Energy Conservation for the Euler Equations with Boundaries

by

Jack William Daniel Skipper

Thesis

Submitted for the degree of

Doctor of Philosophy

Mathematics Institute

The University of Warwick

July 2018

Contents

Acknowledgments

Firstly, my thanks must go to my supervisors, James Robinson and José Rodrigo, for their patient help, suggestions and useful guidance.

Secondly, thanks to my fellow MASDOC students for the constructive criticism and fun events. Also, to the MASDOC directors for the help, feedback and attention to detail throughout the MASDOC course. Further, my thanks to the Engineering and Physical Sciences Research Council for funding my study through MASDOC.

I would like to thank my RSG supervisors Florian Theil and Tim Sullivan, for the skills they helped develop on presentation of mathematics throughout RSG that has hopefully made this thesis a better read.

I am honored to thank my MASDOC cohort for their help, especially Jamie Lukins for the hours of proof-reading of my terrible English and Matthew Egginton for help with LaTeX.

I would like to thank my family for their support through my education to get me here now and my friends and girlfriend, Catherine, for keeping the days fun and memorable throughout the PhD.

Finally, thanks to the Warwick Ultimate Frisbee team, as without the sporting release I would not be sane today.

Dedicated to Bob Skipper.

Declarations

I declare that, to the best of my knowledge, the material contained in this dissertation is original and my own work except where otherwise indicated, cited, or commonly known.

The material in this dissertation is submitted to the University of Warwick for the degree of Doctor of Philosophy, and has not been submitted to any other university or for any other degree.

Abstract

In this thesis we study energy conservation for the incompressible Euler equations that model non-viscous fluids. This has been a topic of interest since Onsager conjectured regularity conditions for solutions to conserve energy in 1949. Very recently the full conjecture has been resolved in the case without boundaries.

We first perform a study of the different conditions used to ensure energy conservation for domains without boundaries. Results are presented in Chapter 2, as well as an analysis of the similarities between the weakest of these conditions and the conditions we use later with a boundary.

We then study the time regularity in Chapter 3 and present a detailed proof for energy conservation without boundaries imposing the conditions $u \in L^3(0,T;L^3)$ and

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_0^T \int |u(x+y) - u(x)|^3 \,\mathrm{d}x\,\mathrm{d}t = 0.$$

In Chapters 4 and 5 we consider the easiest case of a flat finite boundary corresponding to the domain $\mathbb{T}^2 \times \mathbb{R}_+$. In Chapter 4 we use an extension argument and impose a condition of continuity at the boundary to prove energy conservation under the conditions that $u \in L^3(0,T;L^3(\mathbb{T}^2 \times \mathbb{R}_+))$,

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_0^T \int_{\mathbb{T}^2} \int_{|y|}^{\infty} |u(x+y) - u(x)|^3 \,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}t = 0,$$

$u \in L^3(0,T;L^\infty(\mathbb{T}^2 \times [0,\delta)))$ and u is continuous at the boundary. We then improve this result further by making it a local method in Chapter 5 and use a different definition of a weak solution where there is no pressure term involved.

Chapter 6 considers various different definitions of weak solutions for the incompressible Euler equations on a bounded domain. We study the relations between these varying definitions with and without pressure terms. We then use the recent work of Bardos & Titi (2018), who showed energy conservation with pressure terms included, to get a condition for energy conservation when we consider a weak solution without reference to the pressure term.

Chapter 1

Introduction

In this thesis we will focus on the incompressible Euler equations, which model the movement of non-viscous fluids. Denoting by u the velocity field of the fluid and by p the scalar function corresponding to the pressure we can write the equations for the pair (u, p) as

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0, \quad \nabla \cdot u = 0.$$

We study the equation on domains without a boundary ($\mathbb{R}^d$ or $\mathbb{T}^d$ with $d \geq 2$) and on domains Ω with a boundary. In the last case we impose the no-flux boundary condition, that is, $u \cdot n = 0$ on $\partial\Omega$, where n is the outward normal to the boundary. We will study whether these solutions conserve energy, that is, for every $t \in [0, T]$ we have $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$ and what are the weakest regularity conditions needed to ensure this energy conservation.

If we consider sufficiently smooth solutions u that are C^1 on $\mathbb{R}^d$ or $\mathbb{T}^d$ with $d \geq 2$, or solutions u that are C^1 on a domain Ω with a Lipschitz boundary (satisfying $u \cdot n = 0$ on $\partial\Omega$ on the boundary) then an easy integration-by-parts argument shows that energy is conserved. However, when considering weak solutions, as defined in Section 3.1, we only have $u \in C_{\mathrm{w}}(0, T; L^2) \cap L^3(0, T; L^3)$ and we do not have the regularity needed to perform these operations. We therefore need to regularise the equation first, manipulate the now smooth terms and then converge back to the original weak solution imposing necessary conditions on the solution so that energy is conserved.

Onsager's conjecture (1949) was originally formulated into two parts. Firstly, a 'positive' part stating that weak solutions satisfying a Hölder continuity condition of order greater than one third in space should conserve energy, that is, if a weak solution $u(\cdot, t) \in C^{1/3+\varepsilon}$ for some $\varepsilon > 0$ then u will conserve energy. Secondly, a 'negative' part conjecturing that there exists solutions $u(\cdot, t) \in C^{1/3-\varepsilon}$ that do not conserve energy. Nowadays the conjecture is formulated to consider the weak solution u with regularity in both space and time as below.

Conjecture 1.1 *Onsager's Conjecture for a weak solution $u \in C_{\mathrm{w}}(0, T; H_\sigma)$ of the Euler equations states that:*

- *('Positive part') if $u \in L^3(0, T; C^{1/3+\varepsilon})$ for some $\varepsilon > 0$ then u conserves energy, that is, for every $t \in [0, T]$ we have $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$*
- *('Negative part') and for every $\varepsilon > 0$ there exists least one solution $u \in L^3(0, T; C^{1/3-\varepsilon})$ that does not conserve energy for ε made arbitrarily small but positive.*

The 'negative' part has been resolved in the very recent works of Isett (2016) and Buckmaster et al. (2016) where solutions are constructed prescribing an arbitrary energy profile and we will not consider this problem here. Bardos & Titi (2009) and Bardos et al. (2012) constructed explicit shear solutions in $L^\infty(0, \infty; L^2)$ that do conserve energy and Bardos & Titi (2013) notes that this shows that low regularity of a weak solution does not imply energy dissipation (or creation). The case where the solutions have regularity exactly $u(\cdot, t) \in C^{1/3}$ is still open.

We will focus on the first part of this conjecture, i.e. conditions for energy conservation, and will consider the problem in several different domains.

1.1 Historical results

The majority of the studies on energy conservation for the incompressible Euler equations have been carried out on the domains $\mathbb{R}^d$ or $\mathbb{T}^d$. The important property of these domains is that they have no boundary and thus no boundary conditions that complicate the calculations. For the next few paragraphs we will go through the past work on $\mathbb{R}^d$ or $\mathbb{T}^d$ gradually weakening the conditions for energy conservation until we will obtain the condition

$$\lim_{q\to\infty}\int_0^T 2^q\|\Delta_q u\|_{L^3}^3\,\mathrm{d}t = 0,$$

where Δ_q performs a smooth restriction of u into Fourier modes of order 2^q (see Chapter 2).

The first proof of energy conservation for weak solutions was given by Eyink (1994) on the torus. The method, taking inspiration from the ideas of Onsager (1949), involved studying a Fourier formulation of the equation and writing the solution as a Fourier series. He then studies the solution in dyadic Fourier modes and observes that if the series representing the nonlinear term converges absolutely then the order of the sums can be commuted and we have energy conservation. This is linked to looking at the energy flux between different scales in Fourier space and the main proof revolves around controlling the energy flux to the large Fourier modes. Since controlling the large scale Fourier modes means that we are imposing conditions on the small scale fluctuations of the function this imposes a differentiability condition on the function. Energy conservation is obtained assuming that the solution satisfies $u(\cdot,t)\in C_\star^\alpha$ for $\alpha>1/3$ with a uniform bound for $t\in[0,T]$. A definition of the space $C_\star^\alpha$ equivalent to that of Eyink's is as follows: expand u as the Fourier series

$$u=\sum_{k\in\mathbb{Z}^3}\hat{u}_k\mathrm{e}^{\mathrm{i}k\cdot x},$$

imposing conditions to ensure that u is real ($\hat{u}_k=\overline{\hat{u}_{-k}}$) and is divergence free ($k\cdot\hat{u}_k=0$); then $u\in C_\star^\alpha(\mathbb{T}^3)$ if

$$\sum_{k\in\mathbb{Z}^3}|k|^\alpha|\hat{u}_k|<\infty.$$

Requiring $u\in C_\star^\alpha$ with $\alpha>1/3$ is a stronger condition than the one-third Hölder continuity conjectured by Onsager.

Subsequently Constantin, E, & Titi (1994) gave a short proof of energy conservation, in the framework of Besov spaces (but still on the torus), under the weaker assumption that

$$u \in L^3(0,T;B^{\alpha}_{3,\infty}) \qquad \text{with} \qquad \alpha > 1/3. \tag{1.1}$$

As $C^\alpha \subset B^{\alpha}_{3,\infty}$ this proves the 'positive' part of Onsager's Conjecture with no boundary. Here $B^s_{p,r}$ denotes a Besov space as defined in Bahouri et al. (2011) and Lemarié-Rieusset (2002). The α in $B^{\alpha}_{p,r}$ corresponds to the amount of 'differentiable regularity' of the function, but measured in other spaces to have different control of the 'integrability'.

This method involved regularising the weak formulation of the equation by mollification and then noting that we have energy conservation if we can permute the mollification operator with the product operator. This is a similar problem to that studied by Eyink (1994) before, however, here we must find conditions on the solution u to re-order integrals rather than sums. Here the remainder terms left over from permuting the product of solutions with mollification are similar to the energy flux problem in the work by Eyink (1994). They study the properties of the solution u at small scales and observe that with the regularity $u \in L^3(0,T;B^{\alpha}_{3,\infty})$ with $\alpha > 1/3$ one can control the remainder terms and show that they tend to zero at small scales.

Duchon & Robert (2000) showed that solutions satisfying a weaker regularity condition still conserve energy. They derived a local energy equation that contains a term $D(u)$ representing the dissipation or production of energy caused by the lack of smoothness of u; this term can be seen as a local version of Onsager's original statistically averaged description of energy dissipation used to motivate the original conjecture. Here the term $D(u)$ is of the form

$$D(u)(x) = \lim_{\varepsilon\to 0} \tfrac{1}{4} \int \nabla\varphi_\varepsilon(\xi) \cdot u(x-\xi) - u(x)|u(x-\xi) - u(x)|^2 \,\mathrm{d}\xi$$

where φ_ε is a 'nice mollifier' as defined later in (3.3). They showed that if u satisfies

$$\frac{1}{|\xi|}\int |u(t,x+\xi)-u(t,x)|^3 \,\mathrm{d}x \le C(t)\sigma(|\xi|), \tag{1.2}$$

where $\sigma(a)\to 0$ as $a\to 0$ and $C\in L^1(0,T)$, then $\|D(u)\|_{L^1(0,T,L^1(\mathbb{T}^3))}=0$ and hence the kinetic energy is conserved. The condition in (1.2) is weaker than (1.1). A detailed review examining this and further work relating to Onsager's conjecture is given by Eyink & Sreenivasan (2006).

More recently energy conservation was shown by Cheskidov et al. (2008) when u lies in the space $L^3(0,T;B^{1/3}_{3,c(\mathbb{N})})$, where $B^{1/3}_{3,c(\mathbb{N})}$ is a subspace of $B^{1/3}_{3,\infty}$. We can introduce two operators S_q and Δ_q which perform smooth restrictions of functions in Fourier space, with the full definitions given in Chapter 2. S_q restricts to all Fourier modes below 2^q and Δ_q restricts to the modes of order 2^q. Using these operators we can define the space

$$B^{1/3}_{3,\infty}(\mathbb{R}^d) := \left\{ f : f\in\mathcal{S}' \quad \text{and} \quad \|S_0 f\|_{L_3} + \left\| 2^{\frac{q}{3}}\|\Delta_q f\|_{L^3}\right\|_{l^\infty(q,\mathbb{N})} < \infty \right\},$$

further, we can define the subspace

$$B^{1/3}_{3,c(\mathbb{N})} := \left\{ f : f\in B^{1/3}_{3,\infty} \quad \text{and} \quad \lim_{q\to\infty} 2^{\frac{q}{3}}\|\Delta_q f\|_{L^3} = 0 \right\}.$$

The space of $L^3(0,T;B^{1/3}_{3,\infty})$ is important with regards to Onsager's conjecture as for any dimension $d\ge 2$ it has the correct scaling for energy conservation and is coined an Onsager-critical space in Shvydkoy (2010). For some function space B we say that it is Onsager-critical if denoting velocity by U, length by X and time by T we have the relation

$$(\dim \|\cdot\|_B)^3 = TU^3X^{d-1}.$$

This scaling comes from studying the term

$$\int_0^t \int_\Omega \nabla\cdot(u\otimes u)\cdot u \,\mathrm{d}x\,\mathrm{d}t,$$

which we obtain by testing the Euler equations with u and integrating over space and time. We see that we have three velocity terms, one integration in time, an integration over all of space and then one derivative and so we obtain the scaling TU^3X^{d-1} on the right hand side.

In fact Cheskidov et al. (2008) showed that energy conservation holds for solutions satisfying the still weaker condition

$$\lim_{q\to\infty}\int_0^T 2^q\|\Delta_q u\|_{L^3}^3\,\mathrm{d}t = 0. \tag{1.3}$$

In a follow-up paper Shvydkoy (2009) (see also Shvydkoy, 2010) states that this condition is equivalent to

$$\lim_{|y|\to 0}\frac{1}{|y|}\int_0^T\int |u(x+y)-u(x)|^3\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{1.4}$$

and proves a local energy balance under this condition. Here the method involves using the Fourier definition of Besov spaces, splitting the solution into a sum of Δ_q Fourier modes, and truncating the series to regularise the equation. Then with similar methods to Eyink (1994) and Constantin, E, & Titi (1994) a flux term is derived and is split up into two terms corresponding to different scales. Here a bound is obtained treating the small scales and large scales separately. With this method they were able to relax the condition for energy conservation still further. We observe that condition (1.2) has similar form to (1.4), yet explicitly separates the limit and the integrability in time. This makes (1.4) less restrictive than (1.2) which will be shown in Chapter 2 along with a proof of the equivalence of condition (1.3) to condition (1.4).

When discussing the problem on a bounded domain Ω with the condition $u\cdot n = 0$ on the boundary it becomes harder to prove energy conservation. While it is now well understood how the potential dissipation or creation of energy could be generated by the local interaction of the fluid (or high wave numbers in Fourier space) there is now the added complication of how to regularise the equation and keep the boundary data and the incompressibility condition. In all the previous works the incompressibility of the solution is preserved by the regularising techniques used and this significantly helps with the calculations as the pressure term no longer appears. Further, the regularising techniques involve a non-local operator and thus more refined methods would have to be used to maintain the boundary data.

Recently Bardos & Titi (2018) have considered the case of energy conservation on a bounded domain with a C^2 boundary and showed energy conservation to hold for solutions u, where $u \in L^3(0,T;C^\alpha(\bar{\Omega}))$ for $\alpha > 1/3$, to give a proof of Onsager's conjecture on a bounded domain. Their definition of a weak solution uses smooth, compactly supported test functions ψ, but without any assumption of incompressibility; therefore the pressure is included in the weak formulation. More precisely, a pair (u,p) is a weak solution if for $\psi \in C_c^\infty(\Omega \times (0,T))$

$$\langle u, \partial_t \psi \rangle_\Omega + \langle u \otimes u : \nabla \psi \rangle_\Omega + \langle p : \nabla \cdot \psi \rangle_\Omega = 0, \quad \text{in} \quad L^1(0,T),$$

where $\langle \cdot, \cdot \rangle_\Omega$ denotes the L^2 inner-product over Ω. In the analysis of the equation in this formulation estimates for p are required. To obtain these estimates they use the fact that u and p are connected via an elliptic equation. Namely, p weakly solves

$$-\Delta p = \partial_i \partial_j (u_i u_j) \quad \text{in} \quad \Omega, \quad \text{and} \quad \nabla p \cdot n = -(u_j \partial_j u_i) n_i \quad \text{on} \quad \partial\Omega,$$

where we sum the components over the repeated indices. They only assume that $\psi \in C_c^\infty(\Omega \times (0,T))$ and do not include incompressibility in the test functions. This allows the use of smooth cut-off functions to be used to restrict u and then mollification can be applied to the restricted u and so it can be used as a test function as now smooth and compactly supported.

A similar method to Constantin, E, & Titi (1994) is then applied, however, for the non-linear and pressure terms extra remainder terms are produced from the gradient of the smooth cut-off function. Here using the boundary conditions, that $u \cdot n = 0$ on the boundary and that $u \in L^3(0,T;C^\alpha(\bar{\Omega}))$ for $\alpha > 1/3$, these remainder terms are shown to vanish.

1.2 Outline

In this thesis we will present work proving energy conservation on domains with a boundary; however we will use incompressible test functions and so no pressure term appears in the weak formulation of the equation. In Chapter 2 we will present preliminary work that compares previous conditions used to impose energy conservation on domains without a boundary. The main result of Proposition 2.9 was stated before in Shvydkoy (2009), but the proof is a new proof.

In Chapter 3 we present a new proof of energy conservation on domains without a boundary, i.e. for $\mathbb{R}^d$ or $\mathbb{T}^d$ for $d \geq 2$. Here we concentrate on a new method to rigorously derive sufficient time regularity of $J_\varepsilon u$ so that it can be used as a test function. We focus on techniques that are easily extendable to a domain with a boundary with the extra steps presented in Chapter 5. We then adapt some of the ideas from Duchon & Robert (2000) and give a direct new proof that energy conservation follows on the whole domain under the condition that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \nabla\varphi_\varepsilon(\xi)\cdot(v(x+\xi)-v(x))|v(x+\xi)-v(x)|^2\,\mathrm{d}\xi\,\mathrm{d}x \to 0$$

as $\varepsilon \to 0$, where φ is a radial mollifier, as defined in (3.3).

Given this condition it is relatively simple to show energy conservation under the assumption (1.4), which we do in Theorem 3.9, and under the alternative condition

$$\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^3}{|x-y|^{4+\delta}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t < \infty, \qquad \delta > 0, \tag{1.5}$$

which is equivalent to requiring $u \in L^3(0,T;W^{\alpha,3}(\mathbb{R}^3))$ for some $\alpha > 1/3$ (Theorem 3.10). Although energy conservation of a weak solution was known under assumption (1.4) the proofs in this section are new and we obtain a new result of energy conservation using assumption (1.5).

In Chapter 4 we use condition (1.4) to analyse energy conservation in the domain $D_+ := \mathbb{T}^2 \times \mathbb{R}_+$. We show that if (u,p) is a weak solution, in the sense of Definition 4.1, on D_+ where p is a distribution defined on the boundary only, that is, $p \in \mathcal{D}'(\partial D_+ \times [0,T])$ then (u_R, p) is a weak solution, in the sense of Definition 4.1, on D_-. Here u_R is an appropriately 'reflected' version of u, and that $u_E := u + u_R$ almost everywhere, is a weak solution, in the sense of Definition 3.2, on $D := \mathbb{T}^2 \times \mathbb{R}$. It follows that energy is conserved for u_E under condition (1.4); from here we deduce conservation of energy for u under the condition

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \int_{\mathbb{T}^2} \int_{|y|}^{\infty} |u(t,x+y) - u(t,x)|^3 \, \mathrm{d}x_3 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}t = 0.$$

This condition is very similar to the best known condition for the spatial domains $\mathbb{R}^3$ or $\mathbb{T}^3$ with just an extra restriction so that it is only acting on the interior of the domain. We require additional assumptions near ∂D_+, where we assume that $u \in L^3(0,T; C^0(\mathbb{T}^2 \times [0,\delta))$ for some $\delta > 0$, see Theorem 4.6. Though this was the first proof of energy conservation with a boundary it uses a global extension (from $\mathbb{T}^2 \times \mathbb{R}_+$ to $\mathbb{T}^2 \times \mathbb{R}$) which is a potential drawback when trying to generalise to more domains, though this is solved in Chapter 5. The conditions needed here for energy conservation are weaker than those in Bardos & Titi (2018) and give a good indication of the conditions for energy conservation that should be aimed for in other bounded domains.

In Chapter 5 we use u_r a locally 'reflected' version of u and obtain energy conservation with the same assumptions as before on the domain D_+. However, here we use incompressible test functions that also satisfy the no-flux boundary conditions and so there are no pressure terms appearing in the weak formulation of the equation. Further, the extension is done locally around the boundary which may be beneficial when considering extending this result to bounded domains. Again the proofs in this section are all new as we generalise our result from Chapter 4.

Finally, in Chapter 6 we consider the recent results in Bardos & Titi (2018). We compare their definition of a weak solution (which includes the pressure) and the definition of a weak solution assuming incompressibility of the test functions, where no pressure term is involved. We show that assuming $u \in L^3(0,T; C^\delta)$ for some $\delta > 0$ is enough to show that a solution defined using incompressible test functions only, so that no reference to a pressure term appears in the weak formulation, will also be a distributional solution, as defined in Bardos & Titi (2018). We can therefore apply the results in Bardos & Titi (2018) to the pressure-less definition of a weak solution to obtain energy conservation of our weak solution on a bounded domain with a C^2 boundary, where $u \in L^3(0, T; C^\alpha)$ for some $\alpha > 1/3$. Here we apply standard techniques to obtain this new result.

Chapter 2

A study of Energy conservation Conditions

Here we want to study the conditions discussed in Chapter 1 that have been used to show energy conservation. We will show that conditions (1.3) and (1.4) are equivalent and are the weakest conditions known to guarantee energy conservation.

We will define the Besov spaces $B^s_{p,r}$ and $B^s_{p,c(\mathbb{N})}$ (as a subspace of $B^s_{p,\infty}$) using the Littlewood-Paley decomposition. Then, for functions defined on $\mathbb{R}^d$, we give a proof of equivalence of the conditions (1.3), namely

$$\lim_{q\to\infty}\int_0^T 2^q\|\Delta_q u\|_{L^3}^3\,\mathrm{d}t = 0 \tag{2.1}$$

and (1.4), namely

$$\lim_{|y|\to 0}\frac{1}{|y|}\int_0^T\int |u(x+y)-u(x)|^3\,\mathrm{d}x\,\mathrm{d}t = 0$$

in Proposition 2.9. In order to prove the equivalence of the above conditions we will introduce the general definition of a Besov space and the notation we will be using.

Firstly, we will recall some important function spaces; the interested reader may consult Bahouri et al. (2011), for example.

Definition 2.1 (Schwartz space) *(Page 22-23, Robinson, Rodrigo, & Sadowksi (2016)) The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the space of all smooth functions f on $\mathbb{R}^d$ such that*

$$p_{k,\alpha}(f) := \sup_{x\in\mathbb{R}^d} |x|^k |\partial^\alpha u(x)|.$$

is finite for every choice of $k \in \mathbb{N}$ and $\alpha \geq 0$ where α is a multi-index (i.e. an element of $\mathbb{N}^d$).

This is the space of smooth functions where we have decay faster than any polynomial for every derivative. From this space we can define the associated dual space of tempered distributions.

Definition 2.2 (Tempered distributions) *(Page 22-23, Robinson, Rodrigo, & Sadowksi (2016)) We define the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ as the space of all bounded linear functionals T on $\mathcal{S}(\mathbb{R}^d)$ that are continuous in the sense that $T(f_n) \to 0$ as $n \to \infty$ if $(f_n) \in \mathcal{S}(\mathbb{R}^d)$ with $p_{k,\alpha}(f_n) \to 0$ as $n \to \infty$ for all k and α (as defined in Definition 2.1).*

We can now recall the definition of the Fourier transform on $\mathbb{R}^d$ from Bahouri et al. (2011) as

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi} f(x)\,\mathrm{d}x,$$

and define the inverse Fourier transform

$$\mathcal{F}^{-1}f(x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi} f(\xi)\,\mathrm{d}\xi,$$

where $\mathcal{F}, \mathcal{F}^{-1} \colon \mathcal{S} \to \mathcal{S}$.

Bahouri et al. (2011), page 22, proves that the Fourier transform is an automorphism of S' and an automorphism of L^2 and so we can extend $\mathcal{F}, \mathcal{F}^{-1} \colon L^2 \to L^2$ and can further extend them to act on tempered distributions.

We recall the definition in $\mathbb{R}^d$ of S_j, Δ_j, the usual building blocks of a Littlewood-Paley decomposition (see Lemarié-Rieusset (2002) and Bahouri et al. (2011)). We choose a smooth function $\Psi \in C_0^\infty(B_1(0))$ such that $\Psi(\xi) = 1$ for

$|\xi| \le \frac{1}{2}$ and let $\psi(\xi) = \Psi(\frac{\xi}{2}) - \Psi(\xi)$. We then have the useful properties that

$$\Psi(\xi) + \sum_{j\ge 0} \psi(\frac{\xi}{2^j}) = 1, \tag{2.2}$$

and further, for $|i-j| \ge 2$ implies that $\mathrm{supp}(\psi(\frac{\xi}{2^j})) \cap \mathrm{supp}(\psi(\frac{\xi}{2^i})) = \emptyset$.

Using ψ and Ψ as building blocks we and can now define, for $j \in \mathbb{Z}$,

$$\Delta_j u(x) := \mathcal{F}^{-1}\left(\psi\left(\frac{\xi}{2^j}\right)\mathcal{F}u\right)(x) = 2^{dj}\int h(2^j z)u(x-z)\,\mathrm{d}z, \tag{2.3}$$

$$S_j u(x) := \mathcal{F}^{-1}\left(\Psi\left(\frac{\xi}{2^j}\right)\mathcal{F}u\right)(x) = 2^{dj}\int \tilde{h}(2^j z)u(x-z)\,\mathrm{d}z, \tag{2.4}$$

where we have set $h := \mathcal{F}^{-1}\psi$ and $\tilde{h} := \mathcal{F}^{-1}\Psi$; notice that the integral of h over $\mathbb{R}^d$ is zero. Indeed, we see that

$$\mathcal{F}(f)(0) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} x\cdot 0} f(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f\,\mathrm{d}x,$$

thus

$$\int_{\mathbb{R}^d} h\,\mathrm{d}x = \mathcal{F}(h)(0) = \mathcal{F}(\mathcal{F}^{-1}\psi)(0) = \psi(0) = 0.$$

Due to (2.2) we have

$$S_0 + \sum_{j=1}^{\infty} \Delta_j = Id, \tag{2.5}$$

where we make sense of this decomposition in terms of tempered distributions. We can now make use of these operators S_j and Δ_j to define Besov spaces.

Definition 2.3 (Besov space) *Let $s \in \mathbb{R}$, $1 \le p, r \le \infty$. Then we define*

$$B^s_{p,r}(\mathbb{R}^d) := \left\{ f : f \in \mathcal{S}' \quad \text{and} \quad \|S_0 f\|_{L_p} + \left\| 2^{qs}\|\Delta_q f\|_{L^p} \right\|_{l^r(q,\mathbb{N})} < \infty \right\},$$

with the norm

$$\|f\|_{B^s_{p,r}} := \|S_0 f\|_{L_p} + \left\| 2^{qs}\|\Delta_q f\|_{L^p} \right\|_{l^r(q,\mathbb{N})}.$$

We can use S_j to define a useful subspace of tempered distributions $\mathcal{S}'_0$ where we have restricted the space so that Fourier transform of the tempered distribution is locally integrable around 0 thus controlling the low frequencies.

Definition 2.4 *The space $\mathcal{S}_0'(\mathbb{R}^n)$ is the space of tempered distributions $\mathcal{S}'$ such that for $f \in \mathcal{S}'$*

$$\lim_{m \to -\infty} S_m f = 0 \quad in \quad S'.$$

Using this definition and the operator Δ_j we can define the homogeneous Besov spaces. Here, unlike the non-homogeneous case, the the sum in the l^r norm is now over the space $\mathbb{Z}$.

Definition 2.5 (Homogeneous Besov space) *Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. Then we define*

$$\dot{B}^s_{p,r}(\mathbb{R}^d) := \left\{ f : f \in \mathcal{S}_0' \quad and \quad \left\| 2^{qs} \|\Delta_q f\|_{L^p} \right\|_{l^r(q,\mathbb{Z})} < \infty \right\},$$

with the semi-norm

$$\|f\|_{\dot{B}^s_{p,r}} := \left\| 2^{qs} \|\Delta_q f\|_{L^p} \right\|_{l^r(q,\mathbb{Z})}.$$

Finally we consider a subspace of $B^s_{p,\infty}$ that appears when considering the Onsager's conjecture; see Cheskidov et al. (2008), Shvydkoy (2009) and Shvydkoy (2010).

Definition 2.6 *The subspace $B^s_{p,c(\mathbb{N})}$ has the $B^s_{p,\infty}$ norm and is defined by*

$$B^s_{p,c(\mathbb{N})} := \{u : u \in B^s_{p,\infty} \quad and \quad \lim_{q \to \infty} 2^{qs} \|\Delta_q u\|_{L^p} = 0\}.$$

The use of Besov spaces allows for a more refined control of the function's differential regularity and integrability, thus with this greater control problems involving a critical regularity can be approached with more ease.

For certain exponents we obtain some well known spaces (see for example Bahouri et al. (2011), Chapter 2):

- The Sobolev space $H^s = B^s_{2,2}$.
- The non-integer Hölder spaces we have $C^{[s],s-[s]} = B^s_{\infty,\infty}$.

- However, for $s \in \mathbb{N}$ the space $B^s_{\infty,\infty}$ is strictly larger than C^s and $C^{s-1,1}$.

In order to show that for $s \notin \mathbb{N}$ that $C^{[s],s-[s]} = B^s_{\infty,\infty}$ at least for $s \in (0,1)$ we can use the definition of Besov spaces as defined in Peetre (1976) who use a difference quotient and for $s \in (0,1)$

$$B^s_{p,q} := \left\{ f : f \in L^p \text{ and } \left(\int_{\mathbb{R}^d} \left(\frac{\|f(\cdot + h) - f(\cdot)\|_{L^p}}{|h|^s} \right)^q \frac{\mathrm{d}h}{|h|^d} \right)^{\frac{1}{q}} < \infty \right\}, \tag{2.6}$$

and in the case where $q = \infty$ one requires

$$B^s_{p,\infty} = \left\{ f : f \in L^p \text{ and } \sup_h \frac{\|f(\cdot + h) - f(\cdot)\|_{L^p}}{|h|^s} < \infty \right\}.$$

We see that if we want the $B^s_{\infty,\infty}$ space then we can consider the space

$$B^s_{\infty,\infty} = \left\{ f : f \in C^0 \text{ and } \sup_h \frac{\|f(\cdot + h) - f(\cdot)\|_{L^\infty}}{|h|^s} < \infty \right\}$$

and see that this is the definition of the space C^s for $s \in (0,1)$.

In Bahouri et al. (2011) (Theorem 2.36), they prove, for $s \in (0,1)$ and $(p,r) \in [1,\infty]^2$, the equivalence of the norm $\|\cdot\|_{\dot{B}^s_{p,r}}$, defined in Definition 2.5 and the norm

$$\|f\|_X := \left(\int_{\mathbb{R}^d} \left(\frac{1}{|y|^s} \left(\int_{\mathbb{R}^d} |f(x+y) - f(x)|^p \, \mathrm{d}x \right)^{1/p} \right)^r \frac{\mathrm{d}y}{|y|^d} \right)^{1/r},$$

which shows the equivalence of the space defined by (2.6) (for the homogeneous case) and the space given by Definition 2.5.

We conclude this introduction with useful embeddings. Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$; then for any $s \in \mathbb{R}$, the space $B^s_{p_1,r_1}$, is continuously embedded in

$$B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}.$$

2.1 Relations between Energy Conservation Conditions

With the use of Besov spaces we will now study the various conditions that have been used to ensure energy conservation on the solutions to the incompressible Euler equations, on $\mathbb{R}^d$.

Firstly, we will study the original energy conservation result for Onsager's conjecture by Constantin, E, & Titi (1994) and compare it to the later condition of Cheskidov et al. (2008). We will show that Cheskidov et al. (2008) was an improvement of Constantin, E, & Titi (1994).

Proposition 2.7 *If* $u \in L^3\left(0,T;B^{\alpha}_{3,\infty}\right)$ *for some* $\alpha > 1/3$ *then* $u \in L^3\left(0,T;B^{1/3}_{3,c(\mathbb{N})}\right)$.

Proof We know that for $\varepsilon = \alpha - 1/3 > 0$

$$2^{\frac{q}{3}}\|\Delta_q f\|_{L^3} = 2^{\alpha q}\|\Delta_q f\|_{L^3}2^{-\varepsilon q}.$$

Thus

$$\sup_{q\geq q_0}\{2^{\frac{q}{3}}\|\Delta_q f\|_{L^3}\} = \sup_{q\geq q_0}\{2^{\alpha q}\|\Delta_q f\|_{L^3}2^{-\varepsilon q}\} \leq \sup_{q\geq q_0}\{2^{-\varepsilon q}\}\sup_{q\geq q_0}\{2^{\alpha q}\|\Delta_q f\|_{L^3}\}.$$

Taking limits as $q_0 \to \infty$ we obtain that

$$\limsup_{q\to\infty}\{2^{\frac{q}{3}}\|\Delta_q f\|_{L^3}\} \leq \lim_{q\to\infty}\{2^{-\varepsilon q}\}\sup_{q}\{2^{\alpha q}\|\Delta_q f\|_{L^3}\}.$$

Assuming that $f \in B^{\alpha}_{3,\infty}$ then $\sup_q\{2^{\alpha q}\|\Delta_q f\|_{L^3}\}$ is finite and since $\lim_{q\to\infty}\{2^{-\varepsilon q}\} = 0$ we have

$$\lim_{q\to\infty}\{2^{\frac{q}{3}}\|\Delta_q f\|_{L^3}\} = 0$$

and so we are done. □

Proposition 2.8 *If* $u \in L^3\left(0,T;B^{\frac{1}{3}}_{3,c(\mathbb{N})}\right)$ *then* u *satisfies* (2.1), *namely*

$$\lim_{q\to\infty}\int_0^T 2^q\|\Delta_q u\|^3_{L^3}\,\mathrm{d}t = 0.$$

Proof For $u \in L^3\left(0,T;B^{\frac{1}{3}}_{3,c(\mathbb{N})}\right)$ we have by definition,

$$\int_0^T \left(\sup_q\{2^{\frac{q}{3}}\|\Delta_q u\|_{L^3}\}\right)^3 \mathrm{d}t < \infty \quad \text{and} \quad \lim_{q\to\infty} 2^q\|\Delta_q u(t)\|_{L^3}^3 = 0 \text{ for a.e. } t.$$

Setting $f_q(t) := 2^q\|\Delta_q u\|_{L^3}^3$, this gives $\lim_{q\to\infty} f_q(t) = 0$ a.e. Further setting

$$g(t) := \left(\sup_q\left\{2^{\frac{q}{3}}\|\Delta_q u\|_{L^3}\right\}\right)^3$$

then $f_q(t) \leq g(t)$ a.e. t and $g(t)$ is integrable thus we can apply the Dominated Convergence Theorem (DCT) to finish the proof. □

The next proposition was stated but not proved in Shvydkoy (2010). It involves the equivalence of the best condition currently known used to ensure energy conservation in terms of the dyadic decomposition and the condition that we want to consider in the coming chapters in terms of integrals.

Proposition 2.9 *For $u \in L^3(0,T;L^3(\mathbb{R}^d))$, and $s \in (0,1)$ the condition* (1.3)*, namely*

$$\lim_{q\to\infty}\int_0^T \left(2^{sq}\|\Delta_q u\|_{L^3}\right)^3 \mathrm{d}t = 0$$

is equivalent to condition (1.4)*, namely*

$$\lim_{|y|\to 0}\frac{1}{|y|^{3s}}\int_0^T \|u(\cdot+y)-u(\cdot)\|_{L^3}^3 \,\mathrm{d}x\,\mathrm{d}t = 0.$$

The proof is based on the analysis in Bahouri et al. (2011) (Theorem 2.36). For the equivalence of conditions (1.3) and (1.4) on $\mathbb{R}^d$, we will use the case where $s = 1/3$. Further, note that the equivalence of similar expressions could be shown for different L^p, $L^q\left(\frac{\mathrm{d}y}{|y|^d}\right)$ norms and not just for $p = 3$ and $q = \infty$, yet for simplicity we restrict to the case we actually use in the thesis.

Proof We will split the proof into two parts and start by showing, for $s \in (0,1)$, that if u satisfies

$$\lim_{q\to\infty}\left\|2^{sq}\|\Delta_q u\|_{L^3}\right\|_{L^3([0,T])} = 0$$

then

$$\lim_{|y|\to 0} \left\| \frac{1}{|y|^s} \|u(\cdot + y) - u(\cdot)\|_{L^3} \right\|_{L^3([0,T])} = 0.$$

We notice that the terms in the limit are all non-negative so it suffices to find an upper bound that goes to zero in the limit. Firstly using (2.5) we obtain the bound,

$$\|u(\cdot + y) - u(\cdot)\|_{L^3} \leq \|S_0 u(\cdot + y) - S_0 u(\cdot)\|_{L^3} + \sum_{q=1}^{\infty} \|\Delta_q u(\cdot + y) - \Delta_q u(\cdot)\|_{L^3}. \quad (2.7)$$

We will now calculate one bound for the first term and two bounds for the second term, one of which we will use for small q and the other for large q.

For the $S_0 u$ terms we notice that $S_0 u = S_1 S_0 u$ by looking at the supports of $\Psi(\cdot)$ and $\Psi(2\cdot)$; more precisely, we notice that $\Psi(2\cdot) = 1$ over the support of $\Psi(\cdot)$ so we obtain

$$S_0 u(x + y) - S_0 u(x) = S_1 S_0 u(x + y) - S_1 S_0 u(x).$$

We can then use the definition of S_1, given by (2.4), in terms of $\tilde{h}$, to obtain

$$S_1 S_0 u(x + y) - S_1 S_0 u(x) = 2^d \int_{\mathbb{R}^d} \left[\tilde{h}(2(x + y - z)) - \tilde{h}(2(x - z)) \right] S_0 u(z)\, \mathrm{d}z.$$

We can then use the Fundamental Theorem of Calculus to obtain,

$$S_0 u(x + y) - S_0 u(x) = \sum_{i=1}^{d} 2y_i \int_{\mathbb{R}^d} \left(\int_0^1 2^d [\partial_i \tilde{h}]\, (2x + 2\zeta y - 2z)\ \mathrm{d}\zeta \right) S_0 u(z)\, \mathrm{d}z.$$

Since $\|2^d \partial_i \tilde{h}\, (2(\cdot + \zeta y))\, \|_{L^1}$ is uniformly bounded independently of ζ and y, so we can apply Young's convolution inequality,

$$\begin{aligned} \|S_0 u(\cdot + y) - S_0 u(\cdot)\|_{L^3} &\leq C|y| \sup_{i=1,\dots,d} \left[\int_0^1 \|2^d \partial_i \tilde{h}\, (2(\cdot + \zeta y))\, \|_{L^1}\, \mathrm{d}\zeta \right] \|S_0 u\|_{L^3} \quad (2.8) \\ &\leq C|y| \|S_0 u\|_{L^3}. \end{aligned}$$

For the second term of (2.7) we can simply bound using the triangle inequality and the translation invariance of the norm:

$$\|\Delta_q u(\cdot+y)-\Delta_q u(\cdot)\|_{L^3}\leq 2\|\Delta_q u\|_{L^3}. \tag{2.9}$$

We will use this bound for large q later on. We can also bound the second term of (2.7) using a similar method to the S_0 term. First note that by considering the supports of $\psi(2^q\cdot)$ and $\psi(2^{q'}\cdot)$ we obtain

$$\Delta_q=\sum_{|q'-q|\leq 1}\Delta_q\Delta_{q'}.$$

Thus, using the definition of Δ_q, given by (2.3), in terms of h, to obtain

$$\begin{aligned}\Delta_q u(x+y)-\Delta_q u(x) &= \sum_{|q'-q|\leq 1}\big(\Delta_q\Delta_{q'}u(x+y)-\Delta_q\Delta_{q'}u(x)\big)\\ &= \sum_{|q'-q|\leq 1}2^{qd}\int_{\mathbb{R}^d}\left[h(2^q(x+y-z))-h(2^q(x-z))\right]\Delta_{q'}u(z)\,\mathrm{d}z.\end{aligned}$$

We can then use the Fundamental Theorem of Calculus to obtain

$$\begin{aligned}&\Delta_q u(x+y)-\Delta_q u(x)\\ &\qquad=\sum_{|q'-q|\leq 1}\sum_{i=1}^{d}2^q y_i\int_{\mathbb{R}^d}\int_0^1 2^{qd}\left[\partial_i h\right]\left(2^q(x+\zeta y-z)\right)\mathrm{d}\zeta\,\Delta_{q'}u(z)\,\mathrm{d}z.\end{aligned}$$

As $\|2^{qd}[\partial_i h]\left(2^q\cdot+2^q\zeta y\right)\|_{L^1}$ is uniformly bounded independently of q,ζ and y we can apply Young's convolution inequality to obtain

$$\|\Delta_q u(\cdot+y)-\Delta_q u(\cdot)\|_{L^3}\leq C2^q|y|\sum_{|q'-q|\leq 1}\|\Delta_{q'}u\|_{L^3}. \tag{2.10}$$

We will use this bound for small q later on.

Combining estimate (2.8) and estimate (2.9) for large q and estimate (2.10) for small q to bound (2.7) and choosing to split the sum for small $q\leq k$ and large

$q > k$ we obtain,

$$\|u(\cdot + y) - u(\cdot)\|_{L^3} \leq C \left(|y| \|S_0 u\|_{L^3} + \sum_{1 \leq q \leq k} |y| 2^q \sum_{|q'-q| \leq 1} \|\Delta_{q'} u\|_{L^3} \right.$$
$$\left. + 2 \sum_{q > k} \|\Delta_q u\|_{L^3} \right).$$

Now we see that

$$\sum_{1 \leq q \leq k} |y| 2^q \sum_{|q'-q| \leq 1} \|\Delta_{q'} u\|_{L^3} = \sum_{1 \leq q \leq k} |y| 2^q \left(\|\Delta_{q-1} u\|_{L^3} + \|\Delta_q u\|_{L^3} + \|\Delta_{q+1} u\|_{L^3} \right)$$
$$\leq C \sum_{0 \leq q \leq k+1} |y| 2^q \|\Delta_q u\|_{L^3}$$

and so we obtain

$$\|u(\cdot + y) - u(\cdot)\|_{L^3} \leq C \left(|y| \|S_0 u\|_{L^3} + \sum_{0 \leq q \leq k+1} |y| 2^q \|\Delta_q u\|_{L^3} + 2 \sum_{q > k} \|\Delta_q u\|_{L^3} \right).$$

If we choose k such that $|y| \approx 2^{-k}$ then we obtain the bound

$$\leq C \left(2^{-k} \|S_0 u\|_{L^3} + \sum_{0 \leq q \leq k+1} 2^{-qs} 2^{q-k} 2^{qs} \|\Delta_q u\|_{L^3} + \sum_{q > k} 2^{-qs} 2^{qs} \|\Delta_q u\|_{L^3} \right),$$

which yields

$$\leq C|y|^s \left(2^{-k(1-s)} \|S_0 u\|_{L^3} + \sum_{0 \leq q \leq k+1} 2^{(1-s)(q-k)} 2^{qs} \|\Delta_q u\|_{L^3} + \sum_{q > k} 2^{s(k-q)} 2^{qs} \|\Delta_q u\|_{L^3} \right).$$

Therefore,

$$I := \left\| \frac{\|u(\cdot + y) - u(\cdot)\|_{L^3}}{|y|^s} \right\|_{L^3(0,T)} \leq C \left\| 2^{-k(1-s)} \|S_0 u\|_{L^3} \right.$$
$$\left. + \sum_{0 \leq q \leq k+1} 2^{(1-s)(q-k)} 2^{qs} \|\Delta_q u\|_{L^3} + \sum_{q > k} 2^{s(k-q)} 2^{qs} \|\Delta_q u\|_{L^3} \right\|_{L^3(0,T)},$$

which becomes

$$I \le C\Bigg(2^{-k(1-s)}\big\|\|S_0u\|_{L^3}\big\|_{L^3(0,T)} + \sum_{0\le q\le k+1} 2^{(1-s)(q-k)}\big\|2^{qs}\|\Delta_q u\|_{L^3}\big\|_{L^3(0,T)}$$
$$+\sum_{q>k} 2^{s(k-q)}\big\|2^{qs}\|\Delta_q u\|_{L^3}\big\|_{L^3(0,T)}\Bigg).$$

To simplify the analysis of the second and third terms we will define,

$$K(r) = \begin{cases} 2^{(s-1)r}, & r \ge 0 \\ 2^{(s-1)r} + 2^{sr}, & r = -1 \\ 2^{sr}, & r < -1 \end{cases}$$

obtaining,

$$I \le C\Bigg(2^{-k(1-s)}\big\|\|S_0u\|_{L^3}\big\|_{L^3(0,T)} + \sum_{q} K(k-q)\big\|2^{qs}\|\Delta_q u\|_{L^3}\big\|_{L^3(0,T)}\Bigg).$$

We can split the sum into the parts where $q < \frac{k}{2}$ and $q \ge \frac{k}{2}$ and obtain

$$I \le C\Bigg(2^{-k(1-s)}\big\|\|S_0u\|_{L^3}\big\|_{L^3(0,T)} + \sum_{q<\frac{k}{2}} K(k-q)\big\|2^{qs}\|\Delta_q u\|_{L^3}\big\|_{L^3(0,T)} \tag{2.11}$$
$$+\sum_{q\ge\frac{k}{2}} K(k-q)\big\|2^{qs}\|\Delta_q u\|_{L^3}\big\|_{L^3(0,T)}\Bigg).$$

For the second term of (2.11) for $q < \frac{k}{2}$ we have that $k-q > 0$ and so observe that $K(k-q) = 2^{(s-1)(k-q)}$ so

$$\sum_{q<\frac{k}{2}} K(k-q) \le C2^{(s-1)\frac{k}{2}}.$$

Then using Hölder's inequality for l^1 and l^∞, we obtain

$$\sum_{q<\frac{k}{2}} K(k-q)\big\|2^{qs}\|\Delta_q u\|_{L^3}\big\|_{L^3(0,T)} \le C2^{(s-1)\frac{k}{2}} \sup_{q<\frac{k}{2}} \big\|2^{qs}\|\Delta_q u\|_{L^3}\big\|_{L^3(0,T)}. \tag{2.12}$$

For the third term in (2.11) we can split the sum up and show that it is finite and bounded independently of k. We see that

$$\sum_{q\geq\frac{k}{2}} K(k-q) = \sum_{\frac{k}{2}\leq q\leq k} 2^{(s-1)(k-q)} + \sum_{k+1<q} 2^{s(k-q)} + K(-1).$$

Using the formula for a sum of a geometric series we see that

$$\sum_{\frac{k}{2}\leq q\leq k} 2^{(s-1)(k-q)} = 1 + 2^{(s-1)} + \cdots + 2^{(s-1)(\frac{k}{2})} = \frac{1-2^{\frac{k}{2}(s-1)}}{1-2^{s-1}} \leq \frac{1}{1-2^{s-1}}$$

and

$$\sum_{k+1<q} 2^{s(k-q)} = \frac{1}{1-2^{-s}}.$$

We can apply these bounds and the $K(-1)$ estimate to obtain

$$\sum_{q\geq\frac{k}{2}} K(k-q)\big\|2^{qs}\|\Delta_q u\|_{L^3}\big\|_{L^3(0,T)} \leq C\sup_{q\geq\frac{k}{2}}\big\|2^{qs}\|\Delta_q u\|_{L^3}\big\|_{L^3(0,T)}. \tag{2.13}$$

Using (2.12) and (2.13) we see that by taking limits as $|y|\to 0$, using that $|y|\approx 2^{-k}$, we obtain

$$\lim_{|y|\to 0}\left\|\frac{\|u(\cdot+y)-u(\cdot)\|_{L^3}}{|y|^s}\right\|_{L^3(0,T)} \leq C\lim_{k\to\infty}\Bigg(2^{-k(1-s)}\big\|\|S_0u\|_{L^3}\big\|_{L^3(0,T)} + 2^{(s-1)\frac{k}{2}}\sup_{q<\frac{k}{2}}\big\|2^{qs}\|\Delta_q u\|_{L^3}\big\|_{L^3(0,T)} + C\sup_{q\geq\frac{k}{2}}\big\|2^{qs}\|\Delta_q u\|_{L^3}\big\|_{L^3(0,T)}\Bigg). \tag{2.14}$$

The first term will go to zero in the limit as $k\to\infty$ as for $s\in(0,1)$ we have $2^{-k(1-s)}\to 0$ and $\|\|S_0u\|_{L^3}\|_{L^3([0,T])}$ is bounded as $u\in L^3\left([0,T];L^3\right)$. From the assumption (1.3) we have

$$\sup_q \|2^{sq}\|\Delta_q u\|_{L^3}\|_{L^3(0,T)} < \infty \quad \text{and} \quad \lim_{q\to\infty}\|2^{sq}\|\Delta_q u\|_{L^3}\|_{L^3(0,T)} = 0$$

as the existence of the limit guarantees that the supremum is finite. Therefore, since $(s-1)<0$, in the limit as $k\to\infty$ the second term of (2.14) vanishes. For the third term of (2.14) we can use the assumption on u above to see that this term will vanish in the limit as well and so we are done.

Now we want to prove the reverse direction and show that for $s \in (0,1)$ if u satisfies

$$\lim_{|y|\to 0} \left\| \frac{1}{|y|^s} \|u(\cdot + y) - u(\cdot)\|_{L^3} \right\|_{L^3([0,T])} = 0$$

then

$$\lim_{q\to\infty} \left\| 2^{sq} \|\Delta_q u\|_{L^3} \right\|_{L^3([0,T])} = 0.$$

We start off with the definition of $\Delta_q u(x)$, given by (2.3), in terms of h, and use that the integral of h over $\mathbb{R}^d$ is zero, to obtain a difference of u as follows

$$\Delta_q u(x) = 2^{qd} \int_{\mathbb{R}^d} h(2^q y) u(x+y)\,\mathrm{d}y = 2^{qd} \int_{\mathbb{R}^d} h(2^q y)(u(x+y) - u(x))\,\mathrm{d}y.$$

Thus using Minkowski's inequality,

$$\begin{aligned}\|2^{qs}\|\Delta_q u\|_{L^3}\|_{L^3([0,T])} \le& \left\| 2^{qd} \int_{\mathbb{R}^d} 2^{qs} |h(2^q y)| \|u(\cdot + y) - u(\cdot)\|_{L^3}\,\mathrm{d}y \right\|_{L^3([0,T])} \\ \le& 2^{qd} \int_{\mathbb{R}^d} 2^{qs} |h(2^q y)|\, \| \|u(\cdot + y) - u(\cdot)\|_{L^3} \|_{L^3([0,T])}\,\mathrm{d}y.\end{aligned}$$

Now using the substitution $z = 2^q y$ we obtain

$$\begin{aligned}&\|2^{qs}\|\Delta_q u\|_{L^3}\|_{L^3([0,T])} \\ &\qquad \le \int_{\mathbb{R}^d} |z|^s |h(z)| \left\| \frac{2^{qs}}{|z|^s} \left\| u\left(\cdot + \frac{z}{2^q}\right) - u(\cdot) \right\|_{L^3} \right\|_{L^3([0,T])} \mathrm{d}z. \quad (2.15)\end{aligned}$$

We can split the integral in (2.15) into two parts where $|\frac{z}{2^q}| \le \delta$ and $|\frac{z}{2^q}| > \delta$ such that

$$\int_{|\frac{z}{2^q}|>\delta} + \int_{|\frac{z}{2^q}|\le\delta} |z|^s |h(z)| \left\| \frac{2^{qs}}{|z|^s} \left\| u\left(\cdot + \frac{z}{2^q}\right) - u(\cdot) \right\|_{L^3} \right\|_{L^3([0,T])} \mathrm{d}z := I + II.$$

Recalling the assumption

$$\lim_{|y|\to 0}\left\|\frac{1}{|y|^s}\|u(\cdot+y)-u(\cdot)\|_{L^3}\right\|_{L^3([0,T])}=0,$$

we can fix $\varepsilon>0$ and choose $\delta>0$ such that for all q

$$\begin{aligned}II &= \int_{|\frac{z}{2^q}|\le\delta}|z|^s|h(z)|\left\|\frac{2^{qs}}{|z|^s}\left\|u\left(\cdot+\frac{z}{2^q}\right)-u(\cdot)\right\|_{L^3}\right\|_{L^3([0,T])}\,\mathrm{d}z\\ &\le \int_{|\frac{z}{2^q}|\le\delta}|z|^s|h(z)|\varepsilon\,\mathrm{d}z.\end{aligned}$$

Then as $|z|^s|h(z)|$ has finite integral over $\mathbb{R}^d$ so we can bound by $C\varepsilon$. For I we see that,

$$\begin{aligned}I &= \int_{|\frac{z}{2^q}|>\delta}|z|^s|h(z)|\left\|\frac{2^{qs}}{|z|^s}\left\|u\left(\cdot+\frac{z}{2^q}\right)-u(\cdot)\right\|_{L^3}\right\|_{L^3([0,T])}\,\mathrm{d}z\\ &= \int_{|\frac{z}{2^q}|>\delta}|h(z)|2^{qs}\left\|\left\|u\left(\cdot+\frac{z}{2^q}\right)-u(\cdot)\right\|_{L^3}\right\|_{L^3([0,T])}\,\mathrm{d}z\\ &\le C2^{qs}\|u\|_{L^3(0,T;L^3)}\int_{|\frac{z}{2^q}|>\delta}|h(z)|\,\mathrm{d}z.\end{aligned}$$

Taking the limit as $q\to\infty$ of both sides of (2.15) we obtain

$$\begin{aligned}\lim_{q\to\infty}\|2^{qs}\|\Delta_q u\|_{L^3}\|_{L^3([0,T])} &\le \lim_{q\to\infty}(I+II)\le C\varepsilon+\lim_{q\to\infty}I\\ &\le C\varepsilon+\lim_{q\to\infty}\left(C2^{qs}\|u\|_{L^3(0,T;L^3)}\int_{|\frac{z}{2^q}|>\delta}|h(z)|\,\mathrm{d}z\right).\end{aligned}$$

Now, since $h\in\mathcal{S}$ we know that there exists $C>0$ such that

$$|h(z)|\le\frac{C}{1+|z|^{s+d+2}}$$

and so

$$\int_{|\frac{z}{2^q}|>\delta}|h(z)|\,\mathrm{d}z\le C\int_{|\frac{z}{2^q}|>\delta}\frac{|z|^{d-1}}{1+|z|^{s+d+2}}\,\mathrm{d}|z|\le C\frac{1}{2^{q(s+1)}}.$$

Thus we obtain that $\lim_{q\to\infty}I=0$ and we are done.

□

2.2 Conclusion

Both conditions (1.3) and (1.4) are the weakest known conditions to imply that a weak solution of the Euler equations satisfies energy conservation and we have shown that these conditions are equivalent on $\mathbb{R}^d$

Condition (1.3) uses Fourier techniques and so though a powerful tool in $\mathbb{R}^d$ it only works in this case; when considering solutions on a domain with a boundary this condition would not be useful. The importance of this equivalence result lies in the fact that condition (1.4) only treats the functions in real space and so similar conditions to (1.4) can be generated that make sense in a domain with a boundary. We will, however, still have to define boundary conditions for the function. This suggests that a version of condition (1.4), modified around the boundary, would be a suitable condition to guarantee energy conservation for a weak solution on a bounded domain.

Chapter 3

Energy Conservation in the Absence of Boundaries

In this chapter we will treat the incompressible Euler equations on a domain without boundaries: $\mathbb{R}^3$, $\mathbb{T}^3$, or one of the hybrid domains $\mathbb{T} \times \mathbb{R}^2$ or $\mathbb{T}^2 \times \mathbb{R}$. We write D in what follows to denote any one of these domains, being careful to highlight any differences required in the definitions/arguments required to deal with the periodic or hybrid cases.

We will show that if $u \in L^3(0,T;L^3(D))$ is a weak solution of the Euler equations (as in Definition 3.2) that satisfies

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_0^T \int_D |u(t,x+y) - u(t,x)|^3 \,\mathrm{d}x\,\mathrm{d}t = 0,$$

then energy is conserved on $[0,T]$, by which we mean that $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ for all $t \in [\cdot, T]$.

Further, we will also show energy conservation under the condition that $u \in L^3(0,T;W^{\alpha,3}(\mathbb{R}^3))$ for any $\alpha > 1/3$, i.e. if u is a weak solution of the Euler equations on the whole space that satisfies $u \in L^3(0,T;L^3(D))$ and

$$\int_D \int_D \frac{|u(x) - u(y)|^3}{|x-y|^{3+3\alpha}} \,\mathrm{d}x \,\mathrm{d}y < \infty,$$

then energy is conserved.

Here the main part of the work is to show the necessary time regularity needed to use a regularised solution, $J_\varepsilon u$, as a test function. Although this approach has been used repeatedly in the literature [Eyink (1994), Constantin, E, & Titi (1994), Duchon & Robert (2000), Cheskidov et al. (2008), Shvydkoy (2010) and Bardos & Titi (2018)] the issue of the time regularity of $J_\varepsilon u$ is usually sidestepped. Shvydkoy (2010) treats the time regularity of $J_\varepsilon u$ in detail using Fourier analysis techniques. Here we address this issue without the use of Fourier techniques so that the method can be generalised to domains with boundaries, as done in Chapter 5.

It is an interesting result that by regularising the solution u in space only one also gains Lipschitz regularity in time which is enough to use a mollified solution as a test function and further, enough to manipulate the terms involving the time derivative. We also rigorously show how to 'regularise the equation'.

As a result we are able to prove energy conservation without any Fourier techniques and under the weakest condition currently known.

The work in this chapter and Chapter 4 is an extension of Robinson et al. (2018a).

3.1 Weak solutions of the Euler equations

For vector-valued functions f, g and matrix-valued functions F, G we use the notation

$$\langle f, g\rangle = \int_D f_i(x)g_i(x)\,\mathrm{d}x \quad \text{and} \quad \langle F : G\rangle = \int_D F_{ij}(x)G_{ij}(x)\,\mathrm{d}x,$$

employing Einstein's summation convention (sum over repeated indices).

We use the notation $\mathcal{D}(D)$ to denote the collection of C^∞ functions with compact support in D, and $\mathcal{S}(D)$ for the collection of all C^∞ functions with Schwartz-like decay in the unbounded directions of D, e.g. for $\mathbb{T}^2 \times \mathbb{R}$ we require

$$\sup_{x\in\mathbb{T}^2\times\mathbb{R}} |x_3|^k |\partial^\alpha \phi| < \infty,$$

for all $\alpha, k \geq 0$, where α is a multi-index over all the spatial variables (x_1, x_2, x_3) and $k \in \mathbb{N}$. Note that in the periodic directions the requirement of 'compact support' is trivially satisfied. The spaces $\mathcal{D}_\sigma(D)$ and $\mathcal{S}_\sigma(D)$ consist of all divergence-free elements of the spaces $\mathcal{D}(D)$ or $\mathcal{S}(D)$.

We denote by $H_\sigma(D)$ the closure of $\mathcal{D}_\sigma(D)$ in the norm of $L^2(D)$; this coincides with the closure of $\mathcal{S}_\sigma(D)$ in the same norm.

Elements of $H_\sigma(D)$ are divergence free in the sense of distributions, i.e.

$$\langle u, \nabla\phi \rangle = 0 \qquad \text{for all} \quad \phi \in \mathcal{D}(D);$$

but in fact this equality holds for all $\phi \in \mathcal{S}(D)$, and even for all $\phi \in H^1(D)$: indeed, since $\mathcal{S}_\sigma(D)$ is dense in $H_\sigma(D)$, for any $u \in H_\sigma(D)$ we can find $(u_n) \in \mathcal{S}_\sigma(D)$ such that $u_n \to u$ in $L^2(D)$, and then for any $\phi \in H^1(D)$ we have

$$\langle u, \nabla\phi \rangle = \lim_{n\to\infty} \langle u_n, \nabla\phi \rangle = \lim_{n\to\infty} \langle \nabla \cdot u_n, \phi \rangle = 0$$

(cf. Lemma 2.11 in Robinson, Rodrigo, & Sadowksi, 2016, for example).

In a slight abuse of notation we denote by $C_{\mathrm{w}}([0,T]; H_\sigma)$ the collection of all functions $u\colon [0,T] \to H_\sigma(D)$ that are weakly continuous into L^2 (rather than H_σ), i.e., the map

$$t \mapsto \langle u(t), \phi \rangle$$

is continuous for every $\phi \in L^2(D)$. Note that $C_{\mathrm{w}}([0,T]; H_\sigma) \subset L^\infty(0,T; H_\sigma)$.

We take as our space-time test functions the elements of

$$\mathcal{S}_\sigma^T := \{\psi \in C^\infty(D \times [0,T]) : \ \psi(\cdot, t) \in \mathcal{S}_\sigma(D) \text{ for all } t \in [0,T]\}.$$

We choose these functions to take values in $\mathcal{S}_\sigma$ (rather than in $\mathcal{D}_\sigma$) since the property of compact support is not preserved by the Helmholtz decomposition, whereas such a decomposition respects Schwartz-like decay.

Lemma 3.1 *Any $\psi \in \mathcal{S}$ can be decomposed as $\psi = \phi + \nabla\chi$, where $\phi \in \mathcal{S}_\sigma$ and $\chi \in \mathcal{S}$, and moreover there exists C_s, independent of ψ such that*

$$\|\phi\|_{H^s} + \|\nabla\chi\|_{H^s} \leq C_s\|\psi\|_{H^s} \tag{3.1}$$

for each $s \geq 0$.

Proof (Cf. Theorem 2.6 and Exercise 5.2 in Robinson et al., 2016.) Since $\psi \in \mathcal{S}$ we can write ψ in Fourier space, using a hybrid of Fourier series in the periodic directions and the Fourier transform in the unbounded directions. In the periodic directions we will consider $\mathbb{T}^2$ to be the periodised region $[-1/2, 1/2]^2$ and thus $(x_1, \cdots, x_n) \in \mathbb{T}^n \mapsto (k_1, \cdots, k_n) \in \mathbb{Z}^n$. To ensure that u is real valued we impose that $\hat{u}(k) = \overline{\hat{u}(-k)}$ for the components of k in $\mathbb{Z}$. Further, if we are considering the fully periodic case ($D = \mathbb{T}^d$), then $\hat{u}(0) = 0$ so that u has zero mean.

For example, in the case $D = \mathbb{T}^2 \times \mathbb{R}$ we have

$$\psi(x) = \int_{-\infty}^{\infty} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \hat{u}(k)\mathrm{e}^{2\pi \mathrm{i} k\cdot x}\, \mathrm{d}k_3,$$

and we can set

$$\phi(x) = \int_{-\infty}^{\infty} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \left(I - \frac{k\otimes k}{|k|^2}\right) \hat{u}(k)\mathrm{e}^{2\pi \mathrm{i} k\cdot x}\, \mathrm{d}k_3,$$

and

$$\chi(x) = \int_{-\infty}^{\infty} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \frac{k\cdot\hat{u}(k)}{|k|^2}\mathrm{e}^{2\pi \mathrm{i} k\cdot x}\, \mathrm{d}k_3;$$

in the fully periodic case we omit the $k \otimes k/|k|^2$ term when $k = 0$. It is easy to check that these functions have the stated properties. □

Assuming that u is a smooth solution of the Euler equations

$$\partial_t u + (u\cdot\nabla)u + \nabla p = 0 \qquad \nabla\cdot u = 0$$

if we multiply by an element of $\mathcal{S}_\sigma^T$ and integrate by parts in space and time then we obtain (3.2) below; the pressure term vanishes since ψ is divergence free and we have decay in the unbounded directions and we have periodic boundary conditions in the periodic directions. Requiring only (3.2) to hold we obtain our definition of a weak solution.

Definition 3.2 (Weak Solution) *We say that $u \in C_{\rm w}([0,T];H_\sigma)$ is a weak solution of the Euler equations on $[0,T]$, arising from the initial condition $u(0) \in H_\sigma$, if*

$$\langle u(t),\psi(t)\rangle - \langle u(0),\psi(0)\rangle - \int_0^t \langle u(\tau),\partial_t\psi(\tau)\rangle\,\mathrm{d}\tau = \int_0^t \langle u(\tau)\otimes u(\tau) : \nabla\psi(\tau)\rangle\,\mathrm{d}\tau \quad (3.2)$$

for every $t \in [0,T]$ and every $\psi \in \mathcal{S}_\sigma^T$.

We note here that replacing $\mathcal{S}_\sigma^T$ by $\mathcal{D}_\sigma^T$ leads to an equivalent definition (via a simpler version of the argument of Lemma 3.3, below).

Throughout this thesis we let φ be a radial scalar function in $C_c^\infty(B(0,1))$ with $\int_{\mathbb{R}^3}\varphi = 1$ and for any $\varepsilon > 0$ we set $\varphi_\varepsilon(x) = \varepsilon^{-3}\varphi(x/\varepsilon)$. Then for any function f we define the mollification of f as $J_\varepsilon f := \varphi_\varepsilon \star f$ where $\star$ denotes convolution. Thus

$$J_\varepsilon f(x) = \varphi_\varepsilon \star f(x) := \int_{\mathbb{R}^3}\varphi_\varepsilon(x-y)f(y)\,\mathrm{d}y = \int_{B(0,\varepsilon)}\varphi_\varepsilon(y)f(x-y)\,\mathrm{d}y. \quad (3.3)$$

In the periodic directions we extend f by periodicity in this integration. We insist that φ is radially symmetric since this ensures that the operation of mollification satisfies the 'symmetry property', that is, for $u \in L^p$ and $v \in L^q$ with $1/p + 1/q = 1$, we have

$$\langle \varphi_\varepsilon \star u, v\rangle = \langle u, \varphi_\varepsilon \star v\rangle, \quad (3.4)$$

(see Majda & Bertozzi (2002), for example).

Our aim in the next section is to show the validity of the following two equalities that follow from the definition of a weak solution in (3.2). The first is

$$\langle u(t), J_\varepsilon u(t)\rangle - \langle u(0), J_\varepsilon u(0)\rangle - \int_0^t \langle u(\tau), \partial_t J_\varepsilon u(\tau)\rangle \,\mathrm{d}\tau \\ = \int_0^t \langle u(\tau)\otimes u(\tau) : \nabla J_\varepsilon u(\tau)\rangle \,\mathrm{d}\tau; \quad (3.5)$$

this amounts to using $J_\varepsilon u$, a mollification of the solution u, as a test function in the definition of a weak solution (3.2): we need to show that there is sufficient time regularity to do this, which we do in Section 3.2. The second is

$$\int_0^t \langle \partial_t J_\varepsilon u(\tau), u(\tau)\rangle \,\mathrm{d}\tau = -\int_0^t \langle \nabla\cdot J_\varepsilon[u(\tau)\otimes u(\tau)], u(\tau)\rangle \,\mathrm{d}\tau, \quad (3.6)$$

assuming that $u \in L^3(0,T;L^3)$. One could see this heuristically as a "mollification of the equation" tested with u; we will show that this can be done in a rigorous way in Section 3.2.1. We can then add these equations and take the limit as $\varepsilon \to 0$ to obtain the equation for conservation (or otherwise) of energy (Section 3.2.2).

3.2 Using $J_\varepsilon u$ as a test function

We will show that if u is a weak solution then in fact (3.2) holds for a larger class of test functions with less time regularity. We denote by $C^{0,1}([0,T];H_\sigma)$ the space of Lipschitz functions from $[0,T]$ into H_σ.

Lemma 3.3 *If u is a weak solution of the Euler equations in the sense of Definition 3.2 then* (3.2) *holds for every $\psi \in \mathcal{L}_\sigma$, where*

$$\mathcal{L}_\sigma := L^1(0,T;H^3) \cap C^{0,1}([0,T];H_\sigma).$$

Proof We will extend to $f \in \mathcal{L}_\sigma$ using a density argument. We first note that for a fixed u we can write (3.2) as $E(\psi) = 0$ for every $\psi \in \mathcal{S}_\sigma^T$, where

$$E(\psi) := \langle u(t), \psi(t)\rangle - \langle u(0), \psi(0)\rangle - \int_0^t \langle u(\tau), \partial_t \psi(\tau)\rangle \, \mathrm{d}\tau - \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla \psi(\tau)\rangle \, \mathrm{d}\tau \tag{3.7}$$

and observe that E is linear in ψ. Further, we observe that $\mathcal{S}_\sigma^T$ is dense in $\mathcal{L}_\sigma$ and therefore for an $f \in \mathcal{L}_\sigma$ there exists a sequence of functions $\psi_n \in \mathcal{S}_\sigma^T$ such that

$$\|f - \psi_n\|_{\mathcal{L}_\sigma} := \|f - \psi_n\|_{L^1(0,T;H^3)} + \|f - \psi_n\|_{C^{0,1}([0,T];L^2)} \leq \frac{1}{n}$$

and ψ_n is a Cauchy sequence under the $\mathcal{L}_\sigma$ norm.

We need to show that for $\psi \in \mathcal{S}_\sigma^T$ that $|E(\psi)| \leq C\|\psi\|_{\mathcal{L}_\sigma}$ and will proceed term-by-term. For the first two terms of (3.7) we have

$$\begin{aligned} |\langle u(t), \psi(t)\rangle - \langle u(0), \psi(0)\rangle| &\leq 2\|u\|_{L^\infty(0,T;L^2)}\|\psi\|_{L^\infty(0,T;L^2)} \\ &\leq 2\|u\|_{L^\infty(0,T;L^2)}\|\psi\|_{C^{0,1}(0,T;L^2)}, \end{aligned}$$

using the fact that $u \in C_{\mathrm{w}}([0,T]; H_\sigma)$. For the last term of (3.7) we observe that

$$\begin{aligned} \left|\int_0^t \langle u(\tau) \otimes u(\tau) : \nabla \psi(\tau)\rangle \, \mathrm{d}\tau\right| &\leq T\|u\|^2_{L^\infty(0,T;L^2)}\|\nabla \psi\|_{L^1(0,T;L^\infty)} \\ &\leq T\|u\|^2_{L^\infty(0,T;L^2)}\|\psi\|_{L^1(0,T;H^3)} \end{aligned}$$

using the general Sobolev inequalities from Evans (1998), Chapter 5, here for spatial dimension three but for higher dimensions more derivatives will be needed. Finally for the third term of (3.7)

$$\left|\int_0^t \langle u(\tau), \partial_\tau \psi(\tau)\rangle \, \mathrm{d}\tau\right| \leq \|u\|_{L^\infty(0,T;L^2)}\|\partial_\tau \psi\|_{L^1(0,T;L^2)}$$

and we want to show that $\|\partial_\tau \psi\|_{L^1(0,T;L^2)} \leq C\|\psi\|_{C^{0,1}(0,T;L^2)}$. We see that

$$\begin{aligned}
\|\partial_\tau \psi\|_{L^1(0,T;L^2)} &= \int_0^T \left[\int |\partial_\tau \psi(x,\tau)|^2 \, \mathrm{d}x\right]^{\frac{1}{2}} \mathrm{d}\tau \\
&= \int_0^T \left[\int \left|\lim_{h\to 0} \frac{\psi(x,\tau+h)-\psi(x,\tau)}{h}\right|^2 \mathrm{d}x\right]^{\frac{1}{2}} \mathrm{d}\tau \\
&= \int_0^T \left[\int \lim_{h\to 0} \left|\frac{\psi(x,\tau+h)-\psi(x,\tau)}{h}\right|^2 \mathrm{d}x\right]^{\frac{1}{2}} \mathrm{d}\tau.
\end{aligned}$$

We can now use Dominated Convergence Theorem since $\psi \in \mathcal{S}_\sigma^T$ and so

$$\begin{aligned}
\|\partial_\tau \psi\|_{L^1(0,T;L^2)} &= \int_0^T \lim_{h\to 0} \frac{1}{|h|} \left[\int |\psi(x,\tau+h)-\psi(x,\tau)|^2 \, \mathrm{d}x\right]^{\frac{1}{2}} \mathrm{d}\tau \\
&\leq T \sup_{t\in[0,T]} \lim_{h\to 0} \frac{1}{|h|} \left[\int |\psi(x,\tau+h)-\psi(x,\tau)|^2 \, \mathrm{d}x\right]^{\frac{1}{2}} \mathrm{d}\tau \\
&\leq T\|\psi\|_{C^{0,1}(0,T;L^2)}.
\end{aligned}$$

It follows that

$$|E(\psi)| \leq C\|u\|_{L^\infty(0,T;L^2)}\|\psi\|_{C^{0,1}([0,T];L^2)} + C\|u\|^2_{L^\infty(0,T;L^2)}\|\psi\|_{L^1(0,T;H^3)}.$$

We want to show that $E(f) := \lim_{n\to\infty} E(\psi_n) = 0$. As E is linear and for $\psi \in \mathcal{S}_\sigma^T$ we have that $|E(\psi)| \leq C\|\psi\|_{\mathcal{L}_\sigma}$ then for $m > n \geq N$ we have that

$$|E(\psi_n) - E(\psi_m)| = |E(\psi_n - \psi_m)| \leq C\|\psi_n - \psi_m\|_{\mathcal{L}_\sigma} \leq \frac{C}{n}$$

and so $E(\psi_n)$ is a Cauchy sequence and so by completeness of $\mathbb{R}$ it converges and as $E(\psi_n) = 0$ for all n thus $E(f) = 0$ for any $f \in \mathcal{L}_\sigma$. $\square$

We now study the time regularity of u when paired with a sufficiently smooth function that is not necessarily divergence free.

Lemma 3.4 *If u is a weak solution of the Euler equations then*

$$|\langle u(t) - u(s), \phi\rangle| \leq C|t-s| \quad \textit{for all} \quad \phi \in H^3(D), \tag{3.8}$$

where C depends only on $\|u\|_{L^\infty(0,T;L^2)}$ and $\|\phi\|_{H^3}$.

Proof We use Lemma 3.1 to decompose $\phi \in \mathcal{S}(D)$ as $\phi = \eta + \nabla\sigma$, where $\eta \in \mathcal{S}_\sigma(D)$, $\sigma \in \mathcal{S}(D)$, we have

$$\|\nabla\eta\|_{L^\infty} \leq \|\nabla\eta\|_{H^2} \leq \|\eta\|_{H^3} \leq C\|\phi\|_{H^3},$$

using (3.1) and the fact that $H^2(D) \subset L^\infty(D)$. Since $u(t)$ is incompressible for every $t \in [0,T]$, we have

$$\langle u(t) - u(s), \phi\rangle = \langle u(t) - u(s), \eta + \nabla\sigma\rangle = \langle u(t) - u(s), \eta\rangle.$$

Since $\eta \in \mathcal{S}_\sigma$ and $\partial_t\eta = 0$ it follows from Definition 3.2 of a weak solution at times t and s that

$$\langle u(t) - u(s), \phi\rangle = \int_s^t \langle u(\tau) \otimes u(\tau) : \nabla\eta\rangle \, \mathrm{d}\tau$$

and hence

$$|\langle u(t) - u(s), \phi\rangle| \leq \|u\|^2_{L^\infty(0,T;L^2)}\|\nabla\eta\|_{L^\infty}|t-s| \leq C\|u\|^2_{L^\infty(0,T;L^2)}\|\phi\|_{H^3}|t-s|, \tag{3.9}$$

which gives (3.8) for all $\phi \in \mathcal{S}$.

We now want to extend to $\phi \in H^3(D)$. Let $\psi \in \mathcal{S}$ and $\phi \in H^3$ such that, using density, there exists $\varepsilon > 0$ such that $\|\psi - \phi\|_{H^3} \leq \varepsilon$ then we have

$$\begin{aligned}|\langle u(t) - u(s), \phi\rangle| &\leq |\langle u(t) - u(s), \phi - \psi\rangle| + |\langle u(t) - u(s), \psi\rangle| \\ &\leq C\|u\|_{L^\infty(0,T;L^2)}\|\phi - \psi\|_{H^3} + |\langle u(t) - u(s), \psi\rangle| \leq C\varepsilon + |\langle u(t) - u(s), \psi\rangle|.\end{aligned}$$

We can now use (3.9) to see that

$$
\begin{aligned}
|\langle u(t)-u(s),\phi\rangle| &\le C\varepsilon + C\|u\|^2_{L^\infty(0,T;L^2)}\|\psi\|_{H^3}|t-s| \\
&\le C\varepsilon + C\|u\|^2_{L^\infty(0,T;L^2)}\|\psi-\phi\|_{H^3}|t-s| + C\|u\|^2_{L^\infty(0,T;L^2)}\|\phi\|_{H^3}|t-s| \\
&\le C\varepsilon + C\|u\|^2_{L^\infty(0,T;L^2)}\|\phi\|_{H^3}|t-s|.
\end{aligned}
$$

Since we can make ε arbitrarily small we are done. □

A striking corollary of this weak continuity in time is that a mollification *in space alone* yields a function that is Lipschitz continuous *in time*.

Corollary 3.5 *Given a solution u of the Euler equations we have $J_\varepsilon u \in \mathcal{L}_\sigma$ for any $\varepsilon > 0$; in particular the function $J_\varepsilon u(x,t)$ is Lipschitz continuous in t as a function into $L^2(D)$:*

$$\|J_\varepsilon u(\cdot,t) - J_\varepsilon u(\cdot,s)\|_{L^2} \le C_\varepsilon \|u\|^2_{L^\infty(0,T;L^2)}|t-s|. \tag{3.10}$$

Proof Take $f \in L^2(D)$ with $\|f\|_{L^2(D)} = 1$, and let $\phi = J_\varepsilon f$. Then $\phi \in H^3(D)$, and using the symmetry property (3.4) we have

$$
\begin{aligned}
\langle u(t)-u(s),\phi\rangle &= \langle u(t)-u(s), J_\varepsilon f\rangle \\
&= \langle (J_\varepsilon u(t) - J_\varepsilon u(s)), f\rangle.
\end{aligned}
$$

Since we have $\|\phi\|_{H^3} \le C_\varepsilon \|f\|_{L^2} = C_\varepsilon$ it follows from Lemma 3.4 using the bound (3.9) that

$$|\langle J_\varepsilon u(t) - J_\varepsilon u(s), f\rangle| \le C_\varepsilon \|u\|^2_{L^\infty(0,T;L^2)}|t-s|.$$

Since this holds for every $f \in L^2(D)$ with $\|f\|_{L^2(D)} = 1$ we obtain the inequality (3.10) and $J_\varepsilon u \in C^{0,1}([0,T];L^2)$.

As mollification commutes with differentiation it follows that $J_\varepsilon u$ is divergence free. Finally, since $u \in L^\infty(0,T;L^2)$, we observe that $J_\varepsilon u \in L^\infty(0,T;H^3)$ and

$$\|J_\varepsilon u\|_{L^1(0,T;H^3)} \le T\|J_\varepsilon u\|_{L^\infty(0,T;H^3)}$$

as $[0,T]$ is bounded. $\square$

We have now obtained the results needed to use $J_\varepsilon u$ as a test function in the definition of a weak solution. We can combine the results of Lemma 3.3 and the previous Corollary 3.5 so use $J_\varepsilon u$ as a test function and obtain

$$\langle u(t), J_\varepsilon u(t)\rangle - \langle u(0), J_\varepsilon u(0)\rangle - \int_0^t \langle u(\tau), \partial_t J_\varepsilon u(\tau)\rangle \,\mathrm{d}\tau \\ = \int_0^t \langle u(\tau)\otimes u(\tau) : \nabla J_\varepsilon u(\tau)\rangle \,\mathrm{d}\tau;$$

we have validated equation (3.5), the first of the two equalities we need.

3.2.1 Mollifying the equation

We will now derive (3.6). The idea is to test with a mollified test function and move the mollification from the test function onto the terms involving u; all terms are then smooth enough to allow for an integration by parts.

Lemma 3.6 *If u is a weak solution of the Euler equations from Definition 3.2 then*

$$\int_0^t \langle \partial_t J_\varepsilon u, \phi\rangle \,\mathrm{d}\tau = -\int_0^t \langle \nabla\cdot J_\varepsilon[u\otimes u], \phi\rangle \,\mathrm{d}\tau \tag{3.11}$$

for every $t\in[0,T]$ and any $\phi\in\mathcal{S}^T_\sigma$.

Proof Take $\phi\in\mathcal{S}^T_\sigma$, and use $\psi := \varphi_\varepsilon\star\phi$ as the test function in the weak formulation (3.2). Then

$$\langle u(t), (\varphi_\varepsilon\star\phi)(t)\rangle - \langle u(0), (\varphi_\varepsilon\star\phi)(0)\rangle - \int_0^t \langle u(\tau), \partial_t[\varphi_\varepsilon\star\phi](\tau)\rangle \,\mathrm{d}\tau \\ = \int_0^t \langle u(\tau)\otimes u(\tau) : \nabla[\varphi_\varepsilon\star\phi](\tau)\rangle \,\mathrm{d}\tau.$$

Since we have chosen φ to be even we have that $\langle \varphi_\varepsilon\star u, v\rangle = \langle u, \varphi_\varepsilon\star v\rangle$ (see (3.4)) and therefore we can move the derivatives and mollification onto the terms involving u. We will do this in detail for the term on the right-hand side, since it is the most complicated; the other terms follow similarly. We obtain

$$\int_0^t \langle u(\tau)\otimes u(\tau) : \nabla[\varphi_\varepsilon \star \phi](\tau)\rangle \,\mathrm{d}\tau = \int_0^t \langle u(\tau)\otimes u(\tau) : \varphi_\varepsilon \star \nabla\phi(\tau)\rangle \,\mathrm{d}\tau$$
$$= \int_0^t \langle J_\varepsilon[u(\tau)\otimes u(\tau)] : \nabla\phi(\tau)\rangle \,\mathrm{d}\tau = -\int_0^t \langle \nabla\cdot J_\varepsilon[u(\tau)\otimes u(\tau)], \phi(\tau)\rangle \,\mathrm{d}\tau.$$

This implies that

$$\langle J_\varepsilon u(t), \phi(t)\rangle - \langle J_\varepsilon u(0), \phi(0)\rangle - \int_0^t \langle J_\varepsilon u(\tau), \partial_t\phi(\tau)\rangle \,\mathrm{d}\tau$$
$$= -\int_0^t \langle \nabla\cdot J_\varepsilon[u(\tau)\otimes u(\tau)] : \phi(\tau)\rangle \,\mathrm{d}\tau.$$

Since $J_\varepsilon u$ and ϕ are both absolutely continuous in time, the integration-by-parts formula

$$\langle J_\varepsilon u(t), \phi(t)\rangle - \langle J_\varepsilon u(0), \phi(0)\rangle - \int_0^t \langle J_\varepsilon u(\tau), \partial_t\phi(\tau)\rangle \,\mathrm{d}\tau = \int_0^t \langle \partial_t J_\varepsilon u(\tau), \phi(\tau)\rangle \,\mathrm{d}\tau$$

finishes the proof. □

We now show that (3.11) holds for a much larger class of functions than $\phi \in \mathcal{S}_\sigma^T$ under some additional integrability conditions on u.

Lemma 3.7 *If u is a weak solution of the Euler equations from Definition 3.2 and in addition $u \in L^3(0,T;L^3)$ then* (3.11) *holds for any $\phi \in L^3(0,T;L^3)\cap C_{\mathrm{w}}(0,T;H_\sigma)$.*

(Recall that we use $C_{\mathrm{w}}(0,T;H_\sigma)$ to denote H_σ-valued functions that are weakly continuous into L^2.)

Proof First we will obtain from (3.11) an equation that holds for all test functions ψ from the space $\mathcal{S}(D\times[0,T])$, not just for $\psi \in \mathcal{S}_\sigma^T$. For this we will use the Leray projection $\mathbb{P}$, (see Robinson, Rodrigo, & Sadowksi (2016), for example), the projection onto divergence-free vector fields. Since for any $\psi \in \mathcal{S}(D\times[0,T])$ we have $\mathbb{P}\psi \in \mathcal{S}_\sigma$, it follows from (3.11) that

$$\int_0^t \langle \partial_t J_\varepsilon u + \nabla\cdot J_\varepsilon[u\otimes u], \mathbb{P}\psi\rangle \,\mathrm{d}\tau = 0.$$

Since $\mathbb{P}$ is symmetric ($\langle \mathbb{P}g, f\rangle = \langle g, \mathbb{P}f\rangle$) and $\mathbb{P}\partial_t J_\varepsilon u = \partial_t J_\varepsilon u$ (since $\mathbb{P}$ commutes with derivatives and $J_\varepsilon u$ is incompressible) we obtain

$$\int_0^t \langle \partial_t J_\varepsilon u + \mathbb{P}(\nabla \cdot J_\varepsilon[u \otimes u]), \psi\rangle \,\mathrm{d}\tau = 0 \qquad \text{for every} \quad \psi \in \mathcal{S}(D \times [0,T]).$$

Since $J_\varepsilon u$ is Lipschitz in time (as a function from $[0,T]$ into H_σ) its time derivative $\partial_t J_\varepsilon u$ exists almost everywhere (see Theorem 5.5.4 in Albiac & Kalton (2016), for example) and is integrable; we can therefore deduce using the Fundamental Lemma of the Calculus of Variations ($u \in L^2(\Omega)$ with $\int_\Omega u \cdot \psi = 0$ for all $\psi \in C_c^\infty(\Omega)$ implies that $u = 0$ almost everywhere in Ω, see e.g. Lemma 3.2.3 in Jost & Li-Jost (1998)) that for almost every $(x,t) \in D \times [0,T]$

$$\partial_t J_\varepsilon u + \mathbb{P}(\nabla \cdot J_\varepsilon(u \otimes u)) = 0.$$

Observing that $\mathbb{P}\nabla \cdot J_\varepsilon(u \otimes u) \in L^{3/2}(0,T;L^{3/2})$ and that $\partial_t J_\varepsilon u$ has the same integrability since $\partial_t J_\varepsilon u = -\mathbb{P}\nabla \cdot J_\varepsilon(u \otimes u)$, we can now multiply this equality by any choice of function $\phi \in L^3(0,T;L^3) \cap C_{\mathrm{w}}(0,T;H_\sigma)$ and integrate:

$$\begin{aligned}\int_0^t \langle \partial_t J_\varepsilon u, \phi\rangle \,\mathrm{d}\tau &= -\int_0^t \langle \mathbb{P}\nabla \cdot J_\varepsilon[u \otimes u], \phi\rangle \,\mathrm{d}\tau \\ &= -\int_0^t \langle \nabla \cdot J_\varepsilon[u \otimes u], \mathbb{P}\phi\rangle \,\mathrm{d}\tau = -\int_0^t \langle \nabla \cdot J_\varepsilon[u \otimes u], \phi\rangle \,\mathrm{d}\tau,\end{aligned}$$

where we have used the fact that $\mathbb{P}\phi = \phi$ since $\phi(t) \in H_\sigma$ for every $t \in [0,T]$. □

Note that the condition on $u \in L^3(0,T;L^3)$ is stronger than necessary for the proof but since Theorem 3.9 will need this condition the above result will suffice for our purposes.

We can now use u as a test function in (3.11) and thereby obtain equation (3.6), the second of the equalities we need.

3.2.2 Energy Conservation

We can now add equations (3.5) and (3.6) to obtain

$$\begin{aligned}\langle u(t), J_\varepsilon u(t)\rangle - \langle u(0), J_\varepsilon u(0)\rangle \\ = \int_0^t \langle u(\tau)\otimes u(\tau) : \nabla J_\varepsilon u(\tau)\rangle - \langle \nabla\cdot J_\varepsilon[u(\tau)\otimes u(\tau)], u(\tau)\rangle \,\mathrm{d}\tau, \quad (3.12)\end{aligned}$$

valid for any $u \in L^3(0,T;L^3)\cap C_w(0,T;H_\sigma)$ that is a weak solution to the Euler equations.

In order to proceed we will need the following identity. We note that its validity is entirely independent of the Euler equations, but relies crucially on the fact that φ is radial and that the function v is weakly incompressible, so in H_σ.

Lemma 3.8 *Suppose that* $v \in L^3 \cap H_\sigma$ *and define*

$$\mathcal{J}_\varepsilon(v) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \nabla\varphi_\varepsilon(\xi)\cdot(v(x+\xi)-v(x))|v(x+\xi)-v(x)|^2 \,\mathrm{d}\xi\,\mathrm{d}x.$$

Then

$$\tfrac{1}{2}\mathcal{J}_\varepsilon(v) = \langle \nabla\cdot J_\varepsilon[v(\tau)\otimes v(\tau)], v(\tau)\rangle - \langle v(\tau)\otimes v(\tau) : \nabla J_\varepsilon v(\tau)\rangle.$$

Proof We have

$$\mathcal{J}_\varepsilon(v) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_i\varphi_\varepsilon(\xi)(v_i(x+\xi)-v_i(x))(v_j(x+\xi)-v_j(x))(v_j(x+\xi)-v_j(x))\,\mathrm{d}\xi\,\mathrm{d}x.$$

Expanding the expression for $\mathcal{J}_\varepsilon(v)$ yields

$$\begin{aligned}&\int_{\mathbb{R}^3}\Bigg\{\int_{\mathbb{R}^3}\partial_i\varphi_\varepsilon(\xi)v_i(x+\xi)v_j(x+\xi)v_j(x+\xi)\,\mathrm{d}\xi - \int_{\mathbb{R}^3}\partial_i\varphi_\varepsilon(\xi)v_i(x)v_j(x)v_j(x)\,\mathrm{d}\xi \\ &+\int_{\mathbb{R}^3}\partial_i\varphi_\varepsilon(\xi)v_i(x+\xi)v_j(x)v_j(x)\,\mathrm{d}\xi - \int_{\mathbb{R}^3}\partial_i\varphi_\varepsilon(\xi)v_i(x)v_j(x+\xi)v_j(x+\xi)\,\mathrm{d}\xi \\ &+2\left[\int_{\mathbb{R}^3}\partial_i\varphi_\varepsilon(\xi)v_i(x)v_j(x+\xi)v_j(x)\,\mathrm{d}\xi - \int_{\mathbb{R}^3}\partial_i\varphi_\varepsilon(\xi)v_i(x+\xi)v_j(x+\xi)v_j(x)\,\mathrm{d}\xi\right]\Bigg\}\,\mathrm{d}x.\end{aligned}$$

Note that the second term is zero since φ_ε has compact support, and the third term is zero since v is incompressible. For the fourth term we can change variables and set $\eta = x + \xi$ to obtain

$$-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_{\eta_i}\varphi_\varepsilon(\eta - x)v_i(x)v_j(\eta)v_j(\eta)\,\mathrm{d}\eta\,\mathrm{d}x.$$

As $\partial_i\varphi_\varepsilon$ is an odd function we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_i\varphi_\varepsilon(x - \eta)v_i(x)v_j(\eta)v_j(\eta)\,\mathrm{d}\eta\,\mathrm{d}x,$$

which becomes

$$\int_{\mathbb{R}^3} v_i(x)\partial_{x_i}\left[\int_{\mathbb{R}^3}\varphi_\varepsilon(x-\eta)v_j(\eta)v_j(\eta)\,\mathrm{d}\eta\right]\,\mathrm{d}x = \int_{\mathbb{R}^3} v_i(x)\partial_i(J_\varepsilon[v_jv_j])\,\mathrm{d}x = 0,$$

where again the term becomes zero as we use the incompressibility of v. A similar calculation for the first term gives

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_i\varphi_\varepsilon(\xi)v_i(x+\xi)v_j(x+\xi)v_j(x+\xi)\,\mathrm{d}\xi\ \mathrm{d}x = \int_{\mathbb{R}^3}\partial_i(J_\varepsilon[v_iv_jv_j])(x)\ \mathrm{d}x = 0,$$

using periodicity and decay at ∞. For the final two terms similar calculations yield

$$2\int_{\mathbb{R}^3}[v_j\partial_iJ_\varepsilon(v_jv_j) - v_jv_i\partial_iJ_\varepsilon(v_j)]\,\mathrm{d}x = 2[\langle\nabla\cdot J_\varepsilon[v\otimes v], v\rangle - \langle v\otimes v : \nabla J_\varepsilon v\rangle]$$

and the result follows. □

Note that here again the assumption that $v \in L^3$ is stronger than needed but will hold when we use the result in Theorem 3.9.

We now want to look at the limit as $\varepsilon \to 0$ of both sides of (3.12) and see what condition on the solution is needed for the right hand side of (3.12) to converge to zero. While we only need to show energy conservation on the time interval $[0, t]$ we will present the argument for a general interval $[t_1, t_2]$ as it does not require any additional arguments.

Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. We can take the difference of (3.12) with itself once with $t = t_1$ then $t = t_2$ and then take the difference of these two equations to obtain

$$
\begin{aligned}
\langle u(t_2), J_\varepsilon u(t_2)\rangle &- \langle u(t_1), J_\varepsilon u(t_1)\rangle \\
&= \int_{t_1}^{t_2} \langle u(\tau)\otimes u(\tau) : \nabla J_\varepsilon u(\tau)\rangle - \langle \nabla\cdot J_\varepsilon[u(\tau)\otimes u(\tau)], u(\tau)\rangle \,\mathrm{d}\tau \\
&= -\frac{1}{2}\int_{t_1}^{t_2} \mathcal{J}_\varepsilon(u)\,\mathrm{d}t.
\end{aligned}
$$

Therefore talking the limit as $\varepsilon \to 0$, since $u \in C_{\mathrm{w}}([0,T]; H_\sigma)$ we obtain

$$
\|u(t_2)\|_{L^2}^2 - \|u(t_1)\|_{L^2}^2 = -\frac{1}{2}\lim_{\varepsilon\to 0}\int_{t_1}^{t_2} \mathcal{J}_\varepsilon(u)\,\mathrm{d}t.
$$

Hence any condition on u that guarantees that

$$
\lim_{\varepsilon\to 0}\int_{t_1}^{t_2} \mathcal{J}_\varepsilon(u)\,\mathrm{d}t \to 0 \qquad \text{as} \qquad \varepsilon \to 0 \tag{3.13}
$$

ensures energy conservation. We give two such conditions in the next section.

3.3 Two spatial conditions for energy conservation in the absence of boundaries

First we provide another proof (cf. Shvydkoy, 2009) of energy conservation under condition (1.4). Here, we will consider a different commutator, namely, $\mathcal{J}_\varepsilon(u)$, as we show (3.13), without requiring Fourier analysis.

Theorem 3.9 *If $u \in L^3(0,T; L^3(D))$ is a weak solution of the Euler equations that satisfies*

$$
\lim_{|y|\to 0}\frac{1}{|y|}\int_0^T\int_D |u(t,x+y)-u(t,x)|^3\,\mathrm{d}x\,\mathrm{d}t = 0,
$$

then energy is conserved on $[0,T]$.

Proof We take t_1, t_2 with $0 \le t_1 \le t_2 \le T$, and consider the integral of $|\mathcal{J}_\varepsilon(u)|$ over $[t_1, t_2]$; our aim is to show that this is zero in the limit as $\varepsilon \to 0$. We start by noticing that

$$\int_{t_1}^{t_2} |\mathcal{J}_\varepsilon(u)|\,\mathrm{d}t \le \int_{t_1}^{t_2}\int_D\int_{\mathbb{R}^3} \frac{1}{\varepsilon^4}\left|\nabla\varphi\left(\frac{\xi}{\varepsilon}\right)\right| |u(x+\xi)-u(x)|^3\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.$$

We can then change variables $\xi = \eta\varepsilon$ and obtain,

$$\int_{t_1}^{t_2} |\mathcal{J}_\varepsilon(u)|\,\mathrm{d}t \le \int_{t_1}^{t_2}\int_D\int_{\mathbb{R}^3} \frac{1}{\varepsilon}\left|\nabla\varphi\left(\eta\right)\right| |u(x+\varepsilon\eta)-u(x)|^3\,\mathrm{d}\eta\,\mathrm{d}x\,\mathrm{d}t.$$

Using Fubini's Theorem we can exchange the order of the integrals:

$$\int_{t_1}^{t_2} |\mathcal{J}_\varepsilon(u)|\,\mathrm{d}t \le \int_{\mathbb{R}^3}\int_{t_1}^{t_2}\int_D \frac{|u(x+\varepsilon\eta)-u(x)|^3}{|\varepsilon\eta|}\,\mathrm{d}x\,\mathrm{d}t\,|\eta|\left|\nabla\varphi\left(\eta\right)\right|\,\mathrm{d}\eta.$$

Taking limits as ε goes to zero

$$\lim_{\varepsilon\to 0}\int_{t_1}^{t_2} |\mathcal{J}_\varepsilon(u)|\,\mathrm{d}t \le \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\int_{t_1}^{t_2}\int_D \frac{|u(x+\varepsilon\eta)-u(x)|^3}{|\varepsilon\eta|}\,\mathrm{d}x\,\mathrm{d}t\,|\eta|\left|\nabla\varphi\left(\eta\right)\right|\,\mathrm{d}\eta.$$

We are finished if we can exchange the outer integral and limit. This can be done using the Dominated Convergence Theorem. To do this we define the non-negative function,

$$f(y) = \frac{1}{|y|}\int_{t_1}^{t_2}\int_D |u(x+y)-u(x)|^3\,\mathrm{d}x\,\mathrm{d}t.$$

By assumption $\limsup_{|y|\to 0} f(y) = 0$, thus for any $\varepsilon > 0$, we have $\sup_{y\in B_0(\varepsilon)} f(y) \le K$ for some $K = K(\varepsilon)$. Further, $\mathrm{supp}(\varphi)$ is compact. Combining these facts we obtain a dominating integrable function

$$g(\eta) := K|\eta|\left|\nabla\varphi\left(\eta\right)\right|,$$

and the result follows. □

We now show how the general condition in (3.13) allows for a simple proof of energy conservation when $u \in L^3(0,T;W^{\alpha,3}(\mathbb{R}^3))$ for any $\alpha > 1/3$. The use of condition (3.14) below to characterise this space is due independently to Aronszajn, Gagliardo, and Slobodeckij, see Di Nezza, Palatucci, & Valdinoci (2012), for example.

Theorem 3.10 *If u is a weak solution of the Euler equations on the whole space that satisfies $u \in L^3(0,T;W^{\alpha,3}(\mathbb{R}^3))$ for some $\alpha > 1/3$, i.e. if $u \in L^3(0,T;L^3(\mathbb{R}^3))$ and*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^3}{|x-y|^{3+3\alpha}}\,\mathrm{d}x\,\mathrm{d}y < \infty, \tag{3.14}$$

then energy is conserved.

Proof First observe that for $\alpha > 1/3$ the space $W^{\alpha,3}$ has a factor $|x-y|^{4+\delta}$ in the denominator of (3.14), where $\delta = 3\alpha - 1 > 0$.

As in the previous proof, our starting point is that

$$\int_{t_1}^{t_2} |\mathcal{J}_\varepsilon(u)|\,\mathrm{d}t \leq \int_{t_1}^{t_2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{\varepsilon^4}\left|\nabla\varphi\left(\frac{\xi}{\varepsilon}\right)\right| |u(x+\xi)-u(x)|^3\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.$$

We can write

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{\varepsilon^4}\left|\nabla\varphi\left(\frac{y-x}{\varepsilon}\right)\right| |u(y)-u(x)|^3\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_{t_1}^{t_2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{\varepsilon^4}\left|\nabla\varphi\left(\frac{y-x}{\varepsilon}\right)\right| |u(y)-u(x)|^3\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_{t_1}^{t_2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|y-x|^{4+\delta}}{\varepsilon^4}\left|\nabla\varphi\left(\frac{y-x}{\varepsilon}\right)\right| \frac{|u(y)-u(x)|^3}{|y-x|^{4+\delta}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \leq cK_\varphi\varepsilon^\delta \int_{t_1}^{t_2}\int\int \frac{|u(y)-u(x)|^3}{|y-x|^{4+\delta}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = c\varepsilon^\delta,
\end{aligned}$$

since $\|\nabla\varphi\|_{L^\infty} \leq K_\varphi$ and the integrand is only non-zero within the support of φ, i.e. where $|y-x| \leq 2\varepsilon$. Energy conservation now follows. □

3.4 Conclusion

In this chapter we have presented an explicit method to prove energy conservation of weak solutions of the incompressible Euler equations where there are no boundaries. Though this method was presented for the spacial domains D, of dimension 3, we can generalise to any dimension $d \geq 2$ with all the potential d-dimensional hybrid domain variations of $\mathbb{T}$ and $\mathbb{R}$. Here we have focused on a method that does not use Fourier techniques and so is easier to generalise when considering the problem with a boundary. Here we have given a complete proof to show the time regularity of the regularised solution and so justify the use of $J_\varepsilon u$ as a test function. The method to obtain the required time regularity is easy to generalise when considering a domain with a boundary and these steps are done in Chapter 5. As shown in Chapter 2 the final condition we use to prove energy conservation is the weakest condition known and as defined without using Fourier techniques so we will consider similar conditions in domains with boundaries to prove energy conservation.

Chapter 4

Energy Balance on $\mathbb{T}^2 \times \mathbb{R}_+$

We will now present the first result that concerns a domain where we do have a boundary that is flat and has finite area. Here we consider the domain $\mathbb{T}^2 \times \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$, with a weak form of incompressibility that encodes the non-flux boundary condition prescribed for C^1 solutions, that for a solution u, $u \cdot n = u \cdot e_3 = 0$ on $\partial(\mathbb{T}^2 \times \mathbb{R}_+) = \mathbb{T}^2 \times \{0\}$. We will consider a weak formulation of the equation that only prescribes a boundary pressure term that is a distribution on the boundary. We then show that such a solution u can be extended to a weak solution u_E on the boundary-free domain $D := \mathbb{T}^2 \times \mathbb{R}$. (Note that in this chapter we reserve D for this particular domain.)

This extension is then shown to have the property that it solves the weak formulation of the equations in the lower domain $\mathbb{T}^2 \times \mathbb{R}_-$, where $\mathbb{R}_- = (-\infty, 0]$, where the boundary pressure term is the same as before. However, the normal component at the boundary is now in the opposite direction and thus u_E solves the weak formulation of the equations on D, as considered in the previous Chapter 3 (Definition 3.2).

Finally, we can relate the condition for energy conservation for u_E back to conditions on u and obtain the result that if $u \in L^3(0, T; L^3(\mathbb{T}^2 \times \mathbb{R}_+))$ is a weak solution of the Euler equations that satisfies $u \in L^3(0, T; C^0(T^2 \times [0, \delta])$ for some

$\delta > 0$ and

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{|y|}^{\infty} |u(t,x+y) - u(t,x)|^3 \,\mathrm{d}x_3 \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}t = 0;$$

then u conserves energy on $[t_1, t_2]$.

Throughout this chapter as we will be dealing with functions supported on D_-, D_+ and D. For ease of notation if we have functions v with support D_- and w with support D_+ then will define $v + w$ with support D as the sum of functions $\tilde{v}$ and $\tilde{w}$, both with support D, where $\tilde{v}$ and $\tilde{w}$ are the functions v and w respectively where they have been extended by zero.

4.1 Weak solutions on $D_+ := \mathbb{T}^2 \times \mathbb{R}_+$

In this section we will specify the class of weak solutions we will be studying on D_+ but first we will need to define and explain the properties of the function spaces we will be using; for more details one can study Chapter 2 of Robinson, Rodrigo, & Sadowksi (2016).

We define $\mathcal{S}(D_+)$ and $\mathcal{S}_\sigma(D_+)$ by restricting functions in $\mathcal{S}(\mathbb{T}^2 \times \mathbb{R})$ and $\mathcal{S}_\sigma(\mathbb{T}^2 \times \mathbb{R})$ to D_+; this means that we have Schwartz-like decay in the unbounded direction, and that the functions have a smooth restriction to the boundary.

We let

$$\mathcal{S}_{n,\sigma}(D_+) := \{\phi \in \mathcal{S}(D_+) : \nabla \cdot \phi = 0 \text{ and } \phi_3 = 0 \text{ on } \partial D_+\} \tag{4.1}$$

and define $H_\sigma(D_+)$ to be the completion of $\mathcal{S}_{n,\sigma}(D_+)$ in the norm of $L^2(D_+)$. Functions in $H_\sigma(D_+)$ are weakly divergence free in that they satisfy

$$\langle u, \nabla\phi \rangle = 0 \qquad \text{for every} \quad \phi \in H^1(D_+); \tag{4.2}$$

that this holds for every $\phi \in H^1(D_+)$ and not only for $\phi \in \mathcal{D}(D_+)$ (which can be proved exactly as in Section 3.1 for the domain D) will be useful in what follows. [In this case, although there is a boundary, for any $\phi \in \mathcal{S}_{n,\sigma}(D_+)$ we have vanishing normal component. This means that the potential boundary term that would appear in applying Gauss's formula vanishes and so we obtain weak incompressibility. This is shown in Lemma 2.11 in Robinson, Rodrigo, & Sadowksi (2016).]

For functions in H_σ we see that they have a vanishing normal component in trace sense given by the Gauss formula. That is, the normal component of u is well defined on the boundary as a bounded linear functional on the space of traces of functions in H^1 given by

$$0 = \langle \nabla \cdot u, v \rangle_{D_+} + \langle u, \nabla v \rangle_{D_+} = \int_{\partial D_+} (u \cdot n) v \, \mathrm{d}x$$

for $u \in H_\sigma$ and $v \in H^1$ (since ∂D_+ is Lipschitz).

As before, in a slight abuse of notation we denote by $C_{\mathrm{w}}([0,T]; H_\sigma(D_+))$ the collection of all functions $u \colon [0,T] \to H_\sigma(D_+)$ that are weakly continuous into $L^2(D_+)$.

We define

$$\mathcal{S}_\sigma^T(D_+) := \{\psi \in C^\infty(D_+ \times [0,T]) : \ \psi(\cdot, t) \in \mathcal{S}_\sigma(D_+) \text{ for every } t \in [0,T]\},$$

which will be our space of test functions; note that these functions are smooth and incompressible, but there is no restriction on their values on ∂D_+.

To obtain a weak formulation of the equations on D_+ we consider first a smooth solution u with pressure p that satisfies the Euler equations

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0 & \text{in } D_+ \\ \nabla \cdot u = 0 & \text{in } D_+ \\ u \cdot n = 0 & \text{on } \partial D_+, \end{cases}$$

where n is the outward normal to ∂D_+, so that the third equation is in fact $u_3 = 0$ on ∂D_+. We can now multiply the first line by a test function $\phi \in \mathcal{S}_\sigma^T$ and integrate over space and time to give

$$\int_0^t \langle \partial_t u + \nabla \cdot (u \otimes u) + \nabla p, \phi \rangle_{D_+} \, \mathrm{d}\tau = 0.$$

We can now integrate by parts and obtain

$$\begin{aligned} \langle u(t), \phi(t) \rangle_{D_+} - \langle u(0), \phi(0) \rangle_{D_+} - \int_0^t \langle u, \partial_t \phi \rangle_{D_+} \, \mathrm{d}\tau - \int_0^t \langle u \otimes u : \nabla \phi \rangle_{D_+} \, \mathrm{d}\tau \\ - \langle u_3, u \cdot \phi \rangle_{\partial D_+ \times [0,t]} \, \mathrm{d}\tau - \int_0^t \langle p, \nabla \cdot \phi \rangle_{D_+} \, \mathrm{d}\tau + \langle p, \phi \cdot n \rangle_{\partial D_+ \times [0,t]} = 0. \end{aligned}$$

We notice that as $u_3 = 0$ on ∂D_+ and $\nabla \cdot \phi = 0$ in D_+ the two terms involving these expressions vanish and we have

$$\begin{aligned} \langle u(t), \phi(t) \rangle_{D_+} - \langle u(0), \phi(0) \rangle_{D_+} - \int_0^t \langle u, \partial_t \phi \rangle_{D_+} \, \mathrm{d}\tau \\ - \int_0^t \langle u \otimes u : \nabla \phi \rangle_{D_+} \, \mathrm{d}\tau + \langle p, \phi \cdot n \rangle_{\partial D_+ \times [0,t]} = 0. \end{aligned}$$

Since we have not restricted the values of ϕ on ∂D_+ we have a contribution from the boundary, namely

$$\langle p, \phi_3 \rangle_{\partial D_+ \times [0,t]}.$$

We therefore require $p \in \mathcal{D}'(\partial D_+ \times [0, T])$ in our definition of a weak solution.

Definition 4.1 (Weak Solution on D_+) *A weak solution of the Euler equations on $D_+ \times [0, T]$ is a pair (u, p), where $u \in C_{\mathrm{w}}([0, T]; H_\sigma(D_+))$ and $p \in \mathcal{D}'(\partial D_+ \times [0, T])$ such that*

$$\begin{aligned} \langle u(t), \phi(t) \rangle_{D_+} - \langle u(0), \phi(0) \rangle_{D_+} - \int_0^t \langle u(\tau), \partial_t \phi(\tau) \rangle_{D_+} \, \mathrm{d}\tau \\ = \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla \phi(\tau) \rangle_{D_+} \, \mathrm{d}\tau - \langle p, \phi \cdot n \rangle_{\partial D_+ \times [0,t]}, \quad (4.3) \end{aligned}$$

for every $t \in [0, T]$ and for every $\phi \in \mathcal{S}_\sigma^T(D_+)$.

Note that in the final term, $\phi \cdot n = -\phi_3$.

4.2 Half plane reflection map

We introduce an extension $u_E := u + u_R$ a.e. that takes a vector field u defined in D_+ to one defined on the whole of D. Essentially we extend 'by reflection', with appropriate sign changes to ensure that u_R, the 'reflection' of u, is a weak solution on $D_- := \mathbb{T}^2 \times \mathbb{R}_-$. We can then show that u_E is a weak solution on the whole of D (in the sense of Definition 3.2).

Given a vector-valued function $f\colon D_\pm \to \mathbb{R}^3$ we define $f_R\colon D_\mp \to \mathbb{R}^3$ by

$$f_R(x,y,z) := \begin{pmatrix} f_1(x,y,-z) \\ f_2(x,y,-z) \\ -f_3(x,y,-z) \end{pmatrix} \tag{4.4}$$

extending f and f_R by zero beyond their natural domain of definition, we set

$$f_E(x,y,z) := \begin{cases} f(x,y,z) + f_R(x,y,z) & z \neq 0 \\ \frac{1}{2}(f(x,y,z) + f_R(x,y,z)) = (f_1(x,y,0), f_2(x,y,0), 0) & z = 0. \end{cases}$$

Clearly $f_E = f + f_R$ almost everywhere.

Lemma 4.2 *If $u \in H_\sigma(D_+)$ then $u_R \in H_\sigma(D_-)$ and $u_E \in H_\sigma(D)$.*

Proof Since $u \in H_\sigma(D_+)$ there exists $u_n \in \mathcal{S}_{n,\sigma}(D_+)$ (see (4.1)) such that $u_n \to u$ in $L^2(D_+)$. Clearly $u_{n,R} \in \mathcal{S}_{n,\sigma}(D_-)$ and $u_{n,R} \to u_R$ in $L^2(D_-)$. Therefore $u_R \in H_\sigma(D_-)$. Further, $u_n + u_{n,R}$ trivially belongs to $S_\sigma(D)$ and is divergence free. Since $u_n + u_{n,R}$ converges to u_E in $L^2(D)$ we obtain the desired result. □

Now we will show that, with an appropriate choice of the pressure, u_R is a weak solution of the Euler equations in the lower half space D_-. Note that we do not need to extend the pressure distribution p.

Theorem 4.3 *If (u, p) is a weak solution to the Euler equations on D_+ then (u_R, p) is a weak solution in D_-, i.e.*

$$\langle u_R(t), \phi(t)\rangle_{D_-} - \langle u_R(0), \phi(0)\rangle_{D_-} - \int_0^t \langle u_R(\tau), \partial_t\phi(\tau)\rangle_{D_-}\,\mathrm{d}\tau$$
$$= \int_0^t \langle u_R(\tau)\otimes u_R(\tau) : \nabla\phi(\tau)\rangle_{D_-}\,\mathrm{d}\tau - \langle p, \phi\cdot n\rangle_{\partial D_-\times[0,t]}, \quad (4.5)$$

for every $t \in [0, T]$ and for every $\phi \in \mathcal{S}^T_\sigma(D_-)$.

Note that, as always, n represents the outward normal to the domain (here D_-), and therefore in the final term we have $\phi \cdot n = \phi_3$.

Proof Notice first that any $\phi \in \mathcal{S}^T_\sigma(D_-)$ can be written as ψ_R, where $\psi = \phi_R$ and $\phi_R \in \mathcal{S}^T_\sigma(D_+)$. Now, the change of variables $(x_1, x_2, x_3) \to (y_1, y_2, -y_3)$ in the linear term yields

$$\langle u_R, \psi_R\rangle_{D_-} = \langle u, \psi\rangle_{D_+}.$$

For the nonlinear term one can check case-by-case, with the same change of variables, that

$$\int_{D_-} [(u_R)_i(u_R)_j\partial_j(\psi_R)_i](x)\,\mathrm{d}x = \int_{D_+} [u_i u_j \partial_j \psi_i](y)\,\mathrm{d}y.$$

Finally for the pressure term we have

$$\langle p, \psi\cdot n\rangle_{\partial D_+} = \langle p, \psi_3\rangle = -\langle p, \phi_3\rangle = \langle p, \phi\cdot n\rangle_{\partial D_-},$$

since $\psi_3(x, y, 0) = -\phi_3(x, y, 0)$. □

By adding (4.3) and (4.5) it follows that u_E is a weak solution of the Euler equations on D.

Corollary 4.4 *The extension u_E is a weak solution of the Euler equations on D in the sense of Definition 3.2.*

Proof For $\zeta \in \mathcal{S}_\sigma^T$ we can use $\zeta|_{D_+}$ as a test function in (4.3) and $\zeta|_{D_-}$ in (4.5) and add the two equations to obtain

$$\langle u_E(t), \zeta(t)\rangle_D - \langle u_E(0), \zeta(0)\rangle_D - \int_0^t \langle u_E(\tau), \partial_t\zeta(\tau)\rangle_D \,\mathrm{d}\tau = \int_0^t \langle u_E(\tau) \otimes u_E(\tau) : \nabla\zeta(\tau)\rangle_D \,\mathrm{d}\tau,$$

where the pressure terms have cancelled due to the opposite signs of the normal in the two domains; but this is now the definition of a weak solution of the Euler equations in D, given by Definition 3.2. □

Since u_E is a weak solution of the incompressible Euler equations on D, Corollary 3.9 guarantees that if $u_E \in L^3(0, T; L^3(D))$ and

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_0^T \int_D |u_E(t, x+y) - u_E(t, x)|^3 \,\mathrm{d}x \,\mathrm{d}t = 0, \tag{4.6}$$

then u_E conserves energy on $[t_1, t_2]$. Due to the definition of u_E this implies that

$$\|u_E(t_2)\|^2_{L^2(D)} - \|u_E(t_1)\|^2_{L^2(D)} = 2\|u(t_2)\|^2_{L^2(D_+)} - 2\|u(t_1)\|^2_{L^2(D_+)} = 0,$$

i.e. we obtain energy conservation for u, as a solution on D_+. We now find conditions on u alone (rather than $u_E = u + u_R$) that guarantee that (4.6) is satisfied.

4.3 Energy Conservation on D_+

Here we will prove our main result in Theorem 4.6: energy conservation on D_+ under certain assumptions on the weak solution u. The bulk condition we need for u to conserve energy is similar to the condition needed for Theorem 3.9, where we had no boundary.

Lemma 4.5 *Let $u \in L^3(0,T;L^3(D_+))$ be a weak solution of the Euler equations on D_+ such that*

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{|y|}^{\infty} |u(t,x+y)-u(t,x)|^3 \,\mathrm{d}x_3\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t = 0; \tag{4.7}$$

then (4.6) *holds if and only if*

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{-|y|}^{|y|} |u_E(t,x+y)-u_E(t,x)|^3 \,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}t = 0. \tag{4.8}$$

Proof We can split the integral over D in (4.6) into three sub-integrals over the regions $A := \{x|x_3 > |y|\}$, $B := \{x|x_3 < -|y|\}$ and $C := \{x||x_3| \le |y|\}$. We have

$$|u_E(t,x+y)-u_E(t,x)|^3 = [\mathbb{I}_A(x)+\mathbb{I}_B(x)+\mathbb{I}_C(x)]\ |u_E(x+y)-u_E(x)|^3.$$

For $\int_A$, since $x_3 > 0$ and $x_3 + y_3 > 0$ then u_E is in fact u, thus

$$\begin{aligned}\lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{-\infty}^{\infty} \mathbb{I}_A(x)|u_E(t,x+y)-u_E(t,x)|^3 \,\mathrm{d}x_3\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t \\ = \lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{|y|}^{\infty} |u(t,x+y)-u(t,x)|^3 \,\mathrm{d}x_3\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t = 0\end{aligned}$$

by (4.7). For $\int_B$ a very similar argument holds with the extra changes of variables $x_3 \mapsto -\xi_3$ and $(y_1,y_2,y_3) \mapsto (\zeta_1,\zeta_2,-\zeta_3)$; then using the notation $\tilde{x} = (x_1,x_2)$ we have

$$\begin{aligned}&\lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{-\infty}^{\infty} \mathbb{I}_B(x)|u_E(t,x+y)-u_E(t,x)|^3 \,\mathrm{d}x_3\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t \\ &= \lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{\infty}^{-|y|} |u_E(t,x+y)-u_E(t,x)|^3 \,\mathrm{d}x_3\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t \\ &= \lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{|y|}^{\infty} |u(t,\tilde{x}+\tilde{y},\xi_3-y_3)-u(t,\tilde{x},\xi_3)|^3 \,\mathrm{d}\xi_3\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t \\ &= \lim_{|\zeta|\to 0} \frac{1}{|\zeta|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{|\zeta|}^{\infty} |u(t,\tilde{x}+\tilde{\zeta},\xi_3+\zeta_3)-u(t,\tilde{x},\xi_3)|^3 \,\mathrm{d}\xi_3\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t,\end{aligned}$$

which converges to zero by (4.7). This leaves only the integral over the region C, which is (4.8). □

Thus to show energy conservation we have reduced the problem to showing that

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{-|y|}^{|y|} |u_E(t,x+y) - u_E(t,x)|^3 \, \mathrm{d}x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_1 \, \mathrm{d}t = 0.$$

We will impose the a condition of continuity on a strip around the boundary to deal with the presence of this term, that is, there exists a $\delta > 0$ such that $u \in L^3(0,T;C^0(\mathbb{T}^2 \times [0,\delta]))$.

We note that, since $\partial D_+ = \mathbb{T}^2$ is compact, for each $t \in [0,T]$ there exists a non-decreasing function $w_t : [0,\infty) \to [0,\infty)$ with $w_t(0) = 0$ and continuous at 0, such that

$$|u(t,x+z) - u(t,x)| < w_t(|z|) \tag{4.9}$$

whenever $x \in \partial D_+$ and $x + z \in D$ with $|z| \le \frac{\delta}{2}$.

We have assumed so far that $u \in C_{\mathrm{w}}([0,T]; H_\sigma(D_+))$ and so have $u \cdot n = 0$ on the boundary in a trace sense. We are now assuming this extra continuity on this strip around the boundary and so now have $u \cdot n = 0$ on the boundary point-wise. We can now provide conditions on u to ensure energy conservation.

Theorem 4.6 *Let $u \in L^3(0,T;L^3(D_+))$ be a weak solution of the Euler equations on D_+ that satisfies $u \in L^3(0,T;C^0(\mathbb{T}^2 \times [0,\delta])$ for some $\delta > 0$, and*

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{|y|}^{\infty} |u(t,x+y) - u(t,x)|^3 \, \mathrm{d}x_3 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}t = 0;$$

then u conserves energy on $[t_1,t_2]$.

Proof By considering Lemma 4.5 we just need to show that

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_{t_1}^{t_2} \iint_{\mathbb{T}^2} \int_{-|y|}^{|y|} |u_E(t,x+y) - u_E(t,x)|^3 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \,\mathrm{d}t = 0.$$

We have assumed that $u \in L^3(0,T;C^0(\mathbb{T}^2 \times [0,\delta])$, thus $u \in L^3(0,T;L^\infty(\mathbb{T}^2 \times [0,\delta])$ and so

$$u_E \in L^3(0,T;L^\infty(\mathbb{T}^2 \times [-\delta,\delta]).$$

Then, since for all $|y| < \delta$ we have

$$\frac{1}{|y|} \iint_{\mathbb{T}^2} \int_{-|y|}^{|y|} |u_E(t,x+y) - u_E(t,x)|^3 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \leq C \sup_{x\in\mathbb{T}^2\times[0,\delta)} |u(t)|^3,$$

we can move the limit inside the time integral using the Dominated Convergence Theorem, and it suffices to show that

$$\lim_{|y|\to 0} \frac{1}{|y|} \iint_{\mathbb{T}^2} \int_{-|y|}^{|y|} |u_E(t,x+y) - u_E(t,x)|^3 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 = 0$$

for almost every $t \in (t_1,t_2)$. As, $u \in L^3(0,T;C^0(\mathbb{T}^2 \times [0,\delta])$

$$u_E \in L^3(0,T;C^0(\mathbb{T}^2 \times [-\delta,\delta]),$$

this is because $u \cdot n = 0$ on the boundary point-wise and so the boundary values are the same for u and u_R. Now fix t and let $x' = (x_1,x_2,0)$; then

$$\begin{aligned}|u_E(t,x'+x_3+y) - u_E(t,x'+x_3)| \leq& |u_E(t,x'+x_3+y) - u_E(t,x') \\ &+ u_E(t,x') - u_E(t,x'+x_3)| \\ \leq& w_t(|y+x_3|) + w_t(|x_3|) \leq 2w_t(2|y|)\end{aligned}$$

and thus

$$\begin{aligned}\frac{1}{|y|}\iint_{\mathbb{T}^2}\int_{-|y|}^{|y|} & |u_E(t,x+y)-u_E(t,x)|^3 \,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ & \leq C\frac{1}{|y|}\iint_{\mathbb{T}^2}\int_{-|y|}^{|y|}|w_t(2|y|)|^3\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ & \leq C\frac{1}{|y|}|\mathbb{T}^2||y||w_t(2|y|)|^3 \to 0\end{aligned}$$

as $|y| \to 0$, which is what we required. □

We note that all the conditions for this theorem are satisfied by a weak solution u that satisfies

$$|u(x,t)-u(y,t)| \leq Cf(x_3)|x-y|^{\alpha}$$

for $\alpha > \frac{1}{3}$ and $f \in L^3(0,\infty)$.

4.4 Conclusion

This method is a consequence of the choice of extension and the symmetry of the domain; furthermore, we do not need any estimates on the pressure. We further see that for the bulk of the solution away from the boundary effects the same assumptions as before on $\mathbb{R}^3$ or $\mathbb{T}^3$ are required. Here we need some interesting and quite natural extra assumptions near and on the boundary, namely boundedness and continuity at the boundary.

However, it is not obvious how to extend this method to an arbitrary bounded domain as we would have to show that one can solve, for any region Ω, an outer weak solution problem on Ω^c, with a perscribed boundary pressure term, that has the required regularity to apply the theory in Chapter 3 on $\mathbb{R}^3$.

We also have this boundary pressure term and it would be nice to remove this from the proof. In Chapter 5 we will use a similar reflection map for the solution but only in a local strip outside of the boundary and obtain energy conservation under the same conditions as in this chapter but we use a definition of a weak solution that does not require the boundary pressure term, or indeed any pressure in the definition.

Chapter 5

Energy Conservation on $\mathbb{T}^2 \times \mathbb{R}_+$ via a Local Extension

It was noted that although the last chapter gave a simple proof of energy conservation on the domain $\mathbb{T}^2 \times \mathbb{R}_+$ under a weak set of conditions, the argument there had a few drawbacks; we had a boundary pressure term appearing in the definition of a weak solution and it would be difficult to generalise the method to a weak solution of the Euler equations on an arbitrary bounded domain.

In this chapter we will fix all these drawbacks. We will consider the same domain D_+ but now we remove the boundary pressure term from the definition of a weak solution. We only set a local extension and do not require the extension to solve any form of equation, only to keep incompressibility and boundary conditions. We still rely on the same conditions on the solution u as introduced in the previous chapter.

With this local method and no boundary pressure term it should be easier to generalise the argument to an arbitrary bounded domain.

The work in this chapter is an extension of Robinson et al. (2018b).

5.1 Pressure-less weak solutions on $D_+ := \mathbb{T}^2 \times \mathbb{R}_+$

The spaces used here are similar to the ones defined in Sections 3.1 and 4.1 so we will just point out the slight differences here.

We define the space of test functions

$$\mathcal{S}_{n,\sigma}(D_+ \times [0,T]) := \{\psi \in \mathcal{S}(D_+ \times [0,T]) \colon \\ \nabla \cdot \psi(\cdot, t) = 0,\ \psi \cdot n = 0 \,\text{on}\, \partial D_+ \,\forall t \in [0,T]\}.$$

Here we have added the restriction that $\psi \cdot n = 0$ on ∂D_+ to the space of test functions. This allows us to remove the pressure term entirely in the weak formulation of the solution.

To obtain a weak formulation on D_+ assume that we have a smooth solution u with pressure p that satisfies the incompressible Euler equations

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0 & \text{in}\, D_+ \\ \nabla \cdot u = 0 & \text{in}\, D_+ \\ u \cdot n = 0 & \text{on}\, \partial D_+, \end{cases}$$

where n is the outer normal to ∂D_+, so with this domain the third equation is in fact $u_3 = 0$ on ∂D_+. We can multiply (inner product) the first line by a vector valued test function $\phi \in \mathcal{S}_{n,\sigma}(D_+ \times [0,T])$ and integrate over all space and the time interval $(0,t)$ to obtain

$$\int_0^t \langle \partial_t u + \nabla \cdot (u \otimes u) + \nabla p, \phi \rangle_{D_+} \,\mathrm{d}\tau = 0,$$

where $\langle \cdot, \cdot \rangle_{D_+}$ denotes the L^2-inner product in space. We can now integrate by parts and obtain

$$\langle u(t), \phi(t)\rangle_{D_+} - \langle u(0), \phi(0)\rangle_{D_+} - \int_0^t \langle u, \partial_t \phi\rangle_{D_+} \,\mathrm{d}\tau - \int_0^t \langle (u \otimes u) : \nabla \phi\rangle_{D_+} \,\mathrm{d}\tau$$
$$- \int_{\partial D_+ \times [0,t]} u_3 u \cdot \phi \,\mathrm{d}S_x \,\mathrm{d}\tau - \int_0^t \langle p, \nabla \cdot \phi\rangle_{D_+} \,\mathrm{d}\tau + \int_{\partial D_+ \times [0,t]} p\phi_3 \,\mathrm{d}S_x \,\mathrm{d}\tau = 0.$$

We notice that both $u_3 = 0$ and $\phi_3 = 0$ on ∂D_+; further, we have that $\nabla \cdot \phi = 0$ in D_+ and so the three terms involving these expressions vanish and we have

$$\langle u(t), \phi(t)\rangle_{D_+} - \langle u(0), \phi(0)\rangle_{D_+} - \int_0^t \langle u, \partial_t \phi\rangle_{D_+} \,\mathrm{d}\tau - \int_0^t \langle (u \otimes u) : \nabla \phi\rangle_{D_+} \,\mathrm{d}\tau = 0.$$

Thus we have a weak formulation of the equations where there are no pressure terms appearing.

Definition 5.1 (Pressure-less Weak Solution on D_+) *A weak solution of the Euler equations on $D_+ \times [0,T]$ is a vector-valued function $u\colon D_+ \times [0,T] \to \mathbb{R}^3$ where $u \in C_{\mathrm{w}}([0,T]; H_\sigma(D_+))$ such that*

$$\langle u(t), \psi(t)\rangle_{D_+} - \langle u(0), \psi(0)\rangle_{D_+} - \int_0^t \langle u(\tau), \partial_t \psi(\tau)\rangle_{D_+} \,\mathrm{d}\tau = \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla \psi(\tau)\rangle_{D_+} \,\mathrm{d}\tau, \tag{5.1}$$

for every $t \in [0,T]$ and for all $\psi \in \mathcal{S}_{n,\sigma}(D_+ \times [0,T])$.

Here we have obtained a different definition of a weak solution on D_+. In Definition 4.1 we have the extra boundary pressure term. We will compare different definitions of weak solutions in Chapter 6 but do not know the exact relation between weak solutions given by these definitions or whether they are equivalent.

As in Chapter 3 we use the same definition of mollification, though when we want to regularise a function we will need that function to be defined on all of $D := \mathbb{T}^2 \times \mathbb{R}$. Thus if we apply mollification to a function only defined on D_+ what we are implicitly doing is extending by zero to the entirety of D and then mollifying.

5.2 Localised half plane reflection map

In Chapter 4 we used the extension u_E and showed many useful properties of this extension. However, for this chapter we will use a restricted form of the reflection map introduced in Chapter 4. We will restrict to a region within distance δ of the boundary to obtain u_r, then get a restricted extension $u_e := u + u_r$ almost everywhere. We will then show that $J_\varepsilon u_e$ and $J_\varepsilon J_\varepsilon u_e$ are incompressible in D_+ and have the required boundary conditions to be a test function. In order to prove these properties of u_r and u_e we have to define a few more spaces with unusual boundary conditions.

For ease of notation, for any set $I \subset \mathbb{R}$ we define $D_I := \{x \in D : x_3 \in I\}$.

To prove that $J_\varepsilon u_e$ and $J_\varepsilon J_\varepsilon u_e$ are incompressible in D_+ we have to define some function spaces which will treat the boundary $x_3 = 0$ differently to the new boundary we have created at $x_3 = -\delta$.

Definition 5.2 *We define the spaces of functions that are zero on the lower boundary $x_3 = -\delta$*

$$\mathcal{D}_l(D_{[-\delta,0]}) := \{\psi \in C^\infty(D_{[-\delta,0]})\colon \psi = 0 \textit{ for } x_3 = -\delta\},$$

$$\mathcal{S}_l(D_{[-\delta,\infty)}) := \{\psi \in \mathcal{S}(D_{[-\delta,\infty)})\colon \psi = 0 \textit{ for } x_3 = -\delta\},$$

$$H^1_l(D_{[-\delta,0]}) := \textit{the completion of } \mathcal{D}_l(D_{[-\delta,0]}) \textit{ in the } H^1(D_{[-\delta,0]}) \textit{ norm.}$$

and

$$H^1_l(D_{[-\delta,\infty)}) := \textit{the completion of } \mathcal{S}_l(D_{[-\delta,\infty)}) \textit{ in the } H^1(D_{[-\delta,\infty)}) \textit{ norm.}$$

Further, we define the incompressible spaces of functions with zero normal compo-

nent on the upper boundary

$$\mathcal{D}_{u,\sigma}(D_{[-\delta,0]}) := \{\psi \in C^\infty(D_{[-\delta,0]})\colon\ \mathrm{div}\psi = 0 \textit{ and } \psi\cdot n = 0 \textit{ for } x_3 = 0\}$$

and

$$H_{u,\sigma}(D_{[-\delta,0]}) := \textit{the completion of } \mathcal{D}_{u,\sigma}(D_{[-\delta,0]}) \textit{ in the } L^2(D_{[-\delta,0]}) \textit{ norm.}$$

Further, we define

$$\mathcal{S}_\sigma(D_{[-\delta,\infty)}) := \{\phi \in \mathcal{S}(D_{[-\delta,\infty)}) : \nabla\cdot\phi = 0\}$$

and

$$\tilde{H}_\sigma(D_{[-\delta,\infty)}) := \textit{the completion of } \mathcal{S}_\sigma(D_{[-\delta,\infty)}) \textit{ in the } L^2(D_{[-\delta,\infty)}) \textit{ norm.}$$

It is important to notice that we have used the notation $H^1_l(D_{[-\delta,0]})$ to emphasise that the boundary values are not zero on both boundaries but only on the lower boundary $x_3 = -\delta$.

Functions in $H_{u,\sigma}(D_{[-\delta,0]})$ are weakly divergence free with respect to functions in $H^1_l(D_{[-\delta,0]})$ in that they satisfy

$$\langle u, \nabla\phi\rangle_{D_{[-\delta,0]}} = 0 \quad \text{for every} \quad \phi \in H^1_l(D_{[-\delta,0]}).$$

The boundary terms we would expect when integrating by parts vanish as u has zero normal component at the top boundary and ϕ vanishes at the lower boundary. Indeed, since $\mathcal{D}_{u,\sigma}(D_{[-\delta,0]})$ is dense in $H_{u,\sigma}(D_{[-\delta,0]})$, for any $u \in H_{u,\sigma}(D_{[-\delta,0]})$ we can find $(u_k) \in \mathcal{D}_{u,\sigma}(D_{[-\delta,0]})$ such that $u_k \to u$ in $H^1(D_{[-\delta,0]})$, and then for any $\phi \in H^1_l(D_{[-\delta,0]})$ we have

$$\langle u, \nabla\phi\rangle_{D_{[-\delta,0]}} = \lim_{k\to\infty}\langle u_k, \nabla\phi\rangle_{D_{[-\delta,0]}} = \lim_{k\to\infty}\langle \nabla\cdot u_k, \phi\rangle_{D_{[-\delta,0]}} = 0.$$

We have no boundary terms in the integration-by-parts above as $u_k \cdot n = 0$ on ∂D_+ for all k and $\phi = 0$ for $\{x_3 = -\delta\}$. Further, we notice that if $v \in \tilde{H}_\sigma(D_{[-\delta,\infty)})$ then

$$\langle v, \nabla\phi\rangle_{D_{[-\delta,\infty)}} = 0 \quad \text{for every} \quad \phi \in H_l^1(D_{[-\delta,\infty)}).$$

Given a vector-valued function $f\colon D_+ \to \mathbb{R}^3$ we define f_R and f_E as in Section 4.2. We then define $g_e := \mathbb{I}_{D_{[-\delta,\infty)}}(x)g_E(x)$ for some $\delta > 0$ so this can be considered as just a local extension of width δ from the boundary. This naturally gives rise to the definition $g_r := \mathbb{I}_{[-\delta,\infty)}(x)g_R(x)$.

We have defined this extension so that after mollification it should have all the properties of a test function. This will allow us to use it as a test function so we can regularise the equation and manipulate the terms.

Note that for any $\delta > 0$ from the definition of u_e that

$$\|u_e\|_{L^p(D_{[-\delta,\infty)})} \le \|u_e\|_{L^p(D)} \le C\|u\|_{L^p(D_+)}.$$

Lemma 5.3 *If $v \in H_\sigma(D_+)$ then $v_e \in \tilde{H}_\sigma(D_{[-\delta,\infty)})$ and for $\delta > 2\varepsilon$*

1. *$\|v_e\|_{L^p(D_{[-\delta,\infty)})} \le C\|v\|_{L^p(D_+)}$, with C independent of δ,*
2. *$J_\varepsilon(v_e)$ and $J_\varepsilon(J_\varepsilon(v_e))$ are incompressible in D_+, and*
3. *$J_\varepsilon(v_e) \cdot n = 0$ and $J_\varepsilon(J_\varepsilon(v_e)) \cdot n = 0$ on ∂D_+.*

Proof To show that $v_e \in \tilde{H}_\sigma(D_{[-\delta,\infty)})$, notice that since $v \in H_\sigma(D_+)$ the reflection $v_R \in H_\sigma(D_-)$, and therefore $v_r \in H_{u,\sigma}(D_{[-\delta,0]})$ as v_R satisfies the appropriate boundary conditions at $x_3 = 0$. We can then perform the same steps as in proving Lemma 4.2.

For part 1 we see that

$$\|v_e\|_{L^p(D_{[-\delta,\infty)}} = \|v + v_R\|_{L^p(D_{[-\delta,\infty)})} \le \|v\|_{L^p(D_+)} + \|v_R\|_{L^p(D_{[-\delta,\infty)})} \le C\|v\|_{L^p(D_+)}$$

as $\|v_R\|_{L^p(D_-)} = \|v\|_{L^p(D_+)}$. For part 2 we see that the extension is weakly incompressible since it is in $\tilde{H}_\sigma(D_{[-\delta,\infty)})$ by Lemma 4.2 and so $J_\varepsilon(v_e)$ is strongly incompressible in D_+. To show this note that $v_e \in \tilde{H}_\sigma(D_{[-\delta,\infty)})$ and so

$$\langle v_e, \nabla\phi\rangle_D = \langle v_e, \nabla\phi\rangle_{D_{[-\delta,\infty)}} = 0 \text{ for all } \phi \in \mathcal{S}_l(D_{[-\delta,\infty)})$$

where

$$\phi \in \mathcal{S}_l(D_{[-\delta,\infty)}) := \{\phi \in S(D_{[-\delta,\infty)}) : \phi = 0 \text{ for } x_3 = -\delta\}.$$

We can let $\phi = J_\varepsilon\eta$ or $J_\varepsilon J_\varepsilon\eta$ for any $\eta \in \mathcal{S}(D_{[-(\delta-2\varepsilon),\infty)})$ and extend it by zero to all of D so we keep that $\phi = 0$ for $x_3 = -\delta$ and thus

$$0 = \langle v_e, \nabla J_\varepsilon\eta\rangle_{D_{[-\delta,\infty)}} = \langle J_\varepsilon v_e, \nabla\eta\rangle_{D_{[-(\delta-2\varepsilon),\infty)}} = \langle \nabla\cdot J_\varepsilon v_e, \eta\rangle_{D_{[-(\delta-2\varepsilon),\infty)}}.$$

Notice that we need $\delta - 2\varepsilon > 0$. We have that $J_\varepsilon v_e$ is strongly incompressible in D_+. Similarly for $J_\varepsilon J_\varepsilon v_e$.

For part 3 we will first show that $J_\varepsilon(v_e)_3 = 0$ on ∂D_+. Note that this is the same as $J_\varepsilon((v_e)_3) = 0$. Our extension is locally an odd function in the region of width δ in the third component and φ_ε is an even function thus the integral over the ball centered around the boundary is zero. Since $J_\varepsilon v_e$ is still odd the same argument works for $J_\varepsilon J_\varepsilon v_e$. Here, as before, we need $\delta > 2\varepsilon$ so it is an odd extension. □

We now consider various convergence results for $J_\varepsilon(u_e)$ and $J_\varepsilon J_\varepsilon(u_e)$.

Lemma 5.4 *If $u \in L^p(D_+)$ with $1 \le p < \infty$ then $\|J_\varepsilon(u_e) - u\|_{L^p(D_+)} \to 0$ and $\|J_\varepsilon J_\varepsilon(u_e) - u\|_{L^p(D_+)} \to 0$.*

Proof Since we are only integrating over D_+ we have

$$\|J_\varepsilon J_\varepsilon(u_e) - u\|_{L^p(D_+)} = \|J_\varepsilon J_\varepsilon(u_e) - u_e\|_{L^p(D_+)} \le \|J_\varepsilon J_\varepsilon(u_e) - u_e\|_{L^p(D)}.$$

Then as mollification converges in $L^p(D)$ we are finished. For more details see Majda & Bertozzi (2002) page 98. □

For the next lemma we will show that the reflection map can be moved across the L^2 inner-product. The reflection map given by (4.4) can be extended to functions defined on D so for $f : D \to \mathbb{R}^3$ we define $f_R : D \to \mathbb{R}^3$, where we reflect back from D to all of D again. Notice that in the proof of the lemma for $v_r = \mathbb{I}_{D_{[-\delta,\infty)}} v_R$ we need $\gamma \leq \delta - \varepsilon$ to keep the symmetry used in the proof.

Lemma 5.5 *Fix $\delta > 0$, and let u and v be arbitrary vector fields on D. Let $v_r = \mathbb{I}_{D_{[-\delta,\infty)}} v_R$ then for $\gamma \leq \delta$ we have*

$$\langle u, v_r \rangle_{\mathbb{T}^2 \times (-\gamma,\gamma)} = \langle u_r, v \rangle_{\mathbb{T}^2 \times (-\gamma,\gamma)}.$$

Further, if $\varepsilon, \gamma > 0$ satisfy $\gamma \leq \delta - \varepsilon$ we have

$$J_\varepsilon(f_r)(x) = [J_\varepsilon(f)]_r(x)$$

and thus

$$\langle J_\varepsilon u, J_\varepsilon v_r \rangle_{\mathbb{T}^2 \times (-\gamma,\gamma)} = \langle J_\varepsilon u_r, J_\varepsilon v \rangle_{\mathbb{T}^2 \times (-\gamma,\gamma)}.$$

Proof The first part is just a simple change of variables of x_3 to $-\xi_3$, using the symmetric domain of integration and the simple reflection map. Using the notation $x = (\tilde{x}, x_3)$ we can use the change of variables $x_3 = -\xi_3$ so that

$$\begin{aligned}\langle u, v_r \rangle_{\mathbb{T}^2 \times (-\gamma,\gamma)} &= \iint_{\mathbb{T}^2} \int_{-\gamma}^{\gamma} u_i(\tilde{x}, x_3) v_{ri}(\tilde{x}, x_3) \, \mathrm{d}x_3 \, \mathrm{d}\tilde{x} \\ &= \iint_{\mathbb{T}^2} \int_{-\gamma}^{\gamma} u_i(\tilde{x}, -\xi_3) v_{ri}(\tilde{x}, -\xi_3) \, \mathrm{d}\xi_3 \, \mathrm{d}\tilde{x}.\end{aligned}$$

We can use the maps of $u \mapsto u_r$ and $v_r \mapsto v$ in the region $\mathbb{T}^2 \times (-\delta, \delta)$. For $i = 1, 2$

$$\iint_{\mathbb{T}^2} \int_{-\gamma}^{\gamma} u_i(\tilde{x}, -\xi_3) v_{ri}(\tilde{x}, -\xi_3) \, \mathrm{d}\xi_3 \, \mathrm{d}\tilde{x} = \iint_{\mathbb{T}^2} \int_{-\gamma}^{\gamma} u_{ri}(\tilde{x}, \xi_3) v_i(\tilde{x}, \xi_3) \, \mathrm{d}\xi_3 \, \mathrm{d}\tilde{x}$$

while for $i = 3$ we have that the minus sign moves across and so have

$$\iint_{\mathbb{T}^2} \int_{-\gamma}^{\gamma} u_3(\tilde{x}, -\xi_3) v_{3r}(\tilde{x}, -\xi_3) \,\mathrm{d}\xi_3 \,\mathrm{d}\tilde{x} = \iint_{\mathbb{T}^2} \int_{-\gamma}^{\gamma} -u_3(\tilde{x}, -\xi_3) - v_{3r}(\tilde{x}, -\xi_3) \,\mathrm{d}\xi_3 \,\mathrm{d}\tilde{x}$$

$$= \iint_{\mathbb{T}^2} \int_{-\gamma}^{\gamma} u_{3r}(\tilde{x}, \xi_3)(v_3(\tilde{x}, \xi_3) \,\mathrm{d}\xi_3 \,\mathrm{d}\tilde{x}$$

and so $\langle u, v_r \rangle_{\mathbb{T}^2 \times (-\gamma,\gamma)} = \langle u_r, v \rangle_{\mathbb{T}^2 \times (-\gamma,\gamma)}$.

For the second part we just have to show that $J_\varepsilon(f_r) = J_\varepsilon(f)_r$ in the region where $\gamma \leq \delta - \varepsilon$ as we can then apply the first part to obtain the result. As J_ε acts component-wise this is easy to see. For $i = 1, 2$

$$\begin{aligned}
[J_\varepsilon(f_r)]_i(x) &= \int_{B_\varepsilon(0)} \varphi_\varepsilon(\tilde{y}, y_3)(f_r)_i(\tilde{x} - \tilde{y}, x_3 - y_3) \,\mathrm{d}y_3 \,\mathrm{d}\tilde{y} \\
&= \int_{B_\varepsilon(0)} \varphi_\varepsilon(\tilde{y}, y_3) f_i(\tilde{x} - \tilde{y}, -x_3 + y_3) \,\mathrm{d}y_3 \,\mathrm{d}\tilde{y} \\
&= \int_{B_\varepsilon(0)} \varphi_\varepsilon(\tilde{y}, -\xi_3) f_i(\tilde{x} - \tilde{y}, -x_3 - \xi_3) \,\mathrm{d}\xi_3 \,\mathrm{d}\tilde{y} \\
&= \int_{B_\varepsilon(0)} \varphi_\varepsilon(\tilde{y}, \xi_3) f_i(\tilde{x} - \tilde{y}, -x_3 - \xi_3) \,\mathrm{d}\xi_3 \,\mathrm{d}\tilde{y} = ([J_\varepsilon(f)]_r)_i(x)
\end{aligned}$$

since φ_ε is a radial function. For $i = 3$ the calculation is similar but we have to deal with an extra minus sign in the map of $f_r \mapsto f$, we obtain

$$\begin{aligned}
[J_\varepsilon(f_r)]_i(x) &= \int_{B_\varepsilon(0)} \varphi_\varepsilon(\tilde{y}, y_3)(f_r)_i(\tilde{x} - \tilde{y}, x_3 - y_3) \,\mathrm{d}y_3 \,\mathrm{d}\tilde{y} \\
&= \int_{B_\varepsilon(0)} \varphi_\varepsilon(\tilde{y}, y_3) - f_i(\tilde{x} - \tilde{y}, -x_3 + y_3) \,\mathrm{d}y_3 \,\mathrm{d}\tilde{y} \\
&= - \int_{B_\varepsilon(0)} \varphi_\varepsilon(\tilde{y}, -\xi_3) f_i(\tilde{x} - \tilde{y}, -x_3 - \xi_3) \,\mathrm{d}\xi_3 \,\mathrm{d}\tilde{y} \\
&= - \int_{B_\varepsilon(0)} \varphi_\varepsilon(\tilde{y}, \xi_3) f_i(\tilde{x} - \tilde{y}, -x_3 - \xi_3) \,\mathrm{d}\xi_3 \,\mathrm{d}\tilde{y} = ([J_\varepsilon(f)]_r)_i(x).
\end{aligned}$$

This finishes off the proof. □

5.3 Using $J_\varepsilon J_\varepsilon(u_e)$ as a test function

We will show that if u is a weak solution then in fact (5.1) holds for a larger class of test functions with less time regularity. We denote by $C^{0,1}([0,T];H_\sigma(D_+))$ the space of Lipschitz functions from $[0,T]$ into $H_\sigma(D_+)$. Most of the argument in this section follow that from Chapter 3, with minor changes and generalisations needed because of the boundary.

Lemma 5.6 *If u is a pressure-less weak solution of the Euler equations on D_+ as in Definition 5.1, then* (5.1) *holds for every $\psi \in \mathcal{L}_{n,\sigma}$, where $\mathcal{L}_{n,\sigma}$ is the completion of $\mathcal{S}_{n,\sigma}$ under the $L^1(0,T;H^3) \cap C^{0,1}([0,T];H_\sigma)$ norm.*

The proof is the same as the proof in Chapter 3 except we use the density of $\mathcal{S}_{n,\sigma}$ in $\mathcal{L}_{n,\sigma}$ with respect to the norm

$$\|\cdot\|_{L^1(0,T;H^3)} + \|\cdot\|_{C^{0,1}([0,T];L^2)}.$$

We now study the time regularity of u when paired with a sufficiently smooth function that is not necessarily divergence free.

Lemma 5.7 *If u is a pressure-less weak solution on D_+ from Definition 5.1 then*

$$|\langle u(t) - u(s), \psi\rangle_{D_+}| \leq C|t-s| \quad \textit{for all} \quad \psi \in \mathcal{S}(D_+), \tag{5.2}$$

where C depends only on $\|u\|_{L^\infty(0,T;L^2)}$ and $\|\psi\|_{H^3}$. Further, we have

$$|\langle u(t) - u(s), \psi\rangle_{D}| \leq C|t-s| \quad \textit{for all} \quad \psi \in \mathcal{S}(D). \tag{5.3}$$

The proof is the same as in Chapter 3 except here, since we are considering a domain with boundary, the Helmholtz–Weyl decomposition of a vector-valued function $\psi \in \mathcal{S}(D_+)$ into $\psi = \eta + \nabla\sigma$ gives an $\eta \in \mathcal{S}_{n,\sigma}(D_+)$. This can be seen in Chapter 2 theorem 2.16 of Robinson, Rodrigo, & Sadowksi (2016). Note that as the support of u is D_+ we have that

$$|\langle u(t) - u(s), \psi\rangle_D| \le C|t-s| \quad \text{for all} \quad \psi \in \mathcal{S}(D),$$

which gives (5.3).

Remark The inequalities hold for $\psi \in H^3(D_+)$ for (5.2) and $\psi \in H^3(D)$ for (5.3) as the constant C depends on the H^3 norm of ψ. Thus we can use density to extend this lemma to these larger spaces of functions.

Corollary 5.8 *Let u be a pressure-less weak solution on D_+ from Definition 5.1. Fix fix $\varepsilon > 0$ and $\delta > 0$ such that $\delta > 2\varepsilon$, then the functions $J_\varepsilon(u_e)(x,\cdot)$ and $J_\varepsilon J_\varepsilon(u_e)(x,\cdot)$ are Lipschitz continuous in t as a function into $L^2(D_+)$:*

$$\|J_\varepsilon(u_e)(\cdot,t) - J_\varepsilon(u_e)(\cdot,s)\|_{L^2(D_+)} \le C_\varepsilon|t-s|, \tag{5.4}$$

and

$$\|J_\varepsilon J_\varepsilon(u_e)(\cdot,t) - J_\varepsilon J_\varepsilon(u_e)(\cdot,s)\|_{L^2(D_+)} \le C_\varepsilon|t-s|. \tag{5.5}$$

Furthermore, $J_\varepsilon(u_e), J_\varepsilon J_\varepsilon(u_e) \in \mathcal{L}_{n,\sigma}$.

Proof First to prove (5.5) set $v = u_e(t) - u_e(s)$ and we see that

$$\|J_\varepsilon J_\varepsilon v\|_{L^2(D_+)} \le \|J_\varepsilon J_\varepsilon v\|_{L^2(D_{[-\delta,\infty)})} \le \|J_\varepsilon v\|_{L^2(D_{[-\delta,\infty)})}$$

and for (5.4) we see that

$$\|J_\varepsilon v\|_{L^2(D_+)} \le \|J_\varepsilon v\|_{L^2(D_{[-\delta,\infty)})};$$

then notice that

$$\begin{aligned}\|J_\varepsilon v\|_{L^2(D_{[-\delta,\infty)})} =& \|J_\varepsilon([u(t)-u(s)] + [u_r(t) - u_r(s)])\|_{L^2(D_{[-\delta,\infty)})} \\ \le& 2\|J_\varepsilon([u(t)-u(s)])\|_{L^2(D_{[-\delta,\infty)})}.\end{aligned}$$

Using a generalisation of Lemma 5.7 for $\psi \in H^3$, let $\psi = J_\varepsilon f$ for $f \in L^2(D)$ with

$\|f\|_{L^2(D)} = 1$; we obtain

$$|\langle u(t) - u(s), J_\varepsilon f\rangle_D| = |\langle J_\varepsilon(u(t) - u(s)), f\rangle_D| \\ \leq \|u\|_{L^\infty(0,T;L^2(D_+))}\|\varphi_\varepsilon\|_{W^{3,1}}|t-s|\|f\|_{L^2}.$$

We can then take the supremum over all f with $\|f\|_{L^2} = 1$ over both sides to finish off the Lipschitz in time bound and obtain (5.4) and (5.5).

We now need to prove the other properties required to be elements of the space $\mathcal{L}_{n,\sigma}$ for both $J_\varepsilon u_e$ and $J_\varepsilon J_\varepsilon u_e$. Finally, since $u \in L^\infty(0,T;L^2)$, we observe that both $J_\varepsilon u_e$ and $J_\varepsilon J_\varepsilon(u_e) \in L^\infty(0,T;H^3)$ and

$$\|J_\varepsilon(J_\varepsilon(u_e))\|_{L^1(0,T;H^3)} \leq T\|J_\varepsilon(J_\varepsilon(u_e))\|_{L^\infty(0,T;H^3)}$$

as $[0,T]$ is bounded (similary for $J_\varepsilon u_e$).

We see from Lemma 5.3 that both $J_\varepsilon u_e$ and $J_\varepsilon J_\varepsilon(u_e)$ are divergence free and $J_\varepsilon(J_\varepsilon(u_e)) \cdot n$ and $J_\varepsilon(u_e) \cdot n$ vanish on ∂D_+, proving that both $J_\varepsilon u_e$ and $J_\varepsilon J_\varepsilon u_e$ are in $\mathcal{L}_{n,\sigma}$. □

This section and in particular Corollary 5.8 now allows us to use $J_\varepsilon J_\varepsilon(u_e)$ as a test function in the weak formulation of the Euler equations and we have shown the sufficent regularity of $J_\varepsilon(u_e)$ needed to manipulate terms in the future.

5.4 Manipulating equation

Since $J_\varepsilon J_\varepsilon(u_e) \in \mathcal{L}_\sigma$ it follows from Lemma 5.6 that

$$\langle u(t), J_\varepsilon J_\varepsilon(u_e)(t)\rangle_{D_+} - \langle u(0), J_\varepsilon J_\varepsilon(u_e)(0)\rangle_{D_+} - \int_0^t \langle u(\tau), \partial_t J_\varepsilon J_\varepsilon(u_e)(\tau)\rangle_{D_+}\,\mathrm{d}\tau \\ = \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla J_\varepsilon J_\varepsilon(u_e)(\tau)\rangle_{D_+}\,\mathrm{d}\tau. \quad (5.6)$$

Let $v \in L^p(D_+)$ with support in D_+ and extend by zero to all of D; then

$$\begin{aligned}
\langle v, J_\varepsilon J_\varepsilon(u_e)\rangle_{D_+} &= \int_{D_+} v \cdot J_\varepsilon J_\varepsilon(u_e)\,\mathrm{d}x = \int_D \mathbb{I}_{D_+} v \cdot J_\varepsilon J_\varepsilon(u_e)\,\mathrm{d}x \\
&= \int_D \int_D \varphi_\varepsilon(x-y)\mathbb{I}_{D_+} v(x) \cdot J_\varepsilon(u_e)(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_D \int_D \varphi_\varepsilon(x-y)\mathbb{I}_{D_+} v(x)\,\mathrm{d}x \cdot J_\varepsilon(u_e)(y)\,\mathrm{d}y \\
&= \int_D J_\varepsilon(\mathbb{I}_{D_+} v(x)) \cdot J_\varepsilon(u_e)(y)\,\mathrm{d}y \\
&= \int_{D_{[-\varepsilon,\infty)}} J_\varepsilon(v(x)) \cdot J_\varepsilon(u_e)(y)\,\mathrm{d}y \\
&= \langle J_\varepsilon v, J_\varepsilon(u_e)\rangle_{D_{[-\varepsilon,\infty)}}.
\end{aligned}$$

Using this in (5.6) we obtain

$$\begin{aligned}
&\langle J_\varepsilon(u)(t), J_\varepsilon(u_e)(t)\rangle_{D_{[-\varepsilon,\infty)}} \\
&\quad - \langle J_\varepsilon(u)(0), J_\varepsilon(u_e)(0)\rangle_{D_{[-\varepsilon,\infty)}} - \int_0^t \langle J_\varepsilon(u)(\tau), \partial_t J_\varepsilon(u_e)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau \\
&\qquad = \int_0^t \langle J_\varepsilon(u(\tau)\otimes u(\tau)) : \nabla J_\varepsilon(u_e))(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau. \quad (5.7)
\end{aligned}$$

We want to take limits as $\varepsilon \to 0$ of (5.7) and show that the L.H.S becomes

$$\frac{1}{2}\left(\|u(t)\|^2_{L^2(D_+)} - \|u(0)\|^2_{L^2(D_+)}\right).$$

and thus, if we show the R.H.S. converges to zero we will have energy conservation. Here we will use the Lipschitz in time regularity of $J_\varepsilon u_e$ shown in Corollary 5.8 to manipulate the term with the time derivative in the L.H.S. of (5.7). We will then use Lemma 5.5 to show that the remainder term converges to zero.

Note that for the first two terms in the L.H.S of (5.7) we can use Lemma 5.4

to obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big(\langle J_\varepsilon(u)(t), J_\varepsilon(u_e)(t)\rangle_{D_{[-\varepsilon,\infty)}} - \langle J_\varepsilon(u)(0), J_\varepsilon(u_e)(0)\rangle_{D_{[-\varepsilon,\infty)}}\Big)\\
&\quad= \lim_{\varepsilon\to 0}\Big(\langle J_\varepsilon(u)(t), J_\varepsilon(u_e)(t)\rangle_{D} - \langle J_\varepsilon(u)(0), J_\varepsilon(u_e)(0)\rangle_{D}\Big)\\
&\quad= \langle u(t), u_e(t)\rangle_D - \langle u(0), u_e(0)\rangle_D = \|u(t)\|^2_{L^2(D_+)} - \|u(0)\|^2_{L^2(D_+)}.
\end{aligned}$$

For the last term on the L.H.S. of (5.7) linearity implies

$$\begin{aligned}
\int_0^t \langle J_\varepsilon(u)(\tau), \partial_t J_\varepsilon(u_e)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau &= \int_0^t \langle J_\varepsilon(u)(\tau), \partial_t J_\varepsilon(u)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau\\
&\quad+ \int_0^t \langle J_\varepsilon(u)(\tau), \partial_t J_\varepsilon(u_r)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau.
\end{aligned}$$

As $J_\varepsilon(u) \in C^{0,1}([0,T]; H_\sigma)$ thus

$$\begin{aligned}
2\int_0^t \langle J_\varepsilon(u)(\tau), \partial_t J_\varepsilon(u)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau &= \int_0^t \partial_t\langle J_\varepsilon(u)(\tau), J_\varepsilon(u)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau\\
&= \|J_\varepsilon u(t)\|^2_{L^2(D_{[-\varepsilon,\infty)})} - \|J_\varepsilon u(0)\|^2_{L^2(D_{[-\varepsilon,\infty)})},
\end{aligned}$$

and taking limits gives

$$\lim_{\varepsilon\to 0}\int_0^t \langle J_\varepsilon(u)(\tau), \partial_t J_\varepsilon(u)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau = \frac{1}{2}(\|u(t)\|^2_{L^2(D_+)} - \|u(0)\|^2_{L^2(D_+)}).$$

Therefore the L.H.S. of (5.7) will converge to what we want as long as

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^t \langle J_\varepsilon(u)(\tau), \partial_t J_\varepsilon(u_r)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau\\
&\qquad= \lim_{\varepsilon\to 0}\int_0^t\int_{\mathbb{T}^2}\int_{-\varepsilon}^{\varepsilon} J_\varepsilon(u)(\tau)\cdot\partial_t J_\varepsilon(u_r)(\tau)\,\mathrm{d}x\,\mathrm{d}\tau = 0 \qquad (5.8)
\end{aligned}$$

and so this remainder term must vanish in the limit. From Lemma 5.5 we see that

$$\int_0^t\int_{\mathbb{T}^2}\int_{-\varepsilon}^{\varepsilon} J_\varepsilon(u)(\tau)\cdot\partial_t J_\varepsilon(u_r)(\tau)\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t\int_{\mathbb{T}^2}\int_{-\varepsilon}^{\varepsilon} J_\varepsilon(u_r)(\tau)\cdot\partial_t J_\varepsilon(u)(\tau)\,\mathrm{d}x\,\mathrm{d}\tau,$$

which implies that

$$
\begin{aligned}
2\int_0^t \int_{\mathbb{T}^2}\int_{-\varepsilon}^{\varepsilon} J_\varepsilon(u)(\tau)\cdot\partial_t J_\varepsilon(u_r)(\tau)\,\mathrm{d}x\,\mathrm{d}\tau &= \int_0^t \partial_t \int_{\mathbb{T}^2}\int_{-\varepsilon}^{\varepsilon} J_\varepsilon(u)(\tau)\cdot J_\varepsilon(u_r)(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_{\mathbb{T}^2}\int_{-\varepsilon}^{\varepsilon} J_\varepsilon(u)(t)\cdot J_\varepsilon(u_r)(t)\,\mathrm{d}x - \int_{\mathbb{T}^2}\int_{-\varepsilon}^{\varepsilon} J_\varepsilon(u)(0)\cdot J_\varepsilon(u_r)(0)\,\mathrm{d}x.
\end{aligned}
$$

Taking limits as $\varepsilon \to 0$ gives us (5.8).

We are left with the R.H.S. of (5.7) and have the term

$$
\lim_{\varepsilon\to 0}\left(\int_0^t \langle J_\varepsilon(u\otimes u)(\tau) : \nabla J_\varepsilon(u_e)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau\right) =: \lim_{\varepsilon\to 0} I.
$$

We can write

$$
\begin{aligned}
I = &\int_0^t \langle J_\varepsilon(u_e\otimes u)(\tau) : \nabla J_\varepsilon(u_e)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau \\
&\qquad + \int_0^t \langle J_\varepsilon((u-u_e)\otimes u)(\tau) : \nabla J_\varepsilon(u_e)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau.
\end{aligned}
$$

For the second term we notice that the intersection of the supports of u and $u - u_e$ is just the boundary, a set of measure zero and so $(u - u_e) \otimes u = 0$ almost everywhere and the second term vanishes. For the first term we use an identity to commute the mollification with the product which is similar to that used by Eyink (1994) and also used by Constantin, E, & Titi (1994), Cheskidov et al. (2008), Shvydkoy (2009) and Shvydkoy (2010). Here, however, we have two different functions in the product rather than the same function twice. The same identity used below is independently used in Bardos & Titi (2018). We will use the identity

$$
J_\varepsilon(u_e\otimes u) = r_\varepsilon(u_e,u) - (u_e - J_\varepsilon(u_e))\otimes(u - J_\varepsilon(u)) + J_\varepsilon u_e \otimes J_\varepsilon u \tag{5.9}
$$

with

$$
r_\varepsilon(u_e,u) := \int_D \varphi_\varepsilon(y)(u_e(x-y)-u_e(x))\otimes(u(x-y)-u(x))\,\mathrm{d}y.
$$

As $r_\varepsilon(u, u_e)$ expands to

$$\int_D \varphi_\varepsilon(y)[u_e(x-y) \otimes u(x-y) - u_e(x) \otimes u(x-y) - u(x) \otimes u_e(x-y) + u_e(x) \otimes u(x)] \, \mathrm{d}y$$
$$= J_\varepsilon(u_e \otimes u) - u_e \otimes J_\varepsilon u - u \otimes J_\varepsilon u_e + u_e \otimes u$$

we see the validity of (5.9).

Therefore we obtain

$$I = \int_0^t \langle [r_\varepsilon(u_e, u) - (u_e - J_\varepsilon(u_e)) \otimes (u - J_\varepsilon(u)) + J_\varepsilon u_e \otimes J_\varepsilon u] : \nabla J_\varepsilon(u_e)(\tau) \rangle_{D_{[-\varepsilon,\infty)}} \, \mathrm{d}\tau.$$

First we consider the term

$$\int_0^t \langle J_\varepsilon u_e \otimes J_\varepsilon u : \nabla J_\varepsilon(u_e)(\tau) \rangle_{D_{[-\varepsilon,\infty)}} \, \mathrm{d}\tau.$$

If we integrate by parts any potential boundary terms vanish as the support of $J_\varepsilon u$ is in $D_{[-\varepsilon,\infty)}$ and so we obtain

$$-\frac{1}{2} \int_0^t \int_{D_{[-\varepsilon,\infty)}} (\nabla \cdot J_\varepsilon u) |J_\varepsilon(u_e)|^2 \, \mathrm{d}x \, \mathrm{d}\tau,$$

note that this term is zero by incompressibility.

5.4.1 Remainder terms vanish in the limit

We are now left with the remainder terms

$$\int_0^t \langle [r_\varepsilon(u_e, u) - (u_e - J_\varepsilon u_e) \otimes (u - J_\varepsilon u)] : \nabla J_\varepsilon(u_e)(\tau) \rangle_{D_{[-\varepsilon,\infty)}} \, \mathrm{d}\tau. \tag{5.10}$$

As $(\nabla\varphi)_\varepsilon$ is an odd function its integral is zero and therefore we can add

$$\int_D (\nabla\varphi)_\varepsilon(y) \otimes (-u_e)(x))\,\mathrm{d}y$$

to $\nabla J_\varepsilon(u_e)$ to obtain

$$\nabla J_\varepsilon(u_e) = \int_D (\nabla\varphi_\varepsilon)(y) \otimes (u_e(x-y) - u_e(x))\,\mathrm{d}y. \tag{5.11}$$

Firstly we can write $r_\varepsilon(u, u_e)$ in full as

$$\begin{aligned}
&\int_0^t \langle r_\varepsilon(u_e, u) : \nabla J_\varepsilon(u_e)(\tau)\rangle_{D_{[-\varepsilon,\infty)}}\,\mathrm{d}\tau \\
&\qquad = \int_0^t \bigg\langle \int_D \varphi_\varepsilon(y)(u_e(x-y) - u_e(x)) \otimes (u(x-y) - u(x))\,\mathrm{d}y : \\
&\qquad\qquad\qquad \int_D (\nabla\varphi_\varepsilon)(z) \otimes (u_e(x-z) - u_e(x))\,\mathrm{d}z \bigg\rangle_{D_{[-\varepsilon,\infty)}} \mathrm{d}\tau.
\end{aligned}$$

Bringing the modulus inside the integral and using the change of variables $z = \varepsilon\xi$, $y = \varepsilon\eta$ we have

$$\begin{aligned}
&\left| \int_0^t \int_{D_{[-\varepsilon,\infty)}} r_\varepsilon(u_e, u) : \nabla J_\varepsilon(u_e)(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \right| \\
&\qquad \leq \int_0^t \bigg\langle \int_{B_1(0)} |\varphi(\eta)||u_e(x-\varepsilon\eta) - u_e(x)||u(x-\varepsilon\eta) - u(x)|\,\mathrm{d}\eta : \\
&\qquad\qquad\qquad \int_{B_1(0)} \frac{1}{\varepsilon}|\nabla\varphi(\xi)||u_e(x-\varepsilon\xi) - u_e(x)|\,\mathrm{d}\xi \bigg\rangle_{D_{[-\varepsilon,\infty)}} \mathrm{d}\tau.
\end{aligned}$$

Then we can use Fuibini's theorem, Minkowski's inequality and Hölder's inequality to obtain

$$
\begin{aligned}
\left|\int_0^t \int_{D_{[-\varepsilon,\infty)}} r_\varepsilon(u_e, u) : \nabla J_\varepsilon(u_e)(\tau) \,\mathrm{d}x \,\mathrm{d}\tau\right| & \qquad (5.12)\\
\leq \int_{B_1(0)} |\varphi(\eta)| \|u_e(\cdot - \varepsilon\eta) - u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} & \\
\times \|u(\cdot - \varepsilon\eta) - u(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} \,\mathrm{d}\eta & \\
\times \frac{1}{\varepsilon} \int_{B_1(0)} |(\nabla\varphi)(\xi)| \|u_e(\cdot - \varepsilon\xi) - u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} \,\mathrm{d}\xi. &
\end{aligned}
$$

We will show that the right-hand side of (5.12) tends to zero as $\varepsilon \to 0$ in the proof of Theorem 5.10 to prove energy conservation.

For the other term in (5.10) we have

$$\int_0^t \langle (u_e - J_\varepsilon u_e) \otimes (u - J_\varepsilon u) : \nabla J_\varepsilon(u_e)(\tau) \rangle_{D_{[-\varepsilon,\infty)}} \,\mathrm{d}\tau.$$

This becomes

$$
\begin{aligned}
&\int_0^t \langle (u_e - J_\varepsilon u_e) \otimes (u - J_\varepsilon u) : \nabla J_\varepsilon(u_e)(\tau) \rangle_{D_{[-\varepsilon,\infty)}} \,\mathrm{d}\tau \\
&\qquad = \int_0^t \int_{D_{[-\varepsilon,\infty)}} \int_D \varphi_\varepsilon(z)(u_e(x - z) - u_e(x)) \,\mathrm{d}z \otimes \\
&\int_D \varphi_\varepsilon(y)(u(x - y) - u(x)) \,\mathrm{d}y \int_D (\nabla\varphi_\varepsilon)(w) \otimes (u_e(x - w) - u_e(x)) \,\mathrm{d}w \,\mathrm{d}x \,\mathrm{d}\tau,
\end{aligned}
$$

where again we have used (5.11) for the $\nabla J_\varepsilon(u_e)$ term. Thus following similar steps as before with the change of variables $z = \eta\xi$, $y = \varepsilon\zeta$, $w = \varepsilon\xi$ we have

$$
\begin{aligned}
\left|\int_0^t \langle (u_e - J_\varepsilon u_e) \otimes (u - J_\varepsilon u) : \nabla J_\varepsilon(u_e)(\tau) \rangle_{D_{[-\varepsilon,\infty)}} \,\mathrm{d}\tau\right| & \qquad (5.13)\\
\leq \int_{B_1(0)} |\varphi(\eta)| \|u_e(\cdot - \varepsilon\eta) - u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} \,\mathrm{d}\eta & \\
\times \int_{B_1(0)} |\varphi(\zeta)| \|u(\cdot - \varepsilon\zeta) - u(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} \,\mathrm{d}\zeta & \\
\times \frac{1}{\varepsilon} \int_{B_1(0)} |(\nabla\varphi)(\xi)| \|u_e(\cdot - \varepsilon\xi) - u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} \,\mathrm{d}\xi. &
\end{aligned}
$$

We will show that the right-hand side of (5.13) tends to zero as $\varepsilon \to 0$ in the proof of Theorem 5.10 to prove energy conservation.

In order to deal with the limit as ε tends to zero both in (5.12) and (5.13) we will use the Dominated Convergence Theorem to move the limit inside these integrals.

Lemma 5.9 *Let u be solution to the pressure-less incompressible Euler equations on $\mathbb{T}^2 \times \mathbb{R}_+$ as in Definition 5.1, with $u \in L^3(0,T;L^3)$. Assume that u satisfies:*

- *The interior condition,*

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_0^T \int_{D_{[|y|,\infty)}} |u(x+y) - u(x)|^3 \,\mathrm{d}x \,\mathrm{d}t = 0.$$

- *The boundary condition, that there exists a $\delta > 0$ such that*

$$u \in L^3(0,T;L^\infty(\mathbb{T}^2 \times [0,\delta)).$$

Then the limit as $\varepsilon \to 0$ can be moved inside the integrals over η, ξ, ζ in (5.12) *and* (5.13).

Proof We are going to apply the Dominated Convergence Theorem on five similar terms in (5.12) and (5.13) which simplify to:

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{2/3}} \int_{B_1(0)} |\varphi(\eta)| \|u_e(\cdot - \varepsilon\eta) - u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} \times \|u(\cdot - \varepsilon\eta) - u(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} \,\mathrm{d}\eta, \quad (5.14)$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{1/3}} \int_{B_1(0)} |\varphi(\eta)| \|u_e(\cdot - \varepsilon\eta) - u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} \,\mathrm{d}\eta,$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{1/3}} \int_{B_1(0)} |\varphi(\zeta)| \|u(\cdot - \varepsilon\zeta) - u(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} \,\mathrm{d}\zeta$$

and two terms of the form

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{1/3}}\int_{B_1(0)}|(\nabla\varphi)(\xi)|\|u_e(\cdot-\varepsilon\xi)-u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))}\,\mathrm{d}\xi. \tag{5.15}$$

The last three can be treated similarly so we shall take (5.15) as an example. We can split the domain of the L^3 norm into $D_{[\varepsilon,\infty)}$ and $D_{[-\varepsilon,\varepsilon]}$. Considering the $D_{[\varepsilon,\infty)}$ part we see that with $\xi\in B_1(0)$ then $\xi\varepsilon\in B_\varepsilon(0)$ and so $u_e(x-\xi\varepsilon)-u_e(x)=u(x-\xi\varepsilon)-u(x)$. We can define the non-negative function

$$f(y)=\frac{1}{|y|}\int_0^t\int_{D_{[|y|,\infty)}}|u(x+y)-u(x)|^3\,\mathrm{d}x\,\mathrm{d}t$$

and notice that from the assumption that $\lim_{|y|\to 0}f(y)=0$ we have, for sufficiently small $\varepsilon>0$, that $\sup_{y\in B_\varepsilon(0)}f(y)\le K$ for some $K=K(\varepsilon)$. Since the $\mathrm{supp}(\nabla\varphi)$ is compact and the function is bounded, we obtain a dominating integrable function

$$h(\xi):=CK^{1/3}|\nabla\varphi(\xi)|$$

and so

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{1/3}}&\int_{B_1(0)}|(\nabla\varphi)(\xi)|\|u_e(\cdot-\varepsilon\xi)-u_e(\cdot)\|_{L^3(0,t;L^3(D_{[\varepsilon,\infty)}))}\,\mathrm{d}\xi\\&=\int_{B_1(0)}|(\nabla\varphi)(\xi)|\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{1/3}}\|u_e(\cdot-\varepsilon\xi)-u_e(\cdot)\|_{L^3(0,t;L^3(D_{[\varepsilon,\infty)}))}\,\mathrm{d}\xi.\end{aligned}$$

We are left with showing that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{1/3}}&\int_{B_1(0)}|(\nabla\varphi)(\xi)|\|u_e(\cdot-\varepsilon\xi)-u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\varepsilon]}))}\,\mathrm{d}\xi\\&=\int_{B_1(0)}|(\nabla\varphi)(\xi)|\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{1/3}}\|u_e(\cdot-\varepsilon\xi)-u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\varepsilon]}))}\,\mathrm{d}\xi.\end{aligned}$$

We assumed that $u\in L^3(0,T;L^\infty(\mathbb{T}^2\times[0,\varepsilon))$ and so we know that

$$u_e\in L^3(0,T;L^\infty(\mathbb{T}^2\times(-\varepsilon,\varepsilon))).$$

Thus in the region $\mathbb{T}^2 \times (-\varepsilon, \varepsilon)$ we can define the non-negative function

$$
\begin{aligned}
g(y) &= \frac{1}{\varepsilon^{1/3}} \left[\int_0^t \int_{\mathbb{T}^2 \times (-\varepsilon,\varepsilon)} |u_e(x + y\varepsilon) - u_e(x)|^3 \, \mathrm{d}x \, \mathrm{d}t \right]^{1/3} \\
&\leq \frac{C}{\varepsilon^{1/3}} |\mathbb{T}^2|^{1/3} \varepsilon^{1/3} \left[\int_0^t \sup_{x \in \mathbb{T}^2 \times (-2\varepsilon, 2\varepsilon)} |u_e(x)|^3 \, \mathrm{d}t \right]^{1/3}.
\end{aligned}
$$

Thus we obtain the dominating integrable function

$$
l(\xi) := C \|u\|_{L^3(0,T;L^\infty(D_{[-\varepsilon,\varepsilon]}))} |\nabla \phi(\xi)|
$$

and so we can bring the limit inside the integral.

For (5.14) we proceed as before, splitting $D_{[-\varepsilon,\infty)}$ into $D_{[\varepsilon,\infty)}$ and $D_{[-\varepsilon,\varepsilon]}$. As there are two terms involving u in (5.14) we have to consider cross terms. Namely

$$
\begin{aligned}
&\|u_e(\cdot - \varepsilon\eta) - u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} \|u(\cdot - \varepsilon\eta) - u(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} \\
&= \|u(\cdot - \varepsilon\eta) - u(\cdot)\|^2_{L^3(0,t;L^3(D_{[\varepsilon,\infty)}))} && (5.16) \\
&+ \|u_e(\cdot - \varepsilon\eta) - u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\varepsilon]}))} \|u(\cdot - \varepsilon\eta) - u(\cdot)\|_{L^3(0,t;L^3(D_{[\varepsilon,\infty)}))} && (5.17) \\
&+ \|u_e(\cdot - \varepsilon\eta) - u_e(\cdot)\|_{L^3(0,t;L^3(D_{[\varepsilon,\infty)}))} \|u(\cdot - \varepsilon\eta) - u(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\varepsilon]}))} && (5.18) \\
&+ \|u_e(\cdot - \varepsilon\eta) - u_e(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\varepsilon]}))} \|u(\cdot - \varepsilon\eta) - u(\cdot)\|_{L^3(0,t;L^3(D_{[-\varepsilon,\varepsilon]}))}. && (5.19)
\end{aligned}
$$

For the terms over $D_{[-\varepsilon,\varepsilon]}$ we can define the integrable function

$$
\tilde{l}(\xi) := C \|u\|_{L^3(0,T;L^\infty(D_{[-\varepsilon,\varepsilon]}))} |\varphi(\xi)|^{1/2}
$$

and the terms over $D_{[\varepsilon,\infty)}$ we can define the integrable function

$$
\tilde{h}(\xi) := C K^{1/3} |\varphi(\xi)|^{1/2}.
$$

Then for (5.16) we can use $\tilde{h}^2$, for (5.17) and (5.18) we can use $\tilde{h}\tilde{l}$ and finally for (5.19) we can use $\tilde{l}^2$. Thus for (5.14) we can define the dominating integrable function

$$m(\xi) := \tilde{h}^2(\xi) + 2\tilde{h}(\xi)\tilde{l}(\xi) + \tilde{l}^2(\xi)$$

and so bring the limit inside the integral over η in (5.14) and thus we are done. □

In order to manipulate the terms in (5.12) and (5.13) we have to deal with differences in u and u_e near the boundary. To control these terms in a strip around the boundary we will use similar techniques to Chapter 4 and add the assumption that there exists a $\delta > 0$ such that $u \in L^3(0, T; C^0([0, \delta)))$.

Theorem 5.10 (Energy Conservation) *Let u be solution to the pressure-less incompressible Euler equations on $\mathbb{T}^2 \times \mathbb{R}_+$ from Definition 5.1 with $u \in L^3(0, T; L^3)$. Assume that u satisfies the three conditions:*

- *The interior condition,*

$$\lim_{|y|\to 0} \frac{1}{|y|} \int_0^T \int_{D_{[|y|,\infty)}} |u(x+y) - u(x)|^3 \,\mathrm{d}x \,\mathrm{d}t = 0. \tag{5.20}$$

- *The boundary condition, that there exists a $\delta > 0$ such that*

$$u \in L^3(0, T; C^0(\mathbb{T}^2 \times [0, \delta)).$$

Then u conserves energy on $[0, T]$.

Proof It suffices to show that both (5.12) and (5.13) vanish in the limit as $\varepsilon \to 0$. First we want to bring the limit inside the integrals which is shown in Lemma 5.9. We have reduced the problem to showing that

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon} \|u(\cdot - \varepsilon\eta) - u(\cdot)\|^3_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} &< \infty, \\
\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \|u_e(\cdot - \varepsilon\eta) - u_e(\cdot)\|^3_{L^3(0,t;L^3(D_{[-\varepsilon,\infty)}))} &= 0
\end{aligned}$$

for almost every η.

Again splitting the domain of the L^3 norm and considering, for the interior region $D_{[\varepsilon,\infty)}$, then for $\eta\varepsilon \in B_\varepsilon(0)$ both lines above reduce to showing that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \|u(\cdot - \varepsilon\eta) - u(\cdot)\|^3_{L^3(0,t;L^3(D_{[\varepsilon,\infty)}))} = 0.$$

With the change of variables $y = \varepsilon\eta$ for $\eta \in B_1(0)$ we have

$$\lim_{|y|\to 0} \frac{1}{|y|} \|u(\cdot - y) - u(\cdot)\|^3_{L^3(0,t;L^3(D_{[\varepsilon,\infty)}))} = 0$$

and this is controlled by the interior condition (5.20).

We now need to show that

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon} \|u(\cdot - \varepsilon\eta) - u(\cdot)\|^3_{L^3(0,t;L^3(\mathbb{T}^2\times(-\varepsilon,\varepsilon))} < \infty, \tag{5.21}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \|u_e(\cdot - \varepsilon\eta) - u_e(\cdot)\|^3_{L^3(0,t;L^3(\mathbb{T}^2\times(-\varepsilon,\varepsilon)))} = 0. \tag{5.22}$$

For (5.22) as there exists a $\delta > 0$ such that $u \in L^3(0,T;C^0(\mathbb{T}^2 \times [0,\delta))$ we have, similarly to Theorem 4.6, that $u \cdot n = 0$ pointwise on the boundary; the boundary values are the same for u and u_r and so $u_e \in L^3(0,T;C^0(\mathbb{T}^2 \times (-\delta,\delta))$. We fix t and let $x' = (x_1, x_2, 0)$; we have, using w_t defined in (4.9), that

$$\begin{aligned}|u_e(t,x'+x_3+\varepsilon\eta) - u_e(t,x'+x_3)| \leq& |u_e(t,x'+x_3+\varepsilon\eta) - u_e(t,x') \\ &+ u_e(t,x') - u_e(t,x'+x_3)| \leq w_t(|\varepsilon\eta + x_3|) + w(t,|x_3|) \leq 2w_t(2\varepsilon|\eta|)\end{aligned}$$

and thus

$$\begin{aligned}\frac{1}{|\varepsilon|} \iint_{\mathbb{T}^2} \int_{-\varepsilon}^{\varepsilon} &|u_e(t,x-\varepsilon\eta) - u_e(t,x)|^3 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \\ &\leq C\frac{1}{\varepsilon} \iint_{\mathbb{T}^2} \int_{-\varepsilon}^{\varepsilon} |w_t(2\varepsilon|\eta|)|^3 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \\ &\leq C\frac{1}{\varepsilon} |\mathbb{T}^2| \varepsilon |w_t(2\varepsilon|\eta|)|^3 \to 0\end{aligned}$$

as $\varepsilon \to 0$ for almost every t.

For (5.21) we have assumed that $u \in L^3(0,T;L^\infty(\mathbb{T}^2 \times [0,\delta)))$ and so

$$u \in L^3(0,T;L^\infty(\mathbb{T}^2 \times (-\delta,\delta))).$$

We see that

$$\begin{aligned}\frac{1}{\varepsilon}\|u(\cdot-\varepsilon\eta)-u(\cdot)\|^3_{L^3(0,t;L^3(\mathbb{T}^2\times(-\varepsilon,\varepsilon))}\\ \leq&\frac{C}{\varepsilon}|\mathbb{T}^2|\varepsilon\|u(\cdot-\varepsilon\eta)-u(\cdot)\|_{L^3(0,t;L^\infty(\mathbb{T}^2\times(-\varepsilon,\varepsilon))}\\ =&C|\mathbb{T}^2|\|u\|_{L^3(0,t;L^\infty(\mathbb{T}^2\times(-\delta,\delta))}\leq C.\end{aligned}$$

Thus taking the limit as $\varepsilon \to 0$ we have that the limit is bounded.

We have shown that the remainder terms (5.12) and (5.13) both vanish in the limit and so we obtain energy conservation. □

5.5 Conclusion

We have shown energy conservation of solutions u of the Euler equations on the domain D_+, where u satisfies the same conditions as in Chapter 4. Our method does not depend on the dimension and so analogous methods hold in $\mathbb{T}^{d-1} \times \mathbb{R}_+$ for $d \geq 2$.

Importantly, there is no pressure terms appearing in the definition of a weak solution so there are no complications of the existence of this term and the regularity it will need.

Further, this method is completely local around the boundary. Thus all of the methods used would generalise to an arbitrary domain as long as an extension for that domain can be defined that has the properties we needed, noticeably for small $\varepsilon > 0$;

1. $\|v_e\|_{L^p(D_{[-\delta,\infty)})} \leq C\|v\|_{L^p(D_+)}$,

2. $J_\varepsilon(v_e)$ and $J_\varepsilon J_\varepsilon(v_e)$ are incompressible in D_+,

3. $J_\varepsilon(v_e) \cdot n = 0$ and $J_\varepsilon J_\varepsilon(v_e) \cdot n = 0$ on ∂D_+.

A possibility would be to flatten the boundary and use this extension, then regularise in the flat domain and finally map back to the original domain. This may cause extra difficulties with the nice properties of mollification which we will have to account for.

Chapter 6

Energy Conservation in Ω

In the previous chapter we were able to show energy conservation of solutions to the Euler equations with a local extension and without reference to the pressure in the weak formulation of the equations. However, we have only considered the case of a flat boundary with finite area. Here we now want to discuss potential ways of approaching the same problem on a generic bounded domain.

In the recent work of Bardos & Titi (2018), energy conservation was shown for weak solutions to the Euler equations, as given by Definition 6.2, where they did not have incompressible test functions. Without incompressibility of the test functions the pressure explicitly appears in the weak formulation of the equations and estimates on the pressure are needed. To show energy conservation they assumed that the solution u satisfied the extra regularity condition that

$$u \in L^3(0,T;C^\alpha(\bar{\Omega})) \quad \text{for} \quad \alpha > 1/3 \quad \text{with a} \quad C^2 \quad \text{boundary} \quad \partial\Omega.$$

This proves Onsager's conjecture for bounded domains with a C^2 boundary.

In the previous chapter we considered weak solutions of the Euler equations with incompressible test functions and did not have a pressure term appearing in the equations and so no estimates of the pressure term are necessary. However, on a bounded domain there are many choices of definition for a weak solution depending on the family of test functions used. We have the option of including incompressibility or not but also a choice of boundary conditions: compact support, just zero normal component or arbitrary boundary values. Further, we can choose to use, or not, the Leary projection, described in Robinson, Rodrigo, & Sadowksi (2016) for example, and the option of compact support, or not, of the test functions in time as well.

Here we will introduce our preferred Definition 6.1, the generalisation of a weak solution of the Euler equations as in Chapter 5, to a bounded domain. Here we use incompressible test functions in the definition with vanishing normal components. We will then introduce the definition used by Bardos & Titi (2018) and show that if we assume that

$$u \in L^3(0, T; C^\delta(\bar{\Omega})) \quad \text{for} \quad \delta > 0$$

then if u is a solution for Definition 6.1, which we have been using, then it is also a solution for Bardos & Titi (2018). Thus we have energy conservation under the same conditions as Bardos & Titi (2018) for Definition 6.1.

6.1 Different Definitions of Weak Solution

We will define a weak formulation of the Euler equations on a bounded domain Ω with at least a Lipschitz boundary so the normal n is well defined. We will follow similar steps to Chapter 5, where we considered the spatial domain D_+ but, as the domain is now bounded, we will not require Schwartz-like decay and smooth functions will be enough. Here we will briefly remind the reader of the spaces used.

First, we define the space of test functions

$$C^\infty_{n,\sigma}(\Omega \times [0,T]) := \{\psi \in C^\infty(\Omega \times [0,T]) \colon u(\cdot, t) \in C^\infty_{n,\sigma}(\Omega) \ \forall t \in [0,T]\},$$

where

$$C^\infty_{n,\sigma}(\Omega) := \{\psi \in C^\infty(\Omega) \colon \nabla \cdot \psi = 0 \text{ and } \psi \cdot n = 0 \text{ on } \partial\Omega\}$$

and the space $H_\sigma(\Omega)$ as

$$H_\sigma(\Omega) := \text{the completion of } C^\infty_{n,\sigma}(\Omega) \text{ in the } L^2(\Omega) \text{ norm.}$$

Note that functions in $H_\sigma(\Omega)$ are weakly divergence free with respect to $H^1(\Omega)$ similarly to (4.2). The construction and properties of $H_\sigma(\Omega)$ can be found in Section 2.2 of Robinson, Rodrigo, & Sadowksi (2016).

We can suppose that u is a smooth solution to the Euler equations, take the inner-product of the equations with $\psi \in C^\infty_{n,\sigma}(\Omega\times[0,T])$ and integrate over $\Omega\times[0,T]$. We can then perform integration-by-parts on both the time and spacial derivatives and use the incompressibility and boundary conditions of both u and ψ to simplify the equation. We can then ask what is the minimum regularity of u that is needed to make sense of the equation and by doing this we derive a weak solution of the Euler equations on Ω in a way that involves no pressure term; we will call this the pressure-less form of a weak solution.

Definition 6.1 (Pressure-less form) *A weak solution on $\Omega \subset \mathbb{R}^3$ of the Euler equations on $[0,T]$ is a vector-valued function $u \in C_{\mathrm{w}}([0,T]; H_\sigma(\Omega))$ such that*

$$\langle u(t), \psi(t)\rangle_\Omega - \langle u(0), \psi(0)\rangle_\Omega - \int_0^t \langle u(\tau), \partial_t\psi(\tau)\rangle_\Omega \,\mathrm{d}\tau = \int_0^t \langle u(\tau)\otimes u(\tau) : \nabla\psi(\tau)\rangle_\Omega \,\mathrm{d}\tau,$$

for every $t \in [0,T]$, for any $\psi \in C^\infty_{n,\sigma}(\Omega \times [0,T])$.

Another definition is presented in the introduction of Bardos & Titi (2018) where they include the full pressure term in the weak formulation. Here they use a class of test functions that are not incompressible, but are compactly supported in $\Omega \times (0,T)$.

For the next definition of a weak solution we need to define, for any domain Ω, the space of distributions on Ω, denoted $\mathcal{D}'(\Omega)$, as the space of all continuous linear functionals on $\mathcal{D}(\Omega)$. The definition used by Bardos & Titi (2018) is as follows

Definition 6.2 (Bardos & Titi, 2017) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a C^2 boundary, $\partial\Omega$. Then (u,p) is a weak solution of the incompressible Euler* *equations in* $\Omega \times (0,T)$ *if* $u \in L^\infty(0,T;H_\sigma(\Omega))$, $p \in \mathcal{D}'(\Omega \times [0,T])$ *and*

$$\langle u, \partial_t \phi\rangle_\Omega + \langle u \otimes u : \nabla\phi\rangle_\Omega + \langle p, \nabla\cdot\phi\rangle_\Omega = 0, \quad \textit{in} \quad L^1(0,T). \tag{6.1}$$

for every test vector field $\phi \in \mathcal{D}(\Omega \times (0,T))$.

The a-priori regularity of p is not made explicit in Bardos & Titi (2018) so here we have assumed that $p \in \mathcal{D}'(\Omega \times [0,T])$ as the weakest condition possible to allow us to write down the equation and make sense of the last term on the left-hand side of (6.1). However, as a consequence of the above definition p is a weak solution of the elliptic boundary-value problem

$$-\Delta p = \partial_i\partial_j(u_iu_j) \text{ in } \Omega \text{ and } \nabla p \cdot n = -(u_j\partial_j u_i)n_i \text{ on } \partial\Omega \tag{6.2}$$

for almost every t. As Bardos & Titi (2018) assume that $u \in L^3(0,T;C^\alpha)$ for $\alpha > 1/3$ to prove energy conservation, this allows them to use the work in Chapters 5 and 6 of Krylov (1996) to show that $p \in L^{3/2}(0,T;C^\alpha)$ which solves (6.2). [In fact using Krylov (1996) it is enough to have the extra assumption that $u \in L^3(0,T;C^\delta)$ for $\delta > 0$ for $p \in L^{3/2}(0,T;C^\delta)$ to solve (6.2).]

Bardos & Titi (2018) chose test functions compactly supported in time and chose u to have only L^∞ regularity in time so do not include the time end-points, that is, the terms of the form

$$\langle u(t), \psi(t)\rangle_\Omega - \langle u(0), \psi(0)\rangle_\Omega,$$

that occur in Definition 6.1.

We will show that if we restrict our class of solutions from $L^\infty(0,T;H_\sigma(\Omega))$ to $C_{\mathrm{w}}([0,T];H_\sigma(\Omega))$ one can restrict the Definition 6.2 to Definition 6.3 below.

We define the class of test functions that are compactly supported in space but not in time as

$$C^\infty([0,T];\mathcal{D}(\Omega)) := \{\psi \in C^\infty(\Omega \times [0,T]) \colon \operatorname{supp}(\psi(\cdot,t)) \subset\subset \Omega \text{ for every } t\}.$$

Take $\psi \in C^\infty([0,T];\mathcal{D}(\Omega))$ and a sequence of smooth functions χ_n compactly supported and approaching 1 on the interval $[0,t]$ in the limit as $n \to \infty$, that is, $\chi_n \in C^\infty([0,T])$ with values between zero and 1

$$\chi_n(\tau) = \begin{cases} 1 & \tau \in [1/n, t-1/n], \\ 0 & \text{in a neighbourhood of } 0 \,\&\, t \\ \text{smooth,} & \text{otherwise.} \end{cases}$$

Then $\phi_n := \chi_n \psi$ yields a sequence of $\phi_n \in \mathcal{D}(\Omega \times [0,T])$ that tends to $\psi \in C^\infty([0,T];\mathcal{D}(\Omega))$. If we substitute $\phi_n = \chi_n \psi$ in (6.1) we obtain

$$\langle u, \partial_t(\chi_n \psi)\rangle_\Omega + \langle u \otimes u : \nabla(\chi_n \psi)\rangle_\Omega + \langle p, \nabla \cdot (\chi_n \psi)\rangle_\Omega = 0, \quad \text{in} \quad L^1(0,T). \tag{6.3}$$

For the second and third terms χ_n is independent of the spatial variables and so

$$\lim_{n\to\infty} \int_0^t \langle u \otimes u : \nabla(\chi_n \psi)\rangle_\Omega \,\mathrm{d}\tau = \lim_{n\to\infty} \int_0^t \chi_n \langle u \otimes u : \nabla \psi\rangle_\Omega \,\mathrm{d}\tau = \int_0^t \langle u \otimes u : \nabla \psi\rangle_\Omega \,\mathrm{d}\tau$$

and

$$\lim_{n\to\infty} \int_0^t \langle p, \nabla \cdot (\chi_n \psi)\rangle_\Omega \,\mathrm{d}\tau = \lim_{n\to\infty} \int_0^t \chi_n \langle p, \nabla \cdot \psi\rangle_\Omega \,\mathrm{d}\tau = \int_0^t \langle p, \nabla \cdot \psi\rangle_\Omega \,\mathrm{d}\tau.$$

For the first term in (6.3) we have

$$\int_0^t \langle u, \partial_t(\chi_n \psi)\rangle_\Omega \,\mathrm{d}\tau = \int_0^t \langle u, \chi_n \partial_t \psi\rangle_\Omega + \langle u, \psi \partial_t \chi_n\rangle_\Omega \,\mathrm{d}\tau,$$

where the limit of the first term in the R.H.S. becomes

$$\lim_{n\to\infty} \int_0^t \langle u, \chi_n \partial_t \psi\rangle_\Omega \,\mathrm{d}\tau = \lim_{n\to\infty} \int_0^t \chi_n \langle u, \partial_t \psi\rangle_\Omega \,\mathrm{d}\tau = \int_0^t \langle u, \partial_t \psi\rangle_\Omega \,\mathrm{d}\tau.$$

For the second term we notice that $\partial_t \chi_n = 0$ in the region $(1/n, t-1/n)$; however, in the regions $[0, 1/n]$ and $[t-1/n, t]$ we see that the function χ_n goes from a value

1 to value 0. Using the Fundamental Theorem of Calculus then

$$\int_0^{1/n} \partial_t \chi_n(\tau)\,\mathrm{d}\tau = 1 \quad \text{and} \quad \int_{t-1/n}^{t} \partial_t \chi_n(\tau)\,\mathrm{d}\tau = -1$$

for all n. Thus as $n \to \infty$ these terms become approximations to the identity and so

$$\lim_{n\to\infty} \int_0^{1/n} \partial_t \chi_n(\tau) f(\tau)\,\mathrm{d}\tau = f(0) \quad \text{and} \quad \int_{t-1/n}^{t} \partial_t \chi_n(\tau) g(\tau)\,\mathrm{d}\tau = -g(t).$$

Using this we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^t \partial_t \chi_n \langle u, \psi\rangle_\Omega \,\mathrm{d}\tau \\
&\qquad = \lim_{n\to\infty} \left[\int_0^{1/n} \partial_t \chi_n \langle u(\tau), \psi(\tau)\rangle_\Omega \,\mathrm{d}\tau - \int_{t-1/n}^{t} \partial_t \chi_n \langle u(\tau), \psi(\tau)\rangle_\Omega \,\mathrm{d}\tau \right] \\
&\qquad = \langle u(0), \psi(0)\rangle_\Omega - \langle u(t), \psi(t)\rangle_\Omega.
\end{aligned}$$

Here we are converging to a point t and so need to assume enough regularity on u so that $\langle u(t), \psi(t)\rangle_\Omega$ makes sense. Therefore we need to restrict from u in $L^\infty(0,T;H_\sigma)$ to u that is weakly continuous in time, that is, $C_{\mathrm{w}}([0,T];H_\sigma)$. We will therefore restrict Definition 6.2 so that it treats the solutions in time in a similar way to Definition 6.1

Definition 6.3 (Full Pressure Compact Support form) *Let Ω be a bounded subset of $\mathbb{R}^3$, with a C^2 boundary, $\partial\Omega$. Then (u,p) is a weak solution of the incompressible Euler equations in $\Omega \times (0,T)$ if $u \in C_{\mathrm{w}}([0,T];H_\sigma(\Omega))$, $p \in \mathcal{D}'(\Omega \times [0,T])$ such that*

$$\begin{aligned}
&\langle u(t), \psi(t)\rangle_\Omega - \langle u(0), \psi(0)\rangle_\Omega - \int_0^t \langle u(\tau), \partial_t \psi(\tau)\rangle_\Omega \,\mathrm{d}\tau \\
&\qquad = \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla \psi(\tau)\rangle_\Omega \,\mathrm{d}\tau + \int_0^t \langle p(\tau), \nabla \cdot \psi(\tau)\rangle_\Omega \,\mathrm{d}\tau, \quad (6.4)
\end{aligned}$$

for every $t \in [0,T]$ for any $\psi \in C^\infty([0,T];\mathcal{D}(\Omega))$.

6.2 Energy Conservation on Ω

From results of Bardos & Titi (2018) if we consider a solution u in Definition 6.3 on a domain Ω with a C^2 boundary with $u \in L^3(0,T;C^\alpha(\bar{\Omega}))$ for $\alpha > 1/3$ then energy is conserved. Here we will discuss the relation between Definition 6.3 and 6.1 and show that if u is a 'pressure less' weak solution, as in Definition 6.1 then it is a weak solution in the sense of distributions as in Definition 6.3 and thus we can apply the result of Bardos & Titi (2018) to such a weak solution.

Firstly, we will derive the weak formulation of equation (6.2).

Lemma 6.4 *Let u be a weak solution, as in Definition 6.3 and if p solves*

$$\int_0^t \langle u \otimes u : \nabla(\nabla\sigma)\rangle_\Omega \,\mathrm{d}\tau + \int_0^t \langle p, \Delta\sigma\rangle_\Omega \,\mathrm{d}\tau = 0 \tag{6.5}$$

for all $\sigma \in C^\infty(0,T;C^\infty(\Omega))$ then p is a weak solution of equation (6.2), i.e. the elliptic boundary-value problem

$$-\Delta p = \partial_i\partial_j(u_iu_j) \;\; in \;\; \Omega \;\; and \;\; \nabla p \cdot n = -\frac{1}{2}(u_j\partial_j u_i)n_i \;\; on \;\; \partial\Omega.$$

Proof For the first term of (6.5) we can use integration-by-parts in the spatial variable to obtain

$$\begin{aligned}\int_0^t \langle u \otimes u : \nabla\nabla\sigma\rangle_\Omega \,\mathrm{d}\tau = &\int_0^t \int_{\partial\Omega} (u\cdot n)(u\cdot\nabla\sigma)\,\mathrm{d}x\,\mathrm{d}\tau \\ &- \int_0^t \langle \nabla\cdot(u\otimes u), \nabla\sigma\rangle_\Omega \,\mathrm{d}\tau = -\int_0^t \langle \nabla\cdot(u\otimes u), \nabla\sigma\rangle_\Omega \,\mathrm{d}\tau\end{aligned}$$

as $u \cdot n = 0$ on the boundary. We can integrate-by-parts again and obtain

$$\begin{aligned}-\int_0^t \langle \nabla\cdot(u\otimes u), \nabla\sigma\rangle_\Omega \,\mathrm{d}\tau = &-\int_0^t \int_{\partial\Omega} [\nabla\cdot(u\otimes u)\cdot n]\sigma\,\mathrm{d}x\,\mathrm{d}\tau \\ &+ \int_0^t \int_\Omega \nabla\cdot[\nabla\cdot(u\otimes u)]\sigma\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{6.6}$$

A similar calculation with the second term of (6.5) using the boundary condition that $\nabla\sigma \cdot n = 0$ gives that

$$\int_0^t \langle p, \Delta\sigma\rangle_\Omega \,\mathrm{d}\tau = -\int_0^t \int_{\partial\Omega} [\nabla p \cdot n]\sigma \,\mathrm{d}x \,\mathrm{d}\tau + \int_0^t \int_\Omega [\Delta p]\sigma \,\mathrm{d}x \,\mathrm{d}\tau. \tag{6.7}$$

If we then compare the first term on the R.H.S. of (6.6) with the first term of the R.H.S. of (6.7) and similarly for the second terms we get the two equations in (6.2) and so we see that (6.5) is the weak formulation of (6.2). □

Using the lemma above, we can show the relation between Definition 6.1 and 6.3.

Theorem 6.5 *Suppose that u is a weak solution in the sense of Definition 6.1 with*

$$u \in L^3(0, T; C^\delta) \quad \textit{for} \quad \delta > 0.$$

Let p satisfy

$$\int_0^t \langle u \otimes u : \nabla(\nabla\sigma)\rangle_\Omega \,\mathrm{d}\tau + \int_0^t \langle p, \Delta\sigma\rangle_\Omega \,\mathrm{d}\tau = 0$$

for all $\sigma \in C^\infty(0, T; C^\infty(\Omega))$. Then the pair (u, p) is a distributional solution in the sense of Definition 6.3.

Proof Take a weak solution u as in Definition 6.1, i.e u solves

$$\langle u(t), \phi(t)\rangle_\Omega - \langle u(0), \phi(0)\rangle_\Omega - \int_0^t \langle u(\tau), \partial_t\phi(\tau)\rangle_\Omega \,\mathrm{d}\tau = \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla\phi(\tau)\rangle_\Omega \,\mathrm{d}\tau,$$

for every $t \in [0, T]$, for any $\phi \in C^\infty_{n,\sigma}(\Omega \times [0, T])$. Now we take p satisfying (6.5) and want to show that the pair (u, p) satisfies

$$\begin{aligned}\langle u(t), \psi(t)\rangle_\Omega - \langle u(0), \psi(0)\rangle_\Omega - \int_0^t \langle u(\tau), \partial_t\psi(\tau)\rangle_\Omega \,\mathrm{d}\tau \\ = \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla\psi(\tau)\rangle_\Omega \,\mathrm{d}\tau + \int_0^t \langle p(\tau), \nabla \cdot \psi(\tau)\rangle_\Omega \,\mathrm{d}\tau,\end{aligned}$$

for all $\psi \in C^\infty([0, T]; \mathcal{D}(\Omega))$. Given any $\psi \in C^\infty([0, T]; \mathcal{D}(\Omega))$, we can perform a Helmholtz decomposition as explained in Robinson, Rodrigo, & Sadowksi (2016) so that

$$\psi = \phi + \nabla\sigma \quad \text{where} \quad \phi, \sigma \in C^\infty([0,T]; C^\infty(\Omega))$$

with $\nabla \cdot \phi(\cdot, t) = 0$ and $\phi(\cdot, t) \cdot n = 0$ and $\nabla\sigma(\cdot, t) \cdot n = 0$ on the boundary $\partial\Omega$.

We obtain $\phi, \sigma \in C^\infty([0,T]; C^\infty(\Omega))$ because when we construct the Helmholtz decomposition we must solve two elliptic equations. From standard elliptic regularity theory as $\psi(t, \cdot) \in C^\infty(\Omega)$ we obtain $\phi(t, \cdot), \sigma(t, \cdot) \in C^\infty(\Omega)$. As $\partial_t^n \psi(t, \cdot) \in C^\infty(\Omega)$ for any $n \in \mathbb{N}$ we further obtain $\partial_t^n \phi(t, \cdot), \partial_t^n \sigma(t, \cdot) \in C^\infty(\Omega)$. Finally, we note that when considering $\partial_t^n \psi$ as we have time derivatives that are independent of the spacial elliptic problem we can commute the derivatives and obtain that $\phi, \sigma \in C^\infty([0,T]; C^\infty(\Omega))$.

When we use this decomposition in (6.4) and expand every term out after using weak incompressibility of u we obtain

$$\begin{aligned} &\langle u(t), \phi(t) \rangle_\Omega - \langle u(0), \phi(0) \rangle_\Omega - \int_0^t \langle u(\tau), \partial_t \phi(\tau) \rangle_\Omega \,\mathrm{d}\tau \\ &\quad = \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla\phi(\tau) \rangle_\Omega \,\mathrm{d}\tau + \int_0^t \langle u(\tau) \otimes u(\tau) : \nabla(\nabla\sigma)(\tau) \rangle_\Omega \,\mathrm{d}\tau \\ &\qquad + \int_0^t \langle p(\tau), \nabla \cdot \phi(\tau) \rangle_\Omega \,\mathrm{d}\tau + \int_0^t \langle p(\tau), \Delta\sigma(\tau) \rangle_\Omega \,\mathrm{d}\tau. \end{aligned}$$

Using Definition 6.1 this simplifies to

$$\int_0^t \langle u \otimes u : \nabla\nabla\sigma \rangle_\Omega \,\mathrm{d}\tau + \int_0^t \langle p, \Delta\sigma \rangle_\Omega \,\mathrm{d}\tau = 0,$$

for all $\sigma \in C^\infty([0,T]; C^\infty(\Omega))$. We see that if p solves this equation then equation (6.4) would hold. From Lemma 6.4 this weak equation is the same as (6.2). As we have assumed that $u \in L^3(0, T; C^\delta)$ for $\delta > 0$ then using Krylov (1996) we can solve (6.2) to obtain a $p \in L^{3/2}(0, T; C^\delta)$ and so equation (6.4) holds and we are done. □

6.3 Conclusion

We have shown that sufficiently regular solutions in the sense of Definition 6.1 are solutions in the sense of Bardos & Titi (2018). In particular this applies to solutions that are in

$$u \in L^3(0,T;C^{\varepsilon}(\bar{\Omega})) \quad \text{for} \quad \varepsilon > 0.$$

Their result therefore implies energy conservation for this class of weak solutions under the assumption that $u \in L^3(0,T;C^{\alpha}(\bar{\Omega}))$ for $\alpha > 1/3$ with a C^2 boundary $\partial\Omega$. It is interesting to note, however this condition has a little stronger regularity than that suggested by the analysis in Chapters 4 and 5, which suggests that the set of conditions in Theorem 5.10 should be sufficient. Further, the proof of energy conservation for strong solutions works for any bounded domain with a Lipschitz boundary and so this suggest that the C^2 boundary condition could be improved as well.

Chapter 7

Further Work

In Chapter 2, we compared the different regularity conditions used to show energy conservation on domains in the absence of boundaries. We were able to show that the conditions of u in $L^3(0,T;L^3)$ and

$$\lim_{|y|\to 0}\frac{1}{|y|}\int_0^T\int |u(x+y)-u(x)|^3\,\mathrm{d}x\,\mathrm{d}t=0,$$

are equivalent to the weakest known conditions.

We noticed that this condition can be generalised to bounded domains and then by adding extra continuity conditions of the solution u near the boundary, we were able to show energy conservation for weak solutions of the Euler equations on the domains $\mathbb{T}^{d-1}\times\mathbb{R}^+$ for $d\geq 2$, in Chapters 3, 4 and 5, with the set of conditions:

- the interior condition,

$$\lim_{|y|\to 0}\frac{1}{|y|}\int_0^T\int_{D_{[|y|,\infty)}} |u(x+y)-u(x)|^3\,\mathrm{d}x\,\mathrm{d}t=0,$$

- the boundary condition, that there exists a $\delta>0$ such that

$$u\in L^3(0,T;C^0(\mathbb{T}^2\times[0,\delta)),$$

Where in fact we do not use the continuity in the full strip $\mathbb{T}^2\times[0,\delta)$ but only use the fact that the function defined point-wise and is continuous at ∂D. We were able to prove this in two ways. Firstly, in Chapter 4, we used a global extension which solved the equations on $\mathbb{T}^{d-1} \times \mathbb{R}$ with a boundary pressure term in the weak formulation of the equations. Then, in Chapter 5, we were able to improve this by removing the boundary pressure term from the weak formulation and also we were able to use only a local extension around the boundary.

In recent work of Bardos & Titi (2018) energy conservation was shown for weak solutions of the Euler equations on a bounded C^2 domain Ω under the condition

$$u \in L^3(0, T; C^\alpha(\bar{\Omega})) \quad \text{for} \quad \alpha > 1/3.$$

In Chapter 6 we considered their definition, which involved a pressure term, and compared it to a definition without the pressure term, that is, a natural generalisation of the weak formulation we used earlier in Chapter 5. This allowed us to show that we could apply their results to our weak formulation.

However, these results have left some unanswered questions. We see that the conditions used for the domain $\mathbb{T}^{d-1} \times \mathbb{R}^+$, in Chapters 4 and 5, are weaker than the conditions needed by Bardos & Titi (2018) and it is natural to ask whether their conditions can be weakened to make them more similar to our set of conditions used for the domain $\mathbb{T}^{d-1} \times \mathbb{R}^+$. Further, the proof of energy conservation for strong solutions only needs Lipschitz regularity for the domain yet for the work of Bardos & Titi (2018) a C^2 boundary was needed, leaving scope for improvement.

One possible way to do the might be to flatten a C^1 boundary with charts and use the local extension we considered in Chapter 5 to extend the function on each chart. We could then regularise by mollification on each chart to smooth the function in the flat domain and finally map back to the original domain. This method may cause extra difficulties, since the nice properties of mollification will be altered, which we will have to account for. However, this method would be a potential way to improve the regularity conditions for energy conservation and could extend the argument to all C^1 bounded domains.

The method of Bardos & Titi (2018) used a cut-off function to restrict the solution away from the boundary before regularisation by mollification, so the test function remained within the domain. This method, and any other similar approaches, need this C^2 boundary regularity for suitable convergence of the derivative of the cut-off function to the normal of the boundary. However, similar approaches could be used to weaken the regularity needed to the conditions we used in Chapters 4 and 5.

Classical local energy balance equations for strong solutions to the Euler equations exist: for a region $U \subset \Omega$ we have the energy balance equation

$$\frac{1}{2}\partial(|u|^2) = -\nabla \cdot \left(u \left[\frac{1}{2}|u|^2 + p \right] \right).$$

Here the change in energy of a solution in the region U is given by the flux of a term through the boundary of that region. In Shvydkoy (2010) weak formulations of this energy balance were shown for solutions that satisfy (1.4) on the region U. This further suggests that our set of conditions in Chapters 4 and 5 would be enough, on a bounded domain, to ensure energy conservation. It would be interesting to investigate whether this local energy flux can be used to give another method to prove energy conservation in general bounded domains. Potentially using our regularity conditions from Chapter 5 in this case.

Bibliography

F. Albiac & N.J. Kalton (2016) *Topics in Banach space theory.* Second edition. Springer-Verlag.

H. Bahouri, J. Chemin, & R. Danchin (2011) *Fourier analysis and nonlinear partial differential equations.* Springer-Verlag.

C. Bardos, L. Székelyhidi & E. Wiedemann (2014) Non-uniqueness for the Euler equations: the effect of the boundary. *Russian Mathematical Surveys* **69**, 189.

C. Bardos & E.S. Titi (2009) *Loss of smoothness and energy conserving rough weak solutions for the 3d Euler equations.* arXiv preprint arXiv:0906.2029.

C. Bardos & E.S. Titi (2013) Mathematics and turbulence: where do we stand?. *Journal of Turbulence* **14**, 42–76.

C. Bardos & E.S. Titi (2018) Onsager's Conjecture for the Incompressible Euler Equations in Bounded Domains. *Archive for Rational Mechanics and Analysis 228.1*, 197–207.

C. Bardos, E.S. Titi, & E. Wiedemann (2012) The vanishing viscosity as a selection principal for the Euler equations: the case of 3D shear flow. *Comptes Rendus Mathematique* **350**, 757–760.

T. Buckmaster, C. De Lellis, L. Székelyhidi, & V. Vicol (2016) *Onsager's conjecture for admissible weak solutions.* arXiv preprint arXiv:1701.08678.

A. Cheskidov, P. Constantin, S. Friedlander, & R. Shvydkoy (2008) Energy conservation and Onsager's conjecture for the Euler equations. *Nonlinearity* **21**, 1233–1252.

P. Constantin, W. E, & E. Titi (1994) Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Communications in Mathematical Physics* **165**, 207–209.

J. Duchon & R. Robert (2000) Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity* **13**, 249–255.

L. Evans (1998) *Partial differential equations.* American Mathematical Society.

G. Eyink (1994) Energy dissipation without viscosity in ideal hydrodynamics I. Fourier analysis and local energy transfer. *Physica D: Nonlinear Phenomena* **78**, 222–240.

G. Eyink & K. Sreenivasan (2006) Onsager and the theory of hydrodynamic turbulence. *Reviews of modern physics* **78**, 87–135.

P. Isett (2016) *A Proof of Onsager's Conjecture.* arXiv preprint arXiv:1608.08301.

J. Jost & X. Li-Jost (1998) *Calculus of variations.* Cambridge University Press.

N. Krylov (1996) *Lectures on Elliptic and Parabolic Equations in Hölder Spaces.* American Mathematical Soc.

P.G. Lemarié-Rieusset (2002) *Recent developments in the Navier-Stokes problem.* CRC Press.

P.-L. Lions (1997) *Mathematical topics in fluid mechanics, volume 1: Incompressible models.* Oxford University Press. **50**:B81–B81.

A. Majda & A. Bertozzi (2002) *Vorticity and incompressible flow.* Cambridge University Press.

E. Di Nezza, G. Palatucci, & E. Valdinoci (2012) Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136**, 521-573.

L. Onsager (1949) Statistical hydrodynamics. *Il Nuovo Cimento (1943-1954)* **6**, 279–287.

J. Peetre (1976) *New thoughts on Besov spaces.* Mathematics Department, Duke University.

J.C. Robinson, J.L. Rodrigo, & W. Sadowski (2016) *The three-dimensional Navier–Stokes equations.* Cambridge University Press.

J.C. Robinson, J.L. Rodrigo, & J. Skipper (2018a) *Energy conservation in the 3D Euler equations on* $\mathbb{T}^2 \times \mathbb{R}_+$. In C.L. Fefferman, J.C.Robinson, & J.L.Rodrigo (Eds.) *Partial Differential Equations in Fluid Mechanics* LMS Lecture Notes. Cambridge University Press, Cambridge, UK.

J.C. Robinson, J.L. Rodrigo, & J. Skipper (2018b) *Energy conservation for the 3D Euler equations on* $\mathbb{T}^2 \times \mathbb{R}_+$ *for weak solution defined without reference to the pressure. Asymptotic Analysis*, to appear.

R. Shvydkoy (2009) On the energy of inviscid singular flows. *J. Math. Anal. Appl.* **349**, 583–595.

R. Shvydkoy (2010) Lectures on the Onsager conjecture. *Discrete Contin. Dyn. Syst. Ser. S* **3**, 473–496.