Boundaries for CAT(0) Groups

by

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Declaration

I declare that, to the best of my knowledge, this thesis is my own work except where explicitely stated otherwise. I confirm that this thesis has not been submitted for a degree at any other university.

Abstract

In this thesis we construct a boundary ∂G for an arbitrary CAT(0) group G. This boundary is compact and invariant under group isomorphisms. It carries a canonical (possibly trivial) G-action by homeomorphisms. For each geometric action of G on a CAT(0) space X there exists a canonical G-equivariant continuous map $\hat{\tau} : \partial G \to \partial X$. If G is a word-hyperbolic CAT(0) group, its boundary ∂G coincides with the usual Gromov boundary. If G is free abelian of rank k, its boundary is homeomorphic to the sphere S^{k-1}. For product groups of the types $G \times \mathbb{Z}^k$ and $G \times H$, where G and H are non-elementary word-hyperbolic CAT(0) groups, the boundary is worked out explicitly. Finally, we prove that the marked length spectrum associated to a geometric action of a torsion-free word-hyperbolic group on a CAT(0) space determines the isometry type of the CAT(0) space up to an additive constant.

Introduction

Finding the (Gromov) boundary of a word-hyperbolic CAT(0) group G is fairly easy. It is canonically homeomorphic to the visual boundary of any CAT(0) space that carries a geometric action by G, because the homeomorphism type of the visual boundary of any δ-hyperbolic CAT(0) space is invariant under quasi-isometries (see e.g. [BH99], [CDP90] or [GH90] for details). For word-hyperbolic groups many interesting features are connected to this boundary. (See the survey paper [KBe02] by Kapovich and Benakli for a good overview.) So, can one find a similar boundary with similarly interesting features for groups which are in some sense only "semi-hyperbolic" – say for groups that act geometrically on CAT(0) spaces (see [Gro87] and [Gro93] for inspiration)? At first sight the answer is "no"! In [CK00] Croke and Kleiner show that the above result for word-hyperbolic groups does not hold for arbitrary CAT(0) groups: In fact, they give an example of two compact, non-positively curved euclidean 2-complexes X_1 and X_2 which are homeomorphic, yet, their universal covers $\tilde{X}_1$ and $\tilde{X}_2$ have non-homeomorphic visual boundaries. But perhaps one can find another notion of "boundary" for arbitrary CAT(0) groups... and still retain some of the features which make the Gromov boundary interesting.

Several notions of boundary for a larger class of groups have been studied in the past for various purposes (see again [KBe02]). We recall a few of them, which are based on geometric, rather than analytic or stochastic ideas. The construction given by Floyd in [Flo80] applies to all finitely generated groups, and for word-hyperbolic ones it coincides with the usual boundary. However, it just yields a point as the boundary of a free abelian group. In [Ban95] Bandmann defines a boundary also for arbitrary finitely generated groups. His construction is based on ideas similar to those used to construct the Tits metric on the boundary of a CAT(0) space. Consequently, for a word-hyperbolic group this boundary is a discrete space. In [Bes96] Bestvina gives a set of axioms for the boundary of a group, and studies the relation between the group and its boundary on a (co-)homological level. Every CAT(0) group has a boundary that satisfies these axioms. But only the shape of this boundary is well-defined by these axioms, not the homeomorphism type. (Note that errors in [Bes96] have been found and corrected by Swenson in [Swe99].) In [KS96] Kapovich and Short introduce an elementary version of a boundary for finitely generated groups. This boundary is just a point set without topology. Nevertheless, in the case of a CAT(0) group there is a canonical bijection between this boundary and the set of rational points in the boundary presented in this thesis. In [Hru02] Hruska showed that the construction of a boundary for word-hyperbolic groups can be extended to groups that act geometrically on CAT(0) spaces which satisfy both the "Isolated Flats Property" and the "Relative Fellow Traveller Property" (see [Hru02] for exact definitions). This larger class of groups contains e.g. all geometrically finite subgroups of Isom $(\mathbb{H}^n)$, but neither the group in the Croke-Kleiner example nor $F_2 \times \mathbb{Z}$.

To overcome the difficulties posed by the Croke-Kleiner example we introduce the boundary of a CAT(0) group G as a "universal initial object" in the following sense. Let $\mathcal{G}$ be a "suitable" class of geometric actions by groups on CAT(0) spaces. Let G be a group, and let Γ be the set of all G-actions that belong to the class $\mathcal{G}$. For any G-action $\rho \in \Gamma$ on a CAT(0) space X we obtain a canonical map τ_ρ from the set G^∞ of infinite order elements in G into the visual boundary ∂X, by mapping each $g \in G^\infty$ to the positive endpoint of one of its axes. Taking the preimage of the canonical uniform structure on ∂X under this map τ_ρ gives rise to a canonical uniform structure U_ρ on G^∞ associated to each ρ. We define the *boundary uniformity* $U_G^{\mathcal{G}}$ on G^∞ to be the least upper bound of the family $\mathcal{U}_G^{\mathcal{G}} := \{U_\rho \,|\, \rho \in \Gamma\}$ of uniformities. In other words, $U_G^{\mathcal{G}}$ is the coarsest uniform structure on the set G^∞ such that the canonical map τ_ρ is uniformly continuous for each $\rho \in \Gamma$. Note that the boundary uniformity on G^∞ is in general neither Hausdorff nor complete. So, the *boundary* $\partial^{\mathcal{G}} G$ of G is defined as the Hausdorff completion of the set G^∞ with respect to the boundary uniformity $U_G^{\mathcal{G}}$, which makes it well-defined up to isomorphism. Extending τ_ρ by continuity yields a canonical uniformly continuous map $\hat{\tau}_\rho : \partial G \to \partial X$ for each G-action $\rho \in \Gamma$ on a CAT(0) space X; and the uniform structure on $\partial^{\mathcal{G}} G$ is the coarsest one for which all the maps $\hat{\tau}_\rho$ are uniformly continuous.

Some of the typical features of the boundary of a word-hyperbolic group are still present in this more general context. For example, the boundary $\partial^{\mathcal{G}} G$ of an arbitrary CAT(0) group G is compact, it carries a canonical (possibly trivial) G-action by homeomorphisms, and it is invariant under group isomorphisms. On the other hand, it is not clear whether the boundary of an arbitrary CAT(0) group necessarily coincides with the boundary of any subgroup of finite index (and the suspicion is that it does not). One of the most significant differences arises when one compares the boundary $\partial^{\mathcal{G}} G$ of a CAT(0) group G to the visual boundary of a CAT(0) space carrying a geometric G-action. For word-hyperbolic groups these two boundaries are the same up to homeomorphism. But for product groups of the form $H \times \mathbb{Z}$ for example, where H is a non-elementary word-hyperbolic CAT(0) group, $\partial^{\mathcal{G}}(H \times \mathbb{Z})$ has (up to some detail) an additional direct factor $C(H)$ when compared to the visual boundary of any CAT(0) space that carries an $H \times \mathbb{Z}$-action. This factor $C(H)$ can be described (up to homeomorphism) as a compact, convex subspace of an infinite dimensional real vector space in the following way: Let Θ be the set of all H-actions that belong to the class $\mathcal{G}$, and let $\rho \in \Theta$ be fixed. Then $C(H)$ is given as the closure in $\mathbb{R}^\Theta$ of the image $\operatorname{im} \kappa_\rho$, where the map $\kappa_\rho : H^\infty \to \mathbb{R}^\Theta$ sends each infinite order element h to the vector $(\frac{|h|_{\tilde{\rho}}}{|h|_\rho})_{\tilde{\rho} \in \Theta}$. Thus, the space $C(H)$ reflects in some way the multitude of different marked length spectra that can occur for H-actions in the class $\mathcal{G}$. This is one aspect we want to discuss further in [Ini], especially in the context of work by Bonahon in [Bon88] and [Bon91].

We indicated above that the boundary of G depends on $\mathcal{G}$, the chosen "suitable" class of geometric actions of groups on CAT(0) spaces. For the remaining part of this thesis we will take $\mathcal{G}$ to be the class of all geometric actions of groups on CAT(0) spaces, and suppress any further usage of the superscript $\mathcal{G}$. However, it is worth noting that one could base the boundary construction equally well on, say, the class of all free and geometric actions by groups on simply-connected, non-positively curved symmetric spaces. This is another aspect we plan to discuss further in [Ini].

The involvement of marked length spectra in the boundaries of CAT(0) groups raises a question which is also interesting in its own right: What does it mean for two CAT(0) spaces X_1 and X_2 if they both carry geometric actions by the same word-hyperbolic group G such that the associated marked length spectra coincide? Are they necessarily isometric? This question may seem far fetched at first sight, but it is the analogue of the following conjecture for Riemannian manifolds, which was stated by Burns and Katok in the problem session report [BuK85]: Suppose a compact manifold M carries two Riemannian metrics of negative curvature such that the associated marked length spectra coincide; then there exists a diffeomorphism of M that takes one metric to the other. For closed negatively curved surfaces it was proved by Otal in [Ota90], and independently by Croke in [Cro90], that the marked length spectrum indeed determines the isometry type of the metric (see also the paper by Croke, Fathi and Feldman [CFF92] for further results on surfaces). But so far in dimension greater than 2 the conjecture is proved only in special cases, which require for example that one of the manifolds in question is a locally symmetric space (see [Ham90] or [Bour95], [Bour96]). So, what is the answer to the above question in the context of CAT(0) spaces? We prove that in the case of CAT(0) spaces that carry a geometric action by a torsion-free word-hyperbolic group the marked length spectrum determines the isometry type up to an additive constant $k \geq 0$; and examples show that one cannot expect this constant to be equal to 0 in general.

This thesis is organized as follows: In the first section of Chapter 1 some well-known facts like the Svarč-Milnor Lemma and the Flat Torus Theorem are recorded for further reference. In Section 2 we prove a version of the Splitting Theorem for actions of product groups on CAT(0) spaces that is suitable for the purposes of Chapter 3 to 5. In the third section of Chapter 1 we show that the translation lengths associated to a geometric action of a word-hyperbolic group on a CAT(0) space are additive up to a constant, which will be used in Chapter 3 to prove the convexity of the space C_ρ mentioned above. In Chapter 2 we define the boundary of a CAT(0) group G (up to homeomorphism) and work out some of its basic properties. We show in particular that this boundary coincides with the usual Gromov boundary if G is word-hyperbolic, and with the sphere S^{k-1} if G is free abelian of rank k. In these two special cases the family $\mathcal{U}_G$ of uniformities on G^∞ contains one uniform structure only. However, this is not true in general. In the first section of Chapter 3 we build on an example by Bowers and Ruane (see [BR96a]) to show that the family $\mathcal{U}_{F_2 \times \mathbb{Z}}$ contains uncountably many different uniform structures. There are basically two ways in which these different uniformities arise, and we examine them separately in Sections 2 and 3. Then, in the last section of Chapter 3, we work out the boundary of product groups $G \times \mathbb{Z}$, where G is a word-hyperbolic CAT(0) group. This result is generalized in Chapter 4 to product groups of the type $G \times \mathbb{Z}^k$, where $k \in \mathbb{N}$ is arbitrary. In Chapter 5 we work out the boundary of product groups $G \times H$, where both G and H are word-hyperbolic CAT(0) groups. Finally, in Chapter 6 we prove that the marked length spectrum associated to a geometric action of a word-hyperbolic group G on a CAT(0) space X determines the isometry type of X up to an additive constant.

In this thesis we will use some standard facts about uniform spaces, about CAT(0) spaces and group actions upon them, and about word-hyperbolic groups. Since well-written textbooks are available in all three areas, we do not include proofs of these facts, but rather give suitable references. For background information on uniform spaces we refer to [Bou89a] and [Bou89b]. Material about CAT(0) spaces can be found in [BH99], [Bal95] or [BGS85]; and word-hyperbolic groups are discussed, for example, in [CDP90] or [GH90].

Chapter 1

Preliminaries

This chapter is divided into three sections. In the first section we recall some well-known facts about CAT(0) spaces. In the second section we prove a version of the Splitting Theorem for geometric actions by product groups on CAT(0) spaces, which we will need later on. In Section 3 we give our own proof of the fact that the translation lengths for a geometric action of a word-hyperbolic group on a CAT(0) space are approximately additive. This fact may be known to experts, but apparently a proof has not been published.

1.1 Some Preliminary Facts

The aim of this section is to recall some facts, which have become common knowledge in the area of CAT(0) spaces. Proofs of these facts can be found in [BH99].

Proposition 1.1.1 (Švarc-Milnor Lemma) *Let G be a group that acts geometrically on a geodesic space X. Then G is finitely generated. Moreover, if $\mathcal{A}$ is a finite system of generators for G and $d_{\mathcal{A}}$ the associated word-metric on G, then the map $g \mapsto g.x_0$ is a quasi-isometry for any choice of basepoint $x_0 \in X$.*

Let X be a CAT(0) space. Recall that the *displacement function* $d_g : X \to \mathbb{R}_0^+$ of an isometry g of X is defined by $d_g(x) := d(x, g.x)$. The infimum of the displacement function d_g on X is called the *translation length* of g. An isometry g of X is called *semi-simple*, if its displacement function d_g attains its infimum at some $x \in X$. The set of points in X, where d_g attains this infimum, is called the *minimal set* $\mathrm{Min}(g)$ of g. An isometry g of X is called *parabolic* if its minimal set $\mathrm{Min}(g)$ is empty. An isometry g of X is called *elliptic* if it has a fixed point in X, or equivalently, if it is semi-simple and $|g| = 0$. Finally, an isometry g of X is called *hyperbolic* if it is semi-simple and $|g|$ is strictly positive.

Theorem 1.1.2 (Flat Torus Theorem, [BH99]) *Let A be a free abelian group of rank n acting properly by semi-simple isometries on a CAT(0) space X. Then the following is true:*

(i) $\mathrm{Min}(A) := \bigcap_{g \in A} \mathrm{Min}(g)$ *is non-empty and splits as a product $Y \times \mathbb{E}^n$.*

(ii) Every $g \in A$ leaves $\mathrm{Min}(A)$ *invariant and respects the product decomposition; g acts as the identity on the first factor Y and as a translation on the second factor $\mathbb{E}^n$.*

(iii) The quotient of each n-flat $\{y\} \times \mathbb{E}^n$ by the action of A is an n-torus.

(iv) If an isometry of X normalizes A, then it leaves $\mathrm{Min}(A)$ *invariant and preserves the product decomposition.*

Recall that an isometry g of a metric space X is called a *Clifford translation* if its displacement function d_g is a constant function on X.

Theorem 1.1.3 ([BH99], II.6.15) *Let X be a CAT(0) space. If g is a non-trivial Clifford translation of X, then X splits as a product $X = Y \times \mathbb{R}$, and $g(y,t) = (y, t + |g|)$ for all $y \in Y$ and all $t \in \mathbb{R}$. Moreover, if X splits as a product $X' \times X''$, every Clifford translation of X preserves this splitting and is the product of a Clifford translation of X' and a Clifford translation of X''.*

Recall that two geodesic lines $c : \mathbb{R} \to X$ and $c' : \mathbb{R} \to X$ in a CAT(0) space X are called *asymptotic*, if there exists a constant $T \geq 0$ such that $d(c(t), c'(t)) \leq T$ for all $t \in \mathbb{R}$.

Proposition 1.1.4 (Flat Strip Theorem, [BH99]) *Let X be a CAT(0) space, and let $c : \mathbb{R} \to X$ and $c' : \mathbb{R} \to X$ be geodesic lines. If c and c' are asymptotic, then the convex hull of $c(\mathbb{R}) \cup c'(\mathbb{R})$ is isometric to a flat strip $\mathbb{R} \times [0, w] \subset \mathbb{E}^2$.*

1.2 A Splitting Theorem

In this section we prove a variation of a well-known Splitting Theorem for CAT(0) groups. For Riemannian manifolds of non-positive curvature similar theorems were proved by Gromoll and Wolf [GW71], and Lawson and Yau [LY72]. For CAT(0) spaces a Splitting Theorem was proved by Baribaud in [Bar93]. The version presented here is also motivated by the special cases discussed in [BR96a] and [Rua99]. Since we do not make any assumptions about geodesic extension properties of the underlying CAT(0) space, our statement is more general, but also slighly weaker than the usual statement. We need this version in the following chapters to study the boundary of various products of CAT(0) groups:

Theorem 1.2.1 *Let $G = G_1 \times G_2$ be a group that acts geometrically on some CAT(0) space X, and suppose that the centre of G_2 is finite. Then there exists a non-empty, closed, convex, G-invariant subspace $X' \subset X$ that splits as a product $X_1 \times X_2$ such that G_1 acts geometrically on X_1 and trivially on X_2, and G_2 acts geometrically on X_2 and by Clifford translations on X_1.*

In order to prove this theorem we need three lemmas. The first one – together with its proof – follows what Lemma II.6.24 in [BH99] should state and prove. However, what Lemma II.6.24 does state, is not correct, as the following remark explains:

Remark 1.2.2 Here is the statement of Lemma II.6.24: "Let X be a proper CAT(0) space. Let $\Gamma = \Gamma_1 \times \Gamma_2$ be a group acting properly on X by isometries, and suppose that the centre of Γ_2 is finite. Let $C \subset X$ be a closed, convex, Γ_1-invariant subspace and suppose that there exists a compact subset $K \subset X$ such that $C \subset \Gamma.K$. Then there exists a compact subset $K' \subset C$ such that $C = \Gamma_1.K'$." The following is a counter-example for this statement: Take $\Gamma_1 := \mathbb{Z}$ and $\Gamma_2 := F_2$, where F_2 is the free group of rank 2. Let $\mathcal{A} := \langle a, b \rangle$ be the standard system of generators for F_2, and $T := C_{\mathcal{A}}(F_2)$ the associated Cayley graph. Then T is a 4-valent tree. Hence, the Euclidean product $\mathbb{R} \times T$ is a proper CAT(0) space. Take $X := \mathbb{R} \times T$, and consider the $\mathbb{Z} \times F_2$-action ρ given by $(z, g) * (r, t) := (r + z, g.t)$. Note that ρ is geometric. Take $C := X$. Then C is a closed, convex, Γ_1-invariant subspace of X. Since ρ is geometric, there exists a compact subset $K \subset X$, whose

Γ-translates cover C. However, $\Gamma_1.K'$ is a proper subspace of C for any compact subset $K' \subset C$. Note also that one can construct similar counter-examples, where C is a proper subspace of X.

Lemma 1.2.3 *Let $G = G_1 \times G_2$ be a group that acts geometrically on some CAT(0) space X, and suppose that the centre of G_2 is finite. Then for each $x \in X$ the action of G_1 restricted to the closed convex hull $C(G_1.x)$ of the orbit $G_1.x$ is cocompact.*

Proof: We assume that the action of G_1 restricted to $C(G_1.x)$ is not cocompact, and deduce a contradiction. By hypothesis there exists a compact subset $K \subset X$ such that $C(G_1.x) \subset G.K$. If the action of G_1 restricted to $C(G_1.x)$ is not cocompact, we can find a sequence of points x_n in $C(G_1.x)$ such that $d(x_n, G_1.K) \to \infty$ as $n \to \infty$. Note that there exist $\beta_n \in G_2$ so that $\beta_n.x_n \in G_1.K$ for each $n \in \mathbb{N}$ – say $\beta_n.x_n \in \gamma_n.K$ with $\gamma_n \in G_1$.

Claim 1: After passing to a subsequence, we may assume that all the β_n are distinct.

Proof of Claim 1: Note that $\gamma_n^{-1}.x_n$ lies in $\beta_n^{-1}.K$. Therefore, we get

$$\begin{aligned} d(\beta_n^{-1}.K, K) \;&\geq\; d(\gamma_n^{-1}.x_n, K) - \operatorname{diam}(K) \;=\; \dots \\ \dots = \;& d(x_n, \gamma_n.K) - \operatorname{diam}(K) \;\geq\; d(x_n, G_1.K) - \operatorname{diam}(K) \to \infty \end{aligned}$$

This proves Claim 1.

Claim 2: For each $\beta \in G_2$ the sequence $d_{\beta_n\beta\beta_n^{-1}}$ of displacement functions is uniformly bounded on K as n varies.

Proof of Claim 2: Note that the displacement function d_β is a convex, continuous function that is constant on $G_1.x$. Hence it is bounded from above on $C(G_1.x)$ by some constant r. This implies for each $x \in K$:

$$\begin{aligned} & d(\beta_n\beta\beta_n^{-1}.x, x) \\ & \leq d(\beta_n\beta\beta_n^{-1}.x, \beta_n\beta\beta_n^{-1}\gamma_n^{-1}\beta_n.x_n) + d(\beta_n\beta\gamma_n^{-1}.x_n, \gamma_n^{-1}\beta_n.x_n) + d(\gamma_n^{-1}\beta_n.x_n, x) \\ & \leq \operatorname{diam}(K) + d(\gamma_n^{-1}\beta_n\beta.x_n, \gamma_n^{-1}\beta_n.x_n) + \operatorname{diam}(K) \\ & \leq \operatorname{diam}(K) + r + \operatorname{diam}(K), \end{aligned}$$

which proves Claim 2.

Consider an element $\beta' \in G_2$. By Claim 2 the displacement functions $d_{\beta_n\beta'\beta_n^{-1}}$ are uniformly bounded on K as n varies. Since G acts properly on X, we conclude that the set $\{\beta_n\beta'\beta_n^{-1}\}_{n\in\mathbb{N}}$ is finite. So, passing to a subsequence, we can assume that $\beta_i\beta'\beta_i^{-1} = \beta_j\beta'\beta_j^{-1}$ for all $i, j \in \mathbb{N}$. We apply this argument successively to a finite set of generators of G_2, and get a subsequence of distinct elements β_n such that $\beta_i\beta\beta_i^{-1} = \beta_j\beta\beta_j^{-1}$ for all $\beta \in G_2$ and all $i, j \in \mathbb{N}$. Thus, we have found an infinite sequence of distinct elements $\beta_1^{-1}\beta_n$ that lie in the centre of G_2. But the centre of G_2 is finite by hypothesis. □

The following technical lemma is quoted from [BH99], II.2.15, where a proof can be found.

Lemma 1.2.4 *Consider three geodesic lines $c_i : \mathbb{R} \to X$ for $i = 1, 2, 3$ in a CAT(0) space X. Suppose that the union of each pair of these lines is isometric to the union of two parallel lines in $\mathbb{E}^2$. Let $p_{i,j}$ be the projection of $c_i(\mathbb{R})$ to $c_j(\mathbb{R})$. Then the map $p_{1,3}^{-1} \circ p_{2,3} \circ p_{1,2}$ is the identity on $c_1(\mathbb{R})$.*

A proof of the following lemma is left to the reader as an exercise.

Lemma 1.2.5 (Sandwich Lemma) *Let X be a CAT(0) space. For a closed subspace $C \subset X$ and a point $x \in X$ let $d_C(x) := \inf\{d(x, c) \mid c \in C\}$ denote the distance from x to C. Suppose that C_1 and C_2 are complete, convex subspaces of X such that the restriction of d_{C_1} to C_2 is constant – equal to w say – and the restriction of d_{C_2} to C_1 is also constant. Then the convex hull of $C_1 \cup C_2$ is isometric to $C_1 \times [0, w]$.*

Now, we prove Theorem 1.2.1. To do so, we adapt the proof of the Splitting Theorem given in [BH99], II.6.21, to fit our hypotheses.

Proof of Thm. 1.2.1: The proof is divided into several steps. Firstly, let $\mathcal{S}$ be the set of non-empty, closed, convex, G_1-invariant subspaces of X. Let $\mathcal{M}$ be the subset of $\mathcal{S}$ that contains precisely those subspaces which are minimal with respect to inclusion. Note that each $C \in \mathcal{M}$ is necessarily the closed, convex hull of the G_1-orbit of some $x \in X$.

Claim 1: The set $\mathcal{M}$ is non-empty.

Proof of Claim 1: Let $C(G_1.x')$ be the closed convex hull of some orbit $G_1.x'$ in X, and let $\mathcal{C}$ be the set of non-empty, closed, convex, G_1-invariant subspaces of $C(G_1.x')$. $\mathcal{C}$ is partially ordered with respect to inclusion. We consider an ordered chain $\{C_\lambda\}_{\lambda\in\Lambda}$ of elements in $\mathcal{C}$ which is decreasing. By Lemma 1.2.3 there is a compact subset $K \subset X$, whose G_1-translates cover $C(G_1.x')$. Since each C_λ is G_1-invariant, $C_\lambda \cap K$ is non-empty for each $\lambda \in \Lambda$. Hence, we get $(\bigcap_{\lambda\in\Lambda} C_\lambda) \cap K \neq \emptyset$, which implies that $\bigcap_{\lambda\in\Lambda} C_\lambda \in \mathcal{C}$ is a lower bound for the chain $\{C_\lambda\}_{\lambda\in\Lambda}$. It follows from Zorn's Lemma that $\mathcal{C}$ contains a minimal element C. Thus, $\mathcal{M}$ is non-empty.

Secondly, we consider two elements C_1, C_2 of $\mathcal{M}$. For $i = 1,2$ let $p_i : X \to C_i$ denote the projection of X onto C_i, and let $d = d(C_1, C_2) := \inf\{d(x_1, x_2) \mid x_1 \in C_1, x_2 \in C_2\}$.

Claim 2: There exists a unique isometry ϕ of $C_1 \times [0, d]$ onto the convex hull of $C_1 \cup C_2$ such that $\phi(x, 0) = x$ and $\phi(x, d) = p_2(x)$.

Proof of Claim 2: The function $d_{C_1} : x \mapsto d(x, p_1(x))$ is convex and G_1-invariant on C_2. If there are points $x, y \in C_2$ such that $d_{C_1}(x) < d_{C_1}(y)$, then $\{z \in C_2 \mid d_{C_1}(z) \leq d_{C_1}(x)$ is a proper, non-empty, closed, convex, G_1-invariant subspace of C_2. This contradicts the minimality of C_2. Hence, d_{C_1} is constant on C_2. Analogously, we see that d_{C_2} is constant on C_1. Therefore, we can apply the Sandwich Lemma 1.2.5. This proves Claim 2.

Thirdly, we consider three elements C_1, C_2, C_3 of $\mathcal{M}$. Let $p_{i,j} : C_i \to C_j$ denote the projection of C_i onto C_j. Note that each $p_{i,j}$ is G_1-equivariant, i.e. we have $p_{i,j}(g.x) = g.p_{i,j}(x)$ for all $g \in G_1$ and all $x \in C_i$.

Claim 3: The maps $p_{1,3}$ and $p_{2,3} \circ p_{1,2}$ from C_1 to C_3 are the same.

Proof of Claim 3: By Claim 2, each $p_{i,j}$ is an isometry from C_i onto C_j. Hence $g := p_{1,3}^{-1} \circ p_{2,3} \circ p_{1,2}$ is an isometry of C_1 onto itself. Its displacement function $x \mapsto d_g(x) := d(g.x, x)$ is convex and G_1-invariant. As in the proof of Claim 2, we conclude that d_g is constant. Hence g is a Clifford translation of C_1. Suppose g is non-trivial. Then C_1 splits as a product $X = Y \times \mathbb{R}$, because of Prop. 1.1.3. Let $y \in Y$, and consider the geodesic line $c : \mathbb{R} \to C_1$, where $c(r) := (y, r)$, and its projections $p_{1,2}c$ to C_2 and $p_{1,3}c$ to C_3. Because of Claim 2, we can apply Lemma 1.2.4, which implies that $p_{1,3}^{-1} \circ p_{2,3} \circ p_{1,2}(y,0) = (y,0)$. It follows that g is trivial, since d_g is constant on C_1. This completes the proof of Claim 3.

Fourthly, we fix $X_1 \in \mathcal{M}$, and let $p : X \to X_1$ be the projection. Note that $p(g.x) = g.p(x)$ for all $g \in G_1$ and all $x \in X_1$. Note also that all the G_2-translates of X_1 are again in $\mathcal{M}$. The G_2-action on X induces a map $\psi : G_2 \times X_1 \to X_1$ by $(\beta, x) \mapsto \beta * x := p(\beta.x)$.

Claim 4: ψ is a group action of G_2 on X_1 by Clifford translations.

Proof of Claim 4: Let $\beta \in G_2$ be given. Claim 2 implies that the map $x \mapsto \beta * x$ is an isometry of X_1. The associated displacement function $x \mapsto d_\beta(x) := d(\beta * x, x)$ is convex and G_1-invariant. As before, this implies that d_β is constant. Hence $x \mapsto \beta * x$ is a Clifford translation of X_1. We check that ψ is a group action: Let $\alpha, \beta \in G_2$ and $x \in X_1$ be given. Let p' be the projection from the translate $(\alpha\beta).X_1$ to the translate $\alpha.X_1$. Then Claim 3 implies

$$(\alpha\beta) * x \;=\; p(\alpha\beta.x) \;=\; (p \circ p')(\alpha\beta.x) \;=\; p(\alpha.p(\beta.x)) \;=\; \alpha * (\beta * x).$$

This completes the proof of Claim 4.

Fifthly, we consider the subspace $X' := \bigcup\{C \,|\, C \in \mathcal{M}\}$ of X. Note that any two distinct elements of $\mathcal{M}$ are disjoint. Thus, for each $x \in X'$ there exists a unique $C_x \in \mathcal{M}$ such that $x \in C_x$

Claim 5: X' is a non-empty, closed, convex, G-invariant subspace of X.

Proof of Claim 5: As seen above, X' is non-empty, convex and G-invariant. It remains to check that X' is closed. Let x_n be a sequence of points in X' converging to $x \in X$. Passing to a subsequence, we can assume that $d(x_n, x_{n+1}) \leq \frac{1}{2^n}$. Let C_n be the unique element in $\mathcal{M}$ that contains x_n, and let $p_n : X \to C_n$ be the projection. Denote the restriction to C_1 of the composition $p_n \circ \cdots \circ p_2$ by P_n. Then for each $z \in C_1$ the sequence $(P_n(z))_n$ is a Cauchy sequence, because

$$d(P_n(z), P_{n+k}(z)) \leq \sum_{i=n}^{n+k-1} d(C_i, C_{i+1}) \leq \sum_{i=n}^{n+k-1} d(x_i, x_{i+1}) \leq \frac{1}{2^{n-1}}.$$

Therefore, the sequence $P_n : C_1 \to X$ of maps converges uniformly on C_1 to a limit map $P : C_1 \to X$. Since each P_n is a G_1-equivariant isometry, the limit map P is a G_1-equivariant isometry, too. Hence, $P(C_1)$ lies in $\mathcal{M}$. To show that $x \in P(C_1)$, consider the sequence $z_n \in C_1$ defined by $P_n(z_n) = x_n$. We have

$$\begin{aligned} d(P(z_n), x) &\leq d(P(z_n), x_n) + d(x_n, x) \\ &= d(P(z_n), P_n(z_n)) + d(x_n, x) \leq \frac{1}{2^{n-1}} + d(x_n, x). \end{aligned}$$

Since $P(C_1)$ is closed, it follows that $x \in P(C_1)$, which completes the proof of Claim 5.

Finally, we consider on $\mathcal{M}$ the distance function $d(C_1, C_2)$ as defined in Claim 2. Using Claim 2 and 3, it is straightforward to show that d is in fact a metric. Let X_2 be the metric space $(\mathcal{M}, d)$.

Claim 6: There is a G-action by isometries on $X_1 \times X_2$ such that G_1 acts geometrically on X_1 and trivially on X_2; and G_2 acts geometrically on X_2 and by Clifford translations on X_1. Moreover, the map $\varphi : X' \to X_1 \times X_2$ given by $x \mapsto (p(x), C_x)$ is a G-equivariant isometry.

Proof of Claim 6: We get the desired G-action on $X_1 \times X_2$ as follows: The G_1-action on X_1 is the restriction of the action on X. The G_1-action on X_2 is the trivial action. The G_2-action on X_1 is ψ. The G_2-action on X_2 is given by $\gamma_2.C_x := C_{\gamma_2.x}$. If we endow $X_1 \times X_2$ with this action, the map φ is obviously G-equivariant by construction. To see that it is an isometry, consider $x, x' \in X'$, and let $p' : X \to C_{x'}$ be the projection. We have

$$\begin{aligned} d(x, x')^2 &= d(p'(x), x')^2 + d(C_x, C_{x'})^2 \\ &= d(pp'(x), p(x'))^2 + d(C_x, C_{x'})^2 \\ &= d(p(x), p(x'))^2 + d(C_x, C_{x'})^2, \end{aligned}$$

where Claim 2 implies the first and second equality, and Claim 3 the third one. Because φ is a G-equivariant isometry, the G-action on $X_1 \times X_2$ is geometric. In particular, it follows that G_2 acts geometrically on X_2. This completes the proof of Claim 6, as well as the proof of the Splitting Theorem. $\square$

1.3 Additivity of Translation Lengths

The aim of this section is to prove the following theorem:

Theorem 1.3.1 (Additivity of Translation Lengths) *Let G be a word-hyperbolic group, and let ρ be a geometric G-action on a CAT(0) space X. Let $g, h \in G^\infty$ be such that the points g^+ and h^- in ∂G are not the same. Then there exists a constant $C > 0$ such that for all sufficiently large $i, j \in \mathbb{N}$ we have:*

$$|g^i|_\rho + |h^j|_\rho - C \leq |g^i h^j|_\rho \leq |g^i|_\rho + |h^j|_\rho + C.$$

We need to prepare the proof of Theorem 1.3.1. The following lemma was noted by Swenson in [Swe93].

Lemma 1.3.2 *Let G be a CAT(0) group. If for some $g, h \in G^\infty$ the points g^+ and h^+ in ∂G are the same, then there exist $n, m \in \mathbb{N}$ such that $g^n = h^m$.*

Proof: Let ρ be a geometric action of G on some CAT(0) space X. Let $c_g : \mathbb{R} \to X$ and $c_h : \mathbb{R} \to X$ be axes for g and h respectively. Fix $x := c_g(0)$, and $y := c_h(0)$. By hypothesis, the geodesic rays $c_g(\mathbb{R}^+)$ and $c_h(\mathbb{R}^+)$ are asymptotic. Therefore, there exists an $i(n) \in \mathbb{N}$ for each $n \in \mathbb{N}$ such that

$$\begin{aligned} d(g^{-i(n)} h^n.y, y) &\leq d(g^{-i(n)} h^n.y, x) + d(x, y), \\ &\leq d(h^n.y, g^{i(n)}.x) + d(x, y), \\ &\leq 2\, d(x, y) + |g|_\rho. \end{aligned}$$

Because ρ is a proper action, there is an infinite sequence $(n_\nu)_\nu$ of natural numbers such that

$$g^{-i(n_\nu)} h^{n_\nu} = g^{-i(n_\mu)} h^{n_\mu}$$

holds for all $\nu, \mu \in \mathbb{N}$. In particular, we can find a $\nu \in \mathbb{N}$ such that

$$g^{i(n_\nu) - i(n_1)} = h^{n_\nu - n_1},$$

where both exponents are positive. □

We recall Gromov's "approximation by trees" for a δ-hyperbolic space, a proof of which can be found in [CDP90] or [Bow91], for example.

Theorem 1.3.3 (Approximation by Trees) *Let X be a δ-hyperbolic space with a basepoint x_0. For $i = 1, \dots, n$ let x_i be a point in $X \cup \partial X$, and c_i the geodesic segment or geodesic ray joining x_0 to x_i. Set $Z := c_1 \cup \cdots \cup c_n$. Then there exists a simplicial tree T with basepoint t_0, and a continuous mapping $f : (Z, x_0) \to (T, t_0)$, such that the following holds:*

(i) For $i = 1, \dots, n$ the restriction of f to c_i is an isometry.

(ii) There exists a constant $C = C(\delta, n)$ such that for all $z, z' \in Z$

$$d_X(z, z') - C \leq d_T(f(z), f(z')) \leq d_X(z, z').$$

As an immediate consequence of Theorem 1.3.3 we get in particular:

Lemma 1.3.4 *For all $z, z' \in Z$ the Gromov product satisfies*

$$(z, z')_{x_0} \leq (f(z), f(z'))_{t_0} \leq (z, z')_{x_0} + C.$$

Using the above facts, we can prove the following slightly modified version of Theorem 1.3.1. Clearly, this modified version implies Theorem 1.3.1, because for each $x \in X$ and each $g \in G^\infty$ the sequence $\frac{1}{n} d(x, g^n.x)$ converges to $|g|_\rho$ as n goes to infinity.

Proposition 1.3.5 *Let G be a word-hyperbolic group, and let ρ be a geometric G-action on a CAT(0) space X. Let $g, h \in G^\infty$ be such that the points g^+ and h^- in ∂G are distinct. Then there exists a constant $C > 0$ such that for some $x \in X$, for all sufficiently large $i, j \in \mathbb{N}$, and for all $n \in \mathbb{N}$ we have:*

$$(*) \qquad |g^i|_\rho + |h^j|_\rho - C \leq \frac{1}{n} d(x, (g^i h^j)^n.x) \leq |g^i|_\rho + |h^j|_\rho + C.$$

Proof: Since X carries a geometric action of the word-hyperbolic group G, X is δ-hyperbolic for some $\delta \geq 0$. Let x_0 be a point on an axis $c_h : \mathbb{R} \to X$ of h.

We fix $i, j \in \mathbb{N}$, and set $x_n := (g^i h^j)^n.x_0$ for each $n \in \mathbb{N}_0$. We want to prove inequality (*) for $x = x_0$ by induction on n. Note that we have by definition

$$d(x_0, x_{n+1}) = d(x_0, x_n) + d(x_n, x_{n+1}) - 2\,(x_0, x_{n+1})_{x_n}.$$

Therefore suitable estimates for $d(x_n, x_{n+1})$ and $(x_0, x_{n+1})_{x_n}$ are needed. These will be given in Claim 3 and Claim 4 respectively.

Claim 1: Both $C_1 := (g^+, h^-)_{x_0}$ and $C_2 := (g^-, h^+)_{x_0}$ are finite.

Proof of Claim 1: If C_1 is infinite, then the boundary points g^+ and h^- are the same, which contradicts the hypothesis. If C_2 is infinite, then the points g^- and h^+ in ∂G are the same. By Lemma 1.3.2, this implies that $g^{-n} = h^m$ for some $n, m \in \mathbb{N}$. Thus, g^{-1} and h have parallel axes. Hence the points g^+ and h^- are the same, which again contradicts the hypothesis. This proves Claim 1.

For $n \in \mathbb{N}_0$ define $y_n := (g^i h^j)^n g^i.x_0$. Let $c_g : \mathbb{R} \to X$ be an axis of g.

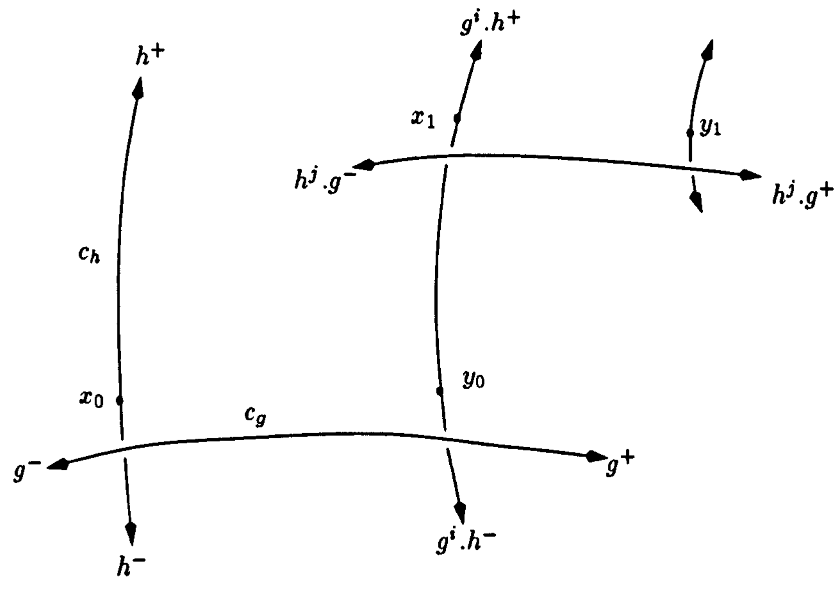

Figure 1.1: Set-up for the proof of Prop. 1.3.5

Take a constant $C_3 \geq d(x_0, c_g(\mathbb{R}))$. Note that we have for all $n \in \mathbb{N}_0$

$$i\,|g|_\rho - 2C_3 \leq d(x_n, y_n) \leq i\,|g|_\rho + 2C_3.$$

Claim 2: Suppose that i and j are sufficiently large. Then there exists a constant $C_4 \geq 0$, which depends on C_2 and δ only, such that for each $n \in \mathbb{N}_0$

$$(x_n, x_{n+1})_{y_n} \leq C_4.$$

Proof of Claim 2: Obviously, it is enough to prove Claim 2 for $n = 0$. Consider the geodesics $c_1 := [y_0, x_0]$, $c_2 := [y_0, g^-]$ and $c_3 := [y_0, g^i.h^+]$ issuing from y_0. Define $Z := c_1 \cup \cdots \cup c_3$. Let the tree T, the constant $C(\delta, 3)$, and the mapping $f : (Z, y_0) \to (T, f(y_0))$, be given according to Gromov's Approximation Theorem 1.3.3. Let $w' \in c_g(\mathbb{R})$ be such that $d(x_0, w') \leq C_3$. Since X is CAT(0), there exists a point w on c_2 such that $d(w, w') \leq d(y_0, g^i.w') \leq C_3$. Therefore, we have $d(x_0, w) \leq 2\,C_3$, and hence $d_T(f(x_0), f(w)) \leq 2\,C_3$ by Theorem 1.3.3. Thus, we get

$$\begin{aligned}(f(g^-), f(x_0))_{f(y_0)} &\geq (f(w), f(x_0))_{f(y_0)} \\ &\geq d_T(f(x_0), f(y_0)) - d_T(f(x_0), f(w)) \\ &\geq d(x_0, y_0) - 2C_3 \geq i\,\left|g\right|_\rho - 4C_3.\end{aligned}$$

Recall that $(f(g^-), f(g^i.h^+))_{f(y_0)} \leq C_2 + C(\delta, 3)$ by Lemma 1.3.4. So, if i is large enough such that $i\,\left|g\right|_\rho - 4\,C_3 > C_2 + C(\delta, 3)$, then

$$(f(g^-), f(x_0))_{f(y_0)} \geq (f(g^-), f(g^i.h^+))_{f(y_0)}.$$

Moreover, note that x_1 lies on c_3 with $d(y_0, x_1) = j\,|h|_\rho$. So, if j is large enough such that $j\,\left|h\right|_\rho > C_2 + C(\delta, 3)$, then

$$(f(g^i.h^+), f(x_1))_{f(y_0)} \geq (f(g^-), f(g^i.h^+))_{f(y_0)}.$$

Since T is a tree, these two inequalities imply that

$$(f(x_0), f(x_1))_{f(y_0)} = (f(g^-), f(g^i.h^+))_{f(y_0)}.$$

Applying Lemma 1.3.4, we can conclude that

$$\begin{aligned}(x_0, x_1)_{y_0} &\leq (f(x_0), f(x_1))_{f(y_0)} \\ &= (f(g^-), f(g^i.h^+))_{f(y_0)} \\ &\leq (g^-, g^i.h^+)_{y_0} + C(\delta, 3) \;=\; C_2 + C(\delta, 3) \;=:\; C_4.\end{aligned}$$

Claim 3: Suppose that i and j are sufficiently large. Then there exists a constant $C_5 \geq 0$, which depends on C_2, C_3 and δ only, such that for all $n \in \mathbb{N}_0$ we have

$$\left|g^i\right|_\rho + \left|h^j\right|_\rho - C_5 \leq d(x_n, x_{n+1}) \leq \left|g^i\right|_\rho + \left|h^j\right|_\rho + C_5.$$

Proof of Claim 3: Obviously, it is enough to prove the statement for $n = 0$. By definition, we have $d(x_0, x_1) = d(x_0, y_0) + d(y_0, x_1) - 2\,(x_0, x_1)_{y_0}$. Suppose i and j are large enough. Then Claim 2 and the triangle inequality imply

$$\begin{aligned}d(x_0, x_1) &\leq i\,\left|g\right|_\rho + 2\,C_3 + j\,\left|h\right|_\rho, \\ d(x_0, x_1) &\geq i\,\left|g\right|_\rho - 2\,C_3 + j\,\left|h\right|_\rho - 2\,C_4.\end{aligned}$$

Thus, we can take $C_5 := 2\,C_3 + 2\,C_4$.

Claim 4: Suppose that i and j are sufficiently large. Then there exists a constant $C_6 \geq 0$, which depends on C_1 and δ only, such that we have for all $n \in \mathbb{N}$

$$(**) \qquad (x_0, x_{n+1})_{x_n} \leq C_6.$$

Proof of Claim 4: For any given $n \in \mathbb{N}$ consider the geodesics $c_1 := [x_n, x_0]$, $c_2 := [x_n, x_{n-1}]$, $c_3 := [x_n, (g^i h^j)^n.h^-]$, $c_4 := [x_n, x_{n+1}]$, $c_5 := [x_n, y_n]$ and $c_6 := [x_n, (g^i h^j)^n.g^+]$ in X issuing from x_n. Define $Z_n := c_1 \cup \cdots \cup c_6$. Let the tree T_n, the constant $C(\delta, 6)$, and the mapping $f_n : (Z_n, x_n) \to (T_n, f_n(x_n))$, be given according to Gromov's Approximation Theorem 1.3.3. We want to prove inequality $(**)$ for $C_6 := C_1 + C(\delta, 6)$ by induction on n. To do so we will check inductively that

$$(f_n(x_0), f_n(x_{n+1}))_{f_n(x_n)} = (f_n((g^i h^j)^n.h^-), f_n((g^i h^j)^n.g^+))_{f_n(x_n)},$$

provided that i and j are sufficiently large. Then inequality (**) follows immediately because of Lemma 1.3.4. We begin by recording three inequalities (a) to (c), which hold for any given $n \in \mathbb{N}$, and all sufficiently large i and j: Firstly, Lemma 1.3.4 and Claim 2 imply that

$$\begin{aligned}(f_n(y_{n-1}), f_n(x_{n-1}))_{f_n(x_n)} \geq (y_{n-1}, x_{n-1})_{x_n} = \dots \\ \dots = d(x_n, y_{n-1}) - (x_{n-1}, x_n)_{y_{n-1}} \geq j \left| h \right|_\rho - C_4.\end{aligned}$$

Note that y_{n-1} lies on c_3 with $d(y_{n-1}, x_n) = j\,|h|_\rho$. Thus, if j satisfies $j \left| h \right|_\rho - C_4 \geq C_1 + C(\delta, 6)$, Lemma 1.3.4 yields inequality (a):

$$\begin{aligned}(f_n((g^i h^j)^n.h^-), f_n(x_{n-1}))_{f_n(x_n)} \geq (f_n(y_{n-1}), f_n(x_{n-1}))_{f_n(x_n)} \geq \dots \\ \dots \geq C_1 + C(\delta, 6) \geq (f_n((g^i h^j)^n.h^-), f_n((g^i h^j)^n.g^+))_{f_n(x_n)}.\end{aligned}$$

Secondly, Lemma 1.3.4 and Claim 2 imply that

$$\begin{aligned}(f_n(x_{n+1}), f_n(y_n))_{f_n(x_n)} \geq (x_{n+1}, y_n)_{x_n} = \dots \\ \dots = d(x_n, y_n) - (x_n, x_{n+1})_{y_n} \geq i \left| g \right|_\rho - 2\,C_3 - C_4.\end{aligned}$$

Thus, if i satisfies $i \left| g \right|_\rho - 2\,C_3 - C_4 \geq C_1 + C(\delta, 6)$, Lemma 1.3.4 yields inequality (b):

$$(f_n(x_{n+1}), f_n(y_n))_{f_n(x_n)} \geq C_1 + C(\delta, 6) \geq (f_n((g^i h^j)^n.h^-), f_n((g^i h^j)^n.g^+))_{f_n(x_n)}.$$

Thirdly, since X is CAT(0), there is a point w_n on c_6 such that $d(w_n, y_n) \leq 2\,C_3$. This implies that

$$\begin{aligned}(f_n((g^i h^j)^n.g^+), f_n(y_n))_{f_n(x_n)} \geq (f_n(w_n), f_n(y_n))_{f_n(x_n)} \geq \dots \\ \dots \geq d_{T_n}(f_n(x_n), f_n(y_n)) - 2\,C_3 \geq i \left| g \right|_\rho - 4\,C_3.\end{aligned}$$

So, if i satisfies $i\,\left|g\right|_{\rho}-4\,C_3\geq C_1+C(\delta,6)$, Lemma 1.3.4 yields inequality (c):

$$
\begin{aligned}
(f_n((g^ih^j)^n.g^+),f_n(y_n))_{f_n(x_n)} &\geq C_1+C(\delta,6)\geq\dots\\
&\cdots\geq(f_n((g^ih^j)^n.g^+),f_n((g^ih^j)^n.h^-))_{f_n(x_n)}.
\end{aligned}
$$

Now, we can prove Claim 4 with the following induction argument: Let $n=1$. Suppose that i and j are large enough for Claims 2 and 3, and inequalities (a) to (c), to hold. Then the inequalities (a) to (c) imply that

$$(f_1(x_0),f_1(x_2))_{f_1(x_1)}=(f_1(g^ih^j.g^+),f_1(g^ih^j.h^-))_{f_1(x_1)},$$

which yields the desired inequality ($**$) for $n=1$ because of Lemma 1.3.4. Now, suppose that inequality ($**$) holds for some $n\geq 1$. Using Lemma 1.3.4, Claim 3 and the induction hypothesis, we get

$$
\begin{aligned}
&(f_{n+1}(x_0),f_{n+1}(x_n))_{f_{n+1}(x_{n+1})}\\
&\qquad\geq d_{T_{n+1}}(f_{n+1}(x_n),f_{n+1}(x_{n+1}))-(f_{n+1}(x_0),f_{n+1}(x_{n+1}))_{f_{n+1}(x_n)}\\
&\qquad\geq d(x_n,x_{n+1})-(x_0,x_{n+1})_{x_n}-C(\delta,6)\\
&\qquad\geq i\,\left|g\right|_{\rho}+j\,\left|h\right|_{\rho}-C_5-C_6-C(\delta,6).
\end{aligned}
$$

Thus, if i and j satisfy $i\,\left|g\right|_{\rho}+j\,\left|h\right|_{\rho}-C_5-C_6-C(\delta,6)\geq C_1+C(\delta,6)$, Lemma 1.3.4 yields inequality (d):

$$
\begin{aligned}
(f_{n+1}(x_0),f_{n+1}(x_n))_{f_{n+1}(x_{n+1})} &\geq C_1+C(\delta,6)\geq\dots\\
&\cdots\geq(f_{n+1}((g^ih^j)^{n+1}.g^+),f_{n+1}((g^ih^j)^{n+1}.h^-))_{f_{n+1}(x_{n+1})}.
\end{aligned}
$$

Together the inequalities (a) to (d) imply for all sufficiently large i and j that

$$
\begin{aligned}
&(f_{n+1}(x_0),f_{n+1}(x_{n+2}))_{f_{n+1}(x_{n+1})}\\
&\qquad=(f_{n+1}((g^ih^j)^{n+1}.g^+),f_{n+1}((g^ih^j)^{n+1}.h^-))_{f_{n+1}(x_{n+1})}.
\end{aligned}
$$

Hence, by Lemma 1.3.4, the desired inequality ($**$) holds for $n+1$, too. This completes the proof of Claim 4.

Now, we complete the proof of Prop. 1.3.5. We verify inequality $(*)$ for $C := C_5 + 2\,C_6$ by induction on n, provided that i and j are sufficiently large: Let $n = 1$. Then inequality $(*)$ follows immediately from Claim 3. Suppose that inequality $(*)$ holds for some $n \geq 1$. By definition, we have

$$d(x_0, x_{n+1}) \;=\; d(x_0, x_n) + d(x_n, x_{n+1}) - 2\,(x_0, x_{n+1})_{x_n}.$$

Applying the induction hypothesis and Claims 3 and 4, we can deduce

$$\begin{aligned} d(x_0, x_{n+1}) &\leq n\,(\,|\,g^i\,|_\rho + |\,h^j\,|_\rho + C) + (\,|\,g^i\,|_\rho + |\,h^j\,|_\rho + C_5) \\ &\leq (n+1)\,(\,|\,g^i\,|_\rho + |\,h^j\,|_\rho) + (n+1)\,C, \end{aligned}$$

and

$$\begin{aligned} d(x_0, x_{n+1}) &\geq n\,(\,|\,g^i\,|_\rho + |\,h^j\,|_\rho - C) + (\,|\,g^i\,|_\rho + |\,h^j\,|_\rho - C_5) - 2\,C_6 \\ &\geq (n+1)\,(\,|\,g^i\,|_\rho + |\,h^j\,|_\rho) - n\,C - C_5 - 2\,C_6. \end{aligned}$$

Therefore, inequality $(*)$ holds for $n+1$, too. This completes the proof of Proposition 1.3.5. □

Chapter 2

The Boundary of a CAT(0) Group

This chapter is divided into two sections. In the first section we define the boundary of an arbitrary CAT(0) group. In the second section we prove some basic properties of this boundary. In particular, we check that for word-hyperbolic CAT(0) groups the boundary as defined in Section 1 is the same as the well-known Gromov boundary; and that the boundary of the free abelian group $\mathbb{Z}^n$ is homeomorphic to the sphere S^{n-1}.

2.1 The Boundary Construction

Let G be a CAT(0) group. The construction of a boundary for G can be outlined as follows: Each geometric action ρ of G on a CAT(0) space canonically induces a uniform structure U_ρ on the set G^∞ of infinite order elements in G. By considering all geometric actions of G on CAT(0) spaces, we obtain a family $\{U_{\rho_\lambda}\}_{\lambda\in\Lambda}$ of uniform structures on G^∞. This family of uniformities has a least upper bound: a uniform structure U on G^∞, which we shall call the *boundary uniformity.* The boundary ∂G will be defined as the Hausdorff completion of G^∞ with respect to the boundary uniformity U.

Firstly, we consider a fixed geometric action ρ of G on a CAT(0) space X. Since X carries a geometric group action, it is proper and complete. So, its visual boundary ∂X is a compact topological space (see e.g. [BH99], Chap. II.8, for details). Since each compact topological space carries a canonical uniform structure that is compatible with its topology (see e.g. [Bou89a], II.4.1, for details), we get

Proposition 2.1.1 *There is exactly one uniformity $U_{\partial X}$ compatible with the topology of ∂X. The entourages of $U_{\partial X}$ are precisely all the neighbourhoods of the diagonal Δ in $\partial X \times \partial X$. Furthermore, ∂X endowed with $U_{\partial X}$ is a complete uniform space.*

Let G^∞ denote the set of infinite order elements in G. Recall that each $\gamma \in G^\infty$ acts on X via ρ as a hyperbolic isometry (see e.g. [BH99], Chap. II.6, for details). Thus, we can define a canonical map $\tau_\rho : G^\infty \to \partial X$, by mapping each $\gamma \in G^\infty$ to the positive endpoint $\gamma^+ \in \partial X$ of one of its axes. Note that this map is well-defined, since all the axes of a hyperbolic isometry of a CAT(0) space are parallel. Thus, taking the inverse image of the uniformity $U_{\partial X}$ on ∂X under τ_ρ, we can assign to ρ a canonical uniform structure U_ρ on G^∞. In other words, U_ρ is the coarsest uniformity on the set G^∞ such that the map τ_ρ is uniformly continuous.

Secondly, consider the set Γ of all geometric G-actions on CAT(0) spaces. By the previous paragraph there exists a family $\mathcal{U}_G := \{U_\rho\}_{\rho\in\Gamma}$ of uniform structures on G^∞. For some types of CAT(0) groups all these uniformities coincide (see Sec. 3), but in general this is far from being true (see Example 3.1.1). Nevertheless, as a direct application of Prop. 4 in [Bou89a], II.2, we get:

Proposition 2.1.2 *The family $\mathcal{U}_G = \{U_\rho\}_{\rho\in\Gamma}$ of uniformities on G^∞ has a least upper bound U_G in the partially ordered set of all uniformities on G^∞. A fundamental system of entourages of U_G is given by the set of all finite intersections*

$$V_{\rho_1} \cap \cdots \cap V_{\rho_n},$$

where V_{ρ_i} is an entourage of U_{ρ_i}.

We call the least upper bound U_G of the family $\mathcal{U}_G$ of uniform structures on G^∞ the *boundary uniformity* of G. Note that according to Prop. 4 in [Bou89a], II.2., the boundary uniformity can be characterized as follows:

Proposition 2.1.3 *The boundary uniformity U_G is the coarsest uniform structure on the set G^∞ such that the map $\tau_\rho : G^\infty \to \partial X$ is uniformly continuous for each geometric G-action ρ on a CAT(0) space X. Moreover, if h is a mapping from a uniform space Z into the set G^∞, then h is uniformly continuous as a map into the uniform space (G^∞, U_G), if and only if the map $\tau_\rho \circ h$ is uniformly continuous for each $\rho \in \Gamma$.*

It is easy to see that in general G^∞ is neither Hausdorff nor complete with respect to U_G. Nevertheless, each uniform space has a Hausdorff completion (see [Bou89a], II.3, for details), which is well-defined up to isomorphism:

Proposition 2.1.4 *There exists a complete Hausdorff uniform space $\hat{G}$ and a uniformly continuous map $\iota : (G^\infty, U_G) \to \hat{G}$ having the following property (P):*

(P): Given any uniformly continuous map f of (G^∞, U_G) into a complete Hausdorff uniform space Z, there is a unique uniformly continuous map $g : \hat{G} \to Z$ such that $f = g \circ \iota$.

If (j, Y) is another pair consisting of a complete Hausdorff uniform space Y and a uniformly continuous map $j : (G^\infty, U_G) \to Y$ having property (P), then there is a unique isomorphism $\phi : \hat{G} \to Y$ such that $j = \phi \circ \iota$.

The map $\iota : (G^\infty, U_G) \to \hat{G}$ is called the *canonical map* of (G^∞, U_G) into its Hausdorff completion $\hat{G}$; and the image $\iota(G^\infty)$ is called the *Hausdorff uniform space* associated to (G^∞, U_G). Now, we can define the boundary of G as follows:

Definition 2.1.5 (Boundary) The *boundary* ∂G of the CAT(0) group G is defined as the Hausdorff completion $\hat{G}$ of G^∞ with respect to the boundary uniformity U_G.

Finally, we record some facts about ∂G, which are immediate consequences of the construction (see [Bou89a], II.3, for details):

Proposition 2.1.6 *(i) The subspace $\iota(G^\infty)$ is dense in ∂G.*

(ii) The graph of the equivalence relation $\iota(g) = \iota(g')$ is the intersection of all the entourages of (G^∞, U_G).

(iii) The boundary uniformity U_G on G^∞ is the inverse image under ι of the uniform structure of ∂G (or that of $\iota(G^\infty)$).

(iv) The entourages of $\iota(G^\infty)$ are the images under $\iota \times \iota$ of the entourages of (G^∞, U_G); and the closures in $\partial G \times \partial G$ of the entourages of $i_G(G^\infty)$ form a fundamental system of entourages of ∂G.

(v) For each geometric G-action ρ on a CAT(0) space X the map $\tau_\rho : G^\infty \to \partial X$ extends to a canonical uniformly continuous map $\hat{\tau}_\rho : \partial G \to \partial X$ such that $\tau_\rho = \hat{\tau}_\rho \circ \iota$.

(vi) The uniformity of ∂G is the coarsest for which all the canonical maps $\hat{\tau}_\rho : \partial G \to \partial X$ associated to a geometric G-action ρ on a CAT(0) space X are uniformly continuous.

(vii) We can identify $\iota(G^\infty)$ with the image in $\prod_{\rho\in\Gamma} \partial X_\rho$ of G^∞ under the product map $(\tau_\rho)_{\rho\in\Gamma}$, where X_ρ is the CAT(0) space carrying the G-action ρ.

(viii) We can identify ∂G with the closure in $\prod_{\rho\in\Gamma} \partial X_\rho$ of the image of G^∞ under $(\tau_\rho)_{\rho\in\Gamma}$.

2.2 Basic Properties of ∂G

The aim of this section is to prove some basic properties for the boundary we have defined in the previous section. We prove that it is compact, that it is invariant under isomorphisms, and that it carries a canonical G-action. We show also that for word-hyperbolic CAT(0) groups the boundary as defined in Section 1 is the same as the usual Gromov-boundary; and that the boundary of the free abelian group $\mathbb{Z}^n$ is homeomorphic to the sphere S^{n-1}.

Proposition 2.2.1 *The boundary ∂G of a CAT(0) group G is compact.*

Proof: Recall that a uniform space is said to be *precompact* if its Hausdorff completion is compact. In order to prove the proposition it is enough to show that G^∞ endowed with the boundary uniformity U_G is a precompact uniform space. By Prop. 2.1.3, U_G is the coarsest uniformity on G^∞ such that the map

$\tau_\rho : G^\infty \to \partial X$ is uniformly continuous for each geometric G-action ρ on a CAT(0) space X. Thus, we can apply Prop. 3 of [Bou89a], II.4.3: The uniform space (G^∞, U_G) is precompact if and only if $\tau_\rho(G^\infty)$ is a precompact subset of ∂X for each ρ. Obviously, the latter holds, since each ∂X is compact. □

Proposition 2.2.2 *Let $\phi : H \to G$ be a monomorphism of a group H into a CAT(0) group G such that $\phi(H)$ is a subgroup of finite index in G. Then H is a CAT(0) group, and there is a unique map $\hat{\phi} : \partial H \to \partial G$ such that*

$$\begin{array}{ccc} (H^\infty, U_H) & \xrightarrow{\phi} & (G^\infty, U_G) \\ {\scriptstyle \iota_H}\downarrow & & \downarrow{\scriptstyle \iota_G} \\ \partial H & \xrightarrow{\hat{\phi}} & \partial G. \end{array}$$

is a commuting diagram of uniformly continous maps.

Proof: Suppose that X be a CAT(0) space that carries a geometric G-action ρ. Since $\phi(H)$ is of finite index in G, the restriction of ρ to the subgroup $\phi(H)$ is a geometric $\phi(H)$-action on X. Set $\phi_*\rho(h, x) := \rho(\phi(h)).x$. Since ϕ is a monomorphism, $\phi_*\rho$ is a geometric action by H on X. Hence, H is a CAT(0) group. Note that for each geometric G-action ρ on a CAT(0) space X the maps $\tau_{\phi_*\rho}$ and $\tau_\rho \circ \phi$ from (H^∞, U_H) into ∂X coincide. Since $\tau_{\phi_*\rho}$ is uniformly continuous, Prop. 2.1.3 implies that the map $\phi : (H^\infty, U_H) \to (G^\infty, U_G)$ is uniformly continuous, too. According to Prop. 15 in [Bou89a], II.3.7, ϕ has a unique uniformly continuous extension $\hat{\phi}$ such that the above diagram commutes. □

We get the statements (i) to (iv) of the following corollary as an immediate consequence of Prop. 2.2.2. The proof of statement (v) is straightforward.

Corollary 2.2.3 *Let G be a CAT(0) group.*

(i) The inclusion of a finite index subgroup H into a CAT(0) group G induces a canonical uniformly continuous map from ∂H to ∂G.

(ii) Each group isomorphism $\phi : G \to H$ induces a canonical isomorphism $\partial\phi : \partial G \to \partial H$.

(iii) The action of the automorphism group Aut (G) *on* G *induces a canonical* Aut (G)*-action on* ∂G *by isomorphisms.*

(iv) The action of G *on itself by conjugation induces a canonical* G*-action on* ∂G *by isomorphisms.*

(v) Let ρ *be a geometric* G*-action on a CAT(0) space* X*. With respect to the canonical* G*-action on* ∂G*, and the* G*-action on* ∂X *that is induced by* ρ*, the canonical map* $\hat{\tau}_\rho : \partial G \to \partial X$ *is* G*-equivariant.*

Finally, we look at two special types of CAT(0) groups for which the boundary construction given in Section 1 is particularly easy to understand, namely word-hyperbolic CAT(0) groups and free abelian groups of finite rank. We will see that in the case of these two types the family $\mathcal{U}_G$ contains just one uniform structure.

Proposition 2.2.4 *Let* G *be a word-hyperbolic CAT(0) group. Then the boundary* ∂G *as defined above is* G*-equivariantly isomorphic to the usual Gromov boundary* $\partial_{Gr} G$ *of* G*.*

Proof: We will use several standard results in the theory of word-hyperbolic groups. Details of these results can be found in [CDP90], [GH90], or [BH99], for example. Let $\mathcal{A}$ be a finite set of generators for G, and let $C_{\mathcal{A}}(G)$ be the associated Cayley graph endowed with the word-metric $d_{\mathcal{A}}$. We can identify the Gromov boundary $\partial_{Gr} G$ with $\partial C_{\mathcal{A}}(G)$. Let $g \in G^\infty$ be given. It follows from Chap. 9, Thms. 3.3 and 3.4, in [CDP90], for example, that g acts as a hyperbolic isometry on $C_{\mathcal{A}}(G)$. By definition this means that the map $g^* : \mathbb{Z} \to C_{\mathcal{A}}(G)$ defined by $g^*(n) := g^n.e$ is a quasi-isometry. Since $C_{\mathcal{A}}(G)$ is δ-hyperbolic, there exists a biinfinite geodesic line $c_g : \mathbb{R} \to C_{\mathcal{A}}(G)$ in a uniformly bounded distance from $\operatorname{im} g^*$. We define a map $\tau : G^\infty \to \partial_{Gr} G$ by $\tau(g) := c_g(+\infty)$. Let ρ be a geometric G-action on a CAT(0) space X, and let $x \in X$ be a basepoint. By the Švarc-Milnor Lemma the map $g \mapsto g.x$ gives rise to a G-equivariant quasi-isometry $\psi : C_{\mathcal{A}}(G) \to X$, which induces an isomorphism $\bar{\psi} : \partial_{Gr} \to \partial X$. For each $g \in G^\infty$ the quasi-isometry ψ maps c_g into a uniformly bounded neighbourhood of an axis a_g of g in X. Hence, the following diagram commutes:

$$\begin{array}{ccc} G^\infty & \xrightarrow{\mathrm{id}_G} & G^\infty \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau_\rho} \\ \partial_{Gr} G & \xrightarrow{\bar{\Psi}} & \partial X. \end{array}$$

Thus, if U_0 is the inverse image of the uniform structure on $\partial_{Gr} G$ under τ, the two uniformities U_0 and U_ρ on G^∞ are the same. Therefore, the family $\mathcal{U}_G$ consists of just one uniform structure, namely U_0. Since the image $\tau(G^\infty)$ is dense in $\partial_{Gr} G$ (see e.g. [BR96b] for details), ∂G is isomorphic to $\partial_{Gr} G$. Obviously, this isomorphism is G-equivariant. $\square$

Proposition 2.2.5 *The boundary of the free abelian group $\mathbb{Z}^n$ of rank n is isomorphic to the sphere S^{n-1}. The canonical $\mathbb{Z}^n$-action on $\partial \mathbb{Z}^n$ is trivial.*

Proof: Throughout this proof we identify the euclidean space $\mathbb{E}^n$ with $\mathbb{R}^n$. Let $z_1, \ldots, z_n$ be a basis for $\mathbb{Z}^n$. Then each $g \in \mathbb{Z}^n$ can be uniquely written as product $\prod_{i=1}^n z_i^{\zeta_i(g)}$, where $\zeta_i(g) \in \mathbb{Z}$ for $i = 1, \ldots, n$. We define a map $\zeta : \mathbb{Z}^n \to \mathbb{E}^n$ by assigning the point $\zeta(g) := (\zeta_1(g), \ldots, \zeta_n(g))$ in $\mathbb{E}^n$ to each $g \in \mathbb{Z}^n$.

Firstly, we consider the standard $\mathbb{Z}^n$-action ρ_0 on the euclidean space $\mathbb{E}^n$, where each z_i acts by translation along the i-th canonical basis vector e_i of $\mathbb{E}^n$. If the visual boundary $\partial \mathbb{E}^n$ is identified with the space of geodesic rays issuing from the origin, the canonical map τ_{ρ_0} from $(\mathbb{Z}^n)^\infty = \mathbb{Z}^n \setminus \{e\}$ into $\partial \mathbb{E}^n$ can be described as follows: Each $g \in (\mathbb{Z}^n)^\infty$ is mapped to the unique geodesic ray that issues from the origin and passes through $\zeta(g)$. Therefore, the image $\tau_{\rho_0}((\mathbb{Z}^n)^\infty)$ is the dense subspace of $\partial \mathbb{E}^n$ that consists of all the geodesic rays issuing from the origin with rational direction. Note that the Hausdorff completion of $(\mathbb{Z}^n)^\infty$ with respect to the uniformity U_{ρ_0} is isomorphic to $\partial \mathbb{E}^n$, i.e. the sphere S^{n-1}. Note also that the intersection of all the entourages of U_{ρ_0} contains precisely those pairs $(g, h) \in (\mathbb{Z}^n)^\infty \times (\mathbb{Z}^n)^\infty$, for which there is a rational $q > 0$ such that $\zeta(g) = q\zeta(h)$.

Secondly, we consider an arbitrary geometric action ρ of $\mathbb{Z}^n$ on a CAT(0) space X. The Flat Torus Theorem 1.1.2 implies that the minimal set $\mathrm{Min}(\mathbb{Z}^n) \subset X$ is non-empty. Hence, we can choose a basepoint $x \in \mathrm{Min}(\mathbb{Z}^n)$. Let C_x be the closed convex hull of the $\mathbb{Z}^n$-orbit of x in X. The Flat Torus Theorem states that C_x is isometric to $\mathbb{E}^n$. We identify C_x with $\mathbb{E}^n$ such that x is identified with the origin, and regard $\partial\mathbb{E}^n$ as a subspace of ∂X. The Flat Torus Theorem states furthermore that each z_i acts as a translation on $\mathbb{E}^n \equiv C_x$, say along a vector v_i, such that the quotient of $\mathbb{E}^n$ by the $\mathbb{Z}^n$-action is an n-torus. Thus, the vectors $v_1, \ldots, v_n$ are linearly independent; and the canonical map $\tau_\rho : (\mathbb{Z}^n)^\infty \to \partial\mathbb{E}^n \subset \partial X$ can be described as follows: τ_ρ maps each $g \in (\mathbb{Z}^n)^\infty$ to the unique geodesic ray in $\mathbb{E}^n$ that issues from the origin and passes through $\sum_{i=1}^n \zeta_i(g)v_i$. Since $v_1, \ldots, v_n$ are linearly independent, the induced uniformity U_ρ on $(\mathbb{Z}^n)^\infty$ is the same as the uniformity U_{ρ_0}. Hence, the least upper bound of the family $\mathcal{U}_{\mathbb{Z}^n}$ is the uniformity U_{ρ_0} itself; and consequently, the boundary $\partial\mathbb{Z}^n$ is isomorphic to the sphere S^{n-1}.

Since the action of $\mathbb{Z}^n$ on itself by conjugation is trivial, the canonical $\mathbb{Z}^n$-action on $\partial\mathbb{Z}^n$ is trivial, too. □

Chapter 3

The Boundary of G $\times$ $\mathbb{Z}$

The aim of this chapter is to study the boundary of product groups of the form $G \times \mathbb{Z}$, where G is a non-elementary word-hyperbolic CAT(0) group.

3.1 Distinct Uniformities in $\mathcal{U}_{F_2 \times \mathbb{Z}}$: Examples

In the previous chapter we only considered the boundary construction for CAT(0) groups G where the family $\mathcal{U}_G$ consists of just one uniform structure. In the examples of this section we will look at various geometric actions of $F_2 \times \mathbb{Z}$ on CAT(0) spaces, which all give rise to distinct uniformities on $(F_2 \times \mathbb{Z})^\infty$.

It was first proved by Bowers and Ruane in [BR96a] that – in the language of the present thesis – the family $\mathcal{U}_{F_2 \times \mathbb{Z}}$ contains at least two distinct uniformities. Essentially we recall their argument in the next example:

Example 3.1.1 Let $\mathcal{A} := \langle a, b \rangle$ be the standard set of generators for F_2, and $T := \Gamma_{\mathcal{A}}(F_2)$ the associated Cayley graph. T is a 4-valent tree. Hence both T and the Euclidean product $T \times \mathbb{R}$ are CAT(0) spaces. Each pair $(p, q) \in \mathbb{R}^2$ uniquely determines a homomorphism $\varphi_{(p,q)}$ in Hom$(F_2, \mathbb{R})$ via $\phi_{(p,q)}(a) := p$ and $\varphi_{(p,q)}(b) := q$, and vice versa. Therefore, we can define for each pair $(p, q) \in \mathbb{R}^2$ (or alternatively for each homomorphism in Hom$(F_2, \mathbb{R})$) a geometric action $\rho_{(p,q)}$ of $F_2 \times \mathbb{Z}$ on $T \times \mathbb{R}$ by

$$(a, 0)*(x, r) := (a.x, r+p), \quad (b, 0)*(x, r) := (b.x, r+q), \quad (e, 1)*(x, r) := (x, r+1),$$

where the action $\rho := (g,x) \mapsto g.x$ is the standard action of F_2 on T. We call ρ the *F_2-factor action* and the pair (p,q) the *shift parameters* associated to the action $\rho_{(p,q)}$. We define $U_{(p,q)}$ to be the uniformity on $(F_2 \times \mathbb{Z})^\infty$ induced by $\rho_{(p,q)}$. Note that for any pair (p,q) of shift parameters $U_{(p,q)}$ is a metrizable uniformity. We can identify the visual boundary $\partial(T \times \mathbb{R})$ with the suspension $\Sigma(\partial F_2) \equiv ([-\infty,+\infty] \times \partial F_2)/\sim$ such that for each action $\rho_{(p,q)}$ the associated canonical map $\tau_{(p,q)}$ from $(F_2 \times \mathbb{Z})^\infty$ to $\partial(T \times \mathbb{R})$ is given by

$$(g,z) \mapsto \tau_{(p,q)}(g,z) = [\frac{z + \varphi_{(p,q)}(g)}{|g|_\rho}, g^+].$$

For example, the following figure illustrates that $\tau_{(2,0)}(ab,1) = [\frac{3}{2},(ab)^+]$. Note that $\tan(\alpha) = \frac{3}{2}$.

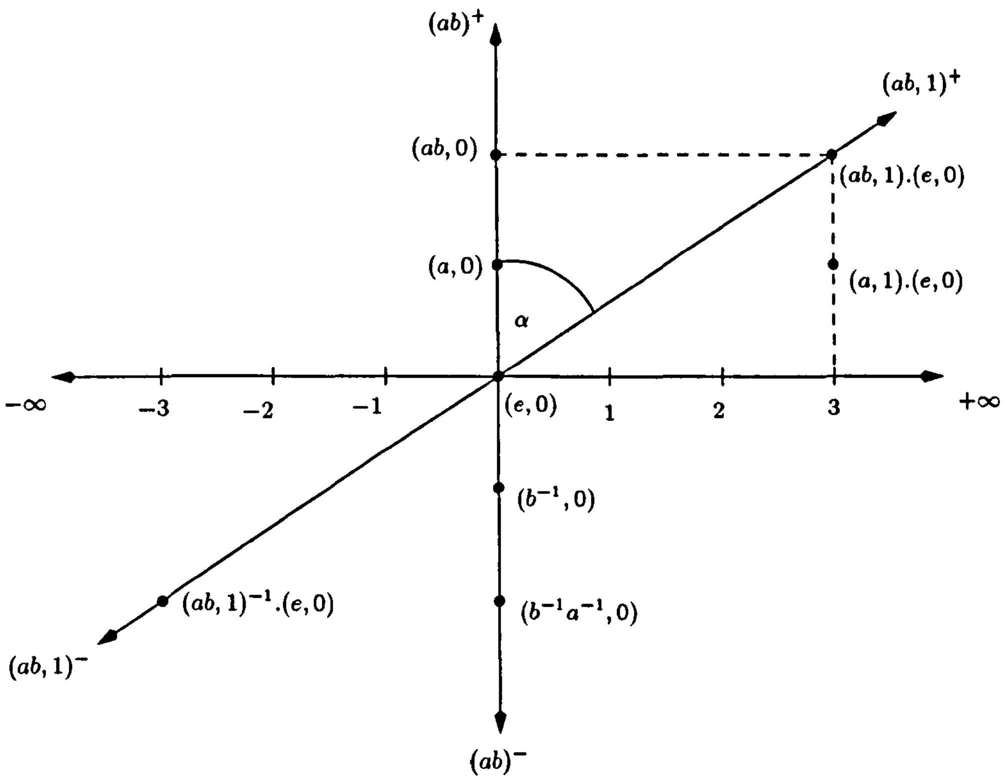

Figure 3.1: Illustrating $\tau_{(2,0)} : (F_2 \times \mathbb{Z})^\infty \to \Sigma(\partial F_2)$

We check that $U_{(p,q)}$ coincides with $U_{(\tilde{p},\tilde{q})}$, if and only if $(p,q) = (\tilde{p},\tilde{q})$: Obviously, $U_{(p,q)}$ coincides with $U_{(\tilde{p},\tilde{q})}$ if $(p,q) = (\tilde{p},\tilde{q})$. So, suppose that the uniformities $U_{(p,q)}$ and $U_{(\tilde{p},\tilde{q})}$ coincide. Let $\nu_j \in \mathbb{N}$ and $\mu_j \in \mathbb{Z}$ be such that $(\frac{\mu_j}{\nu_j})_j$ converges to $\frac{1}{2}(1-p+q)$. Consider the sequences $(c_j)_j := (a^j b^j, j)_j$ and $(d_j)_j := (a^{\nu_j}, \mu_j)_j$ in $(F_2 \times \mathbb{Z})^\infty$. The images of $(c_j)_j$ and $(d_j)_j$ under the canonical maps $\tau_{(p,q)}$ and $\tau_{(\tilde{p},\tilde{q})}$ have the following limits:

$$
\begin{aligned}
\lim (\tau_{(p,q)}(c_j))_j &= [\tfrac{1}{2}(1+p+q), a^+],\\
\lim (\tau_{(p,q)}(d_j))_j &= [\tfrac{1}{2}(1+p+q), a^+],\\
\lim (\tau_{(\tilde{p},\tilde{q})}(c_j))_j &= [\tfrac{1}{2}(1+\tilde{p}+\tilde{q}), a^+],\\
\lim (\tau_{(\tilde{p},\tilde{q})}(d_j))_j &= [\tfrac{1}{2}(1+2\tilde{p}-p+q), a^+].
\end{aligned}
$$

Thus, the alternating sequence $(s_j)_j$ formed of $(c_j)_j$ and $(d_j)_j$ is a Cauchy sequence with respect to $U_{(p,q)}$. Since $U_{(p,q)}$ and $U_{(\tilde{p},\tilde{q})}$ coincide, $(s_j)_j$ is also a Cauchy sequence with respect to $U_{(\tilde{p},\tilde{q})}$. This implies $\frac{1}{2}(1+\tilde{p}+\tilde{q}) = \frac{1}{2}(1+2\tilde{p}-p+q)$, or equivalently,

$$\tilde{q}-\tilde{p} = q-p.$$

An analogous argument based on the sequence $(c'_j)_j := (a^j b^{2j}, j)_j$ and a sequence $(d'_j)_j := (a^{\nu'_j}, \mu'_j)_j$, where $(\frac{\mu'_j}{\nu'_j})_j$ converges to $\frac{1}{2}(1-p+2q)$, yields

$$2\tilde{q}-\tilde{p} = 2q-p.$$

Hence, if the uniformities $U_{(p,q)}$ and $U_{(\tilde{p},\tilde{q})}$ on $(F_2 \times \mathbb{Z})^\infty$ coincide, the corresponding pairs (p,q) and $(\tilde{p},\tilde{q})$ of shift parameters must be the same. □

Note that in the above example we did not need to change the F_2-factor action at all to create uncountably many distinct uniformities on $(F_2 \times \mathbb{Z})^\infty$. It was enough to vary the shift parameters (p,q). However, there is another way in which more uniformities in $\mathcal{U}_{F_2\times\mathbb{Z}}$ can arise – even if we consider actions with trivial shift parameters only: We can vary the marked length spectrum of the F_2-factor action.

Example 3.1.2 For each ε with $\frac{1}{2} \geq \varepsilon \geq 0$ we construct a CAT(0) space T_ε, by modifying T as follows: Let $\Delta(ABC)$ be the convex hull in $\mathbb{E}^2$ of a triangle with side lengths $(AC) = (CB) = \frac{1}{2-\sqrt{2}}\varepsilon$ and $(AB) = \frac{\sqrt{2}}{2-\sqrt{2}}\varepsilon$. Let V be a vertex in T. Suppose A_V is the point on the incoming a-edge at V such that $d_T(A_V, V) = \frac{1}{2-\sqrt{2}}\varepsilon$; and B_V is the point on the outgoing b-edge at V such that $d_T(B_V, V) = \frac{1}{2-\sqrt{2}}\varepsilon$. Then we can glue $\Delta(ABC)$ to T such that the side (AC) is glued isometrically onto $[A_V, V]$, and the side (BC) onto $[B_V, V]$. Gluing a copy of $\Delta(ABC)$ to T in that manner at each vertex V, we obtain a space T_ε that is CAT(0) with respect to the induced path metric.

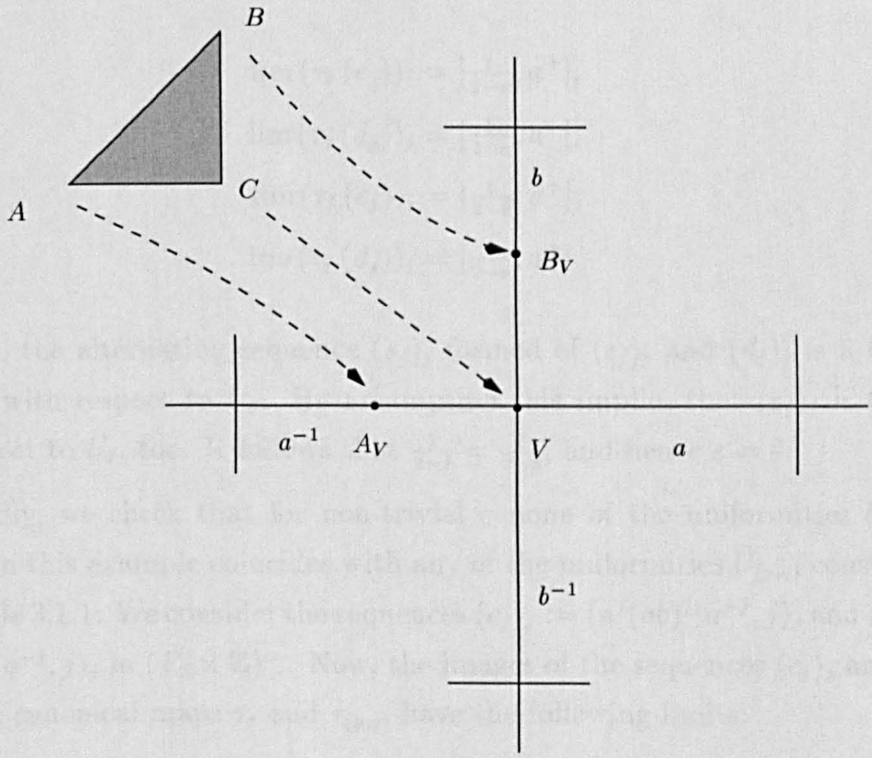

Figure 3.2: Constructing T_ε

Clearly, the geometric F_2-action on T induces a geometric F_2-action on T_ε. Note that the marked length spectrum of this F_2-action on T_ε is different for each ε. For example, the translation length of ab equals $2 - \varepsilon$. For each ε with $\frac{1}{2} \geq \varepsilon \geq 0$ we define a geometric action ρ_ε of $F_2 \times \mathbb{Z}$ on $T_\varepsilon \times \mathbb{R}$ by

$$(a,0) * (x,r) := (a.x,r), \quad (b,0) * (x,r) := (b.x,r), \quad (e,1) * (x,r) := (x,r+1).$$

Note that each action ρ_ε has trivial shift parameters. Let U_ε denote the uniformity on $(F_2 \times \mathbb{Z})^\infty$ that is induced by ρ_ε.

Firstly, we want to see that two uniformities U_ε and $U_{\tilde\varepsilon}$ on $(F_2 \times \mathbb{Z})^\infty$ coincide, if and only if $\varepsilon = \tilde\varepsilon$: Suppose that U_ε and $U_{\tilde\varepsilon}$ are the same. Let $\nu_j, \mu_j \in \mathbb{N}$ be such that $(\frac{\mu_j}{\nu_j})_j$ converges to $\frac{1}{2-\varepsilon}$, and consider the sequences $(c_j)_j := (a^j(ab)^j a^{-j}, j)_j$ and $(d_j)_j := (a^{\nu_j}, \mu_j)_j$ in $(F_2 \times \mathbb{Z})^\infty$. If the visual boundary $\partial(T_\varepsilon \times \mathbb{R})$ is identified with $\Sigma\,(\partial F_2)$ in the same manner as before, the images of the sequences $(c_j)_j$ and $(d_j)_j$ under the canonical maps τ_ε and $\tau_{\tilde\varepsilon}$ have the following limits:

$$\begin{aligned}
\lim\,(\tau_\varepsilon\,(c_j))_j &= [\tfrac{1}{2-\varepsilon}, a^+],\\
\lim\,(\tau_\varepsilon\,(d_j))_j &= [\tfrac{1}{2-\varepsilon}, a^+],\\
\lim\,(\tau_{\tilde\varepsilon}\,(c_j))_j &= [\tfrac{1}{2-\tilde\varepsilon}, a^+],\\
\lim\,(\tau_{\tilde\varepsilon}\,(d_j))_j &= [\tfrac{1}{2-\varepsilon}, a^+].
\end{aligned}$$

Therefore, the alternating sequence $(s_j)_j$ formed of $(c_j)_j$ and $(d_j)_j$ is a Cauchy sequence with respect to U_ε. By assumption this implies that $(s_j)_j$ is Cauchy with respect to $U_{\tilde\varepsilon}$, too. It follows that $\frac{1}{2-\tilde\varepsilon} = \frac{1}{2-\varepsilon}$, and hence $\varepsilon = \tilde\varepsilon$.

Secondly, we check that for non-trivial ε none of the uniformities U_ε constructed in this example coincides with any of the uniformities $U_{(p,q)}$ constructed in Example 3.1.1: We consider the sequences $(c_j)_j := (a^j(ab)^{2j}a^{-j}, j)_j$ and $(d_j)_j := (a^j(a^2b^2)^j a^{-j}, j)_j$ in $(F_2 \times \mathbb{Z})^\infty$. Now, the images of the sequences $(c_j)_j$ and $(d_j)_j$ under the canonical maps τ_ε and $\tau_{(p,q)}$ have the following limits:

$$\begin{aligned}
\lim\,(\tau_\varepsilon\,(c_j))_j &= [\tfrac{1}{4-2\varepsilon}, a^+],\\
\lim\,(\tau_\varepsilon\,(d_j))_j &= [\tfrac{1}{4-\varepsilon}, a^+],\\
\lim\,(\tau_{(p,q)}\,(c_j))_j &= [\tfrac{1}{4}(1+2p+2q), a^+],\\
\lim\,(\tau_{(p,q)}\,(d_j))_j &= [\tfrac{1}{4}(1+2p+2q), a^+].
\end{aligned}$$

Thus, for any pair of shift parameters (p,q) the alternating sequence formed of $(c_j)_j$ and $(d_j)_j$ is a Cauchy sequence with respect to $U_{(p,q)}$. However, it is not a Cauchy sequence with respect to U_ε, unless $\varepsilon = 0$. □

In this section we saw that distinct uniformities on $(F_2 \times \mathbb{Z})^\infty$ can arise in two particular ways: One can fix an F_2-factor action and vary shift parameters as done in Example 3.1.1; or one can vary the marked length spectrum of the F_2-factor action and consider trivial shift parameters only, as done in Example 3.1.2. In order to understand the boundary construction for $G \times \mathbb{Z}$ in full generality, it is helpful to look at these two ways separately first. We will do so in the two following sections.

3.2 Fixing a G-Factor Action

In Example 3.1.1 we saw that the family $\mathcal{U}_{F_2 \times \mathbb{Z}}$ contains more than one uniform structure, because despite a fixed F_2-factor action we could vary shift parameters. In this section we want to study this effect more generally.

Throughout this section we consider a fixed non-elementary word-hyperbolic CAT(0) group G, and a fixed geometric action ρ of G on a CAT(0) space X. Also, we fix a basis $(\omega_1, \ldots, \omega_n)$ of Hom $(G, \mathbb{R})$. (By convention this basis is empty if Hom $(G, \mathbb{R})$ is trivial.) For each $g \in G$ we abbreviate $\omega(g) := (\omega_1(g), \ldots, \omega_n(g)) \in \mathbb{R}^n$. For each $\mu \in \mathbb{R}^n$ we get a geometric $G \times \mathbb{Z}$-action ρ_μ on $X \times \mathbb{R}$ by

$$(g, z) * (x, r) \;:=\; (g.x, r + z + \langle \mu, \omega(g) \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^n$. As in the previous section, we call μ the *vector of shift parameters* of ρ_μ. By $\tau_\mu : (G \times \mathbb{Z})^\infty \to \partial(X \times \mathbb{R})$ we denote the canonical map associated to ρ_μ, and by U_μ the uniform structure that is induced on $(G \times \mathbb{Z})^\infty$ via τ_μ. Varying shift parameters gives rise to a family $\mathcal{U}^\rho_{G \times \mathbb{Z}} := \{U_\mu\}_{\mu \in \mathbb{R}^n}$ of uniform structures on $(G \times \mathbb{Z})^\infty$. Note that in general $\mathcal{U}^\rho_{G \times \mathbb{Z}}$ is a proper subfamily of the family $\mathcal{U}_{G \times \mathbb{Z}}$ which is based on all $G \times \mathbb{Z}$-actions. We denote the least upper bound of the family $\mathcal{U}^\rho_{G \times \mathbb{Z}}$ by $U^\rho_{G \times \mathbb{Z}}$, and call the Hausdorff completion of $(G \times \mathbb{Z})^\infty$ with respect to $U^\rho_{G \times \mathbb{Z}}$ the *ρ-boundary* $\partial_\rho(G \times \mathbb{Z})$ of $G \times \mathbb{Z}$. As in the case of the general construction, $\partial_\rho(G \times \mathbb{Z})$ is a compact space; and there exists a canonical uniformly continuous map $\hat{\tau}_\mu$ from $\partial_\rho(G \times \mathbb{Z})$ to $\partial(X \times \mathbb{R})$ for each vector $\mu \in \mathbb{R}^n$ of shift parameters.

We can identify $\partial(X \times \mathbb{R})$ with the suspension $\Sigma\,(\partial G)$ such that the canonical map τ_μ from $(G \times \mathbb{Z})^\infty$ to $\partial(X \times \mathbb{R})$ associated to ρ_μ is given by

$$(g,z) \mapsto \tau_\mu(g,z) = \begin{cases} [\frac{z + \langle \mu, \omega(g) \rangle}{|g|_\rho}, g^+], & \text{if } g \in G^\infty, \\ [-\infty, \xi], & \text{if } g \notin G^\infty \text{ and } z < 0, \\ [+\infty, \xi], & \text{if } g \notin G^\infty \text{ and } z > 0, \end{cases}$$

where $\xi \in \partial G$ is arbitrary (see Fig. 3.1). Moreover, we can identify the ρ-boundary $\partial_\rho(G \times \mathbb{Z})$ with the closure in $\prod_{\mu \in \mathbb{R}^n} \Sigma(\partial G)$ of the image of $(G \times \mathbb{Z})^\infty$ under the product map $(\tau_\mu)_{\mu \in \mathbb{R}^n}$. Thus, $(\tau_\mu)_{\mu \in \mathbb{R}^n}$ is precisely the canonical map ι from $(G \times \mathbb{Z})^\infty$ into its Hausdorff completion $\partial_\rho(G \times \mathbb{Z})$.

Let the map $\phi_\rho : G^\infty \to \mathbb{R}^n$ be given by

$$g \mapsto \phi_\rho(g) := (\frac{\omega_1(g)}{|g|_\rho}, \ldots, \frac{\omega_n(g)}{|g|_\rho}).$$

Define the space S_ρ as the closure of im ϕ_ρ in $\mathbb{R}^n$. We set $M_\rho^s := ([-\infty, +\infty] \times \partial G \times S_\rho)/\sim$, where $(t, \xi, s) \sim (t', \xi', s')$ if and only if $[\, t = +\infty$ and $t' = +\infty\,]$, or $[\, t = -\infty$ and $t' = -\infty\,]$, or $[\, t = t'$ and $\xi = \xi'$ and $s = s'\,]$. (Here the superscript s stands for *shift.*) In the remainder of this section we want to show that M_ρ^s is canonically isomorphic to the ρ-boundary $\partial_\rho(G \times \mathbb{Z})$. Note that this implies in particular that $\partial_\rho(G \times \mathbb{Z})$ depends only on the marked length spectrum of ρ.

Theorem 3.2.1 *Let G be a non-elementary word-hyperbolic CAT(0) group, and let ρ be a geometric action by G on a CAT(0) space X. Let M_ρ^s be defined as above. Then the following is true:*

(i) The ρ-boundary $\partial_\rho(G \times \mathbb{Z})$ is canonically isomorphic to M_ρ^s.

(ii) The canonical $G \times \mathbb{Z}$-action on $\partial_\rho(G \times \mathbb{Z})$ is given by

$$(g, z) * [t, \xi, s] \;=\; [t, g.\xi, s].$$

(iii) For each vector $\mu \in \mathbb{R}^n$ of shift parameters the canonical map $\hat{\tau}_\mu$ from

$\partial_\rho(G \times \mathbb{Z})$ *to* $\partial(X \times \mathbb{R}) \equiv \Sigma(\partial G)$ *is given by:*

$$\hat{\tau}_\mu([t,\xi,s]) = [\frac{t + \langle \mu, s\rangle}{|g|_\rho}, \xi].$$

In order to prove this theorem we need some lemmas.

Lemma 3.2.2 *The image* im ϕ_ρ *is bounded in* $\mathbb{R}^n$.

Proof: We show that there exists a constant $C > 0$ such that $\left|\frac{\omega_i(g)}{|g|_\rho}\right| \leq C$ for all $g \in G^\infty$ and all $1 \leq i \leq n$. Let $\mathcal{A}$ be a finite system of generators for G, and let $|\cdot|_\mathcal{A}$ the corresponding word metric on G. If we set $c := \max\{\,|\omega_i(a)| \mid a \in \mathcal{A};\ 1 \leq i \leq n\,\}$ then $|\omega_i(g)| \leq c\,|g|_\mathcal{A}$ holds for all $g \in G$ and all $1 \leq i \leq n$. Let X be the CAT(0) space underlying the action ρ, and let $x \in X$ be a basepoint. By the Svarc-Milnor-Lemma 1.1.1 there exist constants $\lambda \geq 1$ and $\varepsilon > 0$ such that $\frac{1}{\lambda}|g|_\mathcal{A} - \varepsilon \leq d_X(x, g.x)$ for all $g \in G$. Since ρ is cocompact, there is a constant $d > 0$ such that for each $g \in G^\infty$ one can find a conjugate $\tilde{g}$ for which the distance between some axis $a_{\tilde{g}}$ of $\tilde{g}$ and x is less than d. It follows for each $k \in \mathbb{N}$ and each $1 \leq i \leq n$ that

$$|\omega_i(\tilde{g}^k)| \leq c\,|\tilde{g}^k|_\mathcal{A} \leq c\lambda\,(d_X(x,\tilde{g}^k.x) + \varepsilon) \leq c\lambda\,(|\tilde{g}^k|_\rho + 2d + \varepsilon),$$

and hence

$$|\omega_i(\tilde{g})| \leq c\lambda\,|\tilde{g}|_\rho + \frac{1}{k}c\lambda\,(2d + \varepsilon).$$

Thus, we can take $C := c\lambda$, and conclude that

$$|\omega_i(g)| = |\omega_i(\tilde{g})| \leq C\,|\tilde{g}|_\rho = C\,|g|_\rho.$$

□

Lemma 3.2.3 *The space* S_ρ *is a compact, convex subspace of* $\mathbb{R}^n$.

Proof: By Lemma 3.2.2 im ϕ_ρ is a bounded subspace of $\mathbb{R}^n$. Hence, its closure S_ρ in $\mathbb{R}^n$ is compact. In order to prove that S_ρ is convex, we consider the uniformly continuous map $H : [0,1] \times S_\rho \times S_\rho \to \mathbb{R}^n$ given by

$$(\vartheta, x, y) \mapsto H(\vartheta, x, y) := \vartheta\, x + (1 - \vartheta)\, y.$$

It is enough to show that H maps the dense subspace $([0,1] \cap \mathbb{Q}) \times \mathrm{im}\ \phi_\rho \times \mathrm{im}\ \phi_\rho$ of $[0,1] \times S_\rho \times S_\rho$ into S_ρ. Let $\vartheta \in [0,1] \cap \mathbb{Q}$ and $x, y \in \mathrm{im}\ \phi_\rho$ be given. For each $g \in G^\infty$ we abbreviate $\phi_\rho(g)$ by $\bar{g}$. Take $g, h \in G^\infty$ such that $\bar{g} = x$ and $\bar{h} = y$. We distinguish between two cases. Case 1: Suppose $g^+ \neq h^-$. Set $\eta := \frac{|g|_\rho}{|h|_\rho}$. Take $p_j, q_j, s \in \mathbb{N}$ and $r \in \mathbb{N}_0$ such that $(q_j)_j$ goes to infinity, $(\frac{p_j}{q_j})_j$ converges to η, and $\frac{r}{s} = \vartheta$. For each $j \in \mathbb{N}$ set $g_j := g^{q_j r} h^{p_j(s-r)}$. Since $g^+ \neq h^-$, we can apply Theorem 1.3.1 about the additivity of translation lengths. It follows that the translation length $|g_j|_\rho$ is non-trivial for all sufficiently large j. Moreover, it follows for all $i = 1, \ldots, n$ and all sufficiently large j that

$$\frac{\omega_i(g^{q_j r} h^{p_j(s-r)})}{|g^{q_j r}|_\rho + |h^{p_j(s-r)}|_\rho + C'} \leq \frac{\omega_i(g_j)}{|g_j|_\rho} \leq \frac{\omega_i(g^{q_j r} h^{p_j(s-r)})}{|g^{q_j r}|_\rho + |h^{p_j(s-r)}|_\rho - C'},$$

where $C' \geq 0$ is some constant. Thus, we get

$$\frac{\frac{r}{s}\,\omega_i(g) + \frac{s-r}{s}\frac{p_j}{q_j}\,\omega_i(h)}{\frac{r}{s}\,|g|_\rho + \frac{s-r}{s}\frac{p_j}{q_j}\,|h|_\rho + \frac{C'}{sq_j}} \leq \frac{\omega_i(g_j)}{|g_j|_\rho} \leq \frac{\frac{r}{s}\,\omega_i(g) + \frac{s-r}{s}\frac{p_j}{q_j}\,\omega_i(h)}{\frac{r}{s}\,|g|_\rho + \frac{s-r}{s}\frac{p_j}{q_j}\,|h|_\rho - \frac{C'}{sq_j}},$$

which implies

$$\begin{aligned}\lim_{j\to\infty} \frac{\omega_i(g_j)}{|g_j|_\rho} &= \frac{\vartheta\,\omega_i(g) + (1-\vartheta)\,\eta\,\omega_i(h)}{\vartheta\,|g|_\rho + (1-\vartheta)\,\eta\,|h|_\rho} = \frac{\sigma\,\omega_i(g) + (1-\sigma)\,\eta\,\omega_i(h)}{|g|_\rho} \\ &= \vartheta\,\frac{\omega_i(g)}{|g|_\rho} + (1-\vartheta)\,\frac{\omega_i(h)}{|h|_\rho} = \vartheta\,\bar{g} + (1-\vartheta)\,\bar{h}.\end{aligned}$$

Hence, $(\bar{g}_j)_j$ is a Cauchy sequence in S_ρ, whose limit is $H'(\vartheta, x, y)$. Since S_ρ is compact, $H'(\vartheta, x, y)$ lies in S_ρ for Case 1. Case 2: Suppose that $g^+ = h^-$. (Warning: In this case it may happen that $h = g^{-1}$, which means $\bar{h} = -\bar{g}$ in S_ρ. If so, $(\overline{g^p h^q})$ equals either $\bar{g}$ or 0 or $-\bar{g}$ for all $p, q \in \mathbb{N}_0$.) Since G is non-elementary, there exists a conjugate $h_1 \in G^\infty$ of h such that $h_1^- \neq h^-$. Thus, we can apply the argument from Case 1 to g and h_1. Note that $\bar{h}_1 = \bar{h}$. Hence, $H'(\vartheta, x, y)$ lies in S_ρ for Case 2, too. $\square$

Example 3.2.4 Let $\mathcal{A} = \langle a, b\rangle$ be the standard set of generators for F_2, and let ρ be the standard action of F_2 on the 4-valent tree T. Let ω_1 be the homomorphism from F_2 to $\mathbb{R}$ that is given by $\omega_1(a) = 1$ and $\omega_1(b) = 0$, and let ω_2 be the homomorphism from F_2 to $\mathbb{R}$ that is given by $\omega_2(a) = 0$ and $\omega_2(b) = 1$. Note that (ω_1, ω_2) is a basis for Hom $(F_2, \mathbb{R})$. S_ρ is looks like: $\square$

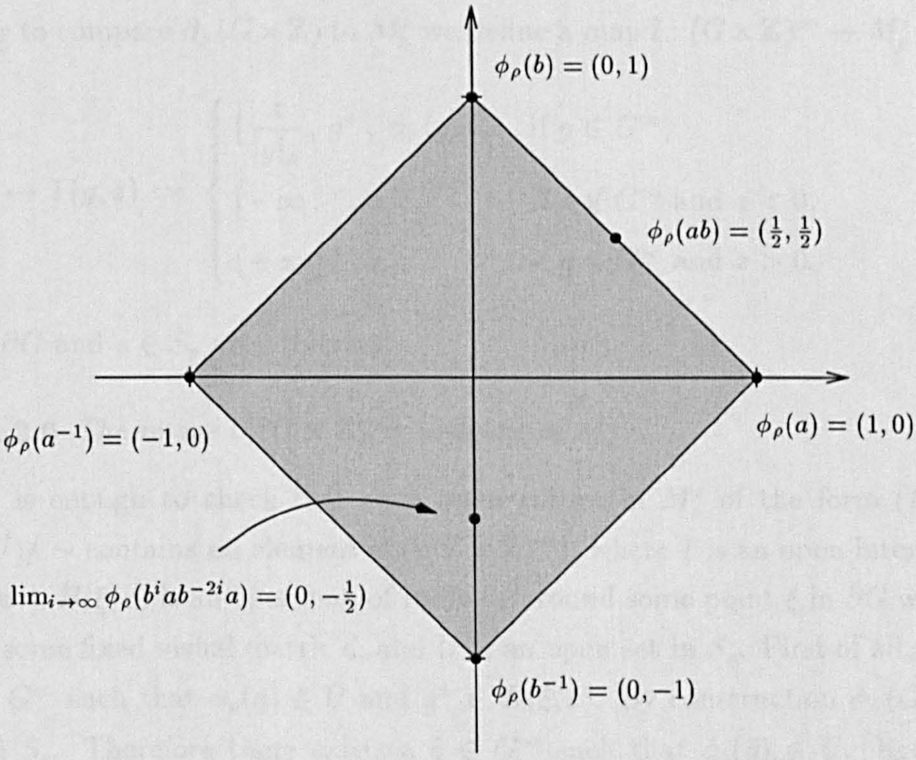

Figure 3.3: The space S_ρ for $\rho : F_2 \times T \to T$

Lemma 3.2.5 *Let X be a compact space, and U its canonical uniform structure. Suppose that $\{V_\lambda\}_{\lambda\in\Lambda}$ is a fundamental system of entourages for U. Then the suspension ΣX of X is compact. For each ε with $1 > \varepsilon > 0$ and each $\lambda \in \Lambda$ let $W_{(\lambda,\varepsilon)}$ consist precisely of those pairs $([t, x], [t', x'])$ in $\Sigma X \times \Sigma X$, for which $[t > \frac{1}{\varepsilon}$ and $t' > \frac{1}{\varepsilon}]$, or $[t < -\frac{1}{\varepsilon}$ and $t' < -\frac{1}{\varepsilon}]$, or $[|t - t'| < \varepsilon$ and $(x, x') \in V_\lambda]$ holds. Then the system $\{W_{(\lambda,\varepsilon)}\}$ is a fundamental system of entourages for the canonical uniformity on ΣX.*

Proof: It is a standard fact that the suspension of a compact space is compact, too. In particular, this implies that ΣX carries a unique uniform structure that is compatible with the given topology. It is straightforward to check that the system $\{W_{(\lambda,\varepsilon)}\}$ satisfies the axioms of a fundamental system of entourages. Moreover, it is obvious that the topology induced by $\{W_{(\lambda,\varepsilon)}\}$ coincides with the topology on ΣX. Therefore, $\{W_{(\lambda,\varepsilon)}\}$ is a fundamental system of entourages for the canonical uniformity on ΣX. □

In order to compare $\partial_\rho(G\times\mathbb{Z})$ to M_ρ^s we define a map $\tilde{\iota}:(G\times\mathbb{Z})^\infty\to M_\rho^s$ by

$$(g,z)\mapsto\tilde{\iota}(g,z):=\begin{cases}[\frac{z}{|g|_\rho}\,,\,g^+\,,\,\phi_\rho(g)], & \text{if } g\in G^\infty,\\ [-\infty\,,\,\xi\,,\,s], & \text{if } g\notin G^\infty \text{ and } z<0,\\ [+\infty\,,\,\xi\,,\,s], & \text{if } g\notin G^\infty \text{ and } z>0,\end{cases}$$

where $\xi\in\partial G$ and $s\in S_\rho$ are arbitrary.

Lemma 3.2.6 *The image $\tilde{\iota}((G\times\mathbb{Z})^\infty)$ is dense in M_ρ^s.*

Proof: It is enough to check that each open subset in M_ρ^s of the form $(I\times B(\xi,\varepsilon)\times U)/\sim$ contains an element of $\tilde{\iota}((G\times\mathbb{Z})^\infty)$, where I is an open interval in $[-\infty,+\infty]$, $B(\xi,\varepsilon)$ is an open ball of radius ε around some point ξ in ∂G with respect to some fixed visual metric d, and U is an open set in S_ρ. First of all, we find a $g\in G^\infty$ such that $\phi_\rho(g)\in U$ and $g^+\in B(\xi,\varepsilon)$: By construction $\phi_\rho(G^\infty)$ is dense in S_ρ. Therefore there exists a $\tilde{g}\in G^\infty$ such that $\phi_\rho(\tilde{g})\in U$. Recall that the set of rational boundary points is dense in ∂G (see e.g. [BR96b] for a proof). Hence, there exists an $h\in G^\infty$ such that $d(\xi,h^+)<\frac{\varepsilon}{2}$. Without loss of generality we can assume that $\tilde{g}^+\neq h^-$. For otherwise Lemma 1.3.2 implies that h^+ is the only rational point in $B(\xi,\frac{\varepsilon}{2})$, which contradicts G being non-elementary. It follows from this assumption and the dynamics of the action of h on ∂G (see e.g. [CDP90], Ch. 11, Prop. 2.4) that there exists an $m\in\mathbb{N}$ such that $d(h^+,h^m.\tilde{g}^+)<\frac{\varepsilon}{2}$. Note that $\phi_\rho(\tilde{g})=\phi_\rho(h^m\tilde{g}h^{-m})$ and $h^m.\tilde{g}^+=(h^m\tilde{g}h^{-m})^+$. We take $g:=h^m\tilde{g}h^{-m}$. Obviously, there exist $z\in\mathbb{Z}$ and $q\in\mathbb{N}$ such that $\frac{z}{q\,|g|_\rho}\in I$. Since $(g^q)^+=g^+$ and $\phi_\rho(g^q)=\phi_\rho(g)$, we conclude that $\tilde{\iota}(z,g^q)$ lies in $(I\times B(\xi,\varepsilon)\times U)/\sim$. □

Proof of Thm. 3.2.1: Throughout this proof we will identify $\partial(X\times\mathbb{R})$ with $\Sigma(\partial G)$, and regard $\partial_\rho(G\times\mathbb{Z})$ as a closed subspace in $\prod_{\mu\in\mathbb{R}^n}\Sigma(\partial G)$. Thus, the product map $(\tau_\mu)_{\mu\in\mathbb{R}^n}$ is the canonical map ι from $((G\times\mathbb{Z})^\infty, U^\rho_{G\times\mathbb{Z}})$ into its Hausdorff completion $\partial_\rho(G\times\mathbb{Z})$. For each $\mu\in\mathbb{R}^n$ we define a map $\tilde\tau_\mu$ from M^s_ρ to $\Sigma(\partial G)$ by

$$[t,\xi,s] \mapsto \tilde\tau_\mu([t,\xi,s]) := [t+\langle\mu,s\rangle,\xi].$$

It is easy to check that

$$\begin{tikzcd}
(G\times\mathbb{Z})^\infty \arrow[r, "\tilde\iota"] \arrow[d, "\iota=(\tau_\mu)_\mu"'] & M^s_\rho \arrow[d, "\tilde\tau_\mu"] \\
\partial_\rho(G\times\mathbb{Z}) \arrow[r, "pr_\mu"] & \Sigma(\partial G)
\end{tikzcd}$$

is a commuting diagram of set maps. Let $\psi := (\tilde\tau_\mu)_\mu$ be the product map from M^s_ρ to $\prod_{\mu\in\mathbb{R}^n}\Sigma(\partial G)$. It is straightforward to check that ψ is uniformly continuous. Let ψ' be the restriction of ψ to the subspace $\tilde\iota((G\times\mathbb{Z})^\infty)$.

Claim 1: ψ' is a bijection from $\tilde\iota((G\times\mathbb{Z})^\infty)$ onto $\iota((G\times\mathbb{Z})^\infty)$.

Proof of Claim 1: We show that ψ' has an inverse. Define a map $\bar\psi$ from $\iota((G\times\mathbb{Z})^\infty)$ to $\tilde\iota((G\times\mathbb{Z})^\infty)$ by

$$\iota(g,z) \mapsto \bar\psi(\iota(g,z)) := \tilde\iota(g,z).$$

We check that $\bar\psi$ is well-defined: Let $(g',z')\in(G\times\mathbb{Z})^\infty$ be such that $\iota(g,z)=\iota(g',z')$, i.e. $\tau_\mu(g,z)=\tau_\mu(g',z')$ for all $\mu\in\mathbb{R}^n$. This means in particular that $\tau_{e_i}(g,z)=\tau_{e_i}(g',z')$ for all $i=1,\dots,n$, where e_i is the i-th unit vector in $\mathbb{R}^n$; and that $\tau_0(g,z)=\tau_0(g',z')$. We distinguish between two cases. Case 1: Suppose that $g\in G^\infty$. Since $-\infty<\frac{z}{|g|_\rho}<+\infty$, it follows that $g'\in G^\infty$, too. We conclude that $\frac{z}{|g|_\rho}=\frac{z'}{|g'|_\rho}$, $g^+={g'}^+$ and $\frac{\omega_i(g)}{|g|_\rho}=\frac{z+\omega_i(g)}{|g|_\rho}-\frac{z}{|g|_\rho}=\frac{z'+\omega_i(g')}{|g'|_\rho}-\frac{z'}{|g'|_\rho}=\frac{\omega_i(g')}{|g'|_\rho}$ for each $i=1,\dots,n$. Thus, $\bar\psi$ is well-defined in Case 1. Case 2: Suppose that $g\notin G^\infty$. Then $z<0$ (resp. $z>0$) implies $g'\notin G^\infty$ and $z'<0$ (resp. $z'>0$). Therefore, $\bar\psi$ is well-defined in Case 2, too. Finally, it is straightforward to check that $\bar\psi$ is the inverse of ψ'.

Claim 2: $\bar\psi$ is uniformly continuous.

Proof of Claim 2: Let d be a fixed visual metric on ∂G throughout this proof of Claim 2. We consider the following fundamental systems of entourages on

$\iota((G\times\mathbb{Z})^\infty)$ and $\tilde{\iota}((G\times\mathbb{Z})^\infty)$: For any ε with $1>\varepsilon>0$ let W_ε consist precisely of those pairs $([t,\xi],[t',\xi'])$ in $\Sigma(\partial G)\times\Sigma(\partial G)$, that satisfy $[\,|t-t'|<\varepsilon$ and $d(\xi,\xi')<\varepsilon\,]$, or $[\,t>\frac{1}{\varepsilon}$ and $t'>\frac{1}{\varepsilon}\,]$, or $[\,t<-\frac{1}{\varepsilon}$ and $t'<-\frac{1}{\varepsilon}\,]$. It is straightforward to check that the system $\{W_\varepsilon\}$ is a fundamental system of entourages for $\Sigma(\partial G)$. For any ε with $1>\varepsilon>0$ and any finite number $\nu_1,\ldots,\nu_m$ of vectors in $\mathbb{R}^n$ let $W_{(\nu_1,\ldots,\nu_m,\varepsilon)}$ consist precisely of those pairs $(([t_\mu,\xi_\mu])_\mu,([t'_\mu,\xi'_\mu])_\mu)$ in $(\prod_{\mu\in\mathbb{R}^n}\Sigma(\partial G))\times(\prod_{\mu\in\mathbb{R}^n}\Sigma(\partial G))$ for which $([t_{\nu_j},\xi_{\nu_j}],[t'_{\nu_j},\xi'_{\nu_j}])$ lies in W_ε for each $j=1,\ldots,m$. Then, by construction, the trace of the system $\{W_{(\nu_1,\ldots,\nu_m,\varepsilon)}\}$ is a fundamental system of entourages for the subspace $\iota((G\times\mathbb{Z})^\infty)$. For any ε with $1>\varepsilon>0$ let V_ε consist precisely of those pairs $([t,\xi,s],[t'\xi',s'])$ in $M^s_\rho\times M^s_\rho$ that satisfy $[\,t>\frac{1}{\varepsilon}$ and $t'>\frac{1}{\varepsilon}\,]$, or $[\,t<-\frac{1}{\varepsilon}$ and $t'<-\frac{1}{\varepsilon}\,]$, or $[\,|t-t'|<\varepsilon$ and $d(\xi,\xi')<\varepsilon$ and $|s-s'|_\infty<\varepsilon\,]$, where $|\cdot|_\infty$ is the ℓ_∞-norm on $\mathbb{R}^n$. According to Lemma 3.2.5, the system $\{V_\varepsilon\}$ is a fundamental system of entourages for M^s_ρ. Let ε with $1>\varepsilon>0$ be given. In order to prove that $\bar{\psi}$ is uniformly continuous it is enough to check that there exists an $\tilde{\varepsilon}>0$ such that $\bar{\psi}\times\bar{\psi}$ maps the trace of $W_{(0,e_1,\ldots,e_n,\tilde{\varepsilon})}$ into V_ε: Let $C>0$ be the constant given in the proof of Lemma 3.2.2, i.e. we have $|s|_\infty<C$ for all $s\in S_\rho$. Take $1>\tilde{\varepsilon}>0$ such that $\tilde{\varepsilon}<\frac{\varepsilon}{2}$ and $\tilde{\varepsilon}<\frac{\varepsilon}{1+(C+1)\varepsilon}$. Suppose $(\iota(g,z),\iota(g',z'))$ in $W_{(0,e_1,\ldots,e_n,\tilde{\varepsilon})}$. We distinguish between three cases:

Case 1: Suppose $\frac{z}{|g|_\rho}>\frac{1}{\tilde{\varepsilon}}$ and $\frac{z'}{|g'|_\rho}>\frac{1}{\tilde{\varepsilon}}$. Then $\frac{z}{|g|_\rho}>\frac{1}{\varepsilon}$ and $\frac{z'}{|g'|_\rho}>\frac{1}{\varepsilon}$. Hence, $(\tilde{\iota}(g,z),\tilde{\iota}(g',z'))=(\bar{\psi}(\iota(g,z)),\bar{\psi}(\iota(g',z')))$ lies in V_ε in Case 1.

Case 2: Suppose $\frac{z}{|g|_\rho}<-\frac{1}{\tilde{\varepsilon}}$ and $\frac{z'}{|g'|_\rho}<-\frac{1}{\tilde{\varepsilon}}$. Then $\frac{z}{|g|_\rho}<-\frac{1}{\varepsilon}$ and $\frac{z'}{|g'|_\rho}<-\frac{1}{\varepsilon}$. Hence, $(\bar{\psi}(\iota(g,z)),\bar{\psi}(\iota(g',z')))$ lies in V_ε in Case 2.

Case 3: Suppose $\left|\frac{z}{|g|_\rho}-\frac{z'}{|g'|_\rho}\right|<\tilde{\varepsilon}$ and $d(g^+,g'^+)<\tilde{\varepsilon}$. Then we can assume without loss of generality that $\left|\frac{z}{|g|_\rho}\right|<\frac{1}{\tilde{\varepsilon}}-C$. For otherwise we have either $\frac{z}{|g|_\rho}>\frac{1}{\tilde{\varepsilon}}-C$ and $\frac{z'}{|g'|_\rho}>\frac{1}{\tilde{\varepsilon}}-C-1$, or $\frac{z}{|g|_\rho}<-\frac{1}{\tilde{\varepsilon}}+C$ and $\frac{z'}{|g'|_\rho}<-\frac{1}{\tilde{\varepsilon}}+C+1$; and $(\bar{\psi}(\iota(g,z)),\bar{\psi}(\iota(g',z')))$ lies in V_ε analogously to Case 1 or Case 2 above. The assumption implies for each $i=1,\ldots,n$ that

$$\left|\frac{z+\omega_i(g)}{|g|_\rho}\right| \leq \left|\frac{z}{|g|_\rho}\right| + \left|\frac{\omega_i(g)}{|g|_\rho}\right| < \frac{1}{\tilde{\varepsilon}}-C+C = \frac{1}{\tilde{\varepsilon}}.$$

Hence, $\left|\frac{z+\omega_i(g)}{|g|_\rho}-\frac{z'+\omega_i(g')}{|g'|_\rho}\right|<\tilde{\varepsilon}$ holds for each $i=1,\ldots,n$, from which it follows that

$$\left|\frac{\omega_i(g)}{|g|_\rho}-\frac{\omega_i(g')}{|g'|_\rho}\right| \leq \left|\frac{z+\omega_i(g)}{|g|_\rho}-\frac{z'+\omega_i(g')}{|g'|_\rho}\right| + \left|\frac{-z}{|g|_\rho}-\frac{-z'}{|g'|_\rho}\right| \leq 2\tilde{\varepsilon} < \varepsilon.$$

Thus, $(\bar{\psi}(\iota(g,z)), \bar{\psi}(\iota(g',z')))$ lies in V_ε in Case 3; and $\bar{\psi}$ is uniformly continuous.

Now, we prove the theorem. The subset $\iota((G\times\mathbb{Z})^\infty)$ is dense in $\partial_\rho(G\times\mathbb{Z})$ by construction, and the subset $\tilde{\iota}((G\times\mathbb{Z})^\infty)$ is dense in M_ρ^s by Lemma 3.2.6. Because ψ is uniformly continuous, so is ψ'. It follows from Claim 1 and Claim 2 that ψ' extends to an isomorphism from M_ρ^s onto $\partial_\rho(G\times\mathbb{Z})$. Since this extension is unique, it must coincide with ψ. Hence, ψ is an isomorphism. Finally, the two remaining statements (ii) and (iii) of the theorem are obvious. $\square$.

3.3 Actions with Trivial Shift Parameters

In this section we continue to study the boundary of groups $G\times\mathbb{Z}$, where G is a non-elementary word-hyperbolic CAT(0) group. But contrary to the previous section, in which we fixed a G-factor action to understand the effect of non-trivial shift parameters, the aim of this section is to consider all possible G-factor actions in conjunction with trivial shift parameters.

Again we begin with some notation. Throughout this section let G be a fixed non-elementary word-hyperbolic CAT(0) group. By Γ we denote the set of all geometric G-actions on CAT(0) spaces. For each G-action $\rho\in\Gamma$ on a CAT(0) space X, we let ρ_{ts} be the product action $(g,z)*(x,r) := (g.x, r+z)$ by $G\times\mathbb{Z}$ on $X\times\mathbb{R}$. (The subscript *ts* stands for *trivial shift.*) We define U_ρ^{ts} to be the uniform structure that is induced on the set $(G\times\mathbb{Z})^\infty$ by ρ_{ts}. Varying the G-factor action gives rise to a family $\mathcal{U}_{G\times\mathbb{Z}}^{ts} := \{U_\rho^{ts}\}_{\rho\in\Gamma}$ of uniform structures on $(G\times\mathbb{Z})^\infty$. We denote the least upper bound of this family by $U_{G\times\mathbb{Z}}^{ts}$, and call the Hausdorff completion of $(G\times\mathbb{Z})^\infty$ with respect to $U_{G\times\mathbb{Z}}^{ts}$ the *trivial-shift-boundary* $\partial_{ts}(G\times\mathbb{Z})$ of $G\times\mathbb{Z}$. Note that, once again, $\partial_{ts}(G\times\mathbb{Z})$ is a compact space; and that for each G-factor action ρ on a CAT(0) space X there exists a canonical uniformly continuous map $\hat{\tau}_\rho^{ts}$ from $\partial_{ts}(G\times\mathbb{Z})$ into $\partial(X\times\mathbb{R})$. If we identify $\partial(X\times\mathbb{R})$ with $\Sigma(\partial G)$ as before, the canonical map τ_ρ^{ts} from $(G\times\mathbb{Z})^\infty$ into $\partial(X\times\mathbb{R})$ is given by

$$(g,z) \mapsto \tau_\rho^{ts}(g,z) = \begin{cases} [\frac{z}{|g|_\rho}, g^+], & \text{if } g \in G^\infty, \\ [-\infty, \xi], & \text{if } g \notin G^\infty \text{ and } z < 0, \\ [+\infty, \xi], & \text{if } g \notin G^\infty \text{ and } z > 0, \end{cases}$$

where $\xi \in \partial G$ is arbitrary.

Remark 3.3.1 In general, the family $\mathcal{U}^{ts}_{G\times\mathbb{Z}}$ is a proper subfamily of the family $\mathcal{U}_{G\times\mathbb{Z}}$ of uniform structures on $(G \times \mathbb{Z})^\infty$, on which the proper boundary construction is based. (See $F_2 \times \mathbb{Z}$ in Example 3.1.2.) However, we will see in the next section that, if G is a CAT(0) group whose abelianization is finite, then the families $\mathcal{U}^{ts}_{G\times\mathbb{Z}}$ and $\mathcal{U}_{G\times\mathbb{Z}}$ coincide. In this case $\partial_{ts}(G\times\mathbb{Z})$ is canonically isomorphic to $\partial(G\times\mathbb{Z})$. An easy example for a word-hyperbolic CAT(0) group with a finite abelianization is the $(2,3,7)$-triangle group.

Before we examine $\partial_{ts}(G \times \mathbb{Z})$, we clarify the relation between the uniform structure U^{ts}_ρ associated to a $G \times \mathbb{Z}$-action ρ_{ts} and the marked length spectrum MLS_ρ associated to the corresponding G-factor action ρ. Example 3.1.2 shows that the uniformities U^{ts}_ρ and $U^{ts}_{\tilde\rho}$ associated to ρ and $\tilde\rho$ in Γ are generally distinct. We can generalize the idea of this example to the following proposition:

Proposition 3.3.2 *Let G be a word-hyperbolic group, and let ρ and $\tilde\rho$ be two geometric G-actions on CAT(0) spaces. Then the following are equivalent:*

(i) The marked length spectra MLS_ρ and $MLS_{\tilde\rho}$ associated to ρ and $\tilde\rho$ are the same up to a positive scaling factor.

(ii) The uniformities U^{ts}_ρ and $U^{ts}_{\tilde\rho}$ on the set $(G \times \mathbb{Z})^\infty$ coincide.

Proof: On the one hand, it is an immediate consequence of Lemma 3.2.5, and the above characterization of the maps τ^{ts}_ρ and $\tau^{ts}_{\tilde\rho}$ from $(G \times \mathbb{Z})^\infty$ to $\Sigma(\partial G)$, that U^{ts}_ρ and $U^{ts}_{\tilde\rho}$ coincide, provided that MLS_ρ and $MLS_{\tilde\rho}$ are the same up to a positive scaling factor.

On the other hand, suppose that the marked length spectra MLS_ρ and $MLS_{\tilde\rho}$ are not the same up to a positive scaling factor. We want to deduce that the uniformities U^{ts}_{ρ} and $U^{ts}_{\tilde\rho}$ are distinct. To do so, we construct a sequence in $(G\times\mathbb{Z})^{\infty}$ that is Cauchy with respect to U^{ts}_{ρ}, but not with respect to $U^{ts}_{\tilde\rho}$. Fix an infinite order element $g\in G$, and set $c := \frac{|g|_\rho}{|g|_{\tilde\rho}}$. Let $c\tilde\rho$ be the geometric G-action on the rescaled CAT(0) space $c\cdot\tilde X$, where $\tilde X$ is the CAT(0) space underlying the action $\tilde\rho$. Note that by the first paragraph the uniformities $U^{ts}_{\tilde\rho}$ and $U^{ts}_{c\tilde\rho}$ coincide. By construction we have $|g|_\rho = |g|_{c\tilde\rho}$. But by hypothesis MLS_ρ and $MLS_{c\tilde\rho}$ cannot be the same. Therefore, there exists an infinite order element $h\in G$ such that $|h|_\rho \neq |h|_{c\tilde\rho}$.

Claim 1: The points g^+ and h^- in ∂G are not the same.

Proof of Claim 1: If g^+ and h^- are the same points in ∂G, then by Lemma 1.3.2 there exist $n,m\in\mathbb{N}$ such that $g^n = h^{-m}$. This implies $|h|_\rho = \frac{n}{m}|g|_\rho = \frac{n}{m}|g|_{c\tilde\rho} = |h|_{c\tilde\rho}$, which is a contradiction.

We set $\alpha := \frac{|h|_\rho}{|g|_\rho}$ and $\beta := \frac{|h|_{c\tilde\rho}}{|g|_{c\tilde\rho}} = \frac{|h|_{c\tilde\rho}}{|g|_\rho}$, and define sequences $(a_i)_i$ and $(b_i)_i$ in $(G\times\mathbb{Z})^{\infty}$ as follows:

$$a_i := (g^{2i}, 2i), \qquad \text{and} \qquad b_i := (g^i h^i, i+\alpha_i),$$

where α_i is the integer part of the real number $i\alpha$. To complete the proof of the proposition it is enough to show:

Claim 2: The alternating sequence formed of $(a_i)_i$ and $(b_i)_i$ is a Cauchy sequence with respect to U^{ts}_{ρ}, but not with respect to $U^{ts}_{c\tilde\rho}$.

Proof of Claim 2: By construction, we have

$$\lim_{i\to\infty}\frac{2i}{|g^{2i}|_\rho} = \frac{1}{|g|_\rho}, \qquad \text{and} \qquad \lim_{i\to\infty}\frac{2i}{|g^{2i}|_{c\tilde\rho}} = \frac{1}{|g|_{c\tilde\rho}} = \frac{1}{|g|_\rho}.$$

However, according to Claim 1 we can apply Theorem 1.3.1, and conclude that

$$\begin{aligned}
\lim_{i\to\infty}\frac{i+\alpha_i}{|g^ih^i|_\rho} &= \lim_{i\to\infty}\frac{i+\alpha_i}{|g^i|_\rho+|h^i|_\rho} = \lim_{i\to\infty}\frac{i+\alpha_i}{i\,(1+\alpha)\,|g|_\rho} = \frac{1}{|g|_\rho},\\
\lim_{i\to\infty}\frac{i+\alpha_i}{|g^ih^i|_{c\tilde\rho}} &= \lim_{i\to\infty}\frac{i+\alpha_i}{|g^i|_{c\tilde\rho}+|h^i|_{c\tilde\rho}} = \lim_{i\to\infty}\frac{i+\alpha_i}{i\,(1+\beta)\,|g|_\rho} = \frac{1+\alpha}{(1+\beta)\,|g|_\rho}.
\end{aligned}$$

Thus, both the sequence $(\tau^{ts}_\rho(a_i))_i$ and the sequence $(\tau^{ts}_\rho(b_i))_i$ converge to $[\frac{1}{|g|_\rho}, g^+]$. However, the sequence $(\tau^{ts}_{c\tilde\rho}(a_i))_i$ converges to $[\frac{1}{|g|_\rho}, g^+]$, and the sequence $(\tau^{ts}_{c\tilde\rho}(b_i))_i$

converges to $[\frac{1+\alpha}{(1+\beta)|g|_\rho}, g^+]$, where $\alpha \neq \beta$ by definition. $\square$

Now, we begin with the study of the trivial-shift-boundary $\partial_{ts}(G \times \mathbb{Z})$. For the remainder of this section let $\rho \in \Gamma$ be a fixed G-action. We define a map $\kappa_\rho : G^\infty \to \mathbb{R}^\Gamma$ by

$$g \mapsto \kappa_\rho(g) := \Big(\frac{|g|_{\tilde\rho}}{|g|_\rho}\Big)_{\tilde\rho\in\Gamma}.$$

We denote the closure of im κ_ρ in $\mathbb{R}^\Gamma$ by C_ρ^{ts}. For $\tilde\rho \in \Gamma$ let $\mathrm{pr}_{\tilde\rho}$ be the projection of $\mathbb{R}^\Gamma$ onto the $\tilde\rho$-component. For any $c \in C_\rho^{ts}$ we abbreviate $\mathrm{pr}_{\tilde\rho}(c)$ by $c_{\tilde\rho}$. We define $M_\rho^{ts} := ([-\infty, +\infty] \times \partial G \times C_\rho^{ts})/\sim$, where $(t,\xi,c) \sim (t',\xi',c')$ if and only if $[\,t = \infty$ and $t' = \infty\,]$, or $[\,t = -\infty$ and $t' = -\infty\,]$, or $[\,t = 0$ and $t' = 0$ and $\xi = \xi'\,]$, or $[\,t = t'$ and $\xi = \xi'$ and $c = c'\,]$. In the remainder of this section we will prove the following theorem:

Theorem 3.3.3 *Let G be a non-elementary word-hyperbolic CAT(0) group, and let ρ be a geometric action by G on a CAT(0) space. Let M_ρ^{ts} be defined as above. Then the following is true:*

(i) The trivial-shift-boundary $\partial_{ts}(G \times \mathbb{Z})$ is canonically isomorphic to M_ρ^{ts}.

(ii) The canonical $G \times \mathbb{Z}$-action on $\partial_{ts}(G \times \mathbb{Z})$ is given by

$$(g,z) * [t,\xi,c] \;=\; [t, g.\xi, c].$$

(iii) For each G-factor action $\tilde\rho$ on a CAT(0) space X the canonical map $\hat\tau_{\tilde\rho}$ from $\partial_{ts}(G \times \mathbb{Z})$ to $\partial(X \times \mathbb{R}) \equiv \Sigma(\partial G)$ is given by:

$$\hat\tau_{\tilde\rho}([t,\xi,c]) \;=\; [\frac{t}{c_{\tilde\rho}}, \xi].$$

In order to prove this theorem we need some lemmas:

Lemma 3.3.4 *Let ρ and ρ' be two geometric actions of a group G on CAT(0) spaces X, and X' respectively. Then there exists a constant $k \geq 1$ such that for each $g \in G^\infty$*

$$\frac{1}{k}|g|_\rho \;\leq\; |g|_{\rho'} \;\leq\; k\,|g|_\rho.$$

Proof: Let $d_{\mathcal{A}}$ be a word-metric on G, and let $x \in X$ and $x' \in X'$ be basepoints. The Svarc-Milnor-Lemma implies that there are constants $\lambda \geq 1$ and $\varepsilon \geq 0$ such that for each $n \in \mathbb{N}$ we get the following two inequalities:

$$\frac{1}{\lambda} d_X(x, g^n.x) - \varepsilon \leq d_{\mathcal{A}}(e, g^n) \leq \lambda\, d_X(x, g^n.x) + \varepsilon,$$
$$\frac{1}{\lambda} d_{X'}(x', g^n.x') - \varepsilon \leq d_{\mathcal{A}}(e, g^n) \leq \lambda\, d_{X'}(x', g^n.x') + \varepsilon.$$

Combining these inequalities we get

$$d_{X'}(x', g^n.x') \leq \lambda\, d_{\mathcal{A}}(e, g^n) + \lambda\varepsilon \leq \lambda^2 d_X(x, g^n.x) + 2\lambda\varepsilon,$$

and therefore

$$|g|_{\rho'} = \lim_{n\to\infty} \frac{1}{n} d_{X'}(x', g^n.x') \leq \lambda^2 \lim_{n\to\infty} \frac{1}{n} d_X(x, g^n.x) = \lambda^2 |g|_\rho.$$

The other inequality of the statement follows analogously. Thus, we can take $k := \lambda^2$. □

Lemma 3.3.5 *The space C^{ts}_ρ is a compact, convex subspace of $\mathbb{R}^\Gamma$.*

Proof: By Lemma 3.3.4, there exists a constant $k(\tilde{\rho}) \geq 1$ for each $\tilde{\rho} \in \Gamma$ such that $\frac{1}{k(\tilde{\rho})} \leq \kappa_\rho(g)_{\tilde{\rho}} \leq k(\tilde{\rho})$ for all $g \in G^\infty$. Therefore, Tychonoff's Theorem implies that the image $\kappa_\rho(G^\infty)$ lies in a compact subset of $\mathbb{R}^\Gamma$. Hence, its closure C^{ts}_ρ is compact. In order to prove that C^{ts}_ρ is convex, we consider the map $H : [0,1] \times C^{ts}_\rho \times C^{ts}_\rho \to \mathbb{R}^\Gamma$ given by

$$(\vartheta, x, y) \mapsto H(\vartheta, x, y) := \vartheta\, x + (1 - \vartheta)\, y.$$

It is straightforward to show that H is uniformly continuous. Therefore, it is enough to show that H maps the dense subspace $([0,1] \cap \mathbb{Q}) \times \operatorname{im} \kappa_\rho \times \operatorname{im} \kappa_\rho$ of $[0,1] \times C^{ts}_\rho \times C^{ts}_\rho$ into C^{ts}_ρ. This can be done as in the proof of Lemma 3.2.3. □

In order to compare $\partial_{ts}(G \times \mathbb{Z})$ with M^{ts}_ρ we define a map $\tilde{\imath} : (G \times \mathbb{Z})^\infty \to M^{ts}_\rho$

by

$$(g,z) \mapsto \tilde{\iota}(g,z) := \begin{cases} [\frac{z}{|g|_\rho}, g^+, \kappa_\rho(g)], & \text{if } g \in G^\infty, \\ [-\infty, \xi, c], & \text{if } g \notin G^\infty \text{ and } z < 0, \\ [+\infty, \xi, c], & \text{if } g \notin G^\infty \text{ and } z > 0, \end{cases}$$

where $\xi \in \partial G$ and $c \in C_\rho^{ts}$ are arbitrary.

Lemma 3.3.6 *The image $\tilde{\iota}((G \times \mathbb{Z})^\infty)$ is dense in M_ρ^{ts}.*

Proof: The proof is analogous to that of Lemma 3.2.6. □

Proof of Thm. 3.3.3: Throughout this proof we will regard $\partial_{ts}(G \times \mathbb{Z})$ as a closed subspace of $\prod_{\tilde{\rho}\in\Gamma} \Sigma(\partial G)$, so that the product map $(\tau_{\tilde{\rho}}^{ts})_{\tilde{\rho}\in\Gamma}$ is the canonical map ι from $((G \times \mathbb{Z})^\infty, U_{G\times\mathbb{Z}}^{ts})$ into its Hausdorff completion $\partial_{ts}(G \times \mathbb{Z})$. For each $\tilde{\rho} \in \Gamma$ we define a map $\tilde{\tau}_{\tilde{\rho}}^{ts}$ from M_ρ^{ts} to $\Sigma(\partial G)$ by

$$[t, \xi, c] \mapsto \tilde{\tau}_{\tilde{\rho}}^{ts}([t, \xi, c]) := [\frac{t}{c_{\tilde{\rho}}}, \xi].$$

It is easy to check that

$$\begin{array}{ccc} (G \times \mathbb{Z})^\infty & \xrightarrow{\tilde{\iota}} & M_\rho^{ts} \\ {\scriptstyle \iota = (\tau_{\tilde{\rho}}^{ts})_{\tilde{\rho}}} \downarrow & & \downarrow {\scriptstyle \tilde{\tau}_{\tilde{\rho}}^{ts}} \\ \partial_{ts}(G \times \mathbb{Z}) & \xrightarrow{pr_{\tilde{\rho}}} & \Sigma(\partial G) \end{array}$$

is a commuting diagram of set maps. Let $\psi := (\tilde{\tau}_{\tilde{\rho}}^{ts})_{\tilde{\rho}}$ be the product map from M_ρ^{ts} to $\prod_{\tilde{\rho}\in\Gamma} \Sigma(\partial G)$, and let ψ' be the restriction of ψ to $\tilde{\iota}((G \times \mathbb{Z})^\infty)$ in M_ρ^{ts}.

Claim 1: ψ is uniformly continuous.

Proof of Claim 1: It is enough to show that $\tilde{\tau}_{\tilde{\rho}}^{ts}$ is uniformly continuous for each $\tilde{\rho} \in \Gamma$. Throughout this proof of Claim 1, let d be a fixed visual metric on ∂G. We consider the following fundamental systems of entourages for the uniform structure on $\Sigma(\partial G)$ and on M_ρ^{ts}: For any ε with $1 > \varepsilon > 0$ let W_ε consist precisely of those pairs $([t, \xi], [t', \xi'])$ in $\Sigma(\partial G) \times \Sigma(\partial G)$ that satisfy $[\,|t - t'| < \varepsilon$

and $d(\xi,\xi') < \varepsilon$], or [$t > \frac{1}{\varepsilon}$ and $t' > \frac{1}{\varepsilon}$], or [$t < -\frac{1}{\varepsilon}$ and $t' < -\frac{1}{\varepsilon}$]. Then, by Lemma 3.2.5, the system $\{W_\varepsilon\}$ is a fundamental system of entourages for $\Sigma(\partial G)$. For any ε with $1 > \varepsilon > 0$ and any finite number $\rho_1, \dots \rho_m$ of actions in Γ let $V_{(\rho_1,\dots,\rho_m,\varepsilon)}$ consist precisely of those pairs $([t,\xi,c],[t',\xi',c'])$ in $M_\rho^{ts} \times M_\rho^{ts}$ that satisfy [$t > \frac{1}{\varepsilon}$ and $t' > \frac{1}{\varepsilon}$], or [$t < -\frac{1}{\varepsilon}$ and $t' < -\frac{1}{\varepsilon}$], or [$|t| < \varepsilon$ and $|t'| < \varepsilon$ and $d(\xi,\xi') < \varepsilon$], or [$|t-t'| < \varepsilon$ and $d(\xi,\xi') < \varepsilon$ and $|c_{\rho_j} - c'_{\rho_j}| < \varepsilon^2$ for all $j = 1,\dots,m$]. Analogously to the proof of Lemma 3.2.5 it is straightforward to show that the system $\{V_{(\rho_1,\dots,\rho_m,\varepsilon)}\}$ is a fundamental system of entourages for M_ρ^{ts}. Let $\tilde\rho \in \Gamma$ and ε with $1 > \varepsilon > 0$ be given. In order to prove that $\tilde\tau_{\tilde\rho}^{ts}$ is uniformly continuous, it is enough to check that there exists an $\tilde\varepsilon > 0$ such that $\tilde\tau_{\tilde\rho}^{ts} \times \tilde\tau_{\tilde\rho}^{ts}$ maps $V_{(\tilde\rho,\tilde\varepsilon)}$ into W_ε. Set $K := k(\tilde\rho)$, where $k(\tilde\rho)$ is defined as in the proof of Lemma 3.3.5. Thus, $K \geq 1$, and $\frac{1}{K} < c_{\tilde\rho} < K$ for each $c \in C_\rho^{ts}$. Take $1 > \tilde\varepsilon > 0$ small enough such that $\tilde\varepsilon < \min\{\frac{\varepsilon}{2K^2}, \frac{\varepsilon}{1+\varepsilon}\}$. Suppose that $([t,\xi,c],[t',\xi',c'])$ lies in $V_{(\tilde\rho,\tilde\varepsilon)}$. Then there are four cases to consider.

Case 1: Suppose $t > \frac{1}{\tilde\varepsilon}$ and $t' > \frac{1}{\tilde\varepsilon}$. Then we get $\frac{t}{c_{\tilde\rho}} > \frac{1}{K\tilde\varepsilon} > \frac{1}{\varepsilon}$, and analogously $\frac{t'}{c'_{\tilde\rho}} > \frac{1}{\varepsilon}$. Therefore, $(\tilde\tau_{\tilde\rho}^{ts}([t,\xi,c]), \tilde\tau_{\tilde\rho}^{ts}([t',\xi',c']))$ lies in W_ε.

Case 2: Suppose $t < -\frac{1}{\tilde\varepsilon}$ and $t' < -\frac{1}{\tilde\varepsilon}$. Then we get $\frac{t}{c_{\tilde\rho}} < -\frac{1}{\varepsilon}$ and $\frac{t'}{c'_{\tilde\rho}} < -\frac{1}{\varepsilon}$; hence $(\tilde\tau_{\tilde\rho}^{ts}([t,\xi,c]), \tilde\tau_{\tilde\rho}^{ts}([t',\xi',c']))$ lies in W_ε.

Case 3: Suppose $|t| < \tilde\varepsilon$ and $|t'| < \tilde\varepsilon$ and $d(\xi,\xi') < \tilde\varepsilon$. Then $d(\xi,\xi') < \varepsilon$ and

$$\left|\frac{t}{c_{\tilde\rho}} - \frac{t'}{c'_{\tilde\rho}}\right| \leq \left|\frac{t}{c_{\tilde\rho}}\right| + \left|\frac{t'}{c'_{\tilde\rho}}\right| < 2K\tilde\varepsilon < \varepsilon.$$

Hence, $(\tilde\tau_{\tilde\rho}^{ts}([t,\xi,c]), \tilde\tau_{\tilde\rho}^{ts}([t',\xi',c']))$ lies in W_ε.

Case 4: Suppose $|t-t'| < \tilde\varepsilon$ and $d(\xi,\xi') < \tilde\varepsilon$ and $|c_{\tilde\rho} - c'_{\tilde\rho}| < \tilde\varepsilon^2$. Then we have $d(\xi,\xi') < \varepsilon$. Moreover, we can assume without loss of generality that $|t| \leq \frac{1}{\tilde\varepsilon}$. For otherwise the triangle inequality implies that either $t > \frac{1}{\tilde\varepsilon}$ and $t' > \frac{1}{\tilde\varepsilon} - 1$, or $t < -\frac{1}{\tilde\varepsilon}$ and $t' < -\frac{1}{\tilde\varepsilon} + 1$, from which it follows analogously to Case 1 or 2 above that $(\tilde\tau_{\tilde\rho}^{ts}([t,\xi,c]), \tilde\tau_{\tilde\rho}^{ts}([t',\xi',c']))$ lies in W_ε. Assuming $|t| < \frac{1}{\tilde\varepsilon}$, we get

$$\begin{aligned}\left|\frac{t}{c_{\tilde\rho}} - \frac{t'}{c'_{\tilde\rho}}\right| \leq \left|\frac{t}{c_{\tilde\rho}} - \frac{t}{c'_{\tilde\rho}} + \frac{t}{c'_{\tilde\rho}} - \frac{t'}{c'_{\tilde\rho}}\right| &\leq |t| \left|\frac{c'_{\tilde\rho} - c_{\tilde\rho}}{c_{\tilde\rho}\, c'_{\tilde\rho}}\right| + \frac{1}{c'_{\tilde\rho}} |t-t'| \\ &< \frac{1}{\tilde\varepsilon} K^2 \tilde\varepsilon^2 + K\tilde\varepsilon < \varepsilon.\end{aligned}$$

Thus, $(\tilde{\tau}^{ts}_{\tilde{\rho}}([t,\xi,c]), \tilde{\tau}^{ts}_{\tilde{\rho}}([t',\xi',c']))$ lies in W_ε. This completes the proof of Claim 1.

Claim 2: ψ' is a bijection from $\tilde{\iota}\left((G\times\mathbb{Z})^\infty\right)$ onto $\iota\left((G\times\mathbb{Z})^\infty\right)$.

Proof of Claim 2: We show that ψ' has an inverse. Define a map $\bar{\psi}$ from $\iota\left((G\times\mathbb{Z})^\infty\right)$ to $\tilde{\iota}\left((G\times\mathbb{Z})^\infty\right)$ by

$$\iota(g,z) \mapsto \bar{\psi}(\iota(g,z)) := \tilde{\iota}(g,z).$$

We need to check that $\bar{\psi}$ is well-defined: Let $(g',z') \in (G\times\mathbb{Z})^\infty$ be such that $\iota(g,z) = \iota(g',z')$, i.e. $\tau^{ts}_{\tilde{\rho}}(g,z) = \tau^{ts}_{\tilde{\rho}}(g',z')$ for all $\tilde{\rho}\in\Gamma$. We distinguish between two cases. Case 1: Suppose that $g\in G^\infty$. Since $-\infty < \frac{z}{|g|_\rho} < +\infty$, it follows that $g'\in G^\infty$, too. We conclude that $\frac{z}{|g|_\rho} = \frac{z'}{|g'|_\rho}$, $g^+ = {g'}^+$ and $\frac{|g|_{\tilde{\rho}}}{|g|_\rho} = \frac{z}{|g|_\rho}\frac{|g|_{\tilde{\rho}}}{z} = \frac{z'}{|g'|_\rho}\frac{|g'|_{\tilde{\rho}}}{z'} = \frac{|g'|_{\tilde{\rho}}}{|g'|_\rho}$ for each $\tilde{\rho}\in\Gamma$. Thus, $\bar{\psi}$ is well-defined in Case 1. Case 2: Suppose that $g\notin G^\infty$. Then $z<0$ (resp. $z>0$) implies $g'\notin G^\infty$ and $z'<0$ (resp. $z'>0$). Therefore, $\bar{\psi}$ is well-defined in Case 2, too. It is straightforward to check that $\bar{\psi}$ is the inverse of ψ'.

Claim 3: $\bar{\psi}$ is uniformly continuous.

Proof of Claim 3: Throughout this proof of Claim 3, let d be a fixed visual metric on ∂G. We consider the following fundamental systems of entourages $\iota\left((G\times\mathbb{Z})^\infty\right)$ and $\tilde{\iota}\left((G\times\mathbb{Z})^\infty\right)$: For any $\varepsilon>0$ with $1>\varepsilon>0$ let $\tilde{W}_\varepsilon$ consist precisely of those pairs $([t,\xi],[t',\xi'])$ in $\Sigma(\partial G)\times\Sigma(\partial G)$ that satisfy $[\,|t-t'|<\varepsilon^4$ and $d(\xi,\xi')<\varepsilon]$, or $[t>\frac{1}{\varepsilon}$ and $t'>\frac{1}{\varepsilon}]$, or $[t<-\frac{1}{\varepsilon}$ and $t'<-\frac{1}{\varepsilon}]$. Then the system $\{\tilde{W}_\varepsilon\}$ is a fundamental system of entourages for $\Sigma(\partial G)$. For any $\varepsilon>0$ with $1>\varepsilon>0$ and any finite number $\rho_1,\ldots,\rho_m$ of actions in Γ let $\tilde{W}_{(\rho_1,\ldots,\rho_m,\varepsilon)}$ consist precisely of those pairs $((x_{\tilde{\rho}})_{\tilde{\rho}}, (x'_{\tilde{\rho}})_{\tilde{\rho}})$ in $\prod_{\tilde{\rho}\in\Gamma}\Sigma(\partial G)\times\prod_{\tilde{\rho}\in\Gamma}\Sigma(\partial G)$ for which $(x_{\rho_j}, x'_{\rho_j})$ lies in $\tilde{W}_\varepsilon$ for each $j=1,\ldots,m$. It is straightforward to check that the trace of the system $\{\tilde{W}_{(\rho_1,\ldots,\rho_m,\varepsilon)}\}$ is a fundamental system of entourages for the subspace $\iota((G\times\mathbb{Z})^\infty)$. For any $\varepsilon>0$ with $1>\varepsilon>0$ and any finite number $\rho_1,\ldots\rho_m$ of actions in Γ let $\tilde{V}_{(\rho_1,\ldots,\rho_m,\varepsilon)}$ consist precisely of those pairs $([t,\xi,c],[t',\xi',c'])$ in $M^{ts}_\rho\times M^{ts}_\rho$ that satisfy $[t>\frac{1}{\varepsilon}$ and $t'>\frac{1}{\varepsilon}]$, or $[t<-\frac{1}{\varepsilon}$ and $t'<-\frac{1}{\varepsilon}]$, or $[\,|t|<\varepsilon$ and $|t'|<\varepsilon$ and $d(\xi,\xi')<\varepsilon]$, or $[\,|t-t'|<\varepsilon$ and $d(\xi,\xi')<\varepsilon$ and $|c_{\rho_j}-c'_{\rho_j}|<\varepsilon$ for all $j=1,\ldots,m]$. It is straightforward to check that the trace of the system $\{\tilde{V}_{(\rho_1,\ldots,\rho_m,\varepsilon)}\}$ is a fundamental system of entourages for the subspace $\tilde{\iota}((G \times \mathbb{Z})^\infty)$. Let a finite number $\rho_1, \ldots, \rho_m$ of actions in Γ and $1 > \varepsilon > 0$ be given. In order to prove that $\bar{\psi}$ is uniformly continuous, it is enough to check that there exists an $\tilde{\varepsilon} > 0$ such that $\bar{\psi} \times \bar{\psi}$ maps each $(\iota(g,z), \iota(g',z'))$ in $\tilde{W}_{(\rho,\rho_1,\ldots,\rho_m,\tilde{\varepsilon})}$ into $\tilde{V}_{(\rho_1,\ldots,\rho_m,\varepsilon)}$. Set $K := \max\{ k(\rho_j) \mid j = 1, \ldots, m\}$, where each $k(\rho_j)$ is defined as in the proof of Lemma 3.3.5. Thus, we have $K \geq 1$; and $\frac{1}{K} \leq c_{\rho_j} \leq K$ for each $c \in C_\rho^{ts}$ and each $j = 1, \ldots, m$. Take $1 > \tilde{\varepsilon} > 0$ small enough such that $\tilde{\varepsilon} + \tilde{\varepsilon}^4 < \varepsilon$ and $\tilde{\varepsilon} < \frac{\varepsilon}{(1+\varepsilon)K}$ and $K(\tilde{\varepsilon} + \tilde{\varepsilon}^3) < \varepsilon$. We distinguish between three cases:

Case 1: Suppose $\frac{z}{|g|_\rho} > \frac{1}{\tilde{\varepsilon}}$ and $\frac{z'}{|g'|_\rho} > \frac{1}{\tilde{\varepsilon}}$. Then we have $\frac{z}{|g|_\rho} > \frac{1}{\varepsilon}$ and $\frac{z'}{|g'|_\rho} > \frac{1}{\varepsilon}$. Hence $(\tilde{\iota}(g,z), \tilde{\iota}(g',z')) = (\bar{\psi}(\iota(g,z)), \bar{\psi}(\iota(g',z')))$ lies in $\tilde{V}_{(\rho_1,\ldots,\rho_m,\varepsilon)}$.

Case 2: Suppose $\frac{z}{|g|_\rho} < -\frac{1}{\tilde{\varepsilon}}$ and $\frac{z'}{|g'|_\rho} < -\frac{1}{\tilde{\varepsilon}}$. Then we get $\frac{z}{|g|_\rho} < -\frac{1}{\varepsilon}$ and $\frac{z'}{|g'|_\rho} < -\frac{1}{\varepsilon}$. Hence, $(\bar{\psi}(\iota(g,z)), \bar{\psi}(\iota(g',z')))$ lies in $\tilde{V}_{(\rho_1,\ldots,\rho_m,\varepsilon)}$.

Case 3: Suppose $|\frac{z}{|g|_\rho} - \frac{z'}{|g'|_\rho}| < \tilde{\varepsilon}^4$ and $d(g^+, g'^+) < \tilde{\varepsilon}$. Then we have $|\frac{z}{|g|_\rho} - \frac{z'}{|g'|_\rho}| < \varepsilon$ and $d(g^+, g'^+) < \varepsilon$. We can assume without loss of generality that $|\frac{z}{|g|_\rho}| < \frac{1}{K\tilde{\varepsilon}}$. For otherwise the triangle inequality implies either $\frac{z}{|g|_\rho} \geq \frac{1}{K\tilde{\varepsilon}}$ and $\frac{z'}{|g'|_\rho} \geq \frac{1}{K\tilde{\varepsilon}} - 1$, or $\frac{z}{|g|_\rho} \leq -\frac{1}{K\tilde{\varepsilon}}$ and $\frac{z'}{|g'|_\rho} \leq -\frac{1}{K\tilde{\varepsilon}} + 1$, from which it follows analogously to Case 1 or 2 above that $(\bar{\psi}(\iota(g,z)), \bar{\psi}(\iota(g',z')))$ lies in $\tilde{V}_{(\rho_1,\ldots,\rho_m,\varepsilon)}$. Note that this assumption implies $\left| \frac{z}{|g|_{\rho_j}} \right| < \frac{1}{\tilde{\varepsilon}}$ for each $j = 1, \ldots, m$. So, we get $|\frac{z}{|g|_{\rho_j}} - \frac{z'}{|g'|_{\rho_j}}| < \tilde{\varepsilon}^4$ for each $j = 1, \ldots, m$ by hypothesis. There are two subcases to consider. Case 3.1: Suppose $|\frac{z}{|g|_\rho}| \leq \tilde{\varepsilon}$ or $|\frac{z'}{|g'|_\rho}| \leq \tilde{\varepsilon}$. Then the triangle inequality implies that $|\frac{z}{|g|_\rho}| < \tilde{\varepsilon} + \tilde{\varepsilon}^4 < \varepsilon$ and $|\frac{z'}{|g'|_\rho}| < \tilde{\varepsilon} + \tilde{\varepsilon}^4 < \varepsilon$. Hence, $(\bar{\psi}(\iota(g,z)), \bar{\psi}(\iota(g',z')))$ lies in $\tilde{V}_{(\rho_1,\ldots,\rho_m,\varepsilon)}$. Case 3.2: Suppose $|\frac{z}{|g|_\rho}| > \tilde{\varepsilon}$ and $|\frac{z'}{|g'|_\rho}| > \tilde{\varepsilon}$. Then we conclude for each $j = 1, \ldots, m$ that

$$\begin{aligned}
\left| \frac{|g|_{\rho_j}}{|g|_\rho} - \frac{|g'|_{\rho_j}}{|g'|_\rho} \right| &= \left| \frac{z}{|g|_\rho} \frac{|g|_{\rho_j}}{z} - \frac{z}{|g|_\rho} \frac{|g'|_{\rho_j}}{z'} + \frac{z}{|g|_\rho} \frac{|g'|_{\rho_j}}{z'} - \frac{z'}{|g'|_\rho} \frac{|g'|_{\rho_j}}{z'} \right| \\
&\leq \left| \frac{|g|_{\rho_j}}{z} - \frac{|g'|_{\rho_j}}{z'} \right| \left| \frac{z}{|g|_\rho} \right| + \left| \frac{z}{|g|_\rho} - \frac{z'}{|g'|_\rho} \right| \left| \frac{|g'|_{\rho_j}}{z'} \right| \\
&\leq \left| \frac{\frac{z'}{|g'|_{\rho_j}} - \frac{z}{|g|_{\rho_j}}}{\frac{z}{|g|_{\rho_j}} \frac{z'}{|g'|_{\rho_j}}} \right| \left| \frac{z}{|g|_\rho} \right| + \left| \frac{z}{|g|_\rho} - \frac{z'}{|g'|_\rho} \right| \left| \frac{|g'|_{\rho_j}}{z'} \right| \\
&\leq \frac{K^2}{\tilde{\varepsilon}^2} \tilde{\varepsilon}^4 \frac{1}{K\tilde{\varepsilon}} + \tilde{\varepsilon}^4 \frac{K}{\tilde{\varepsilon}} < \varepsilon.
\end{aligned}$$

Thus, $(\bar{\psi}(\iota(g,z)), \bar{\psi}(\iota(g',z')))$ lies in $\tilde{V}_{(\rho_1,\ldots,\rho_m,p)}$; and $\bar{\psi}$ is uniformly continuous. This completes the proof of Claim 3.

Finally, we can prove the theorem: The subset $\iota((G \times \mathbb{Z})^\infty)$ is dense in $\partial_{ts}(G \times \mathbb{Z})$ by construction, and the subset $\tilde{\iota}((G \times \mathbb{Z})^\infty)$ is dense in M_ρ^{ts} by Lemma 3.3.6. Therefore, the Claims 1 to 3 imply that ψ' extends to an isomorphism from M_ρ^{ts} onto $\partial_{ts}(G \times \mathbb{Z})$. Since this extension is unique, it must coincide with ψ. Hence, ψ is an isomorphism. The statements (ii) and (iii) are an obvious consequence of the above. □

3.4 The Boundary of $G \times \mathbb{Z}$

The aim of this section is to work out the boundary of groups $G \times \mathbb{Z}$, where G is a non-elementary word-hyperbolic CAT(0) group. Moreover, we show that for each geometric action of $G \times \mathbb{Z}$ the associated canonical map from $\partial(G \times \mathbb{Z})$ into the visual boundary of the underlying CAT(0) space is a homotopy equivalence.

We need some notation: Throughout this section we consider a fixed non-elementary word-hyperbolic CAT(0) group G. We fix a basis $(\omega_1, \ldots, \omega_n)$ of $\mathrm{Hom}(G, \mathbb{R})$. (By convention this basis is empty if $\mathrm{Hom}(G,\mathbb{R})$ is trivial.) So, throughout this section n denotes the rank of $\mathrm{Hom}(G,\mathbb{R})$. For each $g \in G^\infty$ we abbreviate $\omega(g) := (\omega_1(g), \ldots, \omega_n(g)) \in \mathbb{R}^n$. Let Γ be the set of all geometric actions by G on CAT(0) spaces. For each $\mu \in \mathbb{R}^n$ and each action $\rho \in \Gamma$ on a CAT(0) space X we define a geometric $G \times \mathbb{Z}$-action ρ_μ on $X \times \mathbb{R}$ by

$$(g,z) * (x,r) := (\rho(g,x), r + z + \langle \mu, \omega(g) \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^n$. By τ_{ρ_μ} we denote the canonical map from $(G \times \mathbb{Z})^\infty$ to $\partial(X \times \mathbb{R})$ associated to ρ_μ; and by U_{ρ_μ} the uniform structure that is induced on $(G \times \mathbb{Z})^\infty$ via τ_{ρ_μ}. We identify $\partial(X \times \mathbb{R})$ with the suspension $\Sigma(\partial G)$ as before, so that τ_{ρ_μ} is given by

$$(g,z) \mapsto \tau_{\rho_\mu}(g,z) := \begin{cases} [\frac{z+\langle \mu, \omega(g)\rangle}{|g|_\rho}, g^+], & \text{if } g \in G^\infty, \\ [-\infty, \xi], & \text{if } g \notin G^\infty \text{ and } z < 0, \\ [+\infty, \xi], & \text{if } g \notin G^\infty \text{ and } z > 0, \end{cases}$$

where $\xi \in \partial G$ is arbitrary.

The following proposition shows that in order to study the boundary of $G \times \mathbb{Z}$ it is enough to consider $G \times \mathbb{Z}$-actions of the above type. It is motivated by a special case of a result by Bowers and Ruane in [BR96a], namely, that the visual boundary of a CAT(0) space carrying a geometric $G \times \mathbb{Z}$-action is homeomorphic to $\Sigma(\partial G)$.

Proposition 3.4.1 *Let $\bar{\rho}$ be a geometric action of $G \times \mathbb{Z}$ on a CAT(0) space $\bar{X}$. Then $\partial\bar{X}$ can be identified with $\Sigma(\partial G)$. Moreover, there exists an action $\rho \in \Gamma$ on a CAT(0) space X, and a vector $\mu \in \mathbb{R}^n$, such that*

$$\begin{array}{ccc}
((G\times\mathbb{Z})^\infty, U_{\bar{\rho}}) & \xrightarrow{\mathrm{id}_{(G\times\mathbb{Z})^\infty}} & ((G\times\mathbb{Z})^\infty, U_{\rho_\mu})^\infty \\
{\scriptstyle \tau_{\bar{\rho}}}\downarrow & & \downarrow{\scriptstyle \tau_{\rho_\mu}} \\
\{(g,z)^+ \mid (g,z)\in(G\times\mathbb{Z})^\infty\} & \xrightarrow{\mathrm{id}_{\Sigma(\partial G)}} & \{(g,z)^+ \mid (g,z)\in(G\times\mathbb{Z})^\infty\} \\
{\scriptstyle \subset}\downarrow & & \downarrow{\scriptstyle \subset} \\
\partial\bar{X} \equiv \Sigma(\partial G) & \xrightarrow{\mathrm{id}_{\Sigma(\partial G)}} & \partial(X\times\mathbb{R}) \equiv \Sigma(\partial G)
\end{array}$$

is a commutative diagram of uniformly continuous maps. In particular, the uniformities $U_{\bar{\rho}}$ and U_{ρ_μ} on $(G\times\mathbb{Z})^\infty$ coincide.

For the proof of this proposition we need the following lemma:

Lemma 3.4.2 *The centre $C(G)$ of G is finite.*

Proof: It is a consequence of [GH90], Cor. 36, for example, that if each element in $C(G)$ has finite order, then $C(G)$ is finite. So, let us assume that $C(G)$ contains an element g of infinite order, and deduce a contradiction. Since G is non-elementary there exists an element $h \in G^\infty$ such that neither $h^+ = g^+$ nor $h^+ = g^-$. If the elements g and h do not generate a free abelian subgroup of rank 2 in G, then there exist $p, q \in \mathbb{Z} \setminus \{0\}$ such that $g^p h^q = e$. The latter implies that the axes of g and h are parallel (up to orientation), and hence either $h^+ = g^+$ or $h^+ = g^-$. Therefore, g and h must generate a $\mathbb{Z}^2$-isomorphic subgroup of G. But G cannot have a $\mathbb{Z}^2$-isomorphic subgroup, since it is word-hyperbolic (see e.g. [CDP90], Chap. 10, Cor. 7.3). Thus, $C(G)$ is finite. □

Proof of Prop. 3.4.1: The previous lemma allows us to apply the Splitting Theorem 1.2.1: There exists a non-empty, closed, convex, $G \times \mathbb{Z}$-invariant subspace $\bar{X}_1 \times \bar{X}_2$ of $\bar{X}$ such that G acts geometrically on $\bar{X}_1$ and by (possibly trivial) Clifford translations on $\bar{X}_2$, and $\mathbb{Z}$ acts geometrically on $\bar{X}_2$ and trivially on $\bar{X}_1$. We denote the G-action on $\bar{X}_1$ by $\bar{\rho}_1$. Let $\mathrm{Min}(\mathbb{Z}) := \bigcap_{z \in \mathbb{Z}} \mathrm{Min}(z) \subset \bar{X}_2$, where $\mathrm{Min}(z)$ is the set of points in $\bar{X}_2$ at which the displacement function $d(\cdot, z.\cdot)$ for $z \in \mathbb{Z}$ attains its minimum. By the Flat Torus Theorem 1.1.2, $\mathrm{Min}(\mathbb{Z})$ is non-empty and splits as a product $Y \times \mathbb{R}$. The $\mathbb{Z}$-action on $\bar{X}_2$ leaves $\mathrm{Min}(\mathbb{Z})$ invariant. Each $z \in \mathbb{Z}$ acts as the identity on Y-factor and as a non-trivial translation on the $\mathbb{R}$-factor. We denote the translation distance for the action of $1 \in \mathbb{Z}$ on $\mathbb{R}$ by η. Each $g \in G$ centralizes $\mathbb{Z}$. Therefore, the Flat Torus Theorem implies that the action of each $g \in G$ on $\bar{X}_2$ leaves $\mathrm{Min}(\mathbb{Z})$ invariant, and respects the product decomposition $Y \times \mathbb{R}$. Recall that g acts as a Clifford translation on $\mathrm{Min}(\mathbb{Z})$; and that $\mathrm{diam}(Y)$ is finite, because the $\mathbb{Z}$-action on $\bar{X}_2$ is cocompact. Hence, by Thm. 1.1.3, g acts trivially on the Y-factor, and as a translation on the $\mathbb{R}$-factor. It follows that the G-action on the $\mathbb{R}$-factor induces a homomorphism $\psi \in \mathrm{Hom}(G, \mathbb{R})$. Let $y \in Y$, and consider the subspace $Z := \bar{X}_1 \times \{y\} \times \mathbb{R}$ in $\bar{X}$. By the above, Z is a non-empty, closed, convex, $G \times \mathbb{Z}$-invariant subspace of $\bar{X}$. This implies in particular that any infinite geodesic ray that issues from a basepoint in Z towards infinity lies entirely in Z. We conclude that $\partial Z = \partial \bar{X}$, which allows us to identify $\partial \bar{X}$ with $\Sigma(\partial G)$. Let $\bar{\rho}'$ denote the induced $G \times \mathbb{Z}$-action on Z. Note that $\bar{\rho}'$ can be described expicitely by

$$(g, z) * (x, y, r) := (\bar{\rho}_1(g, x), y, r + \eta z + \psi(g)),$$

for any $x \in \bar{X}_1$ and any $r \in \mathbb{R}$. Hence, the canonical map $\tau_{\bar{\rho}}$ from $(G \times \mathbb{Z})^\infty$ into

$\partial\bar{X} = \partial Z \equiv \Sigma(\partial G)$ is given by

$$(g,z) \mapsto \tau_{\bar{\rho}}(g,z) := \begin{cases} [\dfrac{\eta\, z + \psi(g)}{|g|_{\bar{\rho}_1}}, g^+], & \text{if } g \in G^\infty, \\ [-\infty, \xi], & \text{if } g \notin G^\infty \text{ and } z < 0, \\ [+\infty, \xi], & \text{if } g \notin G^\infty \text{ and } z > 0, \end{cases}$$

where $\xi \in \partial G$ is arbitrary. Thus, we can obtain the desired $(G \times \mathbb{Z})$-action ρ_μ on a CAT(0) space $X \times \mathbb{R}$ as follows: Let $\frac{1}{\eta} \cdot \bar{X}_1$ be the CAT(0) space $\bar{X}_1$ with the metric rescaled by the scalar factor $\frac{1}{\eta}$, and let $\frac{1}{\eta} \cdot \bar{\rho}_1$ the geometric G-action on $\frac{1}{\eta} \cdot \bar{X}_1$ that is induced by the G-action $\bar{\rho}_1$ on $\bar{X}_1$. Set $X := \frac{1}{\eta} \cdot \bar{X}_1$ and $\rho := \frac{1}{\eta} \cdot \bar{\rho}_1$. Take $\mu \in \mathbb{R}^n$ such that $\sum_{i=1}^n \mu_i\, \omega_i = \frac{1}{\eta}\, \psi$, i.e. $\sum_{i=1}^n \mu_i\, \omega_i(g) = \frac{1}{\eta}\, \psi(g)$ for each $g \in G$. Then it is straightforward to check that the canonical map τ_{ρ_μ} from $(G \times \mathbb{Z})^\infty$ into $\partial(X \times \mathbb{R}) \equiv \Sigma(\partial G)$ associated to ρ_μ coincides with $\tau_{\bar{\rho}}$. □

A geometric G-action ρ as given by Prop. 3.4.1 is said to be a *G-factor action*, and a vector $\mu \in \mathbb{R}^n$ a *vector of shift parameters*, associated to the $G \times \mathbb{Z}$-action $\bar{\rho}$. Note that a G-factor action associated to a geometric $G \times \mathbb{Z}$-action is in general not unique: Indeed, according to Example 6.0.17 there exist two distinct geometric F_2-actions ρ and σ on CAT(-1) spaces X and Y such that $MLS_\rho = MLS_\sigma$. Hence, both ρ and σ are F_2-factor actions associated to the $F_2 \times \mathbb{Z}$-action ρ_0 on $X \times \mathbb{R}$.

For the remainder of this section let $\rho \in \Gamma$ be a fixed action. Recall that the maps $\phi_\rho : G^\infty \to \mathbb{R}^n$ and $\kappa_\rho : G^\infty \to \mathbb{R}^\Gamma$ are given by

$$g \mapsto \phi_\rho(g) := \Big(\frac{\omega_1(g)}{|g|_\rho}, \ldots, \frac{\omega_n(g)}{|g|_\rho}\Big), \quad \text{and} \quad g \mapsto \kappa_\rho(g) := \Big(\frac{|g|_{\tilde{\rho}}}{|g|_\rho}\Big)_{\tilde{\rho}}.$$

We define a map $\theta_\rho : G^\infty \to \mathbb{R}^\Gamma \times \mathbb{R}^n$ by

$$g \mapsto \theta_\rho(g) := (\kappa_\rho(g), \phi_\rho(g)),$$

and denote the closure of the image im θ_ρ in $\mathbb{R}^\Gamma \times \mathbb{R}^n$ by C_ρ. Let $\mathrm{pr}_{\mathbb{R}^n}$ be the projection of $\mathbb{R}^\Gamma \times \mathbb{R}^n$ onto $\mathbb{R}^n$. For each $\tilde{\rho} \in \Gamma$ let $\mathrm{pr}_{\tilde{\rho}}$ be the projection of $\mathbb{R}^\Gamma \times \mathbb{R}^n$ onto the $\tilde{\rho}$-component of $\mathbb{R}^\Gamma$. For any $c \in C_\rho$ we abbreviate $\mathrm{pr}_{\mathbb{R}^n}(c)$ by

$c^s = (c_1^s, \ldots, c_n^s)$, where the superscript s stands for *shift*, and $\mathrm{pr}_{\bar{\rho}}(c)$ by $c_{\bar{\rho}}$.

Definition 3.4.3 The *ρ-model* M_ρ of the boundary $\partial(G \times \mathbb{Z})$ is defined by $M_\rho := ([-\infty, +\infty] \times \partial G \times C_\rho)/\sim$, where $(t, \xi, c) \sim (t', \xi', c')$ if and only if $[t = t' = -\infty]$, or $[t = t' = +\infty]$, or $[t = t' = 0$ and $c^s = c'^s = 0$ and $\xi = \xi']$, or $[t = t'$ and $\xi = \xi'$ and $c = c']$.

Note that if $\mathrm{rk}\,(\mathrm{Hom}\,(G, \mathbb{R})) = 0$ then $\mathrm{pr}_{\mathbb{R}^0}(C_\rho)$ is just a trivial vector space. Obviously, in this case C_ρ coincides with C_ρ^{ts}, and the ρ-model M_ρ is the same as the space M_ρ^{ts} considered in the previous section.

Theorem 3.4.4 *Let G be a non-elementary word-hyperbolic CAT(0) group, and let ρ be a geometric action by G on a CAT(0) space. Let M_ρ be defined as above. Then the following is true:*

(i) The boundary $\partial(G \times \mathbb{Z})$ is canonically isomorphic to M_ρ.

(ii) The canonical $G \times \mathbb{Z}$-action on $\partial(G \times \mathbb{Z})$ is given by

$$(g, z) * [t, \xi, c] \;=\; [t, g.\xi, c].$$

(iii) Let $\bar{\rho}$ be a geometric $G \times \mathbb{Z}$-action on a CAT(0) space $\bar{X}$. Let $\tilde{\rho}$ be a G-factor action and $\mu \in \mathbb{R}^n$ a vector of shift parameters associated to $\bar{\rho}$. Then the canonical map $\hat{\tau}_{\bar{\rho}}$ from $\partial(G \times \mathbb{Z})$ to $\partial\bar{X} \equiv \Sigma\,(\partial G)$ is given by:

$$\hat{\tau}_{\bar{\rho}}\,([t, \xi, c]) \;=\; [\frac{t + \langle \mu, c^s \rangle}{c_{\bar{\rho}}}, \xi],$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^n$.

(iv) For any geometric $G \times \mathbb{Z}$-action $\bar{\rho}$ on a CAT(0) space $\bar{X}$ the canonical map $\hat{\tau}_{\bar{\rho}}$ from $\partial(G \times \mathbb{Z})$ to $\partial\bar{X}$ is a homotopy equivalence.

Note that for the case $\mathrm{rk}\,(\mathrm{Hom}\,(G, \mathbb{R})) = 0$ the statements (i) to (iii) of this theorem have already been proved in the previous section. In order to prove the theorem in full generality we need some preparatory lemmas:

Lemma 3.4.5 *The space C_ρ is a compact, convex subspace of $\mathbb{R}^\Gamma \times \mathbb{R}^n$.*

Proof: By Lemma 3.3.4, there exists a constant $k(\tilde{\rho}) \geq 1$ for each $\tilde{\rho} \in \Gamma$ such that $\frac{1}{k(\tilde{\rho})} \leq \kappa_\rho(g)_{\tilde{\rho}} \leq k(\tilde{\rho})$ for all $g \in G^\infty$. By Lemma 3.2.2, there exists a constant $C > 0$ such that $|\phi_\rho(g)|_\infty < C$ for all $g \in G^\infty$. Therefore, Tychonoff's Theorem implies that the image $\theta_\rho(G^\infty)$ lies in a compact subset of $\mathbb{R}^\Gamma \times \mathbb{R}^n$. Hence, its closure C_ρ is compact. In order to prove that C_ρ^{ts} is convex, we consider the map $H : [0,1] \times C_\rho \times C_\rho \to \mathbb{R}^\Gamma \times \mathbb{R}^n$ given by

$$(\vartheta, x, y) \mapsto H(\vartheta, x, y) := \vartheta\, x + (1 - \vartheta)\, y.$$

It is straightforward to show that H is uniformly continuous. Therefore, it is enough to show that H maps the dense subspace $([0,1] \cap \mathbb{Q}) \times \operatorname{im} \theta_\rho \times \operatorname{im} \theta_\rho$ of $[0,1] \times C_\rho \times C_\rho$ into C_ρ. This can be done as in the proof of Lemma 3.2.3. □

In order to compare $\partial(G \times \mathbb{Z})$ with M_ρ we define a map $\tilde{\iota} : (G \times \mathbb{Z})^\infty \to M_\rho$ by

$$(g, z) \mapsto \tilde{\iota}(g, z) := \begin{cases} [\, \frac{z}{|g|_\rho} \,,\, g^+ \,,\, \theta_\rho(g)\,], & \text{if } g \in G^\infty, \\ [\, -\infty \,,\, \xi \,,\, c\,], & \text{if } g \notin G^\infty \text{ and } z < 0, \\ [\, +\infty \,,\, \xi \,,\, c\,], & \text{if } g \notin G^\infty \text{ and } z > 0, \end{cases}$$

where $\xi \in \partial G$ and $c \in C_\rho$ are arbitrary.

Lemma 3.4.6 *The image $\tilde{\iota}((G \times \mathbb{Z})^\infty)$ is dense in M_ρ.*

Proof: The proof is analogous to that of Lemma 3.2.6. □

Proof of Theorem 3.4.4: According to Prop. 3.4.1 it is enough to prove the statements (iii) and (iv) of Theorem 3.4.4 for $G \times \mathbb{Z}$-actions of the type $\tilde{\rho}_\mu$, where $\tilde{\rho} \in \Gamma$ and $\mu \in \mathbb{R}^n$. Furthermore, we can regard $\partial(G \times \mathbb{Z})$ as the closure of the image $\operatorname{im}((\tau_{\tilde{\rho}_\mu})_{(\tilde{\rho},\mu)})$ in $\prod_{(\tilde{\rho},\mu) \in \Gamma \times \mathbb{R}^n} \Sigma(\partial G)$ throughout this proof, since the boundary uniformity $U_{G \times \mathbb{Z}}$ on $(G \times \mathbb{Z})^\infty$ is the same as the least upper bound of the family $\{U_{\tilde{\rho}_\mu} \mid \tilde{\rho} \in \Gamma, \mu \in \mathbb{R}^n\}$. Thus, the product map $(\tau_{\tilde{\rho}_\mu})_{\tilde{\rho}_\mu}$ is the canonical map ι from $((G \times \mathbb{Z})^\infty, U_{G \times \mathbb{Z}})$ into its Hausdorff completion $\partial(G \times \mathbb{Z})$.

For each $\tilde{\rho} \in \Gamma$ and each $\mu \in \mathbb{R}^n$ we define a map $\tilde{\tau}_{\tilde{\rho}_\mu}$ from M_ρ to $\Sigma(\partial G)$ by

$$[t, \xi, c] \mapsto \tilde{\tau}_{\tilde{\rho}_\mu}([t, \xi, c]) := [\frac{t + \langle \mu, c^s \rangle}{c_{\tilde{\rho}}}, \xi].$$

It is easy to check that

$$\begin{array}{ccc} (G \times \mathbb{Z})^\infty & \xrightarrow{\tilde{\iota}} & M_\rho \\ {\scriptstyle \iota = (\tau_{\tilde{\rho}_\mu})_{(\tilde{\rho},\mu)}} \downarrow & & \downarrow {\scriptstyle \tilde{\tau}_{\tilde{\rho}_\mu}} \\ \partial(G \times \mathbb{Z}) & \xrightarrow{pr_{(\tilde{\rho},\mu)}} & \Sigma(\partial G) \end{array}$$

is a commuting diagram of set maps. Let $\psi := (\tilde{\tau}_{\tilde{\rho}_\mu})_{(\tilde{\rho},\mu)}$ be the product map from M_ρ to $\prod_{(\tilde{\rho},\mu) \in \Gamma \times \mathbb{R}^n} \Sigma(\partial G)$, and let ψ' be the restriction of ψ to $\tilde{\iota}((G \times \mathbb{Z})^\infty)$.

Claim 1: ψ is uniformly continuous.

Proof of Claim 1: It is enough to show that $\tilde{\tau}_{\tilde{\rho}_\mu}$ is uniformly continuous for each $(\tilde{\rho}, \mu) \in \Gamma \times \mathbb{R}^n$. Throughout this proof of Claim 1, let d be a fixed visual metric on ∂G. We consider the following fundamental systems of entourages for the uniform structure on $\Sigma(\partial G)$ and on M_ρ: For any ε with $1 > \varepsilon > 0$ let W_ε consist precisely of those pairs $([t, \xi], [t', \xi'])$ in $\Sigma(\partial G) \times \Sigma(\partial G)$ that satisfy $[\,|t - t'| < \varepsilon$ and $d(\xi, \xi') < \varepsilon\,]$, or $[\,t > \frac{1}{\varepsilon}$ and $t' > \frac{1}{\varepsilon}\,]$, or $[\,t < -\frac{1}{\varepsilon}$ and $t' < -\frac{1}{\varepsilon}\,]$. Then, by Lemma 3.2.5, the system $\{W_\varepsilon\}$ is a fundamental system of entourages for $\Sigma(\partial G)$. For any ε with $1 > \varepsilon > 0$ and any finite number $\rho_1, \dots, \rho_m$ of actions in Γ let $V_{(\rho_1, \dots, \rho_m, \varepsilon)}$ consist precisely of those pairs $([t, \xi, c], [t', \xi', c'])$ in $M_\rho \times M_\rho$ that satisfy $[\,t > \frac{1}{\varepsilon}$ and $t' > \frac{1}{\varepsilon}\,]$, or $[\,t < -\frac{1}{\varepsilon}$ and $t' < -\frac{1}{\varepsilon}\,]$, or $[\,|t| < \varepsilon$ and $|t'| < \varepsilon$ and $|c^s|_\infty < \varepsilon$ and $|c'^s|_\infty < \varepsilon$ and $d(\xi, \xi') < \varepsilon\,]$, or $[\,|t - t'| < \varepsilon$ and $d(\xi, \xi') < \varepsilon$ and $|c^s - c'^s|_\infty < \varepsilon$ and $|c_{\rho_j} - c'_{\rho_j}| < \varepsilon^2$ for each $j = 1, \dots, m\,]$. Analogously to the proof of Lemma 3.2.5 it is straightforward to show that the system $\{V_{(\rho_1, \dots, \rho_m, \varepsilon)}\}$ is a fundamental system of entourages for M_ρ. Let $(\tilde{\rho}, \mu) \in \Gamma \times \mathbb{R}^n$ and ε with $1 > \varepsilon > 0$ be given. In order to prove that $\tilde{\tau}_{\tilde{\rho}_\mu}$ is uniformly continuous, it is enough to check that there exists an $1 > \tilde{\varepsilon} > 0$ such that $\tilde{\tau}_{\tilde{\rho}_\mu} \times \tilde{\tau}_{\tilde{\rho}_\mu}$ maps $V_{(\tilde{\rho}, \tilde{\varepsilon})}$ into W_ε. Set $K := k(\tilde{\rho})$, where $k(\tilde{\rho})$ is defined as in the proof of Lemma 3.3.5. Thus, $K \geq 1$, and $\frac{1}{K} < c_{\tilde{\rho}} < K$ for each $c \in C_\rho$. Set $M := |\mu|_\infty$. Let $C > 0$ be given as in the proof of Lemma 3.2.2, i.e. $|c^s|_\infty < C$ for all $c \in C_\rho$. Take

$1 > \tilde{\varepsilon} > 0$ small enough such that

$$\tilde{\varepsilon} < \min\{\frac{\varepsilon}{K + nMC\varepsilon}, \frac{\varepsilon}{2K(1+nM)}, \frac{\varepsilon}{1+\varepsilon}, \frac{\varepsilon}{K^2 + K + nM(CK^2+K)}\}.$$

Suppose that $([t, \xi, c], [t', \xi', c'])$ lies in $V_{(\tilde{\rho}, \tilde{\varepsilon})}$. Then there are four cases to consider.

Case 1: Suppose $t > \frac{1}{\tilde{\varepsilon}}$ and $t' > \frac{1}{\tilde{\varepsilon}}$. Then we get

$$\frac{t + \langle \mu, c^s \rangle}{c_{\tilde{\rho}}} > (\frac{1}{\tilde{\varepsilon}} - n\,M\,C)\,\frac{1}{K} > \frac{1}{\varepsilon},$$

and analogously $\frac{t' + \langle \mu, c'^s \rangle}{c'_{\tilde{\rho}}} > \frac{1}{\varepsilon}$. Therefore, $(\tilde{\tau}_{\tilde{\rho}_\mu}([t, \xi, c]), \tilde{\tau}_{\tilde{\rho}_\mu}([t', \xi', c']))$ lies in W_ε for Case 1.

Case 2: Suppose $t < -\frac{1}{\tilde{\varepsilon}}$ and $t' < -\frac{1}{\tilde{\varepsilon}}$. Then we get

$$\frac{t + \langle \mu, c^s \rangle}{c_{\tilde{\rho}}} < (-\frac{1}{\tilde{\varepsilon}} + n\,M\,C)\,\frac{1}{K} < -\frac{1}{\varepsilon},$$

and analogously $\frac{t' + \langle \mu, c'^s \rangle}{c'_{\tilde{\rho}}} < -\frac{1}{\varepsilon}$. Therefore, $(\tilde{\tau}_{\tilde{\rho}_\mu}([t, \xi, c]), \tilde{\tau}_{\tilde{\rho}_\mu}([t', \xi', c']))$ lies in W_ε for Case 2.

Case 3: Suppose $|t| < \tilde{\varepsilon}$ and $|t'| < \tilde{\varepsilon}$ and $|c^s|_\infty < \tilde{\varepsilon}$ and $|c'^s|_\infty < \tilde{\varepsilon}$ and $d(\xi, \xi') < \tilde{\varepsilon}$. Then we get $d(\xi, \xi') < \varepsilon$ and

$$\begin{aligned} \left|\frac{t + \langle \mu, c^s \rangle}{c_{\tilde{\rho}}} - \frac{t' + \langle \mu, c'^s \rangle}{c'_{\tilde{\rho}}}\right| &\leq \left|\frac{t + \langle \mu, c^s \rangle}{c_{\tilde{\rho}}}\right| + \left|\frac{t' + \langle \mu, c'^s \rangle}{c'_{\tilde{\rho}}}\right| \\ &< 2\,K\,(\tilde{\varepsilon} + n\,M\,\tilde{\varepsilon}) < \varepsilon. \end{aligned}$$

Hence, $(\tilde{\tau}_{\tilde{\rho}_\mu}([t, \xi, c]), \tilde{\tau}_{\tilde{\rho}_\mu}([t', \xi', c']))$ lies in W_ε for Case 3.

Case 4: Suppose $|t - t'| < \tilde{\varepsilon}$ and $d(\xi, \xi') < \tilde{\varepsilon}$ and $|c^s - c'^s|_\infty < \tilde{\varepsilon}$ and $|c_{\tilde{\rho}} - c'_{\tilde{\rho}}| < \tilde{\varepsilon}^2$. Then we have $d(\xi, \xi') < \varepsilon$. Moreover, we can assume without loss of generality that $|t| < \frac{1}{\tilde{\varepsilon}}$. For otherwise the triangle inequality implies that either $t \geq \frac{1}{\tilde{\varepsilon}}$ and $t' > \frac{1}{\tilde{\varepsilon}} - 1$, or $t \leq -\frac{1}{\tilde{\varepsilon}}$ and $t' < -\frac{1}{\tilde{\varepsilon}} + 1$, from which it follows analogously to Case

1 or 2 above that $(\tilde{\tau}_{\tilde{\rho}\mu}([t,\xi,c]),\tilde{\tau}_{\tilde{\rho}\mu}([t',\xi',c']))$ lies in W_ε. Assuming $|t| < \frac{1}{\tilde{\varepsilon}}$, we get

$$\begin{aligned}
&\Big|\frac{t+\langle\mu,c^s\rangle}{c_{\tilde{\rho}}} - \frac{t'+\langle\mu,c'^s\rangle}{c'_{\tilde{\rho}}}\Big| \le \Big|\frac{t}{c_{\tilde{\rho}}} - \frac{t'}{c'_{\tilde{\rho}}}\Big| + \Big|\frac{\langle\mu,c^s\rangle}{c_{\tilde{\rho}}} - \frac{\langle\mu,c'^s\rangle}{c'_{\tilde{\rho}}}\Big| \\
&\le \Big|\frac{t}{c_{\tilde{\rho}}} - \frac{t}{c'_{\tilde{\rho}}} + \frac{t}{c'_{\tilde{\rho}}} - \frac{t'}{c'_{\tilde{\rho}}}\Big| + \Big|\frac{\langle\mu,c^s\rangle}{c_{\tilde{\rho}}} - \frac{\langle\mu,c^s\rangle}{c'_{\tilde{\rho}}} + \frac{\langle\mu,c^s\rangle}{c'_{\tilde{\rho}}} - \frac{\langle\mu,c'^s\rangle}{c'_{\tilde{\rho}}}\Big| \\
&\le |t|\,\Big|\frac{c'_{\tilde{\rho}}-c_{\tilde{\rho}}}{c_{\tilde{\rho}}\,c'_{\tilde{\rho}}}\Big| + \frac{1}{c'_{\tilde{\rho}}}\,|t-t'| + |\langle\mu,c^s\rangle|\,\Big|\frac{c'_{\tilde{\rho}}-c_{\tilde{\rho}}}{c_{\tilde{\rho}}\,c'_{\tilde{\rho}}}\Big| + \frac{1}{c'_{\tilde{\rho}}}\,|\langle\mu,c^s-c'^s\rangle| \\
&< \frac{1}{\tilde{\varepsilon}}K^2\tilde{\varepsilon}^2 + K\tilde{\varepsilon} + n\,M\,C\,K^2\tilde{\varepsilon}^2 + K\,n\,M\,\tilde{\varepsilon} < \varepsilon.
\end{aligned}$$

Therefore, $(\tilde{\tau}_{\tilde{\rho}\mu}([t,\xi,c]),\tilde{\tau}_{\tilde{\rho}\mu}([t',\xi',c']))$ lies in W_ε for Case 4, too. This completes the proof of Claim 1.

Claim 2: ψ' is a bijection from $\tilde{\iota}\,((G\times\mathbb{Z})^\infty)$ onto $\iota\,((G\times\mathbb{Z})^\infty)$.

Proof of Claim 2: We show that ψ' has an inverse. Define a map $\bar{\psi}$ from $\iota\,((G\times\mathbb{Z})^\infty)$ to $\tilde{\iota}\,((G\times\mathbb{Z})^\infty)$ by

$$\iota(g,z) \mapsto \bar{\psi}(\iota(g,z)) := \tilde{\iota}(g,z).$$

We need to check that $\bar{\psi}$ is well-defined: Let $(g',z') \in (G\times\mathbb{Z})^\infty$ be such that $\iota(g,z) = \iota(g',z')$, i.e. $\tau_{\tilde{\rho}\mu}(g,z) = \tau_{\tilde{\rho}\mu}(g',z')$ for all $(\tilde{\rho},\mu) \in \Gamma\times\mathbb{R}^n$. We distinguish between two cases. Case 1: Suppose that $g \in G^\infty$. Since $-\infty < \frac{z}{|g|_\rho} < +\infty$, it follows that $g' \in G^\infty$, too. We conclude that $g^+ = g'^+$, as well as $\frac{z}{|g|_{\tilde{\rho}}} = \frac{z'}{|g'|_{\tilde{\rho}}}$ and $\frac{z+\omega_i(g)}{|g|_{\tilde{\rho}}} = \frac{z'+\omega_i(g')}{|g'|_{\tilde{\rho}}}$ for all $\tilde{\rho} \in \Gamma$ and all $i = 1,\ldots,n$. Therefore, we get $\frac{z}{|g|_\rho} = \frac{z'}{|g'|_\rho}$ and $\phi_\rho(g) = \phi_\rho(g')$. If both $\frac{z}{|g|_\rho} = 0$ and $\phi_\rho(g) = 0$, then we already have $[\frac{z}{|g|_\rho}, g^+, \theta_\rho(g)] = [\frac{z'}{|g'|_\rho}, g'^+, \theta_\rho(g')]$. Otherwise, if $\frac{z}{|g|_\rho} \neq 0$, then $\frac{z}{|g|_{\tilde{\rho}}} \neq 0$ and $\frac{z'}{|g'|_{\tilde{\rho}}} \neq 0$ for all $\tilde{\rho} \in \Gamma$; and hence $\frac{|g|_{\tilde{\rho}}}{|g|_\rho} = \frac{z}{|g|_\rho}\frac{|g|_{\tilde{\rho}}}{z} = \frac{z'}{|g'|_\rho}\frac{|g'|_{\tilde{\rho}}}{z'} = \frac{|g'|_{\tilde{\rho}}}{|g'|_\rho}$. And if $\frac{\omega_i(g)}{|g|_\rho} \neq 0$ for some i, then $\frac{\omega_i(g)}{|g|_{\tilde{\rho}}} \neq 0$ and $\frac{\omega_i(g')}{|g'|_{\tilde{\rho}}} \neq 0$ for all $\tilde{\rho} \in \Gamma$; and hence $\frac{|g|_{\tilde{\rho}}}{|g|_\rho} = \frac{\omega_i(g)}{|g|_\rho}\frac{|g|_{\tilde{\rho}}}{\omega_i(g)} = \frac{\omega_i(g')}{|g'|_\rho}\frac{|g'|_{\tilde{\rho}}}{\omega_i(g')} = \frac{|g'|_{\tilde{\rho}}}{|g'|_\rho}$. Thus, $\bar{\psi}$ is well-defined in Case 1. Case 2: Suppose that $g \notin G^\infty$. Then $z < 0$ (resp. $z > 0$) implies $g' \notin G^\infty$ and $z' < 0$ (resp. $z' > 0$). Therefore, $\bar{\psi}$ is well-defined in Case 2, too. It is straightforward to check that $\bar{\psi}$ is the inverse of ψ'. This proves Claim 2.

Claim 3: $\bar{\psi}$ is uniformly continuous.

Proof of Claim 3: Throughout this proof of Claim 3, let d be a fixed visual metric on ∂G. We consider the following fundamental systems of entourages for $\iota((G\times\mathbb{Z})^\infty)$ and for $\tilde{\iota}((G\times\mathbb{Z})^\infty)$: For any $\varepsilon>0$ with $1>\varepsilon>0$ let $\tilde{W}_\varepsilon$ consist precisely of those pairs $([t,\xi],[t',\xi'])$ in $\Sigma(\partial G)\times\Sigma(\partial G)$ that satisfy $[\,|t-t'|<\varepsilon^4$ and $d(\xi,\xi')<\varepsilon]$, or $[t>\frac{1}{\varepsilon}$ and $t'>\frac{1}{\varepsilon}]$, or $[t<-\frac{1}{\varepsilon}$ and $t'<-\frac{1}{\varepsilon}]$. Then the system $\{\tilde{W}_\varepsilon\}$ is a fundamental system of entourages for $\Sigma(\partial G)$. For any $\varepsilon>0$ with $1>\varepsilon>0$ and any finite number $(\rho_1,\mu_1),\dots,(\rho_m,\mu_m)$ of actions in $\Gamma\times\mathbb{R}^n$ let $\tilde{W}_{(\rho_1,\mu_1,\dots,\rho_m,\mu_m,\varepsilon)}$ consist precisely of those pairs $((x_{(\tilde{\rho},\mu)})_{(\tilde{\rho},\mu)},(x'_{(\tilde{\rho},\mu)})_{(\tilde{\rho},\mu)})$ in $(\prod_{(\tilde{\rho},\mu)\in\Gamma\times\mathbb{R}^n}\Sigma(\partial G))\times(\prod_{(\tilde{\rho},\mu)\in\Gamma\times\mathbb{R}^n}\Sigma(\partial G))$ for which $(x_{(\rho_j,\mu_j)},x'_{(\rho_j,\mu_j)})$ lies in $\tilde{W}_\varepsilon$ for each $j=1,\dots,m$. Then, by construction, the trace of the system $\{\tilde{W}_{(\rho_1,\mu_1,\dots,\rho_m,\mu_m,\varepsilon)}\}$ is a fundamental system of entourages for the subspace $\iota((G\times\mathbb{Z})^\infty)$. For any ε with $1>\varepsilon>0$ and any finite number $\rho_1,\dots\rho_m$ of actions in Γ let $\tilde{V}_{(\rho_1,\dots,\rho_m,\varepsilon)}$ consist precisely of those pairs $([t,\xi,c],[t',\xi',c'])$ in $M_\rho\times M_\rho$ that satisfy $[t>\frac{1}{\varepsilon}$ and $t'>\frac{1}{\varepsilon}]$, or $[t<-\frac{1}{\varepsilon}$ and $t'<-\frac{1}{\varepsilon}]$, or $[\,|t|<\varepsilon$ and $|t'|<\varepsilon$ and $|c^s|_\infty<\varepsilon$ and $|c'^s|_\infty<\varepsilon$ and $d(\xi,\xi')<\varepsilon]$, or $[\,|t-t'|<\varepsilon$ and $d(\xi,\xi')<\varepsilon$ and $|c^s-c'^s|_\infty<\varepsilon$ and $|c_{\rho_j}-c'_{\rho_j}|<\varepsilon$ for all $j=1,\dots,m]$. Then it is straightforward to check that the trace of the system $\{\tilde{V}_{(\rho_1,\dots,\rho_m,\varepsilon)}\}$ is a fundamental system of entourages for the subspace $\tilde{\iota}((G\times\mathbb{Z})^\infty)$. Let a finite number $\rho_1,\dots,\rho_m$ of actions in Γ and $1>\varepsilon>0$ be given. In order to prove that $\bar{\psi}$ is uniformly continuous, it is enough to check that there exists an $1>\tilde{\varepsilon}>0$ such that $\bar{\psi}\times\bar{\psi}$ maps each $(\iota(g,z),\iota(g',z'))\in\bigcap_{j=0}^m\bigcap_{i=0}^n\tilde{W}_{(\rho_j,e_i,\tilde{\varepsilon})}$, with $\rho_0:=\rho$, $e_0:=0\in\mathbb{R}^n$, and e_i being the i-th unit vector in $\mathbb{R}^n$, into $\tilde{V}_{(\rho_1,\dots,\rho_m,\varepsilon)}$. Set $K:=\max\{\,k(\rho_j)\mid j=1,\dots,m\,\}$, where each $k(\rho_j)$ is defined as in the proof of Lemma 3.3.5. Thus, we have $K\geq 1$; and $\frac{1}{K}\leq c_{\rho_j}\leq K$ for each $c\in C_\rho$ and each $j=1,\dots,m$. Let $C>0$ be given as in the proof of Lemma 3.2.2, i.e. $|c^s|_\infty<C$ for all $c\in C_\rho$. Take $1>\tilde{\varepsilon}>0$ small enough such that $\tilde{\varepsilon}+2\tilde{\varepsilon}^4<\varepsilon$ and $\frac{1}{K\tilde{\varepsilon}}-(C+1)>\frac{1}{\varepsilon}$ and $K\tilde{\varepsilon}-K^2C\tilde{\varepsilon}^2+K\tilde{\varepsilon}^3<\varepsilon$ and $2K^2C\tilde{\varepsilon}^2+2K\tilde{\varepsilon}^3<\varepsilon$. We distinguish between three cases:

Case 1: Suppose $\frac{z}{|g|_\rho}>\frac{1}{\tilde{\varepsilon}}$ and $\frac{z'}{|g'|_\rho}>\frac{1}{\tilde{\varepsilon}}$. Then $\frac{z}{|g|_\rho}>\frac{1}{\varepsilon}$ and $\frac{z'}{|g'|_\rho}>\frac{1}{\varepsilon}$. Hence $(\tilde{\iota}(g,z),\tilde{\iota}(g',z'))=(\bar{\psi}(\iota(g,z)),\bar{\psi}(\iota(g',z')))$ lies in $\tilde{V}_{(\rho_1,\dots,\rho_m,\varepsilon)}$ for Case 1.

Case 2: Suppose $\frac{z}{|g|_\rho}<-\frac{1}{\tilde{\varepsilon}}$ and $\frac{z'}{|g'|_\rho}<-\frac{1}{\tilde{\varepsilon}}$. Then $\frac{z}{|g|_\rho}<-\frac{1}{\varepsilon}$ and $\frac{z'}{|g'|_\rho}<-\frac{1}{\varepsilon}$. Hence $(\bar{\psi}(\iota(g,z)),\bar{\psi}(\iota(g',z')))$ lies in $\tilde{V}_{(\rho_1,\dots,\rho_m,\varepsilon)}$ for Case 2.

Case 3: Suppose $|\frac{z}{|g|_\rho}-\frac{z'}{|g'|_\rho}|<\tilde{\varepsilon}^4$ and $d(g^+,g'^+)<\tilde{\varepsilon}$. Then we have

$d(g^+, g'^+) < \varepsilon$. Moreover, we can assume without loss of generality that $|\frac{z}{|g|_\rho}| < \frac{1}{K\tilde{\varepsilon}} - C$. For otherwise the triangle inequality implies that either $\frac{z}{|g|_\rho} \geq \frac{1}{K\tilde{\varepsilon}} - C > \frac{1}{\varepsilon}$ and $\frac{z}{|g|_\rho} \geq \frac{1}{K\tilde{\varepsilon}} - (C+1) > \frac{1}{\varepsilon}$, or analogously $\frac{z}{|g|_\rho} < -\frac{1}{\varepsilon}$ and $\frac{z}{|g|_\rho} < -\frac{1}{\varepsilon}$, from which it follows as in Case 1 or 2 above that $(\bar{\psi}(\iota(g,z)), \bar{\psi}(\iota(g',z')))$ lies in $\tilde{V}_{(\rho_1,\dots,\rho_m,\varepsilon)}$. Assuming $|\frac{z}{|g|_\rho}| < \frac{1}{K\tilde{\varepsilon}} - C$, we get for each $i = 1, \dots, n$ and each $j = 1, \dots, m$ that

$$\begin{aligned}
|\frac{z}{|g|_{\rho_j}}| &< K(\frac{1}{K\tilde{\varepsilon}} - C) \leq \frac{1}{\tilde{\varepsilon}}, \\
|\frac{z+\omega_i(g)}{|g|_\rho}| \leq |\frac{z}{|g|_\rho}| + |\frac{\omega_i(g)}{|g|_\rho}| &< \frac{1}{K\tilde{\varepsilon}} - C + C = \frac{1}{\tilde{\varepsilon}}, \\
|\frac{z+\omega_i(g)}{|g|_{\rho_j}}| \leq |\frac{z}{|g|_{\rho_j}}| + |\frac{\omega_i(g)}{|g|_{\rho_j}}| &< K(\frac{1}{K\tilde{\varepsilon}} - C) + KC = \frac{1}{\tilde{\varepsilon}}.
\end{aligned}$$

By hypothesis, this implies $|\frac{z}{|g|_{\rho_j}} - \frac{z'}{|g'|_{\rho_j}}| < \tilde{\varepsilon}^4$, $|\frac{z+\omega_i(g)}{|g|_\rho} - \frac{z'+\omega_i(g')}{|g'|_\rho}| < \tilde{\varepsilon}^4$ and $|\frac{z+\omega_i(g)}{|g|_{\rho_j}} - \frac{z'+\omega_i(g')}{|g'|_{\rho_j}}| < \tilde{\varepsilon}^4$ for each $i = 1, \dots, n$ and each $j = 1, \dots, m$. It follows from the triangle inequality that $|\frac{\omega_i(g)}{|g|_\rho} - \frac{\omega_i(g')}{|g'|_\rho}| < 2\tilde{\varepsilon}^4$ and $|\frac{\omega_i(g)}{|g|_{\rho_j}} - \frac{\omega_i(g')}{|g'|_{\rho_j}}| < 2\tilde{\varepsilon}^4$ for each $i = 1, \dots, n$ and each $j = 1, \dots, m$. There are two subcases to consider. Case 3.1: Suppose $[\,|\frac{z}{|g|_\rho}| \leq \tilde{\varepsilon}$ or $|\frac{z'}{|g'|_\rho}| \leq \tilde{\varepsilon}\,]$ and $[\,|\frac{\omega_i(g)}{|g|_\rho}| \leq \tilde{\varepsilon}$ or $|\frac{\omega_i(g')}{|g'|_\rho}| \leq \tilde{\varepsilon}\,]$ for all $i = 1, \dots, n$. Then we conclude with the triangle inequality that both $|\frac{z}{|g|_\rho}| < \tilde{\varepsilon} + \tilde{\varepsilon}^4 < \varepsilon$ and $|\frac{z'}{|g'|_\rho}| < \tilde{\varepsilon} + \tilde{\varepsilon}^4 < \varepsilon$; as well as both $|\frac{\omega_i(g)}{|g|_\rho}| < \tilde{\varepsilon} + 2\tilde{\varepsilon}^4 < \varepsilon$ and $|\frac{\omega_i(g')}{|g'|_\rho}| < \tilde{\varepsilon} + 2\tilde{\varepsilon}^4 < \varepsilon$ for all $i = 1, \dots, n$. Hence, $(\bar{\psi}(\iota(g,z)), \bar{\psi}(\iota(g',z')))$ lies in $\tilde{V}_{(\rho_1,\dots,\rho_m,\varepsilon)}$ in subcase 3.1. Case 3.2: Suppose $[\,|\frac{z}{|g|_\rho}| \geq \tilde{\varepsilon}$ and $|\frac{z'}{|g'|_\rho}| \geq \tilde{\varepsilon}\,]$ or $[\,|\frac{\omega_i(g)}{|g|_\rho}| \geq \tilde{\varepsilon}$ and $|\frac{\omega_i(g')}{|g'|_\rho}| \geq \tilde{\varepsilon}\,]$ for some $i = 1, \dots, n$. If $|\frac{z}{|g|_\rho}| \geq \tilde{\varepsilon}$ and $|\frac{z'}{|g'|_\rho}| \geq \tilde{\varepsilon}$ holds, we deduce for each $j = 1, \dots, m$ that

$$\begin{aligned}
|\frac{|g|_{\rho_j}}{|g|_\rho} - \frac{|g'|_{\rho_j}}{|g'|_\rho}| &= |\frac{z}{|g|_\rho}\frac{|g|_{\rho_j}}{z} - \frac{z}{|g|_\rho}\frac{|g'|_{\rho_j}}{z'} + \frac{z}{|g|_\rho}\frac{|g'|_{\rho_j}}{z'} - \frac{z'}{|g'|_\rho}\frac{|g'|_{\rho_j}}{z'}| \\
&\leq |\frac{|g|_{\rho_j}}{z} - \frac{|g'|_{\rho_j}}{z'}|\,|\frac{z}{|g|_\rho}| + |\frac{z}{|g|_\rho} - \frac{z'}{|g'|_\rho}|\,|\frac{|g'|_{\rho_j}}{z'}| \\
&\leq |\frac{\frac{z'}{|g'|_{\rho_j}} - \frac{z}{|g|_{\rho_j}}}{\frac{z}{|g|_{\rho_j}}\frac{z'}{|g'|_{\rho_j}}}|\,|\frac{z}{|g|_\rho}| + |\frac{z}{|g|_\rho} - \frac{z'}{|g'|_\rho}|\,|\frac{|g'|_{\rho_j}}{z'}| \\
&\leq \frac{K^2}{\tilde{\varepsilon}^2}\tilde{\varepsilon}^4(\frac{1}{K\tilde{\varepsilon}} - C) + \tilde{\varepsilon}^4\frac{K}{\tilde{\varepsilon}} < \varepsilon.
\end{aligned}$$

Analogously, if $|\frac{\omega_i(g)}{|g|_\rho}| \geq \tilde{\varepsilon}$ and $|\frac{\omega_i(g')}{|g'|_\rho}| \geq \tilde{\varepsilon}$ holds for some i, we deduce for each

$j = 1, \ldots, m$ that

$$\Big| \frac{|g|_{\rho_j}}{|g|_\rho} - \frac{|g'|_{\rho_j}}{|g'|_\rho} \Big| < \frac{K^2}{\tilde{\varepsilon}^2} 2\tilde{\varepsilon}^4 C + 2\,\tilde{\varepsilon}^4 \frac{K}{\tilde{\varepsilon}} < \varepsilon.$$

Thus, $(\bar{\psi}(\iota(g,z)), \bar{\psi}(\iota(g',z')))$ lies in $\tilde{V}_{(\rho_1,\ldots,\rho_m,p)}$ in subcase 3.2, too. This completes the proof of Claim 3.

We prove the statements (i) to (iii) of Theorem 3.4.4: The subset $\iota((G \times \mathbb{Z})^\infty)$ is dense in $\partial(G \times \mathbb{Z})$ by construction, and the subset $\tilde{\iota}((G \times \mathbb{Z})^\infty)$ is dense in M_ρ by Lemma 3.2.6. Therefore, the Claims 1 to 3 imply that ψ' extends to an isomorphism from M_ρ onto $\partial(G \times \mathbb{Z})$. Since this extension is unique, it must coincide with ψ. Hence, ψ is an isomorphism. The statements (ii) and (iii) are an obvious consequence of the above.

It remains to prove statement (iv) for $G \times \mathbb{Z}$-actions of the type $\tilde{\rho}_\mu$, where $\tilde{\rho} \in \Gamma$ is a G-factor action and $\mu \in \mathbb{R}^n$ a vector of shift parameters. Let $\tilde{\rho} \in \Gamma$ and $\mu \in \mathbb{R}^n$ be given. We want to show that the canonical map $\hat{\tau}_{\tilde{\rho}_\mu}$ from $\partial(G \times \mathbb{Z})$ to $\Sigma(\partial G)$ is a homotopy equivalence. According to Lemma 3.4.5, there exists a homotopy $H : [0,1] \times C_\rho \to C_\rho$ that contracts C_ρ to a basepoint $\bar{c} \in C_\rho$. We define a map $f : \Sigma(\partial G) \to \partial(G \times \mathbb{Z})$ by

$$[t, \xi] \mapsto f([t,\xi]) := [t, \xi, \bar{c}].$$

Obviously, f is continuous. We show that f is a homotopy inverse for $\hat{\tau}_{\tilde{\rho}_\mu}$. Firstly, we check that $\hat{\tau}_{\tilde{\rho}_\mu} \circ f$ is homotopic to $\mathrm{id}_{\Sigma(\partial G)}$. Define $F_1 : [0,1] \times \Sigma(\partial G) \to \Sigma(\partial G)$ by

$$(\vartheta, [t,\xi]) \mapsto F_1(\vartheta, [t,\xi]) := [(1-\vartheta)\,t + \vartheta\,\frac{t + \langle \mu, \bar{c}^s \rangle}{\bar{c}_{\tilde{\rho}}}, \xi].$$

Clearly, F_1 is continuous. Moreover, we have $F_1(0,[t,\xi]) = [t,\xi]$, as well as $F_1(1,[t,\xi]) = [\frac{t+\langle\mu,\bar{c}^s\rangle}{\bar{c}_{\tilde{\rho}}}, \xi] = (\hat{\tau}_{\tilde{\rho}_\mu} \circ f)([t,\xi])$. Secondly, we check that $f \circ \hat{\tau}_{\tilde{\rho}_\mu}$ is homotopic to $\mathrm{id}_{\partial(G \times \mathbb{Z})}$. Define $F_2 : [0,1] \times \partial(G \times \mathbb{Z}) \to \partial(G \times \mathbb{Z})$ by

$$(\vartheta, [t,\xi,c]) \mapsto F_2(\vartheta, [t,\xi,c]) := [(1-\vartheta)\,t + \vartheta\,\frac{t + \langle \mu, c^s \rangle}{c_{\tilde{\rho}}}, \xi, H(\vartheta, c)].$$

Note that there exists a constant $K \geq 1$ such that $\frac{1}{K} < c_{\tilde{\rho}} < K$ for all $c \in C_\rho$.

Hence, F_2 is continuous. Moreover, we have $F_2(0,[t,\xi,c]) = [t,\xi,c]$, as well as $F_2(1,[t,\xi,c]) = [\frac{t+\langle\mu,c^s\rangle}{c_{\bar{p}}},\xi,\bar{c}] = (f \circ \hat{\tau}_{\bar{p}_\mu})([t,\xi,c])$. This completes the proof of statement (iv). $\square$

Chapter 4

The Boundary of $\mathbf{G} \times \mathbb{Z}^{\mathbf{k}}$

In this chapter we will generalize the results of Section 4 of the last chapter to product groups of the form $G \times \mathbb{Z}^k$, where G is a non-elementary word-hyperbolic CAT(0) group, and $k \in \mathbb{N}$.

We begin with some notation: As before, we consider a fixed non-elementary word-hyperbolic CAT(0) group G. We fix a basis $(\omega_1, \ldots, \omega_n)$ of $\mathrm{Hom}(G, \mathbb{R})$. So, n denotes again the rank of $\mathrm{Hom}(G, \mathbb{R})$. For each $g \in G^\infty$ we abbreviate $\omega(g) := (\omega_1(g), \ldots, \omega_n(g)) \in \mathbb{R}^n$. Let Γ be the set of all geometric actions by G on CAT(0) spaces. For each matrix $\mu \in \mathbb{R}^{k \times n}$ and each action $\rho \in \Gamma$ on a CAT(0) space X we define a geometric $G \times \mathbb{Z}^k$-action ρ_μ on $X \times \mathbb{R}^k$ by

$$(g, z) * (x, r) := (\rho(g, x), r + z + \mu \cdot \omega(g)),$$

where $\mu \cdot \omega(g) \in \mathbb{R}^k$ is the product of the matrix $\mu \in \mathbb{R}^{k \times n}$ with the vector $\omega(g) \in \mathbb{R}^n$, and $z := (z_1, \ldots, z_k) \in \mathbb{Z}^k$ and $r := (r_1, \ldots, r_k) \in \mathbb{R}^k$. By τ_{ρ_μ} we denote the canonical map from $(G \times \mathbb{Z}^k)^\infty$ to $\partial(X \times \mathbb{R}^k)$ associated to ρ_μ; and by U_{ρ_μ} the uniform structure that is induced on $(G \times \mathbb{Z}^k)^\infty$ via τ_{ρ_μ}. We can identify $\partial(X \times \mathbb{R}^k)$ with the join $\partial G * S^{k-1} = ([0, \infty] \times \partial G \times S^{k-1})/ \sim$, where $(t, \xi, \zeta) \sim (t', \xi', \zeta')$ if and only if $[t = 0$ and $t' = 0$ and $\xi = \xi']$ or $[t = \infty$ and $t' = \infty$ and $\zeta = \zeta']$ or

$[t = t'$ and $\xi = \xi'$ and $\zeta = \zeta']$, so that τ_{ρ_μ} is given by

$$(g,z) \mapsto \tau_{\rho_\mu}(g,z) := \begin{cases} [\frac{|z+\mu\cdot\omega(g)|}{|g|_\rho}, g^+, (z+\mu\cdot\omega(g))^+], & \text{if } g \in G^\infty, \\ [\infty, \xi, z^+], & \text{if } g \notin G^\infty. \end{cases}$$

Here $|\cdot|$ is the euclidean norm in $\mathbb{R}^k$, and $\xi \in \partial G$ is arbitrary. Furthermore, for $z \in \mathbb{R}^k \setminus \{0\}$ we denote by z^+ the point in $S^{k-1} \equiv \partial\mathbb{R}^k$ that is determined by the geodesic ray in $\mathbb{R}^k$ issuing from the origin and passing through z (for $0 \in \mathbb{R}^k$ we denote by 0^+ an arbitrary point in S^{k-1}). Note that we have $\tan(\alpha) = \frac{|z+\mu\cdot\omega(g)|}{|g|_\rho}$ in the following figure.

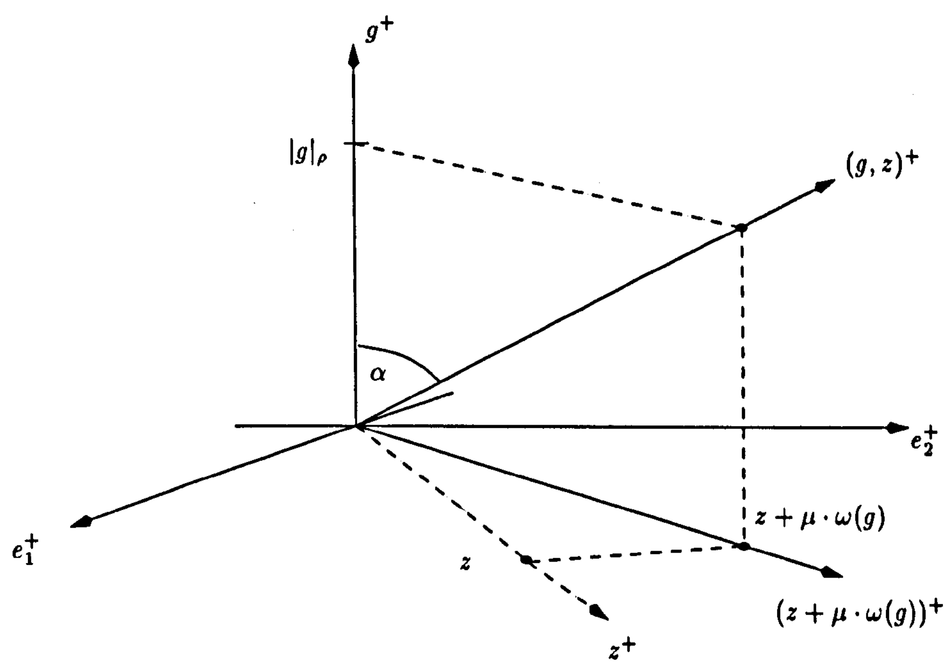

Figure 4.1: Illustrating $\tau_{\rho_\mu} : (G \times \mathbb{Z}^k)^\infty \to \partial G * S^{k-1}$

The following proposition shows that it is enough to consider $G \times \mathbb{Z}^k$-actions of the above type. It is motivated by a result by Bowers and Ruane in [BR96a], namely, that the visual boundary of a CAT(0) space carrying a geometric $G \times \mathbb{Z}^k$-action is homeomorphic to $\partial G * S^{k-1}$.

Proposition 4.0.7 *Let $\bar{\rho}$ be a geometric action of $G \times \mathbb{Z}^k$ on a CAT(0) space $\bar{X}$. Then $\partial\bar{X}$ can be identified with $\partial G * S^{k-1}$. Moreover, there exists an action $\rho \in \Gamma$ on a CAT(0) space X, a matrix $\mu \in \mathbb{R}^{k \times n}$, and a homeomorphism $\varphi : \partial G * S^{k-1} \to \partial G * S^{k-1}$ such that*

$$\begin{tikzcd}
((G \times \mathbb{Z}^k)^\infty, U_{\bar{\rho}}) \arrow[r, "\mathrm{id}_{(G\times\mathbb{Z}^k)^\infty}"] \arrow[d, "\tau_{\bar{\rho}}"'] & ((G \times \mathbb{Z}^k)^\infty, U_{\rho_\mu})^\infty \arrow[d, "\tau_{\rho_\mu}"] \\
\{ (g,z)^+ \mid (g,z) \in (G \times \mathbb{Z}^k)^\infty \} \arrow[r, "\varphi"] \arrow[d, "\subset"'] & \{ (g,z)^+ \mid (g,z) \in (G \times \mathbb{Z}^k)^\infty \} \arrow[d, "\subset"] \\
\partial\bar{X} \equiv \partial G * S^{k-1} \arrow[r, "\varphi"] & \partial(X \times \mathbb{R}^k) \equiv \partial G * S^{k-1}
\end{tikzcd}$$

is a commutative diagram of uniformly continuous maps. In particular, the uniformities $U_{\bar{\rho}}$ and U_{ρ_μ} on $(G \times \mathbb{Z}^k)^\infty$ coincide.

Proof: By Lemma 3.4.2 the centre of G is finite. Therefore, we can apply the Splitting Theorem 1.2.1: There exists a non-empty, closed, convex, $G \times \mathbb{Z}^k$-invariant subspace $\bar{X}_1 \times \bar{X}_2$ of $\bar{X}$ such that G acts geometrically on $\bar{X}_1$ and by (possibly trivial) Clifford translations on $\bar{X}_2$, and $\mathbb{Z}^k$ acts geometrically on $\bar{X}_2$ and trivially on $\bar{X}_1$. We denote the G-action on $\bar{X}_1$ by $\bar{\rho}_1$. Let $\mathrm{Min}(\mathbb{Z}^k) := \bigcap_{z\in\mathbb{Z}^k} \mathrm{Min}(z) \subset \bar{X}_2$, where $\mathrm{Min}(z)$ is the set of points in $\bar{X}_2$ at which the displacement function $d(\cdot, z.\cdot)$ for $z \in \mathbb{Z}^k$ attains its minimum. By the Flat Torus Theorem 1.1.2, $\mathrm{Min}(\mathbb{Z}^k)$ is non-empty and splits as a product $Y \times \mathbb{R}^k$. The $\mathbb{Z}^k$-action on $\bar{X}_2$ leaves $\mathrm{Min}(\mathbb{Z}^k)$ invariant. Each $z \in \mathbb{Z}^k$ acts as the identity on Y-factor and as a non-trivial translation on the $\mathbb{R}^k$-factor. Say, each basis element $e_i \in \mathbb{Z}^k$ acts as a translation by a vector v_i on $\mathbb{R}^k$. Note that $\mathcal{B} := (v_1, \ldots, v_k)$ is a basis for $\mathbb{R}^k$, because according to the Flat Torus Theorem 1.1.2 the quotient of each k-flat $\{y\} \times \mathbb{R}^k$ by the $\mathbb{Z}^k$-action is a k-torus. Let V be the matrix in $\mathbb{R}^{k \times k}$ whose i-th column equals v_i. Each $g \in G$ centralizes $\mathbb{Z}^k$. Therefore, the Flat Torus Theorem implies that the action of each $g \in G$ on $\bar{X}_2$ leaves $\mathrm{Min}(\mathbb{Z}^k)$ invariant, and respects the product decomposition $Y \times \mathbb{R}^k$. Recall that g acts as a Clifford translation on $\mathrm{Min}(\mathbb{Z}^k)$; and that $\mathrm{diam}(Y)$ is finite, because the $\mathbb{Z}^k$-action on $\bar{X}_2$ is cocompact. Hence, by Thm. 1.1.3, g acts trivially on the Y-factor, and as a translation on the $\mathbb{R}^k$-factor. It follows that the G-action on the

$\mathbb{R}^k$-factor induces a homomorphism $\psi \in \mathrm{Hom}(G, \mathbb{R}^k)$. Let $y \in Y$, and consider the subspace $Z := \bar{X}_1 \times \{y\} \times \mathbb{R}^k$ in $\bar{X}$. By the above, Z is a non-empty, closed, convex, $G \times \mathbb{Z}^k$-invariant subspace of $\bar{X}$. This implies in particular that any infinite geodesic ray that issues from a basepoint in Z towards infinity lies entirely in Z. We conclude that $\partial Z = \partial \bar{X}$, which allows us to identify $\partial \bar{X}$ with $\partial G * S^{k-1}$. Let $\bar{\rho}'$ denote the induced $G \times \mathbb{Z}^k$-action on Z. Then $\bar{\rho}'$ can be described expicitely by

$$(g, z) * (x, y, r) := (\bar{\rho}_1(g, x), y, r + V \cdot z + \psi(g)),$$

for any $x \in \bar{X}_1$ and any $r \in \mathbb{R}^k$. Hence, the canonical map $\tau_{\bar{\rho}}$ from $(G \times \mathbb{Z}^k)^\infty$ into $\partial \bar{X} = \partial Z \equiv \partial G * S^{k-1}$ is given by

$$\tau_{\bar{\rho}}(g, z) := \begin{cases} [\frac{|V \cdot z + \psi(g)|}{|g|_{\bar{\rho}_1}}, g^+, (V \cdot z + \psi(g))^+], & \text{if } g \in G^\infty, \\ [\infty, \xi, (V \cdot z + \psi(g))^+], & \text{if } g \notin G^\infty, \end{cases}$$

where $\xi \in \partial G$ is arbitrary. Thus, we can obtain the desired $(G \times \mathbb{Z}^k)$-action ρ_μ on a CAT(0) space $X \times \mathbb{R}^k$ as follows: Set $X := \bar{X}_1$ and $\rho := \bar{\rho}_1$. Take $\mu \in \mathbb{R}^{k \times n}$ such that $V^{-1} \cdot \psi(g) = \mu \cdot \omega(g)$ for each $g \in G$. Then the canonical map τ_{ρ_μ} from $(G \times \mathbb{Z}^k)^\infty$ into $\partial(X \times \mathbb{R}^k) \equiv \partial G * S^{k-1}$ associated to ρ_μ is given by

$$\tau_{\rho_\mu}(g, z) := \begin{cases} [\frac{|z + V^{-1} \cdot \psi(g)|}{|g|_\rho}, g^+, (z + V^{-1} \cdot \psi(g))^+], & \text{if } g \in G^\infty, \\ [\infty, \xi, (z + V^{-1} \cdot \psi(g))^+], & \text{if } g \notin G^\infty, \end{cases}$$

where $\xi \in \partial G$ is arbitrary. It is straightforward to show that the map $\varphi' : \mathrm{im}\, \tau_{\bar{\rho}} \to \mathrm{im}\, \tau_{\rho_\mu}$ given by

$$\tau_{\bar{\rho}}(g, z) \mapsto \tau_{\rho_\mu}(g, z)$$

is a homeomorphism. Since $\mathrm{im}\, \tau_{\bar{\rho}}$ is dense in $\partial \bar{X}$, and $\mathrm{im}\, \tau_{\rho_\mu}$ is dense in $\partial(X \times \mathbb{R}^k)$, φ' extends to a homeomorphism from $\partial \bar{X}$ onto $\partial(X \times \mathbb{R}^k)$. □

For a $G \times \mathbb{Z}^k$-action $\bar{\rho}$ let ρ, μ and φ as given by Prop. 4.0.7. Then ρ is said to be a *G-factor action*, μ a *matrix of shift parameters* and φ a *normalizing homeomorphism* associated to $\bar{\rho}$. Note that, contrary to the case $k = 1$ of the last chapter, the normalizing homeomorphism φ is in general not the identity.

For the remainder of this section let $\rho \in \Gamma$ be a fixed action. Recall from the previous chapter that the maps $\phi_\rho : G^\infty \to \mathbb{R}^n$ and $\kappa_\rho : G^\infty \to \mathbb{R}^\Gamma$ are given by

$$g \mapsto \phi_\rho(g) := \left(\frac{\omega_1(g)}{|g|_\rho}, \ldots, \frac{\omega_n(g)}{|g|_\rho}\right), \quad \text{and} \quad g \mapsto \kappa_\rho(g) := \left(\frac{|g|_{\tilde\rho}}{|g|_\rho}\right)_{\tilde\rho}.$$

As in the last chapter, we define a map $\theta_\rho : G^\infty \to \mathbb{R}^\Gamma \times \mathbb{R}^n$ by

$$g \mapsto \theta_\rho(g) := (\kappa_\rho(g), \phi_\rho(g)),$$

and denote the closure of the image im θ_ρ in $\mathbb{R}^\Gamma \times \mathbb{R}^n$ by C_ρ. Let $\mathrm{pr}_{\mathbb{R}^n}$ be the projection of $\mathbb{R}^\Gamma \times \mathbb{R}^n$ onto $\mathbb{R}^n$. For each $\tilde\rho \in \Gamma$ let $\mathrm{pr}_{\tilde\rho}$ be the projection of $\mathbb{R}^\Gamma \times \mathbb{R}^n$ onto the $\tilde\rho$-component of $\mathbb{R}^\Gamma$. For any $c \in C_\rho$ we abbreviate $\mathrm{pr}_{\mathbb{R}^n}(c)$ by $c^s = (c_1^s, \ldots, c_n^s)$, where the superscript s stands for *shift*, and $\mathrm{pr}_{\tilde\rho}(c)$ by $c_{\tilde\rho}$.

Definition 4.0.8 Let $\overline{\mathbb{R}^k}$ be $\mathbb{R}^k$ compactified by its visual boundary $\partial\mathbb{R}^k \equiv S^{k-1}$. The *$\rho$-model* M_ρ of the boundary $\partial(G\times\mathbb{Z}^k)$ is defined by $M_\rho := (\overline{\mathbb{R}^k}\times\partial G\times C_\rho)/\sim$, where $(v,\xi,c) \sim (v',\xi',c')$ if and only if $[\,|v| = |v'| = \infty$ and $v = v'\,]$, or $[\,v = v' = 0$ and $c^s = c'^s = 0$ and $\xi = \xi'\,]$, or $[\,v = v'$ and $\xi = \xi'$ and $c = c'\,]$.

Theorem 4.0.9 *Let G be a non-elementary word-hyperbolic CAT(0) group, and let ρ be a geometric action by G on a CAT(0) space. Let M_ρ be defined as above. Then the following is true:*

(i) The boundary $\partial(G \times \mathbb{Z}^k)$ is canonically isomorphic to M_ρ.

(ii) The canonical $G \times \mathbb{Z}^k$-action on $\partial(G \times \mathbb{Z}^k)$ is given by

$$(g,z) * [v,\xi,c] = [v, g.\xi, c].$$

(iii) Let $\bar\rho$ be a geometric $G \times \mathbb{Z}^k$-action on a CAT(0) space $\bar X$. Let $\tilde\rho$ be a G-factor action, $\mu \in \mathbb{R}^{k\times n}$ a matrix of shift parameters and φ a normalizing homeomorphism associated to $\bar\rho$. Then the canonical map $\hat\tau_{\bar\rho}$ from $\partial(G\times\mathbb{Z}^k)$ to $\partial\bar X \equiv \partial G * S^{k-1}$ is given by:

$$\hat\tau_{\bar\rho}([v,\xi,c]) = \varphi^{-1}\left(\left[\frac{|v+\mu\cdot c^s|}{c_{\tilde\rho}}, \xi, (v+\mu\cdot c^s)^+\right]\right).$$

(iv) *For any geometric $G\times\mathbb{Z}^k$-action $\tilde{\rho}$ on a CAT(0) space $\bar{X}$ the canonical map $\hat{\tau}_{\tilde{\rho}}$ from $\partial(G\times\mathbb{Z}^k)$ to $\partial\bar{X}$ is a homotopy equivalence.*

We need to prepare the proof of this theorem. In order to compare $\partial(G\times\mathbb{Z}^k)$ with M_ρ we define a map $\tilde{\iota} : (G\times\mathbb{Z}^k)^\infty \to M_\rho$ by

$$(g,z) \mapsto \tilde{\iota}(g,z) := \begin{cases} [\frac{1}{|g|_\rho} z\,,\, g^+\,,\, \theta_\rho(g)], & \text{if } g\in G^\infty, \\ [z^+\,,\,\xi\,,\,c], & \text{if } g\notin G^\infty, \end{cases}$$

where $\xi\in\partial G$ and $c\in C_\rho$ are arbitrary.

Lemma 4.0.10 *The image $\tilde{\iota}((G\times\mathbb{Z}^k)^\infty)$ is dense in M_ρ.*

Proof: The proof is analogous to that of Lemma 3.2.6. □

Proof of Theorem 4.0.9: According to Prop. 4.0.7 it is enough to prove the statements (iii) and (iv) of Theorem 4.0.9 for $G\times\mathbb{Z}^k$-actions of the type $\tilde{\rho}_\mu$, where $\tilde{\rho}\in\Gamma$ and $\mu\in\mathbb{R}^{k\times n}$. Furthermore, we can regard $\partial(G\times\mathbb{Z}^k)$ as the closure of the image $\operatorname{im}((\tau_{\tilde{\rho}_\mu})_{(\tilde{\rho},\mu)})$ in $\prod_{(\tilde{\rho},\mu)\in\Gamma\times\mathbb{R}^{k\times n}}\partial G * S^{k-1}$ throughout this proof, since the boundary uniformity $U_{G\times\mathbb{Z}^k}$ on $(G\times\mathbb{Z}^k)^\infty$ is the same as the least upper bound of the family $\{U_{\tilde{\rho}_\mu} \mid \tilde{\rho}\in\Gamma,\, \mu\in\mathbb{R}^{k\times n}\}$. Thus, the product map $(\tau_{\tilde{\rho}_\mu})_{\tilde{\rho}_\mu}$ is the canonical map ι from $((G\times\mathbb{Z}^k)^\infty, U_{G\times\mathbb{Z}^k})$ into its Hausdorff completion $\partial(G\times\mathbb{Z}^k)$. For each $\tilde{\rho}\in\Gamma$ and each $\mu\in\mathbb{R}^{k\times n}$ we define a map $\tilde{\tau}_{\tilde{\rho}_\mu}$ from M_ρ to $\partial G * S^{k-1}$ by

$$[v,\xi,c] \mapsto \tilde{\tau}_{\tilde{\rho}_\mu}([v,\xi,c]) := [\frac{|v+\mu\cdot c^s|}{c_{\tilde{\rho}}}\,,\,\xi\,,\,(v+\mu\cdot c^s)^+].$$

It is easy to check that

$$\begin{array}{ccc} (G\times\mathbb{Z}^k)^\infty & \xrightarrow{\tilde{\iota}} & M_\rho \\ {\scriptstyle \iota = (\tau_{\tilde{\rho}_\mu})_{(\tilde{\rho},\mu)}}\downarrow & & \downarrow{\scriptstyle \tilde{\tau}_{\tilde{\rho}_\mu}} \\ \partial(G\times\mathbb{Z}^k) & \xrightarrow{pr_{(\tilde{\rho},\mu)}} & \partial G * S^{k-1} \end{array}$$

is a commuting diagram of set maps. Let $\psi := (\tilde{\tau}_{\tilde{\rho}_\mu})_{(\tilde{\rho},\mu)}$ be the product map from M_ρ to $\prod_{(\tilde{\rho},\mu)\in\Gamma\times\mathbb{R}^{k\times n}}\partial G * S^{k-1}$, and let ψ' be the restriction of ψ to $\tilde{\iota}((G\times\mathbb{Z}^k)^\infty)$.

Claim 1: ψ is uniformly continuous.

Proof of Claim 1: It is enough to show that $\tilde{\tau}_{\tilde{\rho}_\mu}$ is uniformly continuous for each $(\tilde{\rho}, \mu) \in \Gamma \times \mathbb{R}^{k \times n}$. Throughout this proof of Claim 1, let d be a fixed visual metric on ∂G. Let the metric $d_{S^{k-1}}$ be the metric on the unit sphere S^{k-1} that is induced by the euclidean metric on $\mathbb{R}^k$. We will regard $d_{S^{k-1}}$ as a metric on $\partial\mathbb{R}^k \equiv S^{k-1}$. We consider the following fundamental systems of entourages for the uniform structure on $\partial G * S^{k-1}$ and on M_ρ: For any ε with $1 > \varepsilon > 0$ let W_ε consist precisely of those pairs $([t, \xi, \zeta], [t', \xi', \zeta'])$ in $\partial G * S^{k-1} \times \partial G * S^{k-1}$ that satisfy $[\, |t - t'| < \varepsilon$ and $d(\xi, \xi') < \varepsilon$ and $d_{S^{k-1}}(\zeta, \zeta') < \varepsilon\,]$, or $[\, t > \frac{1}{\varepsilon}$ and $t' > \frac{1}{\varepsilon}$ and $d_{S^{k-1}}(\zeta, \zeta') < \varepsilon\,]$, or $[\, t < \varepsilon$ and $t' < \varepsilon$ and $d(\xi, \xi') < \varepsilon\,]$. Then it is straightforward to check that the system $\{W_\varepsilon\}$ is a fundamental system of entourages for $\partial G * S^{k-1}$. For any ε with $1 > \varepsilon > 0$ and any finite number $\rho_1, \ldots, \rho_m$ of actions in Γ let $V_{(\rho_1, \ldots, \rho_m, \varepsilon)}$ consist precisely of those pairs $([v, \xi, c], [v', \xi', c'])$ in $M_\rho \times M_\rho$ that satisfy $[\, |v| > \frac{1}{\varepsilon}$ and $|v'| > \frac{1}{\varepsilon}$ and $d_{S^{k-1}}(v^+, v'^+) < \varepsilon\,]$, or $[\, |v| < \varepsilon$ and $|v'| < \varepsilon$ and $|c^s|_\infty < \varepsilon$ and $|c'^s|_\infty < \varepsilon$ and $d(\xi, \xi') < \varepsilon\,]$, or $[\, |v - v'| < \varepsilon^2$ and $d(\xi, \xi') < \varepsilon$ and $|c^s - c'^s|_\infty < \varepsilon^2$ and $|c_{\rho_j} - c'_{\rho_j}| < \varepsilon^2$ for each $j = 1, \ldots, m\,]$. Analogously to the proof of Lemma 3.2.5 one can show that the system $\{V_{(\rho_1, \ldots, \rho_m, \varepsilon)}\}$ is a fundamental system of entourages for M_ρ. Let $(\tilde{\rho}, \mu) \in \Gamma \times \mathbb{R}^{k \times n}$ and ε with $1 > \varepsilon > 0$ be given. In order to prove that $\tilde{\tau}_{\tilde{\rho}_\mu}$ is uniformly continuous, it is enough to check that there exists an $1 > \tilde{\varepsilon} > 0$ such that $\tilde{\tau}_{\tilde{\rho}_\mu} \times \tilde{\tau}_{\tilde{\rho}_\mu}$ maps $V_{(\tilde{\rho}, \tilde{\varepsilon})}$ into W_ε. Set $K := k(\tilde{\rho})$, where $k(\tilde{\rho})$ is defined as in the proof of Lemma 3.3.5. Thus, $K \geq 1$, and $\frac{1}{K} < c_{\tilde{\rho}} < K$ for each $c \in C_\rho$. Set $M := \sup |\mu \cdot x|$, where the supremum is taken over all $x \in \mathbb{R}^n$ with $|x| = 1$. Note that $|x| < n\,|x|_\infty$ for all $x \in \mathbb{R}^n$. Let $C > 0$ be given as in the proof of Lemma 3.2.2, i.e. $|c^s|_\infty < C$ for all $c \in C_\rho$. Take $1 > \tilde{\varepsilon} > 0$ small enough such that $nMC\tilde{\varepsilon} < 1$ and $\frac{K\tilde{\varepsilon}}{1 - nMC\tilde{\varepsilon}} < \varepsilon$ and $\frac{nMC\tilde{\varepsilon}}{1 - nMC\tilde{\varepsilon}} < \frac{1}{3}\varepsilon$ and $\tilde{\varepsilon} < \frac{1}{3}\varepsilon$ and $K(1 + nM)\tilde{\varepsilon} < \varepsilon$ and $\frac{K\tilde{\varepsilon}}{1 - (1 + nM)\tilde{\varepsilon}^3} < \varepsilon$ and $K\tilde{\varepsilon}(K + \tilde{\varepsilon} + nM\tilde{\varepsilon}) < \varepsilon$ hold. Suppose that $([v, \xi, c], [v', \xi', c'])$ lies in $V_{(\tilde{\rho}, \tilde{\varepsilon})}$. Then there are three cases to consider.

Case 1: Suppose $|v| > \frac{1}{\tilde{\varepsilon}}$ and $|v'| > \frac{1}{\tilde{\varepsilon}}$ and $d_{S^{k-1}}(v^+, v'^+) < \tilde{\varepsilon}$. Then we get

$$\frac{|v + \mu \cdot c^s|}{c_{\tilde{\rho}}} > \left(\frac{1}{\tilde{\varepsilon}} - n\,M\,C\right)\frac{1}{K} > \frac{1}{\varepsilon},$$

and analogously $\frac{|v'+\mu\cdot c'^s|}{c'_{\tilde{\rho}}} > \frac{1}{\varepsilon}$. Moreover, we have

$$d_{S^{k-1}}(v^+, (v+\mu\cdot c^s)^+) \leq \frac{|\mu\cdot c^s|}{|v|-|\mu\cdot c^s|} \leq \frac{n\,M\,C\,\tilde{\varepsilon}}{1-n\,M\,C\,\tilde{\varepsilon}} < \frac{1}{3}\varepsilon,$$

and analogously $d_{S^{k-1}}(v'^+, (v'+\mu\cdot c'^s)^+) < \frac{1}{3}\varepsilon$. So, the triangle inequality implies that $d_{S^{k-1}}((v+\mu\cdot c^s)^+, (v'+\mu\cdot c'^s)^+) \leq \varepsilon$. Thus, $(\tilde{\tau}_{\tilde{\rho}_\mu}([v,\xi,c]), \tilde{\tau}_{\tilde{\rho}_\mu}([v',\xi',c']))$ lies in W_ε for Case 1.

Case 2: Suppose $|v| < \tilde{\varepsilon}$ and $|v'| < \tilde{\varepsilon}$ and $|c^s|_\infty < \tilde{\varepsilon}$ and $|c'^s|_\infty < \tilde{\varepsilon}$ and $d(\xi,\xi') < \tilde{\varepsilon}$. Then we get $d(\xi,\xi') < \varepsilon$. Moreover, we deduce

$$\frac{|v+\mu\cdot c^s|}{c_{\tilde{\rho}}} \leq K\,(\tilde{\varepsilon}+n\,M\,\tilde{\varepsilon}) < \varepsilon,$$

and analogously $\frac{|v'+\mu\cdot c'^s|}{c'_{\tilde{\rho}}} \leq \varepsilon$. Hence, $(\tilde{\tau}_{\tilde{\rho}_\mu}([v,\xi,c]), \tilde{\tau}_{\tilde{\rho}_\mu}([v',\xi',c']))$ lies in W_ε for Case 2.

Case 3: Suppose $|v-v'| < \tilde{\varepsilon}^2$ and $d(\xi,\xi') < \tilde{\varepsilon}$ and $|c^s - c'^s|_\infty < \tilde{\varepsilon}^2$ and $|c_{\tilde{\rho}} - c'_{\tilde{\rho}}| < \tilde{\varepsilon}^2$. Then we have $d(\xi,\xi') < \varepsilon$. We can assume without loss of generality that both $|v+\mu\cdot c^s| \geq \tilde{\varepsilon}$ and $|v'+\mu\cdot c'^s| \geq \tilde{\varepsilon}$. For otherwise, suppose that $|v+\mu\cdot c^s| < \tilde{\varepsilon}$, say. Then we deduce that both $\frac{1}{c_{\tilde{\rho}}}|v+\mu\cdot c^s| < \varepsilon$ and $\frac{1}{c'_{\tilde{\rho}}}|v'+\mu\cdot c'^s| < \varepsilon$, because of

$$\begin{aligned}\frac{1}{c_{\tilde{\rho}}}\,|v'+\mu\cdot c'^s| &\leq \frac{1}{c_{\tilde{\rho}}}\,(|v+\mu\cdot c^s| + |v'-v| + |\mu\cdot(c'^s - c^s)|)\\ &\leq K\,(\tilde{\varepsilon}+\tilde{\varepsilon}^2+n\,M\,\tilde{\varepsilon}^2) < \varepsilon;\end{aligned}$$

and conclude that $(\tilde{\tau}_{\tilde{\rho}_\mu}([v,\xi,c]), \tilde{\tau}_{\tilde{\rho}_\mu}([v',\xi',c']))$ lies in W_ε as in Case 2. The assumption implies that

$$\begin{aligned}d_{S^{k-1}}((v+\mu\cdot c^s)^+, (v'+\mu\cdot c'^s)^+) &\leq \frac{1}{\tilde{\varepsilon}}\,(|v-v'| + |\mu\cdot(c^s - c'^s)|)\\ &\leq \frac{1}{\tilde{\varepsilon}}\,(\tilde{\varepsilon}^2+n\,M\,\tilde{\varepsilon}^2) < \varepsilon.\end{aligned}$$

We can assume furthermore without loss of generality that $|v+\mu\cdot c^s| \leq \frac{1}{\tilde{\varepsilon}}$. For otherwise we get that both $\frac{1}{c_{\tilde\rho}}|v+\mu\cdot c^s| > \frac{1}{\varepsilon}$ and $\frac{1}{c'_{\tilde\rho}}|v'+\mu\cdot c'^s| > \frac{1}{\varepsilon}$, because

$$\begin{aligned}\frac{1}{c'_{\tilde\rho}}|v'+\mu\cdot c'^s| &\geq \frac{1}{c'_{\tilde\rho}}\left(|v+\mu\cdot c^s| - |v'-v| - |\mu\cdot(c'^s-c^s)|\right)\\ &\geq \frac{1}{K}\left(\frac{1}{\tilde\varepsilon} - \tilde\varepsilon^2 - n\,M\,\tilde\varepsilon^2\right) > \frac{1}{\varepsilon},\end{aligned}$$

and therefore $(\tilde\tau_{\tilde\rho_\mu}([v,\xi,c]), \tilde\tau_{\tilde\rho_\mu}([v',\xi',c']))$ lies in W_ε as in Case 1. This assumption allows us to conclude that

$$\begin{aligned}&\left|\frac{|v+\mu\cdot c^s|}{c_{\tilde\rho}} - \frac{|v'+\mu\cdot c'^s|}{c'_{\tilde\rho}}\right|\\ &\quad\leq |v+\mu\cdot c^s|\left|\frac{1}{c_{\tilde\rho}} - \frac{1}{c'_{\tilde\rho}}\right| + \big||v+\mu\cdot c^s| - |v'+\mu\cdot c'^s|\big|\,\frac{1}{c'_{\tilde\rho}}\\ &\quad\leq |v+\mu\cdot c^s|\left|\frac{c'_{\tilde\rho}-c_{\tilde\rho}}{c_{\tilde\rho}c'_{\tilde\rho}}\right| + \left(|v-v'| + |\mu\cdot(c^s-c'^s)|\right)\frac{1}{c'_{\tilde\rho}}\\ &\quad\leq \frac{1}{\tilde\varepsilon}K^2\,\tilde\varepsilon^2 + (\tilde\varepsilon^2 + nM\tilde\varepsilon^2)\,K \leq \varepsilon.\end{aligned}$$

Therefore, $(\tilde\tau_{\tilde\rho_\mu}([v,\xi,c]), \tilde\tau_{\tilde\rho_\mu}([v',\xi',c']))$ lies in W_ε for Case 3. This completes the proof of Claim 1.

Claim 2: ψ' is a bijection from $\tilde\iota\,((G\times\mathbb{Z}^k)^\infty)$ onto $\iota\,((G\times\mathbb{Z}^k)^\infty)$.

Proof of Claim 2: We show that ψ' has an inverse. Define a map $\bar\psi$ from $\iota\,((G\times\mathbb{Z}^k)^\infty)$ to $\tilde\iota\,((G\times\mathbb{Z}^k)^\infty)$ by

$$\iota(g,z) \mapsto \bar\psi(\iota(g,z)) := \tilde\iota(g,z).$$

We need to check that $\bar\psi$ is well-defined: Let $(g',z') \in (G\times\mathbb{Z}^k)^\infty$ be such that $\iota(g,z) = \iota(g',z')$, i.e. $\tau_{\tilde\rho_\mu}(g,z) = \tau_{\tilde\rho_\mu}(g',z')$ for all $(\tilde\rho,\mu)\in\Gamma\times\mathbb{R}^{k\times n}$. There are two cases to consider. Case 1: Suppose that $g\in G^\infty$. We distinguish between two subcases. Subcase 1.1: Suppose that $z=0$. Then $z'=0$ and $g^+ = g'^+$, because for $\mu=0$ we get $\frac{|z|}{|g|_\rho} = \frac{|z'|}{|g'|_\rho}$ and $g^+ = g'^+$. Moreover, let $E_{ji}(\lambda)\in\mathbb{R}^{k\times n}$ be such that all its components are 0, except for the one in the j-th row and i-th column, which is λ. It follows for all $i=1,\ldots,n$ and all $\rho\,\tilde\in\,\Gamma$ that $\frac{|\omega_i(g)|}{|g|_{\tilde\rho}} = \frac{|E_{1i}(1)\cdot\omega(g)|}{|g|_{\tilde\rho}} = \frac{|E_{1i}(1)\cdot\omega(g')|}{|g'|_{\tilde\rho}} = \frac{|\omega_i(g')|}{|g'|_{\tilde\rho}}$. So, if $|\omega_i(g)|=0$ for all $i=1,\ldots,n$, then $\tilde\iota(g,z) = \tilde\iota(g',z')$. Otherwise, if $|\omega_i(g)|\neq 0$ for some i, then $\frac{|\omega_i(g)|}{|g|_{\tilde\rho}}\neq 0$ and

$\frac{|\omega_i(g')|}{|g'|_{\tilde\rho}} \neq 0$ for all $\tilde\rho \in \Gamma$. Hence, we get $\frac{|g|_{\tilde\rho}}{|g|_\rho} = \frac{|\omega_i(g)|}{|g|_\rho}\frac{|g|_{\tilde\rho}}{|\omega_i(g)|} = \frac{|\omega_i(g')|}{|g'|_\rho}\frac{|g'|_{\tilde\rho}}{|\omega_i(g')|} = \frac{|g'|_{\tilde\rho}}{|g'|_\rho}$. Furthermore, for each i with $|\omega_i(g)| \neq 0$ we get $(\omega_i(g), 0, \ldots, 0)^+ = (E_{1i}(1) \cdot \omega(g))^+ = (E_{1i}(1) \cdot \omega(g'))^+ = (\omega_i(g'), 0, \ldots, 0)^+$; and therefore $\frac{\omega_i(g)}{|g|_\rho} = \frac{\omega_i(g')}{|g'|_\rho}$ for all $i = 1, \ldots, n$. Thus, we have proved $\tilde\iota(g,z) = \tilde\iota(g',z')$ in Subcase 1.1. Subcase 1.2: Suppose that $z \neq 0$. For $\mu = 0$ we conclude that $\frac{|z|}{|g|_{\tilde\rho}} = \frac{|z'|}{|g'|_{\tilde\rho}}$ for all $\tilde\rho \in \Gamma$, that $g^+ = g'^+$ and that $z^+ = z'^+$. It follows that $\frac{1}{|g|_\rho}z = \frac{1}{|g'|_\rho}z'$ and that $\frac{|g|_{\tilde\rho}}{|g|_\rho} = \frac{|g'|_{\tilde\rho}}{|g'|_\rho}$. Let $j = 1, \ldots, k$ be such that $z_k \neq 0$. For each $i = 1, \ldots, n$ there exists an $\varepsilon_i > 0$ such that $|z + E_{ki}(\varepsilon_i) \cdot \omega(g)| \neq 0$. This implies $\frac{|z+E_{ki}(\varepsilon_i)\cdot\omega(g)|}{|g|_\rho} = \frac{|z'+E_{ki}(\varepsilon_i)\cdot\omega(g')|}{|g'|_\rho}$ and $(z + E_{ki}(\varepsilon_i) \cdot \omega(g))^+ = (z' + E_{ki}(\varepsilon_i) \cdot \omega(g'))^+$. Therefore, we have $\frac{1}{|g|_\rho}(z + E_{ki}(\varepsilon_i) \cdot \omega(g)) = \frac{1}{|g'|_\rho}(z' + E_{ki}(\varepsilon_i) \cdot \omega(g'))$, from which we conclude that $\frac{\omega_i(g)}{|g|_\rho} = \frac{\omega_i(g')}{|g'|_\rho}$ for each $i = 1, \ldots, n$. Thus, we have proved $\tilde\iota(g,z) = \tilde\iota(g',z')$ in Subcase 1.2; and $\bar\psi$ is well-defined in Case 1. Case 2: Suppose that $g \notin G^\infty$. Then $g' \notin G^\infty$, and $z^+ = z'^+$. Therefore, $\bar\psi$ is well-defined in Case 2, too. It is straightforward to check that $\bar\psi$ is the inverse of ψ'. This proves Claim 2.

Claim 3: $\bar\psi$ is uniformly continuous.

Proof of Claim 3: Throughout this proof of Claim 3, let d be a fixed visual metric on ∂G. Let the metric $d_{S^{k-1}}$ be the metric on the unit sphere S^{k-1} that is induced by the euclidean metric on $\mathbb{R}^k$. We will regard $d_{S^{k-1}}$ as a metric on $\partial\mathbb{R}^k \equiv S^{k-1}$. We consider the following fundamental systems of entourages for $\iota((G \times \mathbb{Z}^k)^\infty)$ and for $\tilde\iota((G \times \mathbb{Z}^k)^\infty)$: For any $\varepsilon > 0$ with $1 > \varepsilon > 0$ let $\tilde W_\varepsilon$ consist precisely of those pairs $([t,\xi,\zeta],[t',\xi',\zeta'])$ in $(\partial G * S^{k-1}) \times (\partial G * S^{k-1})$ that satisfy $[\,|t-t'| < \varepsilon^5$ and $d(\xi,\xi') < \varepsilon$ and $d_{S^{k-1}}(\zeta,\zeta') < \varepsilon^5]$, or $[t > \frac{1}{\varepsilon}$ and $t' > \frac{1}{\varepsilon}$ and $d_{S^{k-1}}(\zeta,\zeta') < \varepsilon^5]$, or $[t < \varepsilon^5$ and $t' < \varepsilon^5$ and $d(\xi,\xi') < \varepsilon]$. Then the system $\{\tilde W_\varepsilon\}$ is a fundamental system of entourages for $\partial G * S^{k-1}$. For any $\varepsilon > 0$ with $1 > \varepsilon > 0$ and any finite number $(\rho_1,\mu_1), \ldots, (\rho_m,\mu_m)$ of actions in $\Gamma \times \mathbb{R}^{k\times n}$ let $\tilde W_{(\rho_1,\mu_1,\ldots,\rho_m,\mu_m,\varepsilon)}$ consist precisely of those pairs $((x_{(\tilde\rho,\mu)})_{(\tilde\rho,\mu)}, (x'_{(\tilde\rho,\mu)})_{(\tilde\rho,\mu)})$ in $(\prod_{(\tilde\rho,\mu)\in\Gamma\times\mathbb{R}^{k\times n}} \partial G * S^{k-1}) \times (\prod_{(\tilde\rho,\mu)\in\Gamma\times\mathbb{R}^{k\times n}} \partial G * S^{k-1})$ for which $(x_{(\rho_j,\mu_j)}, x'_{(\rho_j,\mu_j)})$ lies in $\tilde W_\varepsilon$ for each $j = 1, \ldots, m$. Then, by construction, the trace of the system $\{\tilde W_{(\rho_1,\mu_1,\ldots,\rho_m,\mu_m,\varepsilon)}\}$ is a fundamental system of entourages for the subspace $\iota((G \times \mathbb{Z}^k)^\infty)$. For any ε with $1 > \varepsilon > 0$ and any finite number $\rho_1, \ldots \rho_m$ of actions in Γ let $\tilde V_{(\rho_1,\ldots,\rho_m,\varepsilon)}$ consist precisely of those pairs $([v,\xi,c],[v',\xi',c'])$ in $M_\rho \times M_\rho$ that satisfy $[\,|v| > \frac{1}{\varepsilon}$ and $|v'| > \frac{1}{\varepsilon}$ and $d_{S^{k-1}}(v^+, v'^+) < \varepsilon]$, or $[\,|v| < \varepsilon$ and $|v'| < \varepsilon$ and

$|c^s|_\infty < \varepsilon$ and $|c'^s|_\infty < \varepsilon$ and $d(\xi, \xi') < \varepsilon$], or [$|v - v'| < \varepsilon$ and $d(\xi, \xi') < \varepsilon$ and $|c^s - c'^s|_\infty < \varepsilon$ and $|c_{\rho_j} - c'_{\rho_j}| < \varepsilon$ for all $j = 1, \ldots, m$]. Then it is straightforward to check that the trace of the system $\{\tilde{V}_{(\rho_1, \ldots, \rho_m, \varepsilon)}\}$ is a fundamental system of entourages for the subspace $\tilde{\iota}((G \times \mathbb{Z})^\infty)$. Let a finite number $\rho_1, \ldots, \rho_m$ of actions in Γ and $1 > \varepsilon > 0$ be given. In order to prove that $\bar{\psi}$ is uniformly continuous, it is enough to check that there exists an $1 > \tilde{\varepsilon} > 0$ such that $\bar{\psi} \times \bar{\psi}$ maps each $(\iota(g, z), \iota(g', z')) \in \bigcap_{j=0}^m (\bigcap_{i=1}^n \tilde{W}_{(\rho_j, E_{1i}(1), \tilde{\varepsilon})} \cap \tilde{W}_{(\rho_j, [0], \tilde{\varepsilon})})$, where $\rho_0 := \rho$, $[0]$ is the matrix in $\mathbb{R}^{k \times n}$ with all entries 0, and $E_{1i}(1)$ is as defined above, into $\tilde{V}_{(\rho_1, \ldots, \rho_m, \varepsilon)}$. Set $K := \max\{k(\rho_j) \mid j = 1, \ldots, m\}$, where each $k(\rho_j)$ is defined as in the proof of Lemma 3.3.5. Thus, we have $K \geq 1$; and $\frac{1}{K} \leq c_{\rho_j} \leq K$ for each $c \in C_\rho$ and each $j = 1, \ldots, m$. Let $C > 0$ be given as in the proof of Lemma 3.2.2, i.e. $|c^s|_\infty < C$ for all $c \in C_\rho$. Take $1 > \tilde{\varepsilon} > 0$ small enough such that $KC\tilde{\varepsilon} < 1$ and $2(1 + K)\tilde{\varepsilon} < \varepsilon$ and $K(C + 4)\tilde{\varepsilon}^5 < \varepsilon$ and $K(C + 4)\tilde{\varepsilon}^4 < \frac{1+K}{2}$ and $(4C + 2)\frac{K(C+4)}{(1+K)^2}\tilde{\varepsilon}^3 < \varepsilon$ and $\tilde{\varepsilon} + 4\tilde{\varepsilon}^4 < \varepsilon$ and $2K((1 - KC\tilde{\varepsilon})\tilde{\varepsilon} + \tilde{\varepsilon}^3) < \varepsilon$ and $4K(KC\tilde{\varepsilon}^2 + \tilde{\varepsilon}^3) < \varepsilon$. We distinguish between two cases:

Case 1: Suppose $\frac{|z|}{|g|_\rho} \geq \frac{1}{K\tilde{\varepsilon}} - C$ and $\frac{|z'|}{|g'|_\rho} \geq \frac{1}{K\tilde{\varepsilon}} - C$. Then $\frac{|z|}{|g|_\rho} > \frac{1}{\varepsilon}$ and $\frac{|z'|}{|g'|_\rho} > \frac{1}{\varepsilon}$. We also get $d_{S^{k-1}}(z^+, z'^+) < \tilde{\varepsilon}^5 < \varepsilon$. Hence $(\tilde{\iota}(g, z), \tilde{\iota}(g', z'))$ lies in $\tilde{V}_{(\rho_1, \ldots, \rho_m, \varepsilon)}$ for Case 1.

Case 2: Suppose $\frac{|z|}{|g|_\rho}$ or $\frac{|z'|}{|g'|_\rho}$ is less than $\frac{1}{K\tilde{\varepsilon}} - C$. Without loss of generality we assume that $\frac{|z|}{|g|_\rho} < \frac{1}{K\tilde{\varepsilon}} - C$. (If $\frac{|z'|}{|g'|_\rho} < \frac{1}{K\tilde{\varepsilon}} - C$, the argument goes analogously.) We abbreviate $w_i := E_{1i}(1) \cdot \omega(g)$ and $w'_i := E_{1i}(1) \cdot \omega(g')$ for each $i = 1, \ldots, n$. Also we set $w_0 = w'_0 := 0 \in \mathbb{R}^k$. The assumption implies for all $i = 0, \ldots, n$ and all $j = 0, \ldots, m$ (recall: $\rho_0 := \rho$ and $w_0 = 0 \in \mathbb{R}^k$) that

$$\frac{|z + w_i|}{|g|_{\rho_j}} \leq K\frac{|z| + |w_i|}{|g|_\rho} < K(\frac{1}{K\tilde{\varepsilon}} - C + C) = \frac{1}{\tilde{\varepsilon}}.$$

It follows by hypothesis that at least one of the following conditions (α) and (β)

is satisfied for each $i = 0, \ldots, n$ and each $j = 0, \ldots, m$:

$$(\alpha) \quad \frac{|z + w_i|}{|g|_{\rho_j}} < \tilde{\varepsilon}^5 \quad \text{and} \quad \frac{|z' + w_i'|}{|g'|_{\rho_j}} < \tilde{\varepsilon}^5 \quad \text{and} \quad d(g^+, g'^+) < \tilde{\varepsilon}$$

$$(\beta) \quad \left| \frac{|z + w_i|}{|g|_{\rho_j}} - \frac{|z' + w_i'|}{|g'|_{\rho_j}} \right| < \tilde{\varepsilon}^5 \quad \text{and} \quad d(g^+, g'^+) < \tilde{\varepsilon} \quad \text{and} \ \ldots$$

$$\ldots d_{S^{k-1}}((z + w_i)^+, (z' + w_i')^+) < \tilde{\varepsilon}^5.$$

We distinguish between two subcases. Case 2.1: Suppose that $\frac{|z|}{|g|_\rho} < \tilde{\varepsilon}^5$ and $\frac{|z'|}{|g'|_\rho} < \tilde{\varepsilon}^5$ and $d(g^+, g'^+) < \tilde{\varepsilon}$. We distinguish once more between two subcases. Case 2.1.a: Suppose that $[\, \frac{|\omega_i(g)|}{|g|_\rho} < (1+K)\tilde{\varepsilon}$ and $\frac{|\omega_i(g')|}{|g'|_\rho} < (1+K)\tilde{\varepsilon}\,]$ holds for all $i = 1, \ldots, n$. Then clearly $\frac{|\omega_i(g)|}{|g|_\rho} < \varepsilon$ and $\frac{|\omega_i(g')|}{|g'|_\rho} < \varepsilon$ for all $i = 1, \ldots, n$. Furthermore, we have $\frac{|z|}{|g|_\rho} < \varepsilon$ and $\frac{|z'|}{|g'|_\rho} < \varepsilon$ and $d(g^+, g'^+) < \varepsilon$. Thus, $(\tilde{\iota}(g, z), \tilde{\iota}(g', z'))$ lies in $\tilde{V}_{(\rho_1, \ldots, \rho_m, \varepsilon)}$ for Case 2.1.a. Case 2.1.b: Suppose that $[\, \frac{|\omega_i(g)|}{|g|_\rho} \geq (1 + K)\tilde{\varepsilon}$ or $\frac{|\omega_i(g')|}{|g'|_\rho} \geq (1 + K)\tilde{\varepsilon}\,]$ holds for some $i = 1, \ldots, n$. Then those i, for which this condition is true, satisfy together with any $j = 0, \ldots, m$ condition (β), because condition (α) implies

$$\frac{|\omega_i(g)|}{|g|_{\rho_j}} = \frac{|z + w_i - z|}{|g|_{\rho_j}} < \tilde{\varepsilon}^5 + K\tilde{\varepsilon}^5,$$

and analogously $\frac{|\omega_i(g)|}{|g|_{\rho_j}} < \tilde{\varepsilon}^5 + K\tilde{\varepsilon}^5$. So, condition (β) holds. Note that $\frac{|z + w_i|}{|g|_{\rho_j}} < K(\tilde{\varepsilon}^5 + C)$. Therefore, condition (β) implies that $|\frac{1}{|g|_{\rho_j}}(z + w_i) - \frac{1}{|g'|_{\rho_j}}(z' + w_i')| < K(\tilde{\varepsilon}^5 + C)\tilde{\varepsilon}^5 + \tilde{\varepsilon}^5$, and hence $|\frac{\omega_i(g)}{|g|_\rho} - \frac{\omega_i(g')}{|g'|_\rho}| < (K\tilde{\varepsilon}^5 + KC + 1)\tilde{\varepsilon}^5 + 2K\tilde{\varepsilon}^5 < \varepsilon$. We can use the latter to conclude that both $\frac{|\omega_i(g)|}{|g|_{\rho_j}}$ and $\frac{|\omega_i(g')|}{|g'|_{\rho_j}}$ are greater than $\frac{1+K}{2}\tilde{\varepsilon}$, since we have chosen $\tilde{\varepsilon}$ such that $(K\tilde{\varepsilon}^5 + K(C + 2) + 1)\tilde{\varepsilon}^5 < \frac{1+K}{2}\tilde{\varepsilon}$. We can use it furthermore to conclude for at least one $1 \leq i \leq n$ and any $j = 0, \dots, m$ that

$$\begin{aligned}
&\left| \frac{|g|_{\rho_j}}{|g|_\rho} - \frac{|g'|_{\rho_j}}{|g'|_\rho} \right| \\
&= \left| \frac{|g|_{\rho_j}}{|\omega_i(g)|} \frac{|\omega_i(g)|}{|g|_\rho} - \frac{|g'|_{\rho_j}}{|\omega_i(g')|} \frac{|\omega_i(g)|}{|g|_\rho} + \frac{|g'|_{\rho_j}}{|\omega_i(g')|} \frac{|\omega_i(g)|}{|g|_\rho} - \frac{|g'|_{\rho_j}}{|\omega_i(g')|} \frac{|\omega_i(g')|}{|g'|_\rho} \right| \\
&\leq \frac{|\omega_i(g)|}{|g|_\rho} \left| \frac{\frac{|\omega_i(g')|}{|g'|_{\rho_j}} - \frac{|\omega_i(g)|}{|g|_{\rho_j}}}{\frac{|\omega_i(g)|}{|g|_{\rho_j}} \frac{|\omega_i(g')|}{|g'|_{\rho_j}}} \right| + \frac{|g'|_{\rho_j}}{|\omega_i(g')|} \left| \frac{|\omega_i(g)|}{|g|_{\rho_j}} - \frac{|\omega_i(g')|}{|g'|_{\rho_j}} \right| \\
&< \frac{4C\,(K\tilde{\varepsilon}^5 + K(C+2)+1)}{(1+K)^2\tilde{\varepsilon}^2} \tilde{\varepsilon}^5 + \frac{2\,(K\tilde{\varepsilon}^5 + K(C+2)+1)}{(1+K)\tilde{\varepsilon}} \tilde{\varepsilon}^5 < \varepsilon.
\end{aligned}$$

Obviously, we have $|\frac{1}{|g|_\rho} z - \frac{1}{|g'|_\rho} z'| < 2\tilde{\varepsilon}^5 < \varepsilon$; as well as $d(g^+, g'^+) < \varepsilon$; as well as $|\frac{\omega_i(g)}{|g|_\rho} - \frac{\omega_i(g')}{|g'|_\rho}| < 2(1+K)\tilde{\varepsilon} < \varepsilon$ for all those i, for which the above condition is not true. Therefore, $(\tilde{\imath}(g,z), \tilde{\imath}(g',z'))$ lies in $\tilde{V}_{(\rho_1,\dots,\rho_m,\varepsilon)}$ for Case 2.1.b. Case 2.2: Suppose that $\left| \frac{|z|}{|g|_\rho} - \frac{|z'|}{|g'|_\rho} \right| < \tilde{\varepsilon}^5$ and $d(g^+, g'^+) < \tilde{\varepsilon}$ and $d_{S^{k-1}}(z^+, z'^+) < \tilde{\varepsilon}^5$. Then we have $d(g^+, g'^+) < \varepsilon$. Moreover, for each $i = 0, \dots, n$ and each $j = 0, \dots, m$ we get $|\frac{1}{|g|_{\rho_j}}(z + w_i) - \frac{1}{|g'|_{\rho_j}}(z' + w_i')| < 2\tilde{\varepsilon}^5$ if condition (α) holds for i and j, respectively $|\frac{1}{|g|_{\rho_j}}(z + w_i) - \frac{1}{|g'|_{\rho_j}}(z' + w_i')| < (\frac{1}{\tilde{\varepsilon}} + 1)\tilde{\varepsilon}^5$ if condition (β) holds for i and j. Thus, it follows for each $i = 0, \dots, n$ and each $j = 0, \dots, m$ from the triangle inequality that

$$\begin{aligned}
\left| \frac{\omega_i(g)}{|g|_{\rho_j}} - \frac{\omega_i(g')}{|g'|_{\rho_j}} \right| &\leq \left| \frac{1}{|g|_{\rho_j}}(z + w_i) - \frac{1}{|g'|_{\rho_j}}(z' + w_i') \right| + \left| \frac{1}{|g'|_{\rho_j}} z' - \frac{1}{|g|_{\rho_j}} z \right| \\
&\leq 2(1 + \frac{1}{\tilde{\varepsilon}})\tilde{\varepsilon}^5.
\end{aligned}$$

We consider two subcases. Case 2.2.a: Suppose that $[\frac{|z|}{|g|_\rho} \leq \tilde{\varepsilon}$ or $\frac{|z'|}{|g'|_\rho} \leq \tilde{\varepsilon}]$ and $[\frac{|\omega_i(g)|}{|g|_\rho} \leq \tilde{\varepsilon}$ or $\frac{|\omega_i(g')|}{|g'|_\rho} \leq \tilde{\varepsilon}]$ for all $i = 1, \dots, n$. Then the triangle inequality implies that both $\frac{|z|}{|g|_\rho}$ and $\frac{|z'|}{|g'|_\rho}$ are less than $\tilde{\varepsilon} + (\frac{1}{\tilde{\varepsilon}} + 1)\tilde{\varepsilon}^5 < \varepsilon$; and both $\frac{|\omega_i(g)|}{|g|_\rho}$ and $\frac{|\omega_i(g')|}{|g'|_\rho}$ are less than $\tilde{\varepsilon} + 2(\frac{1}{\tilde{\varepsilon}} + 1)\tilde{\varepsilon}^5 < \varepsilon$. Therefore, $(\tilde{\imath}(g,z), \tilde{\imath}(g',z'))$ lies in $\tilde{V}_{(\rho_1,\dots,\rho_m,\varepsilon)}$ for Case 2.2.a. Case 2.2.b: Suppose that $[\frac{|z|}{|g|_\rho} > \tilde{\varepsilon}$ and $\frac{|z'|}{|g'|_\rho} > \tilde{\varepsilon}]$ or $[\frac{|\omega_i(g)|}{|g|_\rho} > \tilde{\varepsilon}$ and $\frac{|\omega_i(g')|}{|g'|_\rho} > \tilde{\varepsilon}]$ for some $i = 1, \dots, n$. If $|\frac{z}{|g|_\rho}| \geq \tilde{\varepsilon}$ and $|\frac{z'}{|g'|_\rho}| \geq \tilde{\varepsilon}$ holds, we deduce for each $j = 1, \dots, m$ that

$$\begin{aligned}
\Big| \frac{|g|_{\rho_j}}{|g|_\rho} - \frac{|g'|_{\rho_j}}{|g'|_\rho} \Big| &= \Big| \frac{z}{|g|_\rho} \frac{|g|_{\rho_j}}{z} - \frac{z}{|g|_\rho} \frac{|g'|_{\rho_j}}{z'} + \frac{z}{|g|_\rho} \frac{|g'|_{\rho_j}}{z'} - \frac{z'}{|g'|_\rho} \frac{|g'|_{\rho_j}}{z'} \Big| \\
&\leq \Big| \frac{|g|_{\rho_j}}{z} - \frac{|g'|_{\rho_j}}{z'} \Big| \, \Big| \frac{z}{|g|_\rho} \Big| + \Big| \frac{z}{|g|_\rho} - \frac{z'}{|g'|_\rho} \Big| \, \Big| \frac{|g'|_{\rho_j}}{z'} \Big| \\
&\leq \Big| \frac{\frac{z'}{|g'|_{\rho_j}} - \frac{z}{|g|_{\rho_j}}}{\frac{z}{|g|_{\rho_j}} \frac{z'}{|g'|_{\rho_j}}} \Big| \, \Big| \frac{z}{|g|_\rho} \Big| + \Big| \frac{z}{|g|_\rho} - \frac{z'}{|g'|_\rho} \Big| \, \Big| \frac{|g'|_{\rho_j}}{z'} \Big| \\
&\leq \frac{K^2}{\tilde{\varepsilon}^2} (1 + \tfrac{1}{\tilde{\varepsilon}}) \tilde{\varepsilon}^5 \Big(\frac{1}{K \tilde{\varepsilon}} - C \Big) + (1 + \tfrac{1}{\tilde{\varepsilon}}) \tilde{\varepsilon}^5 \frac{K}{\tilde{\varepsilon}} < \varepsilon.
\end{aligned}$$

Analogously, if $|\frac{\omega_i(g)}{|g|_\rho}| \geq \tilde{\varepsilon}$ and $|\frac{\omega_i(g')}{|g'|_\rho}| \geq \tilde{\varepsilon}$ holds for some i, we deduce for each $j = 1, \dots, m$ that

$$\Big| \frac{|g|_{\rho_j}}{|g|_\rho} - \frac{|g'|_{\rho_j}}{|g'|_\rho} \Big| < \frac{K^2}{\tilde{\varepsilon}^2} 2 (1 + \tfrac{1}{\tilde{\varepsilon}}) \tilde{\varepsilon}^5 C + 2 (1 + \tfrac{1}{\tilde{\varepsilon}}) \tilde{\varepsilon}^5 \frac{K}{\tilde{\varepsilon}} < \varepsilon.$$

Thus, $(\bar{\psi}(\iota(g,z)), \bar{\psi}(\iota(g',z')))$ lies in $\tilde{V}_{(\rho_1,\dots,\rho_m,p)}$ in Case 2.2.b, too. This completes the proof of Claim 3.

We prove the statements (i) to (iii) of Theorem 4.0.9: The subset $\iota\,((G \times \mathbb{Z}^k)^\infty)$ is dense in $\partial(G \times \mathbb{Z}^k)$ by construction, and the subset $\tilde{\iota}\,((G \times \mathbb{Z}^k)^\infty)$ is dense in M_ρ by Lemma 4.0.10. Therefore, the Claims 1 to 3 imply that ψ' extends to an isomorphism from M_ρ onto $\partial(G \times \mathbb{Z}^k)$. Since this extension is unique, it must coincide with ψ. Hence, ψ is an isomorphism. The statements (ii) and (iii) are an obvious consequence.

It remains to prove statement (iv) for $G \times \mathbb{Z}^k$-actions of the type $\tilde{\rho}_\mu$, where $\tilde{\rho} \in \Gamma$ is a G-factor action and $\mu \in \mathbb{R}^{k \times n}$ is a matrix of shift parameters. Let $\tilde{\rho} \in \Gamma$ and $\mu \in \mathbb{R}^{k \times n}$ be given. We want to show that the canonical map $\hat{\tau}_{\tilde{\rho}_\mu}$ from $\partial(G \times \mathbb{Z}^k)$ to $\partial G * S^{k-1}$ is a homotopy equivalence. According to Lemma 3.4.5, there exists a homotopy $H : [0,1] \times C_\rho \to C_\rho$ that contracts C_ρ to a basepoint $\bar{c} \in C_\rho$. We define a map $f : \partial G * S^{k-1} \to \partial(G \times \mathbb{Z}^k)$ by

$$[t, \xi, \zeta] \mapsto f([t, \xi, \zeta]) := [v(t, \zeta), \xi, \bar{c}],$$

where $v(t,\zeta) \in \overline{\mathbb{R}^k}$ is defined by $|v(t,\zeta)| = t$ and $v(t,\zeta)^+ = \zeta$. Obviously, f is continuous. We show that f is a homotopy inverse for $\hat{\tau}_{\bar{\rho}_\mu}$. Firstly, we check that $\hat{\tau}_{\bar{\rho}_\mu} \circ f$ is homotopic to $\mathrm{id}_{\partial G * S^{k-1}}$. Define $F_1 : [0,1] \times \partial G * S^{k-1} \to \partial G * S^{k-1}$ by

$$F_1(\vartheta, [t,\xi,\zeta]) := [(1-\vartheta)\,t + \vartheta\,\frac{|v(t,\zeta) + \mu \cdot \bar{c}^s|}{\bar{c}_{\bar{\rho}}}\,,\,\xi\,,\,(v(t,\zeta) + \vartheta\,\mu \cdot \bar{c}^s)^+].$$

Clearly, F_1 is continuous. Moreover, we have $F_1(0,[t,\xi,\zeta]) = [t,\xi,\zeta]$, as well as $F_1(1,[t,\xi,\zeta]) = [\frac{|v(t,\zeta)+\mu\cdot c^s|}{\bar{c}_{\bar{\rho}}}, \xi, (v(t,\zeta) + \mu \cdot \bar{c}^s)^+] = (\hat{\tau}_{\bar{\rho}_\mu} \circ f)([t,\xi])$. Secondly, we check that $f \circ \hat{\tau}_{\bar{\rho}_\mu}$ is homotopic to $\mathrm{id}_{\partial(G\times\mathbb{Z}^k)}$. Define $F_2 : [0,1] \times \partial(G \times \mathbb{Z}^k) \to \partial(G \times \mathbb{Z}^k)$ by

$$F_2(\vartheta, [v,\xi,c]) := [v((1-\vartheta)\,|v| + \vartheta\,\frac{|v + \mu \cdot c^s|}{c_{\bar{\rho}}}, (v + \vartheta\,\mu \cdot c^s)^+)\,,\,\xi\,,\,H(\vartheta, c)].$$

Note that there exists a constant $K \geq 1$ such that $\frac{1}{K} < c_{\bar{\rho}} < K$ for all $c \in C_\rho$. Hence, F_2 is continuous. Moreover, we have $F_2(0,[v,\xi,c]) = [v,\xi,c]$, as well as $F_2(1,[v,\xi,c]) = [v(\frac{|v+\mu\cdot c^s|}{c_{\bar{\rho}}}, (v + \mu \cdot c^s)^+), \xi, \bar{c}] = (f \circ \hat{\tau}_{\bar{\rho}_\mu})([v,\xi,c])$. This completes the proof of statement (iv). $\square$

Chapter 5

The Boundary of G × H

The aim of this chapter is to work out the boundary of groups $G \times H$, where both G and H are non-elementary word-hyperbolic CAT(0) groups. Furthermore, we will show that for each geometric action of $G \times H$ the associated canonical map from $\partial(G \times H)$ into the visual boundary of the underlying CAT(0) space is a homotopy equivalence.

The argumentation in this chapter is very similar to that in the previous two chapters. We begin with some notation: Throughout this section we consider two fixed non-elementary word-hyperbolic CAT(0) groups G and H. Let Γ denote the set of all geometric actions by G on CAT(0) spaces, and Θ the set of all geometric actions by H on CAT(0) spaces. For any action $\rho \in \Gamma$ on a CAT(0) space X, and any action $\sigma \in \Theta$ on a CAT(0) space Y, we define a geometric $G \times H$-action $\rho \times \sigma$ on $X \times Y$ by

$$(g,h) * (x,y) := (\rho(g,x), \sigma(h,y)).$$

We can identify $\partial(X \times Y)$ with the join $\partial G * \partial H = ([0,\infty] \times \partial G \times \partial H)/\sim$, where $(t,\xi,\zeta) \sim (t',\xi',\zeta')$ if and only if $[t = t' = 0$ and $\xi = \xi']$, or $[t = t' = \infty$ and $\zeta = \zeta']$, or $[t = t'$ and $\xi = \xi'$ and $\zeta = \zeta']$; such that with respect to this identification the canonical map $\tau_{\rho\times\sigma} : (G \times H)^\infty \to \partial(X \times Y)$ associated to $\rho \times \sigma$

is given by

$$(g,h) \mapsto \tau_{\rho\times\sigma}(g,h) := \begin{cases} [\frac{|h|_\sigma}{|g|_\rho}, g^+, h^+], & \text{if } g \in G^\infty \text{ and } h \in H^\infty, \\ [0, g^+, \zeta], & \text{if } g \in G^\infty \text{ and } h \notin H^\infty, \\ [\infty, \xi, h^+], & \text{if } g \notin G^\infty \text{ and } h \in H^\infty, \end{cases}$$

where $\xi \in \partial G$ and $\zeta \in \partial H$ are arbitrary. Let $U_{\rho\times\sigma}$ denote the uniform structure on $(G \times H)^\infty$ that is the inverse image of the canonical uniformity on $\partial(X \times Y)$ under $\tau_{\rho\times\sigma}$.

The following proposition shows that in order to study the boundary of $G\times H$ it is enough to consider $G \times H$-actions of the above type. It is motivated by the result by Ruane in [Rua99], namely that the visual boundary of a CAT(0) space carrying a geometric $G \times H$-action is homeomorphic to $\partial G * \partial H$.

Proposition 5.0.11 *Let $\bar{\rho}$ be a geometric action of $G \times H$ on a CAT(0) space $\bar{X}$. Then $\partial\bar{X}$ can be identified with $\partial G * \partial H$. Moreover, there exists a G-action $\rho \in \Gamma$ on a CAT(0) space X, and an H-action $\sigma \in \Theta$ on a CAT(0) space Y, such that*

$$\begin{array}{ccc} ((G\times H)^\infty, U_{\bar{\rho}}) & \xrightarrow{\mathrm{id}_{(G\times H)^\infty}} & ((G\times H)^\infty, U_{\rho\times\sigma}) \\ \tau_{\bar{\rho}}\downarrow & & \downarrow\tau_{\rho\times\sigma} \\ \{(g,h)^+ \mid (g,h) \in (G\times H)^\infty\} & \xrightarrow{\mathrm{id}_{\partial G * \partial H}} & \{(g,h)^+ \mid (g,h) \in (G\times H)^\infty\} \\ \subset\downarrow & & \downarrow\subset \\ \partial\bar{X} \equiv \partial G * \partial H & \xrightarrow{\mathrm{id}_{\partial G * \partial H}} & \partial(X\times Y) \equiv \partial G * \partial H \end{array}$$

is a commutative diagram of uniformly continuous maps. In particular, the uniformities $U_{\bar{\rho}}$ and $U_{\rho\times\sigma}$ on $(G \times H)^\infty$ coincide.

To prove this proposition we need the following lemma.

Lemma 5.0.12 *Let X be a $\bar{\delta}$-hyperbolic CAT(0) space. Then the Hausdorff distance between any two asymptotic geodesic lines $c : \mathbb{R} \to X$ and $c' : \mathbb{R} \to X$ is less than $\bar{\delta} + 1$.*

Proof: By the Flat Strip Theorem 1.1.4 the convex hull of $c(\mathbb{R}) \cup c'(\mathbb{R})$ is isometric to a euclidean strip $\mathbb{R} \times [0, w]$ of width w. Obviously, if $w \geq \bar{\delta} + 1$ we can find a geodesic triangle in this strip such that the $\bar{\delta}$-neighbourhood of two of its sides does not cover the third one. This contradicts the hypothesis that X is $\bar{\delta}$-hyperbolic. □.

Proof of Prop. 5.0.11: According to Lemma 3.4.2 the centre of G is finite. Therefore, we can apply the Splitting Theorem 1.2.1: There exists a non-empty, closed, convex, $G \times H$-invariant subspace $\bar{X}_1 \times \bar{X}_2$ of $\bar{X}$ such that G acts geometrically on $\bar{X}_1$ and by (possibly trivial) Clifford translations on $\bar{X}_2$, and H acts geometrically on $\bar{X}_2$ and trivially on $\bar{X}_1$. Let $\bar{x}$ be a basepoint in $\bar{X}_1 \times \bar{X}_2$. Firstly, note that any infinite geodesic ray issuing from $\bar{x}$ lies entirely in $\bar{X}_1 \times \bar{X}_2$. Therefore, the visual boundary $\partial \bar{X}$ coincides with $\partial(\bar{X}_1 \times \bar{X}_2)$, which allows us to identify $\partial \bar{X}$ with $\partial G * \partial H$. Secondly, we check that the action of G on $\bar{X}_2$ is indeed trivial. Suppose there exists a $g \in G$ that acts as a non-trivial Clifford translation on $\bar{X}_2$. Then, according to Thm. 1.1.3, $\bar{X}_2$ splits as a product $\bar{X}_2 = Y_2 \times \mathbb{R}$. If Y_2 has finite diameter, the visual boundary $\partial \bar{X}_2$ consists of two points. Since $\partial \bar{X}_2 \cong \partial H$, this implies that H is elementary, which contradicts our hypothesis. If the diameter of Y_2 is infinite, then $\bar{X}_2$ contains arbitrarily broad Euclidean strips. But this contradicts Lemma 5.0.12, because $\bar{X}_2$ is δ'-hyperbolic for some $\delta' \geq 0$. Thus, the action of G on $\bar{X}_2$ is trivial. Thirdly, set $X := \bar{X}_1$, and let ρ be the induced geometric G-action on X. Set $Y := \bar{X}_2$, and let σ be the induced geometric H-action on Y. Then it follows immediately from the above that the canonical map $\tau_{\rho\times\sigma}$ from $(G \times H)^\infty$ into $\partial(X \times Y) = \partial \bar{X} \equiv \partial G * \partial H$ associated to the product action $\rho \times \sigma$ coincides with $\tau_{\bar{\rho}}$. □

We will call a pair (ρ, σ) of actions as given by Prop. 5.0.11 a *pair of factor actions* associated to the $G \times H$-action $\bar{\rho}$. Note that similarly to the proof of Prop. 3.3.2 one can show the following: Let ρ and $\tilde{\rho}$ be two G-actions in Γ; and let σ and $\tilde{\sigma}$ be two H-actions in Θ. Then the uniformities $U_{\rho\times\sigma}$ and $U_{\tilde{\rho}\times\tilde{\sigma}}$ on $(G \times H)^\infty$ coincide if and only if there are constants $c_1, c_2 > 0$ such that $MLS_\rho = c_1 \cdot MLS_{\tilde{\rho}}$ and $MLS_\sigma = c_2 \cdot MLS_{\tilde{\sigma}}$. However: Let $c_1 \cdot \rho$ be the geometric G-action obtained by rescaling the CAT(0) space that carries ρ, and let $c_2 \cdot \sigma$ be the geometric H-action obtained by rescaling the CAT(0) space that carries σ.

Then according to the above definition the pair $(c_1 \cdot \rho, c_2 \cdot \sigma)$ is not a pair of factor actions associated to the $G \times H$-action $\rho \times \sigma$ unless $c_1 = c_2$.

For the remainder of this section we fix a G-action $\rho \in \Gamma$ and an H-action $\sigma \in \Theta$. Recall from Chapter 3 that the maps $\kappa_\rho : G^\infty \to \mathbb{R}^\Gamma$ and $\kappa_\sigma : H^\infty \to \mathbb{R}^\Theta$ are given by

$$g \mapsto \kappa_\rho(g) := \Big(\frac{|g|_{\tilde\rho}}{|g|_\rho}\Big)_{\tilde\rho}, \qquad \text{and} \qquad h \mapsto \kappa_\sigma(h) := \Big(\frac{|h|_{\tilde\sigma}}{|h|_\sigma}\Big)_{\tilde\sigma}.$$

As before, we denote the closure of the image $\operatorname{im}\kappa_\rho$ in $\mathbb{R}^\Gamma$ by C_ρ^{ts}, and the closure of the image $\operatorname{im}\kappa_\sigma$ in $\mathbb{R}^\Theta$ by C_σ^{ts}. For each $\tilde\rho \in \Gamma$ (resp. for each $\tilde\sigma \in \Theta$) let $\mathrm{pr}_{\tilde\rho}$ (resp. $\mathrm{pr}_{\tilde\sigma}$) be the projection of $\mathbb{R}^\Gamma$ (resp. $\mathbb{R}^\Theta$) onto the $\tilde\rho$-component of $\mathbb{R}^\Gamma$ (resp. onto the $\tilde\sigma$-component of $\mathbb{R}^\Theta$). For each $b \in C_\rho^{ts}$ (resp. for each $c \in C_\sigma^{ts}$) we abbreviate $\mathrm{pr}_{\tilde\rho}(b)$ by $b_{\tilde\rho}$ (resp. $\mathrm{pr}_{\tilde\sigma}(c)$ by $c_{\tilde\sigma}$).

Definition 5.0.13 The *$\rho \times \sigma$-model $M_{\rho\times\sigma}$* of the boundary $\partial(G \times H)$ is defined by $M_{\rho\times\sigma} := ([0,\infty] \times \partial G \times \partial H \times C_\rho^{ts} \times C_\sigma^{ts})/\sim$, where $(t,\xi,\zeta,b,c) \sim (t',\xi',\zeta',b',c')$ if and only if $[\,t = t' = 0$ and $\xi = \xi'\,]$, or $[\,t = t' = \infty$ and $\zeta = \zeta'\,]$, or $[\,t = t'$ and $\xi = \xi'$ and $\zeta = \zeta'$ and $b = b'$ and $c = c'\,]$.

In the remainder of this chapter we will prove the following theorem:

Theorem 5.0.14 *Let G and H be non-elementary word-hyperbolic CAT(0) groups. Suppose that ρ is a geometric G-action on a CAT(0) space, and σ a geometric H-action on a CAT(0) space. Let $M_{\rho\times\sigma}$ be defined as above. Then the following is true:*

(i) The boundary $\partial(G \times H)$ is canonically isomorphic to $M_{\rho\times\sigma}$.

(ii) The canonical $G \times H$-action on $\partial(G \times H)$ is given by

$$(g,h) * [t,\xi,\zeta,b,c] \;=\; [t, g.\xi, h.\zeta, b, c].$$

(iii) Let $\bar\rho$ be a geometric $G \times H$-action on a CAT(0) space $\bar X$. Let $\tilde\rho \in \Gamma$ and $\tilde\sigma \in \Theta$ such that $(\tilde\rho,\tilde\sigma)$ is a pair of factor actions associated to $\bar\rho$. Then the

canonical map $\hat{\tau}_{\bar{\rho}}$ from $\partial(G \times H)$ to $\partial\bar{X} \equiv \partial G * \partial H$ is given by:

$$\hat{\tau}_{\bar{\rho}}([t,\xi,\zeta,b,c]) = \left[\frac{c_{\bar{\sigma}}}{b_{\bar{\rho}}}t,\xi,\zeta\right].$$

(iv) For any geometric $G \times H$-action $\bar{\rho}$ on a CAT(0) space $\bar{X}$ the canonical map $\hat{\tau}_{\bar{\rho}}$ from $\partial(G \times H)$ to $\partial\bar{X}$ is a homotopy equivalence.

We need to prepare the proof of this theorem: In order to compare $\partial(G \times H)$ with $M_{\rho\times\sigma}$ we define a map $\tilde{\iota} : (G \times H)^\infty \to M_{\rho\times\sigma}$ by

$$(g,h) \mapsto \tilde{\iota}(g,h) := \begin{cases} [\frac{|h|_\sigma}{|g|_\rho}, g^+, h^+, \kappa_\rho(g), \kappa_\sigma(h)], & \text{if } g \in G^\infty \text{ and } h \in H^\infty, \\ [0, g^+, \zeta, b, c], & \text{if } g \in G^\infty \text{ and } h \notin H^\infty, \\ [\infty, \xi, h^+, b, c], & \text{if } g \notin G^\infty \text{ and } h \in H^\infty, \end{cases}$$

where $\xi \in \partial G$, $\zeta \in \partial H$, $b \in C_\rho^{ts}$ and $c \in C_\sigma^{ts}$ are arbitrary.

Lemma 5.0.15 *The image $\tilde{\iota}((G \times H)^\infty)$ is dense in $M_{\rho\times\sigma}$.*

Proof: In order to check that $\tilde{\iota}((G \times H)^\infty)$ is dense in $M_{\rho\times\sigma}$, it is enough to check that each open subset of the form $(I \times B_{\partial G}(\xi,\varepsilon) \times B_{\partial H}(\zeta,\varepsilon) \times U \times V)/\sim$ in $M_{\rho\times\sigma} = ([0,\infty] \times \partial G \times \partial H \times C_\rho^{ts} \times C_\sigma^{ts})/\sim$ contains an element of $\tilde{\iota}((G \times \mathbb{Z})^\infty)$; where I is an open interval in $[0,\infty]$, $B_{\partial G}(\xi,\varepsilon)$ is an open ball of radius $\varepsilon > 0$ around a point ξ in ∂G with respect to some fixed visual metric $d_{\partial G}$, $B_{\partial H}(\zeta,\varepsilon)$ is an open ball of radius $\varepsilon > 0$ around a point ζ in ∂H with respect to some fixed visual metric $d_{\partial H}$, U is an open set in C_ρ^{ts}, and V is an open set in C_σ^{ts}. Firstly, we find a $g \in G^\infty$ such that $\kappa_\rho(g) \in U$ and $g^+ \in B_{\partial G}(\xi,\varepsilon)$: By construction $\kappa_\rho(G^\infty)$ is dense in C_ρ^{ts}. Therefore, there exists a $\tilde{g} \in G^\infty$ such that $\kappa_\rho(\tilde{g}) \in U$. According to [BR96b], for example, the set of rational boundary points is dense in ∂G. Hence, there exists a $g_1 \in G^\infty$ such that $d(\xi, g_1^+) < \frac{\varepsilon}{2}$. Without loss of generality we can assume that $\tilde{g}^+ \neq g_1^-$. For otherwise Lemma 1.3.2 implies that g_1^+ is the only rational point in $B_{\partial G}(\xi, \frac{\varepsilon}{2})$, which contradicts G being non-elementary. According to [CDP90], Ch. 11, Prop. 2.4, for example, $\tilde{g}^+ \neq g_1^-$ implies that there exists an $m \in \mathbb{N}$ such that $d(g_1^+, g_1^m.\tilde{g}^+) < \frac{\varepsilon}{2}$. Clearly, $\kappa_\rho(\tilde{g}) = \kappa_\rho(g_1^m \tilde{g} g_1^{-m})$

and $g_1^m.\tilde{g}^+ = (g_1^m \tilde{g} g_1^{-m})^+$. Therefore, we can take $g := g_1^m \tilde{g} g_1^{-m}$. Analogously, we find an $h \in H^\infty$ such that $\kappa_\sigma(h) \in V$ and $h^+ \in B_{\partial H}(\zeta, \varepsilon)$. Obviously, there exist $p, q \in \mathbb{N}$ such that $\frac{p|h|_\sigma}{q|g|_\rho}$ lies in I. Since we have $(g^q)^+ = g^+$ and $\kappa_\rho(g^q) = \kappa_\rho(g)$, as well as $(h^p)^+ = h^+$ and $\kappa_\sigma(h^p) = \kappa_\sigma(h)$, $\tilde{\iota}(g^q, h^p)$ lies in $(I \times B_{\partial G}(\xi, \varepsilon) \times B_{\partial H}(\zeta, \varepsilon) \times U \times V)/\sim$. □

Proof of Theorem 5.0.14: According to Prop. 5.0.11 it is enough to prove the statements (iii) and (iv) of Theorem 5.0.14 for $G \times H$-actions of the type $\tilde{\rho} \times \tilde{\sigma}$, where $\tilde{\rho} \in \Gamma$ and $\tilde{\sigma} \in \Theta$. Furthermore, we can regard $\partial(G \times H)$ as the closure of the image $\mathrm{im}((\tau_{\tilde{\rho}\times\tilde{\sigma}})_{\tilde{\rho}\times\tilde{\sigma}})$ in $\prod_{\tilde{\rho}\times\tilde{\sigma}\in\Gamma\times\Theta} \partial G * \partial H$ throughout this proof, since the boundary uniformity $U_{G\times H}$ on $(G \times H)^\infty$ is the same as the least upper bound of the family $\{ U_{\tilde{\rho}\times\tilde{\sigma}} \mid \tilde{\rho} \in \Gamma, \tilde{\sigma} \in \Theta \}$. Thus, the product map $(\tau_{\tilde{\rho}\times\tilde{\sigma}})_{\tilde{\rho}\times\tilde{\sigma}}$ is the canonical map ι from $((G \times H)^\infty, U_{G\times H})$ into its Hausdorff completion. For each $\tilde{\rho} \in \Gamma$ and each $\tilde{\sigma} \in \Theta$ we define a map $\tilde{\tau}_{\tilde{\rho}\times\tilde{\sigma}}$ from $M_{\rho\times\sigma}$ to $\partial G \times H$ by

$$[t, \xi, \zeta, b, c] \mapsto \tilde{\tau}_{\tilde{\rho}\times\tilde{\sigma}}([t, \xi, \zeta, b, c]) := [\frac{c_{\tilde{\sigma}}}{b_{\tilde{\rho}}} t\, ,\, \xi\, ,\, \zeta\,].$$

It is easy to check that

$$\begin{tikzcd}
(G \times H)^\infty \arrow[r, "\tilde{\iota}"] \arrow[d, "\iota = (\tau_{\tilde{\rho}\times\tilde{\sigma}})_{\tilde{\rho}\times\tilde{\sigma}}"'] & M_{\rho\times\sigma} \arrow[d, "\tilde{\tau}_{\tilde{\rho}\times\tilde{\sigma}}"] \\
\partial(G \times H) \arrow[r, "pr_{\tilde{\rho}\times\tilde{\sigma}}"] & \partial G * \partial H
\end{tikzcd}$$

is a commuting diagram of set maps. Let $\psi := (\tilde{\tau}_{\tilde{\rho}\times\tilde{\sigma}})_{\tilde{\rho}\times\tilde{\sigma}}$ be the product map from $M_{\rho\times\sigma}$ to $\prod_{\tilde{\rho}\times\tilde{\sigma}\in\Gamma\times\Theta} \partial G * \partial H$, and let ψ' be the restriction of ψ to $\tilde{\iota}((G \times H)^\infty)$ in $M_{\rho\times\sigma}$.

Claim 1: ψ is uniformly continuous.

Proof of Claim 1: It is enough to show that $\tilde{\tau}_{\tilde{\rho}\times\tilde{\sigma}}$ is uniformly continuous for each $G \times H$-action $\tilde{\rho} \times \tilde{\sigma} \in \Gamma \times \Theta$. Throughout this proof of Claim 1, let $d_{\partial G}$ be a fixed visual metric on ∂G, and $d_{\partial H}$ a fixed visual metric on ∂H. We consider the following fundamental systems of entourages for the uniform structure on $\partial G * \partial H$ and on $M_{\rho\times\sigma}$: For any ε, with $1 > \varepsilon > 0$, let W_ε consist precisely of those pairs $([t, \xi, \zeta], [t', \xi', \zeta'])$ in $(\partial G * \partial H) \times (\partial G * \partial H)$ that satisfy $[\,|t - t'| < \varepsilon$

and $d_{\partial G}(\xi,\xi') < \varepsilon$ and $d_{\partial H}(\zeta,\zeta') < \varepsilon$], or [$t < \varepsilon$ and $t' < \varepsilon$ and $d_{\partial G}(\xi,\xi') < \varepsilon$], or [$t > \frac{1}{\varepsilon}$ and $t' > \frac{1}{\varepsilon}$ and $d_{\partial H}(\zeta,\zeta') < \varepsilon$]. Then it is straightforward to check that the system $\{W_\varepsilon\}$ is a fundamental system of entourages for $\partial G * \partial H$. For any ε, with $1 > \varepsilon > 0$, any finite number $\rho_1,\dots,\rho_n$ of G-actions in Γ, and any finite number $\sigma_1,\dots,\sigma_m$ of H-actions in Θ let $V_{(\rho_1,\dots,\rho_n,\sigma_1,\dots,\sigma_m,\varepsilon)}$ consist precisely of those pairs $([t,\xi,\zeta,b,c],[t',\xi',\zeta',b',c'])$ in $M_{\rho\times\sigma} \times M_{\rho\times\sigma}$ that satisfy [$t < \varepsilon$ and $t' < \varepsilon$ and $d_{\partial G}(\xi,\xi') < \varepsilon$], or [$t > \frac{1}{\varepsilon}$ and $t' > \frac{1}{\varepsilon}$ and $d_{\partial H}(\zeta,\zeta') < \varepsilon$], or [$|t-t'| < \varepsilon$ and $d_{\partial G}(\xi,\xi') < \varepsilon$ and $d_{\partial H}(\zeta,\zeta') < \varepsilon$ and $|b_{\rho_i} - b'_{\rho_i}| < \varepsilon^2$ for all $i = 1,\dots,n$ and $|c_{\sigma_j} - c'_{\sigma_j}| < \varepsilon^2$ for all $j = 1,\dots,m$]. Analogously to the proof of Lemma 3.2.5 it is straightforward to check that the system $\{V_{(\rho_1,\dots,\rho_n,\sigma_1,\dots,\sigma_m,\varepsilon)}\}$ is a fundamental system of entourages for the uniformity on $M_{\rho\times\sigma}$. Let $\tilde\rho \times \tilde\sigma$ in $\Gamma \times \Theta$ and ε with $1 > \varepsilon > 0$ be given. In order to prove that $\tilde\tau_{\tilde\rho\times\tilde\sigma}$ is uniformly continuous it is enough to check that there exists an $1 > \tilde\varepsilon > 0$ such that $\tilde\tau_{\tilde\rho\times\tilde\sigma} \times \tilde\tau_{\tilde\rho\times\tilde\sigma}$ maps $V_{(\tilde\rho,\tilde\sigma,\tilde\varepsilon)}$ into W_ε. Let $K \geq 1$ be such that $\frac{1}{K} < b_{\tilde\rho} < K$ and $\frac{1}{K} < c_{\tilde\sigma} < K$ for any $b \in C^{ts}_\rho$, for any $c \in C^{ts}_\sigma$. Take $1 > \tilde\varepsilon > 0$ small enough such that $3K^3\tilde\varepsilon < \varepsilon$ and $\tilde\varepsilon < \frac{\varepsilon}{K^2+\varepsilon}$. Suppose that $([t,\xi,\zeta,b,c],[t',\xi',\zeta',b',c'])$ lies in $V_{(\tilde\rho,\tilde\sigma,\tilde\varepsilon)}$. Then there are three cases to consider:

Case 1: Suppose $t < \tilde\varepsilon$ and $t' < \tilde\varepsilon$ and $d_{\partial G}(\xi,\xi') < \tilde\varepsilon$. Then we have $\frac{c_{\tilde\sigma}}{b_{\tilde\rho}}t < K^2\tilde\varepsilon < \varepsilon$, and analogously $\frac{c'_{\tilde\sigma}}{b'_{\tilde\rho}}t' < \varepsilon$. Moreover, we have $d_{\partial G}(\xi,\xi') < \varepsilon$. Therefore, $(\tilde\tau_{\tilde\rho\times\tilde\sigma}([t,\xi,\zeta,b,c]),\tilde\tau_{\tilde\rho\times\tilde\sigma}([t',\xi',\zeta',b',c']))$ lies in W_ε for Case 1.

Case 2: Suppose $t > \frac{1}{\tilde\varepsilon}$, and $t' > \frac{1}{\tilde\varepsilon}$ and $d_{\partial H}(\zeta,\zeta') < \tilde\varepsilon$. Then we have $\frac{c_{\tilde\sigma}}{b_{\tilde\rho}}t > \frac{1}{K^2\tilde\varepsilon} > \frac{1}{\varepsilon}$ and analogously $\frac{c'_{\tilde\sigma}}{b'_{\tilde\rho}}t > \frac{1}{\varepsilon}$. Moreover, we have $d_{\partial H}(\zeta,\zeta') < \varepsilon$. Therefore, $(\tilde\tau_{\tilde\rho\times\tilde\sigma}([t,\xi,\zeta,b,c]),\tilde\tau_{\tilde\rho\times\tilde\sigma}([t',\xi',\zeta',b',c']))$ lies in W_ε for Case 2.

Case 3: Suppose $|t-t'| < \tilde\varepsilon$ and $d_{\partial G}(\xi,\xi') < \tilde\varepsilon$ and $d_{\partial H}(\zeta,\zeta') < \tilde\varepsilon$ and $|b_{\tilde\rho} - b'_{\tilde\rho}| < \tilde\varepsilon^2$ and $|c_{\tilde\sigma} - c'_{\tilde\sigma}| < \tilde\varepsilon^2$. Then we have $d_{\partial G}(\xi,\xi') < \varepsilon$ and $d_{\partial H}(\zeta,\zeta') < \varepsilon$. We can assume without loss of generality that $t < \frac{1}{\tilde\varepsilon}$. For otherwise the triangle inequality implies that both t and t' are greater than $\frac{1}{\tilde\varepsilon} - 1$, from which we conclude analogously to Case 2 that $(\tilde\tau_{\tilde\rho\times\tilde\sigma}([t,\xi,\zeta,b,c]),\tilde\tau_{\tilde\rho\times\tilde\sigma}([t',\xi',\zeta',b',c']))$ lies in W_ε. Assuming $t < \frac{1}{\tilde{\varepsilon}}$, we get

$$\begin{aligned}
\left| \frac{c_{\tilde{\sigma}}}{b_{\tilde{\rho}}} t - \frac{c'_{\tilde{\sigma}}}{b'_{\tilde{\rho}}} t' \right| &= \left| \frac{c_{\tilde{\sigma}}}{b_{\tilde{\rho}}} t - \frac{c_{\tilde{\sigma}}}{b'_{\tilde{\rho}}} t + \frac{c_{\tilde{\sigma}}}{b'_{\tilde{\rho}}} t - \frac{c'_{\tilde{\sigma}}}{b'_{\tilde{\rho}}} t' \right| \\
&\leq t\, c_{\tilde{\sigma}} \left| \frac{1}{b_{\tilde{\rho}}} - \frac{1}{b'_{\tilde{\rho}}} \right| + \frac{1}{b'_{\tilde{\rho}}} \left| t\, c_{\tilde{\sigma}} - t\, c'_{\tilde{\sigma}} + t\, c'_{\tilde{\sigma}} - t'\, c'_{\tilde{\sigma}} \right| \\
&\leq t\, c_{\tilde{\sigma}} \left| \frac{b'_{\tilde{\rho}} - b_{\tilde{\rho}}}{b_{\tilde{\rho}} b'_{\tilde{\rho}}} \right| + \frac{1}{b'_{\tilde{\rho}}} \left(t \left| c_{\tilde{\sigma}} - c'_{\tilde{\sigma}} \right| + c'_{\tilde{\sigma}} \left| t - t' \right| \right) \\
&< \frac{K}{\tilde{\varepsilon}} K^2 \tilde{\varepsilon}^2 + K \left(\frac{1}{\tilde{\varepsilon}} \tilde{\varepsilon}^2 + K \tilde{\varepsilon} \right) < \varepsilon.
\end{aligned}$$

Hence, $(\tilde{\tau}_{\tilde{\rho}\times\tilde{\sigma}}([t,\xi,\zeta,b,c]), \tilde{\tau}_{\tilde{\rho}\times\tilde{\sigma}}([t',\xi',\zeta',b',c']))$ lies in W_ε for Case 3, too. This completes the proof of Claim 1.

Claim 2: ψ' is a bijection from $\tilde{\iota}((G \times H)^\infty)$ onto $\iota((G \times H)^\infty)$.

Proof of Claim 2: We show that ψ' has an inverse. Define a map $\bar{\psi}$ from $\iota((G \times H)^\infty)$ to $\tilde{\iota}((G \times H)^\infty)$ as follows: Let $\iota(g,h) \in \iota((G \times H)^\infty)$ be given, and set

$$\iota(g,h) \mapsto \bar{\psi}(\iota(g,h)) := \tilde{\iota}(g,h).$$

We check that $\bar{\psi}$ is well-defined: Let $(g',h') \in (G \times H)^\infty$ be such that $\iota(g,h) = \iota(g',h')$, i.e. $\tau_{\tilde{\rho}\times\tilde{\sigma}}(g,h) = \tau_{\tilde{\rho}\times\tilde{\sigma}}(g',h')$ for any $\tilde{\rho} \in \Gamma$ and any $\tilde{\sigma} \in \Theta$. We distinguish between three cases. Case 1: Suppose that $g \in G^\infty$ and $h \in H^\infty$. Then we have $0 < \frac{|h|_{\tilde{\sigma}}}{|g|_{\tilde{\rho}}} < \infty$ for all $\tilde{\rho}$ and $\tilde{\sigma}$. We deduce that $g' \in G^\infty$ and $h' \in H^\infty$; as well as $g^+ = g'^+$, $h^+ = h'^+$ and $\frac{|h|_{\tilde{\sigma}}}{|g|_{\tilde{\rho}}} = \frac{|h'|_{\tilde{\sigma}}}{|g'|_{\tilde{\rho}}}$ for all $\tilde{\rho}$ and $\tilde{\sigma}$. This means in particular $\frac{|h|_\sigma}{|g|_\rho} = \frac{|h'|_\sigma}{|g'|_\rho}$. Moreover, we obtain $\frac{|g|_{\tilde{\rho}}}{|g|_\rho} = \frac{|g|_{\tilde{\rho}}}{|h|_\sigma}\frac{|h|_\sigma}{|g|_\rho} = \frac{|g'|_{\tilde{\rho}}}{|h'|_\sigma}\frac{|h'|_\sigma}{|g'|_\rho} = \frac{|g'|_{\tilde{\rho}}}{|g'|_\rho}$ for each $\tilde{\rho} \in \Gamma$; and analogously $\frac{|h|_{\tilde{\sigma}}}{|h|_\sigma} = \frac{|h'|_{\tilde{\sigma}}}{|h'|_\sigma}$ for each $\tilde{\sigma} \in \Theta$. Thus, $\tilde{\iota}(g,h) = \tilde{\iota}(g',h')$ holds in Case 1. Case 2: Suppose that $g \in G^\infty$ and $h \notin H^\infty$. Then $\tau_{\tilde{\rho}\times\tilde{\sigma}}(g,h) = \tau_{\tilde{\rho}\times\tilde{\sigma}}(g',h')$ implies that $g' \in G^\infty$ with $g^+ = g'^+$ and $h' \notin H^\infty$. Thus, we get $\tilde{\iota}(g,h) = \tilde{\iota}(g',h')$ in Case 2. Case 3: Suppose that $g \notin G^\infty$ and $h \in H^\infty$. Then $\tau_{\tilde{\rho}\times\tilde{\sigma}}(g,h) = \tau_{\tilde{\rho}\times\tilde{\sigma}}(g',h')$ implies that $g' \notin G^\infty$, and $h' \in H^\infty$ with $h^+ = h'^+$. Thus, $\tilde{\iota}(g,h) = \tilde{\iota}(g',h')$ holds in Case 3, too. Hence, $\bar{\psi}$ is well-defined. Finally, it is straightforward to check that $\bar{\psi}$ is the inverse of ψ'.

Claim 3: $\bar{\psi}$ is uniformly continuous.

Proof of Claim 3: Throughout this proof of Claim 3, let $d_{\partial G}$ be a fixed visual metric on ∂G, and $d_{\partial H}$ a fixed visual metric on ∂H. We consider the following fundamental systems of entourages for the uniformity on $\iota((G \times H)^\infty)$, and on $\tilde{\iota}((G \times H)^\infty)$ respectively: For any ε, with $1 > \varepsilon > 0$, let $\tilde{W}_\varepsilon$ consist precisely of those pairs $([t,\xi,\zeta],[t',\xi',\zeta'])$ in $(\partial G * \partial H) \times (\partial G * \partial H)$ that satisfy $[\,|t-t'| < \varepsilon^4$ and $d_{\partial G}(\xi,\xi') < \varepsilon$ and $d_{\partial H}(\zeta,\zeta') < \varepsilon]$, or $[t < \varepsilon$ and $t' < \varepsilon$ and $d_{\partial G}(\xi,\xi') < \varepsilon]$, or $[t > \frac{1}{\varepsilon}$ and $t' > \frac{1}{\varepsilon}$ and $d_{\partial H}(\zeta,\zeta') < \varepsilon]$. Then the system $\{\tilde{W}_\varepsilon\}$ is a fundamental system of entourages for $\partial G \times \partial H$. For any ε, with $1 > \varepsilon > 0$, and any finite number $\rho_1 \times \sigma_1, \ldots, \rho_m \times \sigma_m$ of actions in $\Gamma \times \Theta$ let $\tilde{W}_{(\rho_1\times\sigma_1,\ldots,\rho_m\times\sigma_m,\varepsilon)}$ consist precisely of those pairs $((x_{\tilde{\rho}\times\tilde{\sigma}})_{\tilde{\rho}\times\tilde{\sigma}},(x'_{\tilde{\rho}\times\tilde{\sigma}})_{\tilde{\rho}\times\tilde{\sigma}})$ in $(\prod_{\tilde{\rho}\times\tilde{\sigma}\in\Gamma\times\Theta} \partial G * \partial H) \times (\prod_{\tilde{\rho}\times\tilde{\sigma}\in\Gamma\times\Theta} \partial G * \partial H)$ for which $(x_{\rho_j\times\sigma_j}, x'_{\rho_j\times\sigma_j})$ lies in $\tilde{W}_\varepsilon$ for each $j = 1,\ldots,m$. Then, by construction, the trace of the system $\{\tilde{W}_{(\rho_1\times\sigma_1,\ldots,\rho_m\times\sigma_m,\varepsilon)}\}$ is a fundamental system of entourages for the subspace $\iota((G \times H)^\infty)$. For any ε, with $1 > \varepsilon > 0$, any finite number $\rho_1,\ldots\rho_n$ of actions in Γ, and any finite number $\sigma_1,\ldots,\sigma_m$ of actions in Θ let $\tilde{V}_{(\rho_1,\ldots,\rho_n,\sigma_1,\ldots,\sigma_m\varepsilon)}$ consist precisely of those pairs $([t,\xi,\zeta,b,c],[t',\xi',\zeta',b',c'])$ in $M_{\rho\times\sigma} \times M_{\rho\times\sigma}$ that satisfy $[t < \varepsilon$ and $t' < \varepsilon$ and $d_{\partial G}(\xi,\xi')]$, or $[t > \frac{1}{\varepsilon}$ and $t' > \frac{1}{\varepsilon}$ and $d_{\partial H}(\zeta,\zeta')]$, or $[\,|t-t'| < \varepsilon$ and $d_{\partial G}(\xi,\xi') < \varepsilon$ and $d_{\partial H}(\zeta,\zeta') < \varepsilon$ and $|b_{\rho_i} - b'_{\rho_i}| < \varepsilon$ for all $i = 1,\ldots,n$ and $|c_{\sigma_j} - c'_{\sigma_j}| < \varepsilon$ for all $j = 1,\ldots,m\,]$. Then it is straightforward to check that the system $\{\tilde{V}_{(\rho_1,\ldots,\rho_n,\sigma_1,\ldots,\sigma_m,\varepsilon)}\}$ is a fundamental system of entourages for $M_{\rho\times\sigma}$. Hence, the trace of $\{\tilde{V}_{(\rho_1,\ldots,\rho_n,\sigma_1,\ldots,\sigma_m,\varepsilon)}\}$ is a fundamental system of entourages for the subspace $\tilde{\iota}((G \times H)^\infty)$. Let a finite number $\rho_1,\ldots,\rho_n$ of G-actions in Γ, a finite number $\sigma_1,\ldots,\sigma_m$ of H-actions in Θ, and ε with $1 > \varepsilon > 0$ be given. In order to prove that $\bar{\psi}$ is uniformly continuous it is enough to check that there exists a $\tilde{\varepsilon} > 0$ such that $\bar{\psi} \times \bar{\psi}$ maps $\tilde{W}_{(\rho\times\sigma,\rho_1\times\sigma,\ldots,\rho_n\times\sigma,\tilde{\varepsilon})} \cap \tilde{W}_{(\rho\times\sigma_1,\ldots,\rho\times\sigma_m,\tilde{\varepsilon})}$ into $\tilde{V}_{(\rho_1,\ldots,\rho_n,\sigma_1,\ldots,\sigma_m,\varepsilon)}$. Set $K := \max\{\,k(\rho_i), k(\sigma_j) \mid i = 1,\ldots,n;\, j = 1,\ldots,m\,\}$, where each $k(\rho_i)$, resp. each $k(\sigma_j)$, is defined as in the proof of Lemma 3.4.5. Thus, $K \geq 1$; and we have $\frac{1}{K} < b_{\rho_i} < K$ for any $b \in C^{ts}_\rho$ and any $i = 1,\ldots,n$; respectively $\frac{1}{K} < c_{\sigma_j} < K$ for any $c \in C^{ts}_\sigma$ and any $j = 1,\ldots,m$. Take $1 > \tilde{\varepsilon} > 0$ such that $\tilde{\varepsilon} < \frac{\varepsilon}{K(1+\varepsilon)}$ and $K\tilde{\varepsilon} + \tilde{\varepsilon}^4 < \varepsilon$ and $\frac{\tilde{\varepsilon}}{K} + \tilde{\varepsilon}^3 < \varepsilon$. Suppose $(\iota(g,h),\iota(g',h'))$ lies in $\tilde{W}_{(\rho\times\sigma,\rho_1\times\sigma,\ldots,\rho_n\times\sigma,\tilde{\varepsilon})} \cap \tilde{W}_{(\rho\times\sigma_1,\ldots,\rho\times\sigma_m,\tilde{\varepsilon})}$. We distinguish between three cases:

Case 1: Suppose $\frac{|h|_\sigma}{|g|_\rho} < \tilde{\varepsilon}$ and $\frac{|h'|_\sigma}{|g'|_\rho} < \tilde{\varepsilon}$ and $d_{\partial G}(g^+,g'^+) < \tilde{\varepsilon}$. Then $\frac{|h|_\sigma}{|g|_\rho} < \varepsilon$, $\frac{|h'|_\sigma}{|g'|_\rho} < \varepsilon$ and $d_{\partial G}(g^+,g'^+) < \varepsilon$. So, $(\tilde{\iota}(g,h),\tilde{\iota}(g',h')) = (\bar{\psi}(\iota(g,h)),\bar{\psi}(\iota(g',h')))$ lies in $\tilde{V}_{(\rho_1,\ldots,\rho_n,\sigma_1,\ldots,\sigma_m,\varepsilon)}$.

Case 2: Suppose $\frac{|h|_\sigma}{|g|_\rho} > \frac{1}{\tilde{\varepsilon}}$ and $\frac{|h'|_\sigma}{|g'|_\rho} > \frac{1}{\tilde{\varepsilon}}$ and $d_{\partial H}(h^+, h'^+) < \tilde{\varepsilon}$. Then $\frac{|h|_\sigma}{|g|_\rho} > \frac{1}{\varepsilon}$, $\frac{|h'|_\sigma}{|g'|_\rho} > \frac{1}{\varepsilon}$ and $d_{\partial H}(h^+, h'^+) < \varepsilon$. Hence $(\bar{\psi}(\iota(g,h)), \bar{\psi}(\iota(g',h')))$ lies in $\tilde{V}_{(\rho_1,\ldots,\rho_n,\sigma_1,\ldots,\sigma_m,\varepsilon)}$.

Case 3: Suppose $|\frac{|h|_\sigma}{|g|_\rho} - \frac{|h'|_\sigma}{|g'|_\rho}| < \tilde{\varepsilon}^4$ and $d_{\partial G}(g^+, g'^+) < \tilde{\varepsilon}$ and $d_{\partial H}(h^+, h'^+) < \tilde{\varepsilon}$. Then $|\frac{|h|_\sigma}{|g|_\rho} - \frac{|h'|_\sigma}{|g'|_\rho}| < \varepsilon$, $d_{\partial G}(g^+, g'^+) < \varepsilon$ and $d_{\partial H}(h^+, h'^+) < \varepsilon$. Furthermore, we can make two assumptions without loss of generality: Firstly, we can assume that $\frac{|h|_\sigma}{|g|_\rho} < \frac{1}{K\tilde{\varepsilon}}$ and $\frac{|h'|_\sigma}{|g'|_\rho} < \frac{1}{K\tilde{\varepsilon}}$. For otherwise the triangle inequality implies that both $\frac{|h|_\sigma}{|g|_\rho}$ and $\frac{|h'|_\sigma}{|g'|_\rho}$ are greater than $\frac{1}{K\tilde{\varepsilon}} - 1 > \frac{1}{\varepsilon}$; and $(\bar{\psi}(\iota(g,h)), \bar{\psi}(\iota(g',h')))$ lies in $\tilde{V}_{(\rho_1,\ldots,\rho_n,\sigma_1,\ldots,\sigma_m,\varepsilon)}$ analogously to Case 2. Secondly, we can assume that $\frac{|h|_\sigma}{|g|_\rho} > K\tilde{\varepsilon}$ and $\frac{|h'|_\sigma}{|g'|_\rho} > K\tilde{\varepsilon}$. For otherwise the triangle inequality implies that both $\frac{|h|_\sigma}{|g|_\rho}$ and $\frac{|h'|_\sigma}{|g'|_\rho}$ are smaller than $K\tilde{\varepsilon} + \tilde{\varepsilon}^4 < \varepsilon$; and $(\bar{\psi}(\iota(g,h)), \bar{\psi}(\iota(g',h')))$ lies in $\tilde{V}_{(\rho_1,\ldots,\rho_n,\sigma_1,\ldots,\sigma_m,\varepsilon)}$ analogously to Case 1. These two assumptions imply for each $i = 1,\ldots,n$ that $\tilde{\varepsilon} < \frac{|h|_\sigma}{|g|_{\rho_i}} < \frac{1}{\tilde{\varepsilon}}$ and $\tilde{\varepsilon} < \frac{|h'|_\sigma}{|g'|_{\rho_i}} < \frac{1}{\tilde{\varepsilon}}$ (resp. for each $j = 1,\ldots,m$ that $\tilde{\varepsilon} < \frac{|h|_{\sigma_j}}{|g|_\rho} < \frac{1}{\tilde{\varepsilon}}$ and $\tilde{\varepsilon} < \frac{|h'|_{\sigma_j}}{|g'|_\rho} < \frac{1}{\tilde{\varepsilon}}$). Therefore we have by hypothesis that $\left| \frac{|h|_\sigma}{|g|_{\rho_i}} - \frac{|h'|_\sigma}{|g'|_{\rho_i}} \right| < \tilde{\varepsilon}^4$ and $\left| \frac{|h|_{\sigma_j}}{|g|_\rho} - \frac{|h'|_{\sigma_j}}{|g'|_\rho} \right| < \tilde{\varepsilon}^4$ for each $i = 1,\ldots,n$ and each $j = 1,\ldots,m$. Thus, we get for each $i = 1,\ldots,n$:

$$\begin{aligned}
\left| \frac{|g|_{\rho_i}}{|g|_\rho} - \frac{|g'|_{\rho_i}}{|g'|_\rho} \right| &= \left| \frac{|h|_\sigma}{|g|_\rho} \frac{|g|_{\rho_i}}{|h|_\sigma} - \frac{|h'|_\sigma}{|g'|_\rho} \frac{|g|_{\rho_i}}{|h|_\sigma} + \frac{|h'|_\sigma}{|g'|_\rho} \frac{|g|_{\rho_i}}{|h|_\sigma} - \frac{|h'|_\sigma}{|g'|_\rho} \frac{|g'|_{\rho_i}}{|h'|_\sigma} \right| \\
&= \frac{|g|_{\rho_i}}{|h|_\sigma} \left| \frac{|h|_\sigma}{|g|_\rho} - \frac{|h'|_\sigma}{|g'|_\rho} \right| + \frac{|h'|_\sigma}{|g'|_\rho} \left| \frac{\frac{|h'|_\sigma}{|g'|_{\rho_i}} - \frac{|h|_\sigma}{|g|_{\rho_i}}}{\frac{|h|_\sigma}{|g|_{\rho_i}} \frac{|h'|_\sigma}{|g'|_{\rho_i}}} \right| \\
&< \frac{1}{\tilde{\varepsilon}} \tilde{\varepsilon}^4 + \frac{1}{K\tilde{\varepsilon}} \frac{1}{\tilde{\varepsilon}^2} \tilde{\varepsilon}^4 < \varepsilon,
\end{aligned}$$

and for each $j = 1,\ldots,m$:

$$\begin{aligned}
\left| \frac{|h|_{\sigma_j}}{|h|_\sigma} - \frac{|h'|_{\sigma_j}}{|h'|_\sigma} \right| &= \left| \frac{|g|_\rho}{|h|_\sigma} \frac{|h|_{\sigma_j}}{|g|_\rho} - \frac{|g'|_\rho}{|h'|_\sigma} \frac{|h|_{\sigma_j}}{|g|_\rho} + \frac{|g'|_\rho}{|h'|_\sigma} \frac{|h|_{\sigma_j}}{|g|_\rho} - \frac{|g'|_\rho}{|h'|_\sigma} \frac{|h'|_{\sigma_j}}{|g'|_\rho} \right| \\
&= \frac{|h|_{\sigma_j}}{|g|_\rho} \left| \frac{\frac{|h'|_\sigma}{|g'|_\rho} - \frac{|h|_\sigma}{|g|_\rho}}{\frac{|h|_\sigma}{|g|_\rho} \frac{|h'|_\sigma}{|g'|_\rho}} \right| + \frac{|g'|_\rho}{|h'|_\sigma} \left| \frac{|h|_{\sigma_j}}{|g|_\rho} - \frac{|h'|_{\sigma_j}}{|g'|_\rho} \right| \\
&< \frac{1}{\tilde{\varepsilon}} \frac{1}{K^2\tilde{\varepsilon}^2} \tilde{\varepsilon}^4 + \frac{1}{K\tilde{\varepsilon}} \tilde{\varepsilon}^4 < \varepsilon.
\end{aligned}$$

Hence, $(\bar{\psi}(\iota(g,h)), \bar{\psi}(\iota(g',h')))$ lies in $\tilde{V}_{(\rho_1,\ldots,\rho_n,\sigma_1,\ldots,\sigma_m,\varepsilon)}$. This completes the proof of Claim 3.

Using the above claims, we prove the statements (i) to (iii) of Theorem 5.0.14: The subset $\iota((G \times H)^\infty)$ is dense in $\partial(G \times H)$ by construction, and the subset $\tilde{\iota}((G \times H)^\infty)$ is dense in $M_{\rho\times\sigma}$ by Lemma 5.0.15. Therefore, the Claims 1 to 3 imply that ψ' extends to an isomorphism from $M_{\rho\times\sigma}$ onto $\partial(G \times H)$. Since this extension is unique, it must coincide with ψ. Hence, ψ is an isomorphism. Now, the statements (ii) and (iii) are an obvious.

It remains to prove statement (iv) for product actions $\tilde{\rho} \times \tilde{\sigma}$, where $\tilde{\rho} \in \Gamma$ and $\tilde{\sigma} \in \Theta$. Let $\tilde{\rho} \in \Gamma$ and $\tilde{\sigma} \in \Theta$ be given. We want to show that the canonical map $\hat{\tau}_{\tilde{\rho}\times\tilde{\sigma}}$ from $\partial(G \times H)$ to $\partial G * \partial H$ is a homotopy equivalence. According to Lemma 3.4.5, there exists a homotopy $H_\rho : [0,1] \times C_\rho^{ts} \to C_\rho^{ts}$ that contracts C_ρ^{ts} to a basepoint $\bar{b} \in C_\rho^{ts}$, and a homotopy $H_\sigma : [0,1] \times C_\sigma^{ts} \to C_\sigma^{ts}$ that contracts C_σ^{ts} to a basepoint $\bar{c} \in C_\sigma^{ts}$. We define a map $f : \partial G * \partial H \to \partial(G \times H)$ by

$$[t, \xi, \zeta] \mapsto f([t, \xi, \zeta]) := [t, \xi, \zeta, \bar{b}, \bar{c}].$$

Obviously, f is continuous. We show that f is a homotopy inverse for $\hat{\tau}_{\tilde{\rho}\times\tilde{\sigma}}$: Firstly, we check that $\hat{\tau}_{\tilde{\rho}\times\tilde{\sigma}} \circ f$ is homotopic to $\mathrm{id}_{\partial G * \partial H}$. Define $F_1 : [0,1] \times \partial G * \partial H \to \partial G * \partial H$ by

$$(\vartheta, [t, \xi, \zeta]) \mapsto F_1(\vartheta, [t, \xi, \zeta]) := [(1 + \vartheta(\frac{\bar{c}_{\tilde{\sigma}}}{\bar{b}_{\tilde{\rho}}} - 1))\, t \,,\, \xi \,,\, \zeta\,].$$

Clearly, F_1 is continuous. Moreover, we have $F_1(0, [t, \xi, \zeta]) = [t, \xi, \zeta]$, as well as $F_1(1, [t, \xi, \zeta]) = [\frac{\bar{c}_{\tilde{\sigma}}}{\bar{b}_{\tilde{\rho}}} t, \xi, \zeta] = (\hat{\tau}_{\tilde{\rho}\times\tilde{\sigma}} \circ f)([t, \xi, \zeta])$. Secondly, we check that $f \circ \hat{\tau}_{\tilde{\rho}\times\tilde{\sigma}}$ is homotopic to $\mathrm{id}_{\partial(G\times H)}$. Define $F_2 : [0,1] \times \partial(G \times H) \to \partial(G \times H)$ by

$$(\vartheta, [t, \xi, \zeta, b, c]) \mapsto F_2(\vartheta, [t, \xi, \zeta, b, c]) := \ldots$$
$$\ldots [\,(1 + \vartheta(\frac{c_{\tilde{\sigma}}}{b_{\tilde{\rho}}} - 1))\, t \,,\, \xi \,,\, \zeta \,,\, H_\rho(\vartheta, b) \,,\, H_\sigma(\vartheta, c)].$$

Recall that there is a constant $K \geq 1$ such that $\frac{1}{K} < b_{\tilde{\rho}} < K$ for each $b \in C_\rho^{ts}$. Hence, F_2 is continuous. Moreover, we have $F_2(0, [t, \xi, \zeta, b, c]) = [t, \xi, \zeta, b, c]$, as well as $F_2(1, [t, \xi, \zeta, b, c]) = [\frac{c_{\tilde{\sigma}}}{b_{\tilde{\rho}}} t, \xi, \zeta, \bar{b}, \bar{c}] = (f \circ \hat{\tau}_{\tilde{\rho}\times\tilde{\sigma}})([t, \xi, \zeta, b, c])$. This completes the proof of statement (iv) of Theorem 5.0.14. □

Chapter 6

Marked Length Spectrum and k-Isometry Type

In the previous chapters we have seen that the marked length spectra of factor actions play an important rôle for the boundaries of certain CAT(0) product groups. The aim of this chapter is to prove the following result, which is also interesting in its own right.

Let G be a group, and $K \geq 0$ a constant. A (not necessarily continuous) G-equivariant map f from a metric space (X, d_X) to a metric space (Y, d_Y) is said to be a *G-equivariant K-isometry* if ~~there exists a constant $K > 0$ such that~~ $f(X)$ is K-dense in Y, i.e. for each $y \in Y$ there is an $x \in X$ with $d_Y(y, f(x)) \leq K$, and the inequality

$$d_X(x, x') - K \leq d_Y(f(x), f(x')) \leq d_X(x, x') + K$$

holds for all $x, x' \in X$. If there exists such a map f, the spaces X and Y are called *G-equivariantly K-isometric.*

Theorem 6.0.16 *Let G be a torsion-free non-elementary word-hyperbolic group. Let ρ and ρ' be geometric actions by G on CAT(0) spaces X and X'. Then the marked length spectra MLS_ρ and $MLS_{\rho'}$ associated to ρ and ρ' are the same, if and only if for some constant $K \geq 0$ the spaces X and X' are G-equivariantly K-isometric.*

Before we prove this theorem, we look at the following example. It shows that in general we have to admit a constant K strictly greater than 0. Note that a similar example was already given by Hersonsky and Paulin in [HP97].

Example 6.0.17 Let $\mathcal{A} := \langle a, b \rangle$ be the standard set of generators for F_2, and let $T := C_{\mathcal{A}}(F_2)$ be the associated Cayley graph. Then T is a 4-valent tree. Using T, we construct CAT(-1) spaces T_1 and T_2, which carry geometric F_2-actions ρ_1 and ρ_2 such that the associated marked length spectra MLS_1 and MLS_2 are the same. However, T_1 and T_2 are not isometric. We construct T_1 as follows: Let $\Delta\ (ABC)$ be the convex hull in $\mathbb{H}^2_{\mathbb{R}}$ of a geodesic triangle with side lengths $(AB) = (AC) = (BC) = \frac{1}{2}$. Let V be a vertex in T. Suppose A_V is the point on the incoming a-edge at V such that $d_T(A_V, V) = \frac{1}{2}$; and B_V is the point on the outgoing b-edge at V such that $d_T(B_V, V) = \frac{1}{2}$. Then we can glue $\Delta\ (ABC)$ to T such that the side (AC) is isometrically glued onto $[A_V, V]$ and the side (BC) is isometrically glued onto $[B_V, V]$. Gluing a copy of $\Delta\ (ABC)$ to T at each vertex V in that manner, we obtain T_1. T_1 is CAT(-1) with respect to the induced path metric. Clearly, the geometric F_2-action on T induces a geometric F_2-action ρ_1 on T_1.

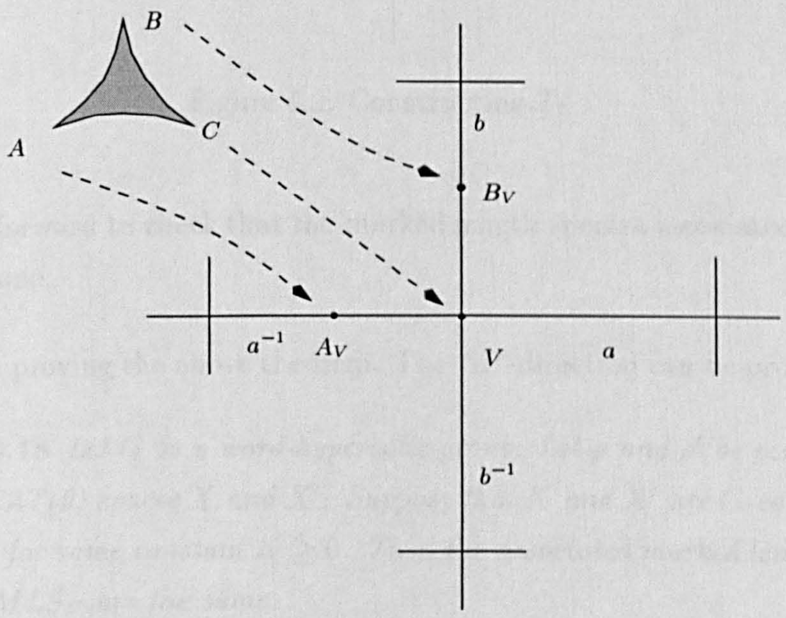

Figure 6.1: Constructing T_1

The space T_2 is constructed as follows: For each vertex V of T let A'_V be the point on the incoming a-edge at V such that $d_T(A'_V, V) = \frac{1}{4}$; and B'_V the point on the outgoing b-edge at V such that $d_T(B'_V, V) = \frac{1}{4}$. We obtain T_2 by gluing $[A'_V, V]$ isometrically onto $[B'_V, V]$ at each vertex V. T_2 is CAT(-1) with respect to the induced path metric; and the geometric F_2-action on T induces a geometric F_2-action ρ_2 on T_2. Obviously, T_1 and T_2 are not isometric. However,

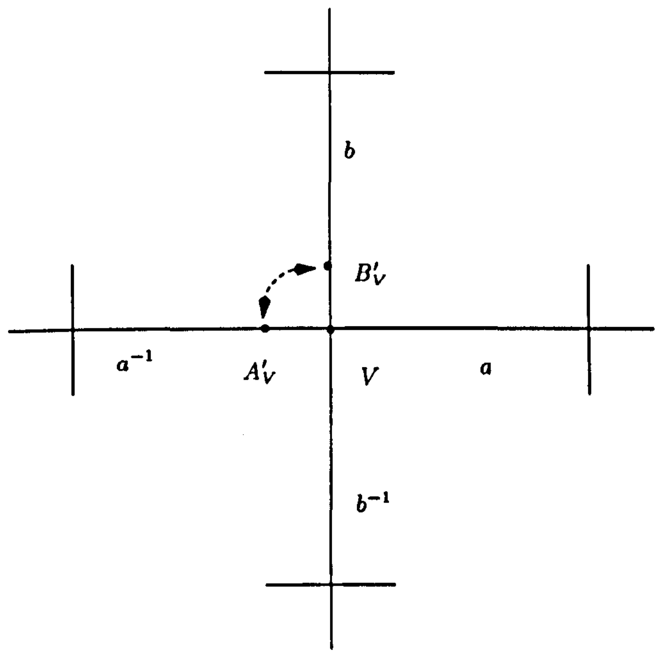

Figure 6.2: Constructing T_2

it is straightforward to check that the marked length spectra associated to ρ_1 and ρ_2 are the same. □

We begin proving the above theorem. The "if"-direction can be proved easily:

Lemma 6.0.18 *Let G be a word-hyperbolic group. Let ρ and ρ' be geometric G-actions on CAT(0) spaces X and X'. Suppose that X and X' are G-equivariantly K-isometric for some constant $K \geq 0$. Then the associated marked length spectra MLS_ρ and $MLS_{\rho'}$ are the same.*

Proof: Let $x \in X$ be a basepoint, and let $f : X \to X'$ be G-equivariant K-

isometry. Then we get for each $g \in G^\infty$ and each $n \in \mathbb{N}$

$$d(g^n.x, x) - K \leq d'(f(g^n.x), f(x)) = d'(g^n.f(x), f(x)) \leq d(g^n.x, x) + K,$$

which implies

$$\left| g \right|_\rho = \lim_{n\to\infty} \frac{1}{n} d(g^n.x, x) = \lim_{n\to\infty} \frac{1}{n} d'(g^n.f(x), f(x)) = \left| g \right|_{\rho'}.$$

$\square$

In the remainder of this chapter we will prove the "only if"-direction. We introduce some notation: Throughout the following we will assume that G is a torsion-free, non-elementary word-hyperbolic group. Let $\mathcal{A}$ be a finite system of generators for G. We denote the induced word-metric on the associated Cayley graph $C_\mathcal{A}(G)$ by $d_\mathcal{A}$. For each $g \in G$ we abbreviate $|g|_\mathcal{A} := d_\mathcal{A}(g, e)$. By $(g, g')_h^\mathcal{A} := \frac{1}{2}(d_\mathcal{A}(g, h) + d_\mathcal{A}(g', h) - d_\mathcal{A}(g, g'))$ we denote the Gromov product for each $g, g', h \in G$. For each $r \in \mathbb{N}$ we set $S(r) := \{g \in G \mid |g|_\mathcal{A} = r\}$ and $B(r) := \{g \in G \mid |g|_\mathcal{A} \leq r\}$. Since G is word-hyperbolic, the geodesic space $C_\mathcal{A}(G)$ is δ-hyperbolic for some constant $\delta \geq 0$. This allows to use Gromov's "approximation by trees":

Theorem 6.0.19 (Approximation by Trees) *Let x_0 be a basepoint in $C_\mathcal{A}(G)$. For $i = 1, \ldots, n$ let x_i be a point in $C_\mathcal{A}(G) \cup \partial C_\mathcal{A}(G)$, and s_i a geodesic segment or geodesic ray joining x_0 to x_i. Set $Z := s_1 \cup \cdots \cup s_n$. Then there exists a simplicial tree T with basepoint t_0, and a continuous mapping $f : (Z, x_0) \to (T, t_0)$, such that the following holds:*

(i) For $i = 1, \ldots, n$ the restriction of f to s_i is an isometry.

(ii) There exists a constant $C(\delta, n)$, which depends on δ and n only, such that for all $z, z' \in Z$

$$d_\mathcal{A}(z, z') - C(\delta, n) \leq d_T(f(z), f(z')) \leq d_\mathcal{A}(z, z').$$

Throughout the following we let the constant C denote $\max\{C(\delta, 3), C(\delta, 4)\}$. We assume without loss of generality that C lies in $\mathbb{N}$. Since $C_\mathcal{A}(G)$ is δ-

hyperbolic, there is a constant $\delta' \geq 0$, which depends on δ only, such that the insize of each geodesic triangle in $C_{\mathcal{A}}(G)$ is less than δ'. Without loss of generality we assume that δ' lies in $\mathbb{N}$, too. Recall that the critical exponent e_G of G is defined by

$$e_G := \limsup_{r\to\infty} \frac{\log \#B(r)}{r}.$$

Definition 6.0.20 (k-Separator) Let $g \in G$ be given. We call an element $h \in S(2k+2C) \cup S(2k+2C+1)$ a *k-separator* for g with respect to $d_{\mathcal{A}}$, if both $(g,h^{-1})_e^{\mathcal{A}} \leq k$ and $(g^{-1},h)_e^{\mathcal{A}} \leq k$ hold.

The above definition may seem rather technical at this stage. Its meaning will become clear later.

Proposition 6.0.21 *There exists a constant $k \in \mathbb{N}$ such that each $g \in G$ with $|g|_{\mathcal{A}} \geq 2k+2C+1$ has a k-separator h in G.*

In order to prove this proposition, we need three lemmas. The proofs of the first and the second one are trivial.

Lemma 6.0.22 *Let $r \in \mathbb{N}$ be given, and let $g \in G$ be such that $|g|_{\mathcal{A}} \geq r$. Suppose $c_g : [0, |g|_{\mathcal{A}}] \to C_{\mathcal{A}}(G)$ be a geodesic segment joining e to g. Then there exists a unique vertex P_g of $C_{\mathcal{A}}(G)$ that lies in $c_g([0,|g|_{\mathcal{A}}]) \cap S(r)$.*

Lemma 6.0.23 *Let $r \in \mathbb{N}$ be given. Then the map $i : S(r) \to S(r)$ given by $g \mapsto g^{-1}$ is a bijection.*

Lemma 6.0.24 *Let $R, r > 0$ with $R > r$, and let $g \in G$ with $|g|_{\mathcal{A}} \geq R$. Set $M^+ := \{h \in S(R) \mid (g,h)_e^{\mathcal{A}} > r\}$ and $M^- := \{h \in S(R) \mid (g,h^{-1})_e^{\mathcal{A}} > r\}$. Then we have*

$$\#M^- = \#M^+ \leq (R-r+1) \cdot \#B(R-r+\delta'+1).$$

Proof: Firstly, we show that $\#M^+ \leq (R-r+1)\cdot \#B(R-r+\delta'+1)$. Let $c_g : [0,|g|_{\mathcal{A}}] \to C_{\mathcal{A}}(G)$ be a geodesic segment joining e to g. Suppose $h \in S(R)$ is such that $\kappa := (g,h^{-1})_e^{\mathcal{A}} > r$. Let $c_{h^{-1}} : [0,R] \to C_{\mathcal{A}}(G)$ be a geodesic segment joining e to h^{-1}. By hypothesis, we have $d_{\mathcal{A}}(c_g(\kappa), c_{h^{-1}}(\kappa)) < \delta'$, because δ' is a bound for the insize of geodesic triangles in $C_{\mathcal{A}}(G)$. Also, we have $d_{\mathcal{A}}(c_{h^{-1}}(\kappa), h^{-1}) < R-r$.

Note that $c_g(\kappa)$ is at most a distance of 1 away from a vertex on $c_g([r,R])$. Hence, there exists a vertex V on $c_g([r,R])$ such that

$$d_{\mathcal{A}}(V,h^{-1}) < R - r + \delta' + 1.$$

Thus, each $h \in M^+$ is contained in at least one ball of radius $R-r+\delta'+1$ around a vertex on $c_g([r,R])$. Since there are exactly $R-r+1$ vertices on $c_g([r,R])$, we obtain

$$\#M^+ \leq (R-r+1) \cdot \#B(R-r+\delta'+1).$$

Secondly, we check that $\#M^+ = \#M^-$. Clearly, we have $h \in M^-$, if and only if $h^{-1} \in M^+$. Therefore, Lemma 6.0.23 implies $\#M^+ = \#M^-$. □

Proof of Prop. 6.0.21: G is non-elementary word-hyperbolic. As shown by Coornaert (see Cor. 5.5 in [Coo93]), this implies that e_G is strictly positive. We assume that Prop. 6.0.21 is false and deduce a contradiction. Suppose for all $k \in N$ there exists a $g \in G$ with $|g|_{\mathcal{A}} \geq 2k + 2C + 1$ such that we have $(g,h^{-1})_e^{\mathcal{A}} > k$ or $(g^{-1},h)_e^{\mathcal{A}} > k$ for all $h \in S(2k+2C) \cup S(2k+2C+1)$. Consider a fixed $k \in N$, and let $g \in G$ be according to our assumption. We define the following subsets of $S(2k+2C)$, respectively of $S(2k+2C+1)$:

$$\begin{aligned}
M_{2k}^+ &:= \{ h \in S(2k+2C) \mid (g^{-1},h)_e^{\mathcal{A}} > k \}, \\
M_{2k}^- &:= \{ h \in S(2k+2C) \mid (g,h^{-1})_e^{\mathcal{A}} > k \}, \\
M_{2k+1}^+ &:= \{ h \in S(2k+2C+1) \mid (g^{-1},h)_e^{\mathcal{A}} > k \}, \\
M_{2k+1}^- &:= \{ h \in S(2k+2C+1) \mid (g,h^{-1})_e^{\mathcal{A}} > k \}.
\end{aligned}$$

According to our assumption every element $h \in S(2k+2C) \cup S(2k+2C+1)$ lies in at least one of these subsets. Therefore, we get

$$\#S(2k+2C) + \#S(2k+2C+1) \leq \#M_{2k}^+ + \#M_{2k}^- + \#M_{2k+1}^+ + \#M_{2k+1}^-.$$

It follows from Lemma 6.0.24 that

$$\begin{aligned}
\#M^+_{2k} + \#M^-_{2k} &\leq 2\,(k + 2C + 1) \cdot \#B(k + 2C + \delta' + 1), \\
\#M^+_{2k+1} + \#M^-_{2k+1} &\leq 2\,(k + 2C + 2) \cdot \#B(k + 2C + \delta' + 2).
\end{aligned}$$

Hence, we get for each $k \in \mathbb{N}$ that

$$\#S(2k + 2C) + \#S(2k + 2C + 1) \leq 4\,(k + 2C + 2) \cdot \#B(k + 2C + \delta' + 2),$$

which implies for each $n \in \mathbb{N}$

$$\begin{aligned}
\#B(2n + 2C + 1) &= \sum_{k=1}^{n} \big(\#S(2k + 2C) + \#S(2k + 2C + 1)\big) + \#B(2C + 1) \\
&\leq \Big(\sum_{k=1}^{n} 4\,(k + 2C + 2) \cdot \#B(k + 2C + \delta' + 2)\Big) + \#B(2C + 1) \\
&\leq 4n \cdot (n + 2C + 3) \cdot \#B(n + 2C + \delta' + 2).
\end{aligned}$$

We can apply another result by Coornaert (see [Coo93], Thm. 7.2), and conclude that there exists constant $c \geq 1$ such that for each $r \in \mathbb{N}$

$$c^{-1} \exp(e_G \cdot r) \leq \#B(r) \leq c \exp(e_G \cdot r).$$

Thus, it follows for all $n \in \mathbb{N}$ that

$$c^{-1} \exp(e_G \cdot (2n + 2C + 1)) \leq 4cn\,(n + 2C + 3) \exp(e_G(n + 2C + \delta' + 2)),$$

which implies $e_G = 0$. This is a contradiction. $\square$

Proposition 6.0.25 *Let $g \in G$ with $|g|_{\mathcal{A}} \geq 2k + 2C$. Suppose that $h \in G$ is a k-separator for g. Then gh has infinite order. Furthermore, for each $n \in \mathbb{N}$ let $c_n : [0, |(gh)^n|_{\mathcal{A}}] \to C_{\mathcal{A}}(G)$ be a geodesic segment joining e to $(gh)^n$. Then there exists a constant $H \geq 0$, which depends on k and δ only, such that for each $n \in \mathbb{N}$ and each $i \in \mathbb{N}$ with $0 < i < n$ the point $(gh)^i$ lies in the H-neighbourhood of c_n.*

We need to prepare the proof of this proposition with some lemmas:

Lemma 6.0.26 *Let x_0 be a basepoint in $C_{\mathcal{A}}(G)$. Let $m \in \{3,4\}$. For $j = 1, \ldots, m$ let x_j be a point in $C_{\mathcal{A}}(G) \cup \partial C_{\mathcal{A}}(G)$, and s_j a geodesic segment or geodesic ray joining x_0 to x_i. Set $Z := s_1 \cup \cdots \cup s_m$. Let the tree T and the map $f : Z \to T$ be given according to the "Approximation by Trees"-Theorem 6.0.19. Then we have for all $z, z' \in Z$*

$$(z,z')^{\mathcal{A}}_{x_0} \le (f(z), f(z'))^{T}_{f(x_0)} \le (z,z')^{\mathcal{A}}_{x_0} + C.$$

Proof: For all $z, z' \in Z$ we have by hypothesis $d_{\mathcal{A}}(x_0, z) = d_T(f(x_0), f(z))$, as well as $d_{\mathcal{A}}(z,z') - C \le d_T(f(z), f(z')) \le d_{\mathcal{A}}(z,z')$. Therefore, we get on the one side

$$\begin{aligned}(f(z), f(z'))^{T}_{f(x_0)} &= \frac{1}{2}\big(d_T(f(x_0), f(z)) + d_T(f(x_0), f(z')) - d_T(f(z), f(z'))\big)\\ &\ge \frac{1}{2}\big(d_{\mathcal{A}}(x_0, z) + d_{\mathcal{A}}(x_0, z') - d_{\mathcal{A}}(z, z')\big)\\ &= (z,z')^{\mathcal{A}}_{x_0},\end{aligned}$$

and on the other side

$$\begin{aligned}(f(z), f(z'))^{T}_{f(x_0)} &= \frac{1}{2}\big(d_T(f(x_0), f(z)) + d_T(f(x_0), f(z')) - d_T(f(z), f(z'))\big)\\ &\le \frac{1}{2}\big(d_{\mathcal{A}}(x_0, z) + d_{\mathcal{A}}(x_0, z') - d_{\mathcal{A}}(z, z') + C\big)\\ &\le (z,z')^{\mathcal{A}}_{x_0} + C.\end{aligned}$$

□

Lemma 6.0.27 *Let $g \in G$ with $|g|_{\mathcal{A}} \ge 2k + 2C$, and let $h \in G$ be a k-separator for g. For each $n \in \mathbb{N}_0$ set $x_n := (gh)^n$ and $y_n := (gh)^n g$. Then we have for all $n \in \mathbb{N}$*

$$(x_n, y_0)^{\mathcal{A}}_{x_0} \ge k + C.$$

Proof: By hypothesis we have for each $n \in \mathbb{N}_0$

$$\begin{aligned} d_{\mathcal{A}}(x_n, y_n) &= |g|_{\mathcal{A}} \geq 2k + 2C & (x_n, x_{n+1})^{\mathcal{A}}_{y_n} &= (g^{-1}, h)^{\mathcal{A}}_e \leq k \\ d_{\mathcal{A}}(y_n, x_{n+1}) &= |h|_{\mathcal{A}} \geq 2k + 2C & (y_n, y_{n+1})^{\mathcal{A}}_{x_{n+1}} &= (h^{-1}, g)^{\mathcal{A}}_e \leq k. \end{aligned}$$

We prove the lemma by induction on n. Obviously, we have

$$(x_1, y_0)^{\mathcal{A}}_{x_0} = d_{\mathcal{A}}(x_0, y_0) - (x_0, x_1)^{\mathcal{A}}_{y_0} \geq 2k + 2C - k \geq k + C.$$

Hence, the desired inequality holds for $n = 1$. So, let us assume that for some $n \geq 1$ we have $(x_n, y_0)^{\mathcal{A}}_{x_0} \geq k + C$. We want to conclude that $(x_{n+1}, y_0)^{\mathcal{A}}_{x_0} \geq k + C$. We will proceed as follows: The assumption gives us a lower bound for $(x_{n+1}, y_1)^{\mathcal{A}}_{x_1}$. This lower bound is used to obtain an upper bound for $(x_{n+1}, y_0)^{\mathcal{A}}_{x_1}$, which in turn yields a lower bound for $(x_{n+1}, x_1)^{\mathcal{A}}_{y_0}$. The latter gives us an upper bound for $(x_{n+1}, x_0)^{\mathcal{A}}_{y_0}$, which we use to get the desired lower bound for $(x_{n+1}, y_0)^{\mathcal{A}}_{x_0}$. To get the upper bound for $(x_{n+1}, y_0)^{\mathcal{A}}_{x_1}$, we consider $Z := [x_1, x_{n+1}] \cup [x_1, y_1] \cup [x_1, y_0]$. Let T be an approximating tree for Z, and f the associated map, as given by Thm. 6.0.19. Then Lemma 6.0.26 implies together with our induction assumption that

$$(f(x_{n+1}), f(y_1))^T_{f(x_1)} \geq (x_{n+1}, y_1)^{\mathcal{A}}_{x_1} \geq k + C,$$

and together with our hypothesis that

$$(f(y_1), f(y_0))^T_{f(x_1)} \leq (y_1, y_0)^{\mathcal{A}}_{x_1} + C \leq k + C.$$

Hence, we get the following upper bound

$$(x_{n+1}, y_0)^{\mathcal{A}}_{x_1} \leq (f(x_{n+1}), f(y_0))^T_{f(x_1)} = (f(y_1), f(y_0))^T_{f(x_1)} \leq k + C.$$

This upper bound gives us a lower bound

$$(x_{n+1}, x_1)^{\mathcal{A}}_{y_0} = d_{\mathcal{A}}(y_0, x_1) - (x_{n+1}, y_0)^{\mathcal{A}}_{x_1} \geq 2k + 2C - (k + C) = k + C.$$

Now, consider $Z' := [y_0, x_{n+1}] \cup [y_0, x_1] \cup [y_0, x_0]$. Let T' be an approximating tree for Z', and f' the associated map, as given by Thm. 6.0.19. Then Lemma 6.0.26

implies together with the last lower bound that

$$(f(x_{n+1}), f(x_1))^T_{f(y_0)} \geq (x_{n+1}, x_1)^{\mathcal{A}}_{y_0} \geq k + C,$$

and together with our hypothesis that

$$(f(x_1), f(x_0))^T_{f(y_0)} \leq (x_1, x_0)^{\mathcal{A}}_{y_0} + C \leq k + C.$$

Hence, we get the following upper bound

$$(x_{n+1}, x_0)^{\mathcal{A}}_{y_0} \leq (f(x_{n+1}), f(x_0))^T_{f(y_0)} = (f(x_1), f(x_0))^T_{f(y_0)} \leq k + C.$$

Finally, this last upper bound yields the desired lower bound:

$$(x_{n+1}, y_0)^{\mathcal{A}}_{x_0} = d_{\mathcal{A}}(y_0, x_0) - (x_{n+1}, x_0)^{\mathcal{A}}_{y_0} \geq 2k + 2C - (k + C) = k + C.$$

□

Lemma 6.0.28 *Let $g \in G$ with $|g|_{\mathcal{A}} \geq 2k + 2C$, and let $h \in G$ be a k-separator for g. For each $n \in \mathbb{N}_0$ set $x_n := (gh)^n$ and $y_n := (gh)^n g$. Then we have for all $n \in \mathbb{N}$*

$$(x_0, y_{n-1})^{\mathcal{A}}_{x_n} \geq k + C.$$

Proof: The proof is done by induction on n – similarly to that of Lemma 6.0.27. Clearly, we have

$$(x_0, y_0)^{\mathcal{A}}_{x_1} = d_{\mathcal{A}}(y_0, x_1) - (x_0, x_1)^{\mathcal{A}}_{y_0} \geq 2k + 2C - k \geq k + C.$$

Hence, the desired inequality holds for $n = 1$. So, let us assume that for some $n \geq 1$ we have $(x_0, y_{n-1})^{\mathcal{A}}_{x_n} \geq k + C$. We want to conclude that $(x_0, y_n)^{\mathcal{A}}_{x_{n+1}} \geq k + C$. We will proceed in a manner similar to Lemma 6.0.27: The assumption gives us a lower bound for $(x_0, y_{n-1})^{\mathcal{A}}_{x_n}$. This lower bound is used to obtain an upper bound for $(x_0, y_n)^{\mathcal{A}}_{x_n}$, which in turn yields a lower bound for $(x_0, x_n)^{\mathcal{A}}_{y_n}$. The latter gives us an upper bound for $(x_0, x_{n+1})^{\mathcal{A}}_{y_n}$, which we use to get the desired lower bound for $(x_0, y_n)^{\mathcal{A}}_{x_{n+1}}$. To get the upper bound for $(x_0, y_n)^{\mathcal{A}}_{x_n}$, we consider

$Z := [x_n, x_0] \cup [x_n, y_{n-1}] \cup [x_n, y_n]$. Let T be an approximating tree for Z, and f the associated map, as given by Thm. 6.0.19. Then Lemma 6.0.26 implies together with our induction assumption that

$$(f(x_0), f(y_{n-1}))^T_{f(x_n)} \geq (x_0, y_{n-1})^{\mathcal{A}}_{x_n} \geq k + C,$$

and together with our hypothesis that

$$(f(y_{n-1}), f(y_n))^T_{f(x_n)} \leq (y_{n-1}, y_n)^{\mathcal{A}}_{x_n} + C \leq k + C.$$

Hence, we get the following upper bound

$$(x_0, y_n)^{\mathcal{A}}_{x_n} \leq (f(x_0), f(y_n))^T_{f(x_n)} = (f(y_{n-1}), f(y_n))^T_{f(x_n)} \leq k + C.$$

This upper bound gives us a lower bound

$$(x_0, x_n)^{\mathcal{A}}_{y_n} = d_{\mathcal{A}}(x_n, y_n) - (x_0, y_n)^{\mathcal{A}}_{x_n} \geq 2k + 2C - (k + C) = k + C.$$

Now, consider $Z' := [y_n, x_0] \cup [y_n, x_n] \cup [y_n, x_{n+1}]$. Let T' be an approximating tree for Z', and f' the associated map, as given by Thm. 6.0.19. Then Lemma 6.0.26 implies together with the last lower bound that

$$(f(x_0), f(x_n))^T_{f(y_n)} \geq (x_0, x_n)^{\mathcal{A}}_{y_n} \geq k + C,$$

and together with our hypothesis that

$$(f(x_n), f(x_{n+1}))^T_{f(y_n)} \leq (x_n, x_{n+1})^{\mathcal{A}}_{y_n} + C \leq k + C.$$

Hence, we get the following upper bound

$$(x_0, x_{n+1})^{\mathcal{A}}_{y_n} \leq (f(x_0), f(x_{n+1}))^T_{f(y_n)} = (f(x_n), f(x_{n+1}))^T_{f(y_n)} \leq k + C.$$

Finally, this last upper bound yields the desired lower bound:

$$(x_0, y_n)^{\mathcal{A}}_{x_{n+1}} = d_{\mathcal{A}}(y_n, x_{n+1}) - (x_0, x_{n+1})^{\mathcal{A}}_{y_n} \geq 2k + 2C - (k + C) = k + C.$$

$\square$

Proof of Prop. 6.0.25: Let $g \in G$ with $|g|_{\mathcal{A}} \geq 2k + 2C$ be given. Suppose that $h \in G$ is a k-separator for g. Then we get

$$d_{\mathcal{A}}(e, gh) = d_{\mathcal{A}}(e, g) + d_{\mathcal{A}}(g, gh) - 2\,(e, gh)_g^{\mathcal{A}} \geq 4k + 4C - 2k > 0.$$

It follows that gh has infinite order in G, because G is torsion-free. Next, let $n \in \mathbb{N}$ and $i \in \mathbb{N}$ with $i < n$ be given. We abbreviate $x_0 := e$, $x_i := (gh)^i$, $x_n := (gh)^n$, $y_{i-1} := (gh)^{i-1}g$ and $y_i := (gh)^i g$ as in the two previous lemmas. Firstly, we find a suitable upper bound for $(x_0, x_n)_{x_i}^{\mathcal{A}}$. To do so, we consider $Z := [x_i, x_0] \cup [x_i, y_{i-1}] \cup [x_i, y_i] \cup [x_i, x_n]$. Let T be an approximating tree for Z, and f the associated map, as given by Thm. 6.0.19. Then Lemma 6.0.26 implies together with Lemma 6.0.27 that

$$(f(x_n), f(y_i))_{f(x_i)}^T \geq (x_n, y_i)_{x_i}^{\mathcal{A}} = (x_{n-i}, y_0)_{x_0}^{\mathcal{A}} \geq k + C,$$

and together with Lemma 6.0.28 that

$$(f(x_0), f(y_{i-1}))_{f(x_i)}^T \geq (x_0, y_{i-1})_{x_i}^{\mathcal{A}} \geq k + C.$$

Lemma 6.0.26 also implies that

$$(f(y_{i-1}), f(y_i))_{f(x_i)}^T \leq (y_{i-1}, y_i)_{x_i}^{\mathcal{A}} + C \leq k + C.$$

Hence, we get the following upper bound

$$(x_0, x_n)_{x_i}^{\mathcal{A}} \leq (f(x_0), f(x_n))_{f(x_i)}^T = (f(y_{i-1}), f(y_i))_{f(x_i)}^T \leq k + C.$$

Secondly, set $\kappa := (x_0, x_i)_{x_n}^{\mathcal{A}}$. Let $c : [0, |x_n|_{\mathcal{A}}] \to C_{\mathcal{A}}(G)$ be a geodesic segment joining x_n to x_0, and $c' : [0, d_{\mathcal{A}}(x_i, x_n)] \to C_{\mathcal{A}}(G)$ a geodesic segment joining x_n to x_i, i.e. we have $c(0) = c'(0) = x_n$, $c(|x_n|_{\mathcal{A}}) = x_0$ and $c'(d_{\mathcal{A}}(x_i, x_n)) = x_i$. Note that $d_{\mathcal{A}}(c(\kappa), c'(\kappa)) \leq \delta'$, because δ' is a bound for the insize. Note furthermore that

$$d_{\mathcal{A}}(x_i, c'(\kappa)) = d_{\mathcal{A}}(x_i, x_n) - (x_0, x_i)_{x_n}^{\mathcal{A}} = (x_0, x_n)_{x_i}^{\mathcal{A}} \leq k + C.$$

Therefore, we get $d_A(x_i, c(\kappa)) \leq k + C + \delta'$. Recall that both C and δ' depend on δ only. Thus, we can take $H := k + C + \delta' + 1$. □

Finally, we prove the following theorem.

Theorem 6.0.29 *Let G be a torsion-free non-elementary word-hyperbolic group. Let ρ and ρ' be two geometric actions by G on CAT(0) spaces X and X'. Let $x \in X$ and $x' \in X'$ be basepoints. Suppose that the marked length spectra MLS_ρ and $MLS_{\rho'}$ associated to ρ and ρ' are the same. Then there exists a constant $K \geq 0$ such that the map $g.x \mapsto g.x'$ from the orbit $G.x$ to the orbit $G.x'$ is a bijective G-equivariant K-isometry.*

Note that this theorem implies the missing "only if"-direction of Theorem 6.0.16, because both G-actions, ρ and ρ', are cocompact. We prepare the proof of Theorem 6.0.29 with some lemmas:

Lemma 6.0.30 *Let G be a torsion-free group that acts geometrically on two CAT(0) spaces X_1 and X_2. Let $x_1 \in X_1$ and $x_2 \in X_2$ be basepoints. Then the map $g.x_1 \mapsto g.x_2$ from the orbit $G.x_1$ to the orbit $G.x_2$ is a G-equivariant bijection.*

Proof: Suppose there are $g_1, g_2 \in G$ such that $g_1.x_1 = g_2.x_1$. Then x_1 is a fixpoint for $(g_2)^{-1}g_1$. Since the G-action on X_1 is proper, $(g_2)^{-1}g_1$ is of finite order in G. Because G is torsion-free, this implies that $(g_2)^{-1}g_1 = e$. Thus, the map $g.x_1 \mapsto g.x_2$ is well-defined. The same argument for $g_1.x_2 = g_2.x_2$ shows that it is injective. Obviously, the map $g.x_1 \mapsto g.x_2$ is surjective and G-equivariant.□

Lemma 6.0.31 *Let X be a $\bar{\delta}$-hyperbolic CAT(0) space. Then the Hausdorff distance between any two asymptotic geodesic lines $c : \mathbb{R} \to X$ and $c' : \mathbb{R} \to X$ is less than $\bar{\delta} + 1$.*

Proof: By the Flat Strip Theorem 1.1.4 the convex hull of $c(\mathbb{R}) \cup c'(\mathbb{R})$ is isometric to a euclidean strip $\mathbb{R} \times [0, w]$ of width w. Obviously, if $w \geq \bar{\delta} + 1$ we can find a geodesic triangle in this strip such that the $\bar{\delta}$-neighbourhood of two of its sides does not cover the third one. This contradicts the hypothesis that X is $\bar{\delta}$-hyperbolic. $\square$.

Lemma 6.0.32 *Let $g \in G$ with $|g|_A \geq 2k + 2C$, and let $h \in G$ be a k-separator for g. Let ρ be a geometric G-action on a CAT(0) space X, and let $x \in X$ be a basepoint. Let $\lambda \geq 1$ and $\varepsilon \geq 0$ be given according to the Švarc-Milnor Lemma 1.1.1 such that the map $g \mapsto g.x$ is a (λ, ε)-quasi-isometry from $C_A(G)$ to X. Then X is $\bar{\delta}$-hyperbolic for some constant $\bar{\delta} \geq 0$. Moreover, there exists a constant $L \geq 0$, which depends on k, δ, λ, ε and $\bar{\delta}$ only, and a geodesic line $c_{gh} : \mathbb{R} \to X$ such that $(gh)^z.x$ lies in the L-neighbourhood of $c_{gh}(\mathbb{R})$ for each $z \in \mathbb{Z}$.*

Proof: It is well-known that Gromov-hyperbolicity is invariant under quasi-isometries (see e.g. [CDP90] or [GH90]). Hence, there exists a constant $\bar{\delta} \geq 0$ such that X is $\bar{\delta}$-hyperbolic. Let $n \in \mathbb{N}$. Let $c_n : [0, d_A((gh)^{-n}, (gh)^n)] \to C_A(G)$ be a geodesic segment joining $(gh)^{-n}$ to $(gh)^n$, and $c'_n : [0, d_A((gh)^{-n}, (gh)^n)] \to X$ the image of c_n under the (λ, ε)-quasi-isometry $g \mapsto g.x$. Thus, c'_n is a (λ, ε)-quasi-geodesic segment joining $(gh)^{-n}.x$ to $(gh)^n.x$. Let $s_n : [0, d((gh)^{-n}.x, (gh)^n.x)] \to X$ be the geodesic segment that joins $(gh)^{-n}.x$ to $(gh)^n.x$ in X. Suppose $z \in \mathbb{Z}$ is such that $-n \leq z \leq n$. Then it follows from Prop. 6.0.25 that $(gh)^z.x$ lies in the $(\lambda H + \varepsilon)$-neighbourhood of c'_n, where $H \geq 0$ depends on k and δ only. Since X is $\bar{\delta}$-hyperbolic, there exists a constant $H' \geq 0$, which depends on λ, ε and $\bar{\delta}$ only, such that c'_n lies in the H'-neighbourhood of s_n. Hence, for each n and each z with $-n \leq z \leq n$ the point $(gh)^z.x$ lies in the $(\lambda H + \varepsilon + H')$-neighbourhood of s_n, say $t_n^z \in [0, d((gh)^{-n}.x, (gh)^n.x)]$ is such that $d(s_n(t_n^z), (gh)^z.x) \leq \lambda H + \varepsilon + H'$. Since X is proper, there exists a sequence $(n_\nu)_\nu$ such that $s_{n_\nu}(t^0_{n_\nu})$ converges to a point $y \in X$. Since the metric on X is convex, we can use a standard Arzela-Ascoli type argument to show that for some subsequence $(n'_\nu)_\nu$ the geodesic segments $[s_{n'_\nu}(t^{-\nu}_{n'_\nu}), s_{n'_\nu}(t^{\nu}_{n'_\nu})]$ converge uniformly on compact subsets to a geodesic line $c_{gh} : \mathbb{R} \to X$ as $\nu \to \infty$. Clearly, $(gh)^z.x$ lies in the $(\lambda H + \varepsilon + H' + 1)$-neighbourhood of c_{gh} for each $z \in Z$. $\square$

Lemma 6.0.33 *Let the hypotheses be the same as in Lemma 6.0.32. Then there exists a constant $M \geq 0$, which depends on k, δ, λ, ε, $\bar{\delta}$ only, and an axis*

$a_{gh} : \mathbb{R} \to X$ of gh such that $(gh)^z.x$ lies in the M-neighbourhood of $a_{gh}(\mathbb{R})$ for each $z \in \mathbb{Z}$.

Proof: Let $c_{gh} : \mathbb{R} \to X$ be given according to the previous Lemma 6.0.32. Since gh is a hyperbolic isometry of X, it has an axis $a_{gh} : \mathbb{R} \to X$. It is straightforward to check that the Hausdorff distance between $c_{gh}(\mathbb{R})$ and $a_{gh}(\mathbb{R})$ is bounded. Therefore, either $c_{gh} : \mathbb{R} \to X$ or $c_{gh}^{-} : \mathbb{R} \to X$, where $c_{gh}^{-}(t) := c_{gh}(-t)$, is asymptotic to $a_{gh} : \mathbb{R} \to X$. This implies by Lemma 6.0.31 that the Hausdorff distance between $c_{gh}(\mathbb{R})$ and $a_{gh}(\mathbb{R})$ is bounded by $\bar{\delta}+1$. Since each $(gh)^z.x$ lies in an L-neighbourhood of $c_{gh}(\mathbb{R})$, it lies in an $(L + \bar{\delta} + 1)$-neighbourhood of $a_{gh}(\mathbb{R})$. Thus, we can take $M := L + \bar{\delta} + 1$. □

Corollary 6.0.34 *Let G be a torsion-free non-elementary word-hyperbolic group. Let ρ be a geometric G-action on a CAT(0) space X. Then there exists a constant $K' \geq 0$ such that any two points $x, y \in X$ lie in the K'-neighbourhood of an axis $a_g : \mathbb{R} \to X$ of some hyperbolic isometry $g \in G$.*

Proof: Let $k \in \mathbb{N}$ be given according to Prop. 6.0.21. Let $x_0 \in X$ be a basepoint. Let $\lambda \geq 1$ and $\varepsilon \geq 0$ be given according to the Švarc-Milnor Lemma 1.1.1 such that the map $g \mapsto g.x_0$ is a (λ, ε)-quasi-isometry from $C_{\mathcal{A}}(G)$ to X. Let $M \geq 0$ be given according to Lemma 6.0.33. Since ρ is cocompact, there exists an $r > 0$ such that the G-translates of B_r cover X, where B_r is the ball of radius r around x_0. Suppose that x, y are two points in X. Firstly, let us assume that $x \in B_r$, and that $y \in g.B_r$ for some $g \in G$. We distinguish between two cases. Case 1: Suppose $|g|_{\mathcal{A}} \geq 2k + 2C + 1$. Then we get $|g|_{\mathcal{A}} \geq 2k + 2C + 1$; and it follows from Prop. 6.0.21 that g has a k-separator h. According to Lemma 6.0.33, both x_0 and $gh.x_0$ lie in the M-neighbourhood of an axis $a_{gh} : \mathbb{R} \to X$ of gh. Note that we have $d_X(x, x_0) \leq 2r$, and

$$\begin{aligned} d_X(y, gh.x_0) \leq\; & d_X(y, g.x_0) + d_X(g.x_0, gh.x_0) \leq \dots \\ & \dots \leq 2r + \lambda\, |h|_\rho + \varepsilon \leq 2r + \lambda\,(2k + 2C + 1) + \varepsilon. \end{aligned}$$

Hence, both x in y lie in the $(M + 2r + \lambda\,(2k + 2C + 1) + \varepsilon)$-neighbourhood of

a_{gh}. Case 2: Suppose $|g|_{\mathcal{A}} < 2k + 2C + 1$. Then we get

$$d_X(x,y) \leq d_X(x,x_0) + d_X(x_0, g.x_0) + d_X(g.x_0, y) \leq 4r + \lambda(2k + 2C + 1) + \varepsilon.$$

Since ρ is cocompact, x lies in the $2r$-neighbourhood of an axis $a_{\tilde{g}}$ of some hyperbolic isometry $\tilde{g} \in G$. It follows that y lies in the $(6r + \lambda(2k + 2C + 1) + \varepsilon)$-neighbourhood of $a_{\tilde{g}}$. Thus, in either of the two cases both x and y lie in the K'-neighbourhood of an axis $a_{\bar{g}}$ of some $\bar{g} \in G$, when $K' := M + 6r + \lambda\,(2k+2C+1) + \varepsilon$. Secondly, assume that $x \in g_1.B_r$ and $y \in g_2.B_r$ for some $g_1, g_2 \in G$. Then both $g_1^{-1}.x$ and $g_1^{-1}.y$ lie in the K'-neighbourhood of an axis $a_{\bar{g}}$ of some $\bar{g} \in G$ by the above argument. Hence, both x and y lie in the K'-neighbourhood of $g_1.a_{\bar{g}}$, and the latter is an axis of $g_1\bar{g}g_1^{-1}$. □

Proof of Thm. 6.0.29: Let $k \in \mathbb{N}$ be given according to Prop. 6.0.21. Let $x \in X$ and $x' \in X'$ be basepoints. Let $\lambda \geq 1$ and $\varepsilon \geq 0$ be given according to the Švarc-Milnor Lemma 1.1.1 such that the maps $g \mapsto g.x$ and $g \mapsto g.x'$ are (λ, ε)-quasi-isometries from $C_{\mathcal{A}}(G)$ to X, respectively to X'. Let $M, M' \geq 0$ be given for ρ, respectively for ρ', according to Lemma 6.0.33. By Lemma 6.0.30 the map $g.x \mapsto g.x'$ from $G.x$ to $G.x'$ is a G-equivariant bijection. In order to prove the theorem it is therefore enough to show that there exists a constant $K \geq 0$ such that $d(x, g.x) - K \leq d'(x', g.x') \leq d(x.gx) + K$ for each $g \in G$. We distinguish between two cases. Case 1: Suppose that $|g|_{\mathcal{A}} \geq 2k + 2C + 1$. Then, by Prop. 6.0.21, g has a k-separator h in G. Because of Lemma 6.0.33, x lies in the M-neighbourhood of an axis $a_{gh} : \mathbb{R} \to X$ of gh in X; and x' lies in the M'-neighbourhood of an axis $a'_{gh} : \mathbb{R} \to X'$ of gh in X'. Hence, there are $t, t' \in \mathbb{R}$ such that $d(x, a_{gh}(t)) \leq M$ and $d'(x', a'_{gh}(t')) \leq M'$. It follows on the one hand that

$$\begin{aligned} d(x, g.x) &\leq d(x, a_{gh}(t)) + d(a_{gh}(t), gh.a_{gh}(t)) + d(gh.a_{gh}(t), gh.x) + d(gh.x, g.x) \\ &\leq \left|gh\right|_{\rho} + 2M + \lambda\,(2k + 2C + 1) + \varepsilon, \end{aligned}$$

and on the other hand that

$$\begin{aligned} |\,gh\,|_{\rho} &= d(a_{gh}(t), gh.a_{gh}(t)) \\ &\leq d(a_{gh}(t), x) + d(x, g.x) + d(g.x, gh.x) + d(gh.x, gh.a_{gh}(t)) \\ &\leq d(x, g.x) + 2M + \lambda\,(2k + 2C + 1) + \varepsilon. \end{aligned}$$

Analogous inequalities are obtained for $d'(x', g.x')$ and $|\,gh\,|_{\rho'}$. Setting $R := 2M + \lambda\,(2k + 2C + 1) + \varepsilon$ and $R' := 2M' + \lambda\,(2k + 2C + 1) + \varepsilon$, we get

$$\begin{aligned} |\,gh\,|_{\rho} - R \leq d(x, g.x) &\leq |\,gh\,|_{\rho} + R \\ |\,gh\,|_{\rho'} - R' \leq d'(x', g.x') &\leq |\,gh\,|_{\rho'} + R'. \end{aligned}$$

Since the marked length spectra for ρ and ρ' coincide, it follows that

$$d(x, g.x) - (R + R') \leq d'(x', g.x') \leq d(x, g.x) + (R + R').$$

Case 2: Suppose that $|g|_{\mathcal{A}} < 2k+2C+1$. Then we get $d(x, g.x) \leq \lambda(2k+2C+1)+\varepsilon$ and $d'(x', g.x') \leq \lambda(2k + 2C + 1) + \varepsilon$. Hence,

$$d(x, g.x) - (R + R') \leq d'(x', g.x') \leq d(x, g.x) + (R + R')$$

holds also in Case 2; and we can take $K := R + R'$. $\square$

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