TITLE

GRAPHS WITH PARALLEL MEAN CURVATURE AND A VARIATIONAL PROBLEM IN CONFORMAL GEOMETRY

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by

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Abstract

This thesis essentially deals with two basic problems, one in Riemannian, the other in Conformal Geometry, described in Part I resp. Part III. Part II can be considered as an interlude serving as a sort of bridge between Riemannian and Comformal Geometry.

The main result of the first part, formulated in Corollaries 1.1.1 and 1.1.2 of Theorem 1.1, states that any graph $\Gamma_f \subset M \times N$ of a map $f : M \to N$ between Riemannian manifolds, with parallel mean curvature, is minimal, provided M is compact or non-compact with zero Cheeger constant. This result generalises the case $M = \mathbb{R}^m$, $N = \mathbb{R}$, independently treated by E. Heinz, S. S. Chern, and H. F. Flanders. Moreover, Theorem 1.2 and Proposition 2.3 show that, for M the m-hyperbolic space — thus with non-zero Cheeger constant — there exists a real-valued function f, the graph of which is a submanifold of $M \times \mathbb{R}$ with parallel mean curvature H satisfying $\|H\| = c$, where c can be any positive constant less than or equal to the ratio of the Cheeger constant and the dimension m. Furthermore, the behaviour of the mean curvature of a graph is studied in some special cases.

The second part deals with the problem of finding a criterion for an immersion between Riemannian manifolds to be a conformal one. Sufficient conditions on the mean curvature, tension field, and ratio of given and induced volume elements in the immersed manifold are derived in Theorem 1. Thereto, a special, "almost conformal" vector field is introduced, which also allows the obtainment of a Liouville-type theorem for harmonic maps.

Part III is devoted to Conformal Geometry. In chapter 1, the conformal geometry of submanifolds of the Möbius space is extensively reviewed by using Elie Cartan's method of moving frames. As the latter method is scarcely used in the literature, it is treated in a quite detailed way, which might seem excessive to those who are more familiar with it. In chapter 2, the generalised Willmore m-submanifolds of the Möbius space S^n are investigated as critical points of a functional integral, formulated in the framework of conformal geometry, which was introduced by M. Rigoli, leading to an Euler-Lagrange equation. This equation generalises the one obtained by R. L. Bryant (for $m = 2$, $n = 3$) and later by Rigoli (for $2 = m \leq n$). Furthermore, a Bernstein-type theorem is formulated

for Willmore hypersurfaces of S^n, involving the hyperbolic conformal Gauss map, which generalises the Bernstein theorem for surfaces of S^3 due to Rigoli. However, in the general case a condition on the hypersurface has to be imposed, which nevertheless is satisfied by Willmore submanifolds with conformal Gauss map being a critical point of another, well-known functional. Finally, chapter 3 deals with the explicit computation of the second-variation formula for a Willmore surface immersed into a space form. The obtained formula reduces to the one of J. L. Weiner in the special case of a minimal surface of the 3-sphere.

Foreword

It is a pleasure to thank Professor Jim Eells for his constant encouragement and his liberal attitude towards research. Moreover, he offered me several very valuable suggestions of problems to study. I am also indebted to Dr. Marco Rigoli for introducing me to the somewhat misterious, old-new field of conformal geometry. Further, I should mention the helpful conversations with Drs. D. M. Duc and R. Tribuzy. The Calouste Gulbenkian Foundation in Lisbon is thanked for its financial support during my stay abroad and the Faculty of Sciences of the University of Lisbon for permitting my leave of absence. Finally, I dearly thank my husband George Rupp, who lost two months of his work in formatting my thesis with LaTeX and supported me decisively in its realisation.

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Contents

Part I

GRAPHS WITH PARALLEL MEAN CURVATURE

Chapter 0

GENERAL REMARKS AND NOTATIONS

Let (M^m, g), (N^n, h) denote two smooth Riemannian manifolds of dimension m, n, equipped with their respective Levi-Civita connections ∇ and ∇'.

If $\phi : M \to N$ is a C^2-map, then $\phi^{-1}TN \to M$ denotes the pull-back of TN by ϕ, i.e. the C^2-vector bundle with fibre at $x \in M$ given by $T_{\phi(x)}N$. The differential $d\phi$ of ϕ is a C^1- 1-form on M with values in $\phi^{-1}TN$. $\phi^{-1}TN$ has a Riemannian metric induced by the metric h of TN. Let $\nabla^{\phi^{-1}}$ denote the induced connection on $\phi^{-1}TN$, i.e. $\nabla^{\phi^{-1}}$ is the unique linear connection on $\phi^{-1}TN$ such that for each smooth section Z of TN and $x \in M$, $X \in T_xM$

$$\nabla_X^{\phi^{-1}}(Z \circ \phi)_x = \nabla'_{d\phi_x(X)} Z_{\phi(x)}. \tag{0.1}$$

The *first fundamental form* of ϕ is the semi-definite 2-covariant tensor field $\phi^* h$. The *second fundamental form* of ϕ is the section $\nabla d\phi$ of the vector bundle $\odot^2 T^*M \otimes \phi^{-1}TN \to M$ given by

$$\nabla d\phi(X,Y) = \nabla_X^{\phi^{-1}}(d\phi(Y)) - d\phi(\nabla_X Y)\,,$$

where X, Y are smooth vector fields on M.
The *tension field* of ϕ is the section of $\phi^{-1}TN$ given by

$$\tau_\phi = \mathrm{trace}_g(\nabla d\phi)\,.$$

ϕ is said to be *harmonic*, if it has vanishing tension field. The map ϕ is said to be *totally geodesic*, if it has vanishing second fundamental form. If $N = \mathbb{R}$, then $\tau_\phi = \Delta\phi$ is the *Laplacian* of ϕ.

Let $U \subset M$, $\Omega \subset \mathbb{R}^m$ be open sets and $x : U \to \Omega$ be a map that defines a co-ordinate system. Using the index range $i, j, k, \ldots \in \{1, \ldots, m\}$ and writing locally the metric g on U as $g(x) = g_{ij} dx^i dx^j$ (here we use the index-summation convention), that is, $g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_g$, and denoting by $[g^{ij}]$ the inverse matrix of $[g_{ij}]$, by $|g|$ the determinant of $[g_{ij}]$, and by ${}^M\Gamma^k_{ij}$ the Christoffel symbols of the Levi-Cevita connection of M, we have the standard expressions

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = {}^M\Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

$${}^M\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\partial}{\partial x^i} g_{lj} + \frac{\partial}{\partial x^j} g_{il} - \frac{\partial}{\partial x^l} g_{ij} \right\}, \quad {}^M\Gamma^k_{ik} = \frac{\partial}{\partial x^i} \log \sqrt{|g|} \tag{0.2}$$

$$\frac{\partial}{\partial x^i} g_{jk} = {}^M\Gamma^r_{ij} g_{rk} + {}^M\Gamma^r_{ki} g_{jr} .$$

If $X = X^k \frac{\partial}{\partial x^k}$ is a smooth vector field on M and $u = u^k \frac{\partial}{\partial x^k}(x) \in T_x M, x \in M$, we have the following formulae

$$\nabla_u X_{(x)} = u^j \left(\frac{\partial}{\partial x^j} X^k + X^i \, {}^M\Gamma^k_{ij} \right) \frac{\partial}{\partial x^k}$$

$$\begin{aligned} \operatorname{div}(X) &= \frac{\partial}{\partial x^k} X^k + X^i \, {}^M\Gamma^k_{ik} && (0.3) \\ &= \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^j} (X^j \sqrt{|g|}). && (0.4) \end{aligned}$$

If $u : M \to \mathbb{R}$ is a C^1-function, then the *gradient* of u on U is given by

$$\nabla u = g^{kl} \frac{\partial u}{\partial x^l} \frac{\partial}{\partial x^k}. \tag{0.5}$$

If M is oriented and x is an orientation-preserving chart, then the volume element of (M, g) is given by $dV_g = \sqrt{|g|} dx^1 \wedge \ldots \wedge dx^m$.

Let $V \subset N$, $\Omega' \subset \mathbb{R}^n$ be open sets and $y : V \to \Omega'$ be a co-ordinate system on N. Then, using the index range $\alpha, \beta, \ldots \in \{1, \ldots, n\}$, we have, on V, $h(y) = h_{\alpha\beta} dy^\alpha dy^\beta$. Denoting by $[h^{\alpha\beta}]$ the inverse matrix of $[h_{\alpha\beta}]$, by ${}^N\Gamma^\gamma_{\alpha\beta}$ the Christoffel symbols of the Levi-Civita connection of N, the first and second fundamental forms of $\phi : M \to N$ on U are given by (assuming that $\phi(U) \subset V$)

$$(\phi^* h)_{ij} = \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} h_{\alpha\beta}$$

$$(\nabla d\phi)^\gamma_{ij} = \frac{\partial^2 \phi^\gamma}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij} \frac{\partial \phi^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} ,$$

and the tension field of ϕ by

$$\tau_\phi^\gamma = g^{ij}\left(\frac{\partial^2\phi^\gamma}{\partial x^i\partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial\phi^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\right). \tag{0.6}$$

Thus harmonic maps are locally solutions of a system of second-order semi-linear elliptic partial differential equations. From regularity theory of solutions of elliptic equations we know that C^2-harmonic maps of smooth Riemannian manifolds are smooth [Mo/66]. In particular, totally geodesic C^2-maps are smooth. Such maps carry geodesics of M to geodesics of N.

Note: if $N = \mathbb{R}$, Eq. (0.6) takes the following form

$$\Delta\,\phi = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^j}(g^{jk}\sqrt{|g|}\frac{\partial\phi}{\partial x^k}). \tag{0.7}$$

Now assume that $\phi : (M^m, g) \to (N^n, h)$ is an isometric immersion, i.e. $g = \phi^* h$. Then the *mean curvature* H of ϕ is exactly

$$H = \frac{1}{m}\tau_\phi\,.$$

Let $V \to M$ denote the normal bundle of ϕ. Then $\phi^{-1}TN = d\phi(TM) \oplus V$, where the direct sum is an orthogonal one. The second fundamental form $\nabla d\phi$ is a section of $\odot^2 T^*M \otimes V$ and H is a section of V.

If Z is a section of $\phi^{-1}TN$, we will denote by $Z^\top$ and $Z^\perp$ the orthogonal projections of Z on the vector bundles $d\phi(TM)$ and V, respectively. V has an induced Riemannian metric from the one of $\phi^{-1}TN$. The induced connection $\nabla^\perp$ on V is given by

$$\nabla^\perp_X Z = \left(\nabla^{\phi^{-1}}_X Z\right)^\perp$$

for each C^1-section Z of V and $X \in T_xM, x \in M$.

ϕ is said to be a *minimal* immersion, if $H = 0$. That is, ϕ is minimal, if and only if ϕ is harmonic.

ϕ is said to have *constant mean curvature* , if the norm $\|H\|$ of H in V (which is equal to the norm in $\phi^{-1}TN$) is constant.

If ϕ is an isometric immersion of class C^2, then ϕ is said to have *parallel mean curvature* , if H is a parallel C^1-section of V, i.e.

$$\nabla^\perp H = 0\,.$$

Since $\forall x \in M$ and $X \in T_xM$, $d\|H\|_x^2(X) = 2\left\langle \nabla_X^\perp H, H \right\rangle_{h(\phi(x))}$, if ϕ has parallel mean curvature, then it also has constant mean curvature. For $n = m + 1$ the converse is also true.

Given an isometric immersion into a Euclidean space $\phi : (M^m, \phi^* h) \to (\mathbb{R}^n, h)$, the corresponding *Gauss map* of ϕ, $\gamma_\phi : (M, \phi^* h) \to G(n, m)$, where $G(n,m)$ is the Grassmannian manifold of m-spaces through the origin in $\mathbb{R}^n$, is given by $\gamma_\phi(x) = d\phi_x(T_xM)$. Considering $G(n,m)$ with its usual Riemannian structure (see e.g. Ref. [Ko-No/69]), we have the following relation between the mean curvature H of ϕ and the tension field τ_{γ_ϕ} of γ_ϕ due to Ruh and Vilms [Ru-Vi/70] (see also Ref. [Ee-Le/83])

$$\tau_{\gamma_\phi} = m\nabla^\perp H \ .$$

This equality means the following:
$\forall x \in M$ $\tau_{\gamma_\phi}(x) \in T_{\gamma_\phi(x)}G(n,m)$ and, using the canonical identification of

$$T_{\gamma_\phi(x)}G(n,m) \simeq \left(\gamma_\phi(x)\right)^* \otimes \left(\gamma_\phi(x)\right)^\perp = \left(d\phi_x(T_xM)\right)^* \otimes \left(d\phi_x(T_xM)\right)^\perp ,$$

we have

$$\tau_{\gamma_\phi}(x)\big(d\phi_x(X)\big) = m\nabla_X^\perp H_{(x)}, \ \forall X \in T_xM \ .$$

Hence, ϕ has parallel mean curvature, iff γ_ϕ is a harmonic map.

On the vector bundles $\otimes T^*M \otimes \phi^{-1}TN$, $\odot^2 T^*M \otimes V$, ..., i.e. on tensor products of Riemannian vector bundles, we will employ the usual induced Riemannian metrics which at each fibre are the Hilbert-Schmidt inner products. In general, if $\xi : W \to M$ is a vector bundle over a manifold M, then $C^k(W)$ denotes the vector space of C^k-sections of W.

Note that we are using the following sign for the curvature tensor of (M, g)

$$R^M(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

and that, if $P = [e_1, e_2]$ is a plane of T_xM, where e_1, e_2 is an orthonormal basis of P, the sectional curvature of (M, g) of the plane P is given by

$$K_x(P) = K(e_1, e_2) = \left\langle R_x^M(e_1, e_2)e_1, e_2 \right\rangle_g \ .$$

Two very well-known functionals in Riemannian Geometry are the functional volume, applied to isometric immersions, and the functional energy, applied to maps between two Riemannian manifolds.

Let $\phi : M^m \to (N, h)$ be an immersion of an m-manifold M into a Riemannian manifold (N, h). For each oriented compact domain $D \subset M$ (and we will denote by D the interior of $\overline{D}$, that is, $\overline{D} = D \cup \partial D$) the *volume* of ϕ on D is given by

$$V_D(\phi) = \int_D 1 dV_{\phi^* h} ,$$

where $dV_{\phi^* h}$ is the volume element of $(D, \phi^* h)$.

Let $(\phi_t)_{t \in (-\epsilon, \epsilon)}$ be a smooth variation of ϕ such that the vector variation $W = \frac{\partial \phi_t}{\partial t}|_{t=0} \in C^\infty(\phi^{-1}TN)$ has compact support in D. Then it is well-known that

$$\frac{\partial}{\partial t} V_D(\phi_t)|_{t=0} = - \int_D m \langle H_\phi, W \rangle_h dV_{\phi^* h} ,$$

where H_ϕ is the mean curvature of ϕ.

That is, the Euler-Lagrange equation of this variational problem reads $H_\phi = 0$, i.e. the critical points of V_D are the minimal immersions.

If ϕ is a critical point of V_D, then (see Refs. [Si/68], [Sp/79]) the *Hessian* of V_D at ϕ satisfies

$$\mathrm{Hess} V_D(\phi)(W, W) = \frac{\partial^2}{\partial t^2} V_D(\phi_t)|_{t=0} = \int_D \left\langle J_\phi(W^\perp), W^\perp \right\rangle_h dV_{\phi^* h}. \qquad (0.8)$$

Here

$$J_\phi(W^\perp) = -\Delta^\perp W^\perp - A(W^\perp) - \left(\mathrm{Ricci}_\phi(W^\perp)\right)^\perp$$

with $\mathrm{Ricci}^N_\phi(W^\perp)_x = \sum_{i=1}^m R^N_{\phi(x)}(d\phi_x(X_i), W^\perp_x) d\phi_x(X_i), \forall x \in D$, where $X_1, \ldots, X_m$ is an orthonormal basis of $(T_x M, \phi^* h)$, R^N is the curvature tensor of (N, h), $(\ )^\perp$ denotes the orthogonal projection of $\phi^{-1}TN$ onto the normal bundle V of ϕ, A is the element of $C^\infty(\otimes V^* \otimes V)$ given by $A_x(W^\perp_x) = \sum_{i,j} \left\langle \nabla d\phi_x(X_i, X_j), W^\perp_x \right\rangle_h \nabla d\phi_x(X_i, X_j)$, and where $\Delta^\perp$ denotes the Laplacian in the normal bundle:

$$\Delta^\perp W^\perp_x = \sum_{i=1}^m \nabla^{\perp 2} W^\perp_x(X_i, X_i) = \sum_{i=1}^m \nabla^\perp_{X_i} \nabla^\perp_{X_i} W^\perp_x - \nabla^\perp_{\nabla_{X_i} X_i} W^\perp_x$$

(assuming that the X_i are extended as local sections of TM defined on a neighbourhood of x, constituting a local frame of M).

Note that we are using the opposite sign of the Laplacian of Eells and Lemaire [Ee-Le/83] for sections of Riemannian vector bundles, and the sign of the Laplacian of functions adopted by Chavel [Cha/84].

ϕ is said to be *(strictly) volume-stable* in D, if Hess $V_D(\phi)(W, W) \geq 0$ (> 0), for

all $W \in C^\infty(\phi^{-1}TN)\backslash\{0\}$ with compact support contained in D.

The differential operator $J_\phi : C^\infty(V) \to C^\infty(V)$ is the Jacobi operator and is L^2-selfadjoint strongly elliptic [Si/68]. A section W in $C^\infty(V)$ is said to be a Jacobi field on D, if $J_\phi(W) = 0$ on D. If Z is a Killing vector field on (N,h), that is, Z is a vector field on N such that the Lie derivative $L_Z h$ of h along Z is zero, then $(\phi^{-1}Z)^\perp$ is a Jacobi field on D.

If ϕ is an immersion of a hypersurface M^m into (N^{m+1},h), then Eq. (0.8) is simpler. Let ν denote a unit normal to ϕ on D. Then $W^\perp = u\nu$ with $u \in C_c^\infty(D)$, that is, u is a smooth function on D with compact support contained in D. In this case Eq. (0.8) reduces to

$$\begin{aligned} \mathrm{Hess}V_D(\phi)(W,W) &= \int_D u\big(-\Delta u - (R + \|\nabla d\phi\|^2)u\big)dV_{\phi^*h} \qquad (0.9) \\ &= \int_D \big(\|\nabla u\|^2 - (R + \|\nabla d\phi\|^2)u\big)dV_{\phi^*h} \ , \end{aligned}$$

where $R_x = \mathrm{Ricci}_\phi(\nu_x,\nu_x) = \sum_{i=1}^{m} \left\langle R^N_{\phi(x)}(d\phi_x(X_i),\nu_x)d\phi_x(X_i),\nu_x \right\rangle_h$. It is well-known [Fi-Sch/80] (Lemma 1, Th. 1) [Si/68] [Sm/65] that ϕ is strictly volume-stable on D, iff there are no Jacobi fields defined in a subdomain $D' \underset{\neq}{\subset} D$ which are zero on $\partial D'$.

If we have a map $\phi : (M^m,g) \to (N^n,h)$ between two Riemannian manifolds, for each compact oriented domain $\overline{D} \subset M$ the *energy* of ϕ on D is given by

$$E_D(\phi) = \tfrac{1}{2}\int_D \|d\phi\|^2 dV_g \ ,$$

where dV_g is the volume element of (D,g).
If ϕ is an isometric immersion, then $E_D(\phi) = \frac{m}{2}V_D(\phi)$.
It is well-known that ϕ is a critical point of E_D, iff ϕ is a harmonic map. If ϕ is a critical point of E_D, then, for a variation $(\phi_t)_{t\in(-\epsilon,\epsilon)}$ of ϕ, such that $W = \frac{\partial\phi_t}{\partial t}\big|_{t=0} \in C^\infty(\phi^{-1}TN)$ has compact support contained in D,

$$\begin{aligned} \mathrm{Hess}E_D(\phi)(W,W) &= \frac{\partial^2}{\partial t^2}E_D(\phi_t)\big|_{t=0} = \int_D \left\langle -\Delta W - \mathrm{Ricci}^N_\phi(W),W \right\rangle_h dV_g \\ &= \int_D \left(\|\nabla W\|^2 - \left\langle \mathrm{Ricci}^N_\phi(W),W \right\rangle_h\right)dV_g \ , \qquad (0.10) \end{aligned}$$

where Δ is the Laplacian on $\phi^{-1}TN$ and $\mathrm{Ricci}^N_\phi(W)_x = \sum_{i=1}^{m} R^N_{\phi(x)}(d\phi_x(X_i),W)d\phi_x(X_i)$ with $X_1,\dots,X_m$ an orthonormal basis of (T_xM,g) [Ee-Le/83].

A harmonic map ϕ is said to be ***energy-stable*** , if, for every oriented compact domain $\overline{D} \subset M$ and all $W \in C^\infty(\phi^{-1}TN)$ with compact support in D, $\mathrm{Hess}E_D(\phi)(W,W) \geq 0$.

From Eq. (0.10) it follows obviously that, if (N,h) has non-positive sectional curvatures, any harmonic map $\phi : (M,g) \to (N,h)$ is energy-stable.

For a minimal isometric immersion $\phi : (M, \phi^*h) \to (N,h)$, the following relation between $\mathrm{Hess}V_D(\phi)$ and $\mathrm{Hess}E_D(\phi)$ holds, for $W \in C_c^\infty(V)$ (see Ref. [Fe/85]):

$$\mathrm{Hess}E_D(\phi)(W,W) = \mathrm{Hess}V_D(\phi)(W,W) + 2\int_D \|(\nabla^{\phi^{-1}}W)^T\|^2 dV_{\phi^*h} .$$

On a Riemannian manifold can be defined some very important constants, viz. Cheeger, isoperimetric, and Sobolev constants. These constants may provide estimates of eigenvalues and eigenfunctions for the Laplacian operator on domains of M (relative to the Dirichlet problem). One can find an extensive study on these constants in Refs. [Cha/84], [Be-Ga-Ma/71]. In this manuscript we are only going to use the Cheeger constant, the definition of which we give here.

Let (M^m, g) be a non-compact oriented Riemannian manifold with dimension $m \geq 2$, and possibly having a boundary. The ***Cheeger constant*** of M is the non-negative number

$$\mathfrak{h}(M) = \inf_D \frac{A(\partial D)}{V(D)} ,$$

where D ranges over all open submanifolds of M with compact closure in M and smooth boundary, $V(D)$ is the volume of D, and $A(\partial D)$ is the area of the boundary of D.

Due to a result of Yau [Ya/75] (see also Ref. [Cha/84], Theorem 5, page 98), in the definition of $\mathfrak{h}(M)$ it suffices to let D range over open submanifolds of M that are connected. We note that, if M were compact (without boundary), the constant $\mathfrak{h}(M)$ defined as above would be zero. In fact, there is a different definition of the Cheeger constant for a compact manifold (see Ref. [Cha/84]), but we are not going to need it.

The simplest example of complete non-compact Riemannian manifolds with Cheeger constant equal to zero are the ***simple*** Riemannian manifolds, i.e. the Riemannian manifolds (M^m, g) such that there exists a diffeomorphism $\phi : (M,g) \to (\mathbb{R}^m, <,>)$ onto $\mathbb{R}^m$ satisfying $\lambda g \leq \phi^* <,> \leq \mu g$ for some positive constants λ, μ. But there exist also complete Riemannian manifolds diffeomorphic to $(\mathbb{R}^m, <,>)$

with positive Cheeger constant, as for example the m-hyperbolic space. In fact, if (M^m, g) is a complete simply connected Riemannian m-dimensional manifold with sectional curvatures bounded from above by K, where K is a negative constant, then [Ya/75]

$$\mathfrak{h}(M) \geq (m-1)\sqrt{-K}. \qquad (0.11)$$

This result was obtained by using Bishop's comparison theorem to arrive at $\Delta r \geq (m-1)\sqrt{-K}$, where r is the distance function to a fixed point in (M, g), integrating Δr, and using Stokes' theorem.

Another way of estimating $\mathfrak{h}(M)$ is the following inequality due to Cheeger (see e.g. Ref. [Cha/84], theorem 3, page 95)

$$\lambda(D) \geq \frac{1}{4}\mathfrak{h}^2(D), \quad \forall D \subset M \text{domain},$$

where $\lambda(D)$ is the first eigenvalue for the Dirichlet problem in the domain D.

Using this fact and an estimate of the lowest Dirichlet eigenvalue of the geodesic disk of radius δ in the m-hyperbolic space H^m of constant sectional curvature $K = -1$, one can see that $\mathfrak{h}(H^m) = m - 1$ (see Ref. [Cha/84], page 96), so inequality (0.11) is sharp.

Chapter 1

THE MEAN CURVATURE OF A GRAPH

1.1 Introduction

In 1955 Heinz [He/55] proved that, if $z = z(x,y)$ is a surface of $\mathbb{R}^3$ defined for $x^2 + y^2 \leq R^2$ with mean curvature satisfying $\|H\| \geq a > 0$, then $R \leq \frac{1}{a}$. Thus, in particular, if z is defined in all $\mathbb{R}^2$, then $\inf_{\mathbb{R}^2} \|H\| = 0$, which implies that, if z has constant mean curvature, z must be a minimal surface of $\mathbb{R}^3$. In 1965 Chern (see Ref. [Ch/65], Cor. of Th. 1) and, independently, Flanders [Fla/66] obtained the same result for hypersurfaces of $\mathbb{R}^{n+1}$ defined by the equation $z = z(x_1, \ldots, x_n)$.

One can formulate a generalisation of the above problem as follows:
Given two smooth Riemannian manifolds (M,g), (N,h) and a smooth map $f : M \to N$, the graph of f, $\Gamma_f = \{(x, f(x)) : x \in M\}$, is a m-submanifold of the product $M \times N$ of co-dimension n. We take on $M \times N$ the Riemannian metric product $g \times h$ and on Γ_f the induced one.

Question (Eells) *Assume that Γ_f has parallel mean curvature. Does this imply Γ_f to be a minimal submanifold of $M \times N$?*

The basic idea of Chern and Flanders to tackle this question, in the particular cases mentioned above, was to find a way of writing the mean curvature of Γ_f as a divergence of a bounded vector field on M which involves first derivatives of f. This procedure suggests us, in the general case, to relate the mean curvature of Γ_f to the second fundamental form of f. As we will see, the relation between the mean curvature of Γ_f and the tension field of f is more relevant in some special cases, for example when f is an isometry or even a conformal map, a Riemannian

submersion, a harmonic morphism, or when $n = 1$.
In the general case we are going to impose a condition on the Riemannian manifold (M, g) that positively answers the above question (see Cor. 1.1.2). Moreover, we will also show that the absence of this condition conjures up counter-examples (see Th. 1.2).

Let us consider Γ_f as an embedding

$$\begin{aligned} \Gamma_f : M &\to (M \times N, g \times h) \\ x &\to (x, f(x)) . \end{aligned}$$

So we have two Riemannian metrics on M, viz. g and the one induced by Γ_f,

$$\Gamma_f^*(g \times h) = g + f^*h ,$$

which makes $\Gamma_f : (M, g + f^*h) \to (M \times N, g \times h)$ an isometric immersion.
Let ∇ and ∇^* denote the Levi-Civita connections on (M, g) and $(M, g + f^*h)$, respectively. Let V be the normal bundle of Γ_f in $\Gamma_f^{-1}T(M \times N) = TM \times f^{-1}TN$ and $\nabla^* d\Gamma_f \in C^\infty(\odot^2 T^*M \otimes V)$ be the second fundamental form of the immersion Γ_f. The mean curvature of Γ_f is the section

$$H = \frac{1}{m}\mathrm{trace}_{(g+f^*h)}(\nabla^* d\Gamma_f)$$

of V. Let $\nabla df \in C^\infty(\odot^2 T^*M \otimes f^{-1}TN)$ be the second fundamental form of the map f and τ_f its tension field, when M is considered with the metric g. We denote by $\nabla^{f^{-1}}$ and $\nabla^{\Gamma_f^{-1}}$ the induced connections on $f^{-1}TN$ and $\Gamma_f^{-1}T(M \times N)$, respectively, and $\nabla^\perp$ denotes the connection on the normal bundle V. Let $(,)^\perp$ and $(,)^\top$ denote the orthogonal projections of $TM \times f^{-1}TN$ on V and on $d\Gamma_f(TM)$, respectively, relative to the metric $g \times h$.
In general, there is no natural way to relate the Levi-Civita connections ∇ and ∇^* of resp. (M, g) and $(M, g + f^*h)$, but we have the following relation among the connections ∇, $\nabla^{f^{-1}}$, and $\nabla^{\Gamma_f^{-1}}$:
if $X \in C^\infty(TM)$, $U \in C^\infty(f^{-1}TN)$, then (X, U) given by $(X, U)_x = (X_x, U_x)$, $\forall x \in M$, is an element of $C^\infty(\Gamma_f^{-1}T(M \times N))$ and we have

$$\nabla_Y^{\Gamma_f^{-1}}(X, U) = \left(\nabla_Y X, \nabla_Y^{f^{-1}} U\right), \quad \forall Y \in C^\infty(TM) . \tag{1.1}$$

To prove Eq. (1.1) we only have to consider the property Eq. (0.1).

1.2 The General Case

Next we are going to derive an expression for the mean curvature of Γ_f and its covariant derivative in V.

Let $(X_i)_{1\leq i\leq m}$ be a local orthonormal frame of (M,g). Defining

$$\tilde{g}_{ij} := \langle X_i, X_j\rangle_{g+f^*h} = \delta_{ij} + \langle df(X_i), df(X_j)\rangle_h \, , \; \forall i,j \in \{1,\ldots,m\} \tag{1.2}$$

and denoting by $[\tilde{g}^{ij}]_{1\leq i,j\leq m}$ the inverse of the matrix $[\tilde{g}_{ij}]_{1\leq i,j\leq m}$ we have

$$mH = \sum_{i,j=1}^{m} \tilde{g}^{ij}\nabla^* d\Gamma_f(X_i, X_j)\, .$$

Let $(\overline{X}_i)_{1\leq i\leq m}$ be a local orthonormal frame of $(M, g+f^*h)$. Then, $(\overline{X}_i, df(\overline{X}_i))_{1\leq i\leq m}$ is a local orthonormal frame of $d\Gamma_f(TM)$. Next we define the following sections $W \in C^\infty(f^{-1}TN), Z \in C^\infty(TM)$

$$W = \mathrm{trace}_{(g+f^*h)}(\nabla df) \tag{1.3}$$

$$Z = \sum_{i,j=1}^{m} \tilde{g}^{ij}\, \langle W, df(X_i)\rangle_h\, X_j\, . \tag{1.4}$$

We note that Z is well defined over all M and that another way to write Z is

$$Z = \sum_{i=1}^{m} \left\langle W, df(\overline{X}_i)\right\rangle_h \overline{X}_i\, . \tag{1.5}$$

Then we can formulate the following lemma:

Lemma 1.1 $\forall X, Y \in C^\infty(TM)$

(i) $\nabla^* d\Gamma_f(X,Y) = \left(0, \nabla df(X,Y)\right)^\perp$

(ii) $mH = (-Z, W - df(Z)) = (0, W)^\perp$

(iii) $m\nabla_X^{\Gamma_f^{-1}} H = \left(0, \nabla_X^{f^{-1}} W - \nabla df(X,Z)\right) - \left(\nabla_X Z, df(\nabla_X Z)\right)$
$m\nabla_X^\perp H = \left(0, \nabla_X^{f^{-1}} W - \nabla df(X,Z)\right)^\perp$

Proof. Using Eq. (1.1) we have

$$\begin{aligned}
\nabla^* d\Gamma_f(X,Y) &= \nabla_X^{\Gamma_f^{-1}}\left(d\Gamma_f(Y)\right) - d\Gamma_f(\nabla_X^* Y) \\
&= \nabla_X^{\Gamma_f^{-1}}\left(Y, df(Y)\right) - \left(\nabla_X^* Y, df(\nabla_X^* Y)\right) \\
&= \left(\nabla_X Y, \nabla_X^{f^{-1}}(df(Y))\right) - \left(\nabla_X^* Y, df(\nabla_X^* Y)\right) \\
&= \left(\nabla_X Y - \nabla_X^* Y, \nabla df(X,Y) + df(\nabla_X Y - \nabla_X^* Y)\right) \\
&= d\Gamma_f(\nabla_X Y - \nabla_X^* Y) + \left(0, \nabla df(X,Y)\right)\, .
\end{aligned}$$

Since $\nabla^* d\Gamma_f(X,Y) \in C^\infty(V)$, we get (i). Thus, we have

$$\begin{aligned} mH &= \sum_{i,j=1}^m \tilde{g}^{ij}\nabla^* d\Gamma_f(X_i,X_j) = \Big(0, \sum_{i,j=1}^m \tilde{g}^{ij}\nabla df(X_i,X_j)\Big)^\perp \\ &= \big(0, \mathrm{trace}_{(g+f^*h)}(\nabla df)\big)^\perp \\ &= (0,W)^\perp = (0,W) - (0,W)^\top . \end{aligned}$$

Since $(0,W)^\top = \sum_{i=1}^m \big\langle (0,W),(X_i,df(X_i))\big\rangle_{g\times h}(X_i,df(X_i)) = \sum_{i=1}^m \big\langle W, df(X_i)\big\rangle_h \cdot (X_i, df(X_i))$,

$$\begin{aligned} mH &= (0,W) - \Big(\sum_{i=1}^m \big\langle W, df(X_i)\big\rangle_h X_i, \sum_{i=1}^m \big\langle W, df(X_i)\big\rangle_h df(X_i)\Big) \\ &= (0,W) - (Z, df(Z)) , \end{aligned}$$

which gives (ii).

Finally, differentiating the latter expression and using Eq. (1.1) we obtain

$$\begin{aligned} m\nabla_X^{\Gamma_f^{-1}} H &= \big(0, \nabla_X^{f^{-1}} W\big) - \big(\nabla_X Z, \nabla_X^{f^{-1}}(df(Z))\big) \\ &= \big(0, \nabla_X^{f^{-1}} W\big) - \big(\nabla_X Z, \nabla df(X,Z) + df(\nabla_X Z)\big) \\ &= \big(0, \nabla_X^{f^{-1}} W - \nabla df(X,Z)\big) - \big(\nabla_X Z, df(\nabla_X Z)\big) . \ \heartsuit \end{aligned}$$

The following lemma will often be used.

Lemma 1.2 *Let $x \in M$, $X \in T_xM$, and $s \in T_{f(x)}N$. Then $(X,0),(0,s) \in T_xM \times T_{f(x)}N$ and*

(i) $s = 0$ iff $(0,s)^\perp = 0$

(ii) $(X,0) \in V_x$ iff $X = 0$.

Proof. At the point x we have

$$\begin{aligned} (0,s)^\perp &= (0,s) - (0,s)^\top = (0,s) - \sum_{i=1}^m \big\langle (0,s),(X_i, df_x(X_i))\big\rangle_{g\times h}(X_i, df_x(X_i)) \\ &= \Big(-\sum_{i=1}^m \big\langle s, df_x(X_i)\big\rangle_h X_i\,,\ s - \sum_{i=1}^m \big\langle s, df_x(X_i)\big\rangle_h df_x(X_i)\Big) . \qquad (1.6) \end{aligned}$$

If $(0,s)^\perp = 0$, then the first component of the vector in Eq. (1.6) is also zero. Therefore, since $(X_i)_{1\le i\le m}$ is a basis of T_xM, $\big\langle s, df_x(X_i)\big\rangle_h = 0$, $\forall i \in \{1,\ldots,m\}$, and the vector in Eq. (1.6) becomes

$$0 = (0,s)^\perp = (0,s) .$$

That is, $z = 0$ and (i) is proved. Now we prove (ii):
If $(X,0) \in V_x$, then $\forall Y \in T_xM \quad \langle (X,0),(Y,df_x(Y))\rangle_{g\times h} = 0$, so $\langle X,Y\rangle_g = 0$. Hence $X = 0$. ♡

In Ref. [Ee/79] it was pointed out that Γ_f is minimal, iff

$$\mathrm{id} : (M, g + f^*h) \to (M, g) \qquad f : (M, g + f^*h) \to (N, h) \tag{1.7}$$

are both harmonic maps.
In fact, since $\Gamma_f = (\mathrm{id}, f) : (M, g + f^*h) \to (M \times N, g \times h)$, $mH = (\tau^*_{\mathrm{id}}, \tau^*_f)$, where $\tau^*_{\mathrm{id}}, \tau^*_f$ are the tension fields of the maps id and f in Eq. (1.7).
The system (1.7) can be reduced to an equivalent equation.

Proposition 1.1 *The following statements are equivalent:*

(*i*) Γ_f *is minimal,*

(*ii*) $f : (M, g + f^*h) \to (N, h)$ *is harmonic,*

(*iii*) $W = (trace)_{(g+f^*h)}(\nabla df) = 0$.

Also, Γ_f *is a totally geodesic submanifold of* $M \times N$ *, iff* $f : (M, g) \to (N, h)$ *is a totally geodesic map.*

Proof. From Lemma 1.1 (ii) $mH = (-Z, W - df(Z))$, hence $-Z = \tau^*_{\mathrm{id}}$ and $W - df(Z) = \tau^*_f$. Therefore, if (i) holds, then (ii) and (iii) obviously hold.
If (ii) holds, that is $\tau^*_f = 0$, then $mH = (\tau^*_{\mathrm{id}}, 0)$. So, as $H \in C^\infty(V)$ and from Lemma 1.2 (ii), $H = 0$.
If (iii) holds, then $mH = -(Z, df(Z)) \in V \cap d\Gamma_f(TM)$. So $H = 0$.
The last statement follows immediately from Lemmas 1.1(i) and 1.2(i). ♡

To prove the main theorem of part one of this work we recall the following formula (see Ref. [Ee-Le/78], page 9):
Given a map $\phi : (P_1, g_1) \to (P_2, g_2)$ between Riemannian manifolds, we have

$$\mathrm{div}_{g_1}(d\phi \cdot \tau_\phi) = \|\tau_\phi\|^2_{g_2} + \left\langle d\phi, \nabla^{\phi^{-1}} \tau_\phi \right\rangle , \tag{1.8}$$

where $d\phi \cdot \tau_\phi$ is the vector field of TP_1 given by

$$\langle d\phi \cdot \tau_\phi, X\rangle_{g_1} = \langle d\phi(X), \tau_\phi\rangle_{g_2} , \ \forall X \in C^\infty(TP_1) ,$$

and where $\langle\ ,\ \rangle$ is the induced Riemannian metric in the vector bundle $\otimes T^*P_1 \otimes \phi^{-1}TP_2$, that is, $\forall x \in M$, $\left\langle d\phi, \nabla^{\phi^{-1}} \tau_\phi \right\rangle(x) = \sum_{i=1}^m \left\langle d\phi_x(e_i), \nabla^{\phi^{-1}}_{e_i} \tau_{\phi(x)} \right\rangle_h$, where $e_1, \ldots, e_p$ is an orthonormal basis of T_xP_1.
In particular, if ϕ is an isometric immersion, then, since τ_ϕ is orthogonal to $d\phi(TP_1)$, Eq. (1.8) becomes

$$\left\langle d\phi, \nabla^{\phi^{-1}} \tau_\phi \right\rangle = -\|\tau_\phi\|^2_{g_2} . \tag{1.9}$$

This formula can easily be computed directly, too.

Theorem 1.1 *Assume that Γ_f has parallel mean curvature. Let $c = \|H\|_{g\times h}$ (c is a constant). Then, for each oriented compact domain $\overline{D} \subset M$, we have*

$$c \leq \frac{1}{m}\frac{A(\partial D)}{V(D)} .$$

where $V(D)$ is the volume of D and $A(\partial D)$ is the area of ∂D, relative to the metric g.

Proof. From Lemma 1.1 (iii) we have $\forall X \in C^\infty(TM)$

$$0 = m\nabla^\perp_X H = \left(0, \nabla^{f^{-1}}_X W - \nabla df(X, Z)\right)^\perp ,$$

hence, from Lemma 1.2 (i),

$$\nabla^{f^{-1}}_X W = \nabla df(X, Z) .$$

From Lemma 1.1 (iii)

$$m\nabla^{\Gamma_f^{-1}}_X H = -\left(\nabla_X Z, df(\nabla_X Z)\right) . \tag{1.10}$$

From the latter equation we can prove now that

$$m\left\langle \nabla^{\Gamma_f^{-1}} H, d\Gamma_f \right\rangle = -\mathrm{div}_g(Z) \text{ on } M . \tag{1.11}$$

Let $x_0 \in M$ and $X_1, \ldots, X_m$ be a local orthonormal frame of (M, g), defined in a neighbourhood of x_0 and satisfying $\nabla X_i(x_0) = 0$, $\forall i = 1, \ldots, m$. Such frames can be constructed using parallel transport in (M, g).

Then, $Z = \sum_{i,j=1}^{m} \tilde{g}^{ij} \langle W, df(X_i)\rangle_h X_j$ in a neighbourhood of x_0.
Since $\nabla X_i(x_0) = 0$, we have at the point x_0

$$\begin{aligned}\nabla_{X_i} Z &= \sum_{k,p=1}^{m} \nabla_{X_i}\left(\tilde{g}^{kp} \langle W, df(X_k)\rangle_h X_p\right) \\ &= \sum_{k,p=1}^{m} d\left(\tilde{g}^{kp} \langle W, df(X_k)\rangle_h\right)(X_i)\, X_p \,,\end{aligned}$$

so $\forall i, j$

$$\begin{aligned}&\left\langle (\nabla_{X_i} Z, df(\nabla_{X_i} Z))\,,\, (X_j, df(X_j))\right\rangle_{g\times h} = \\ &= \sum_{k,p=1}^{m} \left\langle d\left(\tilde{g}^{kp} \langle W, df(X_k)\rangle_h\right)(X_i)\,(X_p, df(X_p))\,,\,(X_j, df(X_j))\right\rangle_{g\times h} \\ &= \sum_{k,p=1}^{m} \tilde{g}_{pj} d\left(\tilde{g}^{kp} \langle W, df(X_k)\rangle_h\right)(X_i)\,,\end{aligned}$$

and, therefore, from Eq. (1.10)

$$\begin{aligned}m\left\langle \nabla^{\Gamma_f^{-1}} H, d\Gamma_f\right\rangle(x_0) &= \sum_{i,j=1}^{m} m\tilde{g}^{ij}\left\langle \nabla^{\Gamma_f^{-1}}_{X_i} H, d\Gamma_f(X_j)\right\rangle_{g\times h} \\ &= \sum_{i,j=1}^{m} -\tilde{g}^{ij}\left\langle (\nabla_{X_i} Z, df(\nabla_{X_i} Z))\,,\,(X_j, df(X_j))\right\rangle_{g\times h} \\ &= \sum_{i,j,k,p=1}^{m} -\tilde{g}^{ij}\tilde{g}_{pj} d\left(\tilde{g}^{kp} \langle W, df(X_k)\rangle_h\right)(X_i) \\ &= \sum_{i,k,p=1}^{m} -\delta_{ip} d\left(\tilde{g}^{kp} \langle W, df(X_k)\rangle_h\right)(X_i) \\ &= \sum_{i,k=1}^{m} -d\left(\tilde{g}^{ki} \langle W, df(X_k)\rangle_h\right)_{x_0}(X_i)\,.\end{aligned}$$

Since $\sum_{k=1}^{m} \tilde{g}^{ki} \langle W, df(X_k)\rangle_h = \langle Z, X_i\rangle_g$, $\forall i = 1, \ldots, m$ in a neighbourhood of x_0,

$$\begin{aligned}m\left\langle \nabla^{\Gamma_f^{-1}} H, d\Gamma_f\right\rangle(x_0) &= \sum_{i=1}^{m} -d\left(\langle Z, X_i\rangle_g\right)_{x_0}(X_i) = \sum_{i=1}^{m} -\left\langle \nabla_{X_i} Z, X_i\right\rangle_g (x_0) \\ &= -\mathrm{div}_g(Z)(x_0)\end{aligned}$$

and we have proved Eq. (1.11).
On the other hand, from Eq. (1.9) we have

$$\left\langle \nabla^{\Gamma_f^{-1}} H, d\Gamma_f\right\rangle = -m\|H\|^2_{g\times h} = -mc^2\,.$$

So Eq. (1.11) gives

$$m^2c^2 = \mathrm{div}_g(Z) \ \text{ on } M\,. \tag{1.12}$$

Let $D \subset M$ be an oriented compact domain and dV_g, dA_g denote the respective volume elements of D and ∂D relative to the metric g. Applying Stokes' theorem we get

$$\begin{aligned} m^2c^2V(D) &= \int_D m^2c^2 dV_g = \int_D \mathrm{div}_g(Z)dV_g \\ &= \int_{\partial D} \langle Z, \vec{n}\rangle_g \, dA_g \,, \end{aligned}$$

where $\vec{n}$ is the outward unit normal of ∂D.
From the Schwarz inequality $|\langle Z, \vec{n}\rangle_g| \leq \|Z\|_g \|\vec{n}\|_g = \|Z\|_g$ and Lemma 1.1(ii), we obtain

$$mc = m\|H\|_{g\times h} = \|(-Z, W - df(Z))\|_{g\times h} \geq \|Z\|_g \,.$$

Hence

$$m^2c^2V(D) \leq \int_{\partial D} |\langle Z, \vec{n}\rangle_g| dA_g \leq \int_{\partial D} mc dA_g = mcA(\partial D) \,,$$

so $c \leq \dfrac{1}{m}\dfrac{A(\partial D)}{V(D)}$. ♡

Corollary 1.1.1 *If (M, g) is an oriented non-compact Riemannian manifold and $f : M \to N$ is a smooth map such that Γ_f has parallel mean curvature H, then*

$$\|H\|_{g\times h} \leq \frac{1}{m}\mathfrak{h}(M) \,.$$

Corollary 1.1.2 *If (M, g) is an oriented, compact (without boundary) manifold or an oriented non-compact Riemannian manifold with Cheeger constant equal to zero (see Ch. 0 for definition), then for any Riemannian manifold (N, h) and any map $f : (M, g) \to (N, h)$, if the graph $\Gamma_f : (M, g + f^*h) \to (M \times N, g \times h)$ is an immersion with parallel mean curvature, it is in fact a minimal submanifold of $M \times N$.*

In Chapter 0 we recalled that, if (M, g) is a simply connected Riemannian m-dimensional manifold with sectional curvatures bounded from above by K, where K is a negative constant, then $\mathfrak{h}(M) \geq (m-1)\sqrt{-K}$, and that, if M is the m-hyperbolic space, $\mathfrak{h}(M) = m - 1$. Therefore, in such cases Cor. 1.1.2 cannot be applied. Moreover, we will give next an explicit example which shows that the condition on the Cheeger constant of (M, g) is a fundamental criterion for a graph with parallel mean curvature to be minimal.

Theorem 1.2 *Consider the 2-dimensional hyperbolic space (H^2, g), where H^2 is the unit open disk of $\mathbb{R}^2$ with centre at the origin and g is the Riemannian metric on H^2 given by*

$$g = \frac{4|dx|^2}{(1-|x|^2)^2} . \tag{1.13}$$

The function $f : H^2 \to \mathbb{R}$ given by

$$f(x) = \int_0^{r(x)} \sqrt{\frac{1}{2}(\cosh(r) - 1)}dr ,$$

where $r(x) = \log\left(\frac{1+|x|}{1-|x|}\right)$ is the distance function from the origin in H^2, is smooth on all H^2, and $\Gamma_f \subset H^2 \times \mathbb{R}$ has constant mean curvature $\|H\| = \frac{1}{2}$.

Proof. It follows from Lemma (1.3), to be given and proved in the next section, that we only have to verify if f satisfies the equation $\mathrm{div}_g\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|^2}}\right) = 1$.
First we calculate the Christoffel symbols of the m-hyperbolic space (H^m, g), where g is given by Eq. (1.13). Defining the identity map $x : H^m \to \mathbb{R}^m$ as a coordinate system, we have $\frac{\partial}{\partial x^i} = e_i$, where $e_1, \ldots, e_m$ is the canonic basis of $\mathbb{R}^m$. Let $g_{ij} = g(e_i, e_j) = \frac{4}{(1-|x|^2)^2}\delta_{ij}$ and $[g^{ij}]$ be the inverse matrix of $[g_{ij}]$, that is, $g^{ij} = \frac{(1-|x|^2)^2}{4}\delta_{ij}$. Then, using Eq. (0.2), we obtain

$$\Gamma_{ij}^k = \frac{2}{1-|x|^2}(\delta_{kj}x_i + \delta_{ik}x_j - \delta_{ij}x_k) .$$

Now we prove that f is smooth.
$\forall x \in H^2\backslash\{0\}$, $u \in T_xH^2 = \mathbb{R}^2$, we have $df_x(u) = \sqrt{\frac{1}{2}(\cosh(r(x)) - 1)}dr_x(u)$. Note that $\cosh(r(x)) = \frac{1+|x|^2}{1-|x|^2}$ and that $dr_x(u) = \frac{2}{1-|x|^2}\frac{<x,u>}{|x|}$. So

$$df_x(u) = \frac{2 < x, u >}{(1-|x|^2)^{3/2}} . \tag{1.14}$$

Now we show that $\frac{\partial f}{\partial x^i}(0) = 0$ for $i = 1, 2$.

$$\lim_{h\to 0}\left|\frac{f(he_i) - f(0)}{h}\right| = \lim_{h\to 0}\left|\frac{1}{h}\int_0^{2\tanh^{-1}(|h|)} \sqrt{\frac{1}{2}(\cosh t - 1)}dt\right| .$$

Since $\lim_{r\to 0}\frac{\tanh^{-1}(r)}{r} = 1$ and $tanh^{-1} : (-1,1) \to (-\infty, +\infty)$ is an increasing function, we have $\forall\delta > 0, \exists\epsilon > 0$ such that, $\forall h : 0 < |h| < \epsilon$, $\frac{\tanh^{-1}(|h|)}{|h|} < 1 + \delta$ and, $\forall t \in [0, 2\tanh^{-1}(|h|)]$, $\sqrt{\frac{1}{2}(\cosh t - 1)} < \delta$. Hence,

$$\left|\frac{1}{h}\int_0^{2\tanh^{-1}(|h|)} \sqrt{\frac{1}{2}(\cosh t - 1)}dt\right| < \frac{\delta}{|h|}2\tanh^{-1}(|h|) < 2\delta(1+\delta) .$$

So we have proved that Eq. (1.14) also holds for $x = 0$, which proves the smoothness of f on all H^2.
Finally we calculate $\mathrm{div}_g\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}}\right)$.
The vector fields $\tilde{e}_i(x) = \frac{1-|x|^2}{2}e_i$, $i = 1, \ldots, m$ form an orthonormal frame of (H^m, g). So
$\nabla f_x = \sum_{i=1}^m df_x(\tilde{e}_i)\tilde{e}_i = \frac{\sqrt{1-|x|^2}}{2}x$ and $\|\nabla f\|_g^2 = \frac{|x|^2}{1-|x|^2}$.
Using formula (0.3) we get, for $m = 2$,

$$
\begin{aligned}
&\mathrm{div}_g\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}}\right) = \\
&= \ \mathrm{div}_g\left(\frac{1-|x|^2}{2}x\right) = \sum_{k=1}^m \frac{\partial}{\partial x^k}\left(\frac{1-|x|^2}{2}x_k\right) + \sum_{i,k=1}^m \frac{1-|x|^2}{2}x_i\Gamma_{ik}^k \\
&= \ \sum_{k=1}^2\left(-x_kx_k + \frac{1-|x|^2}{2}\right) + \sum_{i,k=1}^2 \frac{1-|x|^2}{2}\,\frac{2}{1-|x|^2}x_i(\delta_{kk}x_i + \delta_{ik}x_k - \delta_{ik}x_k) \\
&= \ -|x|^2 + 1 - |x|^2 + 2|x|^2 = 1\ .\ \heartsuit
\end{aligned}
$$

Remark 1.1 As a consequence of Cor. 1.1.2, Prop. 1.1, and Hopf's maximum principle (see for example Ref. [Cha/84]), if M is an oriented compact manifold, $N = \mathbb{R}^n$, and Γ_f has parallel mean curvature, then f is a constant map.

Remark 1.2 In Sec. 1 we presented the result of Chern [Ch/65] on the mean curvature of a graph as a starting point for the main theorem of this section. This result was a corollary of a theorem in his paper quoted above, which we reproduce here:

Theorem (Chern) *Let P be a compact piece of an oriented hypersurface of dimension m with smooth boundary ∂P which is immersed in a Euclidean space of dimension $m+1$. Suppose the mean curvature $\sigma_1 \geq c > 0$. Let a be a fixed unit vector which makes an angle $\leq \frac{\pi}{2}$ with all normals of M. Then $mcV_a \leq L_a$, where V_a is the volume of the orthogonal projection of P and L_a that of ∂P in the hyperplane perpendicular to a. If M is defined by the equation $z = F(x_1, \ldots, x_m)$, for $x_1^2 + \ldots + x_m^2 < R$, then $cR \leq 1$.*

The above case seems, at first sight, much more general than a graph, but, in fact, it is essentially the same, as we are going to explain in detail.
The condition "*...Let a be a fixed unit vector which makes an angle $\leq \frac{\pi}{2}$ with all*

normals of $P\ldots$" means the following:

Let us assume that the angles are $< \frac{\pi}{2}$. Denote by $\Phi : P^m \to \mathbb{R}^{m+1}$ the immersion of P into the $m+1$- dimensional Euclidean space, and let $\phi : P \to [a]^\perp \simeq \mathbb{R}^m$ denote the composition of Φ with the orthogonal projection of $\mathbb{R}^{m+1}$ onto $[a]^\perp$. That is, $\phi(x) = \Phi(x) - \langle\Phi(x), a\rangle\, a$, $\forall x \in P$. Then ϕ is also an immersion of P, as follows straightforwardly from our assumption concerning the angles.

V_a and L_a in Chern's theorem are resp. the volume and area of P and ∂P relative to the metric induced by the immersion ϕ of P into $[a]^\perp \simeq \mathbb{R}^m$. Writing now $\mathbb{R}^{m+1} = [a]^\perp \times [a]$, then

$\Phi : P \to \mathbb{R}^{m+1} = [a]^\perp \times [a]$ is given by

$\Phi(x) = (\phi(x), <\Phi(x), a> a) \simeq (\phi(x), <\Phi(x), a>)$, since $\phi(x) + <\Phi(x), a> a = \Phi(x)$.

Thus Φ can be written as $\Phi(x) = (\phi(x), f(x))$, where $f : P^m \to \mathbb{R} \simeq [a]$ is a smooth map and $\phi : P^m \to \mathbb{R}^m \simeq [a]^\perp$ is an immersion. We can consider Φ as a *parametrisation* of a graph, where the first component of Φ is the isometric immersion $\phi : P^m \to \mathbb{R}^m$ instead of the identity map, which is the case of a graph.

In the same spirit, we can also improve our main theorem (1.1) for the case of a *parametrisation* of a graph:

Let (M^m, g), (N^n, h) be smooth Riemannian manifolds and P^m an m-dimensional manifold. Let $\Phi = (\phi, f) : (P, \phi^*g + f^*h) \to (M \times N, g \times h)$ be an isometric immersion with components ϕ and f, such that $\phi : (P^m, \phi^*g) \to (M^m, g)$ is an isometric immersion and $f : P \to N$ is a map. Let H be the mean curvature of the isometric immersion Φ. Then Th. 1.1 can be reformulated as follows:

Theorem 1.1' *If Φ has parallel mean curvature, then, for each compact oriented domain $\overline{D} \subset P$, we have*

$$c \leq \frac{1}{m}\frac{A(\partial D)}{V(D)},$$

where $c = \|H\|_{g \times h}$ (constant), $V(D)$ and $A(\partial D)$ are resp. the volume of D and the area of ∂D relative to the metric ϕ^*g.

The proof of this theorem is analogous to the one of Th. 1.1, with some obvious changes of notation.

1.3 Co-Dimension One

If the graph Γ_f is a hypersurface of $M \times N$, that is, N is of dimension one, we can obtain an estimate for the infimum of the norm of the mean curvature of Γ_f, without needing to impose the assumption of Γ_f having parallel mean curvature, as in the general case.

Let us suppose that N is oriented and of dimension one. Let Y be a unit vector field defined on all (N, h). Define $\omega := \sqrt{1 + \|df\|^2}$, where $\|df\|$ is the norm of df in Riemannian vector bundle $\otimes T^*M \otimes f^{-1}TN$. Denote by ∇f the smooth section of TM given by $\langle \nabla f_x, u \rangle_g = \langle df_x(u), Y_x \rangle_h$, $\forall x \in M$, $u \in T_xM$. Thus, $\|\nabla f\|_g = \|df\|$ and $\nu = \frac{1}{\omega}(-\nabla f, Y)$ is a unit normal of Γ_f.
In this case it is easy to derive an expression for the matrix $[\tilde{g}^{ij}]$ (here we use the same notations as in Sec. 1.2). Denoting $p_i = \langle df(X_i), Y \rangle_h$, we have

$$\tilde{g}_{ij} = \delta_{ij} + p_i p_j \quad \text{and} \quad \tilde{g}^{ij} = \delta_{ij} - \frac{1}{\omega^2} p_i p_j \ .$$

Observe that $\omega^2 = |\tilde{g}_{ij}|$. So,

$$\begin{aligned} mH &= \sum_{i,j=1}^{m} \tilde{g}^{ij} \nabla^* d\Gamma_f(X_i, X_j) \\ &= \sum_{i=1}^{m} \nabla^* d\Gamma_f(X_i, X_i) - \sum_{i,j=1}^{m} \frac{1}{\omega^2} p_i p_j \nabla^* d\Gamma_f(X_i, X_j) \ . \end{aligned}$$

From Lemma 1.1 (i), we have

$$\nabla^* d\Gamma_f(X_i, X_j) = \Big\langle (0, \nabla df(X_i, X_j)) \, , \, \nu \Big\rangle_h \nu = \frac{1}{\omega} \Big\langle \nabla df(X_i, X_j) \, , \, Y \Big\rangle_h \nu \ .$$

Hence,

$$\begin{aligned} m < H, \nu >_{g \times h} &= \sum_{i,j=1}^{m} \tilde{g}^{ij} \Big\langle \frac{1}{\omega} \nabla df(X_i, X_j) \, , \, Y \Big\rangle_h = \frac{1}{\omega} \mathrm{trace}_{(g + f^* h)} \Big\langle \nabla df(\cdot, \cdot), Y \Big\rangle_h \\ &= \frac{1}{\omega} \Big\langle \tau_f - \sum_{i,j=1}^{m} \frac{1}{\omega^2} p_i p_j \nabla df(X_i, X_j) \, , \, Y \Big\rangle_h \ . \end{aligned} \tag{1.15}$$

For $M = \mathbb{R}^m$ and $N = \mathbb{R}$, this expression is equal to the one obtained by Flanders [Fla/66].

Lemma 1.3

$$m < H, \nu >_{g \times h} = div_g \left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|_g^2}} \right) \ .$$

In particular, $\mathrm{div}_g\left(\frac{\nabla f}{\sqrt{1-\|\nabla f\|_g^2}}\right) = mc$ *(c constant), iff* $\|H\|_{g\times h} = |c|$.

Proof. Let $x_0 \in M$, and $X_1, \ldots, X_m$ be a local orthonormal frame of (M, g) in a neighbourhood of x_0, such that $\nabla X_i(x_0) = 0$. Then, at the point x_0,

$$\begin{aligned}
\mathrm{div}_g\left(\frac{\nabla f}{\omega}\right)(x_0) &= \sum_{i=1}^{m}\left\langle \nabla_{X_i}\left(\frac{\nabla f}{\omega}\right), X_i\right\rangle_g = \sum_{i=1}^{m} d\left(\left\langle \frac{\nabla f}{\omega}, X_i\right\rangle_g\right)(X_i)\\
&= \sum_{i=1}^{m}\left\langle \nabla^{f^{-1}}_{X_i}\left(\frac{1}{\omega}df(X_i)\right), Y\right\rangle_h\\
&= \sum_{i=1}^{m}\left\langle d\left(\frac{1}{\omega}\right)(X_i)df(X_i) + \frac{1}{\omega}\nabla df(X_i, X_i)\,,\, Y\right\rangle_h\\
&= \left\langle \sum_{i=1}^{m} -\frac{1}{2\omega^3} d\|df\|^2(X_i)\, df(X_i) + \frac{1}{\omega}\tau_f\,,\, Y\right\rangle_h\\
&= \left\langle -\frac{1}{\omega^3}\sum_{i=1}^{m} <\nabla_{X_i} df, df> df(X_i) + \frac{1}{\omega}\tau_f\,,\, Y\right\rangle_h\\
&= \left\langle -\sum_{i,j=1}^{m}\frac{1}{\omega^3} <\nabla df(X_i, X_j), df(X_j)>_h df(X_i) + \frac{1}{\omega}\tau_f\,,\, Y\right\rangle_h\\
&= \left\langle -\frac{1}{\omega^3}\sum_{i,j=1}^{m} <\nabla df(X_i, X_j), Y>_h\, < df(X_j), Y>_h df(X_i) + \frac{1}{\omega}\tau_f\,,\, Y\right\rangle_h\\
&= \left\langle -\frac{1}{\omega^3}\sum_{i,j=1}^{m} p_i p_j \nabla df(X_i, X_j) + \frac{1}{\omega}\tau_f\,,\, Y\right\rangle_h\\
&= m < H, \nu >_{g\times h}(x_0) \quad \text{(from Eq. (1.18))}. \ \heartsuit
\end{aligned}$$

Let $\|\nabla df\|$ denote the norm of ∇df in $\odot^2 T^*M \otimes f^{-1}TN$.

Proposition 1.2

(a) If $\overline{D} \subset M$ *is an oriented compact domain of* M, *then*

$$\min_{\overline{D}}\|H\|_{g\times h} \leq \frac{1}{m}\frac{A(\partial D)}{V(D)},$$

where $A(\partial D)$ *and* $V(D)$ *are resp. the area of* ∂D *and the volume of* D, *relative to the metric* g. *In particular, if* (M, g) *is a compact manifold or non-compact with Cheeger constant equal to zero, then* $\inf_M\|H\|_{g\times h} = 0$.

(b) If (M, g) *is a connected, oriented, complete Riemannian manifold and* $\frac{\|\nabla f\|_g}{\sqrt{1+\|\nabla f\|_g^2}}$ *is integrable in* (M, g), *then there exists a* $x \in M$, *such that* $H_x = 0$. *Moreover, if* $< H, \nu >_{g\times h}$ *is contained in* $[0, +\infty)$ *or in* $(-\infty, 0]$, *then* $H \equiv 0$.

Proof. (a) Let $c = \min_D \|H\|$. Clearly we may suppose $c \neq 0$. Since D is connected,

$$\begin{array}{rcll} \|H\|_x & = & <H_x, \nu_x>_{g \times h}, & \forall x \in D \\ & \text{or} & & \\ \|H\|_x & = & -<H_x, \nu_x>_{g \times h}, & \forall x \in D\,. \end{array} \tag{1.16}$$

Denoting by dV_g and dA_g the volume elements of D and ∂D, respectively, and by $\vec{n}$ the outward unit normal of ∂D, and applying Lemma 1.3 and Stokes' theorem, we obtain

$$\begin{aligned} cV(D) &\leq \int_D \|H\|_{g\times h} dV_g = \left|\int_D <H,\nu>_{g\times h} dV_g\right| = \\ &= \frac{1}{m}\left|\int_D \mathrm{div}_g(\frac{\nabla f}{\omega}) dV_g\right| = \frac{1}{m}\left|\int_{\partial D} <\frac{\nabla f}{\omega}, \vec{n}>_g dA_g\right| \\ &\leq \frac{1}{m}\int_{\partial D} \|\frac{\nabla f}{\omega}\|_g dA_g\,. \end{aligned}$$

As $\|\frac{\nabla f}{\omega}\|_g \leq 1$, $\quad c \leq \frac{1}{m}\frac{A(\partial D)}{V(D)}$.

(b). Suppose that $H_x \neq 0$, $\quad \forall x \in M$. Note that, in this case, Eq. (1.16) holds on all M.

Let $\theta = *\frac{<d\Pi(\cdot), Y>}{\omega}$, where $* : \bigwedge^1 T^*M \to \bigwedge^{m-1} T^*M$ is the *Hodge operator* .

Then, $d\theta = d*\frac{<d\Pi(\cdot), Y>}{\omega} = \mathrm{div}_g(\frac{\nabla f}{\omega}) dV_g$. Since we are supposing that $\|\frac{\nabla f}{\omega}\|_g$ is integrable and since $*$ is an orthogonal isomorphism between the Riemannian vector bundles $\bigwedge^1 T^*M$ and $\bigwedge^{m-1} T^*M$, $\|\theta\|$ is integrable on (M, g), i.e. θ is an integrable $(m-1)$-form of (M, g). By applying the extended Stokes' theorem of Gaffney-Yau (see Ref. [Ya/76], lemma of Sec. 1) to θ, we may take a sequence of compact domains B_i of M, such that $B_i \subset B_{i+1}$, $\quad \forall i$, $\quad \bigcup B_i = M$, and $\lim_{i\to+\infty}\int_{B_i} d\theta = 0$, that is,

$$\lim_{i\to+\infty}\int_{B_i} \mathrm{div}_g(\frac{\nabla f}{\omega}) dV_g = 0\,.$$

Therefore, we can conclude from Eq. (1.16) and Lemma 1.3

$$\lim_{i\to+\infty}\int_{B_i} \|H\|_{g\times h} dV_g = 0\,.$$

Consequently, $\int_{B_i} \|H\|_{g\times h} dV_g = 0$, $\forall i$, i.e. $H \equiv 0$, which is a contradiction. If we suppose that $<H,\nu>_{g\times h}$ is contained in $[0, +\infty)$ or in $(-\infty, 0]$, then, again, Eq. (1.16) holds on all M, which implies $H \equiv 0$ as well. ♡

Remark 1.3 In Prop. 1.2(b)(i) we could only require the weaker condition

$$\lim_{R\to+\infty}\frac{1}{R}\int_{B_R(x_0)}\frac{\|\nabla f\|_g}{\sqrt{1+\|\nabla f\|_g^2}}dV_g=0$$

for some $x_0 \in M$, where $B_R(x_0)$ is the geodesic ball of (M,g) with centre x_0 and radius R. In fact, the Stokes' theorem of Yau still holds with this condition (see Appendix of Ref. [Ya/76]).

1.4 Graphs of Isometric Immersions, Conformal Maps, Riemannian Submersions, and Harmonic Morphisms

In Sec. 2 we have seen that for a map $f:(M,g)\to(N,h)$, Γ_f to be minimal is in general not equivalent to $f:(M,g)\to(N,h)$ be harmonic. However, we will treat some cases where the equivalence does hold.

A map $\phi:(P_1,g_1)\to(P_2,g_2)$ between two Riemannian manifolds is said to be (weakly) conformal, if $\phi^*g_2=\rho^2 g_1$, where $\rho:P_1\to\mathbb{R}$ is a smooth map. If $\dim P_1>\dim P_2$, then ϕ is constant. If ρ is a non-zero constant, ϕ is said to be a homothetic map and, in particular, an isometric immersion, if $\rho=1$.
If $\rho(x)\neq 0\ \ \forall x\in P_1$, i.e. ϕ is an immersion, then we have the following well-known relation [Ho-Os/82] between τ_ϕ, the tension field of $\phi:(P_1,g_1)\to(P_2,g_2)$ and H_ϕ, the mean curvature of the isometric immersion $\phi:(P_1,\phi^*g_2)\to(P_2,g_2)$:

$$mH_\phi=\frac{1}{\rho^2}\tau_\phi+\frac{m-2}{\rho^2}d\phi(w)\,, \tag{1.17}$$

where $m=\dim(P_1)$ and $w=\nabla_{g_1}\log\rho$. We recall that a Riemannian manifold (M,g) is said to be (strongly) parabolic, if it admits no non-constant subharmonic functions f (i.e. $\Delta f\geq 0$) that are bounded from above.

Proposition 1.3 *Let $f:(M^m,g)\to(N^n,h)$ be a conformal map with $f^*h=\lambda^2 g$. Let H be the mean curvature of Γ_f. Then we have:*

(a)

$$mH = \left(0, (1+\lambda^2)^{-1}\tau_f\right)^{\perp} .$$

In particular, if $x_0 \in M$, $H_{x_0} = 0$, iff $\tau_f(x_0) = 0$. Therefore, Γ_f is a minimal submanifold of $(M \times N, g \times h)$, iff $f : (M, g) \to (N, h)$ is a harmonic map (and in this case, for $m \neq 2$, f is a homothetic map).

(b) If $m = 2$, or f is an isometric immersion or, more generally, f is a homothetic map, then Γ_f has parallel mean curvature, iff Γ_f is minimal.

(c) If $m \neq 2$ and Γ_f has parallel mean curvature, then

$$\Delta\left((1+\lambda^2)^{-1}\right) = \frac{2m^2}{m-2}c^2 ,$$

with $c = \|H\|$ (constant). Consequently,

(i) if (M, g) is parabolic or if λ has a minimum on $M\backslash\partial M$ for $m \geq 3$, then Γ_f is minimal.

(ii) if (M^m, g) is complete, connected, and oriented, and $m \geq 3$, then for $vol(M, g) < +\infty$ Γ_f is minimal, and for $vol(M, g) = +\infty$ $(1+\lambda^2)^{-1} \notin L^p(M, g)$, $\forall p \in (1, +\infty)$.

Proof. Since $f^*h = \lambda^2 g$, $\Gamma_f^*(g \times h) = g + f^*h = (1+\lambda^2)g = \mu^2 g$, where $\mu : M \to [1, +\infty)$ is a smooth map. It follows from Eq. (1.17) that

$$mH = \mu^{-2}\tau_{\Gamma_f} + (m-2)\mu^{-2}\left(w, df(w)\right) ,$$

with τ_{Γ_f} the tension field of $\Gamma_f : (M, g) \to (M \times N, g \times h)$ and with $w = \nabla_g \log \mu$. Thus, $\tau_{\Gamma_f} = (\tau_{\mathrm{id}}, \tau_f) = (0, \tau_f)$. Hence,

$$mH = \mu^{-2}(0, \tau_f) + (m-2)\mu^{-2}\left(w, df(w)\right) . \tag{1.18}$$

So $mH = (mH)^{\perp} = (0, \mu^{-2}\tau_f)^{\perp}$ and (a) is proved by applying Lemma 1.2(i). If f is harmonic, i.e. $H = 0$ (from (a)), and $m \neq 2$, then Eq. (1.18) gives $w = 0$, that is, f is a homothetic map (see also Ref. [Ee-Le/83]).
Next we prove (b). If $m = 2$ or f is a homothetic map (i.e $w = 0$), we obtain from Eq. (1.18)

$$mH = \mu^{-2}(0, \tau_f) .$$

So, applying formula (1.1) we have, $\forall X \in C^\infty(TM)$,

$$m\nabla_X^{\Gamma_f^{-1}} H = \left(0, \nabla_X^{f^{-1}}(\mu^{-2}\tau_f)\right) \tag{1.19}$$

and

$$m\nabla_X^{\perp} H = \left(0, \nabla_X^{f^{-1}}(\mu^{-2}\tau_f)\right)^{\perp} . \tag{1.20}$$

Hence, applying Lemma 1.2(i) to Eq. (1.20), we conclude that $\nabla_X^{\perp} H = 0$ iff $\nabla_X^{f^{-1}}(\mu^{-2}\tau_f) = 0$, which is equivalent to $\nabla_X^{\Gamma_f^{-1}} H = 0$ due to Eq. (1.19). Using Eq. (1.9) for $\phi = \Gamma_f$, we get $\nabla^{\Gamma_f^{-1}} H = 0$ iff $H = 0$, and we have proved (b). In order to obtain (c) we are first going to prove the following formula:

$$\langle \tau_f, df(\cdot)\rangle_h = \frac{2-m}{2} d\lambda^2 . \tag{1.21}$$

Let us fix $x_0 \in M$, and let $X_1, \ldots, X_m$ be a local orthonormal frame of (M, g) defined in a neighbourhood of x_0 and satisfying $\nabla X_i(x_0) = 0, \ \forall i = 1, \ldots, m$.
At x_0 we have, $\forall i, j, k \in \{1, \ldots, m\}$,

$$\begin{aligned}
\left\langle \nabla df(X_i, X_j), df(X_k)\right\rangle_h (x_0) &= \left\langle \nabla_{X_i}^{f^{-1}}(df(X_j)), df(X_k)\right\rangle_h \\
&= d\left(< df(X_j), df(X_k) >_h \right)(X_i) - \left\langle df(X_j), \nabla df(X_i, X_k)\right\rangle_h \\
&= \delta_{jk} d\lambda^2(X_i) - \left\langle df(X_j), \nabla df(X_i, X_k)\right\rangle_h .
\end{aligned}$$

Performing a cyclic permutation on the indices i, j, k we get

$$\begin{aligned}
\left\langle \nabla df(X_i, X_j), df(X_k)\right\rangle_h &= \delta_{jk} d\lambda^2(X_i) - \left\langle df(X_j), \nabla df(X_i, X_k)\right\rangle_h \\
\left\langle \nabla df(X_k, X_i), df(X_j)\right\rangle_h &= \delta_{ij} d\lambda^2(X_k) - \left\langle df(X_i), \nabla df(X_k, X_j)\right\rangle_h \\
\left\langle \nabla df(X_j, X_k), df(X_i)\right\rangle_h &= \delta_{ki} d\lambda^2(X_j) - \left\langle df(X_k), \nabla df(X_j, X_i)\right\rangle_h ,
\end{aligned}$$

and so, at the point x_0,

$$\left\langle \nabla df(X_i, X_j), df(X_k)\right\rangle_h = \frac{1}{2}\left\{\delta_{jk} d\lambda^2(X_i) - \delta_{ij} d\lambda^2(X_k) + \delta_{ki} d\lambda^2(X_j)\right\} .$$

Hence, for $i = j$

$$\left\langle \nabla df(X_i, X_i), df(X_k)\right\rangle_h = \delta_{ik} d\lambda^2(X_i) - \frac{1}{2} d\lambda^2(X_k) ,$$

and so

$$\begin{aligned}
\langle \tau_f, df(X_k)\rangle_h (x_0) &= \sum_{j=1}^{m} \left\langle \nabla df(X_j, X_j), df(X_k)\right\rangle_h \\
&= d\lambda^2(X_k) - \frac{m}{2} d\lambda^2(X_k) = \frac{2-m}{2} d\lambda^2_{x_0}(X_k) .
\end{aligned}$$

Supposing that Γ_f has parallel mean curvature we have, from Eq. (1.12) in the proof of Th. 1.1,

$$m^2c^2 = \mathrm{div}_g(Z)\ ,$$

where Z is the vector field of M, given in Eq. (1.4). Next we are going to prove that

$$m^2c^2 = \frac{m-2}{2}\,\Delta\,(\mu^{-2}) \text{ on } M\ . \tag{1.22}$$

Let $x_0 \in M$ and $X_1,\ldots,X_m$ be a local orthonormal frame of (M,g) defined in a neighbourhood of x_0 and satisfying $\nabla X_i(x_0) = 0$, $\forall i = 1,\ldots,m$. Then,

$$Z = \sum_{i,j=1}^{m} \tilde{g}^{ij} < W, df(X_i) >_h X_j \quad \text{and} \quad W = \sum_{i,j=1}^{m} \tilde{g}^{ij}\nabla df(X_i,X_j)$$

in a neighbourhood of x_0. Since $\tilde{g}_{ij} = \mu^2\delta_{ij}$, we have

$$W = \sum_{i=1}^{m}\mu^{-2}\nabla df(X_i,X_i) = \mu^{-2}\tau_f$$

and

$$Z = \sum_{i=1}^{m}\mu^{-4} < \tau_f, df(X_i) >_h X_i = \sum_{i=1}^{m}\mu^{-4}\frac{(2-m)}{2}d\lambda^2(X_i)\,X_i\ ,$$

and at the point x_0

$$\begin{aligned}
m^2c^2 &= \mathrm{div}_g\Big(\sum_{i=1}^{m}\mu^{-4}\frac{(2-m)}{2}d\lambda^2(X_i)\,X_i\Big)(x_0)\\
&= \sum_{i,k=1}^{m}\frac{2-m}{2}\left\langle\nabla_{X_k}(\mu^{-4}d\mu^2(X_i)\,X_i)\,,\,X_k\right\rangle_g(x_0)\\
&= \sum_{i,k=1}^{m}\frac{2-m}{2}d(\mu^{-4}d\mu^2(X_i))_{x_0}(X_k) < X_i, X_k >_g\\
&= \sum_{i=1}^{m}\frac{2-m}{2}d\big(\mu^{-4}d\mu^2(X_i)\big)_{x_0}(X_i)\\
&= \frac{m-2}{2}\sum_{i=1}^{m}\nabla d(\mu^{-2})_{x_0}(X_i,X_i)\\
&= \frac{m-2}{2}\,\Delta\,(\mu^{-2})(x_0)\ .
\end{aligned}$$

Thus, Eq. (1.22) holds, that is, $\Delta(\mu^{-2}) = \frac{2m^2}{m-2}c^2$ with $0 < \mu^{-2} \leq 1$. So,

$$\text{for } m \geq 3 \begin{cases} \Delta(\mu^{-2}) & \geq\ 0\\ \mu^{-2} & \leq\ 1\end{cases} \tag{1.23}$$

and

$$\text{for } m = 1 \begin{cases} \Delta(\mu^{-2}) & \leq\ 0\\ \mu^{-2} & >\ 0\ .\end{cases}$$

Hence, if (M,g) is parabolic, μ must be constant, and, therefore, $0 = \Delta(\mu^{-2}) = \frac{2m^2}{m-2}c^2$, i.e. Γ_f is minimal.
For $m \geq 3$, if λ has a minimum on $M\backslash\partial M$, then μ^{-2} has a maximum on $M\backslash\partial M$. As $\Delta\mu^{-2} \geq 0$, it follows from Hopf's maximum principle, applied on a bounded domain of M where that maximum is attained (see e.g. Ref. [Au/82], page 96), that (c) (i) holds.
Now we prove (c)(ii). From Eq. (1.23) we have $\mu^{-2}\,\Delta\,(\mu^{-2}) \geq 0$. So, from Th. 3 of Ref. [Ya/76], we have either $\int_M(\mu^{-2})^p dV_g = +\infty$, $\forall p \in (0,+\infty)\backslash\{1\}$, or μ is constant. Thus, if the volume of (M,g) is finite, we conclude from $0 < \mu^{-2} \leq 1$ that $\mu^{-2} \in L^p(M,g)$, $\forall p \in (1,+\infty)$. Therefore, μ is constant and Γ_f is minimal. Let us now suppose that the volume of (M,g) is infinite. If $\mu^{-2} \in L^p(M,g)$ for some $p \in (1,+\infty)$ were true, then μ would be constant. Since μ cannot be equal to zero, this would imply that the volume of (M,g) is finite, which is a contradiction. So, in this case, $\mu^{-2} \notin L^p(M,g)$, $\forall p \in (1,+\infty)$. $\heartsuit$

Remark 1.4 Prop. 1.3(c) means that, if $m \geq 3$, $\mathrm{vol}(M,g) = +\infty$, and Γ_f has non-zero parallel mean curvature, then, $\forall p \in (1,+\infty)$, $\left(1 + \frac{\|H\|^2}{m}\right)^{-p}$ cannot be integrable, nor have a maximum.

Now we study the graphs of Riemannian submersions and harmonic morphisms. Henceforth, until the end of this section, we assume that (M,g) and (N,h) are boundaryless manifolds.

Let $f : (M^m,g) \to (N^n,h)$ be a map. For each $x \in M$, we denote $T_x^V M := \mathrm{Ker} df_x$ and $T_x^H M := (T_x^V M)^\perp$, its orthogonal complement in (T_xM,g). The elements of $T_x^V M$ and $T_x^H M$ are called vertical resp. horizontal tangent vectors of M at the point x. Let us denote by $(\ )^V$ and $(\ )^H$ the orthogonal projections of T_xM on $T_x^V M$ resp. $T_x^H M$.
The map f is said to be ***horizontally conformal*** , if, $\forall x \in M$ such that $df_x \neq 0$, $df_x : T_x^H M \to T_{f(x)}N$ is a conformal, linear isomorphism. For such maps we have (see Ref. [Ee-Le/83])

$$\forall x \in M,\ u,v \in T_x^H M, \quad < df_x(u), df_x(v) >_h = \frac{2e_f(x)}{n} < u,v >_g \ ,$$

where $e_f = \frac{1}{2}\|df\|^2$ is the energy density of f.

A map $f:(M,g)\to(N,h)$ is said to be a *harmonic morphism*, if, for any harmonic function ϕ defined on an open set V of N, the composition $\phi\circ f$ is harmonic on $f^{-1}(V)$.
The following proposition, which we will use later on, is due to Fuglede [Fu/78] and Ishiara [Is/79] (see also Ref. [Ee-Le/83]).

Proposition 1.4 *A map $f:(M^m,g)\to(N^n,h)$ is a harmonic morphisms, iff it is a harmonic and horizontally conformal map. If f is non-constant, it is a submersion on an open dense subset of M (and so $m\geq n$). If at a point x* rank$df_x<n$, *then* $df_x=0$.

Let $f:(M^m,g)\to(N^n,h)$ be a submersion. Then, $T^VM\to M$ and $T^HM\to M$ are smooth vector bundles. Hence, in the neighbourhood of each point of M we may take an orthonormal frame $X_1,\ldots,X_n,X_{n+1},\ldots,X_m$ of (M,g), such that $X_1,\ldots,X_n\in C^\infty(T^HM)$ and $X_{n+1},\ldots,X_m\in C^\infty(T^VM)$.
For all $y\in f(M)$, the fibre $F_y=f^{-1}(y)$ of f at the point y is a submanifold of M of dimension $m-n$ with $T_x(F_y)=T^V_xM$.
Let the inclusion map $\imath_y:F_y\to(M,g)$ be an isometric immersion. Its second fundamental form satisfies

$$\nabla d\imath_y(X_x,Y_x)=\left(\nabla_XY_x\right)^H,\quad \forall X,Y\in C^\infty(T^VM),\ \forall x\in F_y\,,$$

where ∇ is the Levi-Civita connection of (M,g).
Thus, the tension field of $\imath_y$ is given by

$$\tau_{\imath_y}(x)=\sum_{j=n+1}^{m}\left(\nabla_{X_j}X_j\right)^H_x\quad\in T^H_xM=\left(T_x(F_y)\right)^\perp \tag{1.24}$$

and is equal to $m-n$ times the mean curvature of the fibre F_y.
Since $f\circ\imath_y$ is constant on F_y, $\nabla d(f\circ\imath_y)=0$, we get [Ba-Ee/81], using the composition law,

$$\nabla df_x(X,Y)=-df_x\left(\nabla d\imath_y(X,Y)\right),\quad \forall X,Y\in T_x(F_y)\,. \tag{1.25}$$

The submersion f is said to be *Riemannian*, if, $\forall x\in M$, $df_x:T^H_xM\to T_{f(x)}N$ is an isometry. Here we recall the following results about Riemannian submersions and harmonic morphisms (see Ref. [Ee-Le/83] for further references).

Proposition 1.5 *Let $f : (M,g) \to (N,h)$ be a submersion. Then:*

(a) f has totally geodesic fibres, iff $\nabla df|_{T^V M \times T^V M} = 0$.

If, moreover, f is Riemannian, then also (b) and (c) hold:

(b) $\nabla df|_{T^H M \times T^H M} = 0$.

(c) the following conditions are equivalent:

(i) f has minimal fibres;

(ii) f is harmonic;

(iii) f is a harmonic morphism.

If, on the other hand, f is a harmonic morphism, then (d) and (e) hold:

(d) if $n = 2$, the fibres are minimal.

(e) if $n \geq 3$, the following conditions are equivalent:

(i) the fibres are minimal;

(ii) ∇e_f is vertical everywhere;

(iii) the mean curvature of the horizontal distribution, which is the vertical vector field given by $\frac{1}{n}\left(\sum_{i=1}^{n} \nabla_{X_i} X_i\right)^V$, is equal to $\dfrac{\nabla e_f}{2e_f}$.

Let now $f : (M^m, g) \to (N^n, h)$ be a Riemannian submersion. From now on $X_1, \ldots, X_n, X_{n+1}, \ldots, X_m$ denotes a local frame of (M,g), such that $X_1, \ldots, X_n \in C^\infty(T^H M)$ and $X_{n+1}, \ldots, X_m \in C^\infty(T^V M)$. Note that from Prop. 1.5 we have

$$\tau_f = \sum_{i=1}^{m} \nabla df(X_i, X_i) = \sum_{i=n+1}^{m} \nabla df(X_i, X_i) \,. \tag{1.26}$$

On $(T^H M, g)$ we have an induced connection ∇^H which is given by:

$$\nabla_X^H Z_x = \left(\nabla_X Z_x\right)^H, \quad \forall Z \in C^\infty(T^H M),\ X \in T_x M \,.$$

Proposition 1.6 *Let $f : (M^m, g) \to (N^n, h)$ be a Riemannian submersion, and denote by $\mathcal{H}\tau_f$ the section of $T^H M$ given by*

$$\mathcal{H}\tau_f = \left(df|_{T^H M}\right)^{-1}(\tau_f) \,.$$

Then we have:

(a) $\forall y \in f(M)$, $x \in F_y$, $\quad \mathcal{H}\tau_f(x) = -\tau_{i_y}(x)$.
In particular, $\|\tau_f(x)\|_h = \|\tau_{i_y}(x)\|_g$. Thus, the fibres of f have constant mean curvature, iff the norm of the tension field of f is constant in each fibre.

(b) $\forall y \in f(M)$, $x \in F_y$, $X \in T_x(F_y) = T_x^V M$,

$$\left(\nabla_X^{\imath_y^{-1}} \tau_{\imath_y}\right)_x^H = -\nabla_X^H (\mathcal{H}\tau_f)_x .$$

In particular, the fibres of f have parallel mean curvature, iff $\mathcal{H}\tau_f$ is a parallel section of $T^H M$ along the vertical vector fields.

(c) $\forall X \in C^\infty(T^H M)$, $df\left(\nabla_X(\mathcal{H}\tau_f)\right) = \nabla_X^{f^{-1}} \tau_f$.

Proof. Let $y \in f(M)$. From Eq. (1.24) we have

$$\tau_{\imath_y}(x) = \sum_{i=n+1}^{m} \left(\nabla_{X_i} X_i\right)_x^H, \ \forall x \in F_y ,$$

and from Eqs. (1.26),(1.25)

$$\begin{aligned} \tau_f(x) &= \sum_{i=n+1}^{m} \nabla df_x(X_i, X_i) = -df_x\left(\nabla d(\imath_y)_x(X_i, X_i)\right) \\ &= -df_x(\tau_{\imath_y}) . \end{aligned}$$

Since $\tau_{\imath_y}(x) \in T_x^H M$, from the definition of $\mathcal{H}\tau_f$ we get $-\tau_{\imath_y}(x) = \mathcal{H}\tau_f(x)$. Now let $X \in T_x(F_y)$. Then,

$$\nabla_X^{\imath_y^{-1}} (\tau_{\imath_y})_x = -\nabla_X^{\imath_y^{-1}} \left(\imath_y^{-1}(\mathcal{H}\tau_f)\right)_x = -\nabla_X(\mathcal{H}\tau_f)_x .$$

Since $\left(T_x(F_y)\right)^\perp = T_x^H M$, $\forall x \in F_y$, the connection ${}^F\nabla^\perp$ of $(T(F_y))^\perp$ is exactly equal to $\left(\nabla^{\imath_y^{-1}}\right)^H$. Thus, we have

$${}^F\nabla_X^\perp (\tau_{\imath_y})_x = \left(\nabla_X^{\imath_y^{-1}} (\tau_{\imath_y})_x\right)^H = -\nabla_X^H (\mathcal{H}\tau_f)_x ,$$

and we have proved (b).

Now we prove (c). For all $X \in C^\infty(TM)$,

$$\begin{aligned} df(\nabla_X \mathcal{H}\tau_f) &= \nabla_X^{f^{-1}}\left(df(\mathcal{H}\tau_f)\right) - \nabla df(X, \mathcal{H}\tau_f) \\ &= \nabla_X^{f^{-1}} \tau_f - \nabla df(X, \mathcal{H}\tau_f) . \end{aligned}$$

So, from Prop. 1.5(b) we get

$$df(\nabla_X \mathcal{H}\tau_f) = \nabla_X^{f^{-1}} \tau_f, \ \forall X \in C^\infty(T^H M) . \ \triangledown$$

Next we study the mean curvature H of the graph Γ_f of f.
Note that the $\tilde{g}_{ij}$ (given in Eq. (1.2)) are in this case given by

$$\tilde{g}_{ij} = \begin{cases} 2\delta_{ij} & \text{for } i,j \leq n \\ \delta_{ij} & \text{for } i \geq n+1 \text{ or } j \geq n+1 \,. \end{cases} \tag{1.27}$$

Proposition 1.7 *Let $f : (M^m, g) \to (N^n, h)$ be a Riemannian submersion. Then we have:*

(a) $mH = (0, \tau_f)^\perp$.
In particular, for any point $x \in M$, $\tau_f(x) = 0$, iff $H_x = 0$, and so the following conditions are equivalent:
(i) Γ_f is a minimal submanifold of $M \times N$.
(ii) f is harmonic.
(iii) f is a harmonic morphism.
(iv) the fibres of f are minimal.

(b) The following conditions are equivalent:
(i) Γ_f has constant mean curvature.
(ii) $\|\tau_f\|_h$ is constant.
(iii) the fibres of f have constant mean curvature, the norm of which is the same for all fibres.

(c) if Γ_f has parallel mean curvature, then $\nabla_X^{f^{-1}} \tau_f = 0, \ \forall X \in C^\infty(T^H M)$.

Proof. From Lemma 1.1(ii),

$$mH = -\big(Z, df(Z)\big) + (0, W) = (0, W)^\perp \,,$$

where W and Z are given by Eqs. (1.3) resp. (1.4). From Eq. (1.27),

$$W = \sum_{i=1}^{n} \frac{1}{2} \nabla df(X_i, X_i) + \sum_{i=n+1}^{m} \nabla df(X_i, X_i) \,,$$

and from Prop. 1.5 and Eq. (1.26) we have

$$W = \sum_{i=n+1}^{m} \nabla df(X_i, X_i) = \tau_f \,.$$

Hence,

$$Z = \sum_{i=1}^{n} \frac{1}{2} < \tau_f, df(X_i) >_h X_i \,.$$

Thus, we have $mH = (0, \tau_f)^\perp$, and (a) is proved by applying Lemma 1.2(i) and Prop. 1.5(c).

The above expression for Z gives $df(Z) = \sum_{i=1}^{n} \frac{1}{2} < \tau_f, df(X_i) >_h df(X_i)$. Since $df_x : T_x^H M \to T_{f(x)}N$ is an isometry, $df_x(X_1), \ldots, df_x(X_n)$ is an orthonormal basis of $T_{f(x)}N$. Thus,

$$df(Z) = \frac{1}{2}\tau_f = \frac{1}{2}W ,$$

that is,

$$Z = (df|_{T^H M})^{-1}(\frac{1}{2}\tau_f) = \frac{1}{2}\mathcal{H}\tau_f .$$

Therefore, $mH = -(Z, df(Z)) + (0, 2df(Z)) = (- Z, df(Z))$, and so

$$\begin{aligned} \|mH\|^2_{g\times h} &= \|Z\|^2_g + \|df(Z)\|^2_h = 2\|Z\|^2_g \\ &= 2\|df(Z)\|^2_h = \frac{1}{2}\|\tau_f\|^2_h . \end{aligned}$$

Consequently, Γ_f has constant mean curvature, iff $\|\tau_f\|_h$ is constant, and (b) follows from Prop. 1.6(a).

Finally, we prove (c). From Lemma 1.1(iii) we have, $\forall X \in C^\infty(TM)$,

$$\nabla^\perp_X H = (0\, ,\, \nabla^{f^{-1}}_X W - \nabla df(Z, X))^\perp .$$

If Γ_f has parallel mean curvature, then, from Lemma 1.2(i), $\nabla^{f^{-1}}_X W = \nabla df(Z, X)$, that is,

$$\nabla^{f^{-1}}_X \tau_f = \nabla df(\frac{1}{2}\mathcal{H}\tau_f, X) .$$

Using Prop. 1.5(b) we obtain

$$\nabla^{f^{-1}}_X \tau_f = 0, \ \ \forall X \in C^\infty(T^H M) . \ \heartsuit$$

Let $f : (M^m, g) \to (N^n, h)$ be a harmonic morphism. We are now going to study the mean curvature H of Γ_f.

Let $U = \{x \in M : df_x \neq 0\}$. From Prop. 1.4, if f is not constant, U is an open dense subset of M, and, $\forall x \in U$, df_x is a submersion with

$$< df_x(u), df_x(v) >_h = \frac{2e_f}{n} < u, v >_g, \ \ \forall u, v \in T_x^H M .$$

Proposition 1.8 *Let $f:(M^m,g)\to(N^n,h)$ be a harmonic morphism. Then:*
(a) $\forall x\in M\backslash U$, $H_x=0$.
(b) $\forall x\in U$,

$$mH_x=\Big(0\,,\,\frac{-2e_f(x)}{n+2e_f(x)}df_x(\tau_{\iota_y}(x))\Big)^{\perp}\,,$$

where $y=f(x)$ and $\iota_y:F_y\to(M,g)$ is the inclusion map of the fibre F_y of f at y. In particular, $H_x=0$, $\forall x\in F_y$, iff the fibre F_y is a minimal $(m-n)$-submanifold of (M,g). So, Γ_f is a minimal submanifold of $M\times N$, iff the fibres of $f|_U$ are minimal.

Proof. Let $x_0\in M\backslash U$, and $X_1,\ldots,X_m$ be an orthonormal basis of $(T_{x_0}M,g)$. Then, $\hat g_{ij}(x_0)=<X_i,X_j>_g+<df_{x_0}(X_i),df_{x_0}(X_j)>_h=\delta_{ij}$. From Lemma 1.1(ii) we have $mH_{x_0}=(0,W_{x_0})^{\perp}$, where $W_{x_0}=\sum_{i,j=1}^m\hat g^{ij}(x_0)\nabla df_{x_0}(X_i,X_j)=\sum_{i=1}^m\nabla df_{x_0}(X_i,X_i)=\tau_f(x_0)$. Since f is harmonic (Prop. 1.4), $H_{x_0}=0$.
On U, $f:U\to N$ is a submersion. Let $X_1,\ldots,X_n,X_{n+1},\ldots,X_m$ be a local orthonormal frame of (M,g), such that $X_1,\ldots,X_n\in C^\infty(T^HM)$ and $X_{n+1},\ldots,X_m\in C^\infty(T^VM)$. As f is a horizontally conformal map (Prop. 1.4),

$$\begin{aligned}\hat g_{ij}&=<X_i,Xj>_g+<df(X_i),df(X_j)>_h\\&=\begin{cases}\delta_{ij}(1+\frac{2e_f}{n}) & \text{for } i,j\le n\\ \delta_{ij} & \text{for } i\ge n+1 \text{ or } j\ge n+1\,.\end{cases}\end{aligned}$$

From Lemma 1.1(ii), $mH=(0,W)^{\perp}$, where $W=\sum_{i,j=1}^m\hat g^{ij}\nabla df(X_i,X_j)=\sum_{i=1}^n\frac{n}{n+2e_f}\nabla df(X_i,X_i)+\sum_{i=n+1}^m\nabla df(X_i,X_i)$. On the other hand, since f is harmonic,

$$0=\tau_f=\sum_{i=1}^n\nabla df(X_i,X_i)+\sum_{i=n+1}^m\nabla df(X_i,X_i)\,.$$

Thus,

$$\begin{aligned}W&=-\sum_{i=n+1}^m\frac{n}{n+2e_f}\nabla df(X_i,X_i)+\sum_{i=n+1}^m\nabla df(X_i,X_i)\\&=\frac{2e_f}{n+2e_f}\sum_{i=n+1}^m\nabla df(X_i,X_i)\,.\end{aligned}$$

From Eqs. (1.25),(1.24) we have, $\forall x\in U$,

$$\sum_{i=n+1}^m\nabla df_x(X_i,X_i)=-df_x\Big(\sum_{i=n+1}^m(\nabla_{X_i}X_i)^H\Big)=-df_x(\tau_{\iota_y}(x))\,,$$

where $y=f(x)$. Therefore,

$$W_x=\frac{-2e_f(x)}{n+2e_f(x)}df_x(\tau_{\iota_y}(x))\,,$$

and so

$$mH_x = \left(0\,,\ \frac{-2e_f(x)}{n+2e_f(x)} df_x(\mathcal{T}_{1_y}(x))\right)^{\perp}.$$

Since $\mathcal{T}_{1_y}(x) \in T_x^H M$, $df_x(\mathcal{T}_{1_y}(x)) = 0$, iff $\mathcal{T}_{1_y}(x) = 0$. Using Lemma 1.2(i) we get $H_x = 0$, iff $\mathcal{T}_{1_y}(x) = 0$. ♡

Applying Prop. 1.5(d)(e) we obtain immediately:

Corollary 1.8.1 *If $n = 2$, Γ_f is a minimal submanifold of $M \times N$. If $n \geq 3$, Γ_f is minimal, iff ∇e_f restricted to U is a vertical vector field for the submersion $f|_U$.*

Chapter 2

STABILITY OF A MINIMAL GRAPH AND A GENERALISED EQUATION FOR NON-PARAMETRIC HYPERSURFACES WITH CONSTANT MEAN CURVATURE

2.1 Some Remarks on the Stability of a Minimal Graph

Given a map $f : (M^m, g) \to (N^n, h)$ between Riemannian manifolds, such that the graph of f, $\Gamma_f : (M, g + f^*H) \to (M \times N, g \times h)$, is a minimal immersion, we may wonder when it is volume-stable or energy-stable.

In Ch. 0 we have given a brief introduction on the stability of volume and energy functionals, from which we may conclude at once that, if (M, g) and (N, h) have non-positive sectional curvatures, then minimal graphs are energy-stable, but not necessarily volume-stable, like e.g. in the case of Ex. 2.1 in this section. However, the latter does hold, when (M, g) has non-negative sectional curvatures and $\dim N = 1$, as we will show to be an immediate consequence of a result obtained by Barbosa [Bar/78]. He studied the Jacobi fields on a domain D of a minimal hypersurface for the case $R \geq 0$ in the expression (0.9) for the Hessian of the volume fuctional V_D, obtaining the following result:

Theorem (Barbosa) *Let $\phi : M^m \to (\bar{M}^{m+1}, \bar{l})$ be an isometric minimal immersion, ν a unit normal vector field to M, $D \subset M$ a domain with compact closure, and $\bar{X}$ be a Killing vector field on $\bar{M}$. Assume that $\bar{M}$ has non-negative sectional curvatures. Then we have:*

(a) If $<\bar{X},\nu>_{\bar{g}}\ >0$ on $\bar{D}$, then ϕ is strictly volume-stable on D.

(b) If there exists a domain $D'\subset D$, such that $<\bar{X},\nu>_{\bar{g}}\ =0$ on $\partial D'$, then ϕ is not strictly volume-stable on D.

This theorem has an immediate application to graphs with co-dimension one. Suppose that N is oriented and one-dimensional. Let Y be a unit section along all N. In Ch. 1, Sec. 3, we remarked that $\nu=\frac{1}{\omega}(-\nabla f,Y)$, where $\omega=\sqrt{1+\|\nabla f\|_g^2}$ is a unit normal to the graph $\Gamma_f : M\to(M\times N, g\times h)$. Then, $\bar{X}=(0,Y)\in C^\infty(T(M\times N))$ is a parallel vector field and, therefore, also a Killing vector field. Moreover, it satisfies $<\bar{X},\nu>_{g\times h}\ =\frac{1}{\omega}>0$.

Proposition 2.1 *If (M,g) is a Riemannian manifold with non-negative sectional curvatures, and if Γ_f is minimal, then, for each compact domain $D\subset M$, Γ_f is volume-stable on D.*

Remark 2.1 Also Barbosa [Bar/78] mentioned this consequence for the case $M^m=\mathbb{R}^m$, which was already a well-known result.

Example 2.1 Micallef [Mi/84] observed that the example given by Osserman [Os/69] of the map $f:\mathbb{R}^2\to\mathbb{R}^2$ reading

$$f(x,y)=\frac{1}{2}(e^x-3e^{-x})\big(\cos(\frac{y}{2}),-\sin(\frac{y}{2})\big)$$

has a graph $\Gamma_f:\mathbb{R}^2\to\mathbb{R}^4$ which is minimal and, moreover, energy-stable, but not volume-stable.

Besides, this example shows that there are minimal graphs of functions $f:\mathbb{R}^2\to\mathbb{R}^2$ which are not linear planes of $\mathbb{R}^4$, i.e. the *Bernstein Theorem* does not hold for graphs of co-dimension ≥ 2. This was already to be expected from the work of Lawson and Osserman [La-Os/77], which gave a negative answer to the uniqueness, regularity, and even existence of solutions to the minimal surface system for co-dimension ≥ 2.

Remark 2.2 At this point we should recall the theorem of Bernstein, since it concerns minimal graphs. It states that, if $f:\mathbb{R}^m\to\mathbb{R}$ is a smooth function, such that the graph $\Gamma_f\subset\mathbb{R}^{m+1}$ is a minimal hypersurface of $\mathbb{R}^{m+1}$ and $m\leq 7$, then f is a linear function. The case $m=2$ was proved by Bernstein in 1927 [B/27] and reproved by Fleming in 1962 [Fl/62], who used a new technique. This

method constituted a basis for the proofs of the cases $m = 3$ (de Giorgi [DG/65]), $m = 4$ (Almegren [Al/66]), and $m \leq 7$ (Simons [Si/68]). For $m \geq 8$ the theorem is no longer true, i.e. there exist complete analytic minimal graphs of sufficiently high dimension (from $m = 8$ upwards) that are not hyperplanes (Bombieri, de Giorgi, and Giusti [Bo-DG-Gi/69]).

A minimal graph of a map from $\mathbb{R}^m$ to $\mathbb{R}$ is a solution of a differential equation, viz. the minimal-hypersurface equation (see next section). In general, a Bernstein-type problem amounts to determining when the domain of a solution of a certain differential equation is sufficiently large (for a given metric) in order to conclude that the solution is a trivial one. Given a minimal submanifold of a Riemannian manifold (and minimal means being a solution of a certain differential equation), a Bernstein-type problem would be to find out when that submanifold is a totally geodesic one. This problem can be solved, if we require the minimal submanifold to be volume-stable and/or impose a rigidity condition. For example, a condition on the Gauss map of a surface or on the total scalar curvature of a hypersurface may lead to the desired result. It seems surprising that the original Bernstein Theorem only holds for $m \leq 7$. Stability is not sufficient to make the theorem hold, since, for all m, a minimal graph of $\mathbb{R}^{m+1}$ is stable. A reason why it fails for $m \geq 8$ appears to originate in the way the total scalar curvature grows. This conjecture was pointed out and justified by do Carmo and Peng [DC-Pe/80]. Their result is the following:

Theorem (do Carmo,Peng) *Let $x : M \to \mathbb{R}^{m+1}$ be a complete stable minimal hypersurface of $\mathbb{R}^{m+1}$, K the scalar curvature of M with the induced metric, and $B_R(p)$ a geodesic ball of M with centre in a fixed point p and radius R. Thus, if*

$$\lim_{R \to +\infty} \frac{\int_{B_R(p)} |K| dM}{R^{2+2q}} = 0 \, , \ q < \sqrt{\frac{2}{m}} \, , \tag{2.1}$$

then $x(M)$ is a hyperplane of $\mathbb{R}^{m+1}$. In particular, if the total curvature of x, i.e. $\int_M |K| dM$, is finite, the conclusion holds.

On the other hand, Miranda [Mir/67] proved that, for a minimal graph of $\mathbb{R}^{m+1}$,

$$\lim_{R \to +\infty} \frac{\int_{B_R(p)} |K| dM}{R^{m-2}} < +\infty \, . \tag{2.2}$$

So, from Eqs. (2.1,2.2), the Bernstein Theorem holds for $m \leq 5$. Moreover, the authors [DC-Pe/80] conclude that counter-examples to it, for higher dimensions,

should then have infinite total curvature, approaching infinity at least quadratically in the geodesic distance from a fixed point in M.

A similar Bernstein-type problem has been formulated by Schoen, Simon, and Yau [Sch-Si-Ya/75] for a stable minimal hypersurface M^m of a space N^{m+1} with non-negative constant sectional curvatures, imposing the condition $\lim_{R\to+\infty} R^{-q}\mathrm{vol}_M(B_R(p)) = 0$, for some $q \in (0, 4+\sqrt{\frac{8}{m}})$, where $B_R(p)$ denotes a geodesic ball of M or the intersection of a ball of N with M. This condition is satisfied for minimal graphs of $\mathbb{R}^{m+1}$, when $m \leq 5$, too.

If a map $f : \mathbb{R}^m \to \mathbb{R}^n$ has minimal graph $\Gamma_f \subset \mathbb{R}^{m+n}$, we cannot expect Γ_f to be linear, as shows Ex. 2.1. Nevertheless, we can find some conditions in order to obtain a Bernstein-type problem for $n \geq 2$. As we recalled in Ch. 0, $\Gamma_f : (\mathbb{R}^m, g+f^*h) \to (\mathbb{R}^{m+n}, g\times h)$ with g, h the resp. Euclidean metrics of $\mathbb{R}^m, \mathbb{R}^n$, has parallel mean curvature, iff its Gauss map $\gamma_{\Gamma_f} : (\mathbb{R}^m, g+f^*h) \to G(m+n, m)$ is harmonic. Using this fact and studying the regular balls of the Grassmannian manifolds, Hildebrandt, Jost, and Widman [Hi-Jo-Wi/80] (see also Ref. [Hi/85]) got the following Bernstein Theorem:

Theorem (Hildebrandt, Jost, Widman) *Suppose that the C^2-functions $z^i = f^i(x)$, $i = m+1, \ldots, m+n$, $x \in \mathbb{R}^m$ define a non-parametric m-dimensional manifold X of $\mathbb{R}^{m+n}$ which has parallel mean-curvature field. Suppose also that the tangent planes of X do not differ too much from the "horizontal plane" $z^{m+1} = 0, \ldots, z^{m+n} = 0$. More precisely, suppose that there is a number β_0 with*

$$0 < \beta_0 < \cos^{-p}\left(\frac{\pi}{2\sqrt{\kappa p}}\right), \quad p = \min\{m,n\}, \quad \kappa = \begin{cases} 1 & \textit{if } p = 1 \\ 2 & \textit{if } p \geq 2, \end{cases} \tag{2.3}$$

such that

$$\det|\bar{g}_{ij}| = \det|\delta_{ij} + < df(e_i), df(e_j) >| \leq \beta_0^2, \tag{2.4}$$

where $e_1, \ldots, e_m$ is the canonic basis of $\mathbb{R}^m$. Then, $f : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map.

Therefore, if $\| df \|$ is bounded by a conveniently chosen positive constant, that is, $g + f^*h$ is a "small" deformation of the metric g, and if Γ_f has parallel mean curvature, then f is in fact a linear map. Besides, Hildebrandt et al. observed that, if $n = 1$, then $p = \kappa = 1$, and so condition (2.3) does not impose any restriction on β_0 and condition (2.4) becomes $\|\nabla f\|_g \leq$ constant, which results in Moser's weak Bernstein Theorem [Mos/61], reading: any entire C^2-solution $f(x)$, $x \in \mathbb{R}^m$, of the

minimal-surface equation $\mathrm{div}_g\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}}\right)=0$, with $\sup_{R^n}\|\nabla f\|_g<\infty$, is necessarily a linear function. The above theorem of Hildebrandt et al. is a particular case of their main result in Ref. [Hi-Jo-Wi/80], which is a Liouville-type theorem for harmonic maps of simple or compact Riemannian manifolds with range contained in a regular ball.

2.2 The Equation for a Non-Parametric Hypersurface of $(M\times \mathbb{R}, g\times h)$ with Constant Mean Curvature: Some Remarks on Regularity of Solutions

Let (M^m, g) be a m-dimensional Riemannian manifold and $f: M \to \mathbb{R}$ be a smooth function. Let h be the Euclidean metric of $\mathbb{R}$. From Eq. (1.15) and Lemma 1.3 we know that the mean curvature H of the graph $\Gamma_f \subset (M\times\mathbb{R}, g\times h)$ is given by

$$\begin{aligned} H &= \frac{1}{m\sqrt{1+\|\nabla f\|_g^2}}\left(\mathrm{trace}_{(g+f^*h)}\nabla df\right)\nu \\ &= \frac{1}{m\sqrt{1+\|\nabla f\|_g^2}}\left(\Delta f-\frac{1}{1+\|\nabla f\|_g^2}\sum_{i,j=1}^{m}\nabla df(X_i,X_j)df(X_i)df(X_j)\right)\nu \\ &= \frac{1}{m}\mathrm{div}_g\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}}\right)\nu\,, \end{aligned}$$

where $\nu=\frac{(-\nabla f,1)}{\sqrt{1+\|\nabla f\|_g^2}}$ is a unit normal to Γ_f and $X_1,\dots,X_m$ is a local orthonormal frame of (TM,g). So Γ_f has constant mean curvature with $\|H\|=|c'|$, iff $\mathrm{div}_g\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}}\right)=mc'$. Thus, the equation

$$\mathrm{div}_g\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}}\right)=c \quad \text{(constant)} \tag{2.5}$$

or, equivalently,

$$\mathrm{trace}_{(g+f^*h)}(\nabla df)=c\sqrt{1+\|\nabla f\|_g^2} \tag{2.6}$$

is the equation for *non-parametric* hypersurfaces of $M\times\mathbb{R}$ — i.e. for graphs of maps from M to $\mathbb{R}$ — with constant mean curvature. For $c=0$ it becomes the equation for a minimal graph. More generally, if in Eqs. (2.5,2.6) we replace c by a

function $mH(x)$, we get the equation for non-parametric hypersurfaces of $M \times \mathbb{R}$ with prescribed mean curvature, given at each point $x \in M$ by $H(x)\nu$.

Let $x : D \subset M \to \Omega \subset \mathbb{R}^m$ be a coordinate system of M and let g_{ij}, g^{ij}, and $|g|$ be as given in Ch. 0. Note: throughout this section we will use the index-summation convention. Then, we have

$$\mathrm{trace}_{(g+f^*h)}(\nabla df) = \bar{g}^{ij} \nabla df\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$

where $[\bar{g}^{ij}]$ denotes the inverse matrix of $[\bar{g}_{ij}]$ with

$$\bar{g}_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_{g+f^*h} = g_{ij} + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}. \tag{2.7}$$

We can easily verify that

$$\bar{g}^{ij} = g^{ij} - \frac{g^{is} g^{tj} \frac{\partial f}{\partial x^s}\frac{\partial f}{\partial x^t}}{1 + g^{kl}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^l}}. \tag{2.8}$$

Since we have $\nabla df\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial^2 f}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial f}{\partial x^k}$, $\nabla f = g^{kl}\frac{\partial f}{\partial x^l}\frac{\partial}{\partial x^k}$, $\left\langle \nabla f, \frac{\partial}{\partial x^j}\right\rangle_g = \frac{\partial f}{\partial x^j}$, and $\|\nabla f\|_g^2 = g^{kl}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^l}$, Eq. (2.6) is, in this coordinate system, given by

$$\left(g^{ij} - \frac{g^{is}g^{tj}\frac{\partial f}{\partial x^s}\frac{\partial f}{\partial x^t}}{1 + g^{kl}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^l}}\right)\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right) = c\left(1 + g^{kl}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^l}\right)^{\frac{1}{2}}. \tag{2.9}$$

This equation is of the form

$$Qu = a^{ij}(x, u, Du)\frac{\partial^2 u}{\partial x^i \partial x^j} + b(x, u, Du), \tag{2.10}$$

where $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ is a C^2-function of the variable $x \in \mathbb{R}^m$, $Du = \left(\frac{\partial u}{\partial x^1}, \ldots, \frac{\partial u}{\partial x^m}\right)$, and where the coefficients of Q are the functions a^{ij}, $b : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ defined for all values $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^m$.

Equation (2.10) is called a second-order quasi-linear differential equation. In our case, these coefficients are given by

$$a^{ij}(x, z, p) = a^{ij}(x, p) = g^{ij}(x) - \frac{g^{is}(x)g^{tj}(x)p_s p_t}{1 + g^{kl}(x)p_k p_l}, \quad p = (p_1, \ldots, p_m), \tag{2.11}$$

and

$$b(x, z, p) = -{}^M\Gamma^k_{ij}(x)\left(g^{ij}(x) - \frac{g^{is}(x)g^{tj}(x)p_s p_t}{1 + g^{kl}(x)p_k p_l}\right)p_k - c(1 + g^{kl}p_k p_l)^{\frac{1}{2}}. \tag{2.12}$$

Note that

$$\left[a^{ij}(x, p)\right] = \left[g_{ij}(x) + p_i p_j\right]^{-1}. \tag{2.13}$$

Denote by $G(x) : \mathbb{R}^m \to \mathbb{R}^m$ the self-adjoint, positive-definite linear operator given by $< G(x)e_i, e_j >= g_{ij}(x)$, and by $B(p)$ the self-adjoint, semi-definite, non-negative linear operator given by $< B(p)e_i, e_j >= p_i p_j$. Then, $[a^{ij}(x,p)]$ represents the matrix (in the canonic basis of $\mathbb{R}^m$) of the self-adjoint positive definite operator $(G(x)+B(p))^{-1}$. Hence, Eq. (2.9) is a ***second-order quasi-linear elliptic differential equation*** in all $\Omega \times \mathbb{R} \times \mathbb{R}^m$ [Gil-Tr/83]. However, it is not uniformly elliptic in all $\Omega \times \mathbb{R} \times \mathbb{R}^m$, as we will see in the following.

Let $\lambda(x,p) > 0$ and $\Lambda(x,p) > 0$ denote the minimum resp. maximum eigenvalues of $[a^{ij}(x,p)]$. Then, $\lambda^{-1}(x,p)$ and $\Lambda^{-1}(x,p)$, are the maximum resp. minimum eigenvalues of $[g_{ij}(x) + p_i p_j]$. Denote by $\check{\alpha}(x)$ and $\hat{\alpha}(x)$ the minimum resp. maximum eigenvalues of $[g_{ij}(x)]$. Then, we have the two inequalities

$$\begin{aligned} \frac{1}{\Lambda(x,p)} &= \min_{\|u\|=1}\left(g_{ij}(x)u^i u^j + < u,p >^2\right) \\ &\le \min_{\|u\|=1}\left(\hat{\alpha}(x) + < u,p >^2\right) = \hat{\alpha}(x)\,, \end{aligned} \tag{2.14}$$

because there exists a u with $u \perp p$ (we are supposing $m \ge 2$), and

$$\begin{aligned} \frac{1}{\lambda(x,p)} &= \max_{\|u\|=1}\left(g_{ij}(x)u^i u^j + < u,p >^2\right) \\ &\ge \max_{\|u\|=1}\left(\check{\alpha}(x) + < u,p >^2\right) = \check{\alpha}(x) + \|p\|^2\,. \end{aligned} \tag{2.15}$$

Note that in Eqs. (2.14,2.15) we have equalities, if $g_{ij}(x) = \alpha(x)\delta_{ij}$ for some positive function $\alpha(x)$ $(\equiv \check{\alpha}(x) \equiv \hat{\alpha}(x))$. Thus,

$$\frac{\Lambda(x,p)}{\lambda(x,p)} \ge \frac{\check{\alpha}(x) + \|p\|^2}{\hat{\alpha}(x)}\,. \tag{2.16}$$

So, $\frac{\Lambda}{\lambda}$ is not bounded on all $\Omega \times \mathbb{R} \times \mathbb{R}^m$, which proves that $\mathcal{Q}$ is non-uniformly elliptic on all $\Omega \times \mathbb{R} \times \mathbb{R}^m$, being only uniformly elliptic on an open subset $\mathcal{U}$ with $P^1(\mathcal{U}) \subset \Omega$ and $P^3(\mathcal{U}) \subset \mathbb{R}^m$ both bounded.

Let us now write Eq. (2.10) in the form $\mathcal{Q}u(x) = F(x,u,Du,D^2u) = 0$, where $D^2u := \left(\frac{\partial^2 u}{\partial x^1 \partial x^1}, \frac{\partial^2 u}{\partial x^1 \partial x^2}, \ldots, \frac{\partial^2 u}{\partial x^m \partial x^m}\right)$, and where

$$F(x,z,p,r) = F(x,p,r) = a^{ij}(x,z,p)r_{ij} + b(x,z,p)$$

is a C^∞-function on $\Omega \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m^2}$. If now $u_0 \in C^2(\Omega)$ is a solution of $\mathcal{Q}u = 0$, we conclude, since $\mathcal{Q}$ is elliptic on Ω at u_0 and $[a^{ij}(x,u_0,Du_0)]$ is a positive-definite matrix for all $x \in \Omega$, and using a well-known regularity theorem on second-order

differential operators (see e.g. Ref. [Au/82], page 86, Th. 3.56), that u_0 is smooth on Ω.

We can improve this regularity property, starting from Eq. (2.5). Let us suppose that M is oriented. In a local coordinate system $x : D \to \overline{\Omega} \subset \mathbb{R}^m$, assumed to be orientation-preserving, we have

$$\frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}} = \frac{g^{kl}\frac{\partial f}{\partial x^l}}{\sqrt{1+g^{st}\frac{\partial f}{\partial x^s}\frac{\partial f}{\partial x^t}}}\frac{\partial}{\partial x^k} .$$

Using Eq. (0.4) we get

$$\operatorname{div}_g\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}}\right) = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^k}\left(\frac{\sqrt{|g|}g^{kl}\frac{\partial f}{\partial x^l}}{\sqrt{1+g^{st}\frac{\partial f}{\partial x^s}\frac{\partial f}{\partial x^t}}}\right) .$$

Thus, Eq. (2.5) is, in this local coordinate system, given by

$$\frac{\partial}{\partial x^k}\left(\frac{\sqrt{|g|}g^{kl}\frac{\partial f}{\partial x^l}}{\sqrt{1+g^{st}\frac{\partial f}{\partial x^s}\frac{\partial f}{\partial x^t}}}\right) = c\sqrt{|g|} . \tag{2.17}$$

This equation is of the *divergence form*

$$\begin{aligned} \mathfrak{Q}u &= \frac{\partial}{\partial x^k}\Big|_{(x)}\big(A^k(x,u,Du)\big) + B(x,u,Du) \\ &= \operatorname{div}_{(x)}\big(A^k(x,u,Du)\big) + B(x,u,Du) , \end{aligned} \tag{2.18}$$

where

$$A^k(x,z,p) = A^k(x,p) = \frac{\sqrt{|g|}g^{kl}(x)p_l}{\sqrt{1+g^{st}p_sp_t}} \tag{2.19}$$

and

$$B(x,z,p) = B(x) = -c\sqrt{|g|}(x) . \tag{2.20}$$

A C^1-function $u : \overline{\Omega} \to \mathbb{R}$ is said to be a weak solution of Eq. (2.17), if, $\forall \phi \in \mathcal{D}(\Omega)$ (i.e. $\phi \in C^\infty(\Omega)$ with compact support in Ω),

$$\int_\Omega\left(\frac{\sqrt{|g|}g^{kl}\frac{\partial u}{\partial x^l}}{\sqrt{1+g^{st}\frac{\partial u}{\partial x^s}\frac{\partial u}{\partial x^t}}}\frac{\partial\phi}{\partial x^k} + c\sqrt{|g|}\phi\right)dx^1\wedge\ldots\wedge dx^m = 0 . \tag{2.21}$$

For $f = u\circ x : D \subset M \to \mathbb{R}$, Eq. (2.21) is equivalent to

$$\int_D\left(\frac{g^{kl}\frac{\partial f}{\partial x^l}}{\sqrt{1+g^{st}\frac{\partial f}{\partial x^s}\frac{\partial f}{\partial x^t}}}\frac{\partial\phi}{\partial x^k} + c\phi\right)dV_g = 0 , \quad \forall\phi\in\mathcal{D}(D) , \tag{2.22}$$

that is, to

$$-\int_D \left\langle \frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}}, \nabla\phi \right\rangle_g dV_g = \int_D c\phi dV_g\,, \quad \forall\phi \in \mathcal{D}(D)\,. \tag{2.23}$$

We call a C^1-function $f : M \to I\!R$ a *weak solution* of the equation for non-parametric hypersurfaces of $(M \times I\!R, g \times h)$ with constant mean curvature, and we write

$$\mathrm{div}_g \left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}} \right) = c\,, \tag{2.24}$$

if, $\forall\phi \in \mathcal{D}(M)$,

$$-\int_M \frac{<\nabla\phi, \nabla f>_g}{\sqrt{1+\|\nabla f\|_g^2}} dV_g = \int_M c\phi dV_g\,. \tag{2.25}$$

More generally, we obtain weak solutions of the equation for non-parametric hypersurfaces of $(M \times I\!R, g \times h)$ with prescribed mean curvature, replacing everywhere the constant c by the function $mH(x)$.

In order to be able to apply the regularity theory of Morrey [Mo/54], we have to write Eq. (2.24) as the Euler-Lagrange equation of a variational problem. Thereto we use the method of Gulliver [Gu/83,Gu/74] of characterising submanifolds with prescribed mean curvature as critical points of a sum of two functionals.

We consider the volume functional for graphs of C^1-functions $f : M \to I\!R$ on a compact domain $\overline{D}$, given by

$$V(f,\overline{D}) = V_{\overline{D}}(\Gamma_f) = \int_D 1 dV_{(g+f^*h)} = \int_D \sqrt{1+\|\nabla f\|_g^2} dV_g\,.$$

The function f is a critical point of $V(\cdot,\overline{D})$, if, for any variation $f_t = f + tW$ with $W : D \to I\!R$ a C^1-map and $t \in (-\epsilon,\epsilon)$ with compact support in D, we have

$$\frac{\partial}{\partial t} V(f_t,\overline{D})\big|_{t=0} = 0\,. \tag{2.26}$$

We are going to calculate explicitly the l.h.s. of Eq. (2.26). Let $x \in M$ and $X_1,\ldots,X_m$ be an orthonormal basis of (T_xM, g). Then,

$$\begin{aligned}
&\frac{\partial}{\partial t}\sqrt{1+\|\nabla f_t\|_g^2}(x)\big|_{t=0} = \\
&= \frac{\frac{\partial}{\partial t}\langle \nabla f_t(x), \nabla f_t(x)\rangle_g\big|_{t=0}}{2\sqrt{1+\|\nabla f\|_g^2}} = \sum_{i=1}^m \frac{\frac{\partial}{\partial t}\left(d(f_t)_x(X_i)\right)^2\big|_{t=0}}{2\sqrt{1+\|\nabla f\|_g^2}} \\
&= \sum_{i=1}^m \frac{\frac{\partial}{\partial t}\left\{\left(df_x(X_i)\right)^2 + 2t dW_x(X_i) df_x(X_i) + t^2\left(dW_x(X_i)\right)^2\right\}\big|_{t=0}}{2\sqrt{1+\|\nabla f\|_g^2}}
\end{aligned}$$

$$= \sum_{i=1}^{m} \frac{dW_x(X_i)df_x(X_i)}{\sqrt{1+\|\nabla f\|_g^2}} ,$$

which is a continuous map in the variable x. Hence,

$$\frac{\partial}{\partial t} V(f_t, D)\big|_{t=0} = \int_D \frac{\partial}{\partial t}\sqrt{1+\|\nabla f_t\|_g^2}\big|_{t=0} dV_g = \int_D \frac{<\nabla W, \nabla f>_g}{\sqrt{1+\|\nabla f\|_g^2}} dV_g . \qquad (2.27)$$

Observe that, if f is C^2 and W has compact support in D,

$$\frac{\partial}{\partial t} V(f_t, D)\big|_{t=0} = -\int_D \mathrm{div}_g \left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}} \right) W dV_g .$$

Now let us suppose that $\overline{D}$ is sufficiently small, say contractible. Then, $\overline{D} \times I\!R$ is also contractible. Therefore, all closed forms on $D \times I\!R$ are exact. In particular, there exists a $\alpha \in C^{\infty}\left(\bigwedge^m T^*(\overline{D} \times I\!R)\right)$, such that $d\alpha = (-1)^m c dV_{g \times h}$, where h is the Euclidean metric of $I\!R$ and $dV_{g\times h}$ is the volume element of $D \times I\!R$.
Consider the following functional defined for C^1-functions $f : \overline{D} \to I\!R$:

$$C(f, \overline{D}) = \int_D \Gamma_f^* \alpha ,$$

where $\Gamma_f^*\alpha$ is the continuous m-form of $\overline{D}$, given by

$$\Gamma_f^* \alpha_x(u_1, \ldots, u_m) = \alpha_{(x,f(x))}\Big((u_1, df_x(u_1)), \ldots, (u_m, df_x(u_m)) \Big) .$$

Let $W : \overline{D} \to I\!R$ be a C^1-function with $W\big|_{\partial D} = 0$. Next we calculate

$$\frac{\partial}{\partial t} C(f + tW, D)\big|_{t=0} = \frac{\partial}{\partial t} \int_D \left\langle \Gamma_{f+tW}^* \alpha, dV_g \right\rangle dV_g \big|_{t=0}$$

with $<,>$ the Hilbert-Schmidt Riemannian metric on $\bigwedge^m T^*M$. Fix $X_1, \ldots, X_m$ as an orthonormal frame of (TM, g), defined on all $\overline{D}$, and with the same orientation as $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$. Since D is compact and Γ_f is an embedding, we can define, for each $i \in \{1, \ldots, m\}$, a C^1-vector field Z^i on all $\overline{D} \times I\!R$, such that

$$Z^i_{(x,f(x))} = \Big(X_i(x), df_x(X_i(x)) \Big) , \ \forall x \in \overline{D} .$$

Also, let $\tilde{W} \in C^1(T(\overline{D} \times I\!R))$, such that $\tilde{W}_{(x,f(x))} = (0, W_x)\ \forall x \in \overline{D}$. We remark that, trivially,

$$\overset{M\times I\!R}{\nabla}_{Z^i} \tilde{W}_{(x,f(x))} = \Big(0, dW_x(X_i(x)) \Big) . \qquad (2.28)$$

Denote by $\theta : (-\epsilon, \epsilon) \times D \to D \times I\!R$ the C^1-map given by $\theta(t, x) = (x, f(x) + tW_x)$, and by $\tilde{Z}^i$ the C^1-section of $\theta^{-1}T(D \times I\!R)$ given by

$$\tilde{Z}^i_{(t,x)} = Z^i_{(x,f(x))} \quad \in \quad T_xM \times I\!R = T_{\theta(t,x)}(\bar{D} \times I\!R)\ , \quad \forall t \in (-\epsilon, \epsilon)\ ,\ x \in \bar{D}\ .$$

Then, we have

$$\nabla^{\theta^{-1}}_{\frac{\partial}{\partial t}} \tilde{Z}^i_{(0,x)} = 0\ , \tag{2.29}$$

where $\frac{\partial}{\partial t}$ denotes the smooth section of $T((-\epsilon, \epsilon) \times \bar{D})$, such that $\frac{\partial}{\partial t}(t, x) = (1, 0)$. Now let us fix $x \in D$. Then,

$$\begin{aligned} &\left\langle \Gamma^*_{f+tW}\,\alpha(x), dV_t(x) \right\rangle = \Gamma^*_{f+tW}\,\alpha(x)\big(X_1(x), \ldots, X_m(x)\big) = \\ &= \alpha_{(x,f(x)+tW_x)}\Big(\big(X_1, df_x(X_1) + t\,dW_x(X_1)\big), \ldots, \big(X_m, df_x(X_m) + t\,dW_x(X_m)\big)\Big)\ . \end{aligned}$$

Next we have to determine the following limit:

$$\begin{aligned} \lim_{t \to 0} \frac{1}{t} \Big\{ &\alpha_{(x,f(x)+tW_x)}\Big(\big(X_1, df_x(X_1)\big) + t\big(0, dW_x(X_1)\big), \ldots, \big(X_m, df_x(X_m)\big) + t\big(0, dW_x(X_m)\big)\Big) \\ &- \alpha_{(x,f(x))}\Big(\big(X_1, df_x(X_1)\big), \ldots, \big(X_m, df_x(X_m)\big)\Big)\Big\}\ . \end{aligned} \tag{2.30}$$

The first term in the limit can be evaluated as

$$\begin{aligned} &\alpha_{(x,f(x)+tW_x)}\Big(\big(X_1, df_x(X_1)\big) + t\big(0, dW_x(X_1)\big), \ldots, \big(X_m, df_x(X_m)\big) + t\big(0, dW_x(X_m)\big)\Big) = \\ &= \alpha_{(x,f(x)+tW_x)}\Big(\big(X_1, df_x(X_1)\big), \ldots, \big(X_m, df_x(X_m)\big)\Big) \\ &\quad + t \sum_{i=1}^m \alpha_{(x,f(x)+tW_x)}\Big(\big(X_1, df_x(X_1)\big), \ldots, \big(0, dW_x(X_i)\big), \ldots, \big(X_m, df_x(X_m)\big)\Big) \\ &\quad + \sum_{k \geq 2} t^k \Phi_k(x, t)\ , \end{aligned}$$

where $\Phi_k(x, t)$ is a continuous function in $t \in (-\epsilon, \epsilon)$. Therefore,

$$\begin{aligned} &(2.30) = \\ &= \lim_{t \to 0} \frac{1}{t} \Big\{ \alpha_{(x,f(x)+tW_x)}\Big(\big(X_1, df_x(X_1)\big), \ldots, \big(X_m, df_x(X_m)\big)\Big) \\ &\qquad\qquad - \alpha_{(x,f(x))}\Big(\big(X_1, df_x(X_1)\big), \ldots, \big(X_m, df_x(X_m)\big)\Big)\Big\} \\ &\quad + \sum_{i=1}^m \lim_{t \to 0} \alpha_{(x,f(x)+tW_x)}\Big(\big(X_1, df_x(X_1)\big), \ldots, \big(0, dW_x(X_i)\big), \ldots, \big(X_m, df_x(X_m)\big)\Big) \\ &\quad + \sum_{k \geq 2} \lim_{t \to 0} t^{k-1} \Phi_k(x, t) \\ &= \frac{\partial}{\partial t} \left(\alpha_{(x,f(x)+tW_x)}\big(Z^1_{(x,f(x))}, \ldots, Z^m_{(x,f(x))}\big)\right) \Big|_{t=0} \\ &\quad + \sum_{i=1}^m \alpha_{(x,f(x))}\Big(\big(X_1, df_x(X_1)\big), \ldots, \big(0, dW_x(X_i)\big), \ldots, \big(X_m, df_x(X_m)\big)\Big) \end{aligned}$$

$$= d\big(\theta^{-1}\alpha(\tilde{Z}^1,\ldots,\tilde{Z}^m)\big)_{(0,x)}\big(\frac{\partial}{\partial t}\big)$$
$$+\sum_{i=1}^{m}\alpha_{(x,f(x))}\big(Z^1_{(x,f(x))},\ldots,(0,dW_x(X_i)),\ldots,Z^m_{(x,f(x))}\big)\,,$$

where $\theta^{-1}\alpha\in C^1\big(\bigwedge^m\theta^{-1}(T(\bar{D}\times\mathbb{R}))^*\big)$ is the alternating m-tensor given by

$$\theta^{-1}\alpha_{(t,x)}(z_1,\ldots,z_m)=\alpha_{\theta(t,x)}(z_1,\ldots,z_m)\,,$$

$\forall(t,x)\in(-\epsilon,\epsilon)\times D$ and $z_i\in T_{\theta(t,x)}(\bar{D}\times\mathbb{R})=\theta^{-1}\big(T(\bar{D}\times\mathbb{R})\big)_{(t,x)}$. Let ∇ denote the connection of the vector bundle $\bigwedge^m\theta^{-1}(T(\bar{D}\times\mathbb{R}))^*$. Using Eqs. (2.28,2.29) we have

$$
\begin{aligned}
(2.30) &= \\
&= \nabla_{\frac{\partial}{\partial t}}(\theta^{-1}\alpha)_{(0,x)}\big(\tilde{Z}^1_{(0,x)},\ldots,\tilde{Z}^m_{(0,x)}\big)\\
&\quad+\sum_{i=1}^{m}(\theta^{-1}\alpha)_{(0,x)}\big(\tilde{Z}^1_{(0,x)},\ldots,\nabla^{\theta^{-1}}_{\frac{\partial}{\partial t}}\tilde{Z}^i_{(0,x)},\ldots,\tilde{Z}^m_{(0,x)}\big)\\
&\quad+\sum_{i=1}^{m}\alpha_{(x,f(x))}\big(Z^1_{(x,f(x))},\ldots,(0,dW_x(X_i)),\ldots,Z^m_{(x,f(x))}\big)\\
&= \nabla_{d\theta(\frac{\partial}{\partial t})}\alpha_{\theta(0,x)}\big(\tilde{Z}^1_{(0,x)},\ldots,\tilde{Z}^m_{(0,x)}\big)\\
&\quad+\sum_{i=1}^{m}\alpha_{(x,f(x))}\big(Z^1_{(x,f(x))},\ldots,(0,dW_x(X_i)),\ldots,Z^m_{(x,f(x))}\big)\\
&= \nabla_{\tilde{W}}\alpha_{(x,f(x))}\big(Z^1_{(x,f(x))},\ldots,Z^m_{(x,f(x))}\big)\\
&\quad+\sum_{i=1}^{m}\alpha_{(x,f(x))}\big(Z^1_{(x,f(x))},\ldots,\overset{M\times\mathbb{R}}{\nabla}_{Z^i}\tilde{W}_{(x,f(x))},\ldots,Z^m_{(x,f(x))}\big)\\
&= d\big(\alpha(Z^1,\ldots,Z^m)\big)_{(x,f(x))}\big(\tilde{W}_{(x,f(x))}\big)\\
&\quad-\sum_{i=1}^{m}\alpha_{(x,f(x))}\big(Z^1_{(x,f(x))},\ldots,\overset{M\times\mathbb{R}}{\nabla}_{\tilde{W}}Z^i_{(x,f(x))},\ldots,Z^m_{(x,f(x))}\big)\\
&\quad+\sum_{i=1}^{m}\alpha_{(x,f(x))}\big(Z^1_{(x,f(x))},\ldots,\overset{M\times\mathbb{R}}{\nabla}_{Z^i}\tilde{W}_{(x,f(x))},\ldots,Z^m_{(x,f(x))}\big)\\
&= d\big(\alpha(Z^1,\ldots,Z^m)\big)_{(x,f(x))}\big(\tilde{W}_{(x,f(x))}\big)\\
&\quad+\sum_{i=1}^{m}\alpha_{(x,f(x))}\big(Z^1_{(x,f(x))},\ldots,[Z^i,\tilde{W}]_{(x,f(x))},\ldots,Z^m_{(x,f(x))}\big)\\
&= L_{\tilde{W}}\alpha_{(x,f(x))}\big(Z^1_{(x,f(x))},\ldots,Z^m_{(x,f(x))}\big)\,.
\end{aligned}
$$

Using now the following formula for a k-form $\theta\in C^1\big(\bigwedge^k T(M\times\mathbb{R})^*\big)$,

$$
\begin{aligned}
d\theta(Y_1,\ldots,Y_{k+1}) &= \textstyle\sum_{i=1}^{k+1}(-1)^{i-1}d\big(\theta(Y_1,\ldots,\hat{Y}_i,\ldots,Y_{k+1})\big)(Y_i)\\
&\quad+\textstyle\sum_{i<j}(-1)^{i+j}\theta\big([Y_i,Y_j],Y_1,\ldots,\hat{Y}_i,\ldots,\hat{Y}_j,\ldots,Y_{k+1}\big)\,,
\end{aligned}
$$

we get $L_{\tilde{W}}\alpha = \imath_{\tilde{W}} \circ d\alpha + d(\imath_{\tilde{W}}\alpha)$. Hence,

$$
\begin{aligned}
(2.30) = & \\
= \ & L_{\tilde{W}}\alpha_{(x,f(x))}\big(Z^1_{(x,f(x))},\dots,Z^m_{(x,f(x))}\big)\\
= \ & \imath_{\tilde{W}} \circ d\alpha_{(x,f(x))}(Z^1,\dots,Z^m) + d(\imath_{\tilde{W}}\alpha)_{(x,f(x))}(Z^1,\dots,Z^m)\\
= \ & d\alpha_{(x,f(x))}\big(\tilde{W}_{(x,f(x))}, Z^1_{(x,f(x))},\dots,Z^m_{(x,f(x))}\big) + d(\imath_{\tilde{W}}\alpha)_{(x,f(x))}\big(Z^1_{(x,f(x))},\dots,Z^m_{(x,f(x))}\big)\\
= \ & (-1)^m c\,dV_{g\times h}(x,f(x))\Big((0,W_x),\big(X_1,df_x(X_1)\big),\dots,\big(X_m,df_x(X_m)\big)\Big)\\
& + d(\imath_{\tilde{W}}\alpha)_{(x,f(x))}\Big(\big(X_1,df_x(X_1)\big),\dots,\big(X_m,df_x(X_m)\big)\Big)\\
= \ & (-1)^m c\,dV_{g\times h}(x,f(x))\Big((0,W_x),\big(X_1,df_x(X_1)\big),\dots,\big(X_m,df_x(X_m)\big)\Big)\\
& + \Gamma_f^*\big(d(\imath_{\tilde{W}}\alpha)\big)_x\big(X_1(x),\dots,X_m(x)\big)\,.
\end{aligned}
$$

Hence, $\frac{\partial}{\partial t}\left\langle \Gamma^*_{f+tW}\alpha(x), dV_g(x)\right\rangle\big|_{t=0}$ exists and gives a function continuous in the varaiable $x \in \overline{D}$. Thus,

$$
\begin{aligned}
\frac{\partial}{\partial t}C(f+tW,\overline{D})\big|_{t=0} &= \int_{\overline{D}} \frac{\partial}{\partial t}\left\langle \Gamma^*_{f+tW}\alpha(x), dV_g(x)\right\rangle\big|_{t=0} dV_g\\
&= \int_{\overline{D}}(-1)^m c\,dV_{g\times h}(x,f(x))\Big((0,W_x),\big(X_1,df_x(X_1)\big),\dots,\big(X_m,df_x(X_m)\big)\Big)dV_g\\
&\quad + \int_{\overline{D}}\Gamma_f^*\big(d(\imath_{\tilde{W}}\alpha)\big)\,.
\end{aligned}
$$

We cannot claim that $\Gamma_f^*\big(d(\imath_{\tilde{W}}\alpha)\big) = d\big(\Gamma_f^*(\imath_{\tilde{W}}\alpha)\big)$, because Γ_f is only C^1 and not C^2. So we cannot use Stokes' theorem directly. However, one may approximate f uniformly up to first derivatives by smooth functions on D and then prove the following, more general, Stokes' theorem (see e.g. Ref. [Ma/79]):

$$
\int_{\overline{D}}\Gamma_f^*\big(d(\imath_{\tilde{W}}\alpha)\big) = \int_{\partial D}\Gamma_f^*|_{\partial D}(\imath_{\tilde{W}}\alpha)\,.
$$

Since $\tilde{W}_{(x,f(x))} = (0,W_x) = 0\,, \ \ \forall x \in \partial D$, we get

$$
\begin{aligned}
&\frac{\partial}{\partial t}C(f+tW,\overline{D})\big|_{t=0} =\\
&= \int_{\overline{D}}(-1)^m c\,dV_{g\times h}(x,f(x))\Big((0,W_x),\big(X_1,df_x(X_1)\big),\dots,\big(X_m,df_x(X_m)\big)\Big)dV_g\,.
\end{aligned}
$$

In order to compute $dV_{g\times h}(x,f(x))\Big((0,W_x),\big(X_1,df_x(X_1)\big),\dots,\big(X_m,df_x(X_m)\big)\Big)dV_g$, which does not depend on the choice of the orthormal basis $X_1(x),\dots,X_m(x)$ of (T_xM,g), we may choose one, such that $df_x(X_i) = 0\ ,\ \ \forall i \geq 2$. Then, we get straightforwardly

$$
dV_{g\times h}(x,f(x))\Big((0,W_x),\big(X_1,df_x(X_1)\big),\dots,\big(X_m,df_x(X_m)\big)\Big) = (-1)^m W_x\,.
$$

Hence,

$$\frac{\partial}{\partial t}C(f+tW,\overline{D})\big|_{t=0} = \int_D cW\,dV_g\,. \tag{2.31}$$

Adding Eqs. (2.27) and (2.31) we obtain

$$\frac{\partial}{\partial t}(V+C)(f+tW,\overline{D})\big|_{t=0} = \int_D \left(\left\langle \frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}}, \nabla W\right\rangle_g + cW\right) dV_g\,. \tag{2.32}$$

A C^1-function $f:\overline{D}\to I\!R$ is said to be a critical point of the functional $V+C$, if $\frac{\partial}{\partial t}(V+C)(f+tW,\overline{D})\big|_{t=0}=0$ for any C^1-function $W:\overline{D}\to I\!R$ with compact support in D. Thus, a weak solution of Eq. (2.24) is a critical point of $V+C$ and vice versa. Here we should remark that this result also holds in the more general case of prescribed mean curvature, replacing everywhere the constant c by the function $mH(x)$.

Next we write the functional $V+C$ in local coordinates in order to verify that it fulfils the required conditions of Morrey.

For a C^1-function $f:\overline{D}\to I\!R$

$$(V+C)(f,\overline{D}) = \int_D \left(\sqrt{1+\|\nabla f\|_g^2} + \left\langle \Gamma_f^*\alpha, dV_g\right\rangle\right) dV_g\,.$$

In local coordinates, $\alpha\in\bigwedge^m T^*(\overline{D}\times I\!R)$ takes the form

$$\alpha = \lambda\,dx^1\wedge\ldots\wedge dx^m + \sum_{i_1<\ldots<i_{m-1}} \lambda^{i_1\ldots i_{m-1}}\,dx^{i_1}\wedge\ldots\wedge dx^{i_{m-1}}\wedge dt\,,$$

where $\lambda, \lambda^{i_1\ldots i_{m-1}}:\overline{D}\times I\!R\to I\!R$ are C^∞-functions. Then,

$$\begin{aligned}\left\langle \Gamma_f^*\alpha, dV_g\right\rangle &= \frac{1}{\sqrt{|g|}}\Gamma_f^*\alpha\Big(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^m}\Big)\\ &= \frac{1}{\sqrt{|g|}}\alpha\Big(\big(\frac{\partial}{\partial x^1}, df(\frac{\partial}{\partial x^1})\big),\ldots,\big(\frac{\partial}{\partial x^m}, df(\frac{\partial}{\partial x^m})\big)\Big)\\ &= \frac{1}{\sqrt{|g|}}\Big(\lambda + \sum_{i_1<\ldots<i_{m-1}}(-1)^{m-i_m}\lambda^{i_1\ldots i_{m-1}}\frac{\partial f}{\partial x^{i_m}}\Big)\,,\end{aligned}$$

where, for each group of $(m-1)$ indices $i_1<\ldots<i_{m-1}$, i_m denotes the remaining one to complete $\{1,\ldots,m\}$. So we obtain

$$\begin{aligned}&(V+C)(f,\overline{D}) =\\ &= \int_\Omega \left\{\sqrt{|g|}\sqrt{1+g^{st}\frac{\partial f}{\partial x^s}\frac{\partial f}{\partial x^t}} + \Big(\lambda + \sum_{i_1<\ldots<i_{m-1}}(-1)^{m-i_m}\lambda^{i_1\ldots i_{m-1}}\frac{\partial f}{\partial x^{i_m}}\Big)\right\} dx^1\wedge\ldots\wedge dx^m\\ &= I(f\circ x^{-1},\overline{\Omega})\,,\end{aligned}$$

where I is the functional, acting on C^1-functions $u : \overline{\Omega} \to I\!R$, given by

$$I(u, \overline{\Omega}) = \int_{\Omega} \Psi(x, u(x), Du(x))dx$$

with $\Psi : \Omega \times I\!R \times I\!R^m \to I\!R$ the smooth function

$$\Psi(x,z,p) = \sqrt{|g|}(x)\sqrt{1+g^{st}(x)p_sp_t} + \left(\lambda(x,z) + \sum_{i_1<\dots<i_{m-1}} (-1)^{m-i_m}\lambda^{i_1\dots i_{m-1}}(x,z)p_{i_m}\right) .$$

A C^1-function $u : \overline{\Omega} \to I\!R$ is a critical point of I, if, for any C^1-function W : $\overline{\Omega} \to I\!R$ with compact support in Ω, $\frac{\partial}{\partial t}I(u + tW, \overline{\Omega})\big|_{t=0} = 0$. So, u is a critical point of I, iff $f = u \circ x : \overline{D} \to I\!R$ is a critical point of $V + C$. Let us fix a C^1-function $u(x)$ and let $\mathcal{G}$ be a bounded domain of $\Omega \times I\!R \times I\!R^m$ of the form $\mathcal{G} = \{(x,z,p) : x \in \Omega\ ,\ \ |z - u(x)| < h\ ,\ \ \|p - p(x)\| < h\}$, where $p(x)$ is some continuous $I\!R^m$-valued function of x and h is a positive constant. Deriving the function Ψ w.r.t. the variable p, we get

$$\begin{aligned}\frac{\partial^2\Psi}{\partial p_i \partial p_j}(x,z,p) &= \frac{g^{ij}(x)\big(1+g^{ks}(x)p_kp_s\big) - g^{kj}(x)g^{si}(x)p_kp_s}{(1+g^{ks}(x)p_kp_s)^{\frac{3}{2}}}\sqrt{|g|}(x) \\ &= a^{ij}(x,p)\frac{\sqrt{|g|}(x)}{\sqrt{1+g^{ks}(x)p_kp_s}}\ ,\end{aligned}$$

$\forall x \in \Omega,\ z \in I\!R,\ p \in I\!R^m$, where $[a^{ij}(x,p)]$ is positive-definite matrix (see Eqs. (2.11,2.13)). Then, $\forall (x,z,p) \in \mathcal{G}$, we have

$$\frac{\partial^2\Psi}{\partial p_i \partial p_j}(x,z,p)\mu_i\mu_j > 0\ ,\quad \forall \mu \in I\!R^m\backslash\{0\}\ . \tag{2.33}$$

The fulfilment of this inequality means that Ψ is strictly convex with respect to the variable p, and is in accordance with Morrey's condition on $\Psi(x,z,p)$ to be the integrand of a regular variational problem near $u(x)$. Moreover, it is equivalent to what nowadays is called the ellipticity condition of the Euler-Lagrange operator $\mathcal{Q}(u) = \mathrm{div}_{(x)} D_p\Psi(x,u,Du) - D_z\Psi(x,u,Du)$ (cf. Ref. [Gil-Tr/83], page 289). Morrey (see Ref. [Mo/54], page 158) proved that, if $u : \overline{\Omega} \to I\!R$ is a C^1-function and is a critical point of $I(\cdot,\overline{\Omega})$, then, since Ψ is smooth, u is smooth on Ω. Moreover, if $u|_{\partial\Omega}$ is smooth, then u is smooth on $\overline{\Omega}$. Thus, we have the desired regularity property:

Proposition 2.2 *Let (M,g) be a smooth Riemannian manifold and $f : M \to I\!R$ be a C^1-function which is a weak solution of Eq. (2.24). Then, f is smooth on all $M\backslash\partial M$. If $f|_{\partial M}$ is also smooth, then f is smooth on all M.*

2.3 Existence of Graphs of Functions on the m-Hyperbolic Space with Given Constant Mean Curvature

In the previous section we have derived some regularity properties of graphs of maps $f : M^m \to \mathbb{R}$ with constant mean curvature c. From Sec. 1.1 we also know that, if M is non-compact and oriented, this constant cannot exceed the ratio of the Cheeger constant $\mathfrak{h}(M)$ and the dimension m, and that, if M is compact (without boundary) and oriented, c can only be zero. Supposing that $\mathfrak{h}(M) \neq 0$, we may pose the following question:

Question *Given a constant c' with $0 \leq c' \leq \frac{1}{m}\mathfrak{h}(M)$, does there exist a map $f : M \to \mathbb{R}$, such that $\Gamma_f \subset M \times \mathbb{R}$ has constant mean curvature equal to c'?*

In Th. 1.2 we only gave a positive answer for the case of the two-dimensional hyperbolic space with c' assuming its extreme value $\frac{1}{2}$. Here we consider the more general case of the hyperbolic space of arbitrary dimension $m \geq 2$, $H^m = (B^m, g)$, where B^m is the unit open disk with centre O in $\mathbb{R}^m$ and where g is the complete metric given by

$$g = \frac{4|dx|^2}{(1 - |x|^2)^2} \, .$$

We recall (see Ch. 0) that H^m has constant curvature equal to -1 and that $\mathfrak{h}(H^m) = m - 1$.

Proposition 2.3 *For each $c \in [1 - m, m - 1]$, the function $f : H^m \to \mathbb{R}$ given by*

$$f(x) = \int_0^{r(x)} \frac{\frac{c}{(\sinh r)^{m-1}} \int_0^r (\sinh t)^{m-1} dt}{\sqrt{1 - \left(\frac{c}{(\sinh r)^{m-1}} \int_0^r (\sinh t)^{m-1} dt\right)^2}} dr \, ,$$

where

$$r(x) = \log\left(\frac{1 + |x|}{1 - |x|}\right) \, ,$$

is smooth on all H^m, and $\Gamma_f \subset H^m \times \mathbb{R}$ has constant mean curvature given by $\|H\| = \frac{|c|}{m}$.
In particular, if $m = 2$ and $c = 1$, f can be written as

$$f(x) = \int_0^{r(x)} \sqrt{\frac{1}{2}(\cosh r - 1)} dr \, .$$

Proof. According to Sec. 2.2, one only has to verify that f satisfies the differential equation (2.5). Of course, we are not going to execute such simple but tiresome arithmetic. Instead, we will show how the above expression for f is obtained. The procedure to be followed is to solve for f in Eq. (2.5) as a function of the intrinsic distance $r(x)$ in H^m from the origin, thereby considering c as a varying parameter. Using the expressions for the Christoffel symbols of (H^m, g) computed in the proof of Th. 1.2, we see that the distance function $r : H^m \to \mathbb{R}$, $r(x) = \log\left(\frac{1+|x|}{1-|x|}\right) = 2\tanh^{-1}(|x|)$, has the following properties: $\forall x \neq 0$, $\nabla r = \frac{1-|x|^2}{2}\frac{x}{|x|}$, where the gradient of r is w.r.t. the metric g. Hence, $\|\nabla r\|_g = 1$ and $\Delta r = (m-1)\coth r$. We observe that r^2 is smooth.

Let us write $f = h o r$ with $h : \mathbb{R}_0^+ \to \mathbb{R}$.

Then, $\nabla f = h' o r \nabla r$, and Eq. (2.5) applied to f becomes equivalent to $(\forall x \neq 0)$

$$\begin{aligned} c &= \mathrm{div}_g\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|_g^2}}\right) = \mathrm{div}_g\left(\frac{h'or\nabla r}{\sqrt{1+(h'or)^2}}\right) \\ &= \frac{h'or\,\Delta\, r}{\sqrt{1+(h'or)^2}} + \left\langle \nabla\left(\frac{h'or}{\sqrt{1+(h'or)^2}}\right), \nabla r\right\rangle_g \\ &= \frac{h'or\,\Delta\, r}{\sqrt{1+(h'or)^2}} + h'or\left\langle \nabla\left(\frac{1}{\sqrt{1+(h'or)^2}}\right), \nabla r\right\rangle_g + \frac{\langle \nabla(h'or), \nabla r\rangle_g}{\sqrt{1+(h'or)^2}} \\ &= \frac{h'or\,\Delta\, r}{\sqrt{1+(h'or)^2}} - \frac{1}{2}h'or\left\langle \frac{\nabla(1+(h'or)^2)}{(1+(h'or)^2)^{\frac{3}{2}}}, \nabla r\right\rangle_g + \frac{h''or < \nabla r, \nabla r >_g}{\sqrt{1+(h'or)^2}} \\ &= \frac{h'or\,\Delta\, r}{\sqrt{1+(h'or)^2}} - \frac{(h'or)^2 h''or\|\nabla r\|_g^2}{(1+(h'or)^2)^{\frac{3}{2}}} + \frac{h''or\|\nabla r\|_g^2}{\sqrt{1+(h'or)^2}}\,. \end{aligned}$$

Using the above properties of r we get

$$\begin{aligned} & c\big(1+(h'or)^2\big)^{\frac{3}{2}} = \\ &= (m-1)\coth r\,(h'or)\big(1+(h'or)^2\big) - (h'or)^2 h''or + h''or\big(1+(h'or)^2\big) \\ &= (m-1)\coth r\,(h'or)\big(1+(h'or)^2\big) + h''or\,. \end{aligned}$$

With the substitution $\omega(r) = h'(r)$, the equation becomes

$$\omega' = c(1+\omega^2)^{\frac{3}{2}} - (m-1)\coth r\,\omega(1+\omega^2)\,,\ \ \forall r > 0\,. \tag{2.34}$$

The next step is to reduce this differential equation to a linear one through several changes of variables. First we write Eq. (2.34) as

$$\frac{\omega'\omega}{(1+\omega^2)^{\frac{3}{2}}} = c\omega - (m-1)\coth r\frac{\omega^2}{(1+\omega^2)^{\frac{1}{2}}}\,.$$

Let $y = \frac{1}{(1+\omega^2)^{\frac{1}{2}}} \in (0,1]$. Then, $\omega = \pm\frac{\sqrt{1-y^2}}{y}$. Taking first ω non-negative, we get

$$\text{Eq. (2.34)} \iff -y' = c\frac{\sqrt{1-y^2}}{y} - (m-1)\coth r \frac{1-y^2}{y^2}y .$$

Thus,

$$-y'y = c\sqrt{1-y^2} - (m-1)\coth r\,(1-y^2) .$$

Let $v = y^2 \in (0,1]$. Then,

$$\text{Eq. (2.34)} \iff -\frac{1}{2}\frac{v'}{\sqrt{1-v}} = c - (m-1)\coth r\,\sqrt{1-v} .$$

Finally, let $u = \sqrt{1-v} \in [0,1)$. Hence,

$$\text{Eq. (2.34)} \iff u' = c - (m-1)\coth r\, u , \qquad (2.35)$$

which equation is linear. Let us first suppose $c = 1$. Then, the general solution of Eq. (2.35) is given by

$$\begin{aligned}
u(r) &= e^{-\int_{r_0}^{r}(m-1)\coth t dt}\left(\int_{r_0}^{r} e^{(m-1)\int_{r_0}^{s}\coth t dt}ds + u_0\right)\\
&= e^{-(m-1)(\log\sinh r - \log\sinh r_0)}\left(\int_{r_0}^{r} e^{(m-1)(\log\sinh s - \log\sinh r_0)}ds + u_0\right)\\
&= \frac{(\sinh r_0)^{m-1}}{(\sinh r)^{m-1}}\left(\frac{1}{(\sinh r_0)^{m-1}}\int_{r_0}^{r}(\sinh s)^{m-1}ds + u_0\right)\\
&= \frac{1}{(\sinh r)^{m-1}}\int_{r_0}^{r}(\sinh s)^{m-1}ds + u_0\frac{(\sinh r_0)^{m-1}}{(\sinh r)^{m-1}} .
\end{aligned}$$

Let us now put $r_0 = u_0 = 0$. Then, we have

$$u(r) = \frac{1}{(\sinh r)^{m-1}}\int_0^r (\sinh s)^{m-1}ds , \quad \forall r > 0 . \qquad (2.36)$$

Next we prove that $u \in [0,1)$ with $u(0) = 0$, and, moreover, that $\sup_{r\in(0,+\infty)} u(r) = \lim_{r\to+\infty} u(r) = \frac{1}{m-1}$.

Obviously, u is positive and, with l'Hospital's rule, $u(0) = \lim_{r\to 0} u(r) =$
$= \lim_{r\to 0}\frac{(\sinh r)^{m-1}}{(m-1)(\sinh r)^{m-2}\cosh r} = \lim_{r\to 0}\frac{\tanh r}{m-1} = 0$.

If $u(r)$ attains a local maximum at some $r_0 \in (0,+\infty)$, then $u'(r_0) = 0$. From Eq. (2.35) we have $u(r_0) = \frac{\tanh r_0}{m-1}$. Thus, $u(r_0) < \frac{1}{m-1} \le 1$. On the other hand, if there are no local maxima, then, necessarily, $\sup_{r\in(0,+\infty)} u(r) = \lim_{r\to+\infty} u(r)$. So we only

have to calculate this limit. With partial integration,

$$\begin{aligned}
&\int_0^r (\sinh s)^{m-1} ds = \\
&= \left[\cosh s(\sinh s)^{m-2}\right]_0^r - (m-2)\int_0^r \cosh^2 s(\sinh s)^{m-3} ds \\
&= \cosh r(\sinh r)^{m-2} - (m-2)\int_0^r (1+\sinh^2 s)(\sinh s)^{m-3} ds \\
&= \cosh r(\sinh r)^{m-2} - (m-2)\int_0^r (\sinh s)^{m-3} ds - (m-2)\int_0^r (\sinh s)^{m-1} ds .
\end{aligned}$$

Thus,

$$\int_0^r (\sinh s)^{m-1} ds = \frac{1}{m-1}\cosh r(\sinh r)^{m-2} - \frac{m-2}{m-1}\int_0^r (\sinh s)^{m-3} ds$$

and

$$\begin{aligned}
\frac{\int_0^r (\sinh s)^{m-1} ds}{(\sinh r)^{m-1}} &= \frac{1}{m-1}\coth r - \frac{(m-2)\int_0^r (\sinh s)^{m-3} ds}{(m-1)(\sinh r)^{m-1}} \\
&= \frac{1}{m-1}\coth r - \frac{m-2}{(m-1)\sinh^2 r}\,\frac{\int_0^r (\sinh s)^{m-3} ds}{(\sinh r)^{m-3}} .
\end{aligned}$$

Since $\forall p$, $\frac{\int_0^r (\sinh s)^p ds}{(\sinh r)^p}$ is a bounded function on $r \in [0, +\infty)$, we have

$$\lim_{r\to+\infty} \frac{\int_0^r (\sinh s)^{m-1} ds}{(\sinh r)^{m-1}} = \frac{1}{m-1}\lim_{r\to+\infty}\coth r = \frac{1}{m-1} .$$

Therefore,

$$\sup_{r\in[0,+\infty)} u(r) = \frac{1}{m-1} , \tag{2.37}$$

which is not a maximum. So, $0 \le u(r) < \frac{1}{m-1}$, $\forall r \in [0, +\infty)$ and $u(r)$ satisfies Eq. (2.35) for $c = 1$. Let now c be an arbitrary constant. Then, the function $\tilde{u}(r) = cu(r)$ is a solution of Eq. (2.35), but we have to impose $\tilde{u}(r) \in [0, 1)$. From Eq. (2.37) we conclude that c must satisfy $0 \le c \le m-1$. That is, $\forall 0 \le c \le m-1$, the function

$$\tilde{u}(r) = c\,\frac{\int_0^r (\sinh s)^{m-1} ds}{(\sinh r)^{m-1}}$$

fulfils the condition specified in Eq. (2.35).

In terms of the original function f, we have

$$f(x) = h(r(x)) = \int_0^{r(x)} \frac{\frac{c}{(\sinh r)^{m-1}}\int_0^r (\sinh t)^{m-1} dt}{\sqrt{1 - \left(\frac{c}{(\sinh r)^{m-1}}\int_0^r (\sinh t)^{m-1} dt\right)^2}} dr ,$$

which solves Eq. (2.5). If we had chosen ω non-positive, we would have obtained the same expression for f, but now with $1 - m \le c \le 0$. Obviously, f is smooth

on $H^m\setminus\{0\}$. Let us now investigate the behaviour of f close to the origin. Near $t=0$ we have the following Taylor expansions:
$\sinh t = t + \frac{t^3}{6} + O(t^5) = t(1+\frac{t^2}{6}+O(t^4))$,
$(1+t)^m = 1 + mt + \Theta(t^2)$,
$\frac{1}{\sqrt{1+t}} = 1 - \frac{t}{2} + \Theta(t^2)$, $\frac{1}{1-t} = 1 + t + \Theta(t^2)$,
where $\Theta(t^k)$ and $O(t^k)$ are analytic functions of the form
$\Theta(t^k) = \sum_{n\geq 0} \frac{a^{k+n}}{(k+n)!} t^{k+n}$, $O(t^k) = \sum_{n\geq 0} \frac{a^{k+2n}}{(k+2n)!} t^{k+2n}$.
Then we have
$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{t^2}{2} + O(t^4)$, $\frac{1}{1-t^2} = 1 + t^2 + O(t^4)$, and
$(\sinh t)^{m-1} = t^{m-1}\big(1+\frac{t^2}{6}+O(t^4)\big)^{m-1} = t^{m-1}\big(1+\frac{m-1}{6}t^2\big) + O(t^{m+3})$.
Hence,

$$\frac{1}{(\sinh s)^{m-1}}\int_0^s (\sinh t)^{m-1}dt = \frac{\frac{s^m}{m} + \frac{m-1}{m-2}\frac{s^{m+2}}{6} + O(s^{m+4})}{s^{m-1}\big(1+\frac{m-1}{6}s^2 + O(s^4)\big)} =$$
$$= \frac{\frac{s}{m} + \frac{m-1}{m-2}\frac{s^3}{6} + O(s^5)}{1+\frac{m-1}{6}s^2 + O(s^4)}$$
$$= \Big(\frac{s}{m} + \frac{m-1}{m+2}\frac{s^3}{6} + O(s^5)\Big)\Big(1 - s^2\Big(\frac{m-1}{6} + O(s^2)\Big) + O(s^4)\Big)$$
$$= s\Big(1-\frac{m-1}{6}s^2\Big)\Big(\frac{1}{m} + \frac{m-1}{m+2}\frac{s^2}{6}\Big) + O(s^5)$$
$$= \frac{s}{m}\Big(1 - \frac{m-1}{m+2}\frac{s^2}{3}\Big) + O(s^5)\,.$$

For A close to zero, $\frac{A}{\sqrt{1-A^2}} = A(1+\frac{1}{2}A^2) + O(A^5)$. Putting

$$A = \frac{c}{(\sinh s)^{m-1}}\int_0^s (\sinh t)^{m-1}dt = c\frac{s}{m}\Big(1 - \frac{m-1}{m+2}\frac{s^2}{3}\Big) + O(s^5)\,,$$

we have $O(A^5) = O(s^5)$ and

$$\frac{A}{\sqrt{1-A^2}} =$$
$$= \Big(c\frac{s}{m}\Big(1-\frac{m-1}{m+2}\frac{s^2}{3}\Big) + O(s^5)\Big)\Big(1+\frac{1}{2}\Big(\frac{cs}{m}\Big(1-\frac{m-1}{m+2}\frac{s^2}{3}\Big)+O(s^5)\Big)^2\Big) + O(s^5)$$
$$= c\frac{s}{m}\Big(1 + s^2\Big(\frac{c^2}{2m^2} - \frac{m-1}{3(m+2)}\Big)\Big) + O(s^5)\,.$$

Therefore,

$$\int_0^r \frac{A}{\sqrt{1-A^2}}ds = \frac{c}{m}\frac{r^2}{2} + \frac{c}{m}\frac{r^4}{4}\Big(\frac{c^2}{2m^2} - \frac{m-1}{3(m+2)}\Big) + O(r^6)$$
$$= \frac{c}{m}\frac{r^2}{2} + O(r^4)\,.$$

Consequently,

$$f(x) = \int_0^{r(x)} \frac{A}{\sqrt{1-A^2}} ds = \frac{c}{m}\frac{r^2(x)}{2} + O(r^4(x)) \,.$$

Since $r^2(x)$ is smooth on all H^m, we conclude that $f(x)$ is, too. $\heartsuit$

Remark 2.3 We could not find a non-trivial global solution f of Eq. (2.5) of the type $f(x) = h \circ r(x)$ for $c = 0$. In fact, if in Eq. (2.35) we set $c = 0$, it has as solution $u(r) = \kappa(\sinh r)^{1-m}$ with κ an arbitrary integration constant, which, for $\kappa \neq 0$, tends to $+\infty$ near the origin. Hence, $u(r) \notin (0,1]$. Thus, it seems that we can formulate the following Bernstein-type conjecture:

Conjecture Let $f : H^m \to I\!R$ be a smooth map, such that $\Gamma_f \subset H^m \times I\!R$ is a minimal graph. Then, f is a totally geodesic map.

We also remark that the function f given in Prop. 2.3 has non-bounded $\|\nabla f\|_g$.

REFERENCES

[Al/66] F. J. Almegren Jr., *Some Interior Regularity Theorems for Minimal Surfaces and an Extension of Bernstein's Theorem*, Ann. of Math. 85, 277–292 (1966).

[Au/82] T. Aubin, *Non-Linear Analysis on Manifolds, Monge-Ampère Equations*, Grundl. Math. Wiss. **252**, Springer Verlag New York (1982).

[Ba-Ee/81] P. Baird and J. Eells, *A Conservation Law for Harmonic Maps*, Geometry Symp. Utrecht 1980, Lect. Notes in Math. **894**, 1–25 (1981).

[Bar/78] J. L. Barbosa, *Remarks on Stability of Minimal Surfaces*, An. Acad. Brasil. Ciênc. 50 (3), 295–297 (1978).

[B/27] S. N. Bernstein, *Über ein Geometrisches Theorem und seine Anwendung auf die Partielle Differentialgleichungen vom Elliptischen Typus*, Math. Z. **26**, 551–558 (1927).

[Be-Ga-Ma/71] M. Berger, P. Gauduchon, and E. Mazet, *Le Spectre d'une Variété Riemannienne*, Lect. Notes in Math. **194** (1971).

[Bo-DG-Gi/69] E. Bombieri, E. de Giorgi, and E. Giusti, *Minimal Cones and the Bernstein Problem*, Inventiones Math. **7**, 243–268 (1969).

[Cha/84] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press (1984).

[Ch/65] S. S. Chern, *On the Curvatures of a Piece of Hypersurface in Euclidean Space*, Abh. Math. Sem. Hamburg **29**, 77–91 (1965).

[DG/65] E. de Giorgi, *Una Estensione del Teorema di Bernstein*, Ann. Scuola Norm. Sup. Pisa **19**, 79–85 (1965).

[DC-Pe/80] M. do Carmo and C. K. Peng, *Stable Complete Minimal Hypersurfaces*, Proc. Beijing Symp. on Diff. Geom. and Diff. Eq. (1980).

[Ee/79] J. Eells, *Minimal Graphs*, Manuscripta Math. **28**, 101–108 (1979).

[Ee-Le/78] J. Eells and L. Lemaire, *A Report on Harmonic Maps*, Bull. London Math. Soc. **10**, 1–68 (1978).

[Ee-Le/83] J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*, C.B.M.S. Reg. Conf. Series No. 50 A.M.S. (1983).

[Ee-Sa/64] J. Eells and J. Sampson, *Harmonic Mappings of Riemannian Manifolds*, Am. J. Math. **86**, 109–160 (1964).

[Fe/85] M. J. Ferreira, *Aplicações Ramificadas Conformes de Superfícies de Riemann e Problemas Variacionais*, Ph.D. Thesis, Science Faculty, Univ. of Lisbon (1985).

[Fi-Sch/80] D. Fischer-Colbrie and R. Schoen, *The Structure of Complete Stable Minimal Surfaces in 3-Manifolds of Non-Negative Scalar Curvature*, Comm. Pure and Appl. Math. **XXXIII**, 199–211 (1980).

[Fla/66] H. F. Flanders, *Remark on Mean Curvature*, J. London Math. Soc. **41**, 364–366 (1966).

[Fl/62] W. H. Fleming, *On the Oriented Plateau Problem*, Rend. Circ. Mat. Palermo **11**, 69–89 (1962).

[Fu/78] B. Fuglede, *Harmonic Morphisms between Riemannian Manifolds*, Ann. Inst. Fourier (Grenoble) **28**, 107–144 (1978).

[Ga/54] M. Gaffney, *A Special Stokes' Theorem for Complete Riemannian Manifolds*, Ann. of Math. **60** (1), 141–145 (1954).

[Gil-Tr/83] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer Verlag Berlin , 2nd edition (1983).

[Gu/74] R. Gulliver, *On the Existence of a Hypersurface of Prescribed Mean Curvature with a Given Boundary*, Manuscripta Math. **11**, 15–36 (1974).

[Gu/83] R. Gulliver, *Necessary Conditions for Submanifolds and Currents with Prescribed Mean Curvature*, Sem. on Minimal Submanifolds, Ann. Math. Studies **103**, Princeton Univ. Press, 225–242 (1983).

[He/55] E. Heinz, *Über Flächen mit Eindeutiger Projektion auf eine Ebene deren Krümmungen durch Ungleichungen Eingeschränkt Sind*, Math. Annalen **129**, 451–454 (1955).

[Hi/85] S. Hildebrandt, *Harmonic Mappings of Riemannian Manifolds*, Montecantini 1984, Lect. Notes in Math. **1161**, 1–117 (1985).

[Hi-Jo-Wi/80] S. Hildebrandt, J. Jost, and K. O. Widman, *Harmonic Mappings and Minimal Submanifolds*, Inventiones Math. **62**, 269–298 (1980).

[Ho-Os/82] D. Hoffman and R. Osserman, *The Area of the Generalised Gaussian Image and the Stability of Minimal Surfaces in S^n and $\mathbb{R}^n$*, Math. Annalen **260**, 437–452 (1982).

[Is/79] T. Ishiara, *A Mapping of Riemannian Manifolds which Preserves Harmonic Functions*, J. Math. Kyoto Univ. **19**, 215–229 (1979).

[Ko-No/63/69] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Intersc. Pub. New York, London, Vol. I (1963), Vol. II (1969).

[La-Os/77] H. B. Lawson and R. Osserman, *Non-Existence, Non-Uniqueness, and Irregularity of Solutions to the Minimal Surface System*, Acta Math. **139**, 1–17 (1977).

[Ma/79] A. Machado, *Tópicos de Análise e Geometria*, Vol. II, Textos e Notas do C.M.A.F. **14** (1979).

[Mi/84] M. Micalef, *Stable Minimal Surfaces in Euclidean Space*, J. Diff. Geom. **19**, 57–84 (1984).

[Mir/67] M. Miranda, *Una Maggiorazione Integrale per la Curvatura della Ipersuperfice Minimali*, Rend. del Sem. Mat. della Univ. Padova **38**, 91–107 (1967).

[Mo/54] C. B. Morrey, *Second-Order Elliptic Systems of Differential Equations*, in *"Contributions to the Theory of Partial Differential Equations"*, Ann. of Math. Studies **33**, Princeton Press, 101–159 (1954).

[Mo/66] C. B. Morrey, *Multiple Integrals in Calculus of Variations*, Grundl. Math. Wiss. **130**, Springer Verlag Berlin (1966).

[Mos/61] J. Moser, *On Harnack's Theorem for Elliptic Differential Equations*, Comm. Pure Appl. Math. **14**, 577–591 (1961).

[Os/69] R. Osserman, *A Survey of Minimal Surfaces*, Van Nostrand Math. Studies **25** (1969).

[Ru-Vi/70] E. A. Ruh and J. Vilms, *The Tension Field of the Gauss Map*, Trans. Am. Math. Soc. **149**, 569–573 (1970).

[Sa/86] I. M. C. Salavessa, *Graphs with Parallel Mean Curvature*, I.C.T.P. preprint IC/86/398 Trieste (1986), submitted for publication.

[Sch-Si-Ya/75] R. Schoen, L. Simon, and S. T. Yau, *Curvature Estimates for Minimal Hypersurfaces*, Acta Math. **134**, 275–288 (1975).

[Si/68] J. Simons, *Minimal Varieties in Riemannian Manifolds*, Ann. of Math. **88**, 62–105 (1968).

[Sm/65] S. Smale, *On the Morse Index Theorem*, J. Math. Mech. **14**, 1049–1056 (1965).

[Sp/79] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. IV, Publish or Perish Inc. Berkeley (1979).

[Ya/75] S. T. Yau, *Isoperimetric Constants and their First Eigenvalue of a Compact Manifold*, Ann. Sci. Ec. Norm. Sup. Paris **8**, 487–507 (1975).

[Ya/76] S. T. Yau, *Some Function-Theoretic Properties of Complete Riemannian Manifolds and their Applications to Geometry*, Indiana Univ. Math. J. **25** (7), 659–670 (1976).

Part II

CONFORMAL AND ISOMETRIC IMMERSIONS OF RIEMANNIAN MANIFOLDS [Ri-Sa/87]

1 Introduction

Let $f : M \to N$ be an immersion where M and N are Riemannian manifolds with metrics g resp. h. A natural problem is to study relations between g and $h^* = f^*h$, the induced metric on M via f. For instance, we can try to find out if, under some assumptions on f and the manifolds (M, g), (N, h), the two metrics h^* and g are conformally related, or, *a fortiori* , if $h^* = g$, that is, if f is an isometry. In the present work we give some sufficient conditions to positively answer the former problem and show that a slight strengthening of these provides a necessary and sufficient criterion to solve the latter. In both cases we assume the existence of a special vector field on N, at least in a neighbourhood of $f(M)$, proving, anyhow, that this class of vector fields is large enough to justify their use. As a side product, we present a Liouville-type result for f harmonic and with finite energy (proposition 2). The core of this work is in Sec. 3, in the form of theorems 1,2 and proposition 3. Some applications, in the more transparent case M compact, are given at the end of the same section. In particular, proposition 4 should be compared with the main results of Chern and Hsiung [Ch-Hs/63] and Hsiung and Rhodes [Hs-Rh/69].

2 Preliminaries and Formulae

Let (N, h) be a Riemannian manifold and $U \subset N$ an open set.

Definition A vector field X defined in U is said to be *almost conformal*, if there exist smooth functions $\alpha, \beta : U \to \mathbb{R}$, such that the Lie derivative of h with respect to X, $L_X h$, satisfies

$$2\alpha h \leq L_X h \leq 2\beta h \,. \tag{1}$$

X is said to be *finite*, if $\inf \alpha > -\infty$ and $\sup \beta < +\infty$, and to be *strongly almost conformal*, if it is finite and $\alpha > 0$.

Examples

1. Any conformal vector field X on U is almost conformal.
2. Any homothetic vector field X on U for which $L_X h$ is positive definite is strongly almost conformal. For instance, in $(\mathbb{R}^n, <,>)$ the position vector field X satisfies $L_X <,>= 2 <,>$.

3. A procedure to construct almost conformal vector fields is given by the following:

Proposition 1 *Let $U \subset (N,h)$ be an open set supporting a real function ϕ. Let $X = \nabla\phi$ be its gradient and $\nabla d\phi$ its second fundamental form. Then, $\alpha h \leq \nabla d\phi \leq \beta h$ for some smooth functions $\alpha, \beta : U \to I\!R$,* iff $2\alpha h \leq L_X h \leq 2\beta h$. .

Proof. Recall that, given any vector fields X, Y, Z, $L_X h(Y,Z) = \left\langle \nabla_Y X, Z \right\rangle_h + \left\langle \nabla_Z X, Y \right\rangle_h$. For $X = \nabla\phi$, we have $\left\langle \nabla_Y X, Z \right\rangle_h = \left\langle \nabla_Y(\nabla\phi), Z \right\rangle_h = Y \left\langle \nabla\phi, Z \right\rangle_h - \left\langle \nabla\phi, \nabla_Y Z \right\rangle_h = Y(Z\phi) - \nabla_Y Z(\phi) = \nabla d\phi(Y,Z)$. Therefore, we obtain $L_X h(Y,Z) = 2\nabla d\phi(Y,Z)$. ♡

For instance, let (N,h) be a complete manifold and $B_R(p)$ a *regular ball*,that is, $B_R(p)$ is a geodesic ball of radius R centred at $p \in N$ with the properties:
(i) $\sqrt{k}R < \frac{\pi}{2}$,
(ii) $C(p) \cap B_R(p) = \emptyset$,
where $k = \max\{0, \sup_{B_R(p)} K\}$ with K the sectional curvature of N, and where $C(p)$ is the cut locus of the centre p. Due to a result of Hildebrandt, Kaul, and Widman [Hi-Ka-Wi/77] (see also Ref. [Hi/85], page 66, Th. 5.2, the second fundamental form of the function $\phi = \frac{1}{2}\rho^2$ with $\rho(q) = \mathrm{dist}(q,p)$ satisfies, in $B_R(p)$,

$$a_k(\rho)h \leq \nabla d\phi \ ,$$

where $a_k(t) = t\sqrt{k}\cot(\sqrt{k}t)$ for $0 \leq t < \frac{\pi}{\sqrt{k}}$. Furthermore, if $K \geq w$, $w \leq 0$ on $B_R(p)$, then, in $B_R(p)$,

$$\nabla d\phi \leq a_w(\rho)$$

with $a_w(t) = t\sqrt{-w}\coth(\sqrt{-w}t)$ for $0 \leq t < \infty$. As a consequence, under the above assumptions the vector field $X = \rho\frac{\partial}{\partial\rho} = \frac{1}{2}\nabla(\rho^2)$ is strongly almost conformal on $B_R(p)$. By the Cartan-Hadamard theorem, this is particularly significant, if N is simply connected and with non-positive sectional curvatures. Indeed, in this case any geodesic ball is regular.

The above discussion also justifies the terminology of the following.

Definition A vector field X defined in U is said to be ***strongly convex***, if there exists a $\alpha : U \to I\!R$, such that $\inf_U \alpha > 0$ and

$$L_X h \geq 2\alpha h \ .$$

Again, if $B_R(p)$ is a regular ball in the complete manifold (N,h), then $X=\rho\frac{\partial}{\partial\rho}$ is strongly convex in the geodesic ball $B_R(p)$.

Let (M,g) be a second Riemannian manifold of dimension m and $f:M\to N$ a smooth map. The tension field τ_f of f is defined as ([Ee-Le/83])

$$\tau_f=\mathrm{trace}_g\nabla df\ .$$

Given a strongly convex vector field X in the open set $U\subset N$, we set $\nu=\inf_U\alpha>0$ and suppose $f(M)\subset U$. Now we denote by X_f the vector field along f and by ∇, ∇', and $\nabla^{f^{-1}}$ the connections on TM, TN, and $f^{-1}TN$, respectively. Let Z be the vector field on M defined by

$$<Z_x,Y>_g=<df_x(Y),X_{f(x)}>_h\ ,\quad \forall Y\in T_xM,\,x\in M\ .$$

Fixing $x_0\in M$ and choosing $X_1,\dots,X_m$ as an orthonormal frame of (M,g) defined in a neighbourhood of x_0, such that $\nabla X_i(x_0)=0$, we have, at the point x_0,

$$\begin{aligned}
\langle\tau_f,X_f\rangle_h(x_0) &= \sum_{i=1}^m\left\langle\nabla df_{x_0}(X_i,X_i),X_f\right\rangle_h=\sum_{i=1}^m\left\langle\nabla^{f^{-1}}_{X_i}(df(X_i))_{x_0},X_f\right\rangle_h\\
&= \sum_{i=1}^m\left\{d(\langle df(X_i),X_f\rangle_h)_{x_0}(X_i)-\left\langle df(X_i),\nabla^{f^{-1}}_{X_i}X_f\right\rangle_h\right\}\\
&= \sum_{i=1}^m\left\{d(<Z,X_i>_g)_{x_0}(X_i)-\left\langle df(X_i),\nabla'_{df(X_i)}X\right\rangle_h\right\}\\
&= \mathrm{div}_g(Z)(x_0)-\sum_{i=1}^m\frac{1}{2}L_Xh\big(df(X_i),df(X_i)\big)\\
&\le \mathrm{div}_g(Z)(x_0)-\sum_{i=1}^m\alpha\,(df(X_i),df(X_i))_h\\
&= \mathrm{div}_g(Z)(x_0)-\alpha\|df\|_g^2(x_0)\ .
\end{aligned}$$

So we have obtained the formula

$$<\tau_f,X_f>_h\le\mathrm{div}_g(Z)-\alpha\|df\|_g^2\le\mathrm{div}_g(Z)-\nu\|df\|_g^2\ ,\qquad (2)$$

where $\|df\|_g^2$ is the square of the Hilbert-Schmidt norm of the section $df\in C^\infty(TM^*\otimes f^{-1}TN)$, M being supplied with the metric g.

Supposing next that M is compact, we get by integrating Eq. (2)

$$E(f)\le-\frac{1}{2\nu}\int_M<\tau_f,X_f>_h dV_g\ ,\qquad (3)$$

where $E(f)$ is the energy of f. Observe that, in case $(N,h) = (\mathbb{R}^n, <,>)$, X is the position vector field, and f is an isometry, Eq. (3) transforms into the equality

$$V(M) = -\int_M < H, f > dV_g$$

with H the mean-curvature vector of the immersion f and $V(M)$ the volume of M. Hence, Eq. (3) can be thought to generalise a classical formula of Minkowski on convex bodies.

Furthermore, from Eq. (3) we deduce that, if f is harmonic, i.e. $\tau_f = 0$, then $E(f) = 0$ and f is constant. This result generalises to the non-compact case in the following:

Proposition 2 *Let (M,g) be a complete, non-compact, oriented Riemannian manifold, and $f : M \to U \subset (N,h)$ be a harmonic map of finite energy, where U is an open set supporting a strongly convex vector field X. Let y be some point in M and t the distance function from y. If $\lim_{t\to+\infty} \frac{1}{t^2}\int_{B_t(y)} \|X_f\|_h^2 dV_g = 0$, then f is constant.*

Proof. Let ς be the 1-form dual to the vector field Z on M appearing in Eq. (2), that is, $\varsigma(Y) =< Z, Y >_g$, and let $*$ be the Hodge star operator. Then, $d{*}\varsigma = \mathrm{div}_g(Z)dV_g$ with dV_g the volume element of (M,g), and, since f is harmonic, Eq. (2) gives

$$\nu\|df\|^2 dV_g \leq d{*}\varsigma \,. \tag{4}$$

Let now $\|{*}\varsigma\|$ be the norm of the $(m-1)$-form ${*}\varsigma$. Then, $\|{*}\varsigma\| = \|\varsigma\| = \|Z\|_g$, but, from the definition of Z and the Schwartz inequality, $\|Z\|_g \leq \|df\|_g\|X_f\|_h$. Therefore, applying Hölder's inequality, we have

$$\frac{1}{t}\int_{B_t(y)} \|{*}\varsigma\| dV_g \leq \left\{\int_{B_t(y)} \|df\|_g^2 dV_g\right\}^{\frac{1}{2}} \left\{\frac{1}{t^2}\int_{B_t(y)} \|X_f\|_h^2 dV_g\right\}^{\frac{1}{2}}$$

and, since the energy of f is finite,

$$\frac{1}{t}\int_{B_t(y)} \|{*}\varsigma\| dV_g \leq \sqrt{2E(f)} \left\{\frac{1}{t^2}\int_{B_t(y)} \|X_f\|_h^2 dV_g\right\}^{\frac{1}{2}} \xrightarrow[t\to+\infty]{} 0 \,.$$

By the Gaffney-Yau extension of Stokes' theorem (see the appendix of Ref. [Ya/76]), there exists a sequence of compact domains K_i in M, such that $K_i \subset K_{i+1}$,

$\bigcup_i K_i = M$, and $\int_{K_i} d*\varsigma \xrightarrow[i\to+\infty]{} 0$. Applying this to Eq. (4) we deduce $E(f) = 0$, i.e. f is constant. $\heartsuit$

Remark After careful inspection of the proof of Prop. 2 we conclude that it is sufficient to require U to support a *strictly convex* vector field instead of a strongly convex one. Such vector fields satisfy $L_X h \geq 2\alpha h$ with $\alpha > 0$.

Remark In case $U = B_r(p)$ and $X = \rho\frac{\partial}{\partial\rho}$ as in Sec. 2, Ex. 3, Prop. 2 should be compared with the results of Karp [Kar/82]. Indeed, if N is simply connected, complete, and with non-positive sectional curvatures, ρ^2 is smooth on all of N, $\nabla d(\rho^2) \geq 2h$, and $\|\nabla\rho\|_h = 1$ almost everywhere. Thus, it appears that our assumptions $E(f) < +\infty$ and $\lim_{t\to+\infty}\frac{1}{t^2}\int_{B_t(p)}\|X_f\|_h^2 dV_g = \lim_{t\to+\infty}\frac{1}{t^2}\int_{B_t(p)}(\rho\circ f)^2 dV_g = 0$ play the roles of boundedness of $f(M)$ resp. of moderate volume growth of M in Cor. 4.1.1 of Ref. [Kar/82].

Let now $f : (M, g) \to (N, h)$ be an immersion and M be oriented with $\dim M = m$. We set

$$h^* = f^* h$$

for the pulled-back metric. Let u be the ratio of the volume elements of h^* and g, so that u is the positive function defined by

$$dV_{h^*} = u\, dV_g\ .$$

Then, $\|df\|_g^2$ and u are related by the inequality

$$m u^{\frac{2}{m}} \leq \|df\|_g^2 \tag{5}$$

at any point $y \in M$, with equality holding, iff

$$h^* = \lambda^2 g \tag{6}$$

for some non-zero λ at y. In order to prove these statements, we choose at each point $x \in M$ an orthonormal basis $X_1, \ldots, X_m$ on $T_x M$ which diagonalises h^*, i.e. $< X_i, X_j >_{h^*} = < df_x(X_i), df_x(X_j) >_h = \delta_{ij}\lambda_i$. As $dV_{h^*} = dV_{h^*}(X_1, \ldots, X_m) dV_g$, we have $u = dV_{h^*}(X_1, \ldots, X_m) = \sqrt{\det < X_i, X_j >_{h^*}} = \sqrt{\lambda_1 \cdots \lambda_m}$. From the well-known geometric-arithmetic-mean inequality $(a_1 \cdots a_m)^{\frac{1}{m}} \leq \frac{1}{m}(a_1 + \cdots + a_m)$

for any non-negative real a_i, with equality iff $a_i = a_j \ \ \forall i, j$, and using $\|df\|_g^2 = \lambda_1 + \cdots + \lambda_m$, we obtain Eqs. (5) and (6).

As a final notation, we denote by H the mean-curvature vector of the isometric immersion $f : (M, h^*) \to (N, h)$.

3 Main Results

Given a strongly almost conformal vector field X on U, we define

$$\theta = \frac{\beta}{\alpha} \geq 1 ,$$

where α and β are as in Eq. (1).

Theorem 1 *Let (M, g) be an m-dimensional, with $m \neq 2$, oriented complete Riemannian manifold, $U \subset (N, h)$ be an open set supporting a strongly almost conformal vector field X, and $f : M \to U$ be an immesion satisfying*
(A) $\quad E(f) < +\infty$, *h^* is complete, and* $\|X_f\|_h, u\|H\|_h \in L^2(M, g)$.
If
(i) $\quad (\tau_f - muH \ , \ X_f)_h \geq 0 \quad$ *and*
(ii) $\quad u \leq \theta^{\frac{2}{2-m}} \quad$ for $\quad m \geq 3$,
(iii) $\quad u \geq \theta \quad$ for $\quad m = 1$,
then f is conformal with $h^* = \theta^{\frac{2}{2-m}}$.

Proof. With notations analogous to those used in Sec. 2, we get

$$\langle mH, X_f \rangle_h = \mathrm{div}_{h^*}(W) - \sum_i \frac{1}{2} L_X h\big(df(X_i), df(X_i)\big) \geq \mathrm{div}_{h^*}(W) - m\beta , \qquad (7)$$

where $\{X_i\}$ is an orthonormal basis of (M, h^*) and W is the vector field on M defined by $< W, Y >_{h^*} = \langle df(Y), X_f \rangle_h \ , \ \ \forall Y \in T_x M \ , \ x \in M$.
Multiplying Eq. (7) by u and subtracting the result from Eq. (2), we obtain

$$\langle \tau_f - muH \ , \ X_f \rangle_h \leq \mathrm{div}_g(Z) - u\,\mathrm{div}_{h^*}(W) - \alpha\|df\|_g^2 + m\beta u \ . \qquad (8)$$

Let us consider the case $m \geq 3$. From $u \leq \theta^{\frac{2}{2-m}} \leq 1$ (assumption (ii)) and since $\theta^{\frac{2}{2-m}}$ is the solution of the equation $\theta t = t^{\frac{m}{2}}$, we have $\theta t \leq t^{\frac{m}{2}}$, $\forall t : 0 \leq t \leq \theta^{\frac{2}{2-m}}$. Hence, Eq. (5) and $\theta \geq 1$ give $u \leq \frac{1}{\theta} u^{\frac{m}{2}} \leq \frac{1}{m}\|df\|_g^2 \leq \|df\|_g^2$. Therefore, as

f has finite energy, $u \in L^1(M,g)$, but $u \leq 1$, thus $u^2 \leq u$ and $u \in L^2(M,g)$. Using the definition of W and applying the Schwarz inequality we get $\|W\|_{h^*} \leq \sqrt{m}\|X_f\|_h$. So, as we assumed $\|X_f\|_h \in L^1(M,g)$ (conditions (A)), we obtain $u\|W\|_{h^*} \in L^1(M,g)$ or, equivalently, $\|W\|_{h^*} \in L^1(M,h^*)$. Furthermore, the first part of Eq. (7) and the properties of $L_X h$ yield

$$\langle mH, X_f\rangle_h \leq \mathrm{div}_{h^*}(W) - \alpha m \ .$$

Combining this with Eq. (7) and multiplying by u we arrive at

$$u\,\langle mH, X_f\rangle_h + \alpha m u \leq u\,\mathrm{div}_{h^*}(W) \leq u\,\langle mH, X_f\rangle_h + \beta m u \ .$$

Thus, since $u\|H\| \in L^2(M,g)$, we immediately deduce that $u\,\mathrm{div}_{h^*}(W) \in L^1(M,g)$ or, equivalently, $\mathrm{div}_{h^*}(W) \in L^1(M,h^*)$. Finally, as in the proof of Prop. 2, we have $\|Z\|_g \in L^1(M,g)$. Applying the Gaffney-Yau extension of Stokes' theorem [Ga/54] [Ya/76], we find a sequence of compact domains K_i telescoping to M, such that

$$\int_{K_i} u\,\mathrm{div}_{h^*}(W)dV_g = \int_{K_i} \mathrm{div}_{h^*}(W)dV_{h^*} \underset{i\to+\infty}{\longrightarrow} \int_M \mathrm{div}_{h^*}(W)dV_{h^*} = 0$$

and

$$\int_{K_i} \mathrm{div}_g(Z)dV_g \underset{i\to+\infty}{\longrightarrow} 0 \ .$$

Using assumption (i), integrating Eq. (8) over K_i, and letting $i \to +\infty$ we get

$$\int_M (m\beta u - \alpha\|df\|_g^2)dV_g \geq 0 \ . \tag{9}$$

Now, from Eq. (5) we obtain

$$m\beta u - \alpha\|df\|_g^2 \leq m(\beta u - \alpha u^{\frac{1}{m}}) = m\alpha(\theta u - u^{\frac{1}{m}}) \ ,$$

but assumption (ii) implies $\theta u - u^{\frac{1}{m}} \leq 0$. Hence, from Eq. (9) and the above inequality we conclude $\alpha\|df\|_g^2 = m\beta u$, i.e. $\|df\|_g^2 = m\theta u \leq m u^{\frac{1}{m}}$. This gives equality in Eq. (5). Consequently, Eq. (6) holds. Furthermore, since $\theta u = u^{\frac{1}{m}}$, $u = \theta^{\frac{m}{1-m}}$ and thus

$$h^* = \theta^{\frac{1}{1-m}} g \ . \tag{10}$$

The case $m = 1$ is proved analogously. ♡

Remarks If M is compact, conditions (A) are automatically satisfied and the

condition $\sup_U \beta < +\infty$ can be dropped. Moreover, from the proof of Th. 1, $u\|H\|_h \in L^1(M,g)$ is clearly satisfied, if $\|H\|$ is bounded. This guarantees the convergence of the integral $\int_M \mathrm{div}_{h^*}(W)dV_{h^*}$. Finally, one may substitute this condition by $\|\tau_f\|_h \in L^1(M,g)$ and work out a reasoning similar to the one presented, using now $\mathrm{div}_h(Z)dV_g$. The same remarks apply to the next results.

In what follows, conditions (A), (i), (ii), (iii) always refer to the ones given in Th. 1. As expected, the case $m = 2$ is special.

Proposition 3 *Let (M,g) be an oriented complete surface, $U \subset (N,h)$ an open set supporting a strongly conformal (i.e. $\alpha = \beta$) vector field X, and $f : M \to U$ an immersion satisfying (A) with $u \in L^1(M,g)$. Then, f is conformal,* iff *(i) holds with $m = 2$.*

Proof. A simple modification of the previous proof gives the sufficient part. In fact, in this case we immediately have from Eq. (5) $2u \leq \|df\|^2$, obtaining Eq. (9) as well. As $2\beta u - \alpha\|df\|_g^2 = \alpha(2u - \|df\|_g^2) = 0$, we conclude from Eq. (6) that f is conformal with $h^* = ug$.
Now we prove necessity. Given a conformal immersion $f : (M,g) \to (N,h)$, the following formula is well-known [Ho-Os/82]:

$$mH = \frac{1}{\sigma}\tau_f + \frac{m-2}{\sigma}df(\nabla_g \log\sqrt{\sigma}) , \tag{11}$$

where $m = \dim M$, $h^* = \sigma g$, and ∇_g is the gradient w.r.t. g. So, if M is a surface and f is conformal, then $u = \sigma$ and, from Eq. (11), $\tau_f - 2uH = \tau_f - \tau_f = 0$, which proves necessity of (i). ♡

Theorem 1 and its proof, together with Prop. 3, give:

Theorem 2 *Let (M,g) be an m-dimensional, oriented, complete Riemannian manifold, $U \subset (N,h)$ be an open set supporting a strongly conformal vector fields X, and $f : M \to U$ be an immersion satisfying (A) (with $u \in L^1(M,g)$, if $m = 2$). Then, f is an isometry,* iff

(i) $\langle \tau_f - muH\, ,\, X_f\rangle_h \geq 0$ *and*

(ii) $u \leq 1$, *that is, f is volume decreasing for $m \geq 3$,*

(*iii*) $u \geq 1$, *that is, f is volume increasing for $m = 1$,*

(*iv*) $u = 1$, *that is, f is volume preserving for $m = 2$.*

Proof. Necessity is obvious. As for sufficiency, since X is conformal, $\theta = 1$ and, for $m \geq 3$, formula (10) gives $h^* = g$, i.e. f is an isometry. The other cases are analogous. ♡

Remark Theorem 2 was proved in Ref. [Ri/87] under the assumptions $(N,h) = (\mathbb{R}^n, <,>)$, X is the position vector field, and M is compact.

Consider now the case where a strongly almost conformal vector field X has the additional property $\inf_U \alpha = \nu > 0$. Set $\mu = \sup_U \beta$ and $\bar\theta = \frac{\mu}{\nu} \geq 1$, which is a constant. Replacing θ by $\bar\theta$, α by ν, and β by μ in Th. 1, thus obtaining the corresponding conditions $(\bar{i}), (\bar{ii}), (\bar{iii})$, we can formulate the following strengthened theorem:

Theorem 3 *Let (M,g) be an m-dimensional, with $m \neq 2$, oriented, complete Riemannian manifold, $U \subset (N,h)$ be an open set supporting a strongly almost conformal vector field X with the property $\inf_U \alpha > 0$, and $f : M \to U$ be an immersion satisfying condition (A). If $(\bar{i})$ and $(\bar{ii})$ or $(\bar{iii})$ hold, then f is an isometry and X is homothetic.*

Proof. The proof of Th. 1 goes through till Eq. (10), which now becomes

$$h^* = \bar\theta^{\frac{2}{2-m}} g \, , \tag{12}$$

whence f is a homothety. So in this case $u = \bar\theta^{\frac{m}{2-m}}$. Computing the tension field τ_f, using Eq. (11), we obtain

$$\langle \tau_f - muH \, , \, X_f \rangle_h = \bar\theta^{\frac{2}{2-m}} (1 - \bar\theta^{-1}) \, \langle mH, X_f \rangle_h \, . \tag{13}$$

Combined with Eq. (7) this gives

$$\langle \tau_f - muH \, , \, X_f \rangle_h = \bar\theta^{\frac{2}{2-m}} (1 - \bar\theta^{-1}) \{ \mathrm{div}_{h^*}(W) - \Phi \} \, , \tag{14}$$

where $\Phi \geq m\nu > 0$. Condition $(\bar{i})$ and once more the Gaffney-Yau Stokes' theorem yield

$$0 \leq -\bar\theta^{\frac{2}{2-m}} (1 - \bar\theta^{-1}) \int_M \Phi dV_{h^*} \leq 0 \, , \tag{15}$$

as $\bar{\theta} \geq 1$. Consequently, $\bar{\theta} = 1$, that is, X is homothetic and, from Eq. (12), f is an isometry. ♡

Next we give an application of Theorem 2.

Proposition 4 *Let $i : (N', h') \to (N, h)$ be an isometric immersion of an oriented manifold N', with $\dim N' = m$ and $i(N') \subset U$ an open set in N supporting a conformal vector field X and having the property $\alpha > 0$ on U. Let (M, g) be an m-dimensional, compact, oriented Riemannian manifold and $F : (M, g) \to (N', h')$ be an orientation-preserving harmonic diffeomorphism with ratio of the volume elements u. Let ∇di be the second fundamental tensor of $i : N' \to N$ and H its mean-curvature vector field. Then, F is an isometry,* iff

(1) $\left\langle \mathrm{trace}_g \nabla di(dF, dF) - muH \, , \, X_{i \circ F} \right\rangle_h \geq 0$ *and*

(2) F *is volume decreasing* for $m \geq 3$,

(3) F *is volume preserving* for $m = 2$,

(4) F *is volume increasing* for $m = 1$.

Proof. Let $f = i \circ F$. Since i is an isometric immersion and $\dim N' = \dim M$, a standard composition formula of Eells-Sampson [Ee-Sa/64] gives

$$\tau_f = \tau_F + \mathrm{trace}_g \nabla di(dF, dF) \quad \text{and} \quad H_f = H \, ,$$

where H_f is the mean-curvature vector with respect to f. Moreover, F is harmonic and the ratio u_f of volume elements w.r.t. f satisfies $u_f = u$, which yields

$$\langle \tau_f - muH_f \, , \, X_f \rangle_h = \left\langle \mathrm{trace}_g \nabla di(dF, dF) - muH \, , \, X_{i \circ F} \right\rangle_h \, .$$

Since f is an isometry, iff F is so, the result follows immediately from Th. 2. ♡

Remark Proposition 4 generalises the main result of Hsiung and Rhodes [Hs-Rh/68] (and, earlier, of Chern and Hsiung [Ch-Hs/63]), which in our formulation can be stated in the form:

Let $F : (M, g) \to (N', h')$ be a harmonic, volume-preserving diffeomorphism. Let $x : (M, g) \to (N, h)$ and $i : (N', h') \to (N, h)$ be isometric immersions of compact submanifolds into the Riemannian manifold (N, h) which admits a strongly conformal vector field X. If $\langle \tau_f - mH_f \, , \, X_f \rangle_h \geq 0$, with $f = i \circ F$, then F is an isometry.

REFERENCES

[Ch-Hs/63] S. S. Chern and C. C. Hsiung, *On the Isometry of Compact Submanifolds in Euclidean Space*, Math. Annalen **149**, 278–285 (1963).

[Ee-Le/83] J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*, C.B.M.S. Reg. Conf. Series No. 50 A.M.S. (1983).

[Ga/54] M. Gaffney, *A Special Stokes' Theorem for Complete Riemannian Manifolds*, Ann. of Math. **60** (1), 141–145 (1954).

[Hi-Ka-Wi/77] S. Hildebrandt, H. Kaul, and K. O. Widman, *An Existence Theory for Harmonic Mappings of Riemannian Manifolds*, Acta Math. **138**, 1–16 (1977).

[Hi/85] S. Hildebrandt, *Harmonic Mappings and Minimal Immersions*, Lect. Notes in Math. **1161** (1985).

[Ho-Os/82] D. Hoffman and R. Osserman, *The Area of the Generalised Gaussian Image and Stability of Minimal Surfaces in S^n and $\mathbb{R}^n$*, Math. Annalen **260**, 437–452 (1982).

[Hs-Rh/68] C. C. Hsiung and B. H. Rhodes, *Isometries of Compact Submanifolds of a Riemannian Manifold*, J. Diff. Geom. **2**, 9–24 (1968).

[Ka/82] L. Karp, *Subharmonic Functions on Real and Complex Manifolds*, Math. Z. **179**, 535–554 (1982).

[Ri/87?] M. Rigoli, *On Immersed Compact Submanifolds of Euclidean Space*, to appear in Proc. Am. Math. Soc. (1987?).

[Ri-Sa/87] M. Rigoli and I. M. C. Salavessa, *Conformal and Isometric Immersions of Riemannian Manifolds*, to appear in Math. Z. (1987).

[Ya/76] S. T. Yau, *Some Function-Theoretic Properties of Complete Riemannian Manifolds and their Applications to Geometry*, Indiana Univ. Math. J. **25**, 659–670 (1976).

Part III

A VARIATIONAL PROBLEM AND A RELATED BERNSTEIN-TYPE THEOREM IN CONFORMAL GEOMETRY

Chapter 0

INTRODUCTION

Conformal Geometry is concerned with the properties of figures and objects of S^n, invariant under the action of the Möbius group, that is, invariant under an arbitrary conformal transformation of the sphere S^n equipped with its usual Riemannian structure of constant positive sectional curvature. This geometry was first introduced by Élie Cartan [C/55]. Here, we review in Ch. 1 the geometry of the Möbius space S^n and the induced conformal structure of an immersed submanifold, described by, among others, Schiemangk and Sulanke [Sch-Su/80], Sulanke [Su/81], Bryant [Br/84], and Rigoli [Ri/87], which authors use Cartan's method of *moving frames* . Faithful versions of this method can be found in Refs. [Je/77] and [Su-Šv/80].

Some of the conformal invariants in Riemannian geometry can be interpreted as invariants of conformal geometry. More precisely, we can compare the geometries of submanifolds in the Euclidean space $\mathbb{R}^n$ and of those of the Möbius space S^n, thinking of S^n as $\mathbb{R}^n$ with a point at infinity through stereographic projection. For example, the Willmore integrand for immersed surfaces $F : M \to \mathbb{R}^3$ into the 3-dimensional Euclidean space, which is invariant under conformal transformations of $\mathbb{R}^3$ (plus the "point at infinity"), can be interpreted as the Riemannian version of a conformally invariant 2-form Ω_F on M endowed with the induced conformal structure by the Möbius space S^3. In this way, Bryant [Br/84] studied the Willmore functional and the associated variational problem, deriving its Euler-Lagrange equation. The critical points are called Willmore immersed surfaces. This procedure allowed Rigoli [Ri/87] to generalise in a natural manner the concept of Willmore immersed submanifolds $f : M^m \to S^n$ of the Möbius space S^n as critical points of the variational problem associated with a functional $\mathcal{W}(f)$.

However, he only derived the Euler-Lagrange equation for the case $m = 2$ and n arbitrary. In this work, viz. in Ch. 2, we will solve for the Euler-Lagrange equation for any dimension $m \leq n$. This variational problem is related to the one of a different conformally invariant functional, involving the conformal Gauss map $\gamma_f : M^m \to Q_{n-m}(\mathbb{R}^{n+1})$ for an immersion $f : M^m \to S^n$. This relation was first pointed out by Bryant [Br/84], in the $m = 2$, $n = 3$ case, and by Rigoli [Ri/87], for $m = 2$, $n \leq 3$.

Also, in Ch. 2, we wil solve a Bernstein-type problem for Willmore hypersurfaces of S^n, which generalises the one solved by Rigoli [Ri/86] for surfaces of S^3.

Finally, in Ch. 3, we compute the second variation formula for Willmore surfaces immersed into a space form, in the context of Riemannian geometry. Earlier, this was done by Weiner [We/78] in the particular case where M^2 is a minimal surface of S^3.

Throughout this part we use the index-summation convention on repeated indices.

Chapter 1

THE CONFORMAL GEOMETRY OF SUBMANIFOLDS OF S^n

1.1 The Geometry of the Möbius Space

1.1.A The Infinitesimal Conformal Transformations of $\mathbb{R}^n$ and S^n

Two Riemannian manifolds (M, g) and (N, h) are said to be conformally equivalent, if there exists a diffeomorphism $\phi : M \to N$, such that $\phi^* h = e^{2\rho} g$, where ρ is a function on M. If $(N, h) = (M, g)$, such a diffeomorphism ϕ is called a conformal transformation of (M, g). (M, g) is said to be conformally flat, if it is locally conformally equivalent to a flat Riemannian space. Conformal flatness is well-known to be equivalent to the vanishing of the Weyl conformal curvature tensor, if $\dim M > 3$. For example, all the Riemannian manifolds with constant sectional curvature are conformally flat. A vector field X on M is called conformal (or a conformal infinitesimal transformation), if the local one-parameter group of transformations generated by X consists of local conformal diffeomorphisms. The vector field X is conformal, iff $L_X g = \mu g$, for some function μ on M. The conformal vector fields form a Lie algebra. Then, we have the following well-known results (see e.g. Ref. [Ko–No/63], notes 11,9; Ref. [Ib/85], pages 88,89; Ref. [Ei/64], page 285):

Proposition *The group of all conformal transformations of a connected n-dimensional Riemannian manifold N is a Lie group of dimension less than or equal to* $\frac{(n+1)(n+2)}{2}$, *provided* $n \geq 3$. *Its Lie algebra is isomorphic to the one generated by the complete conformal vector fields on* N. *The Lie algebra of the conformal vector*

fields (not necessarily complete) of any Riemannian manifold of dimension $n = 3$, or of any conformally flat Riemannian manifold of dimension $n > 3$, has dimension equal to $\frac{(n+1)(n+2)}{2}$, and only in these cases.

Thus, for all $n \geq 3$, the dimension of the Lie algebra of the conformal vector fields of S^n and of $\mathbb{R}^n$ attains the maximum value $\frac{(n+1)(n+2)}{2}$. In fact, we may obtain the infinitesimal conformal transformations of the n-sphere from those of the Euclidean space $(\mathbb{R}^n, <,>_n)$ via stereographic projection, which is a conformal diffeomorphism. We recall that a vector field $X = (X^1, \ldots, X^n) : \mathbb{R}^n \to \mathbb{R}^n$ of the n-Euclidean space is conformal, iff it is of the form

$$X^i_{(x^1,\ldots,x^n)} = \xi^j\left(\frac{1}{2}\|x\|^2\delta_{ij} - x^i x^j\right) + D^i_j x^j + a x^i + v^i , \tag{1.1}$$

where $[D^i_j]$ is a given skew-symmetric matrix, ξ^j, a, and v^i are given constants, and where $\|x\|^2 = x^{1^2} + \cdots + x^{n^2}$ [Ib/85] [He/75]. One can prove this by simply checking that such vector fields, which form a vector space of dimension $\frac{(n+1)(n+2)}{2}$, satisfy $L_X <,>_n = \mu <,>_n$, for some function μ. A concise way of writing X given in Eq. (1.1) is

$$X_x = \frac{1}{2}\|x\|^2\xi - <\xi, x>_n x + D(x) + ax + v , \tag{1.2}$$

where $\xi = (\xi^1, \ldots, \xi^n)$, $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$, D is a self-adjoint linear operator, and $a \in \mathbb{R}$. Under the usual identification $\mathbb{R}^n \equiv T_x\mathbb{R}^n$ for each $x \in \mathbb{R}^n$, which identifies the canonic basis e_i with the differential operators $\frac{\partial}{\partial x^i}$, a standard basis of the Lie algebra of these vector fields is given by

$$P_i = \frac{\partial}{\partial x^i} , \quad M_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} , \quad i < j ,$$

$$K_i = \left(\frac{1}{2}\|x\|^2\delta_{ij} - x^i x^j\right)\frac{\partial}{\partial x^j} , \quad D = x^i \frac{\partial}{\partial x^i} .$$

The *linear conformal group* of $\mathbb{R}^n$ is the $(\frac{n(n-1)}{2}+1)$-dimensional group $CO(n) = O(n) \times \mathbb{R}^+$ with composition law given by $(A, r) \circ (B, s) = (AB, rs)$. We can identify $CO(n)$ with the subgroup of the invertible $(n+2) \times (n+2)$ matrices

$$CO(n) \cong \left\{ \begin{bmatrix} r^{-1} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & r \end{bmatrix} : A \in O(n),\ r > 0 \right\} . \tag{1.3}$$

The Lie algebra of $CO(n)$ is given by

$$\begin{aligned} \mathfrak{co}(n) = \mathfrak{o}(n) \times \mathbb{R} &= \{D + aI_n : D \text{ is skew-symmetric}, a \in \mathbb{R}\} \\ &= \left\{ \begin{bmatrix} -a & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & a \end{bmatrix} : D \in \mathfrak{o}(n),\ a \in \mathbb{R} \right\}. \end{aligned}$$

The *affine conformal group of* $\mathbb{R}^n$ is the group of dimension $\frac{n(n+1)}{2} + 1$

$$\mathbb{R}^n \times CO(n) \cong \left\{ (Z, (A, r)) = \begin{vmatrix} r^{-1} & 0 & 0 \\ r^{-1}Z & A & 0 \\ \frac{1}{2}r^{-1}\,{}^tZZ & {}^tZA & r \end{vmatrix} \;:\; \begin{array}{c} A \in O(n) \\ r > 0 \\ Z \in \mathbb{R}^n \end{array} \right\}, \tag{1.4}$$

where Z is a column vector and ${}^tZ$ denotes its transposed, with composition law $(Z, (A, r)) \circ (W, (B, s)) = (Z + rAW, (AB, rs))$, and with Lie algebra

$$\mathbb{R}^n \times \mathfrak{co}(n) \cong \left\{ \begin{vmatrix} -a & 0 & 0 \\ v & D & 0 \\ 0 & {}^tv & a \end{vmatrix} \;:\; D \in \mathfrak{o}(n),\ a \in \mathbb{R},\ v \in \mathbb{R}^n \right\}.$$

The affine conformal group acts transitively on the left on $\mathbb{R}^n$ as $(Z, (A, r))(w) = Z + rA(w)$, $\forall w \in \mathbb{R}^n$, being the group of all conformal transformations of the Euclidean space $\mathbb{R}^n$. This group is also called the group of *similarities* and consists of translations, orthogonal maps, and multiplications by a non-zero constant. In fact, the elements of $\mathbb{R}^n \times \mathfrak{co}(n)$ constitute all the complete conformal vector fields of $\mathbb{R}^n$: the element $(v, D + aI_n)$ is identified with the conformal vector field $X_x = D(x) + ax + v$. The Killing vector fields of $\mathbb{R}^n$, i.e. the vector fields X, such that $L_X <,>_n = 0$, or, equivalently, the ones that generate local one-parameter groups of isometries, are precisely the vector fields of the form $X_x = D(x) + v$ that constitute the elements of $\mathbb{R}^n \times \mathfrak{o}(n)$. Note that the conformal vector fields of the type $X_x = \frac{1}{2}\|x\|^2\xi - <\xi, x>_n x$ are not complete. As we will see, these ones generate conformal transformations defined only on $\mathbb{R}^n \backslash \{p\}$, "mapping" the missing point p to infinity and vice versa, which are also known as origin-preserving inversions. By a theorem of Liouville (see e.g. Ref. [Po/81], page 172), a conformal transformation of $\mathbb{R}^n$ maps a hypersphere or a hyperplane to a hypersphere or a hyperplane, if $n \geq 3$.

On the other hand, the n-sphere S^n is an example where the group of conformal transformations has the maximum dimension $\frac{(n+1)(n+2)}{2}$ and all the conformal

vector fields are complete. Let us now choose the stereographic projection

$$
\begin{array}{rcl}
\sigma : \; S^n \backslash \{N\} & \longrightarrow & \mathbb{R}^n \\
(x^0, x^1, \dots, x^n) & \longrightarrow & \frac{1}{1-x^0}(x^1, \dots, x^n) \, ,
\end{array}
\tag{1.5}
$$

where $N = (1, 0, \dots, 0)$, with inverse

$$
\begin{array}{rcl}
\sigma^{-1} : \; \mathbb{R}^n & \longrightarrow & S^n \backslash \{N\} \\
\omega & \longrightarrow & \left(\frac{\|\omega\|^2 - 1}{1 + \|\omega\|^2}, \frac{2\omega}{1 + \|\omega\|^2} \right) .
\end{array}
$$

The coefficient of conformality of σ is given by $\sigma^* <,>_n = \frac{1}{(1-x^0)^2} <,>_{S^n}$. If X is a vector field of $\mathbb{R}^n$, then the vector field of $S^n \backslash \{N\}$, σ^{-1}-related with X, reading $X'_x = d\sigma^{-1}_{\sigma(x)}\left(X_{\sigma(x)}\right)$, $\forall x \in S^n$, satisfies

$$
L_{X'} <,>_{S^n} = d \log \left((1 - x^0)^2 \right)(X') <,>_{S^n} + (1 - x^0)^2 L_X <,>_n \circ (d\sigma \otimes d\sigma) \, . \tag{1.6}
$$

Thus, X is a conformal vector field of $\mathbb{R}^n$, iff X' is a conformal vector of $S^n \backslash \{N\}$. Explicitly, we have

$$
X'_x = \left(\frac{4 \left\langle \sigma(x), X_{\sigma(x)} \right\rangle_n}{(1 + \|\sigma(x)\|^2)^2} \, , \, \frac{-4 \left\langle \sigma(x), X_{\sigma(x)} \right\rangle_n \sigma(x) + 2 X_{\sigma(x)} (1 + \|\sigma(x)\|^2)}{(1 + \|\sigma(x)\|^2)^2} \right) . \tag{1.7}
$$

If X is a conformal vector field of $\mathbb{R}^n$, we can smoothly extend X' as to be also defined at the point N. In fact, from Eq. (1.2) follows that $d\sigma^{-1}_\omega(X_\omega) \to (0, \xi)$ as $\|\omega\| \to +\infty$. Thus, letting $x \to N$, we have $\|\sigma(x)\|^2 = \frac{1+x^0}{1-x^0} \to +\infty$. Hence,

$$
X'_x \xrightarrow[x \to N]{} (0, \xi) \, .
$$

The group $CO(n)$ acts on S^n via stereographic projection as

$$
\begin{array}{rcl}
CO(n) \times S^n & \longrightarrow & S^n \\
(P, x) & \longrightarrow & \begin{cases} \sigma^{-1}\left(P(\sigma(x))\right) & \text{for } x \neq N \\ N & \text{for } x = N \, . \end{cases}
\end{array}
$$

In the same way $\mathbb{R}^n \times CO(n)$ acts on S^n, whereby keeping N fixed, in other words, keeping the point of $\mathbb{R}^n$ at infinity fixed.

1.1.B The Möbius Group

Now we are going to review the group of conformal transformations of S^n, for $n \geq 2$, also called the Möbius group.

Let Q be the quadratic form given by

$$
Q(x) = -(x^0)^2 + (x^1)^2 + \cdots + (x^{n+1})^2 \, , \quad \text{for } x = (x^0, x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} \, ,
$$

that is, Q is the quadratic form associated with the Lorentz inner product $<\,,\,>$ of $I\!R^{n+2}$ with signature $(-,+,\cdots,+)$. The Lorentz group of dimension $\frac{(n+1)(n+2)}{2}$

$$O(n+1,1)=\{P\in GL(n+2;I\!R):\ \ P \text{ leaves } Q \text{ invariant}\}$$

is the group of the linear automorphisms of $I\!R^{n+2}$ that preserve $<\,,\,>$. Let $\mathcal{L}$ denote the light cone, $\mathcal{L}=Q^{-1}(0)$, and $\mathcal{L}^+$ its connected component

$$\mathcal{L}^+=\{x=(x^0,x^1,\ldots,x^{n+1})\in I\!R^{n+2}:\ \ Q(x)=0,\ x^0>0\}\,,$$

the positive light cone.

Henceforth, we agree on the index range $1\leq A,B,\ldots\leq n$, $0\leq a,b,\ldots\leq n+1$, and we fix a righthanded basis $\{\eta_0,\eta_A,\eta_{n+1}\}$ of $I\!R^{n+2}$ with $\eta_0,\eta_{n+1}\in\mathcal{L}^+$, and such that $<\,,\,>$ is represented in this basis by the matrix

$$S=[S^a_b=<\eta_a,\eta_b>]=\begin{bmatrix}0&0&-1\\0&I_n&0\\-1&0&0\end{bmatrix}\,. \tag{1.8}$$

We can always find such a basis, like for example $\eta_0=\frac{\epsilon_0-\epsilon_{n+1}}{\sqrt{2}}$, $\eta_A=\epsilon_A$, $\eta_{n+1}=\frac{\epsilon_0+\epsilon_{n+1}}{\sqrt{2}}$, where $(\epsilon_a)_{0\leq a\leq n+1}$ denotes the canonic basis of $I\!R^{n+2}$. Note that in this basis Q is given by $Q(x)=-2\bar{x}^0\bar{x}^{n+1}+\bar{x}^A\bar{x}^A$, for $x=\bar{x}^a\eta_a$.

If $P=[P^a_b]\in M_{(n+2)^2}$ is a $(n+2)\times(n+2)$ matrix, we identify P with the element of $GL(n+2,I\!R)$ given by $P(\eta_a)=P^b_a\eta_b$. Then, we have

$$\begin{aligned}{}^tPSP=S\quad &\text{iff}\quad <P_a,P_b>=S^a_b\,,\quad \text{where }\ P_a=\begin{bmatrix}P^0_a\\ \vdots\\ P^{n+1}_a\end{bmatrix}\in I\!R^{n+2}\,,\\ &\text{iff}\quad <P(u),P(v)>=<u,v>\,,\ \ \forall u,v\in I\!R^{n+2}\,.\end{aligned}$$

Thus, we can identify (though not canonically) $O(n+1,1)$ with the group

$$\{P\in M_{(n+2)^2}\ :\ {}^tPSP=S\}\,.$$

Observe that, if η'_a is another basis of $I\!R^{n+2}$, satisfying the same conditions as η_a, the linear map $P:I\!R^{n+2}\to I\!R^{n+2}$, such that $P(\eta_a)=\eta'_a$, is an element of $O(n+1,1)$. Here we remark that some authors prefer to represent the inner product $<\,,\,>$ in the canonic basis ϵ_a, resulting in the matrix

$$\begin{bmatrix}-1&0\\0&I_{n+1}\end{bmatrix}\,.$$

Finally, we note that all elements of $O(n+1,1)$ have determinant equal to ± 1 and that, $\forall P \in O(n+1,1)$, $P(\mathcal{L}) \subset \mathcal{L}$.

It is well-known that $O(n+1,1)$ has four connected components and that its identity component can be identified with (cf. Ref. [Ko-No/69], page 268)

$$G = \{P \in O(n+1,1) \;:\; \det P = 1,\; P(\mathcal{L}^+) \subset \mathcal{L}^+\}\,. \tag{1.9}$$

G acts on the left on $\mathcal{L}^+$ by matrix multiplication as

$$\begin{array}{rccc} \Lambda: & G \times \mathcal{L}^+ & \longrightarrow & \mathcal{L}^+ \\ & (P,x) & \longrightarrow & P(x)\,. \end{array}$$

If $x = \begin{vmatrix} c \\ v \\ s \end{vmatrix}$ is an element of $\mathcal{L}^+$ written in the basis η_a, we get from the equation $Q(x) = 0$

$$\begin{aligned} x &= c \begin{vmatrix} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{vmatrix}, \text{ if } c \neq 0, \\ &\text{or} \\ x &= s \begin{vmatrix} \frac{1}{2}\|\omega\|^2 \\ \omega \\ 1 \end{vmatrix}, \text{ if } s \neq 0, \text{ and, in particular,} \\ x &= s \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}, \text{ if } c = 0 \ (s \neq 0), \end{aligned} \tag{1.10}$$

where $c, s \geq 0$, $\omega = \begin{vmatrix} \omega^1 \\ \vdots \\ \omega^n \end{vmatrix} \in \mathbb{R}^n$, and $\|\omega\|^2 = {}^t\omega\omega$. Thus, it is straightforward to prove that G acts on $\mathcal{L}^+$ transitively on the left.

Let S^n denote the unit sphere of the Euclidean space $\mathbb{R}^{n+1}$. We can identify S^n with the projectivisation of the positive light cone $\mathcal{L}^+$ as follows: the map

$$\begin{array}{rccc} F: & \mathcal{L}^+ & \longrightarrow & S^n \subset \mathbb{R}^{n+1} \\ & \begin{vmatrix} c \\ v \\ s \end{vmatrix} & \longrightarrow & \left(\frac{2s-c}{2s+c}, \frac{2v}{2s+c}\right), \end{array}$$

is a smooth submersion onto S^n. Let $\sim$ denote the relation of equivalence on $\mathbb{R}^{n+1}\backslash\{0\}$ given by $x \sim y$, iff $\exists a \neq 0 : x = ay$. Then, $\mathbb{P}^{n+1} = \mathbb{R}^{n+1}\backslash\{0\}_{/\sim}$ is the

projective space. For $x, y \in \mathcal{L}^+$, we have $x \sim y$, iff $F(x) = F(y)$. Thus, denoting by $\begin{bmatrix} c \\ v \\ s \end{bmatrix}_\sim$ the equivalence class of the element $\begin{bmatrix} c \\ v \\ s \end{bmatrix} \in \mathbb{R}^{n+2}$ in $\mathbb{P}^{n+1}$, the map

$$\begin{array}{rcll}
\kappa = F_{/\sim} : & \mathcal{L}^+_{/\sim} \subset \mathbb{P}^{n+1} & \longrightarrow & S^n \\
& \begin{bmatrix} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{bmatrix}_\sim & \rightarrow & \left(\frac{\|\omega\|^2-1}{\|\omega\|^2+1}, \frac{2\omega}{\|\omega\|^2+1}\right) \\
& \begin{bmatrix} \frac{1}{2}\|\omega\|^2 \\ \omega \\ 1 \end{bmatrix}_\sim & \rightarrow & \left(\frac{1-\frac{1}{4}\|\omega\|^2}{1+\frac{1}{4}\|\omega\|^2}, \frac{\omega}{1+\frac{1}{4}\|\omega\|^2}\right) \\
& [\eta_{n+1}]_\sim = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_\sim & \rightarrow & (1, 0, \ldots, 0) \\
& [\eta_0]_\sim = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_\sim & \rightarrow & (-1, 0, \ldots, 0)
\end{array} \tag{1.11}$$

defines a diffeomorphism with inverse given by

$$\begin{array}{rccc}
\kappa^{-1} : & S^n & \longrightarrow & \mathcal{L}^+_{/\sim} \\
& (z^0, z) & \rightarrow & \begin{bmatrix} 2 - 2z^0 \\ 2z \\ 1 + z^0 \end{bmatrix}_\sim .
\end{array}$$

Definition 1.1 *Embedded into $\mathbb{P}^{n+1}$ in the above way, S^n is called the* Möbius space. *We call $\mathbf{x}_0 = [\eta_0]_\sim$ and $\mathbf{x}_\infty = [\eta_{n+1}]_\sim$ the* origin *resp.* Möbius point *of S^n.*

(See Appendix I for some observations concerning this definition.)

Observe that, composing the map κ defined in Eq. (1.11) with the stereographic projection σ of Eq. (1.5), we obtain the following "diffeomorphism":

$$\begin{array}{rccc}
\sigma \circ \kappa : & \mathcal{L}^+_{/\sim} & \longrightarrow & \mathbb{R}^n \cup \{\infty\} \\
& \begin{bmatrix} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{bmatrix}_\sim & \rightarrow & \omega \\
& \mathbf{x}_0 & \rightarrow & 0 \\
& \mathbf{x}_\infty & \rightarrow & \infty .
\end{array}$$

Since $\Lambda : G \times \mathcal{L}^+ \rightarrow \mathcal{L}^+$ is a transitive action, it induces a transitive (well-defined) action on $\mathcal{L}^+_{/\sim} = S^n$, viz.

$$\begin{array}{rccc}
\Lambda_{/\sim} : & G \times S^n & \longrightarrow & S^n \\
& (P, [x]_\sim) & \longrightarrow & [P(x)]_\sim .
\end{array}$$

Moreover, it is well-known that this action is effective and that G, the identity component of $O(n+1,1)$, is *the group of orientation-preserving conformal transformations* of the n-sphere, considered with a Riemannian structure of constant positive sectional curvature (cf. Refs. [Sch-Su/80] [Ko-No/69]). The group G is called the (positive) *Möbius group*.

Let $G_0 = \{P \in G : \ [P(\eta_0)]_\sim = [\eta_0]_\sim\} = \{P \in G : \ \exists r > 0 : \ P(\eta_0) = r^{-1}\eta_0\}$ be the isotropic subgroup of G at the point $\mathbf{x}_0$. Then, G_0 is represented by

$$G_0 = \left\{ \begin{bmatrix} r^{-1} & {}^tXB & \frac{1}{2}r\,{}^tXX \\ 0 & B & rX \\ 0 & 0 & r \end{bmatrix} \ : \ \begin{matrix} B \in SO(n) \\ X \in \mathbb{R}^n \\ r \in \mathbb{R}^+ \end{matrix} \right\}, \tag{1.12}$$

where X is a column vector. We have that S^n is diffeomorphic to the homogeneous space $G/G_0 = \{PG_0 \ : \ P \in G\}$ of the left-cosets module G_0.

Remark 1.1 Following Ref. [Sch-Su/80], the Möbius group is in fact the group $\tilde{G} = O(n+1,1)/\{\mathrm{id}, -\mathrm{id}\}$ that can be identified with the isotropic group of $\mathcal{L}^+$, $\{P \in O(n+1,1) : \ \ P(\mathcal{L}^+) \subset \mathcal{L}^+\}$, which has two connected components: the identity component G of $O(n+1,1)$ and $\tilde{G}\backslash G$. The group $\tilde{G}$ still acts effectively on S^n (and, of course, transitively). Furthermore, it is, as is $O(n+1,1)$, the group of all conformal orientation-preserving and -non-preserving transformations of the sphere S^n equipped with a Riemannian structure of constant positive sectional curvature. Thus, S^n can also be represented as the homogeneous space $\tilde{G}/\tilde{G}_0$, where

$$\tilde{G}_0 = \left\{ \begin{bmatrix} r^{-1} & {}^tXB & \frac{1}{2}r\,{}^tXX \\ 0 & B & rX \\ 0 & 0 & r \end{bmatrix} \ : \ \begin{matrix} B \in O(n) \\ X \in \mathbb{R}^n \\ r \in \mathbb{R}^+ \end{matrix} \right\}$$

is the isotropic group of the action $\tilde{G}$ on S^n at the point $\mathbf{x}_0$. As for the moment we are only interested in oriented immersed submanifolds of S^n, we only consider the positive Möbius group G.

The Lie algebra $\mathfrak{g}$ of the group G is identified with the tangent space of G at the identity element, that is,

$$\begin{aligned} \mathfrak{g} = \ & T_{\mathrm{id}}G = \{P \in M_{(n+2)^2} \ : \ {}^tPS + SP = 0\} \\ = \ & \left\{ \begin{bmatrix} a & {}^t\xi & 0 \\ v & D & \xi \\ 0 & {}^tv & -a \end{bmatrix} \ : \ \begin{matrix} a \in \mathbb{R} \\ v, \xi \in \mathbb{R}^n \\ D \in \mathfrak{o}(n) \end{matrix} \right\}, \end{aligned} \tag{1.13}$$

and the Lie algebra of G_0 is given by

$$\mathfrak{h} = \left\{ \begin{vmatrix} a & {}^t\xi & 0 \\ 0 & D & \xi \\ 0 & 0 & -a \end{vmatrix} \; : \; a \in \mathbb{R},\ \xi \in \mathbb{R}^n,\ D \in \mathfrak{o}(n) \right\} .$$

The canonic projection of G onto the quotient space G/G_0 is given by

$$\begin{array}{rcl} \Pi : \quad G & \longrightarrow & G/G_0 \simeq S^n \simeq \mathcal{LC}^+_\sim \\ \Pi(P) & = & [P(\eta_0)]_\sim \in \mathbb{P}^{n+1} . \end{array} \tag{1.14}$$

$G_0 \to G \xrightarrow{\Pi} S^n$ is a principal fibre bundle with structure group G_0.

Now we relate the action of the elements of G on $\mathcal{LC}^+_\sim$ to the conformal transformations of S^n, generated by its conformal vector fields. The identity component $\mathbb{R}^n \times CO(n)^+$ of the affine conformal group of $\mathbb{R}^n$ acts on $\mathcal{LC}^+_\sim$ in the same way as on S^n (see Sec. 1.1.A), i.e. the following diagram is commutative:

$$\begin{array}{ccccc}
\mathbb{R}^n \times CO(n)^+ & \times & \mathcal{LC}^+_\sim & \xrightarrow{\Delta/\sim} & \mathcal{LC}^+_\sim \\
\left(\begin{vmatrix} r^{-1} & 0 & 0 \\ r^{-1}Z & A & 0 \\ \frac{1}{2}r^{-1}\,{}^tZZ & {}^tZA & r \end{vmatrix} \right. & , & \left. \begin{vmatrix} c \\ v \\ s \end{vmatrix}_\sim \right) & \longrightarrow & \left| \begin{vmatrix} r^{-1} & 0 & 0 \\ r^{-1}Z & A & 0 \\ \frac{1}{2}r^{-1}\,{}^tZZ & {}^tZA & r \end{vmatrix} \begin{vmatrix} c \\ v \\ s \end{vmatrix} \right|_\sim \\
\downarrow \mathrm{id} & & \downarrow \kappa & & \downarrow \kappa \\
\Big(P = (Z, A, r) & , & x = \left(\frac{2s-c}{2s+c}, \frac{2v}{2s+c} \right) \Big) & \longrightarrow & \begin{cases} \sigma^{-1}(P(\sigma(x))) & \text{if } x \neq N \\ N & \text{if } x = N \end{cases} \\
\mathbb{R}^n \times CO(n)^+ & \times & S^n & \longrightarrow & S^n ,
\end{array} \tag{1.15}$$

where σ and κ are the diffeomorphisms given in Eqs. (1.5) resp. (1.11).

The Lie algebra of G can be decomposed as $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with

$$\mathfrak{g}_{-1} = \left\{ \begin{vmatrix} 0 & 0 & 0 \\ v & 0 & 0 \\ 0 & {}^tv & 0 \end{vmatrix} \; : \; v \in \mathbb{R}^n \right\} \simeq \mathbb{R}^n ,$$

$$\mathfrak{g}_0 = \left\{ \begin{vmatrix} -a & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & a \end{vmatrix} \; : \; D \in \mathfrak{o}(n),\ a \in \mathbb{R} \right\} \simeq \mathfrak{co}(n) = \left\{ D - aI_n : \; \begin{array}{l} D \in \mathfrak{o}(n) \\ a \in \mathbb{R} \end{array} \right\} ,$$

$$\mathfrak{g}_1 = \left\{ \begin{vmatrix} 0 & {}^t\xi & 0 \\ 0 & 0 & \xi \\ 0 & 0 & 0 \end{vmatrix} \; : \; \xi \in \mathbb{R}^n \right\} \simeq (\mathbb{R}^n)^* .$$

Note that $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the Lie algebra of G_0 and $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ is the one of $\mathbb{R}^n \times CO(n)^+$. Let $p = \kappa \circ \Pi : G \to S^n \subset \mathbb{R}^{n+1}$, where $\Pi : G \to \mathcal{LC}^+_\sim \subset \mathbb{P}^{n+1}$ is the projection

given in Eq. (1.14). As G acts on the left on $\mathcal{L}^{\dagger}_{/\sim}$, each $\mathcal{X} \in$ defines a vector field $\mathcal{X}^{*}$ on $\mathcal{L}^{\dagger}_{/\sim}$ given by

$$\mathcal{X}^{*}_{\Pi(P)} = \frac{\partial}{\partial t}\Big|_{t=0}\Big(\Pi\big(\exp(t\mathcal{X})P\big)\Big)\,,\ \forall P \in G\,,$$

where $\exp : \mathfrak{g} \to G$, $\mathcal{X} \to I + \sum_{n\geq 1}\frac{\mathcal{X}^n}{n!}$, is the exponential map of the Lie group G. The vector field $\mathcal{X}^{*}$ corresponds to a vector field $\tilde{\mathcal{X}}^{*}$ on S^n defined by

$$\tilde{\mathcal{X}}^{*}_{\nu(P)} = \frac{\partial}{\partial t}\Big|_{t=0}\Big(\nu\big(\exp(t\mathcal{X})P\big)\Big) = d\kappa_{\Pi(P)}\big(\mathcal{X}^{*}_{\Pi(P)}\big)\,,$$

which is κ-related to $\mathcal{X}^{*}$. Note that the 1-parameter group of diffeomorphisms $\phi_t : \mathcal{L}^{\dagger}_{/\sim} \to \mathcal{L}^{\dagger}_{/\sim}$ generated by $\mathcal{X}^{*}$ is given by $\phi_t\left(\begin{bmatrix} c \\ v \\ s \end{bmatrix}_{\sim}\right) = \left[\exp(t\mathcal{X})\begin{bmatrix} c \\ v \\ s \end{bmatrix}\right]_{\sim}$, and the one generated by $\tilde{\mathcal{X}}^{*}$, $\tilde{\phi}_t : S^n \to S^n$, reads $\tilde{\phi}_t = \kappa \circ \phi_t \circ \kappa^{-1}$. For several typical $\mathcal{X} \in \mathfrak{g}$, we will give the explicit expressions for the conformal transformation $\exp(\mathcal{X}) : \mathcal{L}^{\dagger}_{/\sim} \simeq S^n \to \mathcal{L}^{\dagger}_{/\sim} \simeq S^n$ of S^n.

1) If $\mathcal{X} = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{bmatrix} \in \mathfrak{g}_0$, then $\exp(\mathcal{X}) = \begin{bmatrix} e^{-a} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & e^{a} \end{bmatrix}$, which gives the transformation

$$\begin{array}{cccccc}
\exp(\mathcal{X}) : & \mathcal{L}^{\dagger}_{/\sim} & \longrightarrow & \mathcal{L}^{\dagger}_{/\sim} & & \\
& \begin{bmatrix} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{bmatrix}_{\sim} & \to & \begin{bmatrix} e^{-a} \\ \omega \\ \frac{1}{2}e^{a}\|\omega\|^2 \end{bmatrix}_{\sim} & = & \begin{bmatrix} 1 \\ e^{a}\omega \\ \frac{1}{2}e^{2a}\|\omega\|^2 \end{bmatrix}_{\sim} \\
& \begin{bmatrix} \frac{1}{2}\|\omega\|^2 \\ \omega \\ 1 \end{bmatrix}_{\sim} & \to & \begin{bmatrix} \frac{1}{2}e^{-a}\|\omega\|^2 \\ \omega \\ e^{a} \end{bmatrix}_{\sim} & = & \begin{bmatrix} \frac{1}{2}e^{-2a}\|\omega\|^2 \\ e^{-a}\omega \\ 1 \end{bmatrix}_{\sim} \\
& \mathbf{x}_{\infty} & \to & \begin{bmatrix} 0 \\ 0 \\ e^{ta} \end{bmatrix} & = & \mathbf{x}_{\infty} \\
& \mathbf{x}_{0} & \to & \begin{bmatrix} e^{-ta} \\ 0 \\ 0 \end{bmatrix}_{\sim} & = & \mathbf{x}_{0}\,.
\end{array}$$

Using the diffeomorphisms κ of Eq. (1.11) and the stereographic projection σ of

Eq. (1.5), we have the transformations

$$\begin{array}{rlcl}
\exp(\mathcal{X}): & S^n \subset I\!R^{n+1} & \longrightarrow & S^n \subset I\!R^{n+1} \\
& \left(\frac{\|\omega\|^2-1}{\|\omega\|^2+1}, \frac{2\omega}{\|\omega\|^2+1}\right) & \rightarrow & \left(\frac{e^{2a}\|\omega\|^2-1}{e^{2a}\|\omega\|^2+1}, \frac{2e^a\omega}{e^{2a}\|\omega\|^2+1}\right) \\
& (x^0, x^1, \ldots, x^n) & \rightarrow & \left(\frac{x^0+\tanh a}{x^0\tanh a+1}, \frac{\operatorname{sech} a(x^1,\ldots,x^n)}{x^0\tanh a+1}\right) \\
& N = (1,0,\ldots,0) & \rightarrow & N \\
& S = (-1,0,\ldots,0) & \rightarrow & S\,,
\end{array}$$

and

$$\begin{array}{rccc}
\exp(\mathcal{X}): & I\!R^n \cup \{\infty\} & \longrightarrow & I\!R^n \cup \{\infty\} \\
& \omega & \rightarrow & e^a\omega \\
& \infty & \rightarrow & \infty \\
& 0 & \rightarrow & 0\,.
\end{array}$$

This type of conformal transformation of S^n is called *homothetic* with centres $\mathbf{x}_0$ and $\mathbf{x}_\infty$ (i.e. S resp. N or 0 resp. ∞). The integral curves of $\hat{\mathcal{X}}^*$ through S are constant, because $\mathcal{X}$ belongs to the isotropic algebra $\mathfrak{h}$ of G_0. The integral curve passing through the point $(0,\omega) \in S^n$, with ω a unit vector of $I\!R^n$, is given by $\gamma(s) = (\tanh s, \operatorname{sech} s\,\omega)$, and is a reparametrisation of the great circle in S^n through the points S, $(0,\omega)$, and N (cf. Ref. [Po/81]). Besides, since $\exp(t\mathcal{X}) : I\!R^n \to I\!R^n$ is given by $\exp(t\mathcal{X})(\omega) = e^{ta}\omega$, $\hat{\mathcal{X}}^*$ is σ^{-1}-related to the conformal vector field of $I\!R^n$ $\hat{\mathcal{X}}^*(\omega) = \frac{\partial}{\partial t}\big|_{t=0}\exp(t\mathcal{X})\omega = a\omega$, which is a dilatation. Thus, from Eq. (1.7), we have $\hat{\mathcal{X}}^*_{(x^0,x^1,\ldots,x^n)} = \left(a(1-x^{0^2}), -ax^0(x^1,\ldots,x^n)\right)$, and, from Eq. (1.6), $L_{\hat{\mathcal{X}}^*}(<,>_{S^n})_{(x^0,x^1,\ldots,x^n)} = -2ax^0 <,>_{S^n}$.

2) If $\mathcal{X} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{g}_0$ with $D \in \mathrm{O}(n)$, then $\exp(\mathcal{X}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^D & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $e^D \in SO(n)$. Hence,

$$\begin{array}{rccc}
\exp(\mathcal{X}): & \mathcal{L}_{/\sim}^{+} & \longrightarrow & \mathcal{L}_{/\sim}^{+} \\
& \begin{bmatrix} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{bmatrix}_\sim & \rightarrow & \begin{bmatrix} 1 \\ e^D\omega \\ \frac{1}{2}\|\omega\|^2 \end{bmatrix}_\sim \\
& \mathbf{x}_\infty & \rightarrow & \mathbf{x}_\infty \\
& \mathbf{x}_0 & \rightarrow & \mathbf{x}_0
\end{array}$$

and, using the diffeomorphism κ,

$$\begin{array}{rrcl}\exp(\mathfrak{X}): & S^n & \longrightarrow & S^n \\ & \left(\frac{\|\omega\|^2-1}{\|\omega\|^2+1}, \frac{2\omega}{\|\omega\|^2+1}\right) & \to & \left(\frac{\|\omega\|^2-1}{\|\omega\|^2+1}, \frac{2e^D(\omega)}{\|\omega\|^2+1}\right) \\ & (x^0, x^1, \ldots, x^n) & \to & \left(x^0, e^D(x^1, \ldots, x^n)\right) \\ & N & \to & N \\ & S & \to & S\,,\end{array}$$

which gives a rotation of S^n around the axis N–S. For $(x^0, x^1, \ldots, x^n) \in S^n$, we have $\tilde{\mathfrak{X}}^*_{(x^0,x^1,\ldots,x^n)} = \frac{\partial}{\partial t}\big|_{t=0}\left(x^0, e^{tD}(x^1, \ldots, x^n)\right) = \left(0, D(x^1, \ldots, x^n)\right)$. Furthermore, as $\exp(t\mathfrak{X})$ are obviously isometries of S^n, $\tilde{\mathfrak{X}}^*$ is a Killing vector field of S^n, that is, $L_{\tilde{\mathfrak{X}}^*} <,>_{S^n} = 0$. Now, using the stereographic projection σ, we obtain

$$\begin{array}{rccc}\exp(\mathfrak{X}): & \mathbb{R}^n \cup \{\infty\} & \longrightarrow & \mathbb{R}^n \cup \{\infty\} \\ & \omega & \to & e^D(\omega) \\ & \infty & \to & \infty \\ & 0 & \to & 0\,,\end{array}$$

which gives an isometry (rotation) of $\mathbb{R}^n$, too. Therefore, $\hat{\mathfrak{X}}^*(\omega) = \frac{\partial}{\partial t}\big|_{t=0} \exp(t\mathfrak{X})\omega = D(\omega)$ is a Killing vector field of $\mathbb{R}^n$, σ-related to $\tilde{\mathfrak{X}}^*$.

3) If $\mathfrak{X} = \begin{bmatrix} 0 & 0 & 0 \\ v & 0 & 0 \\ 0 & {}^t v & 0 \end{bmatrix} \in \mathfrak{g}_{-1}$, then $\exp(\mathfrak{X}) = \begin{bmatrix} 1 & 0 & 0 \\ v & I_n & 0 \\ \frac{1}{2}\|v\|^2 & {}^t v & 1 \end{bmatrix}$ gives the transformation

$$\begin{array}{rccc}\exp(\mathfrak{X}): & \mathcal{L}\mathfrak{f}_\sim & \longrightarrow & \mathcal{L}\mathfrak{f}_\sim \\ & \begin{bmatrix} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{bmatrix}_\sim & \to & \begin{bmatrix} 1 \\ v+\omega \\ \frac{1}{2}\|v+\omega\|^2 \end{bmatrix}_\sim \\ & \begin{bmatrix} \frac{1}{2}\|\omega\|^2 \\ \omega \\ 1 \end{bmatrix}_\sim & \to & \begin{bmatrix} \frac{1}{2}\|\omega\|^2 \\ \frac{1}{2}\|\omega\|^2 v + \omega \\ \frac{1}{2}\|\omega\|^2 \frac{1}{2}\|v\|^2 + {}^t v\omega + 1 \end{bmatrix}_\sim = \begin{bmatrix} \frac{1}{2}\|\omega'\|^2 \\ \omega' \\ 1 \end{bmatrix}_\sim \\ & \mathbf{x}_\infty & \to & \mathbf{x}_\infty \\ & \mathbf{x}_0 & \to & \begin{bmatrix} 1 \\ v \\ \frac{1}{2}\|v\|^2 \end{bmatrix}_\sim\,,\end{array}$$

where $\omega' = \|\omega\|^2 \frac{(\frac{1}{2}\|\omega\|^2 v + \omega)}{\|\frac{1}{2}\|\omega\|^2 v + \omega\|^2}$ (put $\omega' = 0$, if $\omega = 0$). Using the diffeomorphism κ,

we have

$$\begin{array}{rccc}\exp(\mathcal{X}): & S^n & \longrightarrow & S^n \\ & \left(\frac{\|\omega\|^2-1}{\|\omega\|^2+1}, \frac{2\omega}{\|\omega\|^2+1}\right) & \to & \left(\frac{\|v+\omega\|^2-1}{\|v+\omega\|^2+1}, \frac{2(v+\omega)}{\|v+\omega\|^2+1}\right) \\ & N & \to & N \\ & S & \to & \left(\frac{\|v\|^2-1}{\|v\|^2+1}, \frac{2v}{\|v\|^2+1}\right) . \end{array}$$

Cartan called this kind of conformal transformation an *elation* with centre at $\mathbf{x}_\infty$, i.e. at the north pole N. The integral curves of $\bar{\mathcal{X}}^*$ are a family of circles passing through the point N and with tangent vector $(N,(0,v))$ (cf. Refs. [C/55], page 176; [Po/81]). With the stereographic projection σ we have

$$\begin{array}{rccc}\exp(\mathcal{X}): & \mathbb{R}^n \cup \{\infty\} & \longrightarrow & \mathbb{R}^n \cup \{\infty\} \\ & \omega & \to & v+\omega \\ & \infty & \to & \infty \\ & 0 & \to & v\,, \end{array}$$

which gives a translation on $\mathbb{R}^n$. Since $\exp(t\mathcal{X}): \mathbb{R}^n \to \mathbb{R}^n$ is given by $\omega \to tv+\omega$, $\hat{\mathcal{X}}^*(\omega) = \frac{\partial}{\partial t}\big|_{t=0}\exp(t\mathcal{X})\omega = v$ is a constant and, in particular, a Killing vector field on $\mathbb{R}^n$ that is σ-related to the vector field $\hat{\mathcal{X}}^*_{(x^0,x')} = \left((1-x^0)<x',v>_n, -<x',v>_n x' + (1-x^0)v\right)$ of S^n. From Eq. (1.6), we have $L_{\hat{\mathcal{X}}^*}(<,>_{S^n})_{(x^0,x')} = -2<x',v>_n<,>_{S^n}$.

4) Finally, if $\mathcal{X} = \begin{bmatrix} 0 & {}^t\xi & 0 \\ 0 & 0 & \xi \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{g}_1$, then $\exp(\mathcal{X}) = \begin{bmatrix} 1 & {}^t\xi & \frac{1}{2}\|\xi\|^2 \\ 0 & I_n & \xi \\ 0 & 0 & 1 \end{bmatrix}$, giving the transformation

$$\begin{array}{rlcl}\exp(\mathcal{X}): & \mathcal{L}\mathfrak{J}_{\sim} & \longrightarrow & \mathcal{L}\mathfrak{J}_{\sim} \\ & \begin{bmatrix} \frac{1}{2}\|\omega\|^2 \\ \omega \\ 1 \end{bmatrix}_{\sim} & \to & \begin{bmatrix} \frac{1}{2}\|\omega+\xi\|^2 \\ \omega+\xi \\ 1 \end{bmatrix}_{\sim} \\ & \begin{bmatrix} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{bmatrix}_{\sim} & \to & \begin{bmatrix} \frac{1}{2}\|\omega\|^2\frac{1}{2}\|\xi\|^2 + {}^t\xi\omega + 1 \\ \frac{1}{2}\|\omega\|^2\xi + \omega \\ \frac{1}{2}\|\omega\|^2 \end{bmatrix}_{\sim} = \begin{cases} \begin{bmatrix} 1 \\ \omega' \\ \frac{1}{2}\|\omega'\|^2 \end{bmatrix}_{\sim} & \text{if } \omega \neq \frac{-2\xi}{\|\xi\|^2} \\ \begin{bmatrix} 0 \\ 0 \\ \frac{2}{\|\xi\|^2} \end{bmatrix}_{\sim} = \mathbf{x}_\infty & \text{if } \omega = \frac{-2\xi}{\|\xi\|^2} \end{cases} \\ & \mathbf{x}_\infty & \to & \begin{bmatrix} \frac{1}{2}\|\xi\|^2 \\ \xi \\ 1 \end{bmatrix}_{\sim} = \begin{bmatrix} 1 \\ \frac{2\xi}{\|\xi\|^2} \\ \frac{2}{\|\xi\|^4} \end{bmatrix}_{\sim} \\ & \mathbf{x}_0 & \to & \mathbf{x}_0\,, \end{array}$$

where $\omega' = \frac{\frac{\omega}{\|\omega\|^2}+\frac{\xi}{2}}{\|\frac{\omega}{\|\omega\|^2}+\frac{\xi}{2}\|^2}$. Using the diffeomorphism κ we get

$$\begin{array}{cccc}
\exp(\mathcal{X}): & S^n & \longrightarrow & S^n \\
& \left(\frac{1-\frac{1}{4}\|\omega\|^2}{1+\frac{1}{4}\|\omega\|^2}, \frac{\omega}{1+\frac{1}{4}\|\omega\|^2}\right) & \to & \left(\frac{1-\frac{1}{4}\|\omega+\xi\|^2}{1+\frac{1}{4}\|\omega+\xi\|^2}, \frac{\omega+\xi}{1+\frac{1}{4}\|\omega+\xi\|^2}\right) \\
& N & \to & \left(\frac{1-\frac{1}{4}\|\xi\|^2}{1+\frac{1}{4}\|\xi\|^2}, \frac{\xi}{1+\frac{1}{4}\|\xi\|^2}\right) \\
& S & \to & S\,.
\end{array}$$

Cartan called also this conformal transformation an elation, with centre $\mathbf{x}_0$ (i.e. at S). With stereographic projection we get

$$\begin{array}{rcl}
\exp\mathcal{X}: \ \mathbb{R}^n\cup\{\infty\} & \longrightarrow & \mathbb{R}^n\cup\{\infty\} \\
\omega \neq \frac{-2\xi}{\|\xi\|^2} & \to & \frac{\frac{\omega}{\|\omega\|^2}+\frac{\xi}{2}}{\|\frac{\omega}{\|\omega\|^2}+\frac{\xi}{2}\|^2} \\
\infty & \to & \frac{2\xi}{\|\xi\|^2} \\
0 & \to & 0 \\
\frac{-2\xi}{\|\xi\|^2} & \to & \infty\,.
\end{array}$$

This is called an inversion on $\mathbb{R}^n$ that keeps the origin fixed. The vector field $\hat{\mathcal{X}}^*(\omega) = \frac{\partial}{\partial t}\big|_{t=0}\exp(t\mathcal{X})\omega = \frac{\partial}{\partial t}\big|_{t=0}\frac{\frac{\omega}{\|\omega\|^2}+\frac{t\xi}{2}}{\|\frac{\omega}{\|\omega\|^2}+\frac{t\xi}{2}\|^2} = \frac{1}{2}\|\omega\|^2\xi - <\omega,\xi>_n \omega$ is a conformal vector field on $\mathbb{R}^n$ and is σ-related to $\hat{\mathcal{X}}^*_{(x^0,x')} = \big(-\frac{1}{2}(1+x^0) <x',\xi>_n, -\frac{1}{2} <x',\xi>_n x' + \frac{1}{2}(1+x^0)\xi\big)$. Moreover, $\hat{\mathcal{X}}^*$ satisfies $L_{\hat{\mathcal{X}}^*}(<,>_{S^n})_{(x^0,x')} = - <\xi,x'>_n <,>_{S^n}$. From the expressions for $L_{\hat{\mathcal{X}}^*}(<,>_{S^n})$ in examples 3) and 4), we conclude that the vector subspace of $\mathfrak{G}$ of dimension $\frac{n(n+1)}{2}$

$$\left\{ \left| \begin{array}{ccc} 0 & -{}^t v & 0 \\ \frac{v}{2} & D & -v \\ 0 & \frac{{}^t v}{2} & 0 \end{array} \right| \ : \ D \in \mathcal{O}(n), \ v \in \mathbb{R}^n \right\}$$

generates all the Killing vector fields of S^n.

1.1.C The Structure Equations of the Möbius Group

First, we recall that assigning a conformal structure to a manifold M means giving a class of conformally equivalent Riemannian metrics. The conformal structure of S^n will be defined by considering it as the homogeneous space G/G_0, using sections of the bundle $\Pi : G \to S^n$, the Maurer-Cartan form of G, and its structure equations. Henceforth, S^n stands for the projectivisation of the light cone $\mathcal{L}^+$, except when we want to refer to the unit sphere of $\mathbb{R}^{n+1}$, which will become clear

from the context.

A basis of the Lie algebra $\mathfrak{g}$ of G is given by the $\frac{(n+1)(n+2)}{2}$ linearly independent matrices

$$P_{(0)} = \begin{bmatrix} 1 & 0\cdots 0 & 0 \\ 0 & & 0 \\ \vdots & 0 & \vdots \\ 0 & & 0 \\ 0 & 0\cdots 0 & -1 \end{bmatrix} \qquad P_{(A,B)} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c c} \begin{array}{ccccccccc} & \overset{B}{\updownarrow} & & & & \overset{A}{\updownarrow} & & & \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} & \begin{array}{l} \\ \leftarrow B \\ \\ \leftarrow A \\ \\ \\ \\ \end{array} \end{array}$$

$$P_{(0,A)} = \begin{array}{c c} \begin{array}{cccccc} & \overset{A}{\updownarrow} & & & & \end{array} & \\ \begin{bmatrix} 0 & 1 & 0\cdots 0 & 0 & 0 \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & 0\cdots 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \\ \leftarrow A \\ \\ \\ \\ \\ \end{array} \end{array} \qquad P_{(A,0)} = \begin{array}{c c} \begin{array}{ccccc} & & & \overset{A}{\updownarrow} & \end{array} & \\ \begin{bmatrix} 0 & 0 & 0\cdots 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ 0 & 0 & 0\cdots 0 & 1 & 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \leftarrow A \\ \\ \end{array} \end{array}$$

with $A > B$. We denote by Φ the *Maurer-Cartan form* of G, i.e. the $\mathfrak{g}$-valued left-invariant 1-form of G given by

$$\Phi_Q(\hat{P}_Q) = P\,, \quad \forall Q \in G,\ P \in \mathfrak{g}\,,$$

where $\hat{P}$ is the left-invariant vector field of G, such that $\hat{P}_{\mathrm{id}} = P$, that is, $\hat{P}_Q = Q \circ P \in T_Q G = Q\mathfrak{g}$. Then,

$$\Phi = \widetilde{P_{(0)}}_{\bullet} P_{(0)} + \sum_{A>B} \widetilde{P_{(A,B)}}_{\bullet} P_{(A,B)} + \sum_{A} \left(\widetilde{P_{(0,A)}}_{\bullet} P_{(0,A)} + \widetilde{P_{(A,0)}}_{\bullet} P_{(A,0)} \right),$$

where $\left(\widetilde{P_{(0)}}_{\bullet}, \widetilde{P_{(A,B)}}_{\bullet}, \widetilde{P_{(0,A)}}_{\bullet}, \widetilde{P_{(A,0)}}_{\bullet}\right)$ are 1-forms dual to the frame of left-invariant vector fields $\left(\widetilde{P_{(0)}}, \widetilde{P_{(A,B)}}, \widetilde{P_{(0,A)}}, \widetilde{P_{(A,0)}}\right)$. Since Φ assumes values on $\mathfrak{g}$, we denote by Φ^a_b, $0 \leq a, b \leq n+1$, the components of Φ. Thus, $\Phi = [\Phi^a_b] \in \mathfrak{g}$ is a matrix of left-invariant 1-forms. From Eq. (1.13), we have $\Phi^c_a S^c_b + S^a_c \Phi^c_b = 0$, which gives the following explicit relations among the components of Φ:

$$\Phi^0_0 = -\Phi^{n+1}_{n+1}\,, \quad \Phi^A_0 = \Phi^{n+1}_A\,, \quad \Phi^0_A = \Phi^A_{n+1}\,, \quad \Phi^A_B = -\Phi^B_A\,, \quad \Phi^{n+1}_0 = \Phi^0_{n+1} = 0\,, \tag{1.16}$$

$\forall A, B \in \{1, \ldots, n\}$.

Moreover, we have $\Phi^0_0 = \widetilde{P_{(0)_e}}$, $\Phi^A_B = \widetilde{P_{(A,B)_e}}$ for $A > B$, $\Phi^A_0 = \widetilde{P_{(A,0)_e}}$, and $\Phi^0_A = \widetilde{P_{(0,A)_e}}$, whence Φ^0_0, Φ^A_0, Φ^0_A, Φ^A_B $(A > B)$ form at each point $P \in G$ a basis of $T^*_P G$.

If we denote by $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ the Lie bracket of $\mathfrak{g}$ given by $[P, \overline{P}] = P \circ \overline{P} - \overline{P} \circ P$, then Φ satisfies the Maurer-Cartan *structure equations* of the group G, reading

$$d\Phi = -\frac{1}{2}[\Phi \wedge \Phi] = -\Phi \wedge \Phi \, .$$

Explicitly,

$$d\Phi^a_b = -\Phi^a_c \wedge \Phi^c_b \, , \ \ \forall 0 \le a, b \le n+1 \, . \tag{1.17}$$

Using the relations in Eq. (1.16), we can reduce these equations to the following ones:

$$\begin{aligned}
d\Phi^0_0 &= -\Phi^0_A \wedge \Phi^A_0 \\
d\Phi^A_0 &= -\Phi^A_0 \wedge \Phi^0_0 - \Phi^A_B \wedge \Phi^B_0 \\
d\Phi^0_A &= -\Phi^0_0 \wedge \Phi^0_A - \Phi^0_B \wedge \Phi^B_A \\
d\Phi^A_B &= -\Phi^A_0 \wedge \Phi^0_B - \Phi^A_C \wedge \Phi^C_B - \Phi^0_A \wedge \Phi^B_0 \, .
\end{aligned} \tag{1.18}$$

A *section* of the bundle $\Pi : G \to G/G_0 = S^n$ given in Eq. (1.14) is a smooth map $s : S^n \to G$, defined on an open set of S^n, such that $\Pi \circ s = \mathrm{id}$ with id the identity map of the domain of definition of s. One calls s also a local G-frame field of S^n. It is well-known that such sections exist on a neighbourhood of any given point of S^n. The maps

$$\begin{array}{cccc}
s: & S^n \backslash \{\mathbf{x}_\infty\} & \longrightarrow & G \\
 & \begin{bmatrix} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{bmatrix}_\sim & \mapsto & \begin{bmatrix} 1 & 0 & 0 \\ \omega & I_n & 0 \\ \frac{1}{2}\|\omega\|^2 & {}^t\omega & 1 \end{bmatrix}
\end{array} \tag{1.19}$$

and

$$\begin{array}{cccc}
\tilde{s}: & S^n \backslash \{\mathbf{x}_0\} & \longrightarrow & G \\
 & \begin{bmatrix} \frac{1}{2}\|\omega\|^2 \\ \omega \\ 1 \end{bmatrix}_\sim & \mapsto & \begin{bmatrix} \frac{1}{2}\|\omega\|^2 & {}^t\omega & 1 \\ \omega & I_n & 0 \\ 1 & 0 & 0 \end{bmatrix}
\end{array} \tag{1.20}$$

are two canonic sections of the bundle $\Pi : G \to S^n$.

With each section $s : S^n \to G$ of Π we associate a $\mathfrak{g}$-valued (local) 1-form on S^n, given by

$$\phi = s^*\Phi \, , \tag{1.21}$$

with components $\phi^a_b = s^*\Phi^a_b$, $\forall 0 \le a, b \le n+1$. Of course, these components satisfy the same relations and structure equations as the ones of Φ in Eqs. (1.16)

and (1.18).

Since Φ satisfies $\Phi_P = P^{-1}dP$, i.e. $\Phi_P(Q) = P^{-1} \circ Q$, $\forall Q \in T_P G = P\mathfrak{g}$, we have, $\forall x \in S^n$, $z \in T_x S^n$, $\phi_x(z) = \Phi_{s(x)}\big(ds_x(z)\big) = \big(s(x)\big)^{-1} ds_x(z)$, that is,

$$\phi = s^{-1}ds \,. \tag{1.22}$$

If we regard, in the basis η_a, the column components s_a of $s = [s_0, s_A, s_{n+1}]$ as $\mathbb{R}^{n+2}$-valued functions $s_a : S^n \to \mathbb{R}^{n+2}$, then we have

$$< s_a, s_b >= S^a_b \,, \tag{1.23}$$

where $S = [S^a_b]$ is the matrix given in Eq. (1.8), and, from Eq. (1.22), we get

$$ds_a = \phi^b_a s_b \,. \tag{1.24}$$

Similarly to the Riemannian terminology, we call the 1-forms ϕ^a_b, which constitute a matrix with values in $\mathfrak{g}$, the *connection forms* corresponding to the moving frame s. Besides, differentiating Eqs. (1.23) and (1.24) would also lead to the relations (1.16) resp. the structure equations (1.18), thereby replacing Φ^a_b by ϕ^a_b. We also observe that, since $\Pi \circ s = [s_0]_\sim$, s_0 represents the "position" vector of s.

Let $s, \tilde{s} : S^n \to G$ be two sections of Π. In the intersection of their domains of definition we have

$$\tilde{s} = sK \tag{1.25}$$

with $K : S^n \to G_0$ a smooth map. Conversely, given such a map K and a section s of Π, the map $\tilde{s} = sK$ is a section of Π. In order to obtain the transformation laws under a change of frame, we compute the components of $\tilde{\phi} = \tilde{s}^*\Phi$ from those of $\phi = s^*\Phi$, using Eq. (1.25). The map K has the explicit form

$$K = \begin{bmatrix} r^{-1} & {}^tXA & \frac{1}{2}r{}^tXX \\ 0 & A & rX \\ 0 & 0 & r \end{bmatrix} , \tag{1.26}$$

where $r : S^n \to \mathbb{R}^+$, $X : S^n \to \mathbb{R}^n$ and $A : S^n \to SO(n)$ are smooth maps. Thus,

$$\begin{aligned} \tilde{s} &= [\tilde{s}_0, \tilde{s}_A, \tilde{s}_{n+1}] = [s_0, s_A, s_{n+1}] \begin{bmatrix} r^{-1} & {}^tXA & \frac{1}{2}r{}^tXX \\ 0 & A & rX \\ 0 & 0 & r \end{bmatrix} \\ &= \left[r^{-1}s_0, X_B A^B_A s_0 + A^B_A s_B, \frac{1}{2}r{}^tXXs_0 + rX_A s_A + rs_{n+1}\right] . \end{aligned} \tag{1.27}$$

From Eqs. (1.22),(1.25), we have

$$\begin{aligned}\tilde{\phi}(\cdot) &= \tilde{s}^{-1}d\tilde{s}(\cdot) = (sK)^{-1}d(sK)(\cdot) = K^{-1}s^{-1}\big(ds(\cdot)K + s\,dK(\cdot)\big)\\ &= K^{-1}s^{-1}ds(\cdot)K + K^{-1}dK(\cdot) = K^{-1}\phi(\cdot)K + K^{-1}dK(\cdot)\,,\end{aligned}$$

that is,

$$\tilde{\phi} = K^{-1}\phi K + K^{-1}dK\,. \tag{1.28}$$

With Eq. (1.16) we obtain, in matrix form,

$$\begin{aligned}\tilde{\phi} &= \begin{vmatrix} \tilde{\phi}^0_0 & \tilde{\phi}^0_A & 0 \\ \tilde{\phi}^A_0 & \tilde{\phi}^A_B & {}^t[\tilde{\phi}^0_A] \\ 0 & {}^t[\tilde{\phi}^A_0] & -\tilde{\phi}^0_0 \end{vmatrix}\\ &= \underbrace{\begin{vmatrix} r & -r\,{}^tX & \frac{1}{2}r\,{}^tXX \\ 0 & {}^tA & -{}^tAX \\ 0 & 0 & r^{-1} \end{vmatrix}}_{K^{-1}} \circ \left\{ \begin{vmatrix} \phi^0_0 & \phi^0_A & 0 \\ \phi^A_0 & \phi^A_B & {}^t[\phi^0_A] \\ 0 & {}^t[\phi^A_0] & -\phi^0_0 \end{vmatrix} \circ \begin{vmatrix} r^{-1} & {}^tXA & \frac{1}{2}r\,{}^tXX \\ 0 & A & rX \\ 0 & 0 & r \end{vmatrix} \right.\\ &\qquad\qquad \left. + \begin{vmatrix} dr^{-1} & d({}^tXA) & d\big(\frac{1}{2}r\,{}^tXX\big) \\ 0 & dA & d(rX) \\ 0 & 0 & dr \end{vmatrix} \right\}.\end{aligned}$$

Working out the above matrix compositions, we obtain the final expression (which is clearly not the entire matrix $\tilde{\phi}$)

$$\begin{bmatrix} \tilde{\phi}^0_0 & \tilde{\phi}^0_A \\ \tilde{\phi}^A_0 & \tilde{\phi}^A_B \end{bmatrix} = \left| \begin{matrix} \left(\phi^0_0 - {}^tX[\phi^A_0] - d\log r\right) & \begin{pmatrix} r\,{}^tXA\phi^0_0 - r\,{}^tX[\phi^A_0]\,{}^tXA + r[\phi^0_A]A + \\ -r\,{}^tX[\phi^A_B]A + \frac{1}{2}r\,{}^tXX\,{}^t[\phi^A_0]A + r\,d({}^tX)A \end{pmatrix} \\ \left(r^{-1}\,{}^tA[\phi^A_0]\right) & \begin{pmatrix} {}^tA[\phi^A_0]\,{}^tXA + {}^tA[\phi^A_B]A + \\ -{}^tAX\,{}^t[\phi^A_0]A + {}^tA\,dA \end{pmatrix} \end{matrix} \right|. \tag{1.29}$$

In particular, we have

$$\tilde{\phi}^A_0 = r^{-1}A^C_A\phi^C_0\,,\ \ \forall A \in \{1,\dots,n\}\,, \tag{1.30}$$

which leads to the transformations

$$\sum_{A=1}^{n}(\tilde{\phi}^A_0)^2 = r^{-2}\sum_{A=1}^{n}(\phi^A_0)^2 \tag{1.31}$$

and

$$\tilde{\phi}^1_0 \wedge \ldots \wedge \tilde{\phi}^n_0 = r^{-n}\phi^1_0 \wedge \ldots \wedge \phi^n_0\,. \tag{1.32}$$

Let us now reconsider for a moment example (1.19). In that case, we have, $\forall x \in S^n \backslash \{\mathbf{x}_\infty\}$ and $z \in T_x S^n$,

$$\phi_x(z) = (s(x))^{-1} ds_x(z) = \begin{vmatrix} 0 & 0 & 0 \\ v & 0 & 0 \\ 0 & {}^t v & 0 \end{vmatrix},$$

where $v = d(\sigma \circ \kappa)_x(z) \in \mathbb{R}^n$ with σ and κ the diffeomorphisms given in Eqs. (1.5) resp. (1.11). So, $\phi_0^A = (\sigma \circ \kappa)^* dv^A$, where dv^A is the projection of $\mathbb{R}^n$ onto the coordinate A. This shows that $\left(\phi_0^A\right)_{1 \le A \le n}$ are linearly independent 1-forms on $S^n \backslash \{\mathbf{x}_\infty\}$. The same conclusion is obtained for the section $\tilde{s}$ defined in Eq. (1.20), using now, instead of σ, the stereographic projection

$$\begin{array}{cccc} \tilde{\sigma}: & S^n \backslash \{S\} & \longrightarrow & \mathbb{R}^n \\ & (x^0, x^1, \ldots, x^n) & \to & \frac{1}{1+x^0}(x^1, \ldots, x^n) \end{array} \tag{1.33}$$

with $S = (-1, 0, \ldots, 0)$ the south pole of S^n.

As the domains of these two particular sections cover all S^n, we conclude from relation (1.30), concerning any pair of sections of the fibre bundle Π, that, for any section $s : S^n \to G$ of $\Pi : G \to S^n$, the 1-forms $\left(\phi_0^A\right)_{1 \le A \le n}$ of S^n are linearly independent. Furthermore, from the transformations rules in Eqs. (1.31) and (1.32) follows that these 1-forms determine a *conformal structure* resp. an *orientation* on S^n. As we see from the above examples of sections, the conformal structure assigned to S^n is the same as the one generated by the Riemannian metric ds^2 of S^n, induced by the Euclidean metric of $\mathbb{R}^{n+1}$. Explicitly, using the section (1.19), we get $\sum_{A=1}^n (\kappa^{-1})^* (\phi_0^A)^2 = \sum_{A=1}^n \sigma^* (dv^A)^2 = \frac{1}{(1-x^0)^2} ds^2$.

1.2 Submanifolds of S^n

Let $f : M^m \to S^n$ be a smooth immersion of an oriented m-manifold M with $m \ge 2$. We will assign to M a conformal structure induced by f from the conformal structure of S^n. In addition to the index ranges given in Sec. 1.1.B, we agree on $1 \le i, j, \ldots \le m\,, \;\; m+1 \le \alpha, \beta, \ldots \le n$.

1.2.A Zeroth-Order G-Frame Fields Along f

Definition 1.2 *A zeroth-order G-frame field along f is a map $e : M \to G$ defined on an open set of M, such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{e} & G \\ & {\scriptstyle f}\searrow & \downarrow \Pi \\ & & S^n \end{array}$$

is commutative. In other words,

$$\Pi \circ e = f \ , \tag{1.34}$$

where $\Pi : G \to S^n$ is the principal bundle of Eq. (1.14).

We can always define a zeroth-order frame e along f in a neighbourhood of each point of M. In fact, if $s : S^n \to G$ is a section of Π, then $e = s \circ f : M \to G$ is such a frame. Observe that e is an immersion, as is clear from Eq. (1.34).
With each zeroth-order frame $e : M \to G$ along f we associate a $\underset{\sim}{\mathfrak{g}}$-valued (local) 1-form on M defined by

$$\phi = e^*\Phi \ \ (= e^{-1}de) \tag{1.35}$$

with components

$$\phi^a_b = e^*\Phi^a_b \ , \ \ 0 \leq a, b \leq n+1 \ .$$

These components satisfy the same relations as the ones of Φ in Eq. (1.16). Let now $\tilde{e} : M \to G$ be another zeroth-order frame along f. Then,

$$\tilde{e} = eK \ , \tag{1.36}$$

where $K : M \to G_0$ is a map defined on an open set of M, and which is of the form (1.26), with $r : M \to \mathbb{R}^+$, $X : M \to \mathbb{R}^n$, and $A : M \to SO(n)$ smooth maps. Conversely, given such a map K and a zeroth-order frame $e : M \to G$ along f, then $\tilde{e}$ defined by Eq. (1.36) is so. Writing $e = [e_0, e_A, e_{n+1}]$ with $e_a : M \to \mathbb{R}^{n+2}$ vector-valued functions, we obtain the same transformation laws as in Eqs. (1.27), (1.28), (1.29), and (1.30) in Sec. 1.1.C, thereby replacing the sections $s, \tilde{s} : S^n \to G$ of Π by the zeroth-order frames $e, \tilde{e} : M \to G$ of Π along f. From Eq. (1.30), we have that, for any two zeroth-order frame fields $e, \tilde{e} : M \to G$ along f, the 1-forms $(\phi^A_0)_{1 \leq A \leq n}$ span T^*M (in the intersection of the domains of $e, \tilde{e}$), iff $(\tilde{\phi}^A_0)_{1 \leq A \leq n}$ does

the same. If we take the sections $s, \tilde{s}$ of Eqs. (1.19) resp. (1.20), then $e = s \circ f : f^{-1}(S^n \setminus \{x_\infty\}) \to G$ and $\tilde{e} = \tilde{s} \circ f : f^{-1}(S^n \setminus \{x_0\}) \to G$ are zeroth-order frames along f, whose domains of definition cover M. Since $\phi_0^A = e^*\Phi_0^A = f^*(s^*\Phi_0^A)$ and the 1-forms $(s^*\Phi_0^A)_{1\leq A\leq n}$ span T^*S^n, the $(\phi_0^A)_{1\leq A\leq n}$ span T^*M. The same conclusion holds for the 1-forms $\tilde{\phi}_0^A = \tilde{e}^*\Phi_0^A$. Summarising, for any zeroth-order G-frame $e : M \to G$ along f, the 1-forms $(\phi_0^A)_{1\leq A\leq n}$ span T^*M.

1.2.B First-Order G-Frame Fields Along f

In order to be able to define a conformal structure on M, we have to perform a first *reduction* of the zeroth-order G-frame field along f given in the previous subsection. There exist formal theories concerning the method of moving frames on submanifolds immersed into homogeneous spaces, which describe in a general context the concept of reduction of frames (see e.g. Refs. [Je/77] [Su-Šv/80] [Su/79]). Here, we will construct explicitly the specialised frames that we will need to define some geometric objects in conformal geometry, following closely the procedure of Refs. [Sch-Su/80] [Br/84] [Ri/87].

Let $x_0 \in M$ and $e : M \to G$ be a zeroth-order G-frame field of $\Pi : G \to S^n$ along $f : M \to S^n$, defined in a neighbourhood of x_0. Let $Z_1, \ldots, Z_m$ be a local linear frame of TM, defined near x_0. For each $x \in U$ with U a suitable neighbourhood of x_0, we consider the $\mathbb{R}^n$ column vectors

$$v_i(x) = \begin{bmatrix} \phi_0^1(Z_i(x)) \\ \vdots \\ \phi_0^n(Z_i(x)) \end{bmatrix} , \quad \forall i = 1, \ldots, m ,$$

where ϕ_0^A is defined in Eq. (1.35). These define smooth maps from U to $\mathbb{R}^n$. As $\phi_0^1, \ldots, \phi_0^n$ span T_x^*M, $V_x = \operatorname{span}\{v_1(x), \ldots, v_m(x)\}$ is an m-dimensional subspace of $\mathbb{R}^n$. Thus, $V = \{(x, v) : \ x \in U, \ v \in V_x\}$ is a smooth vector subbundle of $U \times \mathbb{R}^n$ and the v_i form a linear frame of V. Let $\bar{v}_1, \ldots, \bar{v}_m$ be the orthogonal linear frame of V (relative to the Euclidean metric of $\mathbb{R}^n$), obtained by Gramm-Schmidt orthogonalisation of $v_1, \ldots, v_m$, and $\bar{v}_{m+1}, \ldots, \bar{v}_n$ be a local orthonormal frame of the orthogonal complement of V in $U \times \mathbb{R}^n$, which can be assumed to be defined on all U. Now we define the map ${}^1\!A : U \to O(n)$, such that, for $x \in U$, ${}^1\!A(x) : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal linear map given by ${}^1\!A(x)(\bar{v}_A) = e_A$, $\forall A \in \{1, \ldots, n\}$, with e_A the canonic basis of $\mathbb{R}^n$. Then, ${}^1\!A(x)(V_x) = \mathbb{R}^m \times \{0\}^{n-m}$. Of

course, we may assume that ${}^tA$ takes values in $SO(n)$. Hence,

$$
{}^tA \begin{bmatrix} \phi_0^1 \\ \vdots \\ \phi_0^n \end{bmatrix} = \begin{bmatrix} \varphi^1 \\ \vdots \\ \varphi^n \end{bmatrix} ,
$$

where φ^A are 1-forms on M, such that $\varphi^A = 0 \; \forall A \geq m+1$. Let $K : U \to G_0$ be given by

$$
K(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A(x) & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad \forall x \in U
$$

and $\tilde{e} : M \to G$ by $\tilde{e} = eK$. Then, from the transformation laws in Eq. (1.29) follows that $\tilde{\phi} = \tilde{e}^*\Phi$ satisfies $[\tilde{\phi}_0^A] = {}^tA[\phi_0^A]$, which implies

$$
\tilde{\phi}_0^\alpha = 0 \; , \quad \forall \alpha = m+1, \ldots, n \; .
$$

In particular, $\tilde{\phi}_0^1, \ldots, \tilde{\phi}_0^m$ constitute a basis of T^*M in a neighbourhood of x_0.

Definition 1.3 *A zeroth-order G-frame field $e : M \to G$ of Π along f is said to be of first order at a point $x_0 \in M$, if $\phi_0^\alpha = 0$ at x_0 , $\forall \alpha = m+1, \ldots, n$ with ϕ_i^j given by Eq. (1.35). The frame e is said to be of first order, if it is so at each point of its domain of definition.*

The above construction proves the existence of first-order frames in a neighbourhood of any given point of M.

Remark 1.2 We note that also first-order frames of the type $e = s \circ f$, where s is a section of Π, can be constructed in a neighbourhood of any given point of M. Assume that we start the above construction with a zeroth-order G-frame along f of the form $e = s \circ f$, where $s : S^n \to G$ is a section of Π on a neighbourhood of $f(x_0)$. Then, we define $\hat{Z}_i(f(x)) = df_x(Z_i(x)) \in T_{f(x)}S^n$ and extend $\hat{Z}_i$ on a neighbourhood of $f(x_0)$ in S^n, giving vector fields on S^n. These are linearly independent on a neighbourhood of $f(x_0)$ in S^n. Let $p : S^n \to M$ be a map defined near $f(x_0)$, satisfying $p \circ f = \mathrm{id}_M$. We define

$$
\hat{v}_i(y) = \begin{bmatrix} p^*\phi_0^1(\hat{Z}_i(y)) \\ \vdots \\ p^*\phi_0^n(\hat{Z}_i(y)) \end{bmatrix} , \quad \forall i = 1, \ldots, m \; .
$$

Then, $V_y = \mathrm{span}\{\bar{v}_1(y), \dots, \bar{v}_m(y)\}$ is an m-dimensional subspace of $I\!R^m$ for y in a neighbourhood U of $f(x_0)$ in S^n. Repeating the above construction, but now replacing v_i by $\bar{v}_i$ and x by y, we obtain a map ${}^t\hat{A} : U \to SO(n)$. Defining $\bar{s} : S^n \to G$ by $\bar{s}(y) = s(y)\hat{K}(y)$, with $\hat{K} : U \subset S^n \to G_0$ given by

$$\hat{K}(y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{A}(y) & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

we obtain a section of Π. Thus, $\bar{e} = \bar{s} \circ f$ is a zeroth-order frame of Π along f which satisfies $\bar{e}(x) = e(x)\hat{K}(f(x))$. Moreover, if we denote $K = \hat{K} \circ f : M \to G_0$, we have that $\bar{e} = eK$ satisfies $\bar{\phi}_0^\alpha(x) = \hat{A}_\alpha^B(f(x))\phi_0^B(x) = 0$. Hence, $\bar{e}$ is a first-order frame of the type $s \circ f$ with s a section of Π.

Consider the closed subgroup G_1 of G_0

$$G_1 = \left\{ \begin{bmatrix} r^{-1} & {}^tXA & {}^tYB & \frac{1}{2}r({}^tXX + {}^tYY) \\ 0 & A & 0 & rX \\ 0 & 0 & B & rY \\ 0 & 0 & 0 & r \end{bmatrix} : \begin{array}{l} A \in SO(m) \\ B \in SO(n-m) \\ X \in I\!R^m,\ Y \in I\!R^{n-m} \\ r \in I\!R^+ \end{array} \right\} , \quad (1.37)$$

where X, Y are column vectors.

Let $e, \bar{e} : M \to G$ be zeroth-order frame fields along f which are of first order at a point $x \in M$. Let

$$K = e^{-1}\bar{e} = \begin{bmatrix} r^{-1} & {}^tZC & \frac{1}{2}r{}^tZZ \\ 0 & C & rZ \\ 0 & 0 & r \end{bmatrix} \in G_0 .$$

Writing

$$C = \begin{bmatrix} A & A' \\ B' & B \end{bmatrix} \begin{array}{l} \}\, m \\ \}\, n-m \end{array} ,$$

with the column blocks of widths $\underbrace{}_{m}$ and $\underbrace{}_{n-m}$, we have, from Eq. (1.29), at the point x

$$\begin{bmatrix} \bar{\phi}_0^i \\ 0 \end{bmatrix} = r^{-1}{}^tC \begin{bmatrix} \phi_0^i \\ 0 \end{bmatrix} = r^{-1} \begin{bmatrix} {}^tA[\phi_0^i] \\ {}^tA'[\phi_0^i] \end{bmatrix} ,$$

that is, ${}^tA'[\phi_0^i] = 0$. Therefore, $A' = 0$. Analogously, from the equality $\begin{bmatrix} \phi_0^i \\ 0 \end{bmatrix} = rC \begin{bmatrix} \bar{\phi}_0^i \\ 0 \end{bmatrix}$ we obtain $B' = 0$. So, $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ at the point x, with $(A, B) \in SO(m) \times SO(n-m) \cup O^-(m) \times O^-(n-m)$. If we assume that $(\phi_0^1, \dots, \phi_0^m)$

and $(\tilde\phi_0^i, \ldots, \tilde\phi_0^m)$ define the same orientation on T_xM, then $A \in SO(m)$ and $B \in SO(n-m)$. Writing $Z = (X,Y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, then, at the point x, we have ${}^tZC = {}^tXA + {}^tYB$, ${}^tZZ = {}^tXX + {}^tYY$, and so

$$K = \begin{bmatrix} r^{-1} & {}^tXA & {}^tYB & \frac{1}{2}r({}^tXX + {}^tYY) \\ 0 & A & 0 & rX \\ 0 & 0 & B & rY \\ 0 & 0 & 0 & r \end{bmatrix} \in G_1 . \tag{1.38}$$

Conversely, if $e : M \to G$ is a given zeroth-order frame along f which is of first order at a point $x \in M$, and if $K : M \to G_0$ is a map, such that $K(x) \in G_1$, say like in Eq. (1.38), then $\tilde e : M \to G$ given by

$$\tilde e = eK \tag{1.39}$$

is a zeroth-order frame along f, satisfying, at the point x,

$$\begin{bmatrix} \tilde\phi_0^i \\ \tilde\phi_0^\alpha \end{bmatrix} = r^{-1} \begin{bmatrix} {}^tA & 0 \\ 0 & {}^tB \end{bmatrix} \begin{bmatrix} \phi_0^i \\ 0 \end{bmatrix} = \begin{bmatrix} r^{-1}{}^tA[\phi_0^i] \\ 0 \end{bmatrix} .$$

Thus, $\tilde e$ is a first-order frame at $x \in M$. For this reason, G_1 is called the ***isotropic group*** of the first-order G-frame fields at a point $x \in M$.

Next we give the transformation laws for a change of a first-order G-frame field along f. Let $e, \tilde e : M \to G$ be two first-order frames. Then, $\tilde e = eK$ with $K : M \to G_1$ a map of the form (1.38). Writing $e = [e_0, e_i, e_\alpha, e_{n+1}]$, where $e_a : M \to \mathbb{R}^{n+2}$ are vector-valued functions, we obtain explicitly

$$\begin{aligned} \tilde e &= [\tilde e_0, \tilde e_i, \tilde e_\alpha, \tilde e_{n+1}] = eK \qquad (1.40) \\ &= [r^{-1}e_0, A_i^j(X_je_0 + e_j), B_\alpha^\beta(Y_\beta e_0 + e_\beta), \frac{1}{2}r({}^tXX + {}^tYY)e_0 + rX_je_j + rY_\alpha e_\alpha + re_{n+1}]. \end{aligned}$$

As in Eq. (1.28), $\tilde\phi = K^{-1}\phi K + K^{-1}dK$, giving

$$\tilde\phi = \begin{bmatrix} \tilde\phi_0^0 & \tilde\phi_j^0 & \tilde\phi_\alpha^0 \\ \tilde\phi_0^i & \tilde\phi_j^i & \tilde\phi_\alpha^i \\ 0 & \tilde\phi_j^\alpha & \tilde\phi_\beta^\alpha \end{bmatrix} =$$

$$\left.\begin{array}{ccc}
\begin{pmatrix}\phi_0^0 - {}^tX[\phi_0^i] + \\ -d\log r\end{pmatrix} & \begin{pmatrix} rd({}^tX)A + r(\phi_0^0 - {}^tX[\phi_0^i]){}^tXA + \\ +r([\phi_i^0] - {}^tX[\phi_j^i] - {}^tY[\phi_i^\alpha])A + \\ +\frac{1}{2}r({}^tXX + {}^tYY){}^t[\phi_0^i]A \end{pmatrix} & \begin{pmatrix} rd({}^tY)B + \\ +r(\phi_0^0 - {}^tX[\phi_0^i]){}^tYB + \\ r([\phi_\alpha^0] - {}^tX[\phi_\alpha^i] - {}^tY[\phi_\alpha^\beta])B \end{pmatrix} \\
\left(r^{-1}{}^tA[\phi_0^i]\right) & \begin{pmatrix} {}^tA([\phi_0^i]{}^tX - X{}^t[\phi_0^i]) + \\ +{}^tA[\phi_j^i]A + {}^tAdA \end{pmatrix} & \left({}^tA[\phi_0^i]{}^tYB + {}^tA[\phi_\alpha^i]B\right) \\
0 & \left({}^tB[\phi_j^\alpha]A - {}^tBY{}^t[\phi_0^i]A\right) & \left({}^tB[\phi_\beta^\alpha]B + {}^tBdB\right)
\end{array}\right| \tag{1.41}$$

In particular,

$$[\tilde{\phi}_0^i] = r^{-1}{}^tA[\phi_0^i] \,. \tag{1.42}$$

Thus, we have

$$\tilde{g} = \sum_{i=1}^{m} (\tilde{\phi}_0^i)^2 = r^{-2} \sum_{i=1}^{m} (\phi_0^i)^2 = r^{-2} g \tag{1.43}$$

and

$$d\tilde{V} = \tilde{\phi}_0^1 \wedge \ldots \wedge \tilde{\phi}_0^m = r^{-m} \phi_0^1 \wedge \ldots \wedge \phi_0^m = r^{-m} dV \,. \tag{1.44}$$

The equations (1.43) and (1.44) for first-order G-frame fields along f define a *conformal structure* and an orientation on M, respectively.

Let $e : M \to G$ be a first-order G-frame field along f and $\phi = [\phi_b^a]$ be defined as in Eq. (1.35). The 1-forms ϕ_b^a of M satisfy the same relations as Φ_b^a in Eq. (1.16), with the additional property $\phi_0^\alpha = 0$, $\forall \alpha = m+1, \ldots, n$. The structure equations (1.18) also hold for the components of ϕ. In particular, for each α

$$0 = d\phi_0^\alpha = -\phi_i^\alpha \wedge \phi_0^i \,. \tag{1.45}$$

At this point we recall Cartan's Lemma, because we are going to use it quite often.

Lemma (Cartan) *Let $p \leq m$ and let $\omega_1, \ldots, \omega_p$ be 1-forms on an m-dimensional manifold M that are linearly independent pointwise. Let $\theta_1, \ldots, \theta_p$ be 1-forms on M, such that*

$$\theta_i \wedge \omega_i = 0 \,.$$

Then, there exist functions C_{ij}, such that $C_{ij} = C_{ji}$ and $\theta_i = C_{ij}\omega_j$.

Applying Cartan's Lemma to Eq. (1.45), we have

$$\phi_i^\alpha = h_{ij}^\alpha \phi_0^j \,, \quad \forall \alpha = m+1, \ldots, n \,, \tag{1.46}$$

where h^α_{ij} are smooth functions defined on the domain of definition of e and with the symmetry property

$$h^\alpha_{ij} = h^\alpha_{ji}\ , \quad \forall 1 \leq i,j \leq m\ . \tag{1.47}$$

Hence, the structure equations (1.18) are, in the case of first-order frames, reduced to

$$\begin{aligned}
d\phi^0_0 &= -\phi^0_i \wedge \phi^i_0 \\
d\phi^i_0 &= -\phi^i_0 \wedge \phi^0_0 - \phi^i_j \wedge \phi^j_0 \\
d\phi^0_i &= -\phi^0_0 \wedge \phi^0_i - \phi^0_j \wedge \phi^j_i - h^\alpha_{ik}\phi^0_\alpha \wedge \phi^k_0 \\
d\phi^i_j &= -\phi^i_0 \wedge \phi^0_j - \phi^i_k \wedge \phi^k_j - \phi^0_i \wedge \phi^j_0 + h^\alpha_{ik}h^\alpha_{jl}\phi^k_0 \wedge \phi^l_0 \\
d\phi^0_\alpha &= -\phi^0_0 \wedge \phi^0_\alpha + h^\alpha_{ij}\phi^0_i \wedge \phi^j_0 - \phi^0_\beta \wedge \phi^\beta_\alpha \\
d\phi^\alpha_i &= -h^\alpha_{jk}\phi^k_0 \wedge \phi^j_i - \phi^0_\alpha \wedge \phi^i_0 - h^\beta_{ij}\phi^\alpha_\beta \wedge \phi^j_0 \\
d\phi^\alpha_\beta &= h^\alpha_{ij}h^\beta_{ik}\phi^j_0 \wedge \phi^k_0 - \phi^\alpha_\gamma \wedge \phi^\gamma_\beta\ .
\end{aligned} \tag{1.48}$$

Now we give the transformation law of the h^α_{ij}. Let $\tilde{e} : M \to G$ be another first-order frame and let $\tilde{h}^\alpha_{ij}$ denote the functions as defined in Eq. (1.46), but now relative to the frame $\tilde{e}$. From Eq. (1.41), we have

$$\tilde{\phi}^\alpha_i = B^\beta_\alpha \phi^\beta_j A^j_i - B^\beta_\alpha Y_\beta \phi^j_0 A^j_i \quad , \quad \tilde{\phi}^j_0 = r^{-1} A^k_j \phi^k_0\ .$$

Thus,

$$\begin{aligned}
\tilde{\phi}^\alpha_i &= \tilde{h}^\alpha_{ij}\tilde{\phi}^j_0 = r^{-1}\tilde{h}^\alpha_{ij}A^k_j\phi^k_0 \\
&= B^\beta_\alpha A^j_i(\phi^\beta_j - Y_\beta\phi^j_0) = B^\beta_\alpha A^j_i(h^\beta_{jk}\phi^k_0 - Y_\beta\phi^j_0) \\
&= B^\beta_\alpha(A^j_i h^\beta_{jk} - A^k_i Y_\beta)\phi^k_0\ .
\end{aligned}$$

That is, $\tilde{h}^\alpha_{ij}A^k_j = rB^\beta_\alpha(A^j_i h^\beta_{jk} - A^k_i Y_\beta)$. Multiplying both sides by A^k_l and letting k run over $1, \ldots, m$, we obtain from the orthogonality of A the equation

$$\tilde{h}^\alpha_{ij} = rB^\beta_\alpha A^k_j(A^l_i h^\beta_{lk} - A^k_i Y_\beta)\ . \tag{1.49}$$

1.2.C Second-Order G-Frame Fields Along f

Let $e : M \to G$ be a first-order G-frame field of $\Pi : G \to S^n$ along $f : M \to S^n$. If $\tilde{e} : M \to G$ is any other first-order G-frame, with $\tilde{e} = eK$, where K is of the form (1.38), then, taking the trace in the indices i, j in Eq. (1.49), we obtain

$$\tilde{h}^\alpha_{ii} = rB^\beta_\alpha(h^\beta_{kk} - mY_\beta)\ . \tag{1.50}$$

For each $m+1 \leq \beta \leq n$, let the function $Y_\beta : M \to \mathbb{R}$, defined in the domain of definition of e, be given by

$$Y_\beta = \frac{1}{m} h^\beta_{kk}, \tag{1.51}$$

and $K : M \to G_1$ by

$$K = \begin{bmatrix} 1 & 0 & {}^tY & \frac{1}{2}{}^tYY \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_{n-m} & Y \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{1.52}$$

where $Y = (Y_{m+1}, \ldots, Y_n) : M \to \mathbb{R}^m$ is as in Eq. (1.51). Then, from Eq. (1.50), $\hat{e}$ satisfies

$$\hat{h}^\alpha_{ii} = 0 \, , \ \forall \alpha = m+1, \ldots, n \, .$$

Observe that in this case, due to Eq. (1.42), $\hat{\phi}^i_0 = \phi^i_0$, $\forall i = 1, \ldots, m$, which implies $\hat{g} = g$ and $d\hat{V} = dV$ in Eqs. (1.43), (1.44). Thus, we have just proved that one can define a first-order G-frame field $e : M \to G$ along f in a neighbourhood of each point x of M, with the property $h^\alpha_{ii} = 0$. Moreover, such frames still define all the Riemannian metrics of the conformal structure of M.

Definition 1.4 *A first-order G-frame field $e : M \to G$ along $f : M \to S^n$ is said to be of second order at a point $x \in M$, if it satisfies $h^\alpha_{ii} = 0$ at x, $\forall \alpha = m+1, \ldots, n$, with h^α_{ij} given by Eq. (1.46). The frame e is said to be of second order, if it is so at each point of its domain of definition.*

Consider the closed subgroup of G_1 defined by

$$G_2 = \left\{ \begin{bmatrix} r^{-1} & {}^tXA & 0 & \frac{1}{2}r{}^tXX \\ 0 & A & 0 & rX \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & r \end{bmatrix} : \begin{array}{lcl} A & \in & SO(m) \\ B & \in & SO(n-m) \\ X & \in & \mathbb{R}^m \\ r & \in & \mathbb{R}^+ \end{array} \right\}. \tag{1.53}$$

If $e, \hat{e} : M \to G$ are first-order frames that are of second order at a point $x \in M$, we get, writing $\hat{e} = eK$ with $K : M \to G_1$ of the form (1.38) and using Eq. (1.50), $B^\beta_\alpha Y_\beta = 0$, i.e. $Y_\beta = 0$ at x, $\forall \beta$. Therefore, $K(x) \in G_2$. Conversely, if $e : M \to G$ is a first-order frame which is of second order at a point $x \in M$, and if $K : M \to G_1$ is a map, such that, at x, $K(x) \in G_2$, then from Eq. (1.50) follows that $\hat{e} = eK : M \to G$ is a first-order frame satisfying $\hat{h}^\alpha_{ii}(x) = 0$. Hence, $\hat{e}$ is also of second order at x. Thus, G_2 is the isotropic group of second-order frames at any point x.

Remark 1.3 The frames that we have just called to be of second order are, strictly speaking, not of second order in the terminology of the general theory on reduction of frames (see Refs. [Je/77] [Su-Šv/80]), as was already pointed out in Refs. [Sch-Su/80] [Br/84] [Ri/87]. The construction of our "second-order" frames is more correctly called a *partial second-order reduction*, resulting in more specialised first-order frames, corresponding to the so-called Darboux frames in the Riemannian geometry of submanifolds of the Euclidean space (see Sec. 1.3). Further reductions can only be carried out by imposing some non-degeneracy conditions.

Now we are going to derive functions h^α_{ijk}, p^α_i, p^α_{ik}, h^α_{ijkl}, and H_{ij}, relative to a second-order frame e, that, together with the ϕ^α_i and h^α_{ij}, will be our essential tools in constructing geometric objects (e.g. tensors) of the conformal geometry of M. Differentiating Eq. (1.46) and using the structure equations (1.48) for a first-order frame, we get

$$\begin{aligned} d\phi^\alpha_i &= \left(dh^\alpha_{ij} + h^\alpha_{ij}\phi^0_0 - h^\alpha_{ik}\phi^k_j\right) \wedge \phi^j_0 \\ &= -h^\alpha_{jk}\phi^k_0 \wedge \phi^j_i - \phi^0_\alpha \wedge \phi^i_0 - h^\beta_{ij}\phi^\alpha_\beta \wedge \phi^j_0 \,, \end{aligned}$$

which gives

$$\left(dh^\alpha_{ij} - h^\alpha_{ik}\phi^k_j - h^\alpha_{kj}\phi^k_i + h^\beta_{ij}\phi^\alpha_\beta + h^\alpha_{ij}\phi^0_0 + \delta_{ij}\phi^0_\alpha\right) \wedge \phi^j_0 = 0 \,.$$

Hence, by Cartan's Lemma, we have, for each i, α,

$$dh^\alpha_{ij} - h^\alpha_{ik}\phi^k_j - h^\alpha_{kj}\phi^k_i + h^\beta_{ij}\phi^\alpha_\beta + h^\alpha_{ij}\phi^0_0 + \delta_{ij}\phi^0_\alpha = h^\alpha_{ijk}\phi^k_0 \,, \qquad (1.54)$$

where $h^\alpha_{ijk} = h^\alpha_{ikj}$ are smooth functions. From Eq. (1.47), we have

$$h^\alpha_{ijk} = h^\alpha_{jik} = h^\alpha_{ikj} \,, \quad \forall \alpha = m+1, \ldots, n \,, \quad i, j, k = 1, \ldots, m \,. \qquad (1.55)$$

Taking the trace of Eq. (1.54) in the indices i, j, and noting that $h^\alpha_{ij}\phi^i_j = 0$, we obtain

$$m\phi^0_\alpha = h^\alpha_{iik}\phi^k_0 \,.$$

Defining

$$p^\alpha_k = \frac{1}{m} h^\alpha_{iik} \qquad (1.56)$$

we have

$$\phi^0_\alpha = p^\alpha_k \phi^k_0 \,. \qquad (1.57)$$

Differentiation of this equation yields, with the structure equations (1.48),

$$
\begin{aligned}
d\phi_\alpha^0 &= dp_k^\alpha \wedge \phi_0^k + p_k^\alpha \wedge d\phi_0^k = \left(dp_j^\alpha - p_k^\alpha \phi_j^k + p_j^\alpha \phi_0^0\right) \wedge \phi_0^j && (1.58)\\
&= -p_j^\alpha \phi_0^0 \wedge \phi_0^j + h_{ij}^\alpha \phi_i^0 \wedge \phi_0^j + p_j^\beta \phi_\alpha^\beta \wedge \phi_0^j\,. && (1.59)
\end{aligned}
$$

Combining Eqs. (1.58) and (1.59) we obtain

$$\left(dp_j^\alpha - p_k^\alpha \phi_j^k - h_{kj}^\alpha \phi_k^0 + p_j^\beta \phi_\beta^\alpha + 2p_j^\alpha \phi_0^0\right) \wedge \phi_0^j = 0\,.$$

Hence, from Cartan's Lemma,

$$dp_j^\alpha - p_k^\alpha \phi_j^k - h_{kj}^\alpha \phi_k^0 + p_j^\beta \phi_\beta^\alpha + 2p_j^\alpha \phi_0^0 = p_{jk}^\alpha \phi_0^k\,, \tag{1.60}$$

where p_{ik}^α are smooth functions on the domain of definition of e with the symmetry property

$$p_{ik}^\alpha = p_{ki}^\alpha\,,\ \ \forall \alpha, k, i\,. \tag{1.61}$$

Using Eq. (1.54), we get

$$
\begin{aligned}
h_{ij}^\alpha dh_{ij}^\alpha &= h_{ij}^\alpha h_{ijk}^\alpha \phi_0^k + h_{ij}^\alpha h_{kj}^\alpha \phi_i^k + h_{ij}^\alpha h_{ik}^\alpha \phi_j^k - h_{ij}^\alpha h_{ij}^\beta \phi_\beta^\alpha + \\
&\quad - h_{ij}^\alpha h_{ij}^\alpha \phi_0^0 - h_{ii}^\alpha p_k^\alpha \phi_0^k\,.
\end{aligned}
$$

Since $h_{ij}^\alpha h_{kj}^\alpha$ is symmetric in i, k and ϕ_i^k is anti-symmetric, $h_{ij}^\alpha h_{kj}^\alpha \phi_i^k = 0$. Analogously, $h_{ij}^\alpha h_{ik}^\alpha \phi_j^k = h_{ij}^\alpha h_{ij}^\beta \phi_\beta^\alpha = 0$. Moreover, as e is of second order,

$$h_{ij}^\alpha dh_{ij}^\alpha = -\sum_{i,j,\alpha} (h_{ij}^\alpha)^2 \phi_0^0 + h_{ij}^\alpha h_{ijk}^\alpha \phi_0^k\,.$$

Hence, using the vanishing of $d(h_{ij}^\alpha dh_{ij}^\alpha)$ and the structure equations (1.48), we obtain

$$d(h_{ij}^\alpha h_{ijk}^\alpha) \wedge \phi_0^k = \Big(-3h_{ij}^\alpha h_{ijk}^\alpha \phi_0^0 - \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 \phi_k^0 + h_{ij}^\alpha h_{ijp}^\alpha \phi_k^p \Big) \wedge \phi_0^k\,.$$

Then, Cartan's Lemma yields

$$d(h_{ij}^\alpha h_{ijk}^\alpha) = -3h_{ij}^\alpha h_{ijk}^\alpha \phi_0^0 - \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 \phi_k^0 + h_{ij}^\alpha h_{ijp}^\alpha \phi_k^p + H_{kr} \phi_0^r\,, \tag{1.62}$$

where the H_{kr} are smooth functions with the symmetry property

$$H_{kr} = H_{rk}\,. \tag{1.63}$$

Alternatively, we can express H_{kr} as follows. Differentiating Eq. (1.54) and applying Cartan's Lemma, we get

$$\begin{aligned} dh^\alpha_{ijk} &= h^\alpha_{ijkl}\phi^l_0 + h^\alpha_{irk}\phi^r_j + h^\alpha_{jrk}\phi^r_i + h^\alpha_{ijr}\phi^r_k - 2h^\alpha_{ijk}\phi^0_0 - h^\beta_{ijk}\phi^\alpha_\beta + \\ &\quad -h^\alpha_{ir}h^\beta_{rl}h^\beta_{jk}\phi^l_0 - h^\alpha_{jr}h^\beta_{rl}h^\beta_{ik}\phi^l_0 + h^\alpha_{rl}h^\beta_{ij}h^\beta_{rk}\phi^l_0 + \\ &\quad +\delta_{jk}h^\alpha_{ir}\phi^0_r + \delta_{ik}h^\alpha_{jr}\phi^0_r + \delta_{ij}h^\alpha_{rk}\phi^0_r - h^\alpha_{ik}\phi^0_j - h^\alpha_{jk}\phi^0_i - h^\alpha_{ij}\phi^0_k \,, \end{aligned} \tag{1.64}$$

where h^α_{ijkl} are smooth functions with the symmetry properties

$$h^\alpha_{ijkl} = h^\alpha_{ijlk} = h^\alpha_{jikl} \,. \tag{1.65}$$

Expanding the l.h.s. of Eq. (1.62) and using Eq. (1.64) plus definition (1.56), we obtain

$$H_{kr} = h^\alpha_{ij}h^\alpha_{ijkr} + h^\alpha_{ijk}h^\alpha_{ijr} - mp^\alpha_k p^\alpha_r - 2h^\beta_{kj}h^\alpha_{ji}h^\alpha_{il}h^\beta_{lr} + h^\beta_{kl}h^\alpha_{lr}h^\alpha_{ij}h^\beta_{ij} \,. \tag{1.66}$$

Besides, from Eq. (1.35) we have $de = e \circ \phi$. If we regard, in the basis η_a, the column components e_a of the matrix $e = |e_0, e_i, e_\alpha, e_{n+1}|$ as $\mathbb{R}^{n+2}$-valued functions $e_a : S^n \to \mathbb{R}^{n+2}$, then we get, with Eqs. (1.16),(1.46),(1.57),

$$\begin{aligned} de_0 &= \phi^0_0 e_0 + \phi^i_0 e_i \\ de_k &= \phi^0_k e_0 + \phi^i_k e_i + h^\alpha_{kj}\phi^j_0 e_\alpha + \phi^k_0 e_{n+1} \\ de_\alpha &= p^\alpha_i \phi^i_0 e_0 - h^\alpha_{ij}\phi^i_0 e_j + \phi^\beta_\alpha e_\beta \\ de_{n+1} &= \phi^0_i e_i + p^\alpha_i \phi^i_0 e_\alpha - \phi^0_0 e_{n+1} \,. \end{aligned} \tag{1.67}$$

Finally, the first four structure equations (1.48), rewritten for second-order frames, take the form

$$\begin{aligned} d\phi^0_0 &= -\phi^0_i \wedge \phi^i_0 \\ d\phi^i_0 &= -\phi^i_0 \wedge \phi^0_0 - \phi^i_j \wedge \phi^j_0 \\ d\phi^0_i &= -\phi^0_0 \wedge \phi^0_i - \phi^0_j \wedge \phi^j_i - p^\alpha_j h^\alpha_{ik}\phi^j_0 \wedge \phi^k_0 \\ d\phi^i_j &= -\phi^i_0 \wedge \phi^0_j - \phi^i_k \wedge \phi^k_j - \phi^0_i \wedge \phi^j_0 + h^\alpha_{ik}h^\alpha_{jl}\phi^k_0 \wedge \phi^l_0 \,. \end{aligned}$$

If we define, for each a, b, $\Omega^a_b = -\phi^a_\alpha \wedge \phi^\alpha_b$, then the above structure equations can be written as

$$\begin{aligned} d\phi^0_0 &= -\phi^0_i \wedge \phi^i_0 + \Omega^0_0 \\ d\phi^i_0 &= -\phi^i_0 \wedge \phi^0_0 - \phi^i_j \wedge \phi^j_0 + \Omega^i_0 \\ d\phi^0_i &= -\phi^0_0 \wedge \phi^0_i - \phi^0_j \wedge \phi^j_i + \Omega^0_i \\ d\phi^i_j &= -\phi^i_0 \wedge \phi^0_j - \phi^i_k \wedge \phi^k_j - \phi^0_i \wedge \phi^j_0 + \Omega^i_j \,, \end{aligned} \tag{1.68}$$

where

$$\Omega^0_0 = \Omega^i_0 = 0 \,, \quad \Omega^0_i = -p^\alpha_j h^\alpha_{ik}\phi^j_0 \wedge \phi^k_0 \,, \quad \Omega^i_j = h^\alpha_{ik}h^\alpha_{jl}\phi^k_0 \wedge \phi^l_0 \,. \tag{1.69}$$

The $\frac{(m+1)(m+2)}{2}$ 1-forms ϕ_0^0, ϕ_i^0, ϕ_0^i, ϕ_j^i and 2-forms Ω_0^0, Ω_0^i, Ω_i^0, Ω_j^i are called the ***connection forms*** resp. ***curvature forms*** corresponding to the second-order frame e.

Next we give the transformation laws of the e_a, ϕ_i^a, h_{ij}^a, p_i^a, h_{ijk}^a, p_{ij}^a, and H_{ij}. Let $e, \tilde{e} : M \to G$ be second-order frames along f with $\tilde{e} = eK$, where $K : M \to G_2$ is a map of the form

$$K = \begin{bmatrix} r^{-1} & {}^tXA & 0 & \frac{1}{2}r\,{}^tXX \\ 0 & A & 0 & rX \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & r \end{bmatrix} . \tag{1.70}$$

Then,

$$\begin{aligned} \tilde{e} &= [\tilde{e}_0, \tilde{e}_i, \tilde{e}_\alpha, \tilde{e}_{n+1}] = [e_0, e_i, e_\alpha, e_{n+1}]K \\ &= \left[r^{-1}e_0, A_i^j(X_je_0 + e_j), B_\alpha^\beta e_\beta, \frac{1}{2}r\,{}^tXXe_0 + rX_je_j + re_{n+1}\right] . \end{aligned} \tag{1.71}$$

As in Eq. (1.28), $\tilde{\phi} = K^{-1}\phi K + K^{-1}dK$, which gives

$$\tilde{\phi} = \begin{bmatrix} \tilde{\phi}_0^0 & \tilde{\phi}_j^0 & \tilde{\phi}_\alpha^0 \\ \tilde{\phi}_0^i & \tilde{\phi}_j^i & \tilde{\phi}_\alpha^i \\ 0 & \tilde{\phi}_j^\alpha & \tilde{\phi}_\beta^\alpha \end{bmatrix} =$$

$$\begin{bmatrix} \begin{pmatrix} \phi_0^0 - {}^tX[\phi_0^i] + \\ -d\log r \end{pmatrix} & \begin{pmatrix} rd({}^tX)A + r(\phi_0^0 - {}^tX[\phi_0^i])\,{}^tXA + \\ +r([\phi_i^0] - {}^tX[\phi_j^i] + \frac{1}{2}{}^tXX\,{}^t[\phi_0^i])A \end{pmatrix} & \left(r([\phi_\alpha^0] - {}^tX[\phi_\alpha^i])B\right) \\ \left(r^{-1}\,{}^tA[\phi_0^i]\right) & \left({}^tA([\phi_0^i]\,{}^tX - X\,{}^t[\phi_0^i]) + {}^tA[\phi_j^i]A + {}^tAdA\right) & \left({}^tA[\phi_\alpha^i]B\right) \\ 0 & \left({}^tB[\phi_j^\alpha]A\right) & \left({}^tB[\phi_\beta^\alpha]B + {}^tBdB\right) \end{bmatrix} \tag{1.72}$$

From Eq. (1.49), we have (with now $Y = 0$)

$$\tilde{h}_{ij}^\alpha = rB_\alpha^\beta A_i^k A_j^l h_{kl}^\beta . \tag{1.73}$$

From Eq. (1.57) and the transformation laws of $\tilde{\phi}_0^i = r^{-1}A_i^k\phi_0^k$ and $\tilde{\phi}_\alpha^0 = rB_\alpha^\beta(\phi_\beta^0 - X_i\phi_\beta^i)$ given in Eq. (1.72), we obtain

$$\tilde{p}_i^\alpha = r^2B_\alpha^\beta A_i^k(p_k^\beta + h_{kj}^\beta X_j) . \tag{1.74}$$

In order to derive the transformation law of the h_{ijk}^α, we differentiate Eq. (1.73), and use Eq. (1.54) and the transformation laws (1.72),(1.73), obtaining

$$\begin{aligned}\tilde{h}^{\alpha}_{ijk} &= r^3 B^{\beta}_{\alpha}\big(A^u_i A^v_j A^w_k h^{\beta}_{uvw} - A^u_i A^v_j A^w_k X_v h^{\beta}_{uw} + \\ &\quad - A^u_i A^v_j A^w_k X_u h^{\beta}_{vw} - A^u_i A^v_j A^w_k X_w h^{\beta}_{uv} + \\ &\quad + \delta_{ij} A^v_k X_u h^{\beta}_{uv} + \delta_{ik} A^v_j X_u h^{\beta}_{uv} + \delta_{jk} A^v_i X_u h^{\beta}_{uv}\big).\end{aligned} \tag{1.75}$$

Eqs. (1.73) and (1.75) yield the transformation

$$\tilde{h}^{\alpha}_{ij}\tilde{h}^{\alpha}_{ijk} = r^5 A^w_k \left(h^{\beta}_{uv} h^{\beta}_{uvw} - X_w h^{\beta}_{uv} h^{\beta}_{uv}\right). \tag{1.76}$$

Differentiating Eq. (1.74) and applying Eq. (1.60) to $e, \tilde{e}$, we obtain the transformation law of the $\tilde{p}^{\alpha}_{ij}$, reading

$$\begin{aligned}\tilde{p}^{\alpha}_{ij} &= r^4 B^{\beta}_{\alpha}\big(A^k_i A^t_j p^{\beta}_{kt} + A^k_i A^t_j X_p h^{\beta}_{pkt} - A^k_i A^t_j X_t X_p h^{\beta}_{pk} + \\ &\quad - A^k_i A^t_j X_k X_p h^{\beta}_{pt} - \frac{1}{2} A^k_i A^t_j X_p X_p h^{\beta}_{kt} - 2A^k_i A^t_j X_t p^{\beta}_k + \\ &\quad - 2A^k_i A^t_j X_k p^{\beta}_t + \delta_{ij} X_p X_t h^{\beta}_{pt} + \delta_{ij} X_p p^{\beta}_p\big).\end{aligned} \tag{1.77}$$

Taking the trace of this equation in the indices i, j leads to

$$\tilde{p}^{\alpha}_{ii} = r^4 B^{\beta}_{\alpha}\big(p^{\beta}_{tt} + 2(m-2) X_p p^{\beta}_p + (m-2) X_p X_t h^{\beta}_{pt}\big), \tag{1.78}$$

which, in the particular case $m = 2$, gives

$$\tilde{p}^{\alpha}_{ii} = r^4 B^{\beta}_{\alpha} p^{\beta}_{tt}. \tag{1.79}$$

Differentiation of Eq. (1.76) and application of Eq. (1.62) and the transformation law of the $\tilde{\phi}^{\alpha}_{i}$ in Eq. (1.72) gives

$$\begin{aligned}\tilde{H}_{kt} &= r^4 \Big\{A^r_k A^p_t H_{rp} - 3A^r_k A^p_t h^{\beta}_{uv} h^{\beta}_{uvp} X_r - 3A^r_k A^p_t h^{\beta}_{uv} h^{\beta}_{uvr} X_p + \\ &\quad + \delta_{tk} h^{\beta}_{uv} h^{\beta}_{uvp} X_p + 3\Big(\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2\Big) A^u_k A^p_t X_u X_p - \frac{1}{2}\delta_{kt}\Big(\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2\Big) X_p X_p\Big\}.\end{aligned} \tag{1.80}$$

Combining this with Eq. (1.73), we get

$$\tilde{h}^{\alpha}_{kt}\tilde{H}_{kt} = r^6 B^{\gamma}_{\alpha}\Big(h^{\gamma}_{kt} H_{kt} - 6h^{\gamma}_{kt} h^{\beta}_{uv} h^{\beta}_{uvk} X_t + 3\Big(\sum_{i,j,\beta}(h^{\beta}_{ij})^2\Big) h^{\gamma}_{kt} X_k X_t\Big). \tag{1.81}$$

Finally, we derive the transformation law of the $(m-1)$-form $\phi^{1\ldots\hat{i}\ldots m} := \phi^1_0 \wedge \ldots \wedge \phi^{i-1}_0 \wedge \phi^{i+1}_0 \wedge \ldots \wedge \phi^m_0$, where the missing index i is assumed to be summed

over when appearing repeated in composite expressions. From Eq. (1.72), we have $\tilde{\phi}_0^i = r^{-1}A_i^k\phi_0^k$. Denoting by $\tilde{E}_i$ and E_i the linear frames dual to the co-frames $\tilde{\phi}_0^i$ resp. ϕ_0^i, we get

$$\tilde{E}_i = rA_i^k E_k \ . \tag{1.82}$$

Hence,

$$\phi^{1\ldots\hat{i}\ldots m} = \sum_j \det\left(A^{1\ldots\hat{i}\ldots m}_{1\ldots\hat{j}\ldots m}\right)\tilde{\phi}^{1\ldots\hat{j}\ldots m} \ ,$$

where $A^{1\ldots\hat{i}\ldots m}_{1\ldots\hat{j}\ldots m}$ denotes the submatrix of the matrix $[rA_i^k]$ with row i and column j removed. Since A is orthogonal and from the rule $(-1)^{i+j}\det\left(A^{1\ldots\hat{i}\ldots m}_{1\ldots\hat{j}\ldots m}\right)(rA_k^i) = \delta_{kj}\det[rA_i^k] = \delta_{kj}r^m$, we obtain $(-1)^{i+j}\det\left(A^{1\ldots\hat{i}\ldots m}_{1\ldots\hat{j}\ldots m}\right) = r^{m-1}A_j^i$. Thus, $\phi^{1\ldots\hat{i}\ldots m} = (-1)^{i+j}r^{m-1}A_j^i\tilde{\phi}^{1\ldots\hat{j}\ldots m}$. Multiplying by $(-1)^{i+k}r^{1-m}A_k^i$ and summing over i, we arrive at

$$\tilde{\phi}^{1\ldots\hat{k}\ldots m} = (-1)^{i+k}r^{1-m}A_k^i\phi^{1\ldots\hat{i}\ldots m} \ . \tag{1.83}$$

1.2.D The Generalised Weyl Tensor and Conformally Flat Submanifolds

Given a second-order G-frame $e : M \to G$ along $f : M \to S^n$, one can define the quantities (see Ref. [Ri/87])

$$T^i_{jkl} = h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk} \ , \tag{1.84}$$

where the h^α_{ij} are given by Eqs. (1.46) and (1.47). The T^i_{jkl} satisfy the symmetry relations $T^i_{jkl} = -T^j_{ikl} = -T^i_{jlk} = T^k_{lij}$. Also, from the structure equations (1.68), (1.69) we have $\Omega^i_j = \sum_{k<l} T^i_{jkl}\phi_0^k \wedge \phi_0^l$, i.e. the T^i_{jkl} are the components of the curvature form Ω^i_j relative to the co-frame ϕ_0^k. If $\tilde{e} : M \to G$ is another second-order frame, then from Eq. (1.73) follows

$$\tilde{T}^i_{jkl} = \tilde{h}^\alpha_{ik}\tilde{h}^\alpha_{jl} - \tilde{h}^\alpha_{il}\tilde{h}^\alpha_{jk} = r^2A_i^s A_j^t A_k^u A_l^v T^s_{tuv} \ , \tag{1.85}$$

where r and A_j^i are as in Eqs. (1.70),(1.39). Denote by E_i the frame of M dual to the co-frame ϕ_0^i. Then, from the transformation law of these frames in Eq. (1.82) we conclude that a global tensor $T \in C^\infty\left(\otimes^3 T^*M \otimes TM\right)$ can be defined on M, locally given by

$$T = -T^i_{jkl}\phi_0^k \otimes \phi_0^l \otimes \phi_0^j \otimes E_i \tag{1.86}$$

on a domain of a second-order frame e. Rigoli called $\mathcal{T}$ the ***generalised Weyl tensor.*** Taking the trace of $\mathcal{T}^i_{jkl}$ in the indices i, l, one obtains

$$\mathcal{N}_{jk} := \mathcal{T}^i_{jki} = h^\alpha_{ik} h^\alpha_{ji} \,, \tag{1.87}$$

which defines a global symmetric tensor $\mathcal{N} \in C^\infty\big(\bigodot^2 T^*M\big)$, locally given by

$$\mathcal{N} = \mathcal{N}_{jk} \phi^j_0 \otimes \phi^k_0 \,. \tag{1.88}$$

Note that $\mathcal{N}_{jj} = \sum_{i,j,\alpha} (h^\alpha_{ij})^2$ and that, if $m = 2$, $\mathcal{N} = \frac{1}{2}\mathcal{N}_{jj}(\phi^1_0 \otimes \phi^1_0 + \phi^2_0 \otimes \phi^2_0)$. For any m, one has trivially (cf. Refs. [Sch-Su/80] [Ri/87]) at a point $x \in M$

$$\operatorname{trace} \mathcal{N}(x) = \mathcal{N}_{jj}(x) = 0 \,, \ \text{ iff } \ h^\alpha_{ij} = 0 \,, \ \ \forall i, j, \alpha \,, \ \text{ iff } \ \mathcal{N}(x) = 0 \,. \tag{1.89}$$

In particular, the condition $\mathcal{N}_{jj}(x) = 0$ is conformally invariant, as we can also see directly from the transformation law $\tilde{\mathcal{N}}_{jj} = r^2 \mathcal{N}_{jj}$. A point $x \in M$ is said to be ***umbilic***, if $\mathcal{N}_{jj}(x) = 0$, and the immersion $f : M \to S^n$ is said to be ***Möbius-flat***, if all the points of M are umbilic (see Refs. [Sch-Su/80] [Br/84] [Ri/87]). If x is umbilic, the curvature forms Ω^0_0, Ω^i_0, Ω^0_i, Ω^i_j vanish at x, for any second-order frame. The use of these names becomes clear from the following proposition, first formulated by Schiemangk and Sulanke [Sch-Su/80] (see also [Ri/87]):

Proposition (Schiemangk-Sulanke,Rigoli) *Suppose that M is connected and $m \geq 2$. Then, $\mathcal{N} \equiv 0$, iff there exists a $S^m \subset S^n$, such that $f(M) \subset S^m$. In this case, if, moreover, M is compact, then f is a diffeomorphism of M onto S^m.*

In particular, the map

$$\begin{array}{ccccccc} f: & \mathbb{R}^n & \xrightarrow{\kappa^{-1} \circ \sigma^{-1}} & S^m & \hookrightarrow & S^n = \mathcal{L}^+_\sim \\ & y & \to & \begin{bmatrix} 1 \\ y \\ \frac{1}{2}\|y\|^2 \end{bmatrix}_\sim & \to & \begin{bmatrix} 1 \\ \begin{bmatrix} y \\ 0 \end{bmatrix} \\ \frac{1}{2}\|y\|^2 \end{bmatrix}_\sim \end{array} \tag{1.90}$$

immerses $\mathbb{R}^m$ as a Möbius-flat submanifold into the Möbius space S^n.

1.3 Relation with Riemannian Geometry of Submanifolds of the Euclidean Space

If one considers S^n as $\mathbb{R}^n$ with a point at infinity, one can relate the Riemannian geometry of a submanifold of $\mathbb{R}^n \subset S^n$ and its conformal geometry induced by the one of S^n, which we will describe in the following.

Let us consider the diffeomorphisms

$$\begin{array}{ccc} \mathbb{P}^{n+1} \supset \mathcal{L}^{+}_{\sim} \supset S^n \backslash \{\mathbf{x}_\infty\} & \xrightarrow{i^{-1}} & \mathbb{R}^n \\ \left[\begin{array}{c} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{array}\right]_{\sim} & \xleftarrow{i} & \omega \\ \downarrow \kappa & \nearrow \sigma & \\ \left(\frac{\|\omega\|^2-1}{\|\omega\|^2+1}, \frac{2\omega}{\|\omega\|^2+1}\right) & & \\ S^n \backslash \{N\} \subset \mathbb{R}^{n+1} & & \end{array} \tag{1.91}$$

where σ and κ are given in Eqs. (1.5) resp. (1.11). Through the diffeomorphism $i = \kappa^{-1} \circ \sigma^{-1}$, $\mathbb{R}^n$ is identified with $S^n \backslash \{\mathbf{x}_\infty\}$. In order to use the method of moving frames in $\mathbb{R}^n$, we have to write $\mathbb{R}^n$ as a homogeneous space of a subgroup of G.

The isotropic subgroup of G at $\mathbf{x}_\infty$ is given by

$$\bar{G} = \left\{ \left| \begin{array}{ccc} r^{-1} & 0 & 0 \\ r^{-1}Z & A & 0 \\ \frac{1}{2}r^{-1}\,{}^tZZ & {}^tZA & r \end{array} \right| \; : \; \begin{array}{l} A \in SO(n) \\ Z \in \mathbb{R}^n \\ r > 0 \end{array} \right\}.$$

Let G^* be the subgroup of $\bar{G}$ defined by

$$G^* = \left\{ \left[\begin{array}{ccc} 1 & 0 & 0 \\ Z & A & 0 \\ \frac{1}{2}\,{}^tZZ & {}^tZA & 1 \end{array} \right] \; : \; \begin{array}{l} A \in SO(n) \\ Z \in \mathbb{R}^n \end{array} \right\}. \tag{1.92}$$

The group G^* is isomorphic to the identity component $\mathbb{E}^+(n)$ of the group of the Euclidean motions of $\mathbb{R}^n$, i.e.

$$\mathbb{E}^+(n) = \{(A, Z) : \; A \in SO(n), \; Z \in \mathbb{R}^n\}$$

with structure group defined by $(A, Z) \circ (B, W) = (AB, AW + Z)$, $(A, Z)^{-1} = (A^{-1}, -A^{-1}Z)$, and id $= (I_n, 0)$. This isomorphism is given by

$$\begin{array}{ccc} \mathbb{E}^+(n) & \longrightarrow & G^* \\ (A, Z) & \rightarrow & \left[\begin{array}{ccc} 1 & 0 & 0 \\ Z & A & 0 \\ \frac{1}{2}\,{}^tZZ & {}^tZA & 1 \end{array} \right]. \end{array} \tag{1.93}$$

Under this identification, the action of G^* on $S^n\backslash\{\mathbf{x}_\infty\}$, which is the restriction of the one of G on S^n, is identical to the usual action of $\mathbb{E}^+(n)$ on $\mathbb{R}^n$. In other words, the following diagram is commutative.

$$\begin{array}{ccccc}
\mathbb{E}^+(n) & \times & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\
((A,Z) & , & \omega) & \rightarrow & A\omega + Z. \\
\downarrow & & \downarrow i & & \uparrow i^{-1} \\
\left(\begin{vmatrix} 1 & 0 & 0 \\ Z & A & 0 \\ \frac{1}{2}{}^tZZ & {}^tZA & 1 \end{vmatrix}\right. & , & \left.\begin{vmatrix} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{vmatrix}_\sim\right) & \rightarrow & \begin{bmatrix} 1 \\ Z + A\omega \\ \frac{1}{2}\|Z + A\omega\|^2 \end{bmatrix}_\sim \\
G^* & \times & S^n\backslash\{\mathbf{x}_\infty\} & \longrightarrow & S^n\backslash\{\mathbf{x}_\infty\}\ .
\end{array}$$

As the action of $\mathbb{E}^+(n)$ on $\mathbb{R}^n$ is transitive, the same holds for the action of G^* on $S^n\backslash\{\mathbf{x}_\infty\}$.

The isotropic subgroup of G^* at the origin $\mathbf{x}_0 \in S^n\backslash\{\mathbf{x}_\infty\}$ is given by

$$G_0^* = \left\{ \begin{vmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{vmatrix} \ : \ A \in SO(n) \right\} \tag{1.94}$$

and is isomorphic, via Eq. (1.93), to the isotropic subgroup $SO(n)$ of $\mathbb{E}^+(n)$ at the point $i^{-1}(\mathbf{x}_0) = 0$. Thus, $\mathbb{R}^n \simeq S^n\backslash\{\mathbf{x}_\infty\}$ is diffeomorphic to the homogeneous space G^*/G_0^*. Let $j : G^* \to G$ be the inclusion map. The canonic projection $\overline{\Pi} : G^* \to G^*/G_0^* \simeq \mathbb{R}^n$ is given by

$$\overline{\Pi}(P) = i^{-1}([P(\eta_0)]_\sim)\ , \tag{1.95}$$

that is,

$$\overline{\Pi}\left(\begin{vmatrix} 1 & 0 & 0 \\ Z & A & 0 \\ \frac{1}{2}{}^tZZ & {}^tZA & 1 \end{vmatrix}\right) = i^{-1}\left(\begin{vmatrix} 1 \\ Z \\ \frac{1}{2}\|Z\|^2 \end{vmatrix}_\sim\right) = Z\ .$$

Thus, $\overline{\Pi} = i^{-1} \circ \Pi \circ j$, where $\Pi : G \to G/G_0 \simeq S^n$ is the projection in Eq. (1.14). The Lie algebra of G^*, $\mathfrak{g}^* \simeq \mathbb{R}^n \times \mathfrak{o}(n)$, has basis $\{P_{(A,0)}, P_{(A,B)} : A > B\}$ (see Sec. 1.1.C). The Maurer-Cartan form of G^* is given by

$$\overline{\Phi} = j^*\Phi : TG^* \to \mathfrak{g}^*\ ,$$

where $\Phi : G \to \mathfrak{g}$ is the Maurer-Cartan form of G, and its components $(\overline{\Phi}^a_b)_{0\le a,b\le n+1}$ satisfy the relations

$$\overline{\Phi}^0_0 = -\overline{\Phi}^{n+1}_{n+1} = \overline{\Phi}^0_A = \overline{\Phi}^A_{n+1} = \overline{\Phi}^{n+1}_0 = \overline{\Phi}^0_{n+1} = 0,\ \ \overline{\Phi}^A_0 = \overline{\Phi}^{n+1}_A,\ \ \overline{\Phi}^A_B = -\overline{\Phi}^B_A\ . \tag{1.96}$$

The structure equations of G^* are supplied by the Maurer-Cartan equation $d\mathring{\Phi} = -\mathring{\Phi} \wedge \mathring{\Phi}$, that is, in components, $d\mathring{\Phi}^a_b = -\mathring{\Phi}^a_c \wedge \mathring{\Phi}^c_b$, through Eq. (1.96) reducing to

$$\begin{aligned} d\mathring{\Phi}^A_0 &= -\mathring{\Phi}^A_B \wedge \mathring{\Phi}^B_0 \\ d\mathring{\Phi}^A_B &= -\mathring{\Phi}^A_C \wedge \mathring{\Phi}^C_B . \end{aligned} \tag{1.97}$$

Next we assign to $\mathbb{R}^n \simeq G^*/G^*_0$ a *Riemannian structure*, described in the following. For each (local) section $\rho : \mathbb{R}^n \to G^*$ of the bundle $\Pi : G^* \to \mathbb{R}^n$, i.e. ρ is a map that satisfies $\Pi \circ \rho = \mathrm{id}_{\mathbb{R}^n}$, we take the $\mathfrak{g}^*$-valued 1-form

$$\mathring{\phi} = \rho^* \mathring{\Phi} = \rho^{-1} d\rho . \tag{1.98}$$

The components of $\mathring{\phi}$, $\mathring{\phi}^a_b = \rho^* \mathring{\Phi}^a_b$, satisfy the same relations (1.96) and structure equations (1.97) as the components of $\mathring{\Phi}$. Since $j \circ \rho \circ i^{-1} : S^n \to G$ is a section of the bundle $\Pi : G \to S^n$, we know from Sec. 1.1.C that the 1-forms

$$i^{-1*} \mathring{\phi}^A_0 = i^{-1*} \rho^* \mathring{\Phi}^A_0 = (j \circ \rho \circ i^{-1})^* \Phi^A_0 , \quad 1 \leq A \leq n$$

are linearly independent. Therefore, $\left(\mathring{\phi}^A_0\right)_{1 \leq A \leq n}$ constitute a (local) basis of $T^* \mathbb{R}^n$. On the domain of definition of ρ we take the Riemannian metric

$$dl^2 = \sum_{A=1}^{n} \left(\mathring{\phi}^A_0\right)^2 . \tag{1.99}$$

If $\tilde{\rho} : \mathbb{R}^n \to G^*$ is another section of Π, then, in the intersection of the domains of definition of ρ and $\tilde{\rho}$, we have

$$\tilde{\rho} = \rho K , \tag{1.100}$$

where $K : \mathbb{R}^n \to G^*_0$ is a smooth map of the form

$$K = \begin{vmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{1.101}$$

with $A : \mathbb{R}^n \to SO(n)$. Thus, we get the transformation laws of the components of $\rho = [\rho_0, \rho_A, \rho_{n+1}]$, reading

$$\tilde{\rho} = [\tilde{\rho}_0, \tilde{\rho}_A, \tilde{\rho}_{n+1}] = [\rho_0, A^B_A \rho_B, \rho_{n+1}] ,$$

and of the components of $\mathring{\phi} = \begin{bmatrix} 0 & 0 & 0 \\ \mathring{\phi}^A_0 & \mathring{\phi}^A_B & 0 \\ 0 & {}^t[\mathring{\phi}^A_0] & 0 \end{bmatrix}$, viz.

$$\tilde{\mathring{\phi}} = \tilde{\rho}^* \mathring{\Phi} = \tilde{\rho}^{-1} \circ d\tilde{\rho} = K^{-1} \mathring{\phi} K + K^{-1} dK = \begin{bmatrix} 0 & 0 & 0 \\ {}^t\!A[\mathring{\phi}^A_0] & {}^t\!A[\mathring{\phi}^A_B] A + {}^t\!A dA & 0 \\ 0 & {}^t[\mathring{\phi}^A_0] A & 0 \end{bmatrix} . \tag{1.102}$$

In particular,

$$\tilde{\phi}_0^A = A_A^B \phi_0^B ,$$

whence

$$d\tilde{l}^2 = \sum_{A=1}^{n} (\tilde{\phi}_0^A)^2 = \sum_{A=1}^{n} (\phi_0^A)^2 = dl^2 .$$

Thus, the Riemannian metric defined locally in Eq. (1.99) is a global one in $\mathbb{R}^n$, such that, for any section ρ of Π, the linear frame field $X_1, \ldots, X_n$ dual to the co-frame $\tilde{\phi}_0^1, \ldots, \tilde{\phi}_0^n$ given in Eq. (1.98) is orthonormal. Moreover, due to Eq. (1.96) and the structure equations (1.97), the 1-forms $\tilde{\phi}_B^A$ satisfy

$$\begin{aligned} \tilde{\phi}_B^A &= -\tilde{\phi}_A^B \\ d\tilde{\phi}_0^A &= -\tilde{\phi}_B^A \wedge \tilde{\phi}_0^B . \end{aligned} \tag{1.103}$$

Consequently, the $\tilde{\phi}_B^A$ are the *Levi-Civita* connection forms corresponding to the co-frame $(\tilde{\phi}_0^A)_{1 \leq A \leq n}$. Since $\tilde{\phi}_B^A$ additionally has the property (from Eq. (1.97))

$$d\tilde{\phi}_B^A = -\tilde{\phi}_C^A \wedge \tilde{\phi}_B^C , \tag{1.104}$$

the above Riemannian structure on $\mathbb{R}^n$ is *flat*. In fact, the metric dl^2 is the usual Euclidean one, as we can see by taking the section $\rho = s \circ i$ with $s : S^n \setminus \{x_\infty\} \to G^n$ the map defined in Eq. (1.19). We observe also that, given a (local) right-handed orthonormal frame $X_1, \ldots, X_n$ of $\mathbb{R}^n$, there exists a section $\bar{\rho} : \mathbb{R}^n \to G^n$, such that $\tilde{\phi}_0^A(X_B) = \delta_{AB}$. This section can be chosen as $\bar{\rho} = \rho \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{vmatrix}$ with $A_B^C = (s \circ i)^* \phi_0^C (X_B)$.

Now let $F : M^m \to \mathbb{R}^n$ be an immersion of an oriented m-manifold M with $m \geq 2$.

A map $E : M \to G^n$ defined on an open set of M is called a *G^n-frame field of $\Pi : G^n \to \mathbb{R}^n$ along* F, if $\Pi \circ E = F$. For example, if $\rho : \mathbb{R}^n \to G^n$ is a section of Π, then $E = \rho \circ F$, defined on a conveniently chosen open set of M, is a G^n-frame field of Π along F. If $\tilde{E} : M \to G^n$ is another G^n-frame of Π along F, then $\tilde{E} = EK$ with $K : M \to G_0^n$ is a smooth map defined in the intersection of the two domains. Conversely, given such a map K and a G^n-frame E, then $\tilde{E} = EK$ is also a G^n-frame.

Set $f = i \circ F : M^m \to S^n$, which gives an immersion into the Möbius space S^n.

If $E : M \to G^*$ is a G^*-frame field of $\bar{\Pi}$ along F, then $e = j \circ E : M \to G$ is a zeroth-order G-frame field of $\Pi : G \to S^n$ along f. Summarising, we give the relations among G^*-frames of $\bar{\Pi}$ along F and the corresponding G-frames of Π along f in the following commutative diagram:

$$\begin{tikzcd}
 & G^* \arrow[r, "j"] \arrow[d, "\bar{\Pi}"'] & G \arrow[d, "\Pi"] \\
M \arrow[ur, "E"] \arrow[r, "F"] \arrow[urr, bend left=60, "e"] \arrow[rr, bend right=40, "f"'] & \mathbb{R}^n \arrow[r, "i"] \arrow[u, bend right, "\rho"'] & S^n \arrow[u, bend left, "s"]
\end{tikzcd} \tag{1.105}$$

As in Sec. 1.2, we are now going to construct in a neighbourhood of each point of M a more specialised G^*-frame field. With each G^*-frame field $E : M \to G^*$ of $\bar{\Pi}$ along F we associate the $\mathfrak{g}^*$-valued 1-form

$$\psi = E^*\bar{\Phi} = E^{-1}dE \tag{1.106}$$

on M, with components $\psi^i_j = E^*\bar{\Phi}^i_j$ satisfying the same relations (1.96) and structure equations (1.97) as the ones of $\bar{\Phi}$. If $\tilde{E} : M \to G^*$ is another frame of $\bar{\Pi}$ along F, then

$$\tilde{E} = EK\ , \tag{1.107}$$

where $K : M \to G^*_0$ is as in Eq. (1.101) with $A : M \to SO(n)$. Writing

$$\psi = \begin{vmatrix} 0 & 0 & 0 \\ \psi^A_0 & \psi^A_B & 0 \\ 0 & {}^t[\psi^A_0] & 0 \end{vmatrix}\ ,$$

we get the transformation

$$\begin{aligned} \tilde{\psi} &= \tilde{E}^*\bar{\Phi} = \tilde{E}^{-1}d\tilde{E} = K^{-1}\psi K + K^{-1}dK \\ &= \begin{vmatrix} 0 & 0 & 0 \\ {}^tA[\psi^A_0] & {}^tA[\psi^A_B]A + {}^tA\,dA & 0 \\ 0 & {}^t[\psi^A_0]A & 0 \end{vmatrix}\ . \end{aligned} \tag{1.108}$$

Let $E : M \to G^*$ be any G^*-frame of $\bar{\Pi}$ along F. Take $e = j \circ E : M \to G$, which is a zeroth-order frame of Π along $f = i \circ F : M \to S^n$, and consider the $\mathfrak{g}$-valued 1-form on M given by $\phi = e^*\Phi$. Then,

$$\phi = e^*\Phi = E^*(j^*\Phi) = E^*(\bar{\Phi}) = \psi\ . \tag{1.109}$$

So $\phi_0^a = \psi_0^a$ and, in particular, $(\psi_0^A)_{1\le A\le n}$ are 1-forms on M that span T^*M. Following the procedure of Sec. 1.2.B, we can find a map $K : M \to G_0^*$ of the form (1.101), defined on the domain of E, such that $\tilde{E} = EK$ satisfies

$$\tilde{\psi}_0^a = 0\,, \quad \forall m+1 \le \alpha \le n\,. \tag{1.110}$$

In particular, the $(\tilde{\psi}_0^i)_{1\le i\le m}$ span T^*M. Moreover, for any two G^*-frames $E, \tilde{E} : M \to G^*$ of Π along F that satisfy Eq. (1.110), the map $K : M \to G_0^*$ defined by $\tilde{E} = EK$, i.e. of the form (1.101), satisfies $[\tilde{\psi}_0^A] = {}^t\!A[\psi_0^A]$, as we can see from the transformation laws (1.108). Then, under the assumption that $(\psi_0^1, \dots, \psi_0^m)$ and $(\tilde{\psi}_0^1, \dots, \tilde{\psi}_0^m)$ define the same orientation on M, A is of the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix},$$

where $A_1 \in SO(m)$ and $A_4 \in SO(n-m)$ (cf. Sec. 1.2.B). In other words, K takes values in the closed subgroup of G_0^* given by

$$G_1^* = \left\{ \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \ : \ \begin{matrix} A \in SO(m) \\ B \in SO(n-m) \end{matrix} \right\}. \tag{1.111}$$

Conversely, if $E : M \to G^*$ is a G^*-frame field of Π along F which satisfies $\psi_0^a = 0$, $\forall \alpha$, and $K : M \to G_1^*$ is a map, then $\tilde{E} = EK : M \to G^*$ is a G^*-frame that also satisfies Eq. (1.110).

Definition 1.5 *A G^*-frame field $E : M \to G^*$ of Π along F with the property $\psi_0^a = 0$, $\forall m+1 \le \alpha \le n$, where ψ is given in Eq. (1.106), is called a Darboux frame.*

For a Darboux frame $E : M \to G^*$, set $\psi = E^*\hat{\Phi}$. Then, from Eq. (1.96), we have the relations

$$\psi_0^i = \psi_i^{n+1}\,, \quad \psi_B^A = -\psi_A^B\,, \quad \psi_i^a = 0 \ \text{ else}, \tag{1.112}$$

and, from the Maurer-Cartan equations (1.97) for $\hat{\Phi}$,

$$\begin{aligned} d\psi_0^i &= -\psi_j^i \wedge \psi_0^j \\ d\psi_j^i &= -\psi_k^i \wedge \psi_j^k + \hat{\Omega}_j^i\,, \end{aligned} \tag{1.113}$$

where

$$\hat{\Omega}_j^i = -\psi_a^i \wedge \psi_j^a\,. \tag{1.114}$$

Differentiating $\psi_0^\alpha = 0$, we get from the above equations

$$0 = d\psi_0^\alpha = -\psi_i^\alpha \wedge \psi_0^i - \psi_\beta^\alpha \wedge \psi_0^\beta = -\psi_i^\alpha \wedge \psi_0^i .$$

Applying Cartan's Lemma we obtain

$$\psi_i^\alpha = h_{ij}^\alpha \psi_0^j \tag{1.115}$$

with h_{ij}^α smooth functions satisfying

$$h_{ij}^\alpha = h_{ji}^\alpha . \tag{1.116}$$

These functions are called the ***coefficients of the second fundamental form*** of the immersion $F : M \to \mathbb{R}^n$, relative to the frame E.

If $\tilde{E} = EK : M \to G^n$ is another Darboux frame along F, with $K : M \to G_1^n$ of the form

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \tag{1.117}$$

then the vector components of $\tilde{E}$ transform as

$$\tilde{E} = [\tilde{E}_0, \tilde{E}_i, \tilde{E}_\alpha, \tilde{E}_{n+1}] = [E_0, A_i^j E_j, B_\alpha^\beta E_\beta, E_{n+1}] \tag{1.118}$$

and the components of $\tilde{\psi} = \tilde{E}^* \tilde{\Phi}$ as

$$\tilde{\psi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \tilde{\psi}_0^i & \tilde{\psi}_j^i & \tilde{\psi}_\alpha^i & 0 \\ 0 & \tilde{\psi}_i^\alpha & \tilde{\psi}_\beta^\alpha & 0 \\ 0 & {}^t[\tilde{\psi}_0^i] & 0 & 0 \end{bmatrix} = \tilde{E}^{-1} d\tilde{E} = K^{-1}\psi K + K^{-1} dK =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ {}^tA[\psi_0^i] & {}^tA[\psi_j^i]A + {}^tA dA & {}^tA[\psi_\alpha^i]B & 0 \\ 0 & {}^tB[\psi_i^\alpha]A & {}^tB[\psi_\beta^\alpha]B + {}^tB dB & 0 \\ 0 & {}^t[\tilde{\psi}_0^i]A & 0 & 0 \end{bmatrix} . \tag{1.119}$$

In particular, $\tilde{\psi}_0^i = A_i^j \psi_0^j$ and $\tilde{\psi}_i^\alpha = B_\alpha^\beta \psi_j^\beta A_i^j$, giving the transformation

$$\tilde{h}_{ij}^\alpha = A_i^k A_j^l B_\alpha^\beta h_{kl}^\beta .$$

Also, from Eq. (1.119), we see that

$$dl^2 = \sum_{i=1}^m (\psi_0^i)^2 \tag{1.120}$$

defines a global metric on M, and, from Eq. (1.113), that the ψ^i_j form the *Levi-Civita connection forms* on M, relative to the orthonormal co-frame $(\psi^i_0)_{1\leq i\leq m}$ and with *curvature forms* $\hat{\Omega}^i_j$.

Remark 1.4 First we note that G^*-frames $E : M \to G^*$ of Π along F of the type $E = \rho \circ F$, where $\rho : \mathbb{R}^n \to G^*$ is a section of Π, are defined in a neighbourhood of each point of M. Moreover, we can assume such a frame to be a Darboux one, which can be shown in an analogous way as in Remark 1.2. in Sec. 1.2.B. For such Darboux frames, we have $\psi = E^*\hat{\Phi} = F^*\rho^*\hat{\Phi} = F^*\hat{\phi}$. In particular,

$$dl^2 = \sum_{i=1}^{m}(\psi^i_0)^2 = \sum_{A=1}^{n}(\psi^A_0)^2 = \sum_{A=1}^{n}(F^*\hat{\phi}^A_0)^2 = F^*(dt^2)\ .$$

Thus, the metric dl^2 of M is the one induced by F from the metric dt^2 of $\mathbb{R}^n$. If we take $X_1, \ldots, X_m$ as the local orthonormal frame of (M, dl^2) dual to $\psi^1_0, \ldots, \psi^m_0$, then

$$\delta_{ij} = \psi^i_0(X_j) = F^*(\hat{\phi}^i_0(X_j)) = \hat{\phi}^i_0(dF(X_j))$$

and

$$0 = \psi^\alpha_0 = \hat{\phi}^\alpha_0(dF(\cdot))\ .$$

Let $U_1, \ldots, U_m, U_{m+1}, \ldots, U_n$ be the orthonormal frame of $(\mathbb{R}^n, dt^2)$ dual to $\hat{\phi}^1_0, \ldots, \hat{\phi}^m_0$, $\hat{\phi}^{m+1}_0, \ldots, \hat{\phi}^n_0$. Since

$$\hat{\phi}^i_0(dF_x(X_j)) = \delta_{ij}\ ,\ \hat{\phi}^i_0(U_\alpha(F(x))) = 0\ ,\ \hat{\phi}^\alpha_0(dF_x(X_j)) = 0\ ,\ \hat{\phi}^\alpha_0(U_\beta(F(x))) = \delta_{\alpha\beta}\ ,$$

we conclude that $U_i(F(x)) = dF_x(X_i)$, $\forall x \in M$, $1 \leq i \leq m$ and that $(U_\alpha \circ F)_{m+1\leq\alpha\leq n}$ is an orthonormal frame of the normal bundle to F. Then, since $\hat{\phi}^A_B(U_C) = < dU_B(U_C), U_A >_{\mathbb{R}^n}$ and $\psi^i_j(X_k) = < \nabla_{X_k} X_j, X_i >_{dl^2}$, the second fundamental form of $F : M \to \mathbb{R}^n$ is given by

$$\nabla dF = \sum_{i,j,\alpha} h^\alpha_{ij}\psi^i_0 \otimes \psi^j_0 \otimes U_\alpha \circ F$$

with $h^\alpha_{ij} = \psi^\alpha_i(X_j) = \hat{\phi}^\alpha_i(dF(X_j))$. We can easily verify that the r.h.s. of this equation defines a global tensor on M, by applying the transformation laws given in Eq. (1.119) on another Darboux frame of the type $\tilde{E} = \tilde{\rho}\circ F$, where $\tilde{\rho} : \mathbb{R}^n \to G^*$ is a section of Π.

Finally, we remark that, for such a Darboux frame $E : M \to G^*$, its vector components can be written as ($E_{n+1} \equiv 0$)

$$E_0(x) = \begin{vmatrix} 1 \\ F(x) \\ \frac{1}{2}{}^tF(x)F(x) \end{vmatrix} \; , \; E_i(x) = \begin{vmatrix} 0 \\ A_i(F(x)) \\ F^A(x)A_i^A(F(x)) \end{vmatrix} \; , \; E_\alpha(x) = \begin{vmatrix} 0 \\ A_\alpha(F(x)) \\ F^A(x)A_\alpha^A(F(x)) \end{vmatrix} \; ,$$

where $A : \mathbb{R}^n \to SO(n)$ with vector components $A_i = \begin{vmatrix} A_i^1 \\ \vdots \\ A_i^n \end{vmatrix}$ is a smooth map. Identifying $E(x) \in G^*$ with the element $\big(A(F(x)), F(x)\big) \in \mathbb{E}^+(n)$ via the isomorphism (1.93) corresponds to identifying $E_0(x)$ with $F(x)$, $E_i(x)$ with $A_i(F(x))$, and $E_\alpha(x)$ with $A_\alpha(F(x))$. Then, one can show that $E_i = A_i \circ F = dF(X_i) \in dF(TM)$ and $E_\alpha = A_\alpha \circ F = U_\alpha \circ F$ give rise to orthonormal frames of $dF(TM)$ and its normal bundle, respectively.

In order to simplify the relations that can be derived between the Riemannian geometry of submanifolds of $\mathbb{R}^n$ and their conformal geometry, when considered as submanifolds of S^n, we are going to show how a second-order G-frame field of $\Pi : G \to S^n$ along $f = i \circ F : M \to S^n$ can be constructed from a Darboux frame of $\Pi : G^* \to \mathbb{R}^n$ along $F : M \to \mathbb{R}^n$.
Let $E : M \to G^*$ be a Darboux frame of Π along F and e be the zeroth-order G-frame $e = j \circ E : M \to G$. Then, the vector components e_a of e are identical to the E_a of E and, with the usual notation

$$\psi = E^*\bar{\Phi} \;\; , \;\; \phi = e^*\Phi \; ,$$

Eq. (1.109) holds. In particular, $\phi_0^\alpha = \psi_0^\alpha = 0$, that is, e is a first-order G-frame field of Π along f. Note that the map $f : (M, dl^2) \to S^n$ is conformal, i.e. $dl^2 = \sum_{i=1}^m (\psi_0^i)^2 = \sum_{A=1}^n (\psi_0^A)^2 = \sum_{A=1}^n (E^*\bar{\Phi}_0^A)^2 = \sum_{A=1}^n ((j \circ E)^*\Phi_0^A)^2 = \sum_{A=1}^n (e^*\Phi_0^A)^2$ is an element of the conformal class of metrics of M induced by the one of S^n. Since e is of first order, we have, as in Eqs. (1.46),(1.47),

$$\phi_i^\alpha = h_{ij}^\alpha \phi_0^j \; .$$

As

$$\psi_i^\alpha = \phi_i^\alpha = h_{ij}^\alpha \phi_0^j = h_{ij}^\alpha \psi_0^j \; ,$$

we get, comparing with Eq. (1.115), $h^\alpha_{ij} = \tilde{h}^\alpha_{ij}$, i.e. h^α_{ij} are the coefficients of the second fundamental form of F, relative to the frame E. The Gauss equation (see e.g. Ref. [Ko-No/63]) yields that (M, dl^2) has Riemannian curvature tensor

$$R_{ijkl} = h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk} \tag{1.121}$$

with scalar curvature

$$R = 2\sum_{i<j} R_{ijij} = 2\sum_{i<j,\alpha}\{h^\alpha_{ii}h^\alpha_{jj} - (h^\alpha_{ij})^2\}\ .$$

The mean curvature has coefficients

$$H^\alpha = \frac{1}{m}h^\alpha_{kk}\ , \tag{1.122}$$

that is, if E is a Darboux frame like in Remark 1.4, then $H = H^\alpha U_\alpha$. Let $\tilde{e} = eK : M \to G$ with K as in Eqs. (1.51),(1.52), which is a second-order G-frame of Π along f, as shown in Sec. 1.2.C. Then, we get the transformation laws

$$\tilde{e} = [\tilde{e}_0, \tilde{e}_i, \tilde{e}_\alpha, \tilde{e}_{n+1}] = [E_0, E_i, E_\alpha + H^\alpha E_0, \frac{1}{2}H^\alpha H^\alpha E_0 + H^\alpha E_\alpha + E_{n+1}]$$

and

$$\begin{aligned}\tilde{\phi} = \tilde{e}^*\phi \ &= \begin{bmatrix} \tilde{\phi}^0_0 & \tilde{\phi}^0_j & \tilde{\phi}^0_\alpha \\ \tilde{\phi}^i_0 & \tilde{\phi}^i_j & \tilde{\phi}^i_\alpha \\ 0 & \tilde{\phi}^\alpha_j & \tilde{\phi}^\alpha_\beta \end{bmatrix} \\ &= \begin{bmatrix} 0 & -{}^tY[\psi^\alpha_i] + \frac{1}{2}{}^tYY^t[\psi^i_0] & d({}^tY) - {}^tY[\psi^\alpha_\beta] \\ \psi^i_0 & \psi^i_j & [\psi^i_0]{}^tY + [\psi^i_\alpha] \\ 0 & [\psi^\alpha_i] - Y^t[\psi^i_0] & \psi^\alpha_\beta \end{bmatrix}.\end{aligned}$$

In components, we have

$$\begin{aligned} \tilde{\phi}^0_0 &= 0 \\ \tilde{\phi}^i_0 &= \psi^i_0 \\ \tilde{\phi}^0_i &= \tfrac{1}{m}h^\alpha_{kk}\left(\tfrac{1}{2m}h^\alpha_{ll}\delta_{ij} - h^\alpha_{ij}\right)\psi^j_0 \\ \tilde{\phi}^0_\alpha &= d\left(\tfrac{1}{m}h^\alpha_{kk}\right) - \tfrac{1}{m}h^\beta_{kk}\psi^\beta_\alpha \\ \tilde{\phi}^\alpha_0 &= \psi^\alpha_0 = 0 \\ \tilde{\phi}^i_j &= \psi^i_j \\ \tilde{\phi}^\alpha_i &= \left(h^\alpha_{ij} - \tfrac{1}{m}h^\alpha_{kk}\delta_{ij}\right)\psi^j_0 \\ \tilde{\phi}^\alpha_\beta &= \psi^\alpha_\beta\ . \end{aligned} \tag{1.123}$$

We note that the coefficients $\tilde{h}^\alpha_{ij}$ relative to the second-order frame $\tilde{e}$ are not the ones of the second fundamental form of F, because $\tilde{e}$ is in general not a frame of the Riemannian structure (it may not take values in G^*). Nevertheless, from the above transformation laws we obtain a relation between these coefficients and the h^α_{ij} of the second fundamental form for the Darboux frame E, reading

$$\tilde{h}^\alpha_{ij} = -\frac{1}{m} h^\alpha_{kk}\delta_{ij} + h^\alpha_{ij} = -\delta_{ij}H^\alpha + h^\alpha_{ij} \,. \tag{1.124}$$

The Weyl tensor of (M^m, dl^2), which is invariant under conformal changes of the metric dl^2, has components (for $m > 2$)

$$C_{ijkl} = R_{ijkl} + \frac{1}{m-2}(-\delta_{ik}R_{jl} - \delta_{jl}R_{ik} + \delta_{il}R_{jk} + \delta_{jk}R_{il}) + \frac{R}{(m-1)(m-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \,,$$

where $R_{jk} = R_{ijik}$ are the components of the Ricci tensor. From Eq. (1.124) we deduce that the components $\tilde{\mathcal{T}}^{i}_{jkl}$ of the generalised Weyl tensor $\mathcal{T}$ given by Eq. (1.84), in the second-order frame $\tilde{e}$ along f, are related to the C_{ijkl} through the formula

$$\begin{aligned} C_{ijkl} = \tilde{\mathcal{T}}^{i}_{jkl} \; &+ \; \frac{1}{m-2}\big(- \delta_{il}\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{tk} + \delta_{jl}\tilde{h}^\alpha_{ti}\tilde{h}^\alpha_{tk} - \delta_{jk}\tilde{h}^\alpha_{ti}\tilde{h}^\alpha_{tl} + \delta_{ik}\tilde{h}^\alpha_{tj}\tilde{h}^\alpha_{tl}\big) + \\ &+ \; \frac{1}{(m-1)(m-2)}(\delta_{kj}\delta_{il} - \delta_{ik}\delta_{jl})\tilde{h}^\alpha_{tr}\tilde{h}^\alpha_{tr} \,. \end{aligned}$$

If $\tilde{h}^\alpha_{ij} = 0$, $\forall i, j$, then, by Eq. (1.124),

$$h^\alpha_{ij} = \frac{1}{m} h^\alpha_{kk}\delta_{ij} \,, \quad \forall i, j \,.$$

Thus, the second fundamental form of F has components of the form

$$(\nabla dF)^\alpha = \lambda^\alpha dl^2 \,,$$

where $\lambda^\alpha = \frac{1}{m}h^\alpha_{kk}$. In other words, $F : M \to \mathbb{R}^n$ is a so-called *totally umbilic* immersion.

Supposing that E is a Darboux frame of the type $E = \rho \circ F$ with $\rho : \mathbb{R}^n \to G^*$ a section of Π, and denoting $\mathring{\phi} = \rho^* \mathring{\Phi}$, we have

$$\psi^\alpha_i = E^* \mathring{\Phi}^\alpha_i = F^* \rho^* \mathring{\Phi}^\alpha_i = F^* \mathring{\phi}^\alpha_i \,.$$

Let $X_1, \ldots, X_m$ be the orthonormal frame of (M, dl^2) dual to the forms $\psi^1_0, \ldots, \psi^m_0$ and $U_{m+1}, \ldots, U_n$ be the orthonormal frame of the normal bundle to F dual to

$\tilde\phi_0^{m+1}, \ldots, \tilde\phi_0^n$ (cf. Remark 1.4). Let ∇ denote the Levi-Civita connection of (M, dl^2) and $\nabla^\perp$ the connection of the normal bundle V. These can be related to the Riemannian connection forms ψ_b^a and the corresponding conformal ones $\tilde\phi_b^a$ given in Eqs. (1.123), relative to the second-order frame $\tilde e$, as follows:

$$\begin{aligned}
\psi_j^i(X_k) &= \left\langle \nabla_{X_k} X_j, X_i \right\rangle_{dl^2} \\
\psi_j^\alpha(X_k) &= h_{kj}^\alpha = \left\langle \nabla dF(X_k, X_j), U_\alpha \right\rangle_{dl^2} \\
\langle H, U_\alpha\rangle_{dl^2} &= \frac{1}{m} h_{kk}^\alpha \;, \; \|H\|^2 = \frac{1}{m^2}\sum_\alpha \big(\sum_k h_{kk}^\alpha\big)^2 \\
\psi_\alpha^\beta(X_k) &= \left\langle \nabla_{X_k}^\perp U_\alpha, U_\beta \right\rangle_{dl^2} \qquad (1.125)\\
\tilde\phi_k^0(X_i) &= -\left\langle H, \nabla dF(X_i, X_k) \right\rangle_{dl^2} + \frac{1}{2}\delta_{ik}\|H\|^2 \\
\tilde h_{ij}^\alpha &= \left\langle -\delta_{ij} H + \nabla dF(X_i, X_j), U_\alpha \right\rangle_{dl^2} \\
\operatorname{trace}\tilde N &= \tilde h_{ij}^\alpha \tilde h_{ij}^\alpha = \|\nabla dF\|^2 - m\|H\|^2 .
\end{aligned}$$

Applying Eq. (1.54) to $\tilde e$, we obtain

$$\tilde h_{ijk}^\alpha = \left\langle \nabla_{X_k} \nabla dF(X_i, X_j), U_\alpha \right\rangle_{dl^2} \qquad (1.126)$$

with $\nabla\nabla dF$ the covariant derivative in $\odot^2 T^*M \otimes V$. Further, $\tilde\phi_\alpha^0 = \tilde p_k^\alpha \tilde\phi_0^k$ yields

$$\tilde p_k^\alpha = \tilde\phi_\alpha^0(X_k) = d(\langle H, U_\alpha\rangle_{dl^2})(X_k) - \langle H, U_\beta\rangle_{dl^2}\left\langle \nabla_{X_k}^\perp U_\alpha, U_\beta\right\rangle_{dl^2} = \left\langle \nabla_{X_k}^\perp H, U_\alpha\right\rangle_{dl^2} ,$$

that is,

$$\nabla^\perp H = \tilde p_k^\alpha \psi_0^k \otimes U_\alpha . \qquad (1.127)$$

Using Eq. (1.60) we get

$$\begin{aligned}
\tilde p_{ik}^\alpha \;=\; \Big\langle &\nabla^{\perp 2} H(X_k, X_i) - \left\langle H, \nabla dF(X_k, X_i)\right\rangle_{dl^2} H + \frac{1}{2}\delta_{ik}\|H\|^2 H + \\
&+ \left\langle H, \nabla dF(X_k, X_r)\right\rangle_{dl^2} \nabla dF(X_i, X_r) - \frac{1}{2}\|H\|^2 \nabla dF(X_i, X_k) \;,\; U_\alpha \Big\rangle_{dl^2} .
\end{aligned} \qquad (1.128)$$

Taking the trace of this expression yields

$$\tilde p_{ii}^\alpha = \left\langle \Delta H - m\|H\|^2 H + \tilde A(H), U_\alpha \right\rangle_{dl^2} \qquad (1.129)$$

with $\tilde A(H) = \left\langle H, \nabla dF(X_i, X_r)\right\rangle_{dl^2} \nabla dF(X_i, X_r)$. Finally, from Eq. (1.62), we have

$$\begin{aligned}
\tilde H_{kr} \;=\; &(\|\nabla dF\|^2 - m\|H\|^2)\big(- \left\langle H, \nabla dF(X_k, X_r)\right\rangle_{dl^2} + \frac{1}{2}\delta_{kr}\|H\|^2 \big) \\
&- m\left\langle \nabla_{X_r}^\perp H, \nabla_{X_k}^\perp H\right\rangle_{dl^2} - m\left\langle H, \nabla^2 H(X_r, X_k)\right\rangle_{dl^2} \\
&+ \left\langle \nabla_{X_r}\nabla dF, \nabla_{X_k}\nabla dF\right\rangle + \left\langle \nabla dF, \nabla^2\nabla dF(X_r, X_k)\right\rangle ,
\end{aligned} \qquad (1.130)$$

where $\nabla^2(W)(X,Y) = \nabla_X \nabla_Y W - \nabla_{\nabla_X Y} W$**, in a generic sense.**

Appendix I

We observe that the embedding κ^{-1} of S^n into $\mathbb{P}^{n+1}$, which defines the Möbius space given in Def. 1.1, is not the most standard way of embedding S^n into the projective space. Here we followed Ref. [Po/81], but e.g. in Ref. [Ko-No/63], page 311, the authors chose the embedding

$$\begin{array}{cccc} \mathcal{E}: & S^n & \longrightarrow & \mathcal{L}^{+}_{/\sim} \subset \mathbb{P}^{n+1} \ , \\ & (y, y^{n+1}) & \rightarrow & \begin{bmatrix} \frac{1-y^{n+1}}{\sqrt{2}} \\ y \\ \frac{1+y^{n+1}}{\sqrt{2}} \end{bmatrix}_{\sim} \end{array}$$

which is an isometry, S^n being considered with the metric induced by $\mathbb{R}^{n+1}$ and $\mathbb{P}^{n+1}$ with the metric $2ds^2$ given by

$$p^* ds^2 = \frac{\left(\sum_{i=0}^{n+1} x_i^2\right)\left(\sum_{i=0}^{n+1} dx_i^2\right) - \left(\sum_{i=0}^{n+1} x_i dx_i\right)^2}{\left(\sum_{i=0}^{n+1} x_i^2\right)^2} \ ,$$

where $p : \mathbb{R}^{n+2}\backslash\{0\} \rightarrow \mathbb{P}^{n+1}$ is the canonic projection. In this sense, our map κ^{-1} is not an isometry, but a conformal map: κ^{-1} can be obtained from $\mathcal{E}$ by the formula $\kappa^{-1} = \mathcal{E} \circ R^{-1} \circ \sigma^{-1} \circ \frac{1}{\sqrt{2}}\mathrm{id}_{\mathbb{R}^n} \circ \sigma$, where $R : S^n \rightarrow S^n$ is the rotation $R(y, y^{n+1}) = (y^{n+1}, y)$. So we have $\kappa^{-1*}{}_{(z^0,z)} ds^2_{\mathbb{P}^{n+1}} = \frac{1}{1-z^0} <,>_{S^n}$. Clearly, the conformal structures on S^n by choosing either of these two conformally equivalent embeddings are equal. If, instead of the submersion F on page 83, we had chosen the map $\bar{F}\left(\begin{vmatrix} c \\ v \\ e \end{vmatrix}\right) = \left(\frac{e-c}{e+c}, \frac{\sqrt{2}v}{e+c}\right)$ we would have obtained

$$\begin{array}{cccc} \tilde{\kappa} = \bar{F}_{/\sim}: & \mathcal{L}^{+}_{/\sim} & \longrightarrow & S^n \\ & \begin{bmatrix} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{bmatrix}_{\sim} & \rightarrow & \left(\frac{\left\|\frac{\omega}{\sqrt{2}}\right\|^2-1}{\left\|\frac{\omega}{\sqrt{2}}\right\|^2+1}, \frac{\sqrt{2}\omega}{\left\|\frac{\omega}{\sqrt{2}}\right\|^2+1}\right) \\ & \begin{bmatrix} \frac{1}{2}\|\omega\|^2 \\ \omega \\ 1 \end{bmatrix}_{\sim} & \rightarrow & \left(\frac{1-\left\|\frac{\omega}{\sqrt{2}}\right\|^2}{1+\left\|\frac{\omega}{\sqrt{2}}\right\|^2}, \frac{\sqrt{2}\omega}{1+\left\|\frac{\omega}{\sqrt{2}}\right\|^2}\right) \end{array}$$

with inverse

$$\begin{array}{cccc} \tilde{\kappa}^{-1}: & S^n & \longrightarrow & \mathcal{L}^{+}_{/\sim} \ . \\ & (z^0, z) & \rightarrow & \begin{bmatrix} \frac{1-z^0}{\sqrt{2}} \\ z \\ \frac{1+z^0}{\sqrt{2}} \end{bmatrix}_{\sim} \end{array}$$

$\bar{\kappa}^{-1}$ differs from $\mathcal{E}$ by the rotation R and we have

$$
\begin{array}{cccc}
\sigma \circ \bar{\kappa} : & \mathcal{L}^{+}_{/\sim} & \longrightarrow & \mathbb{R}^n \cup \{\infty\} \ . \\
& \begin{bmatrix} 1 \\ \omega \\ \frac{1}{2}\|\omega\|^2 \end{bmatrix}_{\sim} & \rightarrow & \frac{\omega}{\sqrt{2}} \\
& \mathbf{x}_{\infty} & \rightarrow & \infty
\end{array}
$$

We note that, if we had chosen the map $\bar{\kappa}$ instead of κ, then Eq. (1.15) would not hold anymore, unless we had replaced "id" by "$T \circ \mathrm{id}$" with $T(Z, A, r) = (\frac{Z}{\sqrt{2}}, A, r)$.

We also remark that, if we had chosen the embedding $\bar{\kappa}^{-1}$, the Killing vector fields of S^n would be generated by

$$
\left\{ \begin{bmatrix} 0 & -{}^t v & 0 \\ v & D & -v \\ 0 & {}^t v & 0 \end{bmatrix} \ : \ D \in \mathfrak{O}(n),\ v \in \mathbb{R}^n \right\}.
$$

Chapter 2

VARIATIONAL PROBLEMS IN CONFORMAL GEOMETRY

2.1 Introduction: The Willmore Functional

2.1.A The Riemannian Case

Let M^2 be a closed (i.e. compact and oriented) surface and $f : M \to \mathbb{R}^3$ be an embedding into the Euclidean 3-space. In 1965 Willmore [Wi/65] introduced the functional, since then called *Willmore functional*,

$$\overline{\mathcal{W}}(f) = \int_M H^2 dA\,, \tag{2.1}$$

where H is the scalar mean curvature of f and dA is the volume element of M with metric induced by f. Then, he posed the problem of finding $\inf_f \overline{\mathcal{W}}$, where f ranges over all embeddings of M. Moreover, he also proved [Wi/68] that $\overline{\mathcal{W}}(f) \geq 4\pi$, with equality, iff M^2 is embedded as the standard sphere (see also Ref. [Wi/74]). In 1973 White [Wh/73] pointed out that Blaschke [Bl/29] had observed that, for any immersed surface M^2 of $\mathbb{R}^3$, the quantity $(H^2 - K)$,with K the Gaussian curvature,is invariant under any conformal mapping of the Euclidean 3-space plus the point at infinity. Hence, the integral (also called Willmore functional)

$$\mathcal{W}(f) = \int_M (H^2 - K) dA \tag{2.2}$$

is a conformal invariant. Supposing again that M is closed, then, from the Gauss-Bonnet theorem

$$\int_M K dA = 2\pi\chi(M)$$

with $\chi(M)$ the, topologically invariant, Euler characteristic of M, one obtains

$$\overline{\mathcal{W}}(f) = \mathcal{W} + 2\pi\chi(M)\,. \tag{2.3}$$

Thus, $\overline{W}(f)$ is also conformally invariant, only differing from W by a constant.
If e_1, e_2 is an orthonormal basis of T_xM, $x \in M$, and ν is a unit normal to $df_x(T_xM)$, then, denoting $h_{ij} = \left\langle \nabla df_x(e_i, e_j), \nu \right\rangle_{\mathbb{R}^3}$, we have $H_x = \frac{1}{2}(h_{11} + h_{22})$. From the Gauss equation, we get $K_x = R^M_x(e_1, e_2, e_1, e_2) = h_{11}h_{22} - h_{12}^2 = \det|h_{ij}|$. So,

$$H^2 - K = \frac{1}{4}(h_{11} - h_{22})^2 + h_{12}^2 = \frac{1}{2}\|\nabla df\|^2 - H^2 \,. \tag{2.4}$$

Hence, $H_x^2 - K_x \geq 0$, with equality, iff f is umbilic (see Sec. 1.3) at the point x. Now, it is well-known that, if f is a totally umbilic surface, $f(M)$ is either a part of a plane or a sphere. Since M is closed, $W(f) \geq 0$, with equality, iff $M^2 = S^2$ and f is totally umbilic.

In order to find some possible minima of the functional (2.1) or (2.2) for M^2 a fixed closed surface, one can work out the corresponding variational problem. An immersion $f : M^2 \to \mathbb{R}^3$ is said to be a critical point of $\overline{W}$, if, for any smooth variation $(f_t)_{t\in(-\epsilon,\epsilon)}$ of $f_0 = f$ through immersions, $\frac{\partial}{\partial t}\overline{W}(f_t)\big|_{t=0} = 0$. As a consequence of a more general result of his, Chen [Ch/73a] concluded that f is a critical point of $\overline{W}$, iff

$$\Delta H + 2H(H^2 - K) = 0 \,. \tag{2.5}$$

This equation is the Euler-Lagrange equation for the functional $\overline{W}$ and is invariant under conformal mappings of the Euclidean 3-space. Obviously, the critical points of $\overline{W}$ are identical to the ones of W and they satisfy the same Euler-Lagrange equation (2.5). The functional W has as absolute minimum the value zero, if $M = S^2$ and $f : S^2 \to \mathbb{R}^3$ is totally umbilic. In this case, $\overline{W}(f) = 4\pi$. Willmore also showed that, if M^2 is a torus, Eq. (2.5) is satisfied for an embedding of M into $\mathbb{R}^3$ the image of which is an anchor ring generated by revolving a circle of radius r about the line with distance $\sqrt{2}r$ from its centre, i.e. the torus

$$\left\{ \left((\sqrt{2}r + r\cos u)\cos v, (\sqrt{2}r + r\cos u)\sin v, r\sin u \right) \,:\, u, v \in \mathbb{R} \right\} \,.$$

For such a torus, $W(f) = \overline{W}(f) = 2\pi^2$. However, it is not yet known whether such an immersion is an absolute minimum among all immersions of the torus, only that, if $f(M)$ is a smooth surface of revolution, then $W(f) \geq 2\pi^2$, with equality, iff $f(M)$ is the above anchor ring, as shown by Willmore in Ref. [Wi/72]. It had been conjectured by Willmore [Wi/65] and, *a fortiori*, by Shiohama-Takagi [Sh-Ta/70] that the special anchor rings are the only unknotted tori in $\mathbb{R}^3$ that satisfy Eq.

(2.5), but this turned out to be false due to the above remark of White concerning the work of Blaschke. Since $\mathcal{W}(f)$ is a conformal invariant and the inversions $\mathrm{Inv} : x \in \mathbb{R}^3 \to c^2 \frac{x}{|x|^2}$ are conformal transformations of $\mathbb{R}^3 \cup \{\infty\}$, if $f(M)$ is an anchor ring, then $\mathrm{Inv} \circ f$ also satisfies Eq. (2.5), which gives rise to a special class of tori, called cyclides of Dupin. Later, the above conjecture was modified, claiming that the surfaces of $\mathbb{R}^3$ which differ from these special anchor rings by a conformal transformation of $\mathbb{R}^3 \cup \{\infty\}$ minimise $\mathcal{W}$ among all immersions of the torus into $\mathbb{R}^3$. Weiner [We/78], using a result of Lawson, showed that there exist embeddings of closed surfaces in $\mathbb{R}^3$ with arbitrary genus satisfying Eq. (2.5). In fact, these are images of embedded minimal surfaces in S^3 under stereographic projection onto $\mathbb{R}^3$. Note that this contrasts with the fact that there are no closed minimal surfaces in $\mathbb{R}^3$. The functionals (2.1) and (2.2) can be defined in the same way for immersions $f : M^2 \to \mathbb{R}^n$ of a surface into the Euclidean n-space, where H^2 now denotes the square of the norm of the vector mean curvature H. Chen [Ch/73b] proved the conformal invariance of $(\|H\|^2 - K)dA$ under conformal mappings of $\mathbb{R}^n$ and, moreover, that in the case of M being a closed surface Eq. (2.3) still holds. Then, the functional (2.1) is also conformally invariant. Later he proved [Ch/74] that, for M a closed surface, $\int_M (\|H\|^2 - K)dA \geq 2\pi(2 - \chi(M))$, with equality, iff M is diffeomorphic to a 2-sphere and $f : M \to \mathbb{R}^n$ is totally umbilic. Furthermore, if $n = 4$ and M has non-positive Gauss curvature, then $\int_M \|H\|^2 dA \geq 2\pi^2$, and if $\|H\|^2$ is constant, then equality holds, iff M is the Clifford torus $S^1 \times S^1$. Finally, Weiner generalised the definition of Willmore functional for immersions of surfaces M^2 into a Riemannian n-dimensional manifold (N^n, h) in the following way:
Let $f : M^2 \to (N^n, h)$ be an immersion of a surface with or without boundary and let $G : M \to \mathbb{R}$ be the map given by

$$G_x = \left\langle \nabla df_x(e_1, e_1), \nabla df_x(e_2, e_2) \right\rangle_h - \left\langle \nabla df_x(e_1, e_2), \nabla df_x(e_1, e_2) \right\rangle_h \tag{2.6}$$

with e_1, e_2 an orthonormal basis of (T_xM, f^*h). Chen [Ch/74] called G the "extrinsic scalar curvature" of M and proved that $(\|H\|_h^2 - G)dA$ is invariant under conformal changes of the metric h. By the Gauss equation,

$$K_x = R_x^M(e_1, e_2, e_1, e_2) = G_x + R_{f(x)}^N(df_x(e_1), df_x(e_2), df_x(e_1), df_x(e_2)) , \tag{2.7}$$

that is,

$$K_x = G_x + \overline{K}_{f(x)}$$

with $\overline{K}_{f(x)}$ the sectional curvature of the plane $df_x(T_xM)$ of $T_{f(x)}N$. Then, the functional integral

$$\mathcal{W}(f) = \int_M (\|H\|_h^2 - K + \overline{K}_f)dA \tag{2.8}$$

is conformally invariant. Since $\int_M KdA + \int_{\partial M} \kappa_g ds$, with κ_g the signed geodesic curvature of ∂M, is a topological invariant, the functional integral

$$\overline{\mathcal{W}}(f) = \int_M (\|H\|_h^2 + \overline{K}_f)dA + \int_{\partial M} \kappa_g ds \tag{2.9}$$

is also invariant under conformal changes of the metric h.

In particular, if $N = S^n$, $\sigma : S^n \setminus \{\text{point}\} \to I\!R^n$ is a stereographic projection, and $f : M \to I\!R^n$ is an immersion, then $\mathcal{W}(f) = \mathcal{W}(\sigma^{-1} \circ f)$ and the same holds for the functional $\overline{\mathcal{W}}$. If M is a closed surface, then, by the Gauss-Bonnet theorem, we have $\overline{\mathcal{W}}(f) = \mathcal{W}(f) + 2\pi\chi(M)$. Weiner showed that, if (N, h) has constant sectional curvature and $\overline{\mathcal{W}}(f) < \infty$, then f is a critical point of $\overline{\mathcal{W}}$, iff

$$\Delta\, H - 2\|H\|^2 H + \tilde{A}(H) = 0 \tag{2.10}$$

and

$$H - \kappa^V = 0 \quad \text{on } \partial M\,, \tag{2.11}$$

where κ^V is the normal component of the principal curvature vector of ∂M in N and $\tilde{A}$ is the section of $\otimes V^* \otimes V$, with V the normal bundle to f, given by

$$\tilde{A}_x(U) = \sum_{i,j=1}^{2} \left\langle \nabla df_x(e_i, e_j), U_x \right\rangle_h \nabla df_x(e_i, e_j)\,, \quad \forall U \in V_x\,, \tag{2.12}$$

with e_1, e_2 an orthonormal basis of (T_xM, f^*h_x). Observe that, if M is a closed surface and $N = I\!R^3$, then Eq. (2.10) is equivalent to

$$\Delta H - 2H^3 + \|\nabla df\|^2 H = 0\,,$$

where now H stands for scalar mean curvature. Furthermore, using Eq. (2.4), we obtain that Eq. (2.10) is equivalent to Eq. (2.5).

2.1.B Conformal Interpretation with Further Generalisations

Bryant [Br/84] was the first to study the Willmore functional for immersed surfaces in $I\!R^3$ using the conformal invariance from the outset, by interpreting it as a

functional acting on immersed surfaces of the Möbius space. More precisely, let $f : M^2 \to S^3$ be a smooth immersion of an oriented surface and Ω_f be the 2-form on M given by

$$\Omega_f = \frac{1}{2}\mathrm{trace}\,\mathcal{N}\,dV = \frac{1}{2}\mathcal{N}_{ij}\phi_0^1 \wedge \phi_0^2 , \tag{2.13}$$

where ϕ_0^i and $\mathcal{N}_{ij}$ are given by Eqs. (1.35) resp. (1.88) relative to a second-order G-frame field $e : M \to G$ of $\Pi : G \to S^3$ along f. The 2-form Ω_f is the corresponding 2-form $(H^2 - K)dA$ in the Riemannian geometry of surfaces in $\mathbb{R}^3$. Explicitly, if f takes values in $\mathbb{R}^3 \subset S^3$ (in the sense of Sec. 1.3) and $E : M \to G^*$ is a Darboux frame of $\overline{\Pi} : G^* \to \mathbb{R}^3$ along f, then, as follows from Eqs. (1.121), (1.122), $(H^2 - K)dA$ is in this frame locally written as

$$\begin{aligned} H^2 - K &= \frac{1}{4}(\bar{h}_{11}^3 + \bar{h}_{22}^3)^2 - R_{1212} \\ &= \frac{1}{4}(\bar{h}_{11}^3 + \bar{h}_{22}^3)^2 - (\bar{h}_{11}^3\bar{h}_{22}^3 - (\bar{h}_{12}^3)^2) \\ &= \frac{1}{4}(\bar{h}_{11}^3 - \bar{h}_{22}^3)^2 + (\bar{h}_{12}^3)^2 , \end{aligned}$$

where $\bar{h}_{ij}^3$ are the coefficients of the second fundamental form given in Eqs. (1.115), (1.116). From the Darboux frame E one constructs a second-order G-frame $\tilde{e} : M \to G$ of Π which is related with E in the same way as in Sec. 1.3, yielding, through Eqs. (1.124), (1.123),

$$\begin{aligned} H^2 - K &= \frac{1}{4}(\tilde{h}_{11}^3 - \tilde{h}_{22}^3)^2 + (\tilde{h}_{12}^3)^2 = (\tilde{h}_{11}^3)^2 + (\tilde{h}_{12}^3)^2 \\ &= \frac{1}{2}((\tilde{h}_{11}^3)^2 + (\tilde{h}_{22}^3)^2 + 2(\tilde{h}_{12}^3)^2) = \frac{1}{2}\mathrm{trace}\,\tilde{\mathcal{N}} \end{aligned}$$

and

$$dA = \psi_0^1 \wedge \psi_0^2 = \tilde{\phi}_0^1 \wedge \tilde{\phi}_0^2 = d\tilde{V} .$$

Thus, $\Omega_f = \frac{1}{2}\mathrm{trace}\,\tilde{\mathcal{N}}d\tilde{V} = (H^2 - K)dA$, when written in these frames. Given a compact domain $D \subset M^2$, consider the functional

$$\mathcal{W}_D(f) = \int_D \Omega_f \tag{2.14}$$

acting on immersions $f : M^2 \to S^3$. Such an immersion is said to be a Willmore immersed surface of the Möbius space S^3, if, for any compact domain D and smooth variation $f_t : M \to S^3$ of f through immersions with compact support in D, we have

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\big|_{t=0} = 0 .$$

Bryant calculated the Euler-Lagrange equation for this variational problem, obtaining

$$(p^2_{11} + p^2_{22})dV = 0\,, \tag{2.15}$$

which is conformally invariant, as we can see from the transformation laws (1.72) and (1.79) for second-order G-frames along f. Moreover, we see from relation (1.129) that the above Euler-Lagrange equation represents, in the Riemannian geometry of M as a surface of $\mathbb{R}^3$, the Euler-Lagrange equation (2.10).

This variational problem suggested to Rigoli [Ri/87] a natural way of extending the concept of Willmore surfaces to submanifolds of the Möbius space S^n, as we describe now. Let $f : M^m \to S^n$ be an immersion of an oriented m-dimensional manifold. Then, one can define on M a global m-form

$$\Omega_f = \frac{1}{m}(\mathrm{trace}\,\mathcal{N})^{\frac{m}{2}} dV\,, \tag{2.16}$$

where $\mathcal{N}$ and dV are as in Eqs. (1.88) resp. (1.44), as one can see from the transformation laws for second-order frames. On a domain of a second-order G-frame $e : M \to G$ along f, Ω_f takes the expression

$$\Omega_f = \frac{1}{m}\Big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m}{2}} \phi^1_0 \wedge \ldots \wedge \phi^m_0\,, \tag{2.17}$$

where ϕ^i_0 and h^α_{ij} are given in Eqs. (1.35) resp. (1.46). If f takes values on $\mathbb{R}^n \subset S^n$, the m-form Ω_f has the following interpretation: let $E : M \to G^*$ be a Darboux frame of $\Pi : G^* \to \mathbb{R}^n$ along $f : M \to \mathbb{R}^n$ and $\bar{e} : M \to G$ be the corresponding second-order frame given in Sec. 1.3. Then, using Eq. (1.124), we have

$$\begin{aligned}
\mathrm{trace}\,\tilde{\mathcal{N}} &= \tilde{\mathcal{N}}_{jj} = \sum_{i,j,\alpha}(\tilde{h}^\alpha_{ij})^2 = \frac{1}{m}\left\{m\sum_{i\neq j,\alpha}(\tilde{h}^\alpha_{ij})^2 + m\sum_{i,\alpha}(\tilde{h}^\alpha_{ii})^2\right\}\\
&= \frac{1}{m}\left\{m\sum_{i\neq j,\alpha}(\tilde{h}^\alpha_{ij})^2 + (m-1)\sum_{i,\alpha}(\tilde{h}^\alpha_{ii})^2 + \sum_{j,\alpha}(\tilde{h}^\alpha_{jj})^2\right\}\\
&= \frac{1}{m}\left\{m\sum_{i\neq j,\alpha}(\tilde{h}^\alpha_{ij})^2 + (m-1)\sum_{i,\alpha}(\tilde{h}^\alpha_{ii})^2 - \sum_{j,\alpha}\Big(\sum_{i\neq j}\tilde{h}^\alpha_{ii}\Big)\tilde{h}^\alpha_{jj}\right\}\\
&= \frac{1}{m}\left\{2m\sum_{i<j,\alpha}(\tilde{h}^\alpha_{ij})^2 + \frac{1}{2}(m-1)\sum_{i,\alpha}(\tilde{h}^\alpha_{ii})^2 - \sum_{i\neq j,\alpha}\tilde{h}^\alpha_{ii}\tilde{h}^\alpha_{jj} + \frac{m-1}{2}\sum_{j,\alpha}(\tilde{h}^\alpha_{jj})^2\right\}\\
&= \frac{1}{m}\left\{2m\sum_{i<j,\alpha}(\tilde{h}^\alpha_{ij})^2 + \frac{1}{2}\sum_{i\neq j,\alpha}\left((\tilde{h}^\alpha_{ii})^2 - 2\tilde{h}^\alpha_{ii}\tilde{h}^\alpha_{jj} + (\tilde{h}^\alpha_{jj})^2\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m}\left\{2m\sum_{i<j,\alpha}(\tilde h^\alpha_{ij})^2 + \frac{1}{2}\sum_{i\neq j,\alpha}(\tilde h^\alpha_{ii}-\tilde h^\alpha_{jj})^2\right\}\\
&= \frac{1}{m}\left\{2m\sum_{i<j,\alpha}(\tilde h^\alpha_{ij})^2 + \sum_{i<j,\alpha}(\tilde h^\alpha_{ii}-\tilde h^\alpha_{jj})^2\right\}\\
&= \frac{1}{m}\left\{2m\sum_{i<j,\alpha}(h^\alpha_{ij})^2 + \sum_{i<j,\alpha}(h^\alpha_{ii}-h^\alpha_{jj})^2\right\},
\end{aligned}
$$

where $h^\alpha_{ij} = \tilde h^\alpha_{ij}$ are the coefficients of the second fundamental form of f relative to the Darboux frame E. Since

$$
\begin{aligned}
\sum_{i<j,\alpha}(h^\alpha_{ii}-h^\alpha_{jj})^2 &= \sum_{i<j,\alpha}\left\{(h^\alpha_{ii})^2 - 2h^\alpha_{ii}h^\alpha_{jj} + (h^\alpha_{jj})^2\right\}\\
&= -2\sum_{i<j,\alpha}h^\alpha_{ii}h^\alpha_{jj} + \sum_{i<j,\alpha}\left((h^\alpha_{ii})^2+(h^\alpha_{jj})^2\right)\\
&= -2\sum_{i<j,\alpha}h^\alpha_{ii}h^\alpha_{jj} + \frac{1}{2}\sum_{i,j,\alpha}\left((h^\alpha_{ii})^2+(h^\alpha_{jj})^2\right) - \sum_{i,\alpha}(h^\alpha_{ii})^2\\
&= -2\sum_{i<j,\alpha}h^\alpha_{ii}h^\alpha_{jj} + (m-1)\sum_{i,\alpha}(h^\alpha_{ii})^2,
\end{aligned}
$$

we get

$$
\begin{aligned}
\operatorname{trace}\tilde{\mathcal N} &= \frac{1}{m}\left\{\sum_{i<j,\alpha}-2h^\alpha_{ii}h^\alpha_{jj} + (m-1)\sum_{i,\alpha}(h^\alpha_{ii})^2 + 2m\sum_{i<j,\alpha}(h^\alpha_{ij})^2\right\}\\
&= \frac{1}{m}\left\{2\sum_{i<j,\alpha}\left(-h^\alpha_{ii}h^\alpha_{jj} + m(h^\alpha_{ij})^2\right) + \sum_{i,\alpha}-(h^\alpha_{ii})^2 + \sum_{i,\alpha}m(h^\alpha_{ii})^2\right\}\\
&= \frac{1}{m}\sum_{i,j,\alpha}\left(-h^\alpha_{ii}h^\alpha_{jj} + m(h^\alpha_{ij})^2\right)\\
&= \frac{1}{m}\left\{\sum_{i,j,\alpha}(m-1)h^\alpha_{ii}h^\alpha_{jj} - m\sum_{i,j,\alpha}\left(h^\alpha_{ii}h^\alpha_{jj} - (h^\alpha_{ij})^2\right)\right\}\\
&= m(m-1)\sum_\alpha\left(\frac{1}{m}\sum_i h^\alpha_{ii}\right)\left(\frac{1}{m}\sum_j h^\alpha_{jj}\right) - \sum_{i,j,\alpha}\left(h^\alpha_{ii}h^\alpha_{jj} - (h^\alpha_{ij})^2\right)\\
&= m(m-1)\|H\|^2 - 2\sum_{i<j,\alpha}\left(h^\alpha_{ii}h^\alpha_{jj} - (h^\alpha_{ij})^2\right)\\
&= m(m-1)\|H\|^2 - R,
\end{aligned}
$$

where R is the scalar curvature. Summarising,

$$
\operatorname{trace}\tilde{\mathcal N} = m(m-1)\|H\|^2 - R = \frac{1}{m}\sum_{i<j}\left\{(h^\alpha_{ii}-h^\alpha_{jj})^2 + 2m(h^\alpha_{ij})^2\right\}.
$$

Note that this expression obviously justifies the definition of umbilic point given in Ch. 1. This was also observed by Sulanke [Su/85?], who proved $\operatorname{trace}\tilde{\mathcal N} =$

$\|\nabla df\|^2 - m\|H\|^2 \geq 0$. Besides, the latter equality follows straightforwardly from the above. Moreover, Rigoli showed that the m-form

$$\left(\sum_{i<j}\left((h^\alpha_{ii} - h^\alpha_{jj})^2 + 2m(h^\alpha_{ij})^2\right)\right)^{\frac{m}{2}} \phi^1_0 \wedge \ldots \wedge \phi^m_0 ,$$

written relative to a first-order G-frame $e : M \to G$ along $f : M \to \mathbb{R}^n \subset S^n$ does not depend on the choice of first-order frame and, therefore, defines a conformal invariant in Riemannian geometry.
On each compact domain $\overline{D} \subset M$ we consider the functional

$$\mathcal{W}_D(f) = \int_D \Omega_f \tag{2.18}$$

defined for immersions $f : M^m \to S^n$. Such an immersion is said to be a *Willmore immersed submanifold of the Möbius space S^n*, if f is a critical point of the latter functional. That is, for each compact domain $\overline{D} \subset M$ and smooth variation $v : M^m \times (-\epsilon, \epsilon) \to S^n$ of f through immersions $f_t = v(\cdot, t)$, with compact support on D, i.e. $f_t(x) = f(x)$, $\forall t$, and x outside a compact set of D, we have

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\Big|_{t=0} = 0 . \tag{2.19}$$

In Ref. [Ri/87] Rigoli calculated the Euler-Lagrange equation for this variational problem in the particular case $m = 2$ with $n \geq 3$ arbitrary, obtaining an equation rather similar to the one of Bryant, reading

$$p^\alpha_{11} + p^\alpha_{22} = 0 , \quad \forall\alpha = 3, \ldots, n \tag{2.20}$$

with p^α_{ij} as in Eq. (1.60), relative to a second-order G-frame. The transformation law (1.79) shows that this equation is conformally invariant, i.e. it is independent of the choice of second-order frame. Also, if f takes values in $\mathbb{R}^n$, then the Riemannian equivalent of Eq. (2.20) is Eq. (2.10), as we can see from relation (1.129). We further observe from the proposition in Sec. 1.2.D that, if $f(M) \subset S^m \subset S^n$, then $\mathcal{W}_D(f) = 0$, that is, f is a trivial Willmore submanifold.

In the next section we are going to calculate the Euler-Lagrange equation of the variational problem associated with $\mathcal{W}_D$ acting on immersions $f : M^m \to S^n$ with $2 \leq m \leq n$ arbitrary.

2.2 The Euler-Lagrange Equation for the Willmore Functional $\mathcal{W}$

Let $f : M^m \to S^n$ be an immersion of an oriented m-manifold ($m \geq 2$) into the Möbius space S^n and $\overline{D} \subset M$ be a compact domain. Then,

$$\mathcal{W}_D(f) = \int_D \Omega_f = \frac{1}{m}\int_D \Big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m}{2}} \phi^1_0 \wedge \ldots \wedge \phi^m_0$$

with ϕ^i_0 and h^α_{ij} as in Eqs. (1.35) resp. (1.46), relative to a second-order G-frame $e : M \to G$ of Π along f.

Let $v : \overline{D} \times (-\epsilon, \epsilon) \to S^n$ be a smooth variation of f through immersions $f_t = v(\cdot, t)$, which we assume to have compact support $C \subset D$, i.e. $f_t(x) = f(x)$, $\forall t \in (-\epsilon, \epsilon)$, $x \in \overline{D}\backslash C$. Thus, the variation vector $W \in C^\infty(f^{-1}TS^n)$ given by $W_x = \frac{\partial}{\partial t} f_t(x)\big|_{t=0}$, has compact support in C. Now we are going to compute

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\big|_{t=0} = \frac{\partial}{\partial t}\Big(\int_D \Omega_{f_t}\Big)\big|_{t=0} \,.$$

To that end, we construct smooth maps $e : M \times (-\epsilon, \epsilon) \to G$ defined on $U \times (-\epsilon', \epsilon')$, where $U \subset \overline{D}$ is a neighbourhood of a given point $x_0 \in \overline{D}$ and $0 < \epsilon' \leq \epsilon$, satisfying the properties

$$\begin{array}{ll}\text{(i) } e(x,t) = e(x,0)\,, \ \ \forall x \in U\backslash C',\ t \in (-\epsilon', \epsilon')\,, & \\ \text{(ii) } \forall t \in (-\epsilon', \epsilon')\,, \ \ e_t = e(\cdot, t) : M \to G \text{ is a second-order} & \quad (2.21)\\ \qquad G\text{-frame along } f_t \text{ defined on } U\,, & \end{array}$$

where C' is a compact set, such that $C \subset C' \subset D$. First we take a section $s : S^n \to G$ of $\Pi : G \to S^n$ defined on a neighbourhood of $v(x_0, 0)$ in S^n. Let $\hat{e} = s \circ v : \hat{U} \times (-\hat{\epsilon}, \hat{\epsilon}) \to G$ with $\hat{U}$ a convenient neighbourhood of x_0. Then, $\Pi \circ \hat{e}_t(x) = \Pi \circ \hat{e}(x,t) = v(x,t)$, that is, $\hat{e}_t$ is a zeroth-order frame along f_t which satisfies: for $x \in \hat{U}\backslash C$, $\hat{e}_t(x) = s(v(x,t)) = s(v(x,0)) = \hat{e}_0(x)$. Following the construction of a first-order frame from a zeroth-order one given in Sec. 1.2.B, we denote $\phi(t) = \hat{e}_t^*\Phi$, with components $\phi^a_b(t)$, and take the $\mathbb{R}^n$ vector-valued smooth functions on $\hat{U} \times (-\hat{\epsilon}, \hat{\epsilon})$

$$v_i(x,t) = \begin{vmatrix} \phi^1_0(t)(Z_i(x)) \\ \vdots \\ \phi^n_0(t)(Z_i(x)) \end{vmatrix}\,, \quad 1 \leq i \leq m\,,$$

which we may assume to be linearly independent and orthonormal after Gramm-Schmidt orthogonalisation. Then, $v_i(x,t) = v_i(x,0)$, $\forall (x,t) \in \hat{U}\backslash C \times (-\epsilon, \epsilon)$.

Observe that, as $\hat{a}_t = \hat{e}_0$ on $\hat{U}\backslash C$, $\phi(t) = \phi(0)$ on the same open set. Next we choose $\mathbb{R}^n$ vectors $v_{m+1}(x,t), \ldots, v_n(x,t)$ that form an orthonormal frame orthogonal to the subbundle V of $\hat{U} \times (-\hat{\epsilon}, \hat{\epsilon}) \times \mathbb{R}^n$ with fibre $\mathrm{span}\{v_1(x,t), \ldots, v_m(x,t)\}$ at the point (x,t). We can also assume that $v_\alpha(x,t) = v_\alpha(x,0)$, $\forall x \in \hat{U}\backslash C'$, replacing, if necessary, $v_\alpha(x,t)$ by $v_\alpha(x, \theta(x)t)$ with $\theta : M \to [0,1]$ a smooth function, such that $\theta(x) = 1$ on C and $\theta(x) = 0$ on $M\backslash\mathring{C}'$, where C' is some compact set such that $C \subset \mathring{C}' \subset C' \subset D$. Then, the map

$$K(x,t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A(x,t) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $A : \hat{U} \times (-\hat{\epsilon}, \hat{\epsilon}) \to SO(n)$ is given by ${}^t\!A(x,t)\big(v_A(x,t)\big) = \epsilon_A$, satisfies $K(x,t) = K(x,0)$, $\forall x \in \hat{U}\backslash C'$. Let $\tilde{e} : \hat{U} \times (-\hat{\epsilon}, \hat{\epsilon}) \to G$ by defined by $e(x,t) = \tilde{e}(x,t)K(x,t)$. For each t, $\tilde{e}_t$ is a first-order frame along f_t, identical to $\tilde{e}_0$ on $\hat{U}\backslash C'$. Let $\tilde{\phi}(t) = \tilde{e}_t^*\Phi$ with components $\tilde{\phi}^a_b$, and $\tilde{h}^\alpha_{ij}(\cdot, t)$ be as in Eq. (1.46) relative to the frame $\tilde{e}_t$. Let $\overline{K} : \hat{U} \times (-\hat{\epsilon}, \hat{\epsilon}) \to G_0$ be a smooth map given by

$$\overline{K}(x,t) = \begin{bmatrix} 1 & 0 & {}^tY(x,t) & \frac{1}{2}{}^tY(x,t)Y(x,t) \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_{n-m} & Y(x,t) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $Y_\alpha(x,t) = \frac{1}{m}\tilde{h}^\alpha_{kk}(x,t)$ are the components of Y. Note that, on $\hat{U}\backslash C'$, $\tilde{h}^\alpha_{ij}(x,t) = \tilde{h}^\alpha_{ij}(x,0)$, since $\tilde{e}_t = \tilde{e}_0$. Hence, for all t, $\overline{K}(x,t) = \overline{K}(x,0)$ on $\hat{U}\backslash C'$. Let $e : \hat{U} \times (-\hat{\epsilon}, \hat{\epsilon}) \to G$ be defined by $e(x,t) = e(x,t)\overline{K}(x,t)$. Then, as in Sec. 1.2.C, $e_t : \hat{U} \to G$ is a second-order G-frame along f_t and satisfies $e_t = e_0$ on $\hat{U}\backslash C'$. If we now set $U = \hat{U}$ and $\epsilon' = \hat{\epsilon}$, then $e : U \times (-\epsilon', \epsilon') \to G$ satisfies the conditions (2.21). We observe that using the compactness of D we may assume that the ϵ' given in (2.21) is fixed.

For a map $e : U \times (-\epsilon, \epsilon) \to G$ in the conditions (2.21), we define the $\mathfrak{g}$-valued 1-form on $U \times (-\epsilon, \epsilon)$ given by

$$\phi = e^*\Phi = e^{-1}de \tag{2.22}$$

with components ϕ^a_b satisfying the relations in Eq. (1.16) and the structure equations (1.18). For each $t \in (-\epsilon, \epsilon)$, let $\phi(t)$ denote the $\mathfrak{g}$-valued 1-form on U

$$\phi(t) = e_t^*\Phi \tag{2.23}$$

on U. Then,

$$\phi_{(x,t)} = \phi(t)_x + \overline{\lambda}(x,t)dt \tag{2.24}$$

with the meaning $\phi_{(x,t)}(u,h) = \phi(t)_x(u) + h\overline{\lambda}(x,t)$, $\forall u \in T_xM$, $h \in I\!R$, where $\overline{\lambda} : U \times (-\epsilon,\epsilon) \to \mathfrak{g}$ is a smooth function with components $\overline{\lambda}^a_b$. Thus, $\overline{\lambda}(x,t) = \Phi_{e_t(x)}\left(\frac{\partial}{\partial t}e(x,t)\right)$. From the first property in Eq. (2.21), we have $\frac{\partial}{\partial t}e(x,t) = 0$, $\forall t \in (-\epsilon,\epsilon)$, $x \in U\backslash C'$, which implies

$$\overline{\lambda}^a_b(x,t) = 0 \quad \text{and} \quad \phi^a_{b\,(x,t)} = \phi^a_{b\,(x,0)}\ , \ \ \forall t \in (-\epsilon,\epsilon),\ x \in U\backslash C'\ . \tag{2.25}$$

As for each t, e_t is a second-order frame, $\phi^a_0(t) = 0$. Thus, if we set $\lambda^A_0 = \overline{\lambda}^A_0$, we have

$$\phi^a_{0\,(x,t)} = \lambda^a_0(x,t)dt\ , \tag{2.26}$$

with

$$\lambda^a_0(x,t) = 0\ ,\ \ \forall t \in (-\epsilon,\epsilon),\ x \in U\backslash C'\ . \tag{2.27}$$

Since $\phi^a_i(t)_x = h^a_{ij}(x,t)\phi^j_0(t)_x$, with $h^a_{ii}(x,t) = 0$, $h^a_{ij}(x,t) = h^a_{ji}(x,t)$, and

$$h^a_{ij}(x,t) = h^a_{ij}(x,0)\ ,\ \ \forall t \in (-\epsilon,\epsilon),\ x \in U\backslash C'\ , \tag{2.28}$$

we get

$$\begin{aligned} \phi^a_{i\,(x,t)} &= h^a_{ij}(x,t)\phi^j_0(t)_x + \overline{\lambda}^a_i(x,t)dt \\ &= h^a_{ij}(x,t)\left(\phi^j_{0(x,t)} - \lambda^j_0(x,t)dt\right) + \overline{\lambda}^a_i(x,t)dt\ . \end{aligned} \tag{2.29}$$

This expression can be written in the form

$$\phi^a_{i\,(x,t)} = h^a_{ij}(x,t)\phi^j_{0(x,t)} + \lambda^a_i(x,t)dt\ , \tag{2.30}$$

where $\lambda^a_i : U \times (-\epsilon,\epsilon) \to I\!R$ is a smooth map satisfying

$$\lambda^a_i(x,t) = 0\ ,\ \ \forall t \in (-\epsilon,\epsilon),\ x \in U\backslash C'\ . \tag{2.31}$$

Differentiating Eq. (2.26) and using the structure equations (1.18) and Eqs. (2.30), (2.26), we obtain

$$\begin{aligned} d\lambda^a_0 \wedge dt &= -\phi^a_0 \wedge \phi^0_0 - \phi^a_i \wedge \phi^i_0 - \phi^a_\beta \wedge \phi^\beta_0 \\ &= -\lambda^a_0 dt \wedge \phi^0_0 - h^a_{ij}\phi^j_0 \wedge \phi^i_0 - \lambda^a_i dt \wedge \phi^i_0 - \lambda^\beta_0\phi^a_\beta \wedge dt \\ &= (\lambda^a_0\phi^0_0 + \lambda^a_i\phi^i_0 - \lambda^\beta_0\phi^a_\beta) \wedge dt\ . \end{aligned}$$

By Cartan's Lemma,

$$d\lambda_0^\alpha = \lambda_0^\alpha \phi_0^0 + \lambda_i^\alpha \phi_0^i - \lambda_0^\beta \phi_\beta^\alpha + \mu^\alpha dt \ . \tag{2.32}$$

As $\lambda_0^\alpha, \lambda_i^\alpha$ have support in $C' \times (-\epsilon, \epsilon)$,

$$\mu^\alpha(x,t) = 0 \ , \ \ \forall t \in (-\epsilon,\epsilon), \ x \in U \backslash C' \ . \tag{2.33}$$

Analogously, by differentiating Eq. (2.30) and using the linear independence of $(\phi_0^1, \ldots, \phi_0^m, dt)$, we obtain

$$dh_{ij}^\alpha - h_{ik}^\alpha \phi_j^k - h_{jk}^\alpha \phi_i^k + h_{ij}^\beta \phi_\beta^\alpha + h_{ij}^\alpha \phi_0^0 + \delta_{ij}\phi_\alpha^0 = h_{ijk}^\alpha \phi_0^k + \lambda_{ij}^\alpha dt \tag{2.34}$$

$$d\lambda_i^\alpha - \lambda_0^\alpha \phi_i^0 - \lambda_j^\alpha \phi_i^j + \lambda_i^\beta \phi_\beta^\alpha + \lambda_0^\beta h_{ij}^\alpha h_{jk}^\beta \phi_0^k = \lambda_{ik}^\alpha \phi_0^k + \mu_i^\alpha dt \ , \tag{2.35}$$

where h_{ijk}^α and λ_{ij}^α are smooth functions on $U \times (-\epsilon, \epsilon)$ with the symmetry properties $h_{ijk}^\alpha = h_{ikj}^\alpha = h_{jik}^\alpha$, $\lambda_{ij}^\alpha = \lambda_{ji}^\alpha$ (compare with Eqs. (1.54), (1.55) for a fixed t), and

$$h_{ijk}^\alpha(x,t) = h_{ijk}^\alpha(x,0) \ \ , \ \ \lambda_{ij}^\alpha(x,t) = 0 \ , \ \ \forall t \in (-\epsilon,\epsilon), \ x \in U \backslash C' \ . \tag{2.36}$$

Multiplying both sides of Eq. (2.34) by h_{ij}^α and summing over i, j, we get

$$h_{ij}^\alpha dh_{ij}^\alpha = -h_{ij}^\alpha h_{ij}^\alpha \phi_0^0 + h_{ij}^\alpha h_{ijk}^\alpha \phi_0^k + h_{ij}^\alpha \lambda_{ij}^\alpha dt \ . \tag{2.37}$$

If $\tilde{e} : \tilde{U} \times (-\epsilon, \epsilon) \to G$ is another map in the conditions (2.21), then $\tilde{e}_t = e_t K_t$, where K_t is a map on $U \cap \tilde{U}$ with values in G_2. Obviously, $K : U \cap \tilde{U} \times (-\epsilon,\epsilon) \to G_2$, $(x,t) \to K_t(x)$, is smooth. From the equalities $\phi = e^{-1}de$ and $\tilde{e}(x,t) = e(x,t)K(x,t)$, we obtain $\tilde{\phi} = \tilde{e}^{-1}d\tilde{e} = K^{-1}\phi K + K^{-1}dK$. Writing K as in Eq. (1.70), with now r, X, A, B maps of the variables (x,t), we derive from the latter equation the transformation laws

$$[\tilde{\phi}_0^A] = r^{-1}\mathcal{O}[\phi_0^A] \ \text{ with } \ \mathcal{O} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} .$$

Hence,

$$\begin{aligned} \tilde{\phi}_0^j &= r^{-1}A_j^i \phi_0^i \\ \tilde{\phi}_0^\beta &= \tilde{\lambda}_0^\beta dt = r^{-1}B_\beta^\alpha \lambda_0^\alpha dt \ , \end{aligned} \tag{2.38}$$

which implies the transformations

$$\begin{aligned} \tilde{\lambda}_0^j &= r^{-1}A_j^i \lambda_0^i \\ \tilde{\lambda}_0^\beta &= r^{-1}B_\beta^\alpha \lambda_0^\alpha \ . \end{aligned} \tag{2.39}$$

Furthermore, comparing with Eq. (1.29), we obtain

$$[\tilde{\phi}^A_B] = \mho[\phi^A_0]^t Z\mho + \mho[\phi^A_B]\mho - \mho Z^t[\phi^A_0]\mho + \mho d\mho \ ,$$

where $Z = \begin{bmatrix} X \\ 0 \end{bmatrix}$. So, in particular,

$$\begin{aligned} \tilde{\phi}^\alpha_i &= \tilde{h}^\alpha_{ij}\tilde{\phi}^j_0 + \tilde{\lambda}^\alpha_i dt = B^\beta_\alpha \phi^\beta_0 X_j A^j_i + B^\beta_\alpha \phi^\beta_j A^j_i \\ &= \lambda^\beta_0 B^\beta_\alpha X_j A^j_i dt + B^\beta_\alpha A^j_i (h^\beta_{jk}\phi^k_0 + \lambda^\beta_j dt) \\ &= B^\beta_\alpha A^j_i h^\beta_{jk}\phi^k_0 + (\lambda^\beta_j B^\beta_\alpha A^j_i + \lambda^\beta_0 B^\beta_\alpha A^j_i X_j)dt \ , \end{aligned}$$

whence

$$\tilde{\lambda}^\alpha_i = \lambda^\beta_j B^\beta_\alpha A^j_i + \lambda^\beta_0 B^\beta_\alpha A^j_i X_j \ . \tag{2.40}$$

As a final remark on maps e with the property (2.21), we observe that, given a point $x_0 \in M$, one can always find a variation $(f_t)_{t\in(-\epsilon,\epsilon)}$ of f with compact support C contained in a domain D, such that x_0 lies in the interior of C, and a map e satisfying the conditions (2.21) with arbitrary $\lambda^\alpha_0(\cdot,0)$ as long as $\mathrm{supp}\lambda^\alpha_0(\cdot,0) \subset C' \cap U$. For example, assuming that, near the point x_0, f is of the form $f(x) = \begin{bmatrix} 1 \\ \rho(x) \\ \frac{1}{2}\|\rho(x)\|^2 \end{bmatrix}_{\sim}$, we take the variation $f_t(x) = \begin{bmatrix} 1 \\ \rho(x) + t\Lambda(x) \\ \frac{1}{2}\|\rho(x) + t\Lambda(x)\|^2 \end{bmatrix}_{\sim}$, where $\Lambda(x)$ is an arbitrary $\mathbb{R}^n$-valued function with support C. If we choose the section s given in Eq. (1.19), the map e as constructed above satisfies $\lambda^\alpha_0(x,0) = \phi^\alpha_{0\,(x,0)}(\frac{\partial}{\partial t}) = A^B_\alpha(x,0)d\big(\rho_B(x) + t\Lambda_B(x)\big)_{(x,0)}(\frac{\partial}{\partial t}) = A^B_\alpha(x,0)\Lambda_B(x)$, where $A^B_A(\cdot,0)$ only depends on f, which can take any arbitrary value. If f were of the form $\begin{bmatrix} \frac{1}{2}\|\rho(x)\|^2 \\ \rho(x) \\ 1 \end{bmatrix}_{\sim}$, we would arrive at the same conclusion by taking this time the section $\hat{s}$ of Eq. (1.20).

Proposition 2.1 *Let $f : M^m \to S^n$ be an immersion of an oriented m-manifold into the Möbius space. Then, we have:*
For $m = 2$, f is a Willmore immersed surface, iff [Br/84] [Ri/87]

$$p^\alpha_{jj} = 0 \ , \ \ \forall \alpha = 3, \ldots, n \ .$$

For $m = 4$, f is a Willmore immersed 4-submanifold, iff

$$\mathrm{trace}\mathcal{N}(3p^\alpha_{jj} + h^\alpha_{kj}h^\beta_{ji}h^\beta_{ik}) + 2h^\alpha_{ij}H_{ij} + 12p^\alpha_i h^\gamma_{m}h^\gamma_{\alpha i} = 0 \ , \ \ \forall \alpha = 5, \ldots, n \ .$$

If $m = 3$ or $m = 5$ with the assumption that $trace\mathcal{N}_x \neq 0$, $\forall x \in M$, i.e. f has no umbilic points, or $m > 5$ without any non-degeneracy condition, then f is a Willmore immersed m-submanifold, iff

$$\begin{aligned}&\left(trace\mathcal{N}\right)^{\frac{m-2}{2}}\left((m-1)p^{\alpha}_{jj} + h^{\alpha}_{kj}h^{\beta}_{ji}h^{\beta}_{ik}\right)+\\ &+(m-2)\left(trace\mathcal{N}\right)^{\frac{m-4}{2}}\left(h^{\alpha}_{ij}H_{ij} + 2(m-1)p^{\alpha}_{i}h^{\gamma}_{st}h^{\gamma}_{sti}\right)+\\ &+(m-2)(m-4)\left(trace\mathcal{N}\right)^{\frac{m-6}{2}}h^{\alpha}_{ij}h^{\gamma}_{st}h^{\gamma}_{stj}h^{\nu}_{pq}h^{\nu}_{pqi} \qquad = 0\,, \quad \forall \alpha = m+1, \ldots, n\,,\end{aligned}$$

where the quantities h^{α}_{ij}, h^{α}_{ijk}, p^{α}_{i}, p^{α}_{ij}, H_{ij}, and $trace\mathcal{N} = \mathcal{N}_{jj}$ are as defined respectively in Eqs. (1.46,1.47), (1.54,1.55), (1.58), (1.60,1.61), (1.62,1.63), and (1.87), relative to a second-order G-frame field $e : M \to G$ of $\Pi : G \to S^n$ along f. Note that the above equations are conformally invariant, that is, they do not depend on the choice of second-order frame.

Proof. Let $v : \overline{D} \times (-\epsilon, \epsilon) \to S^n$ be a smooth variation of f through immersions and with compact support $C \subset D$. Let $x_0 \in D$ and $e : U \times (-\epsilon', \epsilon') \to G$ be a map in the conditions (2.21), with U a neighbourhood of x_0 in $\overline{D}$. Then, $\forall t \in (-\epsilon', \epsilon')$, $x \in U$, we have

$$\Omega_{f_t}(x) = \frac{1}{m}\left(\sum_{i,j,\alpha}\left(h^{\alpha}_{ij}(x,t)\right)^2\right)^{\frac{m}{2}} \phi^1_0(t)_x \wedge \ldots \wedge \phi^m_0(t)_x\,.$$

Let Ω be the m-form on $U \times (-\epsilon', \epsilon')$ given by

$$\Omega_{(x,t)} = \frac{1}{m}\left(\sum_{i,j,\alpha}\left(h^{\alpha}_{ij}(x,t)\right)^2\right)^{\frac{m}{2}} \phi^1_{0(x,t)} \wedge \ldots \wedge \phi^m_{0\,(x,t)}\,.$$

Although Ω is only defined on $U \times (-\epsilon', \epsilon')$ and depends on e, its restriction $\Omega_{(x,t)}|_{T_xM} = \Omega_{f_t}(x)$ is a well-defined global m-form on all $\overline{D}$. Let Ψ be the m-form on $U \times (-\epsilon', \epsilon')$ defined as

$$\Psi = \Omega - dt \wedge \left(\imath_{\frac{\partial}{\partial t}}\Omega\right)\,.$$

Then, $\imath_{\frac{\partial}{\partial t}}\Psi = 0$. Thus, $\Psi_{(x,t)} = \Psi_{(x,t)}|_{T_xM} = \Omega_{f_t}(x)$. So, Ψ is a well-defined global m-form on $M \times (-\epsilon, \epsilon)$ and we have

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t)|_{t=0} = \frac{\partial}{\partial t}\left(\int_D \Psi_{(x,t)}|_{T_xM}\right)|_{t=0} = \int_D L_{\frac{\partial}{\partial t}}\Psi\Big|_{\substack{t=0\\T_xM}}\,.$$

Since $L = \imath \circ d + d \circ \imath$, we get

$$L_{\frac{\partial}{\partial t}}\Psi = \imath_{\frac{\partial}{\partial t}}d\Psi + d\imath_{\frac{\partial}{\partial t}}\Psi = \imath_{\frac{\partial}{\partial t}}d\Psi\,.$$

On $U \times (-\epsilon', \epsilon)$,

$$\mathrm{L}_{\frac{\partial}{\partial t}} \Psi = \iota_{\frac{\partial}{\partial t}}\big(d\Omega + dt \wedge d(\iota_{\frac{\partial}{\partial t}} \Omega)\big) = \iota_{\frac{\partial}{\partial t}} d\Omega + d(\iota_{\frac{\partial}{\partial t}} \Omega)(t)$$

and

$$\begin{aligned} d\Omega &= \frac{1}{m} d\left(\Big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m}{2}} \phi^1_0 \wedge \ldots \wedge \phi^m_0\right) \\ &= \frac{1}{m}\left(m\Big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m-2}{2}} h^\beta_{\alpha} dh^\beta_{\alpha} \wedge \phi^1_0 \wedge \ldots \wedge \phi^m_0 + \Big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m}{2}} d(\phi^1_0 \wedge \ldots \wedge \phi^m_0)\right). \end{aligned}$$

Henceforth, we will use the notations

$$\begin{aligned} \phi^{1\ldots m} &= \phi^1_0 \wedge \ldots \wedge \phi^m_0 \\ \phi^{1\ldots\hat{j}\ldots m} &= \phi^1_0 \wedge \ldots \wedge \phi^{j-1}_0 \wedge \phi^{j+1}_0 \wedge \ldots \wedge \phi^m_0 . \end{aligned} \tag{2.41}$$

Using the structure equations (1.18) and Eqs. (2.26),(2.30), we have

$$\begin{aligned} d\phi^{1\ldots m} &= (-1)^{j-1} d\phi^j_0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &= (-1)^{j-1}(-\phi^j_0 \wedge \phi^0_0 - \phi^j_k \wedge \phi^k_0 - \phi^j_\alpha \wedge \phi^\alpha_0) \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &= m\phi^0_0 \wedge \phi^{1\ldots m} + (-1)^j \phi^j_\alpha \wedge \phi^\alpha_0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &= m\phi^0_0 \wedge \phi^{1\ldots m} + (-1)^{j-1}(h^\alpha_{jk}\phi^k_0 + \lambda^\alpha_j dt) \wedge \lambda^\alpha_0 dt \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &= m\phi^0_0 \wedge \phi^{1\ldots m} + (-1)^{j-1}\lambda^\alpha_0 h^\alpha_{jj}\phi^j_0 \wedge dt \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &= m\phi^0_0 \wedge \phi^{1\ldots m} . \end{aligned}$$

From Eq. (2.37), we obtain

$$\Big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m-2}{2}} h^\beta_\alpha dh^\beta_\alpha \wedge \phi^{1\ldots m} = -\Big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m}{2}} \phi^0_0 \wedge \phi^{1\ldots m} + \Big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m-2}{2}} h^\alpha_{\alpha}\lambda^\alpha_{\alpha} dt \wedge \phi^{1\ldots m} .$$

Hence,

$$d\Omega = \Big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m-2}{2}} h^\alpha_\alpha \lambda^\alpha_\alpha dt \wedge \phi^{1\ldots m} .$$

Thus,

$$\iota_{\frac{\partial}{\partial t}} d\Omega = \Big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m-2}{2}} h^\alpha_\alpha \lambda^\alpha_\alpha \phi^{1\ldots m}(t)$$

and

$$\iota_{\frac{\partial}{\partial t}} \Omega_{(x,t)} = \frac{1}{m}\Big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m}{2}} \iota_{\frac{\partial}{\partial t}}(\phi^{1\ldots m})$$

$$
\begin{aligned}
&= \frac{1}{m}\Big(\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2\Big)^{\frac{m}{2}} \imath_{\frac{\partial}{\partial t}}\big((\phi^1_0(t)+\lambda^1_0 dt)\wedge\ldots\wedge(\phi^m_0(t)+\lambda^m_0 dt)\big)\\
&= \frac{1}{m}\Big(\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2\Big)^{\frac{m}{2}} \imath_{\frac{\partial}{\partial t}}\Big\{\sum_k \phi^1_0(t)\wedge\ldots\wedge\phi^{k-1}_0(t)\wedge\lambda^k_0 dt\wedge\phi^{k+1}_0(t)\wedge\ldots\wedge\phi^m_0(t)\Big\}\\
&= \frac{1}{m}\Big(\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2\Big)^{\frac{m}{2}}(-1)^{k-1}\lambda^k_0\phi^{1\ldots\hat{k}\ldots m}(t)\,.
\end{aligned}
$$

Consequently,

$$
d(\imath_{\frac{\partial}{\partial t}}\Omega)\Big|_{\substack{t=0\\TM}} = d\Big(\imath_{\frac{\partial}{\partial t}}\Omega\Big|_{\substack{t=0\\TM}}\Big) = d\left(\frac{1}{m}\Big(\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2\Big)^{\frac{m}{2}}(-1)^{k-1}\lambda^k_0\phi^{1\ldots\hat{k}\ldots m}\right)
$$

and

$$
\imath_{\frac{\partial}{\partial t}}d\Omega\Big|_{\substack{t=0\\TM}} = \Big(\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2\Big)^{\frac{m-2}{2}} h^{\alpha}_{ss}\lambda^{\alpha}_{ss}\phi^{1\ldots m}\,,
$$

where now $\phi^{1\ldots\hat{k}\ldots m}$ $(=\phi^{1\ldots\hat{k}\ldots m}(t=0))$ and $\phi^{1\ldots m}$ are forms on U only, as defined in Eq. (2.41) with ϕ^i_0, h^{α}_{ss} relative to the second-order G-frame $e_0 : U \to G$ of Π along $f = f_0$, and where λ^k_0 and λ^{α}_{ss} are considered as functions only of the variable $x \in U$, fixing $t = 0$. Thus, we have obtained

$$
\begin{aligned}
L_{\frac{\partial}{\partial t}}\Psi\Big|_{\substack{t=0\\TM}} &= \Big(\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2\Big)^{\frac{m-2}{2}} h^{\alpha}_{ss}\lambda^{\alpha}_{ss}\phi^{1\ldots m} + \\
&\quad + d\left(\frac{1}{m}\Big(\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2\Big)^{\frac{m}{2}}(-1)^{k-1}\lambda^k_0\phi^{1\ldots\hat{k}\ldots m}\right)\,. \qquad (2.42)
\end{aligned}
$$

Next we rewrite the relations given in Eqs. (2.32,2.35,2.27,2.31) among the λ^{α}_0, λ^{α}_i, and λ^{α}_{ij} in terms of functions of the variable $x \in U$ only, thereby fixing $t = 0$, which yields

$$
\begin{aligned}
d\lambda^{\alpha}_0 &= \lambda^{\alpha}_0\phi^0_0 + \lambda^{\alpha}_i\phi^i_0 - \lambda^{\beta}_0\phi^{\alpha}_{\beta} &(2.43)\\
d\lambda^{\alpha}_i &= \lambda^{\alpha}_0\phi^0_i + \lambda^{\alpha}_j\phi^j_i - \lambda^{\beta}_i\phi^{\alpha}_{\beta} - \lambda^{\beta}_0 h^{\alpha}_{ij}h^{\beta}_{jk}\phi^k_0 + \lambda^{\alpha}_{ik}\phi^k_0\,, &(2.44)
\end{aligned}
$$

where λ^{α}_0, λ^{α}_i, and λ^{α}_{ij} have support in $C' \cap U \subset D$, and with ϕ^{α}_i, h^{α}_{ij} relative to the second-order G-frame e_0 along f. Now we evaluate the expression

$$
\Big(\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2\Big)^{\frac{m-2}{2}} h^{\alpha}_{ss}\lambda^{\alpha}_{ss}\phi^{1\ldots m}\,. \qquad (2.45)
$$

For the sake of notational simplicity, we define

$$
\|h\| = \sqrt{\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2} = \sqrt{\operatorname{trace} N}\,. \qquad (2.46)
$$

From Eq. (1.54), we have, for positive integer $r \neq 1$ (unless $\|h\| \neq 0$),

$$\begin{aligned} d\|h\|^r &= r\|h\|^{r-2}h^{\gamma}_{st}dh^{\gamma}_{st} = r\|h\|^{r-2}\left(-\|h\|^2\phi^0_0 + h^{\gamma}_{st}h^{\gamma}_{stk}\phi^k_0\right) \\ &= -r\|h\|^r\phi^0_0 + r\|h\|^{r-2}h^{\gamma}_{st}h^{\gamma}_{stk}\phi^k_0 . \end{aligned} \tag{2.47}$$

Using the structure equations (1.48), we obtain

$$d\phi^{1\ldots\hat{j}\ldots m} = (m-1)\phi^0_0 \wedge \phi^{1\ldots\hat{j}\ldots m} + (-1)^{k+1}\phi^k_i \wedge \phi^{1\ldots\hat{k}\ldots m} . \tag{2.48}$$

Starting from Eq. (2.44), we get, for i, j fixed,

$$\begin{aligned} \lambda^{\alpha}_{ij}\phi^{1\ldots m} &= (-1)^{j-1}\lambda^{\alpha}_{ij}\phi^j_0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &= (-1)^{j-1}(\lambda^{\alpha}_{ik}\phi^k_0) \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &= (-1)^{j-1}\left(d\lambda^{\alpha}_i - \lambda^{\alpha}_k\phi^k_i + \lambda^{\beta}_i\phi^{\alpha}_{\beta} - \lambda^{\alpha}_0\phi^0_i + \lambda^{\beta}_0 h^{\alpha}_{ik}h^{\beta}_{kl}\phi^l_0\right) \wedge \phi^{1\ldots\hat{j}\ldots m} . \end{aligned}$$

So,

$$\begin{aligned} (2.45) &= \|h\|^{m-2}h^{\alpha}_{ij}\lambda^{\alpha}_{ij}\phi^{1\ldots m} = \\ &= (-1)^{j-1}\|h\|^{m-2}h^{\alpha}_{ij}d\lambda^{\alpha}_i \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad + (-1)^j\|h\|^{m-2}h^{\alpha}_{ij}\lambda^{\alpha}_k\phi^k_i \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad + (-1)^{j-1}\|h\|^{m-2}h^{\alpha}_{ij}\lambda^{\beta}_i\phi^{\alpha}_{\beta} \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad + (-1)^j\|h\|^{m-2}h^{\alpha}_{ij}\lambda^{\alpha}_0\phi^0_i \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad + \|h\|^{m-2}h^{\alpha}_{ij}h^{\alpha}_{ik}\lambda^{\beta}_0 h^{\beta}_{kj}\phi^{1\ldots m} \\ &= d\left((-1)^{j-1}\|h\|^{m-2}\lambda^{\alpha}_i h^{\alpha}_{ij}\phi^{1\ldots\hat{j}\ldots m}\right) \\ &\quad + (-1)^j h^{\alpha}_{ij}\lambda^{\alpha}_i d(\|h\|^{m-2}) \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad + (-1)^j\|h\|^{m-2}\lambda^{\alpha}_i dh^{\alpha}_{ij} \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad + (-1)^j\|h\|^{m-2}\lambda^{\alpha}_i h^{\alpha}_{ij}d\phi^{1\ldots\hat{j}\ldots m} \\ &\quad + (-1)^j\|h\|^{m-2}h^{\alpha}_{ij}\lambda^{\alpha}_k\phi^k_i \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad + (-1)^{j-1}\|h\|^{m-2}h^{\alpha}_{ij}\lambda^{\beta}_i\phi^{\alpha}_{\beta} \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad + (-1)^j\|h\|^{m-2}h^{\alpha}_{ij}\lambda^{\alpha}_0\phi^0_i \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad + \|h\|^{m-2}h^{\alpha}_{ij}h^{\alpha}_{ik}\lambda^{\beta}_0 h^{\beta}_{kj}\phi^{1\ldots m} . \end{aligned}$$

Using Eqs. (2.47), (1.54), (1.57), (2.48), and (1.60), and assuming $m \neq 3$ unless $\|h\| \neq 0$, we get

(2.45) =

$$
\begin{aligned}
= \;& d\big((-1)^{j-1}\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m}\big) \\
& + (-1)^j h_{ij}^\alpha \lambda_i^\alpha (m-2)\|h\|^{m-4}\big(-\|h\|^2\phi_0^0 + h_{\alpha}^{\gamma} h_{\alpha k}^{\gamma}\phi_0^k\big) \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j \|h\|^{m-2}\lambda_i^\alpha\big(h_{kj}^\alpha \phi_i^k + h_{ik}^\alpha \phi_j^k - h_{ij}^\beta \phi_\beta^\alpha - h_{ij}^\alpha \phi_0^0 - \delta_{ij} p_k^\alpha \phi_0^k + h_{ij,k}^\alpha \phi_0^k\big) \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j \|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha (m-1)\phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j \|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha (-1)^{k+j}\phi_j^k \wedge \phi^{1\ldots\hat{k}\ldots m} \\
& + (-1)^j \|h\|^{m-2} h_{ij}^\alpha \lambda_k^\alpha \phi_i^k \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^{j-1} \|h\|^{m-2} h_{ij}^\alpha \lambda_i^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j \|h\|^{m-2}\lambda_0^\alpha\big(dp_j^\alpha - p_i^\alpha \phi_j^i + p_j^\beta \phi_\beta^\alpha + 2p_j^\alpha \phi_0^0 - p_{j,i}^\alpha \phi_0^i\big) \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + \|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m}
\end{aligned}
$$

$$
\begin{aligned}
= \;& d\big((-1)^{j-1}\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m}\big) \\
& + (-1)^{j-1}(m-2)\|h\|^{m-2} h_{ij}^\alpha \lambda_i^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& - (m-2)\|h\|^{m-4} h_{ij}^\alpha \lambda_i^\alpha h_{\alpha}^{\gamma} h_{\alpha j}^{\gamma} \phi^{1\ldots m} \\
& + (-1)^j \|h\|^{m-2}\lambda_i^\alpha h_{kj}^\alpha \phi_i^k \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j \|h\|^{m-2}\lambda_i^\alpha h_{ik}^\alpha \phi_j^k \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^{j-1} \|h\|^{m-2}\lambda_i^\alpha h_{ij}^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^{j-1} \|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + \|h\|^{m-2}\lambda_j^\alpha p_j^\alpha \phi^{1\ldots m} \\
& - \|h\|^{m-2}\lambda_i^\alpha h_{ij,j}^\alpha \phi^{1\ldots m} \\
& + (-1)^j (m-1)\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^k \|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi_j^k \wedge \phi^{1\ldots\hat{k}\ldots m} \\
& + (-1)^j \|h\|^{m-2} h_{ij}^\alpha \lambda_k^\alpha \phi_i^k \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^{j-1} \|h\|^{m-2} h_{ij}^\alpha \lambda_i^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j \|h\|^{m-2}\lambda_0^\alpha dp_j^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^{j-1} \|h\|^{m-2}\lambda_0^\alpha p_i^\alpha \phi_j^i \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j \|h\|^{m-2}\lambda_0^\alpha p_j^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j 2\|h\|^{m-2}\lambda_0^\alpha p_j^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + \|h\|^{m-2}\lambda_0^\alpha p_{j,j}^\alpha \phi^{1\ldots m} \\
& + \|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m} \; .
\end{aligned}
$$

In the latter expression, we have several simple cancellations, by permuting indices when necessary and using the symmetry properties of the coefficients and forms involved. By applying also Eq. (1.56), we obtain

$$
\begin{aligned}
(2.45) \;=\; & d\Big((-1)^{j-1}\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m}\Big) \\
& - (m-2)\|h\|^{m-4} h_{ij}^\alpha \lambda_i^\alpha h_{\alpha}^\gamma h_{\alpha j}^\gamma \phi^{1\ldots m} \\
& + (1-m)\|h\|^{m-2}\lambda_j^\alpha p_j^\alpha \phi^{1\ldots m} \\
& + (-1)^j\|h\|^{m-2}\lambda_0^\alpha dp_j^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^{j-1}\|h\|^{m-2}\lambda_0^\alpha p_i^\alpha \phi_j^i \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j\|h\|^{m-2}\lambda_0^\alpha p_j^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j 2\|h\|^{m-2}\lambda_0^\alpha p_j^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + \|h\|^{m-2}\lambda_0^\alpha p_{jj}^\alpha \phi^{1\ldots m} \\
& + \|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m} \\
=\; & d\Big((-1)^{j-1}\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m}\Big) \\
& - (m-2)\|h\|^{m-4} h_{ij}^\alpha \lambda_i^\alpha h_{\alpha}^\gamma h_{\alpha j}^\gamma \phi^{1\ldots m} \\
& + (1-m)\|h\|^{m-2}\lambda_j^\alpha p_j^\alpha \phi^{1\ldots m} \\
& + d\Big((-1)^j\|h\|^{m-2}\lambda_0^\alpha p_j^\alpha \phi^{1\ldots\hat{j}\ldots m}\Big) \\
& + (-1)^{j-1}\|h\|^{m-2} p_j^\alpha d\lambda_0^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^{j-1} p_j^\alpha \lambda_0^\alpha d\big(\|h\|^{m-2}\big) \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^{j-1} p_j^\alpha \lambda_0^\alpha \|h\|^{m-2} d\big(\phi^{1\ldots\hat{j}\ldots m}\big) \\
& + (-1)^{j-1}\|h\|^{m-2}\lambda_0^\alpha p_i^\alpha \phi_j^i \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j\|h\|^{m-2}\lambda_0^\alpha p_j^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + (-1)^j 2\|h\|^{m-2}\lambda_0^\alpha p_j^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\
& + \|h\|^{m-2}\lambda_0^\alpha p_{jj}^\alpha \phi^{1\ldots m} \\
& + \|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m}\,.
\end{aligned}
$$

Using again Eqs. (2.43), (2.47), and (2.48), we get

$$
\begin{aligned}
(2.45) \;=\; & d\Big((-1)^{j-1}\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m} + (-1)^j\|h\|^{m-2}\lambda_0^\alpha p_j^\alpha \phi^{1\ldots\hat{j}\ldots m}\Big) \\
& - (m-2)\|h\|^{m-4} h_{ij}^\alpha \lambda_i^\alpha h_{\alpha}^\gamma h_{\alpha j}^\gamma \phi^{1\ldots m} \\
& + (1-m)\|h\|^{m-2}\lambda_j^\alpha p_j^\alpha \phi^{1\ldots m}
\end{aligned}
$$

$$
\begin{aligned}
&+ (-1)^{j-1}\|h\|^{m-2} p_j^\alpha \lambda_0^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\
&+ \|h\|^{m-2} p_j^\alpha \lambda_j^\alpha \phi^{1\ldots m} \\
&+ (-1)^j \|h\|^{m-2} p_j^\alpha \lambda_0^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\
&+ (-1)^j (m-2)\|h\|^{m-2} p_j^\alpha \lambda_0^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\
&+ (m-2)\|h\|^{m-4} p_j^\alpha \lambda_0^\alpha h_{\alpha t}^\gamma h_{\alpha t j}^\gamma \phi^{1\ldots m} \\
&+ (-1)^{j-1}(m-1)\|h\|^{m-2} p_j^\alpha \lambda_0^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\
&+ (-1)^{j-1}\|h\|^{m-2} p_j^\alpha \lambda_0^\alpha (-1)^{k+j} \phi_j^k \wedge \phi^{1\ldots\hat{k}\ldots m} \\
&+ (-1)^{j-1}\|h\|^{m-2} \lambda_0^\alpha p_i^\alpha \phi_j^i \wedge \phi^{1\ldots\hat{j}\ldots m} \\
&+ (-1)^j \|h\|^{m-2} \lambda_0^\alpha p_j^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\
&+ (-1)^j 2\|h\|^{m-2} \lambda_0^\alpha p_j^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\
&+ \|h\|^{m-2} \lambda_0^\alpha p_{jj}^\alpha \phi^{1\ldots m} \\
&+ \|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m}\,,
\end{aligned}
$$

which gives, after some obvious cancellations and rearrangements,

$$
\begin{aligned}
(2.45) \quad = \quad & d\Big((-1)^{j-1}\|h\|^{m-2} \lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m} + (-1)^j \|h\|^{m-2} \lambda_0^\alpha p_j^\alpha \phi^{1\ldots\hat{j}\ldots m}\Big) \\
& - (m-2)\|h\|^{m-4} h_{ij}^\alpha \lambda_i^\alpha h_{\alpha t}^\gamma h_{\alpha t j}^\gamma \phi^{1\ldots m} \\
& + (2-m)\|h\|^{m-2} \lambda_j^\alpha p_j^\alpha \phi^{1\ldots m} \\
& + (m-2)\|h\|^{m-4} p_j^\alpha \lambda_0^\alpha h_{\alpha t}^\gamma h_{\alpha t j}^\gamma \phi^{1\ldots m} \\
& + \|h\|^{m-2} \lambda_0^\alpha p_{jj}^\alpha \phi^{1\ldots m} \\
& + \|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m}\,.
\end{aligned} \tag{2.49}
$$

This expression will also serve for later use. Substituting the factor $\lambda_i^\alpha \phi^{1\ldots m}$ in the second term of the r.h.s. as

$$
\lambda_i^\alpha \phi^{1\ldots m} = (-1)^{i-1} \lambda_i^\alpha \phi_0^i \wedge \phi^{1\ldots\hat{i}\ldots m} = (-1)^{i-1} \lambda_k^\alpha \phi_0^k \wedge \phi^{1\ldots\hat{i}\ldots m} \tag{2.50}
$$

and using Eq. (2.43), we derive

$$
\begin{aligned}
(2.45) \quad = \quad & d\Big((-1)^{j-1}\|h\|^{m-2} \lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m} + (-1)^j \|h\|^{m-2} \lambda_0^\alpha p_j^\alpha \phi^{1\ldots\hat{j}\ldots m}\Big) \\
& + (-1)^i (m-2)\|h\|^{m-4} h_{ij}^\alpha h_{\alpha t}^\gamma h_{\alpha t j}^\gamma \big(d\lambda_0^\alpha - \lambda_0^\alpha \phi_0^0 + \lambda_0^\beta \phi_\beta^\alpha\big) \wedge \phi^{1\ldots\hat{i}\ldots m} \\
& + (2-m)\|h\|^{m-2} \lambda_j^\alpha p_j^\alpha \phi^{1\ldots m} \\
& + (m-2)\|h\|^{m-4} p_j^\alpha \lambda_0^\alpha h_{\alpha t}^\gamma h_{\alpha t j}^\gamma \phi^{1\ldots m}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \|h\|^{m-2}\lambda_0^\alpha p_{jj}^\alpha \phi^{1\ldots m} \\
&\quad + \|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m} \\
&= d\Big((-1)^{j-1}\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m} + (-1)^j \|h\|^{m-2}\lambda_0^\alpha p_j^\alpha \phi^{1\ldots\hat{j}\ldots m}\Big) \\
&\quad + (-1)^i (m-2)\|h\|^{m-4} h_{ij}^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma \, d\lambda_0^\alpha \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^{i-1}(m-2)\|h\|^{m-4} h_{ij}^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma \lambda_0^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^i (m-2)\|h\|^{m-4} h_{ij}^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma \lambda_0^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (2-m)\|h\|^{m-2}\lambda_j^\alpha p_j^\alpha \phi^{1\ldots m} \\
&\quad + (m-2)\|h\|^{m-4} p_j^\alpha \lambda_0^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma \phi^{1\ldots m} \\
&\quad + \|h\|^{m-2}\lambda_0^\alpha p_{j,j}^\alpha \phi^{1\ldots m} \\
&\quad + \|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m} \\
&= d\Big((-1)^{j-1}\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m} + (-1)^j \|h\|^{m-2}\lambda_0^\alpha p_j^\alpha \phi^{1\ldots\hat{j}\ldots m}\Big) \\
&\quad + d\Big((-1)^i (m-2)\|h\|^{m-4} h_{ij}^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma \lambda_0^\alpha \wedge \phi^{1\ldots\hat{i}\ldots m}\Big) \\
&\quad + (-1)^{i-1}(m-2)\|h\|^{m-4} h_{ij}^\alpha \lambda_0^\alpha d\big(h_{\alpha i}^\gamma h_{\alpha j}^\gamma\big) \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^{i-1}(m-2)\|h\|^{m-4} \lambda_0^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma \, dh_{ij}^\alpha \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^{i-1}(m-2)\lambda_0^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma h_{ij}^\alpha \, d\big(\|h\|^{m-4}\big) \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^{i-1}(m-2)\|h\|^{m-4} \lambda_0^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma h_{ij}^\alpha \, d\phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^{i-1}(m-2)\|h\|^{m-4} h_{ij}^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma \lambda_0^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^i (m-2)\|h\|^{m-4} h_{ij}^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma \lambda_0^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (2-m)\|h\|^{m-2}\lambda_j^\alpha p_j^\alpha \phi^{1\ldots m} \\
&\quad + (m-2)\|h\|^{m-4} p_j^\alpha \lambda_0^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma \phi^{1\ldots m} \\
&\quad + \|h\|^{m-2}\lambda_0^\alpha p_{jj}^\alpha \phi^{1\ldots m} \\
&\quad + \|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m} \,.
\end{aligned}
$$

From Eqs. (1.62), (1.54), and (2.47), and assuming $m \neq 5$ (unless $\|h\| \neq 0$ everywhere), we obtain

(2.45) =

$$
\begin{aligned}
& d\Big((-1)^{j-1}\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m} + (-1)^j \|h\|^{m-2}\lambda_0^\alpha p_j^\alpha \phi^{1\ldots\hat{j}\ldots m} \\
&\quad + (-1)^i (m-2)\|h\|^{m-4} h_{ij}^\alpha h_{\alpha i}^\gamma h_{\alpha j}^\gamma \lambda_0^\alpha \phi^{1\ldots\hat{i}\ldots m}\Big) \\
& + (-1)^{i-1}(m-2)\|h\|^{m-4} h_{ij}^\alpha \lambda_0^\alpha \Big(-3h_{\alpha i}^\gamma h_{\alpha j}^\gamma \phi_0^0 - \|h\|^2 \phi_j^0 + h_{\alpha i}^\gamma h_{\alpha p}^\gamma \phi_j^p + H_{jr}\phi_0^r\Big) \wedge \phi^{1\ldots\hat{i}\ldots m}
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{i-1}(m-2)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma\big(h_{kj}^\alpha\phi_i^k+h_{ik}^\alpha\phi_j^k-h_{ij}^\beta\phi_\beta^\alpha-h_{ij}^\alpha\phi_0^0-\delta_{ij}p_k^\alpha\phi_0^k+\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+h_{ijk}^\alpha\phi_0^k\big)\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(-1)^{i-1}(m-2)\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ij}^\alpha\big(-(m-4)\|h\|^{m-4}\phi_0^0+(m-4)\|h\|^{m-6}h_{p\nu}^\nu h_{p\nu k}^\nu\phi_0^k\big)\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(-1)^{i-1}(m-2)\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ij}^\alpha\|h\|^{m-4}\big((m-1)\phi_0^0\wedge\phi^{1\ldots\hat{i}\ldots m}+(-1)^{i+k}\phi_i^k\wedge\phi^{1\ldots\hat{k}\ldots m}\big)\\
&+(-1)^{i-1}(m-2)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ij}^\alpha\phi_0^0\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(-1)^{i}(m-2)\|h\|^{m-4}h_{ij}^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma\lambda_0^\beta\phi_\beta^\alpha\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&-(m-2)\|h\|^{m-2}\lambda_j^\alpha p_j^\alpha\phi^{1\ldots m}\\
&+(m-2)\|h\|^{m-4}p_j^\alpha\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma\phi^{1\ldots m}\\
&+\|h\|^{m-2}\lambda_0^\alpha p_{j,j}^\alpha\phi^{1\ldots m}\\
&+\|h\|^{m-2}h_{ij}^\alpha h_{ik}^\alpha\lambda_0^\beta h_{kj}^\beta\phi^{1\ldots m}\ .
\end{aligned}
$$

$$
\begin{aligned}
(2.45) = \\
= \;& d\big((-1)^{j-1}\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha\phi^{1\ldots\hat{j}\ldots m}+(-1)^{j}\|h\|^{m-2}\lambda_0^\alpha p_j^\alpha\phi^{1\ldots\hat{j}\ldots m}\\
&\quad+(-1)^{i}(m-2)\|h\|^{m-4}h_{ij}^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma\lambda_0^\alpha\phi^{1\ldots\hat{i}\ldots m}\big)\\
&+(-1)^{i}3(m-2)\|h\|^{m-4}h_{ij}^\alpha\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma\phi_0^0\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(-1)^{i}(m-2)\|h\|^{m-2}h_{ij}^\alpha\lambda_0^\alpha\phi_j^0\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(-1)^{i-1}(m-2)\|h\|^{m-4}h_{ij}^\alpha\lambda_0^\alpha h_\alpha^\gamma h_{\alpha p}^\gamma\phi_j^p\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(m-2)\|h\|^{m-4}h_{ij}^\alpha\lambda_0^\alpha H_{ji}\phi^{1\ldots m}\\
&+(-1)^{i-1}(m-2)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{kj}^\alpha\phi_i^k\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(-1)^{i-1}(m-2)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ik}^\alpha\phi_j^k\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(-1)^{i}(m-2)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ij}^\beta\phi_\beta^\alpha\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(-1)^{i}(m-2)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ij}^\alpha\phi_0^0\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&-(m-2)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma p_j^\alpha\phi^{1\ldots m}\\
&+(m-2)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ij,i}^\alpha\phi^{1\ldots m}\\
&+(-1)^{i}(m-2)(m-4)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ij}^\alpha\phi_0^0\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(m-2)(m-4)\|h\|^{m-6}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ij}^\alpha h_{p\nu}^\nu h_{p\nu i}^\nu\phi^{1\ldots m}\\
&+(-1)^{i-1}(m-1)(m-2)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ij}^\alpha\phi_0^0\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(-1)^{k-1}(m-2)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ij}^\alpha\phi_i^k\wedge\phi^{1\ldots\hat{k}\ldots m}\\
&+(-1)^{i-1}(m-2)\|h\|^{m-4}\lambda_0^\alpha h_\alpha^\gamma h_{\alpha j}^\gamma h_{ij}^\alpha\phi_0^0\wedge\phi^{1\ldots\hat{i}\ldots m}
\end{aligned}
$$

$$
\begin{aligned}
&+ (-1)^i (m-2)\|h\|^{m-4} h^\alpha_{ij} h^\gamma_{\alpha t} h^\gamma_{\alpha j} \lambda^\beta_0 \phi^\alpha_\beta \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (2-m)\|h\|^{m-2} \lambda^\alpha_j p^\alpha_j \phi^{1\ldots m} \\
&+ (m-2)\|h\|^{m-4} p^\alpha_j \lambda^\alpha_0 h^\gamma_{\alpha t} h^\gamma_{\alpha j} \phi^{1\ldots m} \\
&+ \|h\|^{m-2} \lambda^\alpha_0 p^\alpha_{jj} \phi^{1\ldots m} \\
&+ \|h\|^{m-2} h^\alpha_{ij} h^\alpha_{ik} \lambda^\beta_0 h^\beta_{kj} \phi^{1\ldots m} \, .
\end{aligned}
$$

This expression can be further simplified, by taking also into acount definition (1.56), so as to yield

$$
\begin{aligned}
(2.45) = \\
= \; & d\Big((-1)^{j-1}\|h\|^{m-2}\lambda^\alpha_i h^\alpha_{ij}\phi^{1\ldots\hat{j}\ldots m} + (-1)^j\|h\|^{m-2}\lambda^\alpha_0 p^\alpha_j \phi^{1\ldots\hat{j}\ldots m} \\
& \quad + (-1)^i (m-2)\|h\|^{m-4} h^\alpha_{ij} h^\gamma_{\alpha t} h^\gamma_{\alpha j} \lambda^\alpha_0 \phi^{1\ldots\hat{i}\ldots m}\Big) \\
& + (-1)^i (m-2)\|h\|^{m-2} h^\alpha_{ij} \lambda^\alpha_0 \phi^0_j \wedge \phi^{1\ldots\hat{i}\ldots m} \\
& + (m-2)\|h\|^{m-4} h^\alpha_{ij} \lambda^\alpha_0 H_{ji} \phi^{1\ldots m} \\
& + m(m-2)\|h\|^{m-4} \lambda^\alpha_0 h^\gamma_{\alpha t} h^\gamma_{\alpha j} p^\alpha_j \phi^{1\ldots m} \\
& + (m-2)(m-4)\|h\|^{m-6} \lambda^\alpha_0 h^\gamma_{\alpha t} h^\gamma_{\alpha j} h^\alpha_{ij} h^\nu_{pv} h^\nu_{pvi} \phi^{1\ldots m} \\
& + (2-m)\|h\|^{m-2} \lambda^\alpha_j p^\alpha_j \phi^{1\ldots m} \\
& + \|h\|^{m-2} \lambda^\alpha_0 p^\alpha_{jj} \phi^{1\ldots m} \\
& + \|h\|^{m-2} h^\alpha_{ij} h^\alpha_{ik} \lambda^\beta_0 h^\beta_{kj} \phi^{1\ldots m} \, .
\end{aligned}
\tag{2.51}
$$

Now we compute separately the term $(2-m)\|h\|^{m-2} p^\alpha_i \lambda^\alpha_i \phi^{1\ldots m}$. Using Eqs. (2.50), (2.43), (1.60), (2.47), and (2.48), we have

$$
\begin{aligned}
\|h\|^{m-2} p^\alpha_i \lambda^\alpha_i \phi^{1\ldots m} &= (-1)^{i-1}\|h\|^{m-2} p^\alpha_i (d\lambda^\alpha_0 - \lambda^\alpha_0 \phi^0_0 + \lambda^\beta_0 \phi^\alpha_\beta) \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&= (-1)^{i-1}\|h\|^{m-2} p^\alpha_i \, d\lambda^\alpha_0 \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^i \|h\|^{m-2} p^\alpha_i \lambda^\alpha_0 \phi^0_0 \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^{i-1}\|h\|^{m-2} p^\alpha_i \lambda^\beta_0 \phi^\alpha_\beta \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&= d\Big((-1)^{i-1}\|h\|^{m-2} p^\alpha_i \lambda^\alpha_0 \phi^{1\ldots\hat{i}\ldots m}\Big) \\
&\quad + (-1)^i \|h\|^{m-2} \lambda^\alpha_0 \, dp^\alpha_i \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^i \lambda^\alpha_0 p^\alpha_i \, d\big(\|h\|^{m-2}\big) \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^i \lambda^\alpha_0 p^\alpha_i \|h\|^{m-2} d\phi^{1\ldots\hat{i}\ldots m}
\end{aligned}
$$

$$
\begin{aligned}
&+ (-1)^{i}\|h\|^{m-2} p_i^{\alpha} \lambda_0^{\alpha} \phi_0^{0} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (-1)^{i-1}\|h\|^{m-2} p_i^{\alpha} \lambda_0^{\beta} \phi_{\beta}^{\alpha} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
= \;& d\big((-1)^{i-1}\|h\|^{m-2} p_i^{\alpha} \lambda_0^{\alpha} \phi^{1\ldots\hat{i}\ldots m}\big) \\
&+ (-1)^{i}\|h\|^{m-2} \lambda_0^{\alpha} p_k^{\alpha} \phi_i^{k} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (-1)^{i-1}\|h\|^{m-2} \lambda_0^{\alpha} p_i^{\beta} \phi_{\beta}^{\alpha} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (-1)^{i-1} 2\|h\|^{m-2} \lambda_0^{\alpha} p_i^{\alpha} \phi_0^{0} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (-1)^{i}\|h\|^{m-2} \lambda_0^{\alpha} h_{ki}^{\alpha} \phi_k^{0} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&- \|h\|^{m-2} \lambda_0^{\alpha} p_{ii}^{\alpha} \phi^{1\ldots m} \\
&+ (-1)^{i-1}(m-2)\|h\|^{m-2} \lambda_0^{\alpha} p_i^{\alpha} \phi_0^{0} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&- (m-2)\|h\|^{m-4} \lambda_0^{\alpha} p_i^{\alpha} h_{st}^{\gamma} h_{sti}^{\gamma} \phi^{1\ldots m} \\
&+ (-1)^{i}(m-1)\|h\|^{m-2} \lambda_0^{\alpha} p_i^{\alpha} \phi_0^{0} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (-1)^{k}\|h\|^{m-2} \lambda_0^{\alpha} p_i^{\alpha} \phi_i^{k} \wedge \phi^{1\ldots\hat{k}\ldots m} \\
&+ (-1)^{i}\|h\|^{m-2} p_i^{\alpha} \lambda_0^{\alpha} \phi_0^{0} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (-1)^{i-1}\|h\|^{m-2} p_i^{\alpha} \lambda_0^{\beta} \phi_{\beta}^{\alpha} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
= \;& d\big((-1)^{i-1}\|h\|^{m-2} p_i^{\alpha} \lambda_0^{\alpha} \phi^{1\ldots\hat{i}\ldots m}\big) \\
&+ (-1)^{i}\|h\|^{m-2} \lambda_0^{\alpha} h_{ki}^{\alpha} \phi_k^{0} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&- \|h\|^{m-2} \lambda_0^{\alpha} p_{ii}^{\alpha} \phi^{1\ldots m} \\
&- (m-2)\|h\|^{m-4} \lambda_0^{\alpha} p_i^{\alpha} h_{st}^{\gamma} h_{sti}^{\gamma} \phi^{1\ldots m} \,.
\end{aligned}
$$

Returning to Eq. (2.51) and substituting the latter expression, we get

$$
\begin{aligned}
&(2.45) = \\
&= d\Big((-1)^{j-1}\|h\|^{m-2} \lambda_i^{\alpha} h_{ij}^{\alpha} \phi^{1\ldots\hat{j}\ldots m} + (-1)^{j}\|h\|^{m-2} \lambda_0^{\alpha} p_j^{\alpha} \phi^{1\ldots\hat{j}\ldots m} \\
&\qquad + (-1)^{i}(m-2)\|h\|^{m-4} h_{ij}^{\alpha} h_{st}^{\gamma} h_{stj}^{\gamma} \lambda_0^{\alpha} \phi^{1\ldots\hat{i}\ldots m} \\
&\qquad + (-1)^{i}(m-2)\|h\|^{m-2} p_i^{\alpha} \lambda_0^{\alpha} \phi^{1\ldots\hat{i}\ldots m}\Big) \\
&\quad + (-1)^{i}(m-2)\|h\|^{m-2} h_{ij}^{\alpha} \lambda_0^{\alpha} \phi_j^{0} \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (m-2)\|h\|^{m-4} h_{ij}^{\alpha} \lambda_0^{\alpha} H_{ji} \phi^{1\ldots m} \\
&\quad + m(m-2)\|h\|^{m-4} \lambda_0^{\alpha} h_{st}^{\gamma} h_{stj}^{\gamma} p_j^{\alpha} \phi^{1\ldots m} \\
&\quad + (m-2)(m-4)\|h\|^{m-6} \lambda_0^{\alpha} h_{st}^{\gamma} h_{stj}^{\gamma} h_{ij}^{\alpha} h_{pv}^{\nu} h_{pvi}^{\nu} \phi^{1\ldots m}
\end{aligned}
$$

$$
\begin{aligned}
&+ (-1)^i(2-m)\|h\|^{m-2}\lambda_0^\alpha h_{ki}^\alpha \phi_k^0 \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (m-2)\|h\|^{m-2}\lambda_0^\alpha p_{ii}^\alpha \phi^{1\ldots m} \\
&+ (m-2)^2\|h\|^{m-4}\lambda_0^\alpha p_i^\alpha h_{st}^\gamma h_{sti}^\gamma \phi^{1\ldots m} \\
&+ \|h\|^{m-2}\lambda_0^\alpha p_{jj}^\alpha \phi^{1\ldots m} \\
&+ \|h\|^{m-2}h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m} \\
= \;& d\Big((-1)^{j-1}\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m} + (-1)^j\|h\|^{m-2}\lambda_0^\alpha p_j^\alpha \phi^{1\ldots\hat{j}\ldots m} \\
&\quad + (-1)^i(m-2)\|h\|^{m-4}h_{ij}^\alpha h_{st}^\gamma h_{stj}^\gamma \lambda_0^\alpha \phi^{1\ldots\hat{i}\ldots m} \\
&\quad + (-1)^i(m-2)\|h\|^{m-2}p_i^\alpha \lambda_0^\alpha \phi^{1\ldots\hat{i}\ldots m}\Big) \\
&+ \lambda_0^\alpha\Big((m-1)\|h\|^{m-2}p_{ii}^\alpha \\
&\qquad + (m-2)\|h\|^{m-4}h_{ij}^\alpha H_{ij} \\
&\qquad + 2(m-1)(m-2)\|h\|^{m-4}p_i^\alpha h_{st}^\gamma h_{sti}^\gamma \\
&\qquad + (m-2)(m-4)\|h\|^{m-6}h_{ij}^\alpha h_{st}^\gamma h_{stj}^\gamma h_{pv}^\nu h_{pvi}^\nu \\
&\qquad + \|h\|^{m-2}h_{kj}^\alpha h_{ji}^\beta h_{ik}^\beta\Big)\,\phi^{1\ldots m}\,.
\end{aligned}
$$

Thus, on U,

$$
\begin{aligned}
L_{f_t}\Psi\big|_{t=0} \;=\;& d\big(\frac{1}{m}(-1)^{k-1}\|h\|^m\lambda_0^k\phi^{1\ldots\hat{k}\ldots m}\big) + \\
&+ d\Big((-1)^{j-1}\|h\|^{m-2}(\lambda_i^\alpha h_{ij}^\alpha - \lambda_0^\alpha p_j^\alpha)\phi^{1\ldots\hat{j}\ldots m}\Big) \\
&+ d\Big((-1)^i(m-2)\|h\|^{m-4}h_{ij}^\alpha h_{st}^\gamma h_{stj}^\gamma \lambda_0^\alpha \phi^{1\ldots\hat{i}\ldots m} \\
&\qquad + (-1)^i(m-2)\|h\|^{m-2}p_i^\alpha \lambda_0^\alpha \phi^{1\ldots\hat{i}\ldots m}\Big) \\
&+ \lambda_0^\alpha\Big((m-1)\|h\|^{m-2}p_{ii}^\alpha \\
&\qquad + (m-2)\|h\|^{m-4}h_{ij}^\alpha H_{ij} \\
&\qquad + 2(m-1)(m-2)\|h\|^{m-4}p_i^\alpha h_{st}^\gamma h_{sti}^\gamma \\
&\qquad + (m-2)(m-4)\|h\|^{m-6}h_{ij}^\alpha h_{st}^\gamma h_{stj}^\gamma h_{pv}^\nu h_{pvi}^\nu \\
&\qquad + \|h\|^{m-2}h_{kj}^\alpha h_{ji}^\beta h_{ik}^\beta\Big)\,\phi^{1\ldots m}\,. \qquad (2.52)
\end{aligned}
$$

If $m = 2$, this equation reduces to

$$
\begin{aligned}
L_{f_t}\Psi\big|_{t=0} \;=\;& d\big(\frac{1}{m}(-1)^{k-1}\|h\|^m\lambda_0^k\phi^{1\ldots\hat{k}\ldots m}\big) + \\
&+ d\Big((-1)^{j-1}\|h\|^{m-2}(\lambda_i^\alpha h_{ij}^\alpha - \lambda_0^\alpha p_j^\alpha)\phi^{1\ldots\hat{j}\ldots m}\Big) \\
&+ \lambda_0^\alpha p_{ii}^\alpha \phi^{1\ldots m}\,,
\end{aligned}
$$

since in this case, for each α, $h^{\alpha}_{kj}h^{\beta}_{ji}h^{\beta}_{ik} = 0$ as a consequence of $h^{\alpha}_{ii} = 0$.
If $m = 4$, Eq. (2.52) takes the form

$$\begin{aligned} L_{\frac{\partial}{\partial t}}\Psi\Big|_{t=0} \;=\; & d\Big(\frac{1}{m}(-1)^{k}\|h\|^{m}\lambda^{k}_{0}\phi^{1\ldots\hat{k}\ldots m}\Big) + \\ & + d\Big((-1)^{j-1}\|h\|^{m-2}(\lambda^{\alpha}_{i}h^{\alpha}_{ij} - \lambda^{\alpha}_{0}p^{\alpha}_{j})\phi^{1\ldots\hat{j}\ldots m}\Big) \\ & + d\Big((-1)^{i}2\lambda^{\alpha}_{0}(h^{\alpha}_{ij}h^{\gamma}_{\alpha}h^{\gamma}_{\alpha j} + \|h\|^{2}p^{\alpha}_{i})\phi^{1\ldots\hat{i}\ldots m}\Big) \\ & + \lambda^{\alpha}_{0}\big(3\|h\|^{2}p^{\alpha}_{ii} + 2h^{\alpha}_{ij}H_{ij} + 12p^{\alpha}_{i}h^{\gamma}_{\alpha}h^{\gamma}_{\alpha i} + \|h\|^{2}h^{\alpha}_{kj}h^{\beta}_{ji}h^{\beta}_{ik}\big)\phi^{1\ldots m}\,. \end{aligned}$$

If $m = 3$ or $m = 5$, Eq. (2.52) only holds at the points where $\|h\| \neq 0$, that is, outside of the set of umbilic points.
Using now the transformation laws (2.39) and (2.40) for the λ^{i}_{0}, λ^{α}_{0}, λ^{α}_{i} under a change of map $e : M \times (-\epsilon, \epsilon) \to G$ in the conditions (2.21), and the transformation laws for second-order G-frames along f given in Sec. 1.2.C, we can easily verify that the local forms

$$\begin{aligned} \lambda^{\alpha}_{0}\Big(&(m-1)\|h\|^{m-2}p^{\alpha}_{ii} + (m-2)\|h\|^{m-4}h^{\alpha}_{ij}H_{ij} + \\ & + 2(m-1)(m-2)\|h\|^{m-4}p^{\alpha}_{i}h^{\gamma}_{\alpha}h^{\gamma}_{\alpha i} + \\ & + (m-2)(m-4)\|h\|^{m-6}h^{\alpha}_{ij}h^{\gamma}_{\alpha}h^{\gamma}_{\alpha j}h^{\nu}_{p\sigma}h^{\nu}_{p\sigma i}\Big)\,\phi^{1\ldots m}\,, \end{aligned} \tag{2.53}$$

$$\lambda^{\alpha}_{0}\|h\|^{m-2}h^{\alpha}_{kj}h^{\beta}_{ji}h^{\beta}_{ik}\phi^{1\ldots m}\,, \tag{2.54}$$

$$(-1)^{k-1}\|h\|^{m}\lambda^{k}_{0}\phi^{1\ldots\hat{k}\ldots m}\,, \tag{2.55}$$

$$(-1)^{i-1}\|h\|^{m-2}(\lambda^{\alpha}_{j}h^{\alpha}_{ij} - \lambda^{\alpha}_{0}p^{\alpha}_{i})\phi^{1\ldots\hat{i}\ldots m}\,, \tag{2.56}$$

$$(-1)^{i}(m-2)\big(\|h\|^{m-4}\lambda^{\alpha}_{0}h^{\alpha}_{ij}h^{\gamma}_{\alpha}h^{\gamma}_{\alpha j} + \|h\|^{m-2}\lambda^{\alpha}_{0}p^{\alpha}_{i}\big)\phi^{1\ldots\hat{i}\ldots m}\,, \tag{2.57}$$

and

$$\begin{aligned} &\lambda^{\alpha}_{0}\big(\|h\|^{m-2}p^{\alpha}_{ii} + (m-2)\|h\|^{m-4}p^{\alpha}_{i}h^{\gamma}_{\alpha}h^{\gamma}_{\alpha i}\big)\phi^{1\ldots m} + \\ &- (m-2)\lambda^{\alpha}_{i}\big(\|h\|^{m-2}p^{\alpha}_{i} + \|h\|^{m-4}h^{\alpha}_{ij}h^{\gamma}_{\alpha}h^{\gamma}_{\alpha j}\big)\phi^{1\ldots m} \end{aligned} \tag{2.58}$$

are well-defined global forms on all $\overline{D}$ (if $m = 3$ or $m = 5$, only away from the umbilic points).
Hence, Eq. (2.52) is of the form

$$L_{\frac{\partial}{\partial t}}\Psi\Big|_{t=0} = d\varsigma + \vartheta$$

with ς and ϑ a globally well-defined $(m-1)$- resp. m-form. Moreover, ς has compact support in $G' \subset D$, just as λ^{α}_{0}, λ^{i}_{0}, and λ^{α}_{i}. Therefore, integrating $L_{\frac{\partial}{\partial t}}\Psi\big|_{t=0}$

over D and applying Stokes' theorem, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\big|_{t=0} = \int_D \lambda_0^\alpha \big(& (m-1)\|h\|^{m-2}p^\alpha_{ii} + (m-2)\|h\|^{m-4}h^\alpha_{ij}H_{ij} + \\ & + 2(m-1)(m-2)\|h\|^{m-4}p^\alpha_i h^\gamma_{\alpha t}h^\gamma_{\alpha t i} + \\ & + (m-2)(m-4)\|h\|^{m-6}h^\alpha_{ij}h^\gamma_{\alpha t}h^\gamma_{\alpha t j}h^\nu_{\rho\sigma}h^\nu_{\rho\sigma i} + \\ & + \|h\|^{m-2}h^\alpha_{kj}h^\beta_{ij}h^\beta_{ik}\ \big)\, \phi^{1\ldots m}\ .\end{aligned}$$

Since λ_0^α may be any smooth function with compact support $C' \subset D$, we conclude that f is a critical point of $\mathcal{W}_D$, iff $\forall\alpha$

$$\begin{aligned}& (m-1)\|h\|^{m-2}p^\alpha_{ii} + (m-2)\|h\|^{m-4}h^\alpha_{ij}H_{ij} + \\ & + 2(m-1)(m-2)\|h\|^{m-4}p^\alpha_i h^\gamma_{\alpha t}h^\gamma_{\alpha t i} + \\ & + (m-2)(m-4)\|h\|^{m-6}h^\alpha_{ij}h^\gamma_{\alpha t}h^\gamma_{\alpha t j}h^\nu_{\rho\sigma}h^\nu_{\rho\sigma i} + \\ & + \|h\|^{m-2}h^\alpha_{kj}h^\beta_{ij}h^\beta_{ik} \ = \ 0\ .\end{aligned}$$

This Euler-Lagrange equation is conformally invariant, i.e. the vanishing of the l.h.s. does not depend on the choice of second-order G-frame field along f. ♡

2.3 The Conformal Gauss Map

In Riemannian geometry, there exist well-known relations between the mean curvature of immersed submanifolds of the Euclidean space and the tension field of their respective Gauss maps, as e.g. the result of Ruh and Vilms quoted in Ch. 0 of Part I, or the somewhat more elaborate result for immersed surfaces due to Hoffman and Osserman [Ho-Os/82]. Something similar can be done for immersed m-submanifolds $f: M^m \to S^n$ of the Möbius space. In Ref. [Br/84], Bryant defined a (hyperbolic) conformal Gauss map for immersions $f: M^2 \to S^3$ as a map $\gamma_f : M^2 \to \mathcal{Q}$, with $\mathcal{Q}$ the hyperboloid of $\mathbb{R}^5$

$$\mathcal{Q} = \{x \in \mathbb{R}^5\ :\ <x,x>= 1\}\ ,$$

given by $\gamma_f(x) = e_3(x)$, where $e: M \to G$ is an arbitrary second-order G-frame along f, defined on a neighbourhood of the point x. From the transformation law (1.71), we see that γ_f is well-defined. In Ref. [Ri/87], Rigoli extended the above definition to the case of an immersion $f : M^m \to S^n$, for any $m \le n$, as follows. Let $G_{n-m}(\mathbb{R}^{n+2})$ denote the Grassmannian manifold of the $n-m$ planes of $\mathbb{R}^{n+1}$. Fix $\mathcal{O} = \mathrm{span}\{\eta_{m+1},\ldots,\eta_n\}$ as the origin of $G_{n-m}(\mathbb{R}^{n+2})$. Note that $\mathcal{O} = \mathrm{span}\{P(\epsilon_{m+1}),\ldots,P(\epsilon_n)\}$, for some $P \in G$, where $\epsilon_0,\epsilon_1,\ldots,\epsilon_n,\epsilon_{n+1}$ is the canonic basis of $\mathbb{R}^{n+2}$. Then, G acts on the left on $G_{n-m}(\mathbb{R}^{n+2})$ by matrix

multiplication. The *conformal Grassmannian* is the open orbit $\mathcal{Q}_{n-m}(\mathbb{R}^{n+2})$ of the origin, reading

$$\mathcal{Q}_{n-m}(\mathbb{R}^{n+2}) = G(\mathcal{O}) = \{\mathrm{span}\{P(\eta_{m+1}),\ldots,P(\eta_n)\} \;:\; P\in G\} \;,$$

which is a submanifold of $G_{n-m}(\mathbb{R}^{n+2})$. The group G acts transitively on $\mathcal{Q}_{n-m}(\mathbb{R}^{n+2})$ and the isotropic subgroup of G at $\mathcal{O}$ is given by

$$H_{\mathcal{O}} = \left\{ \begin{bmatrix} a & {}^tZ & 0 & b \\ X & A & 0 & Y \\ 0 & 0 & B & 0 \\ c & {}^tW & 0 & d \end{bmatrix} \in G \;:\; \begin{array}{l} X,Y,Z,W\in\mathbb{R}^m \\ A\in SO(m),\; B\in SO(n-m) \\ a,b,c,d\in\mathbb{R} \end{array} \right\} . \qquad (2.59)$$

Observe that X,Y,Z,W and a,b,c,d cannot be chosen arbitrarily, but must satisfy the relations

$$\begin{cases} (W=0 \;\vee\; Z=0)\wedge(X=0\;\vee\;Y=0) \\ -aW+{}^tAX-cZ=0 \\ -dZ+{}^tAY-bW=0 \\ dX-AW+cY=0 \\ aY-AZ+bX=0 \end{cases} ,\quad \begin{cases} {}^tXX=2ca \\ {}^tYY=2bd \\ {}^tZZ=2ba \\ {}^tWW=2dc \\ da+bc=1 \;, \end{cases} \qquad (2.60)$$

which can be obtained from the closure of $H_{\mathcal{O}}$ w.r.t. matrix inversion.
Thus, $\mathcal{Q}_{n-m}(\mathbb{R}^{n+2})$ can be identified with the homogeneous space $G/H_{\mathcal{O}}$ with canonic projection $\hat{\Pi} : G \to \mathcal{Q}_{n-m}(\mathbb{R}^{n+2})$ given by $\hat{\Pi}(P) = \mathrm{span}\{P(\eta_{m+1}),\ldots,P(\eta_n)\}$. The conformal Grassmannian $\mathcal{Q}_{n-m}(\mathbb{R}^{n+2})$ has dimension $(n-m)(m+2)$ and carries a pseudo-metric with signature $(\underbrace{-\ldots-}_{n-m},\underbrace{+\ldots+}_{(m+1)(n-m)})$ given by

$$d\ell^2 = -\varsigma^*\Phi^0_\alpha\otimes\varsigma^*\Phi^\alpha_0 - \varsigma^*\Phi^\alpha_0\otimes\varsigma^*\Phi^0_\alpha + \varsigma^*\Phi^i_\alpha\otimes\varsigma^*\Phi^i_\alpha \;, \qquad (2.61)$$

where $\varsigma : \mathcal{Q}_{n-m}(\mathbb{R}^{n+2}) \to G$ is a local section of the principal bundle $\hat{\Pi} : G \to G/H_{\mathcal{O}} \simeq \mathcal{Q}_{n-m}(\mathbb{R}^{n+2})$ and Φ^a_b are the components of the Maurer-Cartan form Φ of G. Denoting $\theta^{0,\alpha} = \varsigma^*\Phi^\alpha_0$, $\theta^{\alpha,0} = \varsigma^*\Phi^0_\alpha$, $\theta^{\alpha,i} = \varsigma^*\Phi^i_\alpha$, ordering the pairs $(\alpha,0)$, (α,i), $(0,\alpha)$, as

$$\begin{array}{l} (\gamma,0) < (\beta,i) < (0,\alpha)\;,\;\; \forall \alpha,\beta,\gamma,i \\ (0,\beta) < (0,\alpha) \iff \beta > \alpha \\ (\beta,j) < (\alpha,i) \iff \beta<\alpha \vee (\beta=\alpha\wedge j<i) \\ (\beta,0) < (\alpha,0) \iff \beta<\alpha \;, \end{array}$$

and representing by the symbols $\hat{A},\hat{B},\ldots$ the $(m+2)(n-m)$ indices $(\alpha,0),(0,\alpha),(\alpha,i)$, one can write $d\ell^2$ as

$$d\ell^2 = g_{\hat{A}\hat{B}}\theta^{\hat{A}}\otimes\theta^{\hat{B}}$$

with

$$
[g_{\bar A\bar B}] = \hat S = \begin{array}{c} \left|\begin{array}{ccc} 0 & 0 & \begin{pmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{pmatrix} \\ 0 & I & 0 \\ \begin{pmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{pmatrix} & 0 & 0 \end{array}\right| \\ \begin{array}{ccc} \underbrace{\qquad\qquad}_{n-m} & \underbrace{\qquad}_{m(n-m)} & \underbrace{\qquad\qquad}_{n-m} \end{array} \end{array} \begin{array}{l} \}\, n-m \\ \}\, m(n-m) \\ \}\, n-m \end{array} .
$$

The Levi-Civita forms $\omega^{\bar A}_{\bar B}$ with respect to the non-orthonormal co-frame $\theta^{\bar A}$ is given by

$$
\begin{aligned} d\theta^{\bar A} &= -\omega^{\bar A}_{\bar B} \wedge \theta^{\bar B} \\ g_{\bar A\bar C}\omega^{\bar C}_{\bar B} + g_{\bar B\bar C}\omega^{\bar C}_{\bar A} &= 0\,. \end{aligned} \tag{2.62}
$$

From these equations and the structure equations (1.18), one obtains the relations

$$
\begin{aligned} &\omega^{\alpha,0}_{\beta,0} = \varsigma^*(\delta^\alpha_\beta \Phi^0_0 + \Phi^\alpha_\beta) \quad , \quad \omega^{\alpha,0}_{\beta,i} = \delta^\alpha_\beta \varsigma^* \Phi^0_i \quad , \quad \omega^{\alpha,0}_{0,\beta} = 0 \\ &\omega^{\alpha,i}_{\beta,k} = \varsigma^*(\delta^\alpha_\beta \Phi^i_k + \delta^i_k \Phi^\alpha_\beta) \quad , \quad \omega^{\alpha,i}_{0,\beta} = \delta^\alpha_\beta \varsigma^* \Phi^0_i \quad , \quad \omega^{0,\alpha}_{0,\beta} = \varsigma^*(\Phi^\alpha_\beta - \delta^\alpha_\beta \Phi^0_0)\,. \end{aligned} \tag{2.63}
$$

The *conformal Gauss map* γ_f of an immersion $f : M^m \to S^n$ is then given by

$$
\begin{array}{rccc} \gamma_f : & M^m & \longrightarrow & Q_{n-m}(\mathbb{R}^{n+2}) \\ & x & \rightarrow & \mathrm{span}\{e_{m+1}(x), \ldots, e_n(x)\}\,, \end{array} \tag{2.64}
$$

where $e = [e_0, e_i, e_\alpha, e_{n+1}] : M \to G$ is a second-order G-frame field of $\Pi : G \to S^n$ along f defined in a neighbourhood of x. From the transformation law (1.71), we see that this map is well-defined. When $m = n - 1$, $Q_1(\mathbb{R}^{n+2})$ can be identified with the projectivisation of the 1-fold hyperboloid $Q = \{x \in \mathbb{R}^{n+2} : < x, x >= 1\}$ supplied with the Lorentz inner product induced by the one of $\mathbb{R}^{n+2}$, still to be denoted by dl^2. In this case it is more practical to use the *hyperbolic conformal Gauss map*, still to be denoted as γ_f, given by

$$
\begin{array}{rccc} \gamma_f : & M^{n-1} & \longrightarrow & Q \\ & x & \rightarrow & e_n\,, \end{array} \tag{2.65}
$$

which generalises the conformal Gauss map for immersed surfaces in S^3 used by Bryant. Rigoli [Ri/87] proved, in the general case, that

$$
\gamma_f^* dl^2 = \mathcal{N} \tag{2.66}
$$

with $\mathcal{N}$ defined in Eq. (1.88), obtaining the following proposition:

Proposition (Bryant,Rigoli) *Let $f : M^m \to S^n$ be an immersion of an m-manifold M endowed with the induced conformal structure. Then, for $x \in M$, $d\gamma_f(x)$ is not injective, iff $\mathcal{N}(x)$ is a degenerate symmetric bilinear map. Let $c(\gamma_f)$ be the set of points x in these conditions. In the case $m = 2$, $\mathcal{N} = \frac{1}{2}(\text{trace}\mathcal{N})g$ (see Eqs. (1.43), (1.89) for notations), whence $c(\gamma_f)$ is the set of umbilic points of f. In the general case, outside $c(\gamma_f)$, γ_f induces a positive definite metric on M that belongs to the conformal class of $M \backslash c(\gamma_f)$, iff $\mathcal{N}$ does so. This is always the case, when $m = 2$.*

Another variational problem, mentioned in Ref. [Ri/87], is the one associated with the functional

$$\eta_D(\rho) = \frac{1}{m}\int_D \left(\text{trace}(\rho^* d\ell^2)\right)^{\frac{m}{2}} dV \tag{2.67}$$

with $\overline{D}$ a compact domain of M, applied to maps $\rho : \overline{D} \to \mathcal{Q}_{n-m}(\mathbb{R}^{n+2})$ with the property (only for m odd) $\text{trace}(\rho^* d\ell^2) \geq 0$, and where the trace and dV are taken relative to any metric belonging to the conformal class of M. We remark that, obviously, definition (2.67) can be generalised to any map $\rho : \overline{D} \to \mathcal{Q}_{n-m}(\mathbb{R}^{n+2})$, replacing $\left(\text{trace}(\rho^* d\ell^2)\right)^{\frac{m}{2}}$ by $|\text{trace}(\rho^* d\ell^2)|^{\frac{m}{2}}$. Moreover, for $m = 2$, $\eta_D(\rho)$ is the energy functional. The functional η is well-defined: given two second-order G-frames along f, say $e, \tilde{e} : M \to G$, from the transformation laws (1.72) and (1.82) we have

$$\begin{aligned}\left(\widetilde{\text{trace}}(\rho^* d\ell^2)\right)^{\frac{m}{2}} d\tilde{V} &= \left(\rho^* d\ell^2(\hat{E}_i, \hat{E}_i)\right)^{\frac{m}{2}} d\tilde{V} \\ &= \left(\text{trace}(\rho^* d\ell^2)\right)^{\frac{m}{2}} dV\,,\end{aligned}$$

where $\tilde{E}_i$ and E_i are the duals of the co-frames $\tilde{\phi}_0^i$ resp. ϕ_0^i.
Thus, from Eq. (2.66), one has $\mathcal{W}(f) = \eta(\gamma_f)$. Rigoli calculated, in the case $2 = m \leq n$, the Euler-Lagrange equation for the functional $\eta(\rho)$ when $\rho = \gamma_f$ (see also Remark 2.1 below). Here we are going to discuss the case where $f : M \to S^n$ is an immersion of a hypersurface into the Möbius space, i.e. $m = n-1$. For convenience, we consider, in this case, the functional (2.67) to act on maps $\rho : \overline{D} \to \mathcal{Q}$ satisfying (only for m odd) $\text{trace}(\rho^* d\ell^2) \geq 0$, where now $d\ell^2$ denotes the induced Lorentz inner product of $\mathcal{Q}$. One can easily derive the Euler-Lagrange equation of this functional, obtaining (for $m \neq 3$) (see Appendix II)

$$\text{trace}\nabla\Big(\left(\text{trace}(\rho^* d\ell^2)\right)^{\frac{m-2}{2}} d\rho\Big) = \tag{2.68}$$

$$\operatorname{trace}\left\{\frac{m-2}{2}\left(\operatorname{trace}(d\rho^* d\ell^2)\right)^{\frac{m-4}{2}} d\left(\operatorname{trace}(\rho^* d\ell^2)\right)\otimes d\rho+\left(\operatorname{trace}(\rho^* d\ell^2)\right)^{\frac{m-2}{2}}\nabla d\rho\right\}=0\,,$$

where M is considered with one of the metrics out of its conformal class, Ω with the induced Lorentz inner product $d\ell^2$, and both with the respective Levi-Civita connections. Let us suppose now that $\rho=\gamma_f : M \to \Omega$ is the hyperbolic conformal Gauss map given in Eq. (2.65). Let $x_0 \in M$ and let $e : M \to G$ be a second-order frame field defined near x_0. Then, $\gamma_f(x) = e_n(x)$ near x_0. From Eq. (1.67), we have

$$d\gamma_f = de_n = p^n_j \phi^j_0 e_0 - h^n_{ij}\phi^j_0 e_i\,.$$

Therefore, as the components of e satisfy Eq. (1.23) (with s_a replaced by e_a), we get, for $u, v \in T_x M$,

$$\begin{aligned}
\gamma_f^* d\ell^2(u,v) &= \;< d\gamma_{f_x}(u), d\gamma_{f_x}(v) > \\
&= \left\langle p^n_j\phi^j_0(u)e_0 - h^n_{ij}\phi^j_0(u)e_i\,,\, p^n_k\phi^k_0(v)e_0 - h^n_{ik}\phi^k_0(v)e_i\right\rangle \\
&= h^n_{ij}h^n_{ik}\phi^j_0(u)\phi^k_0(v)\,,
\end{aligned}$$

that is,

$$\gamma_f^* d\ell^2 = \phi^i_n \otimes \phi^i_n = h^n_{ij}h^n_{ik}\phi^j_0 \otimes \phi^k_0 = N_{jk}\phi^j_0 \otimes \phi^k_0 = N\,,$$

which, by the way, also proves Eq. (2.66). Hence, considering M with the metric $g = \phi^i_0 \otimes \phi^i_0$, we have

$$\operatorname{trace}(\gamma_f^* d\ell^2) = N_{jj} = h^n_{ij}h^n_{ij} \geq 0\,, \tag{2.69}$$

and, in particular,

$$\eta_D(\gamma_f) = \frac{1}{m}\int_D \left(\operatorname{trace}(\gamma_f^* d\ell^2)\right)^{\frac{m}{2}} dV = \frac{1}{m}\int_D (N_{jj})^{\frac{m}{2}} dV = W_D(f)\,.$$

Now we evaluate the Euler-Lagrange equation (2.68) for $\rho = \gamma_f$. To that end we compute $\operatorname{trace}\nabla d\gamma_f$, whereby considering M to be supplied with the Levi-Civita connection ∇ corresponding to the Riemannian metric $g = \phi^i_0 \otimes \phi^i_0$ and Ω with the induced Lorentz metric $d\ell^2$. One can immediately conclude from the structure equations (1.68) that this connection on M is defined by the connections forms

$$\upsilon^i_k = \phi^i_k + \mu_k\phi^i_0 - \mu_i\phi^k_0\,,$$

where $\phi^0_0 = \mu_k\phi^k_0$. The Levi-Civita connection on Ω satisfies $\left(\nabla^{\Omega}_u X\right)_{(e_n)} = d(X)_{(e_n)}(u)$ $-< d(X)_{(e_n)}(u), e_n > e_n$, where $X \in C^\infty(T\Omega)$, $u \in T_{(e_n)}\Omega$, and, on the r.h.s., X

is considered as a map from Ω to $\mathbb{R}^{n+1}$. Let E_i denote the dual of the co-frame ϕ_0^i. Then, $\nabla E_i = v_i^k E_k$. Let $\nabla^{\Omega'}$ denote the pull-back connection on $\gamma_f^{-1}T\Omega$. We have $\nabla d\gamma_f(E_i, E_i) = \nabla_{E_i}^{\Omega'}(d\gamma_f(E_i)) - d\gamma_f(\nabla_{E_i}E_i)$. From Eq. (1.67), we get

$$\begin{aligned} de_n(E_i) &= p_i^n e_0 - h_{ki}^n e_k \\ de_0(E_i) &= \mu_i e_0 + e_i \\ de_k(E_i) &= \phi_k^0(E_i)e_0 + \phi_k^j(E_i)e_j + h_{ki}^n e_n + \delta_{ki}e_{n+1} \,. \end{aligned}$$

Then, by Eqs. (1.60), (1.54), and (1.56),

$$\begin{aligned} d(d\gamma_f(E_i))(E_i) &= d(p_i^n e_0 - h_{ki}^n e_k)(E_i) \\ &= dp_i^n(E_i)e_0 + p_i^n de_0(E_i) - dh_{ki}^n(E_i)e_k - h_{ki}^n de_k(E_i) \\ &= (p_k^n\phi_i^k - 2p_i^n\phi_0^0 + h_{ki}^n\phi_k^0 + p_{ik}^n\phi_0^k)(E_i)e_0 + p_i^n(\mu_i e_0 + e_i) + \\ &\quad - (h_{kj}^n\phi_i^j + h_{ij}^n\phi_k^j - h_{ik}^n\phi_0^0 - \delta_{ik}p_j^n\phi_0^j + h_{ikj}^n\phi_0^j)(E_i)e_k + \\ &\quad - h_{ki}^n(\phi_k^0(E_i)e_0 + \phi_k^j(E_i)e_j + h_{ki}^n e_n + \delta_{ki}e_{n+1}) \\ &= (-m+2)p_i^n e_i - \mu_i p_i^n e_0 + p_k^n\phi_i^k(E_i)e_0 + p_{ii}^n e_0 + \\ &\quad - h_{kj}^n\phi_i^j(E_i)e_k + \mu_i h_{ik}^n e_k - h_{ki}^n h_{ki}^n e_n \,. \end{aligned}$$

Hence,

$$\begin{aligned} \nabla_{E_i}^{\Omega'}(d\gamma_f(E_i)) &= (-m+2)p_i^n e_i - \mu_i p_i^n e_0 + p_k^n\phi_i^k(E_i)e_0 + \\ &\quad + p_{ii}^n e_0 - h_{kj}^n\phi_i^j(E_i)e_k + \mu_i h_{ik}^n e_k \end{aligned}$$

and

$$\begin{aligned} d\gamma_f(\nabla_{E_i}E_i) &= de_n(v_i^k(E_i)E_k) = v_i^k(E_i)(p_k^n e_0 - h_{jk}^n e_j) \\ &= (\phi_i^k(E_i) + \mu_i\phi_0^k(E_i) - \mu_k\phi_0^i(E_i))(p_k^n e_0 - h_{jk}^n e_j) \\ &= (\phi_i^k(E_i) + (-m+1)\mu_k)(p_k^n e_0 - h_{jk}^n e_j) \\ &= p_k^n\phi_i^k(E_i)e_0 + (-m+1)\mu_k p_k^n e_0 - h_{jk}^n\phi_i^k(E_i)e_j + (m-1)\mu_k h_{kj}^n e_j \,. \end{aligned}$$

So, we obtain

$$\nabla d\gamma_f(E_i, E_i) = -(m-2)(p_k^n + \mu_i h_{ik}^n)e_k + (p_{ii}^n + (m-2)\mu_i p_i^n)e_0 \,.$$

Therefore, for $\rho = \gamma_f$, Eq. (2.68) becomes (with notation (2.46) and Eq. (2.47))

$$(2.68) \;=\; \frac{m-2}{2}\|h\|^{m-4}d\|h\|^2 \otimes de_n(E_i, E_i) + \|h\|^{m-2}\nabla d\gamma_f(E_i, E_i)$$

$$
\begin{aligned}
&= \frac{m-2}{2}\|h\|^{m-4}\big(-2\|h\|^2\phi_0^0 + 2h^n_{st}h^n_{stk}\phi_0^k\big)(E_i)\,(p_i^n e_0 - h^n_{ki}e_k) \\
&\quad -(m-2)\|h\|^{m-2}(p_k^n + \mu_i h^n_{ik})e_k + \|h\|^{m-2}(p^n_{ii} + (m-2)\mu_i p_i^n)e_0 \\
&= (m-2)\big(-\|h\|^{m-2}\mu_i + \|h\|^{m-4}h^n_{st}h^n_{sti}\big)(p_i^n e_0 - h^n_{ki}e_k) + \\
&\quad -(m-2)\|h\|^{m-2}(p_k^n + \mu_i h^n_{ik})e_k + \|h\|^{m-2}(p^n_{ii} + (m-2)\mu_i p_i^n)e_0 \\
&= (m-2)\|h\|^{m-4}h^n_{st}h^n_{sti}p_i^n e_0 - (m-2)\|h\|^{m-4}h^n_{st}h^n_{sti}h^n_{ki}e_k + \\
&\quad -(m-2)\|h\|^{m-2}p_k^n e_k + \|h\|^{m-2}p^n_{ii}e_0 \,.
\end{aligned}
$$

Consequently, since e_0, e_k are linearly independent, γ_f is a critical point of η_D, i.e. the expression (2.68) vanishes for $\rho = \gamma_f$, iff

$$
\begin{cases}
(m-2)(\operatorname{trace}\mathcal{N})^{\frac{m-4}{2}}h^n_{st}h^n_{sti}p_i^n + (\operatorname{trace}\mathcal{N})^{\frac{m-2}{2}}p^n_{ii} = 0 \\
\text{and} \\
(m-2)\big((\operatorname{trace}\mathcal{N})^{\frac{m-4}{2}}h^n_{st}h^n_{sti}h^n_{ik} + (\operatorname{trace}\mathcal{N})^{\frac{m-2}{2}}p_k^n\big) = 0\,, \quad \forall k = 1,\dots,m\,.
\end{cases}
\tag{2.70}
$$

The vanishing of the latter system is independent of the choice of second-order frame. Observe that, if $m = 2$ and $n = 3$, this system reduces to the equation $p^3_{ii} = 0$, which is the Euler-Lagrange equation of $\mathcal{W}$.

Remark 2.1 In a private communication (see also Ref. [Ri-Sa/88]), Rigoli demonstrated that, in the most general case ($m \le n$), the conformal Gauss map $\gamma_f : M \to Q_{n-m}(\mathbb{R}^{n+1})$ is a critical point of the functional (2.67), iff

$$
\begin{cases}
(m-2)(\operatorname{trace}\mathcal{N})^{\frac{m-4}{2}}h^\beta_{st}h^\beta_{sti}p_i^\alpha + (\operatorname{trace}\mathcal{N})^{\frac{m-2}{2}}p^\alpha_{ii} = 0\,, \quad \forall\alpha = m+1,\dots,n \\
\text{and} \\
(m-2)\big((\operatorname{trace}\mathcal{N})^{\frac{m-4}{2}}h^\beta_{st}h^\beta_{sti}h^\alpha_{ik} + (\operatorname{trace}\mathcal{N})^{\frac{m-2}{2}}p_k^\alpha\big) = 0\,, \quad \begin{matrix}\forall k = 1,\dots,m \\ \forall\alpha = m+1,\dots,n\,,\end{matrix}
\end{cases}
\tag{2.71}
$$

which generalises Eq. (2.70). This result can be derived in an analogous way to the special case $m = 2$ with $n \ge 2$ arbitrary, treated in Ref. [Ri/87]. Observe also that, for $m = 2$, Eq. (2.71) is identical to the Euler-Lagrange equation of $\mathcal{W}$.

Consequently, if $m = 2$, then γ_f is a critical point of η, iff f is a critical point of $\mathcal{W}$. Now we analyse the general case $m \le n$ arbitrary. Let $f : M^m \to S^n$ be an immersion, such that γ_f is a critical point of the functional η. Then, following the

computations in the proof of Prop. 2.1, we obtain for a variation f_t of f equation (2.49), yielding

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\big|_{t=0} = \int_D \Big\{ & d\big(\tfrac{1}{m}\|h\|^m(-1)^{k-1}\lambda_0^k\phi^{1\ldots\hat{k}\ldots m}\big) \\ & + d\big((-1)^{j-1}\|h\|^{m-2}\lambda_i^\alpha h_{ij}^\alpha\phi^{1\ldots\hat{j}\ldots m} + (-1)^j\|h\|^{m-2}\lambda_0^\alpha p_j^\alpha\phi^{1\ldots\hat{j}\ldots m}\big) \\ & - \lambda_i^\alpha\big((m-2)\|h\|^{m-4}h_{ij}^\alpha h_{st}^\gamma h_{stj}^\gamma + (m-2)\|h\|^{m-2}p_i^\alpha\big)\phi^{1\ldots m} \\ & + \lambda_0^\alpha\Big((m-2)\|h\|^{m-4}p_j^\alpha h_{st}^\gamma h_{stj}^\gamma + \|h\|^{m-2}p_{j,j}^\alpha + \\ & \quad + \|h\|^{m-2}h_{ij}^\beta h_{ik}^\beta h_{kj}^\alpha\Big)\phi^{1\ldots m}\Big\}\,.\end{aligned}$$

Taking into account that the expressions given in Eqs. (2.53–2.58) define tensors, that the λ_i^α, λ_0^A have compact support, and that Eq. (2.71) holds, we obtain, by using Stokes' theorem,

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\big|_{t=0} = \int_D \|h\|^{m-2}\lambda_0^\alpha h_{ij}^\alpha h_{ik}^\beta h_{kj}^\beta\phi^{1\ldots m}.$$

Observe that, for $m = 2$, one has $h_{ij}^\alpha h_{jk}^\beta h_{ki}^\beta = 0$, since $h_{ii}^\alpha = 0$. Hence, f is also a critical point of $\mathcal{W}_D$, as we knew already. For $m > 3$, we conclude that f is a critical point of $\mathcal{W}$, iff $h_{ij}^\alpha h_{jk}^\beta h_{ki}^\beta = 0$, $\forall\alpha$. Note that the condition

$$h_{ij}^\alpha h_{jk}^\beta h_{ki}^\beta = 0\,,\quad \forall\alpha = m+1,\ldots,n \tag{2.72}$$

is conformally invariant, i.e. it does not depend on the second-order frame $e : M \to G$ along f we choose. Furthermore, we observe that, because of Eqs. (1.64), (1.60), condition (2.72) is equivalent to $mp_{kk}^\alpha = h_{iikk}^\alpha$, $\forall\alpha$. Naturally, one can wonder if the converse is also true: if f is a Willmore submanifold satisfying (2.72), is then γ_f a critical point of η? This does not seem to be the case, because in the above expression for $\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\big|_{t=0}$ the λ_0^α can be chosen arbitrarily, but not necessarily the λ_i^α (see Eq. (2.43)). Thus, we conclude

Proposition 2.2 *Let $f : M^m \to S^n$ be an immersion of an oriented m-manifold into the Möbius space. Then,*

For $m = 2$, f is a Willmore immersed surface, iff γ_f is a critical point of η [Ri/87]; for $m > 3$, if γ_f is a critical point of η, then f is a Willmore m-submanifold, iff condition (2.72) holds.

Therefore, condition (2.72) looks quite natural. Moreover, it may have far-reaching geometrical consequences, as we will see in the next section on a conformal Bernstein-type theorem.

2.4 A Conformal Bernstein-type Theorem

In this section we will formulate a Bernstein-type theorem for immersed Willmore hypersurfaces of the Möbius space, which generalises the special case of immersed surfaces in S^3 treated in Ref. [Ri/86].

Let $F : M^2 \to \mathbb{R}^3$ be an oriented Willmore surface immersed into the Euclidean 3-space, i.e. F satisfies Eq. (2.5). Let $\nu_F : M \to \mathbb{R}^3$ be the spherical Gauss map given by $\nu_F(x) = \nu_x$, $\forall x \in M$, where ν_x is the positive unit normal to F. Let $\sigma_F : M \to \mathbb{R}^3$ be the map defined by

$$\sigma_F(x) = \nu_F(x) + HF(x)\ .$$

Then, the following theorem can be formulated [Ri/86]:

Theorem (Rigoli) *Let $F : M^2 \to \mathbb{R}^3$ be a complete, oriented immersed Willmore surface. If there exists an $a \in \mathbb{R}^3$ with $v =< \sigma_F, a >_{\mathbb{R}^3} \neq 0$ on M, then $F(M)$ is either a sphere or a plane.*

This theorem is the analogue of the weak form of the parametric Bernstein theorem, which states that a complete, oriented, minimal immersed surface $F : M^2 \to \mathbb{R}^3$ with spherical Gauss map ν_F lying in a hemisphere of S^2 is a plane. Furthermore, it was reformulated in the conformal geometry of surfaces of S^3 by the same author:

Consider the immersion $f = i \circ F : M^2 \to S^3$ into the Möbius space, where $i : \mathbb{R}^3 \to S^3\backslash\{\mathbf{x}_\infty\}$ is the diffeomorphism as defined in diagram (1.91). Let $E = [E_0, E_1, E_2, E_3, E_4] : M \to G^*$ be a Darboux frame along F of the type described in Remark 1.4. Then, using the identification (1.93), we can consider E_0 and E_3 as vectors of $\mathbb{R}^3$, being $E_0 = F$ and E_3 the positive unit normal to F. Then, in the latter frame, we can write $\sigma_F(x) = (E_3 + HE_0)(x)$. Let $\hat{e} : M \to G$ be the second-order frame constructed from E as described in Sec. 1.3. Thus, $\hat{e}_3 = E_3 + HE_0$. That is, σ_F corresponds to the hyperbolic Gauss map γ_f of f. The following theorem is the conformal version of the previous one:

Theorem (Rigoli) *Let $f : M \to S^3$ be a compact, connected, oriented Willmore surface with hyperbolic conformal Gauss map γ_f. If there exists an $a \in \mathbb{R}^5$, such that $< \gamma_f, a > \neq 0$ on M, then $f(M)$ is a 2-sphere.*

Now we derive a generalisation of this theorem. Let $f : M^{n-1} \to S^n$ be an immersion of a hypersurface into the Möbius space, and let $\gamma_f : M \to \Omega$ be the hyperbolic conformal Gauss map of f defined in Eq. (2.65). Observe that, if M is the Möbius space S^{n-1} and f is the inclusion map given by $f\left(\left[\begin{bmatrix} c \\ \omega \\ s \end{bmatrix}\right]_{\sim}\right) = \left[\begin{bmatrix} c \\ \omega \\ 0 \\ s \end{bmatrix}\right]_{\sim}$, then f is a trivial Willmore hypersurface and $\gamma_f = \eta_n$. In particular, $< \gamma_f, \eta_n > \neq 0$ on all M. The following theorem shows that this property (with an additional condition) characterises the hyperspheres of S^n.

Theorem 2.1 *Suppose $n \neq 4$ and $n \neq 6$. Let $f : M^{n-1} \to S^n$ be a compact, oriented, connected Willmore hypersurface immersed into S^n with hyperbolic conformal Gauss map γ_f. If there exists an $a \in \mathbb{R}^{n+2}$, such that $< \gamma_f, a > \neq 0$ on all M, and if f satisfies the condition (2.72), then $f(M)$ is an $(n-1)$-sphere.*

Proof. Set $m = n - 1$. Obviously, without loss of generality, we may assume $(\gamma_f, a) > 0$ on all M. Let $e : M \to G$ be a second-order G-frame along f and let $\|h\|$ be as in Eq. (2.46), relative to this frame. Consider the local $(m-1)$-form on M given by

$$\begin{aligned} \omega \;=\; & (-1)^{i-1}\|h\|^{m-2}\big((m-1)p_i^n < e_0, a > -h_{ik}^n < e_k, a > \big)\phi^{1\ldots\hat{i}\ldots m} \\ & + (-1)^{i-1}(m-2) < e_0, a > \|h\|^{m-4} h_{ik}^n h_{\alpha}^n h_{\alpha k}^n \phi^{1\ldots\hat{i}\ldots m} \, . \end{aligned}$$

One can straightforwardly verify, using the transformation laws for second-order frames given in Sec. 1.2.C, that ω is a well-defined global $(m-1)$-form on M. Using Eqs. (1.60), (1.62), (1.67), (2.47), and (2.48), we have

$$\begin{aligned} d\omega \;=\; & (-1)^{i-1}(m-1)p_i^n < e_0, a > d\big(\|h\|^{m-2}\big) \wedge \phi^{1\ldots\hat{i}\ldots m} \\ & + (-1)^{i-1}(m-1)\|h\|^{m-2} < e_0, a > dp_i^n \wedge \phi^{1\ldots\hat{i}\ldots m} \\ & + (-1)^{i-1}(m-1)\|h\|^{m-2} p_i^n d(< e_0, a >) \wedge \phi^{1\ldots\hat{i}\ldots m} \\ & + (-1)^{i-1}(m-1)\|h\|^{m-2} < e_0, a > p_i^n d\phi^{1\ldots\hat{i}\ldots m} \\ & + (-1)^{i} h_{ik}^n < e_k, a > d\big(\|h\|^{m-2}\big) \wedge \phi^{1\ldots\hat{i}\ldots m} \\ & + (-1)^{i}\|h\|^{m-2} < e_k, a > dh_{ik}^n \wedge \phi^{1\ldots\hat{i}\ldots m} \\ & + (-1)^{i}\|h\|^{m-2} h_{ik}^n d(< e_k, a >) \wedge \phi^{1\ldots\hat{i}\ldots m} \\ & + (-1)^{i}\|h\|^{m-2} < e_k, a > h_{ik}^n d\phi^{1\ldots\hat{i}\ldots m} \end{aligned}$$

$$
\begin{aligned}
& + (-1)^{i-1}(m-2) < e_0, a > h^n_{ik} h^n_{st} h^n_{stk} d(\|h\|^{m-4}) \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (-1)^{i-1}(m-2)\|h\|^{m-4} < e_0, a > h^n_{ik} d(h^n_{st} h^n_{stk}) \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (-1)^{i-1}(m-2)\|h\|^{m-4} < e_0, a > h^n_{st} h^n_{stk} dh^n_{ik} \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (-1)^{i-1}(m-2)\|h\|^{m-4} h^n_{ik} h^n_{st} h^n_{stk} d(< e_0, a >) \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (-1)^{i-1}(m-2)\|h\|^{m-4} < e_0, a > h^n_{ik} h^n_{st} h^n_{stk} d\phi^{1 \ldots \hat{i} \ldots m} \\
= \; & (-1)^{i}(m-1)p^n_i < e_0, a > (m-2)\|h\|^{m-2} \phi^0_0 \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (m-1)p^n_i < e_0, a > (m-2)\|h\|^{m-4} h^n_{st} h^n_{sti} \phi^{1 \ldots m} \\
& + (-1)^{i-1}(m-1)\|h\|^{m-2} < e_0, a > p^n_k \phi^k_i \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (-1)^{i}(m-1)\|h\|^{m-2} < e_0, a > 2p^n_i \phi^0_0 \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (-1)^{i-1}(m-1)\|h\|^{m-2} < e_0, a > h^n_{ki} \phi^0_k \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (m-1)\|h\|^{m-2} < e_0, a > p^n_{ii} \phi^{1 \ldots m} \\
& + (-1)^{i-1}(m-1)\|h\|^{m-2} p^n_i < e_0, a > \phi^0_0 \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (m-1)\|h\|^{m-2} p^n_i < e_i, a > \phi^{1 \ldots m} \\
& + (-1)^{i-1}(m-1)^2\|h\|^{m-2} < e_0, a > p^n_i \phi^0_0 \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (-1)^{k-1}(m-1)\|h\|^{m-2} < e_0, a > p^n_i \phi^k_i \wedge \phi^{1 \ldots \hat{k} \ldots m} \\
& + (-1)^{i-1} h^n_{ik} < e_k, a > (m-2)\|h\|^{m-2} \phi^0_0 \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& - h^n_{ik} < e_k, a > (m-2)\|h\|^{m-4} h^n_{st} h^n_{sti} \phi^{1 \ldots m} \\
& + (-1)^{i}\|h\|^{m-2} < e_k, a > h^n_{jk} \phi^j_i \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (-1)^{i}\|h\|^{m-2} < e_k, a > h^n_{ij} \phi^j_k \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (-1)^{i-1}\|h\|^{m-2} < e_k, a > h^n_{ik} \phi^0_0 \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + \|h\|^{m-2} < e_i, a > p^n_i \phi^{1 \ldots m} \\
& - \|h\|^{m-2} < e_k, a > h^n_{iki} \phi^{1 \ldots m} \\
& + (-1)^{i}\|h\|^{m-2} h^n_{ik} < e_0, a > \phi^0_k \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (-1)^{i}\|h\|^{m-2} h^n_{ik} < e_j, a > \phi^j_k \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& - \|h\|^{m-2} h^n_{ik} < e_n, a > h^n_{ki} \phi^{1 \ldots m} \\
& - \|h\|^{m-2} h^n_{ii} < e_{n+1}, a > \phi^{1 \ldots m} \\
& + (-1)^{i}\|h\|^{m-2} < e_k, a > h^n_{ik}(m-1)\phi^0_0 \wedge \phi^{1 \ldots \hat{i} \ldots m} \\
& + (-1)^{j}\|h\|^{m-2} < e_k, a > h^n_{ik} \phi^j_i \wedge \phi^{1 \ldots \hat{j} \ldots m}
\end{aligned}
$$

$$
\begin{aligned}
&+ (-1)^i(m-2) < e_0, a > h^n_{ik} h^n_{\alpha t} h^n_{\alpha k}(m-4)\|h\|^{m-4}\phi^0_0 \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (m-2) < e_0, a > h^n_{ik} h^n_{\alpha t} h^n_{\alpha k}(m-4)\|h\|^{m-6} h^n_{uv} h^n_{uvi}\phi^{1\ldots m} + \\
&+ (-1)^i(m-2)\|h\|^{m-4} < e_0, a > h^n_{ik} 3 h^n_{\alpha t} h^n_{\alpha k}\phi^0_0 \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (-1)^i(m-2)\|h\|^{m-2} < e_0, a > h^n_{ik}\phi^0_k \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (-1)^{i-1}(m-2)\|h\|^{m-4} < e_0, a > h^n_{ik} h^n_{\alpha t} h^n_{\alpha j}\phi^j_k \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (m-2)\|h\|^{m-4} < e_0, a > h^n_{ik} H_{ki}\phi^{1\ldots m} \\
&+ (-1)^{i-1}(m-2)\|h\|^{m-4} < e_0, a > h^n_{\alpha t} h^n_{\alpha k} h^n_{jk}\phi^j_i \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (-1)^{i-1}(m-2)\|h\|^{m-4} < e_0, a > h^n_{\alpha t} h^n_{\alpha k} h^n_{ij}\phi^j_k \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (-1)^i(m-2)\|h\|^{m-4} < e_0, a > h^n_{\alpha t} h^n_{\alpha k} h^n_{ik}\phi^0_0 \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&- (m-2)\|h\|^{m-4} < e_0, a > h^n_{\alpha t} h^n_{\alpha t i} p^n_i \phi^{1\ldots m} \\
&+ (m-2)\|h\|^{m-4} < e_0, a > h^n_{\alpha t} h^n_{\alpha k} h^n_{iki}\phi^{1\ldots m} \\
&+ (-1)^{i-1}(m-2)\|h\|^{m-4} h^n_{ik} h^n_{\alpha t} h^n_{\alpha k} < e_0, a > \phi^0_0 \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (m-2)\|h\|^{m-4} h^n_{ik} h^n_{\alpha t} h^n_{\alpha k} < e_i, a > \phi^{1\ldots m} \\
&+ (-1)^{i-1}(m-1)(m-2)\|h\|^{m-4} < e_0, a > h^n_{ik} h^n_{\alpha t} h^n_{\alpha k}\phi^0_0 \wedge \phi^{1\ldots\hat{i}\ldots m} \\
&+ (-1)^{j-1}(m-2)\|h\|^{m-4} < e_0, a > h^n_{ik} h^n_{\alpha t} h^n_{\alpha k}\phi^j_i \wedge \phi^{1\ldots\hat{j}\ldots m} .
\end{aligned}
$$

Taking into account definition (1.56) and the vanishing of h^n_{ii}, we obtain, after several cancellations and obvious rearrangements, the expression

$$
\begin{aligned}
d\omega = \; & 2(m-1)(m-2)\|h\|^{m-4} p^n_i < e_0, a > h^n_{\alpha t} h^n_{\alpha t i}\phi^{1\ldots m} \\
& + (m-1)\|h\|^{m-2} < e_0, a > p^n_{i,i}\phi^{1\ldots m} \\
& + (m-2)(m-4)\|h\|^{m-6} < e_0, a > h^n_{ik} h^n_{\alpha t} h^n_{\alpha k} h^n_{uv} h^n_{uvi}\phi^{1\ldots m} \\
& + (m-2)\|h\|^{m-4} < e_0, a > h^n_{ik} H_{ik}\phi^{1\ldots m} \\
& - \|h\|^{m-2} h^n_{ik} h^n_{ik} < e_n, a > \phi^{1\ldots m} .
\end{aligned}
$$

Since $e_n = \gamma_f$, we can rewrite the latter expression as

$$
\begin{aligned}
d\omega = \; & < e_0, a > \Big\{(m-1)\|h\|^{m-2} p^n_{i,i} + 2(m-1)(m-2)\|h\|^{m-4} p^n_i h^n_{\alpha t} h^n_{\alpha t i} + \\
& \quad + (m-2)(m-4)\|h\|^{m-6} h^n_{ik} h^n_{\alpha t} h^n_{\alpha k} h^n_{uv} h^n_{uvi} \\
& \quad + (m-2)\|h\|^{m-4} h^n_{ik} H_{ik}\Big\}\phi^{1\ldots m} \\
& - \|h\|^m < \gamma_f, a > \phi^{1\ldots m} .
\end{aligned}
$$

If f is a Willmore hypersurface, then, using the Euler-Lagrange equation derived in Prop. 2.1, we obtain

$$d\omega = -\left(< e_0, a > \|h\|^{m-2} h^n_{ij} h^n_{jk} h^n_{ki} + \|h\|^m < \gamma_f, a > \right) \phi^{1 \ldots m} , \tag{2.73}$$

which is a global m-form on M. Now, since f satisfies, by assumption, condition (2.72), application of Stokes' theorem yields

$$0 = \int_M d\omega = - \int_M \|h\|^m < \gamma_f, a > \phi^{1 \ldots m} .$$

As, also by assumption, $\langle \gamma_f, a \rangle > 0$ on all M, necessarily $\|h\| = \sqrt{\sum_{i,j} (h^n_{ij})^2} = 0$. Applying, finally, Eq. (1.89) and the proposition due to Schiemangk-Sulanke quoted in Sec. 1.2.D, we conclude that $f(M)$ is an $(n-1)$-sphere. ♡

Taking into account Prop. 2.2, we obtain the following corollary:

Corollary 2.1.1 *Suppose $n \neq 4$ and $n \neq 6$. Let $f : M^{n-1} \to S^n$ be a compact, oriented, connected Willmore hypersurface immersed in S^n. If γ_f is a critical point of the functional η given in Eq. (2.67) and if there exists an $a \in \mathbb{R}^{n+2}$ such that $< \gamma_f, a > \neq 0$ on all M, then $f(M)$ is an $(n-1)$-sphere.*

We remark that the conclusion of Cor. 2.1.1 can be obtained without the assumption of $f : M^{n-1} \to S^n$ being a Willmore hypersurface, by slightly modifying the proof of Th. 2.1. It is sufficient that γ_f be a critical point of the functional η given in Eq. (2.67). More precisely, we have the following result:

Theorem 2.2 *Suppose $n \neq 4$. Let $f : M^{n-1} \to S^n$ be a compact, oriented, connected immersed hypersurface into S^n with hyperbolic conformal Gauss map $\gamma_f : M \to Q$. If γ_f is a critical point of the functional η given in Eq. (2.67) and if there exists an $a \in \mathbb{R}^{n+2}$, such that $< \gamma_f, a > \neq 0$ on all M, then $f(M)$ is an $(n-1)$-sphere.*

Proof. Set $m = n - 1$. Let $e : M \to G$ be a second-order G-frame along f and $\|h\|$ be as in Eq. (2.46) relative to this frame. Consider the local $(m-1)$-form on M given by

$$\omega = (-1)^{i-1} \|h\|^{m-2} (p^n_i < e_0, a > - h^n_{ik} < e_k, a >) \phi^{1 \ldots \hat{i} \ldots m} .$$

We can easily verify, using the transformation laws for second-order frames, that ω is a well-defined, global $(m-1)$-form on M. Through straightforward computations, similar to the ones in the proof of Th. 2.1, we obtain

$$\begin{aligned} d\omega &= <e_0, a> \big((m-2)\|h\|^{m-4} p_i^n h_{\alpha}^n h_{\alpha i}^n + \|h\|^{m-2} p_{ii}^n\big)\phi^{1-m} \\ &\quad + <e_i, a> (2-m)\big(\|h\|^{m-2} p_i^n + \|h\|^{m-4} h_{ik}^n h_{\alpha}^n h_{\alpha k}^n\big)\phi^{1-m} \\ &\quad - \|h\|^m <\gamma_f, a> \phi^{1-m} \,. \end{aligned}$$

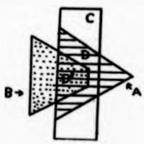
Since γ_f is a critical point of η, Eq. (2.70) holds, i.e.

$$d\omega = -\|h\|^m <\gamma_f, a> \phi^{1-m} \,.$$

Now the conclusion follows as in the proof of Th. 2.1. ♡

Thus, we have obtained two different Bernstein-type theorems with non-empty intersection. This can be visualised diagrammatically as follows. Let

$$\begin{aligned} C &= \{\text{Immersions } f : M^{n-1} \to S^n \text{ satisfying } h_{ij}^n h_{jk}^n h_{ki}^n = 0\} \\ A &= \{\text{Willmore hypersurfaces } f : M^{n-1} \to S^n\} \\ B &= \{\text{Immersions } f : M^{n-1} \to S^n \text{ s.t. } \gamma_f : M \to Q \text{ is a critical point of } \eta\}. \end{aligned}$$



Then we have $B \cap C \subset A$ and $B \cap A \subset C$. Let $D = C \cap A$ and $D' = B \cap A$. Then, $D' \subset D$. On D and B we have the Bernstein-type theorems (perhaps better called rigidity theorems) 2.1 resp. 2.2 with intersection of the domains of validity given by D'. If D' happens to coincide with D, then Th. 2.2 is more general than 2.1. In the case $n = 3$, where C is the set of all immersed surfaces, we have $D' = D = A = B$.

Remark 2.2 We note, reviewing carefully the proof of Th. 2.1, that, by dropping the condition (2.72) on f, one can still arrive at an interesting, though somewhat vague, conclusion. From Eq. (2.73), which holds in any case, we obtain by applying Stokes' theorem

$$0 = \int_M d\omega = -\int_{M \setminus U} \|h\|^{m-2}\big(<e_0, a> h_{ij}^n h_{jk}^n h_{ki}^n + \|h\|^2 <\gamma_f, a> \big)\phi^{1-m} \,,$$

where U is the set of all umbilic points of f. Given a point $x \in M$, the sign or vanishing of the expression

$$\big\langle h_{ij}^n h_{jk}^n h_{ki}^n e_0 + \|h\|^2 \gamma_f \,,\, a \big\rangle (x)$$

is independent of the choice of second-order frame on a neighbourhood of x, as follows from the transformation laws in Sec. 1.2.C. Thus, we can reformulate the above theorem in the following way:

Theorem 2.1' *Let f be as in Th. 2.1, except for condition (2.72). If there exists an $a \in \mathbb{R}^{n+2}$, such that $\left\langle h^n_{ij} h^n_{jk} h^n_{ki} e_0 + \|h\|^2 \gamma , a \right\rangle \geq 0$ on all M, then, necessarily, this inequality implies equality to zero on all M.*

Appendix II

Let $f : M^m \to S^n$ be an immersed hypersurface ($m = n-1$) into the Möbius space and fix a Riemannian metric $g = \phi_0^i \otimes \phi_0^i$ of the conformal class of M. Let $\overline{D}$ be a compact domain on M. Now we calculate the Euler-Lagrange equation of the functional $\eta_D(\rho) = \frac{1}{m}\int_D \left(\mathrm{trace}(\rho^* d\ell^2)\right)^{\frac{m}{2}} dV$ for $\rho : \overline{D} \to Q$ a smooth map, where $Q = \{x \in \mathbb{R}^{n+2} : <x,x> = 1\}$ is endowed with the Lorentz inner product $d\ell^2$ induced by the one of $\mathbb{R}^{n+2}$.
Let $\tilde\rho : (-\epsilon,\epsilon)\times\overline{D} \to Q$, $\tilde\rho(t,\cdot) = \rho_t(\cdot)$, be a variation of $\tilde\rho_0 = \rho$ with compact support in D. Let $W \in C^\infty(\rho^{-1}TQ)$ be defined by $W_x = \frac{\partial}{\partial t}\rho_t(x)|_{t=0}$. Let ∇, ∇' denote the Levi-Civita connections of (M,g) resp. $(Q, d\ell^2)$ and $\nabla^{\rho^{-1}}$, $\nabla^{\tilde\rho^{-1}}$ be the connections of $\rho^{-1}TQ$ resp. $\tilde\rho^{-1}TQ$. Let $x_0 \in D$ and $X_1,\ldots,X_m$ be an orthonormal frame of (M,g) defined near x_0 and satisfying $\nabla X_i(x_0) = 0$. We denote by $(0,X_i)$ and $\frac{\partial}{\partial t}$ the vector fields on $(-\epsilon,\epsilon)\times M$ given by $(0,X_i)_{(t,x)} = (0,X_{i\,x})$ resp. $\frac{\partial}{\partial t}(t,x) = (1,0)$. Then,

$$\overset{(-\epsilon,\epsilon)\times M}{\nabla_{\frac{\partial}{\partial t}}}(0,X_i) = \overset{(-\epsilon,\epsilon)\times M}{\nabla_{(0,X_i)}}\frac{\partial}{\partial t} = 0\,. \qquad \text{(II.1)}$$

Let Z be the vector field on M defined by

$$<Z_x, u>_g = d\ell^2\Big(W, m\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho_x(u)\Big)\,, \quad \forall u \in T_xM\,.$$

At the point x_0 we have, because of Eq. (II.1) and the symmetry of the second fundamental form of $\tilde\rho$,

$$\begin{aligned}
&\frac{\partial}{\partial t} d\ell^2(d\rho_t(X_i), d\rho_t(X_i))_{x_0}|_{t=0} = \frac{\partial}{\partial t} d\ell^2(d\tilde\rho(0,X_i), d\tilde\rho(0,X_i))_{(0,x_0)}\\
&= 2d\ell^2\Big(\nabla^{\tilde\rho^{-1}}_{\frac{\partial}{\partial t}}(d\tilde\rho(0,X_i))_{(0,x_0)}, d\tilde\rho_{(0,x_0)}(0,X_i)\Big)\\
&= 2d\ell^2\Big(\nabla d\tilde\rho_{(0,x_0)}\big(\frac{\partial}{\partial t},(0,X_i)\big), d\tilde\rho_{(0,x_0)}(0,X_i)\Big)\\
&= 2d\ell^2\Big(\nabla^{\tilde\rho^{-1}}_{(0,X_i)}\big(d\tilde\rho(\frac{\partial}{\partial t})\big)_{(0,x_0)}, d\tilde\rho_{(0,x_0)}(0,X_i)\Big) = 2d\ell^2\Big(\nabla^{\rho^{-1}}_{X_i} W_{x_0}, d\rho_{x_0}(X_i)\Big)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(\mathrm{trace}(\rho_t^* d\ell^2)\big)^{\frac{m}{2}}_{x_0}|_{t=0} = \frac{\partial}{\partial t}\Big(d\ell^2(d\rho_t(X_i), d\rho_t(X_i))_{x_0}\Big)^{\frac{m}{2}}|_{t=0}\\
&= m\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}}_{x_0} d\ell^2\Big(\nabla^{\rho^{-1}}_{X_i} W_{x_0}, d\rho_{x_0}(X_i)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= d\ell^2\Big(\nabla^{\rho^{-1}}_{X_i} W_{x_0}, m\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}}_{x_0} d\rho_{x_0}(X_i)\Big) \\
&= d\left\{d\ell^2\Big(W, m\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho(X_i)\Big)\right\}_{x_0}(X_i) \\
&-d\ell^2\Big(W_{x_0}, \nabla^{\rho^{-1}}_{X_i}\Big(m\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho(X_i)\Big)_{x_0}\Big) \\
&= d(<Z, X_i>)_{x_0}(X_i) - d\ell^2\Big(W_{x_0}, \nabla\Big(m\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho\Big)_{x_0}(X_i, X_i)\Big) \\
&= \mathrm{div}_g(Z)_{x_0} - m\, d\ell^2\Big(W_{x_0}, \mathrm{trace}\Big(\nabla\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho\Big)_{x_0}\Big)\ ,
\end{aligned}
$$

where $\nabla\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho$ is the covariant derivative in the vector bundle $\Lambda^1 T^*M \otimes \rho^{-1}TQ$. Since Z has compact support, we obtain, by applying Stokes' theorem,

$$
\frac{\partial}{\partial t}\psi_D(\rho_t)\Big|_{t=0} = -\int_D d\ell^2\Big(W, \mathrm{trace}\Big(\nabla\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho\Big)\Big)dV\ .
$$

Thus, the Euler-Lagrange equation is given by

$$
\mathrm{trace}\Big(\nabla\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho\Big) = 0\ ,
$$

or, equivalently,

$$
\begin{aligned}
&0 = \mathrm{trace}\Big(\nabla\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho\Big)_{x_0} = \nabla\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho_{x_0}(X_i, X_i) \\
&= \nabla^{\rho^{-1}}_{X_i}\left\{\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho(X_i)\right\}_{x_0} \\
&= d\left\{\big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}}\right\}_{x_0}(X_i)\, d\rho_{x_0}(X_i) + \big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} \nabla d\rho_{x_0}(X_i, X_i) \\
&= \mathrm{trace}\left\{\frac{m-2}{2}\big(\mathrm{trace}(d\rho^* d\ell^2)\big)^{\frac{m-4}{2}} d\big(\mathrm{trace}(\rho^* d\ell^2)\big) \otimes d\rho + \big(\mathrm{trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} \nabla d\rho\right\}_{x_0}\ .
\end{aligned}
$$

Chapter 3

THE SECOND VARIATION FOR WILLMORE SURFACES OF A SPACE FORM

Let (N,h) be an n-dimensional Riemannian manifold of constant sectional curvature $\overline{K}$. Then, in this chapter, we will calculate, in the context of Riemannian geometry, the second variation formula for Willmore immersed surfaces $f : M^2 \to (N,h)$. Weiner [We/78] computed this second variation in the particular case where f is a minimal immersion. Our notations and calculations will be similar to his, up to the step where he demands minimality of f to hold. We will proceed without any such assumption. Recall that the curvature tensor $\overline{R}$ of (N,h) satisfies

$$\overline{R}(X,Y)Z = \overline{K}(< Z,X >_h Y - < Z,Y >_h X)\ ,\ \forall X,Y,Z \in C^\infty(TN)\ .$$

If $f : M^2 \to (N,h)$ is an immersion, we denote by $\tilde{A}$ the element of $C^\infty(\otimes V^* \otimes V)$, where V is the normal bundle to f given by Eq. (2.12).

Let $D \subset M$ be a compact domain, I denote $(-\epsilon,\epsilon)$, and $v : M \times I \to N$ be a variation of f through immersions $f_t = v(\cdot,t) : M \to N$ with variation vector $W \in C^\infty(f^{-1}TN)$ given by $W_x = \frac{\partial}{\partial t}v(x,t)\big|_{t=0}$, $\forall x \in M$, which we assume to be compactly supported in D.

Proposition 3.1 *If $f : M^2 \to (N,h)$ is a Willmore immersed surface, then the second variation formula for the variation $v = f_t$ is given by*

$$\frac{\partial^2}{\partial t^2} W_D(f_t)\big|_{t=0} = \int_D \langle J(W),W\rangle_h \, dA_{g_0}\ ,$$

where

$$
\begin{aligned}
J(W)_x = {} & \frac{1}{2}(\Delta + \tilde{A})(\Delta + 2\bar{K} + \tilde{A})(W)_x \\
& - 2\left\langle (\Delta + \bar{K} + \tilde{A})(W)_x, H_x \right\rangle_h H_x - \|H_x\|_h^2 (\Delta + \tilde{A})(W)_x \\
& + 2\left\langle W_x, \nabla df_x(e_i, e_k)\right\rangle_h \overset{V}{\nabla}{}^2 H_x(e_i, e_k) + 2\left\langle H_x, \nabla df_x(e_i, e_k)\right\rangle_h \overset{V}{\nabla}{}^2 W_x(e_i, e_k) \\
& - 4\left\langle H_x, \overset{V}{\nabla}_{e_k} W_x \right\rangle_h \overset{V}{\nabla}_{e_k} H_x + 2\left\langle W_x, \overset{V}{\nabla}_{e_k} H_x \right\rangle_h \overset{V}{\nabla}_{e_k} H_x + 2\left\langle H, \overset{V}{\nabla}_{e_k} H \right\rangle_h \overset{V}{\nabla}_{e_k} W_x \\
& + 2\left\langle \overset{V}{\nabla}_{e_i} W_x, \nabla df_x(e_i, e_k) \right\rangle_h \overset{V}{\nabla}_{e_k} H_x \\
& - 2\left\langle \overset{V}{\nabla}_{e_i} H_x, \overset{V}{\nabla}_{e_j} W_x \right\rangle_h \nabla df_x(e_i, e_j) \\
& + 2\left\langle \overset{V}{\nabla}_{e_i} H_x, \nabla df_x(e_i, e_j) \right\rangle_h \overset{V}{\nabla}_{e_j} W_x \\
& + 2\left\langle W_x, \nabla df_x(e_i, e_k)\right\rangle_h \left\langle \nabla df_x(e_i, e_j), H_x \right\rangle_h \nabla df_x(e_j, e_k) ,
\end{aligned}
$$

with $\overset{V}{\nabla}{}^2 W \in C^\infty(\otimes^2 T^*M \times V)$ *given by*

$$
\overset{V}{\nabla}{}^2 W_x(X_x, Y_x) = \overset{V}{\nabla}_{X_x} \overset{V}{\nabla}_{Y} W_x - \overset{V}{\nabla}_{\nabla^{M_g}_X Y_x} W_x ,
$$

and with e_1, e_2 *an arbitrary orthonormal basis of* (T_xM, g_x).

Corollary 3.1.1 *If* $\dim N = 3$, *then the operator* J *in the above theorem can be simplified to*

$$
\begin{aligned}
J(W) = {} & \frac{1}{2}(\Delta + \tilde{A})(\Delta + 2\bar{K} + \tilde{A})(W) - 3\|H\|_h^2(\Delta + \tilde{A})(W) \\
& + 2\left\langle \nabla df, \overset{V}{\nabla}{}^2 H \right\rangle W + 2\left\langle \nabla df, \overset{V}{\nabla}{}^2 W \right\rangle H \\
& - 2\left\langle \overset{V}{\nabla} W, \overset{V}{\nabla} H \right\rangle H + 2\left\langle \overset{V}{\nabla} H, \overset{V}{\nabla} H \right\rangle W \\
& + 2\left\langle \overset{V}{\nabla} H \otimes \overset{V}{\nabla} W, \nabla df \right\rangle - 2\bar{K}\|H\|_h^2 W + 2\tilde{B}^H(W) ,
\end{aligned}
$$

where $\tilde{B}^H \in C^\infty(\otimes V^* \otimes V)$ *is given by*

$$
\tilde{B}^H_x(W_x) = \left\langle W_x, \nabla df_x(e_i, e_k)\right\rangle_h \left\langle \nabla df_x(e_i, e_j), H_x \right\rangle_h \nabla df_x(e_j, e_k) ,
$$

$2\left\langle \overset{V}{\nabla} H \otimes \overset{V}{\nabla} W, \nabla df \right\rangle$ *is shorthand for* $2\left\langle \overset{V}{\nabla}_{e_i} H, \nabla df(e_i, e_j)\right\rangle_h \overset{V}{\nabla}_{e_j} W$, *and where the inner products denoted by* $<,>$ *are of the Hilbert-Schmidt type.*

Proof. Let $g_t = f_t^* h$, and denote $M_t = (M, g_t)$. We define the vector subbundles T and V of $v^{-1}TN$ as $T_{(x,t)} = d(f_t)_x(T_xM)$ and $V_{(x,t)}$ as its orthogonal complement in $(T_{f_t(x)}N, h)$, $\forall (x,t) \in M \times I$. Thus, $T_{v(x,t)}N = T_{(x,t)} \overset{\perp}{\oplus} V_{(x,t)}$. Then, for each $t \in I$, we have the vector subbundles T_t and V_t of $f_t^{-1}TN$ defined by $T_{t\,(x)} = T_{(x,t)}$ and $V_{t\,(x)} = V_{(x,t)}$, the latter one defining the normal bundle to the isometric immersion $f_t : M_t \to (N, h)$. We denote by $(\ )^T$ and $(\ )^V$ the orthogonal projections of $v^{-1}TN$ onto T resp. V, and by $\pi : M \times I \to M$ the first projection $(x,t) \to x$. The following connections are going to be used: ∇ of (N,h), $\overset{M_t}{\nabla}$ of M_t, $\overset{M_0 \times I}{\nabla}$ of $M_0 \times I$, $\nabla^{v^{-1}}$ of $(v^{-1}TN, h)$, $\nabla^{f_t^{-1}}$ of $(f_t^{-1}TN, h)$, $\overset{V_t}{\nabla}$ of (V_t, h) (and $\overset{V}{\nabla}$ of (V_0, h)), and $\nabla^{\pi^{-1}}$ of $(\pi^{-1}TM, g_0)$.

If $Z \in C^\infty(v^{-1}TN)$, $\forall t \in I$, then Z_t given by $Z_{t\,(x)} = Z_{(x,t)}$ is an element of $C^\infty(f_t^{-1}TN)$, and

$$\nabla^{v^{-1}}_{(u,0)} Z_{(x,t)} = \nabla^{f_t^{-1}}_u Z_{t\,(x)}\ , \quad \forall u \in T_xM\ .$$

If $Y \in C^\infty(V_t)$, then $\overset{V_t}{\nabla}_u Y_x = \left(\nabla^{f_t^{-1}}_u Y_x\right)^V$.

For each $t \in I$,

$$\mathcal{W}_D(f_t) = \int_D (\|H_t\|_h^2 + \bar{K}) dA_{g_t} + \int_{\partial D} \kappa_{g_t} ds_t\ ,$$

where dA_{g_t} is the volume element of (M, g_t), H_t is the mean curvature of $f_t : M_t \to (N,h)$, and κ_{g_t} is the signed geodesic curvature of ∂D. Observe that $\kappa_{g_t} ds_t = \kappa_{g_0} ds_0$, because $f_t(x) = f(x)$ for $x \in \partial D$. Since $dA_{g_t}(x) = \sqrt{\det[g_t(e_i, e_j)]}(x) dA_{g_0}(x)$ with e_1, e_2 an orthonormal basis of (T_xM, g_0), we can write

$$\mathcal{W}_D(f_t) = \int_D (\|H_t\|_h^2 + \bar{K}) \sqrt{\det[g_t(e_i, e_j)]}\, dA_{g_0} + \int_{\partial D} \kappa_{g_0} ds_0\ .$$

So,

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\Big|_{t=0} = \int_D \left\{ \frac{\partial}{\partial t}\|H_t\|^2\Big|_{t=0} + (\|H\|_h^2 + \bar{K}) \frac{\partial}{\partial t}\sqrt{\det[g_t(e_i, e_j)]}\Big|_{t=0} \right\} dA_{g_0}\ .$$

Here and henceforth, we denote by H, either the section of $v^{-1}TN$, $H_{(x,t)} = H_{t\,(x)}$, or the mean curvature H_0 of f, which notation will become clear from the context. Let us fix $x_0 \in M$ and let e_1, e_2 be an orthonormal frame of $M_0 = (M, g_0)$ around x_0 satisfying $\overset{M_0}{\nabla} e_{i\,(x_0)} = 0$. Then, $e_i(x,0) := e_i(x) \in T_xM = (\pi^{-1}TM)_{(x,0)}$ can be extended as a local section of $\pi^{-1}TM$ on a neighbourhood of $(x_0, 0) \in M \times I$,

resulting in $\tilde{e}_i(x,t) \in (\pi^{-1}TM)_{(x,t)} = T_xM$. Clearly, we may assume $\tilde{e}_1(x,t), \tilde{e}_2(x,t)$ to be linearly independent and, through Gramm-Schmidt orthogonalisation, to be orthonormal in (T_xM, g_t). Thus, we obtain sections $\tilde{e}_1, \tilde{e}_2$ of $\pi^{-1}TM$ satisfying

1) $\tilde{e}_i(x,0) = e_i(x)$ and so $\overset{M_0}{\nabla}(\tilde{e}_i(\cdot,0))_{(x_0)} = \overset{M_0}{\nabla} e_{i\,(x_0)} = 0$,

2) $\forall t$, $\tilde{e}_{i_t} := \tilde{e}_i(\cdot,t)$ constitute an orthonormal frame of $M_t = (M, g_t)$.

Let $\frac{\partial}{\partial t}, (\tilde{e}_i, 0), (e_i, 0) \in C^\infty(T(M \times I))$ be the vector fields respectively given by $\frac{\partial}{\partial t}(x,t) = (0,1)$, $(\tilde{e}_i, 0)_{(x,t)} = (\tilde{e}_i(x,t), 0)$, $(e_i,0)_{(x,t)} = (e_i(x), 0)$, $\forall (x,t) \in M \times I$. Then, we have

$$\overset{M_0\times I}{\nabla}_{\frac{\partial}{\partial t}} (e_i, 0)_{(x,t)} = \overset{M_0\times I}{\nabla}_{(e_i,0)} \frac{\partial}{\partial t}_{(x,t)} = 0 \,. \tag{3.1}$$

Henceforth, we denote by ∇dv the second fundamental form of $v : M_0 \times I \to N$ and by $\nabla d(f_t)$ the one of $f_t : M_t \to N$, the latter taking values on V_t.

Using Eq. (3.1), we get, $\forall x \in M$,

$$\begin{aligned}
\frac{\partial}{\partial t}\langle d(f_t)_x(e_i), d(f_t)_x(e_j)\rangle_h\big|_{t=0} &= \frac{\partial}{\partial t}\left\langle dv_{(x,t)}(e_i,0), dv_{(x,t)}(e_j,0)\right\rangle_h\big|_{t=0} \\
&= \left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}(dv(e_i,0))_{(x,0)}\,,\, df_x(e_j)\right\rangle_h + \left\langle df_x(e_i)\,,\, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}(dv(e_j,0))_{(x,0)}\right\rangle_h \\
&= \left\langle \nabla dv_{(x,0)}\Big(\frac{\partial}{\partial t}, (e_i,0)\Big)\,,\, df_x(e_j)\right\rangle_h + \left\langle df_x(e_i)\,,\, \nabla dv_{(x,0)}\Big(\frac{\partial}{\partial t}, (e_j,0)\Big)\right\rangle_h \\
&= \left\langle \nabla^{v^{-1}}_{(e_i,0)}\Big(dv\Big(\frac{\partial}{\partial t}\Big)\Big)_{(x,0)}\,,\, df_x(e_j)\right\rangle_h + \left\langle df_x(e_i)\,,\, \nabla^{v^{-1}}_{(e_j,0)}\Big(dv\Big(\frac{\partial}{\partial t}\Big)\Big)_{(x,0)}\right\rangle_h \\
&= \left\langle \nabla^{f^{-1}}_{e_i} W_x, df_x(e_j)\right\rangle_h + \left\langle df_x(e_i), \nabla^{f^{-1}}_{e_j} W_x\right\rangle_h \,.
\end{aligned}$$

Then, from the multilinear alternating property of the determinant, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\det[g_t(e_i,e_j)]\big|_{t=0}(x) &= \frac{\partial}{\partial t}\det\left[\langle d(f_t)_x(e_i), d(f_t)_x(e_j)\rangle_h\right]\big|_{t=0} \\
&= 2\left\langle \nabla^{f^{-1}}_{e_i} W_x, df_x(e_i)\right\rangle_h = 2\left\langle \nabla^{f^{-1}} W_x, df_x\right\rangle \,,
\end{aligned}$$

where $<,>$ is the Hilbert-Schmidt inner product of $\Lambda^1 T_x^* M_0 \otimes T_{f(x)}N$. Hence,

$$\frac{\partial}{\partial t}\sqrt{\det[g_t(e_i,e_j)]}\big|_{t=0}(x) = \left\langle \nabla^{f^{-1}} W_x, df_x\right\rangle \,.$$

Still considering the Riemannian spaces $M_0 = (M, g_o)$ and $(f^{-1}TN, h)$, we have the equality (cf. Ref. [Ee-Le/83])

$$\left\langle \nabla^{f^{-1}} W_x, df_x\right\rangle dA_{g_0} = < W, \delta df >_h dA_{g_0} + d(W \wedge * df) \,,$$

where $*$ is the Hodge operator in $\Lambda^1 T^*M_0 \otimes f^{-1}TN$. Thus,

$$\begin{aligned}\frac{\partial}{\partial t}W_D(f_t)\big|_{t=0} &= \int_D \left\{\frac{\partial}{\partial t}\|H_t\|^2\big|_{t=0} + (\|H\|_h^2 + \bar{K}) < W, \delta df >_h\right\} dA_{g_0} \\ &\quad + \int_D (\|H\|_h^2 + \bar{K}) d(W \wedge *df) \ . \qquad (3.2)\end{aligned}$$

If $X \in C^\infty(TM)$, and if $\S(X), \theta \in C^\infty(T^*M)$ are given by $\S(X)(Y) =< X, Y >_{g_0}$ resp. $\theta(Y) =< W, df(Y) >_h$, $\forall Y \in C^\infty(TM)$, then

$$\begin{aligned}(W \wedge *df)(X) &= < W, *df(X) >_h = - < W, \S(X) \wedge df(e_1, e_2) >_h \\ &= -\S(X)\wedge < W, df(\cdot) >_h (e_1, e_2) = -\S(X) \wedge \theta(e_1, e_2) = *\theta(X) \ .\end{aligned}$$

Thus, $W \wedge *df = *\theta$ with θ having compact support in D. By applying Stokes' theorem, we get

$$\int_D (\|H\|_h^2 + \bar{K}) d(W \wedge *df) = \int_D (\|H\|_h^2 + \bar{K}) d*\theta = \int_D \|H\|_h^2 d*\theta \ .$$

Furthermore,

$$\|H\|_h^2 d*\theta = d(\|H\|_h^2 *\theta) - d\|H\|_h^2 \wedge *\theta = d(\|H\|_h^2 *\theta) - < d\|H\|_h^2, \theta > dA_{g_0}$$

and

$$< d\|H\|_h^2, \theta >= d\|H\|_h^2(e_i) < W^T, df(e_i) >_h = d\|H\|_h^2 (df^{-1}(W^T)) \ .$$

A further application of Stokes' theorem gives

$$\int_D (\|H\|_h^2 + \bar{K}) d(W \wedge *df) = - \int_D d\|H\|_h^2 (df^{-1}(W^T)) dA_{g_0} \ . \qquad (3.3)$$

Substituting this result in Eq. (3.2) and using the Weitzenböck formula (see e.g. Ref. [Ee-Le/83]) $\delta df = -2H$, we obtain

$$\frac{\partial}{\partial t}W(f_t)\big|_{t=0} = \int_D \left\{\frac{\partial}{\partial t}\|H_t\|_h^2\big|_{t=0} - 2(\|H\|_h^2 + \bar{K}) < W, H >_h - d\|H\|_h^2 (df^{-1}(W^T))\right\} dA_{g_0} \ . \qquad (3.4)$$

Next we calculate $\frac{\partial}{\partial t}\|H_t\|_h^2\big|_{t=0}$.

$\forall x \in D$, $t \in I$,

$$\begin{aligned}V_{(x,t)} \ni H_{(x,t)} &= \tfrac{1}{2}\nabla d(f_t)_x(\tilde{e}_i(x,t), \tilde{e}_i(x,t)) \\ &= \tfrac{1}{2}\left(\nabla^{f_t^{-1}}_{\tilde{e}_i(x,t)}(d(f_t)(\tilde{e}_{i_t})) - d(f_t)\left(\overset{M_t}{\nabla}_{\tilde{e}_i(x,t)}(\tilde{e}_{i_t})\right)\right)^V \\ &= \tfrac{1}{2}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}(dv(\tilde{e}_i, 0))_{(x,t)}\right)^V \\ &= \tfrac{1}{2}\left\{\nabla^{v^{-1}}_{(\tilde{e}_i,0)}(dv(\tilde{e}_i, 0))_{(x,t)} - \left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}(dv(\tilde{e}_i, 0))_{(x,t)}\right)^T\right\}\end{aligned}$$

$$= \tfrac{1}{2}\left\{\nabla^{v^{-1}}_{(\tilde{e}_i,0)}(dv(\tilde{e}_i, 0))_{(x,t)} - \left\langle\nabla^{v^{-1}}_{(\tilde{e}_i,0)}(dv(\tilde{e}_i, 0))_{(x,t)}, dv_{(x,t)}(\tilde{e}_j, 0)\right\rangle_h dv_{(x,t)}(\tilde{e}_j, 0)\right\} \ .$$

Thus,

$$\begin{aligned}
\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H_{(x,0)} =& \\
=& \frac{1}{2}\Big\{\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\big(dv(\tilde{e}_i,0)\big)_{(x,0)} - \Big\langle\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\big(dv(\tilde{e}_i,0)\big)_{(x,0)}, dv_{(x,0)}(\tilde{e}_j,0)\Big\rangle_h dv_{(x,0)}(\tilde{e}_j,0) \\
&- \Big\langle\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\big(dv(\tilde{e}_i,0)\big)_{(x,0)}, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\big(dv(\tilde{e}_j,0)\big)_{(x,0)}\Big\rangle_h dv_{(x,0)}(\tilde{e}_j,0) \\
&- \Big\langle\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\big(dv(\tilde{e}_i,0)\big)_{(x,0)}, dv_{(x,0)}(\tilde{e}_j,0)\Big\rangle_h \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\big(dv(\tilde{e}_j,0)\big)_{(x,0)}\Big\} \\
=& \frac{1}{2}\Big\{\Big(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\big(dv(\tilde{e}_i,0)\big)_{(x,0)}\Big)^V \\
&- \Big\langle\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\big(dv(\tilde{e}_i,0)\big)_{(x,0)}, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\big(dv(\tilde{e}_j,0)\big)_{(x,0)}\Big\rangle_h df_x(e_j) \\
&- \Big\langle\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\big(dv(\tilde{e}_i,0)\big)_{(x,0)}, df_x(e_j)\Big\rangle_h \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\big(dv(\tilde{e}_j,0)\big)_{(x,0)}\Big\}\,. \qquad (3.5)
\end{aligned}$$

Note that, at the point x_0, we have

$$\Big\langle\nabla^{f^{-1}}_{e_i}\big(df(e_j)\big)_{x_0}, df_{x_0}(e_k)\Big\rangle_h = \big\langle\nabla df_{x_0}(e_i,e_j), df_{x_0}(e_k)\big\rangle_h = 0\,. \qquad (3.6)$$

Hence,

$$\begin{aligned}
\Big(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H_{(x_0,0)}\Big)^V =& \frac{1}{2}\Big(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\big(dv(\tilde{e}_i,0)\big)_{(x_0,0)}\Big)^V = \\
=& \frac{1}{2}\Big(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\big(dv(\tilde{e}_i,0)\big)_{(x_0,0)} + \bar{R}^{v^{-1}}_{(x_0,0)}\big((\tilde{e}_i,0),\frac{\partial}{\partial t}\big)dv_{(x_0,0)}(\tilde{e}_i,0) \\
&\qquad - \nabla^{v^{-1}}_{[(\tilde{e}_i,0),\frac{\partial}{\partial t}]}\big(dv(\tilde{e}_i,0)\big)_{(x_0,0)}\Big)^V \\
=& \frac{1}{2}\Big(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\Big(\nabla dv\big(\frac{\partial}{\partial t},(\tilde{e}_i,0)\big) + dv\big(\overset{M_0\times I}{\nabla}_{\frac{\partial}{\partial t}}(\tilde{e}_i,0)\big)\Big)_{(x_0,0)} + \bar{R}_{f(x_0)}\big(df_{x_0}(e_i),W_{x_0}\big)df_{x_0}(e_i) \\
&\qquad - \nabla^{v^{-1}}_{[(\tilde{e}_i,0),\frac{\partial}{\partial t}]}\big(dv(\tilde{e}_i,0)\big)_{(x_0,0)}\Big)^V\,.
\end{aligned}$$

Since, $\forall(x,t)\in M\times I$,

$$\begin{aligned}
\nabla dv_{(x,t)}\big(\frac{\partial}{\partial t},(\tilde{e}_i,0)\big) &= \nabla^{v^{-1}}_{(\tilde{e}_i,0)}\frac{\partial v}{\partial t}_{(x,t)} - dv_{(x,t)}\big(\overset{M_0\times I}{\nabla}_{(\tilde{e}_i,0)}\frac{\partial}{\partial t}\big) \\
&= \nabla^{v^{-1}}_{(\tilde{e}_i,0)}\frac{\partial v}{\partial t}_{(x,t)}\,, \qquad (3.7)
\end{aligned}$$

we obtain

$$\begin{aligned}
\Big(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H_{(x_0,0)}\Big)^V =& \frac{1}{2}\Big(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\big(\frac{\partial v}{\partial t}\big)_{(x_0,0)} + \nabla^{v^{-1}}_{(\tilde{e}_i,0)}\Big(dv\big(\overset{M_0\times I}{\nabla}_{\frac{\partial}{\partial t}}(\tilde{e}_i,0)\big)\Big)_{(x_0,0)} \\
&+ 2KW_{x_0} - \nabla^{v^{-1}}_{[(\tilde{e}_i,0),\frac{\partial}{\partial t}]}\big(dv(\tilde{e}_i,0)\big)_{(x_0,0)}\Big)^V\,. \quad (3.8)
\end{aligned}$$

Observe that, $\forall (x,t) \in M \times I$, $\overset{M_0 \times I}{\nabla}_{\frac{\partial}{\partial t}} (\tilde{e}_i, 0)_{(x,t)} = \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \tilde{e}_{i\,(x,t)}, 0 \right)$ is an element of $T_x M \times \{0\}$. Let us call z_i the section of TM given by

$$z_i(x) = \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \tilde{e}_{i\,(x,0)} \, . \tag{3.9}$$

Then,

$$\nabla^{v^{-1}}_{(\tilde{e}_i,0)} \left(dv \left(\overset{M_0 \times I}{\nabla}_{\frac{\partial}{\partial t}} (\tilde{e}_i, 0) \right) \right)_{(x,0)} = \nabla^{f^{-1}}_{e_i} (df(z_i))_x$$

and

$$[(\tilde{e}_i, 0), \frac{\partial}{\partial t}]_{(x,0)} = \overset{M_0 \times I}{\nabla}_{(\tilde{e}_i,0)} \frac{\partial}{\partial t}_{(x,0)} - \overset{M_0 \times I}{\nabla}_{\frac{\partial}{\partial t}} (\tilde{e}_i, 0)_{(x,0)} = -(z_i(x), 0) \, . \tag{3.10}$$

Thus, Eq. (3.8) can be written as

$$\begin{aligned}
\left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H_{(x_0,0)} \right)^V &= \\
&= \frac{1}{2} \left(\nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_i} W_{x_0} + \nabla^{f^{-1}}_{e_i} (df(z_i))_{x_0} + 2\overline{K} W_{x_0} + \nabla^{f^{-1}}_{z_i} (df(e_i))_{x_0} \right)^V \\
&= \frac{1}{2} \left(\nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_i} W_{x_0} + 2 \nabla df_{x_0}(e_i, z_i) + 2\overline{K} W_{x_0} \right)^V \, .
\end{aligned} \tag{3.11}$$

If $U \in C^\infty(V)$, then

$$\left\langle \nabla df_{x_0}(e_i, z_j), U \right\rangle_h = \left\langle \nabla^{v^{-1}}_{e_i} (df(z_j))_{x_0}, U \right\rangle_h = - \left\langle df_{x_0}(z_j), \nabla^{v^{-1}}_{e_i} U_{x_0} \right\rangle_h$$

and

$$\left\langle \nabla df_{x_0}(e_i, z_j), U \right\rangle_h = < z_j(x_0), e_k >_{g_0} \left\langle \nabla df_{x_0}(e_i, e_k), U \right\rangle_h \, . \tag{3.12}$$

From the equality

$$\delta_{jk} = < \tilde{e}_j(x,t), \tilde{e}_k(x,t) >_{g_t} = \left\langle dv_{(x,t)}(\tilde{e}_j(x,t), 0), dv_{(x,t)}(\tilde{e}_k(x,t), 0) \right\rangle_h \, ,$$

we get, using Eq. (3.7),

$$\begin{aligned}
0 &= \frac{\partial}{\partial t} \langle dv(\tilde{e}_j, 0), dv(\tilde{e}_k, 0) \rangle_h \big|_{(x_0,0)} \\
&= \left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))_{(x_0,0)} \, , \, dv_{(x_0,0)}(\tilde{e}_k, 0) \right\rangle_h + \left\langle dv_{(x_0,0)}(\tilde{e}_j, 0) \, , \, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_k, 0))_{(x_0,0)} \right\rangle_h \\
&= \left\langle \nabla dv_{(x_0,0)}(\frac{\partial}{\partial t}, (\tilde{e}_j, 0)) \, , \, df_{x_0}(\tilde{e}_k) \right\rangle_h + \left\langle df_{x_0} \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} e_{j\,(x_0,0)} \right) \, , \, df_{x_0}(e_k) \right\rangle_h \\
&\quad + \left\langle df_{x_0}(\tilde{e}_j) \, , \, \nabla dv_{(x_0,0)}(\frac{\partial}{\partial t}, (\tilde{e}_k, 0)) \right\rangle_h + \left\langle df_{x_0}(e_j) \, , \, df_{x_0} \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} e_{k\,(x_0,0)} \right) \right\rangle_h \\
&= \left\langle \nabla^{f^{-1}}_{e_j} W_{x_0}, df_{x_0}(e_k) \right\rangle_h + \langle df_{x_0}(z_i), df_{x_0}(e_k) \rangle_h \\
&\quad + \left\langle df_{x_0}(e_j), \nabla^{f^{-1}}_{e_k} W_{x_0} \right\rangle_h + \langle df_{x_0}(e_j), df_{x_0}(z_k) \rangle_h \\
&= \left\langle \nabla^{f^{-1}}_{e_j} W_{x_0}, df_{x_0}(e_k) \right\rangle_h + < z_i(x_0), e_k >_{g_0} + \left\langle df_{x_0}(e_j), \nabla^{f^{-1}}_{e_k} W_{x_0} \right\rangle_h + < e_j, z_k(x_0) >_{g_0} \, .
\end{aligned}$$

Hence,

$$\begin{aligned}\left\langle \nabla df_{x_0}(e_i, z_i), U\right\rangle_h &=< z_i, e_k >_{g_0} \left\langle \nabla df_{x_0}(e_i, e_k), U\right\rangle_h =\\ &= - < z_k, e_i >_{g_0} \left\langle \nabla df_{x_0}(e_i, e_k), U\right\rangle_h - \left\langle df_{x_0}(e_i), \nabla^{f^{-1}}_{e_k} W_{x_0}\right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_k), U\right\rangle_h\\ &\quad - \left\langle df_{x_0}(e_k), \nabla^{f^{-1}}_{e_i} W_{x_0}\right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_k), U\right\rangle_h\\ &= - < z_k, e_i >_{g_0} \left\langle \nabla df_{x_0}(e_i, e_k), U\right\rangle_h + 2\left\langle df_{x_0}(e_i), \nabla^{f^{-1}}_{e_k} W_{x_0}\right\rangle_h \left\langle df_{x_0}(e_i), \nabla^{f^{-1}}_{e_k} U_{x_0}\right\rangle_h\\ &= - < z_k, e_i >_{g_0} \left\langle \nabla df_{x_0}(e_i, e_k), U\right\rangle_h + 2\left\langle \left(\nabla^{f^{-1}}_{e_k} W\right)^T_{x_0}, \nabla^{f^{-1}}_{e_k} U_{x_0}\right\rangle_h\\ &= -\left\langle \nabla df_{x_0}(z_k, e_k), U\right\rangle_h - 2\left\langle \nabla^{f^{-1}}_{e_k}\left(\nabla^{f^{-1}}_{e_k} W\right)^T_{x_0}, U\right\rangle_h .\end{aligned}$$

Consequently,

$$\nabla df_{x_0}(e_i, z_i) = -\left(\nabla^{f^{-1}}_{e_k}\left(\nabla^{f^{-1}}_{e_k} W\right)^T_{x_0}\right)^V .$$

Equation (3.11) thus becomes

$$\begin{aligned}\left(\nabla^{\varphi^{-1}}_{\frac{\partial}{\partial t}} H_{(x_0,0)}\right)^V &= \frac{1}{2}\left(\nabla^{f^{-1}}_{e_i}\nabla^{f^{-1}}_{e_i} W_{x_0} - 2\nabla^{f^{-1}}_{e_i}\left(\nabla^{f^{-1}}_{e_i} W\right)^T_{x_0} + 2KW_{x_0}\right)^V\\ &= \frac{1}{2}\left\{\left(\nabla^{f^{-1}}_{e_i}\left(\nabla^{f^{-1}}_{e_i} W\right)^V_{x_0}\right)^V - \left(\nabla^{f^{-1}}_{e_i}\left(\nabla^{f^{-1}}_{e_i} W\right)^T_{x_0}\right)^V + 2KW^V_{x_0}\right\}\\ &= \frac{1}{2}\left\{\overset{V}{\nabla}_{e_i}\overset{V}{\nabla}_{e_i} W^V_{x_0} + \left(\nabla^{f^{-1}}_{e_i}\left(\nabla^{f^{-1}}_{e_i} W^T\right)^V_{x_0}\right)^V - \left(\nabla^{f^{-1}}_{e_i}\left(\nabla^{f^{-1}}_{e_i} W\right)^T_{x_0}\right)^V + 2KW^V_{x_0}\right\}\\ &= \frac{1}{2}\left\{\Delta W^V_{x_0} + \left(\nabla^{f^{-1}}_{e_i}\left(\nabla^{f^{-1}}_{e_i} W^T\right)^V_{x_0}\right)^V - \left(\nabla^{f^{-1}}_{e_i}\left(\nabla^{f^{-1}}_{e_i} W^V\right)^T_{x_0}\right)^V\right.\\ &\qquad \left. - \left(\nabla^{f^{-1}}_{e_i}\left(\nabla^{f^{-1}}_{e_i} W^T\right)^T_{x_0}\right)^V + 2KW^V_{x_0}\right\} , \qquad (3.13)\end{aligned}$$

where ΔW^V is the Laplacian in the normal bundle V_0 to f. We further have

$$\left(\nabla^{f^{-1}}_{e_i} W^T\right)^V = \left(\nabla^{f^{-1}}_{e_i}\left(df\left(df^{-1}(W^T)\right)\right)\right)^V = \nabla df\left(e_i, df^{-1}(W^T)\right) . \qquad (3.14)$$

Denoting by $\nabla\nabla df$ the covariant derivative of ∇df in the Riemannian bundle $\odot^2 T^*M_0 \otimes V_0$ (M with the metric g_0), we obtain, as $\overset{M_0}{\nabla}_{e_i} e_{i\,x_0} = 0$,

$$\begin{aligned}\overset{V}{\nabla}_{e_i}\left(\nabla df\left(e_i, df^{-1}(W^T)\right)\right)_{x_0} &= \nabla_{e_i}\nabla df_{x_0}\left(e_i, df^{-1}(W^T)\right) +\\ &\quad + \nabla df_{x_0}\left(e_i, \overset{M_0}{\nabla}_{e_i}\left(df^{-1}(W^T)\right)\right) . \qquad (3.15)\end{aligned}$$

Using Codazzi's equation in a space form (N, h) (cf. e.g. Ref. [Ko-No/69]), we get

$$\begin{aligned}\nabla_{e_i}\nabla df_{x_0}(e_i, e_k) &= \nabla_{e_i}\nabla df_{x_0}(e_k, e_i) = \nabla_{e_k}\nabla df_{x_0}(e_i, e_i)\\ &= \overset{V}{\nabla}_{e_k}\left(\nabla df(e_i, e_i)\right)_{x_0} .\end{aligned}$$

Thus,

$$\nabla_{e_i}\nabla df_{x_0}(e_i, e_k) = 2\overset{V}{\nabla}_{e_k} H_{x_0}\,. \tag{3.16}$$

Since $df : (TM, g_a) \to (T_0, h)$ is an isometry of Riemannian bundles and $\left(\nabla^{f^{-1}}\right)^T$ is the connection of (T_0, h), we obtain, by applying Eqs. (3.14), (3.15), and (3.16) to Eq. (3.13),

$$\begin{aligned}\left(\nabla^{\varphi^{-1}}_{\frac{\partial}{\partial t}} H_{(x_0,0)}\right)^V &= \frac{1}{2}\Big\{\Delta W^V_{x_0} + 2\nabla_{df^{-1}(W^T)} H_{x_0} + \nabla df_{x_0}\Big(e_i, df^{-1}\Big(\big(\nabla^{f^{-1}}_{e_i} W^T\big)^T_{x_0}\Big)\Big) \\ &\quad - \left(\nabla^{f^{-1}}_{e_i}\big(\nabla^{f^{-1}}_{e_i} W^V\big)^T_{x_0}\right)^V - \left(\nabla^{f^{-1}}_{e_i}\big(\nabla^{f^{-1}}_{e_i} W^T\big)^T_{x_0}\right)^V + 2KW^V_{x_0}\Big\}\,. \end{aligned} \tag{3.17}$$

If we now replace W^T by $\left(\nabla^{f^{-1}}_{e_i} W^T\right)^T$ in Eq. (2.14), we get

$$\left(\nabla^{f^{-1}}_{e_i}\big(\nabla^{f^{-1}}_{e_i} W^T\big)^T_{x_0}\right)^V = \nabla df_{x_0}\Big(e_i, df^{-1}\Big(\big(\nabla^{f^{-1}}_{e_i} W^T\big)^T\Big)\Big) \tag{3.18}$$

and, analogously,

$$\left(\nabla^{f^{-1}}_{e_i}\big(\nabla^{f^{-1}}_{e_j} W^V\big)^T_{x_0}\right)^V = \nabla df_{x_0}\Big(e_i, df^{-1}\Big(\big(\nabla^{f^{-1}}_{e_j} W^V\big)^T\Big)\Big)\,. \tag{3.19}$$

The latter equation can be evaluated as follows.

$$\begin{aligned}\nabla df_{x_0}\Big(e_i, df^{-1}\Big(\big(\nabla^{f^{-1}}_{e_j} W^V\big)^T\Big)\Big) &= \Big\langle e_k, df^{-1}_{x_0}\Big(\big(\nabla^{f^{-1}}_{e_j} W^V\big)^T\Big)\Big\rangle_{g_0} \nabla df_{x_0}(e_i, e_k) \\ &= \Big\langle df_{x_0}(e_k), \nabla^{f^{-1}}_{e_j} W^V_{x_0}\Big\rangle_h \nabla df_{x_0}(e_i, e_k) \qquad (3.20) \\ &= -\big\langle \nabla df_{x_0}(e_j, e_k), W^V\big\rangle_h \nabla df_{x_0}(e_i, e_k)\,.\end{aligned}$$

Thus, in particular,

$$\left(\nabla^{f^{-1}}_{e_i}\big(\nabla^{f^{-1}}_{e_i} W^V\big)^T_{x_0}\right)^V = -\tilde{A}_{x_0}(W^V_{x_0})\,. \tag{3.21}$$

Substitution of Eqs. (3.18) and (3.21) in Eq. (3.17) yields

$$\begin{aligned}&\left(\nabla^{\varphi^{-1}}_{\frac{\partial}{\partial t}} H_{(x_0,0)}\right)^V = \\ &= \frac{1}{2}\Big\{\Delta W^V_{x_0} + 2\overset{V}{\nabla}_{df^{-1}(W^T)} H_{x_0} + \tilde{A}_{x_0}(W^V_{x_0}) + 2KW^V_{x_0}\Big\}\,. \end{aligned} \tag{3.22}$$

So,

$$\begin{aligned}\frac{\partial}{\partial t}\|H_t\|^2\big|_{t=0}(x_0) &= 2\Big\langle \nabla^{\varphi^{-1}}_{\frac{\partial}{\partial t}} H_{(x_0,0)}, H_{(x_0,0)}\Big\rangle_h \\ &= \Big\langle \Delta W^V_{x_0} + 2\overset{V}{\nabla}_{df^{-1}(W^T)} H_{x_0} + \tilde{A}_{x_0}(W^V_{x_0}) + 2KW^V_{x_0}\,,\, H_{x_0}\Big\rangle_h\,.\end{aligned}$$

Then, due to the equality $2\left\langle \overset{V}{\nabla}_{df^{-1}(W^T)} H_{x_0}, H_{x_0} \right\rangle_h = d\|H\|^2_{h,x_0}(df^{-1}(W^T))$, Eq. (3.4) becomes

$$
\begin{aligned}
&\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\Big|_{t=0} = \\
&\quad = \int_D \Big\{ < \Delta W^V, H >_h + d\|H\|^2_h(df^{-1}(W^T)) + < \tilde{A}(W^V), H >_h + \\
&\qquad\qquad + 2\bar{K} < W^V, H >_h - 2(\|H\|^2_h + \bar{K}) < W^V, H >_h - d\|H\|^2_h(df^{-1}(W^T)) \Big\}\, dA_{g_0} \\
&\quad = \int_D \Big\{ < \Delta W^V, H >_h + < \tilde{A}(W^V), H >_h - 2\|H\|^2_h < W^V, H >_h \Big\}\, dA_{g_0} .
\end{aligned}
$$

Since W has compact support in D, we have (see e.g. Ref. [Ee-Le/83])

$$\int_D < \Delta W^V, H >_h dA_{g_0} = \int_D < W^V, \Delta H >_h dA_{g_0} .$$

Furthermore, $< \tilde{A}(W^V), H >_h = < W^V, \tilde{A}(H) >_h$. Therefore,

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\Big|_{t=0} = \int_D \left\langle \Delta H + \tilde{A}(H) - 2\|H\|^2_h H \,,\, W^V \right\rangle_h dA_{g_0} ,$$

which depends only on the vertical part of W. Hence, f is a critical point of $\mathcal{W}$, for compactly supported variations on D, iff

$$\Delta H + \tilde{A}(H) - 2\|H\|^2_h H = 0 \ \text{ on } D .$$

If we replace in the above derivation $t = 0$ by t arbitrary, we obtain in the same way the equation

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t) = \int_D \left\langle \Delta H_t + \tilde{A}_t(H_t) - 2\|H_t\|^2_h H_t \,,\, \frac{\partial v}{\partial t}(\cdot, t) \right\rangle_h dA_{g_t} ,$$

where $\tilde{A}_t \in C^\infty(\otimes V_t^* \otimes V_t)$ is the tensor defined by Eq. (2.12), relative to the immersion $f_t : M_t \to N$, and where ΔH_t is the Laplacian in the normal bundle V_t.

Now we suppose that $f = f_0 : M_0 \to N$ is a critical point of $\mathcal{W}_D$ and that W is a vertical vector field, i.e. $W \in C^\infty_c(V)$. Then, we calculate the second variation formula for $\mathcal{W}$ at f, that is, we are going to evaluate the expression

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\mathcal{W}_D(f_t)\Big|_{t=0} &= \int_D \frac{\partial}{\partial t}\left\{ \left\langle \Delta H_t + \tilde{A}_t(H_t) - 2\|H_t\|^2_h H_t \,,\, \frac{\partial v}{\partial t}(\cdot, t) \right\rangle_h dA_{g_t} \right\} \Big|_{t=0} \\
&= \int_D \left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \left(\Delta H_t + \tilde{A}_t(H_t) - 2\|H_t\|^2_h H_t \right)\Big|_{t=0} , W \right\rangle_h dA_{g_0} .
\end{aligned}
\tag{3.23}
$$

Let $x_0 \in M$ and let e_1, e_2 be an orthonormal basis of $(T_{x_0}M, g_0)$, which can be extended to form sections $\tilde{e}_1, \tilde{e}_2$ with some additional properties, to be given below, in order to simplify the forthcoming calculations. Let $\bar{E}_i = df_{x_0}(e_i)$ for $i = 1, 2$. Then, $(\bar{E}_1, \bar{E}_2)$ is an orthonormal basis of $(T_{(x_0,0)}, h)$. On the subbundle T of $v^{-1}TN$, a covariant derivative $\overset{T}{\nabla}$ is defined as $\overset{T}{\nabla}_{(u,h)} Z_{(x,t)} = \left(\nabla^{v^{-1}}_{(u,h)} Z_{(x,t)}\right)^T$, $\forall Z \in C^\infty(T)$, $(x,t) \in M \times I$, $(u,h) \in T_{(x,t)}(M \times I)$. In particular, $\overset{T}{\nabla}_{(u,0)} Z_{(x,t)} = \overset{T_t}{\nabla}_u Z_{t\,(x)}$. Let $\gamma : I \to M \times I$ be given by $\gamma(t) = (x_0, t)$. Then, the vector bundle $\gamma^{-1}T$ has base space I and induced covariant derivative $\nabla^{\gamma^{-1}}$. We define the sections $E_1, E2 \in C^\infty(\gamma^{-1}T)$ as to result from parallel-transporting $\bar{E}_1, \bar{E}_2$ on (T, h) along γ. Thus, for each $t \in I$, $E_1(t), E_2(t)$ form an orthonormal basis of $(T_{(x_0,t)}, h)$ satisfying

$$\nabla_1^{\gamma^{-1}} E_{i_t} = 0 \ , \ \ \forall i = 1, 2 \ .$$

Once more, for each $t \in I$, we parallel-transport the vectors $E_i(t)$ of $(T_t)_{x_0}$ in (T_t, h) along geodesics of $M_t = (M, g_t)$ passing through x_0. In this way, we obtain local smooth sections $\tilde{E}_i(\cdot, t)$ of T_t that constitute, at each point $x \in M$, an orthonormal basis $\tilde{E}_1(x,t), \tilde{E}_2(x,t)$ of $(T_{(x,t)}, h)$. The $\tilde{E}_i$ define sections of the bundle T, smooth in the variable (x,t), and satisfy the properties

$$\overset{T}{\nabla}_{(u,0)} \tilde{E}_{i\,(x_0,t)} = \overset{T_t}{\nabla}_u \left(\tilde{E}_i(\cdot, t)\right)_{x_0} = 0 \ , \ \ \forall u \in T_{x_0}M, \ t \in I$$

and

$$\overset{T}{\nabla}_{\frac{\partial}{\partial t}} \tilde{E}_{i\,(x_0,t)} = \overset{T}{\nabla}_{\gamma'(t)} \tilde{E}_{i\,(x_0,t)} = \nabla_1^{\gamma^{-1}} E_{i\,(t)} = 0 \ .$$

Since, $\forall t, x, \quad d(f_t)_x : (T_xM, g_t) \to (T_{(x,t)}, h)$ is an isometry, $\tilde{e}_i(x,t)$ defined by $d(f_t)_x(\tilde{e}_i(x,t)) = \tilde{E}_{i\,(x,t)}$ gives a smooth vector field $\tilde{e}_{i_t} = \tilde{e}_i(\cdot, t)$ of M_t, which is, in fact, the one obtained by parellel transport of $\tilde{e}_i(x_0, t)$ along geodesics of M_t. Thus, for each t, x, $\tilde{e}_1(x,t), \tilde{e}_2(x,t)$ is an orthonormal basis of (T_xM, g_t) satisfying

$$\overset{M_t}{\nabla}_u \left(\tilde{e}_i(\cdot, t)\right)_{x_0} = 0 \quad \text{i.e.} \quad \left(\nabla^{v^{-1}}_{(u,0)}\left(dv(\tilde{e}_i, 0)\right)_{(x_0,t)}\right)^T = 0 \ , \ \ \forall u \in T_{x_0}M, \ t \in I \quad (3.24)$$

and

$$\left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\left(dv(\tilde{e}_i, 0)\right)_{(x_0,t)}\right)^T = 0 \ , \ \ \forall t \in I \ . \qquad (3.25)$$

We denote $e_i(x) = \tilde{e}_i(x, 0)$, $\forall x \in D$ and s_t as in Eq. (3.9). Then, we have

$$\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\left(dv(\tilde{e}_i, 0)\right)_{(x_0,0)} = \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\left(dv(\tilde{e}_i, 0)\right)_{(x_0,0)}\right)^V$$

$$= \left(\nabla dv_{(x_0,0)}\left(\frac{\partial}{\partial t},(\tilde{e}_i,0)\right) + df_{x_0}(z_i)\right)^V$$

$$= \left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\left(dv\left(\frac{\partial}{\partial t}\right)\right)_{(x_0,0)} - dv_{(x_0,0)}\left(\overset{M_0\times I}{\nabla}_{(\tilde{e}_i,0)}\frac{\partial}{\partial t}\right)\right)^V = \left(\nabla^{f^{-1}}_{e_i} W_{x_0}\right)^V. \quad (3.26)$$

Observe that, as a consequence of Eqs. (3.26) and (3.7), we have .

$$\begin{aligned} < z_j(x_0), e_k >_{g_0} &= \langle df_{x_0}(z_j), df_{x_0}(e_k)\rangle_h = \left\langle df_{x_0}\left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\tilde{e}_j\right), df_{x_0}(e_k)\right\rangle_h \\ &= \left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}(dv(\tilde{e}_j,0))_{(x_0,0)} - \nabla dv_{(x_0,0)}\left(\frac{\partial}{\partial t},(\tilde{e}_j,0)\right),\ df_{x_0}(e_k)\right\rangle_h \\ &= -\left\langle \nabla^{f^{-1}}_{e_j} W_{x_0}, df_{x_0}(e_k)\right\rangle_h = \left\langle W_{x_0}, \nabla df_{x_0}(e_j,e_k)\right\rangle_h, \quad (3.27)\end{aligned}$$

whence

$$\nabla df_{x_0}(e_i,z_j) = \left\langle W_{x_0}, \nabla df_{x_0}(e_j,e_k)\right\rangle_h \nabla df_{x_0}(e_i,e_k). \quad (3.28)$$

From Eq. (3.24), we get

$$\Delta H_t(x_0) = \overset{V_t}{\nabla}_{\tilde{e}_{i_t}} \overset{V_t}{\nabla}_{\tilde{e}_{i_t}} H_{t\,(x_0)} = \left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V\right)^V_{(x_0,0)}.$$

Hence, by applying Eq. (3.10), we obtain

$$\left\langle\left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\Delta H\right)_{(x_0,0)}, W\right\rangle_h =$$

$$\begin{aligned} &= \left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V\right)^V_{(x_0,0)}, W\right\rangle_h \\ &= \left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V_{(x_0,0)} - \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V\right)^T_{(x_0,0)}, W\right\rangle_h \\ &= \Big\langle \nabla^{v^{-1}}_{(\tilde{e}_i,0)}\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V_{(x_0,0)} + \bar{R}_{f(x_0)}(df_{x_0}(e_i), W_{x_0})\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V_{(x_0,0)} \\ &\quad + \nabla^{v^{-1}}_{[\frac{\partial}{\partial t},(\tilde{e}_i,0)]}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V_{(x_0,0)} \\ &\quad - \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\left(\left\langle\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V, dv(\tilde{e}_j,0)\right\rangle_h dv(\tilde{e}_j,0)\right)_{(x_0,0)}, W\Big\rangle_h \\ &= \Big\langle \nabla^{v^{-1}}_{(\tilde{e}_i,0)}\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V_{(x_0,0)} - \bar{K}\left\langle W_{x_0}, \overset{V}{\nabla}_{e_i} H_{x_0}\right\rangle_h df_{x_0}(e_i) \\ &\quad + \nabla^{v^{-1}}_{(z_i(x_0),0)}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V_{(x_0,0)} \\ &\quad - \left\langle\nabla^{v^{-1}}_{(\tilde{e}_i,0)}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V, dv(\tilde{e}_j,0)\right\rangle_h \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}(dv(\tilde{e}_j,0))_{(x_0,0)}, W\Big\rangle_h \\ &= \Big\langle \nabla^{v^{-1}}_{(\tilde{e}_i,0)}\nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}H\right)^V_{(x_0,0)} + \nabla^{f^{-1}}_{z_i}\left(\nabla^{f^{-1}}_{e_i}H\right)^V_{x_0} \end{aligned}$$

$$
\begin{aligned}
&\quad - \big\langle \nabla^{f^{-1}}_{e_i}\big(\nabla^{f^{-1}}_{e_i} H\big)^V, df(e_j)\big\rangle_h \nabla^{f^{-1}}_{e_j} W_{x_0}\,,\ W\big\rangle_h \\
&= \Big\langle \nabla^{v^{-1}}_{(\tilde e_i,0)} \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \nabla^{v^{-1}}_{(\tilde e_i,0)} H_{(x_0,0)} - \nabla^{v^{-1}}_{(\tilde e_i,0)} \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \Big[\big\langle \nabla^{v^{-1}}_{(\tilde e_i,0)} H, dv(\tilde e_j,0)\big\rangle_h dv(\tilde e_j,0)\Big]_{(x_0,0)} \\
&\quad + \nabla^{f^{-1}}_{e_i}\big(\nabla^{f^{-1}}_{e_i} H\big)^V_{x_0} + \big\langle \nabla^{f^{-1}}_{e_i} H_{x_0}, \nabla df_{x_0}(e_i,e_j)\big\rangle_h \nabla^{f^{-1}}_{e_j} W_{x_0}\,,\ W\Big\rangle_h \\
&= \Big\langle \nabla^{v^{-1}}_{(\tilde e_i,0)} \nabla^{v^{-1}}_{(\tilde e_i,0)} \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H_{(x_0,0)} + \nabla^{f^{-1}}_{e_i}\big(R_{f^{-1}}(df(e_i),W)H\big)_{x_0} + \nabla^{v^{-1}}_{(\tilde e_i,0)} \nabla^{v^{-1}}_{[\frac{\partial}{\partial t},(e_i,0)]} H_{(x_0,0)} \\
&\quad - \nabla^{v^{-1}}_{(\tilde e_i,0)} \Big[\big\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \nabla^{v^{-1}}_{(\tilde e_i,0)} H, dv(\tilde e_j,0)\big\rangle_h dv(e_j,0) + \big\langle \nabla^{v^{-1}}_{(\tilde e_i,0)} H, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\big(dv(\tilde e_j,0)\big)\big\rangle_h dv(\tilde e_j,0) \\
&\qquad\qquad + \big\langle \nabla^{v^{-1}}_{(\tilde e_i,0)} H, dv(\tilde e_j,0)\big\rangle_h \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\big(dv(\tilde e_j,0)\big)\Big]_{(x_0,0)} \\
&\quad + \nabla^{f^{-1}}_{e_i}\big(\nabla^{f^{-1}}_{e_i} H\big)^V_{x_0} + \big\langle \nabla^{f^{-1}}_{e_i} H_{x_0}, \nabla df_{x_0}(e_i,e_j)\big\rangle_h \nabla^{f^{-1}}_{e_j} W_{x_0}\,,\ W\Big\rangle_h \\
&= \Big\langle \nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_i}\big(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}\big)^V_{x_0} + \nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_i}\Big[\big\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, df(e_j)\big\rangle_h df(e_j)\Big]_{x_0} \\
&\quad - K\nabla^{f^{-1}}_{e_i}\big(< W,H > df(e_i)\big)_{x_0} + \nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_i} H_{x_0} \\
&\quad - \nabla^{f^{-1}}_{e_i}\Big[\big\langle \nabla^{v^{-1}}_{(\tilde e_i,0)} \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0} + R_f(df(e_i),W)H + \nabla^{v^{-1}}_{[\frac{\partial}{\partial t},(\tilde e_i,0)]} H|_{t=0}\,,\ df(e_j)\big\rangle_h df(e_j) \\
&\quad + \big\langle \nabla^{f^{-1}}_{e_i} H, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\big(dv(\tilde e_j,0)\big)|_{t=0}\big\rangle_h df(e_j) + \big\langle \nabla^{f^{-1}}_{e_i} H, df(e_j)\big\rangle_h \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\big(dv(\tilde e_j,0)\big)|_{t=0}\Big]_{(x_0,0)} \\
&\quad + \nabla^{f^{-1}}_{e_i}\big(\nabla^{f^{-1}}_{e_i} H\big)^V_{x_0} + \big\langle \nabla^{f^{-1}}_{e_i} H_{x_0}, \nabla df_{x_0}(e_i,e_j)\big\rangle_h \nabla^{f^{-1}}_{e_j} W_{x_0}\,,\ W\Big\rangle_h \\
&= \Big\langle \nabla^{f^{-1}}_{e_i} \overset{V}{\nabla}_{e_i}\big(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}\big)^V_{x_0} + \nabla^{f^{-1}}_{e_i}\Big[\big\langle \nabla^{f^{-1}}_{e_i}\big(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}\big)^V, df(e_j)\big\rangle_h df(e_j)\Big]_{x_0} \\
&\quad + \nabla^{f^{-1}}_{e_i}\Big[\big\langle \nabla^{f^{-1}}_{e_i} \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, df(e_j)\big\rangle_h df(e_j)\Big]_{x_0} \\
&\quad + \nabla^{f^{-1}}_{e_i}\Big[\big\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, \nabla^{f^{-1}}_{e_i}\big(df(e_j)\big)\big\rangle_h df(e_j)\Big]_{x_0} \\
&\quad + \nabla^{f^{-1}}_{e_i}\Big[\big\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, df(e_j)\big\rangle_h \nabla^{f^{-1}}_{e_i}\big(df(e_j)\big)\Big]_{x_0} - K\nabla^{f^{-1}}_{e_i}\big(< W,H > df(e_i)\big)_{x_0} \\
&\quad + \nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_i} H_{x_0} - \nabla^{f^{-1}}_{e_i}\Big[\big\langle \nabla^{f^{-1}}_{e_i} \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, df(e_j)\big\rangle_h df(e_j)\Big]_{x_0} \\
&\quad + K\nabla^{f^{-1}}_{e_i}\Big[< W,H >_h < df(e_i), df(e_j) >_h df(e_j)\Big]_{x_0} \\
&\quad - \nabla^{f^{-1}}_{e_i}\Big[\big\langle \nabla^{f^{-1}}_{e_i} H, df(e_j)\big\rangle_h df(e_j)\Big]_{x_0} \\
&\quad - \nabla^{f^{-1}}_{e_i}\Big[\big\langle \nabla^{f^{-1}}_{e_i} H, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\big(dv(\tilde e_j,0)\big)|_{t=0}\big\rangle_h df(e_j)\Big]_{x_0} \\
&\quad - \nabla^{f^{-1}}_{e_i}\Big[\big\langle \nabla^{f^{-1}}_{e_i} H, df(e_j)\big\rangle_h \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}\big(dv(\tilde e_j,0)\big)|_{t=0}\Big]_{x_0} + \nabla^{f^{-1}}_{e_i}\big(\nabla^{f^{-1}}_{e_i} H\big)^V_{x_0}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left\langle \nabla^{f^{-1}}_{e_i} H_{x_0}, \nabla df_{x_0}(e_i, e_j) \right\rangle_h \nabla^{f^{-1}}_{e_j} W_{x_0} \ , \ W \Big\rangle_h \\
= \ &\Big\langle \nabla^{f^{-1}}_{e_i} \overset{V}{\nabla}_{e_i} \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}\right)^V_{x_0} - \nabla^{f^{-1}}_{e_i} \left[\left\langle \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}\right)^V, \nabla df(e_i, e_j) \right\rangle_h df(e_j) \right]_{x_0} \\
&+ \nabla^{f^{-1}}_{e_i} \left[\left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, \nabla^{f^{-1}}_{e_i} (df(e_j)) \right\rangle_h df(e_j) \right]_{x_0} \\
&+ \nabla^{f^{-1}}_{e_i} \left[\left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, df(e_j) \right\rangle_h \nabla^{f^{-1}}_{e_i} (df(e_j)) \right]_{x_0} + \nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_i} H_{x_0} \\
&- \nabla^{f^{-1}}_{e_i} \left[\left\langle \nabla^{f^{-1}}_{e_i} H, df(e_j) \right\rangle_h df(e_j) \right]_{x_0} \\
&- \nabla^{f^{-1}}_{e_i} \left[\left\langle \nabla^{f^{-1}}_{e_i} H, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))|_{t=0} \right\rangle_h df(e_j) \right]_{x_0} \\
&- \nabla^{f^{-1}}_{e_i} \left[\left\langle \nabla^{f^{-1}}_{e_i} H, df(e_j) \right\rangle_h \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))|_{t=0} \right]_{x_0} + \nabla^{f^{-1}}_{e_i} \left(\nabla^{f^{-1}}_{e_i} H\right)^V_{x_0} \\
&+ \left\langle \nabla^{f^{-1}}_{e_i} H_{x_0}, \nabla df_{x_0}(e_i, e_j) \right\rangle_h \nabla^{f^{-1}}_{e_j} W_{x_0} \ , \ W \Big\rangle_h .
\end{aligned}
$$

Since W is vertical, we have, on D,

$$
\begin{aligned}
&\left\langle \nabla^{f^{-1}}_{e_i} \left[\left\langle \nabla^{f^{-1}}_{e_i} H, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))|_{t=0} \right\rangle_h df(e_j) \right]_{x_0} , W \right\rangle_h = \\
&= \left\langle \nabla^{f^{-1}}_{e_i} H, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))|_{t=0} \right\rangle_h \left\langle \nabla df(e_i, e_j), W \right\rangle_h .
\end{aligned}
$$

Using the equality $\left\langle \nabla^{f^{-1}}_{e_i} H, df(e_j) \right\rangle_h = -\left\langle H, \nabla df(e_i, e_j) \right\rangle_h$, we get

$$
\begin{aligned}
&\left\langle \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \Delta H\right)_{(x_0, 0)}, W \right\rangle_h = \\
= \ &\Big\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_i} \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}\right)^V_{x_0} + \nabla^{f^{-1}}_{e_i} \left[\left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, df\left(\overset{M_0}{\nabla}_{e_i} e_j\right) \right\rangle_h df(e_j) \right]_{x_0} \\
&+ \nabla^{f^{-1}}_{e_i} \left[\left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, df(e_j) \right\rangle_h \nabla^{f^{-1}}_{e_i} (df(e_j)) \right]_{x_0} + \nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_i} H_{x_0} \\
&- \nabla^{f^{-1}}_{e_i} \left(\nabla^{f^{-1}}_{e_i} H\right)^T_{x_0} \\
&- \left\langle \nabla^{f^{-1}}_{e_i} H_{x_0}, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))|_{t=0}(x_0) \right\rangle_h \nabla df_{x_0}(e_i, e_j) \\
&+ \nabla^{f^{-1}}_{e_i} \left[\left\langle H, \nabla df(e_i, e_j) \right\rangle_h \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))|_{t=0} \right]_{x_0} + \nabla^{f^{-1}}_{e_i} \left(\nabla^{f^{-1}}_{e_i} H\right)^V_{x_0} \\
&+ \left\langle \nabla^{f^{-1}}_{e_i} H_{x_0}, \nabla df_{x_0}(e_i, e_j) \right\rangle_h \nabla^{f^{-1}}_{e_j} W_{x_0} \ , \ W \Big\rangle_h .
\end{aligned}
$$

Since $\overset{M_0}{\nabla}_{e_i} e_{j\,(x_0)} = 0$, clearly

$$
\left\langle \nabla^{f^{-1}}_{e_i} \left[\left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, df\left(\overset{M_0}{\nabla}_{e_i} e_j\right) \right\rangle_h df(e_j) \right]_{x_0} , W \right\rangle_h = 0 .
$$

Hence,

$$\begin{aligned}
&\left\langle \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \Delta H \right)_{(x_0,0)}, W \right\rangle_h = \\
&= \left\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_i} \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0} \right)^{V}_{x_0} + \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{z_i} H_{x_0} + \overset{V}{\nabla}_{z_i} \overset{V}{\nabla}_{e_i} H_{x_0} \right. \\
&\quad + \nabla^{f^{-1}}_{e_i} \left[\left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, df(e_j) \right\rangle_h \nabla^{f^{-1}}_{e_i} (df(e_j)) \right]_{x_0} \\
&\quad - \left\langle \nabla^{f^{-1}}_{e_i} H_{x_0}, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))|_{t=0}(x_0) \right\rangle_h \nabla df_{x_0}(e_i, e_j) \\
&\quad + \nabla^{f^{-1}}_{e_i} \left[\langle H, \nabla df(e_i, e_j) \rangle_h \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))|_{t=0} \right]_{x_0} \\
&\quad \left. + \left\langle \nabla^{f^{-1}}_{e_i} H_{x_0}, \nabla df_{x_0}(e_i, e_j) \right\rangle_h \nabla^{f^{-1}}_{e_j} W_{x_0} \, , \, W \right\rangle_h .
\end{aligned} \tag{3.29}$$

Now, $\forall x \in D$,

$$\left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H_{(x,0)}, df_x(e_j) \right\rangle_h = - \left\langle H_x, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))_{(x,0)} \right\rangle_h .$$

Moreover,

$$\begin{aligned}
\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))_{(x,0)} &= \nabla dv_{(x,0)}\left(\frac{\partial}{\partial t}, (e_j, 0)\right) + df_x(z_j) \\
&= \nabla^{f^{-1}}_{e_j} W_x + df_x(z_j)
\end{aligned} \tag{3.30}$$

and, from Eq. (3.25), $\nabla^{f^{-1}}_{e_j} W_{x_0} + df_{x_0}(z_j) = \overset{V}{\nabla}_{e_j} W_{x_0}$. Thus,

$$\begin{aligned}
&\left\langle \nabla^{f^{-1}}_{e_i} \left[\left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} H|_{t=0}, df(e_j) \right\rangle_h \nabla^{f^{-1}}_{e_i} (df(e_j)) \right]_{x_0} W \right\rangle_h = \\
&= \left\langle -d\left(\left\langle H, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0)) \right\rangle_h \right)_{x_0}(e_i) \, \nabla df_{x_0}(e_i, e_j) \, , \, W \right\rangle_h \\
&\quad - \left\langle H_{x_0}, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j, 0))_{(x_0,0)} \right\rangle_h \left\langle \nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_i} (df(e_j))_{x_0}, W \right\rangle_h \\
&= - \left\langle \nabla^{f^{-1}}_{e_i} H_{x_0}, \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \langle \nabla df_{x_0}(e_i, e_j) \, , \, W \rangle_h \\
&\quad - \left\langle H_{x_0}, \nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_j} W_{x_0} + \nabla df_{x_0}(e_i, z_j) \right\rangle_h \langle \nabla df_{x_0}(e_i, e_j) \, , \, W \rangle_h \\
&\quad - \left\langle H_{x_0}, \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \left\langle \nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_i} (df(e_j))_{x_0}, W \right\rangle_h \\
&= - \left\langle \overset{V}{\nabla}_{e_i} H_{x_0}, \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \langle \nabla df_{x_0}(e_i, e_j), W \rangle_h \\
&\quad - \left\langle H_{x_0}, \nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_j} W_{x_0} \right\rangle_h \langle \nabla df_{x_0}(e_i, e_j), W \rangle_h \\
&\quad - \langle H_{x_0}, \nabla df_{x_0}(e_i, z_j) \rangle_h \langle \nabla df_{x_0}(e_i, e_j), W \rangle_h \\
&\quad - \left\langle H_{x_0}, \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \left\langle \nabla^{f^{-1}}_{e_i} \nabla^{f^{-1}}_{e_i} (df(e_j))_{x_0}, W \right\rangle_h .
\end{aligned}$$

As

$$\left\langle H_{x_0}, \nabla_{e_i}^{f^{-1}} \nabla_{e_j}^{f^{-1}} W_{x_0} \right\rangle_h = \left\langle H_{x_0}, \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h + \left\langle H_{x_0}, \nabla_{e_i}^{f^{-1}} \left(\nabla_{e_j}^{f^{-1}} W_{x_0} \right)^T \right\rangle_h =$$

$$= \left\langle H_{x_0}, \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h - \left\langle \nabla_{e_i}^{f^{-1}} H_{x_0}, \left(\nabla_{e_j}^{f^{-1}} W_{x_0} \right)^T \right\rangle_h$$

$$= \left\langle H_{x_0}, \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h - \left\langle \nabla_{e_i}^{f^{-1}} H_{x_0}, df_{x_0}(e_k) \right\rangle_h \left\langle \nabla_{e_j}^{f^{-1}} W_{x_0}, df_{x_0}(e_k) \right\rangle_h$$

$$= \left\langle H_{x_0}, \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h - \left\langle H_{x_0}, \nabla df_{x_0}(e_i, e_k) \right\rangle_h \left\langle W_{x_0}, \nabla df_{x_0}(e_j, e_k) \right\rangle_h ,$$

and using Eq. (3.16), we have

$$\left\langle \nabla_{e_i}^{f^{-1}} \nabla_{e_i}^{f^{-1}} (df(e_j))_{x_0}, W_{x_0} \right\rangle_h =$$

$$= \left\langle \nabla_{e_i}^{f^{-1}} \left(\nabla df(e_i, e_j) + df\left(\overset{M_0}{\nabla}_{e_i} e_j \right) \right)_{x_0}, W \right\rangle_h$$

$$= \left\langle \overset{V}{\nabla}_{e_i} \left(\nabla df(e_i, e_j) \right)_{x_0} + \nabla df_{x_0}\left(e_i, \overset{M_0}{\nabla}_{e_i} e_j \right) + df_{x_0}\left(\overset{M_0}{\nabla}_{e_i} \overset{M_0}{\nabla}_{e_i} e_j \right), W \right\rangle_h$$

$$= \left\langle \overset{V}{\nabla}_{e_i} \left(\nabla df(e_i, e_j) \right)_{x_0}, W \right\rangle_h = \left\langle \nabla_{e_i} \nabla df_{x_0}(e_i, e_j), W \right\rangle_h$$

$$= 2 \left\langle \overset{V}{\nabla}_{e_j} H_{x_0}, W \right\rangle_h . \tag{3.31}$$

Hence, from the latter two partial calculations and Eq. (3.27), we obtain

$$\left\langle \nabla_{e_i}^{f^{-1}} \left[\left\langle \nabla_{\frac{\partial}{\partial t}}^{f^{-1}} H|_{t=0}, df(e_j) \right\rangle_h \nabla_{e_i}^{f^{-1}} (df(e_j)) \right]_{x_0} W \right\rangle_h =$$

$$= - \left\langle \overset{V}{\nabla}_{e_i} H_{x_0}, \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h$$

$$- \left\langle H_{x_0}, \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h$$

$$+ \left\langle H, \nabla df_{x_0}(e_i, e_k) \right\rangle_h \left\langle W, \nabla df_{x_0}(e_j, e_k) \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h$$

$$- \left\langle H_{x_0}, \nabla df_{x_0}(e_i, e_k) \right\rangle_h \left\langle W, \nabla df_{x_0}(e_k, e_j) \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h$$

$$-2 \left\langle H_{x_0}, \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \left\langle \overset{V}{\nabla}_{e_j} H_{x_0}, W \right\rangle_h$$

$$= - \left\langle \overset{V}{\nabla}_{e_i} H_{x_0}, \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h$$

$$- \left\langle H_{x_0}, \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h$$

$$-2 \left\langle H_{x_0}, \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \left\langle \overset{V}{\nabla}_{e_j} H_{x_0}, W \right\rangle_h . \tag{3.32}$$

From Eq. (3.26), we have

$$\left\langle \nabla^{f^{-1}}_{e_i} H_{x_0}, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j,0))_{(x_0,0)} \right\rangle_h \nabla df_{x_0}(e_i,e_j) = \left\langle \overset{V}{\nabla}_{e_i} H_{x_0}, \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \nabla df_{x_0}(e_i,e_j)\,. \tag{3.33}$$

Using again Eq. (3.26), combined with Eq. (3.31), we get

$$\begin{aligned}
&\left\langle \nabla^{f^{-1}}_{e_i} [\langle H, \nabla df(e_i,e_j)\rangle_h \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j,0))|_{t=0}]_{x_0}, W \right\rangle_h = \\
&= \left\langle \overset{V}{\nabla}_{e_i} H_{x_0}, \nabla df_{x_0}(e_i,e_j) \right\rangle_h \left\langle \overset{V}{\nabla}_{e_j} W_{x_0}, W \right\rangle_h + 2\left\langle H, \overset{V}{\nabla}_{e_j} H_{x_0} \right\rangle_h \left\langle \overset{V}{\nabla}_{e_j} W_{x_0}, W \right\rangle_h \\
&\quad + \langle H, \nabla df_{x_0}(e_i,e_j)\rangle_h \left\langle \nabla^{f^{-1}}_{e_i} \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j,0))|_{t=0} \right)_{x_0}, W \right\rangle_h \,.
\end{aligned}$$

Equations (3.30), (3.19), (3.20), and (3.28) give

$$\begin{aligned}
\left(\nabla^{f^{-1}}_{e_i} \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j,0))|_{t=0} \right)_{x_0} \right)^V &= \left(\nabla^{f^{-1}}_{e_i} \left(\nabla^{f^{-1}}_{e_j} W + df(z_j) \right)_{x_0} \right)^V = \\
&= \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0} + \left(\nabla^{f^{-1}}_{e_i} \left(\nabla^{f^{-1}}_{e_j} W \right)^T_{x_0} \right)^V + \nabla df_{x_0}(e_i, z_j) \\
&= \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0} - \langle \nabla df_{x_0}(e_j,e_k), W\rangle_h \nabla df_{x_0}(e_i,e_k) \\
&\quad + \langle \nabla df_{x_0}(e_j,e_k), W\rangle_h \nabla df_{x_0}(e_i,e_k) \\
&= \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0}\,.
\end{aligned} \tag{3.34}$$

Hence,

$$\begin{aligned}
&\left\langle \nabla^{f^{-1}}_{e_i} [\langle H, \nabla df(e_i,e_j)\rangle_h \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_j,0))|_{t=0}]_{x_0}, W \right\rangle_h = \\
&= \left\langle \overset{V}{\nabla}_{e_i} H_{x_0}, \nabla df_{x_0}(e_i,e_j) \right\rangle_h \left\langle \overset{V}{\nabla}_{e_j} W_{x_0}, W \right\rangle_h + 2\left\langle H, \overset{V}{\nabla}_{e_j} H_{x_0} \right\rangle_h \left\langle \overset{V}{\nabla}_{e_j} W_{x_0}, W \right\rangle_h \\
&\quad + \langle H, \nabla df_{x_0}(e_i,e_j)\rangle_h \left\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0}, W \right\rangle_h \,.
\end{aligned} \tag{3.35}$$

Substituting now Eqs. (3.22), (3.32), (3.33), and (3.35) in Eq. (3.29), we obtain

$$\begin{aligned}
&\left\langle \left(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \Delta H \right)_{(x_0,0)}, W \right\rangle_h = \\
&\frac{1}{2} \left\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_i} \left(\Delta W + 2KW + \tilde{A}(W) \right)_{x_0}, W \right\rangle_h \\
&+ \left\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{z_i} H_{x_0} + \overset{V}{\nabla}_{z_i} \overset{V}{\nabla}_{e_i} H_{x_0}, W \right\rangle_h - \left\langle \overset{V}{\nabla}_{e_i} H_{x_0}, \overset{V}{\nabla}_{e_j} W \right\rangle_h \langle \nabla df_{x_0}(e_i,e_j), W\rangle_h \\
&- \left\langle H, \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \langle H, \nabla df_{x_0}(e_i,e_j)\rangle_h - 2\left\langle H, \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \left\langle \overset{V}{\nabla}_{e_j} H_{x_0}, W \right\rangle_h
\end{aligned}$$

$$-\left\langle \overset{V}{\nabla}_{e_i} H_{x_0}, \overset{V}{\nabla}_{e_j} W_{x_0} \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h + 2\left\langle \overset{V}{\nabla}_{e_i} H_{x_0}, \nabla df_{x_0}(e_i, e_j) \right\rangle_h \left\langle \overset{V}{\nabla}_{e_j} W_{x_0}, W \right\rangle_h$$
$$+2\left\langle H, \overset{V}{\nabla}_{e_j} H_{x_0} \right\rangle_h \left\langle \overset{V}{\nabla}_{e_j} W_{x_0}, W \right\rangle_h + \left\langle H, \nabla df_{x_0}(e_i, e_j) \right\rangle_h \left\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0}, W \right\rangle_h .$$

Next we evaluate the term $\overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{z_i} H_{x_0} + \overset{V}{\nabla}_{z_i} \overset{V}{\nabla}_{e_i} H_{x_0}$. From Eq. (3.27), we have

$$\overset{V}{\nabla}_{z_i} \overset{V}{\nabla}_{e_i} H_{x_0} = < z_i(x_0), e_k >_{g_0} \overset{V}{\nabla}_{e_k} \overset{V}{\nabla}_{e_i} H_{x_0} = \left\langle W_{x_0}, \nabla df_{x_0}(e_i, e_k) \right\rangle_h \overset{V}{\nabla}_{e_k} \overset{V}{\nabla}_{e_i} H_{x_0} .$$

On the other hand,

$$\overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{z_i} H_{x_0} = \overset{V}{\nabla}_{e_i} \left[< z_i, e_k >_{g_0} \overset{V}{\nabla}_{e_k} H_{x_0} \right]_{x_0} =$$
$$= d(< z_i, e_k >_{g_0})_{x_0}(e_i) \overset{V}{\nabla}_{e_k} H_{x_0} + < z_i, e_k >_{g_0} \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_k} H_{x_0}$$
$$= d(< z_i, e_k >_{g_0})_{x_0}(e_i) \overset{V}{\nabla}_{e_k} H_{x_0} + \left\langle W, \nabla df_{x_0}(e_i, e_k) \right\rangle_h \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_k} H_{x_0} .$$

Using Eqs. (3.7), (3.9), (3.10), and (3.26), we have

$$d(< z_i, e_k >_{g_0})_{x_0}(e_i) = d(< df(z_i), df(e_k) >_h)_{x_0}(e_i) =$$
$$= d\Big(\Big\langle df\big(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \tilde{e}_i|_{t=0}\big), df(e_k) \Big\rangle_h \Big)_{x_0}(e_i)$$
$$= d\Big(\Big\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_i, 0))|_{t=0} - \nabla dv\big(\frac{\partial}{\partial t}, (\tilde{e}_i, 0)\big)|_{t=0} , df(e_k) \Big\rangle_h \Big)_{x_0}(e_i)$$
$$= d\Big(\Big\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_i, 0))|_{t=0} - \nabla^{f^{-1}}_{e_i} W, df(e_k) \Big\rangle_h \Big)_{x_0}(e_i)$$
$$= \Big\langle \nabla^{v^{-1}}_{(\tilde{e}_i, 0)} \big(\nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_i, 0)) \big)_{(x_0, 0)}, df_{x_0}(e_k) \Big\rangle_h - \Big\langle \nabla^{f^{-1}}_{e_i} \big(\nabla^{f^{-1}}_{e_i} W\big)_{x_0}, df_{x_0}(e_k) \Big\rangle_h$$
$$+ \Big\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} (dv(\tilde{e}_i, 0))|_{t=0}(x_0), \nabla^{f^{-1}}_{e_i} (df(e_k))_{x_0} \Big\rangle_h - \Big\langle \nabla^{f^{-1}}_{e_i} W, \nabla^{f^{-1}}_{e_i} (df(e_k))_{x_0} \Big\rangle_h$$

$$= \Big\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \nabla^{v^{-1}}_{(\tilde{e}_i, 0)} (dv(\tilde{e}_i, 0))_{(x_0, 0)} + \overline{R}_{f(x_0)}(W_{x_0}, df_{x_0}(e_i)) df_{x_0}(e_i)$$
$$- \nabla^{v^{-1}}_{[\frac{\partial}{\partial t}, \tilde{e}_i]} (dv(\tilde{e}_i, 0))_{(x_0, 0)}, df_{x_0}(e_k) \Big\rangle_h$$
$$- \Big\langle \nabla^{f^{-1}}_{e_i} \big(\nabla^{f^{-1}}_{e_i} W\big)^T_{x_0}, df_{x_0}(e_k) \Big\rangle_h - \Big\langle \nabla^{f^{-1}}_{e_i} \big(\nabla^{f^{-1}}_{e_i} W\big)^V_{x_0}, df_{x_0}(e_k) \Big\rangle_h$$
$$+ \Big\langle \overset{V}{\nabla}_{e_i} W_{x_0}, \nabla df_{x_0}(e_i, e_k) \Big\rangle_h - \Big\langle \nabla^{f^{-1}}_{e_i} W, \nabla df_{x_0}(e_i, e_k) \Big\rangle_h$$

$$= \Big\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \nabla^{v^{-1}}_{(\tilde{e}_i, 0)} (dv(\tilde{e}_i, 0))_{(x_0, 0)} - 2\overline{K} W_{x_0} - \nabla^{f^{-1}}_{z_i} (df(e_i))_{x_0} , df_{x_0}(e_k) \Big\rangle_h$$
$$- \Big\langle \nabla^{f^{-1}}_{e_i} \big(\nabla^{f^{-1}}_{e_i} W\big)^T_{x_0}, df_{x_0}(e_k) \Big\rangle_h + \Big\langle \overset{V}{\nabla}_{e_i} W_{x_0}, \nabla df_{x_0}(e_i, e_k) \Big\rangle_h$$

$$= \left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \nabla^{v^{-1}}_{(\tilde{e}_i,0)}(dv(\tilde{e}_i,0))_{(x_0,0)},\ df_{x_0}(e_k)\right\rangle_h$$
$$- \left\langle \nabla^{f^{-1}}_{e_i}\left(\nabla^{f^{-1}}_{e_i} W\right)^T_{x_0}, df_{x_0}(e_k)\right\rangle_h + \left\langle \overset{V}{\nabla}_{e_i} W_{x_0}, \nabla df_{x_0}(e_i,e_k)\right\rangle_h .$$

From Eq. (3.16), we have

$$\left\langle \nabla^{f^{-1}}_{e_i}\left(\nabla^{f^{-1}}_{e_i} W\right)^T_{x_0}, df_{x_0}(e_k)\right\rangle_h =$$
$$= d\left(\left\langle\left(\nabla^{f^{-1}}_{e_i} W\right)^T, df(e_k)\right\rangle_h\right)_{x_0}(e_i) - \left\langle\left(\nabla^{f^{-1}}_{e_i} W\right)^T_{x_0}, \nabla^{f^{-1}}_{e_i}(df(e_k))_{x_0}\right\rangle_h$$
$$= d\left(\left\langle\nabla^{f^{-1}}_{e_i} W, df(e_k)\right\rangle_h\right)_{x_0}(e_i) = -d\left(\left\langle W, \nabla df(e_i,e_k)\right\rangle_h\right)_{x_0}(e_i)$$
$$= -\left\langle \overset{V}{\nabla}_{e_i} W, \nabla df_{x_0}(e_i,e_k)\right\rangle_h - \left\langle W, \nabla_{e_i}\nabla df_{x_0}(e_i,e_k)\right\rangle_h$$
$$= -\left\langle \overset{V}{\nabla}_{e_i} W, \nabla df_{x_0}(e_i,e_k)\right\rangle_h - 2\left\langle W, \overset{V}{\nabla}_{e_k} H_{x_0}\right\rangle_h ,$$

and from Eq. (3.24), $\nabla^{v^{-1}}_{(\tilde{e}_i,0)}(dv(\tilde{e}_i,0))_{(x_0,t)} \in V_{(x_0,t)}$, $\forall t \in I$. Thus, using Eq. (3.26) we get

$$\left\langle \nabla^{v^{-1}}_{\frac{\partial}{\partial t}} \nabla^{v^{-1}}_{(\tilde{e}_i,0)}(dv(\tilde{e}_i,0))_{(x_0,0)},\ df_{x_0}(e_k)\right\rangle_h =$$
$$= -\left\langle\left(\nabla^{v^{-1}}_{(\tilde{e}_i,0)}(dv(\tilde{e}_i,0))_{(x_0,0)}\right)^V, \nabla^{v^{-1}}_{\frac{\partial}{\partial t}}(dv(\tilde{e}_k,0))_{(x_0,0)}\right\rangle_h = -\left\langle \nabla df_{x_0}(e_i,e_i), \overset{V}{\nabla}_{e_k} W_{x_0}\right\rangle_h$$
$$= -2\left\langle H, \overset{V}{\nabla}_{e_k} W_{x_0}\right\rangle_h .$$

Hence,

$$d(< z_i, e_k >_{g_0})_{x_0}(e_i) =$$
$$= -2\left\langle H, \overset{V}{\nabla}_{e_k} W_{x_0}\right\rangle_h + 2\left\langle \overset{V}{\nabla}_{e_i} W_{x_0}, \nabla df_{x_0}(e_i,e_k)\right\rangle_h + 2\left\langle W, \overset{V}{\nabla}_{e_k} H_{x_0}\right\rangle_h .$$

Consequently,

$$\left\langle \overset{V}{\nabla}_{e_i}\overset{V}{\nabla}_{e_i} H_{x_0} + \overset{V}{\nabla}_{z_i}\overset{V}{\nabla}_{e_i} H_{x_0}, W\right\rangle_h =$$
$$= -2\left\langle H, \overset{V}{\nabla}_{e_k} W_{x_0}\right\rangle_h \left\langle \overset{V}{\nabla}_{e_k} H_{x_0}, W\right\rangle_{x_0} + 2\left\langle \overset{V}{\nabla}_{e_i} W_{x_0}, \nabla df_{x_0}(e_i,e_k)\right\rangle_h \left\langle \overset{V}{\nabla}_{e_k} H_{x_0}, W\right\rangle_h$$
$$+ 2\left\langle W, \overset{V}{\nabla}_{e_k} H_{x_0}\right\rangle_h \left\langle W, \overset{V}{\nabla}_{e_k} H_{x_0}\right\rangle_h + 2\left\langle W, \nabla df_{x_0}(e_i,e_k)\right\rangle_h \left\langle \overset{V}{\nabla}_{e_i}\overset{V}{\nabla}_{e_k} H_{x_0}, W_{x_0}\right\rangle_h .$$

So,

$$\left\langle \left(\nabla_{\frac{\partial}{\partial t}}^{v^{-1}} \triangle H\right)_{(x_0,0)}, W\right\rangle_h =$$

$$= \frac{1}{2}\left\langle \overset{V}{\nabla}_{e_i}\overset{V}{\nabla}_{e_i}\left(\triangle W + 2\bar{K}W + \tilde{A}(W)\right)_{x_0}, W\right\rangle_h$$
$$- 4\left\langle H, \overset{V}{\nabla}_{e_k} W_{x_0}\right\rangle_h \left\langle \overset{V}{\nabla}_{e_k} H_{x_0}, W\right\rangle_h + 2\left\langle \overset{V}{\nabla}_{e_i} W_{x_0}, \nabla df_{x_0}(e_i, e_k)\right\rangle_h \left\langle \overset{V}{\nabla}_{e_k} H_{x_0}, W\right\rangle_h$$
$$+ 2\left\langle W, \overset{V}{\nabla}_{e_k} H_{x_0}\right\rangle_h \left\langle W, \overset{V}{\nabla}_{e_k} H_{x_0}\right\rangle_h + 2\left\langle W, \nabla df_{x_0}(e_i, e_k)\right\rangle_h \left\langle \overset{V}{\nabla}_{e_i}\overset{V}{\nabla}_{e_k} H_{x_0}, W\right\rangle_h$$
$$- 2\left\langle \overset{V}{\nabla}_{e_i} H_{x_0}, \overset{V}{\nabla}_{e_j} W\right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W\right\rangle_h - \left\langle H, \overset{V}{\nabla}_{e_i}\overset{V}{\nabla}_{e_j} W_{x_0}\right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W\right\rangle_h$$
$$+ 2\left\langle \overset{V}{\nabla}_{e_i} H_{x_0}, \nabla df_{x_0}(e_i, e_j)\right\rangle_h \left\langle \overset{V}{\nabla}_{e_j} W_{x_0}, W\right\rangle_h + 2\left\langle H, \overset{V}{\nabla}_{e_i} H_{x_0}\right\rangle_h \left\langle \overset{V}{\nabla}_{e_j} W_{x_0}, W\right\rangle_h$$
$$+ \left\langle H, \nabla df_{x_0}(e_i, e_j)\right\rangle_h \left\langle \overset{V}{\nabla}_{e_i}\overset{V}{\nabla}_{e_j} W_{x_0}, W\right\rangle_h . \tag{3.36}$$

Next we evaluate the term $\left\langle \nabla_{\frac{\partial}{\partial t}}^{v^{-1}}\left(\tilde{A}_t(H_t)\right)_{(x_0,0)}, W\right\rangle_h$. On D, we have

$$\tilde{A}_t(H_t) = \left\langle \nabla df_t(\tilde{e}_{i_t}, \tilde{e}_{i_t}), H_t\right\rangle_h \nabla df_t(\tilde{e}_{i_t}, \tilde{e}_{i_t}) =$$
$$= \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0)), H\right\rangle_h \left(\nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0))\right)^V$$
$$= \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0)), H\right\rangle_h \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0))$$
$$- \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0)), H\right\rangle_h \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0)), dv(\tilde{e}_k,0)\right\rangle_h dv(\tilde{e}_k,0) .$$

Hence,

$$\left\langle \nabla_{\frac{\partial}{\partial t}}^{v^{-1}}\left(\tilde{A}_t(H_t)\right)_{(x_0,0)}, W\right\rangle_h =$$
$$= \left\langle \nabla_{\frac{\partial}{\partial t}}^{v^{-1}}\left[\left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0)), H\right\rangle_h \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0))\right]_{(x_0,0)} - \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0)), H\right\rangle_h \cdot\right.$$
$$\left. \cdot \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0))_{(x_0,0)}, dv_{(x_0,0)}(\tilde{e}_k,0)\right\rangle_h \nabla_{\frac{\partial}{\partial t}}^{v^{-1}}(dv(\tilde{e}_k,0))_{(x_0,0)}, W\right\rangle_h$$
$$= \left\langle \nabla_{\frac{\partial}{\partial t}}^{v^{-1}} \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0))_{(x_0,0)}, H\right\rangle_h \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0))_{(x_0,0)}, W\right\rangle_h$$
$$+ \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0)), \nabla_{\frac{\partial}{\partial t}}^{v^{-1}} H_{(x_0,0)}\right\rangle_h \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0)), W\right\rangle_h$$
$$+ \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0))_{(x_0,0)}, H\right\rangle_h \left\langle \nabla_{\frac{\partial}{\partial t}}^{v^{-1}} \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0))_{(x_0,0)}, W\right\rangle_h$$
$$- \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0))_{(x_0,0)}, H\right\rangle_h \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}}(dv(\tilde{e}_j,0))_{(x_0,0)}, dv_{(x_0,0)}(\tilde{e}_k,0)\right\rangle_h \cdot$$

$$
\begin{aligned}
& \cdot \left\langle \nabla_{\frac{\partial}{\partial t}}^{v^{-1}} \left(dv(\tilde{e}_k, 0) \right)_{(x_0,0)}, W \right\rangle_h \\
= & \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}} \nabla_{\frac{\partial}{\partial t}}^{v^{-1}} \left(dv(\tilde{e}_j, 0) \right)_{(x_0,0)} + \bar{R}_{f(x)}(df_{x_0}(e_i), W_{x_0}) df_{x_0}(e_j) \right. \\
& \left. + \nabla_{[\frac{\partial}{\partial t},(\tilde{e}_i,0)]}^{v^{-1}} \left(dv(\tilde{e}_j, 0) \right)_{(x_0,0)}, H \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h \\
& + \left\langle \nabla df_{x_0}(e_i, e_j), \left(\nabla_{\frac{\partial}{\partial t}}^{v^{-1}} H_{(x_0,0)} \right)^V \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h \\
& + \left\langle \nabla df_{x_0}(e_i, e_j), H \right\rangle_h \left\langle \nabla_{(\tilde{e}_i,0)}^{v^{-1}} \nabla_{\frac{\partial}{\partial t}}^{v^{-1}} \left(dv(\tilde{e}_j, 0) \right)_{(x_0,0)} \right. \\
& \left. + \bar{R}_{f(x)}(df_{x_0}(e_i), W_{x_0}) df_{x_0}(e_j) + \nabla_{[\frac{\partial}{\partial t},(\tilde{e}_i,0)]}^{v^{-1}} \left(dv(\tilde{e}_j, 0) \right)_{(x_0,0)}, W \right\rangle_h .
\end{aligned}
$$

From Eqs. (3.10) and (3.28), we have

$$
\nabla_{[\frac{\partial}{\partial t},(\tilde{e}_i,0)]}^{v^{-1}} \left(dv(\tilde{e}_j, 0) \right)_{(x_0,0)} = \nabla df_{x_0}(z_i, e_j) = \left\langle W, \nabla df_{x_0}(e_i, e_k) \right\rangle_h \nabla df_{x_0}(e_j, e_k) ,
$$

and using Eqs. (3.34) and (3.22), we obtain

$$
\begin{aligned}
& \left\langle \nabla_{\frac{\partial}{\partial t}}^{v^{-1}} \left(\tilde{A}_t(H_t) \right)_{(x_0,0)}, W \right\rangle_h = \\
= & \left\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0} + \delta_{ij} \bar{K} W_{x_0} + \left\langle W, \nabla df_{x_0}(e_i, e_k) \right\rangle_h \nabla df_{x_0}(e_j, e_k) , H \right\rangle_h \cdot \\
& \cdot \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h \\
& + \frac{1}{2} \left\langle \nabla df_{x_0}(e_i, e_j), \Delta W_{x_0} + 2\bar{K} W_{x_0} + \tilde{A}_{x_0}(W_{x_0}) \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h \\
& + \left\langle \nabla df_{x_0}(e_i, e_j), H \right\rangle_h \left\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0} + \delta_{ij} \bar{K} W_{x_0} + \left\langle W, \nabla df_{x_0}(e_i, e_k) \right\rangle_h \nabla df_{x_0}(e_j, e_k) , W \right\rangle_h \\
= & \left\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0}, H \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h + 2\bar{K} < W_{x_0}, H >_h < H_{x_0}, W >_h \\
& + \left\langle \nabla df_{x_0}(e_i, e_k), W \right\rangle_h \left\langle \nabla df_{x_0}(e_j, e_k), H \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h \\
& + \frac{1}{2} \left\langle \tilde{A}_{x_0}(W_{x_0}), \Delta W_{x_0} + 2\bar{K} W_{x_0} + \tilde{A}_{x_0}(W_{x_0}) \right\rangle_h \\
& + \left\langle \nabla df_{x_0}(e_i, e_j), H \right\rangle_h \left\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0}, W \right\rangle_h + 2\bar{K} < H_{x_0}, H >_h (W_{x_0}, W)_h \\
& + \left\langle \nabla df_{x_0}(e_i, e_j), H \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_k), W \right\rangle_h \left\langle \nabla df_{x_0}(e_j, e_k), W \right\rangle_h \\
= & \left\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0}, H \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h + \left\langle \nabla df_{x_0}(e_i, e_j), H \right\rangle_h \left\langle \overset{V}{\nabla}_{e_i} \overset{V}{\nabla}_{e_j} W_{x_0}, W \right\rangle_h \\
& + 2\bar{K} < W_{x_0}, H >_h < H_{x_0}, W >_h + 2\bar{K} < H_{x_0}, H >_h (W_{x_0}, W)_h \\
& + 2 \left\langle \nabla df_{x_0}(e_i, e_k), W \right\rangle_h \left\langle \nabla df_{x_0}(e_j, e_k), H \right\rangle_h \left\langle \nabla df_{x_0}(e_i, e_j), W \right\rangle_h \\
& + \frac{1}{2} \left\langle \tilde{A}_{x_0}(\Delta W_{x_0} + 2\bar{K} W_{x_0} + \tilde{A}_{x_0}(W_{x_0})) , (W_{x_0}) \right\rangle_h . \qquad (3.37)
\end{aligned}
$$

Finally, from Eq. (3.22), we obtain

$$\begin{aligned}
&\left\langle \nabla_{\frac{\partial}{\partial t}}^{v^{-1}} (\|H\|_h^2 H)_{(x_0,0)}, W\right\rangle_h = \\
&= \left\langle \Delta W_{x_0} + 2\overline{K} W_{x_0} + \tilde{A}_{x_0}(W_{x_0})\,,\, H\right\rangle_h < H_{x_0}, W >_h \\
&\quad + \frac{1}{2}\|H\|^2 \left\langle \Delta W_{x_0} + 2\overline{K} W_{x_0} + \tilde{A}_{x_0}(W_{x_0})\,,\, W\right\rangle_h \,. \qquad (3.38)
\end{aligned}$$

Combining Eqs. (3.36), (3.37), and (3.38), we arrive at the final result

$$\left\langle \nabla_{\frac{\partial}{\partial t}}^{v^{-1}} (\,\Delta\, H_t + \tilde{A}_t(H_t) - 2\|H_t\|_h^2 H_t)_{(x_0,0)}\,,\, W\right\rangle_h = \left\langle (J(W))_{x_0}, W_{x_0}\right\rangle_h \,,$$

where $J : C^\infty(V) \to C^\infty(V)$ is the fourth-order differential operator given by

$$\begin{aligned}
J(W)_x = &\ \frac{1}{2}(\,\Delta + \tilde{A})(\,\Delta + 2\overline{K} + \tilde{A})(W)_x \\
&- 2\left\langle (\,\Delta + \overline{K} + \tilde{A})(W)_x, H_x\right\rangle_h H_x - \|H_x\|_h^2(\,\Delta + \tilde{A})(W)_x \\
&+ 2\left\langle W_x, \nabla df_x(e_i, e_k)\right\rangle_h \overset{V}{\nabla}{}^2 H_x(e_i, e_k) + 2\left\langle H_x, \nabla df_x(e_i, e_k)\right\rangle_h \overset{V}{\nabla}{}^2 W_x(e_i, e_k) \\
&- 4\left\langle H_x, \overset{V}{\nabla}_{e_k} W_x\right\rangle_h \overset{V}{\nabla}_{e_k} H_x + 2\left\langle W_x, \overset{V}{\nabla}_{e_k} H_x\right\rangle_h \overset{V}{\nabla}_{e_k} H_x + 2\left\langle H, \overset{V}{\nabla}_{e_j} H\right\rangle_h \overset{V}{\nabla}_{e_j} W_x \\
&+ 2\left\langle \overset{V}{\nabla}_{e_i} W_x, \nabla df_x(e_i, e_k)\right\rangle_h \overset{V}{\nabla}_{e_k} H_x \\
&- 2\left\langle \overset{V}{\nabla}_{e_i} H_x, \overset{V}{\nabla}_{e_j} W_x\right\rangle_h \,, \nabla df_x(e_i, e_j) \\
&+ 2\left\langle \overset{V}{\nabla}_{e_i} H_x, \nabla df_x(e_i, e_j)\right\rangle_h \overset{V}{\nabla}_{e_j} W_x \\
&+ 2\left\langle W_x, \nabla df_x(e_i, e_k)\right\rangle_h \left\langle \nabla df_x(e_i, e_j), H_x\right\rangle_h \nabla df_x(e_j, e_k)\,.
\end{aligned}$$

Thus, we have obtained the second-variation formula

$$\frac{\partial^2}{\partial t^2} \mathcal{W}_D(f_t)\Big|_{t=0} = \int_D (J(W), W)_h \, dA_{g_0}$$

with the operator J given above.

The case $\dim N = 3$ follows straightforwardly. ♡

Remark 3.1 We observe that, if N is the 3-sphere S^3 and $H \equiv 0$ — obviously implying f to be a Willmore surface — then the above expression for J reduces to $\frac{1}{2}(\Delta + \tilde{A}) \circ (\Delta + 2 + \tilde{A})$, which is just the fourth-order, strongly elliptic operator of Weiner [We/78].

REFERENCES

[Bl/29] W. Blaschke, *Vorlesungen über Differentialgeometrie III*, Springer Berlin (1929).

[Br/84] R. L. Bryant, *A Duality Theorem for Willmore Surfaces*, J. Diff. Geom. 20, 23–53 (1984).

[C/55] E. Cartan, *Les Espaces à Connexion Conforme*, Œvres Complètes Partie III Vol. 1, 747–797 (1955).

[Ch/73a] B. Y. Chen, *On a Variational Problem on Hypersurfaces*, J. London Math. Soc. 6 (2), 321–325 (1973).

[Ch/73b] B. Y. Chen, *An Invariant of Conformal Mappings*, Proc. of Am. Math. Soc. 40 (2), 563–564 (1973).

[Ch/73c] B. Y. Chen, *On the Total Curvature of Immersed Manifolds, III: Surfaces in Euclidean 4-Space*, Am. J. Math. 95, 636–642 (1973).

[Ch/74] B. Y. Chen, *Some Conformal Invariants of Submanifolds and their Applications*, Bol. Un. Mat. Ital. 10 (4), 380–385 (1974).

[Ee-Le/83] J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*, C.B.M.S. Reg. Conf. Series No. 50 A.M.S. (1983).

[Ei/64] L. P. Eisenhart, *Riemannian Geometry*, Princeton Univ. Press (1964).

[He/75] R. Hermann, *Gauge Fields and Cartan-Ehresmann Connections*, Part A, Interdisc. Math. Vol. X, Math-Sci Press (1975).

[Ib/85] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Math. and its Appl. Soviet Series, D. Reidel Pub. Comp. (1985).

[Je/77] G. R. Jensen, *Higher-Order Contact of Submanifolds of Homogeneous Spaces*, Lect. Notes in Math. 610, Springer Verlag Berlin (1977).

[Ko-No/63/69] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I (1963), Vol. II (1969), Intersc. Pub. New York, London.

[Os/69] R. Osserman, *A Survey of Minimal Surfaces*, Van Nostrand Math. Studies 25 (1969).

[Po/81] W. Poor, *Differential Geometric Structures*, Mc.Graw-Hill Book Company New York (1981).

[Ri/86] M. Rigoli, *On a Conformal Bernstein-type Theorem*, Rend. Sen. Mat. Univ. Politec. Torino 44 (3) (1986).

[Ri/87] M. Rigoli, *The Conformal Gauss Map of Submanifolds of the Möbius Space*, Ann. Global Anal. Geom. 5 (1987).

[Ri-Sa/88] M. Rigoli and I. M. C. Salavessa, *Submanifolds of the Möbius Space and a Bernstein-type Theorem*, in preparation.

[Sch-Su/80a] Ch. Schiemangk and R. Sulanke, *Submanifolds of the Möbius Space*, Math. Nachr. 96, 165–183 (1980).

[Sh-Ta/70] K. Shiohama and A. Takagi, *A Characterisation of a Standard Torus in E^3*, J. Diff. Geom. 4, 477–485 (1970).

[Su/79] R. Sulanke, *On E. Cartan's Method of Moving Frames*, Coll. Math. Soc. János Bolyai 31, Diff. Geom. Budapest (1979).

[Su/81] R. Sulanke, *Submanifolds of the Möbius Space, II; Frenet Formulas and Curves of Constant Curvatures*, Math. Nachr. 100, 235–247 (1981).

[Su/85?] R. Sulanke, *Submanifolds of the Möbius Space, IV: Conformal Invariants of Immersions into Spaces of Constant Curvature*, preprint Humboldt Univ. Berlin (1985?).

[Su-Šv/80] R. Sulanke and A. Švec, *Zur Differentialgeometrie der Untermannigfaltigkeiten eines Kleinschen Raumes*, Beitr. Alg. Geom. **10**, 63–85 (1980).

[We/78] J. L. Weiner, *On a Problem of Chen, Willmore, et al.*, Indiana Univ. Math. J. **27** (1), 19–35 (1978).

[Wh/73] J. H. White, *A Global Invariant of Conformal Mappings in Space*, Proc. Am. Math. Soc. **38**, 162–164 (1973).

[Wi/65] T. J. Willmore, *Note on Embedded Surfaces*, An. St. Univ. Iasi, s.I.a. Mat. **11B**, 493–496 (1965).

[Wi/68] T. J. Willmore, *Mean Curvature of Immersed Surfaces*, An. St. Univ. Iasi, s.I.a. Mat. **14**, 99–103 (1968).

[Wi/74] T. J. Willmore, *Minimal Conformal Immersions*, Lect. Notes in Math. **392** (1974), Géometrie Différentielle, Coll. Santiago de Compostela 1972.

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