D-MODULES AND PROJECTIVE STACKS

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Abstract. We study twisted D-modules on weighted projective stacks. We determine for which values of the twist and the weight the global sections functor is an equivalence, thus, proving a version of Beilinson-Bernstein Localisation Theorem.

A key observation in the proof of Kazhdan-Lusztig Conjecture by Beilinson and Bernstein is that the (generalised) flag varieties $G/P$ are D-affine. This is known as Beilinson-Bernstein Localisation Theorem. So far these are the only known connected smooth projective D-affine varieties. In particular, Thomsen proves that a toric smooth projective D-affine variety must be a product of projective spaces [15]. On the other hand, Van den Bergh proves that weighted projective spaces are D-affine (they are singular) [16].

The goal of this paper is to re-examine the D-affinity of weighted projective spaces. Instead of looking at them as singular varieties, we consider them as stacks. We give a necessary and sufficient criterion for a weighted projective stack to be D-affine. Our method of proof is also different: Van den Bergh uses Hodges-Smith Criterion for D-affinity [11], while we do a direct calculation.

In section 1 we make general observations about D-affinity on varieties. In section 2 we establish a technical framework for working with twisted D-modules on a smooth projective stack. In section 3 we use this framework to study D-modules on weighted projective stacks.

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1. D-modules on varieties

We work with a connected algebraic variety $X$ over an algebraically closed field $\mathbb{K}$ of characteristics zero in this section. Let $\mathcal{O}_X$ be its sheaf of functions, $\mathcal{D}_X$ its sheaf of differential operators, $D(X) = \mathcal{D}_X(X)$ its global sections. We consider the category of quasicoherent $\mathcal{D}_X$-modules $\mathcal{D}_X - \text{Qcoh}$ and the category of modules over the globally defined differential operators $D(X) - \text{Mod}$. They are connected by the global sections functor

$$\Gamma : \mathcal{D}_X - \text{Qcoh} \rightarrow D(X) - \text{Mod}.$$  

$X$ is called D-affine if $\Gamma$ is an equivalence. Affine varieties are D-affine but the converse statement is not true: the generalised flag variety $G/P$ is a smooth projective D-affine variety [4]. In the light of this result, it is interesting to pose the following question.

**Question:** Classify connected smooth projective D-affine varieties.

It would be interesting to find other examples of such varieties besides $G/P$. Notice that any such example $X$ must have zero Hodge numbers $h^{0,m}(X)$ for $m > 0$ because $\mathcal{O}_X$ is a $\mathcal{D}_X$-module, hence, has no higher cohomology. A glimmering hope for settling this question is the result of Thomsen who classified smooth toric D-affine varieties [15]. Hereby we will explain that some other classes of varieties will not give new examples.

Recall that a variety $X$ is homogeneous if a connected algebraic (not necessarily linear) group $G$ acts transitively on $X$. For a complete variety $X$ it is equivalent to asking that the automorphism group of $X$ acts transitively on $X$ [13]. Such $X$ is necessarily smooth.

**Theorem 1.** Suppose $X$ is a homogeneous complete D-affine variety. Then $X$ is isomorphic to a generalised flag variety.

**Proof.** By Borel-Remmert Theorem [13] $X$ is a product of a partial flag variety and an abelian variety $A$. It remains to notice that $A$ is not D-affine because $R^{\dim A} \Gamma (A, \mathcal{O}_A) \neq 0$ by Serre’s duality, unless $A$ is a point. This would imply that $R^{\dim A} \Gamma (X, \mathcal{O}_X) \neq 0$ that is impossible because $\mathcal{O}_X$ is a $\mathcal{D}_X$-module. Thus, $A$ is a point and $X$ is a generalised flag variety. $\square$

If $\mathbb{K} = \mathbb{C}$ is the field of complex numbers, this result can be slightly improved.

**Theorem 2.** Suppose $X$ is a complex complete D-affine variety and the tangent sheaf $\mathcal{T}_X$ is generated by global sections. Then $X$ is isomorphic to a generalised flag variety.
Proof. Since $X$ is a complete algebraic variety, the global (algebraic) vector fields $g = \Gamma(T_X)$ form a finite dimensional Lie algebra. Let $G$ be an analytic connected simply-connected Lie group with Lie algebra $g$. The group $G$ locally acts on $X$ by the second Lie Theorem. Since $X$ is compact, each element $a \in g$ defines a one-parameter group $\gamma_a(t)$ of (global) diffeomorphisms of $X$. Choosing a real basis $a_1, \ldots, a_k$ of $g$, we can extend the assignment $\text{Exp}_G(t_1a_1) \cdot \text{Exp}_G(t_2a_2) \cdot \ldots \cdot \text{Exp}_G(t_ka_k) \mapsto \gamma_{a_1}(t_1)\gamma_{a_2}(t_2)\ldots\gamma_{a_k}(t_k)$ to a global (real) analytic action of $G$ on $X$.

Since $T_X$ is generated by global sections, each point $x \in X$ lies in the interior of its orbit $G \cdot x$. Hence each point belongs to an open set, entirely within this point’s orbit. By connectedness there is only one orbit, hence, $X \cong G/H$ as analytic manifolds.

By Borel-Remmert Theorem, there exists an abelian variety $A$ such that $X$ is an $A$-fibration over a generalised flag variety $Y$. If $A$ is a point, we are done. If $A$ is not a point, $\text{R}^{\text{dim}A}(A, \mathcal{O}_A) \neq 0$ by Serre’s duality. Thus, the derived push-forward $\text{R}(X \to Y)_*(\mathcal{O}_X)$ has higher cohomology and so does $\mathcal{O}_X$. This is a contradiction.

Observe that $T_X$ is not usually a $\mathcal{D}_X$-module. This would require a flat connection on $T_X$ which is quite rare. For instance, abelian varieties admit a flat connection on $T_X$ as well as any other variety with a trivial tangent sheaf. On the other hand, the only generalised flag variety with a flat connection on $T_X$ is a point.

Corollary 3. If $X$ is complex complete $D$-affine variety and $T_X$ is a $\mathcal{D}_X$-module, then $X$ is the point.

It would be interesting to extend Theorem and Corollary 3 to varieties over an arbitrary algebraically closed field $\mathbb{K}$. Our proof does not work because we use analytic methods.

2. D-modules on smooth projective stacks

The theory of D-modules on stacks is known. Let $Y$ be a smooth algebraic variety with an action of an algebraic group $G$. The quotient stack $[X] = [Y/G]$ admits the standard smooth atlas $G \times Y \to Y$ with the action and projection maps. This atlas extends to a simplicial variety $\mathcal{X}$ where $\mathcal{X}_n = G^n \times Y$, connected by the maps

$$\mathcal{X}(\varphi) : \mathcal{X}_n \to \mathcal{X}_m, \quad \mathcal{X}(\varphi)(g_1, \ldots, g_n, y) = (h_1, \ldots, h_m, h_{m+1} \cdot y)$$
where (with empty products equal to $1_G$)

$$h_i = \prod_{j=\varphi(i)+1} \varphi(i) g_j, \quad h_{m+1} = \prod_{j=\varphi(m)+1} g_j$$

for any non-decreasing function $\varphi : [m] \to [n] = \{0, 1, \ldots, n\}$. For instance, these are the maps for the low dimensional faces (recall that $\partial^m_n : [n-1] \to [n]$ is the increasing map without $i$ in the image):

$$X(\partial^2_2)(g_1, g_2, y) = (g_1, g_2 \cdot y), \quad X(\partial^2_1)(g_1, g_2, y) = (g_1 g_2, y),$$

$$X(\partial^2_0)(g_1, g_2, y) = (g_2, y), \quad X(\partial^1_1)(g, y) = g \cdot y, \quad X(\partial^1_0)(g, y) = y.$$

The category of quasicoherent D-modules on $[X]$ is equivalent to the category of cosimplicial D-modules on $X$ [7, 6.2.2]. Recall that a cosimplicial D-module $V$ consists of a quasicoherent D-module $V_n$ on each $X_n$ together with an isomorphism of D-modules $V(\varphi) : X(\varphi)^* V_m \to V_n$ for any non-decreasing function $\varphi : [m] \to [n]$ such that the simplicial identities hold.

A cosimplicial D-module $V$ can be recovered (up to an isomorphism) from the D-module $V_0$ and the D-module isomorphism $\gamma : p^* V_0 = X(\partial^1_1)^* V_0 \overset{\varphi(\partial^1_0)}{\longrightarrow} V_1 \overset{\varphi(\partial^1_0)^{-1}}{\longrightarrow} X(\partial^1_0)^* V_0 = a^* V_0$.

The simplicial identities in dimension two force the cocycle condition on the isomorphism $\gamma$, coercing $(V_0, \gamma)$ into a strongly equivariant D-module on $Y$. Vice versa, a strongly equivariant D-module on $Y$ can be extended to a cosimplicial D-module on $X$. This shows that the category of quasicoherent D-modules on $[X]$ is equivalent to the category of strongly equivariant quasicoherent D-modules on $Y$.

Further significant clarification is possible. Consider a $D_Y$-module $M$ with a compatible $G$-action, i.e., $g(dm) = g \cdot d \cdot g m$ for all $g \in G, \ d \in D, \ m \in M$. This is sometimes called a weakly equivariant D-module. Such a $G$-action yields an isomorphism of $O_G \otimes D_Y$-modules $\gamma : p^* M \to a^* M$ [10].

The Lie algebra $\mathfrak{g}$ of $G$ acts on $M$ in two ways: via the differential of the action $\mathfrak{g} \to D_Y$ and via the differential of the $G$-action. These two actions coincide if and only if $\gamma : p^* M \to a^* M$ is an isomorphism of $D_G \otimes D_Y$-modules (note that $D_G \otimes D_Y \cong D_{G \times Y}$) [10]. This gives an alternative definition of a strongly equivariant D-module.

The preceding discussion enables us (modulo equivalences of categories) to define a quasicoherent $D_{[X]}$-module as a quasicoherent strongly $G$-equivariant $D_Y$-module.

There are different notions of a projective stack, for instance, a stack whose coarse moduli space is a projective variety. Here we use a more
restrictive notion: a projective stack is a smooth closed substack of a weighted projective stack \([17]\). Let us spell it out. Let \(V = \bigoplus V_k\) be a positively graded \(n + 1\)-dimensional \(K\)-vector space. Naturally we treat it as a \(\mathbb{G}_m\)-module with positive weights by \(\lambda \cdot v_k = \lambda^k v_k\) where \(v_k \in V_k\). Let \(Y\) be a smooth closed \(\mathbb{G}_m\)-invariant subvariety of \(V \setminus \{0\}\). We define a projective stack as the stack \([X] = [Y/\mathbb{G}_m]\). The G.I.T.-quotient \(X = Y/\mathbb{G}_m\) is the coarse moduli space of \([X]\).

Let us describe the category \(\mathcal{O}_{[X]} - \text{Qcoh}\) of quasicoherent sheaves on \([X]\). Choose a homogeneous basis \(e_i\) on \(V\) with \(e_i \in V_{d_i}, i = 0, 1, \ldots, n\). Let \(x_i \in V^*\) be the dual basis. Then \(K[V] = K[x_0, \ldots, x_n]\) possesses a natural grading with \(\deg(x_i) = d_i\). Let \(I\) be the defining ideal of \(Y\).

Let \(\pi_A : \mathbb{A} - \text{Grmod} \to \mathbb{A} - \text{Tors}\) be the quotient functor. Since \(\mathbb{A} - \text{Grmod}\) has enough injectives and \(\mathbb{A} - \text{Tors}\) is dense then there exists a section functor

\[\omega_A : \mathbb{A} - \text{Grmod}/\mathbb{A} - \text{Tors} \to \mathbb{A} - \text{Grmod}\]

which is right adjoint to \(\pi_A\) in the sense that

\[\text{Hom}_{\mathbb{A} - \text{Grmod}}(N, \omega_A(M)) \cong \text{Hom}_{\mathbb{A} - \text{Grmod}/\mathbb{A} - \text{Tors}}(\pi_A(N), M)\].

Recall that \(\pi_A\) is exact, \(\omega_A\) is left exact and \(\pi_A \omega_A \cong \text{Id}_{\mathbb{A} - \text{Grmod}/\mathbb{A} - \text{Tors}}\). We call \(\omega_A \pi_A(M)\) the \(\mathbb{A}\)-saturation of \(M\). We say that a module is \(\mathbb{A}\)-saturated if it is isomorphic to the saturation of a module. It can be seen from the adjunction that an \(\mathbb{A}\)-saturated module is torsion-free and is isomorphic to its own saturation. If \(M\) and \(N\) are \(\mathbb{A}\)-saturated,
then being isomorphic in $A$–Grmod/$A$–Tors is equivalent to being isomorphic in $A$–Grmod.

We need a description of the global sections functor on $[X]$ in these terms:

$$\Gamma : \mathcal{O}_{[X]}^{-\text{Qcoh}} \to \text{VS}_K, \quad \Gamma(M) = \omega_A(M)_0.$$ 

In particular, if $M$ is an $A$-saturated module then

$$\Gamma(\pi_A(M)) = M_0.$$ 

The sheaf $\mathcal{O}_{[X]}(k)$ is defined as $\pi_A(A[k])$ where $A[k]$ is the shifted regular module and the grading is given by $A[k]_m = A_{k+m}$.

In particular, $\Gamma(\mathcal{O}_{[X]}(k)) = A_k$ if $A[k]$ is $A$-saturated which is the case for polynomial rings of more than two variables [2]. A well-known example of a ring, not $A$-saturated (as an $A$-module), is the polynomial ring in one variable $A = \mathbb{K}[x]$. Its $A$-saturation is the Laurent polynomial ring $\mathbb{K}[x, x^{-1}]$ seen as an $A$-module. Finally we will need the push-forward functor

$$\pi_* : \mathcal{O}_{[X]}^{-\text{Qcoh}} \to \mathcal{O}_X^{-\text{Qcoh}},$$

given by associating a sheaf on $X$ to a graded $A$-module. In general, it is not an equivalence. For instance, $\mathcal{O}_{[X]}(k)$ is an invertible sheaf but $\mathcal{O}_X(1) \cong \pi_*(\mathcal{O}_{[X]}(1))$ is not invertible, in general [6].

Let us now describe the (twisted) $D_{[X]}$-modules. Let $\partial_i = \partial/\partial x_i$, $i = 0, 1, \ldots, n$. The Weyl algebra $D(V) = \mathbb{K}\langle x_0, \ldots, x_n, \partial_0, \ldots, \partial_n \rangle$ gets a grading from the $\mathbb{G}_m$-action on $V$: $\deg(x_i) = d_i$, $\deg(\partial_i) = -d_i$. We define the reduced Weyl algebra as

$$\mathbb{D} := \text{End}_{D(V)}(D(V)/ID(V)) \cong \text{I}(ID(V))/ID(V)$$

where

$$\text{I}(ID(V)) = \{w \in D(V) \mid wID(V) \subseteq ID(V)\}$$

is the idealiser of $ID(V)$ in $D(V)$. Notice that $\mathbb{D}$ is graded: $I$ is graded, then $ID(V)$ is graded, then $\text{I}(ID(V))$ is graded, and finally $\mathbb{D}$ is graded. Observe that $A$ is a graded subalgebra of $\mathbb{D}$ since $\mathbb{K}[x_i] \subseteq \text{I}(ID(V))$. It is known that for $w \in D(V)$ [12, 15.5.9]

$$w \in ID(V) \iff w(\mathbb{K}[x_i]) \subseteq I \quad \text{and} \quad w \in \text{I}(ID(V)) \iff w(I) \subseteq I$$

where $w$ acts naturally on polynomials in $I$. This defines an algebra embedding $\mathbb{D} \hookrightarrow \text{End}_K(A)$ whose image lies in $D(Y)$, the ring of differential operators on $A$.

**Proposition 4.** [12, 15.5.13] The map $\phi : \mathbb{D} \to D(Y)$ is an isomorphism.
The element $\sum d_i x_i \partial_i$ belongs to the idealiser $I(\text{ID}(V))$. We call its image in $\mathbb{D}$ the \textit{Euler field}
\[ E = \sum d_i x_i \partial_i + \text{ID}(V). \]
It belongs to $\mathbb{D}_0$ and defines the grading of $\mathbb{D}$ and its subalgebra $A$.

**Lemma 5.** Let $x \in \mathbb{D}$. Then $x \in \mathbb{D}_k$ if and only if $Ex - xe = kx$.

**Proof.** It suffices to check it on the generators:
\[ Ex_i = \sum j d_j x_j \partial_i x_i = x_i E + d_i x_i. \]
Similarly,
\[ E \partial_i = \partial_i E - d_i \partial_i. \]

The Euler field can be used to define gradings on $\mathbb{D}$-modules.

**Lemma 6.** Let $M$ be a $\mathbb{D}$-module. The span $M'$ of all eigenvectors of the Euler field $E$ is a $K$-graded $\mathbb{D}$-submodule of $M$.

**Proof.** Let $m \in M^\lambda$, the $\lambda$-eigenspace of $E$. Using Lemma 5,
\[ Ex_i m = x_i Em + d_i x_i m = (\lambda + d_i) x_i m, \]
so
\[ x_i m \in M^{\lambda + d_i}. \]
Similarly,
\[ E \partial_i m = \partial_i Em - d_i \partial_i m = (\lambda - d_i) \partial_i m \]
and
\[ \partial_i m \in M^{\lambda - d_i}. \]

Let us fix $\lambda \in \mathbb{K}$. In general,
\[ M \geq M' = \oplus_{\mu \in \mathbb{K}} M^\mu \geq M^{(\lambda)} := \oplus_{n \in \mathbb{Z}} M^{\lambda + n}. \]
A $\mathbb{D}$-module $M$ is called $\lambda$-\textit{Euler} if $M = M^{(\lambda)}$. A $\lambda$-Euler $\mathbb{D}$-module $M$ admits a canonical $\mathbb{Z}$-grading given by $M_k = M^{k+\lambda}$. The \textit{category of $\lambda$-Euler $\mathbb{D}$-modules} $\mathbb{D} \text{- Grmod}^\lambda$ is a full subcategory of the category of graded $\mathbb{D}$-modules $\mathbb{D} \text{- Grmod}$. The full subcategory of the torsion (as $A$-modules) modules is denoted $\mathbb{D} \text{- Tors}^\lambda$. Notice as well that the torsion submodule of a graded $\mathbb{D}$-module is a graded $\mathbb{D}$-module and that if, moreover, it is $\lambda$-Euler, then the torsion submodule is $\lambda$-Euler too.
$\mathbb{D}$–$\text{Grmod}^\lambda$ is a locally small category. $\mathbb{D}$–$\text{Tors}^\lambda$ is a Serre subcategory of $\mathbb{D}$–$\text{Grmod}^\lambda$ which is closed under taking arbitrary direct sums. Therefore, $\mathbb{D}$–$\text{Tors}^\lambda$ is a localising subcategory of $\mathbb{D}$–$\text{Grmod}^\lambda$ \footnote{The article refers to a footnote here, but the actual content is not provided.} and the quotient functor

$$\pi^\lambda_{\mathbb{D}} : \mathbb{D}$–$\text{Grmod}^\lambda \to \mathbb{D}$–$\text{Grmod}^\lambda/\mathbb{D}$–$\text{Tors}^\lambda$$

is exact and has a right adjoint section functor

$$\omega^\lambda_{\mathbb{D}} : \mathbb{D}$–$\text{Grmod}^\lambda/\mathbb{D}$–$\text{Tors}^\lambda \to \mathbb{D}$–$\text{Grmod}^\lambda$$

It follows that we have

$$\text{Hom}_{\mathbb{D}$–$\text{Grmod}^\lambda}(N, \omega^\lambda_{\mathbb{D}}(M)) \cong \text{Hom}_{\mathbb{D}$–$\text{Grmod}^\lambda/\mathbb{D}$–$\text{Tors}^\lambda}(\pi^\lambda_{\mathbb{D}}(N), M).$$

**Theorem 7.** The category $\mathcal{D}_{[X]}$–$\text{Qcoh}$ of quasicoherent $D$-modules on the stack $[X]$ is equivalent to the quotient category $\mathbb{D}$–$\text{Grmod}^0/\mathbb{D}$–$\text{Tors}^0$.

**Proof.** The category of $D$-modules on $\overline{Y}$ is just the category of $D(\overline{Y})$-modules since $\overline{Y}$ is affine. The category of weakly $\mathbb{G}_m$-equivariant $D$-modules on $\overline{Y}$ is $D(\overline{Y})$–$\text{Grmod}$. The two actions of the Lie algebra of the multiplicative group $\mathbb{G}_m$ are given by the Euler element $E$ and by the grading. Thus, the category of strongly $\mathbb{G}_m$-equivariant $D$-modules on $\overline{Y}$ is the category of $0$-Euler $D$-modules $D(\overline{Y})$–$\text{Grmod}^0$.

By definition, the category $\mathcal{D}_{[X]}$–$\text{Qcoh}$ is the category of strongly $\mathbb{G}_m$-equivariant $D$-modules on $Y$. Thus, taking sections on the open set $Y$ induces an exact functor

$$\Gamma(Y, \_ ) : \mathcal{D}_{[X]}$–$\text{Qcoh} \to D(Y)$–$\text{Grmod}$$

where $D(Y)$ is the ring of global differential operators on $Y$. Proposition $\footnote{The article refers to a footnote here, but the actual content is not provided.}$ makes the global sections $\Gamma(Y, \mathcal{M})$ into a graded $D$-module via the restriction map $\mathbb{D} \cong D(\overline{Y}) \to D(Y)$. This module is $0$-Euler, because $\mathcal{M}$ is strongly equivariant. Thus, we obtain exact functors

$$\Gamma(Y, \_ ) : \mathcal{D}_{[X]}$–$\text{Qcoh} \to \mathbb{D}$–$\text{Grmod}^0$$

and

$$\pi^0_{\mathbb{D}} \circ \Gamma(Y, \_ ) : \mathcal{D}_{[X]}$–$\text{Qcoh} \to \mathbb{D}$–$\text{Grmod}^0/\mathbb{D}$–$\text{Tors}^0$$

Let us examine the sheafification functor $\mathbb{D}$–$\text{Grmod}^0 \to \mathcal{D}_{[X]}$–$\text{Qcoh}$. The sheafification of an object in $\mathbb{D}$–$\text{Tors}^0$ is supported at $0$. Hence objects in $\mathbb{D}$–$\text{Tors}^0$ give the zero sheaf on $Y$. So it induces a functor on the quotient

$$\sim : \mathbb{D}$–$\text{Grmod}^0/\mathbb{D}$–$\text{Tors}^0 \to \mathcal{D}_{[X]}$–$\text{Qcoh}$$

which is quasiinverse to $\pi^0_{\mathbb{D}} \circ \Gamma(Y, \_ )$. \qed
An inquisitive reader may observe that we have defined the category $\mathcal{D}_{[X]}^{-}\text{Qcoh}$ without defining the object $\mathcal{D}_{[X]}$. Later on we remedy this partially by constructing an object $\mathcal{D}^\lambda_{[X]}$ for each $\lambda \in \mathbb{K}$ so that $\mathcal{D}_{[X]} = \pi^0_\mathcal{D}(\mathcal{D}^0_{[X]})$. Let us define the category $\mathcal{D}^\lambda_{[X]}^{-}\text{Qcoh}$ of twisted $D$-modules on $[X]$ as the quotient $\mathcal{D}^{-}\text{Grmod}^\lambda / \mathcal{D}^{-}\text{Tors}^\lambda$. It is possible to define the category internally and then prove a version of Theorem 7 but we see no value in doing it here.

Given a module $M$ in $\mathcal{D}^{-}\text{Grmod}^\lambda$, we call $\omega^\lambda_\mathcal{D}\pi^\lambda_\mathcal{D}(M)$ the $\mathcal{D}^\lambda$-saturation of $M$. We say that a module is $\mathcal{D}^\lambda$-saturated if it is isomorphic to the $\mathcal{D}^\lambda$-saturation of a module. It can be seen from the adjunction that a $\mathcal{D}^\lambda$-saturated module is torsion-free and is isomorphic to its own saturation.

We shall prove now that an $A$-saturated $\lambda$-Euler $\mathbb{D}$-module is automatically $\mathbb{D}^\lambda$-saturated. This will make our forthcoming calculations easier.

**Lemma 8.** Let $M$ be a $\lambda$-Euler $\mathbb{D}$-module. Then the $\mathbb{D}^\lambda$-saturation of $M$ is an $A$-submodule of its $A$-saturation.

**Proof.** We have a map

$$M \rightarrow \omega^\lambda_\mathcal{D}\pi^\lambda_\mathcal{D}(M)$$

in $\mathbb{D}^{-}\text{Grmod}^\lambda$ [2]. The kernel and cokernel of this map are torsion which implies that

$$\pi_A(\omega^\lambda_\mathcal{D}\pi^\lambda_\mathcal{D}(M)) \cong \pi_A(M).$$

From adjunction, this isomorphism is the image of a map in $A^{-}\text{Grmod}$,

$$\phi : \omega^\lambda_\mathcal{D}\pi^\lambda_\mathcal{D}(M) \rightarrow \omega_A\pi_A(M).$$

We claim that this map is injective. Since $\pi_A(\phi)$ is an isomorphism then $\text{Ker}\phi$ is a torsion $A$-module. Consider $\mathbb{D}\text{Ker}\phi$ (which contains $\text{Ker}\phi$), it is a left $\mathbb{D}$-submodule of $\omega^\lambda_\mathcal{D}\pi^\lambda_\mathcal{D}(M)$. Take $m \in \text{Ker}\phi$ then there exists an integer $N$ such that

$$A_{\geq N}m = 0.$$

For any $d \in \mathbb{D}$ of order $k$ we have

$$A_{\geq N+k}(dm) \leq \mathbb{D}A_{\geq N}m = 0.$$

It follows that it is a torsion submodule of $\omega^\lambda_\mathcal{D}\pi^\lambda_\mathcal{D}(M)$ but $\omega^\lambda_\mathcal{D}\pi^\lambda_\mathcal{D}(M)$ is torsion-free. Hence $\text{Ker}\phi = 0$.

An immediate corollary is the following:

**Corollary 9.** Any $A$-saturated $\lambda$-Euler $\mathbb{D}$-module is $\mathbb{D}^\lambda$-saturated.
Let us give examples of objects in $\mathcal{D}^\lambda_{[X]} - \mathcal{Qcoh}$. The sheaf $\mathcal{O}_{[X]}(k)$ is an object in $\mathcal{D}^k_{[X]} - \mathcal{Qcoh}$. We introduce

$$D^\lambda_{[X]} := \mathbb{D}/\mathbb{D}(\mathbb{E} - \lambda).$$

Another interesting object in $\mathcal{D}^\lambda_{[X]} - \mathcal{Qcoh}$ is

$$\mathcal{D}^\lambda_{[X]} := \pi^\lambda_\mathbb{D}(D^\lambda_{[X]}).$$

It plays the role of the sheaf of twisted differential operators, although $D^\lambda_{[X]}$ is not an algebra because $\mathbb{D}(\mathbb{E} - \lambda)$ is not a two-sided ideal, in general. However, $\mathbb{E}$ is a central element of $\mathbb{D}$, so $D^\lambda_{[X]}_0 := \mathbb{D}/\mathbb{D}(\mathbb{E} - \lambda)$ is an algebra. It plays the role of the algebra of global sections of the twisted differential operators on $[X]$. $D^\lambda_{[X]}$ is a $\mathbb{D} - D^\lambda_{[X]}_0$-bimodule.

In the next section the adjoint functors of global sections and localisation will play a role. This adjoint pair $(\Gamma_\lambda, L_\lambda)$ is defined as:

$$\Gamma_\lambda : \mathcal{D}^\lambda_{[X]} - \mathcal{Qcoh} \to D^\lambda_{[X]}_0 - \text{Mod}, \quad \Gamma_\lambda(\mathcal{M}) := \omega^\lambda_\mathbb{D}(\mathcal{M})_0 = \omega^\lambda_\mathbb{D}(\mathcal{M})^\lambda,$$

$$L_\lambda : D^\lambda_{[X]}_0 - \text{Mod} \to \mathcal{D}^\lambda_{[X]} - \mathcal{Qcoh}, \quad L_\lambda(N) := \pi^\lambda_\mathbb{D}(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]}_0} N).$$

The ways we defined our global sections functors for $\mathcal{D}^\lambda_{[X]} - \mathcal{Qcoh}$ and $\mathcal{O}_{[X]} - \mathcal{Qcoh}$ are not necessarily equivalent. Yet we know that

$$\Gamma_\lambda(\pi^\lambda_\mathbb{D}(M)) \leq \Gamma(\pi_\mathbb{A}(M))$$

as $\mathbb{A}$-modules for any $\lambda$-Euler $\mathbb{D}$-module $M$.

The exposition would be greatly simplified if restricting the section functor $\omega_\mathbb{A}$ to $\mathcal{D}^\lambda_{[X]} - \mathcal{Qcoh}$ were equivalent to $\omega^\lambda_\mathbb{D}$. This explains why we have different global sections functor for different $\lambda$ although geometrically only one is needed. However, to ensure that we obtain $\lambda$-Euler $\mathbb{D}$-modules and not just $\mathbb{A}$-modules we use $\omega^\lambda_\mathbb{D}$.

### 3. D-modules on weighted projective space

In this section we consider $Y = V \setminus \{0\}$, the punctured vector space of dimension at least 2 and $[X] = [Y/\mathbb{G}_m] = [\mathbb{P}(V)]$, the weighted projective stack. In this case $I = \{0\}$, $A = \mathbb{K}[x_0, \ldots, x_n]$ where the degree of $x_i$ is $d_i > 0$ and $\mathbb{D} = \mathbb{K}(x_0, \ldots, x_n, \partial_0, \ldots, \partial_n)$ is the Weyl algebra. Without loss of generality, we assume that $0 < d_0 \leq d_1 \leq \ldots \leq d_n$.

Let us look at the $\mathbb{D}$-module $\Delta$ generated by the delta-function at zero $\delta = \delta_0(x_0, \ldots, x_n)$

$$\Delta = \mathbb{D}\delta \cong \mathbb{D}/(\mathbb{D}x_0 + \mathbb{D}x_1 + \ldots + \mathbb{D}x_n).$$
The linear map
\[ K[\partial_0, \ldots, \partial_n] \rightarrow \Delta, \quad f(\partial_0, \ldots, \partial_n) \mapsto f(\partial_0, \ldots, \partial_n) \cdot \delta \]
is an isomorphism of vector spaces. If we identify \( K[\partial_0, \ldots, \partial_n] \) with \( \Delta \) using this linear map, then \( \partial_i \) acts by multiplication and \( x_i \) acts by derivation \( \partial_j \mapsto -\delta_{i,j} \). In particular,
\[ \mathbf{E} \cdot \delta = \mathbf{E} \cdot 1 = \sum_j d_j x_j \cdot \partial_j = \sum_j -d_j = -\left( \sum_j d_j \right) \delta. \]
Hence, \( \Delta \) is \( k \)-Euler for each integer \( k \). Its canonical \( k \)-Euler grading is given by
\[ \delta \in \Delta_{-\sum_j d_j} = \Delta_{-k-\sum_j d_j}, \quad \partial_i \cdot \delta \in \Delta_{-k-d_i-\sum_j d_j}. \]

Let \( J = (x_0, \ldots, x_n) \subset A \). If \( M \) is a \( D \)-module, \( \tau_A(M) = \{ m \in M \mid \exists k \ J_k m = 0 \} \) is its torsion \( D \)-submodule (a reader can easily verify that if \( J^k m = 0 \), then \( J^{k+1} \partial_i m = 0 \)). The torsion \( D \)-modules are those, supported set theoretically on the zero \( 0 \in V \). By Kashiwara’s theorem, any \( D \)-module supported at 0 is a direct sum of copies of \( \Delta \).

Let us introduce some notations. Suppose that \( M \) and \( N \) are two \( \mathbb{Z} \)-graded \( A \)-modules. We say that an \( A \)-module homomorphism \( f : M \rightarrow N \) has degree \( l \) if \( f(M_i) \subset N_{i+l} \) for all \( i \). Denote by \( \text{Hom}(M, N)_l \) the set of all degree \( l \) \( A \)-module homomorphisms and write
\[ \text{Hom}_A(M, N) = \bigoplus_{l \in \mathbb{Z}} \text{Hom}(M, N)_l. \]

Now let \( \text{Ext}^q(M, N)_l \) be the derived functor of \( \text{Hom}(M, N)_l \) and write
\[ \text{Ext}^q_A(M, N) = \bigoplus_{l \in \mathbb{Z}} \text{Ext}^q(M, N)_l. \]

Artin and Zhang prove \cite{2} that for any graded \( A \)-module \( M \),
\[ \tau_h(M) \cong \lim_{\rightarrow} \text{Hom}_h(A/A_{\geq k}, M), \]
\[ R^1 \tau_h(M) \cong \lim_{\rightarrow} \text{Ext}^1_A(A/A_{\geq k}, M) \]
and that there exists a long exact sequence of \( A \)-modules
\[ 0 \rightarrow \tau_h(M) \rightarrow M \rightarrow \omega_h \pi_h(M) \rightarrow R^1 \tau_h(M) \rightarrow 0 \]
where \( \tau_h(M) \) and \( R^1 \tau_h(M) \) are torsion modules. This implies the following proposition.

**Proposition 10.** A \( \lambda \)-Euler \( D \)-module \( M \) is \( D^\lambda \)-saturated if it is torsion-free and \( \lim_{\rightarrow} \text{Ext}^1(A/A_{\geq k}, M) = 0 \).

The next lemma will prove primordial in the proof that \( \Gamma \lambda L_\lambda \cong \text{Id}_{D_{\lambda}^{X_0} \text{-Mod}} \) for any \( \lambda \) and \( n \geq 2 \).
Lemma 11. For \( n \geq 2 \), \( D^\lambda_{[X]} \) is \( \mathbb{D} \)-saturated.

Proof. Recall that \( D^\lambda_{[X]} = \mathbb{D}/\mathbb{D}(E - \lambda) \). It is easier to compute Ext groups by taking a projective resolution of the left argument than an injective one of the right argument. Since \( \mathbb{A}/\mathbb{A}_{\geq 1} \cong \mathbb{K} \), the first three terms of the Koszul resolution are given by

\[
\ldots \to \bigoplus_{i_0 < i_1} \mathbb{A}(-d_{i_0} - d_{i_1}) \to \bigoplus_{i=0}^n \mathbb{A}(-d_i) \to \mathbb{A} \to \mathbb{A}/\mathbb{A}_{\geq 1} \to 0.
\]

Take away \( \mathbb{A}/\mathbb{A}_{\geq 1} \) and apply \( \text{Hom}_A(\_, D^\lambda_{[X]}) \) to the above exact sequence to get

\[
0 \to D^\lambda_{[X]} \xrightarrow{\phi_1} \bigoplus_{i=0}^n D^\lambda_{[X]}(d_i) \xrightarrow{\phi_2} \bigoplus_{i_0 < i_1} D^\lambda_{[X]}(d_{i_0} + d_{i_1}) \to \ldots
\]

where

\[
\phi_1 : \overline{m} \mapsto (x_i \overline{m})_{i=0}^n
\]

and

\[
\phi_2 : (\overline{m_i})_{i=0}^n \mapsto (x_i \overline{m}_i - x_i \overline{m}_i)_{i_0 < i_1}.
\]

It follows that

\[
\text{Hom}_A(\mathbb{A}/\mathbb{A}_{\geq 1}, D^\lambda_{[X]}) \cong \text{Ker}(\phi_1),
\]

\[
\text{Ext}_A^1(\mathbb{A}/\mathbb{A}_{\geq 1}, D^\lambda_{[X]}) \cong \frac{\text{Ker}(\phi_2)}{\text{Im}(\phi_1)}.
\]

Both \( \text{Hom}_A(\mathbb{A}/\mathbb{A}_{\geq 1}, D^\lambda_{[X]}) \) and \( \text{Ext}_A^1(\mathbb{A}/\mathbb{A}_{\geq 1}, D^\lambda_{[X]}) \) vanish. Let us first compute \( \text{Hom}_A(\mathbb{A}/\mathbb{A}_{\geq 1}, D^\lambda_{[X]}) \). Pick \( \overline{m} \in \text{Ker}(\phi_1) \), then \( x_i \overline{m} = 0 \) for each \( i \), where

\[
\overline{m} = m + \mathbb{D}(E - \lambda).
\]

We can assume \( m \) to be homogeneous, so

\[
x_i m = p_i(E - \lambda)
\]

for some homogeneous \( p_i \in \mathbb{D} \). We want to show that \( p_i \in x_i \mathbb{D} \). Suppose, for a contradiction, that it is not. Then we can write

\[
p_i = x_i m' + f \partial^\beta + LT
\]

where \( m' \in \mathbb{D} \), \( f \in \mathbb{K}[x_0, \ldots, x_n] \) is the highest term which is non-zero by assumption, free of \( x_i \), \( \beta \) the biggest power and \( LT \) are the lower terms using \textsf{DegLex} for the ordering of the monomials in \( \partial \). Without loss of generality, we can assume that \( i \neq 0 \). It follows that

\[
x_i m = x_i m'' + d_0 f x_0 \partial^\beta + LT
\]
since $f \partial^\beta x_0 \partial_0 = f x_0 \partial^\beta x_0 + LT$. But $f x_0$ is not divisible by $x_i$ and we obtain a contradiction. Thus,

$$\text{Hom}_X(A/A_{\geq 1}, D^A_{X}) = 0.$$ 

Similarly, let us show that $\text{Ext}^1_X(A/A_{\geq 1}, D^A_{X})$ vanishes. To proceed, choose $(m_i)_{i=0}^n \in \text{Ker}(\phi_2)$. Then for all $i, j$, there exists a $\theta_{ij} \in \mathbb{D}$ such that

$$x_i m_j = x_j m_i + \theta_{ij}(E - \lambda).$$

Write

$$m_j = x_j m_j' + f \partial^\beta + LT$$

where $m_j' \in \mathbb{D}$, $f \in \mathbb{K}[x_0, \ldots, x_n]$ is the highest term, free of $x_j$, $\beta$ is the highest power and $LT$ are the lower terms using $\text{DegLex}$ for the ordering of the monomials in $\partial$. Let us suppose, for the sake of a contradiction, that $|\beta| \neq 0$. Then without loss of generality, we can assume that $\beta$ is the lowest among all the possible representatives of $m_j$. Write

$$\theta_{ij} = x_j \theta' + g \partial^\gamma + LT$$

where $g \in \mathbb{K}[x_0, \ldots, x_n]$ is the highest term, free of $x_j$. If $g = 0$ then we are done. Suppose that $g \neq 0$ so that

$$x_i x_j m_j' + x_i f \partial^\beta + LT = x_j (m_i + \theta'(E - \lambda)) + g \partial^\gamma (E - \lambda) + LT.$$ 

Again without loss of generality, suppose that $i, j \neq 0$ as $n \geq 2$. By comparing the highest terms, free of $x_j$, we get

$$x_i f \partial^\beta = d_0 g x_0 \partial^\gamma + e_0$$

with $|\gamma| < |\beta|$. Therefore,

$$f \partial^\beta = d_0 \frac{g}{x_i} x_0 \partial^\gamma + e_0 = \frac{g}{x_i} \partial^\gamma (E - \lambda) + LT.$$ 

So $m_j - \frac{g}{x_i} \partial^\gamma (E - \lambda)$ is another representative of $m_j$ which has an index $\gamma$ lower than $\beta$, contrary to our hypothesis. Thus $g = 0$ and

$$m_j = x_j m_j'.$$

For all $i, j$, we have

$$x_i x_j m_j' = x_i x_j m_j' + \theta_{ij}(E - \lambda)$$

which implies that

$$x_i x_j (m_j' - m_j') \in \mathbb{D}(E - \lambda).$$ 

By using the first argument twice, we obtain that for all $i, j$

$$m_j' - m_j' \in \mathbb{D}(E - \lambda).$$
Write
\[ m' := \overline{m_j^i} = \overline{m_i^j} \]
for the residues of \( m_j^i \) and \( m_i^j \). Then for all \( i \),
\[ \overline{m_i^j} = x_i \overline{m^j} \]
Hence,
\[ \text{Ext}^1_\mathbb{A}(\mathbb{A}/\mathbb{A}_\geq 1, D^\lambda_{[X]}) = 0. \]
To finish our proof, for each \( k \) we have a short exact sequence of graded \( \mathbb{A} \)-modules:
\[ 0 \to \mathbb{A}_\geq k/\mathbb{A}_\geq k+1 \to \mathbb{A}/\mathbb{A}_\geq k+1 \to \mathbb{A}/\mathbb{A}_\geq k \to 0 \]
and \( \mathbb{A}_\geq k/\mathbb{A}_\geq k+1 \) is isomorphic to a finite direct sum of copies of \( \mathbb{A}/\mathbb{A}_\geq 1 \).
By applying \( \text{Hom}_\mathbb{A}(\_, D^\lambda_{[X]}) \) to this short exact sequence and by induction on \( k \), we conclude that for all \( k \):
\[ \text{Hom}_\mathbb{A}(\mathbb{A}/\mathbb{A}_\geq k, D^\lambda_{[X]}) = 0, \]
\[ \text{Ext}^1_\mathbb{A}(\mathbb{A}/\mathbb{A}_\geq k, D^\lambda_{[X]}) = 0. \]
Taking direct limit [2] it follows that
\[ \tau_\mathbb{A}(D^\lambda_{[X]}) = 0, \quad \text{and} \quad \lim \text{Ext}^1(\mathbb{A}/\mathbb{A}_\geq k, D^\lambda_{[X]}) = 0. \]
Hence \( D^\lambda_{[X]} \) is \( \mathbb{D}^\lambda \)-saturated by Proposition 10.

The condition on \( n \) in the last proof is necessary. We can prove that
\( D^\lambda_{[X]} \) is not \( \mathbb{D}^\lambda \)-saturated for all \( \lambda \) when \( n = 1 \). For this, it suffices to notice that for \( \lambda = 0 \),
\[ (-d_1 \partial_1, d_0 \partial_0) \in \text{Ker}(\phi_2) \]
but
\[ (-d_1 \partial_1, d_0 \partial_0) \notin \text{Im}(\phi_1) \]
since \( d_0 x_0 \partial_0 = -d_1 x_i \partial_1 + E \).

**Lemma 12.** Let \( n \geq 2 \). If \( \Gamma_\lambda \) is exact then \( \Gamma_\lambda L_\lambda \cong \text{Id}_{D^\lambda_{[X]}_0} \)-Mod

**Proof.** Let \( N \) be a \( D^\lambda_{[X]}_0 \)-module. Take the first two terms of a free resolution of \( N \)
\[ P_1 \to P_0 \to N \to 0 \]
where \( P_i = \bigoplus_{j \in I_i} D^\lambda_{[X]}_0 \) and \( I_i \) is an index set. Since both \( D^\lambda_{[X]} \otimes D^\lambda_{[X]}_0 \)
and \( \pi^\lambda_\mathbb{D} \) are right exact functors, it follows that
\[ \Gamma_\lambda L_\lambda(P_1) \to \Gamma_\lambda L_\lambda(P_0) \to \Gamma_\lambda L_\lambda(N) \to 0 \]
is exact. We can compute the first two terms explicitly:

\[
\Gamma_\lambda L_\lambda(P_i) = (\omega^\lambda_D \pi^\lambda_D(D^\lambda_[X] \otimes D^\lambda_[X]_0 P_i))_0 \\
= (\omega^\lambda_D \pi^\lambda_D(D^\lambda_[X] \otimes D^\lambda_[X]_0 \bigoplus_{j \in I_i} D^\lambda_[X]_0))_0 \\
\cong (\omega^\lambda_D \pi^\lambda_D(\bigoplus_{j \in I_i} D^\lambda_[X])_0 \\
\cong (\omega^\lambda_D \pi^\lambda_D(\bigoplus_{j \in I_i} D^\lambda_[X]))_0
\]

since the tensor product commutes with arbitrary direct sums and that \(D^\lambda_[X] \otimes D^\lambda_[X]_0 \cong D^\lambda_[X]\). The category \(\mathbb{D}^{\lambda\lambda\lambda_{Grmod}}\) is locally noetherian [8, Prop. 4.18]. By a result of Gabriel, the section functor \(\omega^\lambda_D\) commutes with inductive limits and, in particular, with arbitrary direct sums [9, p. 379]. Moreover, \(\pi^\lambda_D\) is left adjoint to \(\omega^\lambda_D\), so \(\pi^\lambda_D\) commutes as well with arbitrary direct sums. This yields the following sequence of natural isomorphisms:

\[
\Gamma_\lambda L_\lambda(P_i) \cong (\omega^\lambda_D \pi^\lambda_D(\bigoplus_{j \in I_i} D^\lambda_[X]))_0 \\
\cong (\bigoplus_{j \in I_i} \omega^\lambda_D \pi^\lambda_D(D^\lambda_[X]))_0 \\
\cong (\bigoplus_{j \in I_i} D^\lambda_[X])_0 \\
\cong (\bigoplus_{j \in I_i} D^\lambda_[X]_0 \\
\cong P_i
\]

since \(D^\lambda_[X]\) is \(\mathbb{D}^{\lambda\lambda\lambda}\)-saturated and that \((\bigoplus)\) \(_0\) commutes with arbitrary direct sums. Thus, we constructed a commutative diagram with exact rows:

\[
\begin{array}{cccc}
P_1 & \longrightarrow & P_0 & \longrightarrow \Gamma_\lambda L_\lambda(N) & \longrightarrow 0 \\
\bigg\downarrow\alpha & & \bigg\downarrow\beta & & \bigg\downarrow\gamma \\
P_1 & \longrightarrow & P_0 & \longrightarrow N & \longrightarrow 0
\end{array}
\]

where \(\alpha\) and \(\beta\) are isomorphisms, so \(\Gamma_\lambda L_\lambda(N) \cong N\) is a natural isomorphism by the four lemma. \(\square\)

**Theorem 13.** Let \(A\) be the \(\mathbb{Z}_{\geq 0}\)-span of all \(d_i\)-s. If \(\lambda \in \mathbb{K} \setminus (-\sum_i d_i - A)\), then the global sections functor \(\Gamma_\lambda : D^\lambda_{[X]}\text{-Qcoh} \rightarrow D^\lambda_{[X]_0}\text{-Mod}\)
is exact. In this case, $\Gamma_\lambda$ defines an equivalence between the quotient category $D^{\lambda}_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda$ and $D^{\lambda}_{[X]_0} - \text{Mod}$.

**Proof.** The category $D^{\lambda}_{[X]} - \text{Qcoh}$ is the quotient category of the category of $\lambda$-Euler modules by the category of torsion modules. The canonical grading on a $\lambda$-Euler module $M$ is given by $M_k = M^{k+\lambda}$. The torsion modules are direct sums of $\Delta$. The global sections functor $\Gamma_\lambda$ is

$$\Gamma_\lambda : M \mapsto \omega^\lambda_D(M)_0 = \omega^\lambda_D(M)^\lambda.$$

We know that $\omega^\lambda_D$ is a left exact functor. Taking $\lambda$-eigenspaces is an exact functor, so we are left to prove that $\Gamma_\lambda$ is right exact. An epimorphism $f : M \to N$ induces the exact sequence

$$\omega^\lambda_D(M) \to \omega^\lambda_D(N) \to \text{coker}(\omega^\lambda_D(f)) \to 0$$

where $\text{coker}(\omega^\lambda_D(f))$ is a torsion $D$-module. Taking the zeroeth graded part, we get the exact sequence

$$\Gamma_\lambda(M) \to \Gamma_\lambda(N) \to \text{coker}(\omega^\lambda_D(f))_0 \to 0.$$

Our restriction on $\lambda$ provides that $\text{coker}(\omega^\lambda_D(f))_0 = 0$. Indeed, if $\lambda \notin \mathbb{Z}$, then $\text{coker}(\omega^\lambda_D(f)) = 0$. If $\lambda \in \mathbb{Z}$, then $\text{coker}(\omega^\lambda_D(f)) = \oplus \Delta$ and $\text{coker}(\omega^\lambda_D(f))_0 = \oplus \Delta^\lambda$. Since the $E$-weights of $\Delta$ are $-\sum_i d_i - A$, $\text{coker}(\omega^\lambda_D(f))_0 = 0$. Hence $\Gamma_\lambda$ is exact.

The kernel $\text{Ker}\Gamma_\lambda$ is the full subcategory of $D^{\lambda}_{[X]} - \text{Qcoh}$ whose objects are those $M$ without non-trivial global sections, i.e., with $\Gamma_\lambda(M) = 0$. Since $\Gamma_\lambda$ is exact, it is a Serre subcategory, and $\Gamma_\lambda$ descends to a functor

$$\tilde{\Gamma}_\lambda : D^{\lambda}_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda \to D^{\lambda}_{[X]_0} - \text{Mod}.$$

and let

$$Q : D^{\lambda}_{[X]} - \text{Qcoh} \to D^{\lambda}_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda$$

be the quotient functor. We claim that $QL_\lambda$ is a quasiinverse of $\tilde{\Gamma}_\lambda$. Now in one direction,

$$\tilde{\Gamma}_\lambda(QL_\lambda)(N) = (\tilde{\Gamma}_\lambda Q)L_\lambda(N) = \Gamma_\lambda L_\lambda(N) \cong N$$

since $\Gamma_\lambda$ is exact. Thus,

$$\tilde{\Gamma}_\lambda QL_\lambda \cong \text{Id}_{D^{\lambda}_{[X]_0} - \text{Mod}}.$$

In the opposite direction, we have a natural transformation

$$QL_\lambda \tilde{\Gamma}_\lambda \to \text{Id}_{D^{\lambda}_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda}.$$
Take an object $\tilde{M}$ in $D^\lambda_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda$. Then there exists an object $\mathcal{M}$ in $D^\lambda_{[X]} - \text{Qcoh}$ such that $\tilde{M} = Q(\mathcal{M})$. Hence,

$$QL_\lambda \tilde{\Gamma}_\lambda(\tilde{M}) = QL_\lambda \Gamma_\lambda(\mathcal{M}) = Q\pi^\lambda_D(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} (\omega^\lambda_D(\mathcal{M}))_0).$$

On a level of a $\lambda$-Euler module $M$ (with its canonical grading), the natural map $D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} M_0 \to M$ gives rise to the long exact sequence

$$0 \to K \to D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} M_0 \to M \to N \to 0$$

where $K$ is its kernel and $N$ is its cokernel. Since $\pi^\lambda_D$ is exact,

$$0 \to \pi^\lambda_D(K) \to \pi^\lambda_D(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} M_0) \to \pi^\lambda_D(M) \to \pi^\lambda_D(N) \to 0$$

is a long exact sequence as well. If $M = \omega^\lambda_D(\mathcal{M})$, applying $\Gamma_\lambda$ yields

$$0 \to \Gamma_\lambda \pi^\lambda_D(K) \to \omega^\lambda_D(\mathcal{M})_0 \to \omega^\lambda_D(\mathcal{M})_0 \to \Gamma_\lambda \pi^\lambda_D(N) \to 0$$

since $\Gamma_\lambda \pi^\lambda_D(\omega^\lambda_D(\mathcal{M})) \cong \omega^\lambda_D(\mathcal{M})_0$ and $\Gamma_\lambda \Lambda \cong Id_{D^\lambda_{[X]_0} - \text{Mod}}$ when $\Gamma_\lambda$ is exact. The middle map

$$\omega^\lambda_D(\mathcal{M})_0 \to \omega^\lambda_D(\mathcal{M})_0$$

is the identity map and hence an isomorphism. It follows that $\pi^\lambda_D(K)$ and $\pi^\lambda_D(N)$ are objects in $\text{Ker}(\Gamma_\lambda)$. Therefore,

$$\pi^\lambda_D(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} \omega^\lambda_D(M)_0) \to \pi^\lambda_D(\omega^\lambda_D(\mathcal{M}))$$

is an isomorphism in $D^\lambda_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda$ and

$$QL_\lambda \tilde{\Gamma}_\lambda(\tilde{M}) \cong Q\pi^\lambda_D(\omega^\lambda_D(\mathcal{M})) \cong Q(M) \cong \tilde{M}.$$

It follows that $QL_\lambda \tilde{\Gamma}_\lambda \cong I_{D^\lambda_{[X]} - \text{Qcoh}/\text{Ker}\Gamma_\lambda}$. \hfill $\square$

We are left to study when $\text{Ker}\Gamma_\lambda$ is a zero category so that $\Gamma_\lambda$ defines an equivalence between the quotient category $D^\lambda_{[X]} - \text{Qcoh}$ and $D^\lambda_{[X]_0} - \text{Mod}$.

**Lemma 14.** Suppose that $\lambda \in \mathbb{Z} \setminus \mathcal{A}$ or that the greatest common divisor $\gcd_i(d_i) \neq 1$. Then $\text{Ker}\Gamma_\lambda \neq 0$. 

Proof. If \( k \in \mathbb{Z} \), then \( \mathcal{O}_{[X]}(k) = \pi^\lambda_D(A[k]) \) is a non-zero \( \mathbb{D}^k \)-saturated (since it is \( A \)-saturated [2]) object of \( \mathcal{D}^k_{[X]} \)-Qcoh because \( 1 \in \mathbb{A}_0 = A[k]_{-k} \) and

\[ \mathbf{E} \cdot 1 = 0 = (-k + k)1. \]

The global sections

\[ \Gamma_k(\mathcal{O}_{[X]}(k)) = \mathbb{A}[-k]_0 = \mathbb{A}_k \]

are non-zero if and only if \( k \in \mathbb{A} \). Thus, if \( \lambda \in \mathbb{Z} \setminus \mathbb{A} \), then \( \mathcal{O}_{[X]}(\lambda) \) is a non-zero object of \( \mathrm{Ker}\Gamma_\lambda \).

Now let us assume that the greatest common divisor \( d \) of \( d_0, \ldots, d_n \) is greater than 1. It easily follows that

\[ \mathbb{D}_1 = \mathbb{D}_2 = \ldots = \mathbb{D}_{d-1} = 0. \]

Let \( M \) be the \( \mathbb{K} \)-vector space with a basis of all formal monomials \( x_0^{a_0} \cdots x_n^{a_n} \), \( a_i \in \mathbb{K} \). It is a \( \mathbb{D} \)-module under the following operations, defined on the monomials by

\[ x_i \cdot x_0^{a_0} \cdots x_n^{a_n} = x_0^{a_0} \cdots x_i^{a_i+1} \cdots x_n^{a_n}, \]
\[ \partial_i \cdot x_0^{a_0} \cdots x_n^{a_n} = a_i x_0^{a_0} \cdots x_{i-1} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n}. \]

Given \( \lambda \in \mathbb{K} \), we consider the \( \mathbb{D} \)-submodule \( N = \mathbb{D} \lambda x_0^{(\lambda-1)/d_0} \). Since

\[ \mathbf{E} \cdot \lambda x_0^{(\lambda-1)/d_0} = d_0 \partial_0 \partial_0 \lambda x_0^{(\lambda-1)/d_0} = (\lambda - 1) \lambda x_0^{(\lambda-1)/d_0}, \]

the module \( N \) is \( \lambda \)-Euler and \( x_0^{(\lambda-1)/d_0} \in N^{\lambda-1} = N_{-1} \) in the canonical \( \lambda \)-Euler grading. Put \( \mathcal{N} = \tau^\lambda_D(N) \). By definition, \( \mathcal{N} \) is torsion-free. Denote by \( \tau^\lambda_D \) the restriction of \( \tau_\lambda \) to \( \mathbb{D} \)-Grmod\( \lambda \). The long exact sequence [2]

\[ 0 \to \tau^\lambda_D(N) \to N \to \omega^\lambda_D \tau^\lambda_D(N) \to R^1\tau^\lambda_D(N) \to 0 \]

reduces to the short exact sequence

\[ 0 \to N \to \omega^\lambda_D \tau^\lambda_D(N) \to R^1\tau^\lambda_D(N) \to 0. \]

But \( R^1\tau^\lambda_D(N) \) is a torsion \( \mathbb{D} \)-module, hence it is a direct sum of copies of \( \Delta \). The \( \mathbf{E} \)-weights of \( N \) are congruent to \(-1\) modulo \( d \) and the \( \mathbf{E} \)-weights of the module \( \Delta \) are congruent to \( 0 \) modulo \( d \). It follows that the short exact sequence splits and

\[ \omega^\lambda_D \tau^\lambda_D(N) \cong N \oplus R^1\tau^\lambda_D(N). \]

Since \( \omega^\lambda_D \tau^\lambda_D(N) \) is torsion free, \( \omega^\lambda_D \tau^\lambda_D(N) \cong N \) and \( R^1\tau^\lambda_D(N) = 0 \). This means that \( N \) is \( \mathbb{D}^\lambda \)-saturated and

\[ \Gamma_\lambda(N) = N_0 = 0. \]

Hence, \( \mathcal{N} \) is a non-zero object in \( \mathrm{Ker}\Gamma_\lambda \). \( \Box \)
In all the other cases the kernel is trivial.

**Lemma 15.** Let us assume that the greatest common divisor $\gcd_i(d_i)$ is equal to 1. If $\lambda \in (\mathbb{k} \setminus \mathbb{Z}) \cup \mathcal{A}$, then $\text{Ker}_\lambda$ is a zero category.

**Proof.** Let $m$ be the least common multiple of $d_0, \ldots, d_n$. Suppose that $M$ is a non-zero object in $\mathcal{D}_{[X]}^\lambda \mathcal{-Qcoh}$. Then $M := \omega_\mathcal{D}^\lambda(M)$ is a non-zero $\lambda$-Euler torsion-free $\mathcal{D}$-module. We need to show that $M_0 \neq 0$. Let us suppose that the contrary is true, i.e., $M_0 = 0$. We proceed to arrive at a contradiction via a sequence of claims.

**Claim 1.** $M_{-mt} = 0$ for any $t \in \mathbb{Z}_{>0}$.

**Proof of Claim:** If $a \in M_{-mt}$, then $x_i^{mt/d_i} \cdot a = 0$ for all $i = 0, \ldots, n$ since it is an element of $M_0$. Hence, $a$ generates a torsion $\mathcal{D}$-submodule of $M$ but $M$ is torsion-free. Hence $a = 0$. $\square$

**Claim 2.** $M_{-mt+kd_i} = 0$ for all $i$ and $0 \leq k \leq \frac{mt}{d_i}$. In particular, $M_{-kd_i} = 0$ for all $k \geq 0$.

**Proof of Claim:** We proceed by induction. The case $k = 0$ is Claim 1. Assume that this is true for $k$, and let us prove it for $k+1$. If $-mt + (k+1)d_i = 0$, then we are done. Otherwise, let us pick a non-zero element $a \in M_{-mt+(k+1)d_i}$. It follows that

$$\partial_i \cdot a \in M_{-mt+kd_i},$$

which is zero by induction. Moreover, $x_i^{-(k+1)+mt/d_i} \cdot a \in M_0$ which is zero again. Since

$$[\partial_i, x_i^{-(k+1)+mt/d_i}] = \left(\frac{mt}{d_i} - (k+1)\right) x_i^{-(k+2)+mt/d_i},$$

we conclude that $x_i^{-(k+2)+mt/d_i} \cdot a = 0$. We can repeat this argument to conclude that $x_i^{-(k+l)+mt/d_i} \cdot a = 0$ for all positive $l$ with $\frac{mt}{d_i} - (k + l) \geq 0$. In particular, $a = x_i^0 \cdot a = 0$. $\square$

**Claim 3.** If $c_0, \ldots, c_k$ are positive integers and $g$ is their greatest common divisor, then there exist integers $r_0 \leq 0$, and $r_1, \ldots, r_k \geq 0$ such that $r_0c_0 + \ldots + r_kc_k = g$.

**Proof of Claim:** Let $l$ be the least common multiple of $c_0, \ldots, c_k$. By the Euclidean algorithm there exist integers $s_0, \ldots, s_k$ such that

$$s_0c_0 + \ldots + s_kc_k = 1.$$ 

Now we can add $-\frac{l}{c_i}c_0 + \frac{l}{c_i}c_i = 0$ for various $i$ to this relations to get integers $r_0, \ldots, r_k$ such that

$$r_0c_0 + \ldots + r_kc_k = 1$$

and $r_1, \ldots, r_k \geq 0$. Inevitably, $r_0 \leq 0$. $\square$
Claim 4. For all integer \(b_0, \ldots, b_l \geq 0\), \(M_{-(b_0d_0 + \ldots + b_ld_l)} = 0\).

Proof of Claim: We proceed by induction on \(l\). The base case \(l = 0\) is Claim 2. Assume this is true for \(l - 1\). In particular, it is true if \(b_i = 0\) for some \(i\).

Let \(g_l = \gcd(d_0, \ldots, d_l)\) and fix a positive integer \(k\). Consider a non-zero element \(a \in M_{-kg_l}\). There exist positive integers \(c_0, c_1, \ldots, c_l\) such that
\[
\partial_i^{c_i} \cdot a = \partial_i^{c_{i-1}} \cdot a = \ldots = \partial_i^{c_1} \cdot a = 0.
\]
Indeed, by Claim 3, there exist \(r_i \leq 0\) and \(r_0, \ldots, r_{i-1}, r_{i+1}, \ldots, r_l \geq 0\) such that
\[
r_0d_0 + \ldots + r ld_l = g_l
\]
Now if \(c_i = -kr_i \geq 0\), then
\[
\partial_i^{c_i} \cdot a \in M_{-ci d_i - kg_l} = M_{-k(r_0d_0 + \ldots + r_{i-1}d_{i-1} + r_{i+1}d_{i+1} + \ldots + r ld_l)} = 0,
\]
by induction. Let us consider the Weyl algebra
\[
\widetilde{D} = \mathbb{K}\langle x_0, \ldots, x_l, \partial_0, \ldots, \partial_l \rangle
\]
and its polynomial subalgebra \(\widetilde{A} = \mathbb{K}[\partial_0, \ldots, \partial_l]\). The \(\widetilde{A}\)-module \(\widetilde{D}a\) is supported at zero, hence, it must be a direct sum of copies of \(\widetilde{A} = \widetilde{D}\delta(\partial_0, \ldots, \partial_l) \cong \mathbb{K}[x_0, \ldots, x_l]\). It follows that
\[
x_0^{b_0} \ldots x_l^{b_l} \cdot a \neq 0 \quad \text{for all } b_0, \ldots, b_l \geq 0.
\]
We want to determine for which \(k\), we can find \(b_0, \ldots, b_l \geq 0\) such that \(x_0^{b_0} \ldots x_l^{b_l} \cdot a \in M_0 = 0\). We get a contradiction and hence \(M_{-kg_l} = 0\) for such \(k\). The condition is that
\[
b_0d_0 + \ldots + b ld_l = kg_l,
\]
i.e. \(kg_l \in \mathbb{Z}_{\geq 0}d_0 + \mathbb{Z}_{\geq 0}d_1 + \ldots + \mathbb{Z}_{\geq 0}d_l\).

In particular, it is true for \(l = n\), i.e., \(M_{-k} = 0\) for all \(k \in \mathcal{A}\). Now let us finish the proof of the theorem. By Schur’s Theorem there exists \(K \geq 0\) such that \(k \in \mathcal{A}\) for all \(k > K\), in particular, \(M_{-k} = 0\) for all \(k > K\). Thus, \(M\) is supported at zero as a \(\mathbb{K}[\partial_0, \ldots, \partial_n]\)-module. By Kashiwara’s Theorem \(M\) is a direct sum of copies of \(\mathcal{A} = \mathbb{K}[x_0, \ldots, x_n]\). If \(\lambda \in \mathbb{K} \setminus \mathbb{Z}\) then \(\mathcal{A}\) is not \(\lambda\)-Euler. Thus, \(M = 0\). Finally, if \(\lambda \in \mathbb{Z}\) then \(\mathcal{A}\) is \(\lambda\)-Euler. Moreover, as a graded module \(M\) is a direct sum of copies of \(\mathcal{A}[\lambda]\). Observe that \(\mathcal{A}[\lambda]_0 = \mathcal{A}_\lambda \neq 0\) if and only if \(\lambda \in \mathcal{A}\). Thus, if \(\lambda \in \mathcal{A}\), then \(M = 0\) as well.

\footnote{The smallest such \(K\) is called the Frobenius number. It is a NP-hard problem to find such \(K\). There is no known closed formula that gives \(K\) as a function of \(d_0, \ldots, d_n\) for \(n \geq 2\).}
Combining the last two claims, we obtain a characterisation of the kernel of the global sections functor.

**Theorem 16.** The greatest common divisor \( \gcd_i(d_i) \) is equal to 1 and \( \lambda \in (K \setminus \mathbb{Z}) \cup A \) if and only if \( \ker \Gamma_\lambda \) is a zero category.

Together with Theorem 13 this gives the following corollaries.

**Corollary 17.** Let us suppose that \( \lambda \in (K \setminus \mathbb{Z}) \cup A \) and \( \gcd(d_0, \ldots, d_n) = 1 \). Then \( \Gamma_\lambda : \mathcal{D}^\lambda_{[X]} \text{-Qcoh} \to D^\lambda_{[X]} \text{-Mod} \) is an equivalence of categories.

In particular, we obtain a necessary and sufficient condition for a weighted projective stack to be D-affine.

**Corollary 18.** The weighted projective stack \([X] = [\mathbb{P}(V)]\) is D-affine if and only if \( \gcd_i(d_i) \) is equal to 1.

**Proof.** D-affinity deals with the case of \( \lambda = 0 \). \( \Gamma_0 \) is exact, and its kernel is zero if and only if \( \gcd_i(d_i) \) is equal to 1. \( \Box \)

A similar functor for varieties

\[
\Gamma^*_\lambda : \mathcal{D}^\lambda_{X} \text{-Qcoh} \to D^\lambda_{[X]} \text{-Mod}
\]

is studied by Van den Bergh [16]. It is instructive to compare it with the push-forward functor

\[
\pi_* : \mathcal{D}^\lambda_{[X]} \text{-Qcoh} \to \mathcal{D}^\lambda_X \text{-Qcoh}.
\]

The functors \( \Gamma^*_\lambda \pi_* \) and \( \Gamma_\lambda \) are naturally equivalent, so we can conclude the final corollary.

**Corollary 19.** Let us suppose that \( \lambda \in K \setminus \mathbb{Z} \cup A \) and \( \gcd_{i \neq j}(d_i) = 1 \) for every \( j \) (the well-formedness condition). Then the push-forward functor \( \pi_* : \mathcal{D}^\lambda_{[X]} \text{-Qcoh} \to \mathcal{D}^\lambda_X \text{-Qcoh} \) is an equivalence of categories.

It can be noticed as well that the condition of well-formedness is not required for a weighted projective stack to be D-affine. We only need the greatest common divisor of its weights to be equal to one to guarantee it. As varieties, this condition was added to prove D-affinity of weighted projective spaces.

**References**


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