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O(2) Equivariant Mode Interactions

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CONTENTS

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Declaration.

I declare that the content of this work, unless otherwise indicated, is my own.

ABSTRACT

This thesis uses and develops methods of equivariant bifurcation theory in the context of O(2) equivariant steady-state and Hopf mode interactions in circumstances where both two and three modes interact. The general actions of the O(2) orthogonal group are applied to these mode interactions and as an illustration the O(2) equivariant steady-state/Hopf/Hopf mode interaction is placed in the context of the Taylor-Couette system.

CHAPTER 0.

Introduction.

§0.0. Equivariant Bifurcation Theory.

The explicit use of group theory to organise the analysis of bifurcations in a dynamical system with given symmetry, has provided a well grounded framework in which the consequences of symmetry can be discussed. This framework, usually referred to as Equivariant Bifurcation Theory, has been developed by a number of authors c.f. Golubitsky and Schaeffer (1984), Sattinger (1983), Golubitsky, Schaeffer and Stewart (1988).

At the heart of Equivariant Bifurcation Theory is "spontaneous symmetry breaking" whereby a new branch of solutions inherits some subgroup of symmetries of the original branch. The theory analyses this phenomenon when the symmetry group is a compact Lie group acting linearly on a system of differential equations with distinguished bifurcation parameter.

In particular, we consider a compact Lie group Γ and system of ODE

$$\dot{x} + g(x,\lambda) = 0 \qquad (1)$$

where $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a smooth mapping, $\lambda \in \mathbb{R}^p$ the distinguished bifurcation parameters, and Γ, acting linearly on $\mathbb{R}^n$, commutes with g,

$$\gamma g(x,\lambda) = g(\gamma x,\lambda) \qquad \forall \ \gamma \in \Gamma . \qquad (2)$$

With the assumptions that

$$g(0,\lambda_0) = 0$$

and the linearization $L \equiv (Dg)_{0,\lambda_0}$ has eigenvalues on the imaginary axis, the loss of stability for the steady-state solution $x = 0$ is guaranteed.

The Γ-equivariance, condition (2), allows us to classify new branches of solutions bifurcating from $x = 0$, as follows. A point x has <u>isotropy subgroup</u>

$$\Sigma_x = \{\gamma \in \Gamma \mid \gamma x = x\} \tag{3}$$

which gives precisely the symmetries of a solution x. Associated with the isotropy subgroup is the <u>fixed point subspace</u>

$$\mathrm{Fix}(\Sigma) = \{y \in \mathbb{R}^n \mid \gamma y = y \quad \forall \quad \gamma \in \Sigma\}, \tag{4}$$

so that a solution to (1) lying in $\mathrm{Fix}(\Sigma)$ and not contained in any smaller fixed point subspace has isotropy Σ. In fact we see that any point lying in the orbit of a point x, i.e. $\Gamma \cdot x$, will have isotropy contained in some conjugate of Σ_x. Strictly speaking, since we consider orbit representations of solutions, we classify solutions by conjugacy classes. Before introducing some of the tools of equivariant bifurcation theory - made available by the presence of a compact group of symmetries - let us introduce the class of bifurcation problem we study.

In any dynamical system the most straightforward way a degeneracy can occur is in the linear terms in the bifurcation equations. This occurs precisely when the linearization L satisfies the above assumptions. In

particular, if the eigenvalues are zero we have steady-state bifurcations, and if purely imaginary, Hopf bifurcations. Currently the eigenspaces associated with these eigenvalues, or the corresponding linearized eigenfunctions, are referred to as modes. As a number of branches go unstable simultaneously there is often a coupling of the modes, via the non-linear terms, giving rise to complicated behaviour. It is these so called mode interactions, in a dynamical system with $O(2)$ symmetry, which concern us. In particular we study $O(2)$ symmetric steady-state/Hopf and steady-state/Hopf/Hopf interactions.

The fundamental complication introduced by symmetry is that eigenvalues of L need not be simple. Differentiating (2) gives

$$\gamma L = L\gamma \qquad \forall\, \gamma \in \Gamma \tag{6}$$

and it follows from (6) that the eigenspaces of L are Γ-invariant subspaces. Typically these subspaces are irreducible representations so that corresponding to each representation there will be a type of steady-state and a type of Hopf bifurcation. c.f. Golubitsky, Schaeffer and Stewart (1988).

When $\Gamma = O(2)$ it is well known that the irreducible representations are one or two dimensional-corresponding to single or double eigenvalues. $O(2)$ mode interactions are, therefore, obtained by considering direct sums of such irreducibles. In varying contexts much is already known about two mode interactions. c.f. Dangelmayr and Armbruster (1986), Guckenheimer (1986), Dangelmayr and Knobloch (1986), Golubitsky and Stewart (1986) and

Chossat (1985). However, still to be considered is the non-standard action for steady-state/Hopf interaction with double-zero and double imaginary eigenvalues $\pm \omega i$ and also all types of three-mode interactions. However, as stated above, we restrict our attention in the three mode case to the study of the steady-state/Hopf/Hopf interaction where we assume the eigenvalues 0, $\pm\omega_1 i$, and $\pm\omega_2 i$ are each double.

In the study of the three mode case we make two assumptions:

(a) $n_1\omega_1 + n_2\omega_2 \neq 0$ for any pair $(n_1, n_2) \in \mathbb{Z}^2$.

(b) the vector field g in (1) is in Birkhoff normal form. (7)

(7)(a) is the standard assumption that the modes are non-resonant, c.f. Takens (1974). The assumption of Birkhoff normal form introduces toroidal symmetry and allows us to compute the orbital asymptotic stabilities of solution types. Although this is a restriction on the dynamic behaviour under consideration as will be seen in §2 there is still 'plenty' going on.

In the study of steady-state/Hopf interaction it is possible to use a Liapunov-Schmidt reduction and although assuming Birkhoff normal form to all orders would allow us to compute stabilities exactly, it remains possible to estimate the stabilities without this assumption. c.f. Stewart (1988).

§0.1. <u>Action of $O(2) \ltimes T^n$</u>.

As suggested above, if a vector field, with linear part L having purely imaginary eigenvalues, is in Birkhoff normal form, then it has toroidal symmetry. This is now standard in the literature c.f. Golubitsky

and Stewart (1985) and Chossat, Golubitsky and Keyfitz (1986). Before stating these results we introduce some of the nomenclature.

Suppose X is a Γ-invariant vector field whose linear part L has eigenvalues 0, $\pm\omega_1 i, \ldots, \pm\omega_k i$ i.e. we assume an equivariant <u>centre manifold reduction</u> has been performed (or if $k = 1$ a Liapunov-Schmidt reduction can be performed instead).

<u>Definition 1</u>. A representation of a group Γ on a vector space V is said to be <u>absolutely irreducible</u> if the only linear mappings on V commuting with L are scalar multiples of the identity. c.f. Golubitsky, Schaeffer and Stewart (1988).

<u>Definition 2</u>. The eigenvalues are <u>non-resonant</u> if

a) $a_1\omega_1 + \ldots + a_k\omega_k \neq 0 \qquad \forall\ a_i \in \mathbb{Q} \qquad (\underline{a} \neq 0)$

and <u>Γ-simple</u> if

b)(i) the real eigenspace W_j corresponding to $\pm\omega_j i$ can be written $W_j = V_j \oplus V_j$ where

Γ acts absolutely irreducibly on V_j

or (ii) Γ acts irreducibly but not absolutely irreducibly on W_j.

Under the assumption of Γ-simplicity a linear change of coordinates gives

$$L = \begin{bmatrix} A_0 & & \\ & \ddots & \\ & & A_k \end{bmatrix}$$

where $$A_j = \begin{bmatrix} 0 & \omega_j I_{n_j} \\ -\omega_j I_{n_j} & 0 \end{bmatrix} \quad \text{for } j = 1,\ldots,k\ ,\ n_j = \dim V_j \qquad (1)$$

and $$A_0 = \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & m & \\ & & & \ddots \\ & & & & 0 \end{bmatrix} \qquad m = \dim W_0 \ .$$

The assumption of non-resonance gives the following toral action. Let $\psi = (\psi_1, \ldots\ldots, \psi_k) \in T^k$ the k dimensional torus. Then

$$R_\theta = \begin{bmatrix} I_m & & & 0 \\ & C_1 & & \\ & & \ddots & \\ 0 & & & C_k \end{bmatrix} \tag{2}$$

where $$C_j = \begin{bmatrix} \cos\psi_j \, I_{nj} & \sin\psi_j I_{nj} \\ -\sin\psi_j \, I_{nj} & \cos\psi_j \, I_{nj} \end{bmatrix} \qquad \text{for } j = 1, \ldots\ldots, k \ .$$

describes the standard T^n toral action extended to $W_0 \oplus \ldots\ldots \oplus W_k$.

Let H_n be the linear space of vector fields whose coefficients are homogeneous polynomials of degree n. Let $\mathrm{ad}\, L(Y) = [L, Y]$ denote the Lie bracket operation and let G_n be a complement for $L(H_n)$ in H_n. In particular let

$$G_n = \{Y \in H_n : (R_\theta)_* Y = Y \quad \forall \psi \in T^k\} \ . \tag{3}$$

Then we have the following:

<u>Theorem 1</u>. Let X be a Γ-invariant vector field whose linear part L has purely imaginary, Γ-simple non-resonant eigenvalues. Then by a vector field change of coordinates X can be put in the normal form

$$L + Y_2 + \ldots\ldots + Y_N + R_{N+1}$$

where R_{N+1} is Γ-invariant and vanishes through order N, and

$$Y_n \in G_n \cap H_n(\Gamma) \quad \text{where} \quad H_n(\Gamma) = \{Y \in H_n : Y \text{ is } \Gamma\text{-invariant}\} \quad .$$

<u>Proof</u> See Golubitsky and Stewart (1985).

A general $\Gamma \times T^k$ diagonal action may now be defined on $W_0 \oplus \ldots \oplus W_k$ by

$$\begin{aligned} &\gamma(w_0, \ldots, v_i, w_i, \ldots, v_k, w_k) \\ &= (\gamma w_0, \ldots, \gamma v_i, \gamma w_i, \ldots, \gamma v_k, \gamma w_k) \quad \gamma \in \Gamma \times T^k \end{aligned} \tag{4}$$

where $(v_i, w_i) \in W_i = V_i \oplus V_i$.

Furthermore we may write

$$\bigoplus_{i=0}^{K} W_i = \mathbb{C}^{2K+1} \tag{5}$$

using the obvious identification. We apply the resulting $O(2) \times T^k$ action in §1 and §2.

<u>Note</u>: As previously remarked the resulting standard $O(2)$ action, $\ell = m = 1$ has been studied by Golubitsky and Stewart (1985). The $O(2)$ action given by $\ell \neq m$ has not.

§0.2 Invariant Theory.

Obtaining a precise description of the Γ-equivariant mappings, $\mathbb{C}^n \to \mathbb{C}^M$ is made possible by the assumption that Γ is compact, allowing us to exploit some standard invariant theory. We state the results for Γ-equivariant smooth germs of maps $(\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ and bear in mind that identifying $\mathbb{R}^2$ with $\mathbb{C}$ as in §0.1 allows us to work with a group action more convenient for the calculations. We begin with some notation. c.f. Golubitsky, Schaeffer and Stewart (1988).

Recall from §0.0 that a Γ-equivariant mapping $g: \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$yg(x) = g(yx) \quad \forall\, y \in \Gamma . \tag{1}$$

Similarly a Γ-invariant function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$f(yx) = f(x) \quad \forall\, y \in \Gamma . \tag{2}$$

We write $\bar{E}(\Gamma)$ for the set of all smooth map-germs at 0 $g : \mathbb{R}^n \to \mathbb{R}^n$ which satisfy (1), and $E(\Gamma)$ for the set of all smooth function-germs at 0 satisfying (2). Observe that $\bar{E}(\Gamma)$ forms a module over the ring $E(\Gamma)$. Similarly, write $\bar{P}(\Gamma)$ for the module of Γ-equivariant polynomial mappings $\mathbb{R}^n \to \mathbb{R}^n$ and $P(\Gamma)$ for the ring of Γ-invariant polynomials $\mathbb{R}^n \to \mathbb{R}$. The central result is the following, due to Weyl (1946).

<u>Theorem 1</u> (Hilbert-Weyl Theorem) Let Γ be a compact Lie group acting on $\mathbb{R}^n$. Then the ring $P(\Gamma)$ is finitely generated.

Notes: (1) The generating set is often referred to as a <u>Hilbert basis</u> and we shall sometimes adopt this terminology.

(2) The basis may contain relations, as is, for example the case for $O(2) = T^m$ acting on $\mathbb{C}^{2m+1}$.

(3) We may, of course, replace $\mathbb{R}^n$ by a general vector space.

There are two other essential theorems due to Schwarz (1975) and Poénaru (1976). In each we let Γ be a compact Lie-group acting on $\mathbb{R}^n$:

<u>Theorem 2</u> (Schwarz) Every function-germ in $E(\Gamma)$ may be expressed as a function-germ of the Hilbert basis of $P(\Gamma)$.

<u>Theorem 3</u> (Poénaru) If $g_0 \ldots, g_m$ generate $\bar{P}(\Gamma)$ over $P(\Gamma)$ then $g_0 \ldots, g_m$ generate $\bar{E}(\Gamma)$ over $E(\Gamma)$.

CHAPTER 1.

The O(2) equivariant steady-state/Hopf mode interaction.

§1.0. Background Results for the steady-state/Hopf interaction.

Studies of the O(2) steady-state Hopf mode interaction, as previously remarked, have been performed in varying contexts. However, it still remains to consider a non-standard action for the above mode interaction with double zero and imaginary eigenvalues $\pm \omega i$. Results of Golubitsky and Stewart (1985), (1986) in reducing the vector field under consideration by the Liapunov-Schmidt method, allow us to state the general form of the $O(2) \ltimes S^1$ bifurcation problem studied here.

As in §0 consider a system of O.D.E.s

$$\dot{x} + g(x,\lambda) = 0 \tag{1}$$

where $x \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$ is a bifurcation parameter and $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth mapping. Suppose also that

$$g(\gamma x,\lambda) = \gamma g(x,\lambda) \qquad \forall\, \gamma \in O(2) \tag{2}$$

i.e. g commutes with the action of $O(2)$. The assumption that $(dg)_{(0,0)}$ has zero or purely imaginary eigenvalues allows us to put the vector field g (here we identify linear vector fields with a linear system of O.D.E.) into Birkhoff normal form. In other words we may assume g commutes with $O(2) \times S^1$ to any <u>finite</u> order. Assuming the stronger condition that g commutes with $O(2) \times S^1$ to <u>all</u> orders we have the following:

<u>Theorem 1</u> Let $\phi : \mathbb{C}^3 \times \mathbb{R}^2 \to \mathbb{C}^3$ be the Liapunov-Schmidt reduced mapping so that zeros of ϕ are in 1-1 correspondence with periodic solutions of (1) of period near $2\pi/|w|$ then

$$\phi(z,\lambda,\tau) = h(z,\lambda) - i(1+\tau)z^*$$

where $h : \mathbb{C}^3 \times \mathbb{R} \to \mathbb{C}^3$ commutes with $O(2) \times S^1$, τ is the perturbed period parameter, and $z^* = (0,z_1,z_2)$, the zero coordinate corresponding to the eigenspace of the zero eigenvalue.

<u>Proof</u> After a change of coordinates the above follows from Proposition 4.3, Golubitsky and Stewart (1986).

The assumption of Birkhoff normal form to all orders allows us to put ϕ in exactly the above form and allows us to compute precisely the stabilities of periodic branches associated with the zeros of (1).

Begin by letting (u_0,λ_0,τ_0) be a solution to $\phi = 0$. Assume h commutes with $O(2) \times S^1$ to all orders so that ϕ takes the above form. Proposition 8.1 of Golubitsky and Stewart (1986) shows that for a periodic solution

$$u(\psi) = (1,\psi).u_0 \ , \quad (1,\psi) \in O(2) \times S^1 \qquad (3)$$

with isotropy Σ , there are associated with the solution at least $d_\Sigma = 6-\dim \Sigma$ zero eigenvalues.

<u>Theorem 2</u> Assuming h is in normal form, then the periodic solution $u(\psi)$ is <u>orbitally asymptotically</u> stable if $6-d_\Sigma$ eigenvalues of $(d\phi)_{(h_0,\lambda_0,\tau_0)}$ that are not forced by the group action to be zero, have positive <u>real</u> parts. The solution is unstable if one of these has negative real part.

<u>Proof</u> See Golubitsky and Stewart (1985).

As remarked earlier it is not necessary to assume Birkhoff normal form to all orders in the computation of stabilities. However, in some systems there exist physical symmetries identifiable with S^1 which imply - as long as a centre manifold reduction can be performed - that h may be put in this form without loss of generality. Such an example is the Taylor-Couette system, see §3.

Finally, performing a Liapunov-Schmidt reduction results in general conditions. Let the complex conjugate eigenvalues of $(dg)_{0,\lambda}$ be $\sigma(\lambda) \pm i\phi(\lambda)$, where $\sigma(0)$, $\phi(0) = 1$ and $\sigma'(0) \neq 0$ - transversal intersection. Let $q(z_0,z_1,z_2,\lambda,\tau)$ be the $O(2) \times S^1$-equivariant complex Liapunov-Schmidt reduced bifurcation problem then, writing $G = G^1 + iG^2$ we have

$$\begin{aligned} &(i) \quad G^1(0) = 0 \ , \quad G^2(0) = 0 \\ &(ii) \quad G_\tau(0,\ldots,0,\tau) = 0 \ , \quad G^2_\tau(0,\ldots,\tau) = -1 \\ &(iii) \quad G_\lambda(0,0) = \sigma'(0) \ . \end{aligned} \tag{4}$$

This follows from Proposition 4.2 Golubitsky and Stewart (1985).

§1.1 Isotropy Subgroups and Fixed Point Subspaces of $O(2) \times S^1$.

Following the discussion in §0.3 of $O(2) \times T^n$ acting on $\mathbb{C} \oplus \mathbb{C}^{2n}$ we may write the action of $O(2) \times S^1$ on $\mathbb{C} \oplus \mathbb{C}^2$ as

TABLE 1

	Z_0	Z_1	Z_2
$\kappa \in O(2)$	Z_0	Z_2	Z_1
$\theta \in O(2)$	$e^{i\ell\theta}Z_0$	$e^{im\theta}Z_1$	$e^{-im\theta}Z_2$
$\psi \in S^1$	Z_0	$e^{i\psi}Z_1$	$e^{i\psi}Z_2$

We now proceed to find the isotropy subgroups for the above action. In doing this we distinguish the following subgroups of $O(2) \times S^1$.

(a) $\mathbb{Z}(\theta,\psi)$ discrete subgroup generated by $(\theta,\psi) \in O(2) \times S^1$

(b) $\mathbb{Z}_\kappa(\theta,\psi)$ discrete subgroup generated by $(\kappa,\theta,\psi) \in O(2) \times S^1$

(c) $S(p,q)$ the continuous subgroup $\{(p\theta,q\theta) \mid \theta \in S^1 \text{ for } p,q \in Q\}$.

It is also helpful to recall that we really seek representatives from each conjugacy class of isotropy subgroups as discussed in §0.1

With the above in mind we list the isotropy subgroups and the corresponding lattice in the following two tables.

Note that for the cyclic group $\mathbb{Z}_\ell(0,0)$ of order 2 we use the shorthand $\mathbb{Z}_\ell$ and that we do <u>not</u> consider any trivial action of $O(2) \times S^1$.

TABLE 2

Isotropy Subgroups and their fixed point subspaces.

	Σ	Fix Σ	dim Fix(Σ)
0.	$O(2) \times S^1$	$\{0\}$	
1.	$\mathbb{Z}_\ell \ltimes \mathbb{Z}(\frac{2\pi}{\ell},0) \times S^1$ ℓ even	$\mathrm{Im}(Z_0) = Z_1 = Z_2 = 0$	1
2.	$S(1,-m)$	$Z_0 = Z_2 = 0$	2
3.*	$\mathbb{Z}_\ell \ltimes \mathbb{Z}(\frac{\pi}{m},\pi)$	$Z_0 = 0$, $Z_1 = Z_2$	2
4.	$\mathbb{Z}_\ell[\ltimes \mathbb{Z}(\pi,m\pi)]$ ℓ even	$\mathrm{Im}(Z_0) = 0$, $Z_1 = Z_2$	3
5.	$\mathbb{Z}_\ell(0,\pi)[\ltimes \mathbb{Z}(\pi,m\pi)] \rtimes \mathbb{Z}_2$ ℓ even	$\mathrm{Im}(Z_0) = 0$, $Z_1 = -Z_2$	3
6.	$\mathbb{Z}(\frac{2\pi}{\ell}, \frac{-2m\pi}{\ell})$ $\ell > 2$, $m \neq 1$	$Z_2 = 0$	4
7(a)	$\mathbb{Z}(\frac{\pi}{m},\pi)$	$Z_0 = 0$	4
(b)	$\mathbb{Z}(\pi,m\pi)$(ℓ even)	$\mathbb{C} \oplus \mathbb{C}^2$	6

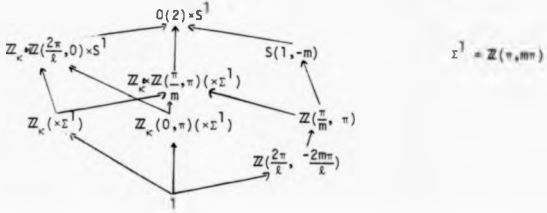
---

* Note that when $m = 1$ and ℓ is even isotropy subgroups 3 and 4 coincide and, of course, have the same three-dimensional fixed point subspace. This, in fact, turns out to be a case of particular interest. Similarly, 7(a) and (b) coincide in this case. Trivially, from Table 2

we write down the lattice (of conjugacy classes) of isotropy subgroups of $O(2) \times S^1$. We write $\Sigma^1 = \mathbb{Z}(\pi, m\pi)$ and note that there are differences according to whether ℓ is even, divisible by 4 and $m = 1$, which are not shown in Table 3.

TABLE 3

Lattice of isotropy subgroups of $O(2) \times S^1$



where arrows indicate containment of one conjugacy class in another.

It remains to verify Table 2. The simplest way to do this is to pick $O(2) \times S^1$ orbit representatives and find their corresponding isotropy subgroups. We assume any trivial action is factored out, i.e. the kernel of the action is factored out. First we look for isotropy subgroups not containing the involution κ. Considering the action of elements in such isotropy subgroups explicitly we have

$$(\theta,\psi).(Z_0,Z_1,Z_2) = (e^{i\ell\theta}Z_0,\ e^{i(m\theta+\psi)}Z_1,\ e^{i(-m\theta+\psi)}Z_2). \qquad (2)$$

By inspection we immediately have for ℓ even the group element $(\pi,m\pi)$ fixes all $Z \in \mathbb{C} \oplus \mathbb{C}^2$ and assuming (θ,ψ) acts non-trivially no more such elements exist, verifying entry 7(b). Suppose $Z_0 \neq 0$, since $\kappa \notin \Sigma$ we may choose either $Z_1 \neq 0$ $(Z_2 = 0)$ or $Z_2 \neq 0$ $(Z_1 = 0)$ and we obtain entry 6. Now, assuming $\kappa \in \Sigma$ we may rotate Z_0 onto the real axis by $\theta \in O(2) \sim \{\kappa\} \times S^1$. Since we have above the only isotropy subgroups Σ s.t. $\kappa \notin \Sigma$ and $Z_0 \neq 0$ in $\mathrm{Fix}(\Sigma)$ we now seek isotropy subgroups containing κ with fixed point subspaces containing (elements with non-zero first coordinate. Such isotropy subgroups will contain elements of the form $\kappa.(\theta,\psi)$ and, again, writing the action explicitly we have

$$\kappa.(\theta,\psi)(Z_0,Z_1,Z_2) = (e^{-i\ell\theta}Z_0, e^{i(m\theta+\psi)}Z_2, e^{-i(m\theta-\psi)}Z_1). \qquad (3)$$

It is immediate that no such elements fix $\mathbb{C} \oplus \mathbb{C}^2$.

We now make the following claim: If Z is fixed by $\kappa.(\theta,\psi)$ then

$$Z \in (O(2) \times S^1) \,.\, (X_0, Z, \pm Z) \quad . \qquad (4)$$

To prove this consider first the group action on the second and third coordinates. (3) implies

$$\begin{aligned} &(a) \qquad e^{i(m\theta+\psi)}Z_2 = Z_1 \\ &(b) \qquad e^{-i(m\theta-\psi)}Z_1 = Z_2 \quad . \end{aligned} \qquad (5)$$

Combining (5)(a) and (b) gives $e^{i2\psi} = 1$ which tells us that $\psi = p\pi$ for some $p \in \mathbb{Z}$. We may now write $\kappa.(\theta,\psi).Z = (e^{-i\ell\theta}Z_0, (-1)^p e^{im\theta}Z_2, Z_2)$

and computing the action of $(-\frac{\theta}{2}, -\frac{m\theta}{2} + p\pi)$ gives $(e^{-i\frac{3}{2}\ell\theta} Z_0, Z_2, (-1)^p Z_2)$. Since we have $Z_0 \in (O(2)\backslash\{\kappa\})\cdot X_0$ choose θ' to obtain the result. □

To verify entries 1, 3, 4 and 5 in Table 2 we must check for conjugacies.

<u>Proposition</u> There exist $\gamma \in O(2)\backslash\{\kappa\} \times S^1$ s.t.

$$\begin{aligned} &\text{(a)} \qquad \gamma(0, Z_1, Z_2) = (0, Z_1, -Z_2) \\ &\text{(b)} \qquad \gamma(X_0, Z, -Z) = (X_0, Z, Z) \quad \text{when } \ell | 4 . \end{aligned} \tag{6}$$

<u>Proof</u> We note that for (6)(a) $\gamma = (\frac{\pi}{2m}, -\frac{\pi}{2})$. Letting $\gamma = (\theta, \psi)$ in (6)(b) we have

$$\begin{aligned} &\text{(a)} \qquad e^{i\ell\theta} = 0 , \\ &\text{(b)} \qquad e^{im\theta + i\psi} = 1 , \\ &\text{(c)} \qquad e^{-im\theta + i\psi} = 1 . \end{aligned} \tag{7}$$

Now (7)(a), (b) and (c) imply there exist $p, q, r \in \mathbb{Z}$ such that

$$\begin{aligned} &\text{(a)} \qquad \ell\theta = 2p\pi , \\ &\text{(b)} \qquad m\theta + \psi = 2q\pi , \\ &\text{(c)} \qquad -m\theta + \psi = (2r+1)\pi . \end{aligned} \tag{8}$$

Combining (8)(b) and (c) gives $2m\theta = (2(q-r)-1)\pi$, and dividing this by 8(a) gives

$$\frac{4m}{\ell} = \frac{(2(q-r)-1)}{p} . \tag{9}$$

It is immediate from (9) that (6)(b) holds iff $\ell|4$.

Finally we check for remaining isotropy subgroups not containing κ. Suppose we have $Z_0 = 0$ in the elements of the fixed-point subspace. Noting that $S(1,m) = \kappa S(1,-m)\kappa^{-1}$ we verify entries 2 and 7(a) in Table 2.

Remark In the case when ℓ and m are odd, and in particular equal to 1, we have $(X_0,X_1,-X_1) \in (O(2) \times S^1) \cdot (iY_0,X_1,X_1)$ which, in this coordinate system, is the orbit representative chosen by Golubitsky and Stewart (1986). Also when ℓ is even this is not a new orbit representative and is therefore not required.

§1.2 Computation of O(2) Invariants and Equivariants.

We are concerned here with computing the $O(2) \times S^1$ equivariant normal form for smooth (germs of) maps $G : \mathbb{C}^3 \to \mathbb{C}^3$. As we have already seen these smooth equivariant maps form a module over the ring of smooth invariant maps and the ring has a finite number of generators so we begin with the following:

Proposition 1 The ring of smooth $O(2) \times S^1$ invariant (germs of) maps $\mathbb{C}^3 \to \mathbb{R}$ is generated by:

$$N_0 = u_0 \,,\; N_1 = u_1+u_2 \,,\; M_1 = u_1u_2 \,,\; O_1 = \mathrm{Re}(z_0^{\mu}(\bar{z},z_2)^{\eta}) \,,$$
$$O_2 = \delta_1 \mathrm{Im}(z_0^{\mu}(\bar{z},z_2)^{\eta})$$

where $u_i = z_i\bar{z}_i$, $\delta_1 = u_1-u_2$

$$\mu = \begin{cases} 2m & \ell \text{ odd} \\ m & \ell \text{ even} \end{cases} \quad \text{and} \quad \eta = \begin{cases} \ell & \ell \text{ odd} \\ \ell/2 & \ell \text{ even.} \end{cases}$$

<u>Proof</u> We prove the above for polynomial maps $\mathbb{C}^3 \to \mathbb{R}$. The result then follows by Schwartz(1975).

A polynomial mapping $\mathbb{C}^3 \to \mathbb{R}$ has the form

$$f(z) = \Sigma\, A_{ab}\, z^{a-b} \tag{1}$$

in multi-index notation.

Now consider the action of $O(2)\backslash\{\kappa\} \times S^1$ on $f: \mathbb{C}^3 \to \mathbb{R}$ recalling that for invariance

$$f(\gamma z) = f(z) \quad \forall \ \gamma \in O(2) \times S^1 \ . \tag{2}$$

From (2), by comparison of coefficients,

$$\begin{array}{lll} \text{(a)} & \theta \in O(2)\backslash\{\kappa\} & \ell(a_0-b_0) + m(a_1-b_1-a_2+b_2) = 0 \\ \text{(b)} & \psi \in S^1 & a_1-b_1+a_2-b_2 = 0 \ . \end{array} \tag{3}$$

Now generators for the ring of invariants correspond to a <u>minimum</u> spanning set for integer solutions of (3) in $(\mathbb{Z}^+)^6$.

Combining (3)(a) and (b) gives

$$\begin{array}{lll} \text{(a)} & (\ell \ \text{odd}) & \ell(a_0-b_0) + 2m(a_1-b_1) = 0 \\ \text{(b)} & (\ell \ \text{even}) & \frac{\ell}{2}(a_0-b_0) + m(a_1-b_1) = 0 \ . \end{array} \tag{4}$$

Concentrating on (4)(a) - noting the same argument will hold for ℓ even - we have, since ℓ and m are assumed coprime

$$\begin{array}{ll} \text{(a)} & a_0-b_0 = 2mj \\ \text{(b)} & a_1-b_1 = -\ell j \end{array} \tag{5}$$

where $j \in \mathbb{Z}$.

Now $j = 0$ gives $V_1 = (1,1,0,0,0,0)$, $V_2 = (0,0,1,1,0,0)$, and $V_3 = (0,0,0,0,1,1)$ belonging to $(\mathbb{Z}^+)^6$. When $j > 0$ we may write (5)(a) and (b) as

$$\begin{array}{ll} \text{(a)} & (a_0^j,0,\ldots,0) + s\,V_1 \\ \text{(b)} & tV_2 - (0,0,0,b_1^j,0,\ldots) \end{array} \quad , \text{ where } s,t \in \mathbb{Z}^+ \qquad (6)$$

which provides us with a further member of our spanning set $(2mj,0,0,\ell j,\ell j,0)$ but since $j > 0$, the minimum such element in $(\mathbb{Z}^+)^6$ is $V_4 = (2m,0,0,\ell,\ell,0)$.

Similarly for $j < 0$ we obtain $(0,-2mj,-\ell j,0,0,-\ell j)$ and since $-j > 0$ we may divide through giving $V_5 = (0,2m,\ell,0,0,\ell)$.

V_1,V_2,V_3,V_4 and V_5 correspond to the invariants u_0,u_1,u_2, $z_0^{2m}(\bar{z}_1z_2)^\ell$ and $\bar{z}_0^{2m}(z_1\bar{z}_2)^\ell$. Recalling that we seek only the real invariants these become

$$u_0,u_1,u_2 \ , \quad \mathrm{Re}(z_0^{2m}(\bar{z}_1z_2)^\ell) \ , \quad \mathrm{Im}(z_0^{2m}((\bar{z}_1z_2)^\ell) \ .$$

Now finally we consider the action of κ . Clearly we obtain the invariants N_0 , N_1 , M_1 and O_1 and since for $f(z) = \mathrm{Im}(z_0^{2m}(\bar{z}_1z_2)^\ell)$ we have $f(\kappa Z) = -f(Z)$ we multiply by the minimum anti-symmetric polynomial $\delta_1 = u_1-u_2$ to obtain invariance.

<u>Theorem 2</u> The module of $O(2) \times S^1$ equivariant smooth (germs of) mappings $\mathbb{C}^3 \to \mathbb{C}^3$ has the following basis over the ring of invariants:

$$\begin{bmatrix} z_0 \\ 0 \\ 0 \end{bmatrix} \ , \quad i\delta_1\begin{bmatrix} z_0 \\ 0 \\ 0 \end{bmatrix} \ , \quad \begin{bmatrix} \bar{z}_0^{\alpha-1}(z_1\bar{z}_2)^\beta \\ 0 \\ 0 \end{bmatrix} \ , \quad i\delta_1\begin{bmatrix} \bar{z}_0^{\alpha-1}(z_1\bar{z}_2)^\beta \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix}0\\z_1\\z_2\end{bmatrix}, \quad i\begin{bmatrix}0\\z_1\\z_2\end{bmatrix}, \quad \begin{bmatrix}0\\u_2z_1\\u_1z_2\end{bmatrix}, \quad i\begin{bmatrix}0\\u_2z_1\\u_1z_2\end{bmatrix}, \quad \begin{bmatrix}0\\z_0^{\alpha}(\bar{z}_1z_2)^{\beta-1}z_2\\\bar{z}_0^{\alpha}(z_1\bar{z}_2)^{\beta-1}\;1\end{bmatrix} \quad i\begin{bmatrix}0\\z_0^{\alpha}(\bar{z}_1z_2)^{\beta-1}z_2\\\bar{z}_0^{\alpha}(z_1\bar{z}_2)^{\beta-1}z_1\end{bmatrix},$$

$$\begin{bmatrix}0\\u_2z_0^{\alpha}(\bar{z}_1z_2)^{\beta-1}z_2\\u_1\bar{z}_0^{\alpha}(z_1\bar{z}_2)^{\beta-1}z_1\end{bmatrix}, \quad i\begin{bmatrix}0\\u_2z^{\alpha}(\bar{z}_1z_2)^{\beta-1}z_2\\u_1\bar{z}_0^{\alpha}(z_1\bar{z}_2)^{\beta-1}z_1\end{bmatrix},$$

where $\alpha = \begin{cases}2\mu & \ell \text{ odd}\\ \mu & \ell \text{ even}\end{cases}$ $\beta = \begin{cases}\ell & \ell \text{ odd}\\ \ell/2 & \ell \text{ even.}\end{cases}$

<u>Proof</u> Recall that Poenaru (1976) says we need only check the above for polynomial maps. We deduce the above from a Lemma and Proposition concerning the relationship between invariants and equivariants of the general action $O(2) \times T^n$ on $\mathbb{C} \oplus \mathbb{C}^{2n}$. Before giving these we generalize some notation. For the polynomial $g : \mathbb{C} \oplus \mathbb{C}^{2n} \to \mathbb{C} \oplus \mathbb{C}^{2n}$ commuting with the action of $O(2) \times T^n$ on $\mathbb{C} \oplus \mathbb{C}^{2n}$, §0.2, write

$$g(z) = (g_0(z),\ldots,g_{2j-1}(z),g_{2j}(z),\ldots,g_{2n}(z)), \quad g_i = \mathbb{C} \oplus \mathbb{C}^{2n} \to \mathbb{C} \tag{7}$$

where $$g(\gamma z) = \gamma g(z) \quad \forall\, \gamma \in O(2) \times T^n . \tag{8}$$

Application of the involution κ gives from (7) and (8),

$$\begin{aligned}&(a) \quad g_0(\kappa z) = \overline{g_0(z)}\\ &(b) \quad g_{2j-1}(\kappa z) = g_{2j}(z) .\end{aligned} \tag{9}$$

(9) (b) implies that we seek for $g_{2j-1} : \mathbb{C} \oplus \mathbb{C}^{2n} \to \mathbb{C}$, $O(2)\backslash\kappa \times T^n$ equivariants over $O(2) \times T^n$ invariants. (9) (a) makes the g_0 equivariants slightly more messy but we are able to extend the following to obtain the required result.

Caveat: In stating and proving the following we are aware that it can be derived from more general results in Algebraic Geometry. However, we give the result and proof appropriate for this setting, rather than introducing unnecessary machinery.

Lemma 3. The entries in the basis for $O(2) \times T^n$ equivariants corresponding to $\mathbb{C}_i$, $i=1,\ldots,2n$, are given by a spanning set over $\mathbb{C}$ of $\frac{\partial}{\partial Z_i} G_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is the Hilbert basis for the ring of $O(2) \times T^n$ invariant polynomials $\mathbb{C}^{2n+1} \to \mathbb{R}$.

Remarks $G_{\mathbb{R}}$ is not 'free' i.e. there exist relations in $G_{\mathbb{R}}$. Although not a necessary condition one consequence of this is that the above spanning set is not unique with the obvious implication for $O(2) \times T^n$ equivariant bifurcation problems that the coefficients in any Taylor expansion differ, depending on the chosen Hilbert basis.

In order to prove Lemma 3 we require the following proposition.

Proposition 4. If I is an $O(2)\backslash(\kappa) \times T^n$ complex invariant then $\frac{\partial I}{\partial Z_i}$ is an $O(2)\backslash(\kappa) \times T^n$ equivariant on $\mathbb{C}_i$. Moreover, the $\mathbb{C}_i$ entries in the basis for the equivariants are given by $\frac{\partial}{\partial Z_i}(I_g)$, where I_g is the Hilbert basis for $O(2)\backslash(\kappa) \times T^n$ invariant ring of polynomials.

<u>Proof</u> Begin by considering the general form for a polynomial $g_i : \mathbb{C} \oplus \mathbb{C}^{2n} \to \mathbb{C}_i$ i.e.

$$g_i(z) = \Sigma A_{ab} z^a \bar{z}^b \tag{11}$$

$O(2)\sim\!\!\ltimes \times T^n$ equivariance implies

(a) $\theta \in O(2)\sim\!\!\ltimes$ $\qquad m_0(a_0-b_0) + \sum_{j=1}^{n} m_{2j-1}(a_{2j-1}-b_{2j-1}-a_{2j}+b_{2j}) = m_i$

(b) $\psi_i \in S^1(\psi_i)$ $\qquad a_i-b_i + a_{i+1}-b_{i+1} = 1$ $\qquad$ (12)

(c) $\psi_j \in S^1(\psi_j)$ $\qquad a_{2j-1}-b_{2j-1}+a_{2j}-b_{2j} = 0 \quad 2j \neq i+1$.

Applying the same reasoning as in Proposition 1 invariance w.r.t. $O(2)\sim\!\!\ltimes \times T^n$ leads to

(a) $m_0(\alpha_0^i-\beta_0^i) + \sum_{j=1}^{n} m_{2j-1}(\alpha_{2j-1}^i-\beta_{2j-1}^i-\alpha_{2j}^i+\beta_{2j}^i) = 0$

(b) $\alpha_{2j-1}^i-\beta_{2j-1}^i+\alpha_{2j}^i-\beta_{2j}^i = 0$. $\qquad$ (13)

But (13)(a) and (b) together are equivalent to

(a) $m_0(\alpha_0^i-\beta_0^i) + \sum_{\substack{j=1 \\ 2j\neq i+1}}^{n} m_{2j-1}(\alpha_{2j-1}^i-\beta_{2j-1}^i-\alpha_{2j}^i+\beta_{2j}^i)$

$\qquad + m_i(\alpha_i^i-(\beta_i^i-1)-\alpha_{i+1}^i+\beta_{i+1}^i) = m_i$

(b) $\alpha_i^i - (\beta_i^i-1) + \alpha_{i+1}^i - \beta_{i+1}^i = 1$ $\qquad$ (15)

(c) $\alpha_{2j-1}^i-\beta_{2j-1}^i + \alpha_{2j}^i-\beta_{2j}^i = 0 \quad i \neq 2j-1$.

Obtaining the equivalences is simply a matter of checking through the possible ways of obtaining m_i in (15)(a). This verifies the first part of the claim. As in Proposition 1 we seek a minimal spanning set to (12) in $(\mathbb{Z}^+)^6$. We have, from (15), that an element of such will take the form

$$V = (\alpha_0^i, \beta_0^i, \ldots, \alpha_{2j-1}^i, \beta_{2j-1}^{i-1}, \alpha_{2j}^i, \beta_{2j}^i, \ldots, \alpha_{2n-1}^i, \beta_{2n-1}^i, \alpha_{2n}^i, \beta_{2n}^i) , \qquad (16)$$

where $(\alpha_0^i, \ldots\ldots, \beta_{2n}^i) \in Ig$.

It is now easy to see that any solution to (12) may be written as V plus linear combinations of elements of Ig . □

Writing G as the generating set for the $O(2) \times T^n$ invariant complex polynomials $\mathbb{C}^{2n+1} \to \mathbb{C}$ we now extend the above result to G . First let us introduce some nomenclature.

Let M_i , $i = 1, \ldots, N$, be a monomial in Ig . Either M_i is κ invariant or not. If not let

$$M_i\kappa = N_i \ . \qquad (17)$$

Writing those invariants that are κ invariant as $\hat{M}_i$ we see that $Ig = \{\hat{M}_1, \ldots, \hat{M}_\ell, M_1, N_1, \ldots, M_m, N_m\}$.

With the above nomenclature we have the following:

<u>Claim 5</u>: G consists of elements of the form

$$\hat{M}_i, \ M_i + N_i, \ M_i - N_i, \ M_i M_j + N_i N_j, \ M_i M_j M_k + N_i N_j N_k \ .$$

<u>Proof</u> It is straightforward to check that invariants of the form

$M_iM_jM_kM_\ell + N_iN_jN_kN_\ell$ are generated by the above. The result then follows by induction. □

Proof of Lemma 3 Begin by taking derivatives of G, in the above claim. Using Proposition 4, after e liminating redundancies, we obtain $\frac{\partial}{\partial Z_i} G$, $i \neq 0$, is a basis for $O(2)\ltimes \times T^n$ equivariant polynomials $\mathbb{C}^{2n+1} \to \mathbb{C}_i$ over $O(2) \times T^n$ invariant polynomials $\mathbb{C}^{2n+1} \to \mathbb{C}$. We obtain a Hilbert basis $G_{\mathbb{R}}$ by taking complex (linear) combinations of G so that complex (linear) combinations of $\frac{\partial}{\partial Z_i} G_{\mathbb{R}}$ will give a spanning set as required. □

Proof of Theorem 2 It is a simple matter to find the equivariants corresponding to g_1 and g_2 from Lemma 3.

To find the equivariants corresponding to g_0 begin by noticing that $\frac{\partial}{\partial Z_0} G_{\mathbb{R}}$ gives $O(2)\ltimes \times S^1$ equivariants corresponding to g_0. (Notice, that (9)(a) rules out the need to consider complex (linear) combinations.) Now, for g_0 we also have $\times$ equivariants which are $O(2)\ltimes \times$ invariant in particular, $i(u_1-u_2)$ is such an invariant Using the above claim we can check this is the only such polynomial. Writing $V = \langle 1, i(u_1-u_2)\rangle$ we have $V \cdot \frac{\partial}{\partial Z_0} G_{\mathbb{R}}$, after eliminating redundancies, gives the basis elements corresponding to g_0. □

§1.3. Solution Branches and corresponding Eigenvalues.

The consequence of the results of §1.2 for the reduced bifurcation problem $\phi : \mathbb{C}^3 \times \mathbb{R}^{n+1} \to \mathbb{C}^3$ discussed at the beginning of this chapter

is that we may write

$$\phi(z,\lambda,\tau) = \begin{bmatrix} P^1 z_0 + P^3 \bar{z}_0^{\alpha-1} (z_1\bar{z}_2)^{\beta} \\ Q^1 z_1 - i(1+\tau)z_1 + Q^5 z_0^{\alpha}(\bar{z}_1 z_2)^{\beta-1} \\ R^1 z_2 - i(1+\tau)z_2 + R^5 \bar{z}_0^{\alpha}(z_1\bar{z}_2)^{\beta-1} \end{bmatrix}_{1,2} \qquad (1)$$

where $\quad P^j = p^j + i\delta_1 p^{j+1}$

$$Q^j = q^j + iq^{j+1} + u_2 q^{j+2} + iu_2 q^{j+3}$$

$$R^j = Q^j_{.\kappa}$$

p^j, q^j are (germs of) mappings $(N_0,\ldots,O_2,\lambda,\ ) \to \mathbb{R}$

and $\quad \alpha = \begin{cases} 2m & \ell \text{ odd} \\ m & \ell \text{ even} \end{cases} \qquad \beta = \begin{cases} \ell & \ell \text{ odd} \\ \ell/2 & \ell \text{ even.} \end{cases}$

As detailed in the introduction we find branching equations by considering the zeros of $\phi|_{Fix(\Sigma)}$. In particular we restrict to a representative of $Fix(\Sigma)$ with $Im(Z_1) = Im(Z_2) = 0$. We obtain the eigenvalues associated with these solution branches by the methods detailed in the latter part of this section. The results are listed in the following table $\forall\ (\ell,m) \neq (2,1)$. The special case $(\ell,m) = (2,1)$ is discussed in §1.4.

TABLE 4.

Branching equations and eigenvalues for solutions with given symmetry.

Isotropy Subgroup: Orbit	Branching equations	Signs of eigenvalues	Multiplicity
$O(2) \times S^1$	None	p^1	2
		$q^1 \pm i(q^2-1-\tau)$	2,2
1.			
$\mathbb{Z}_\kappa \times S^1$	$p^1 = 0$	0	1
$(x_0,0,0)$	at $(x_0^2,0,0,0,0,\lambda)$	$p^1_{N_0}$	1
		$q^1 \pm q^5x_0^2 \pm i(q^2 \pm q^6x_0^2-1-\tau)$	1,1
2.	$q^1 = 0$	0	1
$S(1,-m)$	$q^2 = 1+\tau$	$q^1_{N_1}$	1
$(0,x_1,0)$	at $(0,x_1^2,0,0,0,\lambda)$	$q^3 \pm iq^4$	1,1
		$p^1 \pm ip^2x_1^2$	
3.			
$\mathbb{Z}_\kappa \times \mathbb{Z}(\frac{\pi}{m}, \pi)$	$q^1+q^3x_1^2 = 0$	0	2
$(0,x_1,x_1)$	$q^2+q^4x_1^2 = 1+\tau$	$p^1 \pm p^3x_1^2 \quad m = 1$	1,1
	at $(0,2x_1^2,x_1^4,0,0,\lambda)$	$p^1 \quad m > 1$	2
		$(q^3+2q^1_{N_1})x_1^2$	1
		$-q3x_1^2$	1

4.

$\mathbb{Z}\ (x\mathbb{Z}(\pi,m_\pi))$

(x_0,x_1,x_1)

$$p^1+p^3x_0^{\alpha-2}x_1^{2\beta} = 0$$
$$q^1+q^3x_1^2+q^5x_0^{\alpha-2\beta-2}+q^7x_0^\alpha x_1^{2\beta} = 0$$
$$q^2+q^4x_1^2+q^5x_0^{\alpha-2\beta-2}+q^6x_0^\alpha x_1^{2\beta} = 1+\tau$$
$$\text{at } (x_0^2,2x_1^2,x_1^4,x_0^\alpha x_1^{2\beta},0,\lambda)$$

$$\begin{array}{c} 0[2] \\ q^1_{N_1}[2] \end{array}$$

$$\text{trace} = 2p^1_{N_0}x_0^2 + 2(2q^1_{N_1}+q^3)x_1^2$$
$$\det = 4(p^1_{N_0}(2q^1_{N_2}+q^3) - (q^1_{N_0}+\delta q^5)(2p^1_{N_1}+\delta p^3))x_0^2x_1^2$$
$$\text{trace} = -2(q^3x_1^2+\delta p^3x_1^2+2\delta q^5x_0^2)$$
$$\det = 4x_0^{\alpha-2}x_1^{2\beta-2}\Big[\tfrac{\alpha}{2}(p^3q^3x_1^4+2p^2q^6x_0^2)x_1^2 + \beta(q^3q^5+q^4q^6)x_0^2x_1^2 + \delta(p^3q^5x_1^2 + (q^5)^2x_0^2 + (q^6)^2x_0^2)x_0^2\Big]$$

5.

$\mathbb{Z}_\kappa(0,\pi)x\mathbb{Z}(\pi,m_\pi))$

$(x_0,x_1,-x_1)$

$$p^1 + p^3x_0^{\alpha-2}x_1^{2\beta} = 0$$
$$q^1+q^3x_1^2+(-1)^\beta q^5x_0^\alpha x_1^{2\beta-2}+(-1)^\beta q^7x_0^\alpha x_1^{2\beta} = 0$$
$$q^2+q^4x_1^2+(-1)^\beta q^6x_0^\alpha x_1^{2\beta-2}+(-1)^\beta q^8x_0^\alpha x_1^{2\beta} = 0$$
$$\text{at } (x_0^2,2x_1^2,x_1^4,x_0x_1^2,0,\lambda)$$

$$\begin{array}{c} 0[2] \\ p^1_{N_0}[2] \end{array}$$

$$\text{trace} = -2(q^3x_1^2-\delta p^3x_1^2-2\delta q^5x_0^2)$$
$$\det = -4x_0^{\alpha-2}x_1^{2\beta-2}\Big[\tfrac{\alpha}{2}(p^3q^3x_1^4+2p^2q^6x_0^2)x_1^2+\beta(q^3q^5+q^4q^6)x_0^2x_1^2 - \delta(p^3q^5x_1^2+(q)^2x_0^2 + (q^6)^2x_0^2)x_0^2\Big]$$
$$\text{trace} = 2[p^1_{N_0}x_0^2 + (2q^1_{N_1}+q^3)x_1^2]$$
$$\det = 4(p^1_{N_0}(2q^1_{N_1}+q^3) - (q^1-\delta q^5)(2p^1_{N_1}+\delta p^3))x_0^2x_1^2$$

6.

$\mathbb{Z}(\frac{2\tau}{\ell}, \frac{-2m_\pi}{\ell})$

$(x_0,x_1,0)$

$$p^1 = 0$$
$$p^2x_1^2 = 0$$
$$q^1+q^3x_1^2 = 0$$
$$q^2+q^4x_1^2 = 1+\tau$$

$$\begin{array}{c} 0[2] \\ q^1_{N_1}[2] \end{array}$$

$$\begin{bmatrix} p^1_{N_0}x_0^2 & p^1_{N_1}x_0x_1 \\ q^1_{N_1}x_0x_1 & q^1_{N_1}x_0x_1 \end{bmatrix}$$

7. (a) $\mathbb{Z}(\frac{\pi}{m}, \pi)$	$(q^1+q^3x_1^2)x_1 = 0$	$O[2]$	$\begin{bmatrix} q_{N_1}^1 x_1^2 & (q_{N_1}^1 + q^3)x_0x_1 \\ (q_{N_1}^1+q^3)x_0x_1 & q_{N_1}^1 x_1^2 \end{bmatrix}$
$(0,x_1,x_2)$	$(q^1+q^3x_2^2)x_2 = 0$		
	$q^1+q^3x_2^2 = 1+\tau$		$\begin{bmatrix} p^1+ip^2(x_1^2-x_2^2) & (p^3+ip^4(x_1^2-x_2^2))x_1x_2 \\ (p^3-ip^4(x_1^2-x_2^2) & p^1-ip^2(x_1^2-x_2^2) \end{bmatrix}$

Note: $\delta = 1 \quad \ell = m = 1$

$ 0 \quad$ o.w.

Remarks

(1) Notice that although we list results for ℓ odd and even ($\neq 2$) in Table 4 the branching equations for these two cases are quite different. In the former the branching equations are always even in x_0 and x_1 whereas in the latter, monomials that are odd in x_0 occur. We exploit the first in an application in §3.

(2) For solution type 7 a short calculation shows that if $x_1 \neq x_2 \neq 0$ then $q^3(0, x_1^2+x_2^2, x_1^2x_2^2, 0, 0, \lambda) = 0$. We shall see that $q^3(0) \neq 0$, in §3, is a non-degeneracy condition required to guarantee solutions with lower dimension fixed point subspaces and so a full description of solution type 7 requires singularity theory.

(3) Notice when $4|\ell$ branching equations 4 and 5 coincide as predicted by the group theory.

To facilitate the calculation of the eigenvalues of $(d\phi)_{(x_0, \lambda_0, \tau_0)}$ we bring together a number of results commonly used in the literature c.f. Golubitsky and Stewart (1985), Golubitsky, Stewart and Shaeffer (1987), Buzano and Golubitsky (1983).

The first result hinges on the isotypic decomposition of a vector space X under a compact Lie group Γ. First, write

$$X = V_1 \oplus \ldots \oplus V_k \tag{3}$$

where each V_i is Γ-invariant and is the direct sum of irreducible subspaces of Γ of a given isomorphism type. In other words the action

of Γ on Γ-irreducibles of V_i and V_j, $i \neq j$ is non-isomorphic.

Lemma 1 For a linear map $L : X \to X$ commuting with Γ, we have, using the above notation

$$L(V_i) \subset V_i .$$

Proof See Golubitsky (et al) (1987).

The above will obviously hold for the compact Lie group $\Sigma \subset O(2) \times S^1$ acting on $\mathbb{C}^3$ and the linear map $(d\phi)|_{Fix(\Sigma)}$ which commutes with Σ.

The second result also allows us to make use of the null-eigenvectors of $(d\phi)_{(Z,\lambda,\tau)}$.

Proposition 2 Let $\gamma_t \in \Gamma$ and be differentiable with γ_0 the identity. Assume $\phi(z) = 0$ then $\left.\frac{d(\gamma_t . z)}{dt}\right|_{t=0}$ is a null-eigenvector for $(d\phi)_{(z,\lambda,\tau)}$.

Proof As in Buzano and Golubitsky (1983). Since $\phi(z) = 0$ we have

$$\phi(\gamma_t z) = 0 \tag{4}$$

since ϕ vanishes on $\Gamma \cdot z$. Now, just differentiate (4).

Note: The null-eigenvector is non-zero when the curve γ_t is transverse to Σ which is the generic situation.

Choosing $\gamma_\theta = (\theta,0) \in O(2) \times S^1$ and $(0,\psi) \in O(2) \times S^1$ we obtain, for the action of $O(2) \times S^1$ on $\mathbb{C}^3$, null-eigenvectors of the form:

$$\begin{aligned}
&\text{(a)}\quad (z_0,z_1,z_2) \to \frac{d}{d\theta}\,(e^{1\ell\theta}z_0,e^{1m\theta}z_1,e^{-1m\theta}z_2) = (i\ell z_0,imz_1,-imz_2)\\
&\text{(b)}\quad (z_0,z_1,z_2) \to \frac{d}{d\psi}\,(z_0,e^{i\psi}z_1,e^{1\psi}z_2) = (0,iz_1,iz_2)\ .
\end{aligned} \tag{5}$$

<u>Remarks</u> Lemma 1 allows us to put the 6×6 matrix $d\phi$ into block diagonal form. It is, of course, the case that any of the above relationships would have come out in the wash but in keeping with the general philosophy we allow group theory to both organise and simplify the calculations.

We are now able to detail the calculations of the eigenvalues for $(d\phi)_{(x_0,\lambda_0,\tau_0)}$.

1. For $\Sigma = \mathbb{Z}_< \times S^1$ we decompose $\mathbb{C}^3 = V_0 \oplus V_0' \oplus V_1 \oplus V_2$

where $V_0 = \langle(1,1,0,0,0,0)\rangle$, $V_0' = \langle(1,-1,0,0,0,0)\rangle$,

$V_1 = \langle(0,0,1,0,1,0),(0,0,0,1,0,1)\rangle$ and $V_2 = \langle(0,0,1,0,-1,0),(0,0,0,1,0,-1)\rangle$

We have

	V_0	V_0'	V_1	V_2
$<$	1	-1	1	-1
ψ	1	1	R_ψ	R_ψ

for the actions, which are obviously non-isomorphic.

Now $(d\phi)\big|_{V_1} = \dfrac{\partial\phi_0}{\partial\bar z_0} + \dfrac{\partial\phi_0}{\partial\bar z_0} = p^1 + 2p^1_{N_0}x_0^2$

but $p^1 = 0$ is the branching equation for this isotropy, so that one eigenvalue is $2p^1_{N_0}x_0^2$.

$$(d\phi)\Big|_{V_2} = \frac{\partial\phi_0}{\partial z_0} - \frac{\partial\phi_0}{\partial\bar{z}_0} = p^1 = 0 \ , \quad \text{giving a zero eigenvalue.}$$

Also $(d\phi)\Big|_{V_i}$, $i = 1,2$, takes the form

$$\begin{bmatrix} a_i & b_i \\ \bar{b}_i & \bar{a}_i \end{bmatrix} \tag{6}$$

where $\quad a = \dfrac{\partial\phi_i}{\partial z_1} + \dfrac{\partial\phi_i}{\partial z_2} \ , \qquad b = \dfrac{\partial\phi_i}{\partial\bar{z}_1} + \dfrac{\partial\phi_i}{\partial\bar{z}_2} \ .$

Giving eigenvalues $q^1 \pm q^5x_0^2 \pm i(q^2 \pm q^6x_0^2 - 1 - \tau)$.

2. For $\Sigma = \delta(1,-m)$ we decompose $\mathbb{C}^3 \cong V_0 \oplus V_1 \oplus V_2$ where $V_j = \mathbb{C}$. The action is easily seen to be non-isomorphic. Note, $(d\phi)\Big|_{V_j}$, $j = 0,1,2.$, takes the form

$$\begin{bmatrix} a_j & b_j \\ \bar{b}_j & \bar{a}_j \end{bmatrix} \ , \quad \text{where} \quad a_j = \frac{\partial\phi_j}{\partial z_j} \quad \text{and} \quad b_j = \frac{\partial\phi_j}{\partial\bar{z}_j} \ . \tag{7}$$

Computing the entries we have, after substitution of branching equations,

$a_0 = p^1 + ip^2x_1^2$, $b_0 = 0$, $a_1 = (q^1_{N_1} + iq^2_{N_1})x_1^2 = b_1$ and

$a_2 = (q^3 + iq^4)x_1^2$.

Giving eigenvalues $p^1 \pm ip^2x_1^2$, 0 , $q^1_{N_1}x_1^2$, $(q^3 \pm iq^4)x_1^2$.

3. $\Sigma = \mathbb{Z} \times \mathbb{Z}(\frac{\pi}{m}, \pi)$ we decompose $\mathbb{C}^3$ as in 1 with the following action

	V_0	$V_0^{\perp}$	V_1	V_2
κ	1	-1	1	-1
$(\frac{\pi}{m}, \pi)$	$R_{\frac{2\pi}{m}}$	$R_{\frac{2\pi}{m}}$	1	1

which are again non-isomorphic. Also notice that the null-eigenvector $(0, ix, ix)$ of (5)(b) belongs to $\mathrm{Fix}(\Sigma) = V_1$ so that $(d\phi)|_{V_1}$ has a zero eigenvalue and so we need only consider the trace of $(d\phi)|_{V_1} = \mathrm{Re}(\frac{\partial\phi_1}{\partial z_1} + \frac{\partial\phi_1}{\partial z_2})$. We also have the null-eigenvector $(0, ix, -ix)$ of (5)(a) belongs to V_2 giving a similar result for $(d\phi)|_{V_2}$.

In fact, trace $((d\phi)|_{V_2}) = \mathrm{Re}(\frac{\partial\phi_2}{\partial z_1} - \frac{\partial\phi_2}{\partial z_2}) = -\mathrm{Re}(\frac{\partial\phi_1}{\partial z_1} - \frac{\partial\phi_1}{\partial z_2})$, by applying κ and using equivariance or by straightforward observation. Following 1 we verify the eigenvalues for 3 .

To calculate eigenvalues for solutions 4 and 5 requires a little more effort.

4. The isotropy subgroup $\Sigma = \mathbb{Z}_\kappa$ decomposes $\mathbb{C}^3 = V_1 \oplus V_2$ where

$$V_1 = \langle (z_0+\bar{z}_0,\ z_1+z_2,\ \bar{z}_1+\bar{z}_2,\ 0,\ 0,\ 0) \rangle = \mathrm{Fix}(\Sigma)$$

$$V_2 = \langle (0,\ 0,\ 0,\ z_0-\bar{z}_0,\ z_1-z_2, \bar{z}_1-\bar{z}_2) \rangle = \mathrm{Fix}(\Sigma)^{\perp}.$$

κ acts non-isomorphically on the above decompositions by $+1$, -1 respectively.

The null-eigenvectors in this coordinate system, evaluated on the orbit representative $(x_0, x_1, x_1) \in \mathbb{C}^3$, are

$$\begin{aligned} &\text{(a)} \quad (0,0,0,\ i\ell x_0,\ imx_1,\ -imx_1) \\ &\text{(b)} \quad (0,\ ix_1,\ ix_1,\ 0,0,0) \quad . \end{aligned} \tag{8}$$

(8)(a) reduces finding the non-zero eigenvalues of $dh\big|_{V_2}$ to finding the eigenvalues of the 2×2 matrix

$$\begin{bmatrix} \dfrac{\partial\phi_0}{\partial z_0} - \dfrac{\partial\phi_0}{\partial\bar{z}_0} + \mu\left(\dfrac{\partial\bar{\phi}_1}{\partial z_0} - \dfrac{\partial\bar{\phi}_1}{\partial\bar{z}_0}\right) & \dfrac{\partial\phi_0}{\partial z_1} - \dfrac{\partial\phi_0}{\partial z_2} + \mu\left(\dfrac{\partial\bar{\phi}_1}{\partial z_1} - \dfrac{\partial\bar{\phi}_1}{\partial z_2}\right) \\ 2i\ \mathrm{Im}\left(\dfrac{\partial\phi_1}{\partial z_0} - \dfrac{\partial\phi_1}{\partial\bar{z}_0}\right) & \dfrac{\partial(\phi_1+\bar{\phi}_1)}{\partial z_1} - \dfrac{\partial(\phi_1+\bar{\phi}_1)}{\partial z_2} \end{bmatrix} \tag{9}$$

where $\mu = \dfrac{\ell x_0}{mx_1} = \dfrac{2\beta}{\alpha}\dfrac{x_0}{x_1}$.

Similarly, (8)(b) reduces $dh\big|_{V_1}$ to

$$\begin{bmatrix} \dfrac{\partial\phi_0}{\partial z_0} + \dfrac{\partial\phi_0}{\partial z_0} & \dfrac{\partial\phi_0}{\partial z_1} + \dfrac{\partial\phi_0}{\partial z_2} \\ 2\mathrm{Re}\left(\dfrac{\partial\phi_1}{\partial z_0} + \dfrac{\partial\phi_1}{\partial\bar{z}_0}\right) & 2\mathrm{Re}\left(\dfrac{\partial\phi_1}{\partial z_1} + \dfrac{\partial\phi_1}{\partial z_2}\right) \end{bmatrix} \tag{10}$$

Computing the entries for (10)

$$\frac{\partial\phi_0}{\partial z_0} + \frac{\partial\phi_0}{\partial\bar{z}_0} = 2p^1_{N_0}x_0^2 + (\alpha-1)p^3x_0^{\alpha-2}x_1^{2\beta} + (\alpha p^1_{O_1}+2p^3_{N_0})x_0^{\alpha}x_1^{2\beta} + \alpha p^3_{O_1}x_0^{2\alpha-2}x_1^{4\beta}$$

$$\frac{\partial\phi_0}{\partial z_1} + \frac{\partial\phi_0}{\partial\bar{z}_2} = 2p^1_{N_1}x_1x_0 + 2p^1_{M_1}x_0x_1^2 + \beta p^3x_0^{\alpha-1}x_1^{2\beta-1} + (2p^3_{N_1}x_1^2+\beta p^1_{O_1}x_0^2+2p^3_{M_1}x_1^4)x_0^{\alpha-1}x_1^{2\beta-1}$$
$$+ \beta p^3_{O_1}x_0^{2\alpha-1}x_1^{4\beta-1}$$

$$2\,\mathrm{Re}\left(\frac{\partial\phi_1}{\partial z_0} + \frac{\partial\phi_1}{\partial\bar{z}_0}\right) = 4\Big[q^1_{N_0}x_0x_1 + \frac{\alpha}{2}q^5x_0^{\alpha-1}x_1^{2\beta-1} + q^3_{N_0}x_0x_1^3 + \frac{\alpha}{2}q^1_{O_1}x_0^{\alpha-1}x_1^{2\beta-1}$$
$$+ \frac{\alpha}{2}q^3_{O_1}x_0^{\alpha-1}x_1^{2\beta+3} + q^5_{N_0}x_0^{\alpha+1}x_1^{2\beta-1} + q^7_{N_0}x_0^{\alpha+1}x_1^{2\beta+1}$$
$$+ \frac{\alpha}{2}q^5_{O_1}x_0^{2\alpha-1}x_1^{4\beta-1} + \frac{\alpha}{2}q^7_{O_1}x_0^{2\alpha-1}x_1^{4\beta+1}\Big]$$

$$2\,\mathrm{Re}\left(\frac{\partial\phi_1}{\partial z_1} + \frac{\partial\phi_1}{\partial z_2}\right) = \Big[2q^3x_1^2 + (\beta-1)(q^5+q^7x_1^2)x_0^{\alpha}x_1^{2\beta-2} + 2q^7x_0^{\alpha}x_1^{2\beta}$$
$$+ 4q^1_{N_1}x_1^2 + 4q^1_{M_1}x_1^4 + 4q^3_{N_1}x_1^4 + 2\beta q^1_{O_1}x_0^{\alpha}x_1^{2\beta}$$
$$+ 4q^3_{M_1}x_1^6 + 2\beta q^3_{O_1}x_0^{\alpha}x_1^{2\beta+2} + 4q^5_{N_1}x_0^{\alpha}x_1^{2\beta}$$
$$+ 4q^5_{M_1}x_0^{\alpha}x_1^{2\beta+2} + 2\beta q^5_{O_1}x_0^{2\alpha}x_1^{2\beta-2} + 4q^7_{N_1}x_0^{\alpha}x_1^{2\beta+2}$$
$$+ 4q^7_{M_1}x_0^{\alpha}x_1^{2\beta+4} + 2q^7_{O_1}x_0^{2\alpha}x_1^{2\beta}\Big]$$

Trivially we obtain the trace and to lowest order

$$\det = \Big[4p^1_{N_0}(q^3+2q^1_{N_1})x_0^2x_1^2 + 2(\alpha-1)p^3(q^3+2q^1_{N_1})x_0^{\alpha-2}x_1^{2\beta+2} +$$
$$+ (\alpha-1)(\beta-1)p^3q^5x_0^{2\alpha-2}x_1^{4\beta-2}$$
$$+ 2(2p^1_{N_1}+\beta p^3x_0^{\alpha-2}x_1^{2\beta-2})(2q^1_{N_0}+\alpha q^5x_0^{\alpha-2}x_1^{2\beta-2})x_0^2x_1^2\Big]. \qquad (11)$$

In the case when ℓ is odd we may neglect further terms giving the required entry in Table 4(a) but when ℓ is even the entry is really the Taylor expansion at zero of (11). However, this slight abuse of notation should not cause too many problems.

To compute entries for (9) we have, using branching equations,

$$\frac{\partial\phi_0}{\partial z_0} - \frac{\partial\phi_0}{\partial\bar{z}_0} = -\alpha p^3 x_0^{\alpha-2} x_1^{2\beta}$$

$$\frac{\partial\phi_1}{\partial z_0} - \frac{\partial\phi_1}{\partial\bar{z}_0} = \alpha(q^5+q^7x_1^2)x_0^{\alpha-1}x_1^{2\beta-1} + \alpha i(q^6+q^8x_1^2)x_0^{\alpha-1}x_1^{2\beta-1}$$

$$\frac{\partial\bar{\phi}_1}{\partial z_0} - \frac{\partial\bar{\phi}_1}{\partial\bar{z}_0} = -\left(\overline{\frac{\partial\phi_1}{\partial z_0} - \frac{\partial\phi_1}{\partial\bar{z}_0}}\right)$$

$$\frac{\partial\phi_0}{\partial z_1} - \frac{\partial\phi_0}{\partial z_2} = 2i(p^2x_0x_1 + p^4x_0^{\alpha-1}x_1^{2\beta+1}) + \beta p^3 x_0^{\alpha-1}x_1^{2\beta}$$

$$\begin{aligned}\frac{\partial\phi_1}{\partial z_1} - \frac{\partial\phi_1}{\partial z_2} = -\Big[& q^3x_1^2 + q^7x_0^\alpha x_1^{2\beta} + (\beta+1)(q^5+q^7x_1^2)x_0^\alpha x_1^{2\beta-2} \\ & + i(q^4x_1^2 + q^8x_0^\alpha x_1^{2\beta} + (\beta+1)\,(q^6+q^8x_1^2)x_0^\alpha x_1^{2\beta-2}\Big]\end{aligned}$$

$$\begin{aligned}\frac{\partial\bar{\phi}_1}{\partial z_1} - \frac{\partial\bar{\phi}_1}{\partial z_2} = \Big[& -q^3x_1^2 - q^7x_0^\alpha x_1^{2\beta} + (\beta-1)(q^5+q^7x_1^2)x_0^\alpha x_1^{2\beta-2} \\ & + i(q^4x_1^2 + q^8x_0^\alpha x_1^{2\beta} - (\beta-1)(q^6+q^8x_1^2)x_0^\alpha x_1^{2\beta-2}\Big] \quad .\end{aligned}$$

After some algebra we obtain the given entries. A similar, careful, computation verifies the eigenvalues for solution type 5.

§1.4. The special case $(\ell,m) = (2,1)$

The standard O(2)-Hopf analysis, c.f. Golubitsky and Stewart (1985), or Schecter (1976) and Bajaj (1982) for a particular case with a different approach, finds two primary branches corresponding to a rotating wave with a one-dimensional isotropy subgroup and a standing wave with isotropy subgroup $\mathbb{Z}_2^2$. The initial expectation, in an O(2) mode interaction, is to find all the primary branches at the degeneracy. However, Table 2 shows that when ℓ is even and $m = 1$ the isotropy subgroup corresponding to the standing wave has a three dimensional fixed point subspace and since a primary branch is guaranteed only when $\dim \mathrm{Fix}(\Sigma) = 2$, c.f. Golubitsky and Stewart (1985), these classes of non-standard O(2) actions give new phenomena for O(2)-steady-state/Hopf interactions. In fact, when $\ell = 2$ the branching equations on the three-dimensional fixed point subspace form a $\mathbb{Z}_2$-equivariant corank-two bifurcation problem and the contact equivalent normal forms, up to codimension 5, are known c.f. Dangelmayr and Armbruster(1983). We use these results to simplify the study of the above special case. We begin the analysis of the special case and explanation of the above by consideration of the appropriate branching equations.

Recall the O(2)-equivariant normal form of the reduced bifurcation problem given in §1.3 (1). The branching equations corresponding to the isotropy subgroup $\mathbb{Z}_\kappa \times \mathbb{Z}(\pi,\pi)$ of Table 2 are

$$\begin{aligned}
&(a)\quad p^1x_0 + p^3x_1 = 0\\
&(b)\quad (q^1 + q^3x_1^2 + q^5x_0x_1^{(\ell-2)/2} + q^7x_0x_1^{(\ell+2)/2})x_1 = 0 \qquad\qquad (1)\\
&(c)\quad q^2 + q^4x_1^2 + q^6x_0x_1^{(\ell-2)/2} + q^8x_0x_1^{(\ell+2)/2} = 1 + \tau
\end{aligned}$$

where p^1, q^1 map $(x_0^2, 2x_1^2, x_1^4, x_0x_1^\ell, 0, \lambda) \to \mathbb{R}$ and $\ell \not| 4$.

Remark: The standard method of finding solutions to equations of the above type is to use equation (1)(c) to e liminate τ and then apply the implicit function theorem to conclude the existence of solutions in some neighbourhood of the origin - see §3. However, since the equations are not even in each variable this brute force method becomes messy and (certainly) does not provide a clean route to the qualitative description we seek.

Concentrating on equations (1)(a) and (b) - as we may assume τ to be e liminated - write

$$\begin{aligned} &(a) \quad g_0(x_0, x_1, \lambda) = p^2 x_0 + p^3 x_1^\ell \\ &(b) \quad g_1(x_0, x_1, \lambda) = q^1 + q^3 x_1^2 + q^5 x_0 x_1^{(\ell-2)/2} + q^7 x_0 x_1^{(\ell+2)/2})x_1 \end{aligned} \tag{2}$$

we see that $-g_1(x_0, x_1, \lambda) = g_1(x_0, -x_1, \lambda)$ i.e. g_1 commutes with the $\mathbb{Z}_2$ action $x_1 \to -x_1$. It is now clear that $G(x_0, x_1, \lambda) = (g_0(x_0, x_1, \lambda),$ $g_1(x_0, x_1, \lambda))$ forms a corank two $\mathbb{Z}_2$-equivariant bifurcation problem. Dangelmayr and Armbruster study this problem using the theory of imperfect bifurcations of Golubitsky and Schaeffer, which is developed in full in Golubitsky & Schaeffer (1985). The two basic ideas of the above theory are contact equivalence (with a distinguished bifurcation parameter) and a universal unfolding. The notion of contact equivalence is well known and in this case we also wish the $\mathbb{Z}_2$-symmetry to be preserved. The approoriate definition is:

Definition 1 Bifurcation problems G and G^1 are said to be $\mathbb{Z}_2$-contact

equivalent if

$$G^1(x_0,x_1,\lambda) = T(x_0,x_1,\lambda)G(X(x_0,x_1^2,\lambda),\lambda\Lambda(\lambda),\ x_1^2S^2(x_0,x_1^2,\lambda)) \qquad (3)$$

where $X(0,0,0) = 0$, $S\left.\frac{\partial X}{\partial x_0}\right|_0 > 0$, $\Lambda(0) > 0$ and T is a non-singular $\mathbb{Z}_2$-equivariant matrix,

$$T(x_0,x_1,\lambda) = \begin{bmatrix} T_{11}(x_0,x_1^2,\lambda) & x_1T_{12}(x_0,x_1^2,\lambda) \\ x_1T_{21}(x_0,x_1^2,\lambda) & T_{22}(x_0,x_1^2,\lambda) \end{bmatrix} , \quad (T_1,T_{22})_0 \neq 0 .$$

c.f. Dangelmayr and Armbruster (1982).

An unfolding of a bifurcation problem is a perturbation depending smoothly on some unfolding or imperfection parameters. The unfolding is said to be <u>stable</u> if any other unfolding is contact equivalent and is a universal unfolding if it is stable and contains the minimum number of unfolding parameters for stability. This minimum number is called the <u>codimension</u> of the bifurcation problem. With this definition of codimension it turns out that (1) describes for $(\ell,m) = (2,1)$ a codimension 3 bifurcation problem which is $\mathbb{Z}_2$-contact equivalent to the normal form

$$G(x_0,x_1,\lambda) = \begin{bmatrix} x_0^3 + \varepsilon_1x_0\lambda + \varepsilon_2x_1^2 \\ x_1(x_0 + \varepsilon_3\lambda) \end{bmatrix} \qquad (4)$$

where $\varepsilon_1 = \mathrm{sign}(p^1_{N_0}(0)\quad p^1_\lambda(0))$ $\quad \varepsilon_2 = \mathrm{sign}(p^1_{N_0}(0)\quad p^3(0))$ and

$\varepsilon_3 = \mathrm{sign}(q^5(0)\ q^1_\lambda(0))$ with the unfolding basis

$$\begin{bmatrix}1\\0\end{bmatrix}, \quad \begin{bmatrix}x_0^2\\0\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0\\x_1\end{bmatrix}.$$

Remarks:

(1) When $\ell|4$ $g_1(x_0,x_1,\lambda)$ does not commute with $\mathbb{Z}_2$ action and the bifurcation problem is a general problem $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ - at least on this fixed point subspace. If codim $g \leq 3$ we have a Hilltop bifurcation c.f. Golubitsky and Schaeffer (1985). However, $g_\lambda(0,0) = 0$, a non-degeneracy condition in the above, so that codim $g \geq 4$.

(2) When $\ell \nmid 4$ but $\ell > 2$ we also find codim $g \geq 4$.

It is well known that the Liapunov-Schmidt method reduces study of degenerate Hopf bifurcations to a corank-one $\mathbb{Z}_2$-equivariant bifurcation problem c.f. Golubitsky and Langford (1980) for a singularity theory approach with distinguished bifurcation parameters. Consideration of a steady-state/Hopf interaction without any symmetry reduces via the Liapunov-Schmidt method to the study of a $\mathbb{Z}_2$-equivariant corank-two bifurcation problem c.f. Armbruster et al (1985). Unfortunately the above paper adopts the approach that the two modes should be 'visible' at the degeneracy precluding the study of (1). The philosophy adopted in this thesis is to consider mode interactions primarily as the dynamics arising on the direct sum of eigenspaces corresponding to the individual modes. This interpretation obviously includes (1) for study.

Primarily we are concerned with the unfolding that splits the modes - see §3. for further details. It is clear that the related splitting parameter will be contained in the set of unfolding parameters. It is immediate

from (4) that the unfolding of interest is

$$G_\alpha(x_0,x_1,\lambda_1) = \begin{bmatrix} x_0^3 + \varepsilon_1 x_0\lambda + \varepsilon_2 x_1^2 \\ x_1(x_0 + \varepsilon_3\lambda + \alpha) \end{bmatrix} . \tag{5}$$

Associated with the choices for ε_1, ε_2 and ε_3 are a large number of bifurcation diagrams. We shall see that either the rotating wave solution of $S(1,-1)$ symmetry and the standing wave associated with the above three dimensional fixed point subspace bifurcate simultaneously or the rotating wave solution and the standing wave associated with the isotropy $\mathbb{Z}_\kappa(0,\pi) \times \mathbb{Z}(\pi,m\pi)$, in Table 2 bifurcate simultaneously. Since the two phenomena are essentially the same we concentrate on the former for the bifurcation diagrams.

We begin by considering the Taylor expansion of the steady-state and rotating wave solutions whose branching equations are given in Table 4 1.3:

Steady-state:

$$p_\lambda^1(0)\lambda + p_{N_0}^1(0)x_0^2 + .. = 0$$

Rotating wave:

$$\beta + q_\lambda^1(0)\lambda + q_{N_1}^1 x_1^2 + .. = 0 .$$

The Implicit Function Theorem guarantees solutions

$$\text{(a)} \qquad \lambda = -\frac{p_{N_0}^1(0)}{p_\lambda^1(0)}\, x_0^2 \qquad \text{when } p_{N_0}^1(0) \neq 0 ,\ p_\lambda^1(0) \neq 0$$

$$\text{(b)} \qquad \lambda = -\frac{\beta}{q_\lambda^1(0)} + \frac{q_{N_1}^1(0)}{q_\lambda^1(0)}\, x_1^2 \qquad \text{when } q_\lambda^1(0) \neq 0 ,\ q_{N_1}^1(0) \neq 0 . \tag{6}$$

<u>Note</u>: The unfolding G_α in (5) is equivalent to replacing $q^1(0)$ with β in the branching equations where we set $\alpha(q^1_\lambda(0))\mathrm{sign}(q^5(0)) = \beta$.

To obtain appropriate choices for ε_1 , ε_2 and ε_3 we consider the stability conditions for the above. The eigenvalues associated with the solution types for $(\ell,m) = (2,1)$ can be computed in the same way as those in §1.3. Unfortunately, the rotating wave solution does not have a correspondingly simple isotypic decomposition, leaving us with a 4×4 matrix. However, in any application a numerical analysis could be carried out - we adopt this philosophy in §3 where large matrices providing stabilities are not amenable to an analysis by hand.

Taylor expanding the results of the above computation for the extractable eigenvalues we have:

(0) Trivial Solution : $p^1_\lambda\lambda$ [2] , $\beta + q^1_\lambda\lambda$ [4]

(I) Steady-state : O[1] , $p^1_{N_0} x_0^2$, $\beta + q^1_\lambda\lambda \pm q^5 x_0$ [2,2]

(II) Rotating wave : O[1] , $q^1_{N_1} x_1^2$ [1] , and eigenvalues of

$$\begin{bmatrix} a & 0 & 0 & b \\ 0 & \bar{a} & \bar{b} & 0 \\ 0 & c & d & 0 \\ \bar{c} & 0 & 0 & \bar{d} \end{bmatrix} \quad \text{where} \quad \begin{aligned} a &= p^1 + ip^2x_1^2 \\ b &= p^3 + ip^4x_1^2 \\ c &= r^5 + r^7x_1^2 + i(r^6+r^8x_1^2) \\ d &= (q^3 + iq^4)x_1^2 \end{aligned}$$

Before considering eigenvalues corresponding to solution type (III) let us consider a possibility for ε_1 , ε_2 and ε_3 by combining (6) and (7).

Adopting the convention that eigenvalues should be positive for stability we see that:

(1) $p^1_{N_0} > 0$ is necessary for the steady-state to be bifurcate stably and that the solution bifurcates supercritically if $p^1_\lambda < 0$ when $p^1_{N_0} > 0$.

(2) The rotating wave requires $q^1_{N_0} > 0$ for stability and bifurcates supercritically when $q^1_\lambda < 0$.

This gives:

$$\epsilon_1 = -1 \ , \quad \epsilon_2 = \text{sign}\,(p^3(0)) \ , \quad \epsilon_3 = -\text{sign}\,(q^5(0)) \ . \tag{8}$$

A quick check shows that if the rotating wave and type (III) solution are to concur then $q^5(0) > 0$. Similarly, if the type (IV) solution and the rotating wave were to bifurcate simultaneously then $q^5(0) \leq 0$, at least when condition (8) is imposed.

Since we deal with the reduced equations for the type (III) solution observe that the stabilities on the three dimensional fixed point subspace are given by the eigenvalues of

$$DG_s = \begin{bmatrix} \dfrac{\partial g_0}{\partial x_0} & \dfrac{\partial g_0}{\partial x_1} \\[2ex] \dfrac{\partial g_1}{\partial x_0} & \dfrac{\partial g_1}{\partial x_1} \end{bmatrix} \tag{9}$$

Computing DG_α , with the above conditions imposed, gives

$$\begin{aligned} \text{Trace } DG_\alpha &= 3x_0^2 - x_0 - \alpha \\ \det DG_\alpha &= -2\,\mathrm{sign}(p^3(0))x_1^2 \end{aligned} \tag{10}$$

Now the standard arguments apply for additional bifurcations. In particular if Det > 0 and the Trace changes sign then a Hopf bifurcation occurs. Suppose $p^3(0) < 0$ then DG has real or complex conjugate eigenvalues. It is straightforward to show that for $\alpha < -\frac{1}{12}$ type (III) is always stable - for the reduced system. But for $\alpha \geq -\frac{1}{12}$ the trace may change sign. By the change of stabilities rule we deduce a tertiary Hopf bifurcation occurs. An interesting point here is that the a branch of the type (III) solution is pinched out as α increases there must therefore, be some means of destroying the periodic orbit. In a planar system the only possibility is the collision of the periodic orbit with one or more separatrices of the reduced equations resulting in some form of infinite periodic or "slow" drift bifurcation, c.f. Guckenheimer and Holmes [1983].

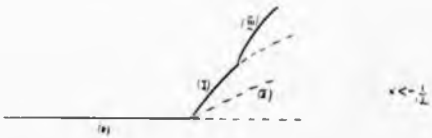


Fig. 1.

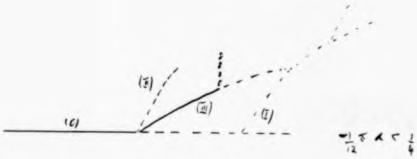


Fig. 2.

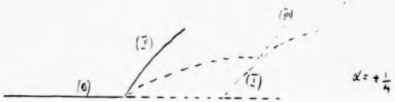


Fig. 3.

CHAPTER 2.

The O(2) equivariant steady-state/Hopf/Hopf mode interaction.

§2.0. Stability result and intended model.

Study of the O(2)-steady-state/Hopf/Hopf mode interaction is a natural extension of either of the two-mode systems. We study the system

$$\dot{x} + g(x,\lambda) = 0 \qquad (1)$$

where $g : \mathbb{R}^{10} \times \mathbb{R} \to \mathbb{R}^{10}$ and g commutes with the O(2) action of §0.1.

The assumptions (7) in §0.0 of non-resonance and Birkhoff normal form imply, §0.1, that g commutes with $O(2) \times T^2$. A further important implication of the assumption of Birkhoff normal form, for applications, is, after identifying $\mathbb{R}^{10} \cong \mathbb{C}^5$:

<u>Theorem 1</u> A solution $Z(t)$ of (1) is orbitally asymptotically stable if the eigenvalues of $(Dg)_{z_0}$ which are not zero or forced by the action of $O(2) \times T^2$ to be zero have positive real parts.

<u>Proof</u> Having made a simple adjustment for the presence of a steady-state mode the result follows from Theorem 5.1, Chossat et al (1987).

In order that such a system might be modelled we use a simple unfolding analogous to that of §1.4. In particular we split the Hopf modes by considering the unfolding $g(z,\lambda,\alpha,\beta)$ where

$$(Dg)_{(0,0,\alpha,\beta)} = \begin{bmatrix} 0 & & & & & & & & & \\ & & & I_\alpha & & & & 0 & & \\ & & & & & & & & & \\ & 0 & & & & & & I_\beta & & \end{bmatrix} \qquad (2)$$

with

$$I_\alpha = \begin{bmatrix} \alpha+i\omega_1 & & & & & & & 0 \\ & \alpha-i\omega_1 & & & & & & \\ & & \alpha+i\omega_1 & & & & & \\ 0 & & & \alpha-i\omega_1 \end{bmatrix} \quad I_\beta = \begin{bmatrix} \beta+i\omega_2 & & & & & & & 0 \\ & \beta-i\omega_2 & & & & & & \\ & & \beta+i\omega_2 & & & & & \\ 0 & & & \beta-i\omega_2 \end{bmatrix}$$

so that the splitting parameters replace the zero linear terms necessary for bifurcation at the origin.

The $O(2) \times T^2$ equivariant vector field g of (1), when restricted to fixed point subspaces of $O(2) \times T^2$, allows us to decouple the correspondingly restricted system of ordinary differential equations into phase and amplitude equations. Applying the above linear unfolding and considering the zeros of the amplitude equations leads to a great variety of solution types. The relationship between the dynamics found in the decoupled vector fields and the full ten-dimensional system depends broadly on the amount of information obtained solely from the amplitude equations. In the simplified system considered, i.e. one that commutes with T^2 to all orders, the steady-states, periodic solutions and invariant tori will persist.

§2.1. Isotropy Subgroups and Fixed Point Subspaces of $O(2) \times T^2$

By the discussion of the action of $O(2) \times T^n$ on $\mathbb{C} \oplus \mathbb{C}^{2n}$ in §0.1, $O(2) \times T^2$ can be assumed to act on $\mathbb{C} \oplus \mathbb{C}^4$ as in Table 5.

TABLE 5.

	z_0	z_1	z_2	z_3	z_4
$\kappa \in O(2)$	$\bar{z}_0$	z_2	z_1	z_4	z_3
$\theta \in O(2)$	$e^{i\ell\theta}z_0$	$e^{im\theta}z_1$	$e^{-im\theta}z_2$	$e^{in\theta}z_3$	$e^{-in\theta}z_4$
$\psi_1 \in S^1\times\{0\}$	1	$e^{i\psi_1}z_1$	$e^{i\psi_1}z_2$	1	1
$\psi_2 \in \{0\}\times S^1$	1	1	1	$e^{i\psi_2}z_3$	$e^{i\psi_2}z_4$

Also recall that by factoring out the kernel of the group action we may assume there is no common divisor for all three of ℓ, m, n .

We proceed as in §1.1 to find the isotropy subgroups and in doing so we distinguish the following subgroups of $O(2) \times T^2$

(a) $\mathbb{Z}(\theta,\psi_1,\psi_2)$ discrete subgroup generated by $(\theta,\psi_1,\psi_2) \in O(2)\backslash\{\kappa\}\times T^2$

(b) $\mathbb{Z}_\kappa(\theta,\psi_1,\psi_2)$ discrete subgroup generated by $(\kappa\theta,\psi_1,\psi_2) \in O(2)\times T^2$ (1)

(c) $S(p,q,r)$ the continuous subgroup $\{(p\theta,q\theta,r\theta) \mid \theta \in S^1,\ p,q,r \in Q\}$.

The possibility that any pair in $\{\ell,m,n\}$ may have a common divisor results in the consideration of the following cases:

$$(\ell,m) = d_1 \ ; \quad (\ell,n) = d_2 \ ; \quad (m,n) = d_3 \ . \tag{2}$$

We write

$$\ell_1 d_1 = \ell \ , \quad m_1 d_1 = m \ ; \quad \ell_2 d_2 = \ell \ , \quad n_2 d_2 = n \ ; \quad m_3 d_3 = m \ , \quad n_3 d_3 = n \ . \tag{3}$$

In the following tables we give the isotropy subgroups after the trivial action of $O(2)$ (Table 5) has been factored out. It is also <u>important</u> to realise that, with (2) in mind, we abuse notation in the following way.

If $d_1 \neq 1$, $d_2 \neq 1$ and $d_3 \neq 1$ then the actions of $O(2)$ restricted to $E_1 = \langle(z_0,z_1,z_2)\rangle$, $E_2 = \langle(z_0,z_3,z_4)\rangle$ or $E_3 = \langle(z_1,z_2,z_3,z_4)\rangle$ have non-empty kernels, K_i, $n = 1,\ldots,3$. Therefore, on a fixed point subspace with appropriate coordinates set to zero we think of the isotropy subgroups as a subgroup of $(O(2) \times T^2)|E_i/K_i$. Clearly the above choice is not unique, consider the possibility $\mathrm{Fix}(\Sigma) \subset E_i \cap E_j$, $i \neq j$.

In this case set $K_{i,j}$ to be the kernel of the action of $O(2)$ restricted to $E_i \cap E_j$. From (3) we have

$$(d_i,d_j) = 1 \quad i \neq j \tag{4}$$

otherwise the triple (ℓ,m,n) would have a common divisor.

$$(4) \Rightarrow d_3|m_1 , \quad d_2|n_3 . \tag{5}$$

We write

$$m = m_4 d_1 d_3 , \quad n = n_4 d_2 d_3 . \tag{6}$$

In this case we think of the isotropy subgroup as a subgroup of $O(2) \times T^2/K_{i,j}$.

TABLE 6.

Isotropy subgroups and fixed point subspaces

Σ	Fix(Σ)	dim Fix(Σ)
0. $O(2)\times T^2$	$\{0\}$	
1. $\mathbb{Z}_\kappa \times T^2 \ltimes \mathbb{Z}(\frac{2\pi}{\ell}, 0, 0)$	$\mathrm{Im}(z_0)=z_1=z_2=z_3=z_4 = 0$	1
2.(a) $S(1,-m_4,0)\times S(0,0,1)$	$z_0=z_2=z_3=z_4 = 0$	2
(b) $S(1,0,-n_4)\times S(0,1,0)$	$z_0=z_1=z_2=z_4 = 0$	2
3.(a) $\mathbb{Z}_\kappa \ltimes \mathbb{Z}(\frac{\pi}{m_4},\pi,0)\times S(0,0,1)$	$z_0=z_3=z_4 = 0,\ z_1 = z_2$	2
(b) $\mathbb{Z}_\kappa \ltimes \mathbb{Z}(\frac{\pi}{n_4},0,\pi)\times S(0,1,0$	$z_0=z_1=z_2 = 0,\ z_3 = z_4$	2
4.(a) $\mathbb{Z}_\kappa\times S(0,0,1)$ $[\times\mathbb{Z}(\pi,m_1\pi,0)]$ ℓ_1 even	$\mathrm{Im}(z_0)=z_3=z_4 = 0,\ z_1 = z_2$	3
(b) $\mathbb{Z}_\kappa\times S(0,1,0)$ $[\times\mathbb{Z}(\pi,0,n_2\pi)]$ ℓ_2 even	$\mathrm{Im}(z_0)=z_1=z_2 = 0\ ,\ z_3 = z_4$	3
5.(a) $\mathbb{Z}_\kappa(0,\pi,0)\times S(0,0,1)[\times\mathbb{Z}(\pi,m_1\pi,0)]$ $4\nmid\ell_1$ ℓ_1 even	$\mathrm{Im}(z_0)=z_3=z_4 = 0,\ z_1 = -z_2$	3
(b) $\mathbb{Z}_\kappa(0,0,\pi)\times S(0,1,0)[\times\mathbb{Z}(\pi,0,n_2\pi)]$ $4\nmid\ell_1$ ℓ_2 even	$\mathrm{Im}(z_0)=z_1=z_2 = 0,\ z_3 = -z_4$	3
6. $S(1,-m_3,-n_3)$	$z_0=z_2=z_4 = 0$	4
7. $S(1,-m_3,\ n_3)$	$z_0=z_2=z_3 = 0$	4
8. $\mathbb{Z}_\kappa \times \mathbb{Z}(\pi,m_3\pi,n_3\pi)$	$z_0 = 0,\ z_1=z_2,\ z_3=z_4$	4
9. $\mathbb{Z}_\kappa(0,0,\pi) \times \mathbb{Z}(\pi,m_3\pi,n_3\pi)$ (n_3 even)	$z_0 = 0,\ z_1=z_2,\ z_3 = -z_4$	4
10. $\mathbb{Z}_\kappa(0,\pi,0) \times \mathbb{Z}(\pi,m_3\pi,n_3\pi)$ (m_3 even)	$z_0 = 0,\ z_1 = -z_2,\ z_3 = z_4$	4
11.(a) $\mathbb{Z}(\frac{2\pi}{\ell_1}, \frac{2\pi m_1}{\ell_1},0)\times S(0,0,1)[\times\mathbb{Z}(\pi,m_1\pi,0)]$ $\ell_1>2$ ℓ_1 even	$z_2=z_3=z_4 = 0$	4
(b) $\mathbb{Z}(\frac{2\pi}{\ell_2},0\ \frac{2\pi n_2}{\ell_2})\times S(0,1,0)[\times\mathbb{Z}(\pi,0,n_2\pi)]$ $\ell_2>2$ ℓ_2 even	$z_1=z_2=z_4 = 0$	4

TABLE 6 (ctd.)

12.(a)	$\mathbb{Z}(\frac{\pi}{m_4},\pi,0) \times S(0,0,1)$	$z_0=z_3=z_4 = 0$	4
(b)	$\mathbb{Z}(\frac{\pi}{n_4},0,\pi) \times S(0,1,0)$	$z_0=z_1=z_2 = 0$	4
13.*	$\mathbb{Z}_\kappa \times \mathbb{Z}(\pi,m\pi,n\pi)$ ℓ even	$\mathrm{Im}(z_0)=0$, $z_1=z_2$, $z_3=z_4$	5
14.*	$\mathbb{Z}_\kappa(0,\pi,0) \times \mathbb{Z}(\pi,m\pi,n\pi)$ (if $4\mid\ell$ then m even) $4\nmid\ell$ ℓ even	$\mathrm{Im}(z_0)=0$, $z_1=-z_2$, $z_3=z_4$	5
15.*	$\mathbb{Z}_\kappa(0,0,\pi) \times \mathbb{Z}(\pi,m\pi,n\pi)$ (if $4\mid\ell$ then n even) $4\nmid\ell$ ℓ even	$\mathrm{Im}(z_0)=0$, $z_1=z_2$, $z_3=-z_4$	5
16.*	$\mathbb{Z}_\kappa(0,\pi,\pi) \times \mathbb{Z}(\pi,m\pi,n\pi)$ $4\nmid\ell$ ℓ even	$\mathrm{Im}(z_0)=0$, $z_1=-z_2$, $z_3=-z_4$	5
17.	$\mathbb{Z}(\frac{\pi}{m_3},\pi,\frac{n_3}{m_3}\pi)$ $(m_3\neq 1)$	$z_0 = z_3 = 0$	6
18.	$\mathbb{Z}(\frac{\pi}{n_3}, \frac{m_3}{n_3}\pi,\pi)$ $(n_3\neq 1)$	$z_0=z_1 = 0$	6
19.	$\mathbb{Z}(\frac{2\pi}{\ell}, -\frac{2m\pi}{\ell}, \frac{2n\pi}{\ell})$, $\ell>2$	$z_2=z_3 = 0$	6
20.	$\mathbb{Z}(\frac{2\pi}{\ell}, \frac{2m\pi}{\ell}, \frac{2n\pi}{\ell})$, $\ell>2$	$z_2=z_4 = 0$	6
21.(a)	$\mathbb{Z}(\pi,m_1\pi,0)$ ℓ_1 even	$z_3=z_4 = 0$	6
(b)	$\mathbb{Z}(\pi,0,n_2\pi)$ ℓ_2 even	$z_1=z_2 = 0$	6
22.	$\mathbb{Z}(\pi,m\pi,n\pi)$ ℓ odd	$z_0 = 0$	8
	ℓ even	$\mathbb{C}^5$	10

Notes: * If ℓ even then $\mathbb{Z}_\kappa(0,\pi,\pi) = \mathbb{Z}_\kappa(\pi,0,0)$ and furthermore for $4\mid\ell$

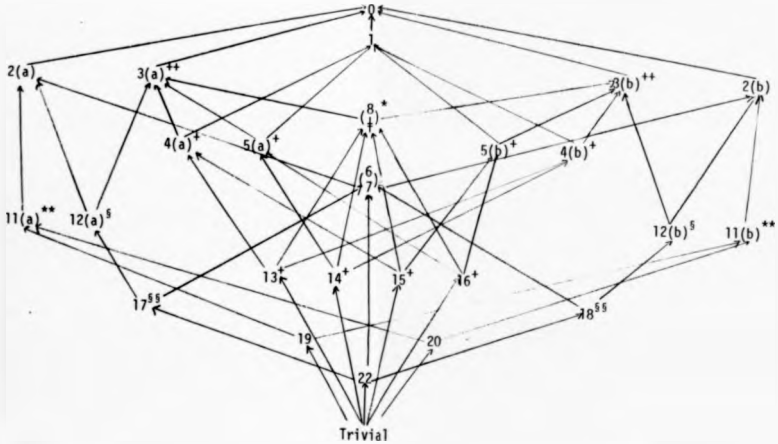
14 = 16 and 13 = 15 when m is even

13 = 14 , 15 = 16 when n is even

13 = 14 = 15 = 16 when n,m odd (for '=' read is conjugate to)

TABLE 7.

Lattice of isotropy subgroups



Containment in Table 7 indicates containment in a representative of a conjugacy class of a coset.

Legends Table 7.

++ 3(a) coincides with 4(a) when $m_1 = 1$ and ℓ_1 even.

3(b) coincides with 4(b) when $m_2 = 1$ and ℓ_2 even.

\+ 4(a) coincides with 5(a) $4 \mid \ell_1$ and $13 = 14$, $15 = 16$.

4(b) coincides with 5(b) $4 \mid \ell_2$ and $13 = 15$, $14 = 16$.

* 8 and $\neq$ not present if $m_1 = 1$ and ℓ_1 even or $m_2 = 1$ and ℓ_2 even.

$\neq$ 9 if n_3 even, 10 if m_3 even.

** 11(a) absent if $\ell_1 \leq 2$. 11(b) absent if $\ell_2 \leq 2$.

§ 12(a) only 'contained' in 2(a) if 3(a) coincides with 4(a).

12(b) only 'contained' in 2(b) if 3(b) coincides with 4(b).

§§ 17 absent if $m_3 = 1$, 18 absent if $n_3 = 1$.

General rule. If an isotropy subgroup is absent then containment just passes along the table in the natural way. e.g. if 17 is absent then 12 is contained directly in 21.

Finally since 21(a) and (b) represent a trivial action of $O(2) \times T^2$ from which we expect to obtain no information we do not include them in Table 7.

To verify Tables 6(a), (b) and (c) it is appropriate to adopt a similar procedure as that for Table 2 §1.1. We enumerate subspaces of $\mathbb{C}^5$ and then check 'algorithmically' which are fixed - eliminating those made redundant by conjugacy i.e. if $\dim \mathrm{Fix}(\Sigma_1) = \dim \mathrm{Fix}(\Sigma_2)$ and $\mathrm{Fix}(\Sigma_2) \in \Gamma\cdot\mathrm{Fix}(\Sigma_1)$ then Σ_1 is conjugate to Σ_2. Entries 1-5(b) follow immediately from Table 2. For the remainder we have $\dim \mathrm{Fix}(\Sigma) = 4$ and either $\kappa \in \Sigma$ or not. First consider the case $\kappa \notin \Sigma$.

To begin suppose $\dim \mathrm{Fix}(\Sigma) = 4$ and the coordinate $z_0 \neq 0$ then entries 11(a)-12(b) follow from Table 2. Now suppose $\dim \mathrm{Fix}(\Sigma) = 4$ and $z = 0$. Writing the action of $(\theta,\psi_1,\psi_2) \in O(2) \times T^2/K_3$ explicitly

$$(\theta,\psi_1,\psi_2)\cdot z = (0, e^{im_3\theta+i\psi_1}z_1, e^{-im_3\theta+i\psi_1}z_2, e^{in_3\theta+i\psi_2}z_3, e^{-in_3\theta+i\psi_2}z_4). \quad (7)$$

Checking through the consequences for (θ,ψ_1,ψ_2) if $z_2 = z_4 = 0$ or $z_2 = z_3 = 0$ gives entries 6 and 7. If $z_3 = z_4 = 0$ then we may also factor through the action by K_1 thus verifying 12(a). Similarly we check 12(b). Suppose now, that $\dim \mathrm{Fix}(\Sigma) > 4$ and $z_0 = 0$.

If $z_1, z_2 \neq 0$ then (7) $\Rightarrow e^{2im_3\theta} = 1$ so that $\theta = \frac{\pi}{m_3}$. Which in turn implies $\psi_1 = \pi$ and $\psi_2 = \pm\frac{n_3}{m_3}\pi$ according as $z_3 = 0$ or $z_4 = 0$. But we see that choosing both is ruled out by conjugacy - just apply κ. This verifies entry 17 and similarly for 18. Entry 21 is obvious.

Next consider $\dim \mathrm{Fix}(\Sigma) > 4$ and $z_0 \neq 0$. Again write the action

of $(\theta,\psi_1,\psi_2) \in O(2)\ltimes \times T^2$ explicitly

$$(\theta,\psi_1,\psi_2)\cdot z = (e^{i\ell\theta}z_0, e^{i(m\theta+\psi_1)}z_1, e^{-im\theta+i\psi_1}z_2, e^{in\theta+i\psi_2}z_3, e^{-in\theta+i\psi_2}z_4) \quad (8)$$

and by inspection we check 19-21(b).

Now consider $\dim \mathrm{Fix}(\Sigma) \geq 4$ with $\kappa \in \Sigma$. Begin by noticing that we may assume z_0 is real by rotation through $\theta \in O(2)\ltimes$. If $z_0 = 0$ we have the explicit action of $O(2) \times T^2/K_3$ is

$$\kappa'(\theta,\psi_1,\psi_2)\cdot z = (0, e^{i(m_3\theta+\psi_1)}z_2, e^{-im_3\theta+i\psi_1}z_1, e^{in_3\theta+i\psi_2}z_4, e^{-in_3\theta+i\psi_2}z_3). \quad (9)$$

As in the verification of Table 2 we deduce from (9) that the fixed point subspace takes the form $(0,z_1,\pm z_1,z_3,\pm z_3)$. Checking for conjugacy i.e. $(\theta,\psi_1,\psi_2) \in O(2)\ltimes \times T^2$ s.t.

$$(\theta,\psi_1,\psi_2)\,(0,z_1,\pm z_1,z_3,z_3) = (0,z_1,\pm z_1,z_3,-z_3) \quad . \qquad (10)$$

Using (7) we obtain

$$2n_3k = m_3(2p+1) \qquad k,p \in \mathbb{Z} \qquad (11)$$

which tells us there is no conjugacy for n_3 even. This verifies entries 8 and 9. Entry (10) follows similarly.

For 13-16 we first show the fixed point subspaces take the form $(x_0,z_1,\pm z_1,z_3,\pm z_3)$. Which follows in the same manner as above. The

presence of non-zero z_0 rules out the possibility of conjugacies except when $\ell|4$. We conjugate by

(i)	$(\frac{\pi}{2}, \frac{-m\pi}{2}, \frac{-n\pi}{2})$	m,n odd	
(ii)	$(\frac{\pi}{2}, 0, \frac{-n\pi}{2})$	m even	(12)
(iii)	$(\frac{\pi}{2}, \frac{-m\pi}{2}, 0)$	n even .	

<u>Note</u> ℓ,m,n can <u>not</u> all be even since we assume no common divisor.

§2.2. <u>The $O(2) \times T^2$ Invariants and Equivariants</u>.

We have already seen that in the group theory we must consider the divisors of the pairs in the triple (ℓ,m,n) and, of course, the same occurs in the computation of $O(2) \times T^2$ equivariant normal forms. (This makes the calculations and in particular the statement of results a little messy.)

We begin by calculating the invariants and then use Lemma 3 of §1.2 - with an appropriate extension to include $\mathbb{C}_0$ equivariants - to obtain the equivariants.

<u>Proposition 1</u> The ring of smooth $O(2) \times T^2$ invariant (germs of) maps $\mathbb{C}^5 \to \mathbb{R}$ has a Hilbert basis:

$N_0 = u_0$, $N_1 = u_1+u_2$, $N_2 = u_3+u_4$, $M_1 = u_1u_2$, $M_2 = u_3u_4$, $C_1 = u_1u_4+u_2u_3$,

$O_1 = \mathrm{Re}[z_0^\alpha(\bar{z}_1z_2)^\beta]$, $O_2 = \delta_1\mathrm{Im}[z_0^\alpha(\bar{z}_1z_2)^\beta]$, $O_3 = \mathrm{Re}[z_0^\gamma(\bar{z}_3z_4)^\sigma]$,

$O_4 = \delta_2\mathrm{Im}[z_0^\gamma(\bar{z}_3z_4)^\sigma]$, $O_5 = \mathrm{Re}[(z_1\bar{z}_2)^\mu(\bar{z}_3z_4)^\eta]$, $O_6 = (\delta_1+\delta_2)\mathrm{Im}[(z_1\bar{z}_2)^\mu(\bar{z}_3z_4)^\eta]$,

$$O_{7,k} = \mathrm{Re}[z_0^{\gamma-k\alpha}(z_1\bar{z}_2)^{k\beta}(\bar{z}_3z_4)^{\sigma}] \ , \ O_{8,k} = (\delta_1+\delta_2)\mathrm{Im}[z_0^{\gamma-k\alpha}(z_1\bar{z}_2)^{k\beta}(\bar{z}_3z_4)^{\sigma}]$$

$$O_{7,k'} = \mathrm{Re}[z_0^{k'\alpha-\gamma}(\bar{z}_1z_2)^{k'\beta}(z_3\bar{z}_4)^{\sigma}] \ , \ O_{8,k'} = (\delta_1+\delta_2)\mathrm{Im}[z_0^{k'\alpha-\gamma}(\bar{z}_1z_2)^{k'\beta}(z_3\bar{z}_4)^{\sigma}]$$

where for definiteness we have $\gamma > \alpha$ with $k \in \delta = \{k \in \mathbb{Z}^+ | \gamma-k\alpha > 0\}$ and k' the least integer s.t. $\gamma-k'\alpha < 0$, and if $j < k'$ s.t. $\gamma = j\alpha$ then $O_{i,j},\ldots,O_{i,k'}$; $i = 7,8$; are redundant, finally

$$\alpha = \begin{cases} 2m_1 & \ell_1 \text{ odd} \\ m_1 & \ell_1 \text{ even} \end{cases} ; \quad \beta = \begin{cases} \ell_1 & \ell_1 \text{ odd} \\ \ell_{1/2} & \ell_1 \text{ even} \end{cases} ; \quad \gamma = \begin{cases} 2n_2 & \ell_2 \text{ odd} \\ n_2 & \ell_2 \text{ even} \end{cases} ;$$

$$\sigma = \begin{cases} \ell_2 & \ell_2 \text{ odd} \\ \ell_{2/2} & \ell_2 \text{ even} \end{cases} ; \qquad \mu = n_3 \ , \qquad \eta = m_3 \ ,$$

and $\delta_1 = u_1-u_2$, $\delta_2 = u_3-u_4$.

<u>Note</u>: we assumed for definiteness that $\gamma > \alpha$, if for a particular action of $O(2) \times T^2$ we have $\alpha > \gamma$ then $O_{7,k} = \mathrm{Re}[z_0^{\alpha-k\gamma}(\bar{z}_1z_2)^{\beta}(z_3\bar{z}_4)^{k\sigma}]$ and similarly for $O_{8,k}$, $O_{7,k'}$ and $O_{8,k'}$.

<u>Proof of Proposition 1</u> We begin, as usual, by writing down the general form of a polynomial $G:\mathbb{C}^5 \to \mathbb{R}$ and consider restrictions imposed by invariance under $O(2) \times T^2$. We have, in multi-index notation,

$$G(z) = \Sigma A_{a,b} Z^a \bar{Z}^b \tag{1}$$

and

$$G(\gamma Z) = G(Z) \qquad \forall \ \gamma \in O(2) \times T^2 \ . \tag{2}$$

Applying $\theta \in O(2)\rtimes\kappa$ gives, after comparing coefficients,

$$\ell(a_0-b_0) + m(a_1-b_1-a_2+b_2) + n(a_3-b_3-a_4+b_4) = 0 \tag{3}$$

and applying $\varphi_1 \in S^1 \times \{0\}$, $\varphi_2 \in \{0\} \times S^1$ gives

$$\begin{aligned} &\text{(i)} \quad a_1-b_1 + a_2-b_2 = 0 \\ &\text{(ii)} \quad a_3-b_3 + a_4-b_4 = 0 \quad . \end{aligned} \tag{4}$$

Combining (3), (4)(i) and (i) we have

$$\ell(a_0-b_0) + 2m(a_1-b_1) + 2n(a_2-b_2) = 0 \ . \tag{5}$$

Setting $a_2-b_2 = 0$ in (5) gives

$$\ell_1 d_1(a_0-b_0) + 2m_1 d_1(a_1-b_1) = 0 \tag{6}$$

and we return to Proposition 1 §1.2 with the pair (ℓ_1,m_1) . Similarly setting $a_1-b_1 = 0$ in (5) and we consider the pair (ℓ_2,n_2) . This gives $N_0,\ldots,M_2$ and $O_1,\ldots,O_4$ as invariant generators. Using the claims in §1.2 it is easy to see that C_1 completes the list of 'simple' invariants. Now, setting $a_0 = b_0$ in (5) we have

$$m_3 d_3(a_1-b_1) + n_3 d_3(n_3-b_3) = 0 \tag{7}$$

since $(m_3,d_3) = 1$ (7) =>

$$\begin{aligned} &\text{(i)} \quad a_1 b_1 = kn_3 \\ &\qquad\qquad\qquad k \in \mathbb{Z} \ . \\ &\text{(ii)} \quad a_3-b_3 = -km \end{aligned} \tag{8}$$

The same reasoning as in Proposition 1 §1.2 now gives the invariant generators O_5 and O_6 .

Finally, assume $a_i \neq b_i$, $i = 0,1,2$ in (5). Recall that finding monomial invariant generators for $O(2)\ltimes \times T^2$ is equivalent to finding a minimum spanning set for the subset of $(\mathbb{Z}^+)^{10}$ defined by (3), (4)(i) and (ii). So far we have the following generators:

$$u_i = (0,\ldots,\underset{i^{th}}{1},\underset{i+1^{st}}{1},0,\ldots 0) \qquad i = 0,\ldots, 4 .$$

$$v_1 = (\alpha,0,0,\beta,\beta,0\ldots,0) \; ; \quad v_2 = (0,\alpha,\beta,0,0,\beta,0,\ldots 0) \; ;$$

$$v_3 = (\gamma,0,\ldots\ldots,0,\sigma,\sigma,0); \; v_4 = (0,\gamma,0,\ldots,0,\sigma,0,0,\sigma) \; , \tag{9}$$

$$v_5 = (0,0,\mu,0,0,\mu,0,\eta,\eta,0), \; v_6 = (0,\ldots,0,\mu,\mu,0,\eta,0,0,\eta) \; .$$

Now we check through the possibilities in (5). First notice that we can <u>not</u> have $a_i - b_i > 0 \;\; \forall\, i$ so begin by assuming $a_0 > b_0$. (10)

Since any positive linear combinations of u_0 in (9) contained in the solution space of (5) are not contributing anything new we may suppose $b_0 = 0$ in (10). Suppose in addition that

$$\begin{aligned} &\text{(i)} \qquad a_1 - b_1 < 0 \\ &\text{(ii)} \qquad a_2 - b_2 < 0 \; . \end{aligned} \tag{11}$$

After assuming that $a_1 - a_2 = 0$, as above, it is easily seen that a solution set of this type is spanned by v_1, v_3 . Similarly, if $a_0 < b_0$, $a_1 > b_1$, $a_2 > b_2$ in (5) then the solution is spanned by u_i , $i = 0,2,4$ and v_2, v_4 . Next assume

$$\begin{aligned} &\text{(i)} && a_0 > b_0 \\ &\text{(ii)} && a_1 > b_1 \\ &\text{(iii)} && a_2 < b_2 \ . \end{aligned} \tag{12}$$

Writing

$$S_1 = \{j,k \in \mathbb{Z}^+ \text{ s.t. } j\gamma - k\alpha > 0\} \ . \tag{13}$$

Inspection shows that solution types (12) in (5) are of the form

$$w_{k,j} = (j\gamma - k\alpha, 0, k\beta, 0, 0, k\beta, 0, j\sigma, j\sigma, 0) \ ; \ k,j \in S_1 \ . \tag{14}$$

Replacing (12)(ii) and (iii) with

$$\begin{aligned} &\text{(i)} && a_1 < b_1 \\ &\text{(ii)} && a_2 > b_2 \end{aligned} \tag{15}$$

and writing $S_2 = \{j,k \in \mathbb{Z}^+ \text{ s.t. } j\gamma - k\alpha < 0\}$ we obtain solutions of the form

$$w'_{k,j} = (k\alpha - j\gamma, 0, 0, k\beta, k\beta, 0, j\sigma, 0, 0, j\sigma) \ ; \ k,j \in S_2 \tag{16}$$

Now for definiteness we set

$$\gamma > \alpha \tag{17}$$

writing

$$w_{k,j} = w_{k,1} + (j-1)v_3 \tag{18}$$

we see that if k' is the least integer s.t. $\gamma - k'\alpha < 0$ and there is <u>no</u> $k < k'$ s.t. $\gamma = k\alpha$ then there are $(k'-1)$ generators of type (14).

Also notice that $w'_{k',i}$ are linear combinations of $w'_{k',1}$ and v_1 and that $w'_{kj,j}$ are linear combinations of $w'_{k',1}$ and v_1. Therefore, we need only consider $w_{1,1}, \ldots, w_{k'-1,1}$ and $w_{k',1}$ as generators for solution types (12).

Notice if $k < k'$ s.t. $\gamma = k\alpha$ (i.e. $k'-1 = k$) then the generators $w_{k'-1,1}$ and $w'_{k',1}$ are redundant.

Finally, replacing (12)(i) with $a_0 < b_0$ and following the above gives invariant generators that are just complex conjugates of (14) and (16).

□

<u>Theorem 1</u> The $O(2) \times T^2$ equivariant smooth (germs of) maps $\mathbb{C}^5 \to \mathbb{C}^5$ take the form

$$G(z) = (g_0(z), g_1(z), g_2(z), g_3(z), g_4(z))$$

where

$$g_0(z) = P^1 z_0 + P^4 \bar{z}_0^{\alpha-1}(z_1\bar{z}_2)^\beta + P^7 \bar{z}_0^{\gamma-1}(\bar{z}_3 z_4)^\sigma + \sum_{k\in S} P_k^{10} \bar{z}_0^{\gamma-k\alpha-1}(\bar{z}_1 z_2)^{12\beta}(z_3\bar{z}_4)^\sigma$$
$$+ P_{k'}^{10} \bar{z}_0^{k'\alpha-\gamma-1}(z_1\bar{z}_2)^{k'\beta}(\bar{z}_3 z_4)^\sigma$$

$$g_1(z) = Q^1 z_2 + Q^7 z_0^\alpha(\bar{z}_1 z_2)^{\beta-1} z_2 + Q^{13}(\bar{z}_1 . z_2)^{\mu-1}(z_3\bar{z}_4)^\eta z_2$$
$$+ \sum_{k\in S} Q_k^{19} \bar{z}_0^{\gamma-k\alpha}(\bar{z}_1 z_2)^{k\beta-1}(z_3\bar{z}_4)^\sigma z_2 + (Q_{k'}^{19} z_0^{k'\alpha-\gamma}(\bar{z}_1 z_2)^{k'\beta-1}(z_3\bar{z}_4)^\sigma z_2$$

$$g_3(z) = R' z_3 + R^7 z_0(\bar{z}_3 z_4)^{\sigma-1} z_4 + R^{13}(z_1\bar{z}_2)^\mu(\bar{z}_3 z_4)^{\eta-1} z_4$$
$$\sum_{k\in S} R_k^{19} z_0^{\gamma-k\alpha}(z_1\bar{z}_2)^{k\beta}(\bar{z}_3 z_4)^{\sigma-1} z_4 + R_{k'}^{19} \bar{z}_0^{k'\alpha-\gamma}(z_1\bar{z}_2)^{k'\beta}(\bar{z}_3 z_4)^{\sigma-1} z_4$$

$$g_1(\kappa z) = g_2(z) \ , \quad g_3(\kappa z) = g_4(z) \ ,$$

where $S = \{k \mid \gamma - k\alpha > 0 \ , \gamma, k, \alpha \ \ \mathbb{Z}^+\}$ if $(k'-1)\alpha = \gamma$ (i.e. $k'-1 \notin S$) then set $P^{10}_{k'} \equiv Q^{19}_{k'} \equiv R^{19}_{k'} \equiv 0$.

$\alpha, \beta, \gamma, \delta, \mu, n$ and k' are defined in Proposition 1.

$$p^j = p^j + i\delta_1 p^{j+1} + i\delta_2 p^{j+2} \qquad P^{10}_k = p^{10}_k + i\delta_1 p^{11}_k + i\delta_2 p^{12}_k \ .$$

$$Q^j = (q^j + iq^{j+1} + u_2(q^{j+2} + iq^{j+3}) + u_4(q^{j+4} + iq^{j+5})$$

$$Q^{19}_k = (q^{19}_k + iq^{20}_k + u_2(q^{21}_k + iq^{22}_k) + u_4(q^{23}_k + iq^{24}_k) \ .$$

$$R^j = r^j + ir^{j+1} + u_2(r^{j+2} + ir^{j+3}) + u_4(r^{j+4} + ir^{j+5})$$

$$R^{19}_k = r^{19}_k + ir^{20}_k + u_2(r^{21}_k + ir^{22}_k) + u_4(r^{23}_k + ir^{29}_k) \ ,$$

and $\delta_1 = u_1 - u_2$, $\delta_2 = u_3 - u_4$.

<u>Proof</u> The equivariants corresponding to g_1, g_2, g_3 and g_4 follow from Proposition 1 and application of Lemma 3 §1.3. To check for equivariants corresponding to g_0 we must again consider those awkward polynomials which are $O(2)\ltimes \times T^2$ invariant but κ equivariant. Again we can use the claim in §1.3 to check that there are only two of these namely $i\delta_1$ and $i\delta_2$. After taking linear combinations we are done. □

§2.3. <u>Amplitude/Phase Equations and (corresponding) Eigenvalues</u>.

As might be expected analysis of $G|\mathrm{Fix}(\Sigma)$ can only be taken so far before it becomes more realistic to consider the results on a case by case basis. In §3 we consider an application to the Taylor-Couette

where it turns out that setting the triple (ℓ,m,n) to $(m,m,m+1)$ is of particular interest. With this in mind we concentrate on this case whenever it is necessary to consider a particular case.

In the following tables we begin by restricting the general $O(2) \times T^2$ equivariant form to the fixed point subspaces - this gives us the reduced O.D.E. We then consider the special case $(m,m,m+1)$, with m odd, giving the amplitude/phase equations and the eigenvalues associated with the zeros of these amplitude equations.

<u>Notes</u>: (1) For convenience we adopt the shorthand convention of writing $g^1(R^1)$ which we think of as evaluated on a fixed point subspace.

(2) In Table 11 we give:

(a) The Taylor expansion about zero when appropriate.

(b) In the difficult solutions 12-15 we highlight only necessary conditions for stability with more information included in the calculations.

TABLE 8.

Reduce O.D.E.

1.	$\dot{z}_0 = p^1 x_0$	steady-state
2.(a)	$\dot{z}_1 = (q^1 + iq^2)z_1$	(m_4)rotating wave
(b)	$\dot{z}_3 = (r^1 + ir^2)z_3$	(n_4)-rotating wave
3.*(a)	$\dot{z}_1 = (q^1 + iq^2)z_1$	(m_4)-standing wave
(b)	$\dot{z}_3 = (r^1 + ir^2)z_3$	(n_4)-standing wave
4.(a)	$\dot{z}_0 = p^1 x_0 + p^4 x_0^{\alpha-1} u_1^{\beta}$	2-torus
	$\dot{z}_1 = (q^1 + iq^2 + u_1(q^3+iq^4))z_1 + (q^7 + iq^8 + u_1(q^9+iq^{10}))x_0^{\alpha} u_1^{\beta-1} z_1$	
(b)	$\dot{z}_0 = p^1 x_0 + p^7 x_0^{\gamma-1} u_3^{\gamma}$	2-torus
	$\dot{z}_3 = (r^1 + ir^2 + u_3(r^5+ir^6))z_3 + (r^7 + ir^8 + u_3(r^{11}+ir^{12}))x_0^{\gamma} u_3^{\sigma-1} z_3$	
5.(a)	$\dot{z}_0 = p^1 x_0 + (-1)^{\beta} p^4 x_0^{\alpha-1} u_1^{\beta}$	2-torus
	$\dot{z}_1 = (q^1 + iq^2 + u_1(q^3+iq^4))z_1 + (-1)^{\beta} (q^7 + iq^8 + u_1(q^9+iq^{10})x_0^{\alpha} u_1^{\beta-1} z_1$	
(b)	$\dot{z}_0 = p^1 x_0 + (-1)^{\gamma} p^7 x_0^{\gamma-1} u_3^{\sigma}$	2-torus
	$\dot{z}_3 = (r^1 + ir^2 + u_3(r^5+ir^6)z_3) + (-1)\ (r^7 + ir^8 + u_3(r^{11}+ir^{12}))x_0^{\gamma} u_3^{\sigma-1} z_3$	
6.	$\dot{z}_0 = 0$	2-torus (superimposed rotating waves)

TABLE 8. (ctd.)

	$\dot{z}_1 = (q^{\bar{1}}+iq^{\bar{2}})z_1$	
	$\dot{z}_3 = (r^1+ir^2)z_3$	
7.	$\dot{z}_0 = 0$	2-torus (interpenetrating rotating waves)
	$\dot{z}_1 = (q^1 + iq^2 + u_4(q^5+iq^6))z_1$	
	$\dot{z}_4 = (r^1 + ir^2)z_4$	
8.**	$\dot{z}_0 = 0$	2-torus (superimposed standing waves)
	$\dot{z}_1 = (Q^1 + Q^{13}\, u_1^{\mu-1}u_3^{\eta}\,)z_1$	
	$\dot{z}_3 = (R^1 + R^{13}\, u_1^{\mu}u_3^{\eta-1})z_3$	
9.**	$\dot{z}_0 = 0$	2-torus (interpenetrating standing waves)
	$\dot{z}_1 = (Q^1 + (-1)^{\eta}\, Q^{13}u_1^{\mu-1}u_3^{\eta})z_1$	
	$\dot{z}_3 = (R^1 + (-1)^{\eta}\, R^{13}u_1^{\mu}u_3^{\eta-1})z_3$	
10.**	$\dot{z}_0 = 0$	2-torus (interpenetrating standing waves)
	$\dot{z}_1 = (Q^1 + (-1)^{\mu}Q^{13}u_1^{\mu-1}u_3^{\eta})z_1$	
	$\dot{z}_3 = (R^1 + (-1)^{\mu}\, R^{13}u_1^{\mu}u_3^{\eta-1})z_3$	

TABLE 8. (ctd.)

11.(a) $\dot{z}_0 = (p^1 + ip^2u_1)z_0$ 2-torus

$\dot{z}_1 = (q^1 + iq^2)z_1$

$\dot{z}_2 = \dot{z}_3 = \dot{z}_4 = 0$

(b) $\dot{z}_0 = (p^1 + ip^2u_3)z_0$ 2-torus

$\dot{z}_3 = (r^1 + ir^2)z_3$

$\dot{z}_1 = \dot{z}_2 = \dot{z}_4 = 0$

12.(a) $\dot{z}_0 = \dot{z}_3 = \dot{z}_4 = 0$ 2-torus

$\dot{z}_1 = ((q^1+q^3u_2) + i(q^2+q^4u_2))z_1$

$\dot{z}_2 = \kappa\dot{z}_1$

(b) $\dot{z}_0 = \dot{z}_1 = \dot{z}_2 = 0$ 2-torus

$\dot{z}_3 = ((r^1+r^5u_4) + i(r^2+r^6u_4))z_5$

$\dot{z}_4 = \kappa\dot{z}_3$

13,14,15 &16.

$$\dot{z}_0 = p^1x_0+a_0p^4x_0^{\alpha-1}u_1^{\beta} + b_0p^7x_0^{\gamma-1}u_3^{\sigma} + \sum_{k\epsilon S} c_0p_k^{10}x_0^{\gamma-k\alpha-1}u_1^{k\beta}\, u_3^{\sigma} + d_0p_{k'}^{10}x_0^{k'\alpha-\gamma-1}u_1^{k'\beta}u_3^{\sigma}$$

$$\dot{z}_1 = (Q^1+a_1Q^7x_0^{\alpha}u_1^{\beta-1} + b_1Q^{13}u_1^{\mu-1}u_3^{\eta} + \sum_{k\epsilon S} c_1Q_k^{19}x_0^{\gamma-k\alpha}u_1^{k\beta-1}u_3 + d_1Q_{k'}^{19}x_0^{k'\alpha-\gamma-1}u_1^{k'\beta-1}u_3^{\sigma})z_1 \quad \text{3-torus}$$

$$\dot{z}_3 = (R^1+a_2R^7x_0^{\gamma}u_3^{\sigma-1} + b_2R^{13}u_1^{\mu}u_3^{\eta-1} + \sum_{k\epsilon S} c_2R_k^{19}x_0^{\gamma-k\alpha}u_1^{k\beta}u_3^{\sigma-1} + d_2R_{k'}^{19}x_0^{k'\alpha-\gamma}u_1^{k'\beta}u_3^{\sigma-1})z_3$$

where

13. $a_0 = \ldots\ldots = d_2 = 1$

TABLE 8. (ctd.)

14. $a_0 = a_1 = (-1)^{\beta}$ $b_0 = a_2 = 1$, $c_0 = c_1 = c_2 = (-1)^{k\beta}$, $d_0 = d_1 = d_2 = (-1)^{k'\beta}$, $b_1 = b_2 = (-1)^{\mu}$

15. $a_0 = a_1 = 1$, $b_0 = a_2 = \ldots. = d_2 = (-1)^{\sigma}$, $b_1 = b_2 = (-1)^{\eta}$

16. $a_0 = a_1 = (-1)^{\beta}$, $b_0 = a_2 = (-1)^{\sigma}$, $c_0 = c_1 = c_2 = (-1)^{k\beta+\sigma}$, $d_0 = d_1 = d_2 = (-1)^{k'\beta+\sigma}$, $b_1 = b_2 = (-1)^{\mu+\eta}$

17.* $\dot{z}_0 = \dot{z}_3 = 0$ 3-torus

$\dot{z}_1 = ((q^1+q^3u_2+q^5u_4) + i(q^2+q^4u_2+q^6u_4))z_1$

$\dot{z}_2 = ((q^1+q^3u_1) + i(q^2+q^4u_1))z_2$

$\dot{z}_4 = ((r^1+r^3u_1) + i(r^2+r^4u_1))z_4$

18.* $\dot{z}_0 = \dot{z}_1 = 0$ 3-torus

$\dot{z}_2 = ((q^1+q^5u_3) + i(q^2+q^6u_3))z_2$

$\dot{z}_3 = ((r^1+r^3u_1+r^5u_4) + i(r^2r^4u_2+r^6u_4))z_3$

$\dot{z}_4 = ((r^1+r^5u_3) + i(r^2+r^6u_3))z_4$

19. $\dot{z}_0 = (p^1 + i(p^2u_1-p^3u_4))z_0$ 3-torus

$\dot{z}_1 = ((q^1+q^5u_4) + i(q^2+q^6u_4))z_1$

$\dot{z}_2 = \dot{z}_3 = 0$

$\dot{z}_4 = ((r^1 + r^3u_1) + i(r^2 + r^4u_1))z_4$

20. $\dot{z}_0 = (p^1 + i(p^2u_1+p^3u_3))z_0$

$\dot{z}_1 = (q^1+iq^2)z_1$

$\dot{z}_2 = \dot{z}_4 = 0$

$\dot{z}_3 = (r^1 + ir^2)z_3$

TABLE 8. (ctd.)

22. $\dot{z}_0 = 0$ 4-torus

$$\dot{z}_1 = Q^1 z_1 + Q^{13}(\bar{z}_1 z_2)^{\mu-1}(z_3\bar{z}_4)^{\eta} z_2$$

$$\dot{z}_2 = \kappa \dot{z}_1$$

$$\dot{z}_3 = R^1 z_3 + R^{13}(z_1\bar{z}_2)^{\mu}(\bar{z}_3 z_4)^{\eta-1} z_4$$

$$\dot{z}_4 = \kappa \dot{z}_3$$

Notes:

* For the special case $\ell = 2$, $m = 1$ 3(a) is absent; for $\ell = 2$, $n = 1$ 3(b) is absent.

** Similarly 8,9 and 10 are absent.

Also notice that 21(a) and (b) are the O(2) steady-state/Hopf equations of §1 and that 22 gives the full equations for the O(2)-Hopf/Hopf system.

The solution types found by considering the zeros of the amplitude in Table 4 are broadly distinguished by the descriptions given in Table 8 but in any physical application a more precise description may be possible. See §3.

TABLE 9. (a)

Reduced amplitude and phase equations for $(\ell,m,n) = (m,m,m+1)$, m odd

	Amplitude Equations	Phase Equations	Evaluated on
1.	$\dot{u}_0 = p^1 u_0$	$\dot{\psi}_0 = 0$	$(u_0,0,\ldots,0,\lambda)$
2.(a)	$\dot{u}_1 = q^1 u_1$	$\dot{\psi}_1 = q^2$	$(0,u_1,0,\ldots,0,\lambda)$
(b)	$\dot{u}_3 = r^1 u_3$	$\dot{\psi}_3 = r^2$	$(0,0,u_3,0,\ldots,0,\lambda)$
3.(a)	$\dot{u}_1 = (q^1+q^3u_1)u_1$	$\dot{\psi}_1 = q^2+q^4$	$(0,2u_1,0,u_1^2,0,\ldots,0,\lambda)$
(b)	$\dot{u}_3 = (r^1+r^5u_3)u_3$	$\dot{\psi}_3 = r^2+r^6$	$(0,0,2u_3,0,u_3^2,0,\ldots,0,\lambda)$
4.(a)$^+$	$\dot{u}_0 = (p^1\ p^4u_1)u_0$	$\dot{\psi}_0 = 0$	$(u_0,2u_1,0,u_1^2,0,\ldots\pm u_0u_1,0,\ldots,\lambda)$
5.(a)$^-$	$\dot{u}_1 = (q^1+q^3u_1\pm q^7u_0\pm q^9u_0u_1)u_1$	$\dot{\psi}_1 = q^2+u_1q^4\pm q^8u_0\pm q^{10}u_0u_1$	
(b)	$\dot{u}_0 = (p^1\ p^7u_0^mu_1^m)u_0$	$\dot{\psi}_0 = 0$	$(u_0,0,2u_3,0,u_3^2,0,\ldots,\ u_0^{m+1}u_1^m,0,\ldots,\lambda)$
	$\dot{u}_3 = (r^1+r^5u_3\ r^7u_0^{m+1}\ r^{11}u_0^{m+1}u_3^m)u_3$	$\dot{\psi}_1 = r^2+r^6u_3\pm r^8u_0^{m+1}\pm r^{11}u_0^{m+1}u_3^m$	
6.	$\dot{u}_1 = q^1u_1$	$\dot{\psi}_1 = q^2$	$(0,u_1,u_3,0,\ldots,0,\lambda)$
	$\dot{u}_3 = r^1u_3$	$\dot{\psi}_3 = r^2$	

TABLE 9. (a) (ctd.)

	Amplitude Equations	Phase Equations	Evaluated on
7.	$\dot{u}_1 = (q^1+q^5u_4)u_1$	$\dot{\psi}_1 = q^2+q^6u_4$	$(0,u_1,u_4,0,\ldots,u_1u_4,0,\ldots,0,\lambda)$
	$\dot{u}_4 = (r^1+r^3u_1)u_4$	$\dot{\psi}_4 = r^2+r^4u_1$	
$8^+,9^-$	$\dot{u}_1 = \mathrm{Re}(Q^1 \pm Q^{13}u_1^m u_3^m)u_1$	$\dot{\psi}_1 = \mathrm{Im}(Q^1 \pm Q^{13}u_1^m u_3^m)$	$(0,2u_1,2u_3,u_1^2,u_3^2,qu_1u_3,0,\ldots\pm u_1^{m+1}u_3^m$
	$\dot{u}_3 = \mathrm{Re}(R^1 \pm R^{13}u_1^{m+1}u_3^{m+1})u_3$	$\dot{\psi}_3 = \mathrm{Im}(R^1 \pm R^{13}u_1^{m+1}u_3^m)u_3$	$\ldots\ldots 0,\lambda)$
11.(b)	$\dot{u}_0 = (p^1+ip^3u_3)u_0$	$\dot{\psi}_0 = 0$	$(u_0,0,u_3,0,\ldots,0,\lambda)$
	$\dot{u}_3 = r^1u_3$	$\psi_3 = r^2$	
12.(a)	$\dot{u}_1 = (q^1+q^3u_2)u_1$	$\dot{\psi}_1 = q^2+q^4u_2$	$(0,u_1+u_2,0,u_1,u_2,0,\ldots,0,\lambda)$
	$\dot{u}_2 = (q^1+q^3u_1)u_2$	$\dot{\psi}_2 = q^2+q^4u_1$	
(b)	$\dot{u}_3 = (r^1+r^5u_4)u_3$	$\dot{\psi}_3 = r^2+r^6u_4$	$(0,0,u_3+u_4,0,u_3u_4,0,\ldots,0,\lambda)$
	$\dot{u}_4 = (r^1+r^5u_3)u_4$	$\dot{\psi}_4 = r^2+r^6u_3$	
13,14 15,16.	$\dot{u}_0 = p^1+(-1)^j p^4u_1+(-1)^\ell p^7u_0^m u_3^m \; u_0$	$\dot{\psi}_0 = 0$	$(u_0,2u_1,2u_3,u_1^2,u_3^2,$
	$+ \sum_{k=1}^{m} (-1)^{kj+\ell} p_k^{10} u_0^{m-k} u_1^k u_3^m$		$2u_1u_3,(-1)^j u_0u_1,0,$

TABLE 9. (a) (ctd.)

Amplitude Equations	Phase Equations	Evaluated on
$\dot{u}_1 = \mathrm{Re}\ \ Q^1+(-1)^j Q^7 u_0+(-1)^\ell Q^{13} u_1^m u_3^m\ \ u_1 + \sum_{k=1}^{m} (-1)^{kj+\ell} Q_k^{19}\ u_0^{m-k+1} u_1^{k-1} u_3^m$	$\dot{\psi}_1 = \mathrm{Im}\ \ Q^1+(-1)^j Q^7 u_0+(-1)^\ell Q^{13} u_1^m u_3^m + \sum_{k=1}^{m} Q_k^{19}\ u_0^{m-k+1} u_1^{k-1} u_3^m$	$(-1)^\ell u_0^{m+1} u_3^m, 0, (-1)^\ell u_1^{m+1} u_3^m,$ $0,\ldots,(-1)^{kj+\ell} u_0^{m-k} u_1^k u_3^m,\ldots\lambda.)$
$\dot{u}_3 = \mathrm{Re}\ \ R^1+(-1)^\ell R^7 u_0^{m+1} u_3^{m-1}+(-1)^\ell R^{13} u_1^{m+1} u_3^{m-1})\ u_3 + \sum_{k=1}^{m} (-1)^{kj+\ell} R_k^{19} u_0^{m-k+1} u_1^k u_3^{m-1}$	$\dot{\psi}_3 = \mathrm{Im}\ \ R^1+(-1)^\ell R^7 u_0^{m+1} u_3^{m-1}+(-1)^\ell R^{13} u_1^{m+1} u_3^{m-1} \sum_{k=1}^{m} (-1)^{kj+\ell} R_k^{19} u_0^{m-k+1} u_1^k u_3^{m-1}$	
where $j =$ 0 for 13, 15 ; 1 for 14, 16	$\ell =$ 0 for 13, 14 ; 1 for 15, 16	
17. $\dot{u}_1 = (q^1 + q^3 u_2 + q^5 u_4) u_1$	$\dot{\psi}_1 = q^2 + q^4 u_2 + q^6 u_4$	$(0, u_1+u_2, u_1 u_2, u_4, 0, u_1 u_4,$
$\dot{u}_2 = (q^1 + q^3 u_1) u_2$	$\dot{\psi}_2 = q^2 + q^4 u_1$	$0,\ldots,0,\lambda)$
$\dot{u}_4 = (r^1 + r^3 u_1) u_4$	$\dot{\psi}_4 = r^2 + r^3 u_1$	
18. $\dot{u}_2 = (q^1 + q^5 u_3) u_2$	$\dot{\psi}_2 = q^2 + q^6 u_3$	$(0, u_2, 0, u_3+u_4, u_3 u_4, u_2 u_3$
$\dot{u}_3 = (r^1 + r^3 u_1 + r^5 u_4) u_3$	$\dot{\psi}_3 = r^2 + r^4 u_1 + r^6 u_4$	$0,\ldots,0,\lambda)$
$\dot{u}_4 = (r^1 + r^5 u_3) u_4$	$\dot{\psi}_4 = r^2 + r^6 u_3$	

TABLE 9. (a) (ctd.)

	Amplitude Equations	Phase Equations	Evaluated on
19.	$\dot{u}_0 = (p^1+i(p^2u_1-p^3u_4))u_0$	$\dot{\psi}_0 = 0$	$(u_0,u_1,0,u_4,0,0,u_1u_4,$
	$\dot{u}_1 = (q^1 + q^5u_4)u_1$	$\dot{\psi}_1 = q^2 + q^6u_4$	$0,\ldots,0,\lambda)$
	$\dot{u}_4 = (r^1 + r^3u_1)u_4$	$\dot{\psi}_4 = r^2 + r^4u_1$	
20.	$\dot{u}_0 = (p^1+i(p^2u_1+p^3u_3))u_0$	$\dot{\psi}_0 = 0$	$(u_0,u_1,0,u_3,0,\ldots,0,\lambda)$
	$\dot{u}_1 = q^1u_1$	$\dot{\psi}_1 = q^2$	
	$\dot{u}_3 = r^1u_3$	$\dot{\psi}_3 = r^2$	

TABLE 9. (b)

Reduced amplitude equations for $(\ell,m,n) = (2,1,2)$

	Amplitude Equations	Evaluated on
1.	$\dot{\rho}_0 = p^1\rho_0$	$(\rho_0^2 0,\ldots\ldots,\lambda)$
2.(a)	$\dot{\rho}_1 = q^1\rho_1$	$(0,\rho_1^2 0,\ldots\ldots,\lambda)$
(b)	$\dot{\rho}_3 = r^1\rho_1$	$(0,0,\rho_3^2,0,\ldots,\lambda)$
3.(b)	$\dot{\rho}_3 = r^1\rho_3$	$(0,0,2\rho_3^2,0,\rho_3^4,0,\ldots,\lambda)$
$4^+,5^-$ (a)	$\dot{\rho}_0 = p^1\rho_0 \pm p^4\rho_1^2$	$(\rho_0^2,2\rho_1^2,0,\rho_1^4,0,\ldots,\pm\rho_0\rho_1^2,0,\ldots,\lambda)$
	$\dot{\rho}_1 = (q^1 + q^3\rho_1^2 \pm q^7\rho_0 \pm q^9\rho_0\rho_1^2)\rho_1$	
(b)	$\dot{\rho}_0 = (p^1 \pm p^7\rho_3^2)\rho_0$	$(\rho_0^2,0,2\rho_3^2,0,\rho_3^4,0,\ldots,\pm\rho_0^2\rho_1^2,0,\ldots,\lambda)$
	$\dot{\rho}_3 = (r^1 + r^5\rho_3^2 \pm r^7\rho_0^2 \pm r^{11}\rho_0^2\rho_3^2)\rho_3$	
6.	$\dot{\rho}_1 = q^1\rho_1$	$(0,\rho_1^2,\rho_3^2,0,\ldots,\lambda)$
	$\dot{\rho}_0 = r^1\rho_3$	

TABLE 9. (b) (ctd.)

	Amplitude Equations	Evaluated on
7.	$\dot{\rho}_1 = (q^1 + q^5\rho_4^2)\,{}_1$	$(0,\rho_1^2,\rho_4^2,0,\ldots,\rho_1^2\rho_4^2,0,\ldots,\lambda)$
	$\dot{\rho}_4 = r^1\rho_4$	
12.(a)	$\dot{\rho}_1 = (q^1+q^3\rho_2^2)\rho_1$ (b) $\dot{\rho}_3 = (r^1+r^5\rho_4^2)\rho_3$	(a) $(0,\rho_1^2+\rho_2^2,0,\rho_1^2\rho_2^2,0,\ldots,\lambda)$
	$\dot{\rho}_2 = (q^1+q^3\rho_1^2)\rho_2$ $\dot{\rho}_4 = (r^1+r^5\rho_3^2)\rho_4$	(b) $(0,0,\rho_3^2+\rho_4^2,0,\rho_3^2\rho_4^2,0,\ldots,\lambda)$
13,14, 15,16	$\dot{\rho}_0 = p^1\rho_0+(-1)^j p^4\rho_1^2+(-1)^\ell p^7\rho_0\rho_3^2+(-1)^{j+\ell}p_1^{10}\rho_1^2\rho_3^2$	$(\rho_0^2,2\rho_1^2,2\rho_3^2,\rho_1^4,\rho_8^4,2\rho_1^2\rho_3^2$
	$\dot{\rho}_1 = \mathrm{Re}(Q^1+(-1)^j Q^7\rho_0+(-1)^\ell Q^{13}\rho_1^2\rho_3^2+(-1)^{j+\ell}Q_1^{19}\rho_0\rho_3^2)\rho_1$	$(-1)^j\rho_0\rho_1^2,0,(-1)^\ell\rho_0^2\rho_3^2,0$
	$\dot{\rho}_3 = \mathrm{Re}(R^1+(-1)^\ell R^7\rho_0^2+(-1)^\ell R^{13}\rho_1^4+(-1)^{j+\ell}R_1^{14}\rho_0\rho_1^2)\rho_3$	$(-1)^\ell\rho_1^4\rho_3^2,0,(-1)^{\ell+j}\rho_0\rho_1^2\rho_3^2,0,\lambda)$
	where $j =$ 0 for 13,15 $\ell =$ 0 for 13,14	
	1 for 14,16 1 for 15,16.	

TABLE 10.

Isotropy Subgroups	Eigenvalues
1. $\mathbb{Z}_\kappa \times T^2$	$0[1],\ p^1_{N_0}[1],\ q^1_\lambda + (q^1_{N_0} \pm q^7)u_0[2,2]$
	$r^1_{N_0}u_0 + r^1_\lambda\ [4]$
2.(a) $S(1,-1,0) \times S(0,0,1)$	$0[1],\ q^1_{N_1}[1],\ q^3[2]\ ,\ p^1_\lambda + p^1_{N_1}u_1[2]$
	$r^1_{N_1}u_1 + r^1_\lambda[2]\ ,\ r^1_\lambda + (r^1_{N_1}+r^3)u_1[2]$
(b) $S(1,0,-m-1) \times S(0,1,0)$	$0[1]\ ,\ p^1_{N_2}u_3 + p^1_\lambda[2]\ ,\ q^1_{N_2}u_3 + q^1_\lambda[2]\ ,$
	$q^1_\lambda + (q^1_{N_2}+q^5)u_3[2]\ ,\ r^5[2],\ r^1_{N_2}[2]$
3.(a) $\mathbb{Z}_\kappa \times \mathbb{Z}(\pi,\pi,0) \times S(0,0,1)$	$0[2],\ -q^3[1],\ (2q^1_{N_1}+q^3)[1],\ p^1_\lambda + (2p^1_{N_1} \pm p^4)u_1[1,1]$
	$r^1_\lambda + (2r^1_{N_1} + r^3)u_1[4]$
(b) $\mathbb{Z}_\kappa \times \mathbb{Z}(\frac{\pi}{m+1},0,\pi) \times S(0,1,0)$	$0[2],\ 2p^1_{N_2}u_3 + p^1_\lambda[2]\ ,\ q^1_\lambda + (2q^1_{N_2}+q^5)u_3[4]$
	$-r^5[1],\ (2r^1_{N_2} + r^5)[1]$

TABLE 10. (ctd.)

Isotropy Subgroups	Eigenvalues
4.(a) $\mathbb{Z}_\kappa \times S(0,0,1)$	$0[2]$
	$T_1^{(a)} = 2(p^1_{N_0}u_0 + (2q^1_{N_1} + q^3)u_1)$
	$D_1^{(a)} = 4(p^1_{N_0}(2q^1_{N_1} + q^3) - (q^1_{N_0} + q^7)(2p^1_{N_1} + p^4))u_0u_1$
	$T_2^{(a)} = -2(q^3u_1 + p^4u_1 + 2q^5u_0)$
	$D_2^{(a)} = 4\ (p^4q^3u_1^2 + ((q^7)^2 + (q^8)^2)u_0^2 + (2p^2q^8 + q^3q^7 + q^4q^8 + p^4p^7)u_0u_1$
	$E^{(a)} = r^1_\lambda + r^1_{N_0}u_0 + (2r^1_{N_1} + r^3)u_1[4]$
4.(b) $\mathbb{Z}_\kappa \times S(0,1,0)$	$0[2]$
	$T_1^{(b)} = 2(p^1_{N_0}u_0 + (2r^1_{N_2} + r^5)u_3)$
	$D_1^{(b)} = 4(p^1_{N_0}(2r^1_{N_2} + r^5) - 2r^1_{N_0}p^1_{N_1})u_0u_3$
	$T_2^{(b)} = -2r^5u_3$
	$D_2^{(b)} = 4u_0^m u_3^{m-1}((m+1)p^7r^5u_3^2 + (2(m+1)p^3r^8 + mr^5r^7 + mr^6r^8)u_0u_3)$
	$E^{(b)} = q^1_\lambda + q^1_{N_0}u_0 + (2q^1_{N_2} + q^5)u_3[4]$

TABLE 10. (ctd.)

Isotropy Subgroups	Eigenvalues
5.(a) $\mathbb{Z}_K(0,\pi,0) \times S(0,0,1)$	$0[2]$
	$T_1^{(a)}$, $D_1^{(a)} + 8(2q^7 p_{N_1}^1 + p^4 q_{N_0}^1)u_0 u_1$
	$-T_2^{(a)} - 4q^3 u_1$, $-D_2^{(a)} + 8(p^4 p^7 u_1^2 + ((q^7)^2 + (q^8)^2)u_0 u_1)$
	$E^{(a)}$ [4]
5.(b) $\mathbb{Z}_K(0,0,\pi) \times S(0,1,0)$	$0[2]$, $T_1^{(b)}$, $D_1^{(b)}$, $T_2^{(b)}$, $-D_2^{(b)}$, $E^{(b)}$
6. $S(1,-m,-m-1)$	$0[2]$, $p_\lambda^1\lambda + p_{N_1}^1 u_1 + p_{N_2}^1 u_3[2]$, $q^3 u_1 + q^5 u_3[2]$
	$r^3 u_1 + r^5 u_3[2]$, $\begin{bmatrix} q_{N_1}^1 u_1 & q_{N_2}^1 \rho_1 \rho_3 \\ r_{N_1}^1 \rho_1 \rho_3 & r_{N_2}^1 u_3 \end{bmatrix}$
7. $S(1,-m,m+1)$	$0[2]$, $p_\lambda^1\lambda + p_{N_1}^1 u_1 + p_{N_2}^1 u_4[2]$, $q^3 u_1 - q^5 u_4[2]$
	$-r^3 u_1 + r^5 u_4[2]$ $\begin{bmatrix} q_{N_1}^1 u_1 + q_{N_1}^5 u_1 u_4 & (q_{N_2}^1 + q^5 + q_{N_1}^5 u_4)\rho_1 \rho_4 \\ (r_{N_1}^1 + r^3 + r_{N_1}^3 u_1)\rho_1 \rho_4 & r_{N_2}^1 u_4 + r_{N_2}^3 u_1 u_4 \end{bmatrix}$

TABLE 10. (ctd.)

Isotropy Subgroups	Eigenvalues
8. $\mathbb{Z}_\kappa \times \mathbb{Z}(\pi,m\pi,(m+1)\pi)$	O[3]
	$E = p^1_\lambda + (2p^1_{N_1} \pm p^4)u_1 + 2p^1_{N_2}u_3$ [1,1]
	$M = \begin{bmatrix} (2q^1_{N_1} + q^3)u_1 & (2q^1_{N_2} + q^5)\rho_1\rho_3 \\ (2r^1_{N_1} + r^3)\rho_1\rho_3 & (2r^1_{N_2} + r^5)u_2 \end{bmatrix}$
	$\text{Trace} = -2\Big[(m+2)\mathrm{Re}(Q^{13})u_1^m u_3^m + q^3u_1 + q^{15}u_1^{m+1}u_3^m + (m+1)\mathrm{Re}(R^{13})u_1^{m+1}u_3^{m-1} + r^5u_3 + r^{17}u_1^{m+1}u_3^m\Big]$
9. $\mathbb{Z}_\kappa(0,0,\pi) \times \mathbb{Z}(\pi,m\pi,(m+1)\pi)$	O[3], E,M
	$\text{Trace} = 2\Big[(m+2)\mathrm{Re}(Q^{13})u_1^m u_3^m - q^3u_1 + q^{15}u_1^{m+1}u_3^m + (m+1)\mathrm{Re}(R^{13})u_1^{m+1}u_3^{m-1} - r^5u_3 + r^{17}u_1^{m+1}u_3^m\Big]$

TABLE 10. (ctd.)

Isotropy Subgroups	Eigenvalues
11.(b) $\mathbf{Z}(\frac{2\pi}{m}, 0, \frac{2\pi(m+1)}{m}) \times S(0,1,0)(m>2)$	$0[2],\ q^1_\lambda + q^1_{N_0}u_0 + q^1_{N_2}u_3[4],\ r^5u_3[2]$
	$\begin{bmatrix} p^1_{N_0}u_0 & p^1_{N_2}\rho_0\rho_3 \\ r^1_{N_0}\rho_0\rho_3 & r^1_{N_2}u_3 \end{bmatrix}$
12.(a) $\mathbf{Z}(\pi,\pi,0) \times S(0,0,1)$	$\begin{bmatrix} q^1_{N_1}u_1 & (q^1_{N_1}+q^3)\rho_1\rho_2 \\ (q^1_{N_1}+q^3)\rho_1\rho_2 & q^1_{N_1}u_2 \end{bmatrix}$, $0[2]$
	$\begin{bmatrix} p^1+ip^2(u_1-u_2) & (p^4+ip^5(u_1-u_2))\rho_1\rho_2 \\ (p^4-ip^5(u_1-u_2))\rho_1\rho_2 & p^1-ip^2(u_1-u_2) \end{bmatrix}$
	$r^1_\lambda + (r^1_{N_1}+r^3)u_1 + r^3u_2[2], r^1_\lambda + (r^1_{N_1}+r^3)u_2 + r^3u_1[2]$
(b) $\mathbf{Z}(\frac{\pi}{m+1}, \pi, 0) \times S(0,0,1)$	$\begin{bmatrix} r^1_{N_2}u_3 & (r^1_{N_2}+r^5)\rho_3\rho_4 \\ (r^1_{N_2}+r^5)\rho_3\rho_4 & r^1_{N_2}u_4 \end{bmatrix}$, $0[2]$
	$p^1_{N_2}[2]$, $q^1_\lambda + (q^1_{N_2}+q^5)u_3 + q^5u_4[2]$
	$q^1_\lambda + (q^1+q^5)u_4+q^5u_3[2]$

TABLE 10. (ctd.)

Eigenvalues

13,14,15 & 16 $\qquad O[2]$,

$$j = \begin{cases} 0, & 13,15 \\ 1 & 14,16 \end{cases}$$

$$\ell = \begin{cases} 0 & 13,14 \\ 1 & 15,16 \end{cases}$$

$$M_1 = \begin{bmatrix} 2p^1_{N_0}u_0 & (2p^1_{N_1} + (-1)^j p^4)\rho_0\rho_1 & 2p^1_{N_2}\rho_0\rho_3 \\ 2(2q^1_{N_0}+(-1)^j q^7)\rho_0\rho_1 & 2(2q^1_{N_1}+q^3)u_1 & 2(2q^1_{N_2}+q^5)\rho_1\rho_3 \\ 2r^1_{N_0}\rho_0\rho_3 & 2(2r^1_{N_1}+r^3)\rho_1\rho_3 & 2(2r^1_{N_2}+r^5)u_3 \end{bmatrix}$$

$$\begin{aligned} \text{Trace} = -2\Big[&(-1)^j p^4 u_1 + (2m+1)(-1)^\ell p^7 u_0^m u_1^m + \sum_{k=1}^{m}(-1)^{kj+\ell} p_k^{10} u_0^{m-k} u_1^{12} u_3^m \\ &+2\,\mathrm{Re}\Big((-1)^j Q^7 u_0 + (-1)^\ell Q^{13} u_1^m u_3^m + \sum_{k=1}^{m}(-1)^{kj+\ell} Q_k^{19} u_0^{m-k+1} u_1^{k-1} u_3^m \\ &+2\,\mathrm{Re}\Big((-1)^\ell R^7 u_0^{m+1} u_3^{m-1} + (-1)^\ell R^{13} u_1^{m+1} u_3^{m-1} + \sum_{k=1}^{m}(-1)^{kj+\ell} R_k^{19} u_0^{m-k+1} u_1^k u_3^{m-1}\Big)\Big] \end{aligned}$$

17. $\mathbb{Z}(\frac{\pi}{m},\pi,(m+1)\frac{\pi}{m})$ $\qquad O[2]$,

$m \neq 1$

$$\begin{bmatrix} q^1_{N_1}u_1 & (q^1_{N_1}+q^3)\rho_1\rho_2 & (q^1_{N_2}+q^5)\rho_1\rho_4 \\ (q^1_{N_1}+q^3)\rho_1\rho_2 & q^1_{N_2}u_2 & q^1_{N_2}\rho_2\rho_4 \\ (r^1_{N_1}+r^3)\rho_1\rho_4 & r^1_{N_1}\rho_2\rho_4 & (r^1_{N_2}+r^5)u_4 \end{bmatrix}$$

$$\begin{bmatrix} p^1 + i(p^2(u_1-u_2)-p^3u_4 & (p^4 + i(p^5(u_1-u_2)-p^6u_4))\rho_1\rho_2 \\ (p^4-i(p^5(u_1-u_2)-p^6u_4))\rho_1\rho_2 & p^1 - (p^2(u_1-u_2)-p^3u_4 \end{bmatrix}$$

$$r^1 + r^2u_2 + r^5u_4 \quad [2]$$

TABLE 10. (ctd.)

Eigenvalues

18. $\mathbb{Z}(\frac{\pi}{m+1}, \frac{m}{m+1}\pi, \pi)$ O[2],

$$\begin{bmatrix} q^1_{N_1}u_2 & (q^1_{N_2} + q^5)\rho_2\rho_3 & q^1_{N_2}\rho_2\rho_4 \\ (r^1_{N_1}+r^3)\rho_2\rho_3 & r^1_{N_2}u_3 & (r^1_{N_2} + r^5)\rho_3\rho_4 \\ r^1_{N_1}\rho_2\rho_4 & (r^1_{N_2} + r^5)\rho_3\rho_4 & r^1_{N_2}u_4 \end{bmatrix}$$

$p^1_{N_1}u_1 + p^1_{N_2}(u_3 + u_4)$ [2]

$q^1 + q^3u_2 + q^5u_3$ [2]

19. $\mathbb{Z}(\frac{2\pi}{m}, 2\pi, \frac{2(m+1)}{m}\pi)$ $m>2$ O[3],

$q^3u_1 - q^5u_4$[2], $-r^3u_1 + r^5u_4$[2]

$$M_3 = \begin{bmatrix} p^1_{N_0}u_0 & p^1_{N_1}\rho_0\rho_1 & p^1_{N_2}\rho_0\rho_4 \\ q^1_{N_0}\rho_0\rho_1 & q^1_{N_1}u_1 & q^1_{N_2}\rho_1\rho_4 \\ r^1_{N_0}\rho_0\rho_4 & r^1_{N_1}\rho_1\rho_4 & r^1_{N_2}u_4 \end{bmatrix}$$

20. $\mathbb{Z}(\frac{2\pi}{m}, -2\pi, \frac{2(m+1)}{m}\pi)$ $m>2$ O[3],

$q^3u_1 + q^5u_3$ 2 , $r^3u_1 + r^5u_4$[2]

M_3.

22. $\mathbb{Z}(\pi, m\pi, (m+1)\pi)$ Not computed.

To calculate the eigenvalues of solutions corresponding to zeros of the amplitude equations we use Lemma 1 §1.3 and extend Proposition 2 §1.3 to the situation here. Again we consider a differentiable curve $\gamma_\theta \in O(2)\ltimes \times T^2$ with γ_0 the identity. We claim that if γ_θ is transverse to the isotropy subgroup Σ_z then the null-vectors corresponding to zeros of the amplitude equation are given by

$$(1)\quad (z_0,z_1,z_2,z_3,z_4) \to \frac{d}{d\psi_1}(z_0,e^{i\psi_1}z_1,e^{i\psi_2}z_2,z_3,z_4) = (0,iz_1,iz_2,0,0)$$

$$(2)\quad (z_0,\ldots,z_4) \to \frac{d}{d\psi_2}(z_0,z_1,z_2,e^{i\psi_2}z_3,e^{i\psi_2}z_4) = (0,\ldots,iz_3,iz_4)$$

$$(3)\quad (z_0,\ldots,z_4) \to \frac{d}{d\theta}(e^{i\ell\theta}z_0,e^{im\theta}z_1,e^{-im\theta}z_2,e^{in\theta}z_3,e^{-in\theta}z_4)$$
$$= i(\ell z_0,mz_1,-mz_2,nz_3,-nz_4)\ .$$

<u>Proof</u> Recall that zeros of the amplitude equations are given by

$$g(z) - i\dot{\psi}z = 0 \qquad (1)$$

where (1) is restricted to a fixed point subspace.

Now, $\dot{\psi}|_{Fix(\Sigma)}$ is just a constant w.r.t. the action of $\gamma_\theta \in O(2)\ltimes \times T^2$ so that applying γ_θ to (1) and using equivariance gives

$$g(\gamma_\theta z) - i\dot{\psi}\gamma_\theta z = 0\ . \qquad (2)$$

Now, differentiating and setting $\theta = 0$, we have

$$(dg)_z \cdot \frac{d(\gamma_\theta z)}{d\theta}\Big|_{\theta=0} - i\dot{\psi}\,\frac{d(\gamma_\theta z)}{d\theta}\Big|_{\theta=0} - i\frac{d\dot{\psi}}{d\theta}\Big|_{\theta=0} z = 0\ . \qquad (3)$$

If $\frac{d\dot{\psi}}{d\theta} = 0$ then we are done. But this is true by the above comment.

To verify entries 1 to 5(b) in Table 10 notice that if in the isotypic decomposition $V_{2j-1} \oplus V_{2j}$, of the form $\langle(0,\ldots,z_{2j-1},\bar{z}_{2j-1},0,\ldots)\rangle \oplus \langle(0,\ldots z_{2j},\bar{z}_{2j},0)\rangle$ we have $z_{2j-1} = z_{2j} = 0$ in $\mathrm{Fix}(\Sigma)$ then $dg|_{V_i}$, $i = 2j-1,2j$ commutes with the action of S^1. This implies $dg|_{V_i}$ is either a rotation matrix or some multiple of one. Whence $dg|_{V_i}$ has purely real or complex conjugate eigenvalues and we need only consider $\mathrm{trace}[dg|_{V_i}]$ to obtain the signs of the eigenvalues. This, plus an obvious extension of Table 4 gives entries 1-5(b).

For entry 6 consider the isotypic decomposition

$$\mathbb{C}^5 = \mathrm{Fix}(\Sigma) \oplus \mathbb{C}_0 \oplus \mathbb{C}_2 \oplus \mathbb{C}_4 \tag{6}$$

where 0,2,4 correspond to the coordinate subscripts in the obvious manner. $dg|_{\mathrm{Fix}(\Sigma)}$ reduces, via the null-eigenvectors (1) and (2), to the given 2×2 matrix. Notice that the action on $\mathbb{C}_i$ is a (multiple) rotation, whence we need only consider $\mathrm{trace}\,(dg|_{\mathbb{C}_i})$, giving the remaining entries. Similarly we deal with entry 7.

For entry 8 we decompose $\mathbb{C}^5 = V_0 \oplus \tilde{V}_0 \oplus V_1 \oplus V_2$ where $V_0 = \langle(z_0 + \bar{z}_0,0,\ldots,0)\rangle$, $\tilde{V}_0 = \langle(0,\ldots,0,z_0-\bar{z}_0,0,\ldots,0)\rangle$, $\mathrm{Fix}(\Sigma) = V_1 = \langle(0,z_1 + z_2,\bar{z}_1 + \bar{z}_2, z_3 + z_4, \bar{z}_3 + \bar{z}_4,0,\ldots,0)\rangle$, and

$$V_2 = \langle(0,\ldots,0,z_1-z_2, \bar{z}_1-\bar{z}_2, z_3-z_4, \bar{z}_3-\bar{z}_4)\rangle .$$

The null-eigenvectors are, in this coordinate system, for $dh|_{V_1}$:

$$\begin{aligned} &\text{(a)} \quad (0,\rho_1,-\rho_1,0,\ldots,0) \\ &\text{(b)} \quad (0,\ldots,0,\rho_3,-\rho_3,0,\ldots,0) \end{aligned} \tag{7}$$

and for $dh|_{V_2}$:

$$(0,\ldots,0,mc_1,-mc_1,(m+1)\rho_3,-(m+1)\rho_3) \ . \tag{8}$$

(7)(a) and (b) reduce $dh|_{V_1}$ to a 2×2 matrix of the form

$$\begin{bmatrix} \mathrm{Re}\ \dfrac{\partial h_1}{\partial z_1} + \dfrac{\partial h_1}{\partial z_2} & \mathrm{Re}\ \dfrac{\partial h_1}{\partial z_3} + \dfrac{\partial h_1}{\partial z_4} \\ \mathrm{Re}\ \dfrac{\partial h_2}{\partial z_1} + \dfrac{\partial h_3}{\partial z_2} & \mathrm{Re}\ \dfrac{\partial h_3}{\partial z_3} + \dfrac{\partial h_3}{\partial z_4} \end{bmatrix} \ . \tag{9}$$

Now, straightforward calculation gives these entries as

$$e_{11} = (2q^1_{N_1} + q^3)u_1 + 2q^5_{N_1}u_1u_3 + \ldots \text{ h.o.t.}$$

$$e_{12} = (2q^1_{N_2} + q^5)\rho_1\rho_3 + \ldots \text{ h.o.t.}$$

$$e_{21} = (2r^1_{N_1} + r^3)\rho_1\rho_3 + \ldots \text{ h.o.t.}$$

$$e_{22} = (2r^1_{N_2} + r^5)u_2 + \ldots \text{ h.o.t.}$$

Here h.o.t. are higher order terms which, given certain non-degeneracy conditions, are not required to determine the signs of the eigenvalues.

We have $dh|_{V_0}$ and $dh|_{\tilde{V}_0}$ equal $\dfrac{\partial h_0}{\partial z_0} \pm \dfrac{\partial h_0}{\partial \bar{z}_0}$ respectively. A short calculation, including Taylor expansion about the origin, gives the signs of the real parts of the eigenvalues. For $dh|_{V_2}$ notice that it has the form of the 4×4 matrix.

$$\begin{bmatrix} a & b & c & d \\ \bar{b} & \bar{a} & \bar{d} & \bar{c} \\ e & f & g & h \\ \bar{f} & \bar{e} & \bar{h} & \bar{g} \end{bmatrix} \tag{10}$$

and that using (8) it is possible to reduce (10) to the 3x3 matrix

$$\begin{bmatrix} a + t\bar{f} & b + t\bar{e} & c + t\bar{h} \\ \bar{b} - t\bar{f} & \bar{a} - t\bar{e} & \bar{d} + t\bar{h} \\ e + \bar{f} & f + \bar{e} & g + \bar{h} \end{bmatrix} \quad \text{where} \quad t = \frac{m\rho_1}{(m+1)\rho_3} \tag{11}$$

$$a = \frac{\partial h_1}{\partial z_1} - \frac{\partial h_1}{\partial z_2} \qquad b = \frac{\partial h_1}{\partial \bar{z}_1} - \frac{\partial h_1}{\partial \bar{z}_2} \qquad c = \frac{\partial h_1}{\partial z_3} - \frac{\partial h_1}{\partial z_4} \qquad d = \frac{\partial h_1}{\partial \bar{z}_3} - \frac{\partial h_1}{\partial \bar{z}_4}$$

$$e = \frac{\partial h_3}{\partial z_1} - \frac{\partial h_3}{\partial z_2} \qquad f = \frac{\partial h_3}{\partial \bar{z}_1} - \frac{\partial h_3}{\partial \bar{z}_2} \qquad g = \frac{\partial h_3}{\partial z_3} - \frac{\partial h_3}{\partial z_4} \qquad h = \frac{\partial h_3}{\partial \bar{z}_3} - \frac{\partial h_3}{\partial \bar{z}_4}$$

computing the above

$$\begin{aligned}
a &= -(n+2)Q^{13}u_1^m u_3^m - q^3 u_1 - q^{15}u_1^{m+1}u_3^m - i(q^4 u_1 + q^{16}u_1^{m+1}u_3^m) \\
b &= +mQ^{13}u_1^m u_3^m - q^3 u_1 - q^{15}u_1^{m+1}u_3^m - i(q^4 u_1 + q^{16}u_1^{m+1}u_3^m) \\
c &= mQ^{13}u_1^m u_3^{m-1}\rho_1\rho_3 - q^5\rho_1\rho_3 - q^{17}u_1^m u_3^m\rho_1\rho_3 - i(q^6\rho_1\rho_3 + q^{18}u_1^m u_3^m\rho_1\rho_3) \\
d &= -mQ^{13}u_1^m u_3^{m-1}\rho_1\rho_3 - q^5\rho_1\rho_3 - q^{17}u_1^m u_3^m\rho_1\rho_3 - i(q^6 + q^{18}u_1^m u_3^m)\rho_1\rho_3 \qquad (12) \\
e &= (m+1)R^{13}u_1^m u_3^{m-1}\rho_1\rho_3 - r^3\rho_1\rho_3 - r^{15}u_1^{m+1}u_3^{m-1}\rho_1\rho_3 - i(r^4 + r^{15}u_1^{m+1}u_3^{m-1})\rho_1\rho_3 \\
f &= -(m+1)R^{13}u_1^m u_3^{m-1}\rho_1\rho_3 - r^3\rho_1\rho_3 - r^{15}u_1^{m+1}u_3^{m-1}\rho_1\rho_3 - i(r^4 + r^{16}u_1^{m+1}u_3^{m-1})\rho_1\rho_3 \\
g &= -(m+1)R^{13}u_1^{m+1}u_3^{m-1} - r^5 u_3 - r^{17}u_1^{m+1}u_3^m - i(r^6 u_3 + r^{18}u_1^{m+1}u_3^m) \\
h &= (m-1)R^{13}u_1^{m+1}u_1^{m-1} - r^5 u_3 - r^{17}u_1^{m+1}u_3^m - i(r^6 u_3 + r^{18}u_1^{m+1}u_3^m) \ .
\end{aligned}$$

To compute the entries for 9 notice that the decomposition $\mathbb{C}^5 = V_0 \oplus \tilde{V}_0 \oplus V_1 \oplus V_2$ is now

$$V_1 = \langle (0, z_1+z_2, \bar{z}_1+\bar{z}_2, z_3-z_4, \bar{z}_3-\bar{z}_4, 0, \ldots, 0) \rangle \quad = \mathrm{Fix}(\Sigma)$$

and $V_2 = \langle (0, \ldots, 0, z_1-z_2, \bar{z}_1-\bar{z}_2, z_3+z_4, \bar{z}_3+z_4) \rangle$ with V_0, $\tilde{V}_0$ as in 8. The null-vectors associated with $dh|_{V_1}$ and $dh|_{V_2}$ are as in 8. For the awkward 4×4 we may reduce to (11) but the entries are now

$$\begin{aligned}
a &= (m+2)Q^{13}u_1^m u_3^m - q^3 u_1 + q^{15}u_1^{m+1}u_3^m - i(q^4 u_1 - q^{16}u_1^{m+1}u_1^m) \\
b &= -mQ^{13}u_1^m u_3^m - q^3 u_1 + q^{15}u_1^{m+1}u_3^m - i(q^4 u_1 - q^{16}u_1^{m+1}u_1^m) \\
c &= -mQ^{13}u_1^m u_3^{m-1}\rho_1\rho_3 - q^5\rho_1\rho_3 + q^{17}u_1^m u_3^m\rho_1\rho_3 - i(q^6 - q^{18}u_1^m u_3^m)\rho_1\rho_3 \\
d &= mQ^{13}u_1^m u_3^{m-1}\rho_1\rho_3 - q^5\rho_1\rho_3 + q^{17}u_1^m u_3^m\rho_1\rho_3 - i(q^6 - q^{18}u_1^m u_3^m)\rho_1\rho_3 \qquad (13) \\
e &= -(m+1)R^{13}u_1^m u_3^{m-1}\rho_1\rho_3 - r^3\rho_1\rho_3 + r^{15}u_1^{m+1}u_3^{m-1}\rho_1\rho_3 - i(r^4 - r^{16}u_1^{m+1}u_3^{m-1})\rho_1\rho_3 \\
f &= (m+1)R^{13}u_1^m u_3^{m-1}\rho_1\rho_3 - r^3\rho_1\rho_3 + r^{15}u_1^{m+1}u_3^{m-1}\rho_1\rho_3 - i(r^4 - r^{16}u_1^{m+1}u_3^{m-1})\rho_1\rho_3 \\
g &= (m+1)R^{13}u_1^{m+1}u_3^{m-1} - r^5 u_3 + r^{17}u_1^{m+1}u_3^m - i(r^6 - r^{18}u_1^{m+1}u_3^{m-1})u_3 \\
h &= -(m-1)R^{13}u_1^{m+1}u_3^{m-1} - r^5 u_3 + r^{17}u_1^{m+1}u_3^m - i(r^6 - r^{18}u_1^{m+1}u_3^{m-1})u_3 \quad .
\end{aligned}$$

The computation for 13, 14, 15 and 16 is similar to above. We may write the isotypic decomposition as $\mathbb{C}^5 = V_{1,j,\ell} \oplus V_{2,j,\ell}$ where

$$V_{i,j,\ell} = \langle (z_0+(-1)^i\bar{z}_0, z_1+(-1)^{i+j}z_2, \bar{z}_1+(-1)^{i+j}\bar{z}_2, z_3+(-1)^{i+\ell}z_4, \bar{z}_3+(-1)^{i+\ell}\bar{z}_4) \rangle \qquad i = 1,2$$

where $j = \begin{matrix} 0 & \text{for } 13, 15 \\ 1 & \text{for } 14, 16 \end{matrix}$ $\qquad \ell = \begin{matrix} 0 & \text{for } 13, 14 \\ 1 & \text{for } 15, 16 \end{matrix}$.

Again $V_{1,j,\ell} = \mathrm{Fix}(I)$ and the null-eigenvectors are:

$$\begin{aligned} &(a) \qquad (0,\rho_1,-\rho_1,0,\ldots\ldots,0) \\ &(b) \qquad (0,\ldots\ldots\ldots\ldots,\rho_3,-\rho_3) \end{aligned} \tag{14}$$

and $$(m\rho_0, m\rho_1, -m\rho_1, (m+1)\rho_3, -(m+1)\rho_3) \tag{15}$$

corresponding to $dh|_{V_{1,j,\ell}}$ and $dh|_{V_{2,j,\ell}}$ respectively.

14(a) and (b) reduces $dh|_{V_{1,j,\ell}}$ to the 3×3 matrix

$$\begin{bmatrix} \frac{\partial h_0}{\partial z_0} + \frac{\partial h_0}{\partial \bar{z}_0} & \frac{\partial h_0}{\partial z_1} + (-1)^j \frac{\partial h_0}{\partial z_2} & \frac{\partial h_0}{\partial z_3} + (-1)^\ell \frac{\partial h_0}{\partial z_4} \\ 2\,\mathrm{Re}\,\frac{\partial h_1}{\partial z_1} + \frac{\partial h_1}{\partial \bar{z}_0} & 2\,\mathrm{Re}\,\frac{\partial h_1}{\partial z_1} + (-1)^j \frac{\partial h_1}{\partial z_2} & 2\,\mathrm{Re}\,\frac{\partial h_1}{\partial z_3} + (-1)^\ell \frac{\partial h_1}{\partial z_4} \\ 2\,\mathrm{Re}\,\frac{\partial h_3}{\partial z_0} + \frac{\partial h_3}{\partial \bar{z}_0} & 2\,\mathrm{Re}\,\frac{\partial h_3}{\partial z_1} + (-1)^j \frac{\partial h_3}{\partial z_2} & 2\,\mathrm{Re}\,\frac{\partial h_3}{\partial z_3} + (-1)^\ell \frac{\partial h_3}{\partial z_4} \end{bmatrix} \tag{16}$$

After a little finicking we reduce $dh|_{V_{2,j,\ell}}$ to $(e_{m,n})$, $m = 1,\ldots,4$; $n = 1,\ldots,4$, where, writing

$$a_1 = \frac{\partial \bar{h}_3}{\partial z_0} - \frac{\partial \bar{h}_3}{\partial \bar{z}_0}, \quad \bar{a}_2 = \frac{\partial h_3}{\partial \bar{z}_1} - (-1)^j \frac{\partial h_3}{\partial \bar{z}_2}, \quad \bar{a}_3 = \frac{\partial h_3}{\partial z_1} - (-1)^j \frac{\partial h_3}{\partial z_2},$$

$$\bar{a}_4 = \frac{\partial h_3}{\partial \bar{z}_3} - (-1)^\ell \frac{\partial h_3}{\partial \bar{z}_4} \text{ and } \alpha_0 = \frac{m\rho_0}{(m+1)\rho_3}, \quad \alpha_1 = \frac{m\rho_1}{(m+1)\rho_3} \text{ we have}$$

$$e_{1,1} = \frac{\partial h_0}{\partial z_0} - \frac{\partial h_0}{\partial \bar{z}_0} + a_1\alpha_0 \ , \ e_{1,2} = \frac{\partial h_0}{\partial z_1} - (-1)^j \frac{\partial h_0}{\partial z_2} + a_2\alpha_0 \ ,$$

$$e_{1,3} = \frac{\partial h_0}{\partial \bar{z}_1} - (-1)^j \frac{\partial h_0}{\partial \bar{z}_2} + a_3\alpha_0 \ , \quad e_{1,4} = \frac{\partial h_0}{\partial z_3} - (-1)^\ell \frac{\partial h_0}{\partial z_4} + a_4\alpha_0$$

$$e_{2,1}(h_1) = \frac{\partial h_1}{\partial z_0} - \frac{\partial h_1}{\partial \bar{z}_0} + a_1\alpha_1 \ , \quad e_{2,2}(h_1) = \frac{\partial h_1}{\partial z_1} - (-1)^j \frac{\partial h_1}{\partial z_2} + a_2\alpha_1 \ ,$$

$$e_{2,3} = \frac{\partial h_1}{\partial \bar{z}_1} - (-1)^j \frac{\partial h_1}{\partial \bar{z}_2} + a_3\alpha_1 \ , \quad e_{2,4} = \frac{\partial h_1}{\partial z_3} - (-1)^\ell \frac{\partial h_1}{\partial z_4} + a_4\alpha_1$$

$$e_{3,n} = e_{2,n}(\bar{h}_1) - 2a_n\alpha_1$$

$$e_{4,1} = \frac{\partial h_3}{\partial z_0} - \frac{\partial h_3}{\partial \bar{z}_0} + a_1 \ , \quad e_{4,2} = \frac{\partial h_3}{\partial z_1} - (-1)^j \frac{\partial h_3}{\partial z_2} + a_2 \ , \quad e_{4,3} = \bar{e}_{4,2}$$

$$e_{4,4} = \frac{\partial h_3}{\partial z_3} - (-1)^\ell \frac{\partial h_3}{\partial z_4} + a_4 \ .$$

We content ourselves with giving the trace of this matrix in Table 11.

For (16) we have the entries, row wise,

$$2p^1_{N_0}u_0 + (-1)^j(p^1_{0_1}+2p^4_{N_0})u_0u_1 + p^4_{0_1}u_0u_1^2 + 2(-1)^\ell\, p^7_{N_0}u_0^{m+1}u_3^m + \ldots \text{ h.o.t.}$$

$$(2p^1_{N_1}+(-1)^j p^4)\rho_0\rho_1+(-1)^j p^1_{0_1\ 0\ 1}u_0+2p^1_{m_1}\rho_0\rho_1u_1+2p^1_{c_1}\rho_0\rho_1u_3 + \ldots + p^{10}_m u_p^m u_3^m + \ldots \text{ h.o.t.}$$

$$2p^1_{N_2}\rho_0\rho_3 + 2p^1_{m_2}\rho_0\rho_3u_3+2p^1_{c_1}\rho_0\rho_3u_1 + \ldots (-1)^\ell\, u_0^m u_1^{m-1}\rho_0\rho_1 + \ldots + \text{h.o.t.}$$

$$2[(2q^1_{N_0}+(-1)^j q^7)\rho_0\rho_1 + 2q^3_{N_0}\rho_0\rho_1u_3+2q^5_{N_0}\rho_0\rho_1u_3+(-1)^j 2q^7_{N_0}\rho_0\rho_1u_0 + \ldots \text{ h.o.t.}]$$

$$2\Big[(2q^1_{N_1}+q^3)u_1+(2q^7_{N_1}+q^9)u_0u_1+(-1)^j q^1_{0_1}u_0u_1+2q^5_{N_1}u_1u_3+2(q^1_{M_1}+q^1_{c_1}u_3)u_1+\dots$$
$$\dots+(-1)^\ell mq^{13}u_1^m u_3^m+\dots+\text{h.o.t.}\Big]$$

$$2\Big[(2q^1_{N_2}+q^5)\rho_1\rho_3+2(q^1_{M_2}u_3+q^1_{c_1}u_1)\rho_1\rho_3+2q^7_{N_2}u_0\rho_1\rho_3+mq^{13}u_1^m u_3^{m-1}\rho_1\rho_3+\dots$$
$$+2mq_1^{19}u_0^m u_3^{m-1}\rho_1\rho_3+\dots+\text{h.o.t.}\Big]$$

$$2\Big[(2r^1_{N_0}+(-1)^\ell r^7 u_0^m u_3^{m-1})\rho_0\rho_3+2r^3_{N_0}\rho_0\rho_3u_1+2r^5_{N_0}\rho_0\rho_3u_3+(-1)^j r^1_{0_1}\rho_0\rho_3u_1+\dots$$
$$\dots+(-1)^\ell 2mr_1^{19}u_1u_3^{m-1}\rho_3\rho_0+\dots+\text{h.o.t}\Big]$$

$$2[(2r^1_{N_1}+r^3)\rho_1\rho_3+(-1)^j r^1_{0_1}u_0\rho_1\rho_3+\dots\text{h.o.t}]$$

$$2[(2r^1_{N_2}+r^5)u_3+\dots+mr_1^{19}u_0^m u_1u_3^{m-1}+\dots\text{h.o.t.}]\ .$$

The entry in Table 11 considers only the terms necessary to obtain the signs of the eigenvalues at zero.

The remaining isotypic decompositions are:

11(b)	$\mathbb{C}^5 = \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \mathbb{C}_4 \oplus \mathrm{Fix}(\Sigma)$;
12(a)	$\mathbb{C}^5 = \mathbb{C}_0 \oplus \mathrm{Fix}(\Sigma_1) \oplus \mathbb{C}_3 \oplus \mathbb{C}_4$;
(b)	$\mathbb{C}^5 = \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \mathrm{Fix}(\Sigma)$;
17, 18	$\mathbb{C}^5 = \mathbb{C}_0 \oplus \mathbb{C}_i \oplus \mathrm{Fix}(\Sigma_j)$ where $i = \begin{cases} 3 & j = 17 \\ 1 & j = 18 \end{cases}$
19	$\mathbb{C}^5 = \mathrm{Fix}(\Sigma) \oplus \mathbb{C}_2 \oplus \mathbb{C}_3$
20	$\mathbb{C}^5 = \mathrm{Fix}(\Sigma) \oplus \mathbb{C}_2 \oplus \mathbb{C}_4$.

Again notice that $dh|_{\mathbb{C}_i}$ commutes with a (multiple) rotation and the remaining computations are straightforward.

CHAPTER 3.

Mode Interactions in the Taylor-Couette system.

§3.0. The Taylor-Couette System.

A natural context in which to set the discussions of the preceeding sections is the Taylor-Couette system. This experiment concerns the fluid flow between two concentric, independently rotating cylinders. Many experimental studies have been made c.f. Andereck, Liu and Swinney (1986) for a recent investigation, and much theoretical analysis has been done on the Navier-Stokes equations governing this system. What concerns us here is a qualitative description of some of the many planforms occurring in the flow, based on the existence of (near) symmetry in the system.

The Euclidean invariance of the Navier-Stokes equations is well known, however, the group of rigid motions is non-compact and therefore not amenable to the analysis discussed here. The standard way to resolve this problem is to impose, so called, periodic boundary conditions resulting in a compact group of symmetries. The details have been dealt with recently by a number of authors c.f. Langford et al (1987), Golubitsky and Langford (1987), Chossat, Demay and Iooss (1987) and for an analogous approach in the context of the Rayleigh-Bénard problem c.f. Buzano and Golubitsky (1983) and Sattinger (1978). We outline the pertinent facts.

The assumption of periodic boundary conditions is motivated by two considerations, both pragmatic in nature. The first is that the assumption of any 'realistic' end boundary conditions results in not (even) being able to obtain an analytic formula for the basic steady-state, Couette flow. Although it is possible to integrate the Navier-Stokes equations numerically c.f. Cliffe and Spence (1985), this gives no further insight regarding the method of pattern selection. Secondly, the primary steady-state bifurcation off Couette flow is the so called Taylor vortex, c.f. Taylor (1923), which exhibits (near) axial periodicity. For pictures of the above flows and those to be discussed, see Figures 1-15, §3.1. The period of the Taylor vortex is set at $2\pi/\alpha$ (α is the wave number) and there are at least two methods in the literature for determining α - we discuss this below.

The basic consequence of the assumption of axial periodicity is the introduction of $O(2)$ symmetry into the problem. Since axial periodicity means axial translations are identified mod $2\pi/\alpha$ and as the Navier-Stokes equations and Couette flow are invariant under these translations as well as a flip (or reflection) about the midplane, we generate the group $O(2)$.

A second consequence of the assumption of axial periodicity is that study of the Navier-Stokes partial differential equations is reduced to an ordinary differential eigenvalue problem c.f. Langford et al (1987). In particular the study takes place on the centre manifold reduced kernel of the linearization of the Navier-Stokes equations about Couette flow. Referring to this kernel as $\ker L$ and the full Navier-Stokes equations with boundary conditions as $\phi = 0$ we have the following:

(I) since Φ commutes with the action of $O(2)$ we have the usual equivariance condition

$$\Phi(\gamma x) = \gamma\Phi(x) \qquad \forall\ \gamma \in O(2) \tag{1}$$

and recalling the observations in §0.0 we expect, if $O(2)$ symmetry is to be broken, to consider double eigenvalues corresponding to flow patterns;

(II) that $\ker L$ is invariant under the action of $O(2)$ and is, of course, the direct sum of the eigenspaces corresponding to the eigenvalues under consideration.

Before describing further $\ker L$ appropriate for a study of steady-state/Hopf/Hopf mode interaction we briefly discuss the method for selecting the wavenumber α of the primary instability

As mentioned above there are at least two methods in the literature for determining α, both providing results in close agreement with each other and experiment. The first, due to Taylor, treats α as a continuous variable and allows the instability to vortices to determine a critical wavenumber α_c which is then fixed. The second used by Chossat, Demay, Iooss (1987) is to choose an axial length h and ignoring the <u>end</u> boundary effects to pick only α s.t. $\alpha = 2\pi k/h$, $k \in \mathbb{Z}$. Although either method would suffice for the qualitative approach presented here we have in mind the latter.

In order that <u>$\ker L$ be invariant</u> under the action of $O(2) \times T^2$ discussed in §0.2 we consider a slightly different interpretation of the symmetries of the system than is current in the literature. Briefly, when (only) a two-mode interaction is considered the fact that the system

has a natural azimuthal symmetry, identifiable with $SO(2)$ in terms of the action on eigenfunctions corresponding to a Hopf mode, gives a symmetry extra to the $O(2)$ already considered. In the case of a steady-state/Hopf interaction, c.f. Golubitsky and Stewart (1986), the symmetry group is $O(2) \times SO(2) \times S^1$. However, it is possible to identify S^1 with $SO(2)$ in a natural way without losing any information conveyed by the $SO(2)$ symmetry. Unfortunately when two Hopf modes are present an identification of the above type means the $SO(2)$ symmetry is realized as a subgroup of T^2, at least on the full space. We begin by asking what the $O(2) \times T^2$ symmetry can tell us, a question which, of course, has been addressed in earlier sections. We then ask what additional information the natural azimuthal symmetry provides.

A primary Hopf instability is the <u>spiral</u>, see Appendices 1 and 2 for computer generated and experimental pictures. Clearly the spirals have different symmetries. Either one can think of identifying spirals by azimuthal rotation or one can move from one spiral to another by axial translation; it is the latter that we begin with. We think of m spirals - m the azimuthal wavenumber - occurring in the axial height of one vortex pair and identify spirals by axial translation. The azimuthal symmetry is introduced in §3.1.

Primarily we study the interaction of a (steady-state) vortex with two (Hopf) spirals, the first having twice the pitch of the second and such that one spiral fits into a vortex pair. It is important to note that there exists the possibility of another primary Hopf instability,

so-called ribbons. This corresponds to the standing wave found in the O(2) Hopf analysis of Golubitsky and Stewart (1985) and the symmetries of this are discussed in §3.1.

With the above in mind the ten-dimensional kernel is spanned by eigenfunctions of the form

$$\begin{aligned} &\bar{U}(r)e^{i\alpha z}, \quad \hat{V}(r)e^{i\alpha z \pm i\psi_1 t + i\phi}, \quad \hat{W}(r)e^{i2\alpha z \pm i\psi_2 t + i2\phi}, \\ &\overline{\bar{U}(r)}e^{-i\alpha z}, \quad (\kappa\bar{V}(r))e^{-i\alpha z \pm i\psi_1 t + i\phi}, \quad (\kappa\hat{W}(r))e^{-i2\alpha z \pm i\psi_2 t + i2\phi}, \end{aligned} \tag{2}$$

where κ is the flip acting by $z \to -z$. Combining this with translations in the axial direction z generates an O(2) action. The vectors $\bar{U}(r),\ldots,\hat{W}(r)$ are unit vectors in the radial direction and ψ_1, ψ_2 are rationally independent eigenvalues. The rotational symmetry, SO(2), is given by the dependence on the planar angle ϕ. For further discussion of the above notation, from different perspectives, see Iooss (1986) and Langford et al (1987).

Writing the above eigenfunctions as $z_0, z_1, \ldots z_4$, where $z_1, \ldots, z_4$ correspond to the positive eigenvalues ψ_1 and ψ_2 respectively, we obtain, after some scaling, the action given in Table 5 §2.1 for $(\ell,m,n) = (1,1,2)$. Before considering in detail the relationship between symmetry and the observed flows, we give some motivation for the study of this highly degenerate codimension 3 phenomenon.

The basic motivation arises from numerous studies already performed. See Figure 1 below for experimental results of Andereck et al. It is a common feature of recent studies that numerous bicritical points exist

for azimuthal wave numbers $m = 0$ and $m = 1,2...$ or $m = 1$ and $m = 2,3$ .... see Figure 2 below. Near-tricritical points also are possible: again see Figure 2 - and it is the tricritical points that are of interest here. Studies of the bicritical points have been made for $m = 0,1$ Golubitsky and Stewart (1986) and for $m = n$, $n + 1$, Chossat, Demay and Iooss (1986). In the jargon of catastrophe theory we consider a possible 'organising centre' for these phenomena. The concept of an organising centre is well known. Briefly, from a geometrical standpoint, if G and G' are two finite codimensional bifurcation problems we say the normal form of G is an organising centre for G' if the universal unfolding of G contains, as a subdiagram, the bifurcation diagram of G' .

It follows that the direct sum of the three eigenspaces associated with the three modes gives one possible organising centre for the tricritical point. Since the direct sum of these eigenspaces is the kernel of the linearised system we refer to this organising centre as the ten-dimensional kernel.

The main point to note from the above studies are that if the three modes are to interact they will do so in the counter-rotating case. We therefore expect our analysis to apply only to this situation. However, in the unfolding we find the codimension two points which may organise behaviour in the co-rotating case and we therefore identify the flows with their symmetries, this is of course as expected.

Finally, note that the tricritical point has not been observed in experiments, the high degree of degeneracy making its observation unlikely. This does not, of course, preclude its consideration as a hidden organising centre.

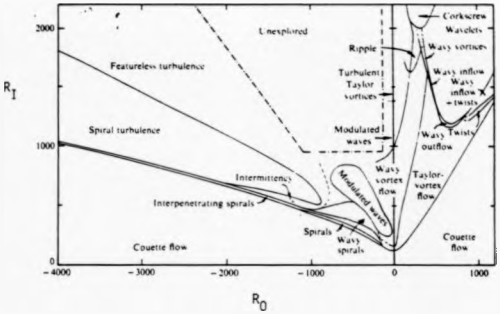


Fig. 1.

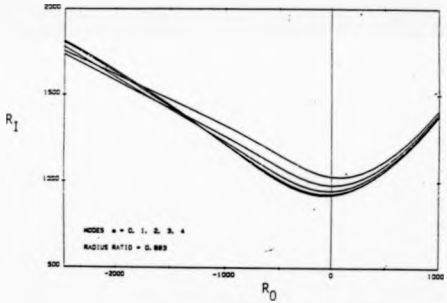


Fig. 2.

(Langford et al 1987)

where R_I and R_O are the Reynolds number proportional to the inner and outer cylinder velocities.

§3.1 The ten-dimensional kernel : flows and symmetries.

Experiment reveals a rich variety of flows in the pre-chaotic Taylor-Couette system. In this section we bring together the analysis of §2.1 and experimental observations in order to detail the symmetries associated with the flow types.

The full symmetry group for the ten-dimensional kernel is $O(2) \times SO(2) \times T^2$. It is therefore necessary to take into account the $SO(2)$ azimuthal symmetry when interpretating the results. In fact the presence of the $SO(2)$ symmetry has the following important consequence:

Lemma 1. Let $u(t)$ be a periodic solution in $\ker L$; then $R_\theta u(t) = u(t+\theta)$, where R_θ is a rotation through θ.

Note: This is an extension of the result given in Golubitsky and Stewart (1985) which states that all periodic solutions in the six-dimensional kernel - the organising centre for the steady-state/Hopf mode interaction - are rotating waves.

Proof. Recall that the ten-dimensional kernel is identified with

$$\ker L = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \quad . \tag{1}$$

where T^2 acts trivially on $\mathbb{C}$, the steady-state kernel and acts by phase shift on each of the Hopf kernels.

Now T^2 commutes with $O(2) \times SO(2)$ and in particular the actions of $O(2) \times SO(2)$ and T^2 commute on $\mathbb{C}^2 \oplus \mathbb{C}^2$. Using the notation of

Table 5, §2.1, we see that $S^1(\psi_1)$ and $S^1(\psi_2)$ are multiples of the $SO(2)$ action restricted to either $\mathbb{C}^2$. □

Writing

$$\Delta_a = \{(\theta,-\theta,0) \in SO(2) \times T^2\}$$
$$\Delta_b = \{(\theta,0,-2\theta) \in SO(2) \times T^2\} \qquad (2)$$

as the subgroups arising from the identification and Tables 11 and 12 detail the symmetries of the fluid states and the bifurcation lattice.

The experimental pictures featured in Appendix 2 give examples of the flow types. We see there how closely the predicted symmetries are followed. The primary stationary flow a Taylor vortex is invariant under the axial flip and azimuthal rotation, of course photographs are not able to reveal the time independence. The spiral vortex or $O(2)$ Hopf rotating wave displays the symmetry expected. It is, of course, an axial rotating wave as seen above but we also see the helical structure means that the consequences of axial translation can be obtained by rotating the cylinders. The $O(2)$ Hopf standing wave solutions, or so called ribbons, correspond to a pattern invariant under the axial flip but unlike the spiral, its time evolution is not given by axial translation. Although it is believed, at the time of writing, that preliminary experimental evidence of Tagg indicates the existence of these solutions, no results have yet been published. For the secondary bifurcations we see that: type 4 branches are axial rotating waves that are invariant under the flip and so are identifiable with the

twisted vortex; type 5 branches are a half period out of phase when flipped in the axial direction and are identifiable with wavy vortices. The tertiary states, which correspond to 2 or 3-torus flows are not all branches in the isotropy lattice. However, they can be predicted from the model. In computing the Floquet exponents, in §3.2, along a branch we expect to see the exponents cross the imaginary axis. Application of the Sacher-Neimark bifurcation theorem (c.f. Marsden and McCracken [1976]) leads to further modulated or quasi-periodic flows. In this context type 12 branches are only tentatively described as wavy spirals. Recall from §2 that such solutions would be seen to exist only when certain non-degeneracy conditions (§3.2) are violated. However, wavy spirals may bifurcate from spirals as exponents cross the imaginary axis. Similarly modulated wavy vortices, a flow not shown in Appendix 2, may be predicted in this fashion. The two torus flows invariant under the double-twisted circle groups are identified with interpenetrating spirals.

Experiment now clearly shows the infinite cylinder approach to have shortcomings, however it is still remarkable that the middle zone of the cylinders still reflects the symmetry assumptions. The pattern formations for other solution types are depicted in Appendix 1.

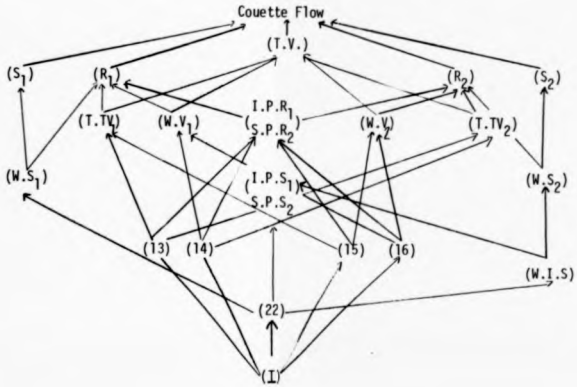
TABLE 11

ISOTROPY SUBGROUPS OBSERVED AND POSSIBLE FLUID STATES

	STATE		ISOTROPY SUBGROUP
0.	Couette flow		$O(2) \times SO(2) \times T^2$
1.	Taylor vortices	(T.V.)	$\mathbb{Z}_\kappa \times SO(2) \times T^2$
2(a)	Spirals	(S_1)	$S(1,-1,0) \times \Delta_a \times S(0,0,1)$
(b)	Spirals	(S_2)	$S(1,0,-2) \times \Delta_b \times S(0,1,0)$
3(a)	Ribbons	(R_1)	$\mathbb{Z}_\kappa \ltimes \mathbb{Z}(\pi,\pi,0) \ltimes \Delta_a \times S(0,0,1)$
(b)	Ribbons	(R_2)	$\mathbb{Z}_\kappa \ltimes \mathbb{Z}(\frac{\pi}{2},0,\pi) \times \Delta_b \times S(0,1,0)$
4(a)	Twisted Taylor vortices	($T.T.V_1$)	$\mathbb{Z}_\kappa \ltimes \Delta_a \times S(0,0,1)$
(b)	Twisted Taylor vortices	($T.T.V_2$)	$\mathbb{Z}_c \ltimes \Delta_b \times S(0,1,0)$
5(a)	Wavy vortices	($W.V_1$)	$\mathbb{Z}_c\,(0,\pi,0) \ltimes \Delta_a \times S(0,0,1)$
(b)	Wavy vortices	($W.V_2$)	$\mathbb{Z}_\kappa\,(0,0,\pi) \times \Delta_b \times S(0,1,0)$
6.	Superimposed spirals	($S.P.S_1$)	$S(1,-1,-2)$
7.	Interpenetrating spirals	($I.P.S_2$)	$S(1,-1,2)$
8.	Superimposed ribbons	(S.P.R.)	$\mathbb{Z}_\kappa \ltimes \mathbb{Z}(\pi,\pi,2\pi)$
9.	Interpenetrating ribbons	(I.P.R.)	$\mathbb{Z}_\kappa(0,0,\pi) \ltimes \mathbb{Z}(\pi,\pi,2\pi)$
12(a)	Wavy spirals	(W.S.)	$\mathbb{Z}(\pi,\pi,0) \times \Delta_a \times S(0,0,1)$
(b)	Wavy spirals	(W.S.)	$\mathbb{Z}(\frac{\pi}{2},0,\pi) \times \Delta_b \times S(0,1,0)$
13.	Unknown		$\mathbb{Z}_\kappa$
14.	"		$\mathbb{Z}_c(0,\pi,0)$
15.	"		$\mathbb{Z}_\kappa(0,0,\pi)$
16.	"		$\mathbb{Z}_c(0,\pi,\pi)$
18.	Wavy interpenetrating spirals	($W.I.S_1$)	$\mathbb{Z}(\frac{\pi}{2},\frac{\pi}{2},\pi)$
22.	Wavy interpenetrating spirals	($W.I.S_2$)	$\mathbb{Z}(\pi,\pi,2\pi)$

TABLE 12

BIFURCATION SEQUENCE DUE TO ISOTROPY LATTICE



§3.2 Bifurcation sequences in the ten-dimensional kernel.

As might be expected there are many possible sequences of bifurcations leading from steady-states to three-torus flows. Recall, from §2.0, that we model a three mode interaction by introducing unfolding parameters (α,β) . In particular we replace q^1 , r^1 (in the equivariant normal form) by $\alpha + q^1$, $\beta + r^1$ respectively. See Fig. 1.

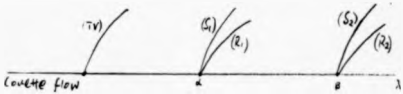


Fig. 1. Schematic of the primary bifurcations in the unfolded system

In Table 13 we list 41 non-degeneracy conditions (and 9 simplifying conditions *); with the caveat that there exist partial conditions distinguishing different 3-torus flows, an $O(2) \times SO(2) \times T^2$-equivariant problem is non-degenerate if the 41 conditions are non-zero.

In Tables 14 and 15 we compute the branching equations and the signs of the real parts of the eigenvalues from Tables 9(a) and 10 §2.3, in terms of the non-degeneracy conditions thus allowing us to decide the direction of branching and the orbital asymptotic stabilities. Since all conditions are to be evaluated at zero we adopt the obvious shorthand in Table 13, e.g. $p^1_{N_0}(0,\ldots,0) = p^1_{N_0}$.

TABLE 13

Non-degeneracy conditions

(1) $p^1_{N_0}$

(2) p^1_λ

(3_1) $q^1_{N_1}$

(3_2) $r^1_{N_2}$

(4_1) q^1_λ

(4_2) r^1_λ

(5_1) $(2q^1_{N_1}+q^3)$

(5_2) $(2r^1_{N_2}+r^5)$

($6^\pm_1$) $(q^1_{N_0}\pm q^7)(2p^1_{N_1}\pm p^4)-p^1_{N_0}(2q^1_{N_1}+q^3)$

(6_2) $2p^1_{N_2}r^1_{N_0}-p^1_{N_0}(2r^1_{N_2}+r^5)$

($7^\pm_1$) $(q^1_{N_0}\pm q^7)p^1_\lambda-q^1_\lambda p^1_{N_0}$

(7_2) $p^1_\lambda r^1_{N_0}-p^1_{N_0}r^1_\lambda$

($8^\pm_1$) $(2p^1_{N_1}\pm p^4)q^1_\lambda-(2q^1_{N_1}+q^3)p^1_\lambda$

(8_2) $2r^1_\lambda p^1_{N_2}-p^1_\lambda(2r^1_{N_2}+r^5)$

(9) $q^1_{N_2}$

(10_1) $q^1_\lambda r^1_{N_2}-r^1_\lambda q^1_{N_2}$

(10_2) $q^1_\lambda r^1_{N_2}-r^1_\lambda(q^1_{N_2}+q^5)$

(11_1) $q^1_{N_1}r^1_{N_2}-q^1_{N_2}r^1_{N_1}$

(11_2) $q^1_{N_1}(r^1_{N_2}+r^3)-(q^1_{N_2}+q^5)r^1_{N_1}$

(12_1) $r^1_\lambda q^1_{N_1}-q^1_\lambda r^1_{N_1}$

(13) $q^1_\lambda(2r^1_{N_2}+r^5)-r^1_\lambda(2q^1_{N_2}+q^5)$

(14) $2q^1_{N_2}+q^5$

(15) $(2q^1_{N_1}+q^3)(2r^1_{N_2}+r^5)-(2r^1_{N_1}+r^3)(2q^1_{N_2}+r^5)$

(16) $r^1_\lambda(2q^1_{N_1}+q^3)-q^1_\lambda(2r^1_{N_1}+r^3)$

(17) q^3

(17_2) r^5

(18_1) $p^1_{N_1}q^1_\lambda-p^1_\lambda q^1_{N_1}$

(19) $r^1_\lambda p^1_{N_2}-p^1_\lambda r^1_{N_2}$

(20) $q^1_\lambda(r^1_{N_1}+r^3)-r^1_\lambda q^1_N$

*(21_1) $p^1_\lambda q^1_{N_2}-q^1_\lambda p^1_{N_2}$

*(21_2) $p^1_\lambda(q^1_{N_2}+q^5)-p^1_{N_2}q^1_\lambda$

*($22^\pm_1$) $2p^1_{N_2}(q^1_{N_0}\pm q^7)-(2q^1_{N_2}+q^5)p^1_{N_0}$

*(23_1) $p^1_{N_1}$

*(23_2) $p^1_{N_2}$

(24) $q^1_\lambda p^1_{N_0}-q^1_{N_0}p^1_\lambda$

*(25) $r^1_{N_1}$

*($26^\pm$) $p^1_\lambda(2r^1_{N_1}+r^3)-r^1_\lambda(2p^1_{N_1}\pm p^4)$

TABLE 13 (ctd.)

$A_i^j = \det M_i$ where $M_i = M\backslash i^{th}$ column

$$\text{s.t. } M = \begin{bmatrix} p^1_\lambda & p^1_{N_0} & 2p^1_{N_0}+(-1)^j p^4 & 2p^1_{N_2} \\ q^1_\lambda & q^1_{N_0}+(-1)^j q^7 & 2q^1_{N_1}+q^3 & 2q^1_{N_2}+q^5 \\ r^1_\lambda & r^1_{N_0} & 2r^1_{N_1}+r^3 & 2r^1_{N_2}+r^5 \end{bmatrix}$$

$B_i = \det N_i$ where $N_i = N\backslash i^{th}$ column

$$\text{s.t. } N = \begin{bmatrix} q^1_\lambda & q^1_{N_1} & q^1_{N_2}+q^5 & q^1_{N_2} \\ r^1_\lambda & r^1_{N_1}+r^3 & r^1_{N_2} & r^1_{N_2}+r^5 \\ r^1_\lambda & r^1_{N_1} & r^1_{N_2}+r^5 & r^1_{N_2} \end{bmatrix}$$

Notes: (1) It is clear that there exist a number of relations between the above conditions. However, to keep the notation simple we introduce non-degeneracy conditions when it is 'legitimate' to do so. i.e. if on a particular branch a combination of coefficients are required to be non-zero for existence or stability then that combination is the given non-degeneracy condition.

(2) In the cases * the conditions are not strictly necessary. See Tables 14 and 15.

TABLE 14

Solution Branches

1. $\lambda = -\frac{(1)}{(2)} u_0$

2(a) $\lambda = -\frac{\alpha}{(4_1)} - \frac{(3_1)}{(4_1)} u_1$

(b) $\lambda = -\frac{\beta}{(4_2)} - \frac{(3_2)}{(4_2)} u_3$

3(a) $\lambda = -\frac{\alpha}{(4_1)} - \frac{(5_1)}{(4_1)} u_1$

(b) $\lambda = -\frac{\beta}{(4_2)} - \frac{(5_2)}{(4_2)} u_3$

4^+(a) $\lambda = \frac{(1)}{(7_1^{\pm})}\alpha - \frac{(6_1^{\pm})}{(7_1)} u_1$ 4(b) $\lambda = \frac{(1)}{(7_2)}\beta - \frac{(6_2)}{(7_2)} u_3$

5^-(a)

$\alpha(2) = (8_1^{\pm})u_1 - (7_1^{\pm})u_0$ $\beta(2) = (8_2)u_3 - (7_2)u_0$

6. $\lambda = \frac{1}{(10_1)} [-\alpha(3_2)+\beta(9_1)-(11_1)u_1]$. $(4_1)\beta-(4_2)\alpha = (12)u_1-(10_1)u_3$

7. $\lambda = \frac{1}{(10_2)} [-\alpha(3_2)+\beta((14)-(9)-(11_2)u_1]$. $(4_1)\beta-(4_2)\alpha = (12)u_1-(10_2)u_4$

8,9 $\lambda = \frac{1}{(13)} [-\alpha(5_2)+\beta(14)-(15)u_1]$, $(4_1)\beta-(4_2)\alpha = (16)u_1-(13)u_3$

12(a) + (b)* Ruled out by (17_1) and (17_2) .

TABLE 14 (ctd.)

$13^+, 15^+$ $\quad \lambda = \frac{1}{A_3^j}[-\alpha(6_2) + \beta(22_1) + A_1^j u_1] \qquad j = \begin{cases} 0 \ ; & 13, 15 \\ 1 \ ; & 14, 16 \end{cases}$

$14^-, 16^-$

$$\alpha(7_2) - \beta(7_1^{\pm}) = A_4^j u_1 + A_3^j u_3$$

$$\alpha(8_2) + \beta[(2)(10_2) + 2(21_1)] = A_3^j u_0 + A_2^j u_1$$

18. $\quad \lambda = \frac{1}{B_2}[(\alpha(5_2) - \beta(14))(17_2) - B_1 u_1]$

$$0 = -(17_2)(\alpha(4_2) + \beta(4_1)) - B_3 u_1 - B_2 u_3$$

$$0 = (17_2)(\alpha(4_2) - \beta(4_1)) - B_4 u_1 - B_2 u_4 \ .$$

---

* As remarked in §1.1 we again have the possibility of a degenerate toral bifurcation occurring if $q^3(0) = 0$ or $r^{\mathfrak{F}}(0) = 0$.

TABLE 15.

Signs of real parts of eigenvalues.

0. $(2)\lambda$ [2], $\alpha+(4_1)\lambda$, [4], $\beta+(4_2)\lambda$ [4].

1. O[1], (1) [1], $\alpha+\frac{(7_1^{\pm})}{(2)}u_0$[2,2] , $\beta+\frac{(7_2)}{(2)}u_0$[4]

2(a) O[1], (3_1) [1], (17_1) [2], $-\frac{(2)}{(4_1)}\alpha+\frac{(18_1)}{(4_1)}u_1$ [2], $\beta-\frac{(4_2)}{(4_1)}\alpha-\frac{(12)}{(4_1)}u_1$ [2]

$\beta-\frac{(4_2)}{(4_1)}\alpha+\frac{(20)}{(4_1)}u_1$ [2]

(b) O[1], (3_2) [1], (17_2)[2], $-\frac{(2)}{(4_2)}\beta+\frac{(19)}{(4_2)}u_3$[2], $\alpha-\frac{(4_1)}{(4_2)}\beta-\frac{(10_1)}{(4_2)}u_3$ [2]

$\alpha-\frac{(4_1)}{(4_2)}\beta-\frac{(10_2)}{(4_2)}u_3$ [2]

3(a) O[2], $-(3_1)$ [1], (5_1) [1], $-\frac{(2)}{(4_1)}\alpha+\frac{(8_1^{\pm})}{(4_1)}u_1$[1,1], $\beta-\frac{(4_2)}{(4_1)}\alpha-\frac{(16)}{(4_1)}u_1$ [4]

(b) O[2], $-(3_2)$ [1], (5_2) [1], $-\frac{(2)}{(4_2)}\beta+\frac{(8_2)}{(4_2)}u_3$[2], $\alpha-\frac{(4_1)}{(4_2)}\beta-\frac{(13)}{(4_2)}u_3$ [4]

4(a) O[2], $T_1^{(a)}=(1)u_0+(5_1)u_1$, $D_1^{(a)}=-(6_1^{+})u_0u$

$T_2^{(a)}=-(2(17_1)+\frac{((8_1^{+})-(8_1^{-}))}{(4_1)})u_1-4((14)-2(9))u_0$, $D_2^{(a)}$

$E^{(a)}=\beta-\frac{(7_2)}{(7_1^{\pm})}\alpha+\ldots$ [4]

(b) O 2 , $T_1^{(b)}=(1)u_0+(5_2)u_3$, $D_1^{(b)}=-(6_2)u_0u_3$

$T_2^{(b)}=-(17_2)$, $D_2^{(b)}$

$E^{(b)}=\alpha+\frac{(24)}{(7_2)}\beta+\ldots.$ [4]

TABLE 15 (ctd.)

5(a) O[2], Trace = $T_1^{(a)}$, Det = $-(6_1^-)u_0u_1$

Trace = $-2(17_1)u_1 + ((8_1^+)-(8_1^-))/(4_1)u_1 + 4((14)-2(9))u_0$

$E^{(a)} = \beta - \frac{(7_2)}{7_1^-}\alpha + \ldots.$ [4] ...

(b) O[2], $T_1^{(b)}$, $D_1^{(b)}$ $T_2^{(b)}$, $-D_2^{(b)}$ $E^{(b)} = \alpha + \frac{(24)}{(7_2)}\beta + \ldots$ [4]

6. O[2], $\frac{1}{(10_1)}((19)\alpha-(21_1)\beta + (-(2)(11_1)+(23_1)(10_1)+(23_2)(12))u_1$ [2]

$+ (17_1)u_1 + ((14)-2(9))u_3$ [2], $+ (\frac{(20)-(12)}{(4_1)})u_1 + (17_2)u_3$ [2]

Trace = $(3_1)u_1 + (3_2)u_2$, Det = $(11_1)u_1u_3$

7. O[2], $\frac{1}{(10_2)}((19)\alpha-(21_2)\beta + (-(2)(11_2)+(23_1)(10_2)+(23_2)(12))u_1$ [2]

$(17_1)u_1-(14)-2(9_1))u_4$ [2], $(\frac{(12)-(20)}{(4_1)})u_1 + (17_2)u_4$ [2]

Trace = $(3_1)u_1 + (3_2)u_2$ Det = $(11_2)u_1u_4$

8. O[3], $E^{\pm} = \frac{1}{(13)}[(9_2)\alpha+(12)(14)-2(23_2)(4_1))\beta+((5_2)(8_1^{\pm})+(14)(26^{\pm}) +$
$+ 2(23_2)(16)u_1]$

T_1 = Trace = $(5_1)u_1 + (5_2)u_3$ D_1 = Det = $(15)u_1u_3$

T_2 = Trace = $-2[3\,q^{13}u_1u_3 + q^3u_1 + 2r^{13}u_1^2 + r^5u_3 + ..]$

9. O(3), E, T_1, D_1, $T_3 = 2[3\,q^{13}u_1u_3 - q^3u_1 + 2r^{13}u_1^2 - r^5u_3 + \ldots]$

Note: Refer to Table 10 §2.3 for remaining eigenvalues.

Obtaining the non-degeneracy conditions necessary for the given branching equations is a straightforward application of the Implicit function theorem. We compute the conditions for solution types 13, 14, 15 and 16, the remainder follow in a similar fashion.

The branching equations associated with these solution types are

$$p^1 + (-1)^j p^4 u_1 + (-1)^\ell p^7 u_0 u_3 + (-1)^{j+\ell} p_1^{10} u_1 u_3 = 0$$

$$\mathrm{Re}(Q^1 + (-1)^j Q^7 u_0 + (-1)^\ell Q^{13} u_1 u_3 + (-1)^{j+\ell} Q_1^{11} u_0 u_3) = 0$$

$$\mathrm{Re}(R^1 + (-1)^\ell R^7 u_0^2 + (-1)^\ell R^{13} u_1^2 + (-1)^{j+\ell} R_1^{14} u_0 u_1) = 0$$

where $j = \begin{cases} 0 & \text{for } 13, 15 \\ 1 & \text{for } 14, 16 \end{cases}$; $\ell = \begin{cases} 0 & \text{for } 13, 14 \\ 1 & \text{for } 15, 16 \end{cases}$ and the

invariant functions are evaluated on

$$(u_0, 2u_1, 2u_3, u_1^2, u_3^2, 2u_1u_3, (-1)^j u_0 u_1, 0, (-1)^\ell u_0^{m+1} u_3^m, 0, (-1)^\ell u_1^{m+1} u_3^m, 0, u_1 u_3, 0, \lambda) .$$

See Table 9(a). Expanding to lowest order, we have

$$0 = p_\lambda^1 \lambda + p_{N_0}^1 u_0 + (2p_{N_1}^1 + (-1)^j p^4) u_1 + 2p_{N_2}^1 u_3 \ldots$$

$$0 = \alpha + q_\lambda^1 \lambda + (q_{N_0}^1 + (-1)^j q^7) u_0 + (2q_{N_1}^1 + q^3) u_1 + (2q_{N_2}^1 + q^5) u_3 + \ldots \qquad (1)$$

$$0 = \beta + r_\lambda^1 \lambda + r_{N_0}^1 u_0 + (2r_{N_1}^1 + r^3) u_1 + (2r_{N_2}^1 + r^5) u_3 .$$

Writing equation (1) as $T^j(u_0, u_1, u_3, \lambda, \alpha, \beta) = 0$ the implicit function theorem guarantees solutions for λ, u_0 and u_3 as functions of u_1

in some neighbourhood of the origin if

$$\det(dT^j_{(u_0,u_1,u_3)})\Big|_{(0)} \neq 0 \ , \quad \det(dT^j_{(\lambda,u_1,u_3)})\Big|_{(0)} \neq 0$$
$$\det(dT^j_{(\lambda,u_0,u_3)})\Big|_{(0)} \neq 0 \ , \quad \det(dT^j_{(\lambda,u_0,u_1)})\Big|_{(0)} \neq 0 \qquad (2)$$

Writing the above as A^j_0 , A^j_1 , A^j_2 and A^j_3 respectively it is straightforward to obtain the entries in Table 12.

We bring together a shorthand analysis of Tables 13 and 14 in Table 15.

TABLE 16

Solution type	Bifurcation point	Conditions (> 0) Supercritical	Stability at bifurcation	Point of Secondary Bifurcation	Conditions (> 0) for Secondary bifurcation	
1. (T.V.)	0	$-\frac{(1)}{(2)}$	$(1), -\alpha, -\beta$	$\lambda_1^{\pm} = \frac{(1)}{(7_1)}\alpha$	$-\frac{(2)}{(7_1^{\pm})}\alpha$	$(1^{\pm})$
				$\lambda_2 = \frac{(1)}{(7_2)}$	$-\frac{(2)}{(7_2)}\beta$	(2)
2(a) (S_1)	$-\frac{\alpha}{(4_1)}$	$-\frac{(3_1)}{(4_1)}$	$(3_1), (17_1), \frac{(-2)}{(4_1)}\alpha$	$\lambda_3 = -\frac{(23_1)}{(18_1)}\alpha$	$\frac{(2)}{(18_1)}\alpha$	(3)
			$\beta - \frac{(4_2)}{(4_1)}\alpha$	$\lambda_4 = \frac{(25_1)\alpha - (3_1)\beta}{(12)}$	$\frac{\beta(4_1) - \alpha(4_2)}{(12)}$	(4)
				$\lambda_5 = \frac{(3_1)\beta - (r^1_{N_1} + r^3)\alpha}{(20)}$	$\frac{\beta(4_1) - \alpha(4_2)}{(20)}$	(5)
2(b) (S_2)	$-\frac{\beta}{(4_2)}$	$-\frac{(3_2)}{(4_2)}$	$(3_2), (17_2), -\frac{(2)}{(4_2)}\beta$	$\lambda_6 = -\frac{(23_2)}{(19)}$	$\frac{(2)}{(19)}\beta$	(6)
			$\alpha - \frac{(4_1)}{(4_2)}\beta$	$\lambda_7 = \frac{(9_1)\beta - (3_2)\alpha}{(10_1)}$	$\frac{\alpha(4_2) - \beta(4_1)}{(10_1)}$	(7)
				$\lambda_8 = \frac{-(3_2)\alpha + \{(14) - (9)\}\beta}{(10_2)}$	$\frac{\alpha(4_2) - \beta(4_1)}{(10_2)}$	(8)

TABLE 16 (ctd.)

3(a)	(R_1)	$-\frac{\alpha}{(4_1)}$	$-\frac{(5_1)}{(4_1)}$	$-(3_1),\ (5_1),\ -\frac{(2)}{(4_1)}\alpha$	$\lambda_q^{\pm} = -\frac{((8_1^{\pm})+(2)(5_1))\alpha}{(8_1^{\pm}).(4_1)}$	$\frac{(2)}{(8_1^{\pm})}\alpha$	$(9^{\pm})$
				$\beta - \frac{(4_2)}{(4_1)}\alpha$	$\lambda_{10} = \frac{\alpha(2r_{N_1}^1+r^3)-\beta(5_1)}{(16)}$	$\frac{\beta(4_1) - \alpha(4_2)}{(16)}$	(10)
3(b)	(R_2)	$-\frac{\beta}{(4_2)}$	$-\frac{(5_2)}{(4_1)}$	$-(3_2),\ (5_2),\ -\frac{(2)}{(4_2)}\beta$	$\lambda_{11} = -\frac{2(23_2)}{(8_2)}\beta$	$\frac{(2)}{(8_2)}\beta$	(11)
				$\alpha - \frac{(4_1)}{(4_2)}\beta$	$\lambda_{12} = \frac{\beta(14) - \alpha(5_2)}{(13)}$	$\frac{\alpha(4_2) - \beta(4_1)}{(13)}$	(12)
4(a)	$(T.T.V_1)$	λ_1^+	$-\frac{(6_1^+)}{(7_1^+)}$	$(1),\ -(6_1^+),\ 2(9)-(14)$	Toral bifurcation	$-(1)(5_1)$	(13)
				$\beta - \frac{(7_2)}{(7_1^+)}$	Toral bifurcation	$-(4_1)[2(4_1)(17_1)+(8_1^+)-(8_1^-)]$ $-[(14)-2(9)]$	(14)
		λ_q^+	$-\frac{(6_1^+)}{(8_1^+)}$	$(5_1),\ -(6_1^+)$ $\left(\frac{(8_1^-)-(8_1^+)-2(17_1)}{(4_1)}\right)$	Bifurcation to solution types (13) or (15) $\lambda_{13} = \frac{1}{A_3^0}[A_1^0(7_2)-A_4^0(6_2))\alpha +$ $(A_4^0(22_1^+)-A_1^0(7_1^-))\beta]$	$\frac{\alpha(7_2) - \beta(7_1^+)}{A_1^0}$	(15)
				$\beta - \frac{1\alpha}{(8_1^+)}[(4_2)(2p_{N_1}^1+p^4)-(2)$ $(2r_{N_1}^1+r^3)]$			

TABLE 16 (ctd.)

4(b) (T.T.V_2)	λ_2	$-\frac{(6_2)}{(7_2)}$	(1), $-(6_2)$, $-(17_2)$, $D_2^{(b)}$ $\alpha + \frac{(24)}{(7_2)}\beta$	Toral bifurcation Bifurcations to solution types (13) or (14)	$-(1)\ (5_2)$	(16)
	λ_{11}	$-\frac{(6_2)}{(8_2)}$	$(5_2, -(6_2)$, $-(17_2)$, $D_2^{(b)}$ $\alpha+\beta\left(\frac{(14)-(2)-2(4_1)(23_2)}{(8_2)}\right)$	$\lambda_{14} = -\frac{\beta(22_1^+) - \alpha(6_2)}{A_3^0}$	$\frac{\alpha(7_2) + \beta(24)}{(4_1)(6_2)-(3_1)(8_2)-(14)(7_2)}$	(17)
5(a) (W.V_1)	λ_1^-	$-\frac{(6_1^-)}{(7_1^-)}$	(1), $-(6_1^-)$ (14)-2(9), $\beta - \frac{(7_2)}{(7_1^-)}\alpha$	Toral bifurcation Toral bifurcation Bifurcations to solution types (14) or (16)	$-(1)\ (5_1)$ $+(4_1)[2(4_1)(17_1)-(8_1^+)+(8^-)]-$ $[(14)-2(9)]$	(18) (19)
	λ_9^-	$-\frac{(6_1^-)}{(8_1^-)}$	(5_1), $-(6_1^-)$ $\frac{(8_1^+)-(8^-)}{(4_1)} - 2(17_1)$ $\beta - \frac{\alpha}{(8_1^-)}[(4_2)(2p_{N_1}^1-p^4)-$ $(2)(2r_{N_1}^1+r^3)]$	$\lambda_{15} = \frac{1}{A_3^1}[(A_1^1(7_2)-A_4^1(6_2))\alpha +$ $(A_4^1(22_1^-)-A_1^1(7_1^+))\beta]$	$\frac{\alpha(7_2) - \beta(7_1^-)}{A_1^1}$	(20)

Table 16 (ctd.)

5(b)	λ_2	$-\frac{(6_2)}{(7_2)}$	$(1),\ -(6_2)$ $-(17_2),\ -D_2^{(b)}$ $\alpha + \frac{(24)}{(7_2)}\beta$	Toral bifurcation Bifurcation to solution types (15) or (16).		
	λ_{11}	$-\frac{(6_2)}{(8_2)}$	$(5_2),\ -(6_2)$ $-(17_2),\ -D_2^{(b)}$ $\alpha + \beta\ \frac{(14).(2)-2(4_1)(23_2)}{(8_2)}$	$\lambda_{16} = \frac{\beta(22_1^{\sim})-\alpha(6_2)}{A_3^1}$	$\frac{\alpha(7_2)-\beta(7_1^{\sim})}{A_3^1}$	(21)
6.	λ_7	$-\frac{(11_1)}{(10_1)}$	$\frac{(19)\alpha + (21_1)\beta}{(10_{11})}$ $(14)-2(9),(17_2)$ $(3_2),\ (11_1)$	Toral bifurcation Bifurcation to solution type (18) $\frac{(17_2)}{B_2B_4}\left[(B_4(5_2)-B_1(4_2))\alpha + (B_1(4_1)-B_4(14))\beta\right]$	$-(17_1)((14)-2(9))$ $-\frac{1}{(4_1)}.(17_2)((2))-12))$ $-(3_1)(3_2)$ $-\frac{1}{(10_1)}\left[\frac{(19)\alpha + (21_1)\beta}{(23_1)(10_1)-(2)(11_1)+(23_2)(12)}\right]$	(22) (23) (24)
	λ_4	$-\frac{(11_1)}{(12_1)}$	$\frac{(12)(25_1)-(23_1)4_2))\alpha+(18)\beta}{(12_1)}$ $(17_1),\ \frac{(20)-(12)}{(4_1)},$ $(3_1),\ (11_1)$			(25)

Table 16 (ctd.)

7. λ_8	$\frac{-(11_2)}{(10_2)}\ \frac{(19)\alpha+(21_2)\beta}{(10_2)}$	Toral bifurcation	$(17_1)\ ((14)-2(9))$	(26)
		Bifurcation to solution type (18).	$-(3_1)(3_2)$	(27)
	$2(9)-(14),\ (17_2),\ (3_2)$			
	(11_2)	$\frac{(17_2)}{B_2B_3}\left[(B_3(5_2)+B_1(4_2))\alpha + (B_1(4_1)-B_3(14))\beta\right]$	$\frac{1}{(10_2)}\left[\frac{(19)\alpha+(21_2)\beta}{(23_1)(10_2)+(23_2)(12)-(2)(11_2)}\right]$	(28)
λ_5	$\frac{(11_2)}{(12)}\ \frac{1}{(10_2)(12)}\left[\alpha((2)(4_2)(11_2)-(2)(3_2)(12)-(10_2)(4_2)(25)) + \beta((23_1)(4_1)(10_2)+((14)-(9))(2)(12)-(4_1)(11_2)(2))\right]$			
	$(17_1),\ \frac{(12)-(20)}{(4_1)},\ (3_1),(11_2)$			

Table 16 (ctd.)

8,9	λ_{10}	$-\frac{(15)}{(13)}$ $\frac{(26^{\pm})\alpha+(8_1^{\pm})\beta}{(16)}$	Toral Bifurcation	$-(5_1)\ (5_2)$	(29)
		(5_1), (15), $-(17_1)$		$\pm(3q^{13}q^3+2r^{12}r^5)<2(3q^3r^5r^{13}q^{13})^{\frac{1}{2}}$	$(30^{\pm})$
	λ_{12}	$\frac{(15)}{(16)}$ $\frac{(8_2)\alpha+((2)(14)-2(23_2)(4_1))\beta}{(13)}$	Bifurcation to solution type		
		(5_2), (15), $-(17_2)$	$(13)^+$, $(15)^+$ or $(14)^-$, $(16)^-$	$D\frac{1}{(13)}[(8_2)\alpha+(12)(14)-2(23_2)(4_1))\beta$	(31)
			$\frac{1}{D}\Big[(2(23_2)(2r_{N_1}^1+r^3)-(5_2)(2p_{N_1}^1\pm p^4))\alpha$ $+((14)(2p_{N_1}^1\ p^4)-2(23_2)(5_1))\beta\Big]$	where $D = (5_2)(8_1^{\pm})+(14)(26^{\pm})+2(23_2)(16)]$.	

Enumeration of all possible bifurcation sequences arising from Table 14, 15 and 16 is clearly an absurd task. However, there is fortunately a sense in which it is pointless. Since we model an experimental system we may make a set of "generic" assumptions either with the experimental observations in mind or by considering the general model. We consider three basic sets of assumptions:

I

1) A codimension one symmetry-breaking steady-state mode interacts with two codimension two symmetry-breaking Hopf modes.

2) The trivial steady-state is asymptotically stable for $\lambda << 0$.

3) A stable steady-state exists.

4) Either a) Two stable rotating waves exist.

b) Both a stable rotating wave and stable standing wave exist.

c) Two stable standing waves exist.

5) Stable mixed modes, both steady-state/Hopf and Hopf/Hopf, exist.

It is important to realise that the above assumptions do not insist that solution types are stable at the point of bifurcation rather, that they either bifurcate stably or become stable by "exchange of stabilities" with a tertiary bifurcation. For instance, inspection of Table 16 shows that rotating waves cannot both bifurcate stably.

Before considering the conditions imposed on the non-degeneracy conditions by assumptions I(a), (b) and (c) some remarks are in order.

1. The first assumption is, of course, saying nothing more than that we consider the "right" singularity with the established non-degeneracy conditions.

2. Since the parameter λ is related to the rotation speed of the concentric cylinders - in the Taylor-Couette system - the second assumption is easily justified.

3. The model coalesces the three modes and we therefore assume that their existence is preserved in the unfolding. The expectation is to observe an initially stable branch which loses stability to a mixed mode solution. Assumptions 4a), b) and c) are partially motivated by the two-mode steady-state/Hopf problem where it has been seen that the stabilities of the rotating and standing wave solutions are mutually exclusive (at least in the codimension two problem).

Let us assume further that solution types S_1 and R_1 may bifurcate stably before S_2 and R_2 thus forcing

$$\beta > \frac{(4_2)}{(4_1)}\alpha \qquad (1)$$

Combining the assumptions with Table 16 we have:

Table 17 $\left(\beta > \frac{(4_2)}{(4_1)} \alpha \right)$

2) $(2) < 0$, $(4_1) < 0$, $(4_2) < 0$; 3) $(1) > 0$

4)a) $(3_1) > 0$, $(17_1) > 0$; b) $(3_1) < 0$ $(5_1) > 0$; c) $(3_1) < 0$; $(5_1) > 0$

$(3_2) > 0$, $(17_2) > 0$ $(3_2) > 0$ $(17_2) > 0$ $(3_2) < 0$, $(5_2) > 0$

5) $\alpha.(7_1^{\pm}) > 0$, $\beta(7_2) > 0$, $\alpha.(18_1) < 0$, $(12) < 0$, $(20) > 0$, $\beta(19) < 0$,

$(10_1) > 0$, $(10_2) > 0$, $\alpha.(8_1^{\pm}) < 0$, $(16) < 0$, $\beta(8_2) < 0$, $(13) > 0$,

$(6_1^{\pm}) < 0$, $(6_2) < 0$, $(11_1) > 0$, $(11_2) > 0$, $(15) > 0$.

It is clear that we could list many more conditions. The above, however, are sufficient to provide the sequences sketched below.

In the diagrams a broken line indicates an unstable solution, an unbroken line a stable solution and four circles a toral bifurcation.

It should be realised that the bifurcation diagrams in no sense form a complete set, not even within the conditions imposed above. In any given context where it has been possible to give the coefficients values via some numerical analysis - here it should be noted that the Taylor-Couette system provides an illustration of such a context but the analysis of §3.2 will be applicable to any system where $O(2) \times T^2$ symmetry is present, Table 16 details all conditions for stabilities and further toral bifurcations down to solution type 9. Obviously the stabilities of the 3-torus flows given by solution types 13,....., 16 require computation of eigenvalues of the appropriate matrices in §2.3 Table 10. For this reason, although we are able to say that solution types 13,... 16 may bifurcate stably, by the exchange of stabilities

rule, a conclusion that they remain stable (e.g. Fig. 4) is not strictly possible without computation of the corresponding eigenvalues. Given, however, the exchange of stabilities rule and the hierachy of bifurcations exhibited in the isotropy lattice the assumption has been made that a set of coefficients and a parameter window exist to allow this behaviour.

Given the above the bifurcation diagrams are intended as representations and consequently assume further trivial conditions not based on any modelling assumptions allowing us to order bifurcations. These are easily extracted from Table 16, for instance for Fig. 1 we assume $\lambda_1^+ < \lambda_1^- = \lambda_2$, $\lambda_4 < \lambda_5$, $\lambda_7 < \lambda_8$.

Two points that are clearly brought out by the bifurcation diagrams:

1) Different solution types can be stable for the same parameter values.

2) Solution types need not necessarily be obtained by smooth exchange from Couette flow.

Point 1 will, of course, be dependent on the values of coefficients. In Fig. 3, to which point 2 applies, it is apparent that spirals can be approached via wavy spirals; a 2-torus flow, or interpenetrating spirals whereas Fig. 1 §3 is more closely modelled by Fig. 2. It is possible that the coefficients for the system will not admit such a possibility.

Finally, it is unfortunate that it is not possible to identify the flow types 13,..., 16. It is, however, possible that further experiment will reveal these flow types. It is also possible that such flows are not realisable in the Couette system; to establish this would require a numerical analysis of the stability conditions as indicated earlier.

I(a)

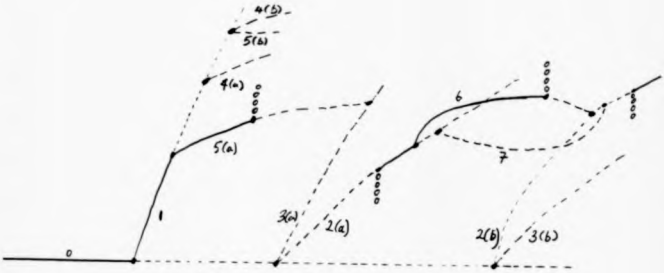


Fig. 1.

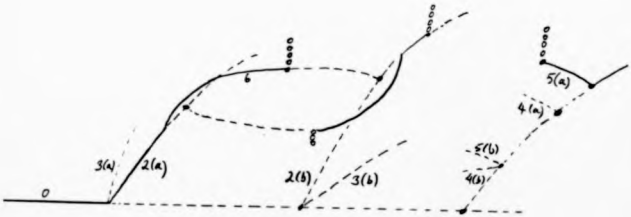


Fig. 2.

I(b)

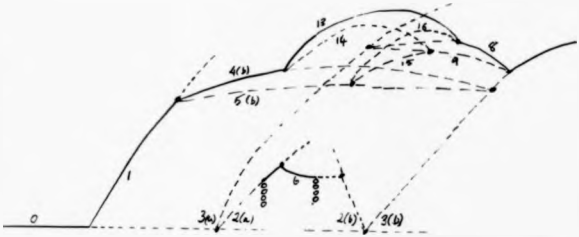

12
14
16
8
9
15
4(b)
5(b)
1
6
0
3(a)
2(a)
2(b)
3(b)

Fig. 3.

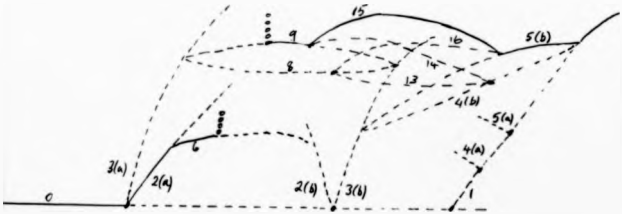

15
9
16
5(b)
8
14
13
4(b)
5(a)
4(a)
6
3(a)
2(a)
0
2(b)
3(b)
1

Fig. 4.

I(c)

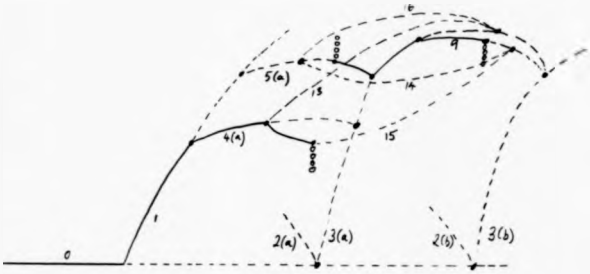


Fig. 5.

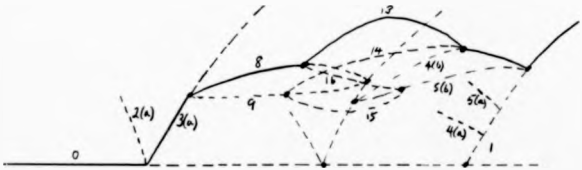


Fig. 6.

Appendix 1.

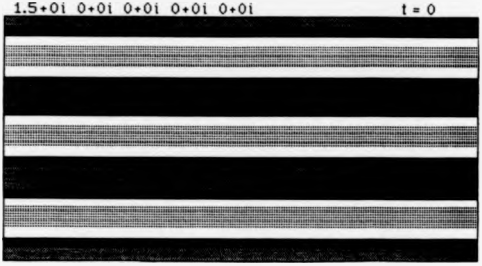
Here we bring together two types of pattern picture, the dotted line picture represents the flow and the shaded pictures the velocity vector field. The programs generated the pictures are given at the end of the appendix. The program generates the two picture types with reference to the eigenfunctions of the ten dimensional kernel, fixing the radial vector as constant and choosing $(z_0, z_1, z_2, z_3, z_4) \in \mathbb{C}^5$ according to the fixed point subspace corresponding to the isotropy subgroup of the pattern.

For each pattern a 3×3 cell is used which, of course, gives 3 periods in each direction. For pattern types 6 et seq. four pictures are given, of the shaded variety, which indicates the time evolution of these 2 and 3-torus flows. Recall that all the time periodic solutions corresponding to symmetry groups higher up the isotropy lattice are axial rotating waves and therefore nothing new would be revealed by similar pictures.

The symmetries of the solution types are perfectly revealed by this method. We see; the diagonal rotation of solution types 6 and 7 ; the modulation of solution types 8 and et seq., along with all the axial flips and phase differences predicted as the symmetries.

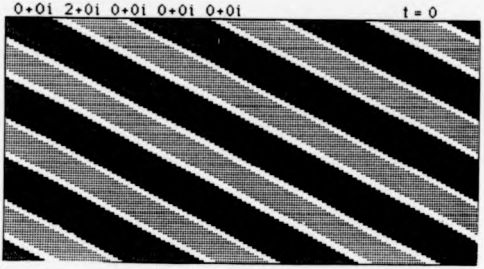
1: Taylor Vortices

1.5+0i 0+0i 0+0i 0+0i 0+0i t = 0



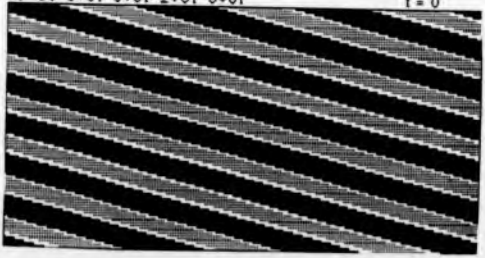
2a: Spiral Vortices 1

0+0i 2+0i 0+0i 0+0i 0+0i t = 0

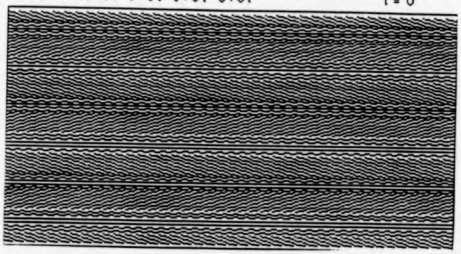


2b: Spiral Vortices 2

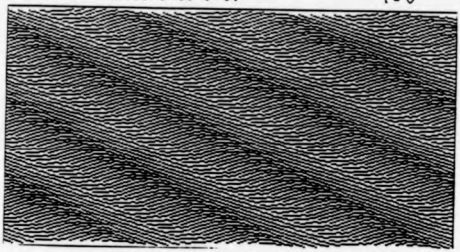
0+0i 0+0i 0+0i 2+0i 0+0i t = 0



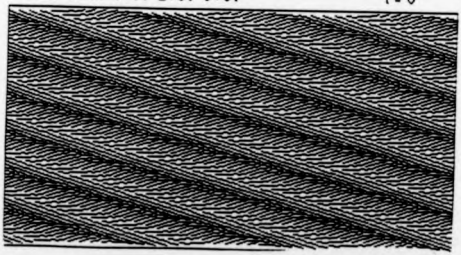
1: Taylor vortices
1.5+0i 0+0i 0+0i 0+0i 0+0i t = 0



2a: Spiral Vortices 1
0+0i 2+0i 0+0i 0+0i 0+0i t = 0

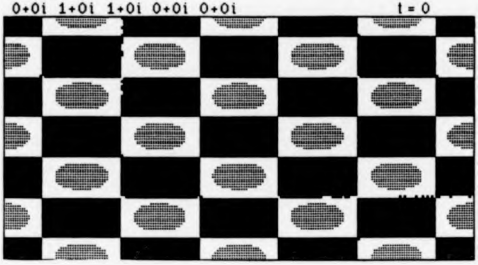


2b: Spiral Vortices 2
0+0i 0+0i 0+0i 2+0i 0+0i t = 0



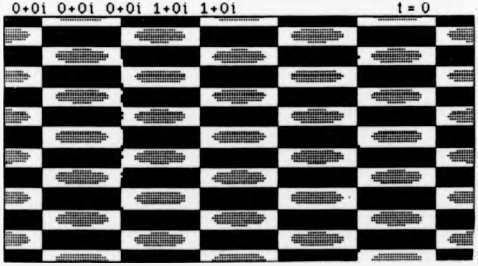
3a: Ribbons 1

0+0i 1+0i 1+0i 0+0i 0+0i t = 0



3b: Ribbons 2

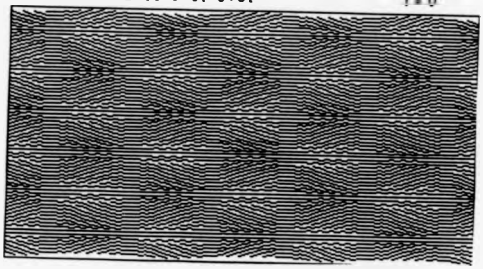
0+0i 0+0i 0+0i 1+0i 1+0i t = 0



3a: Ribbons 1

0+0i 1+0i 1+0i 0+0i 0+0i

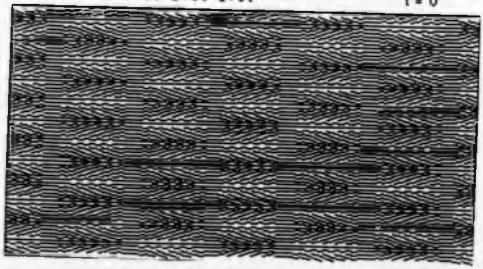
t = 0



3b: Ribbons 2

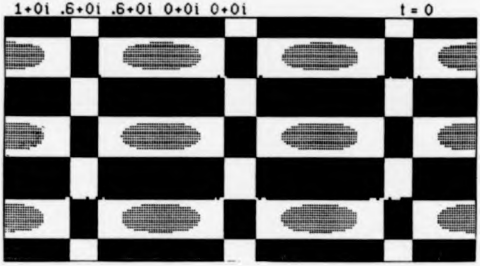
0+0i 0+0i 0+0i 1+0i 1+0i

t = 0



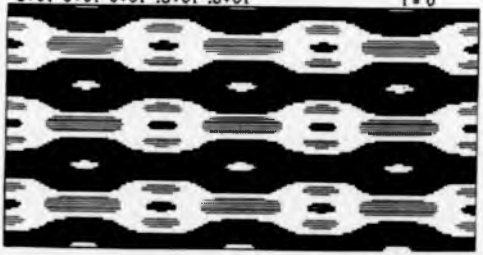
4a: Twisted Vortices 1

1+0i .6+0i .6+0i 0+0i 0+0i t = 0



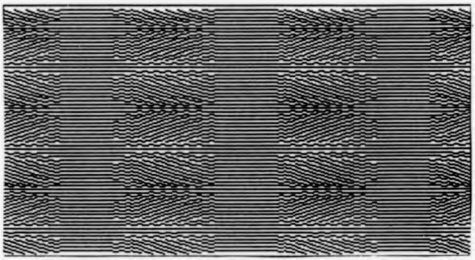
4b: Twisted Vortices 2

1+0i 0+0i 0+0i .6+0i .6+0i t = 0



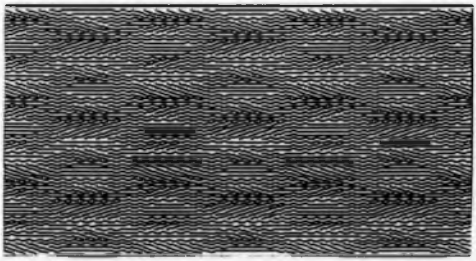
4a: Twisted Vortices 1

1+0i .6+0i .6+0i 0+0i 0+0i t = 0



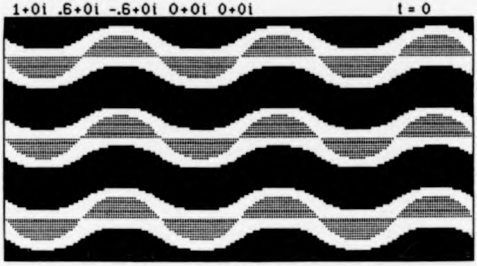
4b: Twisted Vortices 2

1+0i 0+0i 0+0i .6+0i .6+0i t = 0



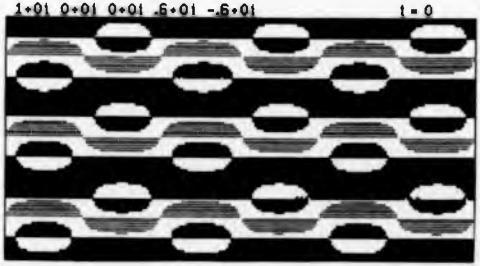
5a: Wavy Vortices 1

1+0i .6+0i -.6+0i 0+0i 0+0i t = 0



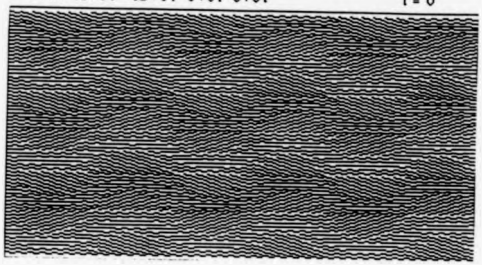
5b: Wavy Vortices 2

1+0i 0+0i 0+0i .6+0i -.6+0i t = 0



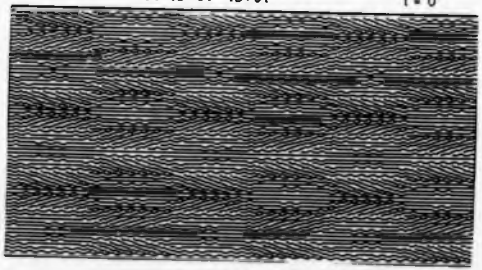
5a: Wavy Vortices 1

1+0i .6+0i -.6+0i 0+0i 0+0i t = 0



5b: Wavy Vortices 2

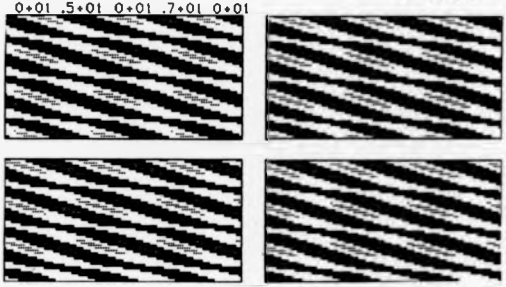
1+0i 0+0i 0+0i .6+0i -.6+0i t = 0



6 Superimposed spirals

0+01 .5+01 0+01 .7+01 0+01

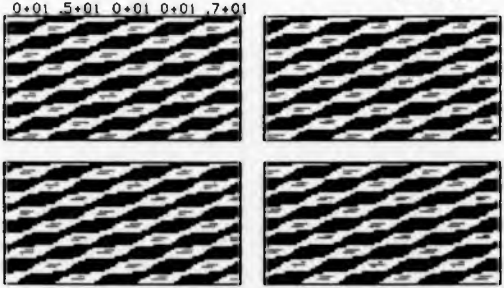
t = 1,2,3,4



7 Interpenetrating spirals

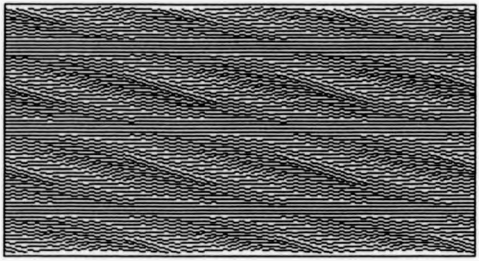
0+01 .5+01 0+01 0+01 .7+01

t = 1,2,3,4



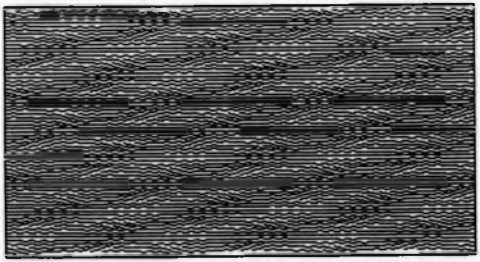
6: Superimposed Spirals

0+0i .5+0i 0+0i .7+0i 0+0i t = 0



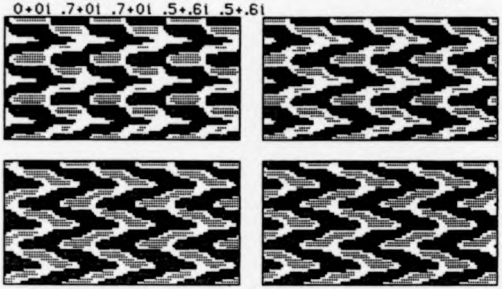
7: Interpenetrating Spirals

0+0i .5+0i 0+0i 0+0i .7+0i t = 0



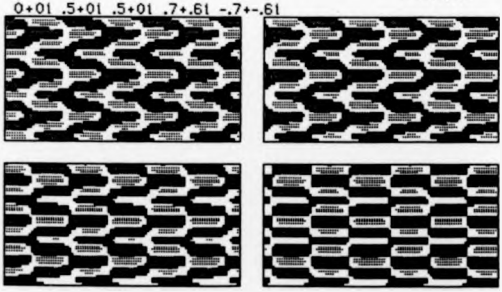
8 Superimposed ribbons t = 1,2,3,4

0+0i .7+0i .7+0i .5+.6i .5+.6i



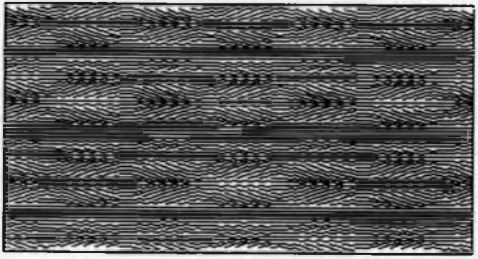
9 t = 1,2,3,4

0+0i .5+0i .5+0i .7+.6i -.7+-.6i



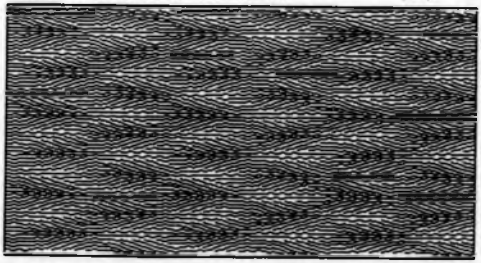
8: Superimposed Ribbons

0+0i .7+0i .7+0i .5+.6i .5+.6i t = 0



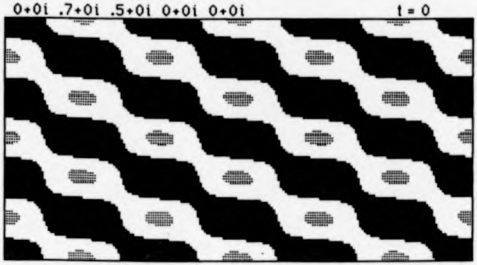
9

0+0i .5+0i .5+0i .7+.6i -.7+-.6i t = 0



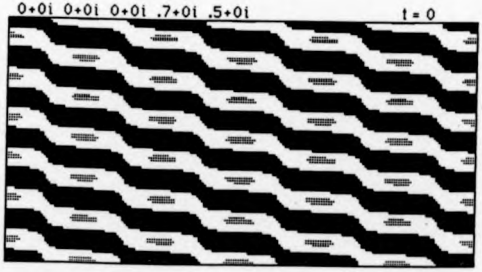
12a

0+0i .7+0i .5+0i 0+0i 0+0i t = 0



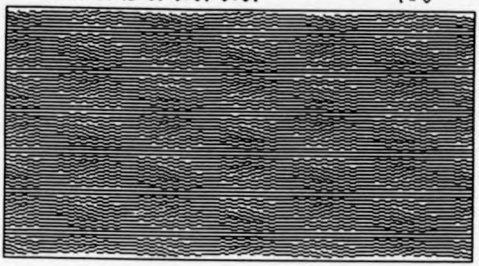
12b

0+0i 0+0i 0+0i .7+0i .5+0i t = 0



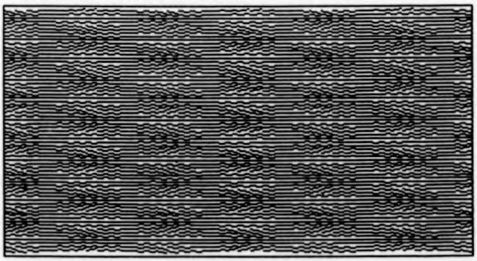
12a

0+0i .7+0i .5+0i 0+0i 0+0i t = 0



12b

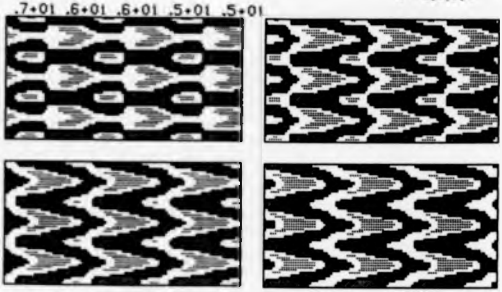
0+0i 0+0i 0+0i .7+0i .5+0i t = 0



13

.7+01 .6+01 .6+01 .5+01 .5+01

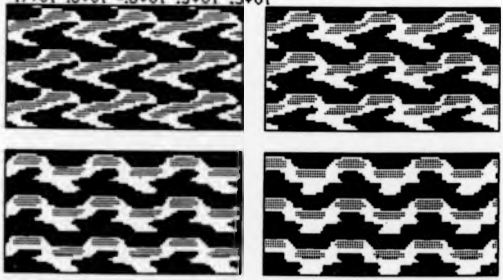
I = 1,2,3,4



14

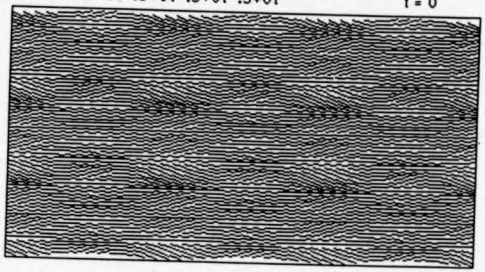
.7+01 .6+01 -.6+01 .5+01 .5+01

I = 1,2,3,4



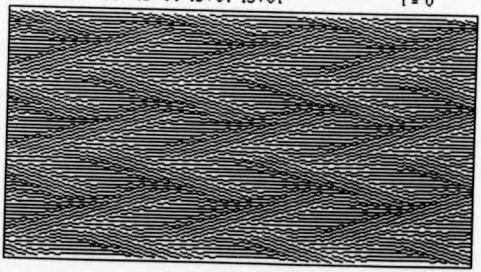
13

.7+0i .6+0i .6+0i .5+0i .5+0i t = 0



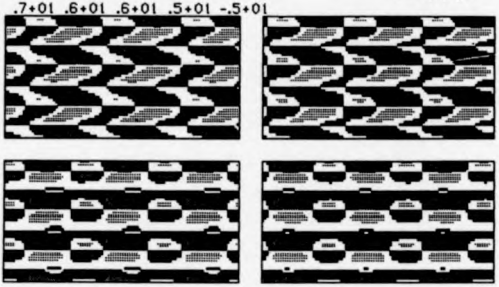
14

.7+0i .6+0i -.6+0i .5+0i .5+0i t = 0



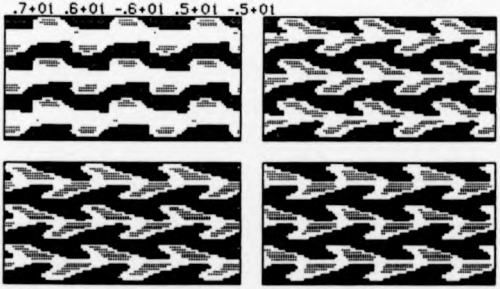
15 t = 1,2,3,4

.7+01 .6+01 .6+01 .5+01 -.5+01



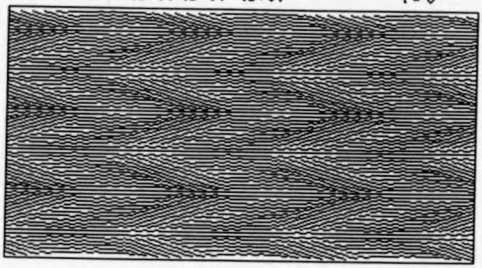
16 t = 1,2,3,4

.7+01 .6+01 -.6+01 .5+01 -.5+01



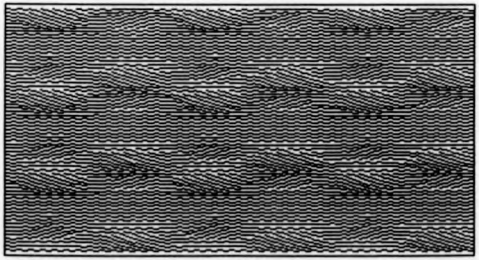
15

.7+0i .6+0i .6+0i .5+0i -.5+0i t = 0



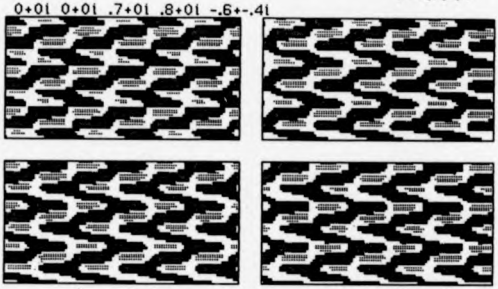
16

.7+0i .6+0i -.6+0i .5+0i -.5+0i t = 0



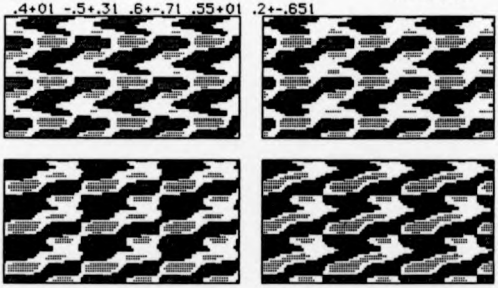
18 t = 1,2,3,4

0+01 0+01 .7+01 .8+01 -.6+-.41



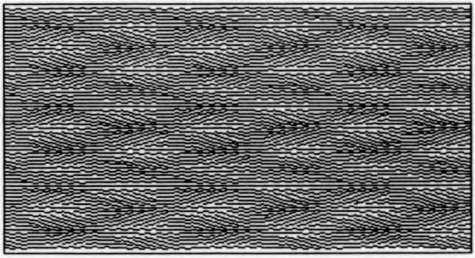
22 t = 1,2,3,4

.4+01 -.5+.31 .6+-.71 .55+01 .2+-.651



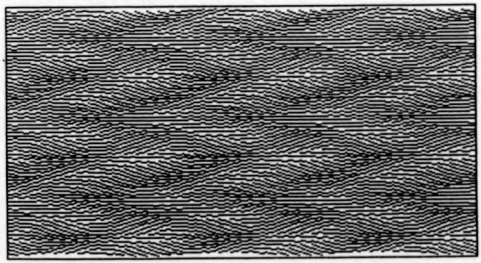
18

0+0i 0+0i .7+0i .8+0i -.6+-.4i t = 0



22

.4+0i -.5+.3i .6+-.7i .55+0i .2+-.65i t = 0



```
pi = 355/113
t = 0
psi = SQR(2)
start:
    CLS
    PRINT "Multiple diagram - 3x3 cells"
    INPUT "Pay or Cow*t?s ""x?s of t?ow ";f$
    IF f$ = "p" OR f$ = "s" THEN STOP
    CLS
    LOCATE 5,1
    INPUT "x0 = ";x0
    INPUT "y0 = ";y0
    INPUT "x1 = ";x1
    INPUT "y1 = ";y1
    INPUT "x2 = ";x2
    INPUT "y2 = ";y2
    INPUT "x3 = ";x3
    INPUT "y3 = ";y3
    INPUT "x4 = ";x4
    INPUT "y4 = ";y4
    PRINT
    INPUT "t = ";t
    CLS
    PRINT "          ";f$
    PRINT "x0=";x0;"+";y0;"i  ";x1;"+";y1;"i  ";x2;"+";y2;"i  ";
    PRINT "x3=";x3;"+";y3;"i  ";x4;"+";y4;"i"
    LOCATE 2,45:PRINT "t = ";t
REM draw box
    PSET (20,30)
    LINE -(20,231)
    LINE -(420,231)
    LINE -(420,30)
    LINE -(20,30)
REM compute z coordinate of velocity vector
    FOR z0 = 0 TO 198 STEP 2
        FOR ph0 = 0 TO 398 STEP 2
        z = 3*pi*z0/100
        phi = 3*pi*ph0/200
        sum = 0
        sum = sum + x0*COS(z)-y0*SIN(z)
        sum = sum + x1*COS(z+phi+t)-y1*SIN(z+phi+t)
        sum = sum + x2*COS(z-phi+t)-y2*SIN(z-phi+t)
        sum = sum + x3*COS(2*z+phi+psi*t)-y3*SIN(2*z+phi+psi*t)
        sum = sum + x4*COS(2*z-phi+psi*t)-y4*SIN(2*z-phi+psi*t)
        IF sum > 1 THEN u = 0
        IF sum >0 AND sum <=1 THEN u = 1
        IF sum > -1 AND sum <= 0 THEN u = 2
        IF sum <= -1 THEN u = 3
        GOSUB pix
        NEXT ph0
    NEXT z0
REM print-out on A4 paper needs 13 extra LPRINTs - delete for US
    LPRINT:LPRINT:LPRINT:LPRINT
    LPRINT:LPRINT:LPRINT:LPRINT
    COPY
    LPRINT:LPRINT:LPRINT: LPRINT:LPRINT
GOTO start

REM draw shaded 2x2 pixel according to axial component z
pix:
    zl=229-z0:pl=20+ph0
    IF u = 3 THEN PSET (pl,zl):RETURN
N   IF u = 2 THEN PSET (pl,zl):PSET(pl+1,zl):PSET(pl,zl+1):PSET(pl+1,zl+1):RETU
    IF u = 0 THEN PSET (pl,zl):PSET(pl+1,zl+1):RETURN
RETURN
```

```
CLS:
PI=355/113
t=0
pr = SQR(2)
hf = 1.5
start:
    LOCATE 2,10
    PRINT "TAYLOR-COUETTE FLOW - vector field"
    LOCATE 5,1
    INPUT "type of flow ";f$
    IF f$ = "s" OR f$ = "S" THEN STOP
    INPUT "x0 = ";x0
    INPUT "y0 = ";y0
    INPUT "x1 = ";x1
    INPUT "y1 = ";y1
    INPUT "x2 = ";x2
    INPUT "y2 = ";y2
    INPUT "x3 = ";x3
    INPUT "y3 = ";y3
    INPUT "x4 = ";x4
    INPUT "y4 = ";y4
    PRINT
    INPUT "t = ";t
    CLS
    PRINT "              ";f$
    PRINT "              ";x0;"+";y0;"i  ";x1;"+";y1;"i  ";x2;"+";y2;"i  ";
    PRINT x3;"+";y3;"i  ";x4;"+";y4;"i"
    LOCATE 25: PRINT "t = "; t
    PSET (20,38)
    LINE -(20,242)
    LINE - (420,242)
    LINE - (420,38)
    LINE - (20,38)
    FOR z0 = 0 TO 198 STEP 3
        FOR ph0 = 0 TO 398 STEP 10
        z = 3*pi*z0/100
        phi = 8*pi*ph0/200
        sum = 0
        sum = sum + x0*COS(z)-y0*SIN(z)
        sum = sum + x1*COS(z+phi*t)-y1*SIN(z+phi*t)
        sum = sum + x2*COS(z-phi*t)-y2*SIN(z-phi*t)
        sum = sum + x3*COS(2*z+phi*psi*t)-y3*SIN(2*z+phi*psi*t)
        sum = sum + x4*COS(2*z-phi*psi*t)-y4*SIN(2*z-phi*psi*t)
        IF ABS(sum) > 8 THEN sum = 8*SGN(sum)
        GOSUB pix
        NEXT ph0
    NEXT z0
    LPRINT:LPRINT:LPRINT:LPRINT:LPRINT:LPRINT:
    LCOPY
    LPRINT:LPRINT:LPRINT:LPRINT:LPRINT:LPRINT:
    GOTO start
STOP

pix:
    z1=237-z0:p1=20+ph0
    LINE (p1,z1)-(p1+9,z1+hf*sum)
RETURN
```

Appendix 2.

Experimental pictures courtesy of Harry Swinney and Randy Tagg.

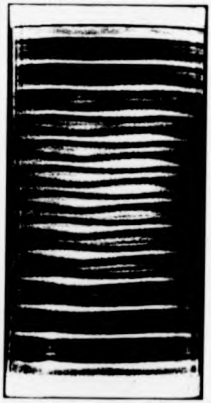
(1) Couette flow; (2) Taylor vortices ; (3) Spiral vortices ; (4) Twisted vortices ; (5) Wavy vortices ; (6) Interpenetrating spirals.

(1)

(2)

(3)

(4)



(5)

(6)

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