WARWICK
THE UNIVERSITY OF WARWICK

Manuscript version: Published Version
The version presented in WRAP is the accepted version.

Persistent WRAP URL:
http://wrap.warwick.ac.uk/96356

How to cite:
The repository item page linked to above, will contain details on accessing citation guidance from the publisher.

Copyright and reuse:
The Warwick Research Archive Portal (WRAP) makes this work of researchers of the University of Warwick available open access under the following conditions.

This article is made available under the Attribution-NonCommercial-NoDerivs 3.0 UK: England & Wales (CC BY-NC-ND 3.0 UK) and may be reused according to the conditions of the license. For more details see: https://creativecommons.org/licenses/by-nc-nd/3.0/

Publisher's statement:
Please refer to the repository item page, publisher's statement section, for further information.

For more information, please contact the WRAP Team at: wrap@warwick.ac.uk

warwick.ac.uk/lib-publications

On the collapse of trial solutions for a damped-driven non-linear Schrödinger equation

Sigurd Assing

Department of Statistics
University of Warwick
e-mail: s.assing@warwick.ac.uk

Astrid Hilbert

Department of Mathematics
Linnéuniversitetet Vaxjo
e-mail: astrid.hilbert@lnu.se

Abstract: We consider the focusing 2D non-linear Schrödinger equation, perturbed by a damping term, and driven by multiplicative noise. We show that a physically motivated trial solution does not collapse for any admissible initial condition although the exponent of the non-linearity is critical. Our method is based on the construction of a global solution to a singular stochastic Hamiltonian system used to connect trial solution and Schrödinger equation.

MSC 2010 subject classifications: Primary 60H10,74H35; secondary 74J30.
Keywords and phrases: Schrödinger equation, singular Hamiltonian system.

1. Motivation

Consider the formal equation,

$$\mathbf{i}\,\partial_t\psi + \Delta_u\psi + |\psi|^2\psi - \Lambda\,\partial_t(|\psi|^2)\psi + \sigma(u,t)\psi \,=\, 0, \tag{1.1}$$

with Cauchy data at $t = 0$, where $[\sigma(u,t),\, (u,t) \in \mathbb{R}^2 \times (0,\infty)]$ is radially symmetric centred Gaussian noise with covariance

$$\langle \sigma(u,t)\sigma(u',t')\rangle \,=\, \frac{D_r}{|u|}\,\delta_0(|u|-|u'|)\delta_0(t-t').$$

This equation was derived in [3] as the isotropic continuum approximation of a model for two-dimensional damped-driven exciton-phonon systems.

Note that (1.1), as derived in Section II of [3], is actually driven by coloured multiplicative noise. But, in Section III of [3], the authors say they would rather approximate the driving noise by space-time white noise which they had justified in [2]. Finally, in order to allow radially symmetric (i.e. isotropic) solutions, they simplified space-time white noise to radially symmetric Gaussian noise as used in the formulation of (1.1)—the reader is referred to [2] for the definition of a parameter D_{white} which can be used to choose a physically meaningful value for D_r. The definition, in physical terms, of the positive damping parameter Λ can be found in [3], too.

In the case of $\Lambda = D_r = 0$, equation (1.1) is identical to the classical focusing (power) non-linear Schrödinger equation, and the power *two* in the non-linearity $|\psi|^2\psi$ is known to be the smallest power-like non-linearity for which blow-up occurs in space dimension $d = 2$.

For example, the wave function,

$$\psi(u,t) \;=\; |t-1|^{-\frac{d}{2}} Q(\frac{u}{t-1})\, e^{\mathbf{i}\frac{|u|^2}{4(t-1)} - \mathbf{i}/(t-1)}, \tag{1.2}$$

where Q is the unique positive radially symmetric solution of

$$\Delta Q - Q + Q^{1+\frac{4}{d}} \;=\; 0 \;\text{ in } \mathbb{R}^d, \quad \text{with } Q(u) \to 0 \text{ as } |u| \to \infty,$$

can be viewed as a solution of

$$\mathbf{i}\,\partial_t\psi + \Delta_u\psi + |\psi|^{\frac{4}{d}}\psi \;=\; 0, \quad \text{in } \mathbb{R}^d \times (0,\infty),$$

with Cauchy data,

$$\psi(u,0) \;=\; Q(u)\, e^{-\mathbf{i}(\frac{|u|^2}{4}-1)}, \quad \text{at } t = 0,$$

which blows up at time $t = 1$. Note that Q is also called *ground state*.

Now observe that

$$\int_{\mathbb{R}^d} |\psi(u,t)|^2\,\mathrm{d}u \;=\; \int_{\mathbb{R}^d} Q(u)^2\,\mathrm{d}u, \quad \text{for all } t \in [0,1),$$

in the above example. Hence, all L^2 - mass is accumulated into blow-up, and, by the shape of Q, this L^2 - mass is concentrated at $u = 0$ at time $t = 1$.

The above described phenomenon, also called the L^2 - concentration phenomenon, is well-known for L^2 - critical Schrödinger equations—the reader is referred to [11],[12],[14] for general results.

Now, in [3], the authors have asked if this phenomenon was possible for solutions of their model for damped-driven exciton-phonon coupled systems, as the balanced energy input could prevent solutions from blow-up.

Of course, (1.1) is hard to solve, and being a formal equation only, its rigorous meaning would need further discussion in the first place. This difficulty was by-passed in [3]. Instead, the authors introduced the following family of wave functions,[1]

$$\psi(u,t) \;\stackrel{\text{def}}{=}\; \frac{\|\psi(\cdot,0)\|_{L^2}}{\sqrt{c_f^{1,2,0}}} \times \frac{1}{x(t)}\, f(\frac{|u|}{x(t)})\, e^{\mathbf{i}\frac{\dot{x}(t)|u|^2}{4x(t)}}, \tag{1.3}$$

[1] See Remark 1.1 for the definition of $c_f^{m,n,p}$.

parametrised by $\|\psi(\cdot,0)\|_{L^2}$, a smooth function $f:\mathbb{R}\to(0,\infty)$ which is rapidly decreasing, and an unknown stochastic process $x=[x(t),\, t\geq 0]$ which plays the role of the width of the corresponding non-linear wave.

Note that f, in contrast to Q used in (1.2), does not have to satisfy any equation. Nevertheless, similar to (1.2), wave functions of this type would blow up, if x starting from a positive value hits zero in finite time, the initial L^2 - mass being preserved in the process. Due to the nature of a blow-up with vanishing width of the wave function, we also call such a behaviour *collapse*.

So, the question asked can be scaled down to the following problems:

a) choose x in a way such that the trial solution (1.3) has something to do with the primary equation (1.1);
b) study whether x chosen this way reaches zero in finite time or not.

To shed some light on a), restricting ourselves to isotropic solutions, we can rewrite (1.1) as

$$\mathbf{i}\,\partial_t\psi+\partial_r^2\psi+\frac{1}{r}\partial_r\psi+|\psi|^2\psi-\Lambda\,\partial_t(|\psi|^2)\psi+\sigma(r,t)\psi\;=\;0,\tag{1.4}$$

reducing the problem to one space dimension with radial coordinate $r=|u|$. Since σ has negative Hölder-regularity, non-trivial solutions of this equation are not smooth in r, but the trial solution (1.3) is. Therefore, (1.3) can at most be an approximate solution having some features of a true solution.

The features chosen in [3] involve the virial coefficient

$$\mathfrak{v}(t)\;=\;\int_{\mathbb{R}^2}|u|^2|\psi(u,t)|^2\,\mathrm{d}u\;=\;2\pi\int_0^\infty r^3\,|\psi(r,t)|^2\,\mathrm{d}r$$

which, when ψ is assumed to solve (1.1), would formally satisfy

$$\ddot{\mathfrak{v}}\;=\;16\mathscr{H}+8\pi\int_0^\infty\partial_r[r^2\,|\psi|^2]\times[\Lambda\partial_t(|\psi|^2)-\sigma]\,\mathrm{d}r\tag{1.5}$$

where

$$\mathscr{H}(t)\;=\;\int_{\mathbb{R}^2}\Big[\,\frac{1}{2}\,|\nabla_u\psi(u,t)|^2-\frac{1}{4}\,|\psi(u,t)|^4\,\Big]\,\mathrm{d}u$$

is the Hamiltonian of the focusing cubic non-linear Schrödinger equation.

The identity (1.5) is standard for the classical focusing cubic non-linear Schrödinger equation, i.e. $\Lambda=D_r=0$, and then it reads $\ddot{\mathfrak{v}}=16\mathscr{H}$. We give a rough sketch of how this identity can be derived. This sketch will make clear where the additional term in (1.5) comes from. All calculations are formal, but they would work when σ is replaced by mollified noise.

Multiplying both sides of equation (1.4) by $\bar{\psi}$ reveals that

$$\partial_t(|\psi|^2)\;=\;-\mathbf{Im}(\,\bar{\psi}\,\partial_r^2\psi+\frac{1}{r}\bar{\psi}\,\partial_r\psi\,),$$

where the operation **Im** takes the imaginary part. As $\dot{\mathfrak{v}} = 2\pi\int_0^\infty r^3\,\partial_t(|\psi|^2)\,\mathrm{d}r$, partial integration yields

$$\dot{\mathfrak{v}} \;=\; 2\pi\int_0^\infty (4r^2\mathbf{Re}\psi\,\partial_r\mathbf{Im}\psi + 4r\mathbf{Re}\psi\,\mathbf{Im}\psi)\,\mathrm{d}r, \tag{1.6}$$

and calculating $\ddot{\mathfrak{v}}$ requires structure of $\partial_t\mathbf{Re}\psi$ and $\partial_t\mathbf{Im}\psi$. Again using (1.4), we find that

$$\partial_t\mathbf{Re}\psi \;=\; -\partial_r^2\mathbf{Im}\psi - \frac{1}{r}\partial_r\mathbf{Im}\psi - |\psi|^2\,\mathbf{Im}\psi + \Lambda\partial_t(|\psi|^2)\mathbf{Im}\psi - \sigma\cdot\mathbf{Im}\psi$$

leading to

$$\begin{aligned} &4r^2\partial_t\mathbf{Re}\psi\,\partial_r\mathbf{Im}\psi + 4r\partial_t\mathbf{Re}\psi\,\mathbf{Im}\psi \qquad (1.7)\\ =&\,4r^2\partial_r\mathbf{Im}\psi\times(-\partial_r^2\mathbf{Im}\psi - \frac{1}{r}\partial_r\mathbf{Im}\psi - |\psi|^2\,\mathbf{Im}\psi + \Lambda\partial_t(|\psi|^2)\mathbf{Im}\psi - \sigma\cdot\mathbf{Im}\psi)\\ &+\,4r\mathbf{Im}\psi\times(-\partial_r^2\mathbf{Im}\psi - \frac{1}{r}\partial_r\mathbf{Im}\psi - |\psi|^2\,\mathbf{Im}\psi + \Lambda\partial_t(|\psi|^2)\mathbf{Im}\psi - \sigma\cdot\mathbf{Im}\psi)\\ =&\,\text{classical part} + [4r^2\,\mathbf{Im}\psi\,\partial_r\mathbf{Im}\psi + 4r(\mathbf{Im}\psi)^2]\times[\Lambda\partial_t(|\psi|^2)-\sigma]\\ =&\,\text{classical part} + 2\partial_r[r^2(\mathbf{Im}\psi)^2]\times[\Lambda\partial_t(|\psi|^2)-\sigma]. \end{aligned}$$

Note that (1.7) is only one part of the time derivative of the integrand in (1.6) when calculating $\ddot{\mathfrak{v}}$. The other part involving $\partial_t\partial_r\mathbf{Im}\psi$ requires a further integration by parts creating more terms justifying the factor 8π in (1.5) without going into the very detail. Of course, all classical parts sum up to $16\mathscr{H}$.

The above calculation shows that, even if it is only formally an identity, (1.5) captures important structure of the underlying non-linear Schrödinger equation. And, when plugging ψ given by (1.3) into (1.5), and then performing the integration against r, this identity turns into an equation for x.

For example,

$$\begin{aligned} \mathfrak{v}(t) \;&=\; 2\pi\int_0^\infty r^3\,|\psi(r,t)|^2\,\mathrm{d}r\\ &=\; 2\pi\int_0^\infty r^3\,\frac{c_0^2}{x(t)^2}\,f(\frac{r}{x(t)})^2\,\mathrm{d}r\\ &=\; 2\pi\int_0^\infty r^3\,x(t)^3\frac{c_0^2}{x(t)^2}\,f(r)^2\,x(t)\,\mathrm{d}r\\ &=\; 2\pi\,c_0^2\,x(t)^2\int_0^\infty r^3\,f(r)^2\,\mathrm{d}r \;=\; 2\pi\,c_0^2\,x(t)^2\,c_f^{3,2,0}, \end{aligned}$$

where $c_0 = \|\psi(\cdot,0)\|_{L^2}/\sqrt{c_f^{1,2,0}}$ is the constant used in (1.3), and $c_f^{3,2,0}$ is short for the integral—see Remark 1.1 below for the definition of general $c_f^{m,n,p}$.

Performing calculations of the above type for all terms on the right-hand side of (1.5) is not deep but exhaustive, and hence further details are omitted.

The authors of [3] performed these calculations and came up with the following equation,

$$\ddot{x} \;=\; \frac{\delta}{x^3} + [\sqrt{2D}\,\frac{\dot{W}}{x^2} - \frac{\gamma\dot{x}}{x^4}\,], \tag{1.8}$$

where $W = [W(t),\, t \geq 0]$ stands for a standard one-dimensional Wiener process.

Remark 1.1. As stated in [3],

$$\delta \;=\; \frac{4}{c_f^{3,2,0}}\,\Big(\,c_f^{1,0,2} - \frac{\|\psi(\cdot,0)\|_{L^2}^2\, c_f^{1,4,0}}{2c_f^{1,2,0}}\,\Big),$$

and

$$\gamma \;=\; 8\Lambda\|\psi(\cdot,0)\|_{L^2}^2\, c_f^{3,2,2}/(c_f^{1,2,0}c_f^{3,2,0}), \quad D \;=\; 32\pi^2 D_r\, c_f^{3,2,2}/(c_f^{3,2,0})^2,$$

using

$$c_f^{m,n,p} \;\stackrel{\mathrm{def}}{=}\; 2\pi\int_0^\infty r^m[f(r)]^n[f'(r)]^p\,\mathrm{d}r.$$

The methodology for solving the first problem, a), as developed in [3]), can therefore be described as follows: for fixed $\|\psi(\cdot,0)\|_{L^2}$ and f, fit the trial solution (1.3) to (1.5) to obtain equation (1.8) for the unknown x. This way, equation (1.1) is more or less replaced by (1.5) subject to a structure condition—the specific form of ψ given by (1.3)—which may put such a ψ in close vicinity of a true solution to (1.1), in particular for well-chosen f.

However, no rigorous analysis has been done to support such a quality of the trial solution (1.3). First, one would have to make rigorous sense of all formal calculations used to motivate both equations (1.5) & (1.8), and, second, one would have to study how close the trial solution (1.3) and true solutions to (1.1) really are, and in what sense.

These open but very interesting problems are beyond the present paper and left for future research. We should nevertheless mention that numerical experiments reported in physics journals confirm a close match of trial solutions and true solutions on short time intervals. So, studying the long-time behaviour of solutions to (1.8), and in particular answering the second problem, b), seems to be a natural next step in the analysis of the original problem. For example, once the answer to b) is known, blow-up of true solutions could be decided by merely comparing functionals of trial solutions and true solutions.

In this paper, we therefore study the physically motivated equation (1.8) and solve the second problem, b), for all parameters of interest. Apart from the above application, these results are interesting in themselves since equation (1.8) describes the dynamics of a perturbed stochastically driven Hamiltonian system with a singular potential (cf. [4, 5, 6, 7, 8] and Remark 2.6 below).

Remark 1.2.

(i) Although the trial solution (1.3) has infinite degrees of freedom, studying it's blow-up through (1.8) means that only the interplay of five one-dimensional parameters has to be considered: $x(0), \dot{x}(0), \delta, \gamma, D$. Here, assuming $\Lambda, D_r > 0$, one also has $\gamma, D > 0$ by Remark 1.1, and since the trial solution has a physical meaning, $x(0) > 0$ should hold, too.

(ii) Note that the parameter δ can have both signs depending on the relationship between $\|\psi(\cdot,0)\|_{L^2}^2$ and integrals of f. Of course, compared with negative δ, the width x of the wave function is less likely to reach zero in finite time when δ is positive, and hence one might expect a phase transition in the behaviour of solutions to (1.8) depending on δ. However, we are going to prove that, for any choice of δ, the width x never collapses to zero in finite time.

2. Results

Motivated by (1.8), we study the degenerated Itô diffusion equation,

$$\begin{aligned} \mathrm{d}x &= y\,\mathrm{d}t\,, \\ \mathrm{d}y &= \frac{a}{x^{\alpha}}\,\mathrm{d}t + [\,\frac{\sqrt{2T\gamma}}{x^{\beta}}\,\mathrm{d}W - \frac{\gamma}{x^{2\beta}}\,y\,\mathrm{d}t\,], \end{aligned} \tag{2.1}$$

where W denotes a one-dimensional Wiener process.

Our goal is to construct a global solution $[x(t), y(t),\, t \geq 0]$ in $\mathbb{H}$, for any initial condition $(x(0), y(0)) \in \mathbb{H} \stackrel{\text{def}}{=} (0,\infty) \times \mathbb{R}$, if both $\gamma, T > 0$, but $a \in \mathbb{R}$ is a real parameter with no restrictions on its sign.

For the purpose of discussing equations as singular as our motivation (1.8), in what follows, we always assume $\alpha, \beta > 0$.

First, note that the infinitesimal operator associated with this equation can formally be written as

$$\mathcal{L} = \frac{\partial H}{\partial y}\partial_x - \frac{\partial H}{\partial x}\partial_y + [\,T\frac{\gamma}{x^{2\beta}}\partial_y^2 - \frac{\gamma}{x^{2\beta}}\,y\,\partial_y\,], \tag{2.2}$$

using the Hamiltonian,

$$H(x,y) = U(x) + \frac{y^2}{2}, \quad \text{with potential} \quad U(x) = \begin{cases} \frac{a}{\alpha-1}\,x^{1-\alpha} & : \quad \alpha \neq 1, \\ -a\log x & : \quad \alpha = 1, \end{cases}$$

so that (2.1) can be interpreted as a damped-driven Hamiltonian system, being forced by noise on an x-depending scale which exactly balances the x-depending dissipation placing the stochastic system at temperature T.

The difficulty we are facing is twofold: first, our potential U has a singularity at zero, and, second, *intensity of noise* $= \sqrt{2T \times \textit{friction}} = \sqrt{2T \times \gamma/x^{2\beta}}$ not only depends on x, but has a singularity at zero, too.

Nevertheless, all coefficients of equation (2.1) are locally Lipschitz continuous, on $\mathbb{H}$. Thus, pathwise-unique solutions exist at least locally until exit of $\mathbb{H}$, and, by standard arguments, for example [13], these local solutions would even be global, if there exists a Lyapunov-type function $f_L : \mathbb{H} \to [0, \infty)$ satisfying

$$\lim_{(x,y)\to\partial\mathbb{H}} f_L(x,y) \;=\; +\infty \quad \text{and} \quad \mathcal{L}f_L \;\leq\; \mathit{const}\,(1+f_L).$$

Of course, for Hamiltonian systems of the above type, the standard choice would be $f_L(x,y) = H(x,y) + x^2/2$, but this function f_L would only satisfy $\lim_{(x,y)\to\partial\mathbb{H}} f_L(x,y) = +\infty$, if $a > 0$ and $\alpha \geq 1$. Furthermore,

$$\mathcal{L}f_L(x,y) \;=\; T\frac{\gamma}{x^{2\beta}} - \frac{\gamma}{x^{2\beta}}y^2 + xy, \quad (x,y) \in \mathbb{H},$$

so that, on the one hand, $\mathcal{L}f_L \leq \mathit{const}\,(1+f_L)$ would not hold true, if $\alpha = 1$, and on the other hand, assuming $a > 0$ and $\alpha > 1$, $\mathcal{L}f_L \leq \mathit{const}\,(1+f_L)$ can only be true, if $\beta \leq (\alpha-1)/2$, leading to

Proposition 2.1. *Assume $a, \gamma, T > 0$, $\alpha > 1, \beta > 0$, and $\beta \leq (\alpha-1)/2$. Then, equation (2.1) has a unique global strong solution in $\mathbb{H}$.*

Unfortunately, this immediate almost trivial result does NOT cover the case of our motivating equation (1.8), where $\alpha = 3$, $\beta = 2$.

So, we tried to find another Lyapunov function of type f_L but could not find one. We studied the literature for methods of how to construct Lyapunov-type functions for singular Hamiltonian systems, but we have not been able to use the methods we found in the case of our system—see Remark 2.6 below for a discussion of the most recent results on systems coming closest to ours.

Also, easy calculations in the deterministic case, i.e. $\gamma = 0$, reveal that, if $a < 0$, then there is no function f_L as above. Worse, the x-component of any solution in $\mathbb{H}$ eventually collapses to zero in finite time, and hence there is no global solution.

We therefore looked for another method to show global existence of solutions to (2.1) for more cases of α, β, and for $a < 0$, even. The proof of our existence result is based on the following lemma which is the key-result of this paper.

Lemma 2.2. *Assume $\gamma, T > 0$, and*

$$\left.\begin{array}{lr} \bullet & \beta > 1/2,\ \alpha \geq \beta \geq \alpha/2 \quad if \quad a > 0, \\ \bullet & \beta > 1/2 \quad if \quad a = 0, \\ \bullet & \alpha > 1,\ \alpha \geq \beta \geq (\alpha+1)/2 \quad if \quad a < 0. \end{array}\right\} \tag{2.3}$$

Consider the product

$$\Omega_+ \;=\; \Omega_- \otimes C([0,\infty)), \quad \mathcal{F}_+ \;=\; \mathcal{F}_- \otimes \mathcal{B}(C([0,\infty))), \quad \mathbf{Q} \;=\; \mathbf{P}_- \otimes \mathbf{P}_W$$

of a probability space $(\Omega_-, \mathcal{F}_-, \mathbf{P}_-)$ and the standard Wiener space $(C([0,\infty)), \mathcal{B}(C([0,\infty))), \mathbf{P}_W)$, and let (x_-, y_-) be a given pair of random variables on $(\Omega_-, \mathcal{F}_-, \mathbf{P}_-)$ satisfying $\mathbf{P}_-(\{x_- > 0\}) = 1$. Extend all random variables on either $(\Omega_-, \mathcal{F}_-, \mathbf{P}_-)$ or $(C([0,\infty)), \mathcal{B}(C([0,\infty)))$, $\mathbf{P}_W)$ to $(\Omega_+, \mathcal{F}_+, \mathbf{Q})$ in the canonical way without changing their notation.

Then there exists a filtration $\mathbb{F}_+ = [\mathcal{F}_t^+,\, t \geq 0]$ of sub-σ-algebras of $\mathcal{F}_+$, a probability measure $\mathbf{P}_+$ on $(\Omega_+, \mathcal{F}_\infty^+)$, a pair $[x_+(t), y_+(t),\, t \geq 0]$ of continuous $\mathbb{F}_+$-adapted processes, and an $\mathbb{F}_+$-Wiener-process W_+, such that

$$\mathbf{P}_+ = \mathbf{Q} \text{ on } \mathcal{F}_0^+,$$

and

$$\begin{aligned} x_+(t) &> 0, \\ x_+(t) &= x_- + \int_0^t y_+(s)\,\mathrm{d}s, \\ y_+(t) &= y_- + a\int_0^t \frac{\mathrm{d}s}{x_+(s)^\alpha} + \sqrt{2T\gamma}\int_0^t \frac{\mathrm{d}W_+(s)}{x_+(s)^\beta} - \gamma\int_0^t \frac{y_+(s)\,\mathrm{d}s}{x_+(s)^{2\beta}}, \end{aligned}$$

for all $t \in [0,1]$, $\mathbf{P}_+$-a.s. Note that $\mathcal{F}_\infty^+$ can be strictly smaller than $\mathcal{F}_+$.

The above result can be used to prove existence of a global weak solution in $\mathbb{H}$ which, by standard arguments, turns into a unique strong solution, as the equation's coefficients are locally Lipschitz on $\mathbb{H}$.

Theorem 2.3. *Assume $\gamma, T > 0$, and (2.3). Then, equation (2.1) has a unique global strong solution in $\mathbb{H}$.*

As a consequence, returning to our motivating example, (1.8), where $\alpha = 3$ and $\beta = 2$, Theorem 2.3 implies that (1.8) always has a positive global solution, regardless the values taken by $\delta \in \mathbb{R}$ and $\gamma, D > 0$.

Corollary 2.4. *If x solves (1.8), then the non-linear wave ψ given by (1.3) would never collapse.*

Now, if non-linear waves of this type exist globally, then the next question is about their long-time behaviour. Could the system (2.1) be an ergodic system with a limiting invariant probability measure? If YES, then the corresponding non-linear waves would 'live' forever. Or would the width x of these waves show transient behaviour leading to flatter and flatter finally disappearing waves? Or would these waves 'pseudo-collapse' in the sense of $\lim_{t\to\infty} x(t) = 0$?

We first discuss possible ergodic behaviour of our system. Recall that noise and dissipation are balanced, and hence

$$\varrho_\star(x,y) \;=\; e^{-H(x,y)/T} \quad \text{satisfies} \quad \mathcal{L}^\star \varrho_\star = 0 \text{ on } \mathbb{H},$$

where $\mathcal{L}^\star$ stands for the (formal) adjoint of $\mathcal{L}$. Therefore, if the density $\varrho_\star$ can be normalised to become the density of a probability measure, then this probability measure could be the system's canonical invariant probability measure.

However, in all cases of global existence as stated in Proposition 2.1 and Theorem 2.3, we always have

$$\int_0^\infty \int_{\mathbb{R}} e^{-H(x,y)/T}\,\mathrm{d}y\mathrm{d}x \;=\; +\infty, \tag{2.4}$$

which means that $\varrho_\star$ cannot be used for constructing an invariant probability measure. Even more, if $a \geq 0$, then no probability measure can be invariant under the conditions of Theorem 2.3, and neither under the conditions of Proposition 2.1, if $\beta > 1/2$.

Indeed, in the proof of Theorem 2.3, the global solution is constructed by patching together solutions on finite intervals as constructed in the proof of Lemma 2.2. However, when $a = 0$, the proof of Lemma 2.2 can be used to obtain a global solution without patching, and we are going to show that the x-component of this global solution is transient. Using a comparison argument, we can also verify this transient behaviour in the case of $a > 0$, leading to

Proposition 2.5. *Assume* $\gamma, T > 0$*. If either* $a \geq 0$ *& (2.3), or* $a > 0$ *&* $1/2 < \beta \leq (\alpha - 1)/2$*, then the* x*-component of any global solution to (2.1) satisfies* $\lim_{t\to\infty} x(t) = +\infty$*, a.s.*

Still, there could be an invariant probability measure under the conditions of Proposition 2.1, if $\beta \leq 1/2$. But, the Lyapunov-type function f_L used to show global existence does not give the bounds needed for existence of an invariant probability measure. So, let us discuss some recent results on ergodic behaviour of singular Hamiltonian systems.

Remark 2.6.

(i) In [4], and as an application of the technique developed in [7], too, singular diffusions associated with operators of type (2.2), but with constant friction, i.e. $\beta = 0$, were constructed via Dirichlet forms, and their ergodic behaviour was studied. However, apart from the non-singular noise and friction terms, one of the crucial conditions is that normalising $e^{-U}\,\mathrm{d}x$ defines a probability measure, and this condition does not apply in our case—see (2.4).

(ii) Recent work in [5] (also see [8] for the higher dimensional case), on systems of type (2.1), again with non-singular noise and friction terms, deals with singular potentials of the form,

$$U(x) \;=\; a_1 x^{\alpha_1} + a_2 x^{-\alpha_2},$$

assuming $a_1, a_2 > 0$, $\alpha_1 > 2$. So, unlike in our case, the integral (2.4) converges, and the canonical invariant measure occurs to be the system's unique invariant measure. More importantly, the authors describe a general method of how to construct a Lyapunov function which gives control over the system's trajectories near the boundary of $\mathbb{H}$. Note that their parameter a_2—the analogue of our parameter a—is assumed to be positive, which makes their potential repulsive at zero.

(iii) The driven Rayleigh-Plesset equation considered in [6] has features being more similar to our equation (2.1). Only looking at the degenerate-diffusion-case (cf. Section 5 in [6]), their potential has no a_1-term either, and the leading singularity is of the form bx^{-3k}, for some $k \geq 1$, but again $b > 0$, so that the singular potential is repulsive at zero, as in [5]. Intensity of noise and friction, though, depend on both x and y, but in an unbalanced way different to ours. The special form of the unbalanced noise and dissipation terms together with the repulsive potential make it possible to find a Lyapunov function good enough to ensure both existence of a global solution and an invariant probability measure.

Under the conditions of Proposition 2.1 or Theorem 2.3 which are NOT covered by Proposition 2.5, we have not been able to construct a Lyapunov function using the method in [5], mainly because of our system's singular noise and friction terms; and we could not use the method given in [6], either, because the structure of their system is very different to ours. The long-time behaviour of solutions to (2.1) remains an open problem in these cases.

However, in all these open cases, we have given relationships between α and β ensuring existence of global solutions, regardless the values taken by $\gamma, T > 0$. But, we cannot tell whether such a global solution $[x(t), y(t), t \geq 0]$ satisfies $\lim_{t\to\infty} x(t) = +\infty$, or $\lim_{t\to\infty} x(t) = 0$, or whether x is recurrent—all three scenarios might be possible, depending on the choice of $\gamma, T > 0$. So far we only know from (4.18) in the proof of Proposition 2.5 that

$$\lim_{t\to\infty} x(t) = +\infty, \quad \text{on } \{\textstyle\int_0^\infty x(t)^{-2\beta}\,\mathrm{d}t \,<\, \infty\}, \quad \text{a.s.},$$

but this set could even have measure zero.

3. Discussion of Conditions

In this section we relate crucial steps in our proofs to the conditions they rely upon, which sheds some light on how essential these conditions actually are.

First, the base step of our construction in the proof of Lemma 2.2 consists in analysing functionals of the solution to equation (4.2), for a given Ornstein-Uhlenbeck process $[\hat{y}(t), t \geq 0]$.

If $\beta \in [0, 1/2]$ then Lemma 4.2 would be wrong because, almost surely, the solution to (4.2) would be a continuous function on the compact interval $[0, \tau]$. We do not see an easy fix for our proofs without a valid Lemma 4.2, and that is the main reason why we want β to be greater than $1/2$ in Lemma 2.2.

In Proposition 2.1, we would get existence of global solutions in cases where $\beta \leq 1/2$, but the arguments leading to Proposition 2.1 give little information on the path-behaviour of these solutions. Therefore, in Proposition 2.5, we extra require $\beta > 1/2$ when $\beta \leq (\alpha - 1)/2$, since the proof of Proposition 2.5 is partly based on the proof of Lemma 2.2.

Second, when $a \neq 0$, the above mentioned base step is followed by a Girsanov transform, and we have to check Novikov's condition. Checking this condition is based on Hölder's inequality which can only be applied if α and β are in the relation $\alpha \geq \beta \geq \alpha/2$.

The third and last crucial step in the proof of Lemma 2.2 is Lemma 4.5. It turns out that, if $a > 0$, the conditions $\beta > 1/2$ and $\alpha \geq \beta \geq \alpha/2$ assumed in the first and second step, respectively, are sufficient for this proof. But, if $a < 0$, two extra conditions, $\alpha > 1$ and $\beta \geq (\alpha + 1)/2$, are required. Note that these extra conditions imply $\beta > 1/2$ needed in the first step.

Furthermore, compared with $\alpha > 1/2$, which is the consequence of condition $\beta > 1/2$, $\alpha \geq \beta \geq \alpha/2$ from case $a > 0$, the extra condition $\alpha > 1$ is in a way counter intuitive for negative a. Indeed, for negative a, a greater power of α in (2.1) should push the trajectories of the x-component further to zero, once x is close to collapse. This push is obviously compensated by stronger fluctuations of the damped noise caused by the other extra condition $\beta \geq (\alpha+1)/2$. The message of the proof of Lemma 4.5 seems to be that global solutions to (2.1) can only exist if α and β are in the right ratio.

We finally discuss the most interesting case $\delta < 0$ of our main application, equation (1.8), where $\alpha = 3$ and $\beta = 2$, so that $\beta = (\alpha+1)/2$, which is at the 'edge' of the condition ensuring global existence. This could mean that, when $\delta < 0$, solutions to (1.8) are 'just' global in the sense that $\lim_{t\to\infty} x(t) = 0$, almost surely or with positive probability, depending on the choice of $\gamma, D > 0$. This behaviour, which the authors called 'pseudo-collapse', has been conjectured and supported by numerical experiments in [3].

However, when $\delta \geq 0$, we know from Proposition 2.5 that solutions are even transient and cannot pseudo-collapse, and hence $\delta \geq 0$ would be a condition for non-pseudo-collapse of the corresponding non-linear wave given by (1.3). Recalling the definition of δ in Remark 1.1, this condition would read

$$\|\psi(\cdot,0)\|_{L^2}^2 \;\leq\; 2c_f^{1,0,2}\, c_f^{1,2,0}/c_f^{1,4,0},$$

which compares the L^2-norm of the wave's initial condition with a product of integrals of f and f'.

A similar but structurally easier condition is well-known for the classical focusing non-linear Schrödinger equation: Weinstein's criterion, [15, Thm. A], says that solutions would never blow up if

$$\|\psi(\cdot,0)\|_{L^2} \;<\; \|Q\|_{L^2},$$

where Q is the ground state used for (1.2). Here, Q satisfies an equation, while f used for (1.3) does not.

4. Proofs

Proof of Lemma 2.2. First, observe that $\beta > 1/2$ in all three cases of (2.3).

Let $[B_t,\, t \geq 0]$ be the coordinate process on $(C([0,\infty)),\, \mathcal{B}(C([0,\infty))), \mathbf{P}_W)$, and define the filtration

$$\mathbb{G}_+ \;=\; [\mathcal{G}_t^+,\, t \geq 0] \quad \text{by} \quad \mathcal{G}_t^+ \;=\; \sigma(x_-, y_-) \vee \sigma(\{B_s : s \leq t\}).$$

Note that $[B_t,\, t \geq 0]$ is a $\mathbb{G}_+$-Wiener-process on $(\Omega_+, \mathcal{F}_+, \mathbf{Q})$.

Of course,

$$\hat{y}(t) \;\stackrel{\text{def}}{=}\; e^{-\gamma t} y_- + \sqrt{2T\gamma} \int_0^t e^{-\gamma(t-s)}\, \mathrm{d}B_s, \quad t \geq 0,$$

satisfies

$$\hat{y}(t) \;=\; y_- + \sqrt{2T\gamma}\,B_t - \gamma\int_0^t \hat{y}(s)\,\mathrm{d}s, \tag{4.1}$$

for all $t \geq 0$, $\mathbf{Q}$-a.s.

Next, for any $\beta > 1/2$,

$$\hat{x}(t) \;\stackrel{\text{def}}{=}\; \Big[\; x_-^{1-2\beta} - (2\beta-1)\int_0^t \hat{y}(s)\,\mathrm{d}s \;\Big]^{\frac{-1}{2\beta-1}}$$

solves

$$\hat{x}(t) \;=\; x_- \;+ \int_0^t \hat{x}(s)^{2\beta}\,\hat{y}(s)\,\mathrm{d}s, \quad \text{for } t < \tau, \tag{4.2}$$

where

$$\tau \stackrel{\text{def}}{=} \inf\{s \geq 0 : \int_0^s \hat{y}(r)\,\mathrm{d}r = \frac{x_-^{1-2\beta}}{2\beta-1}\}.$$

Now, introduce $[T_t,\, t \geq 0]$, defined by

$$T_t \;\stackrel{\text{def}}{=} \begin{cases} \int_0^t \hat{x}(s)^{2\beta}\,\mathrm{d}s & : \quad t < \tau, \\ \quad +\infty & : \quad t \geq \tau, \end{cases}$$

which is an increasing right-continuous $\mathbb{G}_+$-adapted process with right-inverse $A = [A_t,\, t \geq 0]$, defined by

$$A_t \;\stackrel{\text{def}}{=}\; \inf\{s \geq 0 : T_s > t\}.$$

Note that, as a process, A is continuous on $[0, T_\infty)$, but it is also an increasing family of $\mathbb{G}_+$-stopping times.

The above three processes $\hat{x}$, $\hat{y}$, and A, form the basis for our construction of $[x_+(t), y_+(t),\, t \geq 0]$ by time-change:

$$x_+(t) \;\stackrel{\text{def}}{=}\; \hat{x}(A_t), \quad y_+(t) \;\stackrel{\text{def}}{=}\; \hat{y}(A_t), \quad t \geq 0.$$

We are going to prove that, under the conditions of the lemma, the above construction is well-posed, and that the processes x_+, y_+ satisfy the stated equations. The latter requires changing the measure $\mathbf{Q}$ in the case of $a \neq 0$, and well-posedness of all objects has to be established with respect to the new measure, too. The proof involves several lemmas which will only be shown after this proof to avoid distraction from the main ideas.

To start with, recall that $\hat{x}(t)$ is only meaningful for $t < \tau$. Thus, our time-change A should satisfy

$$A_t \;<\; \tau, \quad t \geq 0, \quad \mathbf{Q}\text{-a.s.}, \tag{4.3}$$

and the following property will play an important role in showing this.

Lemma 4.1. *The $\mathbb{G}_+$-stopping time τ satisfies* $\mathbf{Q}(\{\tau < \infty\}) = 1$.

The proof of the next lemma only works for those paths at which the stopping time τ is finite. We also use the explicit definition of $\hat{x}(t)$, and therefore the magnitude of β matters.

Lemma 4.2. *For any $\beta > 1/2$, $\mathbf{Q}(\{\lim_{t\uparrow\tau} T_t = +\infty\}) = 1$.*

As $[T_t,\, t \geq 0]$ is continuous and strictly increasing, on $[0,\tau)$, the last lemma implies that its right-inverse A is continuous and strictly increasing, on $[0,\infty)$, and satisfies (4.3). As a consequence, the pair of processes $[x_+(t), y_+(t),\, t \geq 0]$ is well-defined by time-change, for all $t \geq 0$, on a set in $\mathcal{F}_+$ of $\mathbf{Q}$-measure one. Furthermore, by time-change, equation (4.2) yields

$$x_+(t) \;=\; x_- \;+ \int_0^t y_+(s)\,\mathrm{d}s, \quad t \geq 0, \quad \mathbf{Q}\text{-a.s.} \tag{4.4}$$

Remark 4.3. It follows immediately from the construction of $\hat{x}$ on $[0,\tau)$ that $x_+(t) > 0$, for all $t \geq 0$, $\mathbf{Q}$-a.s.

So, the first two statements of the lemma would be true under the measure $\mathbf{Q}$, and, if $a = 0$, by time-change, equation (4.1) yields (4.7), for all $t \geq 0$, $\mathbf{Q}$-a.s., in a straight forward way[2]. Note that $\beta > 1/2$ has been the only assumption we made, so far, proving the lemma in the case $a = 0$.

In what follows, assume $a \neq 0$.

Then, the equation for $y_+(t)$ requires a measure $\mathbf{P}_+$ different to $\mathbf{Q}$. The next step is to construct this measure.

Introduce the process, $\rho = [\rho(t),\, t \geq 0]$, given by

$$\rho(t) \;\stackrel{\text{def}}{=}\; \exp\{\frac{a}{\sqrt{2T\gamma}}\int_0^t \mathbf{1}_{[0,A_1]}(s)\hat{x}(s)^{2\beta-\alpha}\,\mathrm{d}B_s - \frac{a^2}{4T\gamma}\int_0^t \mathbf{1}_{[0,A_1]}(s)\hat{x}(s)^{4\beta-2\alpha}\,\mathrm{d}s\},$$

which is well-defined since the stochastic integrand is a caglad $\mathbb{G}_+$- adapted process. Since $\alpha \geq \beta \geq \alpha/2$, by Hölder's inequality,

$$\int_0^t \mathbf{1}_{[0,A_1]}(s)\hat{x}(s)^{4\beta-2\alpha}\,\mathrm{d}s \;\leq\; t^{\frac{\alpha-\beta}{\beta}} \cdot (\int_0^{A_1} \hat{x}(s)^{2\beta}\,\mathrm{d}s)^{\frac{2\beta-\alpha}{\beta}} \;=\; t^{\frac{\alpha-\beta}{\beta}} \cdot (T_{A_1})^{\frac{2\beta-\alpha}{\beta}},$$

where $\beta > 1/2$ yields $T_{A_1} = 1$, $\mathbf{Q}$-a.s., by Lemma 4.2, and hence

$$\int \exp\{\frac{a^2}{4T\gamma}\int_0^t \mathbf{1}_{[0,A_1]}(s)\hat{x}(s)^{4\beta-2\alpha}\,\mathrm{d}s\}\,\mathrm{d}\mathbf{Q} \;<\; \infty,$$

for all $t \geq 0$, so that ρ is a $\mathbb{G}_+$-martingale by Novikov's condition.

Copying the proof of Corollary 3.5.2 in [10], one can construct a probability measure $\mathbf{P}_+$ on $\mathcal{G}^+_\infty$ such that

$$\hat{W}_+(t) \;\stackrel{\text{def}}{=}\; B_t \;-a\int_0^t \mathbf{1}_{[0,A_1]}(s)\hat{x}(s)^{2\beta-\alpha}\,\mathrm{d}s/\sqrt{2T\gamma}, \quad t \geq 0,$$

is a $\mathbb{G}_+$-Wiener-process.

[2]More details of how the equations transform under time-change will be given in the more complex case $a \neq 0$.

Remark 4.4.

(i) The measure constructed in the original proof of Corollary 3.5.2 in [10] would be defined on $\sigma(\{B_s : s \geq 0\})$ but our integrand $\mathbf{1}_{[0,A_1]}\hat{x}(\cdot)$ is not $\sigma(\{B_s : s \geq 0\})$-measurable. However, the proof still works when using $\mathcal{G}_\infty^+$ instead. It is not needed that $\mathbb{G}_+$ satisfies the usual conditions.

(ii) The σ-algebra $\mathcal{G}_\infty^+$ may be smaller than $\mathcal{F}_+$.

The measure $\mathbf{P}_+$ does not have to be absolutely continuous w.r.t. $\mathbf{Q}$, but it is on every $\mathcal{G}_t^+$, where $\mathbf{P}_+ = \rho(t) \cdot \mathbf{Q}$. This allows to carry over some $\mathbf{Q}$-a.s. events to $\mathbf{P}_+$-a.s. events by approximation with monotone sequences of events, for example,

$$\mathbf{P}_+(\{\hat{y}(\cdot) \text{ continuous on } [0,\infty)\}) \,=\, 1. \tag{4.5}$$

As a consequence,

$$|\int_0^t \hat{y}(s)\,\mathrm{d}s\,| \,<\, \infty, \quad \text{for all } t \geq 0, \quad \mathbf{P}_+\text{-a.s.},$$

which yields

$$\hat{x}(t) \,>\, 0, \quad \text{for all } t < \tau, \quad \mathbf{P}_+\text{-a.s.}$$

However, the results of both lemmas, 4.1 and 4.2, might not remain true under the new measure $\mathbf{P}_+$.

Indeed, though the definition of $\hat{y}$ is still the same under $\mathbf{P}_+$, the process B is now a Wiener process with drift, and hence the recurrence of the stochastic integral process used to prove Lemma 4.1 might fail to hold. Thus, we have to take into account a positive $\mathbf{P}_+$-probability of the event $\{\tau = \infty\}$, and on this event the proof of Lemma 4.2 does not work. The next lemma gives conditions on the parameters α, β ensuring that T_∞ cannot be finite, on $\{\tau = \infty\}$, $\mathbf{P}_+$-a.s., and this property turns out to be crucial for the rest of the proof.

Lemma 4.5. *Assume $a \neq 0$ and (2.3). Then, $\mathbf{P}_+(\{\lim_{t\uparrow\tau} T_t = +\infty\}) \,=\, 1$.*

All in all, if $a \neq 0$ and (2.3), then there exists a measure $\mathbf{P}_+$ on $\mathcal{G}_\infty^+$ such that the pair of processes $[x_+(t), y_+(t),\, t \geq 0]$ is well-defined, and both equation (4.4) and Remark 4.3 remain true, when the measure $\mathbf{Q}$ is replaced by $\mathbf{P}_+$. Furthermore, equation (4.1) can be written as

$$\hat{y}(t) \,=\, y_- \,+\, a\int_0^t \mathbf{1}_{[0,A_1]}(s)\,\hat{x}(s)^{2\beta-\alpha}\,\mathrm{d}s + \sqrt{2T\gamma}\,\hat{W}_+(t) - \gamma\int_0^t \hat{y}(s)\,\mathrm{d}s,$$

for all $t \geq 0$, $\mathbf{P}_+$-a.s., which gives

$$y_+(t) \,=\, y_- \,+\, a\int_0^t \frac{\mathbf{1}_{[0,A_1]}(A_s)}{x_+(s)^\alpha}\,\mathrm{d}s + \sqrt{2T\gamma}\,\hat{W}_+(A_t) - \gamma\int_0^t \frac{y_+(s)}{x_+(s)^{2\beta}}\,\mathrm{d}s \tag{4.6}$$

since

$$A_t \,=\, \int_0^{A_t} \hat{x}(s)^{-2\beta}\,\mathrm{d}T_s \,=\, \int_0^t x_+(s)^{-2\beta}\,\mathrm{d}s,$$

for all $t \geq 0$, $\mathbf{P}_+$-a.s.

Now, let $\mathbb{F}_+$ be the time-changed filtration given by $\mathcal{F}_t^+ \stackrel{\text{def}}{=} \mathcal{G}_{A_t}^+$, $t \geq 0$, so that

$$\mathbf{P}_+ = \mathbf{Q} \quad \text{on} \quad \mathcal{F}_0^+ = \mathcal{G}_0^+$$

because $\rho(0) = 1$. Also, note that $[x_+(t), y_+(t),\, t \geq 0]$ are $\mathbb{F}_+$-adapted processes which are both $\mathbf{P}_+$-a.s. continuous. Of course, when switching to the filtration $\mathbb{F}_+$, the measure $\mathbf{P}_+$ can be restricted to $\mathcal{F}_\infty^+$ which might be smaller than $\mathcal{G}_\infty^+$.

Next, the continuous local $\mathbb{F}_+$-martingale $M_+(t) \stackrel{\text{def}}{=} \hat{W}_+(A_t)$, $t \geq 0$, has quadratic variation $\langle M_+ \rangle = A$. Since, $\mathbf{P}_+$-a.s., this quadratic variation takes the form, $\int_0^t x_+(s)^{-2\beta}\, \mathrm{d}s$, $t \geq 0$, where the integrand $x_+(s)^{-2\beta}$, $s \geq 0$, is positive and continuous, Theorem II.7.1 in [9] implies that there is an $\mathbb{F}_+$-Wiener-process W_+ on $(\Omega_+, \mathcal{F}_\infty^+, \mathbf{P}_+)$ such that

$$M_+(t) \;=\; \hat{W}_+(A_t) \;=\; \int_0^t \frac{\mathrm{d}W_+(s)}{x_+(s)^\beta}, \quad t \geq 0, \quad \mathbf{P}_+\text{-a.s.}$$

Hence, (4.6) translates into

$$y_+(t) \;=\; y_- \,+\, a\int_0^t \frac{\mathbf{1}_{[0,1]}(s)}{x_+(s)^\alpha}\, \mathrm{d}s + \sqrt{2T\gamma}\int_0^t \frac{\mathrm{d}W_+(s)}{x_+(s)^\beta} - \gamma\int_0^t \frac{y_+(s)}{x_+(s)^{2\beta}}\, \mathrm{d}s, \quad (4.7)$$

for all $t \geq 0$, $\mathbf{P}_+$-a.s., finally proving the lemma. □

Proof of Lemma 4.1. Rewrite (4.1) to obtain

$$\gamma\int_0^s \hat{y}(r)\, \mathrm{d}r \;=\; y_- + \sqrt{2T\gamma}\, B_s - \hat{y}(s), \quad s \geq 0, \quad \mathbf{Q}\text{-a.s.},$$

where

$$\sqrt{2T\gamma}\, B_s - \hat{y}(s) \;=\; -e^{-\gamma s} y_- + \sqrt{2T\gamma}\int_0^s [1 - e^{-\gamma(s-r)}]\, \mathrm{d}B_r$$

by definition of $\hat{y}(s)$. Thus, since $\lim_{s\to\infty} e^{-\gamma s} y_- = 0$, the process $s \mapsto \int_0^s \hat{y}(r)\, \mathrm{d}r$ is almost surely going to hit $x_-^{1-2\beta}/(2\beta - 1)$ in finite time, if the process $s \mapsto \int_0^s [1 - e^{-\gamma(s-r)}]\, \mathrm{d}B_r$ is recurrent.

To show recurrence of this process, we use the representation,

$$\int_0^s [1 - e^{-\gamma(s-r)}]\, \mathrm{d}B_r \;=\; B_s - e^{-\gamma s}\, \tilde{B}\Big(\frac{e^{2\gamma s} - 1}{2\gamma}\Big), \quad s \geq 0, \quad \mathbf{Q}\text{-a.s.},$$

where $[\tilde{B}(t),\, t \geq 0]$ is another Wiener process on $(\Omega_+, \mathcal{F}_+, \mathbf{Q})$—see Thm.II.7.2 in [9]. Taking into account the law of iterated logarithm (cf. Thm.2.9.23 in [10]), i.e., for any one-dimensional standard Wiener process $[W_t,\, t \geq 0]$,

$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t \log\log t}} = 1, \text{ a.s.}, \quad \liminf_{t\to\infty} \frac{W_t}{\sqrt{2t \log\log t}} = -1, \text{ a.s.};$$

we can then conclude that the process

$$s \mapsto B_s - e^{-\gamma s}\,\tilde{B}(\frac{e^{2\gamma s}-1}{2\gamma})$$

is recurrent if

$$\sqrt{2s\log\log s} - e^{-\gamma s}\sqrt{\frac{e^{2\gamma s}-1}{\gamma}\log\log\frac{e^{2\gamma s}-1}{2\gamma}}$$

diverges, when s goes to infinity, which is true. □

Proof of Lemma 4.2. There exists $\Omega_0 \in \mathcal{F}_+$ such that $\mathbf{Q}(\Omega_0) = 1$ and both $\hat{y}(s,\omega)$ is continuous in s as well as $\tau(\omega) < \infty$ for all $\omega \in \Omega_0$. Choose $\omega \in \Omega_0$ and assume that

$$\lim_{t\uparrow\tau(\omega)} T_t(\omega) \;=\; c \;<\; +\infty.$$

Then, for $t < \tau(\omega)$, it follows from the mean value theorem that

$$c - T_t(\omega) \;=\; (\tau(\omega)-t)\,\Big[\,x_-^{1-2\beta} - (2\beta-1)\int_0^{\tilde{t}}\hat{y}(s,\omega)\,\mathrm{d}s\,\Big]^{\frac{-2\beta}{2\beta-1}}$$

where $\tilde{t} \in (t,\tau(\omega))$. Also, by definition of τ and continuity of $\hat{y}(s,\omega)$ in s,

$$x_-^{1-2\beta} - (2\beta-1)\int_0^{\tilde{t}}\hat{y}(s,\omega)\,\mathrm{d}s \;=\; (2\beta-1)(\tau(\omega)-\tilde{t}\,)\;\hat{y}\Big(\tilde{\tilde{t}},\omega\Big)$$

for some $\tilde{\tilde{t}} \in (\,\tilde{t},\tau(\omega))$, again applying the mean value theorem. Thus,

$$c - T_t(\omega) \;=\; O\left((\tau(\omega)-t)^{\frac{-1}{2\beta-1}}\right),\quad t\uparrow\tau(\omega),$$

which means that $c - T_t(\omega)$ should blow up when t goes to $\tau(\omega)$, since $\beta > 1/2$. But, such a blow-up would contradict our assumption of $c < \infty$, proving $\lim_{t\uparrow\tau(\omega)} T_t(\omega) = +\infty$, for all $\omega \in \Omega_0$. □

Proof of Lemma 4.5. First, recall (4.5) and note that, by the same principle, (4.1) is also true, for all $t \geq 0$, $\mathbf{P}_+$-a.s. Then, choose Ω_0 such that $\mathbf{P}_+(\Omega_0) = 1$ and, on Ω_0: $\hat{y}(\cdot)$ is continuous, (4.1) is satisfied for all $t \geq 0$, and

$$\limsup_{t\to\infty}\frac{\hat{W}_+(t)}{\sqrt{2t\log\log t}} = 1,\quad \liminf_{t\to\infty}\frac{\hat{W}_+(t)}{\sqrt{2t\log\log t}} = -1.$$

Here we used both $\beta > 1/2$ and $\alpha \geq \beta \geq \alpha/2$ for validation of Novikov's condition to make sure that $\hat{W}_+$ as defined above Remark 4.4 is a Wiener process.

Now, choose $\omega \in \Omega_0$. To simplify notation, for the rest of this proof, consider all random variables being evaluated at the chosen ω without emphasising.

If $\tau < \infty$, as $\beta > 1/2$, $\lim_{t\uparrow\tau} T_t = +\infty$ can be shown following the arguments used in the proof of Lemma 4.2.

If $\tau = \infty$, using the definitions of $\hat{y}$ and $\hat{W}_+$, a rewrite of equation (4.1) yields

$$\gamma z_t \;=\; y_- - e^{-\gamma t} y_- + a \int_0^t [1 - e^{-\gamma(t-s)}]\, \mathbf{1}_{[0,A_1]}(s) \hat{x}(s)^{2\beta-\alpha}\, \mathrm{d}s + \sqrt{2T\gamma} \int_0^t [1 - e^{-\gamma(t-s)}]\, \mathrm{d}\hat{W}_+(s), \tag{4.8}$$

for all $t \geq 0$, where

$$z_t \stackrel{\text{def}}{=} \int_0^t \hat{y}(s)\, \mathrm{d}s, \quad t \geq 0.$$

Since $\tau = \infty$, and since $t \mapsto z_t$ cannot explode in finite time by our choice of Ω_0, $\hat{x}(\cdot)$ is a positive function on the entire domain $[0, \infty)$. Thus, simple differentiation reveals that the first integral in (4.8) is a monotonously increasing function in t, and hence the behaviour of the function $t \mapsto z_t$ will be different depending on whether this monotone function dominates the stochastic integral or not, when t goes to infinity.

However, by our choice of Ω_0, using the same arguments as in the proof of Lemma 4.1,

$$\limsup_{t\to\infty} \frac{\int_0^t [1 - e^{-\gamma(t-s)}]\, \mathrm{d}\hat{W}_+(s)}{\sqrt{2t \log\log t}} = 1, \quad \liminf_{t\to\infty} \frac{\int_0^t [1 - e^{-\gamma(t-s)}]\, \mathrm{d}\hat{W}_+(s)}{\sqrt{2t \log\log t}} = -1.$$

So, if $a > 0$, then the first integral in (4.8) adds to the upward-fluctuations of the stochastic integral leading to a finite value of τ, and hence the case $a > 0$ cannot occur, once $\tau = \infty$. Therefore, the case $a > 0$ was covered above, only assuming $\beta > 1/2$ and $\alpha \geq \beta \geq \alpha/2$.

Similarly, τ cannot be infinite if $A_1 < \infty$. Indeed, if both was true, then the first integral in (4.8) would be finite, so that the stochastic integral would dominate the behaviour of z_t, when t goes to infinity, again leading to $\tau < \infty$.

All in all, it remains to prove the lemma under three assumptions, $\tau = \infty$, $a < 0$, $A_1 = \infty$, and this is done by showing that, if $\alpha > 1$ and $\beta > \alpha/2$, then $T_\infty < +\infty$ would imply $\beta < (\alpha + 1)/2$.

So, for the rest of this proof, assume $\tau = \infty$, $a < 0$, $A_1 = \infty$, $\alpha > 1$, $\beta > \alpha/2$, $T_\infty < +\infty$. We are going to show that $\beta < (\alpha + 1)/2$.

To begin with, we are going to verify that the first integral in (4.8) would always dominate the stochastic integral, pushing all fluctuations of z_t down to $\lim_{t\to\infty} z_t = -\infty$, eventually.

To see this, using the long-time behaviour of the stochastic integral in (4.8), we first deduce that, for some large enough t_0, there exists $b_0 > 0$ such that, for all $t \geq t_0$,

$$\gamma z_t \;\geq\; y_- - e^{-\gamma t} y_- - |a| \int_0^t \hat{x}(s)^{2\beta-\alpha}\, \mathrm{d}s - b_0 \sqrt{t} \sqrt{\log\log t}\,.$$

Of course, $y_- - e^{-\gamma t} y_-$ is bigger than some negative number, for all $t \geq 0$, and this negative number becomes even smaller when subtracting $|a| \int_0^{t_0} \hat{x}(s)^{2\beta-\alpha}\, \mathrm{d}s$.

Therefore, for all $t \geq t_0$, the above inequality can be written as follows,

$$z_t \;\geq\; -a_0 - \frac{b_0}{\gamma}\sqrt{t_0}\sqrt{\log\log t_0} - \frac{|a|}{\gamma}\int_{t_0}^{t}[c_1 - c_2 z_s]^{-\kappa}\,\mathrm{d}s - \frac{b_0}{\gamma}\int_{t_0}^{t} f(s)\,\mathrm{d}s,$$

writing $f(s)$ for $\frac{\mathrm{d}}{\mathrm{d}s}(\sqrt{t}\sqrt{\log\log t}\,)$, and substituting the definition of $\hat{x}$, so that:

$$c_1 = x_-^{1-2\beta}, \quad c_2 = 2\beta - 1, \quad \kappa = \frac{2\beta-\alpha}{2\beta-1}.$$

Now, consider the ordinary differential equation (ODE),

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{z} \;=\; -\frac{|a|}{\gamma}[c_1 - c_2\tilde{z}]^{-\kappa} - \frac{b_0}{\gamma}f(t).$$

If this equation, when started at t_0 from $-a_0 - \frac{b_0}{\gamma}\sqrt{t_0}\sqrt{\log\log t_0}$, has a unique global solution, then, by standard comparison arguments,

$$z_t \;\geq\; \tilde{z}_t, \quad t \geq t_0,$$

and thus,

$$c_1 - c_2 z_t \;\leq\; p_t \overset{\text{def}}{=} c_1 - c_2\tilde{z}_t, \quad t \geq t_0, \tag{4.9}$$

if $[p_t,\, t \geq t_0]$ was the unique global solution of

$$p_t \;=\; c_1 + c_2 a_0 + c_2\frac{|a|}{\gamma}\int_{t_0}^{t} p_s^{-\kappa}\,\mathrm{d}s + c_2\frac{b_0}{\gamma}\sqrt{t}\sqrt{\log\log t}\,. \tag{4.10}$$

Yet, since $c_1 + c_2 a_0 > 0$ and t_0 was chosen large enough, this ODE (written in integral form) has local solutions, these local solutions are unique on their domain of definition (since the equation's coefficients are locally Lipschitz), and any local solution is monotonously increasing.

So, on its domain of definition, any local solution satisfies

$$p_t \geq c_2\frac{b_0}{\gamma}\sqrt{t}\sqrt{\log\log t},$$

and hence, since $\kappa > 0$,

$$\int_{t_0}^{t} p_s^{-\kappa}\,\mathrm{d}s \;\leq\; (\frac{\gamma}{c_2 b_0})^{\kappa}\int_{t_0}^{t} s^{-\kappa/2}\,\mathrm{d}s,$$

which means that $p_t = c_1 - c_2\tilde{z}_t$, $t \geq t_0$, is indeed the unique global solution of equation (4.10), because blow-up cannot occur in finite time.

Next, since $\kappa \neq 2$, the above inequality asserts

$$\int_{t_0}^{t} p_s^{-\kappa}\,\mathrm{d}s \;\leq\; (\frac{\gamma}{c_2 b_0})^{\kappa}\cdot t^{-\frac{\kappa}{2}+1}, \quad t \geq t_0,$$

which we apply to estimate the right-hand side of (4.10). Here, since $\kappa < 1$, the product $\sqrt{t}\sqrt{\log\log t}$ is dominated by $t^{-\frac{\kappa}{2}+1}$, when t goes to infinity, and therefore (4.10) yields

$$p_t \le c_0\, t^{-\frac{\kappa}{2}+1}, \quad t \ge t_0,$$

for some sufficiently large constant $c_0 > 0$.

Using this bound, (4.9), the definition of $\hat{x}$, and $\kappa > 0$, we obtain that

$$\frac{1}{(c_0\, t^{-\frac{\kappa}{2}+1})^{\kappa}} \le \hat{x}(t)^{2\beta-\alpha}, \quad t \ge t_0,$$

and hence the first integral in (4.8) is bounded below by

$$c_0^{-\kappa} \int_{t_0}^{t} [1 - e^{-\gamma(t-s)}]\, s^{-\kappa(1-\kappa/2)}\, \mathrm{d}s, \quad t \ge t_0.$$

Since $\kappa(1-\kappa/2) > 0$, by l'Hospital,

$$\lim_{t\to\infty} \int_{t_0}^{t} e^{-\gamma(t-s)}\, s^{-\kappa(1-\kappa/2)}\, \mathrm{d}s = 0,$$

and since $\kappa(1-\kappa/2) < 1/2$,

$$\lim_{t\to\infty} \frac{\int_{t_0}^{t} s^{-\kappa(1-\kappa/2)}\, \mathrm{d}s}{\sqrt{2t\log\log t}} = +\infty,$$

finally proving our claim that the first integral in (4.8) would dominate the stochastic integral, for any $a < 0$.

As a consequence, for any $a < 0$, we can now conclude that

$$\lim_{t\to\infty} z_t = -\lim_{t\to\infty} \int_0^t [1 - e^{-\gamma(t-s)}]\, \hat{x}(s)^{2\beta-\alpha}\, \mathrm{d}s = -\infty,$$

and thus, by l'Hospital,

$$\lim_{t\to\infty} \int_0^t e^{-\gamma(t-s)}\, \hat{x}(s)^{2\beta-\alpha}\, \mathrm{d}s = \lim_{t\to\infty} \int_0^t e^{-\gamma(t-s)}\, [c_1 - c_2 z_s]^{-\kappa}\, \mathrm{d}s = 0.$$

Therefore, the long-time behaviour of the right-hand side of (4.8) is fully determined by the long-time behaviour of the function,

$$t \mapsto \int_0^t \hat{x}(s)^{2\beta-\alpha}\, \mathrm{d}s = \int_0^t [c_1 - c_2 z_s]^{-\kappa}\, \mathrm{d}s,$$

and, choosing t_0 large enough, we can conclude that

$$\frac{-|a| - \varepsilon}{\gamma} \int_0^t [c_1 - c_2 z_s]^{-\kappa}\, \mathrm{d}s \le z_t \le \frac{-|a| + \varepsilon}{\gamma} \int_0^t [c_1 - c_2 z_s]^{-\kappa}\, \mathrm{d}s, \quad t \ge t_0,$$

for some $\varepsilon > 0$, such that $-|a| + \varepsilon$ is still negative, leading to

$$c_- \, t^{\frac{1}{\kappa+1}} \;\le\; [c_1 - c_2 z_t] \;\le\; c_+ \, t^{\frac{1}{\kappa+1}}, \quad t \ge t_0,$$

by standard comparison arguments, where $c_+ > c_- > 0$, of course.

Using the definition of $\hat{x}$, the above sandwich-bound translates into

$$\frac{1}{c_+} \, t^{-\frac{2\beta}{(\kappa+1)(2\beta-1)}} \;\le\; \hat{x}^{2\beta}(t) \;\le\; \frac{1}{c_-} \, t^{-\frac{2\beta}{(\kappa+1)(2\beta-1)}}, \quad t \ge t_0,$$

which means that, if $T_\infty < +\infty$, the exponent $2\beta/(\kappa+1)/(2\beta-1)$ would have to be bigger than one, i.e. $\beta < (\alpha+1)/2$. □

Proof of Theorem 2.3. As explained in Section 2 in the paragraph above the theorem, it suffices to show existence of a global weak solution.

Choose an arbitrary initial condition $(x(0), y(0)) \in \mathbb{H}$, set $\Omega_- = \mathbb{R}^2$, $\mathcal{F}_- = \mathcal{B}(\mathbb{R}^2)$, and denote by $\mathbf{P}_-$ the Dirac measure at the point $(x(0), y(0))$. Let (x_-, y_-) be the random variable on $(\Omega_-, \mathcal{F}_-, \mathbf{P}_-)$ induced by the identity on $\mathbb{R}^2$. Observe that $\mathbf{P}_-(\{x_- > 0\}) = 1$ is an immediate consequence of $x(0) > 0$.

Hence, there is a tupel $(\Omega_+, \mathcal{F}^+_\infty, \mathbb{F}_+, \mathbf{P}_+, [x_+(t),\; y_+(t), W_+(t),\, t \ge 0])$ the components of which satisfy the properties stated in the conclusion of Lemma 2.2. Moreover, using $\mathrm{d}x_+(t) = y_+(t)\,\mathrm{d}t$ and (4.7) when multiplying $x_+(t)^\beta$ by $y_+(t)$, we obtain that

$$\begin{aligned} \sqrt{2T\gamma}\, W_+(t) \;&=\; x_+(t)^\beta y_+(t) - x_-^\beta y_- - \beta \int_0^t x_+(s)^{\beta-1}\, y_+(s)^2 \,\mathrm{d}s \\ &-\; a \int_0^t \frac{\mathbf{1}_{[0,1]}\,\mathrm{d}s}{x_+(s)^{\alpha-\beta}} + \gamma \int_0^t \frac{y_+(s)\,\mathrm{d}s}{x_+(s)^\beta}\,, \end{aligned} \tag{4.11}$$

for all $t \ge 0$, $\mathbf{P}_+$-a.s., and hence W_+ can be considered a Wiener process with respect to the filtration $[\sigma(\{(x_+(s), y_+(s)) : s \le t\}),\, t \ge 0]$. Note that the proof of Lemma 2.2 makes clear that no extra sets of measure zero have to be added to this filtration.

The next step is to construct, by induction, a sequence $(\Omega_n, \mathcal{F}_n, \mathbb{F}_n, \mathbf{P}_n, [x_n(t), y_n(t), W_n(t),\, t \ge 0])$, $n = 1, 2, \ldots$, such that

$$\begin{aligned} x_n(t) \;&>\; 0, \\ x_n(t) \;&=\; x(0) + \int_0^t y_n(s)\,\mathrm{d}s, \\ y_n(t) \;&=\; y(0) + a \int_0^t \frac{\mathrm{d}s}{x_n(s)^\alpha} + \sqrt{2T\gamma} \int_0^t \frac{\mathrm{d}W_n(s)}{x_n(s)^\beta} - \gamma \int_0^t \frac{y_n(s)\,\mathrm{d}s}{x_n(s)^{2\beta}}\,, \end{aligned}$$

for all $t \in [0, n]$, $\mathbf{P}_n$-a.s., where $[x_n(t), y_n(t),\, t \ge 0]$ are continuous processes, $\mathbb{F}_n$ stands for the filtration $\mathcal{F}^n_t = \sigma(\{(x_n(s), y_n(s)) : s \le t\})$, $t \ge 0$, and $[W_n(t),\, t \ge 0]$ is an $\mathbb{F}_n$-Wiener process.

Observe that the tupel $(\Omega_+, \mathcal{F}^+_\infty, \mathbb{F}_1, \mathbf{P}_+, [x_+(t),\; y_+(t), W_+(t),\, t \ge 0])$ found in the first part of the proof plays the role of the initial case $n = 1$, of course.

So, fix $n \geq 2$, and suppose that $(\Omega_{n-1}, \mathcal{F}_{n-1}, \mathbb{F}_{n-1}, \mathbf{P}_{n-1}, [x_{n-1}(t),\, y_{n-1}(t), W_{n-1}(t),\, t \geq 0])$ has already been constructed.

Reset $\Omega_- = \Omega_{n-1}$, $\mathcal{F}_- = \mathcal{F}_{n-1}$, $\mathbf{P}_- = \mathbf{P}_{n-1}$, and choose $x_- = x_{n-1}(n-1)$, $y_- = y_{n-1}(n-1)$. Then, again by Lemma 2.2, there is a corresponding tupel $(\Omega_+, \mathcal{F}^+_\infty, \mathbb{F}_+, \mathbf{P}_+,\, [x_+(t),\, y_+(t), W_+(t),\, t \geq 0])$ which we now denote by $(\Omega_n, \mathcal{F}_n, \mathbb{F}_+, \mathbf{P}_n, [x_+(t),\, y_+(t), W_+(t),\, t \geq 0])$.

Remark 4.6. In Lemma 2.2, the filtration $\mathbb{F}_+$ was given by

$$\mathcal{F}^+_t \;=\; \mathcal{G}^+_{A_t}, \quad \text{using} \quad \mathcal{G}^+_t \;=\; \sigma(x_-, y_-) \vee \sigma(\{B_s : s \leq t\}),$$

for the purpose of Remark 4.4. But, when applying Lemma 2.2 in the context of the present proof, we are going to work with

$$\mathcal{G}^+_t \;=\; \sigma(\{(x_{n-1}(s), y_{n-1}(s)) : s \leq n-1\}) \vee \sigma(\{B_s : s \leq t\})$$

instead, without violating the truth of Remark 4.4.

Recall that $[x_{n-1}(t), y_{n-1}(t),\, W_{n-1}(t),\, t \geq 0]$ are extended to $(\Omega_n, \mathcal{F}_n, \mathbf{P}_n)$ in the canonical way without changing their notation. Define, for $t \geq 0$,

$$\begin{aligned} x_n(t) &= x_{n-1}(t \wedge (n-1)) + x_+((t-n+1) \vee 0) - x_{n-1}(n-1), \\ y_n(t) &= y_{n-1}(t \wedge (n-1)) + y_+((t-n+1) \vee 0) - y_{n-1}(n-1), \\ W_n(t) &= W_{n-1}(t \wedge (n-1)) + W_+((t-n+1) \vee 0), \end{aligned}$$

and build a filtration $\hat{\mathbb{F}}_n$ from both $\mathbb{F}_{n-1}$ and $\mathbb{F}_+$ by

$$\hat{\mathcal{F}}^n_t \stackrel{\text{def}}{=} \begin{cases} \mathcal{F}^{n-1}_t & : \; t < n-1, \\ \mathcal{F}^+_{t-n+1} & : \; t \geq n-1. \end{cases}$$

All in all, because of

$$\mathbf{P}_n \;=\; \mathbf{P}_+ \;=\; \mathbf{Q} \;=\; \mathbf{P}_- \otimes \mathbf{P}_W \;=\; \mathbf{P}_{n-1} \otimes \mathbf{P}_W \quad \text{on} \quad \mathcal{F}^+_0,$$

and because $\mathcal{F}^{n-1}_t \subseteq \mathcal{F}^+_0$ (see Remark 4.6), the processes $[x_n(t), y_n(t), W_n(t), t \geq 0]$ would have all properties needed for the induction step, except that W_n is a Wiener process with respect to the filtration $\hat{\mathbb{F}}_n$ which is possibly bigger than $\mathbb{F}_n$.

However, as a consequence of (4.7), we also have

$$y_n(t) \;=\; y(0) + a\int_0^t \frac{\mathbf{1}_{[0,n]}\,\mathrm{d}s}{x_n(s)^\alpha} + \sqrt{2T\gamma}\int_0^t \frac{\mathrm{d}W_n(s)}{x_n(s)^\beta} - \gamma\int_0^t \frac{y_n(s)\,\mathrm{d}s}{x_n(s)^{2\beta}},$$

for all $t \geq 0$, $\mathbf{P}_n$-a.s., leading to the nth-step analogue of (4.11), i.e.,

$$\begin{aligned} \sqrt{2T\gamma}\, W_n(t) = x_n(t)^\beta y_n(t) - x(0)^\beta y(0) - \beta\int_0^t x_n(s)^{\beta-1}\, y_n(s)^2\,\mathrm{d}s \\ - a\int_0^t \frac{\mathbf{1}_{[0,n]}\,\mathrm{d}s}{x_n(s)^{\alpha-\beta}} + \gamma\int_0^t \frac{y_n(s)\,\mathrm{d}s}{x_n(s)^\beta}\,, \end{aligned} \tag{4.12}$$

for all $t \geq 0$, $\mathbf{P}_n$-a.s., so that W_n is $\mathbb{F}_n$-adapted, and therefore it must be an $\mathbb{F}_n$-Wiener process, too.

The next step of the proof consists in constructing a measure on $((\mathbb{R}^2)^{[0,\infty)}, \mathcal{B}((\mathbb{R}^2)^{[0,\infty)}))$ whose finite-dimensional distributions are induced by the laws of the two-dimensional processes $[x_n(t),\, y_n(t),\, t \geq 0]$, $n = 1, 2, \ldots$, in the following way.

For an arbitrary finite sequence of non-negative mutually different numbers $\mathbf{t} = (t_1, \ldots, t_k)$, define

$$\mathbf{P}_{\mathbf{t}}(\Gamma) \;=\; \mathbf{P}_n(\overbrace{\{(x_n(t_1), y_n(t_1), \ldots, x_n(t_k), y_n(t_k)) \in \Gamma\}}^{\textit{cylinder set}}), \quad \Gamma \in \mathcal{B}(\mathbb{R}^{2k}),$$

where n is the smallest integer such that $n \geq \max\{t_1, \ldots, t_k\}$. Then, $\{\mathbf{P}_{\mathbf{t}}\}$ is a consistent family of finite-dimensional distributions in the sense of Kolmogorov. Hence, there is a probability measure $\mathbf{P}$ on $((\mathbb{R}^2)^{[0,\infty)}, \mathcal{B}((\mathbb{R}^2)^{[0,\infty)}))$ satisfying

$$\mathbf{P}_{\mathbf{t}}(\Gamma) \;=\; \mathbf{P}(\{(x(t_1), y(t_1), \ldots, x(t_k), y(t_k)) \in \Gamma\}), \quad \Gamma \in \mathcal{B}(\mathbb{R}^{2k}),$$

where $[x(t),\, y(t),\, t \geq 0]$ denotes the coordinate process on $\Omega = (\mathbb{R}^2)^{[0,\infty)}$. Let $\mathcal{F}$ be the completion of $\mathcal{B}((\mathbb{R}^2)^{[0,\infty)})$ with respect to $\mathbf{P}$, and let $\mathbb{F}$ be the filtration obtained by $\mathbf{P}$-augmentation of $\sigma(\{x(s),\, y(s) : s \leq t\})$, $t \geq 0$.

Note that, almost surely, $[x(t),\, y(t),\, t \geq 0]$ is a pair of continuous $\mathbb{F}$-adapted processes. This is seen by two simple arguments. First, since[3]

$$\mathbf{P}_n(\{t \mapsto (x_n(t), y_n(t)) \text{ continuous on } [0, n]\}) \;=\; 1,$$

each of the events $\{t \mapsto (x(t), y(t)) \text{ continuous on } [0, n]\}$ is in $\mathcal{F}$, $n = 1, 2, \ldots$, and second,

$$\begin{aligned} &\mathbf{P}(\{t \mapsto (x(t), y(t)) \text{ continuous on } [0, n]\}) \\ =\; &\mathbf{P}_n(\{t \mapsto (x_n(t), y_n(t)) \text{ continuous on } [0, n]\}), \end{aligned}$$

for each $n \geq 1$.

In what follows, $[x(t),\, y(t),\, t \geq 0]$ always stands for a fixed continuous version indistinguishable of the coordinate process.

Of course, in a similar way, one shows that $x(t) > 0$, $t \geq 0$, $\mathbf{P}$-a.s., as well as

$$x(t) \;=\; x(0) \;+ \int_0^t y(s)\, \mathrm{d}s, \quad t \geq 0, \quad \mathbf{P}\text{-a.s.} \tag{4.13}$$

Now, introduce

$$\begin{aligned} W(t) \;\; &\overset{\text{def}}{=} \;\; \frac{x(t)^\beta y(t) - x(0)^\beta y(0)}{\sqrt{2T\gamma}} - \beta \int_0^t \frac{x(s)^{\beta-1}\, y(s)^2}{\sqrt{2T\gamma}}\, \mathrm{d}s \\ &- \;\; a \int_0^t \frac{\mathrm{d}s}{\sqrt{2T\gamma}\, x(s)^{\alpha-\beta}} + \gamma \int_0^t \frac{y(s)\, \mathrm{d}s}{\sqrt{2T\gamma}\, x(s)^\beta}\,, \quad t \geq 0, \end{aligned}$$

[3]As this probability can be approximated by probabilities of cylinder sets.

and observe that this process is a Wiener process because, by (4.12), it satisfies

$$\mathbf{P}(\{(W(t_1),\dots,W(t_k))\in\Gamma\}) \;=\; \mathbf{P}_n(\{(W_n(t_1),\dots,W_n(t_k))\in\Gamma\}),$$

for every $\mathbf{t}=(t_1,\dots,t_k)$, $n\geq\max\{t_1,\dots,t_k\}$ and $\Gamma\in\mathcal{B}(\mathbb{R}^k)$, and because the continuity of $[x(t),\, y(t),\, t\geq 0]$ makes it a continuous process. Furthermore, as W_n can be considered a Wiener process with respect to the filtration obtained by $\mathbf{P}_n$-augmentation of $\sigma(\{(x_n(s),y_n(s)) : s\leq t\})$, $t\geq 0$, one can also consider W to be an $\mathbb{F}$-Wiener-process.

Finally, again using the corresponding property on $(\Omega_n,\mathcal{F}_n,\mathbf{P}_n)$, each of the processes $[y(t\wedge n),\, t\geq 0]$ is an $\mathbb{F}$-semimartingale, $n=1,2,\dots$, and hence $[y(t),\, t\geq 0]$ is one, too. As a consequence, by partial integration,

$$\sqrt{2T\gamma}\,W(t) \;=\; \int_0^t x(s)^\beta\,\mathrm{d}y(s) - a\int_0^t \frac{\mathrm{d}s}{x(s)^{\alpha-\beta}} + \gamma\int_0^t \frac{y(s)\,\mathrm{d}s}{x(s)^\beta},$$

for all $t\geq 0$, $\mathbf{P}$-a.s., follows from (4.13) and the definition of $W(t)$. Thus, using the above right-hand side when calculating $\int_0^t x(s)^{-\beta}\,\mathrm{d}W(s)$, eventually proves the theorem. □

Proof of Proposition 2.5. To begin with, assume the conditions of Proposition 2.1 or Theorem 2.3, that is the parameter a could be negative, too.

Let $[x(t),y(t),\, t\geq 0]$ be the solution of (2.1), started at $(x(0),y(0))\in\mathbb{H}$, and driven by a Wiener process $[W(t),\, t\geq 0]$, given on a probability space $(\Omega,\mathcal{F},\mathbf{P})$, i.e.,

$$\begin{aligned} x(t) \;&=\; x(0) + \int_0^t y(s)\,\mathrm{d}s, \\ y(t) \;&=\; y(0) + a\int_0^t \frac{\mathrm{d}s}{x(s)^\alpha} + \sqrt{2T\gamma}\int_0^t \frac{\mathrm{d}W(s)}{x(s)^\beta} - \gamma\int_0^t \frac{y(s)}{x(s)^{2\beta}}\,\mathrm{d}s, \end{aligned}$$

for all $t\geq 0$, $\mathbf{P}$-a.s.

Since the stochastic integral in the above equation is well-defined for all $t\geq 0$, its quadratic variation, $\tilde{A}_t = \int_0^t x(s)^{-2\beta}\,\mathrm{d}s$, is well-defined for all $t\geq 0$, too, and

$$\tilde{T}_t \;=\; \inf\{s\geq 0 : \tilde{A}_s > t\},\quad t\geq 0,$$

can be used as a time-change. Furthermore, since $x(t)>0$, for all $t\geq 0$, $\mathbf{P}$-a.s., this time-change is strictly increasing and continuous on $[0,\tilde{A}_\infty)$, $\mathbf{P}$-a.s., and

$$\mathbf{P}(\{\lim_{t\uparrow\tilde{A}_\infty}\tilde{T}_t = +\infty\}) \;=\; 1. \tag{4.14}$$

So, on the one hand, the time-changed processes, $t\mapsto\tilde{x}(t) = x(\tilde{T}_t)$ and $t\mapsto\tilde{y}(t) = y(\tilde{T}_t)$, are almost surely continuous processes on $[0,\tilde{A}_\infty)$. On the other hand, since

$$\tilde{T}_t \;=\; \int_0^t \tilde{x}(s)^{2\beta}\,\mathrm{d}s,\quad t<\tilde{A}_\infty,\quad \mathbf{P}\text{-a.s.},$$

property (4.14) implies that

$$\limsup_{t\uparrow\tilde{A}_\infty} \tilde{x}(t) \;=\; +\infty, \quad \text{on } \{\tilde{A}_\infty < \infty\}, \quad \mathbf{P}\text{-a.s.}, \tag{4.15}$$

and we are going to show next that, on $\{\tilde{A}_\infty < \infty\}$, $\liminf_{t\uparrow\tilde{A}_\infty} \tilde{x}(t)$ cannot be finite with positive probability, either, even if the parameter a is negative.

Applying the time-change to the above equation yields

$$\begin{aligned} \tilde{x}(t) &= x(0) + \int_0^t \tilde{x}(s)^{2\beta}\, \tilde{y}(s)\,\mathrm{d}s, \\ \tilde{y}(t) &= y(0) + a\int_0^t \tilde{x}(s)^{2\beta-\alpha}\,\mathrm{d}s + \sqrt{2T\gamma}\,\tilde{W}(t) - \gamma\int_0^t \tilde{y}(s)\,\mathrm{d}s, \end{aligned} \tag{4.16}$$

for all $t < \tilde{A}_\infty$, $\mathbf{P}$-a.s., where $[\tilde{W}(t),\, t \geq 0]$ is another Wiener process on a possibly enlarged[4] probability space (cf. Theorem II.7.2' in [9]).

Of course, by the continuity properties of the time-changed processes,

$$\tilde{x}(t) \;=\; \left[\; x(0)^{1-2\beta} - (2\beta-1)\int_0^t \tilde{y}(s)\,\mathrm{d}s \;\right]^{\frac{-1}{2\beta-1}},$$

for at least all $t < \tilde{A}_\infty$, $\mathbf{P}$-a.s., where

$$\tilde{y}(s) \;=\; e^{-\gamma s}y(0) + \sqrt{2T\gamma}\int_0^s e^{-\gamma(s-r)}\,\mathrm{d}\tilde{W}(r) + a\int_0^s e^{-\gamma(s-r)}\,\tilde{x}(r)^{2\beta-\alpha}\,\mathrm{d}r,$$

for all $s < \tilde{A}_\infty$, $\mathbf{P}$-a.s.

The question is now whether different sequences of time points, $(t_n)_{n=1}^\infty$, converging to $\tilde{A}_\infty < \infty$, can lead to different limits of $\tilde{x}(t_n)$, $n \to \infty$, which can only happen if $\int_0^{t_n} \tilde{y}(s)\,\mathrm{d}s$ has different limits for different sequences of time points, which can only happen if $\tilde{y}(t_n)$ has different limits for different sequences of time points.

The key to the answer of this question is writing

$$\tilde{y}(s) \quad \text{as} \quad \hat{y}(s) + ae^{-\gamma s}R_s, \tag{4.17}$$

where $[\hat{y}(s),\, s < \tilde{A}_\infty]$ is indistinguishable of an Ornstein-Uhlenbeck process restricted to $[0, \tilde{A}_\infty)$, and

$$R_s \;=\; \int_0^s e^{\gamma r}\,\tilde{x}(r)^{2\beta-\alpha}\,\mathrm{d}r, \quad s < \tilde{A}_\infty.$$

Note that $s \mapsto R_s$ is almost surely a continuous monotone function on $[0, \tilde{A}_\infty)$, so that $ae^{-\gamma t_n}R_{t_n}$ can only have one limit for any sequence of time points converging to $\tilde{A}_\infty$, on $\{\tilde{A}_\infty < \infty\}$, $\mathbf{P}$-a.s., and the same applies to $\hat{y}(t_n)$, as this function can almost surely be extended to a continuous function on $[0, \infty)$.

[4]By standard convention, the enlarged space is denoted by $(\Omega, \mathcal{F}, \mathbf{P})$, too.

All in all, $\liminf_{t\uparrow\tilde{A}_\infty} \tilde{x}(t)$ has indeed to coincide with $\limsup_{t\uparrow\tilde{A}_\infty} \tilde{x}(t)$, on $\{\tilde{A}_\infty < \infty\}$, $\mathbf{P}$-a.s., proving

$$\lim_{t\to\infty} x(t) \;=\; +\infty, \quad \text{on } \{\tilde{A}_\infty < \infty\}, \quad \mathbf{P}\text{-a.s.,} \tag{4.18}$$

by (4.15), for any parameter $a \in \mathbb{R}$, because $x(t) = \tilde{x}(\tilde{A}_t)$, $t \geq 0$, $\mathbf{P}$-a.s.

In the second part of the proof, we will show that, if $\beta > 1/2$, then the event $\{\tilde{A}_\infty < \infty\}$ has probability one, for any $a \geq 0$, eventually proving the proposition.

First, the Ornstein-Uhlenbeck process used in (4.17) and the process $[\hat{y}(t), t \geq 0]$ used in the proof of Lemma 2.2 have the same law, when started at $y(0)$, justifying the same notation. Therefore,

$$\begin{aligned}\hat{x}(t) \;&=\; \Big[\; x(0)^{1-2\beta} - (2\beta-1)\int_0^t \hat{y}(s)\,\mathrm{d}s \;\Big]^{\frac{-1}{2\beta-1}}, \; t < \tau,\\ \tau \;&=\; \inf\{s \geq 0 : \int_0^s \hat{y}(r)\,\mathrm{d}r = \frac{x(0)^{1-2\beta}}{2\beta-1}\},\end{aligned}$$

and the corresponding objects given in the proof of Lemma 2.2 have the same law, too, when using $x(0)$ instead of x_-, so that

$$\mathbf{P}(\{\tau < \infty\} \cap \{\lim_{t\uparrow\tau} \hat{x}(t) \;=\; +\infty\}) \;=\; 1 \tag{4.19}$$

can easily be derived from Lemma 4.1. Furthermore,

$$\begin{aligned}\hat{x}(t) \;&=\; x(0) + \int_0^t \hat{x}(s)^{2\beta}\,\hat{y}(s)\,\mathrm{d}s,\\ \hat{y}(t) \;&=\; y(0) + \sqrt{2T\gamma}\,\tilde{W}(t) - \gamma\int_0^t \hat{y}(s)\,\mathrm{d}s,\end{aligned} \tag{4.20}$$

for all $t < \tau$, $\mathbf{P}$-a.s.

The next step is to show that, if $a \geq 0$, then $(\tilde{x}(t), \tilde{y}(t))$ satisfying (4.16) dominates $(\hat{x}(t), \hat{y}(t))$ satisfying (4.20), for all $t < \tau \wedge \tilde{A}_\infty$. In fact, since the drift coefficient of equation (4.20), i.e. $(\hat{x}, \hat{y}) \mapsto (\hat{x}^{2\beta}\,\hat{y}, -\gamma\hat{y})$, is quasi-monotonously increasing (cf. Def.3.1 in [1]), and since solutions to (4.20) are pathwise unique up to $\tau \wedge \tilde{A}_\infty$, and since the difference of the drift coefficients of (4.16) and (4.20) is a vector field on $\mathbb{H}$ with non-negative components, it follows from Prop.3.3 in [1] that

$$\tilde{x}(t) \geq \hat{x}(t), \quad \tilde{y}(t) \geq \hat{y}(t), \quad t < \tau \wedge \tilde{A}_\infty, \quad \mathbf{P}\text{-a.s.}$$

Remark 4.7. The results in [1] were obtained for coefficients defined on $\mathbb{R}^d$, but it is an easy exercise to show their validity for coefficients defined on domains like our half-plane, $\mathbb{H}$.

Now, if $\tilde{A}_\infty$ was bigger than τ with positive probability, then the process $t \mapsto \tilde{x}(t)$, being continuous on $[0, \tilde{A}_\infty)$, would blow up before $\tilde{A}_\infty$, by (4.19), which is a contradiction. Thus, $\tilde{A}_\infty$ is almost surely bounded by τ from above leading to $\mathbf{P}(\{\tilde{A}_\infty < \infty\}) = 1$, again by (4.19). □

The authors would like to thank both referees for their very valuable comments!

References

[1] Assing, S., Manthey, R.: The behavior of solutions of stochastic differential inequalities. *Probab. Theory Relat. Fields*, **103**, (1995), 493–514. MR-1360202

[2] Bang, O., Christiansen, P.L., Gaididei, Y.B., If, F., Rasmussen, K.O.: Temperature effects in a nonlinear model of monolayer Scheibe aggregates. *Physical Review E*, **49**, (1994), no. 5, 4627–4635.

[3] Christiansen, P.L., Gaididei, Y.B., Johansson, M., Rasmussen, K.O., Yakimenko, I.I.: Collapse of solitary excitations in the nonlinear Schrödinger equation with nonlinear damping and white noise. *Physical Review E*, **54**, (1996), no. 1, 924–930.

[4] Conrad, F., Grothaus, M.: Construction, ergodicity and rate of convergence of N-particle Langevin dynamics with singular potentials. *Journal of Evolution Equations*, **10**, (2010), 623–662.

[5] Cooke, B., Herzog, D.P., Mattingly, J.C., McKinley, S.A., Schmidler, S.C.: Geometric Ergodicity of Tow-Dimensional Hamiltonian Systems with a Lennard-Jones-like Repulsive Potential. *Commun. Math. Sci.*, **15**, (2017), no. 7, 1987–2015. MR-3717917

[6] Funaki, T., Ohnawa, M., Suzuki, Y., Yokoyama, S.: Existence and uniqueness of solutions to stochastic Rayleigh-Plesset equations. *J. Math. Anal. Appl.*, **425**, (2015), 20–32. MR-3299647

[7] Grothaus, M., Stilgenbauer, P.: A Hypocoercivity Related Ergodicity Method for Singular Distorted Non-Symmetric Diffusions. *Integr. Equ. Oper. Theory*, **83**, (2015), 331–379.

[8] Herzog, D.P., Mattingly, J.C.: Ergodicity and Lyapunov Functions for Langevin Dynamics with Singular Potentials. *ArXiv e-prints*, arxiv:1711.02250v1, (2017), 20pp.

[9] Ikeda, N. and Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. Second edition. North-Holland mathematical library **24**. *North-Holland Publishing*, Amsterdam, 1989. xiv+555 pp. MR-1011252

[10] Karatzas, I. and Shreve, E.: Brownian Motion and Stochastic Calculus. Second edition. Graduate Texts in Mathematics **113**. *Springer Verlag*, New York, 1991. xxiv+470 pp. MR-1121940

[11] Merle, F.: Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity. *Comm. Math. Phys.*, **129**, (1990), no. 2, 223–240. MR-1048692

[12] Merle, F., Tsutsumi, Y.: L^2-concentration of blow up solutions for the nonlinear Schrödinger equation with critical power nonlinearity. *J. Diff. Eq.*, **84**, (1990), 205–214. MR-1047566

[13] Meyn, S.P., Tweedy, R.L.: Stability of Markovian Processes III: Forster-Lyapunov Criteria for Continuous-Time Processes. *Advances in Applied Probability*, **25**, (1993), no. 3, 518–548.

[14] Nawa, H.: Asymptotic and limiting profiles of blowup solutions of the nonlinear Schrödinger Equation with critical power. *Comm. Pure Appl. Math.*, **52**, (1999), no. 2, 193–270. MR-1653454

[15] Weinstein, M.I.: Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.*, **87**, (1983), 567–576. MR-0691044