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Coupling of Brownian motions in Banach spaces[*]

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Abstract

Consider a separable Banach space $\mathcal{W}$ supporting a non-trivial Gaussian measure μ. The following is an immediate consequence of the theory of Gaussian measure on Banach spaces: there exist (almost surely) successful couplings of two $\mathcal{W}$-valued Brownian motions $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ begun at starting points $\mathbf{B}(0)$ and $\widetilde{\mathbf{B}}(0)$ if and only if the difference $\mathbf{B}(0) - \widetilde{\mathbf{B}}(0)$ of their initial positions belongs to the Cameron-Martin space $\mathcal{H}_\mu$ of $\mathcal{W}$ corresponding to μ. For more general starting points, can there be a "coupling at time ∞", such that almost surely $\|\mathbf{B}(t) - \widetilde{\mathbf{B}}(t)\|_{\mathcal{W}} \to 0$ as $t \to \infty$? Such couplings exist if there exists a Schauder basis of $\mathcal{W}$ which is also a $\mathcal{H}_\mu$-orthonormal basis of $\mathcal{H}_\mu$. We propose (and discuss some partial answers to) the question, to what extent can one express the probabilistic Banach space property "Brownian coupling at time ∞ is always possible" purely in terms of Banach space geometry?

Keywords: Banach space; Brownian motion; Cameron-Martin space; coupling; coupling at time ∞; Gaussian measure; Hilbert spaces; Markushevich basis (M-basis); reflection coupling; Schauder basis.
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1 Introduction

When can there be an (almost surely) *successful coupling* of two Brownian motions $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ defined on a separable Banach space $\mathcal{W}$? (When can $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ be made to coincide at and after some random time $\tau < \infty$?) Is a weaker kind of success more widely available? The purpose of this paper is to explore this weaker kind of success, and to raise an interesting open question.

Naturally the answer to the first question depends on the initial displacement of $\mathbf{B}$ relative to $\widetilde{\mathbf{B}}$. Expressed more precisely, given a $\mathcal{W}$-valued Brownian motion $\mathbf{B} = \{\mathbf{B}(t) : t \geq 0\}$ started at 0, for which $x \in \mathcal{W}$ is it possible to construct a second Brownian motion $\widetilde{\mathbf{B}} = \{\widetilde{\mathbf{B}}(t) : t \geq 0\}$ starting at x and such that $\mathbf{B}(\tau + \cdot) = \widetilde{\mathbf{B}}(\tau + \cdot)$ after some random time τ? (The general case $\widetilde{\mathbf{B}}(0) - \mathbf{B}(0) = x$ follows by translation invariance.)

We establish criteria for relative displacements x permitting almost surely successful (classical) coupling, and almost sure coupling *at time ∞* (almost surely $\|\widetilde{\mathbf{B}}(t) - \mathbf{B}(t)\|_{\mathcal{W}} \to 0$ as $t \to \infty$ when $\|\cdot\|_{\mathcal{W}}$ is the norm of $\mathcal{W}$: see Section 4). In both cases the coupling can

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be chosen to be *Markovian*: martingales in the natural filtration of $\mathbf{B}$ (respectively of $\widetilde{\mathbf{B}}$), remain martingales in the joint natural filtration of $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ (see [24]).

The question of successful classical coupling is resolved by recalling the notion of the *Cameron-Martin* space of a Gaussian measure μ on $\mathcal{W}$ (cf. [6]). Given a Gaussian measure μ, there is a standard construction of a Hilbert space $\mathcal{H}_\mu$ densely embedded in $\mathcal{W}$, such that translations of μ by elements of $\mathcal{H}_\mu$ are exactly those that induce translated measures which are absolutely continuous with respect to μ. This theory, together with the well-known Aldous inequality [1, Lemma 3.6], immediately yields the following.

Theorem (See Theorem 3.2 below). *Let $\mathcal{W}$ be a separable Banach space with norm $\|\cdot\|_\mathcal{W}$. Consider a $\mathcal{W}$-valued Brownian motion $\mathbf{B}$ started at 0 and an element $x \in \mathcal{W}$. Another Brownian motion $\widetilde{\mathbf{B}}$ can be constructed to start at x and almost surely meet $\mathbf{B}$ within finite time if and only if the relative initial displacement x lies in $\mathcal{H}_\mu$, for $\mu = \mathcal{L}(\mathbf{B}(1))$ the initial distribution of $\mathbf{B}$. In that case the "fastest possible" coupling time is realized as the hitting time τ of $\langle x, \mathbf{B}(\tau) \rangle_{\mathcal{H}_\mu}$ on $\frac{1}{2}\|x\|_{\mathcal{H}_\mu}^2$.*

If the initial displacement does not lie in $\mathcal{H}_\mu$ then in many cases a weaker form of coupling is still available, namely "coupling at time ∞".

Theorem (See Theorem 4.2 below). *Let $\mathcal{W}$ be a separable Banach space with norm $\|\cdot\|_\mathcal{W}$. Consider a $\mathcal{W}$-valued Brownian motion $\mathbf{B}$ started at 0, such that the associated $\mathcal{H}_\mu$ contains an orthonormal basis which is also a Schauder basis for $\mathcal{W}$. For any $x \in \mathcal{W}$, one can construct another Brownian motion $\widetilde{\mathbf{B}}$ started at x which almost surely couples with $\mathbf{B}$ at time ∞:*

$$\mathbb{P}\left[\left\|\widetilde{\mathbf{B}}(t) - \mathbf{B}(t)\right\|_\mathcal{W} \to 0 \text{ as } t \to \infty\right] = 1. \tag{1.1}$$

The Schauder basis condition is not necessary for coupling at time ∞: we will discuss various extensions and conclude by raising the open Question 5.1: what Banach space geometries imply existence of Brownian coupling at time ∞?

Remark 1.1. When (1.1) holds, in whatever Banach space context, we say that the Brownian motions $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ (begun at $\mathbf{B}(0)$ and $\widetilde{\mathbf{B}}(0)$) *couple at time ∞* almost surely.

The paper is organized as follows. Section 2 surveys the basic theory of Gaussian measures and Brownian motions on Banach spaces. Section 3 treats the case when the initial displacement x is in $\mathcal{H}_\mu$, while the case of $x \in \mathcal{W} \setminus \mathcal{H}_\mu$ is discussed in Section 4 (under some technical assumptions). Section 5 discusses extensions and future work and raises the Open Question 5.1.

2 Gaussian measures and Banach-valued Brownian motion

Recall the following facts about Gaussian measures in Banach space. Proofs can be found for example in [26, 17, 34, 11]. A *Gaussian probability measure μ* on a separable Banach space $\mathcal{W}$ is a Borel measure on $\mathcal{W}$ such that the push-forward $\ell \# \mu$ by any continuous linear functional $\ell \in \mathcal{W}^*$ is a Gaussian probability measure on $\mathbb{R}$. If $\ell \# \mu$ has mean 0 for every ℓ then μ is *centered*. For simplicity, consider only centered Gaussian measures which are *non-degenerate*: $\ell \# \mu$ is non-degenerate for all non-zero ℓ.

The *covariance operator* of μ is a map $\text{Cov}_{\mathcal{H}_\mu} : \mathcal{W}^* \times \mathcal{W}^* \to \mathbb{R}$ defined by

$$\text{Cov}_{\mathcal{H}_\mu}(\ell_1, \ell_2) = \int_\mathcal{W} \ell_1(x)\ell_2(x)\,\mu(\mathrm{d}x).$$

This can be identified with a bounded linear operator $\widehat{\text{Cov}}_{\mathcal{H}_\mu} : \mathcal{W}^* \to \mathcal{W}^{**}$ as follows:

$$\left(\widehat{\text{Cov}}_{\mathcal{H}_\mu} \ell_1\right)(\ell_2) = \text{Cov}_{\mathcal{H}_\mu}(\ell_1, \ell_2). \tag{2.1}$$

In particular Fernique's Theorem ([26, Section III.3]) implies that $\widehat{\mathrm{Cov}}_{\mathcal{H}_\mu}$ is bounded as linear operator $\mathcal{W}^* \to \mathcal{W}$, and $\widehat{\mathrm{Cov}}_{\mathcal{H}_\mu} \ell$ is a barycentre (cf. [17, Section 3.1]):

$$\widehat{\mathrm{Cov}}_{\mathcal{H}_\mu} \ell \quad = \quad \int_{\mathcal{W}} \ell(x) \, x \, \mu(\mathrm{d}\,x) \quad = \quad \int_{\mathcal{W}} \langle \ell, x \rangle_{\mathcal{W}^*;\mathcal{W}} \, x \, \mu(\mathrm{d}\,x) \,. \tag{2.2}$$

Here $\langle \ell, x \rangle_{\mathcal{W}^*;\mathcal{W}} = \ell(x)$ expresses the duality between $\mathcal{W}^*$ and $\mathcal{W}$.

2.1 The Cameron-Martin space

The (non-degenerate) Gaussian probability measure μ is canonically associated with its *Cameron-Martin* space $\mathcal{H}_\mu$; a Hilbert space densely and continuously embedded in the Banach space $\mathcal{W}$, so $\mathcal{H}_\mu \hookrightarrow \mathcal{W}$ ([17, Section 3.2]; [11, Section 4.3]). The triple $(\mathcal{W}, \mathcal{H}_\mu, \mu)$ is an *abstract Wiener space*. More precisely, let

$$\mathring{\mathcal{H}}_\mu \;=\; \{ h \in \mathcal{W} \;:\; \text{there exists } h^* \in \mathcal{W}^* \text{ such that } \mathrm{Cov}_{\mathcal{H}_\mu}(h^*, \ell) = \ell(h), \text{ for all } \ell \in \mathcal{W}^* \} \,. \tag{2.3}$$

Then $\mathring{\mathcal{H}}_\mu$ is a pre-Hilbert space when endowed with the inner product defined by $\langle h, k \rangle_{\mathcal{H}_\mu} = \mathrm{Cov}_{\mathcal{H}_\mu}(h^*, k^*)$. Its Hilbert-space completion under this inner product is the *Cameron-Martin space* of $\mathcal{W}$ furnished with μ. As a Hilbert space, $\mathcal{H}_\mu$ has the norm

$$\|h\|_{\mathcal{H}_\mu}^2 \quad = \quad \langle h, h \rangle_{\mathcal{H}_\mu} \quad = \quad \mathrm{Cov}_{\mathcal{H}_\mu}(h^*, h^*) \,. \tag{2.4}$$

Fernique's Theorem and completeness of $\mathcal{W}$ imply that the embedding $A : \mathcal{H}_\mu \hookrightarrow \mathcal{W}$ is continuous and injective [17, Section 3]. Note that A^*A is the extension of $\mathrm{Cov}_{\mathcal{H}_\mu}$ to $\mathcal{H}_\mu \times \mathcal{H}_\mu$

Remark 2.1. $\mathring{\mathcal{H}}_\mu$ coincides with the range of $\widehat{\mathrm{Cov}}_\mu$ defined in (2.1). Moreover, in view of (2.1) and (2.3), we have $\mathcal{W}^* \hookrightarrow \mathcal{H}_\mu \hookrightarrow \mathcal{W}$ using continuous embeddings of dense image.

In particular, for any $x \in \mathring{\mathcal{H}}_\mu$ there is an element $x^* \in \mathcal{W}^*$ associated to x for which

$$\mathrm{Cov}_{\mathcal{H}_\mu}(x^*, \ell) \quad = \quad \ell(x) \quad = \quad \langle \ell, x \rangle_{\mathcal{W}^*;\mathcal{W}} \,, \quad \text{for all } \ell \in \mathcal{W}^* \,. \tag{2.5}$$

The following result is key for analyzing the possibility of successful Brownian coupling.

Theorem 2.2 (Feldman-Hajek Theorem, see [26, Theorem 3.1]). *For $w \in \mathcal{W}$ define the translation map $T_w : \mathcal{W} \to \mathcal{W}$ by $T_w(x) = x + w$. If $w \in \mathcal{H}_\mu$ then the push-forward measure $T_w \# \mu$ is absolutely continuous with respect to μ, while if $w \notin \mathcal{H}_\mu$ then the push-forward measure and μ are mutually singular. Moreover the join of the probability distributions μ and $T_w \# \mu$ has total mass given by twice the probability that a standard normal random variable exceeds the value $\frac{1}{2} \|w\|_{\mathcal{H}_\mu}$.*

2.2 Brownian motion on a Banach space

Stroock (see [34, Chapter 8]) uses these considerations to define Brownian motion (a process with stationary Gaussian increments independent of the past, and continuous sample paths). Let $K(\mathcal{H}_\mu)$ be the Hilbert space of absolutely continuous functions $h : [0, \infty) \to \mathcal{H}_\mu$ with almost-everywhere defined derivative $\dot{h}$, such that $h(0) = 0$ and $\|h\|_{K(\mathcal{H}_\mu)}^2 = \|\dot{h}\|_{L^2([0,\infty),\mathcal{H}_\mu)}^2 = \int_0^\infty \|\dot{h}\|_{\mathcal{H}_\mu}^2 \, \mathrm{d}\, t < \infty$. Define

$$\mathcal{C}_{\mathcal{W}} \quad = \quad \left\{ \theta : [0, \infty) \to \mathcal{W}, \text{ such that } \theta \text{ is continuous and } \lim_{t \to \infty} \frac{\|\theta(t)\|_{\mathcal{W}}}{t} = 0 \right\},$$

which becomes a Banach space when endowed with the norm

$$\|\theta\|_{\mathcal{C}_{\mathcal{W}}} \quad = \quad \sup_{t \geq 0} \frac{1}{1+t} \|\theta\|_{\mathcal{W}} \,.$$

Each centered non-degenerate Gaussian measure then corresponds to a Brownian motion:

Theorem 2.3 ([34], Theorem 8.6.1)**.** *Given $K(\mathcal{H}_\mu)$ and $\mathcal{C}_\mathcal{W}$ as above, there is a unique measure $\mu_\mathcal{W}$ such that $(\mathcal{C}_\mathcal{W}, K(\mathcal{H}_\mu), \mu_\mathcal{W})$ is an abstract Wiener space.*

Theorem 2.4 ([34], Theorem 8.6.6)**.** *Choose $\Omega = \mathcal{C}_\mathcal{W}$. Let $\mathcal{F}$ be the Borel σ-algebra of $\mathcal{W}$, and let $\mathcal{F}_t$ be the σ-algebra generated by $\{\theta(s) : s \in [0, t]\}$. Then the triple $((\theta(t) : t \geq 0), \mathcal{F}_t, \mu_\mathcal{W})$ is a $\mathcal{W}$-Brownian motion. Conversely, if $((B(t) : t \geq 0), \mathcal{F}_t, \mathbb{P})$ is any $\mathcal{W}$ Brownian motion, then $\mathbb{P}[B(\cdot) \in \mathcal{C}_\mathcal{W}] = 1$ and $\mu_\mathcal{W}$ is the $\mathbb{P}$ distribution of $\omega \mapsto B(\cdot, \omega)$.*

2.3 Decomposition of Brownian motions

Stroock [35] connects $\mathcal{H}_\mu$-orthogonal decompositions to Brownian independence.

Theorem 2.5. *Suppose the Cameron-Martin space $\mathcal{H}_\mu$ admits an orthogonal decomposition into finite-dimensional subspaces $\mathcal{H}_\mu = \mathcal{H}_\mu^{(1)} \oplus \mathcal{H}_\mu^{(2)} \oplus \dots$. The corresponding $\mathcal{W}$-valued Brownian motion $\mathbf{B}$ decomposes as a tuple of independent Brownian motions $\mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2, \dots)$ with $\mathbf{B}_\ell$ living on the $\mathcal{W}$-closure of $\mathcal{H}_\mu^{(\ell)}$; the resulting sum converges not only in $L^2(\mathcal{W}, \mathbb{R})$ but also almost surely in $\mathcal{W}$.*

The proof of Theorem 2.3 uses a fundamental result:

Theorem 2.6 (Banach version of Marcinkiewicz' Theorem)**.** *Let X be a $\mathcal{W}$-valued random variable, measurable with respect to a σ-algebra $\mathcal{F}$. Suppose $X \in L^p(\mathcal{W}, \mathbb{R})$ for some $p \in [1, \infty)$. For any non-decreasing sequence $\{\mathcal{F}_n : n = 1, 2, \dots\}$ of sub-σ-algebras of $\mathcal{F}$,*

$$\lim_{n \to \infty} \mathbb{E}_\mu[X \mid \mathcal{F}_n] \quad = \quad \mathbb{E}_\mu\left[X \,\Big|\, \bigcup_{n=1}^{\infty} \mathcal{F}_n\right] \quad \text{both } \mu\text{–almost surely and also in } L^p(\mathcal{W}, \mathbb{R}).$$

Furthermore, if X is $\bigcup_{n=1}^{\infty} \mathcal{F}_n$-measurable then

$$\lim_{n \to \infty} \mathbb{E}_\mu[X \mid \mathcal{F}_n] \quad = \quad X \quad \text{both } \mu\text{–almost surely and also in } L^p(\mathcal{W}, \mathbb{R}).$$

A proof of Theorem 2.6 is given in [32, Theorems 1.14 and 1.30]; a proof of the original (non-Banach) Marcinkiewicz' Theorem is given in [34, Chapter 5].

We spell out the proof of Theorem 2.5 to help establish notation.

Proof of Theorem 2.5.
Step 1. Separability means that $\mathcal{W}^*$ generates the Borel σ-algebra of $\mathcal{W}$. The elements of $\mathcal{W}^*$ are continuous functions on $\mathcal{W}$, hence are measurable with respect to the Borel σ-algebra $\mathfrak{B}_\mathcal{W}$ of $\mathcal{W}$. Being separable, $\mathcal{W}$ is generated by a countable system of open balls; by separability, each of these balls can be expressed as the intersection of countably many half-spaces of $\mathcal{W}$. Hence $\mathfrak{B}_\mathcal{W}$ is the *smallest* σ-algebra making all maps $\lambda : \theta \mapsto \langle \theta, \lambda \rangle_{\mathcal{W}^*; \mathcal{W}} = \lambda(\theta)$ measurable for all $\lambda \in \mathcal{W}^*$.

Step 2. There is a natural isometry embedding $\mathcal{H}_\mu \to L^2(\mathcal{W}, \mu)$. From (2.3), for every $\lambda \in \mathcal{W}^*$ there is a $h_\lambda \in \mathcal{H}_\mu$ such that $\lambda(h) = \langle \lambda, h \rangle_{\mathcal{W}^*; \mathcal{W}} = \langle h, h_\lambda \rangle_{\mathcal{H}_\mu}$. Therefore, there is a natural isometric injection $\mathcal{I} : \mathcal{W}^* \to L^2(\mathcal{W}, \mu)$ given by $(\mathcal{I}h_\lambda)(\theta) = \langle \theta, \lambda \rangle_{\mathcal{W}, \mathcal{W}^*}$, defined for any $\lambda \in \mathcal{W}^*$, which extends to $\mathcal{I} : \mathcal{H}_\mu \to L^2(\mathcal{W}, \mu)$. Denoting by ι the dense embedding $\mathcal{W}^* \to \mathcal{H}_\mu$, $\|\iota(\lambda)\|_{\mathcal{H}_\mu} = \|\lambda\|_{L^2(\mathcal{W}, \mu)}$. The extension $\mathcal{I} : \mathcal{H}_\mu \to L^2(\mathcal{W}, \mu)$ is the *Paley–Wiener map*.

Step 3. The union of the Paley-Wiener maps of the summands is dense in the Paley-Wiener image of $\mathcal{H}_\mu$. Given the orthogonal decomposition

$$\mathcal{H}_\mu \quad = \quad \bigoplus_{\ell=1}^{\infty} \mathcal{H}_\mu^{(\ell)} \quad = \quad \mathcal{H}_\mu^{(1)} \oplus \mathcal{H}_\mu^{(2)} \oplus \dots, \tag{2.6}$$

let $h_1^{(\ell)}, h_2^{(\ell)}, \dots, h_{k_\ell}^{(\ell)}$ be an orthonormal basis of $\mathcal{H}_\mu^{(\ell)}$. By Steps 1 and 2, since the Paley-Wiener map is an isometry and each $\mathcal{H}_\mu^{(\ell)}$ is finite dimensional, the $\mathcal{I}h_1^{(\ell)}, \mathcal{I}h_2^{(\ell)}, \dots, \mathcal{I}h_{k_\ell}^{(\ell)}$

form an orthonormal basis for the image $\mathcal{I}\mathcal{H}_\mu{}^{(\ell)}$. By (2.6) the following linear span is $L^2(\mathcal{W}, \mu)$–dense in $\mathcal{I}\mathcal{H}_\mu$:

$$\bigcup_{\ell=1}^{\infty} \left\{ \mathcal{I}h_1^{(\ell)}, \mathcal{I}h_2^{(\ell)}, \ldots, \mathcal{I}h_{k_\ell}^{(\ell)} \right\}.$$

Step 4. We use the orthogonal decomposition (2.6) *to establish a filtration of σ-algebras over $\mathcal{W}$: $\mathcal{F}_1 \leq \mathcal{F}_2 \leq \ldots$ is a filtration of σ-algebras on $\mathcal{W}$ given by*

$$\mathcal{F}_n = \sigma \left\{ (\mathcal{I}h_1^{(1)}, \mathcal{I}h_2^{(1)}, \ldots, \mathcal{I}h_{k_1}^{(1)}), \ldots, (\mathcal{I}h_1^{(n)}, \mathcal{I}h_2^{(n)}, \ldots, \mathcal{I}h_{k_n}^{(n)}), \ldots \right\}.$$

Set $\mathcal{F} = \sigma \{\mathcal{F}_1, \mathcal{F}_2 \ldots\}$. By *Step 3*, all the $\mathcal{I}h_\lambda$ functions are measurable with respect to the μ-completion of $\mathcal{F}$. Moreover, by *Step 2*, $\mathcal{I}h_\lambda$ maps θ to $\langle \theta, \lambda \rangle_{\mathcal{W}, \mathcal{W}^*}$, thus $\mathcal{I}h_\lambda$ is also measurable with respect to $\mathfrak{B}_\mathcal{W}$. Furthermore, *Step 1* implies that $\mathfrak{B}_\mathcal{W}$ is the *smallest* σ-algebra with respect to which all such maps are measurable. This implies that $\mathfrak{B}_\mathcal{W}$ is contained in the μ-completion of $\mathcal{F}$.

Step 5. Compute the $\mathcal{F}_n$-conditional expectation of a μ-random choice of $\theta \in \mathcal{W}$. Set

$$S_n(\theta) = \sum_{\ell=1}^{n} \left(\sum_j h_j^{(\ell)} \times \left(\mathcal{I}h_j^{(\ell)} \right)(\theta) \right).$$

On each summand μ induces a Gaussian measure, so compute summand by summand to obtain $S_n(\theta) = \mathbb{E}_\mu[\theta \mid \mathcal{F}_n]$.

The proof of the theorem is completed by combining Theorem 2.6 (the Banach version of Marcinkiewicz' Theorem) with Steps 1–5: both μ–almost surely and in $L^2(\mathcal{W}, \mu)$,

$$\lim_n S_n(\theta) = \lim_n \mathbb{E}_\mu[\theta \mid \mathcal{F}_n] = \mathbb{E}_\mu[\theta \mid \mathcal{F}] = \theta. \qquad \square$$

Remark 2.7. The factors $\mathcal{H}_\mu{}^{(\ell)}$ can also be infinite-dimensional, since infinite-dimensional summands can be expressed as orthogonal direct sums of finite-dimensional subspaces.

3 Coupling of Brownian motion within finite time

Recall that $\mathbf{B}(0) = 0$ and $\widetilde{\mathbf{B}}(0) = x$ and denote by $\mathrm{dist}_{\mathrm{TV}}(\nu, \nu')$ the *total variation distance* between the probability measures ν and ν'. Aldous inequality (cf. [1]) implies

$$\mathbb{P}\left[\widetilde{\mathbf{B}} \text{ meets } \mathbf{B} \text{ by time } t \right] \leq 1 - \mathrm{dist}_{\mathrm{TV}}(\widetilde{\mathbf{B}}(t), \mathbf{B}(t)). \tag{3.1}$$

The Feldman-Hajek Theorem (Theorem 2.2 above), thus enforces a fundamental restriction on the possibility of coupling Brownian motions in Banach space: if $x \notin \mathcal{H}_\mu$ then the distributions of $\mathbf{B}(t)$ and $\widetilde{\mathbf{B}}(t)$ are mutually singular, and thus $\mathbb{P}[\widetilde{\mathbf{B}} \text{ meets } \mathbf{B} \text{ by time } t] = 0$, for all t and all possible couplings. On the other hand, if $x \in \mathcal{H}_\mu$ then there is a *maximal* successful coupling: maximal in the sense that (3.1) becomes an equality for all t.

3.1 Cameron-Martin Reflections in Banach spaces

Given $x \in \mathcal{H}_\mu$, construct the $\mathcal{H}_\mu$-projection $h \mapsto \langle x, h \rangle_{\mathcal{H}_\mu} x / \|x\|_{\mathcal{H}_\mu}^2$. Define the resulting *Cameron-Martin reflection* $R_x(y) : \mathcal{H}_\mu \to \mathcal{H}_\mu$ by

$$R_x(y) = y - 2 \frac{\langle x, y \rangle_{\mathcal{H}_\mu}}{\|x\|_{\mathcal{H}_\mu}^2} x \qquad \text{for } y \in \mathcal{H}_\mu. \tag{3.2}$$

Note that $R_x(x) = -x$. By Theorem 2.5, R_x produces a *reflected* Brownian motion $\widetilde{\mathbf{B}} = x + R_x(\mathbf{B})$: we avoid having to consider the extension of R_x to all of $\mathcal{W}$ by writing $\mathcal{H}_\mu = \mathrm{Ker}(\mathbb{I} - R_x) \oplus \mathrm{Ker}(\mathbb{I} + R_x)$, then applying Theorem 2.5 to decompose $\mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2)$ accordingly (so $\mathbf{B}_1$ is a $\mathrm{Ker}(\mathbb{I} - R_x)$-valued Brownian motion and $\mathbf{B}_2$ is a $\mathrm{Ker}(\mathbb{I} + R_x)$-valued Brownian motion – in fact $\mathbf{B}_1$ is essentially a one-dimensional Brownian motion $\langle x, B \rangle_{\mathcal{H}_\mu} x / \|x\|_{\mathcal{H}_\mu}^2$). Finally set $\widetilde{\mathbf{B}} = x + (\mathbf{B}_1, -\mathbf{B}_2) = x + \mathbf{B} - 2 \langle x, B \rangle_{\mathcal{H}_\mu} x / \|x\|_{\mathcal{H}_\mu}^2$.

Remark 3.1. Lindvall (cf. [27]) introduced *coupling by reflection* for real Brownian motion. [28] adapted it to couple finite-dimensional diffusions. Further generalizations include reflection coupling on Riemannian manifolds and beyond [19, 20, 21, 8, 37].

3.2 Coupling in finite time holds exactly if initial displacement is in $\mathcal{H}_\mu$

We now establish the first and simplest coupling result for Brownian motion in $\mathcal{W}$.

Theorem 3.2. *A $\mathcal{W}$-valued Brownian motion $\mathbf{B}$ with $\mathbf{B}(0) = 0$ can be coupled with another Brownian motion $\widetilde{\mathbf{B}}$ with $\widetilde{\mathbf{B}}(0) = x \neq 0$ if and only if $x \in \mathcal{H}_\mu$, where $\mu = \mathcal{L}(B(1))$ is the Gaussian distribution of the Banach-space-valued random variable $\mathbf{B}(1)$. Moreover, coupling can then succeed almost surely and there is even a maximal coupling time (produced by a Cameron-Martin reflection based on x) which has the distribution of the first time T at which $\langle x, \mathbf{B}(T) \rangle_{\mathcal{H}_\mu} = \frac{1}{2} \|x\|_{\mathcal{H}_\mu}^2$.*

Proof. One direction follows from inequality (3.1) and Theorem 2.2 as noted at the start of Section 3: if $x \notin \mathcal{H}_\mu$, then $\mathbb{P}\left[\mathbf{B} \text{ meets } \widetilde{\mathbf{B}} \text{ within finite time}\right] = 0$.

On the other hand, consider $x \in \mathcal{H}_\mu$. Given the Brownian motion $\mathbf{B}$, construct $\widetilde{\mathbf{B}}$ from $\mathbf{B}$ using the reflection R_x as described in Section 3.1. Thus, until $\widetilde{\mathbf{B}}$ and $\mathbf{B}$ coincide, let

$$\widetilde{\mathbf{B}}(t) \quad = \quad R_x(\mathbf{B}(t)) + x \quad = \quad x + \mathbf{B}(t) - 2 \frac{\langle x, \mathbf{B}(t) \rangle_{\mathcal{H}_\mu}}{\|x\|_{\mathcal{H}_\mu}^2} x. \tag{3.3}$$

Once $\widetilde{\mathbf{B}} = \mathbf{B}$, the two Brownian motions evolve as a single process. By Equation (3.3)

$$\mathbf{A}(t) \quad = \quad \widetilde{\mathbf{B}}(t) - \mathbf{B}(t) \quad = \quad \begin{cases} x - 2 \frac{\langle x, \mathbf{B}(t) \rangle_{\mathcal{H}_\mu}}{\|x\|_{\mathcal{H}_\mu}^2} x & \text{while } \widetilde{\mathbf{B}}(t) \neq \mathbf{B}(t), \\ 0 & \text{once } \widetilde{\mathbf{B}}(t) = \mathbf{B}(t), \end{cases}$$

belongs to the Cameron-Martin space $\mathcal{H}_\mu$, since it is a scalar multiple of $x \in \mathcal{H}_\mu$. Hence,

$$\mathbb{P}\left[\widetilde{\mathbf{B}} \text{ meets } \mathbf{B} \text{ by time } T\right] \quad = $$
$$\mathbb{P}\left[\mathbf{A}(s) = 0 \text{ for some } s \leq T\right] \quad = \quad \mathbb{P}\left[\|\mathbf{A}(s)\|_{\mathcal{H}_\mu} = 0, \text{ for some } s \leq T\right].$$

Now $\mathbf{A}$ is a difference of a Banach-valued Brownian motion and its reflection, stopped once they agree. Therefore its absolute value is a scalar Brownian motion of rate 4, begun at $\|x\|_{\mathcal{H}_\mu}$, stopped on reaching 0:

$$\|\mathbf{A}(t)\|_{\mathcal{H}_\mu} \quad = \quad \begin{cases} \left(1 - 2 \frac{\langle x, \mathbf{B}(t) \rangle_{\mathcal{H}_\mu}}{\|x\|_{\mathcal{H}_\mu}^2}\right) \|x\|_{\mathcal{H}_\mu} & \text{while } \widetilde{\mathbf{B}}(t) \neq \mathbf{B}(t), \\ 0 & \text{once } \widetilde{\mathbf{B}}(t) = \mathbf{B}(t). \end{cases}$$

By the Reflection Principle, the probability of coupling by time t is exactly the total mass of the join of probability distributions as given in Theorem 2.2. Hence this is indeed a maximal coupling. $\qquad \square$

Remark 3.3. As $\mathcal{H}_\mu$ is dense in $\mathcal{W}$, we can produce ε-approximate couplings of Brownian motions begun at any two different starting points in $\mathcal{W}$. Such ε-approximate couplings are reminiscent of the Wasserstein variants of CFTP algorithms introduced in [14] for image restoration, and may be of use in future applications.

4 Coupling at time ∞ and Schauder basis properties

Section 3 suggests the following natural question. When will there exist a coupling between $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ that allows the two Brownian motions almost surely to "*couple at time ∞*", whatever the starting points $\mathbf{B}(0), \widetilde{\mathbf{B}}(0) \in \mathcal{W}$? In other words, when can

$$\mathbb{P}\left[\lim_{t \to \infty} \left\|\widetilde{\mathbf{B}}(t) - \mathbf{B}(t)\right\|_{\mathcal{W}} = 0\right] \quad = \quad 1\,.$$

This question links to the concept of *Schauder bases* for Banach spaces.

4.1 Schauder basis

A Schauder basis for $\mathcal{W}$ is a sequence $(e_k : k = 1, 2, \ldots)$ drawn from a Banach space $\mathcal{W}$, such that each $x \in \mathcal{W}$ admits a *unique* decomposition as the conditionally convergent sum $x = \sum_{k=1}^{\infty} \alpha_k e_k$ for some coefficients $\alpha_k \in \mathbb{R}$ depending on x. In fact [3, pages 110-112] observed that α_k depends continuously (and linearly) on x (see also [29, page 878]). Thus $\alpha_k = \langle e_k^*, x \rangle_{\mathcal{W}^*;\mathcal{W}}$, where each $e_k^* \in \mathcal{W}^*$ depends on the entire Schauder basis, with the following holding as a conditionally $\mathcal{W}$-convergent sum

$$x \quad = \quad \sum_{k=1}^{\infty} \langle e_k^*, x \rangle_{\mathcal{W}^*;\mathcal{W}} \; e_k \,. \tag{4.1}$$

Remark 4.1. The convergence in Equation (4.1) is *conditional* (it depends on the order of the sequence of basis vectors e_n) and must be interpreted using the $\mathcal{W}$-norm topology:

$$\lim_{K \to \infty} \left\| w - \sum_{k=0}^{K} \langle e_k^*, x \rangle_{\mathcal{W}^*;\mathcal{W}} \; e_k \right\|_{\mathcal{W}} = 0\,.$$

A Banach space with a Schauder basis is necessarily separable, but the converse is not true (see the celebrated counterexample of [12]). Separable Banach spaces can admit other types of bases such as the *Markushevich basis* (*M-basis* for short), the *Auerbach basis*, or simply a *finite-dimensional decomposition* ([18, Chapter 1], [7]). M-bases will appear in Section 5, but we will not discuss the other notions here.

4.2 Brownian motions coupled at time ∞

In the following, $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ denote coupled $\mathcal{W}$-valued Brownian motions started at 0 and $x \in \mathcal{W}$ respectively. If $x \in \mathcal{W} \setminus \mathcal{H}_\mu$ then reflection coupling is not well defined!

Theorem 4.2. *Let $\mathcal{W}$ be a separable Banach space possessing a Schauder basis $(e_k : k = 1, 2, \ldots)$ which also forms an orthonormal basis for $\mathcal{H}_\mu$. Then it is possible to construct a coupling at time ∞ of Brownian motions $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ started at any two given starting points, so*

$$\mathbb{P}\left[\left\|\mathbf{B}(t) - \widetilde{\mathbf{B}}(t)\right\|_{\mathcal{W}} \quad \to \quad 0 \text{ as } t \to \infty\right] \quad = \quad 1\,.$$

Remark 4.3. Theorem 4.2 holds for all possible $x \in \mathcal{W}$, though construction details depend on the Schauder expansion (4.1) of x. If $x \notin \mathcal{H}_\mu$ then the coupling is actually maximal, albeit only in a degenerate sense, since the distributions of $\mathbf{B}(t)$ and $\widetilde{\mathbf{B}}(t)$ in (3.1) are then mutually singular for all time $t > 0$.

Proof of Theorem 4.2. Suppose that $x \in \mathcal{W}$ is the initial displacement of $\widetilde{\mathbf{B}}$ relative to $\mathbf{B}$. Because the Schauder basis is orthogonal in $\mathcal{H}_\mu$, there exist $\mathcal{H}_\mu$-orthogonal x_1, x_2, ... with

$$\|x - (x_1 + \ldots + x_n)\|_{\mathcal{W}} \quad \leq \quad 2^{-n-1} \qquad \text{for } n \geq 1\,.$$

It suffices to take $x_n = \sum_{k=r_{n-1}}^{r_n-1} \langle e_k^*, x \rangle_{\mathcal{W}^*;\mathcal{W}} e_k \in \mathcal{H}_\mu$ for a sufficiently rapidly increasing sequence $r_0 = 1 < r_1 < r_2 < \ldots$. By the triangle inequality, for $n \geq 1$,

$$
\begin{aligned}
\|x_{n+1}\|_{\mathcal{W}} &\leq \|x - (x_1 + \ldots + x_n)\|_{\mathcal{W}} + \|(x_1 + \ldots + x_n + x_{n+1}) - x\|_{\mathcal{W}} \\
&\leq 2^{-n-1} + 2^{-n} < 2^{-n+1}. \quad (4.2)
\end{aligned}
$$

The $\mathcal{H}_\mu$-orthogonal finite-dimensional projections $\Pi_n : z \mapsto \sum_{k=r_{n-1}}^{r_n-1} \langle e_k^*, z \rangle_{\mathcal{H}_\mu} e_k$ decompose $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ following Theorem 2.5:

$$
\mathbf{B}(t) = \Pi_1 \mathbf{B}(t) + \Pi_2 \mathbf{B}(t) + \ldots \quad \text{and} \quad \widetilde{\mathbf{B}}(t) = \Pi_1 \widetilde{\mathbf{B}}(t) + \Pi_2 \widetilde{\mathbf{B}}(t) + \ldots .
$$

Thus $\mathbf{B}_n$ is Brownian motion on the finite-dimensional subspace spanned by the elements $e_{r_{n-1}}, \ldots, e_{r_n-1}$. Up to the time of coupling of $\mathbf{B}_n$ and $\widetilde{\mathbf{B}}_n$, construct $\widetilde{\mathbf{B}}_n = R_{x_n} \mathbf{B}_n + x_n$ to be the reflection of B_n started at x_n using the reflection map R_{x_n}.

Let T_n be the time of coupling of $\mathbf{B}_n$ and $\widetilde{\mathbf{B}}_n$; Theorem 3.2 (really the elementary theory of finite-dimensional reflection coupling) implies that T_n is distributed as the first time for scalar Brownian motion to hit 0 when run at rate 1 and started at $\frac{1}{2} \|x_n\|_{\mathcal{H}_\mu}$.

Because real Brownian motion is a continuous martingale it follows that

$$
\mathbb{P}\left[\sup \left\{ \left\| \widetilde{\mathbf{B}}_n(t) - \mathbf{B}_n(t) \right\|_{\mathcal{W}} : 0 \leq t \leq T_n \right\} \geq \frac{1}{2} n(n+1) \|x_n\|_{\mathcal{H}_\mu} \right] = \frac{1}{n(n+1)}. \quad (4.3)
$$

By subadditivity of probability and the bound (4.2),

$$
\mathbb{P}\left[\sup \left\{ \left\| \widetilde{\mathbf{B}}_n(t) - \mathbf{B}_n(t) \right\|_{\mathcal{W}} : 0 \leq t \leq T_n \right\} \geq n(n+1)2^{-n-1} \text{ for some } n \geq N \right]
$$

$$
\leq \sum_{n=N}^{\infty} \frac{1}{n(n+1)} = \frac{1}{N}. \quad (4.4)
$$

For times $t > T_1 \vee \ldots \vee T_{N-1}$, coupling means that $\widetilde{\mathbf{B}}_1(t) = \mathbf{B}_1(t), \ldots, \widetilde{\mathbf{B}}_{N-1}(t) = \mathbf{B}_{N-1}(t)$. Using the triangle inequality, (4.4) implies that, with probability at least $1 - 1/N$,

$$
\sup \left\{ \left\| \widetilde{\mathbf{B}}(t) - \mathbf{B}(t) \right\|_{\mathcal{W}} : t > T_1 \vee \ldots \vee T_{N-1} \right\} \leq
$$

$$
\sum_{n=N}^{\infty} \sup \left\{ \left\| \widetilde{\mathbf{B}}_n(t) - \mathbf{B}_n(t) \right\|_{\mathcal{W}} : t > T_1 \vee \ldots \vee T_{N-1} \right\} \leq \sum_{n=N}^{\infty} n(n+1)2^{-n-1}. \quad (4.5)
$$

Thus, by the first Borel-Cantelli lemma, $\|\widetilde{\mathbf{B}}(t) - \mathbf{B}(t)\|_{\mathcal{W}} \to 0$ almost surely as $t \to \infty$. $\square$

As a corollary, the case when $\mathcal{W}$ is a Hilbert space is now completely settled.

Corollary 4.4. *If $\mathcal{W}$ is a separable Hilbert space, then any $\mathcal{W}$-valued Brownian motion corresponding to a Cameron-Martin space $\mathcal{H}_\mu \hookrightarrow \mathcal{W}$ can be coupled at time ∞ from any two starting points in $\mathcal{W}$.*

Proof. Recall the continuous injection $A : \mathcal{H}_\mu \hookrightarrow \mathcal{W}$. By Prokorov's characterization of Gaussian measures on the Hilbert space $\mathcal{W}$ (see [26, Theorem 2.3 and following remark]), A^*A is positive definite (n.b. A is injective) and trace-class. Its spectral decomposition therefore can be expressed in terms of finite-dimensional spaces, and for any non-zero eigenvalue λ, the renormalized operator A/λ restricted to $\text{Ker}(A^*A - \lambda^2)$ is an isometry. Let $v_1, v_2, \ldots$ be an orthonormal basis of $\mathcal{H}_\mu$ using eigenvectors of A^*A: by an eigenvalue argument, $Av_1, Av_2, \ldots$ are $\mathcal{W}$-orthogonal. By injectivity of A and its dense image, $Av_1, Av_2, \ldots$ form a $\mathcal{W}$-orthogonal basis, hence a Schauder basis. $\square$

For more general Banach spaces, the methods of Theorem 4.2 immediately supply an approach which works at least for some initial displacements $x \in \mathcal{W} \setminus \mathcal{H}_\mu$.

Corollary 4.5. *Given two initial starting points $x \neq y$ in $\mathcal{W}$, suppose it is possible to find orthogonal z_1, z_2 ,... in $\mathcal{H}_\mu$ such that $x - y = z_1 + z_2 + \ldots$, with the sum converging conditionally in $\mathcal{W}$. Then we can construct a coupled pair of Banach-valued Brownian motions $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ starting from x and y which almost surely couple at time ∞.*

Remark 4.6. In [26, Corollary 4.2], Kuo observes that one can always construct Banach spaces $\mathcal{W}_0$ with $\mathcal{H}_\mu \hookrightarrow \mathcal{W}_0 \hookrightarrow \mathcal{W}$ and $\mathcal{W}_0$ strictly containing $\mathcal{H}_\mu$, such that there is an orthonormal basis of $\mathcal{H}_\mu$ which forms a Schauder basis for $\mathcal{W}_0$.

5 Conclusion

In conclusion we discuss an open question, and indicate further lines of research.

5.1 An open question

This paper has shown that Brownian couplings at time ∞ are always possible if $\mathcal{W}$ supports a Schauder basis which is also $\mathcal{H}_\mu$-orthogonal (Theorem 4.2), and in particular that they are always possible in the important special case when $\mathcal{W}$ is a Hilbert space (Lemma 4.4). It is therefore natural to ask

Question 5.1. Given an abstract Wiener space $\mathcal{W}^* \hookrightarrow \mathcal{H}_\mu \hookrightarrow \mathcal{W}$, is it always possible to produce a coupling at time ∞ for two $\mathcal{W}$-valued Brownian motions started from arbitrary starting points in $\mathcal{W}$? (Or if not, then what Banach-space geometry property for $\mathcal{W}^* \hookrightarrow \mathcal{H}_\mu \hookrightarrow \mathcal{W}$ corresponds to the probabilistic property that Brownian coupling at time ∞ is always possible?)

Certainly there are abstract Wiener spaces possessing the property of Brownian coupling at time ∞ which do not possess a $\mathcal{H}_\mu$-orthogonal Schauder basis. Recall that $\mathcal{W}$ has the finite dimensional decomposition property (FDD) if it supports finite-dimensional subspaces $\mathcal{W}^{(n)}$ such that $x = \sum_{n=1}^\infty x_n$ (conditionally convergent) for unique $x_n \in \mathcal{W}^{(n)}$: to be compatible with the Cameron-Martin geometry, we must further require that the $\mathcal{W}^{(n)}$ form an orthogonal decomposition of $\mathcal{H}_\mu$. Note there exist separable Banach spaces $\mathcal{W}$ with FDD but without a Schauder basis [7]. The proof of Theorem 4.2 shows that Brownian couplings at time ∞ are always possible when $\mathcal{W}$ has an FDD compatible with the Cameron-Martin geometry.

Adaptation of work of [36] permits a modest advance on the results described above. We outline the argument briefly.

Recall that an M-basis is a particular kind of biorthogonal system (a sequence of pairs $(x_n, x_n^*)_n \in \mathcal{W} \times \mathcal{W}^*$ is said to be a *bi-orthogonal system* whenever $x_n^*(x_m) = \delta_{n,m}$):

Definition 5.2 (Markushevich basis)**.** *A bi-orthogonal system that is*

1. *fundamental ($\overline{\mathrm{span}}^{\|\cdot\|_\mathcal{W}}\{x_n\}_{n\in\mathbb{N}} = \mathcal{W}$)*
2. *and total ($\overline{\mathrm{span}}^{w^*}\{x_n^*\}_{n\in\mathbb{N}} = \mathcal{W}^*$)*

is a Markushevich basis (M-basis). (Here $\overline{\mathrm{span}}^{w^*}$ is closure of linear span under weak* topology.)

The modest advance is as follows: Brownian coupling at time ∞ is possible if $\mathcal{W}$ supports a *norming* M-basis $(x_n, x_n^*)_n \in \mathcal{W} \times \mathcal{W}^*$, $n = 1, 2, \ldots$, with $\mathcal{H}_\mu$-orthogonal x_n: "norming" means that

$$\|x\| \;=\; \sup\{\langle x, x^* \rangle \;:\; x^* \in (\overline{\mathrm{span}}^{\|\cdot\|_\mathcal{W}}\{x_n^*\}_n) \cap B_{(\mathcal{W}^*, \|\cdot\|_{\mathcal{W}^*})}\}$$

defines a norm on $\mathcal{W}$ such that $\lambda \|x\|_\mathcal{W} \leq \|x\| \leq \|x\|_\mathcal{W}$ for some $0 < \lambda \leq 1$. Here $B_{(\mathcal{W}^*, \|\cdot\|_{\mathcal{W}^*})}$ denotes the unit ball in the Banach space $\mathcal{W}^*$. Note that the argument used

in the proof of Theorem 4.2 will apply if any $x \in \mathcal{W}$ can be represented as the limit of partial sums of orthogonal elements of $\mathcal{H}_\mu$, so this is the objective. [36] establishes this representation in terms of existence of an M-basis for *any* separable Banach space, but without imposing any requirement of $\mathcal{H}_\mu$-orthogonality. We now describe the steps required to adapt [36] to show that the argument can be sufficiently related to the underlying Cameron-Martin geometry to capture enough $\mathcal{H}_\mu$-orthogonality to obtain the required representation.

Step 1: Assume existence of a *norming* M-basis whose elements are orthogonal in $\mathcal{H}_\mu$.

Step 2: Further adjust the norming M-basis to be *bounded* (so that there is a constant $0 < M < \infty$ such that $\sup_n \{\|x_n\|_\mathcal{W} \cdot \|x_n^*\|_{\mathcal{W}^*}\} \leq M$). This uses the approach of [30]: the M-basis is adjusted using Haar unitary matrix transformations on disjoint finite-dimensional subspaces, hence retaining $\mathcal{H}_\mu$-orthogonality while preserving the disjoint subspace decomposition.

Step 3: Now note [18, Lemma 1.40]:

Lemma 5.3. *If $\{x_n, x_n^*\}_n$ is a bounded norming M-basis for $\mathcal{W}$ then there are integers $r(1) < r(2) < ...$ such that for any $x \in \mathcal{W}$ and every $m \in \mathbb{N}$ there is an element v_m in $span\{x_n\}_{n=r(m)+1}^{r(m+1)}$ with*

$$ x \quad = \quad \lim_{m \to \infty} \Big[\sum_{n=1}^{r(m)} \langle x, x_n^* \rangle x_n + v_m \Big]. $$

Remark 5.4. Note that by a result of Fonf (cf. [13]), the above property holds if and only if the M-basis is norming; cf. also [36].

If (selecting a subsequence depending on x if necessary) we could contrive that $v_m \to 0$ in $\mathcal{W}$ for the M-basis obtained in *Steps 1* and *2*, then we would have obtained the required expression of x as the limit of partial sums of orthogonal elements of $\mathcal{H}_\mu$.

Step 4: We seek a block perturbation providing the subsequence required in *Step 3*. A *block perturbation* $\{y_n, y_n^*\}_n$ of $\{x_n, x_n^*\}_n$ amounts (for our purposes) to finding an increasing sequence $\{q_m\}_{m \in \mathbb{N}}$ of positive integers such that for every m

$$\text{span}\{y_n\}_{n=q_m+1}^{q_{m+1}} = \text{span}\{x_n\}_{n=q_m+1}^{q_{m+1}} \quad \text{and} \quad \text{span}\{y_n^*\}_{n=q_m+1}^{q_{m+1}} = \text{span}\{x_n^*\}_{n=q_m+1}^{q_{m+1}}.$$

If it is not possible to find a block perturbation leading to $v_m \to 0$ in $\mathcal{W}$ as described in *Step 3*, then the [36] argument uses further careful constructions of block perturbations of the M-basis, together with the [10] Hilbert space approximation Theorem, to produce a new norming M-basis (depending on x) which can be used to generate successive partial sums which approximate x in $\mathcal{W}$-norm. Because block perturbations are used, these successive partial sums can be expressed in terms of a sequence of orthogonal elements of $\mathcal{H}_\mu$ (depending on x), and so the objective is attained.

We end this section with a couple of remarks about the last proof.

Remark 5.5. We have assumed that we can pick an M-basis for $\mathcal{W}$ whose elements are orthogonal in $\mathcal{H}_\mu$. At present we are not aware of useful sufficient conditions to guarantee this. Indeed we do not know if it is possible *always* to find a $\mathcal{H}_\mu$-orthogonal M-basis for $\mathcal{W}$, though this is implausible.

Remark 5.6. Note that the orthogonal M-basis needs to be *norming*. The norming property is guaranteed when $\mathcal{W}$ is quasi-reflexive (i.e., the canonical image of $\mathcal{W}$ in its second dual has finite co-dimension), following a result of [31] (cf. also [33]). However (by [9]) if $\mathcal{W}$ is not quasi-reflexive then it is always possible to find a total subspace of $\mathcal{W}^*$ that is not norming in $\mathcal{W}$. Hence the norming assumption is necessary when dealing with general separable Banach spaces.

5.2 Further work

Apart from addressing the open question 5.1, one might also ask whether one could additionally couple functionals of the two Banach-valued Brownian motions, such as for example their Lévy stochastic areas. This can be done in the finite-dimensional case [5, 22, 23]. However Lévy stochastic areas are quadratic objects, so it seems likely that only rather limited results can be obtained in infinite-dimensional cases.

[4] have shown that geometric criteria tightly constrain Markovian maximal couplings for (finite-dimensional) smooth elliptic diffusions: in dimension 2 and above, the existence of a Markovian maximal coupling forces the diffusion to be Brownian motion on a simply-connected space of constant curvature, with drift given by a combination of a Killing vectorfield and (in the Euclidean case) a vectorfield related to scaling symmetry. It is evident from Theorem 3.2 that the notion of maximality does not extend usefully to the Banach-space case; but is there a weaker form of optimality which also imposes special requirements in some underlying geometry?

Turning to potential applications, it is natural to speculate about applications of these coupling constructions to multi-scale problems. For example multiresolution image analysis models an image as an infinite hierarchy of features of progressively finer resolution: there are interesting phase transition phenomena linked to image analysis issues (see [25]), while a Coupling-from-the-Past algorithm has been developed for a point process example (see [2]). The "coupling at time ∞" constructions of Theorem 4.2 are suggestive for such problems. Finally, Hairer and others (for example, [15, 16]) have discussed applications of rather specific couplings to certain SPDE. It would be interesting to relate the abstract considerations of the present paper to this applied context.

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