Geometrical Finiteness
for Hyperbolic Groups

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Declaration

Chapter 0, and to a large extent, Chapter 1, are of an expository nature. Chapters 2, 3, and 4 represent original work except where otherwise acknowledged. No part of this thesis has been previously submitted for any degree.
Summary

In this paper, we describe various definitions of geometrical finiteness for discrete hyperbolic groups in any dimension, and prove their equivalence. This generalises what has been worked out in two and three dimensions by Marden, Beardon, Maskit, Thurston and others. We also discuss the nature of convex fundamental domains for such groups. We begin the paper with a discussion of results related to the Margulis Lemma and Bieberbach Theorems.
Acknowledgements

This work was originally an offshoot from the M.Sc. dissertation of Dick Canary and Paul Green, which has been published, in augmented form, as [CEG]. Their paper focusses on other aspects of Thurston's notes [T1], but the start they made on geometrical finiteness was very helpful to me. My greatest debt is to David Epstein, for introducing me to the subject, and for his many suggestions and comments. I would also like to thank the S.E.R.C. for their financial support.
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Abstract.

Let \( \Gamma \) be a group acting properly discontinuously on hyperbolic space \( \mathbb{H}^n \). The aim of this work is to clarify the meaning of "geometrical finiteness" for such groups. In dimension 3, with \( \Gamma \) acting freely, the principal definition is that \( \Gamma \) should possess a "finite-sided fundamental domain". Various definitions in this dimension have been worked out. Marden [Mar] shows that it is equivalent to the statement that the quotient manifold, including its ideal points, can be decomposed into a compact part and standard neighbourhoods of its cusps ("cusp cylinders" and "cusp tori"). Thurston [Th1] introduced two new definitions: that the thick part of the convex core should be compact, or that an \( \epsilon \)-neighbourhood of the convex core should have finite volume. Finally, Beardon and Maskit [BeaM] say that \( \Gamma \) is geometrically finite if and only if its limit set consists of (what we call here) conical limit points and bounded parabolic fixed points.

We shall investigate here how these definitions generalise to arbitrary discrete actions in \( n \) dimensions. This matter has also been considered by Apanasov [Ap1,Ap2], and to some extent by Weilenberg [We].

Our central definition (GF1) will be essentially that of Marden, with appropriate definitions of cusp regions. The Beardon and Maskit description (as they point out in their paper) generalises unchanged (GF2). The use of finite-sided fundamental domains runs into problems when \( n \geq 4 \). The natural generalisation seems to be in terms of what we call "convex cell complexes" (GF3). For the first of Thurston's definitions, we need to clarify what we mean by the "thick part" of an orbifold. The definition chosen here (GF4) does not seem particularly natural, but it proves useful in discussing the final definition (GF5). For this, we impose the additional condition that \( \Gamma \) be finitely generated. We suspect that this is unnecessary, and show it to be unnecessary for manifolds, finite-volume orbifolds, or when \( n \leq 3 \). In the course of this discussion, we give a proof of the existence of an embedded ball in a hyperbolic \( n \)-orbifold, of uniform radius, depending on \( n \), but not on the particular orbifold.

Finally, in Ch.4, we discuss the existence of finite-sided fundamental domains. We give (in principle) a complete description of when a Dirichlet region is finite-sided, and show that in certain special cases, all convex fundamental domains are finite-sided. We give an example (due to Apanasov) of a geometrically finite (henceforth abbreviated to GF) manifold with no finite sided Dirichlet domain.

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0. Introduction.

0.1. Hyperbolic Space.

We begin with a general discussion of hyperbolic geometry in order to induce our terminology and notation. More details may be found in [Bea, Chapter 7].

We shall write \( S^n \) for the unit \( n \)-sphere in euclidean space. We write \( E^n \) for euclidean \( n \)-space, and \( \mathbb{H}^n \) for hyperbolic \( n \)-space. We shall denote the metrics on these spaces by \( d_{euclidean} \), \( d_{euclidean} \), and \( d_{hyperbolic} \) respectively. We shall drop the suffixes where there can be no confusion. In each case, we write \( \text{Isom} X \) for the group of all
isometries of $X$.

We can represent $H^n$ conformally as the open unit ball in $\mathbb{R}^n$ with infinitesimal metric $d_{\text{inf}} = \frac{1}{2} d_{\text{eucl}}$, where $d$ is the euclidean distance from the centre. This is the Poincaré model. The closed unit ball gives a canonical compactification of $H^n$, which we denote by $\mathbb{H}^n$. We write $\mathbb{H}^n$ for the $(n-1)$-sphere of ideal points, so that $\mathbb{H}^n = \mathbb{H}^n \cup \mathbb{H}^0$. Any isometry $\gamma \in \text{Isom} H^n$ can be extended to act conformally on $\mathbb{H}^n$.

Another conformal representation of $H^n$ is as the upper half space in $\mathbb{R}^n$; that is, $H^n = \{ x \in \mathbb{R}^n | x_n > 0 \}$, where $x_n$ is the last coordinate of $x$. The metric is given infinitesimally by $d_{\text{inf}} = \frac{1}{2} d_{\text{eucl}}$. Writing $\mathbb{H}^n = \{ x \in \mathbb{R}^n | x_n = 0 \}$, we may identify $H^n$ as $\mathbb{R}^n \cup \{ 0 \}$, where the ideal point $0$ compactifies $\mathbb{R}^n$ into a ball. Note that if $\gamma \in \text{Isom} H^n$ fixes $0$, then it acts as a euclidean similarity on $\mathbb{H}^n$.

A third model for hyperbolic space we shall use is the Klein model. This consists of the open unit ball with a (non-conformal) Riemannian metric, such that all hyperbolic geodesics correspond to euclidean line segments [see [Bea, Chapter 7]].

We may classify non-trivial isometries $H^n$ into three types, namely elliptic, parabolic and hyperbolic as follows.

Let $\gamma$ be an isometry of $H^n$. We write $\Sigma \gamma$ for the set of fixed points of $\gamma$ in $H^n$. Brouwer's fixed point theorem tells us that $\Sigma \gamma$ must be non-empty.

Suppose that there is some point $x$ in $\Sigma \gamma \cap H^n$. We may take $x$ to be the centre of the ball in the Poincaré model. Then, $\gamma$ acts as a euclidean rotation on the ball, and we see that $\Sigma \gamma$ is a (possibly $0$-dimensional) plane in $H^n$. We call this case elliptic.

If $\gamma$ is not elliptic, then $\Sigma \gamma$ is a subset of $\mathbb{H}^n$. Suppose that $\Sigma \gamma$ consists of just a single point in $\mathbb{H}^n$. We may take this point to be $0$ in the upper half space model, $\mathbb{R}^n$. Now, since $\gamma$ has no fixed point in $\partial \mathbb{H}^n$, it must act as a euclidean isometry of $\partial \mathbb{H}^n$. Moreover, it must preserve setwise each horosphere about $0$. We call this case parabolic.

Suppose that $\gamma$ fixes precisely two points, $x$ and $y$, in $H^n$. Let $l$ be the geodesic joining $x$ to $y$. In this case, $\gamma$ acts as a translation on $l$, and (in general) has a rotational component in the orthogonal direction. We call this case loxodromic, and we call $l$ the loxodromic axis.

Finally, note that if $\gamma$ has three (or more) fixed points in $H^n$, then these must determine a fixed point in $\mathbb{H}^n$, so we are back in the elliptic case.

### 0.2. Groups of Isometries.

Let $\Gamma$ be a subgroup of $\text{Isom} H^n$. It is an elementary result that $\Gamma$ is a discrete subgroup if and only if it acts properly discontinuously on $H^n$, that is to say, each compact subset of $H^n$ meets only finitely many images of itself under $\Gamma$.

In such a discrete group, the finite-order elements are precisely the elliptic isometries. Thus, $\Gamma$ acts freely if and only if it is torsion-free. If $\Gamma$ acts freely, we may form the quotient manifold $M = H^n / \Gamma$ which inherits a complete hyperbolic structure.

More generally, if $\Gamma$ has torsion, the quotient $M = H^n / \Gamma$ is a complete hyperbolic "orbifold", as defined by Thurston [Th1, chapter 13]. That is to say, there is a closed cell complex $\Sigma$ in $M$, such that $M \setminus \Sigma$ is an (incomplete) hyperbolic manifold. The set $\Sigma$ can be defined as the projection of the set of all fixed points of elliptic elements of $\Gamma$, i.e., $\Sigma = \bigcup_{\gamma \in \Gamma} (\Sigma \gamma \cap H^n) / \Gamma$. A neighbourhood of a point of $\Sigma$ may or may not be topologically singular, but it will always be geometrically singular. In an orientable 2-orbifold, for example, $\Sigma$ consists of a discrete set of cone singularities, which may be thought of as points of concentrated positive curvature. We shall call $\Sigma$ the singular set of $M$.

Let $\Gamma \subset \text{Isom} H^n$ be discrete. The action of $\Gamma$ may be extended to $\mathbb{H}_c^n$, and we may define the limit set $A \subset \mathbb{H}_c^n$ as the set of accumulation points of some $\Gamma$-orbit, i.e.,

$$A = \{ y \in \mathbb{H}_c^n | \text{there exist } \gamma_n \in \Gamma \text{ and } x \in H^n \text{ with } \gamma_n x \rightarrow y \}.$$ 

It turns out that this definition is independent of our choice of $x$. Moreover, $A$ is a minimal closed $\Gamma$-invariant set, and $\Gamma$ acts properly discontinuously on its complement $\Omega$ in $\mathbb{H}_c^n$. The set $\Omega = \mathbb{H}_c^n \setminus A$ is called the discontinuity domain. (It is possible for $\Omega$ to be empty.) We may form the quotient orbifold $M_\Gamma = \Omega / \Gamma$. 


of \( \Omega \). Since \( \Gamma \) acts conformally on \( \mathbb{H}^n \), we see that \( M \) inherits a (singular) conformal structure from \( \Omega \). In fact, \( \Gamma \) acts properly discontinuously on \( \mathbb{H}^n \cup \Omega \), so we may write

\[
M_G = (\mathbb{H}^n \cup \Omega) / \Gamma = M \cup M_I.
\]

Note that when \( n = 3 \), \( M \) is a Riemann surface (in general not connected). This fact gives rise to a rich analytical theory in this dimension.

One direction of research in discrete hyperbolic groups, is study to the relationship of various types of "finiteness" — group theoretic, topological and geometric. The simplest group theoretic restriction is to demand that \( \Gamma \) be finitely generated, and ask what this tells us about the topology and geometry of \( M \).

The first result is pure algebraic.

Selberg Lemma \([56]\). Let \( k \) be a field of characteristic 0. Then, any finitely-generated subgroup of \( \text{GL}_n(k) \) is virtually torsion-free, (i.e. contains a torsion-free subgroup of finite index).

For a simpler proof, see \([Cas]\).

Since \( \text{Isom} \mathbb{H}^n \) can be represented as a subgroup of \( \text{GL}_n(k) \), the Selberg Lemma can be applied to finitely-generated subgroups of \( \text{Isom} \mathbb{H}^n \). Geometrically, this tells us that we can restrict attention to the case where \( \Gamma \) acts freely on \( \mathbb{H}^n \). Given this, we may as well assume also that \( \Gamma \) preserves orientation. This latter restriction is solely to simplify the exposition. Thus, for the rest of the introduction, unless otherwise stated, we shall be taking \( \Gamma \) to be a finitely-generated, discrete, torsion-free group of orientation preserving isometries of \( \mathbb{H}^n \).

Beyond the Selberg Lemma, little seems to be known in general. The main thrust of research is in dimension 3, and we shall give a summary of 3-dimensional results in Section 0.3. First, we describe how the 2-dimensional case is trivial from the point of view of finiteness.

Let \( M \) be a complete, orientable, hyperbolic surface with finitely-generated fundamental group. Then, it turns out that \( M \) consists of a compact surface with boundary, together with a finite number of "cusps" and "funnels". A cusp is (isometric to) a horoball in \( \mathbb{H}^3 \), quotiented out by a cyclic parabolic group (FIG 0.1). A funnel consists of a hyperbolic half-space quotiented out by a loxodromic element (FIG 0.2). We see that \( M_I \) is a disjoint union of finitely many circles, which serve to compactify the funnels in \( M_G = M \cup M_I \).

Thus the topological ends of \( M_G \) correspond precisely to the cusps (FIG 0.3). We see that, in any meaningful sense, the geometry of \( M \) is only finitely complicated. This is about the strongest assertion of finiteness one could make.

3. Some 3-dimensional finiteness results.

In this section, we shall give a summary of some finiteness results in 3 dimensions. It is not meant to reflect the historical development of the subject.

Let \( \Gamma \) be a discrete, torsion-free, orientation-preserving subgroup of \( \text{Isom} \mathbb{H}^3 \). Much of the technical complication of the subject arises from having to deal with parabolic subgroups of \( \Gamma \). Suppose that \( \gamma \in \Gamma \) is parabolic with fixed point \( p \). Let \( \Gamma_p \) be the stabiliser of \( p \) in \( \Gamma \). In a discrete group, a parabolic and a loxodromic cannot share a common fixed point. Thus, \( \Gamma_p \) consists entirely of parabolics. We call \( p \) a parabolic fixed point, which we abbreviate to \( p.f.p. \). We can let \( p \) be the point \( 0 \) in the upper half-space model. Now, \( \Gamma_p \) acts freely as a group of isometries on \( \mathbb{H}^3 \). This dimension is special in that such a group must act by translation. We see that \( \Gamma_p \) is isomorphic to either \( \mathbb{Z} \) or \( \mathbb{Z} \oplus \mathbb{Z} \). Taking \( B \) to be any horoball about \( p \), we may form the quotient \( B/\Gamma_p \). If \( \Gamma_p \cong \mathbb{Z} \), then \( B/\Gamma_p \) is a bi-infinite euclidean cylinder, and we call \( B/\Gamma_p \) a \( \mathbb{Z} \)-cusp (FIG 0.4). (Note that a \( \mathbb{Z} \)-cusp is not quite the same as a "cusp cylinder" or "standard cusp region" which will be described later in this section. See Section 2 for details.) If \( \Gamma_p \cong \mathbb{Z} \oplus \mathbb{Z} \) then \( B/\Gamma_p \) is a euclidean torus, and we call \( B/\Gamma_p \) a \( \mathbb{Z} \oplus \mathbb{Z} \)-cusp (FIG 0.5). We may define such cusps to correspond to each orbit of parabolic fixed points. In general, one could expect these cusps to project to a collection of immersed submanifolds in \( M \). However the Margulis Lemma (see Sections 1.2, and 2.2(164)) tells us that (in dimension 3), by taking our horoballs small enough, we can arrange that the cusps be disjoint and embedded in \( M \). We shall write \( cusp(M) \) for the disjoint union of all the cusps.
The construction of this set of disjoint cusps is valid for infinitely-generated groups. From now on, however, we shall insist that \( \Gamma \) be finitely-generated. We first use a purely topological result.

**Theorem (Scott [Sc]).** Let \( M \) be a 3-manifold with finitely-generated fundamental group. Then, there is a compact submanifold \( M_T \) of \( M \), such that the inclusion \( M_T \hookrightarrow M \) induces an isomorphism of fundamental groups.

We call \( M_T \) a topological core for \( M \). With \( M = \mathbb{H}^n / \Gamma \), we deduce immediately that \( \Gamma \) is finitely presented.

In our case, \( M = \mathbb{H}^n / \Gamma \) is an irreducible 3-manifold, that is each embedded 2-sphere in \( M \) bounds a 3-ball. Because of this, we can arrange that \( \partial M_T \) contains no 2-spheres, and then the inclusion of \( M_T \) into \( M \) is a homotopy equivalence. Moreover, there is a bijective correspondence between the boundary components of \( M_T \) and the topological ends of \( M \). We deduce that \( M \) has only finitely many ends. In particular, it contains only finitely many \( \mathbb{Z} \oplus \mathbb{Z} \)-cusps.

In fact (provided that \( \Gamma \) is not cyclic loxodromic), the \( \mathbb{Z} \oplus \mathbb{Z} \)-cusps correspond precisely to the toroidal components of \( \partial M_T \). The remaining ends correspond to components of genus at least 2. The aim now is to understand something of the geometry of these remaining ends, which we shall call "non-cuspidal ends".

Now, a \( \mathbb{Z} \)-cusp is topologically just a product. Thus, we can assume that each \( \mathbb{Z} \)-cusp lies entirely within some non-cuspidal end. The effect of removing the \( \mathbb{Z} \)-cusps would (in general) be to subdivide each such end into smaller pieces, on which we may see qualitatively different behaviour. It is therefore necessary to take account of these \( \mathbb{Z} \)-cusps before going on to consider the geometry. We can do this by applying a relative version of Scott's theorem to \( M' = M / \cusp(M) \).

**Theorem [M e].** Let \( N \) be a 3-manifold with boundary, whose fundamental group is finitely generated. Let \( S \) be a compact submanifold of \( \partial N \). Then, we can find a topological core, \( N_T \), for \( N \) such that \( N_T \cap \partial N = S \).

By using this result, together with an Euler characteristic argument, one may deduce [FM] that there are only a finite number of \( \mathbb{Z} \)-cusps — a result due originally to Sullivan [Sull]. We may now take a core \( M_T \) of \( M' \) which meets each \( \mathbb{Z} \oplus \mathbb{Z} \)-cusp in the bounding torus, and each \( \mathbb{Z} \)-cusp in a compact annular core of its boundary cylinder. Again, we may take the inclusion to be a (relative) homotopy equivalence, so that the topological ends of \( M' \) correspond to the frontier components of \( M_T \). We now look for geometric information about the non-cuspidal ends of \( M' \) (i.e. ends other that \( \mathbb{Z} \oplus \mathbb{Z} \)-cusps).

We have already said that, for \( n = 3 \), \( M_f / \Gamma \) is a Riemann surface. A fundamental result about \( M_f \) is the following.

**Ahlfors' Finiteness Theorem [Ah1, Sull].** Let \( \Gamma \) be a finitely-generated discrete subgroup of \( \text{Isom} \mathbb{H}^3 \). Then \( M_f = \mathbb{H}^3 / \Gamma \) is a Riemann surface of "finite type". That is to say, \( M_f \) is conformally equivalent to a compact surface with finitely many punctures.

(For a proof using deformation theory, see [Sul4].)

Moreover one may show that the punctures of \( M_f \) arise only from parabolic elements of \( \Gamma \); that is, a small loop around a puncture represents a conjugacy class of parabolics in \( \Gamma \).

We want to give Ahlfors' Finiteness Theorem a more geometric interpretation. We can do this by using the convex hull of the limit set — a generalisation of the Nielsen convex region in dimension 2. Let \( Y \) be the smallest convex set in \( \mathbb{H}^3 \) whose closure, \( Y_c \), in \( \mathbb{H}^3 \) contains the limit set \( A \). Then, \( Y_c \) meets \( \mathbb{H}^3 \) precisely in \( A \). Since the construction is equivariant, we may form the quotient \( \tilde{Y} = Y / \Gamma \subset M \), which we call the convex core of \( M \). The nearest point retract of \( \mathbb{H}^3 \) onto \( Y \) extends continuously to all of \( \mathbb{H}^3 \), and therefore given rise to a map from \( M_f \) to \( \tilde{Y} \) (see for example [Thl]). We shall denote by \( q \), the restriction of this map to \( M_f \). Note that \( q(M_f) = \partial \tilde{Y} \).

It is possible for \( \tilde{Y} \) to have empty interior, but if so, then \( \Gamma \) is either abelian or "fuchsian" (i.e. preserves some 2-plane in \( \mathbb{H}^3 \)). Both these cases are completely understood, so we shall assume that the interior of \( \tilde{Y} \) is non-empty. In this case one may show that \( \partial \tilde{Y} \) has the structure of a complete hyperbolic surface in the induced metric. In fact, by applying some kind of smoothing to the nearest point retraction, one may show that \( q \) is homotopic to a quasi-conformal homeomorphism. (EM includes details of this in the case when \( A \) is connected.)
Finiteness Theorem to say that \( \partial \mathcal{Y} \) should have finite 2-dimensional area. (In fact the discussion applies equally well if \( \Gamma \) has torsion, and then \( \partial \mathcal{Y} \) becomes a finite-area orbifold.)

The parabolic cusps of the hyperbolic surface \( \partial \mathcal{Y} \) are essentially the connected components of \( \partial \mathcal{Y} \cap \text{cusp}(M) \). In fact the cusps of \( \partial \mathcal{Y} \) must lie inside \( \mathcal{Z} \)-cusps of \( M \). The remainder of \( \partial \mathcal{Y} \), namely \( \partial \mathcal{Y} \cap M' \), is compact. Then, each component of \( \partial \mathcal{Y} \) corresponds to an end of \( M' \). Such an end is topologically a product, being foliated by components of \( \partial N_r(\mathcal{P}) \) for \( r > 0 \), where \( \mathcal{N}_r(\mathcal{P}) \) is a uniform \( r \)-neighbourhood of \( \mathcal{P} \). We call such ends geometrically finitely \( (GF) \). We see that the GF ends of \( M \) correspond bijectively to components of \( \partial \mathcal{Y} \), and thus to components of \( \partial \mathcal{Y} \cap M' \). (We may think of \( M' \) as the limit of the surfaces \( \partial \mathcal{Y} \).)

If we fix some \( \eta > 0 \), we can modify the topological core \( M' \), so that \( \partial \mathcal{N}_\eta(\mathcal{P}) \cap M' \) becomes a subset of the frontier of \( M' \). That is the frontier components of \( M' \) in \( M' \) that correspond to GF ends coincide with frontier components of \( \partial \mathcal{Y} \cap M' \).

The geometrically finite ends, however, might not account for all the ends of \( M' \). It may be that an end makes no impression on the discontinuity domain \( \Omega \), so that Ahlfors' Finiteness Theorem tells us nothing. Such ends were shown to exist by Bers and Maskit [Ber, Mas], their geometrically infinite nature being made explicit by Greenberg [G]. Jorgensen later described more concrete examples [2]. Thurston [Th2] gives a more general method of construction.

All the non-GF ends constructed so far have been "simply degenerate" as defined by Thurston [Th1 Chapter 9]. A simply degenerate end turns out to be just a product topologically (i.e. homeomorphic to a surface times a half-open interval), but its geometry is infinite. For example, every neighbourhood of the end will contain infinitely many closed geodesics. Bonahon and Otal construct an example of an end containing closed geodesics of arbitrarily small length [BoO]. There are also examples where lengths of closed geodesics have a positive lower bound. In the latter case the end has bounded diameter as one tends to infinity. In general, one may say that the volume of a simply-degenerate end grows at most linearly. This explains why such an end makes no impression on the discontinuity domain — GF ends have exponential growth.

If, as in all the examples constructed so far, each (non-cuspidal) end is either geometrically finite or simply degenerate, we call \( M \) geometrically tame. In this case, \( M \) is topologically finite, i.e. homeomorphic to the interior of a compact manifold with boundary. Moreover, one can show that the limit set of such a group has either zero or full 2-dimensional Lebesgue measure (see [Th1] or [Bo]) — a property conjectured, by Ahlfors, for all finitely-generated discrete groups. There are examples, however, where the limit set has Hausdorff dimension equal to 2, while still having zero 2-dimensional Lebesgue measure [Sul].

It has been conjectured that all finitely-generated discrete groups are geometrically tame. Bonahon [Bo] has proven this under the hypothesis that for any free-product decomposition \( \Gamma = A * B \), there is some parabolic in \( \Gamma \) not conjugate to any element of \( A \) or \( B \). Otal has recently claimed the result for \( \Gamma = \mathbb{Z} * \mathbb{Z} \).

We now restrict attention to the case where all the ends of \( M' = M \setminus \text{cusp}(M) \) are geometrically finite. Then, we call \( M \) "geometrically finite". In this case, we can assume that each end of \( M' \) is bounded by a component of \( \partial N_r(\mathcal{P}) \), which means that we can take the topological core \( M_r^* \) to be equal to \( \partial N_r(\mathcal{P}) \cap M' = N_r(\mathcal{P}) \setminus \text{cusp}(M) \). In other words, geometric finiteness says that \( N_r(\mathcal{P}) \setminus \text{cusp}(M) \) is compact. This is more or less the definition of geometric finiteness (GF1) due to Thurston [Th1 Chapter 9] (see Section 2(GF1)).

Taking the \( r \)-neighbourhood of the convex core allows us to include Fuchsian groups and cyclic loxodromic groups in the discussion, without making special qualifications.

Clearly, \( N_r(\mathcal{P}) \) meets the boundary of any \( \mathcal{Z} \)-cusp in a compact set. From this we see that the intersection of \( N_r(\mathcal{P}) \) with any \( \mathcal{Z} \)-cusp has finite volume. (In fact the intersection will be contained in some \( r \)-neighbourhood of a totally geodesic 2-dimensional cusp — see FIG 0.6.) Since each \( \mathcal{Z} \)-cusp has finite volume, we arrive at Thurston's second definition of geometric finiteness (GF2), namely that \( N_r(\mathcal{P}) \) should have finite volume. (For the definition GF1, it is enough to insist that \( \mathcal{P} \setminus \text{cusp}(M) \) be compact. For GF2, however, it is essential to take some uniform \( \rho \)-neighbourhood of \( \mathcal{P} \), as the example of an infinitely generated Fuchsian group shows.)

If \( M \) had no cusps, one sees that \( M_1 = \Omega/\Gamma \) would give a compactification of \( M \) to \( M_2 \). In the general case, the topological ends of \( M_2 \) correspond precisely to the cusps. In fact, each end of \( M_2 \) has a neighbourhood isometric to one of two standard types — "cusp tori" and "cusp cylinders". Cusp tori are the same as \( \mathcal{Z} \)-cusp, whereas a cusp cylinder is an enlargement of a \( \mathcal{Z} \)-cusp to include a portion of \( M' \).

A fourth description (GF4), due to Beardon and Maskit [Bea,M], demands that the limit set should
be a union of (what we call here) "conical limit points" and "bounded parabolic fixed points". These will be defined in Section 2 (GF2). The notion of a conical limit point (also called a "radial limit point" or "approximation point") originates in \( \text{[H]} \), and has proven useful to the study of dynamics on limit sets.

Finally, the original and simplest definition of geometric finiteness (GF3) demands that \( \Gamma \) should possess a finite-sided convex fundamental polyhedron. This hypothesis was introduced by Ahlfors \([\text{Ah2}]\), where he showed that the limit set of such a group must have either zero or full Lebesgue measure in \( \mathbb{H}^2 \).

It has been known for some time, from the references already cited, that these five definitions are all equivalent in dimension 3. Geometrically finite groups occur frequently as the simplest examples of 3-dimensional hyperbolic groups. It is conjectured that they contain an open dense set of the space of all finitely-generated discrete groups, given the appropriate topology (see \([\text{Sul5}]\)). The hypothesis of geometric finiteness has often been used in the study of the dynamics on limit sets. Sullivan, for example, showed that the limit set of a geometrically finite group is either the whole sphere \( \mathbb{H}^2 \), or else has Hausdorff dimension strictly less than 2 \([\text{Sul3}]\).

0.4. Higher Dimensions.

A natural question to ask is how one should define geometric finiteness in dimensions greater than 3. Most authors have taken geometrical finiteness in this case to mean that the group should possess a finite-sided convex fundamental polyhedron — a direct generalisation of Ahlfors' original definition. However, in dimension 4 and higher, this definition becomes more restrictive than the obvious generalisations of the other four definitions. It seems that these other definitions give rise to a more natural notion of geometric finiteness which we aim to elucidate in this work. All the applications of the traditional geometrical finiteness hypothesis seem to be valid for this slightly more general notion.

The question of defining geometric finiteness in higher dimensions has also been considered by Apanasov \([\text{Ap1, Ap2}]\), as well as by Weilenberg \([\text{We}]\) and Tukia \([\text{Tul}]\). In \([\text{Tu2}]\), Tukia generalises, to dimension \( n \), Sullivan's result about the Hausdorff dimension of the limit set. Thus, the limit set of a GF group is either equal to \( \mathbb{H}^n \), or else has Hausdorff dimension less than \( n - 1 \).

1. The Margulis Lemma and Bieberbach Theorem.

In this section we shall be discussing results related to the Margulis Lemma and Bieberbach Theorems. One form of the Margulis Lemma says the following. Given any positive integer \( n \), we can find some \( c(n) > 0 \) with the following property. Let \( (X, d) \) be any simply connected Riemannian \( n \)-manifold, all of whose sectional curvatures lie in the closed interval \([0, 1]\). Let \( \Gamma \) be any discrete group of isometries acting on \( X \), and \( x \in X \) be any point. Let \( \Gamma, (x) \) be the group generated by those elements of \( \Gamma \) such that \( d(\gamma(x), x) < c(n) \). In symbols, \( \Gamma, (x) = \{ \gamma \in \Gamma \mid d(\gamma(x), x) < c(n) \} \). Then, \( \Gamma, (x) \) is virtually nilpotent (i.e., it contains a nilpotent subgroup of finite index). Moreover, the index of the nilpotent subgroup in \( \Gamma, (x) \) can be bounded by some \( c(n) \) depending only on \( n \). We say that groups of the form \( \Gamma, (x) \) are uniformly virtually nilpotent.

A proof of this result may be found in \([\text{RaGS}]\). In this section, we shall restrict attention to the constant curvature cases, namely \( \mathbb{E}^n \) and \( \mathbb{H}^n \), where we can give a simple proof of the Margulis Lemma. Also, in these cases we may identify the nilpotent subgroup as being generated by elements of small rotational part, and it turns out always to be abelian. This final observation is a consequence of nilpotency, rather than discreteness, so we begin with a discussion of nilpotent groups of isometries in the geometries \( \mathbb{E}^n \) and \( \mathbb{H}^n \). We shall prove that nilpotent subgroups of Isom \( \mathbb{E}^n \), Sim \( \mathbb{E}^n \), and Isom \( \mathbb{H}^n \) are uniformly virtually abelian. This fact seems to be well known, though I know of no explicit reference. However, all the essential ingredients may be found in \([\text{Th1, Chapter 4}]\). We shall go on to show how nilpotent groups arise out of discrete isometry groups. In the course of the discussion we deduce some of the classical Bieberbach Theorems. These results
are also described in [Th2, Chapter 4] and [Wo].

1.1. Nilpotent Implies Virtually Abelian.

Let $S^n$, $E^n$ and $H^n$ denote the unit $n$-sphere, euclidean $n$-space and hyperbolic $n$-space respectively, with metrics $d_{S^n}$, $d_{E^n}$, and $d_{H^n}$. We shall omit the suffixes where there can be no confusion. Let $\text{Isom } X$ denote the entire group of isometries of $X$, and $\text{Sim } E^n$ be the group of euclidean similarities. Throughout, we use the convention on commutators that $[x,y] = xyx^{-1}y^{-1}$.

We shall deal with the three geometries in turn.

1.1(1). Spherical Geometry.

Let

$$U(S^n) = \{ \gamma \in \text{Isom } S^n \mid d(\gamma x, x) < \pi/2 \text{ for all } x \in S^n \}.$$

If we think of $\text{Isom } S^n$ acting on $E^{n+1}$, this says that $\gamma$ lies in $U(S^n)$ if it moves each vector through an acute angle. In other words, $(\gamma u, v) > 0$ for each non-trivial vector $u \in E^{n+1}$, where $(, )$ is the inner product defining the metric on $E^{n+1}$.

Let $\gamma \in \text{Isom } S^n$. By complexifying, we can extend $\gamma$ to act on $C^{n+1}$. Now, $\gamma$ preserves the standard hermitian form on $C^{n+1}$, i.e. the form that restricts to the inner product on $E^{n+1}$. We also use $(, )$ to denote this hermitian form.

Now, let $u \in C^{n+1}$ be any non-trivial complex vector. Write $u = x + iy$, with $x, y \in E^{n+1}$. Then,

$$Re(\gamma u, v) = (\gamma x, x) + (\gamma y, y).$$

If $\gamma \in U(S^n)$, both the terms on the right hand side are non-negative, and at least one is strictly positive. It follows that $\gamma$ lies in $U(S^n)$ if and only if $Re(\gamma v, v) > 0$ for each non-trivial $v \in C^{n+1}$.

We can now prove:

**Lemma 1.1.1** Let $\beta \in U(S^n)$ and $\alpha \in \text{Isom } S^n$. If $\alpha$ commutes with $[\alpha, \beta]$, then $\alpha$ commutes with $\beta$.

**Proof:** Complexifying, we imagine $\alpha$ and $\beta$ acting on $C^{n+1}$. We see that $\alpha$ commutes with $\beta^{-1}\alpha\beta$, so that they are simultaneously diagonalizable. Let $V$ be an eigenspace of $\alpha$. Then $\beta V$ is an eigenspace of $\beta\alpha\beta^{-1}$. If $V \neq \beta V$, then $V$ must intersect non-trivially some other eigenspace $V'$ of $\beta\alpha\beta^{-1}$, orthogonal to $\beta V$. Let $u \in V \cap V'$ be non-zero. Then $\beta u$ lies in $\beta V$, so that $(\beta u, u) = 0$. However, since $\beta$ lies in $U(S^n)$, the discussion immediately prior to the lemma tells us that $Re(\beta u, u) > 0$. This contradiction means that $\beta V = V$. Since $V$ was an arbitrary eigenspace of $\alpha$, we deduce that $\alpha$ and $\beta$ are simultaneously diagonalizable, and hence commute.

$\diamond$

**Corollary 1.1.2** If $T \subset \text{Isom } S^n$ is nilpotent, then $(S^n \cap U(S^n))$ is abelian.

**Proof:** Let $\alpha$ and $\beta$ lie in $S^n \cap U(S^n)$. By a "nested chain of commutators" in $\alpha$ and $\beta$, we mean an expression of the form $d = [c_1, [c_2, \ldots, [c_{n-1}, c_n] \ldots ]$, where each $c_i$ is either $\alpha$ or $\beta$. We take $d$ to be of maximal length, $n$, such that $d \neq 1$. This means that $d$ commutes with both $\alpha$ and $\beta$. It follows that $[c_1, \ldots, [c_{n-1}, c_n] ]$ commutes with $d$. Applying Lemma 1.1.1, with $\alpha = [c_1, \ldots, [c_{n-1}, c_n] ]$ and $\beta = 1$, we deduce that $\alpha$ and $\beta$ commute, so that $d = 1$. We have contradicted the assumption that $n \geq 1$, and so $\alpha$ must commute with $\beta$.

$\diamond$

Let $V$ be an open symmetric neighbourhood of the identity in $\text{Isom } S^n$ such that $V^3 \subset U(S^n)$. There is an upper bound $N(n)$ on the number of disjoint translates of $V$ by $\text{Isom } S^n$ that we can embed in $\text{Isom } S^n$. We deduce that $|\Gamma : (\Gamma \cap U(S^n))| < N(n)$, and so,
Corollary 1.1.3: Nilpotent subgroups of Isom $S^n$ are uniformly virtually abelian.

1.1(11). Euclidean Geometry.

To prepare for the hyperbolic case, it will be useful to consider the group $\text{Sim} E^n$ of euclidean similarities. Let $\Phi$ be the set of parallel classes of (semi-infinite) geodesic rays in $E^n$. We shall embed $\Phi$ as the unit $(n-1)$-sphere in an inner-product space $V(E^n)$, which we can imagine as euclidean space with a preferred basepoint. There is an obvious bijective correspondence between $r$-dimensional subspaces of $V(E^n)$, and foliations of $E^n$ by parallel $r$-planes.

The group $\text{Sim} E^n$ acts isometrically on $\Phi$, so identifying $\Phi$ with $S^{n-1}$ gives us a homomorphism

$$\text{rot} : \text{Sim} E^n \to \text{Isom} S^{n-1}.$$  

We call rot the rotational part of $\gamma$. We define

$$U(E^n) = \{ \gamma \in \text{Sim} E^n | \text{rot } \gamma \in \text{Isom} S^{n-1} \}.$$  

Note that if we embed $E^n$ as a plane in $E^m$, then $U(E^n)$ may be obtained by intersecting $U(E^n)$ with the stabilizer of this plane. This observation will allow us to use induction over dimension. Given $\gamma \in \text{Sim} E^n$, we shall write

$$\text{min } \gamma = \{ x \in E^n | d(x, \gamma) \text{ is minimal} \}.$$  

Then, min $\gamma$ is a plane in $E^n$ on which $\gamma$ acts either trivially or by translation. Of course, min $\gamma$ may consist of just a single fixed point.

Theorem 1.1.4: If $\Gamma \subset \text{Sim} E^n$ is nilpotent, then $(\Gamma \cap U(E^n))$ is abelian.

We shall begin with a lemma.

Lemma 1.1.5: Let $\Gamma$ be an abelian subgroup of $\text{Sim} E^n$. Let $\gamma(\Gamma) = \bigcap_{\gamma \in \Gamma} \text{min } \gamma$. Then, $\gamma(\Gamma)$ is a non-empty, $\Gamma$-invariant plane, on which $\Gamma$ acts by translations.

Proof: If $\Gamma$ is already a translation group, then $\gamma(\Gamma) = E^n$, and we are done. Otherwise, choose any $\gamma \in \Gamma$ which is not a translation. Then, $\text{min } \gamma$ is a proper subspace, and since $\Gamma$ is abelian, it is $\Gamma$-invariant. The result now follows by induction on dimension.

In fact, our plane $\gamma(\Gamma)$ has a natural foliation by (in general) smaller $\Gamma$-invariant planes, namely the set of minimal $\Gamma$-invariant planes. That is to say, each leaf is obtained as the affine span of some $\Gamma$-orbit. This foliation determines a subspace $W_1$ of $V(E^n)$, by taking the set of geodesic rays lying in any one leaf. Now, $W_1$ lies in a larger subspace $W'$ of $V(E^n)$, determined by $\gamma(\Gamma)$ itself. Let $W_2$ be the orthogonal complement of $W_1$ in $W'$, and $W_3$ be the orthogonal complement of $W'$ in $V(E^n)$. This gives us a canonical decomposition $V(E^n) = W_1 \oplus W_2 \oplus W_3$. Let $m_1$ be the dimension of $W_1$. We shall say that the decomposition is trivial if $m_1 = n$ for some $i$.

If $m_1 = n$, then $\Gamma$ is a pure translation group, and the directions of translations span $E^n$. If $m_3 = n$, then each point of $E^n$ is a fixed point of $\Gamma$, thus $\Gamma$ is trivial. If $m_3 = n$, then $\Gamma$ has a unique fixed point in $E^n$. We are now ready for:

Proof of Theorem 1.1.4: Let $\Gamma$ be a nilpotent subgroup of $\text{Sim} E^n$. We shall assume that $\Gamma$ is generated by elements of $U(E^n)$, i.e. that $\Gamma = (\Gamma \cap U(E^n))$. We want to show that $\Gamma$ is abelian.

Let $\xi(\Gamma)$ be the centre of $\Gamma$. From the preceding discussion, $\xi(\Gamma)$ determines a decomposition $W_1 \oplus W_2 \oplus W_3$ of $V(E^n)$. Since this is canonical, it is respected by the whole group $\Gamma$. Thus $\Gamma$ splits as a subgroup of $\text{Sim} E^{n_1} \times \text{Sim} E^{n_2} \times \text{Sim} E^{n_3}$, and the projection of $\Gamma$ onto each component is nilpotent. If the decomposition is non-trivial, we may suppose, by induction on dimension, that each projection of $\Gamma$
is abelian. It then follows that $\Gamma$ itself is abelian. We need therefore deal only with the cases when the decomposition is trivial.

Suppose $m_1 = n$. This means that $Z(\Gamma)$ is a translation group with no non-empty proper invariant plane in $E^n$. Consider any $\gamma \in \Gamma$. Since $\gamma$ commutes with everything in $Z(\Gamma)$, $\min \gamma$ is $Z(\Gamma)$-invariant, and hence equal to $E^n$. It follows that $\gamma$ is a translation of $E^n$. Since translations commute, $\Gamma$ is abelian.

Suppose $m_2 = n$. Now $Z(\Gamma)$ is trivial. Since $\Gamma$ is nilpotent, it is also trivial.

Finally, suppose $m_3 = n$. In this case, $Z(\Gamma)$ has a unique fixed point in $E^n$. This point must be fixed by $T$, so $T$ can be regarded as a subgroup of $\mathbb{R} \times \text{Isom} S^n$, where the first component measures the magnification, and the second, the rotational part of an element. The projection into $\text{Isom} S^n$ is nilpotent and generated by elements of $U(S^n)$. By Corollary 1.1.2, this projection is abelian. We deduce that $\Gamma$ is abelian.

As in the spherical case, for any group $\Gamma$, the index of $(\Gamma \cap U(E^n))$ in $\Gamma$ is finite, and has a bound dependent only on $n$. Thus,

Corollary 1.1.6: Nilpotent subgroups of $\text{Sim} E^n$ are uniformly virtually abelian.

1.1(iii). Hyperbolic Geometry.

We shall write $H^n$ for the ideal $(n - 1)$-sphere at infinity of hyperbolic space $H^n$, and write $\mathbb{E}^n$ for the compactification of hyperbolic space as $H^n \cup \mathbb{E}^n$. By a Möbius transformation on the sphere $S^n$, we mean any map which can be represented as a composition of inversions in $(n - 1)$-spheres. (We are allowing Möbius transformations that reverse orientation.)

We may represent $H^n$, conformally, as a hemisphere $E$ of $S^n$. $\text{Isom} H^n$ then consists of those Möbius transformations which preserve $E$. Let $\gamma$ be a Möbius transformation of $S^n$ with some fixed point $y$. Since $\gamma$ acts conformally, it induces (after scaling) an isometry of the unit tangent space $(T_Y S^n)$ at $y$. Moreover, we may check that if $z$ is any other fixed point of $\gamma$, then the induced isometries on $(T_z S^n)$ and $(T_y S^n)$ are conjugate. Thus, $\gamma$ determines a conjugacy class in $\text{Isom} S^n$, which we shall call $\text{rot} \gamma$. Since our subset $U(S^n)$ of $\text{Isom} S^n$ is invariant under conjugacy, it makes sense to demand that $\text{rot} \gamma$ should lie in $U(S^n)$.

Restricting to $\text{Isom} H^n$, where all Möbius transformations have fixed points, we may define

$$U(H^n) = \{ \gamma \in \text{Isom} H^n | \text{rot} \gamma \subset U(S^n) \}.$$

Theorem 1.1.7: If $\Gamma \subset \text{Isom} H^n$ is nilpotent, then $(\Gamma \cap U(H^n))$ is abelian.

We begin with two lemmas.

Lemma 1.1.8: If $\Gamma \subset \text{Isom} H^n$ is abelian, then $x \Gamma$, the set of points fixed by $\Gamma$, consists of either one or two points in $H^n$, or else is a subspace of $H^n$ (i.e. the closure, in $H^n$, of a plane in $H^n$).

Proof: Let $\gamma$ be any non-trivial element of $\Gamma$. If $\gamma$ is parabolic, then its fixed point is preserved by $\Gamma$, so that $\Gamma$ has a unique fixed point. If $\gamma$ is elliptic, then $\min \gamma$ is a proper $\Gamma$-invariant subspace, and we use induction on dimension. For this, we need to check the 1-dimensional case. But it is easily seen that an abelian group of isometries of the real line must either act trivially, or by translation (thus respecting the two "ideal" points), or else consist of an involution with a single fixed point. Finally, if $\gamma$ is loxodromic, then its axis is $\Gamma$-invariant, and we are immediately reduced to the 1-dimensional case.

Lemma 1.1.9: Suppose $\Gamma \subset \text{Isom} H^n$ is nilpotent, then $\Gamma$ has a fixed point in $H^n$.

Proof: Let $\sigma$ be the set of points fixed by the centre $Z(\Gamma)$. Let $\Gamma' = Z(\Gamma)$ be the subgroup that fixes $\sigma$ pointwise. Since $\sigma$ is canonical with respect to $\Gamma$, $\Gamma'$ is normal in $\Gamma$. Thus $\Gamma/\Gamma'$ is nilpotent, and acts effectively on $\sigma$. 9
From Lemma 1.1.8, we distinguish three possibilities for \( \sigma \). Firstly, if \( \sigma \) is a single point of \( \mathbb{H}^n \), this point is fixed by \( \Gamma \), and we are done. Secondly, if \( \sigma \) is a proper subspace of \( \mathbb{H}^n \), we use induction on dimension. Thus, we may assume that we are in the third case, namely that \( \sigma \) consists of precisely two points, \( z \) and \( y \), in \( \mathbb{H}^n \). If \( \Gamma / \Gamma' \) is trivial, we are done. Therefore we may suppose that \( \Gamma / \Gamma' \) is an involution. This means that there is some \( \gamma \in \Gamma \) that swaps \( z \) and \( y \). Now, each element of \( Z(\Gamma) \) fixes \( z \) and \( y \), and commutes with \( \gamma \). We see that \( Z(\Gamma) \) must fix pointwise the geodesic joining \( z \) and \( y \). This contradicts the definition of \( \sigma \) as \( \text{fix}(\sigma) \).

\[\Diamond\]

Proof of Theorem 1.1.7: By Lemma 1.1.8, \( (\Gamma \cap U(\mathbb{H}^n)) \) fixes some point, \( z \), of \( \mathbb{H}^n \). If \( z \in \mathbb{H}^n \), we are reduced to the spherical case, and if \( z \in \mathbb{H}^n \), we are reduced to the case of euclidean similarities. We observe that our definitions of the rotational part of an isometry (or similarity) are in agreement, so that the theorem follows from Corollary 1.1.2, and Theorem 1.1.4.

\[\Diamond\]

For completeness, we state:

**Corollary 1.1.10:** Nilpotent subgroups of \( \text{Isom} \mathbb{H}^n \) are uniformly virtually abelian.

**Proof:** If \( \Gamma \subseteq \text{Isom} \mathbb{H}^n \) is nilpotent, we need that \( [\Gamma : (\Gamma \cap U(\mathbb{H}^n))] \) is uniformly bounded. But by Lemma 1.1.8, \( \Gamma \) has a fixed point in \( \mathbb{H}^n \), so the result follows from the spherical and euclidean (similarity) cases.

\[\Diamond\]

Note that all the abelian subgroups constructed in this section are normal, since the neighbourhoods \( U(S^n), U(E^n) \) and \( U(\mathbb{H}^n) \) are still conjugacy invariant.

### 1.2. Discrete Subgroups.

In this section, we describe how nilpotent groups occur naturally when considering discrete group actions.

Let \( g \) be a Lie group, and let \( | \cdot | \) be any smooth norm on \( G \), for example, distance from the identity in some Riemannian metric. For any \( g, h \in G \), sufficiently near the identity, we will have \( |g, h| < C|g||h| \), for some constant \( C \). Thus, we can find a bounded symmetric neighbourhood, \( O(G) \) of the identity in \( G \) such that whenever \( g, h \in O(G) \), we have \( |g, h| < C|g||h| \).

**Lemma 1.2.1:** If \( \Gamma \) is a discrete subgroup of \( G \), then \( (\Gamma \cap O(G)) \) is nilpotent.

**Proof:** The elements of \( \Gamma \) have norms bounded by some number \( c > 0 \), and the elements of \( O(G) \) have norms bounded above by some number \( k \). If \( m \) is any integer greater than \( \log \left( \frac{k}{c} \right) \), we see that any \( m \)-fold commutator in elements of \( \Gamma \cap O(G) \) will be trivial. By repeated application of the identity \( [y, s] = [y, [y, s]]/[y, s][y, s] \), we deduce that any \( m \)-fold commutator in \( (\Gamma \cap O(G)) \) is trivial. Thus, \( (\Gamma \cap O(G)) \) is nilpotent.

\[\Diamond\]

The following lemma is a modified version of one to be found in [Th2]. It is relevant to our discussion of the Margulis Lemma. First, we introduce some notation. Given a subset \( X \) of the Lie group \( G \), we write \( \chi \) for those \( g \in G \) expressible as words of length \( r \) in elements of \( X \) together with their inverses \( \{ X^{-1} \} \), i.e., inductively, \( X^r = X \cup \{1\} \cup X^{-1} \), \( X^r = X^{r-1}X \). If \( \Gamma \) is a subgroup of \( G \), we write \( \Gamma_X \) for \( (\Gamma \cap X) \).

**Lemma 1.2.2:** Let \( G \) be a Lie group, with \( W \) a neighbourhood of the identity. Let \( K, i \in N \) be a sequence of symmetric neighbourhoods of the identity. Suppose \( K_1 \) is compact, and \( \{ K_i \} \subseteq K_1 \) for each \( i \). Then, there exists some \( N \in N \) such that for any discrete group \( \Gamma \subseteq G \), \( |\Gamma_X : (\Gamma_X \cap W)| < N \).
Proof: Let $V$ be a neighbourhood of 1 with $V^{-1}V \subseteq W$. Since $K_i$ is compact, there is an upper bound, $k$, on the number of right translates $Vg, g \in K_i$, of $V$, that we can pack disjointly into $G$. Let $N = k + 1$.

Suppose that $\Gamma \subseteq G$ is discrete. Let $(Vg_i, i = 1, \ldots, p)$ be a disjoint packing with $a_i \in \Gamma_{K_i} \cap K_i$, and $p$ maximal. Note that $p \leq k$. Write $\Gamma_N = (\Gamma_{K_i} \cap W)$. We claim that $(\Gamma_{K_i} : i = 1, \ldots, p)$ includes a complete set of cosets for $\Gamma_N$ in $\Gamma_{K_i}$, so that $|\Gamma_{K_i} : \Gamma_N| \leq N$, as required.

To see this, consider $\Gamma_{K_i}$ with $h \in \Gamma_{K_i}$. Write $h = \prod_{i=1}^k g_i$, with $g_i \in \Gamma_i \cap K_i$. If $l \geq k + 1$, consider the collection $(Vh_i, j = 1, \ldots, k + 1)$, where $h_i = \prod_{i=1}^k g_i$, so that $h_i \in (K_i)^{K_i} \subseteq K_i$. These sets cannot all be disjoint. Thus, we can write $h = \alpha \beta g_i$, with $\alpha \beta \in K_i$ and $Vg_i \cap Vg_i \neq \emptyset$. Now, $\alpha \beta g_i^{-1} \in V^{-1}V \subseteq W$, so $\alpha \beta g_i^{-1} \in \Gamma_{K_i}$. Thus, $\Gamma_{K_i} - \Gamma_N (\alpha \beta g_i^{-1}) = \Gamma_{K_i} h_i$. We have reduced the word-length of $h$, so, by induction, $\Gamma_{K_i} - \Gamma_N h_i^h$, with $h_i \in K_i$. But then, $\forall h_i^h \cap Vg_i \neq \emptyset$, for some $g_i$, so that $h_i a_i^{-1} \in W$, and $\Gamma_{K_i} h_i^h = \Gamma_{K_i} a_i$. Hence, $\Gamma_{K_i} = \Gamma_{K_i} a_i$.

We again consider the three geometries in turn.

1.2.1. Spherical Geometry.

We write $U_0(S^n)$ for $O(\{a \in \text{Isom } S^n \mid d(a, a) < \varepsilon \text{ and } \text{rot } a \in U_i\})$, the neighbourhood of the identity $U_i$ at the beginning of Section 1.2. Since this set may be chosen to be arbitrarily small, we may suppose that $U_0(S^n) \subseteq U^n$). We may also suppose that $U_0(S^n)$ is conjugacy invariant. Now if $\Gamma$ is a discrete subgroup of $\text{Isom } S^n$, then $(\Gamma \cap U_0(S^n))$ is nilpotent by Lemma 1.2.1, and thus abelian by Corollary 1.1.2. It is easily checked that $(\Gamma \cap U_0(S^n))$ has a finite index in $\Gamma$, which is bounded as $\Gamma$ varies. Thus we have:

**Lemma 1.2.3 (Jordan Lemma):** Discrete subgroups of $\text{Isom } S^n$ are uniformly virtually abelian.

1.2.2. Euclidean Geometry.

We can assume that $O(\text{Isom } E^n)$ has the form $O(\text{Isom } E^n) = \{a \in \text{Isom } E^n \mid d(a, a) < \varepsilon \text{ and } \text{rot } a \in U_i\}$, where $\varepsilon > 0$, $a$ is some point of $E^n$, and $U_i$ is some neighbourhood of the identity in $\text{Isom } E^{n-1}$ that is contained in $U_i$. For notational convenience we shall identify $U_i$ with the set $U_0(E^n)$ of the Jordan Lemma. We set $U_0(E^n) = \{a \in \text{Isom } E^n \mid \text{rot } a \in U_0(E^n)\}$.

**Proposition 1.2.4:** Suppose that $\Gamma$ is a discrete subgroup of $\text{Isom } E^n$; then $(\Gamma \cap U_0(E^n))$ is abelian.

**Proof:** To begin with, we do not know that $(\Gamma \cap U_0(E^n))$ is finitely generated, so we proceed as follows. Let $D_p = \{a \in \text{Isom } E^n \mid d(a, a) < \varepsilon \}$. Let $g_0$ be the dilation of magnification $p$ about some point $a_0$. Considering $\Gamma_p = (\Gamma \cap U_0(E^n)) \cap D_p$, we see that $\Gamma_p^{-1} \Gamma_p g_0 = \Gamma_p^{-1} \Gamma_p \cap U_0(E^n) \cap D_p = \Gamma_p^{-1} \Gamma_p \cap O(\text{Isom } E^n)$, which is nilpotent (Lemma 1.2.1) and hence abelian (Theorem 1.1.4). Thus, $\Gamma_p$ is abelian for all $r_i$ and so $(\Gamma \cap U_0(E^n)) = \bigcup \Gamma_p$ is abelian.

Again, it is easily seen that $(\Gamma \cap U_0(E^n))$ has bounded index in $\Gamma$, so we have:

**Theorem 1.2.5 (Bieberbach):** Discrete subgroups of $\text{Isom } E^n$ are uniformly virtually abelian.
Note that since \( S'' \) is conjugacy invariant in Isom \( S'' \), the abelian subgroups we produce in this way will be normal. We shall write \( \nu(n) \) for the bound on their index.

We can say a little more about the structure of discrete euclidean groups:

**Proposition 1.2.6** Suppose \( \Gamma \) acts properly discontinuously on \( \mathbb{E}^n \). Then, there is a plane \( \mu \subset \mathbb{E}^n \), preserved by \( \Gamma \), with \( \mu \cap \Gamma \) compact. Moreover, any two such subspaces are parallel, and the action of \( \Gamma \) commutes with the perpendicular translation between them.

**Proof:** If \( \Gamma \) preserves each of two planes \( r_1 \) and \( r_2 \), then it preserves \( r_1 \cap r_2 \). It therefore makes sense to speak of a \( \Gamma \)-invariant plane \( \mu \neq \emptyset \) being minimal.

Let \( \mu_1 \) and \( \mu_2 \) be two such minimal planes. Let \( \lambda(\mu_1, \mu_2) = \{ x \in \mu_1 | d(x, \mu_1) - d(\mu_1, \mu_2) \} \subset \mu_2 \). \( \Gamma \) preserves \( \lambda(\mu_1, \mu_2) \). Hence, by minimality, \( \lambda(\mu_1, \mu_2) = \mu_2 \). It follows easily that \( \mu_1 \) and \( \mu_2 \) must be parallel.

Given any two parallel planes in \( \mathbb{E}^n \), there is a unique perpendicular translation mapping one to the other. Any isometry that preserves these two planes must commute with this translation. It follows that the action of \( \Gamma \) on \( \mathbb{E}^n \) must commute with the perpendicular translation sending \( \mu_1 \) to \( \mu_2 \).

It now remains to show that if \( \Gamma \) acts minimally on \( \mathbb{E}^n \), then it is cocompact. From the Bieberbach theorem, and the discussion of abelian groups in Section 1.1(ii), we can find a normal abelian subgroup \( \Gamma' \), of finite index in \( \Gamma' \), and a plane \( r \subset \mathbb{E}^n \), on which \( \Gamma' \) acts as a cocompact translation group. There are finitely many images, \( \{ r_1, \ldots, r_n \} \), of \( r \) under \( \Gamma' \), each preserved by \( \Gamma' \). Since a cocompact action is minimal, it follows that the \( r_i \) are all parallel. We may now find \( r' \), parallel to \( r \), which represents the centre of mass of the \( r_i \) in any transverse plane. \( \Gamma \) preserves \( r' \), so, by minimality, \( r' = \mathbb{E}^n = r \). Hence, \( \mathbb{E}^n / \Gamma \) is compact. ✷

As in the earlier discussion of the abelian case (Section 1.1(ii)), it is easily seen that the set of minimal planes in \( \mathbb{E}^n \) form a foliation of a larger, canonical subspace.

1.2(iii). Hyperbolic Geometry.

Given \( x \in \mathbb{H}^n \), we write \( \Gamma(x) = \{ \gamma \in \text{Isom} \mathbb{H}^n | d(\gamma x, x) < \epsilon \} \).

Let \( d_1 \) be any Riemannian metric on the unit tangent bundle \( T_1 \mathbb{H}^n \) of \( \mathbb{H}^n \), invariant under the action of Isom \( \mathbb{H}^n \). Given \( x \in \mathbb{H}^n \), we write

\[
I_r(x) = \{ \gamma \in \text{Isom} \mathbb{H}^n | d_1(\gamma x, 0) < \epsilon \text{ for each unit vector based at } x \}.
\]

If \( \Gamma \) is a subgroup of Isom \( \mathbb{H}^n \), we write

\[
\Gamma_r(x) = (\Gamma \cap I_r(x))
\]

and

\[
\Gamma_r(x) = (\Gamma \cap I_r(x)).
\]

Now we may suppose that \( O(\text{Isom} \mathbb{H}^n) \) has the form \( \Gamma_r(x) \) for some \( \epsilon_1 > 0 \) and \( x \in \mathbb{H}^n \). We also assume that \( I_r(x) \subset U(\mathbb{H}^n) \). We now have:

**Proposition 1.2.7** If \( \Gamma \) is a discrete subgroup of Isom \( \mathbb{H}^n \), then \( \Gamma_r(x) = (\Gamma \cap I_r(x)) \) is abelian.

Note that, by homogeneity, this remains true if we fix \( \epsilon_1 \), and choose \( x \) arbitrarily.

We next show that for small \( \epsilon \), \( \Gamma_r(x) \) is virtually abelian. To this end, we take \( I_r(x) \) to be the set \( W \) of Lemma 1.2.2, and the sets \( K_r \) to be \( I_r(x) \). The lemma now tells us that, for some \( N > 0 \),

\[
|\Gamma_{r,n}(x) : (\Gamma_{r,n}(x) \cap I_r(x))| \leq N,
\]

where \( \gamma(n) = 1/N \). Thus,

\[
|\Gamma_{r,n}(x) : (\Gamma_{r,n}(x) \cap I_r(x))| \leq N.
\]
For notational convenience, we shall assume that $N < i/(n)$, the constant of the Bieberbach Theorem, and that we have chosen the metric on $\mathbb{T}$ $\mathbb{H}^n$ so that $e_i = e(n)$. We call $e(n)$ the Margulis constant. In summary, we have:

**Theorem 1.2.8 (Margulis Lemma):** For all $n$, there exist $e(n) > 0$ and $\nu(n) \in \mathbb{N}$ such that if $\Gamma$ is any discrete subgroup of isom $\mathbb{H}^n$, and $z \in \mathbb{H}^n$, then $\Gamma_n(z)$ has an abelian subgroup $\bar{\Gamma}_n(z) \cap \Gamma_n(z)$ of index at most $\nu(n)$.

Note that if $0 < \epsilon < e(n)$, then $\Gamma_n(z) \cap \Gamma_n(z)$ has index at most $\nu(n)$ in $\Gamma_n(z)$. By intersecting all conjugate subgroups to $\Gamma_n(z) \cap \Gamma_n(z)$, we see that $\Gamma_n(z)$ contains a normal abelian subgroup of bounded index, where the bound is independent of the choice of discrete subgroup $\Gamma$.

2. Five Definitions of Geometrical Finiteness.

In this section, we shall give details of the five definitions of geometrical finiteness that we intend to use. First, we clarify a few points of terminology, and notation.

By a (discrete) parabolic group of hyperbolic isometries, $G$, we mean a discrete group which fixes a unique point in $\mathbb{H}^n$, i.e. $\gamma = (p)$, where $p \in \mathbb{H}^n$. In this case, $G$ must contain at least one parabolic with fixed point $p$. Since no loxodromic can share a fixed point with a parabolic in any discrete group, we see that $G$ consists entirely of parabolics and elliptics. We may represent $\mathbb{H}^n$ using the upper half-space model $\mathbb{R}^n_+$ with $p = 0$. It then follows that $G$ acts as a group of euclidean isometries of $\mathbb{R}^n_+$ whose quotient by $G$ is compact. Moreover, any two such planes are parallel. We shall write $\mathcal{P}$ for some choice of such plane, and write $\mathcal{P}$ for the vertical euclidean half-space with $\mathcal{P} \cap \mathbb{R}^n_+ = \mathcal{P}$. Thus, $\mathcal{P}$ is the hyperbolic subspace spanned by $\mathcal{P}$ and $p$. Now, if $\Gamma$ is any discrete subgroup of $\mathbb{H}^n$, and $\gamma \in \Gamma$ is any parabolic, with fixed point $p$, then the stabiliser of $p$ is a parabolic subgroup. We call $p$ a parabolic fixed point, which we abbreviate to p.f.p.

**GF1 Let $\Gamma$ be a discrete group. In Section 0.2, we defined $M_\Gamma$ as the quotient, by $\Gamma$, of $\mathbb{H}^n$ together with the discontinuity domain, that is $M_\Gamma = (\mathbb{H}^n \cup D)/\Gamma$. Thus $M_\Gamma = M \cup M'$, where $M = \mathbb{H}^n / \Gamma$ is a complete hyperbolic manifold, and $M' = \mathbb{H}^n / \Gamma$ consists of "ideal points" of $M$. Where there is more than one group in question, we shall be specific by writing $M(\Gamma)$, $M(\Gamma)$ and $M(\Gamma)$.**

Suppose that $\Gamma$ and $\Gamma'$ are two discrete groups. Suppose $\epsilon$ and $\epsilon'$ are (topological) ends of $M(\Gamma)$ and $M(\Gamma)$ respectively. We say that $\epsilon$ and $\epsilon'$ are equivalent if they admit isometric neighbourhoods. (Here, we use the term "isometric" loosely, in that the orbifolds in question may contain ideal points. In saying that two such orbifolds are isometric, we mean that there is an isometry of the metric parts which extends to a homeomorphism on the ideal points.) Note that if $\Gamma'$ is a parabolic group, then $M(\Gamma')$ has precisely one end (see the discussion below).

**Definition 1: $\Gamma \in GF1$ if $M(\Gamma)$ has finitely many ends, and each such end is equivalent to the end of the quotient of a parabolic group.**

It will be convenient to give this defining a more concrete formulation in terms of the structure of parabolic groups. Let $\Gamma_\mathcal{P}$ be a group, fixing $p = \infty$ in the upper half-space model, $\mathbb{R}^n_+$. Let $\sigma$ be a $\Gamma_\mathcal{P}$-invariant vertical plane with $\sigma / \Gamma_\mathcal{P}$ compact. Let $C(\sigma) = \{ x \in \mathbb{R}^n_+ \cup \partial \mathbb{R}^n_+ | d_{\sigma}(x, \sigma) \geq \epsilon \}$. Thus, $\mathbb{R}^n_+ \cup \partial \mathbb{R}^n_+ \setminus C(\sigma)$ is an open uniform $\sigma$-neighbourhood of $\sigma$ in the euclidean metric on $\mathbb{R}^n_+ \cup \partial \mathbb{R}^n_+$.

Since $\Gamma_\mathcal{P}$ preserves the euclidean metric, the construction is $\Gamma_\mathcal{P}$-equivariant, so we may form the quotient $\bar{C}(\sigma) = C(\sigma) / \Gamma_\mathcal{P}$. Since $\sigma$ is compact, $\mathbb{R}^n_+ \cup \partial \mathbb{R}^n_+ \setminus C(\sigma) / \Gamma_\mathcal{P} = M(\Gamma_\mathcal{P}) \setminus \bar{C}(\sigma)$ is relatively compact in $M(\Gamma_\mathcal{P})$. If we take a sequence $\{ \epsilon_n \}$ tending to infinity, then the sets $M(\Gamma_\mathcal{P}) \setminus \bar{C}(\epsilon_n)$ give a relatively compact exhaustion of $M(\Gamma_\mathcal{P})$. Since each $\bar{C}(\epsilon_n)$ is connected, we see that $M(\Gamma_\mathcal{P})$ has precisely one end. The sets $\bar{C}(\epsilon_n)$ give a neighbourhood base for the end. Given any $\rho$, we call $\bar{C}(\rho)$ a standard parabolic region (FIG 2.1), and the quotient $\bar{C}(\rho)$, a standard cusp (FIG 2.2).
We may now say that $\Gamma$ is $GF1$ if and only if we can write $M = \mathcal{H} \cup \{ \bar{C} \}$, with $\mathcal{H}$ compact, $\bar{C}$ finite, and each $\bar{C} \in \mathcal{C}$ (isometric to) a standard cup (Fig. 2.3). We shall see (Chapter 3, $GF4 \Rightarrow GF2$) that the cusps $\bar{C}$ are in bijective correspondence to the orbits of parabolic fixed points of $\Gamma$.

We write $\mathcal{N}$ for the lift of $\mathcal{H}$ to $\mathcal{H}^\mathfrak{g}$.

**GF2**

Let $p$ be a parabolic fixed point (p.f.p.) of the discrete group $\Gamma$. Let $\Gamma_p = \text{stab}_\Gamma p$ — the stabiliser of $p$. We say that $p$ is a bounded p.f.p. (b.p.f.p.) if $(\bar{A}(\langle p \rangle) \setminus \{ \infty \})/\Gamma_p$ is compact. Let $p = \infty$ in the upper half-space model, and let $\sigma_1$ be a minimal $\Gamma_p$-invariant plane. Then it is not difficult to see that $(\bar{A}(\{ \infty \}) \setminus \{ \infty \})/\Gamma_p$ is compact if and only if $d_{\infty}(y, \sigma_1)$ is bounded as $y$ varies in $\bar{A}(\{ \infty \})$. In other words, $p$ is a b.p.f.p. if and only if $\bar{A}(\{ \infty \}) \subset \bar{A}(\langle p \rangle) \cap \{ \infty \}$ for some $\sigma_1$ (Fig. 2.4).

Let $y \in \bar{A}(\Gamma)$. We say that $y$ is a connected limit point (c.l.p.) if for some (and hence every) geodesic ray $l$ joining some point of $\mathbb{H}^n$ to $y$, the orbit $\Gamma_l$ of $l$ accumulates somewhere in $\mathbb{H}^n$, i.e., $\{ \gamma \in \Gamma | \gamma \cap K \neq \emptyset \} = \infty$ for some compact $K \subset \mathbb{H}^n$. (The terms derive from an alternative description, namely that there should exist a sequence $\gamma_n \in \Gamma$, and a point $x \in \mathbb{H}^n$, with $\gamma_n(x)$ tending to $y$, while remaining a bounded distance from some geodesic ray — see Fig. 2.5.)

**Definition 2:** $\Gamma$ is $GF3$ if every point of $A$ is either a c.l.p. or a b.p.f.p.

We shall see in Ch. 3 that these two classes are, in any discrete group, mutually exclusive. In fact, it is shown in [Sus3] that, in any discrete group, no p.f.p. can also be a c.l.p. It will follow from the discussion in Chapter 3 ($GF4 \Rightarrow GF2$) that in the special case of a geometrically finite group, any p.f.p. is necessarily a b.p.f.p., and thus not a c.l.p.

Beardon and Maskit [BeaM] give several equivalent definitions of c.l.p., including one that makes sense in $\mathbb{H}^n$. This gives $GF2$ as a definition of geometric finiteness intrinsic to the action of $\Gamma$ on $\mathbb{H}^n$.

Finally, we remark that, for a GF group, the convergence of orbits under $\Gamma$ to c.l.p.s can be chosen to be “uniform.” For us, this means that the set $K$, in the definition, can be chosen independently of the point $y$ and the ray $l$. Together with a certain convergence property for the radii of isometric spheres, this implies that the limit set of a GF group has either zero or full spherical Lebesgue measure [see [BeaM, ApI]]. A more geometric proof of this fact is based on the definition $GF5$ (see [Th1]).

**GF3**

Let $\Gamma$ be a discrete group. We have said that the hypothesis that $\Gamma$ should possess a finite-sided fundamental polyhedron is more restrictive than we would like in dimension 4 onwards. In Section 4, we give an example to illustrate this point (at least for the case of Dirichlet domains). However it is possible to modify the criterion so that it works in all dimensions. The idea is that we should allow ourselves more than one polyhedron to constitute a fundamental domain for $\Gamma$. The definition is most clearly expressed in terms of what we shall call "convex cell complexes". A convex cell complex is cell complex in which all the cells are convex, and hence necessarily polyhedra. It need not quite be a CW-complex since we only attach cells along their relative boundaries in hyperbolic space. Thus a finite complex is complete, but not in general compact. We give a more formal description below.

Let $A$ be a subset of $\mathbb{E}^n$. We call $A$ an open (convex) cell if any two distinct points of $A$ lie in the interior of some geodesic segment contained entirely in $A$. Note that by demanding that the two points be distinct, we allow any one-point set as an open cell. We see that the property of being an open cell is closed under taking finite intersections.

**Definition 1:** A collection $A$ of subsets of $\mathbb{E}^n$ is "convex cell complex" if:

1. Each element of $A$ is an open cell,
2. The sets of $A$ are all disjoint,
3. The collection $A$ is locally finite,
4. $\bigcup A = \mathbb{E}^n$,
5. If $A, B \in A$ and $A \cap B \neq \emptyset$, then $B \subset A$.

Let $A$ be such a cell complex, and let $B \in A$. Suppose that $s$ and $y$ are two points of $B$, and suppose that $s \in A$ for some cell $A \in A$. Then, from (5), we see that $y \in A$. Thus, $(A \in A | s \in A)$ is independent
of the choice of \( s \in B \). In particular, from the local finiteness of \( A \), (2), we see that any cell of \( A \) meets the closures of only finitely many other cells.

Now, given two cell complexes \( A \) and \( B \), we call \( B \) a subdivision of \( A \) if each \( B \in \) is a subset of some \( A \in A \). Any two cell complexes \( A' \) and \( A'' \) have a natural common subdivision, namely \( \langle A' \cap A'' \rangle = \{ A' \cap A'' \mid A' \in A', A'' \in A'' \} \). In fact \( \langle A' \cap A'' \rangle \) is minimal with respect to subdivision — if \( B \) is a subdivision of both \( A' \) and \( A'' \), then it is a subdivision of \( \langle A' \cap A'' \rangle \). We also remark that intersecting a cell complex with an affine subspace of \( \mathbb{R}^n \) gives a cell complex in that subspace.

All the above properties are easily verified from the definition above. However, to make the analogy with CW-complexes more explicit, we offer a slightly different description of convex cell complexes as follows.

- Suppose that \( A \subseteq \mathbb{R}^n \) is a convex set. We write \( A \) for the affine span of \( A \), i.e. the smallest subspace of \( \mathbb{R}^n \) containing \( A \). We may define the dimension of \( A \), \( \text{dim} \, A \), to be equal to the dimension of \( A \). We also define \( rA \) and \( cB \) to be, respectively, the relative interior and the relative boundary of \( A \) in \( A \). Note that \( rA \) is always nonempty, provided \( A \) is nonempty. Moreover, it is not difficult to see that \( A \) is an open cell if and only if \( rA = A \).

Let \( K^r \) of \( A \) be the union of all \( r \)-cells with \( r \leq r \), i.e. \( K^r = \bigcup \{ J A \} \). We now claim that, if we know that \( A \) satisfies properties (1)-(4), then property (5) is equivalent to the following:

\[ \text{(5')} \text{ If } A \subseteq \mathbb{R}^n, \text{ then } \text{volume}(A) < 0. \]

Now, let \( A \) be a collection of convex cells of \( \mathbb{R}^n \). We write \( A' \) for set of all \( r \)-cells in \( A \). The \( r \)-skeleton, \( K^r \), of \( A \) is the union of all \( r \)-cells with \( r \leq r \), i.e. \( K^r = \bigcup \{ J A \} \). We now claim that, if we know that \( A \) satisfies properties (1)-(4), then property (5) is equivalent to the following:

\[ \text{(5')} \text{ If } A \subseteq \mathbb{R}^n, \text{ then } \text{volume}(A) < 0. \]

To see \((5') \Rightarrow (5)\), it is enough to note that if one open cell lies in the relative boundary of another, then its dimension must be strictly less. To see \((5) \Rightarrow (5')\) is a little more complicated. Suppose that \( A \subseteq \mathbb{R}^n \) is a collection of convex cells of \( \mathbb{R}^n \). We write \( A' \) for the affine span of \( A \), i.e. the smallest subspace of \( \mathbb{R}^n \) containing \( A \). We may define the dimension of \( A \), \( \text{dim} \, A \), to be equal to the dimension of \( A \). We also define \( rA \) and \( cB \) to be, respectively, the relative interior and the relative boundary of \( A \) in \( A \). Note that \( rA \) is always nonempty, provided \( A \) is nonempty. Moreover, it is not difficult to see that \( A \) is an open cell if and only if \( rA = A \).

Let \( A \) be a collection of convex cells of \( \mathbb{R}^n \). We write \( A' \) for set of all \( r \)-cells in \( A \). The \( r \)-skeleton, \( K^r \), of \( A \) is the union of all \( r \)-cells with \( r \leq r \), i.e. \( K^r = \bigcup \{ J A \} \). We now claim that, if we know that \( A \) satisfies properties (1)-(4), then property (5) is equivalent to the following:

\[ \text{(5')} \text{ If } A \subseteq \mathbb{R}^n, \text{ then } \text{volume}(A) < 0. \]

Now, let \( A \) be a collection of convex cells of \( \mathbb{R}^n \). We write \( A' \) for set of all \( r \)-cells in \( A \). The \( r \)-skeleton, \( K^r \), of \( A \) is the union of all \( r \)-cells with \( r \leq r \), i.e. \( K^r = \bigcup \{ J A \} \). We now claim that, if we know that \( A \) satisfies properties (1)-(4), then property (5) is equivalent to the following:

\[ \text{(5')} \text{ If } A \subseteq \mathbb{R}^n, \text{ then } \text{volume}(A) < 0. \]

Now, let \( A \) be a collection of convex cells of \( \mathbb{R}^n \). We write \( A' \) for set of all \( r \)-cells in \( A \). The \( r \)-skeleton, \( K^r \), of \( A \) is the union of all \( r \)-cells with \( r \leq r \), i.e. \( K^r = \bigcup \{ J A \} \). We now claim that, if we know that \( A \) satisfies properties (1)-(4), then property (5) is equivalent to the following:

\[ \text{(5')} \text{ If } A \subseteq \mathbb{R}^n, \text{ then } \text{volume}(A) < 0. \]
paper, but used only to relate the notion of cell complexes with fundamental domains.

One natural way in which convex cell complexes arise is as follows. Let \( X \) be a discrete subset of \( \mathbb{E}^n \). Given \( x \in X \), we define \( D_X(x) \) to be the set of points in \( \mathbb{E}^n \) nearer to \( x \) than to any other point of \( X \), i.e.

\[
D_X(x) = \{ y \in \mathbb{E}^n \mid d(y, z) < d(y, x) \text{ for all } z \in X \setminus \{x\} \}.
\]

It is easily checked that the collection of sets \( \{D_X(x) \mid x \in X\} \) satisfies all the conditions of being the set of top-dimensional cells of some convex cell complex, namely properties (a)-(d) listed above. Let \( A_X \) be the cell complex, minimal with respect to subdivision with \( A_X \).

An another description of \( A_X \) as follows. Given any finite subset \( Y \) of \( X \), we write \( D_Y(Y) \) to be the set of points \( y \) for which the minimal value of \( d_s,y \) with \( z \in X \) is attained equally at each point \( z \in Y \). Then \( A \) is the set of all \( D_Y(Y) \) as \( Y \) ranges over all finite subsets of \( X \).

Let \( A \) a convex cell complex, with \( A \in A \). We call \( B \in A \) a face of \( A \) if \( B \subseteq A \). We write \( F(A) \) for the set of all faces of \( A \). We call a subset \( B \) of \( A \) a full subcomplex if a face of any element of \( B \) also lies in \( B \). In this case, we write \( |B| \) for the union of all the cells of \( B \). We see that \( |B| \) is a closed subset of \( \mathbb{E}^n \).

We can make sense of the notion of convex cell complex on certain closed subsets of \( \mathbb{E}^n \) by replacing property (4) in the definition by the hypothesis that \( |B| \) is convex.

Suppose that \( A \in A \) for some complex \( A \). Let \( (A) \) be the affine span of \( A \). It is not difficult to see that we may represent \( A \) as an intersection of half-spaces, in \( (A) \), determined by the codimension-one faces of \( A \). Thus each cell of a complex is necessarily the relative interior of a polyhedron, according to the following definition.

Definition: An \( r \)-dimensional polyhedron is a (countable) intersection of closed half spaces of \( \mathbb{E}^r \), \( P = \bigcap_{a \in A} H_a \), where the sets \( H_a \) are locally finite. We insist that \( P \) have non-empty interior in \( \mathbb{E}^r \).

Given such a polyhedron, we may reconstruct a convex cell complex \( S(P) \) on \( P \) by taking, as lower-dimensional faces, the relative interiors of the intersections with \( P \) of the supporting hyperplanes. We call such faces the sides of \( P \). If \( P \) is obtained as the closure of a cell in a convex cell complex, then \( S(P) \) is a subdivision of \( S(P) \).

As an example, consider the tessellation of \( \mathbb{E}^n \) by bi-infinite square prisms (each isometric to \([0,1]^2 \times \mathbb{R}\)). First, the tiles are laid parallelly north-south, then east-west, and so on alternately. Each tile has infinitely many codimension-one faces (in the associated cell complex), but only finitely many codimension-one sides (in fact, four).

So far, we have talked only about cell complexes in Euclidean space. However, all the above discussion is valid with \( \mathbb{E}^n \) replaced by \( H^n \). To see this, we note that in the Klein model for hyperbolic space, hyperbolic and Euclidean convexity coincide.

Now, let \( \Gamma \) be a discrete group acting on \( H^n \). Let \( X \) be a discrete \( \Gamma \)-invariant set, and let \( A_X \) be the complex derived from \( X \), as described above. The complex \( A_X \) has the following properties.

(i) It is \( \Gamma \)-invariant.

(ii) The setwise stabiliser of any cell is finite.

Suppose in particular, that \( X = \bigsqcup_{\alpha} X_\alpha \), where the orbits \( X_\alpha \) are disjoint, and each point \( x_\alpha \) has trivial stabiliser in \( \Gamma \). Then, we call the top-dimensional cells of \( A_X \) (generalised) Dirichlet domains. We write \( A_{\Gamma}(x) \) for the complex \( A_X \), and write \( D_{\Gamma}(x) \) for \( D_X(x) \) — the Dirichlet domain about \( x \). Here, \( x \) represents the finite set \( \{x_1, \ldots, x_n\} \).

More generally, we say that a convex complex \( A \) is associated to \( \Gamma \) if it satisfies the two properties (i)-(ii) above. If we are given such a complex \( A \), we may find a \( \Gamma \)-invariant subdivision \( A_0 \) of \( A \) with the property that if any \( \gamma \in \Gamma \) preserves, setwise, a cell \( A \in A_0 \), then it fixes \( A \) pointwise. This means that \( A_0 \) projects to a cell complex in \( \mathbb{H}^n/\Gamma \).

One may obtain \( A_0 \) as follows. For \( A \in A \), let \( \text{stab}_A \) be the (finite) stabiliser of \( A \) in \( \Gamma \), and let \( \mathcal{U}(A) \) be the set of intersections of \( A \) with a collection of Dirichlet domains for \( \text{stab}_A \). Given any \( \gamma \in \Gamma \setminus \text{stab}_A \), we define \( U(\gamma A) = \gamma U(A) \). Performing this construction for each orbit of top-dimensional cell gives us a \( \Gamma \)-invariant collection of convex sets \( \mathcal{U} = \bigsqcup_{A \in A} U(A) \), satisfying the hypotheses Proposition 2.1. This gives us a subdivision \( A_0 \) of \( A \). Let \( (A, A_0) \) be the common subdivision of both, as defined above. We can
now cut up all the codimension-1 cells in $A$ in a similar way. Applying Proposition 2.1 again gives us a further subdivision $A_0$. Continuing this process inductively gives us, after $n$ steps our required subdivision $A_0$. Note that each cell of $A_0$ is divided into only finitely many pieces in $A_0$.

We can relate the complex $A_0$ to convex fundamental domains. Suppose, for a moment, that $\Gamma$ is orientation-preserving, so that the singular set lies in $K_{n-2}(A_0)$. Let $B^n$ be a set of orbit representatives of $A_0^\Gamma$ under $\Gamma$. Let $B^{n-1} = (\{B^k\})^{n-1}$, the set of codimension-1 faces. Each face in $B^{n-1}$ meets either one or two cells in $B^n$. Those that meet only one are paired under $\Gamma$. The set of face-pairing isometries generate $\Gamma$. If $B^n$ has only one element, say $B$, then $B$ is called a (convex) fundamental polyhedron. In defining geometrical finiteness, it has been usual to demand that the codimension-1 sides and faces of $B$ coincide (the axiom of side-pairings - see [BM]). However, from our point of view, this restriction does not seem particularly natural, and we shall not use it. Note that, if $\Gamma$ is not orientation-preserving, we may have to allow for reflections in codimension-1 faces.

**Definition 3.** If there exists a convex complex cell $A$ on $H^n$, preserved under $\Gamma$, with $\{\gamma \in \Gamma | \gamma A = A\}$ finite for all $A \in A$, and with $\Gamma(A)$ (the set of orbits under $\Gamma$) finite.

In such a case, if we subdivide $A$ to $A_0$ as described above, then $A_0/\Gamma$ will also be finite. Thus, $A_0$ projects to a finite complex in $M$. We may thus rephrase GF3 by saying that $M$ may be represented by a finite complex in which cell is isometric to an open convex set in $H^n$. As stated at the beginning, each cell is attached only along its relative boundary in $H^n$.

**GF4.**

Let $\Gamma$ be a discrete group of isometries of $H^n$. We define $\text{free}(\Gamma) = \{\gamma \in \Gamma | \delta(\gamma) \cap H^n - \delta, \text{finite}\}$ to be the subset of elements acting freely, i.e., without fixed points in $H^n$. Let $0 < \varepsilon < \varepsilon(n)$, where $\varepsilon(n)$ is the Margulis constant. The set thin$^\varepsilon$($M$) = $\{x|d(x, \gamma x) < \varepsilon \text{ for some } \gamma \in \text{free}(\Gamma)\}$ projects to what we shall call the thin part of the quotient orbifold $M$, denoted by thin$^\varepsilon(M)$.

We claim that the connected components of thin$^\varepsilon(M)$ have the form thin$^\varepsilon$($M(G)$), where $G$ is either a parabolic group, i.e., fixes a unique point in $H^n$, or else is, what we shall call here, a "loxodromic group", i.e., it preserves, setwise, a geodesic, whose quotient under $G$ is compact. This is well known in the case where $\Gamma$ is torsion-free, and we can see essentially the same reasoning for our more general situation. For completeness, we give the argument below.

Let $T$ be a component of thin$^\varepsilon(M)$, and let $G$ be the setwise stabilizer of $T$ in $\Gamma$, so that $T = T/G$ is a component of thin$^\varepsilon(M)$, as well known. We first show that $T \subset \text{thin}^\varepsilon(M(G))$. We then show that $G$ is either parabolic or loxodromic, from which it follows that thin$^\varepsilon(M(G))$ is connected, and thus equal to $T$. We can then deduce that $T = \text{thin}^\varepsilon(M(G))$.

For the first part, consider $x \in T$. There is some $\gamma \in \text{free}(\Gamma)$, with $d(x, \gamma x) < \varepsilon$. Let $l$ be the geodesic segment joining $x$ to $\gamma x$. For any $y \in l$, we have $d(y, \gamma y) < d(x, \gamma x) < \varepsilon$. Thus $l \subset \text{thin}^\varepsilon(M(G))$, and so $\gamma x \in T$. Now any element of $\Gamma$ must either preserve $T$ setwise, or map it onto a disjoint component. We deduce that $\gamma \in G$, and so $x \in \text{thin}^\varepsilon(M(G))$.

For the second part, we fix $z \in T$. Now, $\Gamma,\{z\} = \{\gamma \in \Gamma | d(x, \gamma z) < \varepsilon\}$ contains an element of infinite order. From the discussion of the Margulis Lemma in Section 1, we may deduce that $\Gamma,\{z\}$ is either a parabolic or a loxodromic group. Now, let $\gamma \in \Gamma$. We join $x$ to $\gamma x$ by a path $\lambda \subset T$. Consider the groups $\Gamma,\{\lambda(t)\}$, as the parameter $t$ varies. Suppose at some time $t_0$, $\Gamma,\{\lambda(t_0)\}$ changes from one subgroup $G_1$ of $\Gamma$ to another, $G_2 \subset \Gamma$. Choose any $x'$ lying strictly between $x$ and the Margulis constant $\varepsilon(n)$. Then $G_1$ and $G_2$ are both subgroups of $\Gamma,\{\lambda(t_0)\}$. Again from the Margulis lemma, we see that $G_1$ and $G_2$ are either both parabolic with the same fixed point, or loxodromic with the same axis. Thus, if $\Gamma,\{z\}$ is parabolic with fixed point $p$, then $\Gamma,\{z\}$ is also parabolic with fixed point $p$. But, $\Gamma,\{z\} = \Gamma,\{\gamma z\}$, and so $\gamma p = p$. It is clear from the first part of the proof that $G$ contains elements of infinite order, and so we see that $G$ is a parabolic group. In this case, note that if $y \in \text{thin}^\varepsilon(M(G))$, then the geodesic joining $y$ to $p$ lies within thin$^\varepsilon(M(G))$. From this, it is easy to deduce that thin$^\varepsilon(M(G))$ is connected. Similarly, if $\Gamma,\{z\}$ is a loxodromic group, then so is $G$. Again, we may see that thin$^\varepsilon(M(G))$ is connected, since the shortest path from any point of thin$^\varepsilon(M(G))$ to the axis lies within thin$^\varepsilon(M(G))$. This completes the proof of the claim.

If $G$ is parabolic, we call $T = \text{thin}^\varepsilon(M(G))$ a Margulis cusp. If $G$ is loxodromic, we call $T$ a Margulis tube. In the latter case, the quotient of the loxodromic axis is either a short arc, or a short closed geodesic.
which we call the core of the tube. The tube is a regular neighbourhood of the core in the quotient orbifold.

A cross-section of the tube is spherical about its intersection with the axis. In fact it is a finite union of convex sets, since the tube is a finite union of sets of the form \( \{ x \in \mathbb{H}^n \mid d(x, c) < r \} / G \), each of which has convex cross-section.

We shall denote by \( \text{thick}_m(M) \), the closure of the complement of \( \text{thin}_m(M) \), in \( M \). We call \( \text{thick}_m(M) \) the thick part of \( M \).

These definitions are most natural when \( \Gamma \) acts freely. Then, \( \text{thin}_m(M) \) is the set of points with injectivity radius at most \( r \). The definitions for the orbifold case are not standard, but are convenient for our purposes.

To give the fourth definition of geometric finiteness, we need to define the "convex core" of a hyperbolic orbifold. The definition is the same as that given for a hyperbolic 3-manifold in the introduction. Let \( \mathcal{X} \) be the limit set of \( \Gamma \). The "convex hull", \( Y \), of \( \mathcal{X} \) is the minimal closed convex subset of \( \mathbb{H}^n \) whose closure \( Y \) contains \( \mathcal{X} \). The construction of \( Y \) is best seen in the Klein model for hyperbolic space (see [Th1]). From this picture, it is clear that \( \mathcal{X} \times \mathbb{H}^n \) is \( \mathcal{X} \). Since the construction is \( \Gamma \)-equivariant, we may project \( \mathcal{X} \) to a subset, \( \mathcal{X} \), in the quotient orbifold, \( M \). We call \( \mathcal{X} \) the convex core of \( M \) (Fig 2.7).

**Definition 4 :** \( \Gamma \) is GE4 if, for some \( \epsilon < c(n) \), \( \epsilon > 0 \), \( \mathcal{X} \cap \text{thick}_m(M) \) is compact.

We will describe below an alternative way of defining a thick-thin decomposition for orbifolds. The resulting decomposition is identical for manifolds, and qualitatively similar for other orbifolds. The definition is suggested by the following proposition, which we also use in discussing GE5 in Ch.3.

**Proposition 2.1 :** For each \( n \) there is some \( N = N(n) \), such that if \( x \in \mathbb{H}^n \) lies in the interior of \( \text{thick}_m(M) \) (the lift of \( \text{thick}_m(M) \) to \( \mathbb{H}^n \)), then \( \Gamma_n(x) \) is finite.

We begin the proof of Proposition 2.1 with the following lemma.

**Lemma 2.3 :** Let \( G \) be any group, and \( H \leq G \), a subgroup with \( [G : H] = k \). If \( G = \langle A \rangle \), then \( H = \langle H \cap A^{2k+1} \rangle \).

(As in Lemma 1.1, if \( X \subset G \), we denote by \( X^r \) the set of those \( g \in G \) expressible as words of length \( r \) in elements of \( X \times \{1\} \times X^{-1} \).)

**Proof of Lemma :** The proof will be similar to that of Lemma 1.1.

Given any \( h \in H \), we can write \( h = h^r \psi \), with \( \psi \in A \). If \( r > 2k + 1 \), consider the collection \( \{ hh_1 \beta = h \} \), where \( h_1 = \prod_{i=1}^{r-1} h_i \). These cosets cannot all be distinct. Then, \( h = \alpha \beta \gamma \) with \( H \alpha \beta = H \alpha \), \( \alpha \in A^\beta \), \( \beta \neq 1 \), and \( \alpha \beta \in A^{2k+1} \). We can write \( h = \langle \alpha \beta \gamma \rangle \), where \( \alpha \beta \gamma \in A^{2k+1} \), and \( \alpha \beta \gamma \) has shorter word-length than \( h \), so the result follows by induction.

**Proof of Proposition :** Let \( N = 2\nu(n) + 1 \), where \( \nu(n) \) is the bound on index in the Margulis Lemma. We fix \( x \in \text{int} \text{thick}_m(M) \) and consider \( \Gamma, x \). From Chapter 1, we know that \( \Gamma_n(x) \) contains an abelian subgroup \( K \), with \( [\Gamma_n(x) : K] \leq \nu(n) \). Let \( T \) be the torsion subgroup of \( K \). Since \( x \in \text{int} \text{thick}_m(M) \), we have \( I_n(x) \cap T = 0 \), so that \( \Gamma_n \cap I_n(x) = \Gamma \). We write \( I_x \) for \( I_n(x) \), etc.

Let \( n \) be such that \( I_n(x) \subset I \). Then \( I_x \cap K \leq \nu(n) \), where \( \nu(n) = [\Gamma \cap I_n(x) : (\Gamma \cap I_n(x)) \cap K] \). From the lemma,

\[
\Gamma_n \cap K = \left( (\Gamma_n \cap I_n(x)) \cap K \right) \subseteq \left( \Gamma \cap (I_n(x)) \cap K \right) = \left( \Gamma \cap (I \cap K) \right).
\]

But \( I_\infty \cap K \subset T \), so \( [\Gamma_n(x)] \leq \nu(n) \mid T \leq \infty \).

The case when \( \Gamma_n \) is loxodromic is similar.

Suppose that \( \epsilon < c(n)/N(n) \), and let \( F_n(\Gamma) = \{ x \in \mathbb{H}^n \mid \Gamma_n(x) \text{ is finite} \} \). \( F_n(\Gamma) \) is closed in \( \mathbb{H}^n \), since we defined our sets \( I_n(x) \) to be closed. It projects to a set which we denote by \( \text{thin}_m(M) \) in \( M \). We write \( \text{thick}_m(M) \) for the closure of its complement in \( \mathbb{H}^n \). For \( \epsilon < c(n) \), we have the inclusions

\[
\text{thin}_m(M) \subset \text{thick}_m(M) \subset \text{thin}_m(M).
\]
Again, the connected components of \( \text{thin}_\varepsilon(M) \) are of two types - tubes and cusps. In \( GF_4 \Rightarrow GF_1 \), we shall see that if \( M \) is \( GF \), then \( Y \cap \text{thick}_{\varepsilon}(M) \) is compact for any \( \varepsilon > 0 \), arbitrarily small. This fact means that we can reformulate \( GF_4 \) by demanding that \( Y \cap \text{thick}_{\varepsilon}(M) \) be compact for some \( \eta < \varepsilon(n)/N(n) \).

**GF5**

**Definition 5:** \( \Gamma \) is \( GF_5 \) if it is finitely generated and, for some \( \eta > 0 \), the \( \eta \)-neighbourhood, \( N_\eta(Y(\Gamma)) \) of \( Y(\Gamma) \) has finite volume.

We suspect that the assumption of finite-generation is unnecessary. We show this to be the case:

(i) if \( \text{stab}_\gamma(x) \) is bounded for \( x \in \mathbb{H}^n \) (for example, if \( \Gamma \) acts freely); or
(ii) if \( M(\Gamma) \), itself, has finite volume; or
(iii) if \( n \leq 3 \).


The main cycle of proofs will be:

\[ 3 \iff 1 \iff 5 \]

We use \( GF_1 \) as our central definition, since most the facts about geometrically finite groups are most easily deduced from this. We include proofs of \( 1 \Rightarrow 3 \) and \( 1 \Rightarrow 4 \) since they are very much shorter than following the cycle. The only non-geometric input is an appeal to the Selberg Lemma (Chapter 0) which overcomes a technical difficulty in the proof of \( 5 \Rightarrow 4 \).

**GF1 \( \Rightarrow GF_2 \)**

We have \( M(\Gamma) = \hat{\mathcal{H}} \cup (\bigcup \mathcal{C}) \), where \( \hat{\mathcal{H}} \) is the projection of a compact set \( \mathcal{H} \subset \mathbb{H}^n \cup \mathcal{D} \), and \( \mathcal{C} \) is a finite set of cusp regions. Let \( y \in \mathcal{H}(\Gamma) \).

Suppose that \( y \) is the fixed point of a parabolic group \( \Gamma_y \), which stabilises some cusp region \( \mathcal{C} \). Then \( (\mathcal{H} \setminus \{\infty\})/\Gamma_y \) is a closed subset of the relatively compact set \( (\mathbb{H}^n \setminus \mathcal{C})/\Gamma_y \), and is thus compact. We see that, in this case, \( y \) is a b.p.f.p.

So, suppose that \( y \) does not correspond to a cusp region in the way described above. (It is still conceivable, at the stage, that \( y \) may be a p.f.p., though this does not affect the argument.) We must have \( |\mathcal{H}| \geq 2 \), so that the convex hull \( Y \) meets \( \mathbb{H}^n \). We join \( y \) to a point \( x \in Y \cap \mathbb{H}^n \) by a geodesic ray \( I \).

We use \( \rho_x : \mathbb{H}^n \rightarrow K \) to denote the nearest point projection. We may define the nearest point retraction \( \rho_K : \mathbb{H}^n \rightarrow K \), where \( \rho_K(x) = \rho_x(x) \) is the nearest point of \( K \) to \( x \). This map extends continuously to ideal points, \( \rho_K : \mathbb{H}^n \rightarrow K_t \), where \( K_t \) is the closure of \( K \) in \( \mathbb{H}^n \).

We may describe the extension as follows. For \( z \in K \cap \mathbb{H}_\infty \), take \( \rho_K(z) = z \), and for \( z \in \mathbb{H}^n \setminus K \), take \( \rho_K(z) \) to be the unique point such that \( K_t \cap B = \{ \rho_K(z) \} \) for some horoball \( B \) about \( y \). Notice that if, for a pair \( K_t, L \) of convex sets, \( \rho_K(z) = y \in \text{int} K \), then \( \rho_K(z) = y \).

Given a set \( X \subset \mathbb{H}^n \), we shall denote by \( N_r(X) \) the uniform \( r \)-neighbourhood of \( X \), i.e. \( \{ x \in \mathbb{H}^n \mid d(x, X) \leq r \} \). We shall say that two closed convex sets, \( K_1 \) and \( K_2 \), are \( \lambda \)-near (for some \( \lambda > 0 \)) if \( K_1 \subset N_1(K_2) \) and \( K_2 \subset N_1(K_1) \). We show:

**Lemma 3.1:** Given \( \lambda > 0 \), there exists \( L = L(\lambda) > 0 \) such that if \( K_1 \) and \( K_2 \) are \( \lambda \)-near, then \( d(\rho_K(x), \rho_K(y)) < L \) for all \( x \in \mathbb{H}^n \), where \( \rho = \rho_K \).
Proof: Given a triangle $xyz$ in $\mathbb{H}^2$, possibly with $x$ an ideal point, if the angles at $y$ and $z$ are both at least $\pi/4$, then $d(x,z) \leq L_1$, where $L_1$ is some fixed constant. Also, given $\lambda$, we may find $L_2$ so that any two points, a distance no more than $\lambda$ apart, subtend an angle of less than $\pi/4$ at any third point, distant at least $L_2$ from one of them. Let $L = \max(L_1, L_2)$. Then $L > \lambda$.

Suppose now, $x \in \mathbb{H}^2$, with $y_1 = \rho_\alpha(x)$ and $d(y_1, y_2) \geq L$. This means that $x, y_1, y_2$ are all distinct. Since $y_2 \in K_2$, there is a point $y'_2 \in N_\alpha(y_2) \cap K_1$. Similarly, there is some $y'_3 \in N_\alpha(y_3) \cap N_\alpha(y_2)$ (FIG 3.1). By convexity, the line segment $y_2y'_2$ lies in $K_1$. Since $d(x, y_1) = \min$, the angle $x y_1 y'_2$ is at least $\pi/2$. So the angle $y_2 y_1 y'_2$ is at least $\pi/4$. But $d(y_2, y'_2) \geq L_1$, contradicting the fact that $x, y_1, y_2$ form a triangle.

Suppose, more generally, we have a closed convex set $X$, with $K_1 \cap N_\alpha(X)$ and $K_2 \cap N_\alpha(X)$ $\lambda$-near. (Note that $N_\alpha(X)$ is also closed and convex.) Let $y'_2$ be the retraction onto $K_1 \cap N_\alpha(X)$. Let $x \in \mathbb{H}^2$. By the lemma, $d(x, y'_2) \in K_1 \cap N_\alpha(X)$. We must have $\rho_\alpha(x) = \rho_\alpha(y'_2) = \rho_\alpha(z)$, so $z \in \text{int}N_\alpha(X)$. Hence, $\rho_\alpha(z) = \rho_\alpha(x)$. In other words, $\rho_\alpha(x) \in X$ implies that $\rho_\alpha(x) \in \text{int}N_\alpha(X)$. We have shown.

Corollary 3.3: Let $X, K_1, K_2 \subseteq \mathbb{H}^2$ be closed convex subsets. If $K_1 \cap N_\alpha(X)$, $K_2 \cap N_\alpha(X)$ are $\lambda$-near, then $\rho_\alpha^{-1}(K_1 \cap X) \subset \rho_\alpha^{-1}(K_2 \cap N_\alpha(X))$, where $\rho_\alpha$ is the retraction onto $K_1$.

In proving GF2 $\Rightarrow$ GF1, the first step will be to construct standard parabolic regions about each b.p.f.p. We shall arrange that these regions are strictly invariant under $\Gamma$, i.e., they are disjoint, collectively invariant under $\Gamma$, and the stabiliser of each region is equal to the stabiliser of the corresponding p.f.p. They therefore project to disjoint cusp regions in $M$. A priori, there may be infinitely many of these. However, in the second part of the proof, we go on to show that their complement, in $M_\Gamma$, is relatively compact, so that only could only have been finitely many.

Since the construction of standard parabolic regions about b.p.f.p. is valid for any discrete group, we stat it as a separate proposition.

Proposition 3.3: Let $\Gamma$ be a discrete group, and let $P \subseteq \Lambda$ be a $\Gamma$-invariant collection of b.p.f.p. Then, there exists a collection of cusp regions $\{C(p) \mid p \in P\}$, which is strictly invariant under $\Gamma$, i.e., the regions are mutually disjoint, and $C(\gamma p) = \gamma C(p)$ for all $\gamma \in \Gamma$ and $p \in P$.

Proof: If $\Gamma$ is parabolic, the result is trivial. Hence we shall assume that $|\Lambda| \geq 2$ so that the convex hull, $Y$, of $\Lambda$ meets $\mathbb{H}^2$. The retraction $\rho_\alpha$ onto $Y$ is clearly equivariant under the action of $\Gamma$. Let $p \in P$, and let $T(p) \subseteq \mathbb{H}^2$ be a Margulis region about $p$, as defined in Chapter 2 (GF4), i.e., $T(p) = \text{thin}_\alpha$ (stable $p$). The regions $(T(p) \mid p \in P)$ are strictly invariant in the sense defined above. It follows that this is true of the regions $T(p) \cap Y$ and hence $S(p) = \rho_\alpha^{-1}(T(p) \cap Y)$ also. We need therefore only to show that each $S(p)$ contains a standard parabolic region $C(p)$.

Focusing on one such $p = \infty \in \mathbb{R}^2$, with stabiliser $\Gamma_\infty$, we know that $A(\Gamma)\{\infty\} \subset Q_\infty = \{z|d_{\text{eucl}}(z, \sigma) < k\} \subset \partial \mathbb{H}^2$, where $|\sigma|$ is a minimal $\Gamma_\infty$-invariant plane. Let $w: \mathbb{R}^2 \to \mathbb{R}^2$ be vertical euclidean projection. It is not difficult to see that $w(T(p)) = \partial \mathbb{H}^2$. Moreover, we can choose a horoball $B(p)$, about $p$, so that $B(p) \cap \partial \mathbb{H}^2 \subset T(p)$. Since $Y \subseteq \partial \mathbb{H}^2$, we have that $Y \cap B(p) \subset T(p)$.

We have assumed that $\Lambda \neq \{\infty\}$. Hence each point of $\sigma$ lies within some bounded euclidean distance of $\Lambda$. Since $\partial \mathbb{H}^2 \subset \Lambda$, we see that $Y \cap B(p)$ and $\sigma \cap B(p)$ are $\lambda$-near for some $\lambda > 0$ (recalling the notation $\sigma = w^{-1}\sigma$). Let $B(p)$ be the horoball with $\partial B(p)$ a hyperbolic distance $L(\lambda)$ above $\partial B(p)$, i.e., $B(p) \cap \partial \mathbb{H}^2 = N_\lambda(B(p))$. From the corollary to our lemma, we have the inclusions $\rho_\alpha^{-1}(\sigma \cap B(p)) \subset \rho_\alpha^{-1}(Y \cap B(p)) \subset \partial \mathbb{H}^2$. Then, $\rho_\alpha^{-1}(\sigma \cap B(p))$ is a standard parabolic region $C(p)$.

Proof of 2 $\Rightarrow$ 1: Let $\Gamma$ be GF2, and let $P \subseteq \mathbb{H}^2$ be the set of all b.p.f.p. Let $C = \{C(p) \mid p \in P\}$ be the collection of standard parabolic regions constructed as in Proposition 3.3. Let $N$ be the closure, in $\mathbb{H}^2 \cup \Omega$, of the complement $(\mathbb{H}^2 \cup \Omega) \setminus \bigcup_{p \in P} C(p)$. In the quotient, we may write $M_\Gamma = N \cup L$, where $N$ is the projection of $N$, and $\hat{L}$ is a collection of standard cusps. We want to show that $\hat{L}$ is compact.

Let $a$ lie in $N \cap \mathbb{H}^2 \setminus \Sigma$, where $N$ is the lift of $\hat{N}$ to $\mathbb{H}^2 \cup \Omega$, and $\Sigma$ is the singular set. Let $P = \hat{D}(a)$ be the Dirichlet region about $a$. The set $P$ is convex, and its images under $\Gamma$ are locally finite in $\mathbb{H}^2$. Hence,
Suppose $P_3$ consists of a b.p.f.p., $p = \infty$ in $\mathbb{R}_+^n$. $(\mathcal{H} \setminus \text{int}(C(p))) / \Gamma_3$ is compact. Since the images of $P_3 \setminus \text{int}(C(p))$ under $\Gamma_3$ are disjoint and locally finite, $P_3 \setminus \text{int}(C(p))$ must have finite euclidean diameter. We see that $\text{int}(C(p))$ is an open neighborhood of $p$ in $P_3$. Thus, $N \cap P_3 = P_3 \setminus (\bigcup_1 \text{int}(C(p)))$ is a closed subset of $P_3$, and hence a compact subset of $H^n \setminus \Omega$. The closure of $\text{int}(C(p))$ is thus a Dirichlet region in $H^n \setminus \Omega$, which is a fundamental domain for $\Gamma$ acting on this set. It follows that $\hat{N}$ is a quotient of $P_3 \cap N$, and is therefore compact.

Finally, suppose there were an infinite sequence $(C_{n})$ of distinct cusp regions. We take $x_{n} \in \partial C_{n} \cap P_3 \cap H^n$, with $d(x_{n}, y_{n}, x_{n}) < \epsilon(n)$ for some parabolic $y_{n}$ stabilizing $C_{n}$. (Here $\epsilon(n)$ is the Margulis constant.) Margulis constant by some corresponding parabolic element. Taking a subsequence, we have $x_{n} \to y \in H^n \setminus \Omega$. If $y \in H^n$, then $\Gamma(y)$ contains parabolics with different fixed points, contradicting what we know about the structure of $\Gamma(y)$ from Ch.1. If $y \in \Omega$, then $\min\{d(x_{n}, y_{n}) \mid y_{n} \in \text{free}(\Gamma)\} \to 0$, contradicting the choice of $x_{n}$.

$\square$ 2 $\Rightarrow$ 1

$GFS \Rightarrow GF3$

We have $\mathcal{M}_{C} = \hat{N} \cup \{U_{C}\}$, where $\mathcal{C} = \{C_{1}, \ldots, C_{k}\}$. We may write $\mathcal{C} = C_{i}/\Gamma$, where $C_{i} = \{z \mid d_{n}(z, (a_{i})) \geq r_{i} \} \subset R_{+}^{n} \cup \partial R_{+}^{n}$, and $\Gamma_{i}$ is the stabilizer of $C_{i}$. Choose $o_{i} \in \hat{N}$ and $\alpha_{i} \in \text{int}(C_{i}) \cap o_{i}$, where $\alpha_{i}$ is the vertical plane above $\langle a_{i} \rangle$. Let $a_{i} = \{o_{i}, \alpha_{i}, \ldots, \alpha_{i}\}$. Let $\mathcal{A}_{i}(a_{i})$ be the complex defined in Chapter 2, $GF3$ (FIG 3.3). We fix our attention on some $C_{n}$. It is clear that $\mathcal{M}_{C_{n}} = C_{n}/\Gamma_{n}$. Since $\alpha_{n}, C_{n} \subset \mathcal{C} \cap o_{n}$, we see that the highest points of $\mathcal{M}_{C_{n}}$ (those with largest $n$th euclidean coordinate) are precisely the points of $\text{int}(\mathcal{C})$. Moreover, $d_{n}(z, (\alpha_{n})) \geq r_{n}$ for some fixed $r_{n} \geq r_{n}$, the nearest points of $\mathcal{M}_{C_{n}}$ to $a_{n}$ lies in $\mathcal{C}$. Let $C_{n}'$ be the standard region with radius $r_{n}$. Within $C_{n}'$, the complex $\mathcal{A}_{i}(a_{i})$ is identical to that obtained from $a_{i}$ for the group $\Gamma_{n}$, i.e. $C_{n}' \cap \mathcal{A}_{i}(a_{i}) = C_{n}' \cap \mathcal{A}_{i}(o_{i})$. Write $\lambda(y) = \mathcal{A}_{i}(\alpha_{i})$. Since $\alpha_{i}$ fixes $\alpha_{i}$, $\lambda(a_{i})$ is a euclidean product in the directions orthogonal to $\alpha_{i}$. Since $(\alpha_{i}) / \Gamma_{i}$ is compact, $\lambda(a_{i}) / \Gamma_{i}$ must be finite. Rewriting $\mathcal{M}_{C} = N_{n} \cup \{\mathcal{C}_{n}'\}$, with $\mathcal{N}_{n}$ compact, we see that $\lambda(a_{i}) / \Gamma_{i}$ is finite.

$\phi$ 1 $\Rightarrow$ 3

$GF1 \Rightarrow GF3$

Let $K \subset H^n$ be a convex set with non-empty interior. Let $a \in K$, and let $T_{a}(a)$ be the unit tangent space to $H^n$ at $a$. Clearly, $K$ determines a cone in the tangent space to $a$, which intersects the unit tangent space in a subset $T_{a}^{+}(a)$. We define $\omega(K, a)$ to equal $\mu(T_{a}^{+}(a))/\mu(T_{a}(a))$, where $\mu$ is spherical lebesgue measure (FIG 3.4). (Alternatively, $\omega(K, a)$ may defined as the Lebesgue density of $K$ at $a$.) The function $\omega(K, a)$ is strictly positive and lower semicontinuous on $K$. Let $K_{C}$ be the closure of $K$ in $H^n$, and let $K_{C} = K_{C} \cap H^n$. We may extend $\omega$ to a function on $K_{C}$ as follows. If $y \in K_{C} = K_{C} \cap H^n$, we consider the tangent space to $H^n$ at $y$, and measure the proportion of unit tangent vectors to $H^n$ lying in $K_{C}$. Note that on $K_{C}$ it is possible (for example if $K_{C} = \{y\}$) to have $\omega(K, y) = 0$. If this is so, we call $y$ a cusp point of $K_{C}$.

We shall restrict our attention to the case where $K$ is a finite intersection of closed half-spaces, and int$K \neq \emptyset$. We may write $K_{C} = \bigcap_{n} H_{n}$, where each $H_{n}$ is a closed half-space in $H^n$. By $\partial H_{n}$, we shall mean the closure, in $H^n$, of the boundary of $H_{n} \cap H^n$ in $H^n$.

Suppose first, that $\bigcap_{n} \partial H_{n} \neq \emptyset$. If $\bigcap_{n} \partial H_{n}$ contains a point of $H^n$, then it is a (possibly 0-dimensional) plane $\pi \subset H^n$, and in this case, $\omega(K, x)$ takes a fixed, strictly positive value for all $x \in \pi$. If, on the other hand, $\bigcap_{n} \partial H_{n}$ contains only ideal points, it must consist of a single point $y \in H^n$, with $\omega(K, y) > 0$.

More generally, if $K$ is a finite-sided convex polyhedron, by considering all possible intersections of half-spaces $H_{n}$, we may deduce the following:

**Lemma 3.4**: Let $K_{C}$ be a finite-sided convex polyhedron in $H^n$. Then, there exists a finite set $\lambda(K_{C})$ of cusp points in $K_{C}$, and $\delta(K_{C}) > 0$ such that for all $x \in K_{C} \setminus \lambda(K_{C})$, we have $\omega(K, x) > \delta(K_{C})$.

**Proof of 3 $\Rightarrow$ 1**: Let $\Gamma$ be $GF3$. Let $A$ be a $\Gamma$-invariant convex cell complex so that $A / \Gamma$ is finite, and such that for each $A \subset A, \{\gamma A \mid A\}$ is finite. We stated in Chapter 2 (GF3) that any cell of a convex cell
complex meets the closures of only finitely many other cells. Since \( A/\Gamma \) is finite, we may find a fixed constant, \( k_1 \), such that each point of \( H^n \) meets the closures of at most \( k_1 \) cells of \( A \). Also, the orders of \( \text{stab}_A \) are all finite and therefore bounded by some \( k_2 \). Thus, for each \( \gamma \in H^n \) and \( A \in A \), we have \(|\{\gamma \in \gamma A\}| \leq k_2 k_3 \), and so at most \( k_2 k_3 \) faces of any \( A \) can be equivalent under \( \Gamma \). It follows that each cell of \( A \) has only finitely many faces, and in particular, is (the interior of) a finite-sided polyhedron.

By hypothesis, \( A \) is locally finite in \( H^n \). Let \( y \in P \) and \( H_1 \subset H^n \cup \Omega \) be a half-space, containing \( y \) in its interior. We can insist that \( H_1 \) is invariant under \( \text{stab}_y \) and \( \partial H \cap H_1 = \emptyset \) if \( \gamma \notin \text{stab}_y \). Let \( H_2 \) and \( H_3 \) be successively smaller (\( \text{stab}_y \))-invariant half-spaces containing \( y \). Only finitely many cells of \( A \) lie entirely within \( H_1 \). Any cell that meets both \( \partial H_1 \) and \( \partial H_2 \) also meets \( \partial H_1 \cup \partial H_3 \), which is a compact subset of \( H^n \) (FIG 3.5). It follows that only finitely many cells meet \( H_3 \). We have shown that \( A \) is locally finite on \( H^n \cup \Omega \). Hence, every point of \( H^n \) lies in the closure of some top-dimensional cell.

The set of closures of top-dimensional cells consists of the image under \( \Gamma \) of a finite set of polyhedra \( \{P_1, \ldots, P_k\} \). \( M_\zeta \) is thus a quotient of the set \( \bigcup_{\zeta} P_\zeta \setminus \{\zeta\} \).

We shall show below that each \( P_\zeta \) can meet \( A \) only in a finite set of points (a subset of the set of cusp points \( \sigma(P_\zeta) \) of \( P_\zeta \)). When we have done this, the proof of SF1 can be completed as follows. If \( \zeta \) is a standard parabolic region about \( z \in P_\zeta \cap A \), then \( \zeta \) is an open neighbourhood of \( z \) in \( P_\zeta \), which we shall call a "cusp". We choose standard cusp regions for each conjugacy class of b.p.f.p. meeting some polyhedron \( P_\zeta \). By taking these regions small enough, we can ensure that the cusps they form in each polyhedron are disjoint. The closure of each \( P_\zeta \) is compact, and so therefore is the quotient \( M_\zeta = \text{closure}(M_\zeta \setminus \{\zeta\}) \), as required.

We now investigate \( P_\zeta \cap A \). Let \( y \in H^n \) lie in the closure of some polyhedron, \( P_\zeta \). We want to show that either \( y \in \Omega \) or \( y \) is a b.p.f.p. The stabiliser \( \Gamma_y \) of \( y \) cannot contain any loxodromic element, since otherwise, by applying a loxodromic element with \( y \) as repelling fixed point, we would get a contradiction to the local finiteness of \( A \) along the axis of the element. Therefore, \( \Gamma_y \) preserves setwise each horosphere about \( y \), and so \( \Gamma_y \) must be either a finite or a parabolic group. Let \( y = \infty \) in \( H^n \). Let \( P \neq \emptyset \) be the set of closures of top-dimensional cells containing \( y \). We distinguish two cases.

Case 1 : \( y \) is a cusp point of \( P \) (i.e., \( u(P, y) = 0 \)) for each \( P \in \zeta \).

By lemma 3.4, there are only finitely many such points in each polyhedron. It follows that there is a horoball \( B \) for which \( (B \cap P)/\Gamma_y \) is finite. (By \( B \cap P \), we mean \( \{B \cap P \mid p \in P \} \). Thus, if we take \( \partial B \) to be high enough, we can ensure that each \( B \cap P \) is a vertical prism on \( \partial B \) (i.e., isometric to \( (\partial B \cap P) \times [0, \infty] \)). It is still possible that \( \Gamma_y \) may be subdivided into many cells of \( A \). However, we know from the first paragraph of the proof that each polyhedron has only finitely many faces in \( A \). Thus, by raising the level of \( \partial B \) if necessary, we ensure that if \( A \) is a face of some \( P \in \zeta \), then \( A \cap B \) is a vertical prism (possibly empty). It now follows that \( \bigcup (B \cap P) = B \). For if not, consider a codimension-1 face \( A \) that bounds \( \bigcup (B \cap P) \) in \( B \). We know that \( A \) is a vertical prism, and so the (top-dimensional) polyhedron on each side of \( A \) has no finite closure. This contradicts the definition of \( P \).

Suppose now that, for some \( \gamma \in \Gamma_y \), we have \( \gamma \cap B \neq \emptyset \). Since \( B \subset P \), we see that \( \gamma \cap B \) meets some polyhedron \( P \subset \zeta \). Thus, \( P \) is the image under \( \gamma \) of some \( Q \subset \zeta \). From the finiteness of \( P/\Gamma_y \), and of the setwise stabiliser of each polyhedron, we see that \( \gamma \) must lie in one of a finite number of right cosets of \( \Gamma_y \) in \( \Gamma \). This gives us an upper bound on the height of the highest point of \( \gamma \) in \( H^n \). Thus by raising the level of \( \partial B \) if necessary, we can arrange that \( \gamma \cap B = \emptyset \) for all \( \gamma \in \Gamma_y \). Thus, we have found a strictly invariant horoball \( B \subset P \). Now, using \( B \), we want to construct another strictly invariant region, \( C \), which will be either a standard parabolic region or else a half-space, depending on whether \( \Gamma_y \) is parabolic or finite. (In fact, it turns out that the latter case cannot arise in Case 1.)

Let \( \sigma \subset \partial B \) be a minimal \( \Gamma_y \)-invariant subspace. By the finiteness of \( P/\Gamma_y \), there is a bound on the euclidean diameters of the lower-dimensional faces, of the polyhedra in \( P \), which do not contain \( \sigma \). Hence, for some \( r \), we see that \( C \times \{x \mid d_{\infty}(x, \sigma) \geq r\} \) is contained in \( \zeta \). The structure of \( C \cap \partial \zeta \) is independent of the vertical coordinate. Since \( P \) is locally finite on \( H^n \), we see that \( C \) must also be locally finite on \( C \cap \partial \zeta \). Therefore the limit set does not meet \( C \cap \partial \zeta \). If \( \Gamma_y \) is parabolic, then \( y \) is a b.p.f.p., and \( C \) is a standard parabolic region. Note that \( C \) meets \( P \), in an arbitrarily small, neighbourhood of a cusp point. If \( \Gamma_y \) is finite, then \( C \) is a half-space, with \( C \cap P \) finite. Thus \( y \in \Omega \). (Now, it is fairly easy to see that we
get a contradiction to the hypothesis of Case 1, though logically we do not need this.

Case 2: There is some $P \in P$, containing $y$, and with $w(K, y) > 0$.

Let $\delta = \min_{P}(P) > 0$ (see Lemma 3.4). We know that $w(P, y) > \delta$, so that $\Gamma_y$ must be finite (its order bounded by $|\mathcal{L}_w(P)|/\delta$). Since $|\mathcal{L}_w(P)| < \infty$, we can have $w(Q, y) = 0$ for only finitely many $Q \in P$. Otherwise, $w(Q, y) > \delta$. It follows that $P$ is finite. By an argument similar to that in Case 1, we show that some half-space, containing $y$, lies in $P$. Thus, $y \in \Omega$.

Conclusion: We chose an arbitrary $y \in P$. If $y$ lies in the limit set, we must be in Case 1. Then, $y$ is a b.p.f.p., and the standard parabolic regions about $y$ define a base of neighbourhoods for $y$ in $P$. The proof may now be completed as indicated above.

$\Box$

$GF1 \Rightarrow GF4$

Let $\epsilon(n)$ be the Margulis constant, and $\epsilon > 0$ be any number less than $\epsilon(n)$.

Suppose $\Gamma$ is $GF1$. Let $Y(\Gamma) = \text{hull}(\mathcal{L}(\Gamma))$. If $p$ is a b.p.f.p., we may take the corresponding parabolic region $C$ so that $C \cap Y(\Gamma)$ is contained in the corresponding Margulis region. Writing $M = M \cup (\cup C)$, we have that $\mathcal{L}(\Gamma) \cap \text{thick}_{\epsilon}(M)$ is contained in $N$, and is thus compact.

$\Box$

$GF4 \Rightarrow GF2$

We have $\text{thick}_{\epsilon}(M) \cap \mathcal{L}(\Gamma)$ compact for some $\epsilon < \epsilon(n)$.

Let $R$ be a component of thin$^{-\epsilon}(M)$, as defined in Ch2, GF4. If $R$ is the lift of a Margulis tube, then $R$ lies within a uniform neighbourhood of a loxodromic axis. If $R$ is the lift of a Margulis cusp, then it lies in some horoball about the p.f.p. Thus if $I$ is a geodesic ray lying entirely within thin$^{-\epsilon}(M)$, its ideal endpoint is either a loxodromic fixed point, hence a c.l.p., or a p.f.p.

Suppose $y \in \mathcal{L}(\Gamma)$ is neither a parabolic nor a loxodromic fixed point. Join $y$ to $x \in Y(\Gamma)$ by a geodesic ray $I$. The union of those parts of $I$ lying outside thin$^{-\epsilon}(M)$ is unbounded. Thus, its projection $\tilde{I}$ must accumulate in $\text{thick}_{\epsilon}(M) \cap Y(\Gamma)$, and so $y$ is a c.l.p.

It remains to show that any p.f.p. of $\Gamma$ is bounded.

Let $\delta > 0$ be such that $\epsilon + 4\delta < \epsilon(n)$. Let $p = \infty$ in $\mathbb{R}^m$ be a p.f.p. Let $R, R_{s+24}, R_{s+44}$ be the corresponding Margulis regions. The shell $S = (R_{s+24}, \text{int} R_{s})/\Gamma_p$ embeds as a closed subset of $\text{thick}_{\epsilon}(M)$. We show that if $p$ is not bounded, then $S \cap Y(\Gamma)$ is not compact.

Let $\alpha \subseteq \mathbb{H}^n$ be a minimal $\Gamma_p$-invariant plane. Suppose we have a sequence $(x_n)$ in $\mathcal{L}(\Gamma)$ with $d_{\alpha}(x_n, \alpha) \to \infty$. Let $x_n \in \partial R_{s+24}$ be vertically above $x_n$ (FIG 3.6). The hyperbolic ball $N_r(y_n)$ is a subset of $S = R_{s+24, \text{int} R_{s}}$. Taking a subsequence, we can assume that no such ball meets the image of a different ball under $\Gamma_p$. Since $\Gamma_p$ is virtually abelian, it has a torsion-free subgroup of index $k$ (say). Thus, each $(\delta/2)$-ball meets at most $k$ images of itself. It follows that the quotient sequence $x_n \in Y(\Gamma) \cap S$ has no convergent subsequence.

$\Box$

$GF1 \Rightarrow GF6$

Suppose $r$ is an $r$-plane in $\mathbb{H}^n$. Let $\mu_r: \mathbb{H}^n \to r$ be the nearest point retraction. For $X \subseteq r$ and $h > 0$, let $X_h = N_h(r) \cap r$, $N_h(r)$ is the uniform $h$-neighbourhood of $r$. There is a function $f: R \to R_+$, for which $\mu_r(X_h) = f(h)\mu_r(X)$, where $\mu_r$ is the $r$-dimensional volume. In particular, $X_{\infty}$ has finite $n$-volume if and only if $X$ has finite $r$-volume.

Let $\Gamma_p = \text{stab}_{\Gamma_p}$ be an infinite parabolic group, and $\alpha$ a minimal $\Gamma_p$-invariant plane containing $p$. Let $r = \dim r$. We know that $r \geq 2$. Let $C$ be a standard parabolic region. $\Gamma_p$ acts as a cocompact group on $\partial C \cap \alpha$, so we can find a compact $K \subseteq \partial C \cap \alpha \subseteq \mathbb{H}^n$ with $\mathcal{L}(\Gamma_p) = \partial C \cap \alpha$. $K' = \text{hull}(K, p)$ has finite...
Note that $C = \Gamma \cap N_n(a)$ and so the images of $K'\Delta$ under $\Gamma_p$ cover $C \cap N_n(a)$. We see that $\text{vol}_r((C \cap N_n(a))/\Gamma_p) < \infty$.

Now suppose that $\Gamma$ is GF, $\eta > 0$. For each b.p.f.p. $\rho$, we can find a region $C$ about $\rho$ so that $C \cap N_nY \subseteq C \cap N_n(a)$. Thus, $\text{vol}_r((C \cap N_nY)/\Gamma_p) < \infty$. The remainder of $N_{\infty}Y$ is compact, so $\text{vol}_r(N_{\infty}Y) < \infty$.

Finally, to see that $\Gamma$ is finitely generated, note that each cusp is topologically a product orbifold $\hat{C} = \mathbb{B} \times [0,1)$. Hence, $\Gamma = \pi_1\hat{C}$.

$\square$ 1 ⇒ 5

**GF5 ⇒ GF4**

Let $\Gamma$ be GF5. For some $\eta > 0$, $\text{vol}_r(N_{\infty}Y(\Gamma)) < \infty$. Let $\epsilon < \xi(n)$.

Suppose, first, that $\Gamma$ acts freely. Let $\epsilon_1 < \epsilon, \eta$. Take a maximal packing of $N_{\xi}(Y)$ with $\epsilon_1/2$-balls centred in $\text{thick}(M) \cap \hat{Y}$. Since each of these balls is isometric to a standard ball in $\mathbb{H}^n$, this packing is finite. By maximality, the corresponding $\epsilon_1$-balls cover $\text{thick}(M) \cap \hat{Y}$. Thus, $\text{thick}(M) \cap \hat{Y}$ is closed, and covered by finitely many compact sets, and hence is itself compact.

Suppose, now, that $\Gamma$ is any GF5 group. By the Selberg Lemma, $\Gamma$ has a torsion-free subgroup of finite index. This must act freely on $\mathbb{H}^n$, so that the volume of an $\epsilon_1/2$-ball centred on $\text{thick}(M)$ is bounded away from zero. The proof now works as before.

$\square$ 5 ⇒ 4

Let $\Gamma$ be a discrete group of isometries of $\mathbb{H}^n$.

In Ch.2 GF5, we stated three cases in which finite generation is automatically implied by $\text{vol}_r(N_{\infty}(Y)) < \infty$.

**Case (ı) : If there is some number $k$, such that for each $x \in \mathbb{H}^n$, $|\text{stabilizer of } x| \leq k$, then $\Gamma$ is GF.**

**Proof :** Let $\delta < \min(\eta, \epsilon(n)/N(n))$, where $N(n)$ is as defined in Ch.2 GF4. Let $y \in \text{thick}(M) \subseteq \mathbb{H}^n$. We know from Proposition 2.2 that $\Gamma_\delta(y)$ fixes some point of $\mathbb{H}^n$. Hence, $|\Gamma_\delta(y)| \leq k$. $N_{\delta/4}(y)$ meets at most $k$ images of itself under $\Gamma$. Thus, in the quotient, $\text{vol}_r(N_{\delta/4}(\hat{y}))$ is bounded away from zero, for $\hat{y} \in \text{thick}(M)$. The argument is now as for free actions on $\mathbb{H}^n$.

$\square$ Case(ı)

**Case (ıı) :**

**Theorem 3.5 :** A finite-volume complete hyperbolic orbifold (without boundary) is geometrically finite.

Let $\eta < \epsilon = \xi(n)/N(n)$, where $\xi(n)$ is the Margulis constant, and $N(n)$ is as defined in Proposition 2.2. In Ch.2 GF4, we defined $\text{thin}(M)$ to be the projection of the set $\{x \in \mathbb{H}^n \mid |\Gamma_{\epsilon}(x)| = \infty\}$. We saw that GF4 was equivalent to the statement that $\hat{Y}(\Gamma) \cap \text{thick}(M)$ be compact, for any $\eta < \epsilon$. We aim to show here, that if $M$ has finite volume, then $\text{thick}(M)$ is compact for a certain $\delta < \epsilon$. We begin by giving a proof of the following proposition about general hyperbolic orbifolds. We shall then build upon our argument to deduce the main theorem (3.5).

**Proposition 3.6 :** Given $n$, there is some universal $\delta > 0$, such that for any discrete subgroup $\Gamma$ of $\text{Isom}(\mathbb{H}^n)$, there is some $\hat{z} \in \mathbb{H}^n$ for which $\Gamma_{\delta}(\hat{z})$ is trivial, so that $N_{\delta/3}(\hat{z})$ is an embedded hyperbolic $\delta/3$-ball in the quotient.

**Proof :** Let $\Gamma$ be a discrete group, with $M = \mathbb{H}^n/\Gamma$. Given $\hat{z} \in M$, we define the injectivity radius, $\text{inj}(\hat{z}) = \frac{1}{2}\min(\delta(\hat{z}, \gamma \hat{z}) \mid \Gamma \in \Gamma)$. (This is usually defined only for manifolds.) We write $E$ for the singular set of $\Gamma$ in $\mathbb{H}^n$, so that $E = E/\Gamma = \{z \in M \mid \text{inj}(z) = 0\}$. Let $E$ be the union of all the loxodromic axes with
translation distance less than \( \epsilon(n) \). By the Margulis lemma, the collection of such axes is locally finite, so that \( L \) is closed. Its projection, \( L \subset M \) is a disjoint union of arcs and simple closed curves.

Let \( \eta \leq \epsilon = (n)/N(n) \). We define the decomposition of \( M \) into disjoint pieces as follows. Given \( X \subset \mathbb{H}^n \), let \( T_{\eta}(X) = \{ x \in \mathbb{H}^n | \text{stab}_{\eta}(x) = X \} \). Let \( T_{\eta} = \{ T_{\eta}(X) | X \subset \mathbb{H}^n \} \). Clearly, \( T_{\eta} \) is invariant under \( \Gamma \), so it projects to a decomposition \( T_{\eta} \) of \( M \). We shall call the elements of the decomposition \( \eta \)-compartments. Let \( T = T_{\eta}(X) \) be one such \( \eta \)-compartment. \( T \) is non-empty, so from Clh, we know that \( X \) must either consist of one or two points of \( \mathbb{H}^n \), or be a plane in \( \mathbb{H}^n \). We may thus define \( d(T) \) to be the dimension of \( X \), with the convention that \( d(T) = -1 \) if \( X \subset \mathbb{H}^n \) (This is well-defined since, if \( T_{\eta}(X_1) = T_{\eta}(X_2) \), then \( X_1 \) and \( X_2 \) are equivalent under \( L \)). Suppose \( x \in T \) lifts to \( \tilde{x} \in \mathbb{H}^n \). Then, \( \text{stab}_{\eta}(x) \) is a subset of \( \mathbb{H}^n \) if and only if \( \text{stab}_{\eta}(\tilde{x}) \) is infinite. Thus \( d(T) = -1 \) if and only if \( x \in \text{thick}_{\eta}(M) \). We see that \( \text{thick}_{\eta}(M) = \bigcup \{ T \in T_{\eta} | d(T) = -1 \} \), and \( P_{\eta} = T_{\eta}(\mathbb{H}^n) = \{ x \in \mathbb{H}^n | d(x) > \eta/2 \} \).

If \( T = T_{\eta}(X) \in T_{\eta}\{P_{\eta}\} \), (i.e. \( d(T) < \eta \)), we define a smooth unit vector field \( \psi_{\eta} \) on \( T \setminus (\Sigma \cup L) \) as follows. For \( x \in T \setminus (\Sigma \cup L) \), we define \( \psi_{\eta}(x) \) to be the unit tangent vector pointing directly away from \( \text{hull}(X) \), i.e. \( \psi_{\eta}(x) = -\alpha(x) \alpha(0) \), where \( \alpha \) is the geodesic arc from \( x = \alpha(0) \) to the nearest point on \( \text{hull}(X) \). (Note that \( \text{hull}(X) = X \), unless \( \text{stab}_{\eta}(x) \) is an \( \infty \)-loxodromic group, in which case \( \text{hull}(X) \) is the loxodromic axis.) This gives us a well-defined vector \( \psi_{\eta}(x) \) at \( x \in T \). Performing this construction for each \( T \in T_{\eta}\{P_{\eta}\} \), we get a (usually discontinuous) piecewise analytic vector field on \( M \{P_{\eta} \cup \Sigma \cup L\} \). The integral curves are piecewise geodesic.

Now, we fix \( \epsilon = (n)/N(n) \), and choose any \( \delta < \epsilon \). Suppose \( x \in M \{\text{thick}_{\eta}(M) \cup P_{\eta} \cup \Sigma \cup L\} \), so that \( \text{thick}_{\eta}(M) \) is finite and non-trivial. Let \( \beta \) be the integral curve through \( x \) for the vector field \( \psi_{\eta} \). We take \( \beta(0) = x \), and write \( \Gamma_{\delta}(x) \) for \( \Gamma_{\delta}(x) \).

Imagine following the integral curve \( \beta \) in \( \mathbb{H}^n \). At time \( t \), we have \( \Gamma_{\delta}(\beta(t)) \subset \Gamma_{\delta}(\beta(t)) \). Let \( h_{\delta} = \text{hull}_{\mathbb{H}^n}(\beta(t)) \), and \( n_{\delta} = \text{hull}_{\mathbb{H}^n}(\beta(t)) \). Now, \( h_{\delta} \) and \( n_{\delta} \) are both subspaces of \( \mathbb{H}^n \), and \( h_{\delta} \subset n_{\delta} \). Thus, it is easily checked that \( h_{\delta}(\beta(t)) \) always makes an acute angle with \( n_{\delta}(\beta(t)) \). This means that the distance of \( h_{\delta}(t) \) from \( h_{\delta}(t) \) increases at least linearly with \( t \). Now, while \( \Gamma_{\delta}(\beta(t)) \) remains constant and equal to \( \Gamma_{\delta} \), we see that the injectivity radius \( \text{in}(\beta(t)) \) increases steadily, and the derivative of the injectivity radius with respect to \( t \) is non-decreasing. Thus, after a finite distance, at \( \beta(t) \), say, \( \Gamma_{\delta}(\beta(t)) \) must change to a new group \( \Gamma_{t} \). Now \( \Gamma_{\delta} \) and \( \Gamma_{t} \) are both subgroups of \( \Gamma_{\delta}(\beta(t)) \). Since we are moving away from \( \psi_{\delta}(\beta(t)) \), it is easy to see that \( \Gamma_{t} \subset \Gamma_{\delta} \). Again, after another finite distance, \( \Gamma_{t}(\beta(t)) \) changes to a third group \( \Gamma_{\delta} \subset \Gamma_{t} \). Since \( \Gamma_{\delta} \) is finite, it follows that, after a finite number of steps, we shall arrive at a point \( y \), with \( \Gamma_{\delta}(y) = \{1\} \). (Note that we can never run into \( \Sigma \cup L \). Wherever the \( \delta \) is discontinuous along \( \Sigma \cup L \), the vector field is no longer defined.)

It is easy to see, from the form of the components of \( \text{thick}_{\eta}(M) \), that \( \text{thick}_{\eta}(M) \) cannot occupy all of \( M \). \( \Sigma \cup L \) is a lower-dimensional object, so either \( \text{int}\text{thick}_{\eta}(M) \subset P_{\eta} \) or we can find some point \( x \) as above. Either way, there is some \( y \in \mathbb{H}^n \) with \( \Gamma_{\delta}(y) \) trivial.

\( \Diamond \) Prop.3.6

Now, think of \( M \) as cut into \( \delta \)-compartments. We have shown that \( P_{\delta} \neq \emptyset \). In fact, we have shown that any point \( x \in \text{thick}_{\eta}(M) \setminus (\Sigma \cup L) \) can be joined to some \( y \in P_{\delta} \), by a path \( \beta \) with \( \Gamma_{\delta}(\beta(t)) \) monotonically decreasing (i.e. if \( t \geq 0 \), then \( \Gamma_{\delta}(\beta(t)) \subset \Gamma_{\delta}(\beta(t)) \)). Consequently, \( \beta(\delta(t)) \) is monotonically increasing with respect to set inclusion. Each time \( \beta(\delta(t)) \) changes, its dimension must strictly increase. It follows that \( \beta(\delta(t)) \) passes through at most \( \delta \)-compartments of \( M \). Hence, any point of \( \text{thick}_{\eta}(M) \) can be joined to \( P_{\delta} \) by a path that passes through at most \( \delta \) \( \delta \)-compartments.

Now, the collection \( T_{\delta} \) is locally finite. To see this, note that, for any point \( x \in \mathbb{H}^n \), the set \( \{ t \in \mathbb{R} | d(\beta(t), x) < \delta \} \) is finite, by the discreteness of \( L \). So, there are only finitely many candidates for the generating set of any group \( \Gamma_{\delta}(y) \), with \( y \in \mathcal{N}(x) \).

Let \( T_{\delta} = \{ T \in T_{\delta} | d(T) \neq -1 \} \). We know that this covers \( \text{int}\text{thick}_{\eta}(M) \).

Now, suppose that \( M \) has finite volume. We aim to show that \( \text{thick}_{\eta}(M) \) is compact. Consider the \( \delta \)-compartment \( T \in T_{\delta} \). We have \( T = \Gamma_{\delta}(x) \) for some plane \( \sigma \subset \mathbb{H}^n \). Let \( \text{stab}_{\sigma}(x) \) be the subgroup of \( \Gamma \) that fixes \( \sigma \) pointwise. Let \( p = |\text{stab}_{\sigma}(x)| \). For any \( x \in T \), \( \Gamma_{\delta}(x) \subset \text{stab}_{\sigma}(x) \), so \( \mathcal{N}_{\delta}(x) \) meets at most \( p \) images
of itself under \( \Gamma \). This gives a positive lower bound on the volume of any metric \( \delta/4 \)-ball in \( M \), centred on a point of \( T \), namely \( 1/p \) times the volume of a \( \delta/4 \)-ball in \( \mathbb{H}^n \). Since \( M \) has finite volume, we see that \( T \) has finite diameter, and is thus relatively compact.

We now think of \( \text{thick}^\circ_4(M) \) just as a topological space \( W \). We summarise what we know about \( W \). \( W \) has a locally finite cover \( K \) by compact subsets (the closures of the \( \delta \)-compartments). Also, there is a constant \( k = n + 1 \), and a fixed element \( K_0 \) of the cover such that any point of \( W \) can be joined to \( K_0 \) by a path which is covered by at most \( k \) sets from \( K \).

It follows from this, that \( K \) must itself be compact. To see this, think of the elements of the cover as vertices in an abstract graph. Two vertices are joined by an edge if the corresponding sets intersect. The graph has finite diameter (path condition), and each vertex has finite degree (compactness and local finiteness). Thus the graph is finite.

We have shown that \( \text{thick}^\circ_4(M) \) is compact. The discussion of GF4 shows us that \( M \) is GF.

\( \square \) Thm. 3.5

**Case (iii) :** Let \( \Gamma \) be a discrete subgroup of \( \text{Isom} \mathbb{H}^n \), with \( n = 2 \) or 3. If \( \text{vol}_n N_\eta \hat{\Gamma} (\Gamma) < \infty \), for some \( \eta > 0 \), then \( \Gamma \) is GF.

**Proof :** We deal with the case \( n = 3 \) (\( n = 2 \) is similar). We aim to reduce this to Case (i), by showing that \( |\text{stab}_r(x)| \) is bounded for \( x \in \mathbb{H}^n \).

Note that we can assume (by taking an index-2 subgroup if necessary) that \( \Gamma \) is orientation-preserving. We shall also suppose that \( |\Delta(\Gamma)| > 1 \), otherwise \( \Gamma \) is trivially GF. We can take \( \eta \) to be less than \( \epsilon(n) \), the Margulis constant.

Suppose then, that \( |\text{stab}_r(x)| \) is unbounded. From the Jordan Lemma, we can find a sequence \( \{G_i\} \) of finite abelian subgroups of \( \Gamma \), with \( |G_i| \to \infty \). From Lemma 1.3, using the fact that \( n = 3 \), we see that the fixed-point set of each \( G_i \) is a geodesic in \( \mathbb{H}^3 \). (Thus, each \( G_i \) can be assumed to be cyclic of large order.) We can assume that these geodesics are all inequivalent under \( \Gamma \).

Let \( G \to G_i \) be one such group, with fixed-point set \( l \). Since \( |\Delta| > 2 \), we can choose some \( t \in A \setminus l \). Let \( m \) be the perpendicular from \( t \) meeting \( l \) at \( a \). Note that \( m \subset \hat{\mathbb{H}}(\Gamma) \), and that \( G \leq \Gamma_m(a) \), where \( \Gamma_m(a) = \Gamma_a(x) \cap \Gamma_{\text{stab}_r(x)} \), \( \Gamma_m(a) \) has index at most \( \nu(m) \) in \( \Gamma_a(x) \). (For the discussion of the Margulis Lemma Ch 1.) It is not difficult to see that we can find a point \( y \in \mathbb{H} \), with \( \Gamma_m(y) \) trivial, but with \( \Gamma_m(a) \) a non-trivial subgroup of \( \Gamma_m(a) \). Thus, \( l = \mathbb{H}_{\Gamma_m(a)} \), and \( N_{\Delta/\eta}(x) \) meets at most \( \nu(m) \) images of itself under \( \Gamma \). Note that \( N_{\Delta/\eta}(x) \subset N_{\eta/\Delta}(\Gamma) \).

Performing this construction for each group \( G_i \), we obtain a sequence of points \( y_i \in \mathbb{H}^n \). In the quotient, the \( (\eta/\Delta) \)-balls \( N_{\Delta/\eta}(y_i) \) are disjoint (since \( \mathbb{H}_{\Gamma_m(a)} = l_i \)), and their volumes are bounded below. This means that \( N_{\Delta/\eta}(\Gamma) \) must have infinite volume.

\( \square \) Case (iii)


In dimension 3, the central definitions of geometrical finiteness have traditionally been in terms of finite-sided fundamental polyhedra. In particular, the following statements are all equivalent to geometrical finiteness.

1a (1b) : Some (each) convex fundamental polyhedron has finitely many faces.

2a (2b) : Some (each) Dirichlet polyhedron has finitely many faces.

Here we use "face" in the sense of Chapter 2, GF3. This means that each polyhedron meets only finitely many images of itself under the group.

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We can interpret our definition GF as being equivalent to the statement 1a without the assumption of convexity. The remaining definitions, 1b, 2a and 2b, no longer work in higher dimensions, as the following discussion shows.

First, consider on $\mathbb{E}^3$, an infinite cyclic group $\Gamma$ of irrational screw motions with axis $r$, i.e. $\Gamma$ is generated by a translation parallel to $r$ composed with an irrational rotation with $r$ as axis (FIG 4.1). If $a \notin r$, then the Dirichlet domain $D(a)$ about $a$ is infinite-sided.

To see this, let $\Phi$ be the $(n-1)$-sphere of parallel classes of rays in $\mathbb{E}^n$. Suppose that $a \notin r$, and let $l$ be the ray through $a$, perpendicular to $r$. Suppose $D(a)$ is finite-sided. Then, $D(a) = \bigcap_{\gamma \in \Gamma} H_\gamma$, where $H_\gamma$ is the half-space $\{x \in \mathbb{E}^n \mid d(x, a) < d(x, \gamma a)\}$. Note that $l$ is never parallel to $\partial H_\gamma$ for $\gamma \neq 1$. It follows that $l \notin \operatorname{int} \Theta$, where $\Theta \subset \Phi$ is the set of rays lying in $D(a)$. (We may identify $\Phi$ with the set of rays emanating from $a$.) Now, $\Gamma$ acts on $\Phi$ as a non-discrete rotation group fixing $r$. Thus, for any $\gamma \in \Gamma$, we have $l \cap \operatorname{int} \Theta \notin \Theta$, so that $D(a) \cap \gamma D(a) \neq \emptyset$, contradicting the assumption that $D(a)$ is a Dirichlet domain. This proves that $D(a)$ is infinite-sided.

We may get a picture of how the domains $\gamma D(a)$, for $\gamma \in \Gamma$, tessellate $\mathbb{E}^3$, as follows. Let $r$ be some large positive number, and let $S_r = \{x \in \mathbb{E}^3 \mid d(x, r) = r\}$ be the surface of a cylinder of radius $r$ about $r$. Let $S_r$ be the universal cover of $\mathbb{E}^3$. In the induced Riemannian metric, $S_r$ is isometric to $\mathbb{E}^2$. Thus, the tessellation of $\mathbb{E}^3$ determines a CW-decomposition of $\mathbb{E}^3$, invariant under a $\mathbb{Z} \times \mathbb{Z}$ action. In the generic situation, this decomposition is combinatorially equivalent to a regular hexagon tessellation of the plane. As $r$ tends to infinity, the pattern of hexagons changes by an infinite sequence of "Whitehead moves". This process is best described with reference to the quotient torus, $\mathbb{E}^3 / \Gamma \approx \mathbb{E}^2 / \mathbb{Z} \times \mathbb{Z}$. For a generic $r$, the torus is decomposed into two 0-cells, three 1-cells and one 2-cell. As $r$ becomes critical, one of the 1-cells collapses to a single point, giving rise (combinatorially) to a square tessellation of $\mathbb{E}^2$. The 4-valent vertex then splits again into two 3-valent vertices to give another hexagon tessellation (FIG 4.2). The combinatorial structure of the tessellation $\{\gamma D(a) \mid \gamma \in \Gamma\}$, far away from $r$, is thus determined by the sequence of 1-cells which get contracted by Whitehead moves. This sequence is, in turn, determined by the continued fraction expansion of the rotation angle $\theta$, measured as a fraction of a full rotation. (The situation is analogous to following a geodesic in the moduli space of euclidean tori — see [Ser].) The metric structure of each domain $D(a)$ is also related to rational approximation of $\theta$. Clearly, the area of the cross section $\gamma D(a) \cap S_r$ grows linearly with $r$, but its diameter grows much more quickly in the radial direction than in the direction parallel to $r$. The relative rates depend on rational approximations to $\theta$ — the better $\theta$ is approximated, the quicker the cross sections flatten out radially. For a quadratic surd, the radial diameter grows asymptotically like $r^{3/4}$, while the diameter parallel to $r$ grows like $r^{1/4}$.

Now, we may extend our cyclic group, $\Gamma$, to act on $\mathbb{H}^4$ as a parabolic group, with $\mathbb{E}^3 \subset \mathbb{H}^4$ a horosphere about the fixed point $p$. Let $p$ be the 3-plane spanned by $r$ and $p$. If $a \in \mathbb{H}^4 \setminus p$, the Dirichlet domain $D(a)$ will be infinite-sided. (With $p = \infty$ in the upper half-space model, $D(a)$ is a vertical prism on the euclidean Dirichlet domain, $D'$, i.e. $D(a)$ is euclidean-isometric to $D' \times (0, \infty)$.) However, $\Gamma$ is GF with any of the definitions of Chapter 2.

We may now find a half-space, in $\mathbb{H}^4$, disjoint from all its images under $\Gamma$, and disjoint from $p$. This set projects to an embedded half space in the quotient manifold $M$. By removing this half space, and doubling $M$ across the boundary, we get a new manifold $M'$, with fundamental group $\mathbb{Z} \times \mathbb{Z}$. This gives us a geodesically finite action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{H}^4$ with no finite sided Dirichlet domain. This example was constructed by Apanasov.

Remark 1. In the upper half-space model, we may find a sequence $(H_i)$ of such half spaces, disjoint in the quotient, $M$, with diam$_{\infty}(H_i) \to \infty$. We replace each half-space $H_i$ with a copy of $M' \setminus H_i$ to give a new manifold $M'$. On $\mathbb{H}^4$, this gives us a discrete, infinitely generated free group, with no standard horoball about $p$. I do not know of any finitely generated group with this property. In contrast, we know that the GF groups have standard horoballs.

Let $\Gamma$ be a GF group. We have seen that there is no reason to expect a Dirichlet domain for $\Gamma$ to be finite-sided. We say that $p$ is p.f.p. of $\Gamma$ is rational if $\operatorname{stab}_\Gamma(p)$, acting on a standard horosphere, contains a finite-index translation group. Otherwise, we say that $p$ is irrational. We shall show that if $\Gamma$ contains an irrational p.f.p., then $D(a)$ will be infinite-sided if we choose any anywhere on a certain open dense subset of $\mathbb{H}^4$. However, if there are no such p.f.p.s, we show that any convex fundamental domain, $P$, for $\Gamma$ will be
finite-sided polyhedron. Here, we use the term "finite-sided" in the sense of Chapter 2, G3, namely that $P$ should be a finite intersection of half-spaces. It is still possible that $P$ may meet infinitely many images of itself under $\Gamma$. If $P$ happens to be a Dirichlet domain, however, it is fairly easy to see that for any $\gamma \in \Gamma$, we must have $P \cap \gamma P = P \cap \sigma$, for some $\{n - 1\}$-plane $\sigma$. In other words, the "faces" and "sides" of $P$ coincide. Thus, if $\Gamma$ contains no irrational p.f.p.s, then each Dirichlet domain has only finitely many faces. In fact, we shall see that in $\mathbb{H}^3$ and $\mathbb{H}^2$, there can be no irrational p.f.p.s, and that each convex fundamental domain has finitely many faces. This will prove the equivalence of 1a, 1b, 2a and 2b in these dimensions.

Below we give (in principle) a complete description of when a Dirichlet domain is finite-sided. We begin by discussing the euclidean case.

Let $\Gamma$ act discontinuously by isometries on $\mathbb{E}^n$. Suppose the subgroup $\Gamma_1 \leq \Gamma$ acts on the plane $\mathbb{P} \in \mathbb{E}^n$ as a translation group, $i = 1, 2$. The group $\Gamma_1 \cap \Gamma_2$ acts as a translation group on $(\mathbb{P}, \mathbb{P})$. (If $\gamma$ acts by translation on $\mathbb{Q}_1$, and by translation on $\mathbb{Q}_2$, then the two translations are parallel and have the same translation distance. Hence, $\gamma$ acts by translation on $(\mathbb{Q}_1, \mathbb{Q}_2)$.) Therefore makes sense to define $r$ to be the largest plane on which some finite-index subgroup of $\Gamma$ acts as a translation group. Let $\Gamma_0 \leq \Gamma$ be the subgroup of all elements acting by translation on $r$. If $g \in \Gamma$, then $g \Gamma_0 g^{-1}$ is a translation group on $gr$. Thus, $gr = r$, and $g \Gamma_0 g^{-1} = \Gamma_0$, i.e. $\Gamma_0$ is a normal subgroup of $\Gamma$, and $r$ is fixed setwise by $\Gamma$.

**Proposition 4.1** Suppose $a \in \mathbb{E}^n$ is not fixed by any element of $\Gamma$. Then, the Dirichlet domain $D(a)$ is finite sided if $a \in \mathbb{P}$, and infinite sided if $a \notin \mathbb{P}$.

**Proof**: Suppose $a \in \mathbb{P}$. Then, $D(a)$ is a euclidean product, with an orthogonal plane, of the Dirichlet domain $D(a) \cap r$ restricted to $r$. On this subspace, $\Gamma$ has a finite-index translation group, so that any convex fundamental domain is finite-sided (see Lemma 4.2 below).

Suppose $a \notin \mathbb{P}$. Let $\mu$ be a minimal $\Gamma$-invariant affine subspace (see Chapter 1). Note that $\mu$ is a subspace of $\mathbb{P}$. Let $\beta$ be the nearest point in $\mu$ to $a$. Let $l$ be the ray from $b$ through $a$, and let $\sigma$ be the plane $\{(\mu, \alpha) \mid \beta \in \mathbb{P} \}$. We have $l \subseteq D(a)$. To see this, take any $\epsilon \in l$. The images of $\sigma$ under $\Gamma$ all have be a fixed distance from $\mu$. It follows that the nearest image to $\epsilon$ must be $\alpha$ itself, i.e. $\epsilon \in D(a)$.

Suppose $D(a)$ were finite-sided, $D(a) = \bigcap_{\gamma \in \Gamma_0} H_\gamma$, where $G$ is a finite subset of $\Gamma$. Let $\Gamma_1 \leq \Gamma$ be the subgroup of $\Gamma$ that fixes the plane $\sigma$ setwise, and preserves the direction of $l$. (One can see that $\Gamma_1$ is defined independently of the choice of $\mu$, though this is not important for our discussion.) By maximality of $\sigma$, we must have $|\Gamma : \Gamma_1| = \infty$.

If $\Gamma_1$ were trivial, the proof could proceed as in the example of an irrational screw motion described above. $D(a)$ would contain a cone about $l$, and, since the action of $\Gamma$ on the sphere of rays is not discrete (if $E = \Gamma_1$, we could find $\gamma \in \Gamma$ with $t \gamma l$ arbitrarily close (in direction) to $l$, so that $\gamma D(a) \cap D(a) \neq \emptyset$.

To deal with the general case, we write $D(a) = D_1(a) \cap D_2(a)$ with $D_1(a) = \bigcap_{\gamma \in \Gamma_1, \gamma \neq \epsilon} H_\gamma$, and $D_2(a) = \bigcap_{\gamma \in \Gamma_2, \gamma \neq \epsilon} H_\gamma$. As above, $D_2(a)$ contains a cone about $l$, which we can take to be an open spherical cone $C$ centred on $l$. Now, $D^2(a)$ contains $\bigcap_{\gamma \in \Gamma_1, \gamma \neq \epsilon} H_\gamma = D'$, where $D'$ is the Dirichlet domain about $l$ for the group $\Gamma_1$.

Since $\sigma$ is preserved by $\Gamma_1$, $D'$ is a euclidean product, with an orthogonal plane, of the Dirichlet domain $D' \cap \sigma$, of $\Gamma_1 \mid \sigma$ about $a$. Since $\Gamma_1$ fixes the direction of $l$, we see that $D' \cap \sigma$, in turn, is a euclidean product, with the line (l), of the Dirichlet domain $D''$ about $b$ for $\Gamma_1 \mid \sigma$.

As before, $\Gamma_1$ acts as a non-discrete group on the sphere of rays. So, we can find $\gamma \in \Gamma_1$ with $t \gamma l \cap C \neq \emptyset$ (so that $T_1 \gamma l \cap C$ is an infinite ray). For some $g \in \Gamma_1$, we have $\gamma^{-1} b \in g D'$. Now, $g^{-1} \gamma^{-1} l$ is a ray, orthogonal to $b$, emanating from the point $g^{-1} \gamma^{-1} b$ of $D''$. From the previous paragraph, we see that $g^{-1} \gamma^{-1} l$ lies entirely within $D'$. Hence, we see that $l \subseteq T g D' \subseteq \gamma T g D'(a)$.

Now $\gamma^{-1} l \cap C \neq \emptyset$. Since $C$ is a spherical cone about $l$, and $g l$ is parallel to $l$, we see that $g C$ is just a translate of $C$. Thus $\gamma^{-1} l \cap C \neq \emptyset$, and so $T g D'(a) \neq \emptyset$.

We have shown that $l$ intersects both $\gamma g D'(a)$ and $\gamma g D^2(a)$. So, $\gamma g D'(a)$ meets the interior of $D(a)$. But $\gamma \neq 1$, contradicting the assumption that $D(a)$ is a Dirichlet domain.

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One may easily generalise the above proposition to the following.

A collection of generalised Dirichlet domains $\{D_\alpha(a)\}$ contains at least one infinite-sided member if and only if some basepoint $a$, $a \notin \mathbb{P}$. 28
We now want to give a description of when Dirichlet domains for hyperbolic groups are finite-sided. Given a Dirichlet domain, $D(a)$, we need to describe which of the p.f.p.s are contained in the closure, $(D(a))_{c}$. To do this, we introduce a construction analogous to that of Dirichlet domains, for p.f.p.

Let $\Gamma$ be a discrete group, and let $P_{0} \subseteq A$ be an orbit of p.f.p.s. We choose a horoball $B(p)$ about each $p \in P_{0}$, so that the collection $\{B(p) | p \in P_{0}\}$ is strictly invariant. Given $p \in P_{0}$, we define $U(p)$ to be the set of points nearer to $B(p)$ than to any other horoball, i.e., $U(p) = \{x \in H^{n} | d(x, B(p)) < d(x, B(q)) \text{ for all } q \neq p\}$. Let $U = \bigcup \{U(p) | p \in P_{0}\}$. Since $B(p) = \partial B(p)$, the collection $\{B(p) | p \in P_{0}\}$ is determined by the choice of just one horoball $B(p)$. It is easily seen that any choice of $B(p)$ will give rise to the same collection $U$. We also see that $U$ is locally finite, and that $\bigcup U$ is dense in $H^{n}$.

Suppose that $a \in H^{n}$ is not fixed by any $\gamma \in \Gamma$. Then, we see that $(D(a))_{c}$ contains the point p.f.p. $p$, if and only if $a$ lies in the closure $U(p)$ of $U(p)$. This means that $(D(a))_{c} \cap P_{0} = \{p \in P_{0} | a \in U(p)\}$. In particular, we see that $(D(a))_{c} \cap P_{0} = \{p\}$ if and only if $a \in U(p)$. From this, we see that $(D(a))_{c}$ contains only finitely many p.f.p.s in a given orbit, and generically contains only one from each orbit.

Now, the collection $U$ satisfies all the conditions (Chapter 2, GF3, (a)-(d)) to be the set of top-dimensional cells for a convex cell complex. We write $B_{r}(P_{0})$ for unique such complex which is minimal with respect to subdivision (see the construction in Chapter 2, GF3). More generally, suppose that $P \subseteq A$ consists of finitely many orbits of p.f.p.s, $P = \bigcup_{i} P_{i}$. From Chapter 2, GF3, we see that the complexes $B_{r}(P_{i})$ have a minimal common subdivision, $B_{r}(P) = (B_{r}(P_{i}) | i = 1, \ldots, \lambda)$, obtained by intersecting cells from each complex.

If $\Gamma$ acts freely, we may describe the cells of $B_{r}(P)$ as follows. Let $Q \subseteq P$ be finite, and let $A(Q) = \{a \in H^{n} | (D(a))_{c} \cap P = Q\}$. Then $B_{r}(P)$ is the set of all $A(Q)$ as $Q$ ranges over all finite subsets of $P$.

Suppose now that $\Gamma$ is GF. Then there are only finitely many orbits of p.f.p., so we may let $P$ be the set of all parabolic fixed points. In this case, we write $B_{r}$ for $B_{r}(P)$. Note that if for any Dirichlet domain, we have $(D(a))_{c} \cap A = (D(a))_{c} \cap P$.

We are now in a position to describe when a Dirichlet domain for a hyperbolic group is finite-sided. This is only possible when the group is GF. Let $\Gamma$ be a GF group, and let $P \subseteq A$ be the set of all p.f.p.s. To each $p \in P$, we may associate a unique plane $p(p)$ through $p$, which is maximal with the property that some finite-index subgroup of $\text{stab}_{\Gamma}(p)$ act as a translation group on $p(p) \cap \partial B$, for some (and hence each) horoball $B$ about $p$.

Suppose now that the point $a$ is not fixed by any $\gamma \in \Gamma$. Suppose that $a \in U(p)$, so that $(D(a))_{c}$ contains the p.f.p. $p$, but no other point in the orbit of $p$. Let $p = oo$ in $H^{n}$, and let $B$ be a horoball about $p$. Let $l$ be the ray joining $a$ to $p$. If we choose $\partial B$ high enough, we see that $(D(a))_{c} \cap \partial B$ is the Dirichlet domain, about the point $l \cap \partial B$, for the action of $\text{stab}_{\Gamma}(p)$ on $\partial B$. In fact, $(D(a))_{c} \cap \partial B$ is the stable domain; i.e. it is euclidean-isometric to $(D(a))_{c} \cap \partial B) \times [0, \infty)$. Moreover, it is fairly easy to see that the images of $(D(a))_{c}$ under $\text{stab}_{\Gamma}(p)$ cover some standard parabolic region $D(p)$ about $p$. From Proposition 4.1, we see that $D(p)$ meets $(D(a))_{c}$ on only finitely many sides if and only if $a \in \partial B$.

Suppose now that $a$ lies in a top dimensional cell of the complex $B_{r}$. This means that $(D(a))_{c} \cap A = \{a \in D(a) \cap \partial B | \}$ is finite-index in an orbit of p.f.p. Then $(D(a))_{c} \cap \partial B$ is a proper subspace of $H^{n}$. So, in this case, we see that the set of a with $(D(a))_{c}$ in finite-sided contains an open dense subset of $H^{n}$. (In fact we may find a convex cell complex so that the set of a with $D(a)$ finite-sided lies in the $n-1$ skeleton.)

Using the generalisation of Proposition 4.1 stated above, we see that if $a$ lies in a lower dimensional cell of $B_{r}$, we see that $D(a)$ is finite-sided if and only if $a \in (D(a))_{c} \cap \partial B$. Note that the set $(D(a))_{c} \cap \partial B$ is determined by the cell of $B_{r}$ in which $a$ lies.

Finally, we say a few things about general convex fundamental domains.

Let $\Gamma$ act discontinuously on $H^{n}$. Let $X_{1}, \ldots, X_{k}$ be a collection of disjoint open convex subsets of $H^{n}$. Suppose that, as $\gamma \in \Gamma$ and $i = 1, \ldots, k$ vary, the sets $\gamma X_{i}$ are disjoint, locally finite, and their closures cover $H^{n}$. This means that $U = \{\gamma X_{i} | \gamma \in \Gamma, i = 1, \ldots, \lambda\}$ satisfies all the criteria (Chapter 2, GF3, (a)-(d)) to be the set of top-dimensional cells of some cell complex $A$, which we take to be minimal with respect to subdivision. The argument in Chapter 3, GF3 $\Rightarrow$ GF1, (applied to top-dimensional cells) shows that $A$ is locally finite on $H^{n} \cup \partial$. This means that, if we write $X_{i}^{\partial}$ for the closure of $X_{i}$ in $H^{n}$, then $\bigcup X_{i}^{\partial} \setminus A$ is a fundamental domain for the action of $\Gamma$ on $H^{n} \cup \partial$. 
Suppose now, that \( \Gamma \) is GF. By local finiteness, none of the sets \( X^i \cup A \) can meet a c.l.p. So, each \( X^i \cup A \) consists of only (bounded) p.f.ps. We show that each \( X^i \cup A \) is finite, and, moreover, that we can assume that the only standard parabolic regions \( C(p) \) that meet \( X^i \cup A \) are those corresponding to \( p \in \mathbb{R}^3 \cap A \).

The argument is similar to that for local finiteness on \( \mathbb{H}^n \cap A \). Let \( C_1, C_2, C_3 \) be three successively smaller standard parabolic regions about \( \Gamma \). Any set \( X^i \) that meets both \( 3C_1 \) and \( 3C_2 \), meets also \( 3C_3 \cap \text{hull}(2C_1 \cup 2C_2) \), which has compact quotient under \( \text{stab}_p \). Thus \( \{ \gamma \in \Gamma | \gamma X^i \cap C_3 \neq \emptyset \} \) represents only finitely many cosets of the form \( (\text{stab}_p) \gamma \). I.e. \( X^i \cup A \) meets only finitely many elements in the orbit of \( p \).

Shrinking \( C_3 \) further to \( C \), we can assume that any \( \gamma X^i \) with \( \gamma X^i \cap C \neq \emptyset \) has \( p \in \gamma X^i \).

Let \( p = \infty \) in \( \mathbb{R}^3 \), and \( B \), a horoball contained in \( C \). We must have that each \( X^j \cap B \) is a vertical prism on \( X^j \cap \partial B \). We show below that if \( p \) is rational, \( X^j \cap \partial B \) is finite-sided. (It is possible however that \( X^j \cap \partial B \) meets infinitely many other \( \gamma X^i \cap \partial B \) - recall the distinction between "faces" and "sides" made in Chapter 2, GF5.) It follows that if \( \Gamma \) is rational, (i.e. every p.f.p. is rational), then each \( X^j \) is finite-sided. From this, we shall be able to deduce the equivalence, for \( n \leq 3 \), of definitions 1a, 1b, 2a and 2b, mentioned at the start of this chapter.

Lemma 4.2: Let \( \Gamma \) be a discrete group of translations acting on \( \mathbb{E}^n \). Suppose that the open convex sets \( X_1, \ldots, X_n \) together constitute a fundamental domain for \( \Gamma \). Then, each \( X_i \cup A \) (the interior of) a finite-sided polyhedron.

(Note that the orbit of a convex set under a discrete euclidean group is necessarily locally finite, if the sets in the orbit are all disjoint.)

Proof: We know that \( \Gamma \) is a free abelian group. Let \( \{g_1, \ldots, g_n\} \) be a free set of generators. Let \( \Gamma' < \Gamma \) be the subgroup \( < g_1, \ldots, g_\ell > \), so that \( [\Gamma : \Gamma'] = \ell \). The construction of the convex cell complex from \( \{\gamma X_i\} \), enables us to define the set \( \mathbb{F}^{n-1} \) of codimension-1 faces of \( \mathbb{X}_i \). Each \( A \in \mathbb{F}^{n-1} \) corresponds to some \( \gamma X_j \), with \( A = \gamma X_j \cap X_i \). We label \( A \) by the pair \( (\gamma, [\gamma]) \), where \( [\gamma] \) is the coset of \( \Gamma' \) containing \( \gamma \). We claim that if \( A \) and \( B \) in \( \mathbb{F}^{n-1} \) have the same label, then they lie in the same codimension-1 plane of \( \mathbb{E}^n \). We have \( A = \mathbb{X}_i \cap \gamma_1 X_j \), \( B = \mathbb{X}_i \cap \gamma_2 X_j \), with \( \gamma_1 \) and \( \gamma_2 \) differing by twice some translation \( g \in \Gamma \), that is, \( \gamma_2 - \gamma_1 = 2g \). Let \( a \in A, b \in B \). If \( A \) and \( B \) do not lie in the same plane, then the midpoint \( c = (a + b)/2 \) lies in \( \text{hull}(A \cup B) \cap \mathbb{X}_i \). However, for some \( u, v \in \mathbb{X}_j \), we also have \( c = (\gamma_1 u + \gamma_2 v)/2 = \gamma_j (u + v)/2 + uv \in (\gamma_1 + \gamma_2)X_j \), by convexity of \( \mathbb{X}_j \). This gives us the contradiction \( \mathbb{X}_i \cap (\gamma_1 + \gamma_2)X_j \neq \emptyset \).

The tessellation of \( \mathbb{E}^3 \) with square prisms described in Chapter 2, GF3, gives us an example where the \( X_i \) do not have a finite number of faces. Note that the tessellation is invariant under a \( \mathbb{Z} \oplus \mathbb{Z} \) action, acting vertically and in the NW-SE direction. However, this phenomenon cannot occur in euclidean space of dimension less than 3. The only case where we get a non-compact quotient is for an infinite cyclic action on \( \mathbb{E}^2 \). In this case, it is fairly easy to see that if we have a tessellation with finite quotient, then each tile meets only finitely many other tiles. In fact we may classify such tiles according to whether they are compact, or have one or two topological ends (FIG 4.4).

Now, any isometry of \( \mathbb{E}^1 \), or \( \mathbb{E}^2 \), with no fixed point, must be a translation. Thus, any discrete group action on these spaces must have a finite-index translation subgroup. We see that any discrete subgroup of \( \text{Isom} \mathbb{H}^3 \), or of \( \text{Isom} \mathbb{H}^3 \), can have only rational p.f.p.'s. From this, we deduce the equivalence, in three dimensions, of the four descriptions of geometric finiteness stated at the start of the chapter.

The question remains of whether or not a GF group necessarily has a single, convex finite-sided (or finite-faced) fundamental polyhedron. I suspect not, but I do not have a counterexample.

References.


[Thu2] W.P. Thurston, Hyperbolic structures on 3-manifolds I — Surface groups and 3-manifolds which fiber over the circle: preprint.


Figure 0.3: Geometrically finite 2-manifold

Figure 0.4: Z-cusp
Figure 0.6: Upper half-space model, intersection with \( N \). Cross-section of Z - cusp showing.

Figure 0.5: \( Z \oplus Z \) - cusp.
Figure 0.7:
Standard cusp regions

cusp torus

cusp cylinder

Figure 2.1:
A standard parabolic region on the upper half-space model.

\[ n = 3, \quad \Gamma_p \cong \mathbb{Z} \]

\[ \mathbb{H}_c^3 \equiv [\mathbb{R}_x^3, U \mathbb{R}_y^3] \]
\[ \Gamma^r_c = \langle (x, y, t) \mapsto (x, y+1, t) \rangle \]
Figure 2.2:
A standard cusp - the quotient from Figure 2.

Figure 2.3:
A GF1 2-manifold.

\[ M_c = N \cup C \cup C \]
Figure 2.4:
A bounded parabolic fixed point in the boundary of the upper half-space model -
n = 4, \Gamma_p \cong \mathbb{Z}

\mathbb{R}^3 \equiv \partial \mathbb{R}^+
\Gamma_p = \langle (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2 + 1, x_3, x_4) \rangle
\wedge \leq \text{QU}(\infty)

Figure 2.5:
A conical limit point -
upper half-space model.
Figure 2.6:  
A convex cell complex on $\mathbb{E}^3$.

Figure 2.7:  
The convex core of a 2-manifold.
Figure 3.1:
Schematic.

Figure 3.2:
Upper half space model.
Figure 3.3:
Generalised Dirichlet domains on a 2-manifold.

\[ a = (a_0, a_1, a_2) \]

Figure 3.4:
The function \( \omega(K, x) \) for \( K \subseteq \mathbb{R}^2 \).
Figure 3.5:
Upper half space model -- 
\( n = 2 \).

Figure 3.6:
Schematic.
Figure 4.1:
Irrational screw motion on $\mathbb{E}^3$

Figure 4.2:
Whitehead move on a torus
Figure 4.3:
(schematic)

Figure 4.4:
Two singly-periodic tilings of $\mathbb{E}^2$
Geometrical Finiteness for Hyperbolic Groups

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University of Warwick, 1988

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