - xw, \\ w' = -bw + xy + F(t) \end{cases} \tag{6.6}$$

with

$$F(t) = -b(\sigma + r(t)) - r'(t).$$

Thanks to condition (6.4) setting $F_0 = b(\sigma + R_0) + R_0$ yields

$$|F(t)| \leq F_0 \qquad \text{for all} \qquad t \in \mathbb{R}. \tag{6.7}$$

Similar estimates to those in [10] and [27], based on the formulation (6.6), show for each $(\sigma, b) \in (0, \infty)^2$ that the system (6.5) generates a process $S_{\sigma,b}(\cdot, \cdot)$, that there exists pullback attractors $\mathscr{A}_{\sigma,b}(t)$ for every $t \in \mathbb{R}$, and that the uniform attractor $\mathbb{A}_{\sigma,b}$ exists. For the sake of completeness we present explicit estimates here, which will also be used in the next theorem.

Let $v(t) = (x(t), y(t), z(t))$ be a solution of (6.3), and $u(t) = (x(t), y(t), w(t))$. Note that

$$|v| \leq |u| + \sigma + R_0 \quad \text{and} \quad |u| \leq |v| + \sigma + R_0.$$

We have from (6.6), (6.7) and by Cauchy's inequality that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2 + w^2) + \sigma x^2 + y^2 + bw^2 = Fw \leq \frac{b}{2}w^2 + \frac{F_0^2}{2b}.$$

Upon setting $\sigma_0 = \min\{1, \sigma, b/2\}$ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + 2\sigma_0|u|^2 \leq \frac{F_0^2}{b}.$$

This implies for all $t \geq 0$ that

$$|u(t)|^2 \leq |u(0)|^2 e^{-2\sigma_0 t} + \frac{F_0^2}{2\sigma_0 b};$$

hence

$$|u(t)| \le |u(0)|e^{-\sigma_0 t} + \frac{F_0}{\sqrt{2\sigma_0 b}} \le (|v(0)| + \sigma + R_0)e^{-\sigma_0 t} + \frac{F_0}{\sqrt{2\sigma_0 b}}. \tag{6.8}$$

Thus,

$$|v(t)| \le (|v(0)| + \sigma + R_0)e^{-\sigma_0 t} + \frac{F_0}{\sqrt{2\sigma_0 b}} + (\sigma + R_0). \tag{6.9}$$

By (6.8) and (6.9), we have for all $t \ge 0$ that

$$|u(t)|, |v(t)| \le R_1 \quad \text{where} \quad R_1 = |v(0)| + 2(\sigma + R_0) + \frac{F_0}{\sqrt{2\sigma_0 b}}. \tag{6.10}$$

Next we consider the continuity in $\lambda = (\sigma, b)$. For $i \in \{1, 2\}$ let $\lambda_i = (\sigma_i, b_i) \in (0, \infty)^2$ be given and let $v_i(t) = (x_i(t), y_i(t), z_i(t))$ be the corresponding solution of (6.3). Define

$$u_i(t) = (x_i(t), y_i(t), w_i(t)) = (x_i(t), y_i(t), z_i(t) - \sigma_i - r(t)).$$

Let $\bar\lambda = (\bar\sigma, \bar b) = \lambda_1 - \lambda_2$ and $\bar u = (\bar x, \bar y, \bar w) = u_1 - u_2$. Then (6.6) implies

$$\begin{cases} \bar x' = -\bar\sigma x_1 - \sigma_2 \bar x + \bar\sigma y_1 + \sigma_2 \bar y, \\ \bar y' = -\bar y - \bar x w_1 - x_2 \bar w - \bar\sigma x_1 - \sigma_2 \bar x, \\ \bar w' = -\bar b w_1 - b_2 \bar w + \bar x y_1 + x_2 \bar y - \bar b(\sigma_1 + r(t)) - b_2 \bar\sigma. \end{cases}$$

It follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|\bar u|^2 + \sigma_2 \bar x^2 + \bar y^2 + b_2 \bar w^2 &= -\bar\sigma(x_1 + y_1)\bar x - \bar x w_1 \bar y - \bar\sigma x_1 \bar y \\ &\quad - \bar b w_1 \bar w + \bar x y_1 \bar w - [\bar b(\sigma_1 + r(t)) + b_2 \bar\sigma]\bar w. \end{aligned} \tag{6.11}$$

By neglecting $\sigma_2 \bar x^2 + \bar y^2 + b_2 \bar w^2$ on the left-hand side of (6.11) and using estimate (6.10) to bound $|x_1|$, $|y_1|$, $|w_1|$ on the right-hand side, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|\bar u|^2 &\le 2R_1|\bar\lambda||\bar u| + R_1|\bar u|^2 + |\bar\lambda|R_1|\bar u| + |\bar\lambda|R_1|\bar u| + R_1|\bar u|^2 + (|\lambda_1| + R_0 + |\lambda_2|)|\bar\lambda||\bar u| \\ &= (4R_1 + R_0 + |\lambda_1| + |\lambda_2|)|\bar\lambda||\bar u| + 2R_1|\bar u|^2 \\ &\le 2R_2|\bar u|^2 + R_3^2|\bar\lambda|^2, \end{aligned}$$

where R_1 is defined in (6.10) with $v = v_1$,

$$R_2 = R_1 + \frac{1}{8} \quad \text{and} \quad R_3 = R_0 + 4R_1 + |\lambda_1| + |\lambda_2|. \tag{6.12}$$

By Gronwall's inequality, we obtain for $t \ge 0$ that

$$|\bar u(t)|^2 \le |\bar u(0)|^2 e^{2R_2 t} + R_3^2 t e^{2R_2 t}|\bar\lambda|^2. \tag{6.13}$$

Let $\bar v = (\bar x, \bar y, \bar z) = v_1 - v_2$. Therefore, $\bar v = \bar u + (0, 0, \sigma_1 - \sigma_2)$. It follows from (6.13) for $t \ge 0$ that

$$\begin{aligned} |\bar v(t)| &\le |\bar u(t)| + |\bar\lambda| \le |\bar u(0)|e^{R_2 t} + R_3\sqrt{t}e^{R_2 t}|\bar\lambda| + |\bar\lambda| \\ &\le (|\bar v(0)| + |\bar\lambda|)e^{R_2 t} + R_3\sqrt{t}e^{R_2 t}|\bar\lambda| + |\bar\lambda|. \end{aligned}$$

Thus,

$$|\bar{v}(t)| \le e^{R_2 t}\Big\{|\bar{v}(0)| + (2 + R_3\sqrt{t})|\bar{\lambda}|\Big\} \quad \text{for every} \quad t \ge 0. \tag{6.14}$$

We are ready to obtain the continuity of $\mathscr{A}_{\sigma,b}(t)$ and $\mathbb{A}_{\sigma,b}$ as functions of σ and b.

Theorem 6.3. *There is a residual and dense subset Λ_* in $(0,\infty)^2$ such that the functions from $(0,\infty)^2 \to CB(\mathbb{R}^3)$ defined by*

$$(\sigma, b) \mapsto \mathscr{A}_{\sigma,b}(t) \quad \textit{for every} \quad t \in \mathbb{R} \qquad \textit{and} \qquad (\sigma, b) \mapsto \mathbb{A}_{\sigma,b} \tag{6.15}$$

are continuous at every point $(\sigma, b) \in \Lambda_*$.

Proof. Denote $\lambda = (\sigma, b)$. Let $0 < \delta \le \mu$. Suppose $\lambda \in [\delta, \mu]^2$ and $|v(0)| \le M$. Then

$$F_0 \le F_*, \qquad \sigma_0 \ge \sigma_* \qquad \text{and} \qquad b \ge 2\sigma_*$$

where $F_* = \mu(\mu + R_0) + R_0$ and $\sigma_* = \min\{1, \delta/2\}$. Consequently,

$$R_1 \le R_{1,*} \qquad \text{where} \qquad R_{1,*} = M + 2(\mu + R_0) + \frac{F_*}{2\sigma_*}.$$

Similarly denote $\lambda_i = (\sigma_i, b_i)$. Suppose $\lambda_i \in [\delta, \mu]^2$ and $|v_i(0)| \le M$ for $i = 1, 2$. Then

$$R_2 \le R_{2,*} := R_{1,*} + \frac{1}{8} \quad \text{and} \quad R_3 \le R_{3,*} := R_0 + 4R_{1,*} + 2\sqrt{2}\mu.$$

Since $\mathbb{R}^3$ is finite dimensional, then every element of $CB(\mathbb{R}^3)$ is compact. Therefore (6.9) with $\Lambda = [\delta, \mu]^2$ may be used to verify requirement (L2′) and condition (a) of Theorem 4.1 while (6.14) may be used to verify (L3′) and (b).

Let $\delta = 1/n$ and $\mu = n$ for $n \ge 2$. By Theorem 3.3 there is a residual set $\Lambda^p_{*,n}$ in $[1/n, n]^2$ such that the function $[1/n, n]^2 \mapsto CB(\mathbb{R}^3)$ given by $(\sigma, b) \mapsto \mathscr{A}_{\sigma,b}(t)$ is continuous at each point in $\Lambda^p_{*,n}$ for all $t \in \mathbb{R}$. Similarly, by Theorem 4.1 there is a residual set $\Lambda^u_{*,n}$ in $[1/n, n]^2$ such that $(\sigma, b) \mapsto \mathbb{A}_{\sigma,b}$ is continuous at each point in $\Lambda^u_{*,n}$. Let $\Lambda_{*,n} = \Lambda^p_{*,n} \cap \Lambda^u_{*,n} \cap (1/n, n)^2$. Then $\Lambda_{*,n}$ is residual and dense in $(1/n, n)^2$ and the maps given by (6.15) are continuous at every point $(\sigma, b) \in \Lambda_{*,n}$. Set $\Lambda_* = \bigcup_{n=2}^{\infty} \Lambda_{*,n}$. Since each $\Lambda_{*,n}$ is dense in $(1/n, n)^2$, the set Λ_* is dense in $(0,\infty)^2$. Moreover, by Lemma 6.1, Λ_* is residual in $(0,\infty)^2$. We finish noting that the functions defined in (6.15) are continuous at every point $(\sigma, \mu) \in \Lambda_*$. ☐

6.2. The two-dimensional Navier-Stokes equations

We now turn to the two-dimensional Navier–Stokes equations. Let Ω be a bounded, open and connected set in $\mathbb{R}^2$ with C^2 boundary (i.e. $\partial\Omega$ can be represented locally as the graph of a C^2 function). Consider the two-dimensional incompressible Navier–Stokes equations in Ω with no-slip Dirichlet boundary conditions

$$\begin{cases} u_t - \nu\Delta u + (u \cdot \nabla)u = -\nabla p + f & \text{on } \Omega \\ \nabla \cdot u = 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.16}$$

where $u = u(x,t)$ is the Eulerian velocity field, $p = p(x,t)$ is the pressure, $\nu > 0$ is the kinematic viscosity and $f = f(x,t)$ is the body force.

Define

$$\mathscr{V} = \{\, v \in [C_c^\infty(\Omega)]^2 : \nabla \cdot v = 0 \,\}$$

and let H and V be the closures of $\mathscr{V}$ in the norms of $[L^2(\Omega)]^2$ and $[H^1(\Omega)]^2$, respectively. Note that H is a Hilbert space with inner product $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|$ inherited from $[L^2(\Omega)]^2$. Similarly V is a Hilbert space, however, in this case we shall use the norm $v \mapsto \|\nabla v\|$, which is equivalent to the $[H^1(\Omega)]^2$ norm on V due to the Poincaré inequality. The Rellich–Kondrachov Theorem implies V is compactly embedded into H. We denote the dual of V by V^* with the pairing $\langle u, v\rangle$ for $u \in V^*$ and $v \in V$.

Following, for example [26], we write (6.16) in functional form as the equation

$$u_t + \nu A u + B(u,u) = P_L f \tag{6.17}$$

in V^* where $v(t) \in V$. Here P_L is the (Helmholtz–Leray) orthogonal projection from $[L^2(\Omega)]^2$ onto H and A and B are the continuous extensions of the operators given by

$$Au = P_L(-\Delta u) \qquad \text{and} \qquad B(u,v) = P_L((u\cdot\nabla)v) \qquad \text{for} \qquad u, v \in \mathscr{V}$$

such that $A\colon V \mapsto V^*$ and $B\colon V \times V \mapsto V^*$.

Let $\lambda_1 > 0$ be the first eigenvalue of the Stokes operator. With this notation Poincaré's inequality may be written as $\|\nabla v\|^2 \geq \lambda_1 \|v\|^2$ for all $v \in V$. For convenience assume $f(t) \in H$ for all time $t \in \mathbb{R}$ so that $P_L f = f$ in (6.17). When $f \in L^\infty(\mathbb{R}, H)$ further define the Grashof number G as

$$G = \frac{1}{\lambda_1 \nu^2} \big\|f\big\|_{L^\infty(\mathbb{R},H)} \quad \text{where} \quad \big\|f\big\|_{L^\infty(\mathbb{R},H)} = \operatorname{ess\,sup}\big\{\, \|f(t)\| : t \in \mathbb{R} \,\big\}. \tag{6.18}$$

Note when f is time independent, this definition reduces to the definition of Grashof number given, for example, in [27].

As shown in [5] and references therein, when $f \in L^\infty(\mathbb{R}, H)$ the system (6.16) generates a process $S_f(t,s)\colon H \mapsto H$ satisfying Definition 2.1 defined by $S_f(t,s)u_0 = u(t)$ where $u(t)$ is the solution of (6.16) on $[s, \infty)$ with $u(s) = u_0$. Moreover, a pullback attractor $\mathscr{A}_f(t)$ exists for every $t \in \mathbb{R}$ as does a uniform attractor $\mathbb{A}_f$. We therefore have (L1).

To obtain (L2′) and (L3′) we employ bounds on individual solutions in terms of the Grashof number similar to those which show the existence of absorbing sets in H and V in the case when f is time independent. Such estimates may be found in [27] pages 109–111 and also [8, 16, 22, 23, 26] among others. As they are simple we include the relevant calculations in Appendix A.

Theorem 6.4. *Let $f \in L^\infty(\mathbb{R}, H)$ and G be defined as in* (6.18)*. Suppose $u(t) = S_f(t,s)u_0$ where $u_0 \in H$ with $\|u_0\| \leq M$. Then for all $t \geq s$*

$$\nu \int_s^t \|\nabla u(\tau)\|^2 \, \mathrm{d}\tau \leq \|u_0\|^2 + (t-s)\nu^3 \lambda_1 G^2. \tag{6.19}$$

Moreover, there exists a constant $t_0 > 0$, depending only on M, ν, and λ_1, such that $t - s \geq t_0$ implies that

$$\|u(t)\|^2 \leq 2\nu^2 G^2 \qquad \text{and} \qquad \|\nabla u(t)\|^2 \leq \rho(G), \tag{6.20}$$

where $\rho(G)$ is an increasing function of G that also depends on ν and λ_1.

Noting that $L^\infty(\mathbb{R}, H)$ is a complete metric space with respect to the norm described in (6.18), we are now ready to obtain the continuity of $\mathscr{A}_f(t)$ and $\mathbb{A}_f$ as functions of f.

Theorem 6.5. *There is a residual and dense subset Λ_* in $L^\infty(\mathbb{R}, H)$ such that the maps from $L^\infty(\mathbb{R}, H) \mapsto CB(H)$ given by*

$$f \mapsto \mathscr{A}_f(t) \quad \textit{for every} \quad t \in \mathbb{R} \qquad \textit{and} \qquad f \mapsto \mathbb{A}_f \tag{6.21}$$

are continuous at every point $f \in \Lambda_*$.

Proof. Given $n > 0$ let

$$\Lambda_n = \{\, f \in L^\infty(\mathbb{R}, H) : G \le n \,\},$$

where G is the Grashof number defined in (6.18), and let K be the ball of radius $\rho(n)$ in V. We remark that Λ_n is a complete metric space and that K is a compact subset of H. To obtain (L2′) it is enough to show that $\mathscr{A}_f(t) \subset K$ for every $f \in \Lambda_n$ and $t \in \mathbb{R}$. In light of Theorems 11.3 and 2.12 of [5] we recall that

$$\mathscr{A}_\nu(t) = \overline{\bigcup \{\omega(B,t) : \ B \text{ is a bounded set in } H\}},$$

where, according to Definition 2.2 in [5],

$$\omega(B,t) = \bigcap_{\sigma \le t} \overline{\bigcup_{s \le \sigma} S_f(t,s)B}.$$

Now, given any bounded $B \subset H$, there is M large enough such that $u_0 \in B$ implies $\|u_0\| \le M$. From Theorem 6.4 there is t_0 large enough such that

$$\|\nabla S_f(t,s)u_0\| \le \rho(G) \le \rho(n) \qquad \text{whenever} \qquad t - s \ge t_0.$$

Therefore $S_f(t,s)u_0 \in K$ and consequently

$$\omega(B,t) \subseteq \overline{\bigcup_{s \le t - t_0} S_f(t,s)B} \subseteq K.$$

It follows that $\mathscr{A}_f(t) \subseteq K$.

To show that (L3′) holds, let $f_1, f_2 \in \Lambda_n$ and consider the solutions

$$u_1(t) = S_{f_1}(t,s)u_0 \qquad \text{and} \qquad u_2(t) = S_{f_2}(t,s)u_0$$

Then $w = u_1 - u_2$ satisfies

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \nu A w + B(u_1, w) + B(w, u_2) = f \quad \text{where} \quad f = f_1 - f_2. \tag{6.22}$$

The 2D Ladyzhenskaya inequality $\|u\|_{L^4} \le c_L \|u\|^{1/2} \|\nabla u\|^{1/2}$ in conjunction with the Hölder inequality implies that

$$|\langle B(u,v), w\rangle| \le c_B \|u\|^{1/2} \|\nabla u\|^{1/2} \cdot \|\nabla v\| \cdot \|w\|^{1/2} \|\nabla w\|^{1/2} \tag{6.23}$$

where $c_B = c_L^2$. Taking the inner product of (6.22) with w and using (6.36) and (6.23) gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \nu\|\nabla w\|^2 &\le |\langle B(w,u_2),w\rangle| + \|f\|\|w\| \\ &\le c_B\|w\|\|\nabla w\|\|\nabla u_2\| + \lambda_1^{-1/2}\|f\|\|\nabla w\| \\ &\le \frac{\nu}{4}\|\nabla w\|^2 + \frac{c_B^2\|w\|^2\|\nabla u_2\|^2}{\nu} + \frac{\nu}{4}\|\nabla w\|^2 + \frac{\|f\|^2}{\nu\lambda_1}.\end{aligned}$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 \le \frac{2c_B^2\|\nabla u_2\|^2}{\nu}\|w\|^2 + \frac{2\|f\|^2}{\nu\lambda_1}.$$

Applying Gronwall's inequality we obtain

$$\|w(t)\|^2 \le \frac{2\|f\|^2_{L^\infty(\mathbb{R},H)}}{\nu\lambda_1}\int_s^t \exp\Big(\frac{2c_B^2}{\nu}\int_\tau^t \|\nabla u_2(s)\|^2\,\mathrm{d}s\Big)\,\mathrm{d}\tau.$$

Using estimate (6.19) for u_2 we have that

$$\begin{aligned}&\int_s^t \exp\Big(\frac{2c_B^2}{\nu}\int_\tau^t \|\nabla u_2(s)\|^2\,\mathrm{d}s\Big)\,\mathrm{d}\tau \\ &\qquad\le (t-s)\exp\bigg\{\frac{2c_B^2}{\nu^2}\Big(\|u_0\|^2 + (t-s)\frac{\|f_2\|^2_{L^\infty(\mathbb{R},H)}}{\nu\lambda_1}\Big)\bigg\} \le C(t,s)\end{aligned}$$

where

$$C(t,s) = (t-s)\exp\Big\{\frac{2c_B^2}{\nu^2}\big(M^2 + (t-s)\nu^3\lambda_1 n^2\big)\Big\}.$$

It follows that

$$\|S_{f_1}(t,s)u_0 - S_{f_2}(t,s)u_0\|^2 \le \frac{2C(t,s)}{\nu\lambda_1}\|f_1 - f_2\|^2_{L^\infty(\mathbb{R},H)}.$$

Therefore $S_f(t,s)u_0$ is continuous in f uniformly for u_0 in bounded subsets of H. Thus, (L3) and consequently (L3′) holds.

By Theorem 3.3 there is a residual set $\Lambda^p_{*,n}$ in Λ_n such that the maps from $\Lambda_n \to CB(H)$ defined by $f \mapsto \mathscr{A}_f(t)$ for every $t \in \mathbb{R}$ are continuous at each point in $\Lambda^p_{*,n}$. Note that the analysis which proves (L2′) and (L3′) also shows that conditions (a) and (b) of Theorem 4.1 are satisfied. Therefore, there is also a residual set $\Lambda^u_{*,n}$ such that the map defined by $f \mapsto \mathbb{A}_f$ is continuous at each point in $\Lambda^u_{*,n}$. Set $\Lambda_{*,n} = \Lambda^p_{*,n} \cap \Lambda^u_{*,n}$ and

$$\Lambda_* = \bigcup_{n=1}^\infty (\Lambda_{*,n} \cup \Lambda_n^\circ) \qquad \text{where} \qquad \Lambda_n^\circ = \big\{\, f \in L^\infty(\mathbb{R},H) : \|f\|_{L^\infty(\mathbb{R},H)} < \nu^2\lambda_1 n \,\big\}.$$

Arguments identical to those given at the end of the proof for Theorem 6.2 are now sufficient to finish this proof. □

Before closing, let us draw a few consequences from Theorem 6.5.

Corollary 6.6. *Let $\Lambda \subseteq L^\infty(\mathbb{R}, H)$ be closed. There is a residual and dense set Λ_* in Λ such that the maps from $\Lambda \mapsto CB(H)$ given by (6.21) are continuous at every point in Λ_*.*

Proof. While it is not, in general, true that a set which is residual in $L^\infty(\mathbb{R}, H)$ is necessarily residual in Λ, we can argue as follows. First observe that all estimates used in the proof of Theorem 6.6 also hold when considering a smaller collection of forces. Since a closed subset of a complete metric space is also complete, Theorems 3.3 and 4.1 apply equally well to the sets Λ_n given by

$$\Lambda_n = \{ f \in \Lambda : \|f\|_{L^\infty(\mathbb{R},H)} \le \nu^2 \lambda_1 n \}.$$

We therefore obtain a Λ_* residual in Λ that satisfies the desired continuity conditions. ☐

Now consider the autonomous case in which the two-dimensional incompressible Navier–Stokes equations are forced by a time-independent function $f \in H$. Although it is possible to apply Theorem 1.1 using the same analysis as before to show that (G1), (G2) and (G3) hold, we instead apply Corollary 6.6 to obtain the following result.

Corollary 6.7. *There is a residual and dense set Λ_* in H such that the map from $H \mapsto CB(H)$ given by $f \mapsto \mathscr{A}_f$ is continuous at every point $f \in \Lambda_*$.*

Proof. Since the set of time-independent forcing functions may be viewed as a closed subset of $L^\infty(\mathbb{R}, H)$ then there is a residual set Λ_* in H. Since the global attractor $\mathscr{A}_f$ in the autonomous case is identical with the pullback attractor $\mathscr{A}_f(t)$ for all $t \in \mathbb{R}$ when f is time independent (Lemma 1.19 in [5]), we may immediately apply Corollary 6.6 to obtain the desired result. ☐

We close with an example in which we fix the forcing $f = f_0$ where $f_0 \in L^\infty(\mathbb{R}, H)$ and consider the family of attractors parameterized by viscosity ν.

Corollary 6.8. *There is a residual and dense set Λ_* in $(0, \infty)$ such that the maps from $(0, \infty) \mapsto CB(H)$ given by*

$$\nu \mapsto \mathscr{A}_\nu(t) \quad \textit{for every} \quad t \in \mathbb{R} \qquad \textit{and} \qquad \nu \to \mathbb{A}_\nu \tag{6.24}$$

are continuous at every point $\nu \in \Lambda_*$.

Proof. The change of variables $v = \nu^{-1} u$ and $\tau = \nu t$ transforms (6.17) into

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} + Av + B(v, v) = \nu^{-2} f_0. \tag{6.25}$$

For $g \in L^\infty(\mathbb{R}, H)$, denote by $\mathscr{B}_g(\tau)$ the pullback attractor of (6.25) with the right-hand side being replaced by g, and by $\mathbb{B}_g$ the uniform attractor.

Let $\Lambda' = \{ cf_0 : c \in [0, \infty) \}$. Since Λ' is a closed subset of $L^\infty(\mathbb{R}, H)$ then Corollary 6.6 implies there exists a residual (and so dense) set Λ'_* in Λ' such that the maps from $\Lambda' \mapsto CB(H)$ given by

$$Q_1 : g \mapsto \mathscr{B}_g(\tau) \quad \text{for every} \quad \tau \in \mathbb{R} \qquad \text{and} \qquad Q_2 : g \mapsto \mathbb{B}_g \tag{6.26}$$

are continuous at each point in Λ'_*.

Define $\Lambda_* = \{ 1/\sqrt{c} : cf_0 \in \Lambda'_* \text{ and } c > 0 \}$. Since the mapping $\xi : (0,\infty) \to \Lambda' \setminus \{0\}$ given by $\xi(\nu) = \nu^{-2} f_0$ is a continuous bijection, then $\Lambda_* = \xi^{-1}(\Lambda'_* \setminus \{0\})$ is residual and dense in $(0,\infty)$. Then, by (6.26), the maps from $(0,\infty) \to CB(H)$ given by

$$Q_3 = Q_1 \circ \xi : \nu \mapsto \mathscr{B}_{\xi(\nu)}(\tau) \quad \text{for every} \quad \tau \in \mathbb{R} \qquad \text{and} \qquad Q_4 = P_2 \circ \xi : \nu \mapsto \mathbb{B}_{\xi(\nu)} \tag{6.27}$$

are also continuous at each point in Λ_*. Note that

$$\mathscr{A}_\nu(t) = \nu \mathscr{B}_{\xi(\nu)}(\nu t) \qquad \text{and} \qquad \mathbb{A}_\nu = \nu \mathbb{B}_{\xi(\nu)}. \tag{6.28}$$

Since the map

$$(\nu, K) \in (0,\infty) \times CB(H) \mapsto \nu K \text{ is continuous,} \tag{6.29}$$

the continuity of Q_4 in (6.27) and the second identity in (6.28) imply that $\nu \mapsto \mathbb{A}_\nu$ is continuous at each point in Λ_*.

Claim. Given $\tau_0 \in \mathbb{R}$. If $g \in L^\infty(\mathbb{R}, H) \mapsto \mathscr{B}_g(\tau_0)$ is continuous at g_0, then the map $(g,\tau) \mapsto \mathscr{B}_g(\tau)$ is continuous at (g_0, τ_0).

This *Claim* and the continuity of Q_3 in (6.27) imply that the map $\nu \mapsto \mathscr{B}_{\xi(\nu)}(\nu t)$ is continuous at each point in Λ_*, for any $t \in \mathbb{R}$. Combining this fact with the first identity in (6.28) and property (6.29) proves that the map $\nu \mapsto \mathscr{A}_\nu(t)$ is continuous at each point in Λ_*, for any $t \in \mathbb{R}$.

It remains to prove *Claim*. By the triangle inequality,

$$\Delta_H(\mathscr{B}_g(\tau), \mathscr{B}_{g_0}(\tau_0)) \le \Delta_H(\mathscr{B}_g(\tau), \mathscr{B}_g(\tau_0)) + \Delta_H(\mathscr{B}_g(\tau_0), \mathscr{B}_{g_0}(\tau_0)). \tag{6.30}$$

Since $g \to g_0$, we have g belongs to a bounded subset of $L^\infty(\mathbb{R}, H)$. Then the first term on the right-hand side of (6.30) goes to zero as $(g,\tau) \to (g_0,\tau_0)$ by the virtue of Proposition 6.9 below. The second term on right-hand side of (6.30) goes to zero by the assumption $g \mapsto \mathscr{B}_g(\tau_0)$ is continuous at g_0. Thus, the map $(g,\tau) \mapsto \mathscr{B}_g(\tau)$ is continuous at (g_0,τ_0). This finishes the proof of *Claim* and also the proof of this corollary. □

The following result is about the continuity in time of pullback attractors for the Navier–Stokes equations (6.16). It has its own merit, and is stronger than what is needed for the proof in Corollary 6.8.

Proposition 6.9. *Let $\mathscr{A}_{\nu,f}(t)$ be pullback attractors of the Navier–Stokes equations* (6.16) *with $\nu \in (0,\infty)$ and $f \in L^\infty(\mathbb{R}, H)$. Then the map $t \mapsto \mathscr{A}_{\nu,f}(t)$ is locally Hölder continuous on $\mathbb{R}$, uniformly in (ν, f) for ν belonging to any compact subsets of $(0,\infty)$ and f belonging to any bounded subsets of $L^\infty(\mathbb{R}, H)$.*

Proof. Denote by $S_{\nu,f}(t,s)$ the process generated by solutions of the Navier–Stokes equations (6.16). Let $R_0 > 0$ and $\varepsilon_0 \in (0,1)$. Define $\bar{G} = R_0/(\lambda_1 \varepsilon_0^2)$ and

$$\bar{\rho} = \max\{\rho(\bar{G}) : \nu \in [\varepsilon_0, \varepsilon_0^{-1}]\}.$$

Let $D = \bar{B}_V(\bar{\rho})$. Consider $\|f\|_\infty \le R_0$ and $\nu \in [\varepsilon_0, \varepsilon_0^{-1}]$. Same arguments as in Theorem 6.5 for the set K replaced by D, and same as (5.3), we have

$$\mathscr{A}_{\nu,f}(t) = \lim_{s \to -\infty} \overline{S_{\nu,f}(t,s) D}. \tag{6.31}$$

Given $u_0 \in D$, let $u(t) = S_{\nu,f}(t,s)u_0$. By Theorem 6.4, there is $T_0 > 0$ depending on R_0 and ε_0 such that if $t - s \geq T_0$ then

$$\|u(t)\| \leq R_1 = 2\varepsilon_0^{-2}\bar{G}^2, \qquad \|\nabla u(t)\| \leq \bar{\rho}. \tag{6.32}$$

We recall inequality (2.32) in [27] p. 111:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \nu|Au|^2 \leq \frac{2}{\nu}\|f\|^2 + \frac{2c_0}{\nu^3}\|u\|^2\|\nabla u\|^4, \tag{6.33}$$

where $c_0 > 0$ is an appropriate constant.

Consider $t_2 > t_1 > s + T_0$ with $t_2 - t_1 < 1$. Integrating (6.33) in time from t_1 to t_2 gives

$$\nu \int_{t_1}^{t_2} |Au|^2 \mathrm{d}\tau \leq \|\nabla u(t_1)\|^2 + \frac{2c_0}{\nu^3}\int_{t_1}^{t_2} \|u\|^2\|\nabla u\|^4 \mathrm{d}\tau + \frac{2}{\nu}\int_{t_1}^{t_2} \|f\|^2 \mathrm{d}\tau,$$

Using estimates in (6.32) yields

$$\nu \int_{t_1}^{t_2} |Au|^2 \mathrm{d}\tau \leq R_2 = \bar{\rho}^2 + 2c_0\varepsilon_0^{-3}R_1^2\bar{\rho}^4 + 2\varepsilon_0^{-1}R_0^2.$$

Now,

$$u(t_2) - u(t_1) = -\nu\int_{t_1}^{t_2} Au(\tau)\mathrm{d}\tau - \int_{t_1}^{t_2} B(u(\tau),u(\tau))\mathrm{d}\tau + \int_{t_1}^{t_2} f(\tau)\mathrm{d}\tau. \tag{6.34}$$

Then,

$$\begin{aligned}\|u(t_2) - u(t_1)\| &\leq \nu\int_{t_1}^{t_2} \|Au(\tau)\|\mathrm{d}\tau + c_1\int_{t_1}^{t_2} \|u(\tau)\|^{1/2}\|Au(\tau)\|^{1/2}\|\nabla u(\tau)\|\mathrm{d}\tau + R_0(t_2 - t_1)\\ &\leq \nu^{1/2}(t_2 - t_1)^{1/2}\Big(\nu\int_{t_1}^{t_2} \|Au(\tau)\|^2\mathrm{d}\tau\Big)^{1//2}\\ &\quad + c_1R_1^{1/2}\bar{\rho}(t_2 - t_1)^{3/4}\nu^{-1/4}\Big(\nu\int_{t_1}^{t_2} \|Au(\tau)\|^2\mathrm{d}\tau\Big)^{1/4} + R_0(t_2 - t_1).\end{aligned}$$

Above, $c_1 > 0$ is a constant independent of R_0, ε_0. Utilizing estimate (6.31) gives

$$\|u(t_2) - u(t_1)\| \leq \varepsilon_0^{-1/2}R_2^{1/2}(t_2 - t_1)^{1/2} + c_1\varepsilon_0^{-1/4}R_1^{1/2}\bar{\rho}R_2^{1/4}(t_2 - t_1)^{3/4} + R_0(t_2 - t_1).$$

Therefore, there is $M > 0$ depending on R_0 and ε_0 such that

$$\|u(t_2) - u(t_1)\| \leq M(t_2 - t_1)^{1/2}.$$

Consequently,

$$\Delta_H(S_{\nu,f}(t_2,s)D, S_{\nu,f}(t_1,s)D) \leq M(t_2 - t_1)^{1/2}.$$

By this and (6.31), we have

$$\Delta_H(\mathscr{A}_{\nu,f}(t_2), \mathscr{A}_{\nu,f}(t_1)) \leq M(t_2 - t_1)^{1/2}. \tag{6.35}$$

This finishes the proof of our proposition. $\square$

Appendix A

This appendix reproduces the formal *a priori* estimates for the two-dimensional incompressible Navier–Stokes equations stated as Theorem 6.4 in the main body of the paper. These are essentially the estimates that appear in [27] pages 109–111 for the time-independent case that have been adapted for time-dependent body forces $f \in L^\infty(\mathbb{R}, H)$.

Let $u_0 \in H$, $t \geq s$, and set $u(t) = S_f(t,s)u_0$. Taking the inner product of (6.17) with u and using the orthogonality property

$$\langle B(u,v), v\rangle = 0 \tag{6.36}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\|\nabla u\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}. \tag{6.37}$$

Integrating (6.37) in time from s to t yields

$$\nu\int_s^t \|\nabla u(\tau)\|^2\,\mathrm{d}\tau \leq \|u_0\|^2 + (t-s)\frac{\|f\|^2_{L^\infty(\mathbb{R},H)}}{\nu\lambda_1} \tag{6.38}$$

Noting that $\|f\|_{L^\infty(\mathbb{R},H)} = \nu^2\lambda_1 G$ obtains (6.19) in Theorem 6.4.

By Poincaré's inequality we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\lambda_1\|u\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}.$$

Using Gronwall's inequality yields

$$\|u(t)\|^2 \leq \|u_0\|^2 e^{-\nu\lambda_1(t-s)} + \rho_0^2(1-e^{-\nu\lambda_1(t-s)}) \quad \text{where} \quad \rho_0 = \nu G. \tag{6.39}$$

Therefore, for each bounded subset B of H, there exists a time $t_0(B)$ such that

$$\|u(t)\|^2 \leq 2\rho_0^2 \qquad \text{for all} \qquad t \geq t_0(B). \tag{6.40}$$

We have, consequently, obtained the first part of (6.20).

Finally, we recall here the needed estimates for $\|\nabla u(t)\|$. (Again, calculations can be found, for example, in [27] pages 109–111.) Let $\nu > 0$. Recall from (6.39) that $\rho_0 = \nu G$, and define

$$\rho_0' = \rho_0 + 1, \quad m_1 = \lambda_1\rho_0^2 + \frac{{\rho'}_0^2}{\nu}, \quad m_2 = 2\nu\lambda_1^2\rho_0^2, \quad m_3 = \frac{2c_0}{\nu^3}{\rho'}_0^2 m_1,$$

where c_0 is the same as in (6.33). For $R > \rho_0'$, denote

$$t_1(R) = 1 + \frac{1}{\nu\lambda_1}\log\frac{R^2}{2\rho_0+1}.$$

Then, for $\|u_0\| \leq R$ and all $t \geq t_1(R)$,

$$\|\nabla u(t+s)\|^2 \leq \rho(G) \qquad \text{where} \qquad \rho(G) = (m_1+m_2)e^{m_3}. \tag{6.41}$$

Noting that $\rho(G)$ is an increasing function of G such that also depends on ν and λ_1 yields the final part of (6.20) in Theorem 6.4.

Acknowledgments

LTH acknowledges the support by NSF grant DMS-1412796. EJO was supported in part by NSF grant DMS grant DMS-1418928. JCR was supported by an EPSRC Leadership Fellowship EP/G007470/1, as was the visit of EJO to Warwick while on sabbatical leave from UNR.

References

[1] A.V. Babin and M.I. Vishik, *Attractors of evolution equations*, Studies in Mathematics and its Applications, 25. North-Holland Publishing Co., Amsterdam, 1992.

[2] A.V. Babin and S. Yu Pilyugin, Continuous dependence of attractors on the shape of domain, *J. Math. Sci.*, **87** (1997) 3304–3310.

[3] J.E. Billotti and J.P. LaSalle, Dissipative periodic processes, *Bull. Amer. Math. Soc.*, **77** (1971) 1082–1088.

[4] A.N. Carvalho, J.A. Langa, and J.C. Robinson, On the continuity of pullback attractors for evolution processes, *Nonlinear Anal.*, **71** no. 5–6 (2009), 1812–1824.

[5] A.N. Carvalho, J.A. Langa, and J.C. Robinson, *Attractors for infinite-dimensional non-autonomous dynamical systems*, Springer, New York, 2013.

[6] V.V. Chepyzhov and M.I. Vishik, Attractors for equations of mathematical physics, *American Mathematical Society*, Providence, RI, 2002.

[7] I.D. Chueshov, *Introduction to the theory of infinite-dimensional dissipative systems*, AKTA, Kharkiv, 1999.

[8] P. Constantin and C. Foias, *Navier–Stokes Equations*, Chicago: University of Chicago Press, (1988).

[9] J.D. Daron and D. A. Stainforth, On quantifying the climate of the nonautonomous Lorenz-63 model, Chaos, **25** (4), 043103.

[10] C.R. Doering and J.D. Gibbon, *Applied analysis of the Navier-Stokes equations*, Cambridge University Press, Cambridge, 1995

[11] J.K. Hale, *Asymptotic behavior of dissipative systems*, American Mathematical Society, Providence, RI, 1988.

[12] J.K. Hale, X.-B. Lin, and G. Raugel, Upper semicontinuity of attractors for approximations of semigroups and partial differential equations, *Math. Comp.*, **50** (1988) 89–123.

[13] J.K. Hale and G. Raugel, Lower semicontinuity of attractors of gradient systems and applications, *Ann. Mat. Pura Appl.*, **154** (1989) 281–326.

[14] L.T. Hoang, E.J. Olson, and J.C. Robinson (2014) On the continuity of global attractors, *Proc. Amer. Math. Soc.*, **143**:10 (2015) 4389–4395.

[15] P. Kloeden and S. Piskarev, Discrete convergence and the equivalence of equi-attraction and the continuous convergence of attractors, *Int. J. Dyn. Syst. Differ. Equ.*, **1** no. 1 (2007), 38–43.

[16] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, 2nd edition, Gordon and Breach (1969).

[17] O.A. Ladyzhenskaya, *Attractors for semigroups and evolution equations*, Cambridge University Press, Cambridge, 1991.

[18] E.N. Lorenz, Deterministic nonperiodic flow, *J. Atmos. Sci.*, **20** (1963), 130–141.

[19] D. Li and P.E. Kloeden, Equi-attraction and the continuous dependence of attractors on parameters, *Glasg. Math. J.*, **46** (2004), 131–141.

[20] D. Li and P.E. Kloeden, Equi-attraction and the continuous dependence of pullback attractors on parameters, *Stoch. Dyn.*, **4** no. 3 (2004), 373–384.

[21] J.C. Oxtoby, *Measure and category*, second edition, Springer-Verlag, New York-Berlin, 1980.

[22] J.C. Robinson, *Infinite-dimensional dynamical systems*, Cambridge University Press, Cambridge, 2001.

[23] G. R. Sell and Y. You, *Dynamics of Evolutionary Equations*. Applied Mathematical Sciences **143**, Springer, New York (2002).

[24] C.T. Sparrow, *The Lorenz equations: bifurcations, chaos, and strange attractors*, Springer-Verlag, New York (1982).

[25] A.M. Stuart and A.R. Humphries, *Dynamical systems and numerical analysis*, Cambridge University Press, Cambridge, 1996.

[26] R. Temam, *Navier-Stokes equations: Theory and numerical analysis,* 3rd edition,North-Holland, Amsterdam, (1984), or AMS Chelsea Publishing, Providence, RI, (2001).

[27] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Second edition, Springer-Verlag, New York, 1997.

[28] W. Tucker, The Lorenz attractor exists, *C.R. Acad. Sci Paris*, **328** (1999), 1197–1202.

February 28, 2017.