Extending Local Analytic Conjugacies Between Parabolic Fixed Points

by

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Declarations

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which has been accepted for the award of any other degree or diploma of the university or other institute of higher learning, except where due acknowledgement has been made in the text.

Abstract

The focus of this thesis is a study of the extension properties of local analytic conjugacies between simple parabolic fixed points. Any given conjugacy χ itself will generally not have an extension to the immediate basin. However, we show that if both maps belong to a suitable class (which includes polynomial-like maps and rational maps with a simply connected parabolic basin) then for all n large enough $g^{\circ n} \circ \chi$ does have an analytic extension to the immediate parabolic basin.

We begin by by studying qualitative models for the dynamics near a parabolic fixed point, leading us to the Parabolic Flower Theorem. We then construct Fatou coordinates, which conjugate f to the unit translation, and study extension and properties of these maps. By restricting ourselves to the case when the restriction of f to its parabolic basin is a proper map with finitely many critical points we are able to study covering properties of these extended Fatou coordinates. We also introduce the horn map and lifted horn maps and show that the former is a complete invariant of the local analytic conjugacy class.

Working from the covering properties of the horn map, we develop an intuition for how critical orbits of two maps f and g with locally conjugate simple parabolic fixed points should be related. In our main theorem, Theorem 3.1.10, we show that if both maps have a proper parabolic basin and χ is a local analytic conjugacy from (f, z_0) to (g, w_0) then for all n large enough, the map $g^{\circ n} \circ \chi$ has an analytic extension along any curve starting in a region near z_0 contained in the basin of z_0. Under the additional assumption that the immediate basin is simply connected we can then conclude that the map $\chi_n := g^{\circ n} \circ \chi$ has an analytic extension to a semi-conjugacy between the immediate basins whenever n is large enough.

Index of Notation

Subscripts will be omitted where there is only one choice in context. For example $\mathcal{A}$ denotes the full parabolic basin in a context where only one map is under discussion.

Sets and Spaces

$\mathcal{A}_f$	The full parabolic basin of f
$\mathcal{A}_{j,f}$	The j-the parabolic basin of f
$\mathcal{A}^0_{j,f}$	The j-the immediate parabolic basin of f
$\mathcal{A}^0_f$	The immediate parabolic basin of f
$\tilde{\mathcal{A}}_{j,f}$	The impression of the parabolic basin in the repelling Fatou coordinates
$\hat{\mathcal{A}}$	The image of $\tilde{\mathcal{A}}$ under π_-
$\mathbb{C}$	The complex numbers
$\mathbb{C}/\mathbb{Z}$	The quotient cylinder $z \sim z + n$ for $n \in \mathbb{Z}$
$\hat{\mathbb{C}}$	The Riemann sphere
$\mathbb{C}^*, \mathbb{D}^*$	$\mathbb{C} \setminus \{0\}$ and $\mathbb{D} \setminus \{0\}$ respectively
$\mathcal{C}_{\pm,j,f}$	The quotient cylinders of f
$\mathbb{D}$	The unit disc
$\mathbb{D}_r(z)$	The open disc of radius r centred on z
$\mathcal{D}(f)$	The domain of f
$\mathscr{G}_X$	The space of germs on X
$\mathbb{H}_R$	The right half-plane $\{w \in \mathbb{C} \mid \Re(w) > R\}$
I	An interval
$\mathcal{O}^-$	The set of pre-critical and critical points
$\mathcal{O}^+$	The set of post-critical points
$\mathcal{P}_{\pm,j,f}$	Petals of f
$\mathbb{R}$	The real numbers
$\mathbb{R}_{\geq c}$	The subset $\{x \in \mathbb{R} \mid x \geq c\}$
$\mathbb{R}_+$	$\mathbb{R}_{\geq 0}$
$\mathbb{R}_-$	$\mathbb{R}_{\leq 0}$
$S(f)$	The singular value set of f
$\mathbb{T}$	A torus
U, V	Open sets
U_{w_1,w_2}	The domain $\left\{ w \in \mathbb{C} \;\middle|\; \begin{array}{c} \Re(w - w_1) > -\lvert\Im(w - w_1)\rvert \text{ and} \\ \Re(w - w_2) < \lvert\Im(w - w_2) \end{array} \right\}$

$\overline{U}$	The closure of the set U
X, Y	Spaces, typically 1-complex manifolds or a Riemann surfaces
$\mathbb{Z}$	The integers
$\mathbb{Z}/m\mathbb{Z}$	The integers mod m
$[z]$	The equivalence class with representative z
$\Delta_{\pm,j}$	The sector $\left\{ re^{i\theta}\underline{v}_{\pm,j} \;\middle\vert\; R > 0 \text{ and } \lvert\theta\rvert < \frac{\pi}{\nu} \right\}$
$\pi_1(X, x_0)$	The fundamental group of X based at x_0
∂U	The boundary of the set U
$\emptyset$	The empty set

Points and Functions

$\deg f$	The degree of f
$\deg_x(f)$	The local degree of f at x
f, g	Functions, typically holomorphic
(f, x)	The germ of f at x
f_x, f_y	The partial derivatives with respect to x and y of a function $f : \mathcal{D}(f) \subseteq \mathbb{R}^2 \to \mathbb{C}$
$f^{\circ n}$	The n-fold composition $f \circ f \circ \ldots \circ f$
F	A holomorphic map close to unit translation
$F_{\pm,j}$	A map obtained by inverting f around the parabolic point
$\tilde{h}_f$	A lifted horn map of f
$\hat{h}_f$	The horn map of f
$K(\mu)$	The dilatation of μ
P_ν	The map $z \mapsto z + az^{\nu+1}$ for some $a \in \mathbb{C} \setminus \{0\}$
$\Re(z), \Im(z)$	The real and imaginary parts of z respectively
T_n	Translation by $n \in \mathbb{Z}$
$T_\pm$	Translations which satisfy $T_+ \circ \tilde{h}_f = \tilde{h}_g \circ T_-$ on $\tilde{\mathcal{A}}_f^0$
$\underline{v}_{\pm,j}$	Attraction and repulsion vectors of a parabolic fixed point
x, y	Point in X and Y respectively
z	A complex number in the dynamical plane
(z_k)	A sequence of complex numbers
γ	A curve
$\overline{\gamma}$	An analytic continuation of a germ along the curve γ
$\tilde{\gamma}$	A lift of the curve γ under some map
μ	A Beltrami coefficient
π	A covering map, dependent on context
$\pi_1(f)$	The homomorphism of fundamental groups induced by the map f
$\pi_{\pm,j,f}$	The projection maps $\pi_{\pm,j,f} : \mathbb{C} \to \mathcal{C}_{\pm,j,f}$
ϕ	The inversion $z \mapsto \frac{-1}{\nu a z^\nu}$
$\phi^*\mu$	The pullback of μ by ϕ
$\hat{\phi}_\pm$	Conformal isomorphisms between cylinders which satisfy $\hat{\phi}_+ \circ \hat{h}_f = \hat{h}_g \circ \hat{\phi}_-$ on $\hat{\mathcal{A}}_f^0$
Φ	Generalised Fatou coordinates for F
$\tilde{\Phi}_{\pm,j,f}$	The Fatou coordinates of f
$\hat{\Phi}_{\pm,j,f}$	The composition $\pi_{\pm,j,f} \circ \tilde{\Phi}_{\pm,j,f}$
χ	A local analytic conjugacy

χ_n	The composition $g^{\circ n} \circ \chi\vert_{\mathcal{A}_f^0}$
$\psi_{\pm,j}$	The inverse to $\phi\vert_{\Delta_{\pm,j}}$
$\tilde{\Psi}_{-,j,f}$	The extended inverse repelling Fatou coordinates of f
$\partial_z f, \partial_{\bar{z}} f$	$\partial_z f = \frac{1}{2}(f_x - i f_y)$ and $\partial_{\bar{z}} f = \frac{1}{2}(f_x + i f_y)$
$\circ$	Composition

Constants and Parameters

C	A complex constant
N	An integer constant
p/q	A rational number
R	A real constant
s	A sign, $s \in \{\pm\}$
θ	A real number recording an angle
λ	The multiplier of a fixed point
ν	The parabolic multiplicity of a parabolic fixed point
$\subset, \subseteq$	Strict subset inclusion and subset inclusion, respectively

Appendix

This notation only appears in the appendix. Where this conflicts with the previous notation, the following notation is used in the appendix and the previous notation is used in the body of this thesis.

Sets and Spaces

$\mathcal{A}_\infty$	The attracting basin of infinity
$\mathcal{C}_d$	The degree d connectedness locus
$\mathcal{C}_{3,a}$	$\{P \in \mathcal{S}_a \mid J(P) \text{ is connected }\}$
$\tilde{\mathcal{C}}_3$	$\{P_{a,\lambda} \mid J(P_{a,\lambda}) \text{ is connected. }\}$
$\mathcal{C}_{\cong,f_0}$	The space of transit isomorphisms for the map f_0
$\mathcal{F}$	The set of maps fixing 0 with multiplier not equal to 0
$\mathcal{F}_1$	A subspace of $\mathcal{F}$
$\mathcal{G}$	A locally full family at f_0
$\mathcal{HD}$	The space of holomorphic maps on domains of $\hat{\mathbb{C}}$
$J(f)$	The Julia set of f
$J(f,g)$	The generalised Julia set of (f,g)
$K(f)$	The filled Julia set of f
$K(f,g)$	The generalised filled Julia set of (f,g)
$\tilde{K}(P)$	The preimage $\pi^{-1}(\hat{K}(P))$
$\hat{K}(P)$	The image $\hat{\Phi}_{-,P}(K(P))$
$\mathcal{M}$	The Mandelbrot set
poly_d	The space of degree d polynomials up to Möbius conjugacy
Poly_d	The space of degree d polynomials

$\mathbb{S}^1$	The unit circle
$\mathcal{S}$	A suitable submanifold of Poly_d
$\mathcal{S}_a$	The slice $\{\lambda z + az^2 + z^3\} \subset \mathrm{Poly}_3$
$S_{\pm,f}$	Jordan domain on which the pseudo-Fatou coordinates are defined

Points and Functions

(f, g)	An enriched dynamical system, where g is a Lavaurs map of f
g_T	The Lavaurs map $\tilde{\Psi}_{-,f_0} \circ T \circ \tilde{\Phi}_{+,f_0}$
$H(\hat{z})$	The Ecalle height of the point $\hat{z}$
$m(a), M(a)$	The infimum and supremum of the Ecalle heights of points on $\hat{\Phi}_{-,P_a}(J(P_a))$ respectively
P_a	The polynomial $z^3 + az^2 + z$
$P_{a,\lambda}$	$\lambda z + az^2 + z^3$
$\tilde{\mathcal{R}}_f$	The renormalisation of f
T_f	The translation induced by pseudo-Fatou coordinates for f
α	A continuous function on $\mathcal{F}$ such that $f'(0) = \exp(2\pi i\alpha(f))$
A	A continuous map which satisfies $\mathrm{A}(0) = f_0$ and $\alpha(\mathrm{A}(\alpha)) = \alpha$
$\sigma(f)$	The fixed point of f in $\mathcal{F}_1$ near 0
ϕ_f	The natural isomorphism $\phi_f : \mathcal{C}_- \to \mathcal{C}_+$ for a map $f \in \mathcal{F}_1$
$\overset{\phi}{\sim}$	The relation $P \overset{\phi}{\sim} Q$ if and only if $\phi \circ P = Q \circ \phi$

Chapter 1

Introduction

1.1 Motivation of this Study

In mathematics, a dynamical system is a system with a notion of 'time'. We are concerned with those systems which arise from repeatedly applying a holomorphic map to a 1-complex manifold, typically the plane $\mathbb{C}$, the cylinder $\mathbb{C}^*$ or the Riemann sphere $\hat{\mathbb{C}}$.

The simplest possible behaviour that can arise is that of *fixed points*, $x_0 \in \mathcal{D}(f)$ such that $f(x_0) = x_0$. Rather than attempting to understand the whole global dynamics, we might instead ask whether we can understand the dynamics near a fixed point. That is, given a neighbourhood $U \ni x_0$, can we describe the behaviour of $f|_U$?

We consider the behaviour of f near a fixed point x_0 and g near a fixed point y_0 to be 'the same' if there exists a *local analytic conjugacy* from (f, x_0) to (g, y_0). That is, if there exist neighbourhoods $U \ni x_0$, $V \ni y_0$ and a biholomorphic map $\chi : U \to V$ such that $\chi \circ f = g \circ \chi$ wherever both sides are defined.

The study of such local behaviour has been ongoing for more than a century and in several cases the local analytic conjugacy class of (f, x_0) is well-understood. It is determined by the *multiplier* of the fixed point $\lambda = f'(x_0)$, which is well-defined on an arbitrary 1-complex manifold by the chain rule. By Kœnigs' Linearisation Theorem [10], if $|\lambda| \neq 0, 1$ then f is linearisable at x_0. That is, there exists a local analytic conjugacy from (f, x_0) to $(z \mapsto \lambda z, 0)$. This conjugacy is unique up to multiplication by a non-zero constant. If $\lambda = 0$ then f has local degree $d > 1$ at x_0. In this case, by Böttcher's Theorem there exists a local analytic conjugacy from (f, x_0) to $(z \mapsto z^d, 0)$ and this conjugacy is unique up to multiplication by a $d-1$st root of unity.

As a corollary to these two theorems, we see that if x_0 is a fixed point of f and y_0 is a fixed point of g such that $f'(x_0) = g'(y_0) = \lambda$, $\deg_{x_0}(f) = \deg_{y_0}(g)$ and $|\lambda| \neq 1$ then there exists a local analytic conjugacy χ from (f, x_0) to (g, y_0). A question one might then ask is how far the conjugacy χ can be analytically extended in $\mathcal{D}(f)$. Such an extension would be given by the functional equation $\chi \circ f = g \circ \chi$, which we can intuit as rearranging into either $\chi = g^{-1} \circ \chi \circ f$ if $|\lambda| < 1$ (implying that the fixed point is attracting) or $\chi = g \circ \chi \circ f^{-1}$ if $|\lambda| > 1$ (implying that the fixed point is repelling). Therefore such an extension will be obstructed by singular values of either g or f respectively.

In the case of $\lambda \neq 0$, we consider the quotient tori $U^*/f =: \mathbb{T}_f$ and $V^*/g =: \mathbb{T}_g$. The singular values of f leave impressions $\hat{S}(f)_+$ and $\hat{S}(f)_-$ of their forward and backward orbits under f on $\mathbb{T}_f$. A necessary condition for χ to extend to a semi-conjugacy from $f|_{U'}$ to $g|_{V'}$ is that the induced isomorphism $\hat{\chi} : \mathbb{T}_f \to \mathbb{T}_g$ satisfies $\hat{\chi}(\hat{S}(f)_+) \supseteq \hat{S}(g)_+$, if $|\lambda| < 1$, or $\hat{\chi}(\hat{S}(f)_-) \subseteq \hat{S}(g)_-$ if $|\lambda| > 1$.

Since there is only one complex dimension of isomorphisms from $\mathbb{T}_f$ to $\mathbb{T}_g$, in general the former condition cannot be satisfied when g has at least two singular values or when f has fewer singular values than g, and the latter when f has at least two singular values or when g has fewer singular values than f. On the other hand, we see we can always extend when $0 < |\lambda| < 1$ and g has no singular values or when $|\lambda| > 1$ and f has no singular values.

In the case of $\lambda = 0$ we no longer have a reasonable quotient space to consider. However the general consideration that a singular orbit under g must be the image of a singular orbit under f by any semiconjugacy. In particular, this applies to points in the forward orbit that lie in V and U. Since there are $d-1$ local conjugacies between (f, x_0) and (g, y_0), in general none of them will satisfy this requirement on singular orbits. Again, the exception is in the case when g has no singular values other than y_0, in which case any local conjugacy will extend to a basin semiconjugacy.

In this thesis, we focus on the case when λ is a root of unity, or more specifically when $\lambda = 1$. We also restrict ourselves to the *simple parabolic* case, so that (f, x_0) has the local expansion $z + az^2 + o(z^2)$. At first glance, one might expect any attempt at extending a conjugacy from (f, x_0) to (g, y_0) to be obstructed by singular values of g in a similar manner to the cases when $|\lambda| < 1$.

Whereas in the case of $|\lambda| \neq 0, 1$ we could take the quotient U/f in a neighbourhood of the fixed point to obtain a torus, in the case of a simple parabolic fixed point we obtain *two* quotient cylinders, $\mathcal{C}_{\pm,f}$, by quotienting two different overlapping regions adjacent to the fixed point. The cylinder $\mathcal{C}_{+,f}$ is the quotient of the *attracting petal*, a region on which f is injective and all orbits are attracted to the

fixed point in forward time, and the cylinder $\mathcal{C}_{-,f}$ is the quotient of the *repelling petal*, an attracting petal of f^{-1}.

The *horn map*, or *Écalle-Voronin Invariants*, are maps from a domain on the repelling cylinder to the attracting cylinder which record how the domains of the two quotient maps overlap [18]. Alternatively, the horn map can be considered to be the quotient of the return map by f; points near the parabolic point but on the repelling side are pushed away by the dynamics and some will eventually return to the attracting side.

The horn map is a complete invariant of the local analytic conjugacy class. In particular and in contrast to the case when $|\lambda| \neq 1$, this means that the critical orbits contained in the basins of locally conjugate maps must be related. In this thesis we shall show that under suitable restrictions on the maps f and g, if (f, x_0) and (g, y_0) are locally conjugate simple parabolic fixed points then there exists a semiconjugacy between $f|_{\mathcal{A}_f^0}$ and $g|_{\mathcal{A}_g^0}$. Here $\mathcal{A}_f^0$ denotes the *immediate parabolic basin* of x_0, the connected component of $\mathcal{A}_f$ which contains the attracting petal.

1.2 Structure of this Thesis

This thesis is structured as follows. In chapter 2 we develop the ideas that we shall use in stating and proving the results in chapter 3. In the first section of chapter 2, we give a more precise general discussion of the dynamics near a parabolic fixed point of a map f. We begin by introducing the notion of attracting and repelling vectors of a parabolic fixed point, which give a qualitative picture of the dynamics. We then give the Parabolic Flower Theorem, which more precisely describes regions, referred to as *petals* $\mathcal{P}_{\pm,j}$, such that the corresponding vector $\underline{v}_{\pm,j}$ indicates the dynamics of $f|_{\mathcal{P}_{\pm,j}}$.

The middle of this section is dedicated to the construction of Fatou coordinates. Fatou coordinates are a collection of injective maps, $\tilde{\Phi}_{\pm,j}$ from the petals to open subsets of $\mathbb{C}$ which satisfy $\tilde{\Phi}_{\pm,j} \circ f = T_1 \circ \tilde{\Phi}_{\pm,j}$, where $T_1 : z \mapsto z + 1$ is the unit translation. We first consider a map F, defined on a suitable subset of $\mathbb{C}$, which satisfies $|F(w) - (w+1)| < \frac{1}{4}$ and $|F'(w) - 1| < \frac{1}{4}$, and show that in this case there does exist a map Φ which satisfies $\Phi(F(w)) = \Phi(w) + 1$. From here, we can consider inversions of each petal in order to obtain a Fatou coordinate on each one. We conclude the section with the Cylinder Theorem, which makes precise the above description of the quotient spaces $\mathcal{P}_{\pm}/f$, and also study the covering properties of the attracting and inverse repelling Fatou coordinates. The latter results will prove useful in both the study of horn maps and in the proof of the main theorem in

chapter 3.

In the latter section of chapter 2 first discuss *lifted horn maps*, which are the composition of extended inverse Fatou coordinates and extended attracting Fatou coordinates. In this discussion and for the rest of the main body of the thesis we restrict ourselves to the case of a *simple parabolic fixed point*, one of multiplier 1 and parabolic multiplicity 1. We discuss the relation between a lifted horn map of f and the dynamics on the basin, showing how covering properties of the Fatou coordinates relate the singular value of f and a lifted horn map.

We conclude the chapter by discussing the *horn map*, otherwise know as *Écalle-Voronin Invariants*. We show that the first return map from $\mathcal{P}_-$ to $\mathcal{P}_+$ descends to a well-defined holomorphic map defined from a neighbourhood of the ends of the repelling cylinder $\mathcal{C}_-$ to $\mathcal{C}_+$ and how this definition relates to the notation of the *lifted* horn map. We conclude this chapter by showing that if the attracting basin $\mathcal{A}$ is compactly contained in $\mathcal{D}(f)$ then the horn map is a finite-type map.

As noted, in chapter 3 we use the ideas we developed in chapter 2 to state and prove our main result. We begin the first section with a brief discussion of the notion of analytic continuation along a curve and the Monodromy Theorem. We also give a curve lifting lemma for holomorphic functions which we can apply to the extended inverse Fatou coordinates.

In subsection 3.1.2, we show that given a *local analytic conjugacy* χ between two parabolic germs of appropriate maps $(f, 0)$ and $(g, 0)$ we can find an N such that for all $n > N$, $\chi_n := g^{\circ n} \circ \chi$ can be analytically continued along an arbitrary curve contained in the immediate parabolic basin. We can then apply the Monodromy Theorem to conclude that χ_n extends to the whole basin as an analytic semi-conjugacy from f to g.

We conclude the subsection with a simple application, showing that if the attracting parabolic basins of both f and g are simply connected and proper then there exists a pair of proper mutual semi-conjugacies between $f|_{\mathcal{A}_{0,f}}$ and $g|_{\mathcal{A}_{0,g}}$. From this we can conclude that the degrees of the two maps restricted to their immediate basins are equal.

In the second section of this chapter we discuss conjectures on how our previous arguments might be extended with further research. We consider how we might extend the domain of the semi-conjugacy to a global relation, how we might be able to describe exactly when a locally conjugate pair fail to be globally conjugate, how we might be able to extend to the case of multiply connected basins and finally how we might be able to extend to broader classes of maps with non-critical singular values.

Finally, in the appendix we give an example application of the generality of our construction of Fatou coordinates and the relationship of the horn map to the dynamics of a map by giving a translation of Lavaurs' proof that the cubic connectedness locus is not locally connected. Our choice of approach to constructing the Fatou coordinates in chapter 2 allows us to begin discussing the theory of *parabolic implosion* by constructing pseudo-Fatou coordinates for suitable perturbations of a map with a simple parabolic fixed point. We can then relate the dynamics of the perturbation to an *enriched dynamical system* consisting of the original parabolic map and a *Lavaurs map*.

By making a careful choice of the initial parabolic parameter in the cubic connectedness locus and of suitable perturbations we show that the cubic connectedness locus contains a 'comb' structure; a sequence of connected subsets accumulates at the parabolic parameter, each one separated from the rest by a hypercylinder of parameters with disconnected Julia sets.

Chapter 2

Preliminaries

In this chapter our aim is to discuss well-known properties of the dynamics near parabolic fixed points. This groundwork will allow us to understand the ideas and techniques we make use of in Chapter 3.

We begin by discussing the qualitative dynamics near a general parabolic fixed point. Following Milnor [15] we introduce the concepts of attracting and repelling petals and the Parabolic Flower Theorem, which describes how the petals are arranged around the parabolic point.

The second section focuses on the construction of Fatou coordinates following the methods in Shishikura [17]. These are univalent maps defined on the petals which conjugate f to the unit translation. We then use the Fatou coordinates to prove the Cylinder Theorem, which describes the quotient spaces $\mathcal{P}_{\pm,j}/f$.

In the final section we restrict to the case of simple parabolic fixed points to discuss covering properties of Fatou coordinates and horn maps; the map defined between domains on the cylinders induced by the return map from the repelling petal to the attracting petal. It will be of particular importance for Chapter 3 for us to understand the relation between critical points of the map f and critical values of the attracting and inverse repelling Fatou coordinates.

2.1 Parabolic Fixed Points

Consider a holomorphic function f defined from a domain $\mathcal{D}(f) \subseteq \hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. A point $z_0 \in \mathcal{D}(f)$ is called a *parabolic fixed point* of f if $f(z_0) = z_0$ and the multiplier of f at z_0 is a root of unity. For ease of notation we shall generally assume that $z_0 = 0$, so that f has the Taylor expansion

$$f(z) = e^{2\pi i p/q} z + a_2 z^2 + \dots .$$

2.1.1 Attraction and Repulsion Vectors

We begin by assuming that $q = 1$. In this case we can conjugate f on a neighbourhood of 0 to a map with the expansion

$$f(z) = z + az^{\nu+1} + bz^{2\nu+1} + o(z^{2\nu+1})$$

as $z \to 0$, where $a \neq 0$. We shall therefore assume without loss of generality that f has such an expansion. We call ν the *parabolic multiplicity* of the parabolic fixed point 0. Then we will show that the dynamics of f near 0 are approximated by the dynamics of the map $P_\nu : z \mapsto z + az^{\nu+1}$, which can be understood in terms of attraction and repulsion vectors.

Definition 2.1.1 (See Figure 2.1)**.** Let f be a holomorphic function with a parabolic fixed point of multiplier 1 and parabolic multiplicity ν at 0.

A complex number $\underline{v}$ is called a *repulsion vector* if $\nu a \underline{v}^\nu = +1$ and an *attraction vector* if $\nu a \underline{v}^\nu = -1$.

We shall label the repulsion vectors $\underline{v}_{-,j}$ and the attraction vectors $\underline{v}_{+,j}$, where $\underline{v}_{-,0}$ is repelling, $\underline{v}_{+,0} = e^{\pi i/\nu} v_{-,0}$ and

$$\underline{v}_{\pm,j} = e^{2\pi i j/\nu} \underline{v}_{\pm,0}.$$

Before proceeding, let us consider the dynamics of our model map P_ν. If $r > 0$ then P_ν maps $r\underline{v}_{\pm,j}$ to $(r \mp \frac{r^{\nu+1}}{\nu})\underline{v}_{\pm,j}$. If r is small enough then $0 < r - \frac{r^{\nu+1}}{\nu} < r$, whence $P_\nu^{\circ k}(r\underline{v}_{+,j}) \to 0$ as $k \to +\infty$. On the other hand, $r + \frac{r^{\nu+1}}{\nu} > r$ for all $r > 0$ and so $P_\nu^{\circ k}(r\underline{v}_{-,j}) \to \infty$ as $k \to +\infty$. Further, $P_\nu^{-1}(z) = z - az^{\nu+1} + o(z^{\nu+1})$. Thus if r is small enough then $P_\nu^{\circ -k}(r\underline{v}_{-,j}) \to 0$ as $k \to +\infty$.

The aim of this subsection is to show that the dynamics of P_ν are present near any parabolic fixed point; points are attracted towards 0 from the directions $\underline{v}_{+,j}$ in forward time and are attracted towards 0 in backward time from the directions $\underline{v}_{-,j}$. We begin by showing the suitable inversions conjugate f to maps $F_{\pm,j}$ which are close to unit translations.

Lemma 2.1.2.
Let ϕ be the map $\phi : z \mapsto \frac{-1}{\nu a z^\nu}$, defined on a neighbourhood of 0, and let $\Delta_{\pm,j}$ be the sectors

$$\Delta_{\pm,j} = \left\{ re^{i\theta}\underline{v}_{\pm,j} \;\middle|\; r > 0 \text{ and } |\theta| < \frac{\pi}{\nu} \right\}.$$

Then $\phi|_{\Delta_{\pm,j}}$ maps biholomorphically into the slit plane $\mathbb{C} \setminus \mathbb{R}_\mp$, where $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0]$.

Let $\psi_{\pm,j} : \mathbb{C} \setminus \mathbb{R}_{\mp} \to \Delta_{\pm,j}$ *be the inverse to* $\phi|_{\Delta_{\pm,j}}$ *and let*

$$F_{\pm,j} := \phi \circ f \circ \psi_{\pm,j} : \phi(\Delta_{\pm,j} \cap \mathcal{D}(f)) \to \mathbb{C}.$$

Then

$$F_{\pm,j}(w) = w + 1 + \frac{C}{w} + \mathcal{O}\left(\frac{1}{w\sqrt[\nu]{|w|}}\right) \text{ as } |w| \to \infty$$

for some constant $C \in \mathbb{C}$*, and*

$$F'_{\pm,j}(w) = 1 + \mathcal{O}\left(\frac{1}{w^2}\right) \text{ as } |w| \to \infty$$

within the sector $U_{\pm} = \{w \mid \pm\Re(w) > -|\Im(w)|\}$.

Proof. By assumption, f has the expansion

$$f(z) = z + az^{\nu+1} + bz^{2\nu+1} + \mathcal{O}(z^{2\nu+2}) \text{ as } z \to 0.$$

Then

$$f \circ \psi_{\pm,j}(w) = \sqrt[\nu]{\frac{-1}{\nu a w}}\left(1 - \frac{1}{\nu w} + \frac{b/a^2}{(\nu w)^2} + \mathcal{O}\left(\frac{1}{w^{\frac{2\nu+1}{\nu}}}\right)\right) \text{ as } |w| \to \infty.$$

Composing with $\phi : z \mapsto \frac{-1}{\nu a z^{\nu}}$ we see that

$$\begin{aligned} F_{\pm,j}(w) =& w\left(1 - \frac{1}{\nu w} + \frac{b/a^2}{(\nu w)^2} + \mathcal{O}\left(\frac{1}{w^{\frac{2\nu+1}{\nu}}}\right)\right)^{-\nu} \\ =& w\left(1 + \frac{1}{w} + \frac{C}{w^2} + \mathcal{O}\left(\frac{1}{w^{\frac{2\nu+1}{\nu}}}\right)\right). \end{aligned}$$

Thus we have that

$$F_{\pm,j}(w) = w + 1 + \frac{C}{w} + \mathcal{O}\left(\frac{1}{w\sqrt[\nu]{|w|}}\right) \text{ as } |w| \to \infty.$$

For the derivative estimate, consider the function $G : w \mapsto F_{\pm,j}(w) - w - 1$. Notice that for all $w \in U_{\pm}$, $\mathbb{D}_{|w|/2}(w) \subset (\mathbb{C} \setminus R_{\mp})$. For all large $w \in U_{\pm}$ and some $C' > C$, $|F_{\pm,j}(w) - (w+1)| \leq \frac{C}{|w|}$. Hence G maps the disc $\mathbb{D}_{|w|/2}(w)$ into the disc $\mathbb{D}_{2C'/3|w|}(0)$.

By the Cauchy Derivative Estimate, $|G'(w)| \leq \frac{2C'/3|w|}{|w|/2} = \frac{D}{|w|^2}$. Hence $F'_{\pm,j}(w) = 1 + \mathcal{O}(\frac{1}{w^2})$ as claimed. □

In order to complete our picture of dynamics near a parabolic point, we wish to exclude the case when an orbit (z_k), where $f(z_k) = z_{k+1}$, lands directly on the fixed point 0. To this end, we say that an orbit (z_k) converges to 0 *nontrivially* if $z_k \to 0$ as $k \to +\infty$ but $z_k \neq 0$ for all k. The following lemma then tells us that such an orbit behaves like $\underline{v}_{+,j}/\sqrt[\nu]{k}$ as $k \to +\infty$.

Lemma 2.1.3.
Suppose the orbit z_k converges to 0 nontrivially under f.

Then there exists $j \in \mathbb{Z}/\nu\mathbb{Z}$ such that z_k is asymptotic to $\underline{v}_{+,j}/\sqrt[\nu]{k}$ as $k \to +\infty$.

That is, $\lim_{k\to+\infty}\left(z_k\sqrt[\nu]{k}\right)$ exists and is equal to $\underline{v}_{+,j}$ for some j.

Any $\underline{v}_{+,j}$ can occur as such a limit.

Since $f'(0) = 1 \neq 0$, f is injective in a neighbourhood of 0. Thus f^{-1} is well-defined on a neighbourhood of 0. Since

$$f^{-1}(z) = z - az^{\nu+1} + o(z^{\nu+1}) \text{ as } z \to 0$$

we see that $\underline{v}$ is an attraction (resp. repulsion) vector for f if an only if it is a repulsion (resp. attraction) vector for f^{-1}.

Thus this lemma also tells us that if $\ldots \mapsto z_{-2} \mapsto z_{-1}$ is a backward orbit under f converging nontrivially to 0, so $z_k \to 0$ as $k \to -\infty$, then $\lim_{k\to-\infty}\left(z_k\sqrt[\nu]{-k}\right)$ is equal to one of the repulsion vectors $\underline{v}_{-,j}$.

Proof. Let $F_{+,j}$ be as in lemma 2.1.2.

Then we can choose $R > 0$ such that $F_{+,j}$ is defined for all $|w| > R$, $w \notin \mathbb{R}_-$, and

$$|F_{+,j}(w) - (w+1)| < 1/2.$$

In particular, we see that

$$\Re(F_{+,j}(w)) > \Re(w) + 1/2$$

whenever $|w| > R$ and so

$$\Re(\phi(f(z))) > \Re(\phi(z)) + 1/2$$

whenever $|z|$ is sufficiently small.

Let $\mathbb{H}_R$ be the right half-plane $\mathbb{H}_R = \{w \in \mathbb{C} \mid \Re(w) > R\}$ and let $\mathcal{P}_{+,j}$ be the image $\psi_{+,j}(\mathbb{H}_R)$, so $\mathcal{P}_{+,j} = \{z \in \Delta_{+,j} \mid \Re(\phi(z)) > R\}$. Then $f(\mathcal{P}_{+,j}) \subset \mathcal{P}_{+,j}$ and the iterates of f restricted to $\mathcal{P}_{+,j}$ converge uniformly to the constant map $z \mapsto 0$.

Now, suppose that (z_k) is an orbit under f converging nontrivially to 0. Then for all k large enough, $\Re(\phi(z_{k+1})) > \Re(\phi(z_k)) + 1/2$. In particular, there exists some m such that $\Re(\phi(z_m)) > R$. Thus $z_m \in \mathcal{P}_{+,j}$ for some j and so $z_k \in \mathcal{P}_{+,j}$ for all $k \geq m$.

Now consider the sequence $(w_k) = (\phi(z_k))$. By lemma 2.1.2, $w_{k+1} - w_k \to 1$ as $k \to \infty$. Therefore the average

$$\frac{w_k - w_0}{k} = \frac{1}{k}\sum_{j=0}^{k}(w_{j+1} - w_j)$$

also converges to 1 as $k \to \infty$ and so $w_k/k \to 1$ as $k \to \infty$.

Since $1/w_k = -\nu a z_k^\nu$ it follows that $-k\nu a z_k^\nu \to 1$. As $\nu a \underline{v}_{+,j}^\nu = -1$, we see that

$$\frac{k z_k^\nu}{\underline{v}_{+,j}^\nu} \to 1.$$

Taking roots, noting that this is well-defined since we are within $\Delta_{+,j}$, yields the result. Any $\underline{v}_{+,j}$ can occur since $\mathcal{P}_{+,j} = \psi_{+,j}(\mathbb{H}_R)$ is non-empty for all j. $\square$

It is of some interest to note that this lemma shows us that the dynamics near a parabolic fixed point are very 'slow' in comparison to attracting and superattracting fixed points. We can quantify this in the following sense: given a point z which is nontrivially attracted towards 0, we define the function $k_z : \mathbb{R}_{>0} \to \mathbb{Z}_{\geq 0}$ by letting $k_z(r)$ be the least k such that $f^{\circ k}(z)$ lies in the disc of radius $1/r$. Then how k_z varies as $r \to +\infty$ indicates the speed at which z is attracted towards 0. Here and for the remainder of this thesis the notation $f^{\circ k}$ refers to the k-fold iterate of f, $f^{\circ k} = f \circ f \circ \ldots \circ f$ k times.

For attracting fixed points, which have expansions $z \mapsto \lambda z + o(z)$ for some $0 < |\lambda| < 1$, the approach speeds k_z are $\mathcal{O}(\log r)$, whilst for superattracting fixed points, which have expansions $z \mapsto z^d + o(z^d)$ with $d > 1$, the approach speeds are $\mathcal{O}(\log\log r)$.

By contrast, the above lemma shows that for parabolic fixed points the approach speeds are $\mathcal{O}(r^\nu)$. This extreme slowness can present a difficulty in any sort of computation or rendering relating to a parabolic fixed point as it takes vastly more iterates to determine, for example, whether a point lies in the parabolic basin or not.

This lemma also allows us to begin talking about parabolic fixed points with multiplier $e^{2\pi i p/q} \neq 1$. Notice that if f has such a parabolic fixed point at 0 then $f^{\circ q}$ has a fixed point of multiplier $\left(e^{2\pi i p/q}\right)^q = 1$ at 0. Thus we can apply our previous

definitions and results to the q-th iterate.

Lemma 2.1.4.
Suppose that the multiplier λ of f is a qth root of unity.

The number of attraction vectors of $f^{\circ q}$ is a multiple of q.

Proof. Suppose $\underline{v}$ is an attraction vector of $f^{\circ q}$ and that $z_1 \mapsto z_{q+1} \mapsto z_{2q+1} \mapsto \dots$ is an orbit under $f^{\circ q}$ which converges nontrivially to 0 from the direction of $\underline{v}$. Then the image $z_2 \mapsto z_{q+2} \mapsto \dots$ is also an orbit under $f^{\circ q}$ which converges nontrivially to 0 from the direction of $\lambda\underline{v}$. Thus multiplication by $\lambda = e^{2\pi ip/q}$ permutes the attraction vectors and so the number of attraction vectors must be a multiple of q. □

We can therefore define the parabolic multiplicity of a general parabolic fixed point. Suppose that f is a holomorphic map with a parabolic fixed point of multiplier $e^{2\pi ip/q}$ at 0. Then $f^{\circ q}$ is a holomorphic map with a parabolic fixed point of multiplier 1 and parabolic multiplicity m at 0. By lemma 2.1.4, $q|m$ and so we can define the *parabolic multiplicity* of the fixed point 0 of f to be $m/q = \nu$. We define the attraction and repulsion vectors of f to be the attraction and repulsion vectors of $f^{\circ q}$ respectively.

In this way we can transfer discussion of parabolic points with multiplier $e^{2\pi ip/q}$ to the case of those with multiplier 1. Within the context of this thesis this will be a simplifying assumption, but there are contexts in which the symmetry of $f^{\circ q}$ being a q-th iterate is important and this information is lost by assuming that $q = 1$.

2.1.2 The Parabolic Flower Theorem

Now that we understand how an orbit can approach a parabolic fixed point it becomes natural to ask about regions on which the dynamics follow the attraction or repulsion vectors. In the case of the attraction vectors, we could attempt to define the *parabolic basin*s

$$\mathcal{A}_j = \{z \in \mathcal{D}(f) \mid f^{\circ k}(z) \to 0 \text{ nontrivially from the direction of } \underline{v}_{+,j} \text{ as } k \to +\infty\}.$$

However this approach has no counterpart for the repulsion vectors. Moreover the domain $\mathcal{D}(f)$ will generally not be large enough for the basins to behave as one might expect. Later in this chapter we will discuss additional constraints on f so that, for example, the restriction to the immediate basin is surjective. However we

currently allow arbitrary holomorphic maps f and instead look for alternatives to the basins.

The standard approach is to consider *petals*. These are domains on which f is injective with 0 in their boundary, such that any orbit which enters a petal is attracted towards 0 in either forward or backward time. The exact definition of a petal varies between authors, with our definition here following Milnor [15, Definition 10.6, p. 111]. This definition is quite broad, allowing for relatively wild sets to be petals. The sets $\mathcal{P}_{\pm,j}$ from the proof of lemma 2.1.3 are petals.

Definition 2.1.5. Let f be a holomorphic map defined on a neighbourhood of 0, fixing 0 with multiplier $e^{2\pi ip/q}$ and let $\underline{v}$ be an attraction vector of f at 0. An open set $\mathcal{P} \subset \mathcal{D}(f)$ will be called an *attracting petal* of f for the vector $\underline{v}$ if

i $f^{\circ q}$ is injective on $\mathcal{P}$,

ii $f^{\circ q}(\mathcal{P}) \subset \mathcal{P}$ and

iii an orbit (z_k) is eventually contained in $\mathcal{P}$ if and only if it converges to 0 from the direction $\underline{v}$ under $f^{\circ q}$.

An open set $\mathcal{P}$ will be call a *repelling petal* of f for the repulsion vector $\underline{v}$ if it is an attracting petal of f^{-1} for $\underline{v}$. Note the f^{-1} is defined on some neighbourhood of 0.

The main result of this subsection describes how the petals are arranged around the parabolic fixed point. We see that each attraction and repulsion vector can be associated with a unique petal and that we may choose the petals so that either each petal overlaps with its neighbours or so that the petals are pairwise disjoint. We refer to these cases as *fat petals* and *thin petals* respectively.

We obtain the fat petals by taking the image of sets $\pm W$ under the maps $\psi_{\pm,j}$ defined in lemma 2.1.2, where $W = \{x+iy \mid x+|y| > 2R\}$ and R is sufficiently large as to ensure that these are petals. Thin petals are then obtained by removing the closure of the intersection with adjacent petals.

Theorem 2.1.6 (Parabolic Flower Theorem)**.**
Let f be a holomorphic map defined on a neighbourhood of 0*, fixing* 0 *with multiplier $\lambda = e^{2\pi ip/q}$ and parabolic multiplicity ν.*

Then within any neighbourhood U of 0 *there exist $m = \nu q$ attracting petals, $\{\mathcal{P}_{+,j}\}$, and m repelling petals, $\{\mathcal{P}_{-,j}\}$, such that for each $j \in \mathbb{Z}/m\mathbb{Z}$ and $s \in \{\pm\}$, $\mathcal{P}_{s,j}$ is a petal of f for the vector $\underline{v}_{s,j}$.*

Further, these petals can be chosen so that exactly one of the following holds:

- $\{0\} \cup \bigcup_j (\mathcal{P}_{+,j} \cup \mathcal{P}_{-,j})$ *is a neighbourhood of* 0. *If* $m > 1$ *then* $\mathcal{P}_{+,j} \cap \mathcal{P}_{-,j}$ *and* $\mathcal{P}_{+,j} \cap \mathcal{P}_{-,j+1}$ *are connected and simply connected, whilst if* $m = 1$ *then* $\mathcal{P}_{+,0} \cap \mathcal{P}_{-,0}$ *has two connected components, each of which is simply connected. All other intersections are empty in both cases.*

- *The* $\mathcal{P}_{s,j}$*'s are pairwise disjoint.*

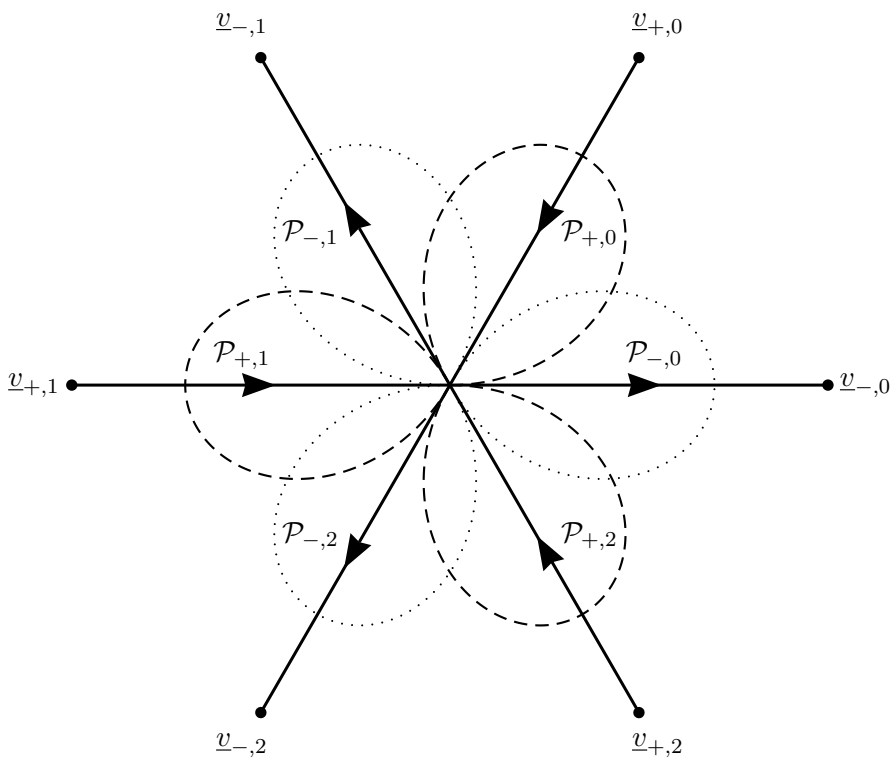

Figure 2.1: A diagram showing the relative positions of fat petals and vectors in the case $m = 3$. The attracting petals are bounded by dashed lines and the repelling petals by dotted lines. The arrows on the vectors $\underline{v}_{\pm,j}$ indicate the direction along which points move under f.

Proof. By passing to the qth iterate, we can assume that $\lambda = 1$ and so $m = \nu$.

First, by lemma 2.1.2, we can choose $R > 0$ so that

$$|F_{s,j}(w) - (w+1)| < 1/2 \text{ for all } |w| > R \text{ , } s \in \{\pm\} \text{ and } j \in \{0, \ldots, \nu - 1\}.$$

Then for all $|w| > R$ it follows that $\Re(F_{s,j}(w)) > \Re(w) + 1/2$ and also that $|\Im(F_{s,j}(w) - w)| < \Re(F_{s,j}(w) - w)$, where the second inequality estimates the slope

of the line from w to $F_{s,j}(w)$ and comes from noting that

$$|F_{s,j}(w) - (w+1)| = |(F_{s,j}(w) - w) - 1| < 1/2.$$

Let $W = \{z = x + iy \mid x, y \in \mathbb{R} \text{ and } x + |y| > 2R\}$, so that $W \supset \mathbb{H}_{2R}$ and $|w| > R$ for all $w \in W$. We can also choose R, hence W, so that $\psi_{s,j}(W) \subset U$ for all $s \in \{\pm\}$ and all j. Suppose $w \in W$. Then

$$\begin{aligned} R < \Re(w) + |\Im(w)| &= (\Re(F_{+,j}(w)) - \Re(F_{+,j}(w)) + \Re(w)) \\ &\quad + |\Im(F_{+,j}(w)) - \Im(F_{+,j}(w)) + \Im(w)| \\ &\leq (\Re(F_{+,j}(w)) + |\Im(F_{+,j}(w))|) \\ &\quad + (|\Im(w) - \Im(F_{+,j}(w))| - (\Re(F_{+,j}(w)) - \Re(w))) \\ &< \Re(F_{+,j}(w)) + |\Im(F_{+,j}(w))| \end{aligned}$$

and so $F_{+,j}(W) \subset W$. Further, every forward orbit under $F_{+,j}$ must eventually enter $\mathbb{H}_{2R}$. By the proof of lemma 2.1.3, $\psi_{+,j}(\mathbb{H}_{2R})$ is an attracting petal for $\underline{v}_{+,j}$. Thus, since every orbit under f which enters $\psi_{+,j}(W)$ enters $\psi_{+,j}(\mathbb{H}_{2R})$, $\mathcal{P}_{+,j} = \psi_{+,j}(W)$ is also an attracting petal for $\underline{v}_{+,j}$.

Similarly, let $-W = \{w \mid -w \in W\}$. Then $F_{-,j}^{-1}(-W) \subset -W$ and so $\mathcal{P}_{-,j} = \psi_{-,j}(-W)$ is a repelling petal of f in the direction $\underline{v}_{-,j}$.

Observe that $\mathcal{P}_{s,j} \subset \Delta_{s,j}$, so the only intersections which might be non-empty are $\mathcal{P}_{+,j} \cap \mathcal{P}_{-,j}$ and $\mathcal{P}_{+,j} \cap \mathcal{P}_{-,j+1}$. From the definitions of ϕ and $\mathcal{P}_{\pm,j}$ we see that $\phi(\mathcal{P}_{+,j} \cap \mathcal{P}_{-,j}) \subset W \cap -W$, and $W \cap -W = \{u + iv \mid u - |v| > 2R\}$ consists of two disjoint v-shaped connected components. We shall call these components V_+ and V_- according to whether $\Im(w)$ is positive or negative respectively.

Moreover, $\phi(\Delta_{+,j} \cap \Delta_{-,j})$, respectively $\phi(\Delta_{+,j} \cap \Delta_{-,j+1})$, is the lower, respectively upper, half-plane. Thus $\phi(\mathcal{P}_{+,j} \cap \mathcal{P}_{-,j}) \subseteq V_-$ and $\phi(\mathcal{P}_{+,j} \cap \mathcal{P}_{-,j+1}) \subseteq V_+$. But $\psi_{j,+}(V_-) \subseteq \mathcal{P}_{+,j} \cap \mathcal{P}_{-,j}$ and $\psi_{j,+}(V_+) \subseteq \mathcal{P}_{+,j} \cap \mathcal{P}_{-,j+1}$.

Hence $\mathcal{P}_{+,j} \cap \mathcal{P}_{-,j} = \psi_{j,+}(V_-)$ and $\mathcal{P}_{+,j} \cap \mathcal{P}_{-,j+1} = \psi_{j,+}(V_+)$. Since $\psi_{j,+}$ is a homeomorphism onto its image and $V_\pm$ are connected and simply connected the first part of the result follows.

To obtain pairwise disjoint petals, simply let $\mathcal{P}'_{+,j} = \mathcal{P}_{+,j} \setminus \overline{(\mathcal{P}_{-,j} \cup \mathcal{P}_{-,j+1})}$ and $\mathcal{P}'_{-,j} = \mathcal{P}_{-,j} \setminus \overline{(\mathcal{P}_{+,j} \cup \mathcal{P}_{+,j-1})}$. By taking the slightly more precise estimate on the slope of the line from w to $F_{s,j}(w)$,

$$|\Im(F_{s,j}(w) - w)| < \frac{\sqrt{3}}{3}\Re(F_{s,j}(w) - w),$$

we see that every orbit under $F_{s,j}$ which enters $V_\pm$ must leave it in both forward and backward time. Thus $z \in \mathcal{P}_{\pm,j}$ if and only if $f^{\circ \pm m}(z) \in \mathcal{P}'_{\pm,j}$ for large enough m and so these sets are also petals. $\square$

2.1.3 Generalised Fatou Coordinates

In order to give a quantitative description of the dynamics we make use of *Fatou coordinates*, which conjugate $f|_{\mathcal{P}_{\pm,j}}$ to the unit translation $T_1 : w \mapsto w+1$. Following Shishikura [17], we first consider the case of a map F which is close to the unit translation.

We first construct a quasiconformal conjugacy ϕ from T_1 to F. We pull back the standard complex structure on F to T_1. We spread this almost-complex structure to all of $\mathbb{C}$ by taking pullbacks under $T_n : w \mapsto w + n$ for all $n \in \mathbb{Z}$. By applying the Measurable Riemann Mapping Theorem to the resultant almost-complex structure and composing with ϕ we obtain our desired conjugacy.

Taking this approach, rather than the more direct limit construction in Milnor [15], yields two major advantages. Firstly, should we wish to consider a family of maps F_λ which vary with some given regularity with λ, then the Ahlfors-Bers Theorem, otherwise know as the Measurable Riemann Mapping Theorem with Dependence on Parameters, tells us that the resulting conjugacies can be chosen to also depend on λ with the same regularity.

Secondly, the class of maps F to which this argument can be applied is larger than simply maps of the form $F_{\pm,j}$ from above. In particular, we can construct *pseudo-Fatou coordinates* for suitable perturbations of a map with a parabolic fixed point. This is a key part of the rich theory of parabolic implosion. We present an example application, making use of both of these properties, in the appendix.

We begin this discussion with a brief review of the theory of quasiconformal maps. As we will only be making use of this theory in a single application, we constrain ourselves to only what is essential. A more comprehensive discussion can be found in Ahlfors [2].

Let us first consider a continuous function $f : U \to \mathbb{C}$ defined on a domain $U \subseteq \mathbb{C}$. Considering $\mathbb{C}$ as an $\mathbb{R}$-vector space, we can identify $\mathbb{C}$ with $\mathbb{R}^2$ via the isomorphism $x + iy \mapsto (x, y)$. We say that f is $\mathbb{R}$-differentiable at a point if it is differentiable as a function $f : U \subset \mathbb{R}^2 \to \mathbb{C}$. Likewise, we define the partial derivatives f_x and f_y of f to be those of $f : U \subset \mathbb{R}^2 \to \mathbb{C}$, if they exist.

Definition 2.1.7. Let $I \subseteq \mathbb{R}$ be an interval and $g : I \to \mathbb{C}$ be a continuous function.

We say that g is *absolutely continuous on* I if, for every $\epsilon > 0$ there exists

$\delta > 0$ such that for every finite sequence of non-intersecting intervals (a_j, b_j) with total length $\sum_j |b_j - a_j| < \delta$, the sum $\sum_j |f(b_j) - f(a_j)| < \epsilon$.

Let $U \subseteq \mathbb{C}$ be a domain and let $f : U \to \mathbb{C}$ be a continuous function.

We say that f is *absolutely continuous on lines* if for any disc D compactly contained in U and any family of parallel lines in D, f is absolutely continuous on almost all of them.

Observe that if g is absolutely continuous on an interval I then g has bounded variation and so is differentiable almost everywhere. It thus follows that if f is absolutely continuous on lines then the partial derivatives f_x and f_y exist almost everywhere. By the following theorem, it follows that if f is an open map then f is $\mathbb{R}$-differentiable almost everywhere. In particular, this holds when f is a homeomorphism.

Theorem 2.1.8.
Let $f : U \to V$ be a continuous open map between domains in $\mathbb{C}$.

If the partial derivatives f_x and f_y exist almost everywhere then f is $\mathbb{R}$-differentiable almost everywhere.

In the discussion of quasiconformal maps, rather than working with the traditional partial derivatives f_x and f_y, it is standard to instead work with the derivatives $\partial_z f$ and $\partial_{\bar{z}} f$, where

$$\partial_z f = \frac{1}{2}\left(f_x - i f_y\right) \text{ and } \partial_{\bar{z}} f = \frac{1}{2}\left(f_x + i f_y\right).$$

We are now able to give a definition of a map f being K-quasiconformal. There are many equivalent definitions, which vary in nature from being primarily analytic to being primarily geometric. Our chosen presentation has the advantage of requiring comparatively few supporting ideas.

Definition 2.1.9. Let $U, V \subseteq \mathbb{C}$ be domains and let $K \in \mathbb{R}$ with $K \geq 1$.

We say that a map $\phi : U \to V$ is *K-quasiconformal* if and only if

i ϕ is a homeomorphism,

ii ϕ is absolutely continuous on lines, and

iii $|\partial_{\bar{z}}\phi| \leq k|\partial_z \phi|$, where $k := \frac{K-1}{K+1}$.

Notice that if ϕ is conformal then $\partial_{\bar{z}}\phi \equiv 0$ and so ϕ is 1-quasiconformal.

We can extend the notion of quasiconformal maps to maps between arbitrary Riemann surfaces by saying that $\phi : X \to Y$ is K-quasiconformal if and only if each of the overlap maps $\varphi \circ \phi \circ \psi^{-1}$ is K-quasiconformal.

As with the notion of quasiconformal maps, there are several different notions which can be used to describe an almost-complex structure. Here, we chose the presentation in terms of Beltrami coefficients in order again reduce the number of other ideas we must introduce.

Definition 2.1.10. Let $U \subseteq \mathbb{C}$ be a domain.

A *Beltrami coefficient* μ on U is an almost-everywhere defined measurable function $\mu : U \to \mathbb{D}$.

The *dilatation* of μ is defined to be

$$K(\mu) := \operatorname*{ess\,sup}_{u \in U} \frac{1 + |\mu(u)|}{1 - |\mu(u)|}.$$

Notice that $K = 1$ if and only if $\mu(u) = 0$ almost everywhere. We denote the 0 coefficient by μ_0.

Given domains U and V, a quasiconformal map $\phi : U \to V$ and a Beltrami coefficient μ on V, we can define a *pullback* $\phi^*\mu$ of μ by ϕ. Briefly, the value of the coefficient $\mu(v)$ parameterises an equivalence class of ellipses in the tangent space T_vV. If $\phi(u) = v$ and ϕ is differentiable at u then $D_u\phi : T_uU \to T_vV$ is a linear map. Further, if $D_u\phi$ is an isomorphism then the preimage of the ellipses corresponding to $\mu(v)$ is a family of ellipses in T_uU. $\phi^*\mu$ then parameterises all such preimage ellipses.

The formula in terms of Beltrami coefficients is given below.

$$\phi^*\mu(u) = \frac{\partial_{\overline{z}}\phi(u) + \mu(\phi(u))\overline{\partial_z\phi(u)}}{\partial_z\phi(u) + \mu(\phi(u))\overline{\partial_{\overline{z}}\phi(u)}}$$

Notice that if ϕ is holomorphic, so $\partial_{\overline{z}}\phi \equiv 0$, this formula simplifies to

$$\phi^*\mu(u) = \mu(\phi(u))\frac{\overline{\partial_{\overline{z}}\phi(u)}}{\partial_{\overline{z}}\phi(u)}.$$

By using the notion of a pullback we can define a *Beltrami differential* on an arbitrary Riemann surface X. The Beltrami differential μ assigns to each chart (U, ψ) on X a Beltrami coefficient μ_ψ on $\psi(U)$ such that if (V, φ) is an overlapping chart then $\mu_\varphi = (\varphi \circ \psi^{-1})^*\mu_\psi$ wherever both sides are defined. Notice that by the above formula $K(\mu_\varphi) = K((\varphi \circ \psi^{-1})^*\mu_\psi)$, hence $K(\mu)$ is well-defined.

Observe that if ϕ is K-quasiconformal then $\phi^*\mu_0$ has dilatation at most

K. The following theorem, the Measurable Riemann Mapping Theorem, can be considered to assert that the converse is true; if a Beltrami coefficient μ has bounded dilatation $K < \infty$ then there exists a quasiconformal map ϕ such that $\phi^*\mu_0 = \mu$.

Theorem 2.1.11 (Measurable Riemann Mapping Theorem)**.**
Let X be a Riemann surface isomorphic to $\hat{\mathbb{C}}, \mathbb{C}$ or $\mathbb{D}$ and let μ be a Beltrami differential on X. Suppose that $K(\mu) = K < \infty$.

Then there exists a K-quasiconformal map $\phi : X \to X$ such that $\mu = \phi^*\mu_0$. This ϕ is unique up to post-composition by an automorphism of X.

We have already seen that if ϕ is conformal then ϕ is 1-quasiconformal and $\phi^*\mu_0 = \mu_0$. The last theorem we shall state in this review states that the converse is true: a 1-quasiconformal map is conformal.

Theorem 2.1.12 (Weyl's Lemma, [2])**.**
Let $U, V \subseteq \mathbb{C}$ be domains and let $\phi : U \to V$ be quasiconformal.

Then ϕ is 1-quasiconformal if and only if ϕ is conformal. Equivalently, $\phi^*\mu_0 = \mu_0$ if and only if ϕ is conformal.

Proposition 2.1.13 ([17, Proposition 2.5.2])**.**
Let $w_1 \in \mathbb{C} \cup \{-\infty\}$ and $w_2 \in \mathbb{C} \cup \{+\infty\}$ such that if both $w_1, w_2 \in \mathbb{C}$ then $\Re(w_2 - w_1) > \frac{5}{2}$.

When $w_1, w_2 \in \mathbb{C}$, let $U = U_{w_1,w_2} \subseteq \mathbb{C}$ be a domain of the form

$$U_{w_1,w_2} = \{w \in \mathbb{C} \mid \Re(w - w_1) > -|\Im(w - w_1)| \text{ and } \Re(w - w_2) < |\Im(w - w_2)|\}.$$

If $w_0 \in \mathbb{C}$ then $U_{-\infty,w_0} = \{w \in \mathbb{C} \mid \Re(w-w_0) < |\Im(w-w_0)\}$ and $U_{w_0,+\infty} = \{w \in \mathbb{C} \mid \Re(w-w_0) > -|\Im$ $U_{-\infty,+\infty} = \mathbb{C}$.

Let $F : U \to \mathbb{C}$ be a holomorphic function which satisfies

$$|F(w) - (w + 1)| < \frac{1}{4} \text{ and } |F'(w) - 1| < \frac{1}{4}.$$

Then there exists an injective holomorphic function $\Phi : U \to \mathbb{C}$ such that

$$\Phi(F(w)) = \Phi(w) + 1$$

for all $w \in U \cap F^{-1}(U)$. This map is unique up to post-composition by a translation.

Figure 2.2. First, we show that F is univalent. Let $w, w' \in U$ and let $\gamma : [0, 1] \to U$ be a smooth curve joining w to w'. Then

$$\begin{aligned}\big|(F(w')-F(w))-(w'-w)\big| &= \left|\int_0^1 (F'(\gamma(t))-1)\gamma'(t)dt\right| \\ &< \left|\int_0^1 \frac{1}{4}\gamma'(t)dt\right| \\ &= \frac{1}{4}|w'-w|\end{aligned}$$

where the inequality comes from the fact that $|F'(w)-1| < \frac{1}{4}$.

Thus $|(F(w')-F(w))-(w'-w)| < \frac{1}{4}|w'-w|$, which can be rearranged to show that $\frac{5}{4}|w'-w| > |F(w')-F(w)| > \frac{3}{4}|w'-w|$. Hence if $w' \neq w$ then $F(w') \neq F(w)$ and so F is univalent.

Let $w_0 \in U$ be a point such that the vertical lines $l = \{w \mid \Re(w) = \Re(w_0)\}$ and $\{w \mid \Re(w) = \Re(w_0) + \frac{5}{2}\}$ are contained in U. Since $\Re(F(w)) < \Re(w) + \frac{5}{4}$, $F(l)$ and $F^{\circ 2}(l)$ are contained in the region bounded by the two vertical lines, hence are contained in U. Since F is univalent it is a homeomorphism onto its image, so $F(l)$ is a simple curve. Thus $l \cup F(l)$ bounds a unique region, $\overline{S}$, which is a vertical strip. Let $S = \overline{S} \setminus F(l)$.

$\overline{S} \cap F(\overline{S}) = F(l)$, since F is univalent. Indeed for all $n, m \in \mathbb{Z}$, we can see that $F^{\circ n}(S) \cap F^{\circ m}(S) \neq \emptyset$ if and only if $n = m$. Since $|F(w) - (w+1)| < 1/4$, $\Re(F^{\circ 2}(w)) > \Re(w) + 3/2$ and so $\inf\{d(w, w') \mid w \in l \text{ and } w' \in F^{\circ 2}(l)\} \geq 3/2$. Since $\Re(F(w)) < \Re(w) + 5/4$ it follows that for all $w \in U$ there exists $n \in \mathbb{Z}$ such that $F^{\circ n}(w) \in \overline{S} \cup F(\overline{S})$. Hence there exists a unique $n \in \mathbb{Z}$ such that $F^{\circ n}(w) \in S$.

Let $\phi : \overline{S}_0 = \{z \mid 0 \leq \Re(z) \leq 1\} \to U$ be the map

$$\phi(x+iy) = w_0 + iy + x(F(w_0+iy) - (w_0+iy)).$$

We claim that ϕ is a quasiconformal map from $\overline{S}_0$ to $\overline{S}$. To show this, we first observe that it is $\mathbb{R}$-differentiable. Therefore it is both absolutely continuous on lines and has well-defined partial derivatives everywhere.

Now we show that ϕ is a homeomorphism, beginning by showing that it is injective. Suppose $\phi(x+iy) = \phi(x'+iy') = w$. Then the straight lines l_1 between $w_0 + iy$ and $F(w_0+iy)$ and l_2 between $w_0 + iy'$ and $F(w_0+iy')$ cross at the point w. Without loss of generality, we shall assume that $y' > y$.

$\frac{\partial F}{\partial y} = iF'(w_0+iy)$ and so in particular $\Im\left(\frac{\partial F}{\partial y}\right) > 3/4 > 0$. Hence, if $y' > y$ then $F(y') > F(y)$. But this is impossible, since $l_1 \cap l_2 \neq \emptyset$ implies that $F(y') \leq F(y)$. Hence ϕ is injective. Also, notice that since ϕ is injective, the line $\phi(\{z \mid 0 \leq \Re(z) \leq 1 \text{ and } \Im(z) = y\})$ can only intersect ∂S at $w_0 + iy$ and $F(w_0+iy)$.

Thus $\phi(\{z \mid 0 < \Re(z) < 1 \text{ and } \Im(z) = y\}) \subset \overline{S}$ and so $\phi(\overline{S}_0) \subseteq \overline{S}$.

Since ϕ is injective it is a homeomorphism onto its image. In particular, $\phi(\overline{S}_0)$ is connected and simply connected. Also, $\Im(\phi(z)) \to \pm\infty$ as $\Im(z) \to \pm\infty$, so we may extend ϕ to a map $\phi : \overline{S}_0 \cup \{\pm i\infty\} \to \overline{S} \cup \{\pm i\infty\}$. Then it follows that $\phi(\partial\overline{S}_0) = \partial\overline{S} = l \cup F(l) \cup \{\pm i\infty\}$. Since $\phi(\overline{S}_0)$ is simply connected we therefore have that $\phi(\overline{S}_0) = \overline{S}$. Thus ϕ is surjective and hence a homeomorphism from $\overline{S}_0$ to $\overline{S}$.

Taking partial derivatives,

$$\frac{\partial\phi}{\partial x} = F(w_0 + iy) - (w_0 + iy) \text{ and } \frac{\partial\phi}{\partial y} = ixF'(w_0 + iy) + i(1 - x).$$

Hence

$$\left|\frac{\partial\phi}{\partial z} - 2\right| = \frac{1}{2}\left|(F(w_0 + iy) - (w_0 + iy + 1)) + x\left(F'(w_0 + iy) - 1\right)\right| \leq \frac{1}{4}$$

and

$$\left|\frac{\partial\phi}{\partial \overline{z}}\right| = \frac{1}{2}\left|(F(w_0 + iy) - (w_0 + iy + 1)) - x\left(F'(w_0 + iy) - 1\right)\right| \leq \frac{1}{4}.$$

Thus $\left|\frac{\partial\phi}{\partial\overline{z}} / \frac{\partial\phi}{\partial z}\right| < 1/3$ and so ϕ is quasiconformal.

Let μ_0 be the 0 coefficient on $\overline{S}$ and let $\mu = \phi^*\mu_0$. Let $T_1 : z \mapsto z + 1$ be unit translation on $\mathbb{C}$ and propagate μ to a Beltrami coefficient defined on $\mathbb{C}$ by $\mu = (T_1^{\circ -n})^*\mu$ on the vertical strip $S_n = \{z \mid n \leq \Re(z) \leq n + 1\}$.

By the Measurable Riemann Mapping Theorem there exists a quasiconformal integrating map $\psi : \mathbb{C} \to \mathbb{C}$ such that $\mu = \psi^*\mu_0$, $\psi(0) = 0$ and $\psi(1) = 1$. Let $T : \mathbb{C} \to \mathbb{C}$ be the map $\psi \circ T_1 \circ \psi^{-1}$. Then

$$T^*\mu_0 = \psi^* T_1^* (\psi^{-1})^*\mu_0 = \psi^* T_1^* \mu = \psi^*\mu = \mu_0,$$

hence T is holomorphic by Weyl's Lemma. Since ψ and T_1 are homeomorphisms, T must be conformal, so an affine map. T_1 does not have any fixed points, so neither does T. Hence T is a translation. $T(0) = \psi \circ T_1 \circ \psi^{-1}(0) = \psi \circ T_1(0) = \psi(1) = 1$. Hence $T = T_1$ is unit translation.

Now we define $\Phi := \psi \circ \phi^{-1} : S \to \mathbb{C}$. $\Phi^*\mu_0 = \psi^*(\phi^{-1})^*\mu_0 = \psi^*\sigma = \mu_0$, so Φ is holomorphic by Weyl's Lemma. Since ψ and ϕ are both homeomorphisms, Φ is univalent.

We can extend Φ to all of U by the functional equation $\Phi(F(w)) = \Phi(w) + 1$. This gives a well-defined univalent function since $U = \bigsqcup_{n\in\mathbb{Z}} F^{\circ n}(S)$ as we have

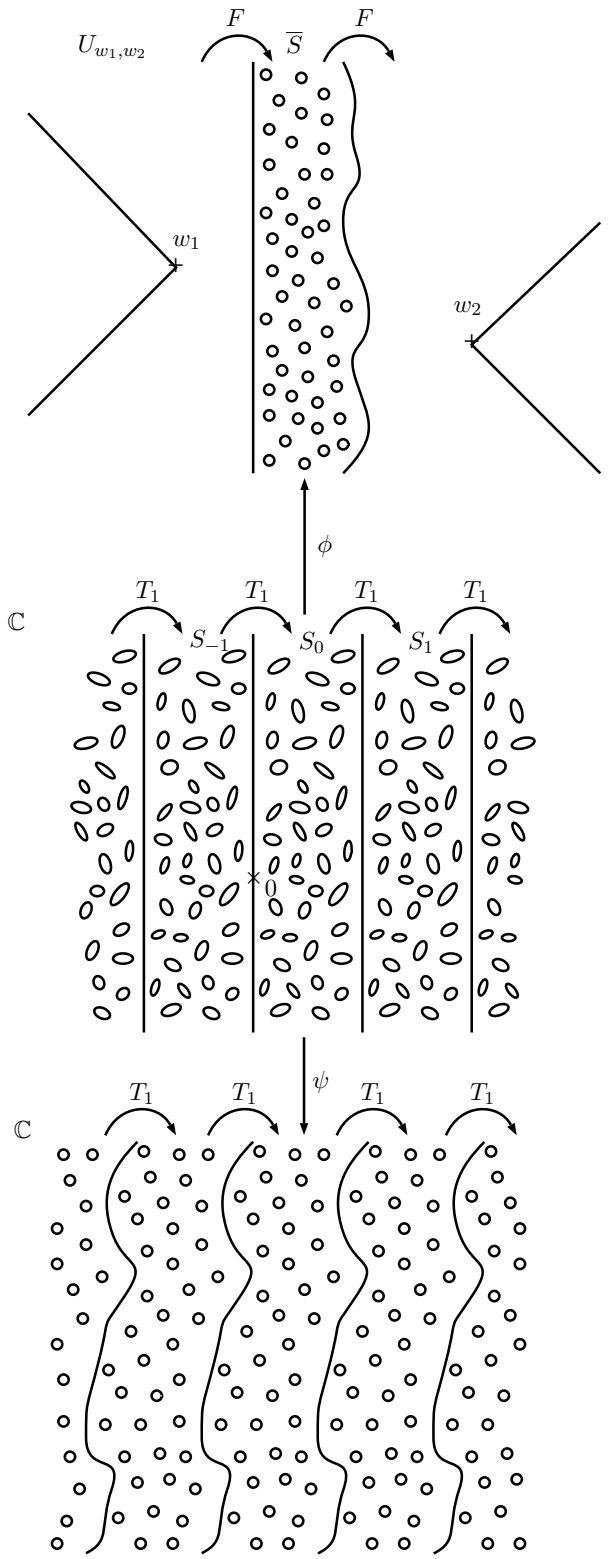

Figure 2.2: Diagram of the proof of Proposition 2.1.13. Under the map ϕ the region S_0 is mapped into S by mapping the straight line between iy and $1 + iy$ to the straight line between $w_0 + iy$ and $F(w_0 + iy)$. The 0 Beltrami coefficient on S is then pulled back under ϕ to give a Beltrami coefficient μ on S_0, which is then spread to $\mathbb{C}$ by $\mu = T_n^* \mu$. $\psi : \mathbb{C} \to \mathbb{C}$ is the integrating map. The composition $\Phi = \psi \circ \phi^{-1}$ preserves the 0 coefficient, hence is holomorphic. We can extend Φ to a holomorphic map defined on all of U by $\Phi(F^{\circ n}(w)) = \Phi(w) + n$.

already shown.

If Ψ is another function satisfying $\Psi(F(w)) = \Psi(w) + 1$, then

$$\begin{aligned}\Psi \circ \Phi^{-1}(z+1) =& \Psi(F(\Phi^{-1}(z))) \\ =& \Psi \circ \Phi^{-1}(z) + 1\end{aligned}$$

and so $\Psi \circ \Phi^{-1}$ commutes with unit translation. We can thus define $\Psi \circ \Phi^{-1} : \mathbb{C} \to \mathbb{C}$ by $\Psi \circ \Phi^{-1}(T_1^{\circ n}(z)) = T_1^{\circ n} \circ \Psi \circ \Phi^{-1}(z)$, which is holomorphic on all of $\mathbb{C}$. Similarly, $\Phi \circ \Psi^{-1}$ also commutes with unit translation. Since $\Phi \circ \Psi^{-1} = (\Psi \circ \Phi)^{-1}$, both maps are affine. Since they commute with translations they must be translations and the claim is proven. $\square$

Before returning to the case of parabolic points, we conclude this subsection by giving an estimate of the generalised Fatou coordinates above. We give this estimate in terms of a larger class of maps, from which we can return to our setting of interest by letting $v(w) = F(w) - w$.

Lemma 2.1.14 ([17, Proposition 2.6.2])**.**
Let $U \subset \mathbb{C}$ be a domain and $\Phi, v : U \to \mathbb{C}$ be holomorphic functions which satisfy

- *Φ is univalent in U.*
- *$|v(w) - 1| < 1/4$ for all $w \in U$.*
- *$\Phi(w + v(w)) = \Phi(w) + 1$ whenever $w, w + v(w) \in U$.*

Then there exist universal constants $R_1, C_1, C_2 > 0$ such that if $U = \{w \in \mathbb{C} \mid |w - w_0| < R\}$ is the disc centred on w_0 of radius R for some $R \geq R_1$, then

$$\left|\Phi'(w_0) - \frac{1}{v(w_0)}\right| \leq C_1\left(\frac{1}{R^2} + |v'(w_0)|\right) \leq \frac{C_2}{R}.$$

Suppose instead that U is the wedge $U = \{w \in \mathbb{C}^* \mid \theta_1 < \arg w < \theta_2\}$, with $\theta_2 < \theta_1 + 2\pi$, and that for some $C, \nu > 0$, for all $w \in U$ we have that $|v'(w)| \leq \frac{C}{|w|^{1+\nu}}$. Then for any $w_0 \in U$ and θ_1', θ_2' with $\theta_1 < \theta_1' < \arg w_0 < \theta_2' < \theta_2$ there exists $R_2, C_3 > 0$ and $\zeta \in \mathbb{C}$ such that

$$\left|\Phi(w) - \int_{w_0}^{w} \frac{dz}{v(z)} - \zeta\right| \leq C_3\left(\frac{1}{|w|} + \frac{C}{|w|^{\nu}}\right),$$

for all w satisfying $\theta_1' < \arg w < \theta_2'$ and $d(w, \mathbb{C} \setminus U) > R_2$. Moreover C_3 depends only on θ_j and θ_j'.

Proof. Without loss of generality, assume $w_0 = 0$. Take $R > 12$. If $|w| < R-2$ then $w + v(w) \in \mathbb{D}_2(w) \subset U$. Consider the function

$$\Phi_w : \mathbb{D} \to \mathbb{C} \ , \ \Phi_w : z \mapsto \frac{\Phi(w+2z) + \Phi(w)}{2\Phi'(w)},$$

which is univalent with $\Phi_w(0) = 0$ and $\Phi_w'(0) = 1$. Applying the Koebe Distortion Theorem we obtain the inequality

$$\frac{|z|}{(1+|z|)^2} \leq \left| \frac{\Phi(w+2z) - \Phi(w)}{2\Phi'(w)} \right| \leq \frac{|z|}{(1-|z|)^2}.$$

In particular, setting $z = \frac{1}{2}v(w)$ we see that

$$\frac{|v(w)|}{(1+|v(w)|/2)^2} \leq \left| \frac{\Phi(w+v(w)) - \Phi(w)}{\Phi'(w)} \right| \leq \frac{|v(w)|}{(1-|v(w)|/2)^2}.$$

Since $|v(w) - 1| < 1/4$ and $\Phi(w + v(w)) - \Phi(w) = 1$ by assumption, we see there exist constants C, C', independent of w, such that $C \leq |\Phi'(w)| \leq C'$ whenever $|w| < R - 2$.

If $|w| < R/2 < 5R/6 < R - 2$ then by Cauchy's derivative estimate we have that $|\Phi''(w)| \leq 3C'/R$. Using Taylor's formula

$$\Phi(w+z) = \Phi(w) + z\Phi'(w) + z^2 \int_0^1 (1-t)\Phi''(w+zt)dt$$

and setting $z = v(w)$ we see that

$$\begin{aligned} |\Phi(w+v(w)) - \Phi(w) - v(w)\Phi'(w)| &= \left| v(w)^2 \int_0^1 (1-t)\Phi''(w+zt)dt \right| \\ |1 - v(w)\Phi'(w)| &\leq |v(w)|^2 3C'/R < 75C'/16R = C''/R \end{aligned}$$

whenever $|w|$ and $|w + v(w)| < R/2$, which is satisfied whenever $|w| < R/2 - 5/4$.

$-(\Phi''(w)v(w) + \Phi'(w)v'(w)) = (1 - \Phi'(w)v(w))'$ and so by Cauchy's estimate if $|w| < R/4 - 5/4$ so $\overline{\mathbb{D}_{R/5}(w)} \subset \mathbb{D}_{R/2-5/4}$ then

$$|\Phi''(w)v(w) + \Phi'(w)v'(w)| \leq C'''/R^2,$$

where $C''' = 5C''$.

Likewise, since $|v(w)| < 5/4$ we can integrate around the contour $|z-w| = R/2$ and apply Cauchy's estimate, integrating around the contour $|z - w| = R/2$ to conclude that $|v'(w)| < 5/R$ and $|v''(w)| < 10/R^2$ whenever $|w| < R/3$. Hence, by

Taylor's formula, $|v'(w)| < |v'(0)| + \frac{50}{4R^2}$ whenever $|w| < 5/4, R/3$.

Rearranging the inequality $|\Phi''(w)v(w) + \Phi'(w)v'(w)| \leq C'''/R^2$, we see that

$$\begin{aligned}|\Phi''(w)| \leq & \frac{1}{|v(w)|}\left(\frac{C'''}{R^2} + |\Phi'(w)v'(w)|\right) \\ \leq & \frac{4}{5}\left(\frac{C'''}{R^2} + C'|v'(0)|\right) = D\left(\frac{1}{R^2} + |v'(0)|\right),\end{aligned}$$

whenever $|w| < 5/4 < R/4$.

Applying Taylor's Formula again, we have

$$|1 - \Phi'(0)v(0)| \leq D(1/R^2 + |v'(0)|) \leq D'/R.$$

Thus

$$|\Phi'(0) - \frac{1}{v(0)}| \leq \frac{4}{3}D(1/R^2 + |v'(0)|) \leq \frac{4}{3}D'/R,$$

since $|v(0)| > 3/4$.

For the second claim, note that there exists a constant

$$C_\theta = \max(\sin(\theta_1' - \theta_1), \sin(\theta_2 - \theta_2'))$$

such that if $\theta_1' < \arg w < \theta_2'$ then $d(w, \mathbb{C} \setminus U) > C_\theta |w|$.

Thus, for all $w \in U$ with $\theta_1' < \arg w < \theta_2'$ and $|w| > R_1/C_\theta$, we have that $\mathbb{D}_{R_1}(w) \subset \mathbb{D}_{C_\theta|w|}(w) \subset U$. Applying the previous result to this region yields the inequality

$$\left|\Phi'(w) - \frac{1}{v(w)}\right| \leq C_1\left(\frac{1}{(C_\theta|w|)^2} + |v'(w)|\right) \leq C'\left(\frac{1}{|w|^2} + \frac{1}{|w|^{1+\nu}}\right).$$

Integrating along a path gives the desired inequality. □

2.1.4 Fatou Coordinates

We can now proceed to proving the existence of Fatou coordinates, which will be central tools for the remainder of this paper. Our first application will be proving the Cylinder Theorem in this subsection, which describes the quotient spaces $\mathcal{P}_{\pm,j}/f$. In the following section we shall use them to define the horn map and lifted horn map. Finally, for a suitable class of maps $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ we prove covering properties which will play a central role in the proof of the main theorem in chapter 3.

Theorem 2.1.15 (Fatou Coordinates)**.**
Let f be a holomorphic map defined in a neighbourhood $\mathcal{D}(f)$ of 0, fixing 0 with

multiplier $e^{2\pi i p/q}$ and parabolic multiplicity ν. Let $\{\mathcal{P}_{\pm,j}\}$ be petals of f.

Then there exist univalent maps $\tilde{\Phi}_{\pm,j} : \mathcal{P}_{\pm,j} \to \mathbb{C}$ which satisfy

$$\tilde{\Phi}_{\pm,j}(f^{\circ q}(z)) = \tilde{\Phi}_{\pm,j}(z) + 1$$

for all $z \in \mathcal{P}_{\pm,j}$. The maps $\tilde{\Phi}_{\pm,j}$ are unique up to post-composition by a translation. The maps $\tilde{\Phi}_{\pm,j}$ are called Fatou coordinates *of f.*

Proof. Let $F_{\pm,j} = \phi \circ f^{\circ q} \circ \psi_{\pm,j}$ be the function defined in lemma 2.1.2.

Then $F_{\pm,j}(w) = w + 1 + \mathcal{O}(\frac{1}{w\sqrt[\nu]{|w|}})$ and $F'_{\pm,j}(w) = 1 + \mathcal{O}(\frac{1}{w^2})$ as $|w| \to \infty$ within $\{w \mid \Re(\pm w) > -|\Im(\pm w)|\}$. In particular, we see that there exists $R > 0$ such that for all $|w| > R/2$, $|F_{\pm,j}(w) - (w+1)| < 1/4$ and $|F'_{\pm,j}(w) - 1| < 1/4$.

Hence $F_{\pm,j}$ satisfies the conditions of proposition 2.1.13 on the set $\pm U_{R,+\infty}$, where $-U_{R,+\infty} = U_{-\infty,-R}$. We can then further choose R large enough that $\psi_{\pm,j}(\pm U_{R,+\infty}) \subset \mathcal{D}(f)$. Thus there exists a univalent map $\Phi_{\pm,j} : \pm U_{R,+\infty} \to \mathbb{C}$ which satisfies $\Phi_{\pm,j}(F_{\pm,j}(w)) = \Phi_{\pm,j}(w) + 1$.

Let $\tilde{\Phi}_{\pm,j} = \Phi_{\pm,j} \circ \phi : \psi_{\pm,j}(\pm U_{R,+\infty}) \to \mathbb{C}$. Then $\tilde{\Phi}_{\pm,j}$ is univalent, since ϕ is univalent on $\Delta_{\pm,j} \supset \psi_{\pm,j}(U_{R,+\infty})$. Further,

$$\begin{aligned}
\tilde{\Phi}_{\pm,j}(f^{\circ q}(z)) =& \Phi_{\pm,j}(\phi(f^{\circ q}(z))) \\
=& \Phi_{\pm,j}(\phi \circ f^{\circ q} \circ \psi_{\pm,j}(\phi(z))) \\
=& \Phi_{\pm,j}(F_{\pm,j}(\phi(z))) \\
=& \Phi_{\pm,j}(\phi(z)) + 1 \\
=& \tilde{\Phi}_{\pm,j}(z) + 1
\end{aligned}$$

We can then obtain $\tilde{\Phi}_{\pm,j} : \mathcal{P}_{\pm,j} \to \mathbb{C}$ by extending via the functional equation $\tilde{\Phi}_{\pm,j}(f^{\circ -q}(z)) = \tilde{\Phi}_{\pm,j}(z) - 1$, noting that f is univalent on $\mathcal{P}_{\pm,j}$ so the resulting function is well-defined and univalent. The uniqueness of $\tilde{\Phi}_{\pm,j}$ follows immediately from the corresponding uniqueness of $\Phi_{\pm,j}$. If $\tilde{\Psi}_{\pm,j}$ are another set of Fatou coordinates then $\tilde{\Psi}_{\pm,j} \circ \tilde{\Phi}_{\pm,j}^{-1} = \Psi_{\pm,j} \circ \phi \circ \phi^{-1} \circ \Phi_{\pm,j}^{-1} = \Psi_{\pm,j} \circ \Phi_{\pm,j}^{-1} = T_c$ for some $c \in \mathbb{C}$. □

Before moving on to our applications, we also give the estimate from Shishikura [17] based on lemma 2.1.14.

Lemma 2.1.16.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$, $F_{\pm,j}$ and $\Phi_{\pm,j}$ be as in theorem 2.1.15.

Then for any $0 < k < 1$,

$$\Phi_{\pm,j}(w) = w - a_{\pm,j} \log w + c_{\pm,j} + o(1)$$

as w *tends to* ∞ *within the sectors* $\pm\{w \mid \Re(w) > R - k|\Im(w)|\}$ *for some constants* $a_{\pm,j}, c_{\pm,j} \in \mathbb{C}$.

Proof. Let $v_{\pm,j}(w) = F_{\pm,j}(w) - w$, which is defined on a domain $\pm U_{R,+\infty}$ for some $R > 0$, so that by lemma 2.1.2

$$v_{\pm,j}(w) = 1 + \frac{a_{\pm,j}}{w} + \mathcal{O}\left(\frac{1}{w\sqrt[\nu]{|w|}}\right) \text{ and } |v'_{\pm,j}(w)| = \mathcal{O}\left(\frac{1}{w^2}\right)$$

as $|w| \to \infty$.

Given $0 < k < 1$, let U_R^k be the sector $\{w \mid \Re(w) > R - k|\Im(w)|\}$. Then there exist constants $R_k, C_k > 0$ such that for all $w \in (\pm U_R^k) \cap (\mathbb{C} \setminus \overline{\mathbb{D}_{R_k}})$ we have that $d(w, \mathbb{C} \setminus \pm U_{R,+\infty}) > C_k|w|$. Then, as in the proof of lemma 2.1.14, for each $w \in \pm U_R^k$ we can apply part i) of lemma 2.1.14 to the discs $\{\omega \mid |w - \omega| < C_k|w|\}$ to obtain the same estimate as part ii) of the lemma,

$$\left|\Phi_{\pm,j}(w) - \int_{w_0}^{w} \frac{d\omega}{v(\omega)} - \zeta\right| \leq C_3 \left(\frac{1 + C_k}{|w|}\right).$$

Since $1/v_{\pm,j}(w) = 1 - \frac{a_{\pm,j}}{w} + \mathcal{O}\left(\frac{1}{w\sqrt[\nu]{|w|}}\right)$, we have that

$$\Phi_{\pm,j}(w) = w - a_{\pm,j} \log(w) + c_{\pm,j} + \mathcal{O}\left(\frac{1}{w}\right).$$

□

We are now ready to prove the Cylinder Theorem, which describes the quotient spaces $\mathcal{P}_{\pm,j}/f =: \mathcal{C}_{\pm,j}$. By showing that $\bigcup_{n\in\mathbb{Z}} T_n(\tilde{\Phi}_{\pm,j}(\mathcal{P}_{\pm,j})) = \mathbb{C}$, where $T_n : z \mapsto z + n$, we can show that each such quotient space is conformally isomorphic to $\mathbb{C}/\mathbb{Z}$. It is important to note that whilst a given choice of Fatou coordinates induces isomorphisms $\mathcal{C}_{\pm,j} \overset{\cong}{\to} \mathbb{C}/\mathbb{Z}$, Fatou coordinates are only unique up to post-composition by a translation there are so there are no natural isomorphisms $\mathcal{C}_{\pm,j} \cong \mathbb{C}/\mathbb{Z}$. Indeed any isomorphism $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}/\mathbb{Z}$ which preserves the ends $\pm i\infty$ can occur as the composition $\mathbb{C}/\mathbb{Z} \overset{\cong^{-1}}{\to} \mathcal{C}_{\pm,j} \overset{\cong}{\to} \mathbb{C}/\mathbb{Z}$.

Of great interest are isomorphisms from the attracting cylinders to the repelling cylinders. These induce an orbit correspondence and can be considered to

describe how an orbit might 'pass through' the parabolic fixed point. The theory of parabolic implosion studies this idea; certain such orbit correspondences (any in the case of a simple parabolic point) can be arbitrarily well realised by orbits of suitable perturbations. This yields a great richness in the structure of parameter space near to parameters with a parabolic point. Appendix A gives such an application, showing that the cubic connectedness locus is not locally connected by studying the structure near a carefully chosen parabolic parameter.

Theorem 2.1.17 (The Cylinder Theorem)**.**
Let f be a holomorphic map, defined in a neighbourhood $\mathcal{D}(f)$ of 0, fixing 0 with multiplier $e^{2\pi i p/q}$ and parabolic multiplicity ν.

Let $\mathcal{P}_{\pm,j}$ be a petal of f.

Then the quotient space $\mathcal{C}_{\pm,j} = \mathcal{P}_{\pm,j}/\sim_f$ is isomorphic to the bi-infinite cylinder $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$, where the relation $\sim_f$ is given by

$$z \sim_f z' \text{ if and only if there exist } n \in \mathbb{Z}_{\geq 0} \text{ such that } f^{\circ nq}(z) = z' \text{ or } f^{\circ nq}(z') = z.$$

Other than for the statement and proof of this theorem we shall use the notation $\mathcal{P}_{\pm,j}/f$ in place of $\mathcal{P}_{\pm,j}/\sim_f$ except where this would cause confusion.

Proof. Let $\Phi_{\pm,j} : \pm U_{R,+\infty} \to \mathbb{C}$ be a map given by proposition 2.1.13, satisfying $\Phi_{\pm,j} \circ F_{\pm,j} = T_1 \circ \Phi_{\pm,j}$. Pick a point $w \in \mathbb{C}$. We wish to show that there exists $n \in \mathbb{Z}$ with $w + n \in \tilde{\Phi}_{\pm,j}(\mathcal{P}_{\pm,j})$. We begin by assuming that $\mathcal{P}_{\pm,j} = \psi_{\pm,j}(\pm U_{R,+\infty})$, which we have previously shown is a petal.

Note first that by lemma 2.1.16, $\Im\Phi_{\pm,j}(x+iy) \to \pm\infty$ as $\Im y \to \pm\infty$. Hence $w \mapsto \Im\Phi_{\pm,j}(w)$ is surjective. Therefore, if no such n exists then either $\tilde{\Phi}_{\pm,j}(\mathcal{P}_{\pm,j})$ is not connected or it is not simply connected. But $\tilde{\Phi}_{\pm,j}(\mathcal{P}_{\pm}) = \Phi_{\pm,j}(U_{R,+\infty})$ and $\Phi_{\pm,j}$ is univalent, so a homeomorphism onto its image. This is a contradiction, hence $w + n \in \tilde{\Phi}_{\pm,j}(\mathcal{P}_{\pm,j})$ for some n.

Therefore, the quotient map $\pi : \tilde{\Phi}_{\pm,j}(\mathcal{P}_{\pm}) \subset \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ is surjective and hence the composition $\pi \circ \tilde{\Phi}_{\pm,j} : \mathcal{P}_{\pm,j} \to \mathbb{C}/\mathbb{Z}$ is surjective. But π is the quotient of $\mathbb{C}$ by T_1 and $\tilde{\Phi}_{\pm,j}$ conjugates $f|_{\mathcal{P}_{\pm,j}}$ to T_1. Hence $\mathcal{C}_{\pm,j} = \mathcal{P}_{\pm,j}/\sim_f \cong \mathbb{C}/\mathbb{Z}$ as claimed.

In general, if $w + n \in \Phi_{\pm,j}(U_{R,+\infty})$ then $f^{\circ mq}(z) \to 0$ from the direction of $\underline{v}_{\pm,j}$, where $\tilde{\Phi}_{\pm,j}(z) = w$. Hence $f^{\circ(n+m)q}(z) \in \mathcal{P}_{\pm,j}$ for some $m \in \mathbb{Z}$ and so $w + (n+m) \in \tilde{\Phi}_{\pm,j}(\mathcal{P}_{\pm,j})$. □

In order to conclude this section by discussing extension properties of Fatou coordinates we first need to somewhat restrict the class of maps we are considering.

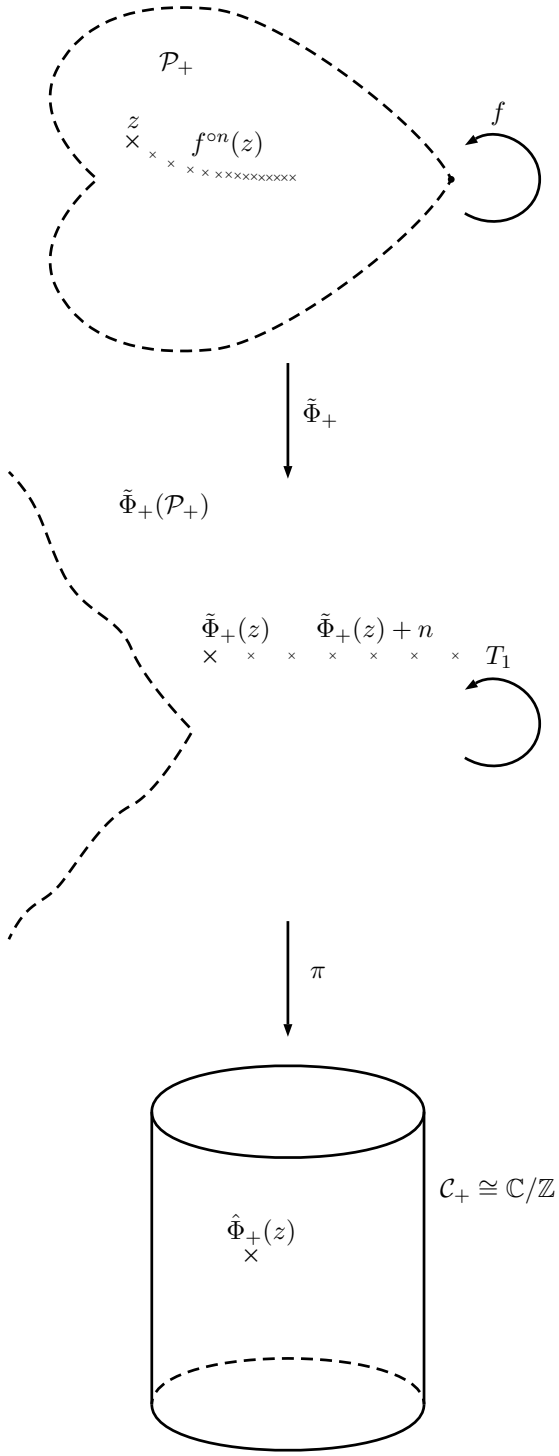

Figure 2.3: A diagram of the relations between orbits under f, orbits under T_1 and points on $\mathcal{C}_+$. $\tilde{\Phi}_+$ maps an orbit under f to an orbit under T_1 and $\hat{\Phi}_+$ maps an orbit under f to a point on $\mathcal{C}$. Thus there exists an isomorphism between $\mathcal{C}_+$ and the image of $\tilde{\Phi}_+(\mathcal{P}_+)$ in $\mathbb{C} \mapsto \mathbb{C}/\mathbb{Z}$.

Definition 2.1.18. Suppose that $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ has a parabolic fixed point of multiplier $e^{2\pi i p/q}$ and parabolic multiplicity ν at 0, where $\mathcal{D}(f) \subseteq \hat{\mathbb{C}}$ is open. Recall

that we previously defined the *parabolic basin* to be

$$\mathcal{A} = \{z \in \mathcal{D}(f) \mid f^{\circ kq}(z) \to 0 \text{ nontrivially as } k \to +\infty\}.$$

We say that f has a *proper parabolic basin* if

i $f|_{\mathcal{A}}$ is proper, and

ii $f|_{\mathcal{A}}$ has finitely many critical points.

The following lemma gives a sufficient condition on the map f for the parabolic basin to be proper; that $\mathcal{A}$ be compactly contained in the domain of definition of f. Notice in particular that if f is either a rational map or polynomial-like map then this condition is satisfied and thus any parabolic basin is proper.

Notice also that if f has a parabolic basin $\mathcal{A}$ which is simply connected, and $f|_{\mathcal{A}}$ is proper, then $f|_{\mathcal{A}}$ has finitely many critical points and hence f has a proper parabolic basin $\mathcal{A}$.

Lemma 2.1.19.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ be a holomorphic map with a parabolic fixed point at 0.

Suppose that the parabolic basin $\mathcal{A}$ is compactly contained in $\mathcal{D}(f)$. That is the closure $\overline{\mathcal{A}}$ is a compact subset of $\mathcal{D}(f)$.

Then f has a proper parabolic basin at 0.

Proof. Note first that $f^{-1}(\overline{\mathcal{A}}) \subseteq \overline{\mathcal{A}}$ and $f^{-1}(\mathcal{A}) \subseteq \mathcal{A}$. Let $K \subset \mathcal{A}$ be a compact set. Then K is closed and so $f^{-1}(K)$ is a closed subset of $\overline{\mathcal{A}}$. $\overline{\mathcal{A}}$ is compact by assumption, hence $f^{-1}(K)$ is compact.

Since $K \subset \mathcal{A}$, $f^{-1}(K) \subset f^{-1}(\mathcal{A}) \subseteq \mathcal{A}$. Hence $f|_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is proper.

Now suppose that $f|_{\mathcal{A}}$ has infinitely many critical points. Then since $\overline{\mathcal{A}}$ is compact, the critical points of f must accumulate at some point in $\overline{\mathcal{A}} \subset \mathcal{D}(f)$. This implies that f is constant, which contradicts the assumption that it has a parabolic fixed point. Hence $f|_{\mathcal{A}}$ has finitely many critical points and so f has a proper parabolic basin at 0. □

For each point $z \in \mathcal{A}$, there exists a unique j such that $f^{\circ kq}(z) \in \mathcal{P}_{+,j}$ for all k large enough. Thus we may divide $\mathcal{A}$ into disjoint open subsets

$$\mathcal{A}_j \coloneqq \{z \in \mathcal{A} \mid f^{\circ kq}(z) \in \mathcal{P}_{+,j} \text{ for all } k \text{ large enough}\}.$$

$\mathcal{P}_{+,j} \subset \mathcal{A}_j$ and we call the connected component $\mathcal{A}_j^0$ containing $\mathcal{P}_{+,j}$ the *j-th immediate basin.* We call the union $\bigcup_j \mathcal{A}_j^0 =: \mathcal{A}^0$ the *immediate basin.* Notice that since $f^{-1}(\mathcal{A}_j) \subseteq \mathcal{A}_j$, if $\mathcal{A}$ is a proper basin then $f|_{\mathcal{A}_j} : \mathcal{A}_j \to \mathcal{A}_j$ is proper.

The requirement that the basin be proper is stronger than we need for the remaining two theorems in this section. However subsequent results will depend on this definition. For now, it suffices to recall that a proper open mapping into a locally compact connected Hausdorff space is surjective. Thus if f has a proper parabolic basin then the restriction to the j-th immediate basin $f|_{\mathcal{A}_j^0} : \mathcal{A}_j^0 \to \mathcal{A}_j^0$ is surjective.

In the case of attracting coordinates, we can extend $\tilde{\Phi}_{+,j}$ to the whole attracting basin $\mathcal{A}_j$ via the functional equation $\tilde{\Phi}_{+,j} \circ f^{\circ q} = T_1 \circ \tilde{\Phi}_{+,j}$. The requirement that the basin be proper ensures that the resulting map is surjective.

Theorem 2.1.20.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ be a holomorphic map with a parabolic fixed point of multiplier $e^{2\pi i p/q}$ and parabolic multiplicity ν at 0. Suppose further that f has a proper parabolic basin at 0 and let $\tilde{\Phi}_{+,j} : \mathcal{P}_{+,j} \to \mathbb{C}$ be attracting Fatou coordinates.

Then each of the maps $\tilde{\Phi}_{+,j}$ may be extended to the entire attracting basin $\mathcal{A}_j$ of the petal $\mathcal{P}_{+,j}$, via the functional equation $\tilde{\Phi}_{+,j}(z) = \tilde{\Phi}_{+,j}(f^{\circ q}(z)) - 1$. The extended maps are holomorphic and surjective, but no longer injective.

Proof. By definition of the parabolic basin and petals, for all $z \in \mathcal{A}$ there exists $k \in \mathbb{Z}_{\geq 0}$ and $j \in \{0, 1, \ldots, q\nu - 1\}$ such that $f^{\circ kq}(z) \in \mathcal{P}_{+,j}$. If $f^{\circ k'q}(z) \in \mathcal{P}_{+,j}$ also then $\tilde{\Phi}_{+,j} \circ f^{\circ kq}(z) = T_{k-k'} \circ \tilde{\Phi}_{+,j} \circ f^{\circ k'q}(z)$. Hence $\tilde{\Phi}_{+,j}(z) := T_{-k} \circ \tilde{\Phi}_{+,j} \circ f^{\circ kq}(z)$ is well-defined. Since k can be chosen to be constant on a small neighbourhood of z, it follows that $\tilde{\Phi}_{+,j}$ is holomorphic.

By the Cylinder Theorem, for all $w \in \mathbb{C}$ there exists $k \in \mathbb{Z}$ such that $w + k \in \tilde{\Phi}_{+,j}(\mathcal{P}_{+,j})$. Since $f^{\circ q}|_{\mathcal{A}_j}$ is proper by assumption, the restriction to the j-th immediate parabolic basin $\mathcal{A}_j^0$ is surjective. Hence $f^{\circ -kq}(\tilde{\Phi}_{+,j}^{-1}(w + k)) \subseteq \tilde{\Phi}_{+,j}^{-1}(w)$ is non-empty and so $\tilde{\Phi}_{+,j}$ is surjective. $\square$

In the case of the repelling coordinates rather than attempting to extend the coordinates themselves we instead consider the inverse maps $\tilde{\Psi}_{-,j} = (\tilde{\Phi}_{-,j})^{-1}$, which satisfy the equation $f^{\circ q} \circ \tilde{\Psi}_{-,j} = \tilde{\Psi}_{-,j} \circ T_1$. The definition of this map is further complicated by the fact that some orbits under $f^{\circ q}$ may escape from the domain of definition $\mathcal{D}(f)$. However, we can be sure that any orbit which starts in the parabolic basin $\mathcal{A}$ will remain there.

By requiring that the parabolic basin be proper we have that $\mathcal{A}$ descends to subset $\tilde{\mathcal{A}}_j$ which is totally invariant under unit translation and whose closure is contained in the domain $\mathcal{D}(\tilde{\Psi}_{-,j})$.

Theorem 2.1.21.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ be a map with a parabolic fixed point of multiplier $e^{2\pi i p/q}$ and parabolic multiplicity ν at 0. Suppose that f has a proper parabolic basin at 0.

Let $\tilde{\Phi}_{-,j} : \mathcal{P}_{-,j} \to \mathbb{C}$ be repelling Fatou coordinates and let

$$\tilde{\Psi}_{-,j} := (\tilde{\Phi}_{-,j})^{-1} : \tilde{\Phi}_{-,j}(\mathcal{P}_{-,j}) \to \hat{\mathbb{C}}$$

be the inverse repelling Fatou coordinates.

Let $\mathcal{A}$ be the parabolic basin of 0 and let $\tilde{\mathcal{A}}_j$ be the set

$$\tilde{\mathcal{A}}_j = \bigcup_{n\in\mathbb{Z}} T_n \circ \tilde{\Phi}_{-,j}(\mathcal{P}_{-,j} \cap \mathcal{A}).$$

This set contains both an upper and lower half-plane.

Then $\tilde{\Psi}_{-,j}$ extends to a holomorphic map $\tilde{\Psi}_{-,j} : \mathcal{D}(\tilde{\Psi}_{-,j}) \to \hat{\mathbb{C}}$ via the functional equation $\tilde{\Psi}_{-,j}(w+1) = f^{\circ q}(\tilde{\Psi}_{-,j}(w))$ and $\tilde{\mathcal{A}}_j \subset \mathcal{D}(\tilde{\Psi}_{-,j})$.

It should be emphasised that the notation $\mathcal{A}_j$ associates a subset of $\mathcal{A}$ with the *attraction* vector $\underline{v}_{+,j}$, whereas the notation $\tilde{\mathcal{A}}_j$ associates a subset of $\mathbb{C}$ with the *repulsion* vector $\underline{v}_{-,j}$. In a sense, $\tilde{\mathcal{A}}_j$ refers to the 'preimage' of the *whole* basin $\mathcal{A}$ in the j-th repelling Fatou coordinate $\tilde{\Phi}_{-,j}$.

Proof. First, suppose that $\tilde{\mathcal{A}}_j$ does not contain an upper half-plane. Then we can find a sequence of points (z_k) in $\mathcal{P}_{-,j}$ such that $\Im(\tilde{\Phi}_{-,j}(z_k)) \to +\infty$ as $k \to \infty$, but $z_k \notin \mathcal{A}$ for all k. Letting $\Phi_{\pm,j}$ be as in theorem 2.1.15, it follows from lemma 2.1.16 that $(w_k = \Phi_{-,j}^{-1} \circ \tilde{\Phi}_{-,j}(z_k))$ is a sequence with $\Im(w_k) \to +\infty$ as $k \to \infty$.

On the other hand, let ϕ be as in lemma 2.1.2. The petals $\mathcal{P}_{\pm,j}$ can be chosen such that $\phi(\mathcal{P}_{+,j+1} \cap \mathcal{P}_{-,j}) \subseteq \{w \in \mathbb{C} \mid \Im(w) > |\Re(w)| + R\}$ for some $R \in \mathbb{R}_{>0}$ as we saw in the proof of theorem 2.1.6. Therefore we must have that $\phi(z_k) < R + 1$ for all k. This is a contradiction, hence $\tilde{\mathcal{A}}_j$ contains an upper half-plane. A similar argument also shows that it contains a lower half-plane.

Now, note that $\tilde{\Phi}_{-,j} : \mathcal{P}_{-,j} \to \mathbb{C}$ is biholomorphic onto its image by theorem 2.1.15 and so $\tilde{\Psi}_{-,j}|_{\tilde{\Phi}_{-,j}(\mathcal{P}_{-,j})} = (\tilde{\Phi}_{-}|_{\mathcal{P}_{-}})^{-1}$ is well-defined and holomorphic.

If $w \in \overline{\tilde{\mathcal{A}}_j}$ then $w - k \in \overline{\tilde{\Phi}_{-,j}(\mathcal{P}_{-,j} \cap \mathcal{A})}$ for some $k \in \mathbb{Z}$ by definition. This implies that $w - k - 1 \in \tilde{\Phi}_{-,j}(\mathcal{P}_{-,j} \cap \overline{\mathcal{A}})$. From the definition of $\mathcal{A}_j$, we see that $f^{\circ q}(\overline{\mathcal{A}}_j) \subseteq \overline{\mathcal{A}}_j$, so therefore $\tilde{\Psi}_{-,j}(w) = f^{\circ q(k+1)} \circ \tilde{\Psi}_{-,j}|_{\tilde{\Phi}_{-,j}(\mathcal{P}_{-,j})} \circ T_{-(k+1)}(w)$ is defined for any fixed $k \in \mathbb{Z}$ large enough.

$f^{\circ kq} \circ \tilde{\Psi}_{-,j}|_{\tilde{\Phi}_{-,j}(\mathcal{P}_{-,j})} \circ T_{-k}$ is the composition of holomorphic maps, so $\tilde{\Psi}_{-,j}$

is holomorphic on a neighbourhood of w. Further,

$$\begin{aligned} f^{\circ k'q} \circ \tilde{\Psi}_{-,j} \circ T_{-k'} &= f^{\circ kq} \circ \tilde{\Psi}_{-,j} T_{k'-k} \circ T_{-k'} \\ &= f^{\circ kq} \circ \tilde{\Psi}_{-,j} \circ T_{-k}, \end{aligned}$$

so $\tilde{\Psi}_{-,j}(w)$ is well-defined.

Hence $\tilde{\Psi}_{-,j}$ is well-defined and holomorphic and $\overline{\tilde{\mathcal{A}}_j} \subset \mathcal{D}(\tilde{\Psi}_{-,j})$. □

2.2 Parabolic Renormalisation

The extended Fatou coordinates and the Cylinder Theorem give us the following pair of commutative diagrams. For simplicity, for the remainder of this paper we shall be restricting ourselves to *simple* parabolic fixed points, those with multiplier 1 and parabolic multiplicity 1. We therefore drop the j from all of our previous notation, so that Fatou coordinates are $\tilde{\Phi}_\pm$ and so on.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] & \mathcal{A} \\ \tilde{\mathcal{A}} \arrow[u, "\tilde{\Psi}_-"] \arrow[r, "T_1"] \arrow[dr, "\pi_-"'] & \tilde{\mathcal{A}} \arrow[u, "\tilde{\Psi}_-"'] \arrow[d, "\pi_-"] \\ & \mathcal{C}_- \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d, "\tilde{\Phi}_+"] & \mathcal{A} \arrow[d, "\tilde{\Phi}_+"'] \\ \mathbb{C} \arrow[r, "T_1"] \arrow[dr, "\pi_+"'] & \mathbb{C} \arrow[d, "\pi_+"] \\ & \mathcal{C}_+ \end{tikzcd} .$$

The two diagrams are linked together by the *lifted horn map*, $\tilde{h}_f : \tilde{\mathcal{A}} \to \mathbb{C}$, and the *horn map*, $\hat{h}_f : \pi_-(\tilde{\mathcal{A}}) =: \hat{\mathcal{A}} \to \mathcal{C}_+$. These maps provide an encoding of the dynamics of f near the parabolic fixed point.

In the first subsection we shall define the *lifted horn map* and also study covering properties of the extended Fatou coordinates of maps with a proper parabolic basin. In the second subsection we shall define the *horn map* and show that when the map has a proper parabolic basin the horn map is a finite type map. We shall also give the theorem of Écalle and Voronin which shows that the horn map up to equivalence uniquely determines the local analytic conjugacy class of a parabolic germ.

2.2.1 Lifted Horn Maps

We begin by studying the covering properties of extended Fatou coordinates. This pair of lemmas, relating to the attracting and inverse repelling Fatou coordinates, are the main reason for our definition of a map having a proper parabolic basin.

They will form key tools in our proof of the main theorem in chapter 3. Our results here offer a slight improvements over those in [4].

Recalling that extended attracting Fatou coordinates were defined by

$$\tilde{\Phi}_+ = T_{-k} \circ \tilde{\Phi}_+ \circ f^{\circ k},$$

one would expect that any singular values of $\tilde{\Phi}_+$, the points in $\mathbb{C}$ which do not have an evenly covered neighbourhood, should be related to singular values of $f|_{\mathcal{A}}$. The following lemma shows that this intuition is correct when f has a proper parabolic basin; $\tilde{\Phi}_+$ is a covering map away from the image under $\tilde{\Phi}_+$ of the backward orbits of the critical points of f.

Lemma 2.2.1 (Compare [4, Proposition 2])**.**
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ be a map with a simple parabolic fixed point at 0 with a proper parabolic basin. Let $\mathcal{C}$ be the set of critical points of $f|_{\mathcal{A}}$ and let $\mathcal{O}^- = \bigcup_{n\geq 0} f^{\circ -n}(\mathcal{C})$. Let $\tilde{\Phi}_+ : \mathcal{A} \to \mathbb{C}$ be an extended attracting Fatou coordinate.

Then the restriction

$$\tilde{\Phi}_+ : \mathcal{A} \setminus (\tilde{\Phi}_+)^{-1}(\tilde{\Phi}_+(\mathcal{O}^-)) \to \mathbb{C} \setminus \tilde{\Phi}_+(\mathcal{O}^-)$$

is a covering map.

Proof. Let $\mathcal{P}_+$ be an attracting petal of f at 0 such that $\tilde{\Phi}_+(\mathcal{P}_+)$ contains a right half-plane $\mathbb{H}_R = \{z \in \mathbb{C} \mid \Re(z) > R\}$. Then $\mathcal{P}_+$ contains a connected component of $\tilde{\Phi}_+^{-1}(\mathbb{H}_R)$.

Let $w \in \mathbb{C} \setminus \tilde{\Phi}_+(\mathcal{O}^-)$ and let $V \subset \mathbb{C} \setminus \tilde{\Phi}_+(\mathcal{O}^-)$ be a round disc containing w of radius $< 1/2$. Let $U \subset \tilde{\Phi}_+^{-1}(V) \subset \mathcal{A}$ be a connected component and let $n \in \mathbb{Z}_{\geq 0}$ and $z \in U$ such that $f^{\circ n}(z) \in \mathcal{P}_+$. Then, taking n larger if necessary, we can assume that $\Re(\tilde{\Phi}_+(f^{\circ n}(z))) > R + 1$. $\tilde{\Phi}_+ \circ f^{\circ n} = T_n \circ \tilde{\Phi}_+$ and so $\tilde{\Phi}_+ \circ f^{\circ n}(U) \subseteq T_n(V)$. $f^{\circ n}(U)$ is connected and $\operatorname{diam}(\tilde{\Phi}_+ \circ f^{\circ n}(U)) \leq \operatorname{diam}(T_n(V)) = \operatorname{diam}(V) < 1$, hence $\tilde{\Phi}_+ \circ f^{\circ n}(U) \subset \mathbb{H}_R$ and so $f^{\circ n}(U) \subset \mathcal{P}_+$.

By definition, $\tilde{\Phi}_+|_U = T_{-n} \circ \tilde{\Phi}_+|_{\mathcal{P}_+} \circ f^{\circ n}|_U$. Since $V \subset \mathbb{C} \setminus \tilde{\Phi}_+(\mathcal{O}^-)$, none of $U, f(U), f^{\circ 2}(U), \ldots, f^{\circ n-1}(U)$ contain any critical points of f. Further, $f|_{\mathcal{A}}$ is proper and U is a connected component of $f^{\circ -n}((\tilde{\Phi}_+|_{\mathcal{P}_+})^{-1} \circ T_n(V))$, hence $f^{\circ n} : U \to f^{\circ n}(U)$ is a covering map. Moreover, V is a disc and T_n and $\tilde{\Phi}_+|_{\mathcal{P}_+}$ are biholomorphic onto their images. Therefore $f^{\circ n}(U)$ is simply connected and thus $f^{\circ n} : U \to f^{\circ n}(U)$ is biholomorphic.

Therefore $\tilde{\Phi}_+|_U = T_{-n} \circ \tilde{\Phi}_+|_{\mathcal{P}_+} \circ f^{\circ n}|_U : U \to V$ is biholomorphic. Since U was an arbitrary connected component of $\tilde{\Phi}_+^{-1}(V)$, V is evenly covered by $\tilde{\Phi}_+$.

Since $V \subset \mathbb{C} \setminus \tilde{\Phi}_+(\mathcal{O}^-)$ was a neighbourhood of an arbitrary point, it follows that $\tilde{\Phi}_+ : \mathcal{A} \setminus (\tilde{\Phi}_+)^{-1}(\tilde{\Phi}_+(\mathcal{O}^-)) \to \mathbb{C} \setminus \tilde{\Phi}_+(\mathcal{O}^-)$ is a covering map as claimed. $\square$

In the case of inverse repelling Fatou coordinates, $\tilde{\Psi}_-$, recall that $\tilde{\mathcal{A}} \subset \mathcal{D}(\tilde{\Psi}_-)$. Our definition of f have a proper parabolic basin only ensures relatively good behaviour within the basin, so we restrict to $\tilde{\Psi}_- : \tilde{\mathcal{A}} \to \mathcal{A}$. After making this restriction we can apply essentially the same intuition; singular values of $\tilde{\Psi}_- = f^{\circ k} \circ \tilde{\Psi}_- \circ T_{-k}$ only arise from post-critical values of f.

If we were instead to consider rational maps, so that $\mathcal{D}(f) = \hat{\mathbb{C}}$, we could extend this lemma to the whole domain of definition of $\tilde{\Psi}_-$, $\mathbb{C}$. In this case, the singular value set of $\tilde{\Psi}_-$, $S(\tilde{\Psi}_-)$, is the *closure* of the post-critical set of f. $\tilde{\Psi}_-$ will have an asymptotic value, a point z which satisfies $\tilde{\Psi}_- \circ \gamma(t) \to z$ but $\gamma(t) \to \infty$ as $t \to 1$ for some curve $\gamma : [0, 1) \to \mathbb{C}$, at the simple parabolic fixed point 0 as well as at least at any other simple parabolic or (super)attracting fixed points.

Lemma 2.2.2 (Compare [4, Proposition 3])**.**
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ be a holomorphic map with a simple parabolic fixed point at 0 with a proper parabolic basin. Let $\mathcal{C}$ be the set of critical points of $f|_{\mathcal{A}}$ and let $\mathcal{O}^+ = \bigcup_{n \in \mathbb{Z}_{>0}} f^{\circ n}(\mathcal{C})$ be the post-critical set.

Let $\tilde{\Psi}_- : \tilde{\mathcal{A}} \to \mathcal{A}$ be the extended inverse repelling Fatou coordinate.

Then the restriction

$$\tilde{\Psi}_- : \tilde{\Psi}_-^{-1}(\mathcal{A} \setminus \mathcal{O}^+) \to \mathcal{A} \setminus \mathcal{O}^+$$

is a covering map.

Proof. Let $z_1 \in \mathcal{A} \setminus \mathcal{O}^+$ and let V be a simply connected neighbourhood of z_1 which is relatively compact in $\mathcal{A} \setminus \mathcal{O}^+$.

Since V is relatively compact in $\mathcal{A}$, we may assume without loss of generality that the attracting petal $\mathcal{P}_+$ satisfies $\mathcal{P}_+ \cap V = \emptyset$. Further we may choose fat petals, so that $\mathcal{P}_+ \cup \mathcal{P}_- \cup \{0\}$ is a neighbourhood of 0 and $\tilde{\Phi}_-(\mathcal{P}_-)$ contains a left half-plane $-\mathbb{H}_R = \{w \in \mathbb{C} \mid \Re(w) < -R\}$.

Let U be a connected component of $\tilde{\Psi}_-^{-1}(V)$. For $n \in \mathbb{Z}_{\geq 0}$ let $U_n = T_{-n}(U)$ and $V_n = \tilde{\Psi}_-(U_n)$. Since $\tilde{\Psi}_- \circ T_1 = f \circ \tilde{\Psi}_-$ it follows that U_n is a connected component of $\tilde{\Psi}_-^{-1}(V_n)$.

V_{n+1} contains no critical points of f for all $n \geq 0$, since $V \cap \mathcal{O}^+ = \emptyset$, hence $f : V_{n+1} \to V_n$ is a covering map. $V = V_0$ is simply connected and therefore it follows by induction that V_{n+1} is simply connected and $f : V_{n+1} \to V_n$ is biholomorphic for all n. Further, since $f|_{\mathcal{A}}$ is proper, V_n is compactly contained in $\mathcal{A} \setminus \mathcal{O}^+$ for all n.

For $n > 0$ let $g_n : V_0 \to V_n$ be the inverse branch to $f^{\circ n}|_{V_n} : V_n \to V_0$. Since $\overline{V}_n \subset \mathcal{A}\backslash\mathcal{O}^+$, which is hyperbolic, the sequence (g_n) forms a normal family. If $z \in V_0$ then $z = \tilde{\Psi}_-(w)$ for some $w \in U_0$ and $g_n(z) = \tilde{\Psi}_- \circ T_{-n}(w)$. If we take n large enough that $w - n \in -\mathbb{H}_R$ then, since $\tilde{\Phi}_-(\mathcal{P}_-) \supset -\mathbb{H}_R$, it follows that $g_n(z) \in \mathcal{P}_-$. Hence for all $z \in V_0$, $g_n(z) \to 0$ as $n \to \infty$.

Since the sequence (g_n) forms a normal family, g_n converges uniformly to the constant map $z \mapsto 0$. In particular, we see that $g_n(V_0) = V_n \subset \mathcal{P}_- \cup \mathcal{P}_+ \cup \{0\}$ for all n large enough. Since $V = V_0 \cap \mathcal{P}_+ = \emptyset$ and $f(\mathcal{P}_+) \subset \mathcal{P}_+$ we see that $V_n \cap \mathcal{P}_+ = \emptyset$ for all $n \in \mathbb{Z}_{\geq 0}$. Also, $0 \notin V_n$ for all $n \in \mathbb{Z}_{\geq 0}$. Hence for all n large enough $V_n \subset \mathcal{P}_-$.

Taking such a sufficiently large n, we have that $\tilde{\Psi}_-|_U : U \to V$ is given by

$$\tilde{\Psi}_-|_U = f^{\circ n}|_{V_n} \circ (\tilde{\Phi}_-|_{\mathcal{P}_-})^{-1} \circ T_{-n}|_U.$$

$\tilde{\Phi}_-$, T_{-n} and $f^{\circ n}|_{V_n}$ are biholomorphic onto their images, hence $\tilde{\Psi}_-|_U : U \to V$ is biholomorphic. Since U was arbitrary, V is evenly covered and thus, since z_1 was arbitrary, $\tilde{\Psi}_- : (\tilde{\Psi}_-)^{-1}(\mathcal{A} \setminus \mathcal{O}^+) \to \mathcal{A} \setminus \mathcal{O}^+$ is a covering map. $\square$

Let f be a holomorphic map with a simple parabolic fixed point at 0 with a proper parabolic basin. A *lifted horn map* is defined to be the composition $\tilde{h}_f := \tilde{\Phi}_+ \circ \tilde{\Psi}_- : \tilde{\mathcal{A}} \to \mathbb{C}$. To understand how this map relates to the dynamics of f we first need to understand exactly how the extended Fatou coordinates relate to the dynamics.

To understand the inverse repelling Fatou coordinate, let us first consider $w \in \tilde{\mathcal{A}}$ and consider the image of the $\mathbb{Z}$-orbit of w under $\tilde{\Psi}_-$, $\{\tilde{\Psi}_-(w+n) \mid n \in \mathbb{Z}\}$. We can assume without loss of generality that $w \in \tilde{\Phi}_-(\mathcal{P}_-)$, so that $f^{\circ n} \circ \tilde{\Psi}_-(w)$ is well-defined for all n by letting $f^{\circ n} = (f|_{\mathcal{P}_-}^{-1})^{\circ -n}$ when $n < 0$.

Then $\{\tilde{\Psi}_-(w+n) \mid n \in \mathbb{Z}\} = \{f^{\circ n}(\tilde{\Psi}_-(w)) \mid n \in \mathbb{Z}\}$ is a bi-infinite orbit under f which tends to 0 as $n \to -\infty$. Distinct $\mathbb{Z}$-orbits map to distinct such bi-infinite orbits and $\tilde{\Psi}_-(w) = \tilde{\Psi}_-(w')$ if and only if the bi-infinite orbit corresponding to $\{w+n\}$ and that corresponding to $\{w'+n'\}$ agree at $n = n' = 0$.

For the attracting Fatou coordinate, suppose that $\tilde{\Phi}_+(z) = \tilde{\Phi}_+(z')$. Then by definition there exist n, n' such that

$$f^{\circ n}(z), f^{\circ n'}(z') \in \mathcal{P}_+ \text{ and } T_{-n} \circ \tilde{\Phi}_+|_{\mathcal{P}_+} \circ f^{\circ n}(z) = T_{-n'} \circ \tilde{\Phi}_+|_{\mathcal{P}_+} \circ f^{\circ n'}(z).$$

By taking a maximum of the two we can assume without loss of generality that $n = n'$. Since T_{-n} and $\tilde{\Phi}_+|_{\mathcal{P}_+}$ are injective, we see that $f^{\circ n}(z) = f^{\circ n}(z')$.

We therefore observe that $\tilde{\Phi}_+$ maps *grand* orbits under f to $\mathbb{Z}$-orbits, with

the set of grand orbits of f contained in $\mathcal{A}$ being mapped bijectively to the set of $\mathbb{Z}$-orbits in $\mathbb{C}$.

Putting the two together, we have that $\tilde{h}_f$ maps a $\mathbb{Z}$-orbit contained in $\tilde{\mathcal{A}}$, $\{w+n\}$, to the $\mathbb{Z}$-orbit in $\mathbb{C}$ which corresponds to the grand orbit under f containing $\{\tilde{\Psi}_-(w+n)\}$. Two distinct $\mathbb{Z}$-orbits contained in $\tilde{\mathcal{A}}$ are mapped to the same $\mathbb{Z}$-orbit in $\mathbb{C}$ if and only if their images under $\tilde{\Psi}_-$ lie in the same grand orbit, which is the case if and only if their images eventually agree.

The covering properties of $\tilde{h}_f$ follow from the covering properties of $\tilde{\Phi}_+$ and $\tilde{\Psi}_-$. By lemmas 2.2.1 and 2.2.2, the restriction

$$\tilde{h}_f = \tilde{\Phi}_+ \circ \tilde{\Psi}_- : \tilde{\Psi}_-^{-1}\left(\tilde{\Phi}_+^{-1}\left(\tilde{\Phi}_+\left(\mathcal{A} \setminus (\mathcal{O}^+ \cup \mathcal{O}^-)\right)\right)\right) \to \tilde{\Phi}_+\left(\mathcal{A} \setminus (\mathcal{O}^+ \cup \mathcal{O}^-)\right)$$

is a covering map.

The critical value set of $\tilde{\Psi}_-$ is $\mathcal{O}^+$ and the critical value set of $\tilde{\Phi}_+$ is $\tilde{\Phi}_+(\mathcal{O}^-)$. Thus the critical value set of $\tilde{h}_f$ is $\tilde{\Phi}_+(\mathcal{O}^+) \cup \tilde{\Phi}_+(\mathcal{O}^-)$, and this is equal to the singular value set of $\tilde{h}_f$.

That is, $w' \in \mathbb{C}$ is a critical value of $\tilde{h}_f$ if and only if $w'+n$ is the image of a critical point of f under $\tilde{\Phi}_+$ for some $n \in \mathbb{Z}$. Also, $w \in \tilde{\mathcal{A}}$ is a critical point of $\tilde{h}_f$ if and only if the bi-infinite orbit under f, $\{\tilde{\Psi}_-(w+n)\}$, contains a critical point of f.

In summary, we have the following lemma.

Lemma 2.2.3.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ be a holomorphic map with a simple parabolic fixed point at 0 with a proper parabolic basin. Let $\tilde{\Phi}_\pm$ be Fatou coordinates of f and let $\tilde{h}_f$ be the lifted horn map $\tilde{\Phi}_+ \circ \tilde{\Psi}_-$.

Then $\tilde{h}_f$ is defined on a domain $\tilde{\mathcal{A}}$, which contains both an upper and lower half plane. The critical value set of $\tilde{h}_f$ is equal to the image of the grand orbits of the critical points of f in $\tilde{\Phi}_+$, and $\tilde{h}_f$ has no other singular values.

2.2.2 The Horn Map

In this subsection we will transfer the previous discussion of the lifted horn map to the cylinders $\mathcal{C}_\pm$. The lifted horn map induces a map called the *horn map* $\hat{h}_f$ between the cylinders, otherwise known as the *Écalle-Voronin invariant* of the parabolic fixed point.

We shall use hats to represent objects on the cylinders, in contrast to our previous tildes to show objects on the plane. Thus $\hat{\Phi}_\pm : \mathcal{P}_\pm \to \mathcal{C}_\pm$ are the quotient maps $\mathcal{P}_\pm \ni z \mapsto [z] \in \mathcal{C}_\pm$ and are equal to the compositions $\pi_\pm \circ \hat{\Phi}_\pm$, $\hat{\mathcal{A}} = \pi(\tilde{\mathcal{A}})$ and so on.

As the latter name suggests, the horn map is invariant under local analytic conjugacy, in the sense that if f and g are defined and holomorphic on a neighbourhood of 0, which is a simple parabolic fixed point of both, and χ is a local analytic conjugacy from $(f, 0)$ to $(g, 0)$ then there exist natural isomorphism $\phi_\pm : \mathcal{C}_{\pm,f} \to \mathcal{C}_{\pm,g}$ such that $\hat{h}_g \circ \phi_- = \phi_+ \circ \hat{h}_f$.

The converse is also true; a pair of isomorphisms $\phi_\pm : \mathcal{C}_{\pm,f} \to \mathcal{C}_{\pm,g}$ which preserve $\pm i\infty$ and satisfy $\hat{h}_g \circ \phi_- = \phi_+ \circ \hat{h}_f$ induce a local analytic conjugacy from $(f, 0)$ to $(g, 0)$. We shall give a proof of this statement at the end of the section.

There are essentially two ways one could define the horn map, either by directly defining where points are mapped or as the map induced by the lifted horn map on the cylinders. The former has the advantage of being more clearly independent of any choice of Fatou coordinate whilst the latter is more clearly well-defined and holomorphic.

Definition 2.2.4 (Figure 2.4)**.** Let f be a holomorphic map with a simple parabolic fixed point at 0 with a proper parabolic basin.

We define the *horn map*, $\hat{h} = \hat{h}_f : \hat{\mathcal{A}}_f \to \mathcal{C}_+$ by the relation

$$\hat{h}_f(\hat{\Phi}_-(z)) = \hat{\Phi}_+(f^{\circ m(z)}(z)),$$

where $m(z) \in \mathbb{Z}_{\geq 0}$ is chosen large enough that the right side is defined.

Lemma 2.2.5.
Let f be a holomorphic map with a simple parabolic fixed point at 0 with a proper parabolic basin.

Let $\tilde{h}_f$ be a lifted horn map and $\hat{h}_f$ be the horn map. Let $\pi_\pm : \mathbb{C} \to \mathcal{C}_\pm$ be the quotient maps induced by the choice of Fatou coordinates.

Then $\hat{h}_f \circ \pi_- = \pi_+ \circ \tilde{h}_f$.

Thus $\hat{h}_f$ is a well-defined holomorphic map.

Before proceeding to the proof we would like to emphasise the choice of articles. Fatou coordinates are unique only up to post-composition by a translation. Therefore if $\tilde{h}_f$ is *a* lifted horn map then so is $T_+ \circ \tilde{h}_f \circ T_-$ for any translations $T_\pm : \mathbb{C} \to \mathbb{C}$.

By contrast, *the* horn map is defined on a subset of the quotient space $\mathcal{P}_-/f$. As our definition showed, there is no choice being made here and so we may speak of *the* horn map. Note however that if we wanted to define horn maps between standard cylinders, such as $\mathbb{C}/\mathbb{Z}$ or $\mathbb{C}^*$, we would have an ambiguity in the isomorphisms $\mathcal{C}_\pm \to \mathbb{C}/\mathbb{Z}$ which would return us to talking about *a* horn map.

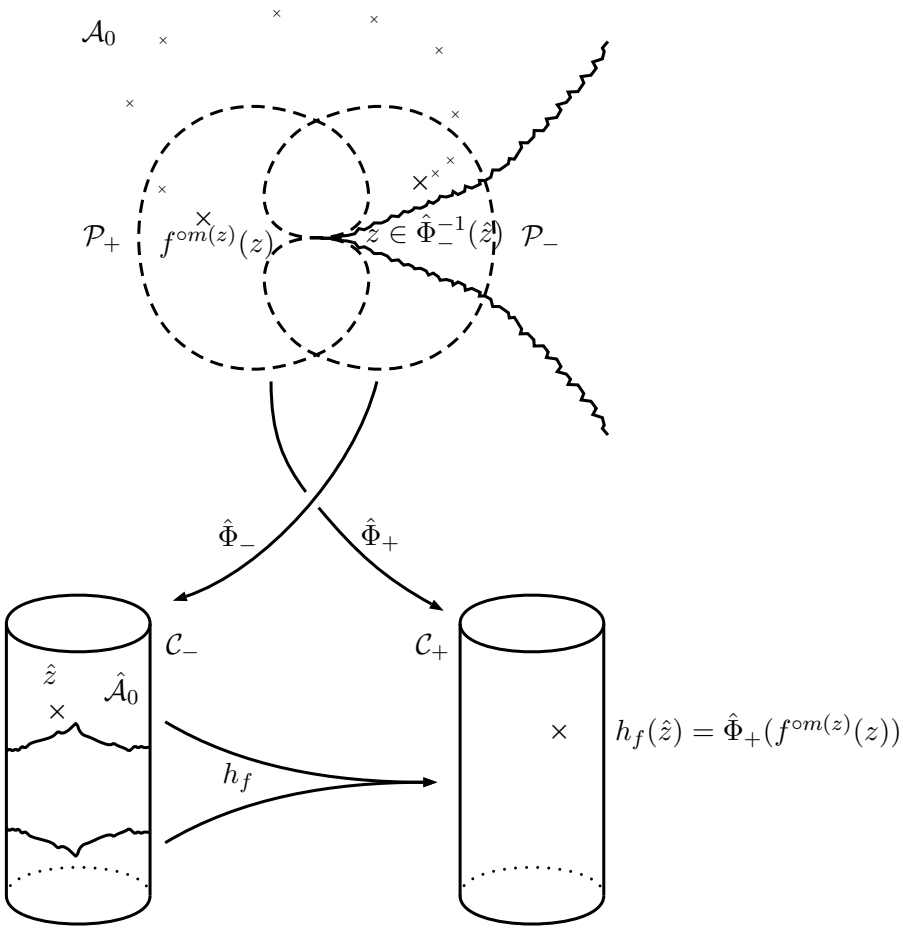

Figure 2.4: An illustration of definition 2.2.4. Given a point $\hat{z} \in \hat{\mathcal{A}}$ let $z \in (\hat{\Phi}_-)^{-1}(\hat{z})$. Taking $m(z)$ sufficiently large, $f^{\circ m(z)}(z) \in \mathcal{P}_+$ and so $\hat{\Phi}_+(f^{\circ m(z)})(z)$ is defined.

Proof. Let $w \in \tilde{\mathcal{A}}$ and let $n \in \mathbb{Z}$ such that $w + n \in \tilde{\Phi}_-(\mathcal{P}_-)$. Let $m \in \mathbb{Z}_{\geq 0}$ such that $f^{\circ m}(\tilde{\Psi}_-(w+n)) \in \mathcal{P}_+$. Then we have that

$$\begin{aligned}
\hat{h}_f \circ \pi_-(w) &= \hat{h}_f \circ \pi_-(w+n) \\
&= \hat{\Phi}_+ \circ f^{\circ m}(\tilde{\Psi}_-(w+n)) \\
&= \pi_+ \circ T_m \circ \tilde{\Phi}_+ \circ \tilde{\Psi}_-(w+n) \\
&= \pi_+ \circ T_m \circ \tilde{h}_f \circ T_n(w) \\
&= \pi_+ \circ T_{m+n} \circ \tilde{h}_f(w) \\
&= \pi_+ \circ \tilde{h}_f(w).
\end{aligned}$$

Here the first and last equalities follow from $\pi_\pm \circ T_1 = \pi_\pm$, the second and fourth are the definitions of $\hat{h}_f$ and $\tilde{h}_f$ respectively and the third and fifth follow from the

fact that $\tilde{\Phi}_\pm \circ f = T_1 \circ \tilde{\Phi}_\pm$.

Since $\hat{h}_f$ is the descent of the holomorphic map $\tilde{h}_f$ under the holomorphic covering maps $\pi_\pm$, $\hat{h}_f$ is well-defined and holomorphic. $\square$

We now study the covering properties of the horn map when f has a proper parabolic basin. To this end we introduce the concept of a finite type map.

Definition 2.2.6. Let W, X be complex 1-manifolds and let $f : W \to X$ be a holomorphic map.

We say that f is of *finite type* if

- f is nowhere locally constant.
- f has no isolated removable singularities. That is, for any holomorphic map $\phi : \mathbb{D}^* \to W$ such that ϕ does not extend analytically to a map $\phi : \mathbb{D} \to W$, $f \circ \phi$ does not extend analytically to a map $f \circ \phi : \mathbb{D} \to X$.
- The singular value set of f, $S(f)$, is finite.
- X is a finite union of compact Riemann surfaces.

If X is connected, we define the degree, $\deg f$ to be $\#f^{-1}(x)$, which is independent of $x \in X \setminus S(f)$.

By our current definitions, the horn map $\hat{h}_f : \hat{\mathcal{A}} \to \mathcal{C}_+$ cannot possibly be a finite type map since $\mathcal{C}_+ \cong \mathbb{C}/\mathbb{Z}$ is not compact. To rectify this we compactify both cylinders $\mathcal{C}_\pm$ by adding the ends $\pm i\infty$, so that now $\mathcal{C}_\pm \cong \hat{\mathbb{C}}$.

Note that if $\mathcal{P}_\pm$ are fat petals then $\hat{\Phi}_\pm(\mathcal{P}_+ \cap \mathcal{P}_-)$ contains a pair of punctured neighbourhoods of $\pm i\infty$. Hence the horn map extends continuously, thus holomorphically, to $\pm i\infty$ by setting $\hat{h}_f(\pm i\infty) = \pm i\infty$.

Theorem 2.2.7.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ be a holomorphic map with a simple parabolic fixed point with a proper parabolic basin at 0. Let $\hat{h}_f : \hat{\mathcal{A}} \to \mathcal{C}_+$ be the horn map of f.

Then h_f is a finite type map.

Proof. By lemma 2.2.3, the critical value set of $\tilde{h}_f$ is

$$\tilde{C} := \{\tilde{c} + n \mid \tilde{\Phi}_+^{-1}(\tilde{c}) \text{ is contained in a critical grand orbit, and } n \in \mathbb{Z}\}$$

and $\tilde{h}_f$ has no other singular values. Since 0 has a proper parabolic basin, $f|_{\mathcal{A}}$ has finitely many critical points and hence finitely many critical grand orbits.

Thus we can find a finite set $\tilde{C}_0 = \{\tilde{c}_1, \ldots, \tilde{c}_m\} \subset \mathbb{C}$ such that

$$\tilde{C} = \{\tilde{c} + n \mid \tilde{c} \in \tilde{C}_0 \text{ and } n \in \mathbb{Z}\}.$$

Therefore $\pi_+(\tilde{C}) = \pi_+(\tilde{C}_0) = \{\pi_+(\tilde{c}_j)\} \subset \mathcal{C}_+$ is a finite set and $\pi_+^{-1}(\pi_+(\tilde{C})) = \tilde{C}$.

We now have the following commutative diagram.

$$\begin{tikzcd}
\tilde{\mathcal{A}} \setminus (\tilde{h}_f^{-1}(\tilde{C})) \arrow[r, "\tilde{h}_f"] \arrow[d, "\pi_-"'] & \mathbb{C} \setminus \tilde{C} \arrow[d, "\pi_+"] \\
\hat{\mathcal{A}} \setminus \pi_-(\tilde{h}_f^{-1}(\tilde{C})) \arrow[r, "\hat{h}_f"] & \mathcal{C}_+ \setminus \pi_+(\tilde{C})
\end{tikzcd} .$$

In this diagram, $\pi_\pm$ and $\tilde{h}_f|_{\tilde{\mathcal{A}}\setminus(\tilde{h}_f^{-1}(\tilde{C}))}$ are covering maps. It follows that $\hat{h}_f|_{\hat{\mathcal{A}}\setminus\pi_-(\tilde{h}_f^{-1}(\tilde{C}))}$ is a covering map. Hence the singular value set of $\hat{h}_f|_{\hat{\mathcal{A}}\setminus\{\pm i\infty\}}$ is contained in $\pi_+(\tilde{C})$ and thus the singular value set of $\hat{h}_f$ is contained in $\pi_+(\tilde{C}) \cup \{\pm i\infty\}$, since $\hat{h}_f^{-1}(\{\pm i\infty\}) = \{\pm i\infty\}$.

$\mathcal{C}_+$ is compact and $\hat{h}_f$ is nowhere locally constant. By our extension to $\pm i\infty$, $\hat{h}_f$ has no isolated removable singularities. Hence $\hat{h}_f$ is a finite type map. □

Amongst the properties of finite type maps, one that is of particular interest to us is that their domains of definition are maximal. As we see in lemma 2.2.9, this maximality is stronger than simply not having an analytic extension: it isn't even possible to extend a finite type map continuously to a point on its boundary. For our purposes, this tells us that the domains $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ are maximal for lifted horn maps and the horn map respectively.

Lemma 2.2.8 ([7, Page 91, Proposition 10])**.**
Let Y be a complex 1-manifold, $W \subset Y$ be open and let $f : W \to X$ be a finite type map.

Let $U \subseteq Y$ be a connected open set with $U \cap \partial W \neq \emptyset$ and let $V \subset X$ be a Jordan domain with $\overline{V} \cap S(f) = \emptyset$.

Then V has infinitely many proper preimages compactly contained in U.

Lemma 2.2.9.
Let Y be a complex 1-manifold, $W \subset Y$ be open let $f : W \to X$ be a finite type map.

Then f does not have a continuous extension to any point $y \in \partial W$. Hence W is a maximal domain of f.

Proof. Let $y \in \partial W$ and let (U_n) be a sequence of connected open sets with $\bigcap_n U_n = \{y\}$.

Let x_1 and x_2 be distinct points in X and let $V_1 \ni x_1$ and $V_2 \ni x_2$ be Jordan domains. By lemma 2.2.8, for each $j \in 1, 2$ and each n, V_j has a proper preimage under f contained in U_n, hence $f^{-1}(x_j) \cap U_n \neq \emptyset$. Thus we can find sequences $(y_{j,n})$ such that $y_{j,n} \in f^{-1}(x_j) \cap U_n$ for all j, n.

Then $(y_{1,n})$ is a sequence with $y_{1,n} \to y$ and $f(y_{1,n}) \to x_1$ as $n \to \infty$, whilst $(y_{2,n})$ is a sequence with $y_{2,n} \to y$ and $f(y_{2,n}) \to x_2$ as $n \to \infty$.

By assumption, $x_1 \neq x_2$. Hence f has no continuous extension to y. □

Proposition 2.2.10.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ be a holomorphic map with a simple parabolic fixed point with a proper parabolic basin at 0.

Then the domain $\hat{\mathcal{A}} \subset \mathcal{C}_-$ is a maximal domain of holomorphy of the horn map $\hat{h}_f$ and a domain $\tilde{\mathcal{A}} \subset \mathbb{C}$ is a maximal domain of holomorphy of a corresponding lifted horn map $\tilde{h}_f$.

Proof. By theorem 2.2.7, $\hat{h}_f : \hat{\mathcal{A}} \to \mathcal{C}_+$ is a finite type map. Hence, by lemma 2.2.9, $\hat{\mathcal{A}}$ is a maximal domain of holomorphy for $\hat{h}_f$.

By lemma 2.2.5, $\hat{h}_f \circ \pi_- = \pi_+ \circ \tilde{h}_f$. Suppose that $\tilde{h}_f$ has an analytic extension to a neighbourhood of a point $w \in \partial\tilde{\mathcal{A}}$. Let $\hat{w} = \pi_-(w)$, noting that by the definition of $\hat{\mathcal{A}}$, $\hat{w} \in \partial\hat{\mathcal{A}}$. Let $\hat{U}$ be a neighbourhood of $\hat{w}$ which is evenly covered by π_- and let $U \subset (\pi_-)^{-1}(\hat{U})$ be the connected component which contains w.

Without loss of generality, we can assume that $\tilde{h}_f$ extends analytically to U. But then $\hat{h}_f$ has an analytic extension to $\hat{U}$, as $\hat{h}_f|_{\hat{U}} = \pi_+ \circ \tilde{h}_f|_U \circ (\pi_-|_U)^{-1}$. This contradicts the maximality of $\hat{\mathcal{A}}$. Hence $\tilde{\mathcal{A}}$ is a maximal domain for $\tilde{h}_f$. □

We conclude this section with a statement and proof of the result that the horn map, up to an equivalence relation, uniquely determines the local analytic conjugacy class of a simple parabolic germ. This result was originally independently proven by Écalle and Voronin.

The version we present here is somewhat restricted by our choice to define the horn map and lifted horn maps only for maps with a proper parabolic basin. Écalle and Voronin's original results applied to arbitrary parabolic germs by defining the horn map only on a neighbourhood of $\{\pm i\infty\}$. The same proof strategy would work in this more general case, we would simply need to be more careful about domains of definition.

Theorem 2.2.11.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ and $g : \mathcal{D}(g) \to \hat{\mathbb{C}}$ be holomorphic maps with simple parabolic

fixed points at 0 *and proper parabolic basins. Let* $\tilde{h}_f$ *and* $\tilde{h}_g$ *be lifted horn maps of* f *and* g *respectively and let* $\hat{h}_f$ *and* $\hat{h}_g$ *be the horn maps of* f *and* g *respectively.*

Let $\tilde{\mathcal{A}}_f^0$ *denote the union of the connected components of* $\tilde{\mathcal{A}}_f$ *which contain an upper or lower half-plane and let* $\hat{\mathcal{A}}_f^0$ *be the image of this set in* $\pi_{-,f}$*, and similarly for* g*.*

Then the following are equivalent:

i There exists a local analytic conjugacy χ *from* $(f, 0)$ *to* $(g, 0)$*.*

ii There exist translations $T_\pm : \mathbb{C} \to \mathbb{C}$ *which satisfy* $T_+ \circ \tilde{h}_f = \tilde{h}_g \circ T_-$ *on* $\tilde{\mathcal{A}}_f^0$*.*

iii There exist conformal isomorphisms $\hat{\phi}_\pm : \mathcal{C}_{\pm,f} \to \mathcal{C}_{\pm,g}$ *with* $\hat{\phi}_s(\pm i\infty) = \pm i\infty$ *for all* $s \in \{\pm\}$ *which satisfy* $\hat{\phi}_+ \circ \hat{h}_f = \hat{h}_g \circ \hat{\phi}_-$ *on* $\hat{\mathcal{A}}_f^0$*.*

In the statement and proof of this theorem, the subscripts denote the map to which the objects are related. Thus $\tilde{\Phi}_{\pm,f}$ are Fatou coordinates of f whilst $\tilde{\Phi}_{\pm,g}$ are Fatou coordinates of g and so on.

Proof. To show that i) implies ii), suppose that a local analytic conjugacy χ exists. Let $\tilde{\Phi}_{\pm,f}$ and $\tilde{\Phi}_{\pm,g}$ be Fatou coordinates which induce $\tilde{h}_f$ and $\tilde{h}_g$. Then $\tilde{\Phi}_{\pm,f}$ and $\tilde{\Phi}_{\pm,g} \circ \chi$ are both sets of Fatou coordinates of f, which we can assume without loss of generality are defined on the same fat petals $\mathcal{P}_{\pm,f}$. Hence by theorem 2.1.15 there exist translations $T_\pm : \mathbb{C} \to \mathbb{C}$ which satisfy $T_\pm \circ \tilde{\Phi}_{\pm,f}|_{\mathcal{P}_{\pm,f}} = \tilde{\Phi}_{\pm,g} \circ \chi|_{\mathcal{P}_{\pm,f}}$.

Then we have that on $\tilde{\Phi}_{-,f}(\mathcal{P}_{+,f} \cap \mathcal{P}_{-,f}) \subseteq \tilde{\mathcal{A}}_f^0$

$$\begin{aligned}
\tilde{h}_g \circ T_- =& \tilde{\Phi}_{+,g} \circ \tilde{\Psi}_{-,g} \circ T_- \\
=& \tilde{\Phi}_{+,g} \circ \chi \circ \tilde{\Psi}_{-,f} \\
=& T_+ \circ \tilde{\Phi}_{+,f} \circ \tilde{\Psi}_{-,f} = T_+ \circ \tilde{h}_f.
\end{aligned}$$

The claim on all of $\tilde{\mathcal{A}}_f^0$ follows from the Uniqueness Theorem and the maximality of $\tilde{\mathcal{A}}_f^0$ by proposition 2.2.10.

Conversely, to show that ii) implies i), if $T_\pm$ exist then let us define conjugacies on the petals, $\chi_\pm : \mathcal{P}_{\pm,f} \to \mathcal{P}_{\pm,g}$, by $\chi_\pm = (\tilde{\Phi}_{\pm,g})^{-1} \circ T_\pm \circ \tilde{\Phi}_{\pm,f}$. Note first that $\chi_\pm \circ f = g \circ \chi_\pm$ wherever both sides are defined. Assuming that we have chosen fat

petals for f, we observe that if $z \in \mathcal{P}_{+,f} \cap \mathcal{P}_{-,f}$ then

$$\begin{aligned}
\chi_+(z) =& (\tilde{\Phi}_{+,g})^{-1} \circ T_+ \circ \tilde{\Phi}_{+,f}(z) \\
=& (\tilde{\Phi}_{+,g})^{-1} \circ T_+ \circ \tilde{\Phi}_{+,f} \circ \tilde{\Psi}_{-,f} \circ \tilde{\Phi}_{-,f}(z) \\
=& (\tilde{\Phi}_{+,g})^{-1} \circ T_+ \circ \tilde{h}_f \circ \tilde{\Phi}_{-,f}(z) \\
=& (\tilde{\Phi}_{+,g})^{-1} \circ \tilde{h}_g \circ T_- \circ \tilde{\Phi}_{-,f}(z) \\
=& (\tilde{\Phi}_{+,g})^{-1} \circ \tilde{\Phi}_{+,g} \circ \tilde{\Psi}_{-,g} \circ T_- \circ \tilde{\Phi}_{-,f}(z) \\
=& (\tilde{\Phi}_{-,g})^{-1} \circ T_- \circ \tilde{\Phi}_{-,f}(z) = \chi_-(z).
\end{aligned}$$

Hence $\chi_+|_{\mathcal{P}_{+,f} \cap \mathcal{P}_{-,f}} = \chi_-|_{\mathcal{P}_{+,f} \cap \mathcal{P}_{-,f}}$ and so we obtain a well-defined holomorphic map $\chi : \mathcal{P}_{+,f} \cup \mathcal{P}_{-,f} \to \mathcal{P}_{+,g} \cup \mathcal{P}_{-,g}$ with $\chi|_{\mathcal{P}_{\pm,f}} = \chi_\pm$. χ extends continuously, hence holomorphically, to 0 by $\chi(0) = 0$. Hence χ is a local analytic conjugacy from $(f, 0)$ to $(g, 0)$.

To show that ii) implies iii), recall that by lemma 2.2.5, $\pi_{+,f} \circ \tilde{h}_f = \hat{h}_f \circ \pi_{-,f}$ and similarly for g. Suppose that $T_\pm$ exist. Since the deck transformations of $\pi_{\pm,f}$ and $\pi_{\pm,g}$ are integer translations, which $T_\pm$ commute with, $T_\pm$ descend to isomorphisms $\hat{\phi}_\pm : \mathcal{C}_{\pm,f} \to \mathcal{C}_{\pm,g}$ which satisfy $\pi_{\pm,g} \circ T_\pm = \hat{\phi}_\pm \circ \pi_{\pm,f}$.

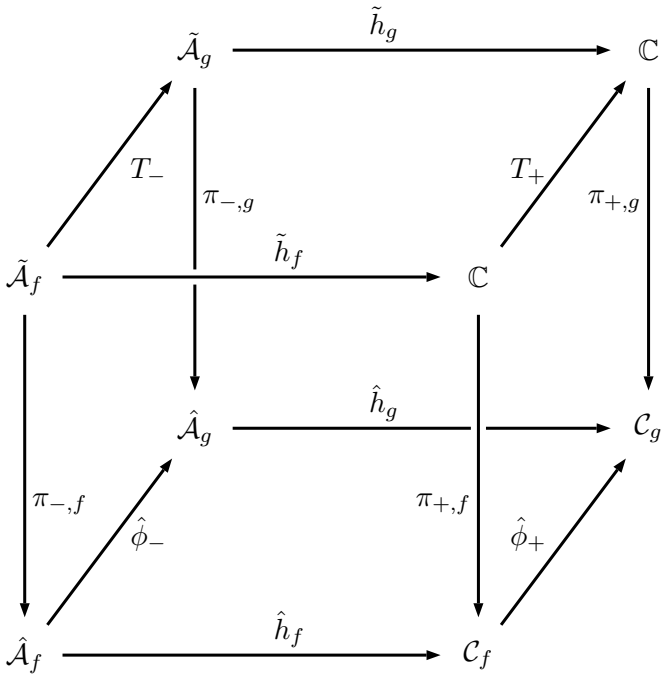

Figure 2.5: A diagram of showing the relations between the horn maps, lifted horn maps, projections and the maps $T_\pm$ and $\hat{\phi}_\pm$. The front and back faces commute by lemma 2.2.5. Given one of $T_\pm$ or $\hat{\phi}_\pm$, we see as above that we can choose the other so that the side faces commute. Diagram chasing, making use of the face that the π's are covering maps, then shows that the top face commutes if and only if the bottom face commutes.

Let $U \subset \tilde{\mathcal{A}}_f^0$ be a small open set such that $\pi_{-,f}|_U$ is biholomorphic onto its

image. Then

$$\begin{aligned}
\hat{\phi}_+ \circ \hat{h}_f|_{\pi_{-,f}(U)} =& \hat{\phi}_+ \circ \hat{h}_f \circ \pi_{-,f} \circ (\pi_{-,f}|_U)^{-1} \\
=& \hat{\phi}_+ \circ \pi_{+,f} \circ \tilde{h}_f \circ (\pi_{-,f}|_U)^{-1} \\
=& \pi_{-,g} \circ T_+ \circ \tilde{h}_f \circ (\pi_{-,f}|_U)^{-1} \\
=& \pi_{-,g} \circ \tilde{h}_g \circ T_- \circ (\pi_{-,f}|_U)^{-1} \\
=& \hat{h}_g \circ \pi_{+,g} \circ T_- \circ (\pi_{-,f}|_U)^{-1} \\
=& \hat{h}_g \circ \hat{\phi}_- \circ \pi_{-,f} \circ (\pi_{-,f}|_U)^{-1} \\
=& \hat{h}_g \circ \hat{\phi}_-|_{\pi_{-,f}(U)}.
\end{aligned}$$

Since U was arbitrary and $\pi_{-,f}$ is a covering map and so surjective, $\phi_+ \circ \hat{h}_f = \hat{h}_g \circ \hat{\phi}_-$ on $\hat{\mathcal{A}}_f^0$.

Finally, to show that iii) implies ii), suppose that $\hat{\phi}_\pm$ exist. Then there exist lifts, $T_\pm$, of $\hat{\phi}_\pm$ under $\pi_{\pm,f}$ and $\pi_{\pm,g}$. By essentially the same diagram chase as in the previous case, it follows that $T_+ \circ \tilde{h}_f = \tilde{h}_g \circ T_-$ on $\tilde{\mathcal{A}}_f^0$. $\square$

Chapter 3

Basin Semi-conjugacies

3.1 Extending Local Analytic Conjugacies

In this chapter we will show how we can extend a local conjugacy between parabolic fixed points of rational maps with simply connected basins to give a semi-conjugacy between the immediate basins. As we remarked in the introduction, the ability to perform such an extension is in sharp contrast to the case with attracting or superattracting fixed points.

Intuitively, if χ is a local analytic conjugacy, defined from a neighbourhood of a fixed point of f to a neighbourhood of a fixed point of g, we can try to extend χ to the basin by the relation $\chi(z) = g^{\circ -m} \circ \chi \circ f^{\circ m}$. However in the attracting or superattracting case we find that we typically can't extend past a preimage of a critical value of g. This is because the relation of being locally analytically conjugate carries no information about the critical orbits of the two maps. Other than special cases, such as conjugation to $z \mapsto \lambda z$ or $z \mapsto z^d$ when we know that g has no critical points in the basin other than the fixed point, we then expect to run into an obstructing critical point and so not be able to obtain a semi-conjugacy on the whole basin.

In contrast to these cases, we saw in theorem 2.2.11 that the horn map and lifted horn maps up to equivalence are local analytic conjugacy invariants. We also saw in lemmas 2.2.1 and 2.2.2, applied in theorem 2.2.7, that the critical values of the horn map are related to critical values of f. We therefore expect to see a relation between the image of the critical orbits under f in χ and the critical orbits under g.

On the other hand, since $\hat{\Phi}_{\pm,f} \circ f = \hat{\Phi}_{\pm,f}$, a horn map equivalence between $\hat{h}_f$ and $\hat{h}_g$ can only determine a local analytic conjugacy up to post-composition by iterates of g. We therefore expect to need to post-compose a local analytic conjugacy

χ by iterates of g in order to have the critical values 'line up'.

In practice rather using the heuristic $\chi_n = g^{\circ n-m} \circ \chi \circ f^{\circ m}$ to perform an analytic extension, we instead consider the pair of functions

$$\chi_n = \begin{cases} (\tilde{\Phi}_{+,g})^{-1} \circ T_+ \circ T_n \circ \tilde{\Phi}_{+,f} \\ \tilde{\Psi}_{-,g} \circ T_- \circ T_n \circ (\tilde{\Psi}_{-,f})^{-1} \end{cases},$$

which are initially both defined on a small region in the intersection of the attracting and repelling petals. By choosing n large enough, equivalent to post-composing χ with a sufficient number of iterates of g, we can ensure that every point of $\mathcal{A}_f^0$ has a neighbourhood on which at least one of these definitions is suitably well-defined. We then perform analytic continuation along arbitrary curves and apply the Monodromy Theorem to obtain the desired result.

Theorem 3.1.10.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ and $g : \mathcal{D}(g) \to \hat{\mathbb{C}}$ be holomorphic maps with simple parabolic fixed points at 0 with proper parabolic basins.

Suppose that χ is a local analytic conjugacy from $(f, 0)$ to $(g, 0)$. Suppose also that the immediate attracting basin $\mathcal{A}_f^0$ is simply connected.

Then there exists $N \in \mathbb{Z}$ such that for all $n \geq N$, $\chi_n := g^{\circ n} \circ \chi|_{\mathcal{A}_f^0}$ extends to an analytic semi-conjugacy from $f|_{\mathcal{A}_f^0}$ to $g|_{\mathcal{A}_g^0}$.

3.1.1 Curves and Analytic Extensions

We begin with a brief overview of the theory of analytic continuation along a curve. The theory behind our definitions, known as sheaf theory, is deep and interesting in its own right but we will only be using a small number of results from it here. The notion of *global analytic functions* provides a concrete way of talking about multi-valued analytic functions and is also very close to our discussion, but beyond the scope of what we need here.

Definition 3.1.1. Let $X \subseteq \hat{\mathbb{C}}$ be a connected domain. Let (f_1, x_1) and (f_2, x_2) be pairs, where $x_j \in X$ and f_j is a holomorphic function defined on some neighbourhood $\mathcal{D}(f_j) \subseteq X$ of x_j.

We say that (f_1, x_1) and (f_2, x_2) are equivalent if and only if $x_1 = x_2$ and $f_1 = f_2$ on a neighbourhood of $x_1 = x_2$. An equivalence class under this relation is called a *germ of an analytic function*, or simply a *germ*. It is customary to use the notation (f, x) to denote the equivalence class, wherever this does not cause confusion.

We denote the set of germs on X by $\mathscr{G}_X$. When topologising $\mathscr{G}_X$ we would like to keep the germs of different holomorphic functions isolated from one another. Our choice of topology will therefore have the property that $\{(f,x) \in \mathscr{G}_X \mid x = x_0\}$ is a discrete set. On the other hand, we would like the evaluation map $(f,x) \mapsto f(x)$ to be continuous.

Let $U \subseteq X$ be a connected open set and let f be a holomorphic function defined on $\mathcal{D}(f) \supseteq U$. We define the set $\Delta_U^f \subset \mathscr{G}_X$ to be $\Delta_U^f := \{(f,x) \mid x \in U\}$. We equip $\mathscr{G}_X$ with the topology generated by all such sets.

As a consequence of our choice of topology, the space $\mathscr{G}_X$ has a natural projection map $\pi : \mathscr{G}_X \to X$ given by $\pi : (f,x) \mapsto x$, which is a local homeomorphism. It should be noted that it is not a covering map; given any domain U we can find a subdomain $U' \subset U$ and a holomorphic function $f : U' \to \hat{\mathbb{C}}$ which does not extend to any point of $\partial U'$. It can then been seen that $\Delta_{U'}^f$ is a connected component of $\mathscr{G}_X$ and its image under π is strictly contained in U, so U is not evenly covered.

We are now ready to give the definition of analytic continuation along a curve. A notable property of this definition is that if the curve γ intersects itself then the value of the continuation need not be the same at both values at the intersection. That is, if $\gamma(t_1) = \gamma(t_2)$ we do *not* require that $\overline{\gamma}(t_1) = \overline{\gamma}(t_2)$.

Definition 3.1.2. Let $\gamma : [0,1] \to X$ be a curve and let $(f, x = \gamma(0))$ be a germ.

An *analytic continuation of* (f,x) *along* γ is a curve $\overline{\gamma} : [0,1] \to \mathscr{G}_X$ such that $\overline{\gamma}(0) = (f,x)$ and $\pi \circ \overline{\gamma} = \gamma$.

One of the most striking properties of holomorphic functions, compared to functions of less smoothness, is the uniqueness of analytic continuation. That is, if $U' \subset U$ are domains in $\hat{\mathbb{C}}$ and $f, g : U \to \hat{\mathbb{C}}$ are holomorphic functions such that $f|_{U'} = g|_{U'}$ then $f = g$. The following theorem tells us that analytic continuation along a curve preserves this uniqueness; given analytic continuations along a curve they either agree along the whole length or never agree.

Note, however, that it is the *germs* which are either equal or completely distinct. It is possible that $f_1(x) = f_2(x)$ but that $(f_1, x) \neq (f_2, x)$ and so $\overline{\gamma}_1(t) \neq \overline{\gamma}_2(t)$ for all $t \in [0,1]$.

Theorem 3.1.3.
Let $\gamma : [0,1] \to X$ be a curve and let (f_1, x) and (f_2, x) be germs at $x = \gamma(0)$.

Let $\overline{\gamma}_1, \overline{\gamma}_2 : [0,1] \to \mathscr{G}_X$ be analytic continuations of (f_1, x) and (f_2, x), respectively, along γ.

Then either $(f_1, x) = (f_2, x)$ and $\overline{\gamma}_1 = \overline{\gamma}_2$ or $\overline{\gamma}_1(t) \neq \overline{\gamma}_2(t)$ for all $t \in [0,1]$.

As a consequence, we may refer to *the* analytic continuation of (f, x) along γ when it exists. This theorem also gives rise to the *principle of the permanence of functional relations.* Suppose that f_1, f_2, g_1, g_2 are holomorphic functions such that the compositions $f_1 \circ g_1$ and $f_2 \circ g_2$ are defined and equal on a neighbourhood of x. Let $\gamma : [0, 1] \to X$ be a curve such that $\gamma(0) = x$. Then if there exists an analytic continuation, $\overline{\gamma}$, of $(f_1 \circ g_1, x)$ along γ then $\overline{\gamma}$ is also the analytic continuation of $(f_2 \circ g_2, x)$ along γ.

The next result we bring up is the Monodromy Theorem, which gives a sufficient condition for the analytic continuations of a germ along two curves with the same end-points to be equal at the ends. A common application of this theorem, which we shall be making ourselves below, is to define an analytic extension of a function to a larger domain.

More explicitly, suppose that f is a holomorphic function defined on a domain U and that $U \subset V$. We could attempt to define an extension of f to V by fixing a base-point $u \in U$. Then, for each $v \in V$, let $\gamma_v : [0, 1] \to V$ be a curve with $\gamma_v(0) = u$ and $\gamma_v(1) = v$. If an analytic continuation $\overline{\gamma}$ of (f, u) along γ exists then we can try to let $(f, v) = \overline{\gamma}_1(1)$. For this to give a well-defined function we would need to know that $\overline{\gamma}_v(1)$ does not depend on the choice of γ_v. The Monodromy Theorem tells us that if there exists an analytic continuation of (f, u) along any curve starting at u and if V is simply connected then this is the case, and so we can define an analytic continuation of f to V.

Theorem 3.1.4 (The Monodromy Theorem)**.**
Let $X \subseteq \mathbb{C}$ be a connected domain. Let $x_0, x_1 \in X$ and let $H : [0, 1] \times [0, 1] \to X$ be a homotopy such that for all $s \in [0, 1]$, $H(0, s) = x_0$ and $H(1, s) = x_1$. For $s \in [0, 1]$, let $\gamma_s : [0, 1] \to X$ be the curve $\gamma_s : t \mapsto H(t, s)$.

Let (f_0, x_0) be a germ at x_0 and suppose that for all $s \in [0, 1]$, (f_0, x_0) has an analytic continuation $\overline{\gamma}_s$ along γ_s.

Then there exists a germ (f_1, x_1) such that $\overline{\gamma}_s(1) = (f_1, x_1)$ for all $s \in [0, 1]$.

Our intention is to perform an analytic continuation of a local analytic conjugacy χ along a curve starting in a petal and contained in the immediate basin. To do so, we will need to know when we can lift a curve γ under the inverse repelling Fatou coordinates.

Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ be a holomorphic map and consider a curve $\gamma : [0, 1] \to \hat{\mathbb{C}}$. A *lift of γ under f* is a curve $\tilde{\gamma} : [0, 1] \to \mathbb{C}$ such that $f \circ \tilde{\gamma} = \gamma$.

Recall the definition of an asymptotic value of f.

Definition 3.1.5. Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ be a holomorphic function.

We say that a point $y \in \hat{\mathbb{C}}$ is an *asymptotic value* of f if there exists a curve $\tilde{\gamma} : [0,1) \to \mathcal{D}(f)$ such that

i $f \circ \tilde{\gamma}(t) \to y$ as $t \to 1$, and

ii for all compact sets $K \subseteq \mathcal{D}(f)$, there exists t_0 such that $\tilde{\gamma}(t) \notin K$ for all $t > t_0$.

In this case, we say that the curve $\gamma(t) \to \infty$ *relative to* $\mathcal{D}(f)$.

Lemma 3.1.6.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ be a holomorphic function and let $\gamma : [0,1] \to \hat{\mathbb{C}}$ be a curve. Suppose that $\gamma([0,1])$ does not intersect the set of asymptotic values of f.

Then for any $x_0 \in f^{-1}(\gamma(0))$ there exists a lift of γ under f, $\tilde{\gamma}$, such that $\tilde{\gamma}(0) = x_0$.

Proof. Let $x_0 \in f^{-1}(\gamma(0))$ be fixed and consider the set

$$S = \left\{ s \in [0,1] \;\middle|\; \begin{array}{c} \text{For all } r \leq s \text{ there exists a lift } \tilde{\gamma} \text{ of } \gamma|_{[0,r]} \text{ with } \tilde{\gamma}(0) = x_0 \\ \text{and every such lift extends to a lift of } \gamma|_{[0,s]} \end{array} \right\}.$$

Note first that since $\{0\} = [0,0]$ and $0 \mapsto x_0$ is a lift of $\gamma|_{[0,0]}$, $0 \in S$ and so S is non-empty. Also S is connected, since if $r < s' < s$ then any lift of $\gamma|_{[0,r]}$ which extends to $[0,s]$ can be restricted to a lift of $\gamma|_{[0,s']}$ and hence $s' \in S$.

Now, suppose that $s \in S$ and let $\tilde{\gamma}$ be a lift of $\gamma|_{[0,s]}$ with $\tilde{\gamma}(0) = x_0$. Let $x = \tilde{\gamma}(s)$ and $y = \gamma(s)$, so that $f(x) = y$. Since f is holomorphic, there exist neighbourhoods $U \ni x$ and $V \ni y$ such that $f : U \backslash x \to V \backslash y$ is a degree $\deg_x(f) < \infty$ covering map. By continuity of γ, there exists a relatively open interval $I \subseteq [0,1]$ of s such that $\gamma(I) \subset V$. Hence there exists a lift $\tilde{\gamma}$ of $\gamma|_I$ such that $\tilde{\gamma}(t) = x$ for all $t \in \gamma^{-1}(y) \cap I$. Therefore $\tilde{\gamma}$ extends to a lift of $\gamma|_{[0,s]\cup I}$ with $\tilde{\gamma}(0) = x_0$ and hence $[0,s] \cup I \subseteq S$, so S is open in $[0,1]$.

Suppose instead that $[0,s) \subseteq S$. Let $s_1 < s_2 < \ldots$ be an increasing sequence of points in S which converges to s. By definition of S, there exists a lift of $\gamma|_{[0,s_1]}$. Moreover, any lift of $\gamma|_{[0,s_k]}$ extends to a lift of $\gamma|_{[0,s_{k+1}]}$. By inductively taking an infinite sequence of such compatible partial lifts we obtain a lift $\tilde{\gamma}$ of $\gamma|_{[0,s)}$.

Let $y = \gamma(s)$, so that $f \circ \tilde{\gamma}(t) \to y$ as $t \to s$. By assumption, y is not an asymptotic value of f, so there exists a compact set $K \subset \mathcal{D}(f)$ with $\tilde{\gamma}(t) \in K$ for all $t \in [0,s)$. Therefore, $\tilde{\gamma}(t)$ has an accumulation point $x \in K$ as $t \to s$ and $f(x) = y$ be continuity.

As before, there exist neighbourhoods $U \ni x$ and $V \ni y$ such that the restriction $f : U \setminus x \to V \setminus y$ is a degree $\deg_x(f) < \infty$ covering map. Further, we

may choose U and V such that U is relatively compact in $\mathcal{D}(f)$, $f(\partial U) = \partial V$ and $f^{-1}(y) \cap \overline{U} = \{x\}$. Since $f \circ \tilde{\gamma}(t) = \gamma(t) \to y$ as $t \to s$, there exists $\epsilon > 0$ with $\gamma(t) \in V$ for all $t \in (s - \epsilon, s)$ and hence $\tilde{\gamma}(t) \in U$ for all $t \in (s - \epsilon, s)$. But $\overline{U}$ is compact and hence the accumulation set of $\tilde{\gamma}(t)$ as $t \to s$ is contained in $f^{-1}(y) \cap \overline{U}$. $\overline{U} \cap f^{-1}(y) = \{x\}$ by assumption, hence $\tilde{\gamma}(t) \to x$ as $t \to s$. Therefore $\tilde{\gamma}$ extends to a lift of $\gamma|_{[0,s]}$ by setting $\tilde{\gamma}(s) = x$ and so $s \in S$, hence S is closed in $[0, 1]$.

S is open and closed in $[0, 1]$, $0 \in S$ so S is non-empty and $[0, 1]$ is connected. Hence $S = [0, 1]$ and so there exists a lift, $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = x_0$. Since $x_0 \in f^{-1}(\gamma(0))$ was arbitrary, the result follows. $\square$

Recall from lemma 2.2.2 that if $\tilde{\Psi}_- : \tilde{\mathcal{A}} \to \mathcal{A} \subset \hat{\mathbb{C}}$ is an extended inverse repelling Fatou coordinate of a parabolic fixed point with a proper parabolic basin then $\tilde{\Psi}_-$ has no asymptotic values. Therefore by lemma 3.1.6, if $\gamma : [0, 1] \to \mathcal{A}$ is a curve with $\gamma(0) \in \mathcal{P}_-$ then there exists a lift of γ under $\tilde{\Psi}_-$, $\tilde{\gamma} : [0, 1] \to \tilde{\mathcal{A}}$, such that $\tilde{\gamma}(0) = \tilde{\Phi}_-(\gamma(0))$.

3.1.2 Analytic Continuation of Local Conjugacies Along Curves

Let us now suppose that $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ and $g : \mathcal{D}(g) \to \hat{\mathbb{C}}$ are holomorphic maps with simple parabolic fixed points at 0 with proper parabolic basins. Suppose that $\chi : (\hat{\mathbb{C}}, 0) \to (\hat{\mathbb{C}}, 0)$ is a local analytic conjugacy from $(f, 0)$ to $(g, 0)$. At this point we will use subscripts to distinguish objects referring to f and those referring to g, so $\tilde{\Phi}_{\pm,f}$ are Fatou coordinates for f and so on.

Let $\mathcal{D}(\chi)$ be an open domain containing 0 on which χ is defined. Then we can find fat petals $\mathcal{P}_{\pm,f}$ of f at 0 such that $\mathcal{P}_{\pm,f} \subset \mathcal{D}(\chi)$. Further, since $\chi : \mathcal{D}(\chi) \to \chi(\mathcal{D}(\chi))$ is a conjugacy from $(f, 0)$ to $(g, 0)$, we see that $\chi(\mathcal{P}_{\pm,f}) =: \mathcal{P}_{\pm,g}$ are fat petals of g at 0. Therefore, in what follows we may assume without loss of generality that χ is well-defined on $\mathcal{P}_{\pm,f}$ and that $\chi(\mathcal{P}_{\pm,f}) = \mathcal{P}_{\pm,g}$.

Our first step in extending χ to a semi-conjugacy is to find out how 'out of step' the orbits of f and g are. We know that points on a critical orbit of f are mapped onto points of a critical orbit of g, since they have the same horn map, but we need to find out where the critical points lie along the two orbits relative to one another.

This key observation is central to the proof of our main result. In particular, the value of N which we obtain from this lemma will be used in Lemma 3.1.8, Proposition 3.1.9 and Theorem 3.1.10.

Lemma 3.1.7 (Key Lemma)**.**
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ and $g : \mathcal{D}(g) \to \hat{\mathbb{C}}$ be holomorphic maps with simple parabolic fixed

points at 0 *with proper parabolic basins. Suppose that* χ *is a local analytic conjugacy from* $(f, 0)$ *to* $(g, 0)$.

Then there exists $N \in \mathbb{Z}$ *with the following property:*
If $\{\ldots, x_{-1}, x_0, x_1, \ldots\} \subset \mathcal{A}_f$ *is a bi-infinite orbit under* f *contained in* $\mathcal{A}_f$ *and* $\{\ldots, x'_{-1}, x'_0, x'_1, \ldots\} \subset \mathcal{A}_g$ *is a bi-infinite orbit under* g *contained in* $\mathcal{A}_g$ *such that for some* $n_+ \in \mathbb{Z}$ *and all* $n \geq n_+$ *it is the case that* $\chi_N(x_n) = x'_n$, *where* $\chi_N := g^{\circ N} \circ \chi$, *then*

i if x_{n_0} *is a critical point of* f *then* $\deg_{x'_n}(g) = 1$ *for all* $n \geq n_0$,

ii if $x'_{n'_0}$ *is a critical point of* g *then* $\deg_{x_n}(f) = 1$ *for all* $n \leq n'_0$.

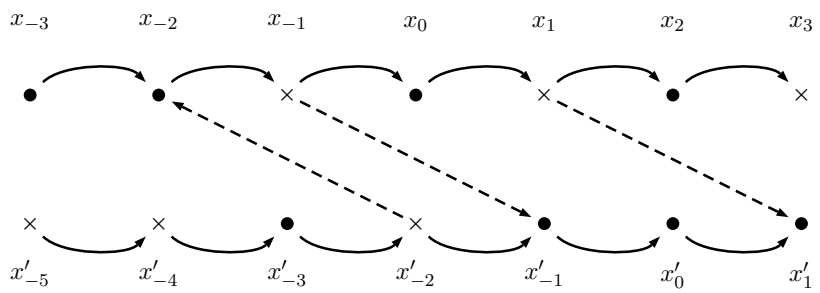

Figure 3.1: An illustration how the critical points might arranged in the two orbits in lemma 3.1.7 for $N = 2$. Here, the crosses indicate critical points and the circles indicate regular points. The orbit (x_n) has no critical points before x_{-1} and the orbit $(x'_{n'})$ has no critical points after x'_{-2}.

Proof. Let $\tilde{h}_f = \tilde{\Phi}_{+,f} \circ \tilde{\Psi}_{-,f}|_{\tilde{\mathcal{A}}_f}$ be a lifted horn map of f. By lemmas 2.2.1 and 2.2.2, $w \in \mathbb{C}$ is a singular value of $\tilde{h}_f$ if and only if $\tilde{\Phi}^{-1}_{+,f}(w)$ is contained in a critical grand orbit of f. Recall that if $\tilde{h}_g = \tilde{\Phi}_{+,g} \circ \tilde{\Psi}_{-,g}|_{\tilde{\mathcal{A}}_g}$ and $T_\pm$ are the translations from theorem 2.2.11 then we have that $T_+ \circ \tilde{h}_f = \tilde{h}_g \circ T_-$. Hence $w \in \mathbb{C}$ is a singular value of $\tilde{h}_f$ if and only if $T_+(w)$ is a singular value of $\tilde{h}_g$. Therefore $z \in \mathcal{P}_{+,f}$ lies in a critical grand orbit of f if and only if $\chi(z)$ lies in a critical grand orbit of g. Further, since f is injective on $\mathcal{P}_{+,f}$ and g is injective of $\mathcal{P}_{+,g}$, χ maps points from distinct grand orbits of f to points of distinct grand orbits of g.

By assumption, f has a proper parabolic basin at 0 and so $\mathcal{A}_f$ contains finitely many critical points of f. Hence each critical grand orbit in $\mathcal{A}_f$ contains finitely many critical points and there are finitely many critical grand orbits. Therefore we can find a finite set of post-critical points P belonging to pairwise disjoint critical grand orbits such that for any critical point $c \in \mathcal{A}_f$ of f, there exists a unique $p \in P$ and $m \in \mathbb{Z}_{>0}$ such that $f^{\circ m}(c) = p$. If $f^{\circ m}(c) = p$ then $f^{\circ m+k}(c) = f^{\circ k}(p)$ for any $k \in \mathbb{Z}_{\geq 0}$ and so we may assume without loss of generality that $P \subset \mathcal{P}_{+,f}$. By the

previous discussion, if $p \in P$ then $\chi(p)$ lies in a critical grand orbit of g. Further, distinct p's lie in distinct critical grand orbits and every critical grand orbit of g contains some $\chi(p)$. Again without loss of generality we may replace each p by some forward iterate in order to assume that for any critical point c' of g there exists $p \in P$ and $m \in \mathbb{Z}_{>0}$ such that $g^{\circ m}(c') = \chi(p)$.

Then we define $N \in \mathbb{Z}_{>0}$ to be

$$N = 1 + \max \left\{ m \in \mathbb{Z}_{>0} \middle| \begin{array}{c} f^{\circ m}(c) = p \text{ for some } p \in P \\ \text{and some critical point } c \in \mathcal{GO}(p) \end{array} \right\}.$$

Now, suppose that $\{x_n\}$ is a bi-infinite orbit under f and that $\{x'_n\}$ is a bi-infinite orbit under g as in the hypotheses. If either $\{x_n\}$ or $\{x'_n\}$ does not contain a critical point then there is nothing to prove. Otherwise, suppose that x_{n_0} is a critical point of f and that $x'_{n'_0}$ is a critical point of g for some $n_0, n'_0 \in \mathbb{Z}$. Then for some $n_1 > n_0$, $x_{n_1} = p \in P$. By our choice of N, $n_1 - n_0 < N$. Also, since $x'_{n'_0}$ is a critical point of g on the same grand orbit as $\chi(p)$, $\chi(p) = x'_{n_1 - N}$ and hence $n'_0 < n_1 - N$. By re-indexing if necessary, we can assume without loss of generality that $n_1 = 0$.

Then we have that $-n_0 < N < -n'_0$ and so $n'_0 < n_0$. Hence, if $n \geq n_0$ then x'_n is not a critical point of g and so $\deg_{x'_n}(g) = 1$. Also, if $n \leq n'_0$ then x_n is not a critical point of f and so $\deg_{x_n}(f) = 1$. $\square$

The following lemma then allows us to perform the analytic continuation along a curve. It tell us that for every $x \in \mathcal{A}_f^0$ there exists a neighbourhood U such that at least one of the following is true:

i All branches of $\tilde{\Psi}_{-,g} \circ T_N \circ T_- \circ (\tilde{\Psi}_{-,f})^{-1}$ are well-defined and holomorphic on U.

ii All branches of $(\tilde{\Phi}_{+,g})^{-1} \circ T_N \circ T_+ \circ \tilde{\Phi}_{+,f}$ are well-defined and holomorphic on U.

We could also choose to remove all mention of Fatou coordinates from the statement of the lemma; N is chosen such that for all $x \in \mathcal{A}_f$ at least one of the following is true:

i x does not lie in the post-critical set of f. That is, there does not exist a critical point $c \in \mathcal{A}_f$ of f and $m \in \mathbb{Z}_{>0}$ such that $f^{\circ m}(c) = x$.

ii If $m_0 \geq N$ and $f^{\circ m_0}(x) \in \mathcal{P}_{+,f}$ then $\bigcup_{m=m_0}^{\infty} \bigcup_{k=0}^{m-N} g^{-k}(\{\chi \circ f^{\circ m}(x)\})$ contains no critical points of g.

Lemma 3.1.8.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ and $g : \mathcal{D}(g) \to \hat{\mathbb{C}}$ be holomorphic maps with simple parabolic fixed points at 0 with proper parabolic basins. Suppose that χ is a local analytic conjugacy from $(f, 0)$ to $(g, 0)$.

Then there exists $N \in \mathbb{Z}$ such that if $\tilde{\Phi}_{\pm,f}$ are Fatou coordinates for f, $\tilde{\Phi}_{\pm,g}$ are Fatou coordinates for g and $T_\pm : \mathbb{C} \to \mathbb{C}$ are the translations which satisfy $\tilde{\Phi}_{\pm,g} \circ \chi = T_\pm \circ \tilde{\Phi}_{\pm,f}$ then every $x \in \mathcal{A}_f$ has a neighbourhood U such that at least one of the following is true:

i U is evenly covered by $\tilde{\Psi}_{-,f}$, or

ii $T_+ \circ T_N \circ \tilde{\Phi}_{+,f}(U)$ is evenly covered by $\tilde{\Phi}_{+,g}$.

Proof. Let $N \in \mathbb{Z}$ be as in lemma 3.1.7 and let $x \in \mathcal{A}_f$. By lemma 2.2.2, if x is not a post-critical point then there exists a neighbourhood U such that U is evenly covered by $\tilde{\Psi}_{-,f}$. Now, suppose that x is a post-critical point of f, so that there exists a critical point c of f and $m \in \mathbb{Z}_{>0}$ such that $f^{\circ m}(c) = x$. Let $\{x_n\}$ be a bi-infinite orbit under f with $x_0 = x$ and $x_{-m} = c$.

Let $v' = T_+ \circ T_N \circ \tilde{\Phi}_{+,f}(x)$ and let $x' \in \tilde{\Phi}_{+,g}^{-1}(v')$. Let $\{x'_n\}$ be a bi-infinite orbit with $x'_N = x'$. Then $\{x_n\}$ and $\{x'_n\}$ satisfy the hypotheses of lemma 3.1.7. x_{-m} is a critical point of f and $m > 0$, hence x'_n is not a critical point of g for all $n \geq N$ and so $x' = x'_N$ is not a pre-critical point of g. Since $x' \in \tilde{\Phi}_{+,g}^{-1}(v')$ was arbitrary, $\tilde{\Phi}_{+,g}^{-1}(v')$ contains no pre-critical points of g and hence, by lemma 2.2.1, v' has a neighbourhood V' which is evenly covered by $\tilde{\Phi}_{+,g}$. Thus, taking U to be the connected component of $(T_+ \circ T_N \circ \tilde{\Phi}_{+,f})^{-1}(V')$ which contains x, we see that ii) holds. □

We are now able to perform the analytic continuation of χ_N.

Proposition 3.1.9.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ and $g : \mathcal{D}(g) \to \hat{\mathbb{C}}$ be holomorphic maps with simple parabolic fixed points at 0 with proper parabolic basins. Suppose that χ is a local analytic conjugacy from $(f, 0)$ to $(g, 0)$.

Then there exists $N \in \mathbb{Z}$ such that if $x_0 \in \mathcal{P}_{+,f} \cap (f|_{\mathcal{P}_{-,f}})^{\circ -N}(\mathcal{P}_{-,f})$ and $\gamma : [0, 1] \to \mathcal{A}_f$ is a curve with $\gamma(0) = x_0$ then there exists an analytic continuation of $(\chi_N := g^{\circ N} \circ \chi, x_0)$ along γ.

Proof. Let N be as in lemma 3.1.8. Notice that since g is injective on a neighbourhood of 0, χ_N is also a local analytic conjugacy from $(f, 0)$ to $(g, 0)$. Further, if we

let $\tilde{\Phi}_{\pm,f}$ and $\tilde{\Phi}_{\pm,g}$ be Fatou coordinates and $T_\pm$ be the translations which satisfy $\tilde{\Phi}_{\pm,g} \circ \chi = T_\pm \circ \tilde{\Phi}_{\pm,f}$ then we have that

$$\begin{aligned}\tilde{\Phi}_{\pm,g} \circ \chi_N =& \tilde{\Phi}_{\pm,g} \circ g^{\circ N} \circ \chi \\ =& T_N \circ \tilde{\Phi}_{\pm,g} \circ \chi = T_N \circ T_\pm \circ \tilde{\Phi}_{\pm,f}.\end{aligned}$$

Therefore, by replacing χ with χ_N and $T_\pm$ with $T_\pm \circ T_N$, we can assume without loss of generality that $N = 0$.

Let $S \subseteq [0,1]$ be the set of $s \in [0,1]$ such that there exists an analytic continuation of (χ, x_0) along $\gamma|_{[0,s]}$. We denote such an extension by $\overline{\gamma} : t \mapsto (\chi_t, \gamma(t))$. Then $0 \in S$, since $0 \mapsto (\chi, x_0)$ is the trivial continuation along $\gamma|_{[0,0]}$.

Suppose that $s \in S$. Let $U = \mathcal{D}(\chi_s)$ and let I be the connected component of $\gamma^{-1}(U)$ which contains s. Then $t \mapsto (\chi_s, \gamma(t))$ is an analytic continuation of $(\chi_s, \gamma(s))$ along $\gamma|_I$ and so the concatenation

$$\overline{\gamma} : t \mapsto \begin{cases} \overline{\gamma}(t) \text{ if } t \in [0,s] \\ (\chi_s, \gamma(t)) \text{ if } t \in I \end{cases}$$

is an analytic continuation of (χ, x_0) along $\gamma|_{[0,s]\cup I}$. Therefore $[0,s] \cup I \subset S$ and so S is open.

Suppose instead that $[0,s) \subseteq S$ and consider the point $\gamma(s)$. By lemma 3.1.8, $\gamma(s)$ has a neighbourhood U such that

i U is evenly covered by $\tilde{\Psi}_{-,f} : \tilde{\mathcal{A}}_f^0 \to \mathcal{A}_f^0$, or

ii $T_+ \circ \tilde{\Phi}_{+,f}(U) = V'$ is evenly covered by $\tilde{\Phi}_{+,g} : \mathcal{A}_g^0 \to \mathbb{C}$.

In the first case, by lemmas 2.2.2 and 3.1.6 there exists a curve $\tilde{\gamma} : [0,s] \to \tilde{\mathcal{A}}_f^0$ which satisfies $\tilde{\Psi}_{-,f} \circ \tilde{\gamma} = \gamma|_{[0,s]}$ and $\tilde{\gamma}(0) = \tilde{\Phi}_{-,f}(x_0) \in \tilde{\Psi}_{-,f}^{-1}(x_0)$. Let $W \subset \tilde{\Psi}_{-,f}^{-1}(U)$ be the connected component which contains $\tilde{\gamma}(s)$. Then $\tilde{\Psi}_{-,f}|_W : W \to U$ is biholomorphic and so we can define $\chi_s := \tilde{\Psi}_{-,g} \circ T_- \circ (\tilde{\Psi}_{-,f}|_W)^{-1}$.

Let $t \in \tilde{\gamma}^{-1}(W) \cap [0,s)$. Since $(\chi \circ \tilde{\Psi}_{-,f}, \tilde{\Phi}_{-,f}(x_0)) = (\tilde{\Psi}_{-,g} \circ T_-, \tilde{\Phi}_{-,f}(x_0))$, by the permanence of functional relations $(\chi_t \circ \tilde{\Psi}_{-,f}, \tilde{\gamma}(t)) = (\tilde{\Psi}_{-,g} \circ T_-, \tilde{\gamma}(t))$. Therefore

$$\begin{aligned}(\chi_t, \gamma(t)) =& (\chi_t \circ \tilde{\Psi}_{-,f} \circ (\tilde{\Psi}_{-,f}|_W)^{-1}, \gamma(t)) \\ =& (\tilde{\Psi}_{-,g} \circ T_- \circ (\tilde{\Psi}_{-,f}|_W)^{-1}, \gamma(t)) = (\chi_s, \gamma(t)),\end{aligned}$$

hence $\overline{\gamma}(t) \in \Delta_U^{\chi_s}$. Since t was arbitrary, $s \mapsto (\chi_s, \gamma(s))$ gives an extension of the analytic continuation of (χ, x_0) along $\gamma|_{[0,s)}$ to $[0,s]$.

In the second case, let $s' \in [0, s)$ such that $\gamma([s', s]) \subset U$. Let U' denote the connected component of $\tilde{\Phi}_{+,g}^{-1}(V')$ which contains $\chi_{s'}(\gamma(s'))$ and define χ_s by $\chi_s := (\tilde{\Phi}_{+,g}|_{U'})^{-1} \circ T_+ \circ \tilde{\Phi}_{+,f}|_U$.

Let $t \in (s', s)$. Then similar to before, since $(\tilde{\Phi}_{+,g} \circ \chi, x_0) = (T_+ \circ \tilde{\Phi}_{+,f}, x_0)$, by the permanence of functional relations $(\tilde{\Phi}_{+,g} \circ \chi_t, \gamma(t)) = (T_+ \circ \tilde{\Phi}_{+,f}, \gamma(t))$. Therefore

$$\begin{aligned}(\chi_t, \gamma(t)) =& ((\tilde{\Phi}_{+,g}|_{U'})^{-1} \circ \tilde{\Phi}_{+,g} \circ \chi_t, \gamma(t)) \\ =& ((\tilde{\Phi}_{+,g}|_{U'})^{-1} \circ T_+ \circ \tilde{\Phi}_{+,f}, \gamma(t)) = (\chi_s, \gamma(t)),\end{aligned}$$

hence $\overline{\gamma}(t) \in \Delta_U^{\chi_s}$. Since t was arbitrary, $s \mapsto (\chi_s, \gamma(s))$ gives an extension of the analytic continuation of (χ, x_0) along $\gamma|_{[0,s)}$ to $[0, s]$.

In either case, we see that $s \in S$ and hence S is closed. Thus S is an open, closed, non-empty subset of $[0, 1]$, which is connected. Hence $S = [0, 1]$ and so (χ, x_0) has an analytic continuation along γ. □

3.1.3 Basin Semiconjugacies

We are now ready to prove the main theorem of this thesis. We have shown that we can extend the local conjugacy χ_N along arbitrary curves in the basin, so we apply the Monodromy Theorem to conclude that χ_N extends to a holomorphic map between basins. The uniqueness of analytic continuation ensures that the resulting map is a semi-conjugacy.

As we mentioned when we introduced the Monodromy Theorem, our approach restricts us to assuming that the immediate basin of f is simply connected, so as to avoid the possibility that an analytic continuation around a closed loop could take different values at the start and end. Weakening this assumption could be a subject of possible further research.

Theorem 3.1.10 (Main Theorem)**.**
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ and $g : \mathcal{D}(g) \to \hat{\mathbb{C}}$ be holomorphic maps with simple parabolic fixed points at 0 with proper parabolic basins.

Suppose that χ is a local analytic conjugacy from $(f, 0)$ to $(g, 0)$. Suppose also that the immediate attracting basin $\mathcal{A}_f^0$ is simply connected.

Then there exists $N \in \mathbb{Z}$ such that for all $n \geq N$, $\chi_n := g^{\circ n} \circ \chi|_{\mathcal{A}_f^0}$ extends to an analytic semi-conjugacy from $f|_{\mathcal{A}_f^0}$ to $g|_{\mathcal{A}_g^0}$.

Proof. $N \in \mathbb{Z}$ be as in proposition 3.1.9 and let $x_0 \in \mathcal{P}_{+,f} \cap (f|_{\mathcal{P}_{-,f}})^{\circ -N}(\mathcal{P}_{-,f})$. Let $x \in \mathcal{A}_f^0$ and let $\gamma_x : [0, 1] \to \mathcal{A}_f^0$ be a curve with $\gamma_x(0) = x_0$ and $\gamma_x(1) = x$.

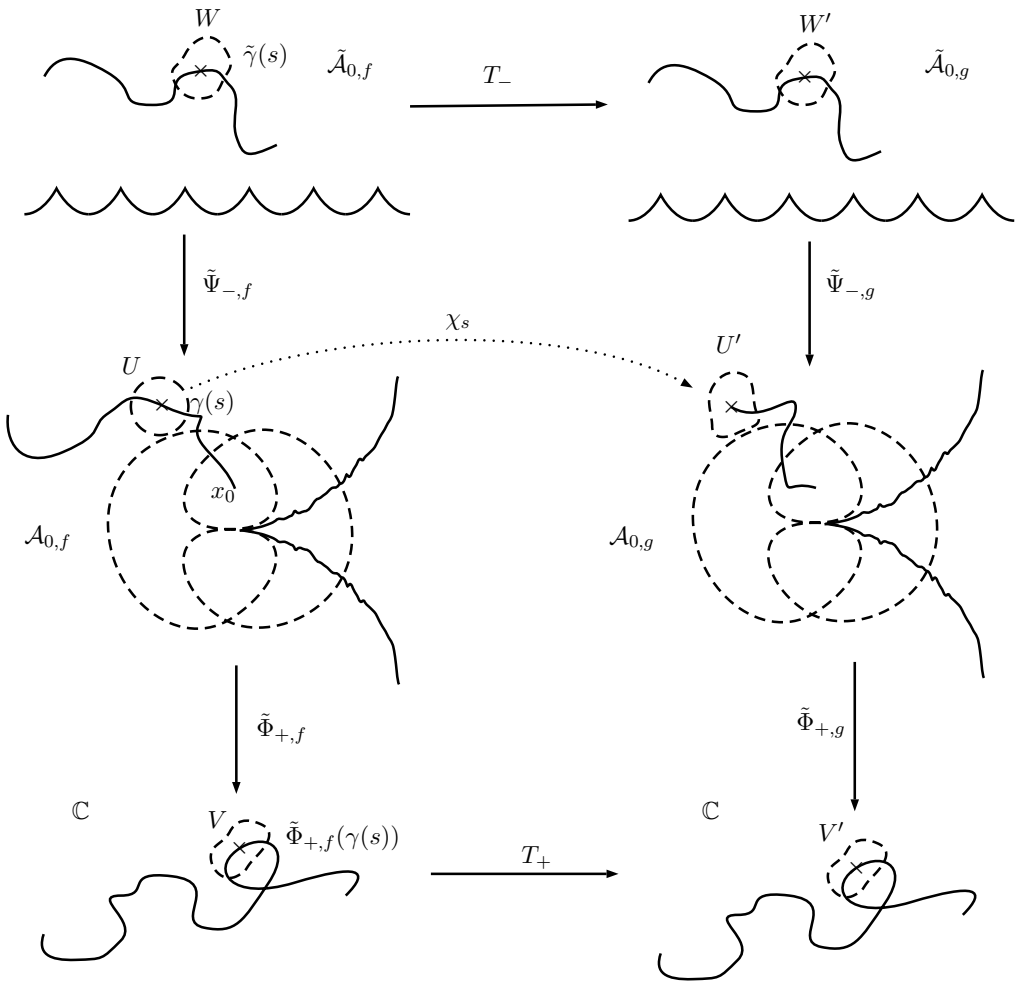

Figure 3.2: An illustration of the proof of proposition 3.1.9. By our choice of N, for all $s \in [0,1]$ at least one of $\tilde{\Psi}_{-,f}|_W : W \to U$ or $\tilde{\Phi}_{+,g}|_{U'} : U' \to V'$ is biholomorphic. Thus we can define $\chi_s : U \to U'$ by either $\chi_s = \tilde{\Psi}_{-,g} \circ T_- \circ (\tilde{\Psi}_{-,f}|_W)^{-1}$ or $\chi_s = (\tilde{\Phi}_{+,g}|_{U'})^{-1} \circ T_+ \circ \tilde{\Phi}_{+,f}$.

By proposition 3.1.9, there exists an analytic continuation, $\overline{\gamma}_x$, of (χ_N, x_0) along γ_x. Since $\mathcal{A}_f^0$ is simply connected and γ_x was arbitrary, by the Monodromy Theorem $\overline{\gamma}_x(1)$ does not depend on the choice of γ_x. Hence, since $x \in \mathcal{A}_f^0$ was arbitrary, χ_N has an analytic extension to all of $\mathcal{A}_f^0$ given by $(\chi_N, x) = \overline{\gamma}_x(1)$.

Since χ_N satisfies $\chi_N \circ f = g \circ \chi_N$ on the open set $\mathcal{P}_{+,f} \cap (f|_{\mathcal{P}_{-,f}})^{\circ -N}(\mathcal{P}_{-,f})$, $\chi_N \circ f|_{\mathcal{A}_f^0} = g|_{\mathcal{A}_g^0} \circ \chi_N$ and so χ_N is a semi-conjugacy from $f|_{\mathcal{A}_f^0}$ to $g|_{\mathcal{A}_g^0}$. Finally,

we note that if $n > N$ then $\chi_n = g^{\circ n-N} \circ \chi_N$ and so

$$\begin{aligned}\chi_n \circ f =& g^{\circ n-N} \circ \chi_N \circ f \\ =& g^{\circ n-N} \circ g \circ \chi_N \\ =& g \circ g^{\circ n-N} \circ \chi_N = g \circ \chi_n.\end{aligned}$$

□

As a first application of this theorem, we show that if the basins of both f and g are proper and simply connected then $\deg f|_{\mathcal{A}_f^0} = \deg g|_{\mathcal{A}_g^0}$. To do this we show that we can extend both $g^{\circ N} \circ \chi$ and $f^{\circ N'} \circ \chi^{-1}$ to a pair of mutual semi-conjugacies. These semi-conjugacies will both be proper and so have a well-defined finite degree. The relation $\chi_N \circ f = g \circ \chi_N$ then tells us that f and g have equal degree when restricted to their respective immediate basins.

In particular, we can apply this result when f and g can both be restricted to polynomial-like maps whose domain of definition contains the parabolic fixed point 0, since then the immediate basin is necessarily simply connected.

Corollary 3.1.11.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ and $g : \mathcal{D}(g) \to \hat{\mathbb{C}}$ be holomorphic maps with simple parabolic fixed points at 0 with proper parabolic basins.

Suppose that there exists a local analytic conjugacy from $(f, 0)$ to $(g, 0)$. Suppose further that both immediate attracting basins $\mathcal{A}_f^0$ and $\mathcal{A}_g^0$ are simply connected.

Then there exists $N \in \mathbb{Z}$ such that for all $n \geq N$, $\chi_{n,+} := g^{\circ n} \circ \chi$ and $\chi_{n,-} := f^{\circ n} \circ \chi^{-1}$ can be extended to a proper *analytic semi-conjugacies from $f|_{\mathcal{A}_f^0}$ to $g|_{\mathcal{A}_g^0}$ and from $g|_{\mathcal{A}_g^0}$ to $f|_{\mathcal{A}_f^0}$ respectively.*

Hence $\deg f|_{\mathcal{A}_f^0} = \deg g|_{\mathcal{A}_g^0}$.

Proof. Consider the local analytic conjugacies χ from $(f, 0)$ to $(g, 0)$ and χ^{-1} from $(g, 0)$ to $(f, 0)$. By theorem 3.1.10 there exist $N_+, N_- \in \mathbb{Z}$ such that $\chi_+ := g^{\circ N_+} \circ \chi$ and $\chi_- := f^{\circ N_-} \circ \chi^{-1}$ extend to holomorphic semi-conjugacies from $f|_{\mathcal{A}_f^0}$ to $g|_{\mathcal{A}_g^0}$ and from $g|_{\mathcal{A}_g^0}$ to $f|_{\mathcal{A}_f^0}$ respectively.

Now, consider the composition $\chi_- \circ \chi_+ : \mathcal{A}_f^0 \to \mathcal{A}_f^0$. Then, on the intersection $\mathcal{P}_{+,f} \cap f^{\circ -(N_+ + N_-)}(\mathcal{P}_{-,f})$ we have that

$$\begin{aligned}\chi_- \circ \chi_+ =& f^{\circ N_-} \circ \chi^{-1} \circ g^{\circ N_+} \circ \chi \\ =& f^{\circ N_- + N_+} \circ \chi^{-1} \circ \chi \\ =& f^{\circ N_- + N_+}.\end{aligned}$$

Therefore $\chi_- \circ \chi_+ = f^{\circ N_- + N_+}|_{\mathcal{A}_f^0}$. By assumption, $f^{\circ N_- + N_+}|_{\mathcal{A}_f^0}$ is proper and hence χ_- and χ_+ are both proper. Thus they have a well-defined finite degree.

$\chi_+ \circ f|_{\mathcal{A}_f^0} = g|_{\mathcal{A}_g^0} \circ \chi_+$ and so $\deg(\chi_+ \circ f|_{\mathcal{A}_f^0}) = \deg(g|_{\mathcal{A}_g^0} \circ \chi_+)$. Hence $\deg(\chi_+)\deg(f|_{\mathcal{A}_f^0}) = \deg(g|_{\mathcal{A}_g^0})\deg(\chi_+)$ and so $\deg(f|_{\mathcal{A}_f^0}) = \deg(g|_{\mathcal{A}_g^0})$. □

3.2 Topics for Further Research

In this final section, we discuss potential topics for further research. We divide our conjectures into two categories. In the first subsection we discuss how theorem 3.1.10 could be extended to give more information about the cases which our present result covers. We consider how we might extend to a global relation when the maps involved are rational and also how getting a sharper minimum for the degree of the semi-conjugacy would reflect on possible pairs of f and g.

In the second subsection we discuss how we might be able to construct basin semi-conjugacies between wider classes of maps. We consider how limitations of our current approach apply when the basin is multiply connected, when the maps f and g are finite type and when they are more general analytic maps.

3.2.1 From Local Conjugacies to Global Properties

We begin by restricting to the case when f and g are rational maps. The following lemma arises as a slight modification of Lemmas 2 and 3 in Buff and Epstein [5].

Lemma 3.2.1.
Let f and g be rational maps. Let Ω_f be a forward invariant Fatou component of f and U_f be an open set with $U_f \cap f(U_f) \cap \partial\Omega_f \neq \emptyset$ and similarly for Ω_g, U_g.

Suppose there exists a holomorphic semi-conjugacy $\chi : \Omega_f \cup U_f \to \Omega_g \cup U_g$ from $f|_{\Omega_f \cup U_f}$ to $g|_{\Omega_g \cup U_g}$.

Then χ has an analytic extension from a neighbourhood of $\overline{\Omega_f}$ to a neighbourhood of $\overline{\Omega_g}$.

We now consider Theorem 1 of Inou [9, Theorem 1].

Theorem 3.2.2.
For $i = 1, 2$, let f_i be a rational map or an entire map. Assume that there exist polynomial-like restrictions $f_i : U_i' \to U_i$ of degree not less than two which are analytically conjugate. Then there exist rational or entire maps g, ϕ_1, ϕ_2 such that for $i \in \{1, 2\}$

$$f_i \circ \phi_i = \phi_i \circ g$$

and g has a polynomial-like restriction $g : V' \to V$ analytically conjugate to $f_1 : U_1' \to U_1$ by ϕ_1.

Furthermore,

- *if both degrees $d_i = \deg f_i$ are finite then g, ϕ_1 and ϕ_2 are also finite of finite degrees. In particular, $d_1 = d_2$.*
- *If f_1 is a polynomial and f_2 is a rational map, then f_2 is Möbius conjugate to a polynomial and after this conjugation so are g, ϕ_1 and ϕ_2.*

Notice that our extension from lemma 3.2.1 may well not be polynomial-like. However, we conjecture that the following alternative condition suffices; the map h must have a Fatou component which is not equal to the whole space on which it is defined, from which we can eliminate possibilities in a similar manner to Section 3 of Inou [9].

Conjecture 3.2.3.
Let f and g be rational maps. Let Ω_f be a forward invariant Fatou component of f and $U_f \supset \overline{\Omega_f}$ be an open set, and similarly for $U_g \supset \overline{\Omega_g} \supset \Omega_g$.

Suppose that there exists a proper holomorphic semi-conjugacy $\chi : U_f \to U_g$.

Then there exist rational maps h, χ_f and χ_g and an open set $U \subset \hat{\mathbb{C}}$ such that

i $f \circ \chi_f = \chi_f \circ h$ *and* $g \circ \chi_g = \chi_g \circ h$,

ii $\chi_f|_U$ *is a proper map from U to U_f and similarly for $\chi_g|_U : U \to U_g$, and*

iii $\chi \circ \chi_f|_U = \chi_g|_U$.

There being semi-conjugacies from h to f and from h to g induces a many-to-many conjugacy-like relationship from f to g. Of interest is also the fact that since h, χ_f and χ_g are all rational they have finite degree and hence $\deg f = \deg g$.

We can apply this conjecture to the setting of Theorem 3.1.10 by noting that χ_n is defined on $\mathcal{A}_f^0 \cup \mathcal{D}(\chi)$ and that $0 \in \mathcal{D}(\chi) \cap f(\mathcal{D}(\chi)) \cap \partial\mathcal{A}_f^0$, so this intersection is non-empty. Applying the previous two conjectures then yields a pair of global semi-conjugacies from h to f and g respectively.

One of the original motivating examples for the entire discussion of semi-conjugacies between parabolic basins came in the form of pairs $h_1 \circ h_2 \neq h_2 \circ h_1$. If both h_1 and h_2 fix 0 with multiplier $\lambda \neq 0$ and $1/\lambda$ respectively then $h_1 \circ h_2$ and $h_2 \circ h_1$ have locally conjugate parabolic fixed points at 0. If $\deg(h_1), \deg(h_2) > 1$

then $h_1 \circ h_2$ and $h_2 \circ h_1$ need not be analytically conjugate, but they are semi-conjugate, since $h_2 \circ (h_1 \circ h_2) = (h_2 \circ h_1) \circ h_2$. We conjecture that this is the only way that two maps f and g with locally conjugate parabolic fixed points can fail to be conjugate and propose a proof strategy to show it.

Recall from the proof of corollary 3.1.11 that if χ_+ is an extension of $g^{\circ N_+} \circ \chi$ and χ_- is an extension of $f^{\circ N_-} \circ \chi_-$ then $\chi_- \circ \chi_+ = f^{\circ n}|_{\mathcal{A}_f^0}$ for some $n \in \mathbb{Z}_{\geq 0}$. Therefore, if both $\deg(\chi_+) < \deg(f) = \deg(g)$ and $\deg(\chi_-) < \deg(f)$ then it follows that $\deg(\chi_- \circ \chi_+) = \deg(f)^n < \deg(f)^2$. Hence either $\chi_- \circ \chi_+ = Id$ and so χ_+ is a conjugacy, or else $\chi_- \circ \chi_+ = f$.

Our approach would be to try to improve the choice of N in proposition 3.1.9 in light of the following lemma.

Lemma 3.2.4.
Let $k, d \in \mathbb{Z}_{\geq 1}$ and let $\phi : \mathbb{D} \to \mathbb{D}$ be a univalent holomorphic map fixing 0. Let $P_d : z \mapsto z^d$ be the dth power map.

Then there exists a holomorphic map $\tilde{\phi} : \mathbb{D} \to \mathbb{D}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\tilde{\phi}"] \arrow[d, "P_{kd}"'] & \mathbb{D} \arrow[d, "P_d"] \\ \mathbb{D} \arrow[r, "\phi"'] & \mathbb{D} \end{tikzcd} .$$

Observe that if $z \in f^{\circ -N}(\mathcal{P}_{-,f}) \cap \mathcal{A}_f^0$ and $n \in \mathbb{Z}_{\geq 0}$ then the local degrees of f and g satisfy $\deg_z(f^{\circ n}) \leq \deg_{\chi_N(z)}(g^{\circ n})$. Moreover, $f^{\circ n}(z)$ is a critical point of χ_N if and only if $\deg_z(f^{\circ n}) < \deg_{\chi_N(z)}(g^{\circ n})$. By choosing N minimal, so that the former inequality is satisfied for all z and n we minimise the degree of χ_N.

3.2.2 Basin Semi-conjugacies on a Wider Class of Maps

Our restriction to simply connected domains allowed us to apply the Monodromy Theorem in the simplest possible way. In order to extend the result to multiply connected domains we would need to make use of the following lemma.

Lemma 3.2.5.
Let $U \subset V \subseteq \mathbb{C}$ be planar domains and let $f : U \to \hat{\mathbb{C}}$ be a holomorphic function defined on U. Let $z_0 \in U$.

Suppose that for all closed loops $\gamma : [0,1] \to V$ with $\gamma(0) = \gamma(1) = z_0$

i there exists an analytic continuation $\overline{\gamma}$ of (f, z_0) along γ and

ii $\overline{\gamma}(1) = (f, z_0)$.

Then there exists an analytic extension of f to V.

Notice that the Monodromy Theorem says that condition ii) is redundant when V is simply connected.

Recall that for any two path-connected topological space X and Y and a continuous map $f : X \to Y$, f induces a homomorphism of fundamental groups $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$. Of interest to us is the fact that the image of $\pi_1(f)$ gives the homotopy classes of the images of loops in X.

By the permanence of functional relations, if $\tilde{\Psi}_{-,f} \circ \tilde{\gamma} = \gamma$ is a closed loop based at $x_0 \in \mathcal{P}_{+,f} \cap (f|_{\mathcal{P}_{-,f}})^{\circ -N}(\mathcal{P}_{-,f})$ and the analytic continuation $\overline{\gamma}$ of (χ_N, x_0) along γ exists then $\overline{\gamma}(1) = \overline{\gamma}(0)$. Thus we arrive at the following conjecture.

Conjecture 3.2.6.
Let $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ and $g : \mathcal{D}(g) \to \hat{\mathbb{C}}$ be holomorphic maps with simple parabolic fixed points at 0 with proper parabolic basins.

Let $\tilde{\Psi}_{-,f}$ be an inverse repelling Fatou coordinate of $(f, 0)$, $x_0 \in \mathcal{P}_{+,f} \cap \mathcal{P}_{-,f}$ and $w_0 = \tilde{\Phi}_{-,f}(x_0)$.

Suppose that χ is a local analytic conjugacy from $(f, 0)$ to $(g, 0)$. Suppose also that the induced homomorphism of fundamental groups

$$\pi_1(\tilde{\Psi}_{-,f}) : \pi_1(\tilde{\mathcal{A}}_f^0, w_0) \to \pi_1(\mathcal{A}_f^0, x_0)$$

is surjective.

Then there exists $N \in \mathbb{Z}$ such that for all $n \geq N$, $\chi_n := g^{\circ n} \circ \chi|_{\mathcal{A}_f^0}$ extends to an analytic semi-conjugacy from $f|_{\mathcal{A}_f^0}$ to $g|_{\mathcal{A}_g^0}$.

We see from the definition of $\tilde{\Psi}_{-,f}$ that $\pi_1(\tilde{\Psi}_{-,f})$ is surjective if and only if $\pi_1(f|_{\mathcal{A}_f^0})$ is surjective. This suggests that $\pi_1(\tilde{\Psi}_{-,f})$ is surjective away from some exceptional maps which satisfy some condition on a component of $\hat{\mathbb{C}} \setminus \mathcal{A}_f^0$ having preimages exclusively mapped by degree > 1. However, we do not currently know the precise nature of such a condition or if it is even possible that $\pi_1(\tilde{\Psi}_{-,f})$ is not surjective.

Finally, we shall say a few words about how far we believe these arguments could possibly be generalised.

Our first consideration would be finite type maps. If the restriction $f|_{\mathcal{A}_f^0}$ has asymptotic values in the immediate basin then so will $\tilde{\Psi}_{-,f}$. This presents a problem in that we can no longer apply lemma 3.1.6 to obtain a lift of an arbitrary curve $\gamma : [0, 1] \to \mathcal{A}_f^0$.

Lemmas 2.2.1 and 2.2.2 would need to change substantially in order to also account for asymptotic values of $\tilde{\Psi}_{-,f}$ and $\tilde{\Phi}_{+,f}$. On the other hand, we conjecture that by modifying the proof of lemma 3.1.7 to also offset the asymptotic values both it and lemma 3.1.8 hold.

We further conjecture that since the asymptotic values of f are isolated it follows that an asymptotic value x of $\tilde{\Psi}_{-,f}$ has a neighbourhood U with a preimage component W conformally isomorphic to a half-plane. Thus if $\gamma(s) = x$ and $\tilde{\gamma}$ is a lift of $\gamma|_{[0,s)}$ then whilst we might not be able to define $\tilde{\gamma}(s)$, we can still define $\tilde{\gamma}|_{[0,s+\epsilon]\setminus\{s\}}$.

If the analytic continuation $\overline{\gamma}$ were defined on $[0, s+\epsilon]$ then we would still have that $(\chi_t \circ \tilde{\Psi}_{-,f}, \tilde{\gamma}(t)) = (\tilde{\Psi}_{-,g} \circ T_-, \tilde{\gamma}(t))$ for all t such that both sides are defined. Thus we can maintain the permanence of functional relations which was the key component in the proof of proposition 3.1.9. Notice also that since $\gamma([0,1])$ is compact the fact that $\tilde{\Phi}_{+,f}$ has asymptotic values has not effect on the proof.

The furthest we might be able to extend our result is to the class of entire functions whose set of singular values is discrete. In this setting lemmas 3.1.7 and 3.1.8 no longer holds. However, if we instead include the existence of an N such that lemma 3.1.8 holds in the hypotheses on f and g then we can carry out the rest of the argument as above.

Finally, if the singular value set of $f|_{\mathcal{A}_f^0}$ is not discrete in $\mathcal{A}_f^0$ then we lose all control of the covering properties of $\tilde{\Phi}_{+,f}$ and $\tilde{\Psi}_{-,f}$. We do not expect that any variation on the argument given could be adapted to this setting.

Appendix A

The Cubic Connectedness Locus is not Locally Connected

In this appendix we present Lavaurs' proof that the cubic connectedness locus is not locally connected. This result provides an application of the generality with which we worked in section 2.2. The result is well known, but to our knowledge the proof is only available in French [12].

The conjecture that the Mandelbrot set, which is the quadratic connectedness locus, is locally connected has many significant consequences. If it were true then by a theorem of Carathéodory the Riemann map $\psi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \mathcal{M}$ would extend to a continuous map $\mathbb{S}^1 \to \partial\mathcal{M}$. From such an extension one could build a topological model $\mathcal{M} \simeq \overline{\mathbb{D}}/\sim$, where the relation $\sim$ collapses down the convex hull of $\psi^{-1}(z)$ for each $z \in \partial\mathcal{M}$.

The set of points for which $\psi^{-1}(z)$ is not a single point is combinatorially well-understood, so if the Mandelbrot set is locally connected then we could study the topology of the Mandelbrot set in terms of a comparatively well-understood and simple model. In particular, by a theorem of Douady and Hubbard [6], if the Mandelbrot set is locally connected then hyperbolic quadratic polynomials are dense in the the space of quadratic polynomials.

By contrast, in higher degrees, and so higher dimension parameter spaces, the implication that local connectivity implies density of hyperbolicity is false. Thus whilst this result in degree 3 provides an interesting contrast with the conjecture that the Mandelbrot set *is* locally connected, it has no deeper consequences on the structure of the parameter space of cubic polynomials.

We structure this appendix as follows. We begin by defining the space Poly_d of degree d polynomials, the quotient space poly_d and the connectedness locus $\mathcal{C}_d$.

We also define the notion of a topological space being locally connected.

In the second we present a brief overview of the theory of parabolic implosion. As we remarked before, proposition 2.1.13 allows us to define Fatou coordinates for maps which are appropriately close to being parabolic. From here we develop the concept of *Lavaurs maps* and *enriched dynamical systems*, which can be considered to be limits of suitable sequences of maps tending towards a map with a parabolic fixed point.

In the third section we present the proof itself. We show that we can find a cubic polynomial P_{a_0} with a parabolic fixed point and a family of Lavaurs maps $\zeta \mapsto g_\zeta$, with $\zeta \in \overline{\mathbb{D}}$, such that (P_{a_0}, g_0) has a generalised superattracting periodic point but $J(P_{a_0}, g_\zeta)$ is not connected for all $\zeta \in \mathbb{S}^1$. From this we are able to deduce first that parameter slices through P_{a_0} have non-locally connected connectedness locus at P_{a_0}. From there, we further show that the full connectedness locus is not locally connected at P_{a_0}.

A.1 The Cubic Connectedness Locus

A.1.1 Spaces of Polynomials

Let $d \in \mathbb{Z}_{\geq 2}$. We define Poly_d to be the space set of degree d polynomials with complex coefficients, so that

$$\mathrm{Poly}_d = \{a_0 + a_1 z + \ldots + a_d z^d \mid a_0, \ldots, a_d \in \mathbb{C}\} \cong \mathbb{C}^d \times \mathbb{C}^*.$$

To any polynomial $P = a_0 + a_1 z + \ldots + a_d z^d \in \mathrm{Poly}_d$ we can naturally associate the map $z \mapsto a_0 + a_1 z + \ldots + a_d z^d$. In what follows, we shall make no distinction between the formal polynomial and the associated polynomial map.

As we are interested in studying the dynamics of polynomial maps we are not interested in distinguishing between polynomials with the same dynamics. To that end, we define the equivalence relation $\sim$ on Poly_d by $P \sim Q$ if and only if there exists an affine map $\phi : z \mapsto az + b$ such that $\phi \circ P = Q \circ \phi$, which we express as $P \overset{\phi}{\sim} Q$. We define the quotient space poly_d to be $\mathrm{Poly}_d / \sim$.

The quotient space poly_d is Hausdorff and locally compact. However if $d > 2$ then there exist $P \in \mathrm{Poly}_d$ such that $P \overset{\phi}{\sim} P$, where $\phi \neq Id$. Then it follows that for a closed neighbourhood U of P, $\{(P, Q) \in U \times U \mid P \sim Q\}$ is not a closed submanifold of $\mathrm{Poly}_d \times \mathrm{Poly}_d$ and so the quotient map $\pi : \mathrm{Poly}_d \to \mathrm{poly}_d$ does not endow poly_d with a complex manifold structure.

To avoid this difficulty, we will typically study a slice $\mathcal{S} \subset \mathrm{Poly}_d$ such that

$\mathcal{S}$ is a complex manifold and $\pi|_{\mathcal{S}} : \mathcal{S} \to \text{poly}_d$ is surjective and finite-to-one. A common choice, although not the one we shall be making, is to consider the set of *monic, centred* polynomials; those for which the coefficient of z^d is 1 and the coefficient of z^{d-1} is 0. In this case, the equivalence relation $\sim$ is induced by the group action of $\langle e^{\frac{2\pi i}{d-1}} \rangle$ on $\mathcal{S}$ by $e^{2\pi ip/(d-1)} \cdot P : z \mapsto e^{2\pi ip/(d-1)} P(e^{-2\pi ip/(d-1)} z)$.

A.1.2 Connectedness Loci

For a polynomial P, let $\mathcal{A}_\infty$ denote the attracting basin of infinity, which is connected by the maximum modulus principle. Then the Julia set of P, $J(P)$ is equal to the boundary $\partial \mathcal{A}_\infty$. Recall the following dichotomy in polynomial dynamics; either $J(P)$ is connected or some critical point of P escapes to infinity.

Theorem A.1.1.
Let $P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial map.

Then exactly one of the following is true. Either

i $J(P)$ is connected and $\mathcal{A}_\infty$ is biholomorphic to a disc, or

ii $J(P)$ has uncountably many connected components and $\mathcal{A}_\infty$ contains a critical point of P.

Notice that z is a critical point of P which escapes to infinity if and only if $\phi(z)$ is a critical point of $\phi \circ P \circ \phi^{-1}$ which escapes to infinity. Therefore if $P \sim Q$ then $J(P)$ is connected if and only if $J(Q)$ is connected.

Definition A.1.2. Let $d \in \mathbb{Z}_{\geq 2}$. We define the *degree d connectedness locus*, $\mathcal{C}_d$, to be the set

$$\mathcal{C}_d = \{[P] \in \text{poly}_d \mid J(P) \text{ is connected. } \}.$$

If $d = 2$ then the map $\pi : \{z^2 + c \mid c \in \mathbb{C}\} \to \text{poly}_2$, from the space of monic centred quadratic polynomials to poly_2 is bijective. Since 0 is the unique critical point of $z \mapsto z^2 + c$, we can identify $\mathcal{C}_2$ with the *Mandelbrot set*, $\mathcal{M} = \{c \in \mathbb{C} \mid 0 \notin \mathcal{A}_\infty(z^2 + c)\}$.

A.1.3 Local Connectivity

We finish this section with the definition of a space being locally connected. There are two distinct notions of a topological space X being locally connected at a point $x \in X$. We will say that X is *locally connected* at x if there exists a neighbourhood system of connected sets, $\mathcal{N}$, at x. That is, for any open set $U \ni x$ there exists

$N \in \mathcal{N}$ such that $N \subseteq U$ and each $N \in \mathcal{N}$ is a neighbourhood of x. We will say that X is *openly locally connected* at x if there exists a neighbourhood system of *open* connected sets, $\mathcal{U}$, at x.

Although the two notions differ pointwise, they are equivalent over the whole space as shown by the following lemma.

Lemma A.1.3.
Let X be a topological space.

Then X is locally connected at x for all $x \in X$ if and only if X is openly locally connected at x for all $x \in X$.

A.2 Parabolic Implosion

In this section we give an overview of the theory of parabolic implosion, which is concerned with the perturbation of parabolic fixed points.

Suppose that $f_0(z) = z + z^2 + \mathcal{O}(z^3)$ is a holomorphic map defined on a neighbourhood of 0 with a simple parabolic fixed point at 0. If we perturb f_0 to a map f then f will, in general, have two fixed points near 0. As show in figure A.1 there are broadly speaking two types of dynamics which can arise. In the first case, one of the fixed points of f is attracting and orbits under f_0 which were attracting towards 0 are perturbed to orbits under f which are attracted towards this fixed point. In this case, the dynamics of f are similar to those of f_0 and little interesting behaviour arises.

In the second case, neither fixed point of f is particularly strongly attracting. Orbits under f_0 which were attracted to 0 instead pass through the "gate" between the two fixed points and out of what was the repelling petal of f_0. This can result is dramatic changes to the global dynamics of the perturbation f compared to f_0. For example, if $f_0(z) = z^2 + 1/4$, which has a parabolic fixed point at $1/2$ then $K(f_0)$ is connected and $J(f_0)$ is a Jordan curve. However, if $\epsilon > 0$ is small then $f_\epsilon(z) = z^2 + 1/4 + \epsilon$ has two complex conjugate repelling fixed points near $1/2$ and the orbit of 0 under f_ϵ passes between them and escapes to infinity. Hence $K(f_\epsilon) = J(f_\epsilon)$ is a Cantor dust.

We can regain some control of the perturbed dynamics by extending our notion of Fatou coordinates to produce *pseudo-Fatou coordinates* for such maps. Unlike Fatou coordinates, pseudo-Fatou coordinates will only be defined onto a region containing a vertical strip, rather than a half-plane. Crucially, whilst a version of the Cylinder Theorem will still hold for perturbed maps, iteration induces a natural transit isomorphism from $\mathcal{C}_+$ to $\mathcal{C}_-$.

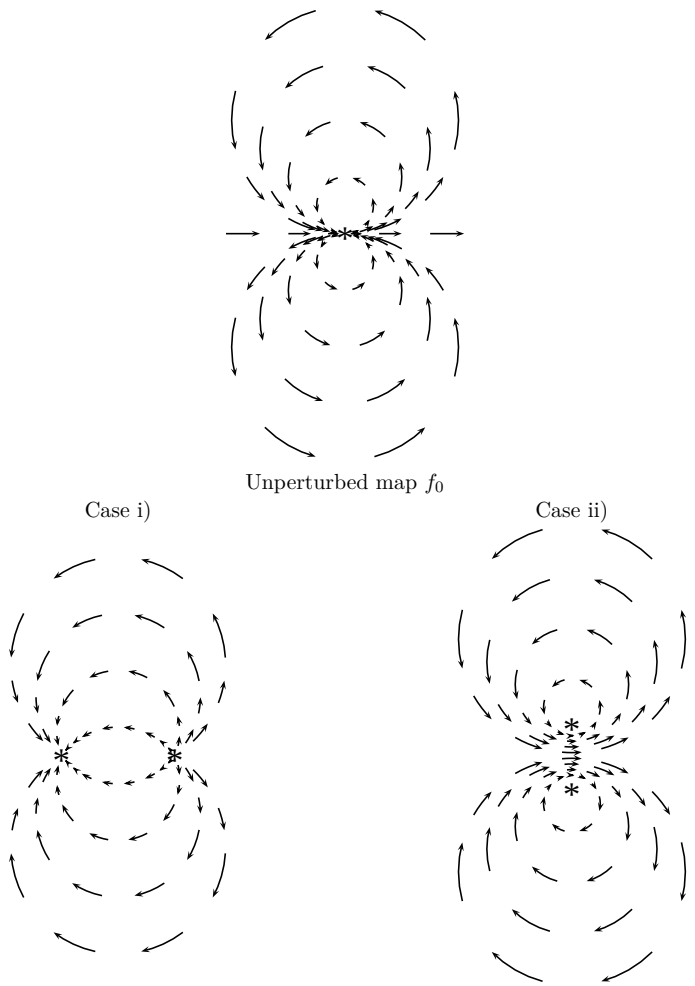

Figure A.1: A diagram showing perturbations of a map f_0 with a simple parabolic fixed point. In case i), the parabolic fixed point is perturbed into a primarily attracting and a primarily repelling fixed point. In case ii), the two fixed points of the perturbed map have local dynamics much closer to rotations.

A simplified intuition of parabolic implosion is then the following. Given a parabolic map f_0 and a transit isomorphism $\phi_0 : \mathcal{C}_{+,f_0} \to \mathcal{C}_{-,f_0}$, there exists a sequence of maps f_k with $f_k \to f_0$, such that $\phi_k \to \phi_0$ where ϕ_k is the natural transit isomorphism of f_k. In order of such a statement to be made meaningful we need to choose normalisations, which induce isomorphisms $\mathcal{C}_{\pm,f} \to \mathbb{C}/\mathbb{Z}$. We will also need to know that for fixed k the choice of f_k can be made continuously in ϕ, for ϕ in a simply connected neighbourhood of ϕ_0.

A.2.1 Topology and Normalisation

We begin by defining a topology on which the maps $f \mapsto \tilde{\Phi}_{\pm,f}$, sending f to its pseudo-Fatou coordinates, will be continuous. The domain of definition of $\tilde{\Phi}_{\pm,f}$ will vary will f, so the following topology will prove to be the most convenient.

Definition A.2.1. Let $\mathcal{HD}$ denote the set of pairs $(\mathcal{D}(f), f)$, where $\mathcal{D}(f) \subseteq \hat{\mathbb{C}}$ is an open set and $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$ is holomorphic.

Let $K \subset \mathcal{D}(f)$ be compact and $\epsilon > 0$. Then we define the neighbourhood $N_{K,\epsilon}(\mathcal{D}(f), f)$ to be

$$N_{K,\epsilon}(\mathcal{D}(f), f) = \{(\mathcal{D}(g), g) \mid K \subset \mathcal{D}(g) \text{ and } \sup_{z \in K} d(g(z), f(z)) < \epsilon\},$$

where $d(.,.)$ is the spherical metric.

The *compact-open topology together with the domain of definition* is the topology on $\mathcal{HD}$ generated by all such neighbourhoods.

Unfortunately this topology is not Hausdorff, as if we have two domains $\mathcal{D}(f) \subset \mathcal{D}'(f)$ and $(\mathcal{D}'(f), f) \in \mathcal{HD}$ then $(\mathcal{D}'(f), f)$ will be contained in any neighbourhood of $(\mathcal{D}(f), f)$.

In order to simplify notation, we will often suppress $\mathcal{D}(f)$ and refer only to f, or $f : \mathcal{D}(f) \to \hat{\mathbb{C}}$. We emphasise, however, that whenever we are working with the compact-open topology together with the domain of definition that all maps have an associated domain of definition and that any restriction or extension of a map is considered to be a different map.

Now, let $\mathcal{F}$ denote the set of maps f such that f is a map defined on a neighbourhood of 0 with $f(0) = 0$ and $f'(0) \neq 0$. For any $f \in \mathcal{F}$ we can write $f'(0) = \exp(2\pi i \alpha(f))$, where $-\frac{1}{2} < \Re(\alpha(f)) \leq \frac{1}{2}$. Let $\mathcal{F}_1 \subset \mathcal{F}$ denote the set

$$\mathcal{F}_1 = \{f \in \mathcal{F} \mid |\arg \alpha(f)| < \pi/4\}.$$

Let $\sigma(f) \in \mathcal{D}(f)$ denote the second fixed point of f near 0.

In what follows, it will be necessary to normalise the Fatou coordinates of a map f_0 with a simple parabolic fixed point relative to one another. We normalise so that

$$\tilde{\Phi}_{+,f_0}(z) - \tilde{\Phi}_{-,f_0}(z) \to 0 \text{ when } z \to 0$$

in $\mathcal{P}_{+,f_0} \cap \mathcal{P}_{-,f_0}$ and $\Im(\tilde{\Phi}_{\pm,f_0}(z)) \to +\infty$. It then follows that if $f_0(z) = z + az^2 + bz^3$ then

$$\tilde{\Phi}_{+,f_0}(z) - \tilde{\Phi}_{-,f_0}(z) \to -2\pi i \left(1 - \frac{b}{a^2}\right) \text{ when } z \to 0$$

in $\mathcal{P}_{+,f_0} \cap \mathcal{P}_{-,f_0}$ and $\Im(\tilde{\Phi}_{\pm,f_0}(z)) \to -\infty$.

A.2.2 Parabolic Implosion in General

The following general statements are propositions 3.2.2 and 3.2.3 from Shishikura [17]. The first concerns the existence and properties of pseudo-Fatou coordinates. More explicitly, by property i), suitable domains for the pseudo-Fatou coordinates exist and property ii) tells us that the coordinates themselves exist.

Property iii) shows that as $f \to f_0$, so that $\alpha(f) \to 0$, the time it takes the travel through the gate tends to infinity, so that the attracting and repelling pseudo-Fatou coordinates differ more and more. Property iv) gives us the continuity of the construction. We see both that the pseudo-Fatou coordinates can be chosen continuously in f and that, with the previous normalisation, the Fatou coordinates of f_0 are limits of the pseudo-Fatou coordinates of f as $f \to f_0$. The domains and maps are illustrated in figure A.2.

Proposition A.2.2.
Let f_0 be a holomorphic map defined on a neighbourhood of 0 with a simple parabolic fixed point at 0.

Then there exists a neighbourhood $\mathcal{N}_0$ of f_0 (in the topology of definition A.2.1) such that if $f \in \mathcal{N}_0 \cap \mathcal{F}_1$ then there exist Jordan domains $S_{\pm,f}$ and analytic functions $\tilde{\Phi}_{\pm,f} : S_{\pm,f} \to \mathbb{C}$ such that

i $S_{\pm,f}$ is bounded by an arc $l_{\pm,f}$ and its image $f(l_{\pm,f})$, such that $l_{\pm,f}$ joins the two fixed points 0 and $\sigma(f)$ of f, $l_{\pm,f} \cap f(l_{\pm,f}) = \{0, \sigma(f)\}$ and $S_{+,f} \cap S_{-,f} = \{0, \sigma(f)\}$.

ii $\tilde{\Phi}_{\pm,f}$ is defined, analytic and injective on a neighbourhood of $S_{\pm,f} \setminus \{0, \sigma(f)\}$.

iii If $z \in S_{+,f} \setminus f(l_{+,f})$ then there exists $n \geq 1$ such that $f^{\circ n}(z) \in S_{-,f} \setminus f(l_{-,f})$ and for the smallest such n

$$\tilde{\Phi}_{-,f}(f^{\circ n}(z)) = \tilde{\Phi}_{+,f}(z) - \frac{1}{\alpha(f)} + n.$$

iv The maps $f \mapsto \tilde{\Phi}_{\pm,f}$ are continuous. Further, when $f \to f_0$ in $\mathcal{F}_1$, the sets $S_{\pm,f} \to S_{\pm,0} \cup \{0\}$ in the Hausdorff metric and $\tilde{\Phi}_{\pm,f} \to \tilde{\Phi}_{\pm,f_0}$ in the compact-open topology with the domain of definition.

We denote the translation $w \mapsto w - \frac{1}{\alpha(f)}$ by T_f. This second proposition formalizes our intuition that this translation, which could be considered to satisfy $\tilde{\Phi}_{-,f} = T_f \circ \tilde{\Phi}_{+,f}$, corresponds to passing through the gate. Properties i) and ii)

define the inverse repelling pseudo-Fatou coordinates of f, $\tilde{\Psi}_{-,f}$, and the renormalisation, $\tilde{\mathcal{R}}_f$, respectively.

Property iii) shows that we can understand the long-term dynamics of f in terms of the composition $T_f \circ \tilde{\mathcal{R}}_f$. Property iv) shows that the construction is continuous and also the renormalisation could be considered to be the pseudo-horn map: $\tilde{\mathcal{R}}_f \to \tilde{h}_{f_0}$ when $f \to f_0$.

Proposition A.2.3.
Let f_0 be a holomorphic map defined on a neighbourhood of 0 with a simple parabolic fixed point at 0. Let $\tilde{\Psi}_{-,f_0}$ be the inverse repelling Fatou coordinate, which is defined on a region

$$Q_0 = \{w \in \mathbb{C} \mid |\arg(-w - \zeta_0)| < \frac{2\pi}{3}\}$$

for $\zeta_0 \in \mathbb{R}$.

Then there exists a neighbourhood $\mathcal{N}_0$ of f_0 such that if $f \in \mathcal{N}_0 \cap \mathcal{F}_1$ then there exist real constants $\zeta_0, \eta_0 > 0$ and analytic maps $\tilde{\Psi}_{-,f} : Q_f \to \hat{\mathbb{C}}$ and $\tilde{\mathcal{R}}_f : \{w \in \mathbb{C} \mid |\Im(w)| > \eta_0\} \to \mathbb{C}$, where

$$Q_f = \left\{ w \in \mathbb{C} \mid |\arg(-w - \zeta_0)| < \frac{2\pi}{3} \text{ and } |\arg(w + \frac{1}{\alpha(f)} - \zeta_0)| < \frac{2\pi}{3} \right\}$$

which satisfy the following:

i $\tilde{\Psi}_{-,f}(Q_f) \subseteq \mathcal{D}(f)$ *and if* $w, w+1 \in Q_f$ *then*

$$\tilde{\Psi}_{-,f}(w+1) = f \circ \tilde{\Psi}_{-,f}.$$

$\tilde{\Psi}_{-,f}(w) \to 0$ *when* $w \in Q_f$ *and* $\Im(w) \to +\infty$ *and* $\tilde{\Psi}_{-,f}(w) \to \sigma(f)$ *when* $w \in Q_f$ *and* $\Im(w) \to -\infty$.

ii If $|\Im(w)| > \eta_0$ *then*

$$\tilde{\mathcal{R}}_f(w+1) = \tilde{\mathcal{R}}_f(w) + 1.$$

This difference $\tilde{\mathcal{R}}_f(w) - w$ *tends to* 0 *when* $\Im(w) \to +\infty$ *and tends to a constant when* $\Im(w) \to -\infty$.

iii If $w \in Q_f$ *with* $|\Im(w)| > \eta_0$ *and* $w' = T_f \circ \tilde{\mathcal{R}}_f(w) + n \in Q_f$ *for some* $n \in \mathbb{Z}$ *then either*

$$f^{\circ n}(\tilde{\Psi}_{-,f}(w)) = \tilde{\Psi}_{-,f}(w') \text{ if } n \geq 0, \text{ or } f^{\circ -n}(\tilde{\Psi}_{-,f}(w')) = \tilde{\Psi}_{-,f}(w) \text{ if } n < 0.$$

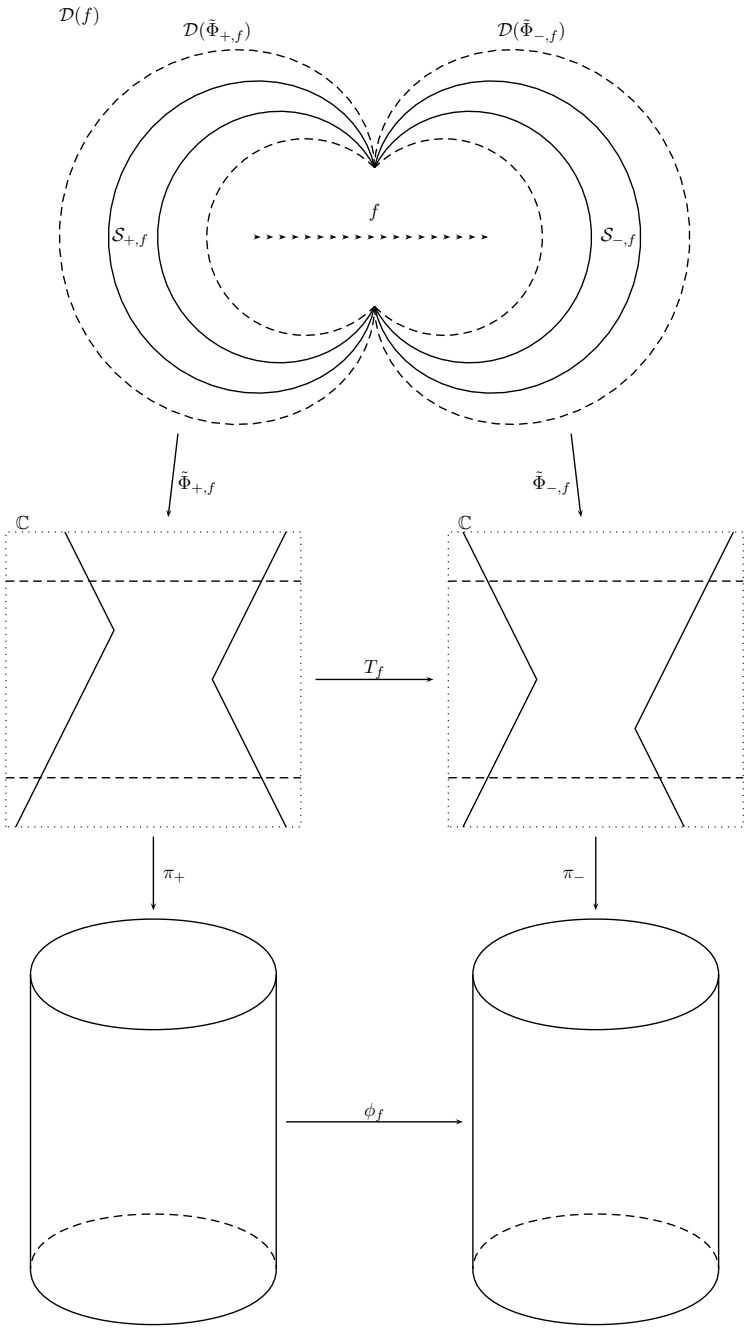

Figure A.2: A diagram showing the relationship between the maps in proposition A.2.2. The lower four maps always commute, $\phi_f \circ \pi_+ = \pi_- \circ T_f$. The upper half commutes up to sufficient iterates of f. That is, $T_f \circ T_n \circ \tilde{\Phi}_{+,f} = \tilde{\Phi}_{-,f} \circ f^{\circ n}$ for suitable large n such that both sides are defined.

iv The maps $f \mapsto \tilde{\Psi}_{-,f}$ and $f \mapsto \tilde{\mathcal{R}}_f$ are continuous and if $f \to f_0$ in $\mathcal{F}_1$ then

$$\tilde{\Psi}_{-,f} \to \tilde{\Psi}_{-,f_0} \text{ and } \tilde{\mathcal{R}}_f \to \tilde{h}_{f_0}.$$

Then convergence of $\tilde{\mathcal{R}}_f$ is uniform with respect to the Euclidean metric.

A.2.3 Parabolic Implosion in Families

For our application we shall be considering slices of the space of cubic polynomials. As such, we restrict our existence results to appropriate families of maps. More explicitly, if $\mathcal{G}$ is a family of holomorphic maps and $f_0 \in \mathcal{G}$ has a simple parabolic fixed point at 0, we shall say that $\mathcal{G}$ is *locally full* at f_0 if there exists a neighbourhood U of 0 and a continuous map $\mathrm{A} : U \to \mathcal{G}$ such that $\mathrm{A}(0) = f_0$ and $\alpha(\mathrm{A}(\alpha)) = \alpha$ for all $\alpha \in U$.

Let $c \in \mathbb{C}$ be a constant. Then if $\mathcal{G}$ is locally full at f_0 there exists $f \in \mathcal{N} \cap \mathcal{F}_1$ and $n \in \mathbb{Z}$ such that

$$c = n - \frac{1}{\alpha(f)}.$$

Moreover, given such an n there exists a sequence $f_k \to f_0$ such that

$$c = n + k - \frac{1}{\alpha(f_k)}, \text{ for all } k \in \mathbb{Z}_{>0}.$$

Let $\phi_0 : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ be the isomorphism induced by translation by c. Then each of the isomorphisms ϕ_{f_k} induced by translation by $-\frac{1}{\alpha(f)}$ is equal to ϕ_0. By proposition A.2.2 iii), ϕ_{f_k} is the natural transit isomorphism associated to the map f_k in terms of the pseudo-Fatou coordinates $\tilde{\Phi}_{\pm,f_k}$. Hence since c and thus ϕ_0 was arbitrary, we see that for any transit isomorphism $\phi_0 : \mathcal{C}_{+,f_0} \to \mathcal{C}_{-,f_0}$ there exists a sequence $f_k \to f$ in $\mathcal{F}_1 \cap \mathcal{G}$ such that $\phi_{f_k} = \phi_0$ in terms of the normalised Fatou coordinates for all $k \in \mathbb{Z}_{>0}$.

By applying the continuity statements of propositions A.2.2 and A.2.3 we can obtain the following proposition, which we shall directly apply to prove the main result of this appendix.

Proposition A.2.4.
Let f_0 be a holomorphic map with a simple parabolic fixed point at 0 and let $\mathcal{G}$ be a locally full family at f_0.

Let $\tilde{\Phi}_{\pm,f_0}$ be normalised Fatou coordinates, which induce conformal isomorphisms $\mathcal{C}_{\pm,f_0} \cong \mathbb{C}/\mathbb{Z}$. Let $\mathcal{C}_{\cong,f_0} \cong \mathbb{C}/\mathbb{Z}$ be the space of transit isomorphisms from $\mathcal{C}_{+,f_0}$ to $\mathcal{C}_{-,f_0}$.

Then for any continuous map $\varphi : \overline{\mathbb{D}} \to \mathcal{C}_{\cong,f_0}$, mapping ζ to an isomorphism $\varphi(\zeta) : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$, there exists a sequence of continuous maps $\zeta \mapsto f_{k,\zeta} \in \mathcal{G} \cap \mathcal{F}_1$ such that

i for each fixed $\zeta \in \overline{\mathbb{D}}$, $f_{k,\zeta} \to f_0$ as $k \to \infty$, and

ii for each $f = f_{k,\zeta}$, $\phi_f : \mathcal{C}_{+,f} \to \mathcal{C}_{-,f}$ is equal to $\varphi(\zeta) : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ in terms of the isomorphisms $\mathcal{C}_{\pm,f} \cong \mathbb{C}/\mathbb{Z}$ induced by the normalised pseudo-Fatou coordinates of f.

Further, if φ is injective then each of the maps $\zeta \mapsto f_{k,\zeta}$ is injective.

Proof. Let $c : \overline{\mathbb{D}} \to \mathbb{C}$ be a lift of φ, so that the following diagram commutes for each $\zeta \in \overline{\mathbb{D}}$.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "T_{c(\zeta)}"] \arrow[d, "\pi_{+,f_0}"'] & \mathbb{C} \arrow[d, "\pi_{-,f_0}"] \\
\mathbb{C}/\mathbb{Z} \arrow[r, "\varphi(\zeta)"] & \mathbb{C}/\mathbb{Z}
\end{tikzcd}$$

Let U be a neighbourhood of 0 and let $\mathrm{A} : U \to \mathcal{G}$ be as in the definition of a locally full family, such that $\mathrm{A}(U) \subseteq \mathcal{N}_0$ from proposition A.2.2. Let $I : \mathbb{C}^* \to \mathbb{C}^*$ be the inversion $z \mapsto 1/z$. Then, since $\overline{\mathbb{D}}$ and hence $c(\overline{\mathbb{D}})$ is compact, there exists $k_0 \in \mathbb{Z}_{>0}$ such that $I \circ T_{k_0} \circ c(\overline{\mathbb{D}}) \subset U \cap \{\alpha \mid |\arg \alpha| < \pi/4\}$.

Let $\zeta \mapsto f_{k,\zeta}$ be the sequence of maps

$$\zeta \mapsto \mathrm{A} \circ I \circ T_{k_0+k} \circ c(\zeta).$$

Then for each fixed ζ, $f_{k,\zeta} \to f_0$ as $k \to \infty$ by construction, since $I \circ T_{k_0+k} \circ c(\zeta) \to 0$ and $\mathrm{A}(0) = f_0$. Statement ii) is then simply a repeat of our previous discussion.

If φ is injective then any lift c of φ will also be injective. Translations and inversion are bijective where defined and A is a right inverse, hence injective. Thus $\zeta \mapsto f_{k,\zeta}$ is injective for all k. □

A.2.4 Lavaurs Maps

We conclude our overview with a brief discussion of Lavaurs maps and enriched dynamical systems. As we have seen, for a map $f \in \mathcal{F}_1 \cap \mathcal{N}_0$ the translation $T_f : w \mapsto w - \frac{1}{\alpha(f)}$ relates orbits passing into the gate through $S_{+,f}$ to orbits passing out of the gate through $S_{-,f}$ via the relation

$$\tilde{\Phi}_{-,f}(f^{\circ n}(z)) = T_n \circ T_f \circ \tilde{\Phi}_{+,f}(z)$$

for some n large enough. Precomposing by $\tilde{\Psi}_{-,f}$, we have that

$$f^{\circ n}(z) = \tilde{\Psi}_{-,f} \circ T_n \circ T_f \circ \tilde{\Phi}_{+,f}(z).$$

If we replace T_f by an arbitrary translation $T_c : w \mapsto w + c$ then we can apply this definition to f_0 to obtain a new function, called a *Lavaurs map* of f_0,

$$g_{T_c} : \mathcal{A} \to \hat{\mathbb{C}} \ , \ g_{T_c} = \tilde{\Psi}_{-,f_0} \circ T_c \circ \tilde{\Phi}_{+,f_0}.$$

$T_n \circ \tilde{\Phi}_{+,f} = \tilde{\Phi}_{+,f} \circ f^{\circ n}$ for all $n > 0$, so we don't need a T_n term in the definition of g_{T_c}.

As we saw in the proof of proposition A.2.4, if $T_c : w \mapsto w + c$ is any translation then there exists $k_0 \in \mathbb{Z}$ and a sequence $f_k \to f_0$ such that $T_c = T_k \circ T_{f_k}$ for all $k > k_0$. By propositions A.2.2 and A.2.3, $\tilde{\Phi}_{+,f_k} \to (\tilde{\Phi}_{+,f_0}, S_{+,f_0})$ and $\tilde{\Psi}_{-,f_k} \to (\tilde{\Psi}_{-,f_0}, \tilde{\Phi}_{-,f_0}(S_{-,f_0}))$. We therefore obtain the following proposition.

Proposition A.2.5.
Let f_0 be a holomorphic map defined on a neighbourhood of 0 with a simple parabolic fixed point at 0 and let $\mathcal{G} \ni f_0$ be a locally full family at f_0.

Let g_T be a Lavaurs map of f_0.

Then there exist sequences $f_k \to f_0$ in $\mathcal{G}$ and $n_k \to \infty$ in $\mathbb{Z}_{\geq 0}$ such that

$$\tilde{\Psi}_{-,f_k} \circ T_{n_k} \circ \tilde{\Phi}_{+,f_k} \to g_T|_{S_{+,f_0}}.$$

From this definition, we can intuitively think of a Lavaurs map g_T as recording what happens to orbits under f_0 after they have taken infinitely many steps to reach the parabolic fixed point from the attracting side and then passed through it to the repelling side. We can formalise this with the notion of an enriched dynamical system (f_0, g_T).

Definition A.2.6. Let f_0 be a holomorphic map defined on a neighbourhood of 0 with a simple parabolic fixed point at 0. Let g_T be a Lavaurs map of f_0.

We say that a pair $(n, m) \in \mathbb{Z}^2$ is *admissible* if either $m = 0$ and $n \geq 0$ or $m > 0$. We equip the set of admissible pairs with the lexicographic ordering, $(n, m) < (n', m')$ if and only if $m < m'$ or $m = m'$ and $n < n'$.

For an admissible pair (n, m), we define the function $(f_0, g_T)^{\circ(n,m)}$ by

$$(f_0, g_T)^{\circ(n,m)}(z) \begin{cases} f_0^{\circ n} \circ g_T^{\circ m}(z) \text{ if } n \geq 0 \\ \tilde{\Psi}_{-,f_0} \circ T_n \circ T \circ \tilde{\Phi}_{+,f_0} \circ g_T^{\circ m-1} \text{ if } m > 0 \end{cases} .$$

Note that both cases agree when n and m are both greater than 0.

We call this construction an *enriched dynamical system.*

We shall say that a sequence f_k converges to (f_0, g_T) if there exists a sequence $n_k \to \infty$ in $\mathbb{Z}$ such that $f_k \to f_0$ and $f_k^{\circ n_k} \to g_T$ as $k \to \infty$.

We can use the dynamics of the enriched dynamical system (f_0, g_T) to understand the dynamics of maps in a sequence $f_k \to (f_0, g_T)$.

We say that a point z_0 is a periodic point of (f_0, g_T) if there exists an admissible pair (n, m) with $m > 0$ such that $(f_0, g_T)^{\circ(n,m)}(z_0) = z_0$. By continuity, if f_k converges to (f_0, g_T) then for all k large enough f_k has a periodic point, z_k, such that $z_k \to z_0$. Further, if z_0 is an attracting fixed point of $(f_0, g_T)^{\circ(n,m)}$ then z_k is an attracting periodic point of f_k for all k large enough.

Alternatively, suppose that f_0 is a polynomial, so that $K(f_0)$ is well defined. Suppose that for some critical point z_0 of f_0 and some admissible pair (n, m), $(f_0, g_T)^{\circ(n,m)}(z_0) \notin K(f_0)$. Then there exists a sequence of critical points z_k of f_k with $z_k \to z_0$ as $k \to \infty$, such that for all k large enough $f_k^{\circ n}(z_k) \to \infty$ as $n \to \infty$. Hence $K(f_k)$ is not connected for all k large enough.

A.3 A Comb of Organ Pipes

We are now ready to begin proving that the cubic connectedness locus is not locally connected. To do so, we shall first consider the family of cubic polynomials

$$\{P_a(z) = z^3 + az^2 + z\},$$

which have a parabolic fixed point of multiplier 1 at 0. If $a \neq 0$ then P_a has a simple parabolic fixed point, whereas if $a = 0$ then the parabolic fixed point has multiplicity 2.

A.3.1 Finding a_0

To begin our proof, we need to find a parameter value a_0 such that the following are true:

i There exists a transit isomorphism ϕ_0 such that $\phi_0 \circ \hat{h}_{P_{a_0}}$ has two superattracting fixed points.

ii There exists a homotopically trivial Jordan curve $\zeta \mapsto \phi_\zeta \in \mathcal{C}_{\cong, P_{a_0}}$ such that for each $\zeta \in \mathbb{S}^1$, at least one critical value of $\phi_\zeta \circ \hat{h}_{P_{a_0}}$ does not lie in $\hat{K}(P_{a_0})$.

If $a = \sqrt{3}$ then P_a has a double critical point, whilst if $a \in (0, \sqrt{3})$ then the two distinct critical points are complex conjugate and P_a is real symmetric. Therefore if one critical point of P_a lies in the immediate parabolic basin then both do.

Consider a real symmetric map f with a parabolic fixed point at 0. By definition of f being real symmetric, $f(\overline{z}) = \overline{f(z)}$. Therefore if $\hat{z} \in \mathcal{C}_{\pm,f}$ and $z, z' \in \hat{\Phi}^{-1}_{\pm,f}(\hat{z})$, so that $f^{\circ m}(z) = z'$, then $f^{\circ m}(\overline{z}) = \overline{z'}$ and hence $\hat{\Phi}_{\pm,f}(\overline{z}) = \hat{\Phi}_{\pm,f}(\overline{z'})$. Thus complex conjugation descends to the cylinders $\mathcal{C}_{\pm,f}$. We shall use the notation $\overline{\hat{z}}$ to denote $\hat{\Phi}_{\pm,f}(\overline{\hat{\Phi}^{-1}_{\pm,f}(\hat{z})})$. We define the *Ecalle height* of a point $\hat{z} \in \mathcal{C}_{\pm,f}$, $H(\hat{z})$, to be $H(\hat{z}) := d(\hat{z}, \overline{\hat{z}})$, where the distance is induced via isomorphism with $\mathbb{C}/\mathbb{Z}$.

Returning to our family P_a with $a \in (0, \sqrt{3}]$, we label the critical values of $\hat{h}_{P_a}$ $\hat{w}_{a,\pm}$, where $\Im(\phi(\hat{w}_{a,+})) > \Im(\phi(\hat{w}_{a,-}))$ for any isomorphism $\phi : \mathcal{C}_{+,f} \to \mathbb{C}/\mathbb{Z}$ induced by a choice of Fatou coordinates. Notice that this labeling is well-defined even though the values of $\Im(\phi(\hat{w}_{a,\pm}))$ vary with the choice of ϕ. Also notice that $\overline{\hat{w}_{a,\pm}} = \hat{w}_{a,\mp}$ and so $H(\hat{w}_{a,+}) = H(\hat{w}_{a,-})$.

For $a \in (0, \sqrt{3}]$, let

$$m(a) := \inf\{H(w) \mid w \in \hat{\Phi}_{-,P_a}(J(P_a))\}, \text{ and}$$
$$M(a) := \sup\{H(w) \mid w \in \hat{\Phi}_{-,P_a}(J(P_a))\}.$$

We shall find a value $a_0 \in (0, \sqrt{3}]$ for which there exists a transit isomorphism ϕ_0 sending the critical values of $\hat{h}_{P_{a_0}}$ to critical points of $\hat{h}_{P_{a_0}}$, so that $\phi_0 \circ \hat{h}_{P_{a_0}}$ has two superattracting fixed points. By real symmetry, the existence of such a transit isomorphism is implied by $\hat{h}_{P_{a_0}}$ having a critical point $\hat{z}_{a_0,+}$ such that $H(\hat{z}_{a_0,+}) = H(\hat{w}_{a_0,\pm})$.

Lemma A.3.1.
Let P_a be the polynomial $P_a(z) = z + az^2 + z^3$.

Then there exists a parameter $a_0 \in (0, \sqrt{3})$ and a pair of complex conjugate critical points, $\hat{z}_{a_0,\pm}$, of $\hat{h}_{P_{a_0}}$ such that $H(\hat{z}_{a_0,\pm}) = H(\hat{w}_{a_0,\pm})$, and also

$$m(a_0) < H(\hat{w}_{a_0,\pm}) < \inf\left(\frac{m(a_0) + M(a_0)}{2}, \frac{3}{2}m(a_0)\right).$$

Proof. If we use our normalisation of the Fatou coordinates of P_a from section A.2, we see that $\tilde{h}_{P_a}(w) - w \to 0$ as $\Im(w) \to +\infty$ and $\tilde{h}_{P_a}(w) - \left(w - 2\pi i\left(1 - \frac{1}{a^2}\right)\right) \to 0$ as $\Im(w) \to -\infty$. Therefore, $\hat{h}'_{P_a}(+i\infty) = 1$ and $\hat{h}'_{P_a}(-i\infty) = e^{4\pi^2(1-1/a^2)}$.

Whilst the values of $\hat{h}'_{P_a}(\pm i\infty)$ depend on the choice of normalisation, the product $p_a = \hat{h}'_{P_a}(+i\infty)\hat{h}'_{P_a}(-i\infty)$ does not. Further, if $\phi : \mathcal{C}_+ \to \mathcal{C}_-$ is any transit isomorphism then $\phi'(+i\infty)\phi'(-i\infty) = 1$. Therefore, the product of the multipliers of the two fixed points of $\phi \circ \hat{h}_{P_a}$ at the ends of the cylinders, $\rho_a\overline{\rho}_a$, is independent of the choice of ϕ.

In particular, if $a \in (0, \sqrt{3}]$ and ϕ is real symmetric then the two multipliers

are complex conjugate. Hence $\rho_a\overline{\rho}_a = e^{4\pi^2(1-1/a^2)} \to 0$ as $a \to 0$ in $(0, \sqrt{3}]$. Thus for a small enough both ends of $\mathcal{C}_{-,P_a}$ are attracting fixed points of $\phi \circ \hat{h}_{P_a}$.

The set of critical points of $\hat{h}_{P_a}$ is equal to $\hat{\Phi}_{-,P_a}(\mathcal{O}^- \cap \mathcal{P}_{-,P_a})$, where $\mathcal{O}^-$ is the set of pre-critical points of P_a. In particular, the critical points of $\hat{h}_{P_a}$ accumulate on $\hat{\Phi}_{-,P_a}(J(P_a))$. On the other hand, if each of the ends of $\mathcal{C}_{-,P_a}$ is an attracting fixed point then each immediate basin must contain a critical value of $\hat{h}_{P_a}$. Therefore if a is small enough then there exist critical points $\hat{z}_{a,\pm}$ such that

$$m(a) < H(\hat{z}_{a,\pm}) < H(\hat{w}_{a,\pm}).$$

On the other hand, $P_{\sqrt{3}}$ has a unique critical point. Therefore $\hat{h}_{P_{\sqrt{3}}}$ has a unique critical value and so $H(\hat{w}_{\sqrt{3},\pm}) = 0$. By continuity of all functions in a, we can apply the intermediate value theorem to conclude that there exists a_0 such that $H(\hat{z}_{a_0,\pm}) = H(\hat{w}_{a_0,\pm})$. Further, since critical point accumulate on $\hat{\Phi}_{-,P_a}(J(P_a))$ and this is not a pair of round circles we can choose $\hat{z}_{a_0,\pm}$ to satisfy

$$H(\hat{z}_{a_0,\pm}) < \inf\left(\frac{m(a_0) + M(a_0)}{2}, \frac{3}{2}m(a_0)\right)$$

as required. □

As with the critical values, we label $\hat{z}_{a_0,\pm}$ such that $\Im(\hat{z}_{a_0,+}) > \Im(\hat{z}_{a_0,-})$. Now, let $\phi_0 : \mathcal{C}_{+,P_{a_0}} \to \mathcal{C}_{-,P_{a_0}}$ be the transit isomorphism which maps $\hat{w}_{a_0,+}$ to $\hat{z}_{a_0,+}$. Then ϕ_0 also maps $\hat{w}_{a_0,-}$ to $\hat{z}_{a_0,-}$. Therefore $\phi_0 \circ \hat{h}_{P_{a_0}}$ has two superattracting fixed points.

We have shown that our first requirement on the choice of a_0 is satisfied. It remains to check the second.

Lemma A.3.2.
Let a_0 and P_{a_0} be as in lemma A.3.1 and let ϕ_0 be as above.

Then there exists a homotopically trivial Jordan curve $\Gamma : \mathbb{S}^1 \to \mathcal{C}_{\cong,P_{a_0}}$, $\Gamma : \zeta \mapsto \phi_\zeta$, such that for each $\zeta \in \mathbb{S}^1$ at least one of $\phi_\zeta(\hat{w}_{a_0,\pm}) \notin \hat{K}(P_{a_0})$.

See Figure A.3. By our choice of a_0, there exists $\hat{z}_0 \in \mathcal{C}_{-,P_{a_0}} \setminus \hat{K}(P_{a_0})$ such that

$$2H(\hat{z}_{a_0,+}) - m(a) < H(\hat{z}_0) < 2m(a).$$

Since P_{a_0} is real symmetric, $z \in K(P_{a_0})$ if and only if $\overline{z} \in K(P_{a_0})$, which implies that $\overline{\hat{z}_0} \in \mathcal{C}_{-,P_{a_0}} \setminus \hat{K}(P_{a_0})$.

Let $\hat{z}_{\frac{1}{4}} = \overline{\hat{z}_0} + \hat{z}_{a_0,+} - \hat{z}_{a_0,-}$. Then $H(\hat{z}_{\frac{1}{4}}) = 2H(\hat{z}_{a_0,+}) - H(\hat{z}_0) < m(a)$ and

so $\hat{z}_{\frac{1}{4}} \in \mathcal{C}_{-,f} \setminus \hat{K}(P_{a_0})$. Since P_{a_0} is a polynomial, $\mathcal{C}_{-,f} \setminus \hat{K}(P_{a_0})$ is connected hence path connected. Let $\gamma : \mathbb{S}^1 \to \mathcal{C}_{-,f}$ be a curve defined piecewise such that

- $\gamma(1) = \hat{z}_0$ and $\gamma(i) = \hat{z}_{\frac{1}{4}}$,
- if $t \in [0, \frac{1}{4}]$ then $\gamma(e^{2\pi i t}) \in \mathcal{C}_{-,P_{a_0}} \setminus \hat{K}(P_{a_0})$,
- if $t \in [\frac{1}{4}, \frac{1}{2}]$ then $\gamma(e^{2\pi i t}) = \hat{z}_{\frac{1}{4}} + 4t$,
- if $t \in [\frac{1}{2}, \frac{3}{4}]$ then $\gamma(e^{2\pi i t}) = \gamma(e^{2\pi i(\frac{3}{4}-t)})$, and
- if $t \in [\frac{3}{4}]$ then $\gamma(e^{2\pi i t}) = \hat{z}_0 - 4t$.

Then γ is a homotopically trivial curve winding once around $\hat{z}_{a_0,+}$.

Let $\tilde{\Gamma} : \mathbb{S}^1 \to \mathcal{C}_{\cong,P_{a_0}}$, denoted $\tilde{\Gamma} : \zeta \mapsto \tilde{\Gamma}_\zeta$, be the curve which satisfies $\tilde{\Gamma}_\zeta(\hat{w}_{a_0,+}) = \gamma(\zeta)$ for all $\zeta \in \mathbb{S}^1$. By construction, if $t \in [\frac{1}{4}, \frac{1}{2}]$ then it follows that $H(\tilde{\Gamma}_{e^{2\pi i t}}(\hat{w}_{a_0,+})) = H(\hat{z}_{\frac{1}{4}}) < m(a)$ and so $\tilde{\Gamma}_\zeta(\hat{w}_{a_0,+}) \in \mathcal{C}_{-,P_{a_0}} \setminus \hat{K}(P_{a_0})$. On the other hand, $\tilde{\Gamma}_1(\hat{w}_{a_0,-}) = \hat{z}_{a_0,-} + \hat{z}_0 - \hat{z}_{a_0,+} = \overline{(\overline{\hat{z}_0} + \hat{z}_{a_0,+} - \hat{z}_{a_0,-})} = \overline{\hat{z}_{\frac{1}{4}}}$ and so $H(\tilde{\Gamma}_1(\hat{w}_{a_0,-})) = H(\hat{z}_{\frac{1}{4}}) < m(a)$. Therefore if $t \in [\frac{3}{4}, 1]$ then $H(\tilde{\Gamma}_{e^{2\pi i t}}(\hat{w}_{a_0,-}) < m(a)$ and so $\tilde{\Gamma}_{e^{2\pi i t}}(\hat{w}_{a_0,-}) \in \mathcal{C}_{-,P_{a_0}} \setminus \hat{K}(P_{a_0})$.

Thus for all $\zeta \in \mathbb{S}^1$ we have that either $\tilde{\Gamma}_\zeta(\hat{w}_{a_0,+}) \in \mathcal{C}_{-,P_{a_0}} \setminus \hat{K}(P_{a_0})$ or $\tilde{\Gamma}_\zeta(\hat{w}_{a_0,-}) \in \mathcal{C}_{-,P_{a_0}} \setminus \hat{K}(P_{a_0})$. By construction, $\tilde{\Gamma}$ is homotopically trivial and winds once around ϕ_0. To complete the proof, we observe that $\mathcal{C}_{-,P_{a_0}} \setminus \hat{K}(P_{a_0})$ is open, hence $\{\phi \in \mathcal{C}_{\cong,P_{a_0}} \mid \phi(\hat{w}_{a_0,s}) \in \mathcal{C}_{-,P_{a_0}} \setminus \hat{K}(P_{a_0})$ for some $s \in \{\pm\}\}$ is open and so we can perturb $\tilde{\Gamma}$ to a Jordan curve $\Gamma : \mathbb{S}^1 \to \mathcal{C}_{\cong,f_{a_0}}$ as in the statement. $\square$

A.3.2 The Parameter Slice is not Locally Connected

Let $\zeta \mapsto \phi_\zeta$ be a continuous injective map from $\overline{\mathbb{D}}$ to $\mathcal{C}_{\cong,P_0}$ such that the restriction to $\{0\} \cup \mathbb{S}^1$ agrees with our previous definition of ϕ_0 and lemma A.3.2. Let $\zeta \mapsto \tau_\zeta$ be a lift to the space of translations of $\mathbb{C}$, so that $\pi_{-,P_0} \circ \tau_\zeta = \phi_\zeta \circ \pi_{+,P_0}$ for all ζ.

We now consider enriched dynamical systems $(P_{a_0}, g_{\tau_\zeta})$. Let $z_{a_0,\pm}$ be the critical points of P_{a_0} such that $\hat{\Phi}_{+,P_{a_0}}(z_{a_0,\pm}) = \hat{w}_{a_0,\pm}$. Then since $\phi_0 \circ \hat{h}_{P_{a_0}}(\hat{z}_{a_0,\pm}) = \hat{z}_{a_0,\pm}$ if follows that if $\tilde{\Psi}_{-,P_{a_0}}(w) = z_{a_0,\pm}$ then $\tau_0 \circ \tilde{h}_{P_{a_0}}(w) = w - n$ for some $n \in \mathbb{Z}$. Since $\tilde{h}_f(w) = \tilde{\Phi}_{+,P_{a_0}}(z_{a_0,\pm})$ we have that $\tilde{\Psi}_{-,P_{a_0}} \circ T_n \circ \tau_0 \circ \tilde{\Phi}_{+,P_{a_0}}(z_{a_0,\pm}) = z_{a_0,\pm}$. Thus $(P_{a_0}, g_{\tau_0})^{\circ(n,1)}(z_{a_0}, \pm) = z_{a_0,\pm}$ and so $z_{a_0,\pm}$ is an attracting periodic point of (P_{a_0}, g_{τ_0}). Therefore, by our previous discussion we see that if $P_k \to (P_{a_0}, g_{\tau_0})$ then for all k large enough P_k has a pair of attracting periodic points $z_{k,\pm}$ such that $z_{k,\pm} \to z_{a_0,\pm}$ as $k \to \infty$.

On the other hand, if $\zeta \in \mathbb{S}^1$ then $\phi_\zeta \circ \hat{h}_{P_{a_0}}(\hat{z}_{a_0,s}) \notin \hat{K}(P_{a_0})$ for some $s \in \{\pm\}$. Therefore if $\tilde{\Psi}_{-,P_{a_0}}(w) = z_{a_0,s}$ then $\tau_\zeta \circ \tilde{h}_{P_{a_0}}(w) \notin \tilde{K}(P_{a_0})$, hence $g_{\tau_\zeta}(z_{a_0,s}) \notin K(P_{a_0})$.

Then again by our previous discussion we see that if $P_k \to (P_{a_0}, g_{\tau_\zeta})$ then for all k large enough $K(P_k)$ and $J(P_k)$ are not connected.

By applying proposition A.2.4 we obtain the following result.

Proposition A.3.3.
For $a \in \mathbb{C}$, let $\mathcal{S}_a \subset \mathrm{Poly}_3$ be the slice

$$\mathcal{S}_a = \{\lambda z + az^2 + z^3 \mid \lambda \in \mathbb{C}\}.$$

Let $\mathcal{C}_{3,a} \subset \mathcal{S}_a$ denote the connectedness locus of $\mathcal{S}_a$, so

$$\mathcal{C}_{3,a} = \{P \in \mathcal{S}_a \mid J(P) \text{ is connected }\}.$$

Then $\mathcal{C}_{3,a_0}$ is not locally connected.

Proof. Let $\zeta \mapsto \phi_\zeta$ be as above, so that $\phi_0 \circ \hat{h}_{P_{a_0}}(\hat{z}_{a_0,\pm}) = \hat{z}_{a_0,\pm}$ and for all $\zeta \in \mathbb{S}^1$ there exists $s \in \{\pm\}$ such that $\phi_\zeta \circ \hat{h}_{P_{a_0}}(\hat{z}_{a_0,s}) \notin \hat{K}(P_{a_0})$.

The family $\mathcal{S}_{a_0}$ is locally full at P_{a_0}. Hence by proposition A.2.4 there exists a sequence of continuous injective maps $\zeta \mapsto P_{k,\zeta}$ such that

i for each fixed $\zeta \in \overline{\mathbb{D}}$, $P_{k,\zeta} \to P_{a_0}$ as $k \to \infty$, and

ii for each $P = P_{k,\zeta}$, $\phi_P : \mathcal{C}_{+,P} \to \mathcal{C}_{-,P}$ is equal to ϕ_ζ in terms of the isomorphisms $\mathcal{C}_{\pm,P} \cong \mathbb{C}/\mathbb{Z}$ induced by the normalised pseudo-Fatou coordinates of P.

By the previous discussion, for all k large enough $P_{k,0}$ has a pair of distinct attracting cycles. Hence, since $P_{k,0}$ is a cubic polynomial and so has only two critical points, $J(P_{k,0})$ is connected for all k large enough.

On the other hand, again by our previous discussion we see that for each $\zeta \in \mathbb{S}^1$ there exists k_ζ such that for all $k > k_\zeta$, $J(P_{k,\zeta})$ is not connected. Furthermore, $\mathcal{C}_{3,a_0}$ is closed, hence $\{\zeta \in \mathbb{S}^1 \mid J(P_{k,\zeta}) \text{ is not connected }\}$ is open for all k. Therefore by compactness of $\mathbb{S}^1$ there exists k_0 such that for all $k > k_0$ and all $\zeta \in \mathbb{S}^1$, $J(P_{k,\zeta})$ is not connected. We can take a maximum to assume without loss of generality that for all $k > k_0$, $J(P_{k,0})$ is connected.

Thus for all $k, k' > k_0$, $P_{k,0} \in \mathcal{C}_{3,a_0}$ is separated from $P_{k',0}$ by the Jordan curves $\zeta \mapsto P_{k,\zeta} \in \mathcal{S}_{a_0} \setminus \mathcal{C}_{3,a_0}$ and $\zeta \mapsto P_{k',\zeta} \in \mathcal{S}_{a_0} \setminus \mathcal{C}_{3,a_0}$ defined on $\mathbb{S}^1 \subset \overline{\mathbb{D}}$. Since $P_{k,0} \to P_{a_0}$ as $k \to \infty$ any neighbourhood of P_{a_0} must contain multiple $P_{k,0}$'s and hence $\mathcal{C}_{3,a_0}$ is not locally connected at P_{a_0}, hence is not locally connected. $\square$

By extending our construction to a small neighbourhood K of a_0 we can show that the full set is not locally connected at a_0. Notice that this does not

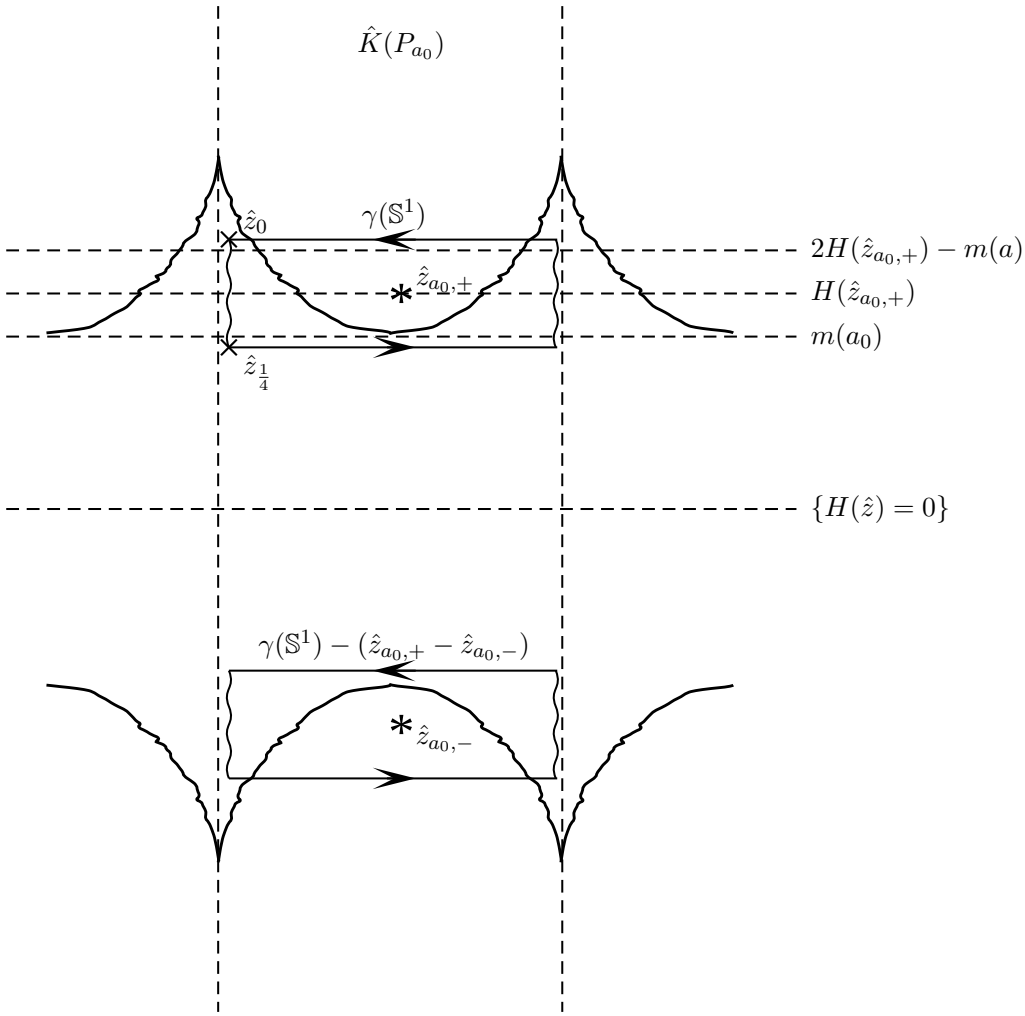

Figure A.3: A diagram illustrating the proof of proposition A.3.3. The transit isomorphism ϕ_0 sends the critical values $\hat{w}_{a_0,\pm}$ of $\hat{h}_{P_{a_0}}$ to the critical points $\hat{z}_{a_0,\pm}$. If $\zeta \in \mathbb{S}^1$ then ϕ_ζ maps $\hat{w}_{a_0,+}$ to $\gamma(\zeta)$ and $\hat{w}_{a_0,-}$ to $\gamma(\zeta) - (\hat{z}_{a_0,+} - \hat{z}_{a_0,-})$. For all $\zeta \in \mathbb{S}^1$, at least one of these points lies in $\mathcal{C}_{-,P_{a_0}} \setminus \hat{K}(P_{a_0})$.

simply follow from each individual slice $\mathcal{C}_{3,a}$ not being locally connected at P_a for a in a neighbourhood of a_0. We need to extend our maps $\zeta \mapsto P_{k,\zeta} = P_{k,a_0,\zeta}$ to injective maps $(a, \zeta) \mapsto P_{k,a,\zeta}$ in order to show that the surfaces $\{P_{k,a,0} \mid a \in K\}$ are separated from one another by hypercylinders $\{P_{k,a,\zeta} \mid a \in K \text{ and } \zeta \in \mathbb{S}^1\}$.

Proposition A.3.4.
Let $\mathcal{S}$ be the space $\{\lambda z + az^2 + z^3 = P_{a,\lambda}\}$ and let $\tilde{\mathcal{C}}_3 \subset \mathcal{S}$ be the connectedness locus $\tilde{\mathcal{C}}_3 = \{P_{a,\lambda} \mid J(P_{a,\lambda})$ is connected. $\}$.

Then $\tilde{\mathcal{C}}_3$ is not locally connected.

Proof. First, note that since $\hat{z}_{a_0,\pm}$ are fixed points of $\phi_0 \circ \hat{h}_{P_{a_0}}$ of multiplier not equal to 1 and $\hat{h}_{P_a}$ varies continuously in a, there exist continuous maps $a \mapsto \hat{z}_{a,\pm}$ sending a to a fixed point $\hat{z}_{a,\pm}$ of $\phi_0 \circ \hat{h}_{P_a}$. Further, $(\phi_0 \circ \hat{h}_{P_a})'(\hat{z}_{a,\pm})$ is continuous in a. Therefore there exists a neighbourhood U_0 of a_0 such that $\hat{z}_{a,\pm}$ are both attracting fixed points of $\phi_0 \circ \hat{h}_{P_a}$ for all $a \in U_0$.

On the other hand, if $\zeta \in \mathbb{S}^1$ then it follows from continuity that for all $n \in \mathbb{Z}_{\geq 0}$, $a \mapsto P_a^{\circ n} \circ g_{\tau_\zeta}(z_{a,\pm})$ is continuous, where $z_{a,\pm}$ are the critical points of P_a and g_{τ_ζ} is the Lavaurs map of P_a corresponding to τ_ζ. In particular, since $P_{a_0}^{\circ n} \circ g_{\tau_\zeta}(z_{a_0,s}) \to \infty$ as $n \to \infty$ for some $s \in \{\pm\}$, there exists a neighbourhood U_ζ of a_0 such that for all $a \in U_\zeta$, $P_a^{\circ n} \circ g_{\tau_\zeta}(z_{a,s}) \to \infty$ as $n \to \infty$. Further, by the continuity of g_{τ_ζ} in ζ and the compactness of $\mathbb{S}^1$, there exists an open neighbourhood $U_{\mathbb{S}^1}$ such that for all $\zeta \in \mathbb{S}^1$ and $a \in U_{\mathbb{S}^1}$ there exists $s \in \{\pm\}$ such that $P_a^{\circ n} \circ g_{\tau_\zeta}(z_{a,s}) \to \infty$ as $n \to \infty$.

Let $K \subset U_0 \cap U_{\mathbb{S}^1}$ be a compact neighbourhood of a_0 homeomorphic to $\overline{\mathbb{D}}$ and consider the map $F : K \times \{|\Re(\alpha)| < 1/2\} \to \mathcal{S}$ given by

$$F : (a, \alpha) \mapsto e^{2\pi i \alpha} z + a z^2 = z^3.$$

We now extend our constructions from the proof of proposition A.2.4. Let $\zeta \mapsto \tau_\zeta$ be a lift of $\zeta \mapsto \phi_\zeta$ to the space of translations. Let $k_0 \in \mathbb{Z}$ be large enough that $|\arg(\tau_\zeta(0) + k_0)| < \pi/4$ for all $\zeta \in \overline{\mathbb{D}}$. We define a sequence of continuous injective maps $K \times \overline{\mathbb{D}} \to \mathcal{S}$ given by

$$(a, \zeta) \mapsto F\left(a, -\frac{1}{\tau_\zeta(0) + k_0 + k}\right) =: P_{k,a,\zeta}.$$

For fixed (a, ζ) we have that $P_{k,a,\zeta} \to (P_a, g_{\tau_\zeta})$ as $k \to \infty$. Then as before, for all $a \in K$ there exists $k_{a,0}$ such that for all $k > k_{a,0}$, $P_{k,a,0}$ has a pair of distinct attracting cycles and so $J(P_{k,a,0})$ is connected. Also, for all $a \in K$ and $\zeta \in \mathbb{S}^1$ there exists $s \in \{\pm\}$ and $k_{a,\zeta}$ such that for all $k > k_{a,\zeta}$, $P_{k,a,\zeta}^{\circ n}(z_{a,s}) \to \infty$ as $n \to \infty$, so $J(P_{k,a,0})$ is not connected.

Further, the two sets $\{a \in K \mid P_{k,a,0}$ has two distinct attracting cycles. $\}$ and $\{(a, \zeta) \in K \times \mathbb{S}^1 \mid J(P_{k,a,\zeta})$ is not connected. $\}$ are open for all k. Hence by compactness there exists κ_0 such that for all $k > \kappa_0$ and all $a \in K$, $J(P_{k,a,0})$ is connected but $J(P_{k,a,\zeta})$ is not connected for all $\zeta \in \mathbb{S}^1$.

Thus for all $k, k' > \kappa_0$, we see that the connectedness locus $\tilde{\mathcal{C}}_3 \subset \mathcal{S}$ contains the surfaces $\{P_{k,a,0} \mid a \in K\}$ and $\{P_{k',a,0} \mid a \in K\}$. These are separated from one another by the sets $\{P_{k,a,\zeta} \mid a \in K \text{ and } \zeta \in \mathbb{S}^1\}$ and $\{P_{k',a,\zeta} \mid a \in K \text{ and } \zeta \in \mathbb{S}^1\}$,

each of which is homeomorphic to the hypercylinder $\mathbb{S}^1 \times \overline{\mathbb{D}}$.

Since the sets $\{P_{k,a,0} \mid a \in K\}$ accumulate on $\{P_a \mid a \in K\}$, we have show that $\tilde{\mathcal{C}}_3$ is not locally connected at P_{a_0}. $\square$

We are almost ready to prove the main theorem. It remains to check that our comb of organ pipes constructed above does not collapse under the quotient map $\pi : \mathcal{S} \to \mathrm{poly}_3$. Our choice of family $\mathcal{S}$ ensures that this is not the case, as the following lemma shows.

Lemma A.3.5.
Let $\mathcal{S} = \{\lambda z + az^2 + z^3 = P_{a,\lambda} \in \mathrm{Poly}_3 \mid a, \lambda \in \mathbb{C}\}$. Let $a_1 \in \mathbb{C}^*$.

Then there exist neighbourhoods K of a_1 and N of 1 such that if

$$C = \{P_{a,\lambda} \in \mathcal{S} \mid a \in K \text{ and } \lambda \in (\{1\} \cup \{e^{2\pi i\alpha} \mid |\arg \alpha| < \pi/4\}) \cap N\}$$

then the projection $\pi : C \to \mathrm{poly}_3$ is injective.

That is, each $P \in C$ is a unique affine conjugacy class representative.

Proof. Suppose that $P, Q \in \mathcal{S}$ and that ψ is an affine conjugacy from P to Q. Then ψ must map fixed points of P to fixed points of Q. In particular, if $P, Q \in C$ then ψ must map a fixed point of P with multiplier in $(\{1\} \cup \{e^{2\pi i\alpha} \mid |\arg \alpha| < \pi/4\}) \cap N$ to the fixed point 0 of Q. We first show that if K and N are small enough then 0 is the only such fixed point of P.

Recall that the *residue fixed point index* of a holomorphic map f at a fixed point z_0 is defined to be

$$\imath(f, z_0) = \frac{1}{2\pi i} \oint \frac{dz}{z - f(z)},$$

integrating around a positively oriented loop winding once around z_0 and around no other fixed points of f.

If γ is a fixed positively oriented loop then $f \mapsto \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z-f(z)}$ is continuous on a neighbourhood of f. Thus, if U is an open set then

$$f \mapsto \sum_{f(z)=z\in U} \imath(f, z)$$

is continuous on a neighbourhood of f.

If z_0 is a fixed point of f of multiplier $\lambda \neq 1$ then $\imath(f, z_0) = \frac{1}{1-\lambda}$, whilst if z_0 is a parabolic fixed point of f locally conjugate to $(z \mapsto z + az^{n+1} + bz^{2n+1} + \mathcal{O}(z^{2n+1}), 0)$ then $\imath(f, z_0) = \frac{b}{a^2}$. Finally, recall that if f is a polynomial then by the Rational Fixed

Point Theorem,

$$\sum_{f(z)=z} \imath(f,z) = 0.$$

Returning to our family of maps, we see that if $a \neq 0$ then $\imath(P_{a,1},0) = \frac{1}{a^2}$, whilst the other fixed point not equal to 0 has residue index $-\frac{1}{a^2}$. For the maps $P_{a,\lambda} = (z \mapsto \lambda z + az^2 + z^3) \in C$, let $\lambda_\sigma(P_{a,\lambda}) = P'_{a,\lambda}(\sigma(P_{a,\lambda}))$ be the multiplier of the other fixed point of $P_{a,\lambda}$ near 0. Then we have that

$$\frac{1}{1-\lambda} + \frac{1}{1-\lambda_\sigma(P_{a,\lambda})} \to \frac{1}{a^2} \text{ as } \lambda \to 1.$$

Letting $\lambda = e^{2\pi i\alpha(a,\lambda)}$ and $\lambda_\sigma(P_{a,\lambda}) = e^{2\pi i\beta(a,\lambda)}$, choosing α and β such that $-1/2 < \Re(\alpha), \Re(\beta) \leq 1/2$, we see that

$$\frac{1}{1-\lambda} \sim \frac{-1}{2\pi i\alpha} \text{ and } \frac{1}{1-\lambda_\sigma} \sim \frac{-1}{2\pi i\beta} \text{ as } \lambda \to 1, \text{ so } \alpha, \beta \to 0.$$

Thus

$$\frac{-1}{2\pi i\alpha} + \frac{-1}{2\pi i\beta} \to \frac{1}{a^2} \text{ as } \alpha, \beta \to 0.$$

$|\arg(\alpha)| < \pi/4$, so $\frac{\pi}{4} < \frac{-1}{2\pi i\alpha} < \frac{3\pi}{4}$ and hence $\Im(\frac{-1}{2\pi i\alpha}) \to +\infty$ as $\alpha \to 0$. Thus $\Im(\frac{-1}{2\pi i\beta}) \to -\infty$ as $\beta \to 0$, hence $|\arg(\beta)| > 3\pi/4$. Therefore there exists a neighbourhood K' of a_0 and N of 1 such that if $P, Q \in C$ and ψ is an affine conjugacy from P to Q then ψ fixes 0.

The coefficient of z^3 on all $P \in \mathcal{S}$ is 1. Hence $(\psi'(0))^2 = 1$ and so either $\psi = Id$ or $\psi : z \mapsto -z$. The latter conjugates $P_{a,\lambda}$ to $P_{-a,\lambda}$. Hence if K is small enough then each $P \in C$ is a unique conjugacy class representative and so $\pi : C \to \text{poly}_3$ is injective. □

Theorem A.3.6.
Let $\mathcal{C}_3 \subset \text{poly}_3$ be the cubic connectedness locus.

Then $\mathcal{C}_3$ is not locally connected.

Proof. By proposition A.3.4, $\tilde{\mathcal{C}}_3$ is not locally connected at P_{a_0}. Let K, N and $C \subset \mathcal{S}$ be as in lemma A.3.5.

Let $(a,\zeta) \mapsto P_{k,a,\zeta}$ be the maps from proposition A.3.4, so that for all k large enough $J(P_{k,a,0})$ is connected by $J(P_{k,a,\zeta})$ is not connected for all $a \in K$, $\zeta \in \mathbb{S}^1$. Let $\mathcal{U} \subset \text{poly}_3$ be a small neighbourhood of $[P_{a_0}]$. We can assume that $\mathcal{U}$ is small enough that if $\tilde{\mathcal{U}} \subset \pi^{-1}(\mathcal{U})$ is the connected component containing P_{a_0} then $\{a \mid P_{a,\lambda} \in \tilde{\mathcal{U}} \text{ for some } \lambda \in \mathbb{C}\} \subset K$ and $\{\lambda \mid P_{a,\lambda} \in \tilde{\mathcal{U}} \text{ for some } a \in \mathbb{C}\} \subset N$.

Let $\mathring{C}$ denote the interior of C. Then by lemma A.3.5, $\pi : \tilde{\mathcal{U}} \cap \mathring{C} \to \mathcal{U}$ is a homeomorphism onto its image. Thus for all $k \neq k'$ the sets $\{[P_{k,a,0}] \mid a \in K\} \cap \mathcal{U}$ and $\{[P_{k',a,0}] \mid a \in K\}$ are separated from one another by the 'organ pipe' sets $\{[P_{k,a,\zeta}] \mid a \in K \text{ and } \zeta \in \mathbb{S}^1\}$ and $\{[P_{k',a,\zeta}] \mid a \in K \text{ and } \zeta \in \mathbb{S}^1\}$. For all k large enough, $\{[P_{k,a,0}] \mid a \in K\} \subset \mathcal{C}_3$ but $\{[P_{k,a,\zeta}] \mid a \in K \text{ and } \zeta \in \mathbb{S}^1\} \cap \mathcal{C}_3 = \emptyset$. Hence $\mathcal{C}_3 \cap \mathcal{U}$ is not connected.

Since $\mathcal{U} \ni [P_{a_0}]$ was arbitrary, $\mathcal{C}_3$ is not locally connected at $[P_{a_0}]$ and hence is not locally connected. $\square$

We conclude this appendix by showing some renderings of relevant parameter spaces and Julia sets, which were produced using Ultra Fractal (https://www.ultrafractal.com/).

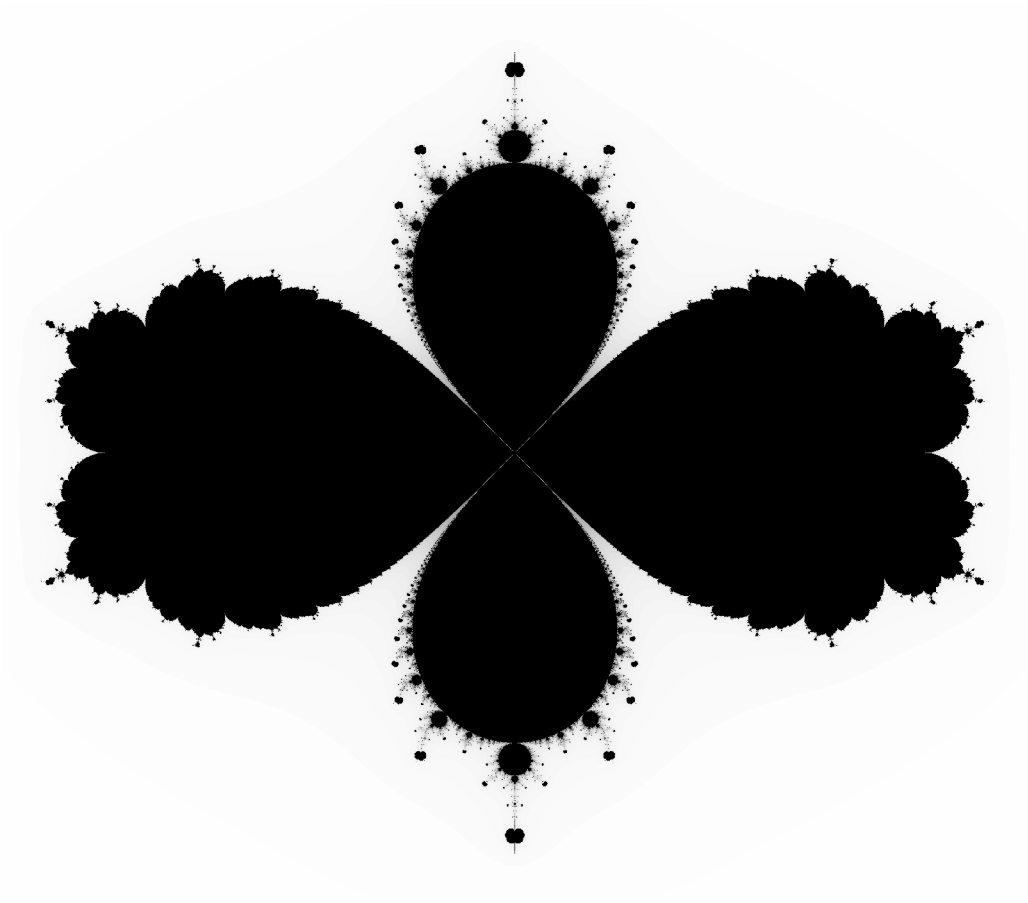

Figure A.4: The connectedness locus in the simple parabolic parameter space $\{z + az^2 + z^3 \mid a \in \mathbb{C}\}$. Parameters in the two left and right 'wings' give rise to maps for which both critical points lie in the immediate basin of the parabolic fixed point, whilst those in the regions that resemble deformed Mandelbrot sets give rise to maps with one attracting cycle. a_0 lies in the right wing.

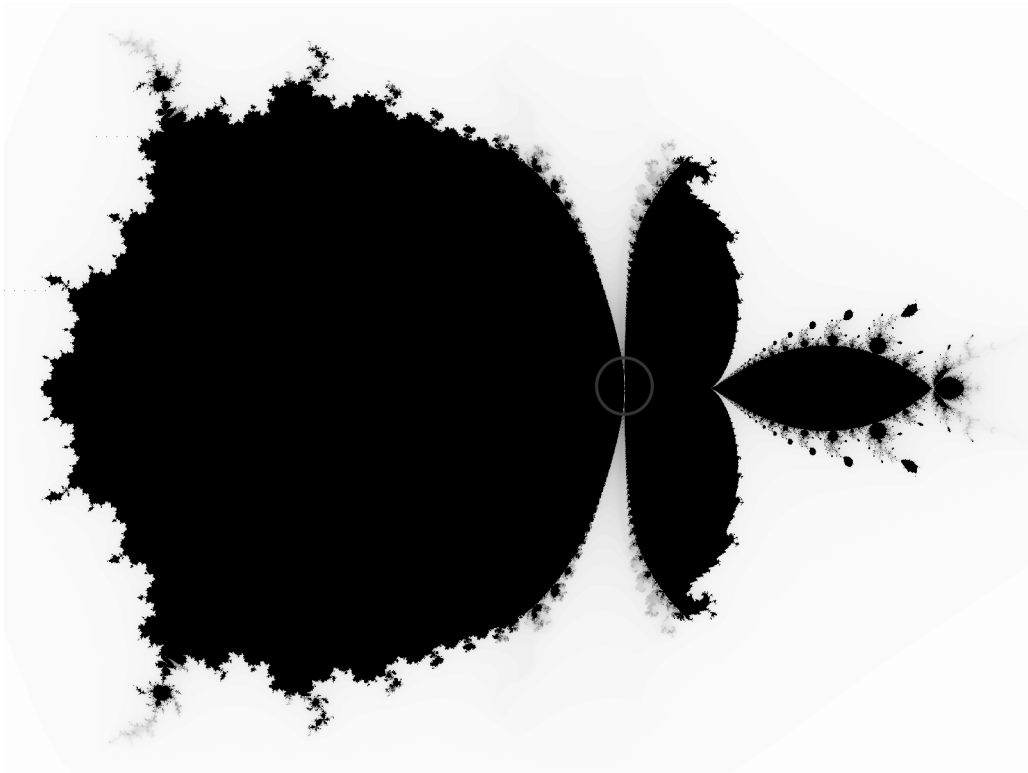

Figure A.5: The connectedness locus in the parameter slice $\mathcal{S}_{1.05}$. The parameter $\lambda = 1$ is circled in grey. The comb intersects this slice in the 'valley' near $\lambda = 1$. The next figure shows a highly zoomed view of this valley.

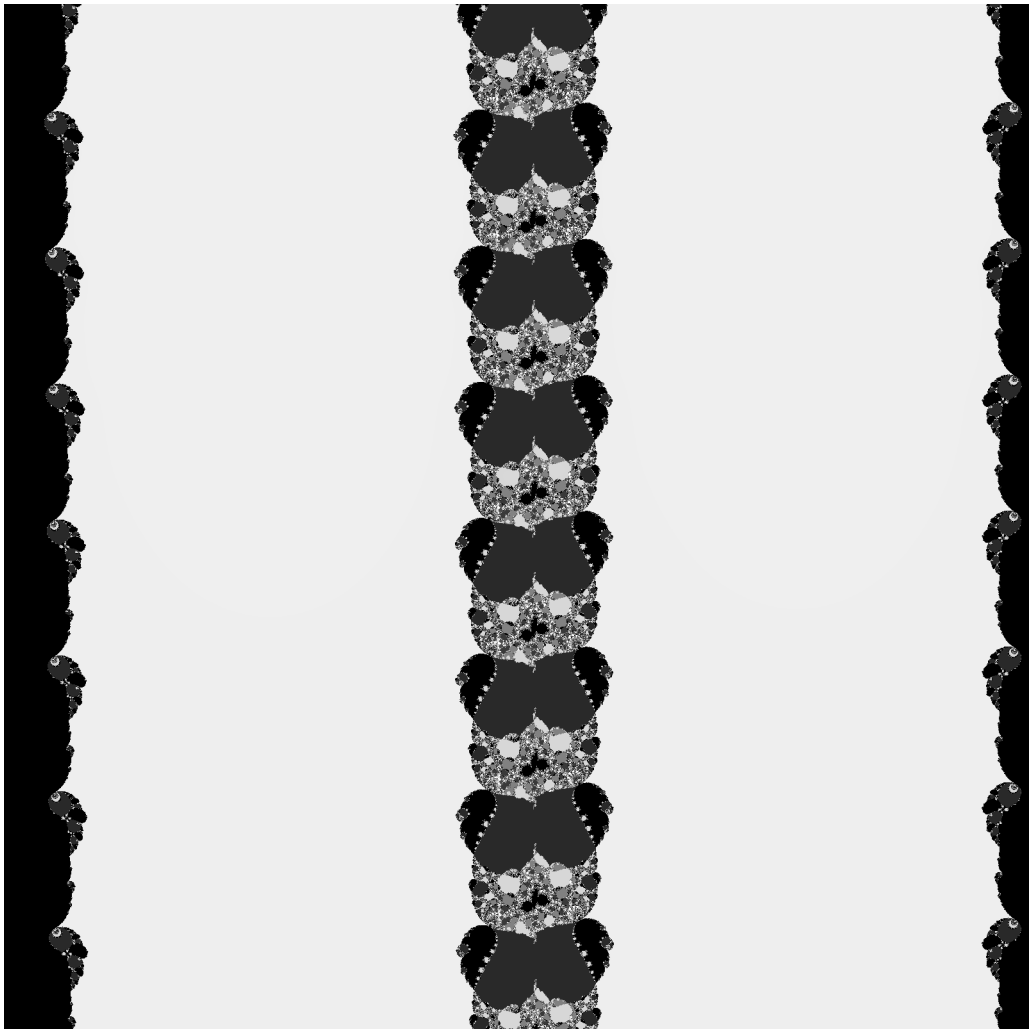

Figure A.6: A zoomed in view showing the repeating structure near $\lambda = 1$. Each repeated unit corresponds to a different lift from $\mathcal{C}_{\cong,P_{1.05}}$. More precisely, this figure is centred on $\lambda = 0.99999583752748095 + 0.0029648654950769685i$ and magnified about $300,000$ times relative to figure A.5.

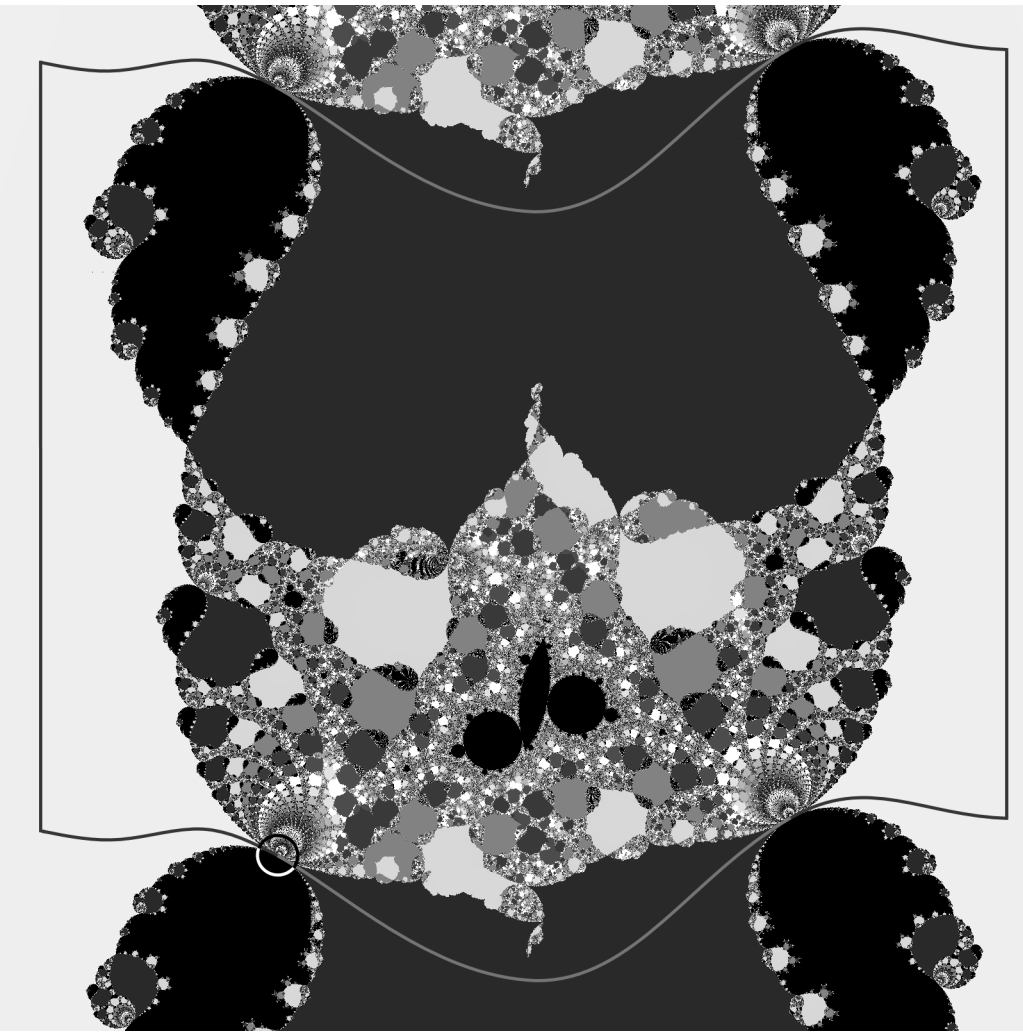

Figure A.7: A zoomed in picture of one of the repeated units. The lens-shaped region in the lower centre between two Mandelbrot-like sets contains $P_{k,0}$. A possible path $t \mapsto P_{k,e^{2\pi i t}}$ has been drawn in grey, which clearly separates the successive units except possibly that the four regions where they nearly touch. There is no significance to the different shades of grey on the path, they are purely to help visibility. The bottom left such region has been circled, and the next figure shows a highly zoomed image of this region.

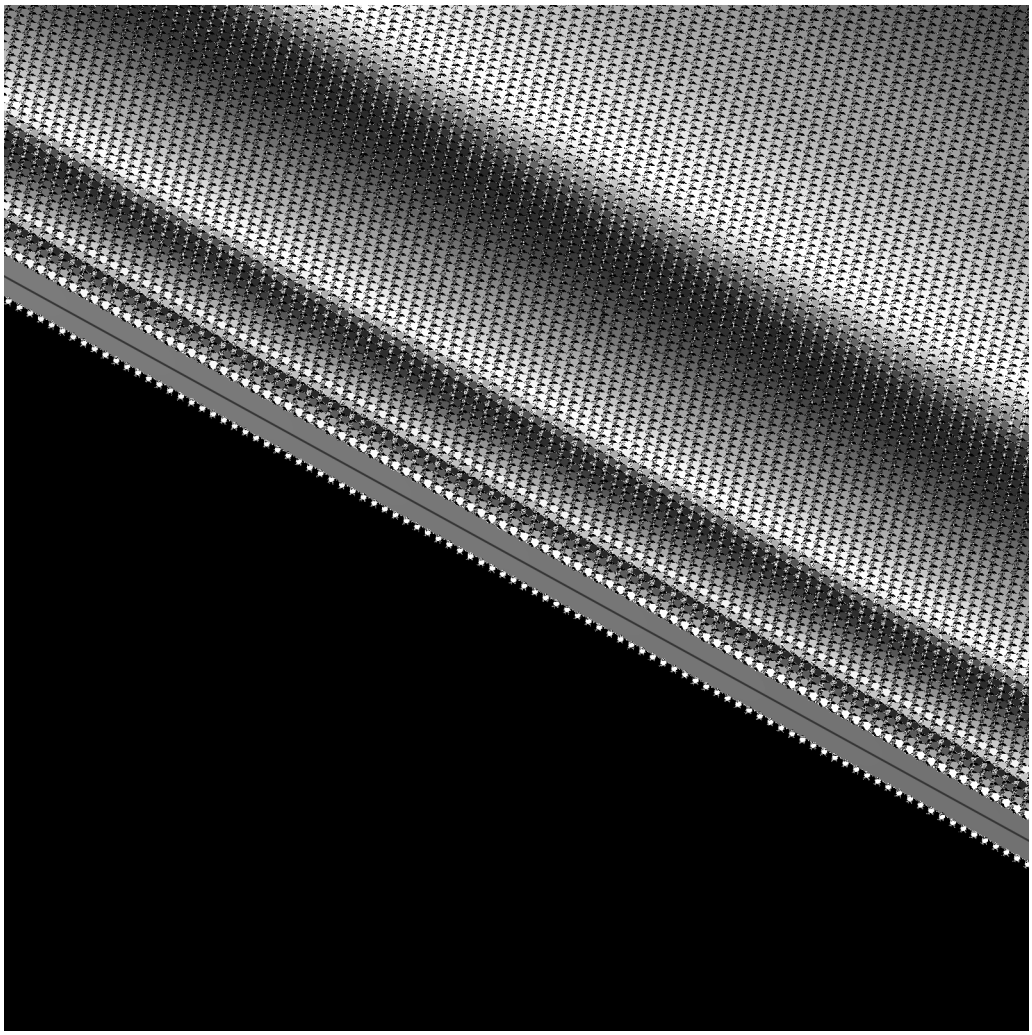

Figure A.8: A further zoom in on the region where the two units almost touch showing that there is a gap between them, with the curve from figure A.7 drawn in. Thus we see that we can indeed find a sequence of Jordan curves converging to $P_{1.05}$ in $\mathcal{S}_{1.05} \backslash \mathcal{C}_{3,1.05}$, each of which bounds a different subset of $\mathcal{C}_{3,1.05}$. More precisely, this image is centred at $\lambda = 0.9999955463215048889095 + 0.0029648369309698688954 6i$ and magnified about $100,000$ times from figure A.7.

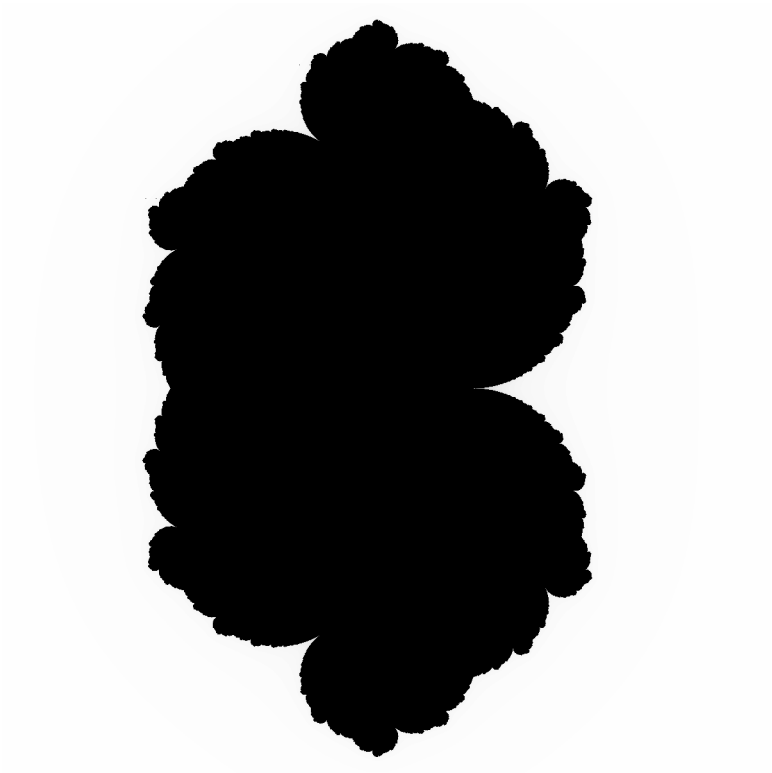

Figure A.9: The Julia set of $P_{1.05,1}$.

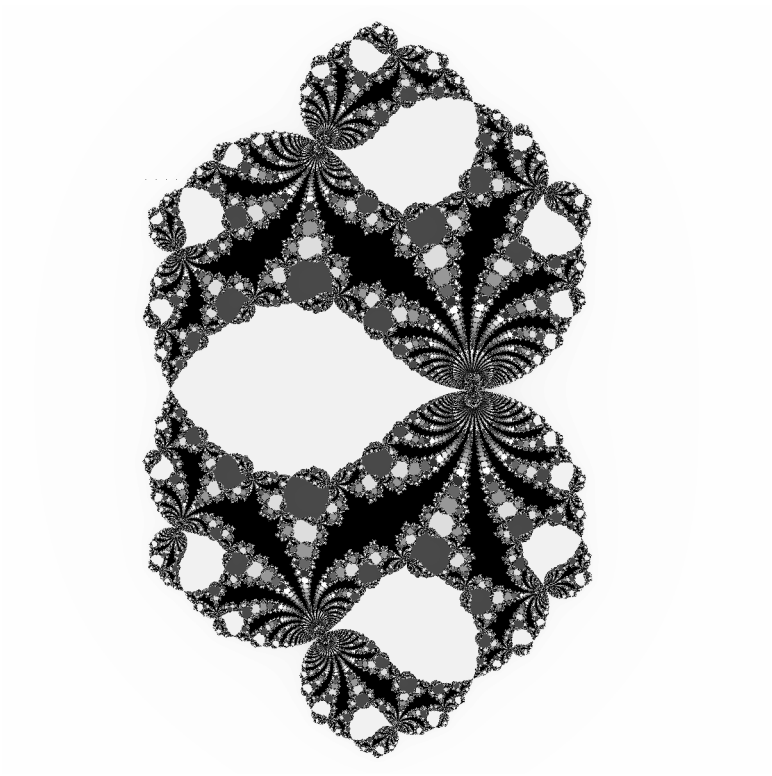

Figure A.10: The Julia set of $P_{k,1.05,0}$. Notice the two distinct high-period attracting cycles, which correspond to the two attracting fixed points of $\phi_0 \circ \hat{h}_{P_{1.05}}$.

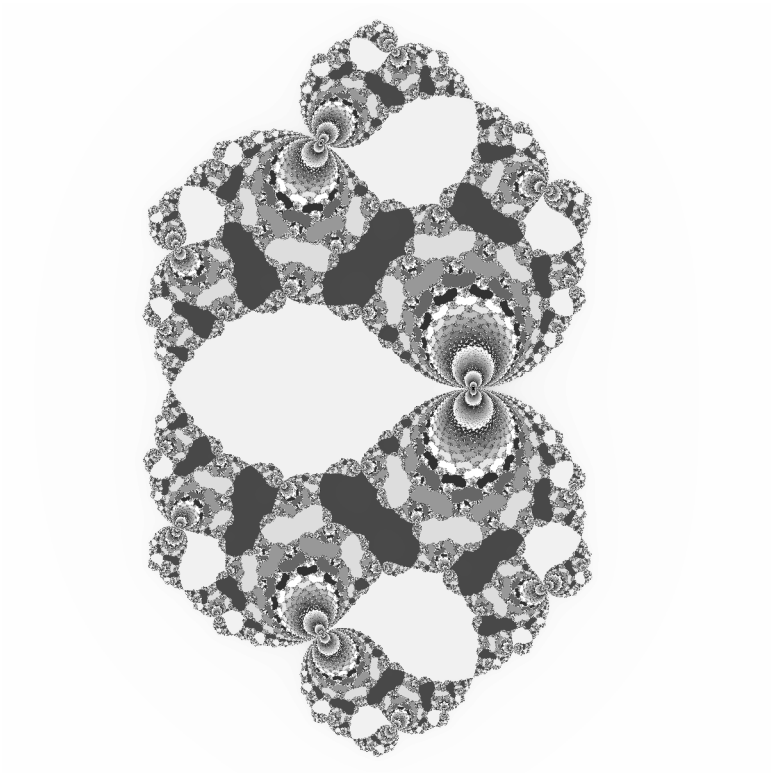

Figure A.11: The Julia set of a possible choice of $P_{k,1.05,1}$. By our choice of $\hat{z}_{1.05,1}$, both critical points of $P_{k,1.05,1}$ escape and so the Julia set is a Cantor dust.

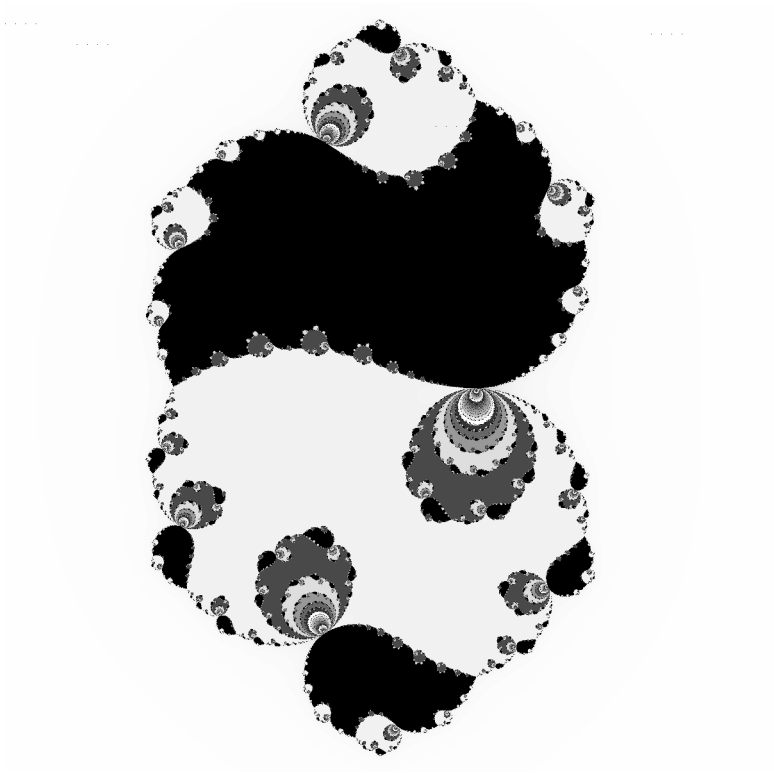

Figure A.12: A nearby Julia set of a polynomial with an attracting fixed point. This corresponds to a transit isomorphism ϕ such that $\phi \circ \hat{h}_{P_{1.05}}$ has an attracting fixed point at one of the ends, $\pm i\infty$. From our proof it is not a priori the case that there exists a ζ such that ϕ_ζ is such a transit isomorphism.

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