A CONTRIBUTION TO THE THEORY OF GROUP LATTICES AND PROJECTIVITIES

Giorgio BUSETTO

Thesis submitted for the degree of Doctor Philosophy in Mathematics.

September 1983.

Mathematics Institute
University of Warwick
Coventry CV47AL

Acknowledgements

I would like to thank my supervisor, dr. S.E. Stonehewer, for his continuous and enthusiastic support during the preparation of this thesis. I am especially indebted to him for the results of chapter 4, that we have obtained together.

I am also indebted to F. Menegazzo, F. Napolitani, G. Zacher for the helpful discussions and comments.

Finally I would like to thank all the graduate students and the members of staff of the Mathematics Institute of the University of Warwick for the two years of really good time I have had there.

Contents

Glossary

If S and S_1 are sets, as usual

$S \subseteq S_1$ means that S is a subset of S_1.

$S \cap S_1$ is the intersection of S and S_1.

$S \cup S_1$ is the set-theoretical union of S and S_1.

$S_1 \setminus S$ is the difference set of S_1 and S, namely the set of those elements belonging to S_1 but not to S.

$x \in S$ means that x belongs to S.

$\{x,y,z......\}$ is the set consisting of the elements x,y,z.........

$\mathbb{N}$ is the set of natural numbers, $\mathbb{Z}$ is the set of integers.

If $m,n \in \mathbb{Z}$, $m \geq n$ means that m is greater or equal to n (in the natural order of $\mathbb{Z}$),whereas $m > n$ means that m is stricly greater than n.

(m,n) is the greatest common divisor of m and n.

$m|n$ means that m divides n.

Through this thesis p will always denote a prime number.

If G is a group,

$H \leq G$ means that H is subgroup of G.

$H < G$ " " " " a proper subgroup of G.

$H \lhd G$ " " " " a normal subgroup of G.

If $H \leq G$,

H^G is the normal closure of H in G.

H_G is the core of H in G.

$N_G(H)$ is the normaliser of H in G.

By $[G/H]$ we shall denote the lattice of subgroups of G containing H.

$N(G)$ is the norm of G.

$Z(G)$ is the centre of G.

If α is an ordinal, $Z_\alpha(G)$ is the α^{th}-term of the upper central series of G. In particular $Z_\omega(G) = \cup_{n\in N} Z_n(G)$.

G' is the derived (or commutator) subgroup of G.

If $n \geq 2$, $G^{(n)}$ denotes the n^{th}-commutator subgroup of G.

If $S \subseteq G$, $<S>$ is the subgroup generated by the elements of S.

If S and S_1 are subgroups of G, $[S,S_1] = <x^{-1}y^{-1}xy \mid x \in S, y \in S_1>$.

If $x \in G$ and $S \leq G$, we shall write $[x,S]$ instead of $[<x>,S]$, while, if x and y are both elements of G, $[x,y]$ denotes the element $x^{-1}y^{-1}xy$.

If $H \leq K \leq G$, $L \leq N_G(H)$, $C_L(K/H) = \{g \in L \mid [k,g] \in H$ for all $k \in K\}$. $C_L(K/H)$ is a subgroup of L, the centraliser in L of K/H.

If $x \in G$, $y \in G$, x^y is the conjugate of x by y, namely $y^{-1}xy$.

If $x \in G$, $|x|$ is the order of x ($|x| = \infty$ if $<x>$ is infinite cyclic).

If $m \in N$, x is said to be a m'-element if $|x|$ is finite and $(|x|,m) = 1$.

If Π is a set of primes a group G is said to be a Π-group if $|x| \neq \infty$ for all $x \in G$ and $(|x|,q) = 1$ for every prime $q \notin \Pi$.

G is said to be of finite exponent m, where $m \in N$, if m is the maximum of the orders of the elements of G. Otherwise G is said to be of infinite exponent.

If G is a p-group (by p-group we mean a $\{p\}$-group), $\Omega_i(G) = \langle x \mid x \in G \text{ and } x^{p^i} = 1\rangle$,

and $\mho_i(G) = \langle x^{p^i} \mid x \in G\rangle$.

If $\{G_i\}_{i\in I}$ is a set of groups, $\mathrm{Dr}_{i\in I}\, G_i$ is the restricted direct product of the G_i's. If $I = \{i_1, \ldots\ldots, i_n\}$ is finite, sometimes we write $\mathrm{Dr}_{i\in I}\, G_i = G_{i_1} \times \ldots\ldots \times G_{i_n}$. Direct products will always be restricted.

C_∞ denotes the (additive) group of $\mathbb{Z}$.

C_{p^n} denotes the (multiplicative) group of complex $(p^n)^{th}$-roots of unity.

C_{p^∞} denotes the Prüfer group relative to p, namely $\bigcup_{n\in\mathbb{N}} C_{p^n}$.

$\equiv$ mod means congruent modulo.

Chapter 1. Introduction, notation and some assumed results.

1.1 Introduction.

If G is a group and H, K are subgroups of G, as usual denote the intersection of H and K by $H \cap K$, and the join of H and K, namely the intersection of all the subgroups of G containing H and K, by $\langle H, K \rangle$. Then the set $L(G)$ of all the subgroups of G endowed with the two operations

$$\cap : L(G) \times L(G) \to L(G)$$
$$(H, K) \to H \cap K$$

and

$$\langle\ ,\ \rangle : L(G) \times L(G) \to L(G)$$
$$(H, K) \to \langle H, K \rangle$$

is a lattice. Following Suzuki ([24], page 31, chapter II), if G and G_1 are groups, by a projectivity $\pi : G \to G_1$ we shall mean a lattice isomorphism from $L(G)$ onto $L(G_1)$. In such a situation we shall often say that G_1 is a projective image of G or that G and G_1 are projective, and, if $H \leq G$, we shall write H^π for the image of H under π. Also, by a projective image X of a subgroup H of some group G we shall implicitly mean that there exist a group G_1 and a projectivity $\pi : G \to G_1$ such that $X = H^\pi$. If G and G_1 are isomorphic groups certainly they are projective, but most of the times the converse is far from being true. Thus the following general question arises naturally: to what extent does the lattice of subgroups of a group determine the group structure? In other words, how much can a projective image of a given group G differ from G? As a matter of fact in most of the cases it is very hard to give a satisfactory answer. This thesis is mainly devoted to building up some tools and techniques which hopefully in some cases could be useful for this task.

Let G and G_1 be groups and $\pi : G \to G_1$ a projectivity. Whereas for an arbitrary subgroup H of G it is in general impossible to describe how H^π behaves inside G_1, a lot, as we shall see, can be said when H is normal in G. And, as the presence of normal subgroups in G is strongly interconnected with the structure of G, hopefully the knowledge of the behaviour inside G_1 of the images under π of the normal subgroups of G would give informations on the structure of G_1 in relation to the structure of G. This thesis is just concerned with normal subgroups and their projective images. The study of this topic has been carried out (in chronological order) at first, in the fifties, by Suzuki ([24] , chapter II, 7) and successively, among the others, by Yakovlef ([25]), Schmidt ([19]), Menegazzo ([12], [13]), Rips ([15]), Zacher ([26],[27]), Napolitani-Zacher ([14]). A major part of this thesis is in fact inspired by results of Schmidt and Menegazzo in [19] and [12] respectively.

If H is a normal subgroup of G, H^π need not be normal in G_1. (As a simple example take for G an elementary abelian group of order 9 and for G_1 the symmetric group on three letters. G and G_1 clearly have isomorphic subgroup lattices.) Thus we may consider the normal closure K^π of H^π in G_1, namely the minimal normal subgroup of G_1 containing H^π, and the core N^π of H^π in G_1, namely the maximal normal subgroup of G_1 contained in H^π. We aim to obtain information about the embedding of H^π in G_1 and to 'measure' its 'deviation' from normality in terms of the structure of K^π/N^π and the action of G_1 on K^π/N^π. We give a brief sketch of the results obtained. The thesis is divided in five chapters. The present chapter is introductory. The second one is inspired, as we said, by a result of Roland Schmidt ([19], Lemma 3.3, (a)) who showed that, in the above notation, if G (and hence G_1) are finite, then N and K are normal in G. This result has proved to be very useful; in fact it implies that π induces in a natural way a projectivity from the group G/N to G_1/N^π and therefore, in order to investigate what happens in G_1 above N^π we are allowed to assume that H^π is core-free in G_1, namely that $N^\pi = 1$. This assumption, as we shall see, has many consequences on the structure of H and H^π and on their embeddings in G and G_1 respectively. The aim of the chapter is to prove Schmidt's result in total generality, removing the hypothesis of finiteness on G (see Theorem 2.1.1).

The third and fourth chapters are dedicated to investigating the structure of H/N and H^π/N^π (by what we have just pointed out, we may assume, without loss of generality, that $N^\pi = 1$.). In this direction Menegazzo has proved the following beautiful result.

Theorem 1.1.1 (Menegazzo, [12]). Let $\pi : G \to G_1$ be a projectivity with G a finite group of odd order. If $H \lhd G$ and H^π is core-free in G_1, then H is abelian.

Since the structure of a projective image of an abelian group is well known (see [24], chapter I, sections 4 and 5), Theorem 1.1.1 gives also many information on H^π; in particular H^π is a metabelian modular group. We recall that a group G is modular if the identity

$$\langle U, V \rangle \cap W = \langle U, V \cap W \rangle$$

is satisfied for all $U, V, W \leq G$ with $W \geq U$.
Abelian groups are clearly modular. However, from the statement of Menegazzo's theorem, two questions arise naturally. Firstly, what happens if G is finite of even order? Menegazzo's proof did not work for groups of even order, but no counterexample was known. Secondly, going even further on, what can we say if we remove the hypothesis of finiteness on G? In chapters 3 and 4 we give answers to these questions. More precisely in chapter 4 we prove, by exhibiting a counterexample, that unfortunately Theorem 1.1.1 is not true for groups of even order. The counterexample consists of two finite 2-groups G and G_1 of the same order 2^{13}, a projectivity $\pi : G \to G_1$ and a non-abelian normal subgroup H of G of order 2^7 such that H^π is core-free in G_1. In the first part of the chapter we also prove that the counterexample is minimal, in a sense that will be specified in the statement of Theorem 4.1.2. The results of chapter 4 have been obtained in collaboration with my supervisor, Dr. S.E. Stonehewer.

Although, as we have seen, (in the usual notation and with $N^\pi = 1$) H need not be abelian, in chapter 3 we prove (see Theorem 3.1.1) that H and H^π are soluble groups of derived length ≤ 3. This result is general, without any finiteness assumption. But we would like to point out that the merit of removing the hypothesis of the finiteness of G

is due essentially to the following powerful recent result by Rips ([15]).

Theorem 1.1.2 (Rips, Zacher [26], Teorema A). Suppose that G and G_1 are groups, $\pi : G \to G_1$ is a projectivity and H is a subgroup of G of finite index in G. Then H^π has finite index in G_1.

This theorem was proved first by Rips. On hearing the statement of the result, before seeing Rips' proof, Zacher found a different much shorter proof.

Theorem 1.1.2 has several useful consequences. One of them, which we will also use in the proof of Theorem 3.1.1 is the following.

Corollary 1.1.3 ([26], Corollario 1). Let G and G_1 be groups, $\pi : G \to G_1$ a projectivity and $H \lhd G$ with G/H infinite cyclic. Then $H^\pi \lhd G_1$.

Using Theorem 1.1.2 and corollary 1.1.3 the proof of Theorem 3.1.1 can be reduced to the case when G and G_1 are finite p-groups and G/H is cyclic. Then the case p odd is settled by Theorem 1.1.1, and it remains to deal with the case $p=2$ which we investigate mainly in Theorem 3.2.3. In the last section of the chapter (see Proposition 3.4.1) we give an example of how this machinery can be applied, assuming that G is soluble, to bound the derived length of G_1 in terms of the derived length of G, improving a similar result by Yakovlev ([25]).

In the last chapter we obtain some information about the actions (in the usual notation and still assuming $N^\pi = 1$) of G on K and of G_1 on K^π (Theorem 5.1.2), in the attempt to generalise to infinite groups a result by R. Schmidt ([19], Theorem 3.4) stating, for G

finite, the existence of series

$$1 = N_0 \leq N_1 \leq \ldots \leq N_t = K$$

and

$$1 = M_0 \leq M_1 \leq \ldots \leq M_s = K^\pi$$

of normal subgroups of G and G_1 respectively, such that N_{i+1}/N_i and M_{i+1}/M_i are cyclic (or even, in certain cases, central in G and G_1 respectively). Unfortunately we have not been able to obtain a general result holding for every group G, but only for a certain class (Theorem 5.3.4). This completes a rough sketch of the contents of the thesis.

In the following section we shall give some more preliminary definitions and state some more preliminary well-known results.

1.2 Preliminaries and some assumed results.

We recall that a subgroup H of a group G is a Dedekind subgroup (modular for some authors) of G if

$$\langle U, H \rangle \cap V = \langle U, H \cap V \rangle \quad \text{for all } U, V \leq G \text{ such that } U \leq V$$

and

$$\langle U, H \rangle \cap V = \langle U \cap V, H \rangle \quad \text{for all } U, V \leq G \text{ such that } H \leq V.$$

Remark 1.2.1. It is clear from the definition of modular group and Dedekind subgroup that a group is modular if and only if all its subgroups are Dedekind subgroups.

A normal subgroup is clearly a Dedekind subgroup and, since the definition of a Dedekind subgroup is purely lattice-thoretical, it follows that the projective image of a Dedekind subgroup is still Dedekind, in particular the projective image of a normal subgroup is a Dedekind subgroup. Closely connected with the notion of Dedekind subgroup we have the notion of quasinormal subgroup.

A subgroup H of a group G is quasinormal in G if $HX = XH$ for all $X \leq G$.

It is easy to see that a quasinormal subgroup is a Dedekind subgroup. Moreover, the connection between these two classes of subgroups is given by the following theorem.

Theorem 1.2.2 (Napolitani, Stonehewer, see [22], Prop. 1). A subgroup H of a group G is quasinormal in G if and only if H is a Dedekind and ascendant subgroup of G.

We recall that H is ascendant in G if there exist an ordinal γ and subgroups H_α for every ordinal $\alpha \leq \gamma$ such that $H_0 = H$, $H_\gamma = G$, $H_\alpha \leq H_\beta$ if $\alpha \leq \beta$, $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$ if α is a limit ordinal and $H_\alpha \lhd H_{\alpha+1}$. H is called subnormal if γ is finite.

Remark 1.2.3. Theorem 1.2.2 implies that a Dedekind subgroup H of a finite p-group G is quasinormal in G. It is also an easy exercise to see that this is still true assuming only G locally nilpotent. For, in order to prove that H is quasinormal in G it is sufficient to show that $hx \in \langle x \rangle H$ for all $x \in G$, $h \in H$. By Proposition 1.2.4 (ii), $H \cap \langle h, x\rangle$ is a Dedekind subgroup of $\langle h,x\rangle$. Since $\langle h,x\rangle$ is nilpotent, $H \cap \langle h,x\rangle$ is quasinormal in

$\langle h, x\rangle$ by Theorem 1.2.2 . Thus we have

$$hx \in (H \cap \langle h, x\rangle)\langle x\rangle = \langle x\rangle(H \cap \langle h,x\rangle) \subseteq \langle x\rangle H,$$

as required.

Dedekind and quasinormal subgroups will play an important role in our treatment. In the following proposition we collect some of their basic properties. The proofs are almost immediate.

Proposition 1.2.4. The following hold:

(i) The join of any number of Dedekind (quasinormal) subgroups is a Dedekind (quasinormal) subgroup.

(ii) If H is a Dedekind (quasinormal) subgroup of a group G and $X \leq G$, then $H \cap X$ is a Dedekind (quasinormal) subgroup of X.

(iii) If $N \lhd G$ and $H \geq N$, H is a Dedekind (quasinormal) subgroup of G if and only if H/N is a Dedekind (quasinormal) subgroup of G/N .

(iv) If a group G is the direct product of the periodic subgroups A_1, A_2 such that $(|a_1|, |a_2|) = 1$ for all $a_1 \in A_1$, $a_2 \in A_2$, then every Dedekind (quasinormal) subgroup of A_i, $i = 1,2$, is a Dedekind (quasinormal) subgroup of G.(This follows immediately from the definition of Dedekind and quasinormal subgroups using the fact that, for all subgroups H of G, we have $H = (H \cap A_1) \times (H \cap A_2)$.)

(v) A maximal subgroup which is quasinormal is normal.

(vi) A periodic quasinormal subgroup H of a group G is normalised by all the elements of G whose order is coprime to the order of every element of H.

In addition we recall three results on quasinormal subgroups, due respectively to Maier-Schmid ([11]), Gross ([5], Lemma 3.1 and [6], Lemma 3.2) and Stonehewer ([21], Lemma 2.1).

Theorem 1.2.5 (Maier-Schmid). A core-free quasinormal subgroup of a finite group G lies in the hypercentre of G.

Lemma 1.2.6 (Gross). Let $G = H\langle x\rangle$ be a finite p-group where H is a core-free quasinormal subgroup of G. Then

(a) $H \cap \langle x\rangle = 1$;

(b) $\Omega_1(G)$ is elementary abelian ;

(c) $\Omega_r(G) = \Omega_r(H)\,\Omega_r(\langle x\rangle)$, $\mho_r(\Omega_r(G)) = 1$ and $H\,\Omega_r(G)/\Omega_r(G)$ is core-free in $G/\Omega_r(G)$ for any positive integer r;

(d) $\Omega_2(G)$ has nilpotency class $\leq p-1$.
Moreover, if $p = 2$, then

(e) $|\langle x\rangle| \geq 2^{n+2}$, where 2^n is the exponent of H ;

(f) $\Omega_2(\langle x\rangle) \leq Z(G)$;

(g) $\Omega_3(G)$ has nilpotency class ≤ 2.

Lemma 1.2.7 (Stonehewer). A quasinormal subgroup H of a group G is normalised by every infinite cyclic subgroup of G which intersects H trivially .

The following proposition shows how some basic group-theoretical properties behave under the action of a projectivity. The proofs can be found in [23] and [24]. Before we state it we recall the definition of P-group .

A group G is a <u>P-group</u> if either it is an elementary abelian p-group or $G = A\langle b\rangle$ where $1 \neq A$ is an elementary abelian p-group, $\langle b\rangle$ has prime order q, $q \mid p-1$ and $a^b = a^r$ for some integer r with $r \not\equiv 1 \bmod p$ for all $a \in A$. If $G = A\langle b\rangle$ is a non-abelian P-group, where A and $\langle b\rangle$ are as above, then $L(G) \cong L(X)$, where X is an elementary abelian p-group isomorphic to $A \times B$, with $|B| = p$. This was already pointed out by Baer (see [24], chapter I, section 3). In particular a P-group is a modular group.

Proposition 1.2.8. Let G and G_1 be groups and $\pi : G \to G_1$ a projectivity. Then the following hold.

(a) (See [24], chapter 1, Theorem 2). If G is cyclic (locally cyclic), G_1 is cyclic (locally cyclic).

(b) (See [24], chapter 1, Theorem 4). If G is the direct product of the periodic subgroups G_λ such that elements of distinct G_λ's have coprime order, then G_1 is the direct product of the G_λ^π's and again elements of distinct G_λ^π's have coprime orders.

(c) (An easy extension to the locally finite case of [23], Theorem 3). If G is a locally finite p-group, then G_1 is also a locally finite p group except in the following cases:

(i) G is isomorphic to the Prüfer group C_{p^∞} and $G_1 \cong C_{q^\infty}$ for some prime $q \neq p$.

(ii) G is cyclic and G_1 is cyclic of q-power order for some prime $q \neq p$.

(iii) G is elementary abelian and G_1 is a non-abelian P-group.

(d) If G is abelian, then G_1 is a metabelian modular group (see [24], chapter 1, Theorems 17,18).

In chapters 2 and 5 we shall need the following stronger and more detailed version of Theorem 1.1.2, which is due to Zacher.

Lemma 1.2.9 (Zacher, [27], Lemmas 3.2, 3.3). Let G and G_1 be groups, $\pi : G \to G_1$ a projectivity, H a normal subgroup of G such that G/H is finitely generated. Then the following hold.

(i) $|(H^\pi)^{G_1} : H^\pi| < \infty$.

(ii) $H^\pi/(H^\pi)_{G_1}$ is a nilpotent group of finite exponent.

(iii) If H^π is not quasinormal in G_1, then $G_1/(H^\pi)_{G_1}$ is periodic and

(a) $G_1/(H^\pi)_{G_1} = P_1^\pi/(H^\pi)_{G_1} \times \dots \times P_t^\pi/(H^\pi)_{G_1} \times K^\pi/(H^\pi)_{G_1}$,
where $t < \infty$, and for $1 \le i \le t$ $P_i^\pi/(H^\pi)_{G_1}$ is a finite non-abelian P-group of order $p_i^{\alpha_i} q_i$, where p_i and q_i are primes, $q_i < p_i$ and $1 \le \alpha_i$. Moreover, elements of distinct direct factors have coprime order;

(b) $H^\pi/(H^\pi)_{G_1} = Q_1^\pi/(H^\pi)_{G_1} \times \dots \times Q_t^\pi/(H^\pi)_{G_1} \times Q^\pi/(H^\pi)_{G_1}$ where $Q_i^\pi = H^\pi \cap P_i^\pi$, $|Q_i^\pi : (H^\pi)_{G_1}| = q_i$, $(Q_i^\pi)^{G_1} = (Q_i^\pi)^{P_i^\pi}$,
$Q^\pi = K^\pi \cap H^\pi$ is quasinormal in G_1 and H^π is quasinormal in $H^\pi K^\pi$;

(c) $(H^\pi)^{G_1}/(H^\pi)_{G_1} = P_1^\pi/(H^\pi)_{G_1} \times \dots \times P_t^\pi/(H^\pi)_{G_1} \times (Q^\pi)^{K^\pi}/(H^\pi)_{G_1}$,
where $(Q^\pi)^{K^\pi}/(H^\pi)_{G_1}$ is nilpotent of finite exponent.

In 1.1 we have defined modular groups. The following theorem, due to Iwasawa, describes the structure of locally finite modular p-groups. We recall that a group is <u>Hamiltonian</u> if it is non-abelian and all its subgroups are normal. A Hamiltonian group is the direct product of a quaternion group of order 8 and a periodic abelian group without elements of order 4.

Theorem 1.2.10 ([24], chapter 1, Theorem 18). A locally finite non-abelian p-group G is modular if and only if either G is Hamiltonian or $G = \langle A, t\rangle$ where A is abelian of finite exponent and, for all $a \in A$, $a^t = a^{1+p^s}$ where s is an integer and $s \ge 2$ if $p = 2$.

Remark 1.2.11 It is easy to deduce from theorem 1.2.10, using an inductive argument, that, if G is a locally finite modular non-Hamiltonian p-group, the map $x \to x^{p^{i-1}}$ is an endomorphism of $\Omega_i(G)$ for all $i \geq 0$ (see [24], chapter 1, page 15).

Finally we introduce the following notation.
Let G be a group and π a projectivity from G to some group G_1. For subgroups X, Y of G such that $X \leq Y$ we shall often denote the subgroups of G $((X^{\pi})^{Y^{\pi}})^{\pi-1}$ and $((X^{\pi})_{Y^{\pi}})^{\pi-1}$ by $X^{\pi,Y}$ and $X_{\pi,Y}$ respectively.

Chapter 2.

On the core and the normal closure of the image of a normal subgroup.

2.1. Introduction.

The aim of this chapter is to show that, when considering problems about a projectivity π of a group G with a normal subgroup H, we may assume that H^π is core-free in G^π. More precisely we will prove the following theorem.

Theorem 2.1.1 Let G and G_1 be groups, $\pi : G \to G_1$ a projectivity and $H \lhd G$. Then $H_{\pi,G}$ and $H^{\pi,G}$ are normal in G.

In particular it follows that π induces a projectivity from the group $G/H_{\pi,G}$ to $G_1/(H^\pi)_{G_1}$ and $H^\pi/(H^\pi)_{G_1}$ is core-free in $G_1/(H^\pi)_{G_1}$.

As mentioned in chapter 1 in the introduction, Theorem 2.1.1 has been proved by R. Schmidt when G, and hence G_1, are finite groups ([19], Lemma 3.3, (a)). However his proof is based on the investigation of the behaviour of minimal normal subgroups under the action of a projectivity and so it is not adaptable to the general case, since minimal normal subgroups do not exist in general. Thus our approach must be different and Lemma 1.2.9 will be an essential tool in the proof. We also need some preliminary results on periodic locally cyclic quasinormal subgroups. We will obtain them in the following section.

2.2 On periodic locally cyclic quasinormal subgroups.

We recall that the norm $N(G)$ of a group G is the intersection of all the normalisers of the subgroups of G. The following result is due to Schenkman ([17]).

Theorem 2.2.1 (Schenkman) $N(G) \leq Z_2(G)$.

For quasinormal subgroups of prime order we have the following simple, but, as we shall see, useful lemma.

Lemma 2.2.2 . Let H be a core-free quasinormal subgroup of prime order of a group G. Then $H \leq N(G)$. In particular, by Theorem 2.2.1, $H \leq Z_2(G)$.

Proof. Let $|H| = p$, say. If x is any element of G such that $\langle x\rangle$ is infinite cyclic or of order coprime to p, then, by Proposition 1.2.4 (vi) and Lemma 1.2.7,

$$x \in N_G(H^g) \quad \text{for all } g \in G . \tag{1}$$

Thus, since H is not normal in G, there exists $y \in G$ of p-power order not normalising H. Fix the element x and set $X = \langle H, y, x \rangle$, $T = H\langle y\rangle$. Then, by (1), $H^X = H^T$ and H is core-free in T. Also T is a p-group and therefore, applying Lemma 1.2.6 (a), (b), (c) to T, it follows that H^X is elementary abelian of order p^2 and $|H^X \cap \langle y\rangle| = p$. H^X contains $p+1$ subgroups of order p. Moreover, by (1), $|X : N_X(H)| = |T : N_T(H)|$ and $|T : N_T(H)| = p$, namely H has p distinct conjugates in X.

Therefore, by (1),

$$x \text{ normalises every subgroup of } H^x . \qquad (2)$$

Let $T_1 = \langle H,z\rangle$, where $\langle z\rangle$ is the Sylow p-subgroup of $\langle yx\rangle$ (note that, since yx does not normalise H, $|yx|$ is finite by Lemma 1.2.7). Again by (1), $H^x = H^{T_1}$ and H is core-free in the p-group T_1. Thus, by Lemma 1.2.6 (c), $H^x \cap \langle z\rangle = H^x \cap \langle yx\rangle$ has order p. Clearly yx centralises $H^x \cap \langle yx\rangle$. Thus, by (2), y normalises, and therefore centralises, $H^x \cap \langle yx\rangle$. Hence x also centralises $H^x \cap \langle yx\rangle$ and so, by (2), $x \in C_G(H^x)$. Therefore

$$H \le C_G \langle x \in G \mid \langle x\rangle \cong C_\infty \text{ or } (|x|, p) = 1\rangle$$

Moreover a quasinormal subgroup of order p clearly normalises the p-subgroups. Therefore H normalises every subgroup of G, namely $H \le N(G)$, as required.

□

Lemma 2.2.3 . Suppose that H is a periodic, locally cyclic, quasinormal subgroup of a group G and $S \le H$. Then S is quasinormal in G.

Proof. By Proposition 1.2.4 (i) we may assume, withour loss of generality, that S is a p-subgroup of H. In order to prove that S is quasinormal in G it is sufficient to show that $S\langle x\rangle = \langle x\rangle S$ for every cyclic subgroup $\langle x\rangle$ of G such that $\langle x\rangle$ is infinite cyclic or of prime power order. If $\langle x\rangle$ is infinite cyclic then, by Lemma 1.2.7, $\langle x \rangle \leq N_G(H)$ and therefore $\langle x \rangle \leq N_G(S)$ since S is characteristic in H. Thus, assume that $\langle x \rangle$ has prime power order q^n, say. If $q \neq p$, since $| \langle H^g, x \rangle : H^g | \mid q^n$ for all $g \in G$, $H/H_{\langle H, x \rangle}$ is a q-group. It follows that $S \leq H_{\langle H, x \rangle}$ and so x normalises S. Suppose, finally, $q = p$ and let $C = \langle S, x \rangle \cap H$. C is quasinormal in $\langle S, x \rangle$ by Proposition 1.2.4(i). As $S \lhd C$, by Theorem 1.2.2 S is ascendant in $\langle S, x \rangle$. It is well-known (see [16], Theorem 2.31 vol. 1) that the join of ascendant p-subgroups is a p-subgroup. Therefore, $S^{\langle S,x \rangle}$, and consequently $\langle S,x \rangle$, are p-groups. Hence C is also a p-group. If $S \leq C_{\langle S,x \rangle}$ then x normalises S. Therefore, suppose $S \geq C_{\langle S,x \rangle}$. Then it will not be restrictive to assume $C_{\langle S,x \rangle} = 1$. As C has finite index in $C\langle x \rangle$, $C\langle x \rangle = \langle S,x \rangle$ is now a finite p-group, and C is a core-free quasinormal subgroup of $C\langle x \rangle$. Also $S = \Omega_i(C)$ for some $i \geq 1$. Applying Lemma 1.2.6 (c) to $C\langle x \rangle$ we get

$$\langle x \rangle S = \langle x \rangle \Omega_i \langle x \rangle \Omega_i(C) = \langle x \rangle \Omega_i(C\langle x \rangle) =$$

$$= \Omega_i(C\langle x \rangle)\langle x \rangle = S\Omega_i \langle x \rangle \langle x \rangle = S\langle x \rangle .$$

The proof is now completed.

□

The following proposition generalises Lemma 2.2.2 . Although this generalisation will not be necessary for our purposes, it has perhaps some interest in the light of Theorem 1.2.5 . Indeed the latter is false, in general, for infinite groups: for example F. Gross ([7])

has constructed a group G containing a non trivial core-free quasinormal subgroup H where, among other properties, $Z(G) = Z_\infty(G)$.
We will bring up again the subject of possible generalisations of Theorem 1.2.5 in Chapter 5. Proposition 2.2.4 goes in the opposite direction.

Proposition 2.2.4 . A core-free, periodic, locally cyclic, quasinormal subgroup H of a group G is contained in $Z_\omega(G)$. More precisely, if S is a p-subgroup of H of order p^n, say, then $S \leq Z_{2n}(G)$.

Proof. Assume $n \geq 1$ and set $\Omega = \Omega_1(S)$. By Lemma 2.2.3 Ω is quasinormal in G, and so $\Omega \leq Z_2(G)$ by Lemma 2.2.2 . It follows that $\Omega^g Z(G) \lhd G$ for all $g \in G$. Thus, since $S \cap Z(G)=1$, $\Omega^G = \Omega^g \times (\Omega^G \cap Z(G))$. Let N/Ω^G be the core of $S\Omega^G/ \Omega^G$ in G/Ω^G. Then $N = N^g =(S^g \cap N)\Omega^G =(S^g \cap N)(\Omega^g \times(\Omega^G \cap Z(G)))$ for all $g \in G$. Ω^G is generated by quasinormal subgroups of order p, hence it is elementary abelian. Moreover $\bigcap_{g\in G} (S^g \cap N) = 1$, as S is core-free in G. Therefore N is residually an elementary abelian p-group, and so N itself is elementary abelian. It follows that $N = \Omega^G$. Thus $S\Omega^G/\Omega^G$ is core-free in G/Ω^G and $|S\Omega^G/\Omega^G| = p^{n-1}$. By induction on n $S\Omega^G/\Omega^G \leq Z_{2(n-1)}(G/\Omega^G)$. As $\Omega^G \leq Z_2(G)$, the result follows.

□

2.3 Proof of Theorem 2.1.1

We show first that

$$H_{\pi,G} \lhd G . \tag{3}$$

We claim that, in order to prove (3), it is not restrictive to assume G/H finitely generated. Indeed, assume that (3) holds whenever G/H is finitely generated. Let now G arbitrary (namely with G/H not necessarily finitely generated). Let Γ be the set of finite subsets of G. For $F \in \Gamma$ set $\Gamma_F = \{G \in \Gamma \mid G \supseteq F\}$. By hypothesis, for $F \in \Gamma$, $H_{\pi,\langle H,F\rangle} \lhd \langle H,F\rangle$ and therefore

$$H_{\pi,G} = \bigcap_{G \in \Gamma_F} H_{\pi\langle H,G\rangle} \lhd \langle H,F\rangle$$

Thus $H_{\pi,G}$ is normalised by every finitely generated subgroup of G, namely $H_{\pi,G} \lhd G$.

Assume then that G/H is finitely generated. For simplicity of notation set $N = H_{\pi,G}$ and suppose, by way of contradiction, that N is not normal in G. Set $M = N^G$. Clearly $M \le H$. By Lemma 1.2.9 (ii) H^π/N^π, and consequently also M^π/N^π, are periodic nilpotent groups of finite exponent. Let Π be the set of primes dividing the exponent of M^π/N^π. $M^\pi/N^\pi = \langle (N^{\langle N,g\rangle})^\pi/N^\pi \mid g \in G\rangle$ and hence $\Pi = \{p \mid p \text{ divides } \exp((N^{\langle N,g\rangle})^\pi/N^\pi) \text{ for some } g \in G\}$ (here we are using the usual fact that if Π is a set of primes and G is a nilpotent group which is the join of periodic Π-subgroups, then G is a Π-group). Therefore, for every $p \in \Pi$ there exists $g_p \in G$ such that

$(N^{<N,g_p>})^\pi/N^\pi$ contains a subgroup R_p^π/N^π, say, of order p. We observe that N, as the image under π^{-1} of the normal subgroup N^π of G_1, is a Dedekind subgroup of G. Thus, by proposition 1.2.4 (i) $N^{<N,g_p>}$, and consequently its projective image $(N^{<N,g_p>})^\pi$, are Dedekind subgroups (of G and G_1 respectively). Besides, $(N^{<N,g_p>})^\pi/N^\pi$ is cyclic, since $<N,g_p>^\pi/N^\pi \cong <g_p>^\pi/<g_p>^\pi \cap N^\pi$ and $<g_p>^\pi$ is cyclic by Proposition 1.2.8 (a).

Suppose now that, for some $p \in \Pi$, R_p^π/N^π is not quasinormal in G_1/N^π. Then we claim that H^π/N^π is not quasinormal in G_1/N^π. Indeed, if this is not the case, as a result of Theorem 1.2.2 and of the fact that H^π/N^π is nilpotent, $(N^{<N,g_p>})^\pi/N^\pi$ is also quasinormal in G_1/N^π. Thus, by Lemma 2.2.3, R_p^π/N^π is quasinormal in G_1/N^π, against the hypothesis. Hence H^π/N^π is not quasinormal in G_1/N^π. Then it follows that G_1/N^π has the structure described in Lemma 1.2.9 (iii). Following the notation introduced in that lemma (with $(H^\pi)_{G_1} = N^\pi$), suppose that $R_p^\pi/N^\pi \le (N^{<N,g_p>} \cap K)^\pi/N^\pi$. The latter is a Dedekind subgroup of K^π/N^π and it is also a subnormal subgroup of $(Q^\pi)^{K^\pi}/N^\pi$, since $(Q^\pi)^{K^\pi}/N^\pi$ is nilpotent. Therefore, by Theorem 1.2.2, $(N^{<N,g_p>} \cap K)^\pi/N^\pi$ is quasinormal in K^π/N^π. Proposition 1.2.4 (iv) then implies that it is in fact quasinormal in G_1/N^π. Thus, as a result of Lemma 2.2.3, R_p^π/N^π is quasinormal in G_1/N^π, again contradicting the assumption. Hence $R_p^\pi/N^\pi \nleq (N^{<N,g_p>} \cap K)^\pi/N^\pi$. Again from Lemma 1.2.9 (iii) (and always using the notation introduced there) it follows that $R_p^\pi/N^\pi = Q_{i_p}^\pi/N^\pi$ for some $1 \le i_p \le t$. Therefore we have shown that

<u>if, for some</u> $p \in \Pi$, R_p^{π}/N^{π} <u>is not quasinormal in</u> G_1/N^{π} <u>then</u> G_1/N^{π} <u>has the structure described in Lemma 1.2.9 (iii) and</u> $R_p^{\pi}/N^{\pi} = Q_{i_p}^{\pi}/N^{\pi}$ <u>for some</u> $1 \le i_p \le t$. (4)

Hence

<u>the only prime in</u> Π <u>dividing</u> $\exp(P_{i_p}^{\pi}/N^{\pi})$ <u>is</u> p, (5)

and

R_p^{π}/N^{π} <u>is the Sylow p-subgroup of</u> H^{π}/N^{π} <u>and</u> M^{π}/N^{π}. (6)

For all $p \in \Pi$ there exists x_p such that R_p^{π} is not normalised by $\langle x_p \rangle^{\pi}$. We show that

p does not divide $|(M^{\pi}/N^{\pi}) \cap (\langle x_p, N\rangle^{\pi}/N^{\pi})|$. (7)

In order to show (7) we distinguish two cases:

(i) R_p^{π}/N^{π} is quasinormal in G_1/N^{π}. Then $[R_p^{\pi}/N^{\pi}, \langle x_p, N\rangle^{\pi}/N^{\pi}]$ is a non identical (because $\langle x_p \rangle^{\pi}$ does not normalise R_p^{π}) p-group (because $(R_p^{\pi})^{G_1}/N^{\pi}$ itself is a p-group, since it is generated by quasinormal subgroups of order p) contained in $(\langle x_p, N\rangle^{\pi}/N^{\pi}) \cap Z(G_1/N^{\pi})$ (Lemma 2.2.2). Therefore the subgroup of order p of the cyclic group $\langle x_p, N\rangle^{\pi}/N^{\pi}$ lies in $Z(G_1/N^{\pi})$. Then, as M^{π}/N^{π} is core-free in G_1/N^{π}, (7) follows.

(ii) R_p^{π}/N^{π} is not quasinormal in G_1/N^{π}. Assume, by way of contradiction, that (7) is false. Then, since, by (6), R_p^{π}/N^{π} is the Sylow p-subgroup of M^{π}/N^{π}, it follows that

$(M^{\pi}/N^{\pi}) \cap (\langle x_p, N\rangle^{\pi}/N^{\pi}) = R_p^{\pi}/N^{\pi}$. Therefore $\langle x_p, N\rangle^{\pi}/N^{\pi}$ centralises R_p^{π}/N^{π}, contradicting the choice of x_p. This completes the proof of (7).

We next show that

<u>for each</u> $p \in \Pi$ <u>there exists</u> $\bar{z}_p \in G_1$ <u>such that</u> $\langle\bar{z}_p\rangle N^{\pi}/N^{\pi}$ <u>does not normalise</u> R_p^{π}/N^{π} <u>and</u> <u>normalises</u> R_s^{π}/N^{π} <u>for each</u> $s \in \Pi$ <u>different from</u> p. (8)

Again, in order to prove (8) we distinguish two cases.

(a) R_p^{π}/N^{π} is not quasinormal in G_1/N^{π}. Then, by (4) and (5) any element $\langle\bar{z}_p\rangle \in P_{p_i}^{\pi}$ such that $\langle\bar{z}_p\rangle N^{\pi}/N^{\pi} \not\leq R_p^{\pi}/N^{\pi}$ satisfies the required conditions.

(b) R_p^{π}/N^{π} is quasinormal in G_1/N^{π}. Then R_p^{π}/N^{π} is normalised by the elements of infinite order or of order coprime to p (Lemma 1.2.7 and Proposition 1.2.4 (iv)). Therefore there exists $\bar{z}_p \in G_1$ such that $\langle\bar{z}_p\rangle N^{\pi}/N^{\pi}$ is a p-group not normalising R_p^{π}/N^{π}. Moreover $\langle\bar{z}_p\rangle N^{\pi}/N^{\pi}$ normalises R_s^{π}/N^{π} if $s \neq p$ by Proposition 1.2.4 (vi) if R_s^{π}/N^{π} is quasinormal in G_1/N^{π}, and by (4) and (5) if R_s^{π}/N^{π} is not quasinormal in G_1/N^{π}. Hence (8) is proved.

Let $y \in G$. We show that

y normalises N. (9)

If $\langle y\rangle/(\langle y\rangle \cap N) \cong C_{\infty}$, since M^{π}/N^{π} is periodic, if follows that $M^{\pi} \cap \langle y, N\rangle^{\pi} = N^{\pi}$ and therefore $M \cap \langle y,N\rangle = N \lhd \langle y,N\rangle$. Thus, suppose that $|\langle y,N\rangle^{\pi}/N^{\pi}|$ is finite. For each prime number r let

$y_r \in G$ such that $\langle y_r \rangle \leq \langle y \rangle$ and $\langle y_r, N\rangle^\pi/N^\pi$ is the Sylow r-subgroup of $\langle y,N\rangle^\pi/N^\pi$. Since $\langle y,N\rangle$ is the join of the subgroups $\langle y_r,N\rangle$, in order to prove (9) it is sufficient to show that y_r normalises N for each r. Set $\langle \bar{y}_r\rangle = \langle y_r\rangle^\pi$. Set also $R^\pi/N^\pi = \langle R_p^\pi/N^\pi \mid p \in \Pi\rangle$. Since $R^\pi/N^\pi \leq H^\pi/N^\pi$ and the latter is nilpotent, R^π/N^π is the direct product of the R_p^π/N^π's. Again we have to split our investigation in two different cases.

(α) $\bar{y}_r$ normalises R^π. Let $\bar{z}$ be the product of the $\bar{z}_p$'s, where the $\bar{z}_p$'s are the elements of G_1 introduced in (8). Set $\langle z\rangle^\pi = \langle \bar{z}\rangle$ and $\langle t_r\rangle^\pi = \langle \bar{z}\bar{y}_r\rangle$. $\langle \bar{y}_r\rangle N^\pi/N^\pi$ normalises the characteristic subgroups R_p^π/N^π of R^π/N^π for all $p \in \Pi$. Therefore, by definition of $\bar{z}$, R_p^π/N^π is neither normalised by $\langle z,N\rangle^\pi/N^\pi$ nor by $\langle t_r,N\rangle^\pi/N^\pi$ for each $p \in \Pi$. Hence, by (7),

$$(\langle z,N\rangle^\pi/N^\pi) \cap (M^\pi/N^\pi) = 1 = (\langle t_r,N\rangle^\pi/N^\pi) \cap (M^\pi/N^\pi).$$

Therefore $M \cap \langle t_r,N\rangle = N \lhd \langle t_r,N\rangle$ and $M \cap \langle z,N\rangle = N \lhd \langle z,N\rangle$, namely N is normalised by $\langle t_r,z\rangle$. Since $\langle t_r,z\rangle^\pi \geq \langle \bar{z}\bar{y}_r, \bar{z}\rangle \geq \langle \bar{y}_r\rangle$, $\langle y_r\rangle \leq \langle t_r,z\rangle$. Thus y_r normalises N.

(β) $\bar{y}_r$ does not normalise R^π. Then there exists $p \in \Pi$ such that $\langle \bar{y}_r\rangle N^\pi/N^\pi$ does not normalise R_p^π/N^π. By (7) p does not divide $|(M^\pi/N^\pi) \cap (\langle \bar{y}_r\rangle N^\pi/N^\pi)|$. Hence, as $\langle \bar{y}_r\rangle N^\pi/N^\pi$ is an r-group, if $p=r$

$$(M^\pi/N^\pi) \cap (\langle \bar{y}_r\rangle N^\pi/N^\pi) = 1. \tag{10}$$

If $p \neq r$, then, by Proposition 1.2.4 (vi) R_p^π/N^π is not quasinormal in G_1/N^π. Then, by (4), $y_r \in P_{i_r}^\pi$ and, by (5), $r \notin \Pi$. Therefore (10) holds even when $p \neq r$. From (10) it follows that $M^\pi \cap \langle y_r, N\rangle^\pi = N^\pi$. So

$$M \cap \langle y_r, N\rangle = N \lhd \langle y_r, N\rangle .$$

This completes the proof of (8). Since y is an arbitrary element of G, it follows that N is normal in G, contradicting the hypothesis that N is not normal in G. Therefore N, i.e. $H_{\pi,G}$, is normal in G.

In order to complete the proof of Theorem 2.1.1 it remains to show that $H^{\pi,G} \lhd G$. Suppose that this is not the case. Then $H^{\pi,G} > (H^{\pi,G})_G \geq H$. Moreover, by applying what we have just proved to the group G_1, the normal subgroup $(H^\pi)^{G_1}$ of G_1 and the projectivity $\pi^{-1}: G_1 \to G$, it follows that $((H^\pi)^{G_1})_{\pi^{-1},G_1} = ((H^{\pi,G})_G)^\pi \lhd G_1$. Thus, since $H^\pi \leq ((H^{\pi,G})_G)^\pi$, we have

$$(H^\pi)^{G_1} \leq ((H^{\pi,G})_G)^\pi < (H^{\pi,G})^\pi = (H^\pi)^{G_1} ,$$

a contradiction. Theorem 2.1.1 is finally proved.

$\square$

Chapter 3.

On the derived length of a normal subgroup with a core-free projective image.

3.1 Introduction.

In the next chapter we will prove, by exibiting a counterexample (see Theorem 4.1.1), that Theorem 1.1.1 is false if we remove the hypothesis that the group G is finite of odd order. However the subgroup H that we will construct in Theorem 4.1.1 is metabelian. Thus, it was natural to ask whether, removing the hypothesis of G finite of odd order in the statement of Theorem 1.1.1 , H is always metabelian. Unfortunately we still do not have an answer to this question. However, in the present chapter we are able to prove the following

Theorem 3.1.1 . Let G and G_1 be groups, $\pi: G \to G_1$ a projectivity and H a normal subgroup of G such that H^π is core free in G_1. Then H and H^π are soluble group of derived length at most 3.

Here, as a result of Theorem 2.1.1 , the hypothesis that H^π is core-free in G_1 is purely for notational convenience. For, Theorem 2.1.1 implies that π induces a projectivity from $G/H_{\pi,G}$ to $G_1/(H^\pi)_{G_1}$ and Theorem 3.1.1 then says that $H/H_{\pi,G}$ and $H^\pi/(H^\pi)_{G_1}$ are soluble groups of derived length at most 3.

We point out that Theorem 3.1.1 has been obtained after the discovery of the counterexample in Theorem 4.1.1. The proof of Theorem 3.1.1 shows how the problem can be reduced to the case where $G = H\langle a\rangle$ is a finite 2-group with $H \cap \langle a\rangle = 1$. We would like to mention that Theorem 2.1.1 is used in this reduction process. Sections 2 and 3 are then devoted to the study of the structure of G and G_1 in Theorem 3.1.1, assuming that $G = H\langle a\rangle$ is a finite 2-group. At the end of section 2 the proof of Theorem 3.1.1 is derived. Section 4 uses the results in section 3 to improve a theorem by Yakovlev (see Proposition 3.4.1), who showed that the projective image of a soluble group of derived length $\leq n$ is soluble of derived lengh $\leq 4n^3 + 14n^2 - 8n$ (see [25], Theorem 4).

3.2 The abelian case for some finite 2-groups.

In this section we shall give a sufficient condition (Theorem 3.2.3) for H to be abelian whenever H is a normal subgroup of a finite 2-group $G = H\langle a\rangle$, $\pi: G \to G_1$ is a projectivity and H^π is core-free in G_1. We point out that Theorem 3.2.3 is the key result, together with Theorem 1.1.1, in order to obtain the more general Theorem 3.1.1.

In the next chapters we shall often make use of some well-known facts occurring in projectivities of certain finite p-groups. We shall state them in the following lemma. Most of these facts are easy consequences of Lemma 1.2.6 on core-free quasinormal subgroups. However, since the statements do not seem to appear explicitly in the literature, we shall indicate how to derive them from Lemma 1.2.6.

Lemma 3.2.1 contains also a result ((xiii)) which is not an easy consequence of Lemma 1.2.6 . It is due to Menegazzo and it will be extremely useful in the proof of Theorem 3.2.3 and 4.1.3 . Since it is not published, we shall give a proof.

Lemma 3.2.1 . Let G and G_1 be finite p-groups, where p is a prime, $1 \neq H \lhd G$ such that $G = H\langle a\rangle$ and let $\pi : G \to G_1$ be a projectivity such that H^π is core-free in G_1. Set $\langle a_1\rangle = \langle a\rangle^\pi$ and suppose that H has exponent p^r . Then

(i) $H \cap \langle a\rangle = 1$, $H^\pi \cap \langle a_1\rangle = 1$;

(ii) for all $i \geq 0$, $\Omega_i(G) = \Omega_i(H)\,\Omega_i\langle a\rangle$ and $\Omega_i(G_1) = \Omega_i(H^\pi)\Omega_i\langle a_1\rangle$;

(iii) for all $i \geq 0$, $H^\pi\Omega_i(G_1)/\Omega_i(G_1)$ is core-free in $G_1/\Omega_i(G_1)$;

(iv) for all $i \geq 0$ $\Omega_{i+1}(G)/\Omega_i(G)$ and $\Omega_{i+1}(G_1)/\Omega_i(G_1)$ are elementary abelian;

(v) for all $i \geq 0$ $\Omega_{i+2}(G)/\Omega_i(G)$ and $\Omega_{i+2}(G_1)/\Omega_i(G_1)$ have nilpotency class $\leq p-1$;

(vi) if $p=2$, for all $i \geq 0$ $\Omega_{i+3}(G)/\Omega_i(G)$ and $\Omega_{i+3}(G_1)/\Omega_i(G_1)$ have nilpotency class ≤ 2 ;

(vii) for all $i \geq 1$ the map $x \to x^{p^{i-1}}$ is an endomorphism of $\Omega_i(G)$; the same power map is an endomorphism of $\Omega_i(G_1)$;

(viii) if $p=2$, $\Omega_2\langle a\rangle \leq Z(G)$ and $\Omega_2\langle a_1\rangle \leq Z(G_1)$;

(ix) if $p=2$, $|a| \geq 2^{r+2}$ (of course H^{π} has exponent 2^r and $|a_1| = |a|$).

Denote the rank of $\Omega_1(G)$ by $m+1$. Then

(x) if $p=2$ or $m \geq 2$, π restricted to $\Omega_1(G)$ is induced by an isomorphism;

(xi) there is a basis $\{ e_0, e_1, \dots, e_m \}$ of $\Omega_1(G)$ such that $\{e_1,\dots,e_m\}$ is a basis of $\Omega_1(H)$, $\langle e_0\rangle = \Omega_1\langle a\rangle$, $e_1^a = e_1$, $e_i^a = e_{i-1}e_i$, for $2 \leq i \leq m$. Also there exists a basis $\{f_0,\dots,f_m\}$ of $\Omega_1(G_1)$ such that $\langle f_i\rangle = \langle e_i\rangle^{\pi}$ for $0 \leq i \leq m$, $f_i^{a_1} \equiv f_{i-1}f_i \mod \langle f_0,\dots,f_{i-2}\rangle$ for $1 \leq i \leq m$ and, moreover, if $p=2$, $f_2^{a_1} = f_1f_2$ (if $m \geq 2$), $f_3^{a_1} = f_1^{\beta}f_2f_3$, $0 \leq \beta \leq 1$ (if $m \geq 3$) ;

(xii) for all $1 \leq i \leq m$ $\Omega_1(H)$ contains exactly one subgroup of order 2^i normalised by a, namely $\langle e_1,\dots, e_{i-1}, e_i\rangle$.

Similarly, for all $0 \leq i \leq m$, $\Omega_1(G_1)$ contains exactly one subgroup of order 2^{i+1} normalised by a_1, namely $\langle f_0, f_1,\dots,f_{i-1}, f_i\rangle$;

(xiii) if $p=2$, $\Omega_1(G) \leq Z(\Omega_r(G))$ and $\Omega_1(G_1) \leq Z(\Omega_r(G_1))$ ([13]).

Proof. By Theorem 1.2.2 H^{π} is quasinormal in G_1. Hence (i), (ii), (iii) follow immediately from Lemma 1.2.6 (a) and (c).

(iv) For all $i \geq 0$ π induces a projectivity from $G/\Omega_i(G)$ to

$G_1/\Omega_i(G_1)$. Thus, as a consequence of (iii) in order to prove (iv) we may assume $i=0$. Then $\Omega_1(G_1)$ is elementary abelian by Lemma 1.2.6 (b). Consequently $\Omega_1(G)$ is a modular finite p-group of exponent p and therefore it is abelian.

(v). As in (iv) we may assume $i=1$. Then Lemma 1.2.6 (d) shows that $\Omega_2(G_1)$ has class $\leq p-1$. Set $K_m = (\Omega_2(H))^{\pi a_1^m \pi^{-1}}$ for every integer m. Since $\Omega_2(H) \lhd G$, K_m is certainly quasinormal in $\Omega_2(G)$. Let $N_m = (K_m)_{\Omega_2(G)}$. It follows from (ii) that

$\Omega_2(G)/N_m = (K_m/N_m)(\Omega_2\langle a\rangle N_m/N_m)$. . Thus Lemma 1.2.6 (d) applied to the group $\Omega_2(G)/N_m$, implies that $\Omega_2(G)/N_m$ has class $\leq p-1$. But, since $(\Omega_2(H))^{\pi}$ is core-free in G_1, $\bigcap_m K_m = 1$ and (v) follows.

(vi). The proof is analogous to the proof of (v) replacing the Ω_2's with Ω_3's and using (g) instead of (c) in Lemma 1.2.6 .

(vii). We use induction on i. For $i=1$ the statement is clearly true. Therefore assume, by inductive hypothesis, that the statement is true for some $i \geq 1$. By (iii) the hypotheses are preserved in the factor groups $G/\Omega_1(G)$, $G_1/\Omega_1(G_1)$. Also, by (iv),

$\Omega_i(G/\Omega_1(G)) = \Omega_{i+1}(G)/\Omega_i(G)$. Thus, if $x, y \in \Omega_{i+1}(G)$, by the inductive hypothesis we have $(xy)^{p^{i-1}} \equiv x^{p^{i-1}} y^{p^{i-1}}$ mod $\Omega_1(G)$. Moreover $x^{p^{i-1}}, y^{p^{i-1}} \in \Omega_2(G)$, which has class $\leq p-1$ by (v) and therefore it is regular, in the sense of Ph. Hall (see [8], Kapitel III, §10).

Hence, as in addition, by (iv), $(\Omega_2(G))'$ is elementary abelian, we have $(xy)^{p^i} = x^{p^i} y^{p^i}$. The proof for $\Omega_{i+1}(G_1)$ is analogous. Thus the statement is true for $i+1$ and (vii) holds.

(viii). By Lemma 1.2.6 (f), $\Omega_2\langle a_1\rangle \le Z(G_1)$. Set $K_m = (H)^{\pi a_1^m \pi^{-1}}$ and $N_m = (K_m)_G$ for every integer m. K_m is quasinormal in G. Thus, again by Lemma 1.2.6 (f), $[\Omega_2\langle a\rangle, G] \le N_m$ for all m. Since H^π is core-free in G_1, $\bigcap_m N_m = 1$. Therefore $\Omega_2\langle a\rangle \le Z(G)$, as required.

(ix). It follows immediately from Lemma 1.2.6 (e).

(x). It is a particular case of the fundamental Theorem of projective geometry, by considering $\Omega_1(G)$ and $\Omega_1(G_1)$ as vector spaces over a field with p elements (see [1], Theorem 2.2.6).

(xi). Clearly $C_{\Omega_1(H^\pi)}\langle a_1\rangle = 1$. Suppose that $|C_{\Omega_1(H)}\langle a\rangle| > p$. Then there exist two distinct subgroups of H of order p, $\langle v\rangle$ and $\langle w\rangle$, say, such that $\langle v\rangle^\pi$ and $\langle w\rangle^\pi$ are core-free quasinormal subgroups of $\Omega_1(H^\pi)\langle a_1\rangle$. It follows that $\langle v\rangle^\pi \times \langle w\rangle^\pi$ induces a cyclic group of automorphisms on $\langle a_1\rangle$ and so $C_{\Omega_1(H^\pi)}\langle a_1\rangle \ne 1$, a contradiction. Therefore $|C_{\Omega_1(H)}\langle a\rangle| = p$. . Consequently there exists a basis $\{e_1, \dots, e_m\}$ of $\Omega_1(H)$ (considered as a vector space over a field with p elements) such that

$$e_1^a = e_1, \quad e_i^a = e_{i-1}\, e_i \qquad \text{for } i \text{ in the range } 2 \le i \le m.$$

Set $\langle e_0\rangle = \Omega_1\langle a\rangle$. For all i in the range $0 \le i \le m$ we have

$\langle e_0, e_1, \dots, e_{i-1}, e_i\rangle = \Omega_1 \langle a, e_1, \dots, e_i\rangle$. Hence, in particular, $\langle e_0, \dots, e_i\rangle^{\pi}$ is normalised by a_1. Moreover, for $1 \le i \le m$, $e_{i-1} \in \langle e_i, a\rangle$ and therefore $\langle e_{i-1}\rangle^{\pi} \le \langle e_i, a\rangle^{\pi}$. Then, considering the further fact that a_1 does not normalise $\langle e_1\rangle^{\pi}$, it follows, for $0 \le i \le m$, that we can find generators f_i of $\langle e_i\rangle^{\pi}$ such that

$$f_i^{a_1} \equiv f_{i-1} f_i \mod \langle f_0, \dots, f_{i-2}\rangle \qquad 2 \le i \le m$$

and

$$f_1^{a_1} = f_0\, f_1 .$$

Thus, if $m \ge 2$ we have

$$f_2^{a_1} = f_0^{\alpha}\, f_1\, f_2 \qquad 0 \le \alpha \le p-1$$

and, if $m \ge 3$

$$f_3^{a_1} = f_0^{\gamma}\, f_1^{\beta}\, f_2\, f_3 \qquad 0 \le \beta, \gamma \le p-1.$$

In order to complete the proof of (xi) we must show that if $m \ge 2$ and $p=2$ we can choose the e_i's and the f_i's subject to the further condition that $\alpha = \gamma = 0$. To obtain this we replace e_i by $e_{i-2}^{-\gamma+\alpha^2} e_{i-1}^{-\alpha} e_i$ for $i \ge 3$, e_2 by $e_1^{-\alpha} e_2$, f_i by $f_{i-2}^{-\gamma+\alpha^2} f_{i-1}^{-\alpha} f_i$ for $i \ge 3$, and f_2 by $f_1^{-\alpha} f_2$.

By (x) π is induced by an isomorphism. Thus for the new e_i's and f_i's we still have $\langle f_i\rangle = \langle e_i\rangle^{\pi}$ for $0 \le i \le m$ and it is also straightforward to check that all the other required conditions are satisfied.

(xii). It is an immediate consequence of (xi).

(xiii). We show first that

$$[\,\Omega_1(G),\ \Omega_r(G)\,] = 1. \tag{1}$$

Suppose, by way of contradiction, that (1) is false and assume also that $|H|$ is minimal with respect to (1) to be false. Let $\{e_0,\dots,e_m\}$, $\{f_0,\dots,f_m\}$ be bases of $\Omega_1(G)$ and $\Omega_1(G_1)$ respectively as in (xi). It follows from (i) that

$$e_0 \notin H^{\pi a_1 \pi^{-1}} . \tag{2}$$

On the other hand $e_1e_0 \in H^{\pi a_1 \pi^{-1}}$ ((xi)) and so

$$e_1 \notin H^{\pi a_1 \pi^{-1}} . \tag{3}$$

Let $K \leq H$ such that $K^{\pi a_1 \pi^{-1}} = (H^{\pi a_1 \pi^{-1}})_G$. $K^{\pi a_1 \pi^{-1}}$ is normalised by a and does not contain e_1. Therefore, by (xii), $K^{\pi a_1 \pi^{-1}} \cap H = 1$ and it implies that $K^{\pi a_1 \pi^{-1}}$ and its projective image K (via the projectivity $\pi a_1^{-1}\pi^{-1}$: $G \to G$) are cyclic groups. Moreover, as $e_1e_0 \in Z(G)$ ((vii)) and (xii)), $\langle e_1e_0\rangle \leq K^{\pi a_1 \pi^{-1}}$ and $\langle e_1\rangle = \langle e_1e_0\rangle^{\pi a_1^{-1} \pi^{-1}} \leq K$. By Theorem 2.1.1 applied to the projectivity $\pi a_1 \pi^{-1} : G \to G$, K is normal in G. Hence $\pi a_1 \pi^{-1}$ induces a projectivity from G/K to $G/K^{\pi a_1 \pi^{-1}}$ and

$H^{\pi a_1\pi^{-1}}/K^{\pi a_1\pi^{-1}}$ is core-free in $G/K^{\pi a_1\pi^{-1}}$. Therefore the groups G/K, $G/K^{\pi a_1\pi^{-1}}$, the projectivity $\pi a_1\pi^{-1}$ and the subgroup H/K of G/K satisfy the hypotheses of the lemma. Then the minimality of $|H|$ implies that $[\Omega_1(G/K), H/K] = 1$. In particular we have

$$[\Omega_1(G), H] \leq \Omega_1(K) = \langle e_1 \rangle . \tag{4}$$

Consider now $\mho_{r-1}(H)$. It is a non-trivial normal subgroup of G contained in $\Omega_1(H)$. Thus $\mho_{r-1}(H) \geq \langle e_1 \rangle$ by (xii). Also, by (vii), $\mho_{r-1}(H) = \{h^{2^{r-1}} \mid h \in H\}$. Therefore there exists $h \in H$ of order 2^r such that

$$h^{2^{r-1}} = e_1 .$$

Then, by (xi),

$$\Omega_1(\langle h \rangle^{\pi a_1\pi^{-1}}) = \langle e_1 \rangle^{\pi a_1\pi^{-1}} = \langle e_0 e_1 \rangle .$$

In particular

$$\langle h \rangle^{\pi a_1\pi^{-1}} \cap H = \langle 1 \rangle .$$

Since $H^{\pi a_1\pi^{-1}}/(H^{\pi a_1\pi^{-1}} \cap H)$ is cyclic of order at most 2^r and $|\langle h \rangle^{\pi a_1\pi^{-1}}| = 2^r$, it follows that

$$H^{\pi a_1\pi^{-1}} = (H \cap H^{\pi a_1\pi^{-1}})\langle h \rangle^{\pi a_1\pi^{-1}} . \tag{5}$$

We next show that

$$[\Omega_1(G), H^{\pi a_1 \pi^{-1}}] = 1 . \tag{6}$$

To see this we observe that, from (4), $\langle h\rangle$ is normalised by $\Omega_1(G)$. Therefore $\langle h\rangle^{\pi a_1 \pi^{-1}}$ is also normalised by $\Omega_1(G)$ (this follows, for instance, from the fact that $|\langle h,x\rangle : \langle h\rangle| \leq 2$ for all $x \in \Omega_1(G)$ and consequently $|\langle h,x\rangle^{\pi a_1 \pi^{-1}} : \langle h\rangle^{\pi a_1 \pi^{-1}}| \leq 2$).
Thus

$$[\langle h\rangle^{\pi a_1 \pi^{-1}}, \Omega_1(G)] \leq H \cap \langle h\rangle^{\pi a_1 \pi^{-1}} = 1 . \tag{7}$$

Moreover, since $H^{\pi a_1 \pi^{-1}}$ is quasinormal in G, $H^{\pi a_1 \pi^{-1}} \cap H$ is quasinormal in H. Hence, by Proposition 1.2.4 (v), $\Omega_1(H)$ normalises $H^{\pi a_1 \pi^{-1}} \cap H$. Then

$$[\Omega_1(H), H^{\pi a_1 \pi^{-1}} \cap H] \leq \langle e_1\rangle \cap H^{\pi a_1 \pi^{-1}} = 1 , \tag{8}$$

by (3) and (4). Now (6) follows from (5), (7), (8), (ii), and (viii).

Let $\langle b\rangle$ be any subgroup of order 2^r containing e_0. Order considerations and (ii) show that

$$\Omega_r(G) = H^{\pi a_1 \pi^{-1}} \langle b\rangle .$$

Considering the fact that, if $\langle b\rangle$ is normalised by $\Omega_1(G)$, then it is centralised by $\Omega_1(G)$, it follows by (6) that

$$\langle b\rangle \text{ is not normalised by } \Omega_1(G) . \tag{9}$$

Order considerations and (3) give the further decomposition

$$\Omega_r(G) = H^{\pi a_1 \pi^{-1}} \langle h\rangle$$

Therefore, by (4) and (6)

$$[\, \Omega_1(G), \Omega_r(G)] = [\Omega_1(G), \langle b\rangle] = \langle e_1\rangle . \tag{10}$$

Set $\langle h_1\rangle = \langle h\rangle^{\pi}$. Since $\langle h\rangle$ is normalised by $\Omega_1(G)$, $\langle h_1\rangle$ is normalised by $\Omega_1(G_1)$. Suppose that $C_{\Omega_1(G_1)}\langle h_1\rangle \geq \langle f_0, f_1, \dots, f_{m-1}\rangle$. Then $f_m^{a_1} \equiv f_m \bmod C_{\Omega_1(G_1)}\langle h_1\rangle$ and so either f_m centralises both $\langle h_1\rangle$ and $\langle h_1^{a_1}\rangle$ or induces on $\langle h_1\rangle$ and $\langle h_1\rangle^{a_1}$ the same power $1+2^{r-1}$ (because $[f_m, h_1] \in \langle h_1\rangle \cap \Omega_1(G_1) = \langle h_1^{2^{r-1}}\rangle$). In both cases

$$[f_m, h_1 h_1^{a_1}] \in \langle h_1^{2^{r-1}} (h_1^{a_1})^{2^{r-1}}\rangle = \langle f_1 f_1^{a_1}\rangle =$$

$$= \langle f_0\rangle = (h_1 h_1^{a_1})^{2^{r-1}} ,$$

by (vii). Therefore $\langle h_1 h_1^{a_1}\rangle$ is a subgroup of order 2^r containing f_0, normalised by $\Omega_1(G_1)$. It implies that its preimage under π is a subgroup of order 2^r containing e_0, normalised by $\Omega_1(G)$, contradicting (9). Thus $\langle f_0, f_1, \dots, f_{m-1}\rangle \nleq C_{\Omega_1(G_1)}\langle h_1\rangle$ and we can find $x \in \langle e_1, e_2, \dots, e_{m-1}\rangle$ such that $x \notin \langle e_1, e_2, \dots, e_{m-2}\rangle$ and $\langle x\rangle^{\pi} \nsubseteq C_{\Omega_1(G_1)}\langle h_1\rangle$. Set $\langle x_1\rangle = \langle x\rangle^{\pi}$. Then

$$[\, h_1,\, x_1 \,] = f_1 \, . \tag{11}$$

By putting $\langle a^{2^l}\rangle = \langle b\rangle$ in (10), where $\langle a^{2^l}\rangle = \Omega_r\langle a\rangle$, it follows from the action of a on $\Omega_1(G)$ that

$$m = 2^l + 1 \quad \text{and} \quad \Omega_1(G) \cap Z(\Omega_r(G)) = \langle e_0, e_1, \dots, e_{m-1}\rangle . \tag{12}$$

The 2^l elements x^{a^j} $0 \le j \le 2^l - 1$ form a basis of $\langle e_1, \dots, e_{m-1}\rangle$. Therefore

$$C_{\langle a\rangle}(\ \langle x\rangle^{\langle x, a^2\rangle}\ ) = 1 \, . \tag{13}$$

Moreover, by (vii) and (ix), $\Omega_1\langle z a^2\rangle = \langle e_0\rangle$ for all $z \in H$. Hence, recalling also that $x \in Z(H)$ by (12), we have

$$\Omega_1 \langle x,\, a^2\rangle = \Omega_1 \langle x, za^2\rangle = \langle e_0\rangle \times \langle x\rangle^{\langle x, a^2\rangle} \, .$$

Thus, in particular,

$$\Omega_1 \langle x_1,\, a_1^2\rangle = \Omega_1 \langle x_1,\, h_1 a_1^2\rangle \, .$$

and so, by (11),

$$f_1 = [h_1, x_1] = [h_1, x_1]^{a_1^2} = [h_1 a_1^2,\, x_1][a_1^2, x_1] \in \Omega_1\langle x_1, a_1^2\rangle \, .$$

Then $e_1 \in \Omega_1 \langle x,a^2\rangle \cap H = \langle x\rangle^{\langle x,a^2\rangle}$, contradicting (13). With this contradiction the proof of (1) is completed.

It remains to show that

$$[\Omega_1(G_1), \Omega_r(G_1)] = 1 .$$

As a consequence of (1) every subgroup of $\Omega_r(G_1)$ is normalised by $\Omega_1(G_1)$. In other words any element $y \in \Omega_1(G_1)$ induces a power automorphism on $\Omega_r(G_1)$. In particular, for any element ω of $\Omega_r(G_1)$ of order 2^r

$$[y,\omega] \in \langle\omega\rangle \cap \Omega_1(G_1) = \langle\omega^{2^{r-1}}\rangle ,$$

and so either y centralises $\langle\omega\rangle$ or induces on $\langle\omega\rangle$ the power $1 + 2^{r-1}$. Then, considering the fact that $\Omega_r(G_1)$ contains at least two cyclic subgroups of order 2^r intersecting trivially (e.g. $\Omega_r\langle a_1\rangle$ and any cyclic subgroup of H^π of order 2^r) and using (vii), it is not hard to see that the power automorphism induced by y on $\Omega_r(G_1)$ is universal and it is either the identity or the power $1 + 2^{r-1}$. It follows that

$$|\Omega_1(G_1) : \Omega_1(G_1) \cap Z(\Omega_r(G_1)| \leq 2 .$$

Moreover, since $\Omega_1(G_1) \cap Z(\Omega_r(G_1)) \lhd G_1$, by (xii)

$$\Omega_1(G_1) \cap Z(\Omega_r(G_1)) \geq \langle f_0, f_1, \ldots, f_{m-1}\rangle ,$$

where $\{f_0, f_1, \dots, f_m\}$ and $\{e_0, e_1, \dots, e_m\}$ are the usual bases of $\Omega_1(G_1)$ and $\Omega_1(G)$ respectively, as in (xi).

Assume now, by way of contradiction, that f_m induces the power $1+2^{r-1}$ on $\Omega_r(G_1)$. Set $\langle a_1^{2^l}\rangle = \Omega_r\langle a_1\rangle$. Then $[f_m, a_1^{2^l}] = f_0$ gives $m = 2^l$. The 2^l elements $e_m^{a^j}$ $0 \le j \le 2^l-1$ form a basis of $\Omega_1(H)$. Therefore

$$C_{\langle a\rangle}(\langle e_m\rangle^{\langle e_m, a^2\rangle}) = 1 . \tag{14}$$

Moreover, by (vii) and (ix), $\Omega_1\langle a^2 z\rangle = \langle e_0\rangle$ for all $z \in H$. In particular $\langle a^2 z\rangle \cap H = 1$. Therefore

$$\langle e_m\rangle^{\langle e_m, a^2\rangle} = \langle e_m\rangle^{\langle e_m, a^2 z\rangle} = \langle e_m, a^2\rangle \cap H = \langle e_m, a^2 z\rangle \cap H .$$

Then

$$\langle f_m, a_1^2\rangle \cap H^\pi = \langle f_m, a_1^2 z_1\rangle \cap H^\pi , \tag{15}$$

for all $z_1 \in H^\pi$. As we have seen in proving (1), there exists $h_1 \in H^\pi$ such that $h_1^{2^{r-1}} = f_1$. Thus

$$f_1 = [h_1, f_m] = [h_1, f_m]^{a_1^2} = [h_1 a_1^2 , f_m][a_1^2 , f_m] \in \langle f_m, a_1^2\rangle \cap H^\pi,$$

by (15). Therefore $e_1 \in \langle e_m, a^2\rangle \cap H = \langle e_m\rangle^{\langle e_m, a^2\rangle}$, contradicting (14). This completes the proof of (xiii).

$\square$

Before we proceed it will also be convenient to state two well-known results about certain modular p-groups. A proof of the first result (part (a) in the following lemma) can be found in [25] (Lemma 3). The second (part (b)) is due to Menegazzo ([13]) and, as it is not published, for completeness reasons we shall give Menegazzo's proof.

Lemma 3.2.2 . Let G be a finite modular p-group, of exponent p^r, where p is a prime.

(a) If $\exp Z(G) = p^r$, then G is abelian.

(b) If G is not Hamiltonian and $G/\Omega_{r-1}(G)$ is not cyclic, then G contains a characteristic abelian subgroup A such that G/A is cyclic and every automorphism of G induces the identity on G/A.

Proof. (a) is proved in [25], Lemma 3.

(b) Assume that G is non-abelian. By Theorem 1.2.10 $G = N\langle t\rangle$ where N is abelian and t induces on N the power $1+p^\lambda$, $p^\lambda > 2$. By hypothesis N has exponent p^r. Let $A = C_G(N)$. A is abelian, G/A is cyclic and $C_G(A) = A$. We now distinguish two cases.

(i) $N/\Omega_{r-1}(N)$ is not cyclic. Let $a = xt^i$ be an element of A, where $x \in N$, and let α be an automorphism of G. We show that $a^\alpha \in A$. Since $t^i \in Z(G) \leq A$, $a^\alpha \in A$ if and only if $x^\alpha \in A$. As $\mho_{r-1}(N)$ ($\cong N/\Omega_{r-1}(N)$) is non-cyclic there exists an element u in N

of order p^r such that $\langle u\rangle \cap \langle x^\alpha\rangle = 1$. $\langle u\rangle$ and $\langle x^\alpha\rangle$ are both normal subgroups of G, therefore $u^{x^\alpha} = u$ and since x^α induces a power automorphism on N, it follows that x^α induces the identity on N. Thus $x^\alpha \in A$ and so A is characteristic. Moreover t^α induces on N^α the power $1+p^\lambda$ and it induces a power on N as well. These powers coincide on $N \cap N^\alpha$, which has exponent p^r (otherwise $N/\Omega_{r-1}(N)$ would be a quotient of the cyclic group $N/N \cap N^\alpha$), and therefore t^α induces on N the power $1+p^\lambda$. Thus $t^{-1} t^\alpha \in A$, as required.

(ii) $N/\Omega_{r-1}(N)$ is cyclic. This forces t to have order p^r . Moreover, since N has exponent p^r and $\langle t\rangle \cap A = C_{\langle t\rangle}(N)$, it follows that $\langle t\rangle \cap A = \langle t^{p^{r-\lambda}}\rangle$ and therefore $A = N\langle t^{p^{r-\lambda}}\rangle$. Let xt^{ip^μ}, $x \in N, \mu \geq r-\lambda$, be an element of A of order p^r. Then, since $(xt^{ip^\mu})^{p^\lambda} = x^{p^\lambda}$, we have

$$(xt^{ip^\mu})^t = xt^{ip^\mu} x^{p^\lambda} = (xt^{ip^\mu})^{1+p^\lambda} .$$

Thus, as A is generated by elements of order p^r, it follows that t induces on A the power automorphism $1+p^\lambda$. Recalling that the group of power automorphismsof an abelian group is in the centre of the whole automorphism group, in order to complete case (ii) it is sufficient to prove that A is characteristic in G. To show this we shall prove that A concides with the subgroup B of G generated by the cyclic normal subgroups of G of order p^r. Clearly $A \leq B$. Conversely, let $\langle b\rangle$ be a cyclic normal subgroup of G of order p^r. We can write $b = t^{p^v} y$, where $y \in N$ and $v \geq 0$. Suppose $v=0$. Then $G = N\langle b\rangle$ and so, by Remark 1.2.11 $\mho_{r-1}(G) = \mho_{r-1}(N)\, \mho_{r-1}\langle b\rangle$.

Also, by the same remark $\mho_{r-1}(G)$ is non-cyclic as $\mho_{r-1}(G) \cong G/\Omega_{r-1}(G)$. Thus, since $\mho_{r-1}(N)$ is cyclic, $\mho_{r-1}\langle b\rangle \cap \mho_{r-1}(N) = 1$. Therefore there exists $h \in N$ of order p^r such that $\langle h\rangle \cap \langle b\rangle = 1$. Recalling that $\langle b\rangle$ induces a group of power automorphisms on N and is normal in G, it follows that b centralises N and so G is abelian, a contradiction. Hence $\nu \geq 1$ and it implies that $|y| = p^r$ (by Remark 1.2.11, in a modular p-group G, $\mho_i(\Omega_i(G)) = 1$ for all $i \geq 0$). As $\langle b\rangle$ is normal in G we have

$$[b,t] = [t^{p^\nu} y,\ t] = [y,\ t] = y^{p^\lambda} \in \langle b\rangle ,$$

and, moreover,

$$\langle y^{p^\lambda}\rangle = \langle b^{p^\lambda}\rangle = \langle (t^{p^\nu} y)^{p^\lambda}\rangle = \langle t^{p^{\nu+\lambda}} y^\rho\rangle$$

for some integer ρ. It follows that $t^{p^{\nu+\lambda}} \in \langle t\rangle \cap \langle y\rangle = 1$, as we have seen before in proving that $\nu \geq 1$. Therefore $p^r \mid p^{\nu+\lambda}$, i.e. $\nu \geq r-\lambda$ and so, finally, $b \in N\langle t^{p^{r-\lambda}}\rangle = A$, as required.

□

We are now in the position to prove the key result of chapter 3.

Theorem 3.2.3 . Let $G = H\langle a\rangle$ be a finite 2-group, where H is a normal subgroup of G of exponent 2^r, $r \geq 1$, and let π be a projectivity from G to some group G_1 such that H^π is core-free in G_1. If $|H/\Omega_{r-1}(H)| \geq 2^3$, then H is abelian.

Proof. Since $H \neq 1$, by Proposition 1.2.8 (c) G_1 is a finite 2-group..

Set $\langle a_1\rangle = \langle a\rangle^{\pi}$ and let $\{e_0,\dots,e_m\}$, $\{f_0,\dots,f_m\}$ be bases of $\Omega_1(G)$ and $\Omega_1(G_1)$ respectively ($\Omega_1(G)$ and $\Omega_1(G_1)$ are elementary abelian by Lemma 3.2.1 (iv)), chosen as in Lemma 3.2.1 (xi). By Lemma 3.2.1 (vii) $H/\Omega_{r-1}(H) \cong \mho_{r-1}(H) \leq \Omega_1(H)$. Hence $m \geq 3$ and from Lemma 3.2.1 (xii) it follows that

$$\mho_{r-1}(H) \geq \langle e_3\rangle \times \langle e_2\rangle \times \langle e_1\rangle \; . \tag{16}$$

Let

$$Q = H \cap H^{\pi a_1 \pi^{-1}} , \quad Q_1 = H \cap H^{\pi a_1^2 \pi^{-1}} . \tag{17}$$

Since $H \cap \langle a\rangle = 1$ (Lemma 3.2.1, (i)), $H^{\pi a_1 \pi^{-1}} \cap \langle a\rangle = 1$, namely $e_0 \notin H^{\pi a_1 \pi^{-1}}$. Therefore, as $e_1 e_0 \in H^{\pi a_1 \pi^{-1}}$ (Lemma 3.2.1 (x) and (xi)),

$$e_1 \notin H^{\pi a_1 \pi^{-1}} \; ; \tag{18}$$

in particular $e_1 \notin Q$. Thus $Q \cap \langle e_3, e_2, e_1\rangle < \langle e_3, e_2, e_1\rangle$.
On the other hand a simple calculation using Lemma 3.2.1 (x),(xi), shows that $\langle e_2 e_1, e_3 e_2 e_1^{\beta}\rangle \leq Q$. Therefore we have

$$Q \cap \langle e_3, e_2, e_1\rangle = \langle e_2 e_1, e_3 e_2 e_1^{\beta}\rangle \; , \tag{19}$$

and

$$Q \cap Q^{a} \cap \langle e_3, e_2, e_1\rangle = \langle e_3 e_1^{1+\beta}\rangle \; . \tag{20}$$

Similarly $e_0 \notin H^{\pi a_1^2 \pi^{-1}}$ and $e_2 e_0 \in H^{\pi a_1 \pi^{-1}}$ (Lemma 3.2.1 (i),(x),(xi)).

Therefore $e_2 \notin H^{\pi a_1^2 \pi^{-1}}$; in particular $e_2 \notin Q_1$. Again Lemma 3.2.1 (x), (xi) also gives $\langle e_3 e_2^{\beta}, e_1\rangle \leq Q_1$. Hence

$$Q_1 \cap \langle e_3, e_2, e_1\rangle = \langle e_3 e_2^{\beta}, e_1\rangle , \tag{21}$$

and

$$Q \cap Q_1 \cap \langle e_3, e_2, e_1\rangle = \langle e_3 e_2^{\beta} e_1\rangle . \tag{22}$$

The lattices $[H/Q]$, $[H/Q_1]$, $[Q/Q \cap Q^a]$ and $[Q/Q \cap Q_1]$ are chains, since they are isomorphic to sublattices of the chain $[G/H]$ and it implies that

$$\begin{gathered} |H : Q\Omega_{r-1}(H)| \leq 2 , \quad |H : Q_1\Omega_{r-1}(H)| \leq 2 , \\ |H : (Q \cap Q^a)\Omega_{r-1}(H)| \leq 4 , \ |H : (Q \cap Q_1)\Omega_{r-1}(H)| \leq 4. \end{gathered} \tag{23}$$

Moreover we have $Q\Omega_{r-1}(H)/\Omega_{r-1}(H) \cong \mho_{r-1}(Q) < \mho_{r-1}(H) \cong H/\Omega_{r-1}(H)$, by Lemma 3.2.1 (vii), (16) and (19). Therefore, by (23),

$$|\mho_{r-1}(Q)| = |H/\Omega_{r-1}(H)|/2 = |\mho_{r-1}(H)|/2 , \tag{24}$$

and so (24) together with (16) and (19) yields

$$\mho_{r-1}(Q) \cap \langle e_3, e_2, e_1\rangle = \langle e_2 e_1, e_3 e_2 e_1^{\beta}\rangle . \tag{25}$$

In the same way, using (21), we get

$$\mho_{r-1}(Q_1) \cap \langle e_3, e_2, e_1 \rangle = \langle e_3 e_2^{\beta}, e_1 \rangle . \tag{26}$$

Moreover, by Lemma 3.2.1 (vii) and by (16), (19), (20) we have

$$(Q \cap Q^a)\Omega_{r-1}(H)/\Omega_{r-1}(H) \cong \mho_{r-1}(Q \cap Q^a) < \mho_{r-1}(Q) < \mho_{r-1}(H) \cong H/\Omega_{r-1}(H).$$

Therefore, by (23)

$$|\mho_{r-1}(Q \cap Q^a)| = |H/\Omega_{r-1}(H)|/4 = |\mho_{r-1}(H)|/4 , \tag{27}$$

and so (27) together with (16) and (20) yields

$$\mho_{r-1}(Q \cap Q^a) \cap \langle e_3, e_2, e_1 \rangle = \langle e_3 e_1^{1+\beta} \rangle . \tag{28}$$

Similarly, using (22),

$$\mho_{r-1}(Q \cap Q_1) \cap \langle e_3, e_2, e_1 \rangle = \langle e_3 e_2^{\beta} e_1 \rangle . \tag{29}$$

Applying Lemma 3.2.1 (vii), by (25), (26), (28) and (29) it follows that <u>there exist</u> $h_1 \in Q_1$, $h_2 \in Q$, $h_3 \in Q \cap Q_1$, $h \in Q \cap Q^a$ <u>such that</u>

$$h_1^{2^{r-1}} = e_1,\ h_2^{2^{r-1}} = e_2 e_1 ,\quad h_3^{2^{r-1}} = e_3 e_2^{\beta} e_1,\quad h^{2^{r-1}} = e_3 e_1^{1+\beta}. \tag{30}$$

Using Lemma 3.2.1 (vii) we have $e_1 = h_1^{2^{r-1}} = (h_1^{2^{r-1}})^a = (h_1^a)^{2^{r-1}} = (h_1[h_1,a])^{2^{r-1}} =$ $= h_1^{2^{r-1}}[h_1,a]^{2^{r-1}} = e_1[h_1,a]^{2^{r-1}}$. Hence $[h_1,a]^{2^{r-1}} = 1$, namely $[h_1,a_1] = \omega_1 \in \Omega_{r-1}(H)$.

Similarly $e_2 = (e_2e_1)^a = (h_2^{2^{r-1}})^a = (h_2^a)^{2^{r-1}} = (h_2[h_2, a])^{2^{r-1}} =$

$= h_2^{2^{r-1}} [h_2,a]^{2^{r-1}} = e_2e_1 [h_2,a]^{2^{r-1}}$. Therefore $[h_2,a]^{2^{r-1}} = e_1$

and Lemma 3.2.1 (vii) together with (30) imply that $[h_2,a] = h_1\omega_2$

where $\omega_2 \in \Omega_{r-1}(H)$. Finally, in the same way,

$e_3e_2^{1+\beta} e_1^{1+\beta} = (e_3e_2^{\beta} e_1)^a = (h_3^{2^{r-1}})^a = (h_3^a)^{2^{r-1}} = (h_3 [h_3,a])^{2^{r-1}} =$

$= h_3^{2^{r-1}} [h_3,a]^{2^{r-1}} = e_3e_2^{\beta} e_1 [h_3,a]^{2^{r-1}}$. Thus $[h_3,a]^{2^{r-1}} = e_2e_1^{\beta} =$

$= (h_2h_1^{1+\beta})^{2^{r-1}}$ and again Lemma 3.2.1 (vii), together with (30), imply

that $[h_3,a] = h_2h_1^{1+\beta}\omega_3$, where $\omega_3 \in \Omega_{r-1}(H)$. Summarizing, the

following relations hold

$$h_1^a = h_1\omega_1,\ h_2^a = h_2h_1\omega_2,\ h_3^a = h_3h_2h_1^{1+\beta}\omega_3, \text{ where } \omega_i \in \Omega_{r-1}(H) \text{ for } 0 \le i \le 3. \tag{31}$$

Since $\langle h_1\rangle \cap Q = \langle h_2\rangle \cap Q_1 = \langle h_2 h_1\rangle \cap Q_1 = \langle h^a\rangle \cap Q = 1$ and since Q and Q_1 are quasinormal in H, recalling that $[H/Q]$, $[H/Q_1]$, $[Q/Q \cap Q_1]$ and $[Q_1/Q \cap Q_1]$ are chains and using Lemma 3.2.1 (vii), yields

$$\begin{aligned}\Omega_i(H) &= \Omega_i(Q_1)\, \Omega_i\langle h_2h_1\rangle = \Omega_i(Q)\, \Omega_i\langle h^a\rangle = \\ &= \Omega_i(Q \cap Q_1)\, \Omega_i\langle h_2\rangle\Omega_i\langle h_1\rangle.\end{aligned} \tag{32}$$

Write $\langle k_i\rangle = \langle h_1\rangle^{\pi a_1 \pi^{-1}}$. Then, by Lemma 3.2.1 (x)

$$k_1^{2^{r-1}} = e_1\, e_0 \notin Q\,. \tag{33}$$

Therefore, by order considerations, we obtain

$$H^{\pi a_1 \pi^{-1}} = Q\langle k_1\rangle\,. \tag{34}$$

We divide the rest of the proof in some steps.

Step 1. If H is a modular group, then H is abelian.

By (34) it follows that $H^{\pi a_1 \pi^{-1}} = Q\langle q k_1\rangle$ for all $q \in Q$.

Thus $|\langle q k_1\rangle / \langle q k_1\rangle \cap Q| = |\langle k_1\rangle/\langle k_1\rangle \cap Q| = |\langle k_1\rangle| = 2^r = \exp(H^{\pi a_1 \pi^{-1}})$,

and it implies that $\langle q k_1\rangle \cap Q = \langle 1\rangle$. Moreover $\langle q k_1\rangle$ is quasinormal in $H^{\pi a_1 \pi^{-1}}$ ($H^{\pi a_1 \pi^{-1}}$ is now a modular group, since it is the projective image of H via the projectivity $\pi a_1 \pi^{-1}$). Hence, for all $q_1 \in Q$ it follows that $\langle q_1, q\, k_1\rangle \cap Q = \langle q_1\rangle \lhd \langle q_1, q k_1\rangle$.

In particular every subgroup of Q is normal in Q and therefore Q is abelian, since it does not contains subgroups isomorphic to the quaternion group (Lemma 3.2.1 (v)). Thus $\langle h\rangle \le Z(QQ^a) = Z(Q\langle h^a\rangle) = Z(H)$ (by (32)) and this forces H to be abelian (Lemma 3.2.2).

We now use induction on $|H|$. By Lemma 3.2.1 (v) we may and shall assume $r \ge 2$.

Step 2. $\Omega_{r-1}(H)$ is abelian. π induces a projectivity from $\Omega_{r-1}(H)\langle a\rangle$ to $\Omega_{r-1}(H^\pi)\langle a_1\rangle$ and

$$|\Omega_{r-1}(H)/\Omega_{r-2}(H)| \geq | \mho_1(H/\Omega_{r-2}(H))| = | H/\Omega_{r-1}(H)| \geq 8 ,$$

by Lemma 3.2.1 (vii) and (v). Thus, by induction, $\Omega_{r-1}(H)$ is abelian.

Step 3. $H' \leq \Omega_1(H) \cap Z(H)$. Since $H/\Omega_1(H) \cong H\Omega_1(G)/\Omega_1(G)$ we have

$$H/\Omega_{r-1}(H) \cong (H/\Omega_1(H))/\Omega_{r-1}(H/\Omega_1(H)) \cong (H\Omega_1(G)/\Omega_1(G))/\Omega_{r-1}(H\Omega_1(G)/\Omega_1(G)).$$

Hence

$$|(H\Omega_1(G)/\Omega_1(G))/\Omega_{r-1}(H\Omega_1(G)/\Omega_1(G))| \geq 2^3 .$$

Therefore, by Lemma 3.2.1 (iii), we can apply induction and it follows that $H\Omega_1(G)/\Omega_1(G) \cong H/\Omega_1(H)$ is abelian. Lemma 3.2.1 (xiii) completes the proof of step 3 .

Step 4. If $|H/\Omega_{r-1}(H)| > 2^3$, H is abelian. Set

$$K^{\pi a_1 \pi^{-1}} = (H^{\pi a_1 \pi^{-1}})_G .$$

By (18) $e_1 \notin K^{\pi a_1 \pi^{-1}} \cap H$. Since $K^{\pi a_1 \pi^{-1}} \cap H$ is normalised by a , from Lemma 3.2.1 (xii) it follows that $K^{\pi a_1 \pi^{-1}} \cap H = 1$. Therefore $K^{\pi a_1 \pi^{-1}}$ and its projective image K (via the projectivity $\pi a_1^{-1} \pi^{-1} : G \to G$) are cyclic groups. On the other hand $e_1 e_0 \in K^{\pi a_1 \pi^{-1}}$, as $e_1e_0 \in H^{\pi a_1 \pi^{-1}} \cap Z(G)$ (Lemma 3.2.1 (viii), (xii)). Hence $e_1 \in K$, in particular $K \neq 1$. Also K, as the preimage of $(H^{\pi a_1 \pi^{-1}})_G$ under the projectivity $\pi a_1 \pi^{-1} : G \to G$ is normal in G (Theorem 2.1.1). Thus $\pi a_1 \pi^{-1}$ induces a projectivity from G/K to $G/K^{\pi a_1 \pi^{-1}}$ and $H^{\pi a_1 \pi^{-1}}/K^{\pi a_1 \pi^{-1}}$ is core-free in $G/K^{\pi a_1 \pi^{-1}}$. Applying Lemma 3.2.1 (vii), since K is cyclic, gives

$$|(H/K)/\Omega_{r-1}(H/K)| = |\mho_{r-1}(H/K)| \geq |\mho_{r-1}(H)K/K| \geq 2^3 .$$

Therefore, by induction, H/K is abelian and step 3 implies that $H' \leq \Omega_1(K) = \langle e_1 \rangle$. Then, by (30), $\langle h_1 \rangle \lhd H$ and hence $\langle k_1 \rangle$ is quasinormal in $H^{\pi a_1 \pi^{-1}}$. Since $Q \lhd H^{\pi a_1 \pi^{-1}}$ and $\langle k_1 \rangle \cap Q = 1$ ((33)), for every $q \in Q$ we have $\langle q, k_1 \rangle \cap Q = \langle q \rangle \lhd \langle q, k_1 \rangle$, namely $\langle k_1 \rangle$ induces a power automorphism on Q which is now abelian since $Q \cap H' = Q \cap \langle e_1 \rangle = 1$ by (19). Moreover k_1 centralises the group of exponent 4 $Q/\Omega_{r-2}(Q)$ (Lemma 3.2.1 (v)). Therefore, from the locally finite modular p-groups structure theorem (Theorem 1.2.10) it follows that $H^{\pi a_1 \pi^{-1}}$ and its projective image H are modular groups. Finally step 3 forces H to be abelian, proving step 4 .

Therefore we may and shall assume that $|H/\Omega_{r-1}(H)| = 8$. Then, from Lemma 3.2.1 (vii) and from (30) it follows that

$$H = \langle \Omega_{r-1}(H), h_3, h_2, h_1 \rangle \tag{35}$$

Step 5. $\Omega_{r-1}(H)$ normalises $\langle h_1\rangle$ and $\langle h_2\rangle$. From step 3 it follows that $\langle h_1^2, h_2^2\rangle \le Z(H)$ and, by (32), $\Omega_{r-1}(H) \le \langle Q \cap Q_1, h_1^2, h_2^2\rangle$. Hence we have

$$\langle h_1, \Omega_{r-1}(H)\rangle' \le \Omega_1\langle Q \cap Q_1, h_1\rangle \le \Omega_1(Q_1) \tag{36}$$

and

$$\langle h_2, \Omega_{r-1}(H)\rangle' \le \Omega_1\langle Q \cap Q_1, h_2\rangle \le \Omega_1(Q). \tag{37}$$

It is clear from (21) and from the action of a on $\Omega_1(H)$ that $\langle e_1\rangle$ is the unique non trivial subgroup of $\Omega_1(Q_1)$ normalised by a. Moreover, by (31), a normalises $\langle h_1, \Omega_{r-1}(H)\rangle$ and hence a normalises $\langle h_1, \Omega_{r-1}(H)\rangle'$. Therefore, by (36),

$$\langle h_1, \Omega_{r-1}(H)\rangle' \le \langle e_1\rangle \le \langle h_1\rangle , \tag{38}$$

and so $\Omega_{r-1}(H)$ normalises $\langle h_1\rangle$.

By (31), (38) and step 2 we have

$$[h_2,\Omega_{r-1}(H)]^a = [h_2h_1\omega_2,\Omega_{r-1}(H)] \le [h_2,\Omega_{r-1}(H)][h_1,\Omega_{r-1}(H)] \le [h_2,\Omega_{r-1}(H)]\langle e_1\rangle.$$

Thus a normalises $\langle h_2,\Omega_{r-1}(H)\rangle'\langle e_1\rangle$ and, assuming that $\langle h_2,\Omega_{r-1}(H)\rangle'$ is not contained in $\langle e_2, e_1\rangle$, it follows that $\langle h_2,\Omega_{r-1}(H)\rangle'$ contains an element of the form $e_3e_2^{n_2}e_1^{n_1}$. Also (30) implies that a^2 normalises $\langle h_2,\Omega_{r-1}(H)\rangle'$.

Therefore $[e_3 e_2^{n_2} e_1^{n_1}, a^2] = e_1 \in \langle h_2, \Omega_{r-1}(H)\rangle' \le \Omega_1(Q)$, by (37), contradicting (19). Hence

$$\langle h_2, \Omega_{r-1}(H)\rangle' \le \langle e_2, e_1\rangle \cap Q = \langle e_2\, e_1\rangle = \Omega_1\langle h_2\rangle ,$$

and so $\Omega_{r-1}(H)$ normalises $\langle h_2\rangle$. This completes the proof of step 5.

Step 6. $\Omega_{r-1}(H) \le Z(H)$. By step 5 $\langle k_1\rangle$ is quasinormal in $\Omega_{r-1}(H^{\pi a_1 \pi^{-1}})\langle k_1\rangle$. Since $Q \lhd H^{\pi a_1 \pi^{-1}}$ and $\langle k_1\rangle \cap Q = 1$ ((33)), it follows that, for all $q \in \Omega_{r-1}(Q)$, $\langle q, k_1\rangle \cap Q = \langle q\rangle \lhd \langle q, k_1\rangle$. Thus

$$k_1 \text{ induces a power automorphism on } \Omega_{r-1}(Q). \qquad (39)$$

Let α be an integer such that $|a^\alpha| = 2^r$. By (33) and (30) $e_1 e_0 = k_1^{2^{r-1}} = a^{\alpha 2^{r-1}} h_1^{2^{r-1}}$. Then Lemma 3.2.1 (vii) implies that $k_1 = a^\alpha h_1 \omega$ where $\omega \in \Omega_{r-1}(G)$. Thus $[k_1, a] = [a^\alpha h_1 \omega, a] =$ $= [h_1\, \omega, a] = [h_1, a]^\omega [\omega, a]$; also $[h_1, a] \in \Omega_{r-1}(H)$ by (31) and $[\omega, a] \in \Omega_{r-1}(G) \cap G' = \Omega_{r-1}(H)$. Therefore $[k_1, a] \in \Omega_{r-1}(H)$ which is abelian by step 2. Together with (39) this implies that k_1 induces the same universal power on $\Omega_{r-1}(Q)$ and on $\Omega_{r-1}(Q^a)$, and hence it induces a power on $\Omega_{r-1}(Q)\Omega_{r-1}(Q^a)$.
Since $h \in Q$, by (32) we obtain $\Omega_{r-1}(Q)\Omega_{r-1}(Q^a) \ge \Omega_{r-1}(Q)\langle h^2\rangle^a = \Omega_{r-1}(H)$.
Therefore

$$k_1 \quad \text{induces a power automorphism on} \quad \Omega_{r-1}(H) . \tag{40}$$

Let $<k_2> = <h_2>^{\pi a_1^2 \pi^{-1}}$. By (30) and Lemma 3.2.1 (x) and (xi),

$$k_2^{2^{r-1}} = e_2\, e_1\, e_0 = h_2^{2^{r-1}}\, a^{\alpha 2^{r-1}} . \tag{41}$$

Then Lemma 3.2.1 (vii) implies that $k_2 = a^{\alpha} h_2\, \omega'$, $\omega' \in \Omega_{r-1}(G)$. From step 5 it follows that $<k_2>$ is quasinormal in $\Omega_{r-1}(Q_1)<k_2>$. Since $<k_2> \cap Q_1 = 1$ ((21), (41)) and Q_1 is normal in $H^{\pi a_1 \pi^{-1}}$, for every $q_1 \in \Omega_{r-1}(Q_1)$ we have $<q_1, k_2> \cap Q_1 = <q_1> \lhd <q_1, k_2>$. In other words

$$k_2 \quad \text{induces a power automorphism on} \quad \Omega_{r-1}(Q_1). \tag{42}$$

Using Lemma 3.2.1 (ii) we can write $\omega = (a^{\alpha})^{2i} b$, $\omega' = (a^{\alpha})^{2j} c$, where b and c are elements of $\Omega_{r-1}(H)$ and i, j are suitable integers. $H<a^{\alpha}>/\Omega_{r-1}(H)$ is abelian, since $(\Omega_r(H)<a^{\alpha}>)' = (\Omega_r(G))' \leq (\Omega_{r-1}(G) \cap H) = \Omega_{r-1}(H)$. Therefore $k_1 \equiv a^{\alpha+2\alpha i} h_1 \mod \Omega_{r-1}(H)$ and $k_2 \equiv a^{\alpha+2\alpha j} h_2 \mod \Omega_{r-1}(H)$. Moreover there exist odd integers δ, γ such that $k_1^{\delta} \equiv a^{\alpha} h_1 \mod \Omega_{r-1}(H)$ and $k_2^{\gamma} \equiv h_2 a^{-\alpha} \mod \Omega_{r-1}(H)$. It follows that $k_2^{\gamma} k_1^{\delta} = h_2\, h_1 \omega''$, where ω'' is an element of the abelian group $\Omega_{r-1}(H)$ and so, since both k_1 and k_2 induce power automorphisms on $\Omega_{r-1}(Q_1)$,

$$h_2\, h_1 \quad \text{induces a power automorphism on} \quad \Omega_{r-1}(Q_1). \tag{43}$$

Moreover, by (31), we have $[h_2\ h_1,\ \Omega_{r-1}(H)]=[h_2\ h_1\ \omega_2,\ \Omega_{r-1}(H)] =$ $= [h_2,\ \Omega_{r-1}(H)]^a \leq \langle e_2\ e_1\rangle^a = \langle e_2\rangle = \Omega_1\ \langle h_2\ h_1\rangle$. Therefore $\langle h_2\ h_1\rangle$ is normalised by $\Omega_{r-1}(H)$ and consequently by $\Omega_{r-1}(Q_1)$; since $\Omega_1\langle h_2\ h_1\rangle = \langle e_2\rangle \nleq Q_1$ (21), it follows from (43) that $[h_2\ h_1,\ \Omega_{r-1}(Q_1)] = 1$. Thus, using (32), we also have $[h_2\ h_1,\ \Omega_{r-1}(H)] = 1$ and consequently $[h_2,\ \Omega_{r-1}(H)]^a =$ $=[h_2\ h_1\ \omega_2,\ \Omega_{r-1}(H)]=[h_2\ h_1,\ \Omega_{r-1}(H)] = 1$. Therefore

$$\Omega_{r-1}(H) \leq Z(\Omega_{r-1}(H)\langle h_2,\ h_1\rangle)\ . \tag{44}$$

In order to complete the proof of step 6, by (35) it is now sufficient to show that h_3 commutes with $\Omega_{r-1}(H)$.
By (32) $\langle h_3,\ \Omega_{r-1}(H)\rangle \leq \langle Q \cap Q_1,\ h_2^2,\ h_1^2\rangle$ and, since $\langle h_1^2,\ h_2^2\rangle \leq Z(H)$, it follows that $\langle h_3,\ \Omega_{r-1}(H)\rangle' \leq \langle Q \cap Q_1,\ h_2^2,\ h_1^2\rangle' \leq Q \cap Q_1$.
Furthermore, by (31) and (44), $[h_3,\Omega_{r-1}(H)]^a=[h_3 h_2 h_1^{1+\beta}\omega_3,\Omega_{r-1}(H)] =$ $= [h_3,\ \Omega_{r-1}(H)]$. Therefore $\langle h_3,\ \Omega_{r-1}(H)\rangle'$ is normalised by a .
On the other hand $Q \cap Q_1$ does not contain any non trivial subgroup normalised by a ((22) and Lemma 3.2.1 (xii)). Thus $\langle h_3,\Omega_{r-1}(H)\rangle' = 1$ and this concludes the proof of step 6 .

Step 7 (final step). H is abelian.

$H = \Omega_{r-1}(H)\ \langle h_3,\ h_2,\ h_1\rangle$ (35). Thus steps 3 and 6 give

$$H' = \langle [h_2, h_3], [h_2, h_1], [h_3, h_1] \rangle , \tag{45}$$

and H' is an elementary abelian normal subgroup of G of order ≤ 8. Hence, by Lemma 3.2.1 (xii),

$$H' \leq \langle e_3, e_2, e_1 \rangle .$$

By (31) and steps 3 and 6, $[h_2, h_1]^a = [h_2 h_1 \omega_2, h_1 \omega_1] = [h_2, h_1]$. Thus, as a result of Lemma 3.2.1 (xii),

$$[h_2, h_1] \in \langle e_1 \rangle . \tag{46}$$

Furthermore, again by (31) and steps 3 and 6, we obtain $[h_3, h_1]^a = [h_3 h_2 h_1^{1+\beta} \omega_3, h_1 \omega_2] = [h_3, h_1][h_2, h_1]$ and hence, by (46), a normalises the elementary abelian group of order ≤ 4 $\langle [h_3, h_1], e_1 \rangle$. Therefore, by Lemma 3.2.1 (xii), $\langle [h_3, h_1], e_1 \rangle \leq \langle e_2, e_1 \rangle$ and, since $\langle h_3, h_1 \rangle \leq Q_1$, it follows from (21) that

$$[h_3, h_1] \in \langle e_2, e_1 \rangle \cap Q_1 = \langle e_1 \rangle . \tag{47}$$

Thus, by (46) and (47), $|H'| \leq 4$ and so again Lemma 3.2.1 (xii) implies that

$$H' \leq \langle e_2, e_1 \rangle . \tag{48}$$

Since $\langle h_3, h_2 \rangle \leq Q$, from (19) and (30) it follows that

$[h_3, h_2] \in \langle e_2, e_1\rangle \cap Q = \langle e_2e_1\rangle \leq \langle h_2\rangle$, hence

$$\langle h_2\rangle \text{ is normalised by } h_3 . \tag{49}$$

Moreover, by (31) and steps 3 and 6, we have

$$[h_3, h_2]^{a^2} = [h_3 h_1, h_2] = [h_3, h_2][h_1, h_2] \tag{50}$$

Since, by (48), a^2 centralises H', (50) implies that

$$[h_2, h_1] = 1. \tag{51}$$

Therefore, by (35), step 6, (49) and (51), it follows that

$$\langle h_2\rangle \text{ and } \langle h_2\rangle^a \text{ are normal in } H.$$

By (31)

$$[h_3, a] \leq \Omega_{r-1}(H) \langle h_1, h_2\rangle = \Omega_{r-1}(H) \langle h_2, h_2^a\rangle$$

and $\Omega_{r-1}(H) \langle h_1, h_2\rangle$ is abelian by step 6 and (51).
Thus h_3 induces on $\langle h_2\rangle$ and $\langle h_2\rangle^a$ the same power.
Hence h_3 induces a power automorphism on $\Omega_{r-1}(H)\langle h_1, h_2\rangle$.
Since $H/\Omega_{r-2}(H)$ is abelian, by (35) and Theorem 1.2.10 it follows that H is a modular group. Finally step 1 forces H to be abelian. This completes the proof of Theorem 3.2.3 .

□

3.3 The general case .

Before proving Theorem 3.1.1 we obtain some more informations on the structure of groups G and G_1, when $G = H\langle a\rangle$ is a finite 2-group and $G_1 = G^\pi$ for some projectivity $\pi : G \to G_1$ such that H^π is core-free in G_1. Before we start investigating on 2-groups we state and prove an unpublished useful result on projectivities of finite p-groups, due to Menegazzo.

Theorem 3.3.1 (Menegazzo [13]). Let G and G_1 be finite p-groups $\pi : G \to G_1$ a projectivity, H a normal abelian subgroup of G such that $G = H\langle a\rangle$ and H^π is core-free in G_1. Then

(a) $H^{\pi,G}$ is a modular p-group,

and

(b) G^π is metabelian.

Proof. Write $p^r = \exp H$ $(r \geq 1)$, $\langle a_1\rangle = \langle a\rangle^\pi$, $\langle a^\beta\rangle = \Omega_r\langle a\rangle$, and let $\{e_0, e_1,\dots,e_m\}$, $\{f_0, f_1,\dots,f_m\}$ be bases of $\Omega_1(G)$ and $\Omega_1(G_1)$ respectively chosen as in Lemma 3.2.1 (xi). In order to prove (a) we show first that

$$\Omega_r\langle a\rangle \ (= \langle a^\beta\rangle) \text{ induces a group of power automorphisms on } H. \quad (52)$$

This is obvious if H is cyclic. Then, suppose H non cyclic and write $s = \min\{i \mid i \in \mathbb{N}$ and $H/\Omega_i(H)$ is cyclic $\}$. $s \geq 1$, as H is not cyclic. By a familiar argument, using Lemma 3.2.1 (xii), there exists

$h \in H$ such that $h^{p^{r-1}} = e_1$. Then, by the choice of the f_i's and e_i's ,

$$\Omega_1(\langle h\rangle^{\pi a_1 \pi^{-1}}) = \langle e_1^{\gamma}\, e_0^{\delta}\rangle \tag{53}$$

where $1 \le \gamma \le p-1$, $1 \le \delta \le p-1$. Therefore, using Lemma 3.2.1 (vii) and (ii), it follows that

$$\langle h\rangle^{\pi a_1 \pi^{-1}} = \langle a^{\beta} h'\rangle \ ,$$

for some $h' \in H$. Set $Q = H \cap H^{\pi a_1 \pi^{-1}}$. $H^{\pi a_1 \pi^{-1}}/Q$ is cyclic of order at most p^r , and $\langle a^{\beta} h'\rangle \cap Q = 1$, by (53). Hence

$$H^{\pi a_1 \pi^{-1}} = Q\langle a^{\beta} h'\rangle \ . \tag{54}$$

Since $H^{\pi a_1 \pi^{-1}}$ is a modular p-group (as the image of the abelian group H under the projectivity $\pi a_1 \pi^{-1}$) and $\langle a^{\beta} h'\rangle \cap Q = 1$ (by (53)), we have

$$\langle q, a^{\beta} h'\rangle \cap Q = \langle q\rangle \trianglelefteq \langle q, a^{\beta} h'\rangle \ .$$

for all $q \in Q$.
In other words $a^{\beta}h'$, and therefore also a^{β}, induce a power automorphism on Q. It follows that $a^{\beta} = (a^{\beta})^a$ induces a power automorphism on Q^a and, furthermore, a^{β} induces the same power on Q and Q^a. Thus a^{β} induces a power automorphism on QQ^a. Since $H \cap \langle a\rangle = 1$ (Lemma 3.2.1(i)), (53) shows that $e_1 \notin H^{\pi a_1 \pi^{-1}}$, in particular

$$e_1 \notin Q \ . \tag{55}$$

Then, as $|H^{\pi a_1 \pi^{-1}}| = |H|$, (54) and order considerations show that

$$H = Q \times \langle h \rangle \qquad (56)$$

In particular H/Q is cyclic, and it implies that $Q^G = QQ^a$. Moreover, by (55) and Lemma 3.2.1 (xii), Q is core-free in G. Therefore, as $2^s = \exp Q$, we have $|Q^G| \geq 2^s|Q| = |Q\,\Omega_s\langle h\rangle| = |\Omega_s(H)|$. Hence

$$QQ^a = \Omega_s(H) ,$$

and we have shown that a^β induces a power automorphism on $\Omega_s(H)$. If $s=r$, a^β induces a power automorphism on H, as required. Suppose $s < r$. π induces a projectivity from $G/\Omega_1(G)$ to $G_1/\Omega_1(G_1)$ and $H^\pi\Omega_1(G_1)/\Omega_1(G_1)$ is core-free in $G_1/\Omega_1(G_1)$ (Lemma 3.2.1 (iii)). Therefore, using induction on $|H|$, we may assume that $\Omega_{r-1}(\langle a\rangle\Omega_1(G)/\Omega_1(G))$ induces a group of power automorphisms on $H\Omega_1(G)/\Omega_1(G)$, namely that a^β induces a power automorphism on $H/\Omega_1(H)$.

Suppose then that a^β acts as the power λ on $H/\Omega_1(H)$, and as the power μ on $\Omega_s(H)$.
Therefore we have $h^{a^\beta} = h^\lambda x$ where $x \in \Omega_1(H)$, $(h^{p^{r-s}})^{a^\beta} = h^{p^{r-s}\mu} = h^{p^{r-s}\lambda}$. Thus $\lambda \equiv \mu \bmod p^s$. $\langle x\rangle$ is normalised by a^β (because $\Omega_1(H) \leq \Omega_s(H)$) and, since $\exp Q = p^s$, by (56) it follows that a^β acts as the power λ on $H/\langle x\rangle$. Suppose first that $x \in \langle h\rangle$. Then $x = h^{\nu p^{r-1}}$ for some integer ν. Set $\lambda' = \lambda + \nu p^{r-1}$; as before for λ, we have $\lambda' \equiv \mu \bmod p^s$. For all $y \in H$ we can write $y = h^i z$ for some integer i and some $z \in Q$.

Then $y^{a^\beta} = (h^i z)^{a^\beta} = h^{i\lambda'} z^\mu = (h^i z)^{\lambda'} = y^{\lambda'}$, namely a^β acts as the power λ' on H. On the other hand, if $x \notin \langle h\rangle$, $\langle x\rangle \cap \langle x\rangle^a = 1$ by Lemma 3.2.1 (xii). Also, a^β acts as the power λ both on $H/\langle x\rangle$ and $H/\langle x\rangle^a$. In particular $h^{a^\beta} = h^\lambda x$ yields $h^{a^\beta}\langle x\rangle^a = (h\langle x\rangle^a)^\lambda =$ $= h^\lambda x\langle x\rangle^a$, and so $x \in \langle x\rangle^a$, a contradiction. This completes the proof of (52). Using the decomposition $\Omega_r(G) = H\Omega_r\langle a\rangle$, (52) guarantees the modularity of $\Omega_r(G)$ if $p\neq 2$, by virtue of Theorem 1.2.10. On the other hand, if $p=2$, Lemma 3.2.1 (v) shows that $\Omega_r(G)/\Omega_{r-2}(G)$ is abelian, and therefore $[H, a^\beta] \leq \Omega_{r-2}(G) \cap H = \Omega_{r-2}(H)$ (since $\Omega_{r-2}(G)$ has exponent at most 2^{r-2} by Lemma 3.2.1 (iv)). This shows, that a^β induces on H a power $\equiv 1 \bmod 4$, and therefore, again by Theorem 1.2.10, $\Omega_r(G)$ is modular. Since $H^{\pi,G}$ is clearly contained in $\Omega_r(G)$, (a) follows. As far as (b) is concerned, observe that $\Omega_r(G_1)$ is a modular non Hamiltonian (by Lemma 3.2.1 (v)) p-group, and $\Omega_r(G_1)/\Omega_{r-1}(G_1)$ is non-cyclic (as $a^{2^\beta} \notin \langle h,\Omega_{r-1}(G)\rangle$ by Lemma 3.2.1 (vii)). Then, as a result of Lemma 3.2.2 (b), $\Omega_r(G_1)$ contains an abelian subgroup A normal in G such that $\Omega_r(G_1)/A \leq Z(G_1/A)$. Since $G_1/\Omega_r(G_1)$ is cyclic, (b) follows.

□

The following result is due to Yakovlev ([25], Lemma 6).

Lemma 3.3.2 . In the hypothesis of Lemma 3.2.1, if B is quasinormal in G and $B \leq H$, then $B \lhd G$.

Remark 3.3.3 . In what follows we need to know that a projective image of a metacyclic 2-group G is still metacyclic. This immediately follows, if $|G| \geq 2^5$, from this result of Blackburn ([8] , Satz 1.1.3 , Kapitel 11) .

Proposition (Blackburn). Let $|G| = 2^n$ with $n \geq 5$. Suppose that, for some integer r such that $5 \leq r \leq n$, every subgroup of G of order 2^{r-1} and 2^r can be generated by two elements. Then G is metacyclic.

On the other hand, if $|G| \leq 2^4$, a direct exam of the few possible cases completes the proof.

□

The situation described in the following lemma is complementary to the one described in Theorem 3.2.3 .

Lemma 3.3.4 . Let $G = H\langle a\rangle$ be a finite 2-group, where H is non trivial normal subgroup of G and let π be a projectivity from G to some group G_1 such that H^π is core-free in G_1. Suppose that $|\Omega_1(H)| \leq 4$. Then

(a) H and H^π are metacyclic modular non Hamiltonian groups, and

(b) G_1 has derived length ≤ 4 .

Proof. Since $H \neq 1$, from Proposition 1.2.8 (c) it follows that G_1 is a 2-group. We immediately observe that, by Lemma 3.2.1 (v), H and H^π are not Hamiltonian. Suppose now first that $\Omega_1(H)$ is cyclic.

Then H, and consequently H^{π} are cyclic groups and also G_1 is metabelian by Ito's Theorem (see [8], Kapitel VI, Satz 4.4). Therefore we may assume that $\Omega_1(H)$ is non-cyclic. Set $\langle a_1\rangle = \langle a\rangle^{\pi}$ and let $\{e_1, e_2\}$, $\{f_1, f_2\}$ be bases of $\Omega_1(H)$ and $\Omega_1(H^{\pi})$ respectively as in Lemma 3.2.1 (xi). Set also $Q = H^{\pi a_1 \pi^{-1}} \cap H$. The same argument used in proving (19) in Theorem 3.2.3 shows that $Q \cap \langle e_1, e_2\rangle = \langle e_1 e_2\rangle$. Thus Q is cyclic. Q is also normal in $H^{\pi a_1 \pi^{-1}}$ and $H^{\pi a_1 \pi^{-1}}/Q$ is cyclic, since $H^{\pi a_1 \pi^{-1}}/Q \cong HH^{\pi a_1 \pi^{-1}}/H \leq G/H$. Therefore $H^{\pi a_1 \pi^{-1}}$ is metacyclic and consequently, by Remark 3.3.3 , its projective image H is also metacyclic. In order to complete the proof of (a) it remains to show that H and H^{π} are modular groups. To show this we observe first that, since $\Omega_2(H)$ is abelian and metacyclic, $|\Omega_2(H)| \leq 16$, and therefore a^4 centralises $\Omega_2(Q) \leq \Omega_2(H)$. Thus $\Omega_2(Q) \lhd H^{\pi a_1 \pi^{-1}}\langle a^4\rangle$. As a consequence of Lemma 3.2.1 (ii) and (ix), $H^{\pi}\langle a_1^4\rangle \lhd G_1$. Hence $H^{\pi a_1 \pi^{-1}}\langle a^4\rangle = H\langle a^4\rangle$ and it follows that $\Omega_2(Q) \lhd H$. By Lemma 3.2.1 (xii), $\Omega_1(H') \leq \langle e_1\rangle$ and therefore $H' \cap Q = 1$. Thus $\Omega_2(Q) \leq Z(H\langle a^4\rangle) = Z(H^{\pi a_1 \pi^{-1}}\langle a^4\rangle)$. In particular $\Omega_2(Q) \leq Z(H^{\pi a_1 \pi^{-1}})$ and, by virtue of Theorem 1.2.10, this is sufficient to guarantee the modularity of $H^{\pi a_1 \pi^{-1}}$ and hence of H and H^{π} .

It remains to prove (b). Set $X^{\pi} = ((H')^{\pi})^{H^{\pi}}$ and let $|H'| = 2^s$, say. We show first that X is an abelian normal subgroup of G. By Lemma 3.3.2, $(H')^{\pi h_1 \pi^{-1}} \lhd G$ for all $h_1 \in H^{\pi}$. Hence X is the join of cyclic normal subgroups of G of order 2^s. Therefore G', and consequently H', centralise X. Moreover X has exponent $2^s = |H'|$, and then, since it is metacyclic, X/H' is cyclic. It follows that X is abelian.

Consider now the group $G/\Omega_s(G)$. π induces a projectivity from $G/\Omega_s(G)$ to $G_1/\Omega_s(G_1)$, $H^\pi\Omega_s(G_1)/\Omega_s(G_1)$ is core-free in $G_1/\Omega_s(G_1)$ (Lemma 3.2.1 (iii)), and $H\Omega_s(G)/\Omega_s(G)$ is abelian, since $H' \leq \Omega_s(G)$. Thus, by Theorem 3.3.1 (b), $G_1/\Omega_s(G_1)$ is a metabelian group. In other words

$$G_1^{(2)} \leq \Omega_s(G_1) \tag{57}$$

and (b) is proved if $s = 0$. Suppose $s > 0$.

We now shift our attention on the group $X\langle a\rangle = Y$ (say). π induces a projectivity from Y to Y^π, X^π is core-free in Y^π (since $1 = (H^\pi)_{G_1} \geq (X^\pi)_{G_1} = (X^\pi)_{H^\pi\langle a_1\rangle} = (X^\pi)_{Y^\pi}$) and X is abelian. Therefore, by Theorem 3.3.1 (a), $(X^\pi)^{Y^\pi}$ is a modular group, i.e.

$$(X^\pi)^{G_1} \text{ is a modular group,} \tag{58}$$

as $(X^\pi)^{Y^\pi} = (X^\pi)^{H^\pi\langle a_1\rangle} = (X^\pi)^{G_1}$. Moreover $X^\pi\Omega_{s-1}(Y^\pi)/\Omega_{s-1}(Y^\pi)$ is core-free in $Y^\pi/\Omega_{s-1}(Y^\pi)$ (Lemma 3.2.1 (iii)) and is non trivial, since $\Omega_{s-1}(Y^\pi)$ has exponent 2^{s-1} (Lemma 3.2.1 (iv)). This implies that $(X^\pi)^{Y^\pi}\Omega_{s-1}(Y^\pi)/\Omega_{s-1}(Y^\pi)$ is non-cyclic and so $(X^\pi)^{Y^\pi}/\Omega_{s-1}((X^\pi)^{Y^\pi})$ is non-cyclic. But $(X^\pi)^{Y^\pi} = (X^\pi)^{G_1}$ and so we have in fact shown that

$$(X^{\pi})^{G_1}/\Omega_{s-1}((X^{\pi})^{G_1}) \text{ is non-cyclic .} \tag{59}$$

We next show that

$$\Omega_s(G_1)/(X^{\pi})^{G_1} \text{ is cyclic.} \tag{60}$$

To see that, we observe that $(H')^{\pi}$ is a core-free cyclic quasinormal subgroup of $\langle (H')^{\pi}, a_1\rangle$. Thus, by Lemma 1.2.6 (c)

$$|(H')^{\pi}(H')^{\pi a_1}| = |(H')^{\pi}\Omega_s\langle a_1\rangle| = |\Omega_s\langle (H')^{\pi}, a_1\rangle| .$$

It follows that $(H')^{\pi}(H')^{\pi a_1}$, and hence also $(X^{\pi})^{G_1}$, contain $\Omega_s\langle a_1\rangle$. Moreover, since $\Omega_s(H)$ is a metacyclic group, as H is, $\Omega_s(H)/H'$ is cyclic. Also $((X^{\pi})^{G_1})^{\pi^{-1}}$ contains $\Omega_s\langle a\rangle$ and H' and, by Lemma 3.2.1 (ii), we have $\Omega_s(G) = \Omega_s(H)\Omega_s\langle a\rangle$. Therefore $\Omega_s(G)/((X^{\pi})^{G_1})^{\pi^{-1}}$ is cyclic and so $\Omega_s(G_1)/(X^{\pi})^{G_1}$ is also cyclic, as required.

Since $(X^{\pi})^{G_1}$ is not Hamiltonian (Lemma 3.2.1 (v)), by (58), (59) and Lemma 3.2.2 (b) , $(X^{\pi})^{G_1}$ possesses a characteristic abelian subgroup A such that $(X^{\pi})^{G_1}/A$ is cyclic and every automorphism of $(X^{\pi})^{G_1}$ induces the identity on $(X^{\pi})^{G_1}/A$. Therefore, by (60), $\Omega_s(G_1)$ is metabelian. Thus, by (57), G_1 has derived length ≤ 4 , and (b) is proved.

□

Combining Theorem 3.2.3 and Lemma 3.3.4 yields:

Theorem 3.3.5 . Let $G = H\langle a\rangle$ be a finite 2-group, where H is a normal subgroup of G and let π be a projectivity from G to some group G_1 such that H^{π} is core-free in G_1. Then

(a) There exists a natural number r such that $\Omega_r(H)$ is abelian and $H/\Omega_r(H)$ is a metacyclic modular non-Hamiltonian group. In particular H has derived length at most 3;

(b) H^{π} has derived length at most 3 ;

(c) G_1 has derived length at most 6.

Proof. If $H = 1$ (a), (b) and (c) trivially hold. Therefore assume $H \neq 1$. Then G_1 is a finite 2-group by Proposition 1.2.8 (c). Let $r = \min\{n \in \mathbb{N} \,|\, |\Omega_1(H/\Omega_n(H)| \leq 4\}$. π induces a projectivity from $G/\Omega_r(G)$ to $G_1/\Omega_r(G_1)$ and $H^{\pi}\Omega_r(G_1)/\Omega_r(G_1)$ is core-free in $G_1/\Omega_r(G_1)$ (Lemma 3.2.1 (iii)). Moreover $H\Omega_r(G)/\Omega_r(G) \cong H/\Omega_r(H)$, as $\Omega_r(G)$ has exponent 2^r (Lemma 3.2.1 (v)). Thus, by Lemma 3.3.4, $H/\Omega_r(H)$ is a metacyclic modular non Hamiltonian group. Hence (a) is proved if $r = 0$. Suppose then $r > 0$. π induces a projectivity from $\Omega_r(H)\langle a\rangle$ to $\Omega_r(H^{\pi})\langle a\rangle^{\pi}$ and

$$|\Omega_r(H)/\Omega_{r-1}(H)| = |\Omega_1(H/\Omega_{r-1}(H)| \geq 8$$

by definition of r. Therefore Theorem 3.2.3 applied to the group $\Omega_r(H)\langle a\rangle$ shows that $\Omega_r(H)$ is abelian. This proves (a) .

By Remark 3.3.3 $H^{\pi}/\Omega_r(H^{\pi})$ is metacyclic. Therefore (b) holds if $r=0$. Assume $r>0$. Then $\Omega_r(H^{\pi})/\Omega_{r-1}(H^{\pi})$ is non-cyclic by definition of r, and $\Omega_r(H^{\pi})$ is a modular 2-group, since $\Omega_r(H)$ is abelian.

Thus, by Lemma 3.2.2 $\Omega_r(H^{\pi})$ contains a characteristic abelian subgroup A such that $\Omega_r(H^{\pi})/A$ is cyclic and every automorphism of $\Omega_r(H^{\pi})$ induces the identity on $\Omega_r(H^{\pi})/A$. Hence, since $(H^{\pi})'\Omega_r(H^{\pi})/\Omega_r(H^{\pi})$ is cyclic, it follows that $(H^{\pi})^{(2)} \leq A$. Therefore $(H^{\pi})^{(3)} = 1$ and (b) follows.

In order to show (c) we observe that π induces a projectivity from $G/\Omega_r(G)$ to $G_1/\Omega_r(G_1)$ and $H\Omega_r(G)/\Omega_r(G) \cong H/\Omega_r(H)$. Thus

$$|\Omega_1(H\Omega_r(G)/\Omega_r(G))| = |\Omega_1(H/\Omega_r(H))| \leq 4$$

by the choice of r. Applying Lemma 3.3.4 to the groups $G/\Omega_r(G)$ and $G_1/\Omega_r(G_1)$ it follows that $G_1/\Omega_r(G_1)$ has derived length at most 4. Moreover, since $\Omega_r(H)$ is abelian, Theorem 3.3.1 (a) applied to the groups $\Omega_r(H)\langle a\rangle$ and $\Omega_r(H^{\pi})\langle a\rangle^{\pi}$ shows that $\Omega_r(\Omega_r(H^{\pi})\langle a\rangle^{\pi}) = (\Omega_r(H^{\pi}))\langle\Omega_r(H), a\rangle^{\pi}$ is a modular group, i.e.

$$\Omega_r(G_1) \text{ is a modular group}$$

by Lemma 3.2.1 (ii). In particular $\Omega_r(G_1)$ is metabelian. Therefore G_1 has derived length ≤ 6. This completes the proof of Theorem 3.3.5 .

$\square$

In order to prove Theorem 3.1.1 we need the following result, due to R. Schmidt ([18], Lemmas 2 and 3).

Lemma (Schmidt) 3.3.6 . Let M be a Dedekind subgroup of the finite group G and suppose that the lattice $[G/M]$ is a chain. Then there are primes p,q such that either G/M_G is a p-group or M is maximal in G and G/M_G is non abelian of order pq.

The following remark, due to Menegazzo ([12], Corollary), will also be useful to us.

Remark 3.3.7 . Let M be a Dedekind subgroup of the group G. Then $M_G = \bigcap_{x \in S} M_{\langle M,x\rangle}$, where

$$S = \{x \in G \mid \langle x\rangle/\langle x\rangle \cap M \text{ is infinite cyclic or has prime power order}\}.$$

We conclude the present section with the proof of Theorem 3.1.1 .

Proof. Denote by S the set $\{x \in G \mid \langle x\rangle/\langle x\rangle \cap H$ has prime power order$\}$. Since $\langle x\rangle^{\pi}/\langle x\rangle^{\pi} \cap H^{\pi}$ has prime power order if and only if $\langle x\rangle/\langle x\rangle \cap H$ has prime power order and it is infinite cyclic if and only if $\langle x,H\rangle/H$ is infinite cyclic (see Proposition 1.2.8 (a)), by Corollary 1.1.3 and Remark 3.3.7 it follows that

$$1 = \bigcap_{x \in S} H^{\pi}_{\langle H,x\rangle^{\pi}} .$$

Also, as a result of Theorem 2.1.1, $H_{\pi,\langle H,x\rangle} \lhd \langle H,x\rangle$ and therefore, in order to prove the theorem, we may assume that G/H is a cyclic p-group.

Hence $|G^{\pi} : H^{\pi}| < \infty$ (Theorem 1.1.2) and therefore G and G^{π} are now finite groups. Moreover, since $[G^{\pi}/H^{\pi}]$ is a chain, excluding the trivial cases $H=1$ or H of prime order, by Lemma 3.3.6 it follows that G_1 is a non abelian q-group for some prime q. Proposition 1.2.8 (c) implies that G is also a q-group and therefore $q=p$. It p is odd then H is abelian (Theorem 1.1.1) and so H^{π} is metabelian (Proposition 1.2.8 (d)). If $p=2$ then Theorem 3.3.5 (a) and (b) applies. We have finally proved Theorem 3.1.1 .

□

3.4 A bound for the derived length of a projective image of a soluble group with given derived length.

In [3] (Problem 40) the following question was posed: If G is a soluble group and π is a projectivity from G to some group G_1, is G_1 also soluble? The answer, for G finite, was obtained by Suzuki ([23], Theorem 12) and Zappa ([28]). The general answer was given by Yakovlev ([25]), who also gave a bound for the derived length of G_1 in terms of the one of G (namely $4n^3+14n^2-8n$ if n is the derived length of G). In the following proposition, using the results previously obtained, we are able to improve Yakovlev's bound.

Proposition 3.4.1. Let G and G_1 be groups, $\pi : G \to G_1$ a projectivity and suppose that G is soluble of derived length $\leq n$. Then G_1 is soluble of derived length $\leq 6n - 4$.

Proof. Clearly we may assume that G is finitely generated. We argue by induction on n. If $n=1$ then G_1 is metabelian by Proposition 1.2.8 (d). Assume $n > 1$. Then, by induction, $(G')^{\pi}$ has derived length at most $6(n-1)-4$. G/G' is a finitely generated abelian group, therefore $G/G' = \langle c_1\rangle G'/G' \times,\ldots,\times \langle c_t\rangle G'/G'$, for suitable $c_i \in G$, $1 \le i \le t$, such that $\langle c_i\rangle/\langle c_i\rangle \cap G'$ is infinite cyclic or has prime power order. Set $H_i = \langle G', c_1,\ldots,c_{i-1},c_{i+1},\ldots,c_t\rangle$ for $1 \le i \le t$. Since $\bigcap_{1\le i\le t} H_i^{\pi} \le G_1'$, in order to prove the statement it is sufficient to show that $G_1^{(6)} \le H_i^{\pi}$ for $1 \le i \le t$. Choose an i in this range. Clearly $G_1 = \langle H_i, c_i\rangle^{\pi}$. Hence, if $\langle c_i\rangle/\langle c_i\rangle \cap G'$ is infinite cyclic, from Corollary 1.1.3 it follows that $H_i^{\pi} \lhd G_1$. Thus, in this case, $G_1' \lhd H_i^{\pi}$. So, suppose that $|\langle c_i\rangle/\langle c_i\rangle \cap G'|$ is a prime power. Then $|G_1 : H_i^{\pi}| < \infty$ (Theorem 1.1.2) and the lattice $[G_1/H_i^{\pi}]$ is a chain. Therefore, if H_i^{π} is not normal in G_1 (the case $H_i^{\pi} \lhd G_1$ is trivial since in that case $G_1' \le H_i^{\pi}$), since H_i^{π} is a Dedekind subgroup of G_1, according to Lemma 3.3.6 we have the following two possibilities:

(a) $G_1/(H_i^{\pi})_{G_1}$ is a non abelian group of order pq, where p and q are prime numbers. In particular $G_1/(H_i^{\pi})_{G_1}$ is metabelian and so $G_1^{(2)} \le H_i^{\pi}$;

(b) $G_1/(H_i^{\pi})_{G_1}$ is a (finite) non-abelian p-group for some prime number p. Set $N_i^{\pi} = (H_i^{\pi})_{G_1}$. By Theorem 2.1.1 N_i is normal in G. Therefore π induces a projectivity from G/N_i to G_1/N_i^{π} and the latter is a finite p-group. Proposition 1.2.8 (c) implies that G/N_i is also a finite p-group. If p is odd then H_i/N_i is abelian (Theorem 1.1.1). Thus, recalling that $G = \langle H_i, c_i \rangle$, from Theorem 3.3.1 (b) and Theorem 3.3.5 (c), it follows that G_1/N_i^{π} is metabelian if p is odd and it has derived length at most 6 if $p=2$. Therefore, in any case, we have $G_1^{(6)} \leq H_i^{\pi}$ and this proves Proposition 3.4.1 .

□

Remark 3.4.2 . The bound obtained in Proposition 3.4.1 almost certainly is <u>not</u> the best possible. Indeed no example is known where G_1 (in the notation of Proposition 3.4.1) has derived length $> n + 1$. However, with the present methods it seems difficult to obtain the best possible bound.

Chapter 4 .

A non abelian normal subgroup with a core-free projective image.

4.1 Introduction and statements of the main results.

In [12] Menegazzo left open the question of whether the hypothesis that G has odd order in the statement of Theorem 1.1.1 is necessary. The main purpose of this chapter is to show that this is in fact the case.

Theorem 4.1.1 . There are finite 2-groups G, G_1, a normal subgroup H of G and a projectivity $\pi: G \to G_1$ such that H^π is core-free in G_1 and H is not abelian.

The groups G and G_1 which we construct in order to prove Theorem 4.1.1 have order 2^{13} and the normal subgroup H has order 2^7. Not surprisingly for groups of this order it has not been easy to establish the existence of a projectivity π from G to G_1. Therefore it is natural to ask if there are smaller and less complicated examples, which would simplify the problem of finding π and proving that it _is_ a projectivity. In fact we have been able to prove that all examples G and G_1 contain sections of order 2^{13} and H always has a (non abelian) quotient of order 2^7. Again this has not been an easy exercise, but we could not reasonably expect these facts to be accepted without proof. Theorems 4.1.2 and 4.1.3 are concerned with these minimality questions. Also, the subgroup H of the group G which we construct has derived length 2. No example seems to be known in which the derived length of H exceeds 2. However, as a result of Theorem 3.1.1, H is always soluble of derived length at most 3. Thus it can reasonably be conjectured that in fact H is always metabelian.

Theorem 4.1.2 . Suppose that G and G_1 are groups, $\pi : G \to G_1$ is a projectivity and $H \lhd G$ with $H/H_{\pi,G}$ non abelian. Then there is a subgroup X of G containing H such that X/H is cyclic and

(i) $X/X_{\pi,X}$ is a finite 2-group of order $\geq 2^{13}$,

(ii) $H/H_{\pi,X}$ is non-abelian of order $\geq 2^7$.

Thus π induces a projectivity $X/H_{\pi,X} \to X^{\pi}/(H^{\pi})_{X^{\pi}}$ and the non-abelian normal subgroup $H/H_{\pi,X}$ has core-free image.

The proof of this theorem quickly reduces to a consideration of finite 2-groups and will then follow from

Theorem 4.1.3 . Suppose that X and X_1 are finite 2-groups, $\pi : X \to X_1$ is a projectivity, $H \lhd X$ and X/H cyclic. If H^{π} is core-free in X_1 and H is non-abelian, then (i) $|X| \geq 2^{13}$ and (ii) $|H| \geq 2^7$.

<u>Deduction of Theorem 4.1.2 from Theorem 4.1.3</u> . Let G, G_1, π and H satisfy the hypotheses of Theorem 4.1.2 . By Remark 3.3.7

$$(H^{\pi})_{G_1} = \bigcap_{x \in S} (H^{\pi})_{\langle H,x\rangle^{\pi}} ,$$

where $S = \{x \in G \,|\, |\langle x\rangle/(\langle x\rangle \cap H)|$ is a prime power or infinite $\}$. However, by Corollary 1.1.3, if $\langle x\rangle$ is infinite and $\langle x\rangle \cap H = 1$, then $\langle x\rangle^{\pi}$ normalises H^{π}. Thus, since $H/H_{\pi,G}$ is non-abelian and

$$H_{\pi,\langle H,x\rangle} \lhd \langle H,x\rangle \ , \qquad (1)$$

by Theorem 2.1.1, there is an element x in G such that $|\langle x\rangle/(\langle x\rangle \cap H)|$ is a prime power and $H/H_{\pi,\langle H,x\rangle}$ is non-abelian. Let $X = \langle H,x\rangle$. Then we see from (1) that π induces a projectivity

$$X/H_{\pi,X} \to X^{\pi}/(H^{\pi})_{X^{\pi}} .$$

We will show that $X/H_{\pi,X}$ is a finite 2-group of order at least 2^{13} and $H/H_{\pi,X}$ has order $\geq 2^7$. (Then $X^{\pi}/(H^{\pi})_{X^{\pi}}$ will have the same order as $X/H_{\pi,X}$, by Proposition 1.2.8 (c)).

Factoring by $H_{\pi,X}$ and $(H^{\pi})_{X^{\pi}}$ in X and X^{π} respectively, we may assume that $H_{\pi,X} = 1$ and $(H^{\pi})_{X^{\pi}} = 1$. Now X/H is cyclic of prime power order p^n say, and clearly $n \geq 1$. Therefore $|X^{\pi}: H^{\pi}|$ is finite by Theorem 1.1.2 . Since H^{π} is core-free in X^{π}, it follows that X^{π} and hence X are finite. If $n = 1$ then H is a maximal subgroup of X and hence H^{π} is a maximal subgroup of X . As the image of a normal subgroup of X, H^{π} is a Dedekind subgroup of X^{π} . It follows from Lemma 3.3.6 that X^{π} is non abelian of order qr, where q and r are primes. This implies that H^{π} and hence H have prime order, contradicting the fact that H is not abelian.

Therefore $n \geq 2$, and, again by Lemma 3.3.6

$$X^{\pi} \text{ is a q-group,}$$

for some prime q. Since X^{π} is not abelian, X is also a q-group (Proposition 1.2.8 (c)) and so $q=p$. Thus X and X^{π} are finite p-groups. By Theorem 1.1.1 we see that $p=2$. Since X/H is cyclic, Theorem 4.1.3 shows that $|X| \geq 2^{13}$ and $|H| \geq 2^{7}$, as required.

□

We prove Theorem 4.1.3 in section 2. Sections 3-6 are devoted to the proof of Theorem 4.1.1, which we now summarize briefly. Theorem 4.1.3 tells us that there is an example proving Theorem 4.1.1 with $G = H\langle a\rangle$, a finite 2-group, and $H \cap \langle a\rangle = 1$ by Lemma 3.2.1 (i). Lemma 3.2.1 (v) does not allow us to take a generalised quaternian group for H. Therefore we choose H such that $\Omega_1(H)$ has rank 2 and then Lemma 3.3.4 (a) tells us that H must be metacyclic and modular. Theorem 4.2.3 tells us that $|H| \geq 2^{7}$ and we choose

$$H = \langle h, q \mid h^{16} = q^{8} = 1,\ h^{q} = h^{9}\rangle \tag{2}$$

of order 2^{7}, consistent with the above and Lemma 3.2.1. Similarly we choose the element a of order 2^{6} and define an action of a on H with $G = H\langle a\rangle$ consistent with the results of Lemma 3.2.1. In order to find a second group G_1 and a projectivity $\pi : G \to G_1$ such that $H_1 = H^{\pi}$ is core-free in G_1, we were able to show that H_1 cannot be abelian or isomorphic to H. Therefore we define

$$H_1 = \langle h_1, q_1 \mid h_1^{16} = q_1^{8} = 1,\ h_1^{q_1} = h_1^{5}\rangle \tag{3}$$

and form a product $G_1 = H_1\langle a_1\rangle$ where $|a_1| = 2^6$ and H_1 is core-free in G_1, again consistent with Lemma 3.2.1 . Every projectivity between finite groups of the same order is induced by an element map. In section 3 we define a bijection $\sigma : G \to G_1$ and in section 4 we show that the image of σ restricted to each subgroup of $E = \langle H, a^2\rangle$ is a subgroup of $E_1 = \langle H_1, a_1^2\rangle$. However, while section 5 establishes the analogous result for all subgroups of G other than the cyclic ones outside E, it is easier for us to abandon element maps in order to handle these latter subgroups where π is defined directly. The short section 6 shows that π is surjective and a projectivity.

Baer's work [2] on projectivities from abelian groups is the starting point of our construction of π. The only other result on projectivities that we have been able to use is the following, due to Schmidt ([19] , Lemma 2.5)).

Lemma 4.1.4 . Let G be a group, Z and H subgroups of G with $Z \leq H$, and suppose that for every subgroup U of G either $U \leq H$ or $Z \leq U$. Let $\overline{Z}$ and $\overline{H}$ be subgroups of the group $\overline{G}$ with the same properties. If τ is a projectivity from H to $\overline{H}$ and σ is an isomorphism from $[G/Z]$ to $[\overline{G}/\overline{Z}]$ such that $U^\sigma = U^\tau$ for all subgroups between Z and H, then the map ρ defined by $U^\rho = U^\tau$ for $U \leq H$ and $U^\rho = U^\sigma$ for $U \nleq H$ is a projectivity from G to $\overline{G}$.

Finally, we recall an elementary fact occurring in modular 2-groups.

<u>In a finite modular 2-group</u> G, $\Omega_2(G) \leq N(G)$ (4)

To see this, let $x \in G$ with $|x| \leq 4$ and let $g \in G$. If $|g| = 2$, then $\langle x,g\rangle$ has order ≤ 8 and $[g,x] = 1$. If g has order ≥ 2, induction on $|g|$ suffices to establish (2) (In fact the hypothesis that G is finite in (4) is not needed).

4.2 Proof of Theorem 4.1.3

Let X and X_1 be finite 2-groups, $\pi : X \to X_1$ a projectivity, $H \triangleleft X$ with X/H cyclic and H^π core-free in X_1 and suppose that H is non-abelian. By hypothesis there is an element a in X such that

$$X = H\langle a\rangle \quad .$$

Write $H_1 = H^\pi$ and $\langle a_1\rangle = \langle a\rangle^\pi$. Since H_1 is core-free in X_1 , $H_1 \cap \langle a_1\rangle = 1$ and so $H \cap \langle a\rangle = 1$.

By Lemma 3.2.1 (iv), $\Omega_1(X)$ and $\Omega_1(X_1)$ are elementary abelian and by (xi) these are bases

$$\{e_0, e_1, \ldots, e_m\} \ , \quad \{f_0, f_1, \ldots, f_m\}$$

of $\Omega_1(X)$ and $\Omega_1(X_1)$, respectively, such that

$$\Omega_1(H) = \langle e_1\rangle \times \ldots \times \langle e_m\rangle \ , \quad \Omega_1(H_1) = \langle f_1\rangle \times \ldots \times \langle f_m\rangle \ ,$$

$$\langle e_0\rangle = \Omega_1\langle a\rangle \ , \quad \langle f_0\rangle = \Omega_1\langle a_1\rangle \ ,$$

$$\langle e_i\rangle^\pi = f_i \ , \quad 0 \le i \le m \ ,$$

$$e_1^a = e_1 \ , \quad e_i^a = e_{i-1}e_i \quad \text{for} \quad 2 \le i \le m \ , \tag{5}$$

$$f_1^{a_1} = f_0 f_1 \ , \quad f_2^{a_1} = f_1 f_2 \quad . \tag{6}$$

<u>Remark.</u> f_0 is the unique involution in the centre of X_1 .

By Lemma 3.2.1 (iv), $\Omega_2(X)/\Omega_1(X)$ and $\Omega_2(X_1)/\Omega_1(X_1)$ are elementary abelian of rank $t+1$, say. Also π induces a projectivity from $X/\Omega_1(X)$ to $X_1/\Omega_1(X_1)$ and $H_1\Omega_1(X_1)/\Omega_1(X_1)$ is core-free in $X_1/\Omega_1(X_1)$ (by Lemma 3.2.1 (iii)). Therefore, by Lemma 3.2.1 (xi), $\Omega_2(X)/\Omega_1(X)$ has a basis $\{c_i\Omega_1(X) \mid 0 \le i \le t\}$ such that

$$<c_0> = \Omega_2<a>\ , \quad c_i \in \Omega_2(H) \quad \text{for} \quad 1 \le i \le t\ ,$$

$$\begin{aligned} c_1^a &\equiv c_1 \mod \Omega_1(H) \\ c_i^a &\equiv c_{i-1}c_i \mod \Omega_1(H)\ , \quad 2 \le i \le t\ , \end{aligned} \tag{7}$$

and there are elements $d_i \in \Omega_2(X_1)$ such that

$$<d_i> = <c_i>^{\pi}\ , \quad 0 \le i \le t\ , \tag{8}$$

$$d_i^{a_1} \equiv d_{i-1}d_i \mod D_{i-2}\ , \quad 1 \le i \le t\ , \tag{9}$$

where, for $-1 \le j \le t$, $D_j = <d_0, d_1, \ldots, d_j, \Omega_1(X_1)>$. (Note that $<d_0> = \Omega_2<a_1>$ and $d_i \in \Omega_2(H_1)$, $1 \le i \le t$. Also it is clear that each D_j is a_1-invariant.)

Denote the exponent of H by 2^r. Then

$$\mho_{r-2}(H\Omega_1(X)/\Omega_1(X))$$

is a non-trivial normal subgroup of $X/\Omega_1(X)$ contained in $H\Omega_1(X)/\Omega_1(X)$.

Therefore, by Lemma 3.2.1(xii) applied to $X/\Omega_1(X)$ and $X_1/\Omega_1(X_1)$,

$$c_1\Omega_1(X) \in \mho_{r-2}(H\Omega_1(X)/\Omega_1(X)) .$$

Also Lemma 3.2.1(vii)(again applied to $X/\Omega_1(X)$ and $X_1/\Omega_1(X_1)$) shows that

$$\mho_{r-2}(H\Omega_1(X)/\Omega_1(X)) = \{h^{2^{r-2}}\Omega_1(X) \mid h \in H\} .$$

So there exists an element $h \in H$ such that

$$h^{2^{r-2}} \equiv c_1 \bmod \Omega_1(X) ,$$

i.e. $$c_1 = h^{2^{r-2}} w ,$$

where $w \in \Omega_1(X)$. Therefore, replacing c_1 by $c_1 w$, we may assume that

$$c_1 = h^{2^{r-2}} . \tag{10}$$

Since $\Omega_2(X)$ is abelian (by Lemma 3.2.1 (v)), substituting for c_1 in (7) and squaring gives

$$(h^{2^{r-1}})^a = h^{2^{r-1}}$$

and hence, by (5),

$$h^{2^{r-1}} = e_1 . \tag{11}$$

Let

$$Q = H \cap H_1^{a_1\pi^{-1}} .$$

It is easy to see that $\langle e_0e_1\rangle^\pi = \langle f_0f_1\rangle$ and thus (5) and (6) show that

$$H_1^{a_1\pi^{-1}} \cap \langle e_0,e_1\rangle = \langle e_0e_1\rangle . \tag{12}$$

Therefore

$$Q \cap \langle e_0,e_1\rangle = 1 . \tag{13}$$

Now $H \lhd X$ shows that $Q \lhd H_1^{a_1\pi^{-1}}$ and since X/H is cyclic, $H_1^{a_1\pi^{-1}}/Q$ is also cyclic. It follows that

$$|H| = |H_1^{a_1\pi^{-1}}| \leq 2^r|Q| .$$

We have $|h| = 2^r$ and $e_1 \notin Q$ (by (13)), and so

$$H = Q\langle h\rangle \quad \text{and} \quad Q \cap \langle h\rangle = 1 . \tag{14}$$

From (11), (5) and (6)

$$\Omega_1(\langle h\rangle^{\pi a_1\pi^{-1}}) = \langle e_0e_1\rangle \tag{15}$$

and therefore $\langle h\rangle^{\pi a_1\pi^{-1}} \cap Q = 1$, again by (13). Thus

$$H_1^{a_1\pi^{-1}} = Q\langle h\rangle^{\pi a_1\pi^{-1}} . \tag{16}$$

In order to prove (i), i.e. $|X| \geq 2^{13}$, we argue by contradiction and assume that $|X|$ is minimal such that (i) is false. We distinguish two cases depending on the exponent of H . The first is not difficult.

<u>Case 1</u> : <u>exponent of $H \geq 2^4$</u> . Then $|a| \geq 2^{r+2} \geq 2^6$ (Lemma 3.2.1 (ix)) and hence $|H| \leq 2^6$ and, by (14), $|Q| \leq 4$. In particular $Q \leq \Omega_2(H)$ and so $Q^X \leq \Omega_2(H)$. By Lemma 3.2.1 (v) $\Omega_2(H)$ is abelian and therefore

$$\Omega_2(H) = Q\Omega_2\langle h\rangle \ . \tag{17}$$

Thus $|Q^X| \leq 16$ and hence any 2-group of automorphisms of Q^X has exponent ≤ 4 . Therefore a^4 centralises Q . Then $Q \triangleleft H_1^{a_1\pi^{-1}}\langle a^4\rangle$. However, by 3.2.1 (ii),

$$\Omega_r(X_1) = \Omega_r(H_1)\Omega_r\langle a_1\rangle = H_1\Omega_r\langle a_1\rangle = H_1^{a_1}\Omega_r\langle a_1\rangle \ .$$

Applying π^{-1} we have

$$H \leq H_1^{a_1\pi^{-1}}\Omega_r\langle a\rangle \leq H_1^{a_1\pi^{-1}}\langle a^4\rangle \ .$$

Thus $Q \triangleleft H$ and H/Q is cyclic by (14). But every normal subgroup ($\neq 1$) of X lying in H contains e_1 (Lemma 3.2.1 (xii)) and so Q is core-free in X . It follows that H is abelian, giving a contradiction.

<u>Case 2</u> : <u>exponent of $H \leq 2^3$</u> . Since $\Omega_2(H)$ is abelian, the exponent of H is 2^3 . Suppose that $|a| \geq 2^6$. Then, by (14), $|Q| \leq 8$.

Now $\Omega_1(H)$ and $H/\Omega_1(H)$ are both abelian (Lemma 3.2.1 (v)) and have order at most 16. Therefore a^8 centralises H and hence also Q. As in case 1, it follows that

$$Q \triangleleft H_1^{a_1\pi^{-1}}\langle a^8\rangle \quad \text{and} \quad H \leq \Omega_3(X) \leq H_1^{a_1\pi^{-1}}\langle a^8\rangle \ .$$

Then $Q \triangleleft H$ and Q is core-free in X. Thus again H is abelian, giving a contradiction.

Therefore we may assume that $|a| = 2^5$ (and H has exponent 2^3). So

$$|Q| \leq 2^4 \ . \tag{18}$$

Let

$$R_1 = R^{\pi} = (H_1)_{H_1\langle a_1^2\rangle} \ .$$

By Lemma 3.2.1 (ix) H_1/R_1 has exponent ≤ 4. Hence $R \geq \mho_2(H)$. If Q has exponent 2^3, then (14) shows that $|\mho_2(H)| \geq 4$. Since there is a unique normal subgroup of order 4 of X lying in $\Omega_1(H)$ (Lemma 3.2.1 (xii)) viz. $\langle e_1, e_2\rangle$, it follows that

$$e_2 \in \mho_2(H) \leq R \ .$$

Then $f_2 \in R_1$. However by (6) $f_2^{a_1^2} = f_0 f_2 \notin H_1$, contradicting $R_1 \leq H_1$ and $R_1^{a_1^2} = R_1$. Thus

$$Q \text{ has exponent } \leq 4 \ .$$

We claim that

$$H' = \langle e_1 \rangle \ . \tag{19}$$

To see this, we observe from Lemma 3.2.1 (v) that $H/\Omega_1(H)$ is abelian. Therefore

$$H' \leq \Omega_1(H) \ . \tag{20}$$

Now let $(H_1^{a_1\pi^{-1}})_X = K^{\pi a_1\pi^{-1}}$. Thus $K \leq H$ and, by Theorem 2.1.1, K (as the preimage of $K^{\pi a_1\pi^{-1}}$ under the projectivity $\pi a_1\pi^{-1}:X \to X$) is normal in X . From (12) $e_1 \notin H_1^{a_1\pi^{-1}}$ and so

$$e_1 \notin K^{\pi a_1\pi^{-1}} \cap H \triangleleft X \ .$$

Then

$$K^{\pi a_1\pi^{-1}} \cap H = 1 \ ,$$

since every non-trivial normal subgroup of X contained in H contains e_1 by (5). It follows that

$$K^{\pi a_1\pi^{-1}} \text{ and } K \text{ are cyclic.} \tag{21}$$

By Lemma 3.2.1 (viii), $e_0 \in Z(X)$ and, by (5) , $e_1 \in Z(X)$. Therefore $e_0e_1 \in Z(X) \cap H_1^{a_1\pi^{-1}}$, by (12) , and so $e_0e_1 \in K^{\pi a_1\pi^{-1}}$. Thus

$$\langle e_1 \rangle = \langle e_0e_1 \rangle^{\pi a_1^{-1}\pi^{-1}} \in K \ . \tag{22}$$

Now consider the projectivity

$$\pi a_1\pi^{-1} : X/K \to X/K^{\pi a_1\pi^{-1}} .$$

By minimality of $|X|$, H/K must be abelian. Then, using (20)

$$H' \le \Omega_1(H) \cap K = \langle e_1\rangle$$

by (21) and (22). Therefore, since we are assuming that H is not abelian, we have proved (19).

Now it follows that $\langle h\rangle \lhd H$. By Lemma 3.2.1 (ii)

$$\Omega_2(X) = \Omega_2(H)\Omega_2\langle a\rangle \quad . \tag{23}$$

Thus, since $\Omega_2\langle a\rangle \le Z(X)$ (Lemma 3.2.1 (viii)), we see that $\langle h\rangle \lhd \langle h,\Omega_2(X)\rangle$. Therefore for any element $x \in \Omega_2(X)$

$$\underline{\langle h,x\rangle \text{ is a modular } 2\text{-group},} \tag{24}$$

by Theorem 1.2.10.

(Here we are using the fact that x centralises h^2 according to Lemma 3.2.1 (v)). Since $\Omega_2(X)$ is invariant under any autoprojectivity, it follows from (24) that

$$\langle\langle h\rangle^{\pi a_1\pi^{-1}},x\rangle \text{ is a modular } 2\text{-group.}$$

However $|x| \leq 4$ and then by (4) x normalises all subgroups of this group; in particular $\langle h\rangle^{\pi a_1 \pi^{-1}}$ is normalised by x. Therefore

$$[\langle h\rangle^{\pi a_1 \pi^{-1}}, \langle x\rangle] \leq \langle h\rangle^{\pi a_1 \pi^{-1}} \cap H = 1 ,$$

by (15). Thus

$$\underline{\langle h\rangle^{\pi a_1 \pi^{-1}} \text{ is centralised by } \Omega_2(X)} . \tag{25}$$

Let $b \in X$ such that $b^4 = e_0$. Suppose, for a contradiction, that $\langle b\rangle$ is normalised by $\Omega_2(X)$. Then

$$[\langle b\rangle, \Omega_2(X)] \leq \langle b\rangle \cap H = 1 . \tag{26}$$

Also, by Lemma 3.2.1 (ii) $\Omega_3(X) = H\Omega_3\langle a\rangle = H\langle b\rangle$ (the latter by order consideration). Thus we obtain

$$\Omega_3(X) = H_1^{a_1 \pi^{-1}} \langle b\rangle^{\pi a_1 \pi^{-1}} = H_1^{a_1 \pi^{-1}} \langle b\rangle ,$$

since $\Omega_1(\langle b\rangle^{\pi a_1 \pi^{-1}}) = \langle e_0\rangle$. Therefore, using (16),

$$\Omega_3(X) = Q\langle h\rangle^{\pi a_1 \pi^{-1}} \langle b\rangle .$$

However, Q has exponent ≤ 4, and so we obtain

$$\Omega_3(X) = \Omega_2(X)\langle h\rangle^{\pi a_1 \pi^{-1}} \langle b\rangle . \tag{27}$$

Now it follows (from (25) and (26)) that $\Omega_2(X)$ lies in the centre of $\Omega_3(X)$. But $H = Q\langle h\rangle$ and this implies that H is abelian, a contradiction. Thus

$$\underline{\text{no cyclic subgroup of order } 8 \text{ of } X \text{, containing } e_0 \text{, is normalised by } \Omega_2(X)} . \tag{28}$$

Write $\langle h_1\rangle = \langle h\rangle^{\pi}$ and let $x_1 \in \Omega_2(X_1)$. By (24)

$$\underline{\langle h_1,x_1\rangle \text{ is a modular 2-group.}} \tag{29}$$

Since $|x_1| \le 4$, $\langle h_1\rangle$ is normalised by x_1 , by (4). Therefore

$$\underline{\langle h_1\rangle \text{ is normalised by } \Omega_2(X_1)} \tag{30}$$

and the group of automorphisms of $\langle h_1\rangle$ induced by $\Omega_2(X_1)$ has order ≤ 2 .

Recall that $D_{t-1} = \langle d_0,d_1,\dots,d_{t-1},\Omega_1(X_1)\rangle$ and suppose for a contradiction that

$$[D_{t-1},h_1] = 1 . \tag{31}$$

Since D_{t-1} is a_1-invariant,

$$[D_{t-1},h_1^{a_1}] = 1 . \tag{32}$$

From (9), $d_t^{a_1} \equiv d_t \mod D_{t-1}$ and so, by (29) and (30),

$$h_1^{d_t} = h_1^{1+4k} \quad , \quad \text{for some } k \; ,$$

and

$$h_1^{a_1 d_t} = h_1^{a_1(1+4k)} \quad .$$

By Lemma 3.2.1 (vii),

$$(h_1 \; h_1^{a_1})^4 = h_1^4 \; h_1^{4a_1} \; (= f_0) \tag{33}$$

and hence $\langle h_1 h_1^{a_1} \rangle$ is normalised by d_t and $\langle h_1 h_1^{a_1}, d_t \rangle$ is modular. Thus, by (31) and (32) $\langle h_1 h_1^{a_1}, x_1 \rangle$ is modular for all $x_1 \in \Omega_2(X_1)$. Applying π^{-1} it follows that

$$\langle \langle h_1 h_1^{a_1} \rangle^{\pi^{-1}}, x \rangle \text{ is modular}$$

for all $x \in \Omega_2(X)$. Therefore, by (4),

$$\langle h_1 h_1^{a_1} \rangle^{\pi^{-1}} \text{ is normalised by } \Omega_2(X) \; .$$

But from (33) $\langle h_1 h_1^{a_1} \rangle^{\pi^{-1}}$ has order 8 and contains e_0 , contradicting (28).

Now we know that $[D_{t-1}, h_1] \neq 1$. Hence, by (30),

$$[D_{t-1}, h_1] = \langle f_1 \rangle \; . \tag{34}$$

Now we can show that $t \geq 3$. For, by Lemma 3.2.1 (xiii)

$$[h_1, \Omega_1(X_1)] = 1 . \tag{35}$$

Also, by Lemma 3.2.1 (vii) (recall that $\langle d_0 \rangle = \Omega_2\langle a_1 \rangle$),

$$[h_1, d_0] = 1 . \tag{36}$$

Furthermore by (8) and (10) $\langle d_1 \rangle = \Omega_2\langle h_1 \rangle$ and so

$$[h_1, d_1] = 1 . \tag{37}$$

Thus (34), (35), (36) and (37) show that

$$t \geq 3 \quad \text{and} \quad |\Omega_2(X)/\Omega_1(X)| \geq 16 .$$

However, from (17), (18) and (23), $|\Omega_2(X)| \leq 2^8$. Therefore, since $\Omega_2(X)$ is abelian, $|\Omega_1(X)| \geq |\Omega_2(X)/\Omega_1(X)|$ and we must have

$$|\Omega_1(X)| = |\Omega_2(X)/\Omega_1(X)| = 16 .$$

Thus $m = t = 3$.

Now, $X/\Omega_2(X)$ is abelian and so, modulo $\Omega_2(X_1)$, X_1 is modular with $\langle a_1 \rangle$ of index 2. It follows that $[h_1, a_1] \in \langle a_1^4 \rangle \Omega_2(X_1)$. Therefore there are integers α_i $(0 \leq i \leq 3)$ such that

$$[h_1, a_1] \equiv a_1^{4\alpha_0} d_1^{\alpha_1} d_2^{\alpha_2} d_3^{\alpha_3} \quad \text{mod } \Omega_1(X) .$$

A straightforward calculation using (9) gives

$$[h_1, a_1^4] \equiv d_0^{\alpha_3} \bmod \Omega_1(X_1) .$$

Since h_1 and a_1^4 belong to $\Omega_3(X_1)$ and $\Omega_3(X_1)/\Omega_1(X_1)$ is abelian (Lemma 3.2.1 (v)) we have

$$\alpha_3 \equiv 0 \bmod 2 .$$

In particular this shows that X_1 is not generated modulo $\Omega_1(X_1)$, by a_1 and h_1 and hence X is not generated, modulo $\Omega_1(X)$, by a and h. Therefore

$$[h,a] \equiv c_1^{\beta_1} c_2^{\beta_2} \bmod \Omega_1(X)$$

for suitable integers β_1, β_2. Recall that the definition of h requires only that $h^2 \equiv c_1 \bmod \Omega_1(X)$. Thus we may replace h by $hc_2^{\beta_1} c_3^{\beta_2}$, and then

$$[h,a] \in \Omega_1(H) . \tag{38}$$

As before, without changing c_1 modulo $\Omega_1(X)$, we may assume that (10) still holds, i.e.,

$$c_1 = h^2 . \tag{39}$$

Now it follows from (38) that, modulo $\Omega_1(X_1)$, $\langle h_1, a_1 \rangle$ is a modular group and so

$$[h_1, a_1] \in \langle a_1^4 , \Omega_1(X_1)\rangle \quad , \tag{40}$$

since h_1 has order 4 modulo $\Omega_1(X_1)$. Also, from (7) and (38), we see that $c_3 \notin \langle c_2, h, a\rangle$ and so

$$\langle c_2, h, a\rangle < X \ .$$

Then by minimality of $|X|$,

$$[c_2, h] = 1 \ . \tag{41}$$

Consider the element

$$x = hc_3a^2$$

belonging to X . We will derive our final contradiction by showing that $\langle c_2, x\rangle$ is a modular group, while $\langle c_2, x\rangle$ is not modular. By Lemma 3.2.1 (vi)

$$\Omega_4(X)/\Omega_1(X) \quad \text{and} \quad \Omega_4(X_1)/\Omega_1(X_1) \quad \text{have class} \le 2 \ . \tag{42}$$

Then (42) shows that

$$\begin{aligned} x^2 &\equiv h^2a^4[a^2, hc_3] \mod \Omega_1(X) \\ &\equiv h^2a^4[a^2, c_3] \mod \Omega_1(X) \ , \quad \text{by (38)}, \\ &\equiv a^4 \mod \Omega_1(X) \ , \quad \text{by (7) and (39)}. \end{aligned}$$

Since $m = 3$, $[a^4, \Omega_1(X)] = 1$ and therefore

$$x^4 = a^8 . \tag{43}$$

Let $\langle x_1 \rangle = \langle x \rangle^\pi$. It is not hard to see that

$$x_1 = h_1^j d_3^i a_1^{2k} ,$$

where i, j, k are odd.

By (42)

$$\begin{aligned} x_1^2 &\equiv h_1^2 [a_1^{2k}, \ h_1^j d_3] \mod \Omega_1(X_1)\langle a_1^4 \rangle \\ &\equiv h_1^2 [a_1^{2k}, d_3] \mod \Omega_1(X_1)\langle a_1^4 \rangle \quad \text{(by (40))} \\ &\equiv 1 \mod \Omega_1(X_1)\langle a_1^4 \rangle \end{aligned}$$

(by (9) and the fact that $d_1 \equiv h_1^2 \mod \Omega_1(X_1)$). Since $\Omega_1(X_1)$ is a 4-dimensional $\langle a_1 \rangle$-module, $[a_1^4, \Omega_1(X_1)] = 1$ and so $x_1^4 \in \langle a_1^8 \rangle$. Since $H_1\langle x_1 \rangle = H_1\langle a_1^2 \rangle$, it follows that $|x_1| \geq |a_1^2| = 16$ and hence

$$\langle x_1^4 \rangle = \langle a_1^8 \rangle . \tag{44}$$

Take $b = a^4$ in (27):

$$\Omega_3(X) = \Omega_2(X)\langle h \rangle^{\pi a_1 \pi^{-1}} \langle a^4 \rangle .$$

By (25), $[\Omega_2(X), \langle h \rangle^{\pi a_1 \pi^{-1}}] = 1$. Thus, $[\Omega_2(X), \langle a^4 \rangle] \neq 1$, otherwise

$[\Omega_2(X),\Omega_3(X)] = 1$, forcing H to be abelian. In fact

$$[\Omega_2(X),\langle a^4\rangle] = \langle e_1\rangle \quad . \tag{45}$$

For, $\Omega_3(X) = H\langle a^4\rangle$. However, by (15), $\langle h\rangle^{\pi a_1 \pi^{-1}} \cap H = 1$ and so $\Omega_3(X) = H\langle h\rangle^{\pi a_1 \pi^{-1}}$, by order considerations. Therefore

$$\begin{aligned}[\Omega_2(X),\Omega_3(X)] &= [\Omega_2(X),H] \\ &= [\Omega_2(H),H] \quad \text{(by Lemma 3.2.1 (ii) and (vii))} \\ &= \langle e_1\rangle \quad ,\end{aligned}$$

by (19).

Thus (45) follows. Since (7) shows that $\langle c_3\rangle^{\langle c_3,a\rangle} = \Omega_2(H)$, $[c_3,a^4] = 1$, otherwise $[\Omega_2(H),\langle a^4\rangle] = 1$, contradicting (45). Therefore, by (45),

$$[c_3,a^4] = e_1 \quad . \tag{46}$$

By (7)

$$c_2^a = c_1c_2w \quad , \qquad c_3^a = c_2c_3w_1 \quad ,$$

where $w, w_1 \in \Omega_1(H)$. Write

$$w = e_1^{i_1}e_2^{i_2}e_3^{i_3} \quad .$$

Then a straightforward calculation using (7), (11), (38), (39), gives

$$c_3^{a^4} = e_1^{1+i_3} c_3 \ .$$

Thus $i_3 = 0$ by (46). Replacing c_2 by $c_2 e_2^{i_1} e_3^{i_2}$ and using (5) we have, therefore,

$$[c_2, a] = c_1 \ .$$

Since c_1 $(= h_1^2)$ is centralised by a, it follows that

$$[c_2, a^2] = c_1^2 = e_1 \ . \qquad (47)$$

In particular $\langle a^2, c_2\rangle/\langle e_0, e_1\rangle$ is abelian and so the projective image $\langle a_1^2, d_2\rangle/\langle f_0, f_1\rangle$ is modular. In this quotient d_2 has order at most 4 (in fact it is 4) and a_1^2 has order 8. Therefore

$$[d_2, a_1^2] \in \langle a_1^8, f_1\rangle \ .$$

On the other hand, $[d_2, a_1^2] \notin \langle a_1^8\rangle$, otherwise $\langle d_2, a_1^2\rangle$ would be modular and hence $\langle c_2, a^2\rangle$ would be modular, forcing c_2 to normalise $\langle a^2\rangle$, which contradicts (47). Thus

$$[d_2, a_1^2] \in f_1\langle a_1^8\rangle \ . \qquad (48)$$

Recall that $x = hc_3a^2$. Then

$$[c_2,x] = [c_2,a^2] \qquad \text{(by (41))}$$
$$= e_1 \ , \tag{49}$$

by (47). Similarly (with $x_1 = h_1^j d_3^i a_1^{2k}$, i,j,k odd)

$$[d_2,x_1] = [d_2,a_1^{2k}][d_2,h_1^j]$$
$$\equiv f_1^2 \bmod \langle a_1^8\rangle \qquad \text{(by (34) and (48))}$$
$$\equiv 1 \bmod \langle a_1^8\rangle \ .$$

Therefore $\langle d_2,x_1\rangle$ is modular, by (44), and hence its preimage $\langle c_2,x\rangle$ is modular. Then c_2 normalises $\langle x\rangle$. But this is incompatible with (43) and (49). This completes the proof of Theorem 4.1.3 (i).

In order to complete the proof of Theorem 4.1.3, we must show (ii) $|H| \geq 2^7$. Suppose, for a contradiction, that $|H| \leq 2^6$. By (i), $|a| \geq 2^7$. We use the notation of (i). If H has exponent $\geq 2^4$, then the argument of Case 1 in (i) shows that H is abelian. On the other hand, if H has exponent $\leq 2^3$, then the argument of the first paragraph of Case 2 in (i) again shows that H is abelian. Therefore we have the desired contradiction.

□

4.3 The groups and the projectivity of Theorem 4.1.1 .

Construction of the groups. We will construct a group G with a normal non-abelian subgroup H, a second group G_1 and a projectivity

$$\pi : G \to G_1$$

such that H^π is core-free in G_1. The groups G and G_1 will be finite of order 2^{13}, H will be metacyclic of order 2^7 and G/H will be cyclic.

Thus let

$$H = \langle h,q \mid h^{16} = q^8 = 1,\ h^q = h^9 \rangle,$$

a split extension of a cyclic group $\langle h \rangle$ of order 16 by a cyclic group $\langle q \rangle$ of order 8. Then

$$H' = \langle h^8 \rangle$$

has order 2. Also H has an automorphism α of order 8 defined by

$$h^\alpha = h^{-1}q^4,\quad q^\alpha = h^2q^{-1}.$$

Therefore there is a split extension G of H by a cyclic group $\langle a \rangle$ of order 64, presented as follows:

$$G = \langle a,h,q \mid a^{64} = h^{16} = q^8 = 1,\ h^q = h^9,\ h^a = h^{-1}q^4,\ q^a = h^2q^{-1} \rangle. \tag{50}$$

This group G has order 2^{13}. The subgroup $\langle a^2, h, q \rangle$ (of order 2^{12}) has class 2 and hence all relations in this subgroup are easy consequences of

$$[h,q] = h^8 \tag{51}$$

$$[a^2,q] = h^4 \tag{52}$$

$$[a^2,h] = h^8. \tag{53}$$

The construction of G_1 proceeds as follows. Let elements b_1 and h_1 generate cyclic groups of order 16 and form their direct product

$$X_1 = \langle b_1\rangle \times \langle h_1\rangle.$$

The relation (53) shows that

$$X = \langle a^4,h\rangle = \langle a^4\rangle \times \langle h\rangle \cong X_1. \tag{54}$$

The subgroup $\langle b_1\rangle$ will be the image under π of $\langle a^4\rangle$; and $\langle h_1\rangle$ will be the image of $\langle h\rangle$ and X_1 the image of X.

The group X_1 has an automorphism β of order 4 defined by

$$b_1^\beta = b_1^{-3}h_1^8,\ h_1^\beta = h_1^5.$$

Thus there exists a split extension Y_1 of X_1 by a cyclic group $\langle q_1\rangle$ of order 8, presented by

$$Y_1 = \langle b_1,h_1,q_1 \mid b_1^{16} = h_1^{16} = q_1^8 = 1,\ h_1^{b_1} = h_1,\ b_1^{q_1} = b_1^{-3}h_1^8, h_1^{q_1} = h_1^5\rangle. \tag{55}$$

This group Y_1 has order 2^{11}. The subgroup $\langle q_1\rangle$ will be the image of $\langle q\rangle$ under π.

We make one final extension of Y_1 by a cyclic group of order 4. First we define a map γ on the generators of Y_1 and show that γ extends to an automorphism of Y_1. Let

$$b_1^\gamma = b_1,\ h_1^\gamma = b_1^{-1} h_1^7 q_1^4,\ q_1^\gamma = h_1^{-2} q_1^{-1}. \tag{56}$$

From the presentation of Y_1 and elementary commutator identities we see that $Y_1' = \langle h_1^4, b_1^4 \rangle$ and

$$\underline{Y_1 \text{ has class } 2}\ . \tag{57}$$

Then it is easy to check that γ preserves the relations of Y_1 and extends to an automorphism. We claim that

$$\underline{\gamma^4 \text{ is conjugation by } b_1.} \tag{58}$$

For,

$$h_1^{\gamma^2} = b_1^{-1}(b_1^{-1}h_1^7q_1^4)^7(h_1^{-2}q_1^{-1})^4 = b_1^{-1}(b_1^{-7}h_1q_1^4)(h_1^8q_1^4) = b_1^8h_1^9$$

and

$$q_1^{\gamma^2} = (b_1^{-1}h_1^7q_1^4)^{-2}(h_1^{-2}q_1^{-1})^{-1} = (b_1^2h_1^2)(h_1^{10}q_1) = b_1^2h_1^{12}q_1.$$

Therefore

$$h_1^{\gamma^4} = b_1^8(b_1^8h_1^9)^9 = h_1 = h_1^{b_1}$$

and

$$q_1^{\gamma^4} = b_1^2(b_1^8h_1^9)^{12}(b_1^2h_1^{12}q_1) = b_1^4h_1^8q_1 = q_1^{b_1}.$$

Hence (58) follows. By the cyclic extension theorem (see, for example, [20], p. 250), there is a group

$$G_1 = Y_1\langle a_1\rangle$$

where $Y_1 \triangleleft G_1, G_1/Y_1$ is cyclic of order 4 and $a_1^4 = b_1$. This group is presented as follows:

$$G_1 = \langle a_1, h_1, q_1 \mid a_1^{64} = h_1^{16} = q_1^8 = 1,\ h_1^{a_1^4} = h_1,\ a_1^{4q_1} = a_1^{-12}h_1^8,$$

$$h_1^{q_1} = h_1^5,\ h_1^{a_1} = a_1^{-4}h_1^7q_1^4,\ q_1^{a_1} = h_1^{-2}q_1^{-1}\rangle. \qquad (59)$$

(Here we have used (55), (56) and (58).) The order of G_1 is 2^{13}, i.e. the same as the order of G. The cyclic subgroup $\langle a_1\rangle$ will be the image of $\langle a\rangle$ under π. We note that

$$a^8 \text{ and } h^8 \text{ lie in the centre of } G \qquad (60)$$

and a_1^{16} lies in the centre of G_1.

Let

$$H_1 = \langle h_1, q_1 \rangle.$$

Here $\langle h_1 \rangle$ has order 16 and $\langle q_1 \rangle$ has order 8. This subgroup H_1 will be the image of H ($\triangleleft\, G$) under π and it is easy to see that

$$\underline{H_1 \text{ is core-free in } G_1.} \tag{61}$$

For,

$$\Omega_1(H_1) = \langle h_1^8, q_1^4 \rangle = W,$$

say. Using the fact that Y_1 (given by (55)) has class 2 and Y_1' has exponent 4, we have

$$(h_1^8)^{a_1} = (a_1^{-4} h_1^7 q_1^4)^8 = a_1^{32} h_1^8$$

and

$$(q_1^4)^{a_1} = (h_1^{-2} q_1^{-1})^4 = h_1^8 q_1^4.$$

Thus

$$W^{a_1} = \langle a_1^{32} h_1^8, h_1^8 q_1^4 \rangle \quad \text{and} \quad W^{a_1^2} = \langle h_1^8, a_1^{32} q_1^4 \rangle.$$

Therefore $W \cap W^{a_1} \cap W^{a_1^2} = 1$, proving (61).

Definition of π.

First we define an element map

$$\sigma : G \to G_1 . \tag{62}$$

Every element of G can be written uniquely in the form

$$a^k h^j q^i , \tag{63}$$

where

$$0 \le k \le 63, \quad 0 \le j \le 15, \quad 0 \le i \le 7. \tag{64}$$

Similarly every element of G_1 can be written in the form

$$a_1^k h_1^j q_1^i , \tag{65}$$

where k, j, i are integers uniquely determined modulo 64, 16, 8 respectively. Writing the elements of G in the form (63), the map (62) is defined by

$$(a^k h^j q^i)^\sigma = a_1^{k'} h_1^{j'} q_1^{i'} , \tag{66}$$

where

$$k' = k(1 + 4i) \tag{67}$$

$$j' = j(1 + 4i) \tag{68}$$

$$i' = \begin{cases} i + 2 \text{ if } i \text{ is odd} \\ i + 4jk \text{ if } i \text{ is even} . \end{cases} \tag{69}$$

Remarks 1. Replacing k, j, i by integers congruent modulo 64, 16, 8 respectively does not change the element (63). Also the right hand sides of (68) and (69) will be unchanged modulo 16, 8 respectively and therefore they can be used as the exponents of h_1 and q_1 in (66). However, the right hand side of (67) will be invariant only modulo 32 and so it can be used as the exponent of a_1 in (66) only when k is even.

2. The term 4jk in the definition of i' should be viewed as a small adjustment to what will shortly emerge as a natural map to consider in order to attempt to construct π.

Next we show that

$$\text{the map } \sigma \text{ is a bijection.} \tag{70}$$

For, suppose that

(i) $k_1(1 + 4i_1) \equiv k_2(1 + 4i_2) \bmod 64$

(ii) $j_1(1 + 4i_1) \equiv j_2(1 + 4i_2) \bmod 16$

(iii) $i_1' \equiv i_2' \bmod 8$.

Suppose that i_1 is odd. Then $i_1' = i_1 + 2$ is odd. So i_2' is odd (by (iii)) and therefore $i_2' = i_2 + 2$ (by (69)). It follows that $i_1 = i_2$ and hence $j_1 = j_2$, $k_1 = k_2$ (from (ii), (i) respectively). Now suppose that i_1 is even. Then $i_1' = i_1 + 4j_1k_1$ is even and so i_2' is even. Thus $i_2' = i_2 + 4j_2k_2$ and (iii) becomes

$$i_1 + 4j_1k_1 \equiv i_2 + 4j_2k_2 \bmod 8. \tag{71}$$

Therefore from (ii) we see that $j_1 = j_2$. Similarly from (i), $k_1 \equiv k_2 \bmod 4$. Thus (71) shows that $i_1 = i_2$ and then $k_1 = k_2$ follows from (i). This establishes (70).

We are now ready to define π. It is easy to see that the elements (63) with k even form a subgroup E of index 2 in G. Similarly the elements (65) with k even form a subgroup E_1 of index 2 in G_1.

Every cyclic subgroup $<a^{k'}h^{j'}q^{i'}>$, with k' odd, is generated by an element of the form ah^jq^i. If K is a subgroup of E or a non-cyclic subgroup of G, define

$$K^\pi = K^\sigma.$$

Otherwise $K = <ah^jq^i>$ and we define (72)

$$K^\pi = <(ah^jq^i)^\sigma>.$$

(We have not checked to see if we can define $K^\pi = K^\sigma$ for all K, because such a calculation would be too tedious.)

4.4 Consideration of π restricted to E.

Cyclic subgroups. Let $B = \langle a^8, h^2, q\rangle$, $B_1 = \langle a_1^8, h_1^2, q_1\rangle$ ($\leq Y_1$). It is clear from (67), (68) and (69) that σ restricts to a bijection from B to B_1. The subgroup B is abelian and homogeneous of exponent 8 with basis $\{a^8, h^2, q\}$. The subgroup B_1 is the split extension of $\langle a_1^8, h_1^2\rangle = \langle a_1^8\rangle \times \langle h_1^2\rangle$ (homogeneous of exponent 8) by $\langle q_1\rangle \cong C_8$, where q_1 conjugates the elements of $\langle a_1^8, h_1^2\rangle$ to their 5th powers, as we easily see from (59). In particular B_1 is a modular group and it is a well-known fact that B and B_1 have isomorphic subgroup lattices. In [2] Baer shows how to construct a bijection from B to B_1 inducing a projectivity. It is not difficult to check that our map σ is Baer's map. However, while σ has its origins in the work of Baer, it is not necessary to check our claim here, because we will prove that $\sigma|_E$ induces a projectivity from E to E_1, and therefore (by restriction) a projectivity from B to B_1.

We show first that

$$\sigma \text{ maps cyclic subgroups of } E \text{ to cyclic subgroups of } E_1. \tag{73}$$

Therefore we need formulas for powers of elements of E and E_1. As we have already pointed out (before (51)), E has class 2. Then for any elements u, v of E,

$$(uv)^n = u^n v^n [v,u]^{n(n-1)/2}. \tag{74}$$

So it is easy to check that

$$(a^{2k}h^jq^i)^\ell = a^{2k_1}h^{j_1}q^{i_1} \tag{75}$$

where
$$\left.\begin{aligned} k_1 &\equiv k\ell \bmod 32 \\ j_1 &\equiv \{j + 2[i(2j-k) + 2jk](\ell-1)\}\ell \bmod 16 \\ i_1 &\equiv i\ell \bmod 8. \end{aligned}\right\} \tag{76}$$

In order to obtain a formula for powers of elements of E_1, we first consider the action of a_1^2 on powers of q_1. We claim that

$$(q_1^i)^{a_1^{2k}} = a_1^{8ki(2i-1)}\, h_1^{-4ki} q_1^i. \tag{77}$$

We prove this by induction on k. When $k = 0$, (77) is trivially true. Therefore suppose that (77) holds for some $k \geq 0$. From (59)

$$q_1^{a_1^2} = (a_1^{-4}h_1^7q_1^4)^{-2}\, (h_1^{-2}q_1^{-1})^{-1}.$$

In order to express the right hand side in the standard form (65), we use

$$[a_1^4, q_1] = a_1^{-16}h_1^8$$

(from (59)). The fact that $Y_1 = \langle a_1^4, h_1, q_1\rangle$ is a class 2 group then gives

$$q_1^{a_1^2} = a_1^8h_1^{-4}q_1.$$

Taking i-th powers, we obtain

$$(q_1^i)^{a_1^2} = a_1^{8i} h_1^{-4i} q_1^i \, [h_1^{-4} q_1, a_1^8]^{i(i-1)/2} = a_1^{8i(2i-1)} h_1^{-4i} q_1^i .$$

Similarly

$$h_1^{a_1^2} = a_1^{32} h_1^9 \tag{78}$$

and so

$$(h_1^{-4ki})^{a_1^2} = h_1^{-4ki} .$$

Now conjugating (77) by a_1^2 gives

$$\begin{aligned}(q_1^i)^{a_1^{2(k+1)}} &= a_1^{8ki(2i-1)} h_1^{-4ki} \; a_1^{8i(2i-1)} h_1^{-4i} q_1^i \\ &= a_1^{8(k+1)i(2i-1)} \; h_1^{-4(k+1)i} \; q_1^i .\end{aligned}$$

Thus (77) holds for all k.

Now let $x_1 = a_1^{2k} h_1^j q_1^i$. Using (77), (78) and the relation $h_1^{q_1} = h_1^5$ (59), it follows that

$$x_1^2 = a_1^{k'} h_1^{j'} q_1^{i'}$$

where

$$\begin{aligned} k' &= 4k[1 + 2i(2i-1) + 8j] \\ j' &= 2(j - 2ji - 2ik - 4jk) \\ i' &= 2i \end{aligned}$$

and

$$x_1^4 = a_1^{k''} h_1^{j''} q_1^{i''} \tag{79}$$

where

$$k'' = 8k[1 + 2i(2i + 1)]$$

$$j'' = 4(j-2ji - 2ik)$$

$$i'' = 4i.$$

The factors of (79) commute and so, <u>if k is odd</u>,

$$x_1^{16} = a_1^{32} \tag{80}$$

has order 2. Modulo $\langle a_1^{32}\rangle$,

$$[h_1, a_1^2] = h_1^8,\ [h_1, q_1] = h_1^4 \text{ and } [q_1, a_1^2] = a_1^8 h_1^{-4},$$

the last by (77). Thus these three commutators all lie in the centre of $E_1/\langle a_1^{32}\rangle$ and since $E_1 = \langle a_1^2, h_1, q_1\rangle$, we see that $E_1/\langle a_1^{32}\rangle$ has class 2.

When k is even, $x_1 \in Y_1$, which also has class 2, by (57). Therefore, using (74) in $E_1/\langle a_1^{32}\rangle$ if k is odd, and in Y_1 if k is even, we have

$$x_1^m = a_1^{2k_0} h_1^{j_0} q_1^{i_0}$$

where

$$\left.\begin{aligned} 2k_0 &\equiv 2km[1 + 2i(m-1)] \begin{cases} \text{mod } 32 \text{ if } k \text{ is odd} \\ \text{mod } 64 \text{ if } k \text{ is even} \end{cases} \\ j_0 &\equiv m\{j-2[i(k+j) + 2jk]\,(m-1)\} \text{ mod } 16 \\ i_0 &\equiv im \text{ mod } 8. \end{aligned}\right\} \tag{81}$$

Now we can begin to establish (73). Let

$$x = a^{2k}h^jq^i. \tag{82}$$

We will show that

$$\langle x\rangle^\sigma = \langle x^\sigma\rangle. \tag{83}$$

When k is even we do this directly. When k is odd, we show first that

$$\langle x\rangle^\sigma \le \langle x^\sigma\rangle\langle a_1^{32}\rangle. \tag{84}$$

However, in this case the exponent of a_1 in x^σ has the form $2k'$, where k' is odd (by (67)), and so $a_1^{32} \in \langle x^\sigma\rangle$, by (80). Thus (84) will imply $\langle x\rangle^\sigma \le \langle x^\sigma\rangle$. Since x and x^σ both have order 32, by (75) and (80), (83) will then follow. (We work modulo $\langle a_1^{32}\rangle$ when k is odd in order to simplify calculations.)

Let ℓ be an integer. We show that there is an integer m such that

$$(x^\ell)^\sigma = (x^\sigma)^m \text{ (modulo } \langle a_1^{32}\rangle \text{ if } k \text{ is odd).}$$

By (75), $x^{\ell} = a^{2k_1}h^{j_1}q^{i_1}$, where k_1, j_1, i_1 satisfy (76). Recalling Remark 1 (after (69)), the form (65) for $(x^{\ell})^{\sigma}$ has

$$a_1 \text{ exponent} = 2k_1(1 + 4i_1), \tag{85}$$

$$h_1 \text{ exponent} = j_1(1 + 4i_1) \tag{86}$$

and

$$q_1 \text{ exponent} = \begin{cases} i_1 + 2 \text{ if } i_1 \text{ is odd} \\ i_1 \text{ if } i_1 \text{ is even} \end{cases} \tag{87}$$

(from (67), (68) and (69)). Now write $x^{\sigma} = a_1^{2k_2}h_1^{j_2}q_1^{i_2}$. Then by (81) the form (65) for $(x^{\sigma})^m$ (for any integer m) has

$$a_1 \text{ exponent} \equiv 2k_2m[1 + 2i_2(m-1)] \begin{cases} \text{mod } 32 \text{ if } k_2 \text{ is odd} \\ \text{mod } 64 \text{ if } k_2 \text{ is even,} \end{cases} \tag{88}$$

$$h_1 \text{ exponent} = m\{j_2-2[i_2(k_2+j_2) + 2j_2k_2](m-1)\} \tag{89}$$

and

$$q_1 \text{ exponent} = i_2m. \tag{90}$$

By (67), (68), (69) we have

$$k_2 = k(1 + 4i) \tag{91}$$

$$j_2 = j(1 + 4i) \tag{92}$$

$$i_2 = \begin{cases} i + 2 \text{ if } i \text{ is odd} \\ i \text{ if } i \text{ is even.} \end{cases} \tag{93}$$

We need to show that the three equations obtained by equating (85), (86), (87) respectively with (88), (89), (90) have a common solution for m. We distinguish three cases.

<u>Case 1</u>. <u>i and ℓ are odd</u>. From (76) and (91) we see that

$$i_1 \text{ and } i_2 \text{ are odd.}$$

Therefore equating (87) and (90) gives

$$i\ell + 2 \equiv (i + 2)m \bmod 8.$$

The solutions of this congruence are

$$m = 3\ell - 2 + 8\lambda, \tag{94}$$

where λ is any integer. Thus $m^2 \equiv \ell^2 \bmod 8$. Now equating (85) and (88) yields

$$k\ell(1+4i\ell) \equiv k(1+4i)m[1+2(i+2)(m-1)] \bmod \begin{cases} 16 \text{ if } k \text{ is odd} \\ 32 \text{ if } k \text{ is even.} \end{cases}$$

Therefore this congruence will hold if we find m satisfying

$$\ell(1+4i\ell) \equiv (1+4i)m\,[1+2(i+2)(m-1)] \bmod 16. \tag{95}$$

(Note that when we consider the case i odd and ℓ even, the congruence to be satisfied by equating (85) and (88) is still (95), and for i even and any ℓ we only replace (i+2) on the right hand side of (95) by i.) Substituting for m (from (94)), (95) reduces to

$$4\lambda \equiv (\ell-1)(i+1) \bmod 8. \tag{96}$$

Equating (86) and (89) gives

$$\{j+2[i(2j-k) + 2jk](\ell-1)\}\ell(1 + 4i\ell)$$

$$\equiv \{j(1+4i)-2[(i+2)(k+j)+2jk](m-1)\}m \bmod 16. \tag{97}$$

(As before observe that when we consider i odd and ℓ even, (97) remains unchanged; and when i is even and ℓ is arbitrary, then we change only (i+2) on the right hand side of (97) to i.) A routine check shows that any choice of λ gives a value of m (from (94)) satisfying (97). Since $(\ell-1)(i+1) \equiv 0$ mod 4, we can take $\lambda = (\ell-1)(i+1)/4$, which satisfies (96) and so there is a solution for m in this case.

Case 2. i odd and ℓ even. Now i_1 is even (76) and i_2 is odd (93). Equating (87) and (90) gives

$$i\ell \equiv (i+2)m \bmod 8.$$

The solutions of this congruence are

$$m = 3\ell + 8\lambda \tag{98}$$

for any integer λ. Again $m^2 \equiv \ell^2$ mod 8. Equating (85) and (88) yields (95) (as previously noted). Substituting for m from (98) reduces (95) to

$$4\lambda \equiv \ell(\ell + i + 1) \bmod 8. \tag{99}$$

Equating (86) and (89) gives (97) (as before) and it is easy to check that any choice of λ in (98) satisfies (97). So it is necessary only to solve (99) for λ. Again $\ell(\ell+i+1) \equiv 0$ mod 4 and we can take $\lambda = \ell(\ell+i+1)/4$.

Case 3. i even. This time i_1 and i_2 are both even (by (76) and (93)). So, equating (87) and (90),

$$i\ell \equiv im \bmod 8. \tag{100}$$

If we recall the remark after (95), setting (85) equal to (88) gives (95) with (i+2) on the right hand side replaced by i. Then (95) reduces to

$$m \equiv \ell + 2i\ell(\ell-1) \bmod 16. \tag{101}$$

Any solution m of this congruence satisfies

$$m \equiv \ell \bmod 8$$

and hence satisfies (100). Finally equating (86) and (89) gives (97) with (i+2) replaced by i on the right hand side (as observed immediately after (97)). Substituting for m from (101) yields

$$ij\ell(\ell-1) \equiv 0 \bmod 4,$$

which is clearly true since i is even. Therefore $m = \ell + 2i\ell(\ell-1)$ is a solution in this case. We have now proved (73).

Arbitrary subgroups. We show now that σ maps every subgroup of E to a subgroup of E_1. The following two results will achieve this. Write $N = \langle a^2,h\rangle$.

Lemma 4.4.1. If U is a subgroup of N and V is a subgroup of E, then $(UV)^\sigma = U^\sigma V^\sigma$.

Proof. Let $u \in U$, $v \in V$. Then $u = a^{2k}h^j$ (by (53)) and $v = a^{2k_1}h^{j_1}q^{i_1}$. Again using (53) we have

$$uv = a^{2k+2k_1}h^{j+8jk_1+j_1}q^{i_1}$$

and hence

$$(uv)^\sigma = a_1^{(2k+2k_1)(1+4i_1)}h_1^{(j+8jk_1+j_1)(1+4i_1)}q_1^m$$

where $m = i_1+2$ if i_1 is odd and $m = i_1$ if i_1 is even. From (78) and the fact that $\langle a_1^2,h_1\rangle$ has class 2, it follows that

$$[h_1^{(j+8jk_1)(1+4i_1)}, a_1^{2k_1(1+4i_1)}] = (a_1^{32}h_1^8)^{jk_1}$$

and so

$$(uv)^\sigma = (a_1^{2k(1+4i_1)}h_1^{j(1+4i_1)})(a_1^{2k_1(1+4i_1)+32jk_1}h_1^{j_1(1+4i_1)}q_1^m).$$

Thus if $j + k_1$ are not both odd, then $(uv)^\sigma = (u^{1+4i})^\sigma v^\sigma$. On the other hand if k_1 is odd, then $v^{16} = a^{32}$ by (75) and (76). If also j is odd, then $a_1^{32jk_1} = a_1^{32}$. Moreover, for any element of G,

$$(a^{32}g)^\sigma = a_1^{32}g^\sigma \tag{102}$$

(by definition of σ). Hence in this case $(uv)^\sigma = (u^{1+4i})^\sigma (v^{17})^\sigma$. Therefore in both cases $(UV)^\sigma = U^\sigma V^\sigma$. $\square$

Now let $N_1 = \langle a_1^2,h_1\rangle$. Then we have

<u>Lemma 4.4.2</u>. σ induces a projectivity from N to N_1.

<u>Proof</u>. From the definition of σ, it is clear that σ restricts to a bijection from N to N_1. We apply Lemma 4.1.4 to N and N_1 (with $\langle a^{32}\rangle$, $X = \langle a^4,h\rangle$ for Z,H respectively and $\langle a_1^{32}\rangle$, $X_1 = \langle a_1^4,h_1\rangle$ for $\bar{Z}$, $\bar{H}$ respectively. By (54), $X \cong X_1$ and $\sigma: a^{4k}h^j \to a_1^{4k}h_1^j$ defines an isomorphism $X \to X_1$. Thus, in particular, σ induces a projectivity from X to X_1.

Similarly $N/\langle a^{32}\rangle \cong N_1/\langle a_1^{32}\rangle$ (by (53) and (78)) and

$$\sigma : \langle a^{32}\rangle a^{2k}h^j \to \langle a_1^{32}\rangle a_1^{2k}h_1^j$$

defines such an isomorphism (by (102)). Suppose that $U \le N$ and $U \nleq X$. Then (75) and (76) show that $\langle a^{32}\rangle \le U$; and similarly if $U_1 \le N_1$ and $U_1 \nleq X_1$, then (80) gives $\langle a_1^{32}\rangle \le U_1$. Thus Lemma 4.1.4 shows that σ induces a projectivity $N \to N_1$. $\square$

Now let K be a subgroup of E. By (51) and (52), $N \lhd E$ and $E = N\langle q\rangle$. So $K = UV$ where $U = K \cap N$ and V is cyclic. By Lemma 4.4.1 $K^\sigma = U^\sigma V^\sigma$, and by Lemma 4.4.2 U^σ is a subgroup of E_1. Also V^σ is a subgroup of E_1, by (73). Again by (73) $(K^\sigma)^{-1} = K$. Therefore

$$U^\sigma V^\sigma = K^\sigma = (K^\sigma)^{-1} = (V^\sigma)^{-1}(U^\sigma)^{-1} = V^\sigma U^\sigma$$

and it follows that K^σ is a subgroup of E_1. We have now shown that

<u>σ (and hence π, by (15)) map each subgroup of E to a subgroup of E_1.</u>

4.5 Consideration of π applied to subgroups outside E.

Let $x = a^k h^j q^i$ where k is odd. Then $x \notin E$, but $|G:E| = 2$ and so $x^2 \in E$. From section 4 we know that $\langle x^2\rangle^\sigma$ is a subgroup of G_1. We will prove next that

$$\langle x^2\rangle^\sigma = \langle (x^\sigma)^2\rangle \quad . \tag{103}$$

For this purpose it suffices to show that

(i) $|x| = |x^\sigma|$ and

(ii) $(x^\sigma)^2 \in \langle x^2\rangle^\sigma$.

Proof of (i). Remembering that k is odd, we easily obtain (from (50))

$$(q^i)^{a^k} = h^{2ki}q^{-i} \quad . \tag{104}$$

Similarly

$$(h^j)^{a^k} = h^{3j-4kj}q^{4j} \quad . \tag{105}$$

Then (104) and (105) give

$$x^2 = a^{2k}h^{4j(1-k)+2ki+8ji}q^{4j} \quad . \tag{106}$$

Since the factors in (106) commute, taking the 8th powers gives

$$x^{16} = a^{16k} \quad . \tag{107}$$

In particular $|x| = |a| = 64$.

Now since k is odd, x^σ has a_1 exponent (in (66)) odd. Therefore consider an element of G_1 of the form $x_1 = a_1^\gamma h_1^\beta q_1^\alpha$ where γ is odd. Using (59) and (77) gives

$$(q_1^\alpha)^{a_1^\gamma} \equiv a_1^{4\alpha(\gamma-1)} h_1^{2\alpha(2\alpha+\gamma+4)} q_1^{-\alpha} \mod \langle a_1^{16} \rangle \quad ; \tag{108}$$

and (59) and (78) give

$$(h_1^\beta)^{a_1^\gamma} \equiv a_1^{-4\beta} h_1^{\beta(4\gamma+3)} q_1^{4\beta} \mod \langle a_1^{16} \rangle \quad . \tag{109}$$

(These congruences can easily be established by induction on γ .)

Then (108) and (109) show that

$$x_1^2 \equiv a_1^{4\alpha\gamma+2\gamma-4\alpha-4\beta} h_1^{4\beta(1+\gamma)+2\alpha\gamma+4\alpha^2+8\alpha+4\alpha\beta} q_1^{4\beta} \mod \langle a_1^{16} \rangle \quad . \tag{110}$$

The factors on the right hand side of (110) commute modulo $\langle a_1^{32} \rangle$ (from (77) and (78)) and hence, taking 8th powers in (110), we obtain

$$x_1^{16} = a_1^{16\gamma+32\beta} \quad .$$

Therefore, since γ is odd,

$$\langle x_1^{16} \rangle = \langle a_1^{16} \rangle$$

and $|x_1| = |a_1| = 64$. This proves (i).

In order to prove (ii) we may work modulo $\langle a_1^{16}\rangle$. For, $a_1^{16} = (a^{16})^\sigma = (x^{16\ell})^\sigma$ (where ℓ is odd, by (107)) and so $a_1^{16} \in \langle x^2\rangle^\sigma$. Recall that $x = a^k h^j q^i$, where k is odd. We have $x^\sigma = a_1^\gamma h_1^\beta q_1^\alpha = x_1$ (say), where

$$\gamma = k(1+4i) \quad , \quad \beta = j(1+4i) \quad , \quad \alpha = \begin{cases} i+2 & \text{if } i \text{ odd} \\ i+4jk & \text{if } i \text{ even} \end{cases} \tag{111}$$

(by (66)). From (106) and (66) we obtain

$$(x^2)^\sigma \equiv a_1^{2k} h_1^{4j(1-k)+2ki+8ji} q_1^{4j} \mod \langle a_1^{16}\rangle \ . \tag{112}$$

We want to show that the congruence

$$((x^2)^\sigma)^\lambda \equiv x_1^2 \mod \langle a_1^{16}\rangle$$

(where x_1^2 is given by (110) and (111)) has an integer solution for λ . Comparing exponents of a_1, h_1, q_1 in (110) and the λ-th power of (112) (noting that the factors on the right-hand side of (112) commute), we must solve

$$k\lambda \equiv 2\alpha\gamma + \gamma - 2\alpha - 2\beta \mod 8 \ , \tag{113}$$

$$(2j(1-k)+ki+4ji)\lambda \equiv 2\beta(1+\gamma)+\alpha\gamma+2\alpha^2+4\alpha+2\alpha\beta \mod 8 \ , \tag{114}$$

$$j\lambda \equiv \beta \mod 2 \ . \tag{115}$$

We substitute for α,β,γ from (111) and note that $k^2 \equiv 1 \bmod 8$, since k is odd. When i is odd the solution of (113) is

$$\lambda \equiv -1-2k(j+1) \mod 8$$

which clearly satisfies (115) and can easily be checked to satisfy (114). When i is even, the solution of (113) is

$$\lambda \equiv 1-2jk \mod 8$$

which again satisfies (114) and (115). Therefore (ii) is true and (103) follows.

Suppose that K is a non-cyclic subgroup of G with $K \not\leq E$. We will show that

$$\underline{K^\sigma \text{ is a subgroup of } G_1}\ . \tag{116}$$

Clearly, K contains an element of the form $x = a^k h^j q^i$ where k is odd. We claim that

$$F = \langle h^8, a^8 \rangle \leq K\ . \tag{117}$$

For, since G/H is cyclic and K is non-cyclic, $K \cap H = 1$. Thus if $h^8 \not\leq K$, then $K \cap H$ contains an element of the form $h^{8j_1}q^4$ in $\Omega_1(H)$. But then K contains

$$[h^{8j_1}q^4, x] = [h^{8j_1}q^4, a^k h^j q^i] = [h^{8j_1}q^4, a^k] = [q^4, a^k] = h^8 ,$$

giving a contradiction. Therefore $h^8 \in K$. Also K contains $x^8 = a^{8k}h^{8i}$, by (106), and so $a^8 \in K$. Then (117) follows.

Now let $F_1 = \langle a_1^8, h_1^8 \rangle$. So $F_1 = F^\sigma$. Also, for all $x \in G$,

$$(Fx)^\sigma = F_1 x^\sigma . \tag{118}$$

For, let $f \in F$. Then $(fx)^\sigma \equiv f^\sigma x^\sigma \bmod \langle a_1^{32} \rangle$ and so $(fx)^\sigma \in F_1 x^\sigma$. Thus, by order considerations, (118) follows. By (60), F lies in the centre of G, and from the presentation of G_1, we see that $F_1 \lhd G_1$. Recall that K is a non-cyclic subgroup of G and that $K \nleq E$. In order to prove (116) we distinguish three cases.

Case 1 : K/F is cyclic. Then $K = \langle F, x \rangle$, where $x = a^k h^j q^i$ and k is odd. It suffices to show that

$$(Fx^{2r+1})^\sigma = F_1 (x^{2r})^\sigma x^\sigma , \tag{119}$$

for any integer r. For, recalling (103), $\langle x^2 \rangle^\sigma = \langle (x^\sigma)^2 \rangle$. Also any generator of $\langle x \rangle$ can be written as x^{2r+1}. Hence if (119) holds, then

$$(x^{2r+1})^\sigma \in F_1 (x^{2r})^\sigma x^\sigma \subseteq F_1 \langle x^\sigma \rangle .$$

Thus

$$\langle x \rangle^\sigma \subseteq F_1 \langle x^\sigma \rangle . \tag{120}$$

Therefore

$$K^\sigma = (F\langle x\rangle)^\sigma \subseteq F_1\langle x^\sigma\rangle$$

by (118) and (120). But by (103) and (118)

$$(F\langle x^2\rangle)^\sigma = F_1\langle (x^\sigma)^2\rangle \quad .$$

Since $F\langle x^2\rangle$ has index 2 in $F\langle x\rangle$ and $F_1\langle (x^\sigma)^2\rangle$ has index 2 in $F_1\langle x^\sigma\rangle$, order considerations show that

$$K^\sigma = (F\langle x\rangle)^\sigma = F_1\langle x^\sigma\rangle \quad .$$

Thus K^σ is a subgroup of G_1 .

To prove (119), we have (from (106))

$$x^2 \equiv a^{2k}h^{2ki}q^{4j} \bmod F \ .$$

Since the factors on the right hand side of this congruence commute (as is easily seen from the presentation (50) of G), it follows that

$$x^{2r} \equiv a^{2kr}h^{2kir}q^{4jr} \bmod F \ .$$

Then (again from (50))

$$x^{2r+1} \equiv a^{2kr+k}h^{-2kir+j}q^{4jr+i} \bmod F \ .$$

Therefore

$$(x^{2r+1})^{\sigma} \equiv a_1^{k(2r+1)(1+4i)} h_1^{(-2kir+j)(1+4i)} q_1^{i_1} \mod F_1$$

where $i_1 = 4jr+i+2$ if i is odd and $i_1 = 4jr+i+4j$ if i is even. It follows that

$$\begin{aligned}(x^{2r+1})^{\sigma} &\equiv a_1^{2kr+k(1+4i)} h_1^{-2kir+j(1+4i)} q_1^{i_1} \mod F_1 \\ &\equiv a_1^{2kr} h_1^{2kir} q_1^{4jr} a_1^{k(1+4i)} h_1^{j(1+4i)} q_1^{i_1-4jr} \mod F_1 \\ &\equiv (x^{2r})^{\sigma} x^{\sigma} \mod F_1 .\end{aligned}$$

We have now proved (119) and hence Case 1 is complete.

<u>Case 2</u> : $K \cap H \leq \langle h^2, q^2 \rangle$. Let $v = a^{k_1} h^{j_1} q^{i_1}$, $w = h^{2j_2} q^{2i_2}$ be elements of G . Since h_1^2 and q_1 commute modulo F_1 , we see that

$$(vw)^{\sigma} \equiv v^{\sigma} w^{\sigma} \mod F_1 . \tag{121}$$

Now $K/K \cap H \cong KH/H$ and therefore $K/K \cap H$ is cyclic and

$$K = V(K \cap H) ,$$

where V is cyclic. Thus from (121) it follows that

$$K^{\sigma} \equiv V^{\sigma}(K \cap H)^{\sigma} \mod F_1 ;$$

i.e. $F_1K^\sigma = F_1V^\sigma(K \cap H)^\sigma$ and so, by (118),

$$K^\sigma = (F_1V^\sigma)(K \cap H)^\sigma \quad . \tag{122}$$

Applying case 1 to FV, we see that $(FV)^\sigma$ is a subgroup of G_1. Also (118) shows that $(FV)^\sigma = F_1V^\sigma$; and from section 4.2 we know that $(K \cap H)^\sigma$ is a subgroup of G_1. Now, by section 4.1 and case 1, K^σ contains all powers, in particular the inverse, of each of its elements. Therefore from (122)

$$K^\sigma = (K^\sigma)^{-1} = (K \cap H)^\sigma(F_1V^\sigma)$$

and hence K^σ is a subgroup of G_1.

<u>Case 3</u> : $K \cap H \nleq \langle h^2,q^2\rangle$. We claim that

$$\langle a^4,h^4,q^4\rangle \leq K \ . \tag{123}$$

For, since $K \cap H \nleq \langle h^2,q^2\rangle$, K contains an element

$$u = h^{j_1}q^{i_1}$$

where at least one of j_1, i_1 is odd. Also, since $K \nleq E$, K contains an element

$$x = ah^jq^i \quad .$$

From (106)

$$x^2 \equiv a^2h^{2i}q^{4j} \bmod F \ .$$

Suppose that i_1 is odd. Then without loss of generality we may assume that $i_1 = 1$. Thus K contains $[u,x^2]$; and modulo F

$$[u,x^2] \equiv [h^{j_1}q,a^2] \equiv [h^{j_1},a^2][q,a^2]$$

$$\equiv [q,a^2] \quad \text{(by (53))}$$

$$\equiv h^4 \quad \text{(by (52))}.$$

Since $F \leq K$, it follows that $h^4 \in K$. Therefore

$$q^4 \in \langle u^4,h^4 \rangle \leq K .$$

Now suppose that i_1 is even. Then j_1 is odd and we may even assume that $j_1 = 1$. Hence $h^4 = u^4 \in K$. Also

$$[u,x] = [hq^{i_1},ah^jq^i] = [h,ah^jq^i]^{q^{i_1}}[q^{i_1},ah^jq^i] .$$

Thus modulo F

$$[u,x] \equiv [h,a][q,a]^{i_1} \equiv h^{-2}q^4(h^2q^{-2})^{i_1} \quad \text{(from (50))}$$

$$\equiv h^{-2+2i_1}q^{4-2i_1} .$$

Therefore $h^{-2}q^{4-2i_1} \in K$. Then K contains

$$u^2h^{-2}q^{4-2i_1} = q^4 .$$

It follows that, for all i_1 ,

$$\langle h^4, q^4 \rangle \leq K .$$

Now K contains x^4 and, by (106), $x^4 = a^4 h^{4i}$. Thus $a^4 \in K$ and (123) follows.

Let $J = \langle a^4, h^4, q^4 \rangle$. Then $J \triangleleft G$. For, from (50) we see that $\langle h^4, q^4 \rangle \triangleleft G$. Also from (52) and (53) a^4 is central in G modulo $\langle h^4, q^4 \rangle$. Similarly $J_1 = J^\sigma \triangleleft G_1$. For, from (55), it follows that

$$\mho_2(Y_1) = \langle a_1^{16}, h_1^4, q_1^4 \rangle \triangleleft G_1 \;;$$

and, modulo $\mho_2(Y_1)$, a_1^4 is central in G_1 .

Let $g \in G$. Then

$$(Jg)^\sigma = J_1 g^\sigma \; . \tag{124}$$

To see this, let $y \in J$. Thus

$$y = a^{4k} h^{4j} q^{4i} \quad \text{and} \quad g = a^{k_1} h^{j_1} q^{i_1} \; .$$

Then

$$yg \equiv a^{4k+k_1} h^{4j+j_1} q^{4i+i_1} \mod F$$

and so

$$(yg)^\sigma \equiv a_1^{(4k+k_1)(1+4i_1)} h_1^{(4j+j_1)(1+4i_1)} q_1^{i_2} \mod F_1$$

where $i_2 = 4i+i_1+2$ if i_1 is odd and $i_2 = 4i+i_1+4k_1j_1$ if i_1 is even by (118). Thus

$$(yg)^\sigma \equiv a_1^{k_1}h_1^{j_1}q_1^{i_2} \equiv g^\sigma \bmod J_1 \quad .$$

Therefore

$$(Jg)^\sigma = \bigcup_{y\in J}(yg)^\sigma \subseteq J_1g^\sigma$$

and (124) follows.

The groups G/J and G_1/J_1 are isomorphic via the map induced by $a \to a_1$, $h \to h_1$, $q \to q_1$ and σ induces this isomorphism. Therefore if $g_1,g_2 \in K$, then

$$\begin{aligned} g_1^\sigma g_2^\sigma \in J_1K^\sigma &= (JK)^\sigma \qquad \text{(by (124))} \\ &= K^\sigma \end{aligned}$$

by (123). Thus K^σ is a subgroup of G_1 .

We have finally proved (116), i.e. for every non cyclic subgroup K of G with $K \nleq E$, K^σ is a subgroup of G_1 .

4.6 Surjectivity of π.

We now know that π , defined by (72), maps each subgroup of G to a subgroup of G_1 of the same order. Let U and V be subgroups of G with $U < V$. Then

$$U^{\pi} < V^{\pi} \quad . \tag{125}$$

For, by (76) and (107), E has exponent 32 and G has exponent 64. Thus suppose that U is cyclic of order 64, generated by $u = ah^jq^i$. Then V is non-cyclic and so $V^{\pi} = V^{\sigma}$. But $u^{\sigma} \in V^{\sigma}$ and so $\langle u^{\sigma}\rangle \le V^{\sigma}$, i.e. $U^{\pi} < V^{\pi}$.

Now suppose that V is cyclic of order 64, generated by $v = ah^jq^i$. Then $U \le E \cap \langle v^2\rangle$ and so

$$U^{\pi} = U^{\sigma} \le \langle v^2\rangle^{\sigma} = \langle (v^{\sigma})^2\rangle \qquad \text{(by (103))}$$
$$< \langle v^{\sigma}\rangle = V^{\pi} \quad .$$

Finally suppose that neither U nor V is cyclic of order 64. Then $U^{\pi} = U^{\sigma} < V^{\sigma} = V^{\pi}$. We have now proved (125).

In order to prove that π is a projectivity from G to G_1 it is sufficient now to show that each subgroup of G_1 occurs in the image of π . This will follow from the following result.

Lemma 4.6.1 Let G, G_1 be finite 2-groups. Suppose that π is a map from the subgroup lattice of G into the subgroup lattice of G_1 such that $U \leq V$ if and only if $U^{\pi} \leq V^{\pi}$ and

(i) $|U| = |U^{\pi}|$, all $U \leq G$,

(ii) U^{π} is cyclic whenever U is cyclic,

(iii) $G^{\pi} = G_1$.

Then π is a projectivity from G to G_1.

Proof. Suppose that the Lemma is false. Choose $K_1 \leq G_1$ with $|K_1|$ minimal subject to

(a) K_1 has no preimage under π and

(b) there is a subgroup $N \leq G$ with $N^{\pi} > K_1$ and $|N^{\pi}:K_1| = 2$.

This choice is possible by (iii). Also N is not cyclic, by (i) and (ii). Therefore there exist maximal subgroups $M_1 \neq M_2$ of N. Let $M = M_1 \cap M_2$. Then $|N:M| = 4$ and so $|N^{\pi}:M^{\pi}| = 4$, by (i). Since $M \triangleleft N$ and $M^{\pi} \triangleleft N^{\pi}$ and N/M, N^{π}/M^{π} are elementary of order 4, it follows that $K_1 \not\geq M^{\pi}$. Let $L_1 = M^{\pi} \cap K_1$. Then $L_1 < M^{\pi}$ and $L_1 \triangleleft N^{\pi}$ with N^{π}/L_1 elementary of order 8. Now $|M^{\pi}:L_1| = 2$ and therefore, by choice of K_1, there is a subgroup $L \leq G$ such that $L^{\pi} = L_1$.

We claim that

$$\underline{\text{there is an element } t \in N \text{ such that } t^2 \notin L.} \tag{126}$$

For, if not, $\mho_1(N) \leq L$ and then $L \triangleleft N$. Since $|N:L| = 8$, N/L is then elementary of order 8. Thus K_1 would have a preimage under π . Then (126) follows.

Let $T = \langle L,t \rangle$. If $T = N$, then $N = \langle M,t \rangle$ and N/M is cyclic, which is not the case. Therefore $T < N$ and $|T:L| = 4$, by (126). Thus $|T^{\pi}:L_1| = 4$, by (i). Now we see that

<u>there is a unique subgroup strictly between T and L</u> .

For, if there were two such subgroups, they would be normal in T and L would be their intersection, showing that T/L is elementary of order 4. But T/L is cyclic by definition.

Now $T^{\pi}/L_1 \leq N^{\pi}/L_1$ and so T^{π}/L_1 is elementary of order 4. Therefore there are three subgroups strictly between T^{π} and L_1 (all of index 4 in N^{π}) and there is only one subgroup strictly between T and L , contradicting our choice of K_1 . □

Returning to the conclusion of the proof of Theorem 4.1.1, we see that all the hypotheses of Lemma 4.6.1 are satisfied by our groups G and G_1 , and the map π , defined in (50), (59) and (72). Therefore we have finally shown that $\pi: G \to G_1$ is a projectivity, $H \triangleleft G$, H^{π} is not abelian, and H^{π} is core-free in G_1 . This completes the proof of Theorem 4.1.1 .

□

Remark. Lemma 4.6.1 does not hold for finite p-groups when p is odd. For, let G be the non-abelian group of order p^3 and exponent p and let G_1 be the elementary abelian p-group of rank 3. It is not difficult to define a map π, from the lattice of subgroups of G to the lattice of subgroups of G_1, which is not a projectivity but which satisfies the hypotheses of Lemma 4.6.1 . .

Chapter 5.

On the embedding of core-free images of normal subgroups.

5.1 Introduction

As usual, let G and G_1 be groups, $H \triangleleft G$ and let $\pi: G \to G_1$ be a projectivity such that H^π is core-free in G_1. As already mentioned in 1.1, R. Schmidt ([19] , Theorem 3.4) has shown that, if G is finite, there exist series

$$1 = N_0 \leq N_1 \leq \ldots . \leq N_t = H^{\pi,G}$$

and

$$1 = M_0 \leq M_1 \leq \ldots . \leq M_s = (H^\pi)^{G_1}$$

of normal subgroups of G and G_1 respectively, such that, for all $0 \leq i \leq t-1$, $0 \leq j \leq s-1$, N_{i+1}/N_i and M_{j+1}/M_j are cyclic, and, even more, central in G and G_1 respectively (i.e. $[N_{i+1}, G] \leq N_i$ and $[M_{j+1}, G_1] \leq M_j$), if H^π is quasinormal in G_1. This chapter is just concerned with the attempt to extend Schmidt's result to infinite groups. We now briefly discuss the results obtained. First of all we recall the definition of series.

Let X be a group and let Σ be a linearly ordered set. Following Robinson ([16] , 1.2), a series in X with ordered type Σ is a set of subgroups of X

$$\mathcal{S} = \{ \Lambda_\sigma , V_\sigma \mid \sigma \in \Sigma \}$$

such that

(a) $X = \bigcup_{\sigma\in\Sigma}(\Lambda_\sigma \setminus V_\sigma)$.

(b) $\Lambda_\tau \leq V_\sigma$ if $\tau < \sigma$.

(c) $V_\sigma \lhd \Lambda_\sigma$.

The subgroups Λ_σ and V_σ are the terms of $\mathcal{S}$, and the groups $\Lambda_\sigma / V_\sigma$ are the factors of $\mathcal{S}$. From the definition of $\mathcal{S}$ it follows that, for $1 \neq x \in X$, there exists a unique $\sigma = \sigma(x)$ in Σ such that

$$x \in \Lambda_{\sigma(x)} \setminus V_{\sigma(x)}.$$

If Y is a group acting on X, $\mathcal{S}$ is said to be Y-invariant if each term of $\mathcal{S}$ is Y-invariant.

Returning to the groups G, G_1, the projectivity π and the normal subgroup H of G, in the light of Schmidt's result the following question arises naturally:

do exist a G-invariant series $\mathcal{S} = \{\Lambda_\sigma, V_\sigma \mid \sigma\in\Sigma\}$ in $H^{\pi,G}$ and a G_1-invariant series $\mathcal{S}_1 = \{\Lambda'_\mu, V'_\mu \mid \mu\in M\}$ in $(H^\pi)^{G_1}$ such that

(i) Λ_σ/V_σ and Λ'_μ/V'_μ are cyclic, (1)

or, if H^π is quasinormal in G_1,

(ii) $[\Lambda_\sigma, G] \leq V_\sigma$ and $[\Lambda'_\mu, G_1] \leq V'_\mu$.

The following recent result due to Napolitani and Zacher ([14] , Satz 2.6), reduces question (1) to the case that H^π is quasinormal in G_1.

Theorem 5.1.1. Let G and G_1 be groups, $\pi : G \to G_1$ a projectivity and $H \triangleleft G$ such that H^{π} is core-free in G_1. If H^{π} is not quasinormal in G_1, then G and G_1 are periodic, $G = (\underset{i\in I}{\mathrm{Dr}}\, P_i) \times K$, $G_1 = (\underset{i\in I}{\mathrm{Dr}}\, P_i^{\pi}) \times K^{\pi}$ where P_i and P_i^{π} are P-groups, and elements of distincts direct factors have coprime order. (Thus, in particular, $H = (H \cap \underset{i\in I}{\mathrm{Dr}}\, P_i) \times (H \cap K)$ and $H \cap K \triangleleft G$). Moreover $(H \cap K)^{\pi}$ is quasinormal in G_1.

From Theorem 5.1.1 and the structure of P-groups it is clear that, in order to answer question (1) it is sufficient to show the existence of series of type (ii) assuming that H^{π} is quasinormal in G_1. Unfortunately we have not been able to answer question (1) in total generality, and our proof holds only for a certain class of groups (see Theorem 5.3.4.). The reason for this is partially due to the fact that it is still not clear to what extent Maier-Schmid theorem (Theorem 1.2.5) holds for infinite groups; and, as a matter of fact, Theorem 1.2.5 is an essential tool in the proof of the above mentioned Schmidt's result. We discuss briefly the relevance of a possible extention to infinite groups of Theorem 1.2.5, in relation with question (1). Although, as we have seen in 2.2, Theorem 1.2.5 is false if we remove from the statement the hypothesis of finiteness of G, the following questions still do not have an answer. Let Q be a core-free quasinormal subgroup of a group X;

does exist an X-invariant series in Q^X whose factors are central in X ? (2)

Is $Q \leq Z_{n_Q}(X)$ for some $n_Q < \infty$ if X is assumed to be finitely generated modulo Q (i.e. $X = \langle Q, x_1, \dots, x_n \rangle$, $n < \infty$) ? (3)

A positive answer to question (3) would lead, using a method described in [16], 8.2, that we will briefly summarize in 5.3, to a positive solution of questions (1) and (2).

It is well known that, if X is finitely generated modulo Q, Q^X is nilpotent of finite exponent ([10]) and $X/C_X(Q^X)$ is periodic ([4]). Therefore question (3) can be split in the following way.

If X is finitely generated modulo the core-free quasinormal subgroup Q and S is the Sylow p-subgroup of Q, is $X/C_X(S^X)$ a p-group ? (4)

If X is finitely generated modulo the core-free quasinormal subgroup Q, is $X/C_X(Q^X)$ finite ? (5)

As far as we know, neither (4) nor (5) have been solved. On the other hand the situation has shown to be easier to handle in the context of projectivities, namely when there exist a group G, a normal subgroup H of G and a projectivity $\pi: G \to X = G^\pi$ such that $Q = H^\pi$. In this case we have been able to solve question (4). More precisely we shall prove the following theorem.

Theorem 5.1.2. Let G and G_1 be groups, $\pi: G \to G_1$ a projectivity and $H \lhd G$ such that G/H is finitely generated and H^π is a core-free quasinormal subgroup of G_1. Let S^π be the Sylow p-subgroup of H^π (recall that H^π is nilpotent of finite exponent by Lemma 1.2.9 (ii)). Then $G/C_G(S^{\pi,G})$ and $G_1/C_{G_1}((S^\pi)^{G_1})$ are p-groups.

As far as question (5) is concerned, it is, unfortunately, still unsettled even in the context of projectivities. It is mainly for this reason that we have obtained an answer to question (1) only for a certain class $\mathcal{A}$ of groups (see 5.3 for the definition of $\mathcal{A}$), class for which question (5) has a positive solution.

In the next section we prove Theorem 5.1.2.

5.2 Proof of Theorem 5.1.2

Since H^π is a periodic nilpotent group (Lemma 1.2.9 (ii)), by Proposition 1.2.8 (b), $S \lhd G$. Therefore S^π is a Dedekind subgroup of G_1. Since $S^\pi \lhd H^\pi$, by Theorem 1.2.2 S^π is quasinormal in G_1. We claim that

$$S^{\pi,G} \text{ and } (S^\pi)^{G_1} \text{ are locally finite p-groups.} \qquad (6)$$

This is clear for $(S^\pi)^{G_1}$, since $(S^\pi)^{G_1}$ is the join of the nilpotent subnormal p-subgroups $(S^\pi)^{v_1}$, as v_1 varies in G_1. ($(S^\pi)^{v_1}$ is subnormal in G_1 by Theorem 1.2.2 and Lemma 1.2.9 (i).) Also, if $S \neq 1$, by Proposition 1.2.4 (vi) and Lemma 1.2.7, there exists a p-element $w_1 \in G_1$

which does not normalise S^{π}. Hence $<(S^{\pi})^{G_1}, w_1>$ is a locally finite non-abelian p-group. Then, by Proposition 1.2.8 (c) it follows that $<(S^{\pi})^{G_1}, w_1>^{\pi^{-1}}$, and consequently $S^{\pi,G}$, are locally finite p-groups. In particular, by Remark 1.2.3,

the preimage under π of every conjugate of S^{π} in G_1 is quasinormal in G. (7)

Also, again by Remark 1.2.3,

every Dedekind subgroup of G (of G_1) contained in $S^{\pi,G}$ (in $(S^{\pi})^{G_1}$) is quasinormal in G (in G_1). (8)

Suppose now that x and y are elements of G such that $|<x>/<x> \cap C_G(S^{\pi,G})| = q^n$ and $|<y>^{\pi}/<y>^{\pi} \cap C_{G_1}((S^{\pi})^{G_1})| = r^m$ where q and r are primes different from p. Assume also that $<x>$ and $<y>$ are infinite cyclic or of prime power order. We will show that

$$<x> \leq C_G(S^{\pi,G}) \tag{9}$$

and

$$<y>^{\pi} \leq C_{G_1}((S^{\pi})^{G_1}). \tag{10}$$

Denote by $S_{i,h}/S_{\pi,<S,h>}$ the group $\Omega_i(S^{\pi,<S,h>}/S_{\pi,<S,h>})$, where $h \in \mathfrak{T} = \{h \in G \mid <h>^{\pi} \text{ is a p-group}\}$ and $i \geq 0$. Assume for the moment that, for all $h \in \mathfrak{T}$ and for $i \geq 1$ we have

$$[x, S_{i,h}] \leq S_{i-1,h} \tag{11}$$

and

$$[\langle y\rangle^{\pi}, S^{\pi}_{i,h}] \leq S^{\pi}_{i-1,h}. \qquad (12)$$

Thus x acts trivially on the quotients $S_{i,h}/S_{i-1,h}$. Since x induces a p'-automorphism on the finite p-group $S^{\pi,\langle S,h\rangle}/S_{\pi,\langle S,h\rangle}$ (since $S^{\pi}/(S^{\pi})_{\langle S,h\rangle^{\pi}}$ has finite index and is core-free in $\langle S,h\rangle^{\pi}/S^{\pi}_{\langle S,h\rangle^{\pi}}$, $\langle S,h\rangle^{\pi}/S^{\pi}_{\langle S,h\rangle^{\pi}}$, and therefore also $\langle S,h\rangle/S_{\pi,\langle S,h\rangle}$, are finite), by [9] 7.10 it follows that

$$[\langle x\rangle, S^{\pi,\langle S,h\rangle}] \leq S_{\pi,\langle S,h\rangle}.$$

Similarly

$$[\langle y\rangle^{\pi}, (S^{\pi})^{\langle S,h\rangle^{\pi}}] \leq (S^{\pi})_{\langle S,h\rangle^{\pi}}.$$

Therefore

$$[\langle x\rangle, S] \leq \bigcap_{h\in\mathfrak{J}} S_{\pi,\langle S,h\rangle} = 1$$

and

$$[\langle y\rangle^{\pi}, S^{\pi}] \leq \bigcap_{h\in\mathfrak{J}} (S^{\pi})_{\langle S,h\rangle^{\pi}} = 1$$

since $\bigcap_{h\in\mathfrak{J}} (S^{\pi})_{\langle S,h\rangle^{\pi}} = 1$ by Proposition 1.2.4 (vi) and Lemma 1.2.7. In particular

$$[\langle x\rangle, S_{\pi,\langle S,h\rangle}] = 1$$

and

$$[\langle y\rangle^{\pi}, (S^{\pi})_{\langle S,h\rangle^{\pi}}] = 1.$$

Therefore $\langle x\rangle$ and $\langle y\rangle^{\pi}$ act trivially on the factors of the series

$$1 \leq S_{\pi,<S,h>} \lhd S^{\pi,<S,h>}$$

and

$$1 \leq (S^{\pi})_{<S,h>^{\pi}} \lhd (S^{\pi})^{<S,h>^{\pi}}$$

respectively. Recalling that, by (6), $S^{\pi,<S,h>}$ and $(S^{\pi})^{<S,h>^{\pi}}$ are locally finite p-groups, using again [9] 7.10, it follows that

$$[x, S^{\pi,<S,h>}] = 1$$

and

$$[<y>^{\pi}, (S^{\pi})^{<S,h>^{\pi}}] = 1.$$

Then, since by Proposition 1.2.4 (vi) and Lemma 1.2.7

$$(S^{\pi})^{G_1} = <(S^{\pi})^{<S,h>^{\pi}} \mid h \in \mathfrak{I}> \text{ and } S^{\pi,G} = <S^{\pi,<S,h>} \mid h \in \mathfrak{I}>,$$

(9) and (10) follow. Hence we are reduced to prove (11) and (12).

We claim that

$<y>^{\pi}$ and $<x>^{\pi}$ normalise every conjugate R^{π} of S^{π}. (13)

This is clear for $<y>^{\pi}$, and for $<x>^{\pi}$ if $<x>^{\pi}$ is infinite cyclic or has order coprime to p, by Proposition 1.2.4 (vi) and Lemma 1.2.7 respectively. On the other hand, if $<x>^{\pi}$ is a p-group, then, from Proposition 1.2.8 (c) it follows that $<x,R>^{\pi}$ is elementary abelian, and so (13) holds even in this case. Similarly

x and y normalise the preimage under π of every conjugate of S^{π}. (14)

Consider now the group $A = <S,h,x,y>$, where $h \in \mathfrak{I}$. From (13) it follows that $(S^{\pi})^{A^{\pi}} = (S^{\pi})^{<S,h>^{\pi}}$ and $(S^{\pi})_{A^{\pi}} = (S^{\pi})_{<S,h>^{\pi}}$. Hence, by Theorem 2.1.1, $S_{\pi,<S,h>}$ and $S^{\pi,<S,h>}$ are normal in A, and therefore

$S_{i,h}$ and $S_{i,h}^{\tau}$ are normal in A and A^{τ} respectively for all $i \geq 0$. Also, as a result of Lemma 1.2.6 (c) applied to the finite p-group $\langle S,h\rangle^{\tau}/(S^{\tau})_{\langle S,h\rangle^{\tau}}$, $S^{\tau}S_{i,h}^{\tau}/S_{i,h}^{\tau}$ is core-free in $\langle S,h\rangle^{\tau}/S_{i,h}^{\tau}$ for all $i \geq 0$. Fix an $i \geq 1$. Since our argument in order to prove (11) and (12) will take place inside the groups $B = \langle S_{i,h}, h, x, y\rangle$ and B^{τ}, factoring by $S_{i-1,h}$ and $S_{i-1,h}^{\tau}$, we may assume, without loss of generality, that $S_{i-1,h} = 1$. Then, in particular, $i=1$. Set $X = \langle \Omega_1(S), h\rangle$.

$$\Omega_1(S^{\tau}) \text{ is now core-free in } X \tag{15}$$

and, since $\Omega_1(S)$ is normal in B,

$$\Omega_1(S^{\tau}) \text{ is quasinormal in } B^{\tau}$$

by (8). Therefore, assuming $\Omega_1(S) \neq 1$ (if $\Omega_1(S) = 1$ there is nothing to prove), from Proposition 1.2.8 (c) it follows that

$$X \text{ is a finite p-group.}$$

Then Lemma 3.2.1 (xii) applied to X and X^{τ} shows that

$$\Omega_1(S) \text{ contains a unique normal subgroup of } X \text{ of order } p. \tag{16}$$

Thus $\Omega_1(S)$ contains a unique minimal normal subgroup N, say, of B. Let M^{τ} be a conjugate of $\Omega_1(S^{\tau})$ in B^{τ} such that $N^{\tau} \not\leq X^{\tau}$. Then $M_B \cap N = 1$ and therefore

$$M_B \cap \Omega_1(S) = 1. \tag{17}$$

Moreover, as a result of Lemma 3.2.1 (ii) and (iv),

$$\Omega_1(X) \text{ and } \Omega_1(X^{\tau}) \text{ are elementary abelian} \tag{18}$$

and

$$\Omega_1(X) = \Omega_1(S) \times \Omega_1\langle h\rangle, \quad \Omega_1(X^{\tau}) = \Omega_1(S^{\tau}) \times \Omega_1(\langle h\rangle^{\tau}). \tag{19}$$

In particular, since $\Omega_1(S) \neq 1$ and $\Omega_1(S^\pi)$ is not normal in X^π, recalling also that $S_{1,h} \leq \Omega_1(X)$, it follows from (19) that

$$S_{1,h} = \Omega_1(X) \quad \text{and} \quad (S_{1,h}^\pi) = \Omega_1(X^\pi).$$

Thus, by (14) and by the definition of B, we see that $M_B = M_{\langle M,h\rangle}$. Furthermore, by (18), h centralises a subgroup of order p^2 of $S_{1,h}$. Therefore, by (19), $M_B \neq 1$ and consequently, by (17), $|M_B| = p$. Hence M_B^π is a core-free quasinormal subgroup of order p of B^π. The same argument used in proving (13) shows that $\langle y\rangle^\pi$ and $\langle x\rangle^\pi$ normalise every conjugate of M_B^π in B^π. Since $\langle h\rangle^\pi$ does not normalise M_B^π ((15) and (18)), $\langle yh\rangle^\pi$ and $\langle xh\rangle^\pi$ do not normalise M_B^π as well. Hence, by Lemma 2.2.2,

$$1 \neq [\langle yh\rangle^\pi, M_B^\pi] \leq \langle yh\rangle^\pi \cap Z(B^\pi) \cap S_{1,h}^\pi$$

and

$$1 \neq [\langle xh\rangle^\pi, M_B^\pi] \leq \langle xh\rangle^\pi \cap Z(B^\pi) \cap S_{1,h}^\pi.$$

On the other hand, by (15) and (19), $S_{1,h}^\pi$ $(= \Omega_1(X^\pi))$ contains a unique subgroup of order p which is normalised by $\langle h\rangle^\pi$, namely $\Omega_1(\langle h\rangle^\pi)$. Thus, necessarily,

$$\Omega_1\langle h\rangle = S_{1,h} \cap \langle xh\rangle$$

and

$$\Omega_1(\langle h\rangle^\pi) = (S_{1,h}^\pi) \cap \langle yh\rangle^\pi.$$

It follows that

$$\langle x\rangle \text{ centralises } \Omega_1\langle h\rangle \tag{20}$$

and

$$\langle y\rangle^\pi \text{ centralises } \Omega_1(\langle h\rangle^\pi). \tag{21}$$

Set $\langle h_1\rangle = \langle h\rangle^\pi$.

Using (19) we can write $S_{1,h} = \Omega_1(S^{\pi h_1^t \pi^{-1}}) \times \Omega_1\langle h\rangle$ and

$S^{\pi}_{1,h} = \Omega_1(S^{\pi h_1^t}) \times \Omega_1(\langle h\rangle^{\pi})$ for every integer t. As t varies we have, by (15),

$$\bigcap_t \Omega_1(S^{\pi h_1^t \pi^{-1}}) = 1 \quad , \qquad \bigcap_t \Omega_1(S^{\pi h_1^t}) = 1.$$

Finally, since $\langle x\rangle$ and $\langle y\rangle^{\pi}$ normalise $\Omega_1(S^{\pi h_1^t \pi^{-1}})$ and $\Omega_1(S^{\pi h_1^t})$ respectively ((14) and (13)), using (20) and (21), (11) and (12) follow.

In order to complete the proof of the theorem it remains to show that

$$G/C_G(S^{\pi,G}) \text{ and } G_1/C_{G_1}((S^{\pi})^{G_1}) \text{ are periodic groups.} \tag{22}$$

Let $\langle g\rangle$ be an infinite cyclic subgroup of G. Let also R^{π} be a conjugate of S^{π} in G_1 and $h \in \mathfrak{T}$. For all $i \geq 1$ denote by $R_{i,h}/R_{\pi,\langle R,h\rangle}$ the group $\Omega_i(R^{\pi,\langle R,h\rangle}/R_{\pi,\langle R,h\rangle})$ and by $T_{i,h}/R_{\pi,\langle R,h\rangle}$ the group $\Omega_1(RR_{i-1,h}/R_{\pi,\langle R,h\rangle})$

As a result of Lemma 1.2.6 (c) applied to the finite p-group $\langle R,h\rangle^{\pi}/R^{\pi}_{\langle R,h\rangle^{\pi}}$, we obtain

$$|R^{\pi}_{i,h} : T^{\pi}_{i,h}| = |R_{i,h} : T_{i,h}| \leq p,$$

and, moreover, $R^{\pi}R^{\pi}_{i-1,h}/R^{\pi}_{i-1,h}$ is core-free in $\langle R,h\rangle^{\pi}/R^{\pi}_{i-1,h}$.

Thus, recalling that $\langle g\rangle^{\pi}$ normalises $R^{\pi}_{i,h}$ and every conjugate of $T^{\pi}_{i,h}$ in $\langle R,h\rangle^{\pi}$ (Lemma 1.2.7) and, similarly, g normalises $R_{i,h}$ and the preimage under π of every conjugate of $T^{\pi}_{i,h}$ in $\langle R,h\rangle^{\pi}$, it follows that

$$[g^{p-1}, R_{i,h}] \leq R_{i-1,h}$$

and

$$[g_1^{p-1}, R^{\pi}_{i,h}] \leq R^{\pi}_{i-1,h},$$

where $\langle g_1\rangle = \langle g\rangle^{\pi}$. As h varies in $\mathfrak{T}$ and R^{π} varies in the set of conjugates of S^π in G_1, the exponents of the groups $R^{\pi,<R,h>}/R_{\pi,<R,h>}$ have a common upper bound (this is because $(S^\pi)^{G_1}$ has finite exponent by Lemma 1.2.9 (i),(ii)). Therefore there exists an integer s such that

$$[g^{(p-1)s}, R] \leq R_{\pi,<R,h>}$$

and

$$[g_1^{(p-1)s}, R^\pi] \leq (R^\pi)_{<R,h>^\pi}$$

for all $h \in \mathfrak{T}$ and for every conjugate R^π of S^π in G_1. Moroever $\bigcap_{h \in \mathfrak{T}} (R^\pi)_{<R,h>^\pi} = 1$, by Proposition 1.2.4 (vi) and Lemma 1.2.7, for every R^π. Therefore, since $(S^\pi)^{G_1}$ is the join of the R^π's, we obtain

$$[g^{(p-1)s}, S^{\pi,G}] = 1$$

and

$$[g_1^{(p-1)s}, (S^\pi)^{G_1}] = 1.$$

This proves (22). The proof of Theorem 5.1.2 is now completed.

□

5.3 On Maier-Schmid theorem in the context of projectivities.

Let $\mathcal{A}$ be the class of groups defined as follows:

a group G belongs to $\mathcal{A}$ if and only if every periodic homomorphic image of a finitely generated subgroup of G is finite.

Note that the class $\mathcal{A}$ is projectively invariant. For, suppose that $G \in \mathcal{A}$, $\pi: G \to G_1$ is a projectivity, F^π is a finitely generated subgroup of G_1

and $N^{\pi} \triangleleft F^{\pi}$ such that F^{π}/N^{π} is periodic. By Lemma 1.2.9 (i), $|N^F : N|$ is finite. Also, since F/N^F is periodic, by hypothesis F/N^F is finite. Therefore $|F : N| < \infty$ and so F^{π}/N^{π} is finite.

In this section we give a positive solution to question (1) stated in 1.1, assuming that the group G belongs to $\mathcal{A}$. We first give a brief summary of the method employed, which is essentially the same as the one described in [16] , 8.2. In fact the next paragraph is entirely taken from [16] , 8.2.

Let X be a group and let $\mathcal{S} = \{\Lambda_{\sigma}, V_{\sigma} \mid \sigma \in \Sigma\}$ be a series in X. $\mathcal{S}$ determines a binary relation $\prec$ on X defined as follows: $x \prec y$ means that either $x = 1$ or $x \neq 1$ and $\sigma(x) \leq \sigma(y)$ (recall that $\sigma(x)$ is the unique element of Σ such that $x \in \Lambda_{\sigma(x)} \setminus V_{\sigma(x)}$). It is easy to see that $\prec$ has the following properties

$$\left.\begin{array}{l}
\text{(i) } x \prec y \text{ and } y \prec z \text{ imply that } x \prec z,\\
\text{(ii) either } x \prec y \text{ or } y \prec x \text{ (possibly both)},\\
\text{(iii) } x \prec 1 \text{ implies } x = 1,\\
\text{(iv) } x \prec y \text{ and } z \prec y \text{ imply } xz^{-1} \prec y,\\
\text{(v) } y \prec x^{y} \text{ imply } y \prec z.
\end{array}\right| \quad (23)$$

Conversely, if $\prec$ is a binary relation on X satisfying (23), it determines a series in X in the following way. Let us define

$$x \sim y \text{ if and only if both } x \prec y \text{ and } y \prec x \text{ hold.}$$

Then $\sim$ is an equivalence relation on G by (i) and (ii). Let Σ be the set

of all $\sim$-equivalence classes on G other than $\{1\}$ (note that $\{1\}$ is a $\sim$-equivalence class by (iii)). Define a linear ordering on Σ as follows:

if $\sigma, \tau \in \Sigma$, then $\sigma < \tau$ if and only if $\sigma \neq \tau$ and there exist $x \in \sigma$ and $y \in \tau$ such that $x \prec y$.

By (i) $\prec$ is well-defined and, by (ii), $\prec$ is a linear ordering on Σ.

If $\sigma \in \Sigma$ let

$$\Lambda_\sigma = \{x \mid x \in G, x \prec y \text{ for some } y \in \sigma\}$$

and

$$V_\sigma = \bigcup_{\tau < \sigma} \Lambda_\tau .$$

It is shown in [16], 8.2, that $\{\Lambda_\sigma, V_\sigma \mid \sigma \in \Sigma\}$ is a series in X. Evidently we have obtained a 1-1 correspondence between series in X and binary relations on X satisfying (23).

Suppose now, in addition, that there is a group G acting on X and denote by x^g the image of $x \in X$ under the action of $g \in G$. If $\mathcal{S}$ is a G-invariant series in X such that G induces the identity on the factors of $\mathcal{S}$, then the binary relation $\prec$ on X determined by $\mathcal{S}$ (in the way defined above) satisfies

$$x \nprec x^{-1}x^g \quad \text{for all } 1 \neq x \in X,\ g \in G. \tag{24}$$

For, $x^{-1}x^g \in V_{\sigma(x)}$, and this implies that either $x^{-1}x^g = 1$ or $\sigma(x^{-1}x^g) < \sigma(x)$. In both cases, by definition of $\prec$, it follow that $x \nprec x^{-1}x^g$. Conversely, if $\prec$ is a binary relation on X satisfying (24) in addition to (23), then the series determined by $\prec$ in the way defined above is G-invariant and G induces the identity on the factors. For, suppose that $1 \neq x \in \Lambda_\sigma$

of all $\sim$-equivalence classes on G other than $\{1\}$ (note that $\{1\}$ is a $\sim$-equivalence class by (iii)). Define a linear ordering on Σ as follows:

if $\sigma, \tau \in \Sigma$, then $\sigma < \tau$ if and only if $\sigma \neq \tau$ and there exist $x \in \sigma$ and $y \in \tau$ such that $x \prec y$.

By (i) $\prec$ is well-defined and, by (ii), $\prec$ is a linear ordering on Σ.

If $\sigma \in \Sigma$ let

$$\Lambda_\sigma = \{x \mid x \in G, x \prec y \text{ for some } y \in \sigma\}$$

and

$$V_\sigma = \bigcup_{\tau < \sigma} \Lambda_\tau .$$

It is shown in [16], 8.2, that $\{\Lambda_\sigma, V_\sigma \mid \sigma \in \Sigma\}$ is a series in X. Evidently we have obtained a 1-1 correspondence between series in X and binary relations on X satisfying (23).

Suppose now, in addition, that there is a group G acting on X and denote by x^g the image of $x \in X$ under the action of $g \in G$. If $\mathcal{S}$ is a G-invariant series in X such that G induces the identity on the factors of $\mathcal{S}$, then the binary relation $\prec$ on X determined by $\mathcal{S}$ (in the way defined above) satisfies

$$x \not\prec x^{-1}x^g \quad \text{for all } 1 \neq x \in X,\ g \in G. \tag{24}$$

For, $x^{-1}x^g \in V_{\sigma(x)}$, and this implies that either $x^{-1}x^g = 1$ or $\sigma(x^{-1}x^g) < \sigma(x)$. In both cases, by definition of $\prec$, it follow that $x \not\prec x^{-1}x^g$. Conversely, if $\prec$ is a binary relation on X satisfying (24) in addition to (23), then the series determined by $\prec$ in the way defined above is G-invariant and G induces the identity on the factors. For, suppose that $1 \neq x \in \Lambda_\sigma$

of all $\sim$-equivalence classes on G other than $\{1\}$ (note that $\{1\}$ is a $\sim$-equivalence class by (iii)). Define a linear ordering on Σ as follows:

if $\sigma, \tau \in \Sigma$, then $\sigma < \tau$ if and only if $\sigma \neq \tau$ and there exist $x \in \sigma$ and $y \in \tau$ such that $x \prec y$.

By (i) $\prec$ is well-defined and, by (ii), $\prec$ is a linear ordering on Σ.

If $\sigma \in \Sigma$ let

$$\Lambda_\sigma = \{x \mid x \in G, x \prec y \text{ for some } y \in \sigma\}$$

and

$$V_\sigma = \bigcup_{\tau < \sigma} \Lambda_\tau .$$

It is shown in [16], 8.2, that $\{\Lambda_\sigma, V_\sigma \mid \sigma \in \Sigma\}$ is a series in X. Evidently we have obtained a 1-1 correspondence between series in X and binary relations on X satisfying (23).

Suppose now, in addition, that there is a group G acting on X and denote by x^g the image of $x \in X$ under the action of $g \in G$. If $\mathcal{S}$ is a G-invariant series in X such that G induces the identity on the factors of $\mathcal{S}$, then the binary relation $\prec$ on X determined by $\mathcal{S}$ (in the way defined above) satisfies

$$x \nprec x^{-1}x^g \quad \text{for all } 1 \neq x \in X,\ g \in G. \tag{24}$$

For, $x^{-1}x^g \in V_{\sigma(x)}$, and this implies that either $x^{-1}x^g = 1$ or $\sigma(x^{-1}x^g) < \sigma(x)$. In both cases, by definition of $\prec$, it follow that $x \nprec x^{-1}x^g$. Conversely, if $\prec$ is a binary relation on X satisfying (24) in addition to (23), then the series determined by $\prec$ in the way defined above is G-invariant and G induces the identity on the factors. For, suppose that $1 \neq x \in \Lambda_\sigma$

for some $\sim$-equivalence class σ. We show that

$$x^{-1}x^{g} \in V_{\sigma} \tag{25}$$

for all $g \in G$. By (24) and by definition of Λ_{σ} , $x^{-1}x^{g} \notin \sigma$. Thus, if $x^{-1}x^{g} \neq 1$ (if $x^{-1}x^{g} = 1$ obviously it belongs to V_{σ}), denoting by $[x^{-1}x^{g}]$ the $\sim$-equivalence class determined by $x^{-1}x^{g}$, we have

$$[x^{-1}x^{g}] < \sigma.$$

Therefore $\Lambda_{[x^{-1}x^{g}]} \leq V_{\sigma}$, and since $x^{-1}x^{g} \in \Lambda_{[x^{-1}x^{g}]}$, (25) follows.

We recall that, if G is a group, a local system $\mathfrak{L}$ of subgroups of G is a collection of subgroups of G such that every finitely generated subgroup of G lies within some member of $\mathfrak{L}$.

The following lemma, whose significance will be shortly clear, is a particular case of Lemma 8.22 in [16] .

Lemma 5.3.1. Let $\mathfrak{L}$ be a local system of subgroups of a group G. Suppose that, for each $H \in \mathfrak{L}$, there is a function $\alpha_{H} : H \times X \to \{0,1\}$. Then there is a function $\alpha: G \times G \to \{0,1\}$ such that, for every finite subset $\{(x_1,y_1),\ldots\ldots,(x_m,y_m)\}$ of $G \times G$, there is an $H \in \mathfrak{L}$ such that (x_i,y_i) $H \times H$ and $\alpha(x_i,y_i) = \alpha_{H}(x_i,y_i)$ for $i = 1,\ldots, m$.

Remark 5.3.2. A binary relation $\prec$ on a set X can be described by means of the function

$$\alpha_{X}: X \times X \to \{0,1\}$$

defined by

$$\alpha_{X}(x,y) = 1 \text{ if } x \prec y ,$$

$\alpha_X(x,y) = 0$ otherwise.

In particular, if X is a group and $\mathfrak{L}$ is a local system of subgroups of X such that for each $Y \in \mathfrak{L}$ there is a binary relation $\prec_Y$ on Y, then Lemma 5.3.1 says that

there is a binary relation $\prec$ on X such that, for every finite subset $\{(x_1,y_1), \ldots, (x_n,y_n)\}$ of X x X there is $Y \in \mathfrak{L}$ such that $x_i, y_i \in Y$ and $x_i \prec y_i$ if and only if $x_i \prec_Y y_i$ for $0 \leq i \leq n$. (26)

Proposition 5.3.3. Let G be a group acting on a group X.

(a) If $\mathfrak{L}$ is a local system of subgroups of X such that for each $Y \in \mathfrak{L}$ there is a G-invariant series $\mathcal{S}_Y$ in Y on whose factors the action induced by G is trivial, then there is a G-invariant series in X with the same property.

(b) If $\mathfrak{L}_1$ is a local system of subgroups of G such that for all $H \in \mathfrak{L}_1$ there is an H-invariant series $\mathcal{S}_H$ in X on whose factors the action induced by H is trivial, then there exists a G-invariant series in X on whose factors the action induced by G is trivial.

Proof (a) For each $Y \in \mathfrak{L}$ the binary relation $\prec_Y$ on Y determined by $\mathcal{S}_Y$ satisfies (23) and (24) (with Y and G for X and G respectively). By Remark 5.3.2 there is a binary relation $\prec$ on X satisfying (26) (with $\mathfrak{L}$ for $\mathfrak{L}$ and X for X). Then, since for each $Y \in \mathfrak{L}$ the binary relation $\prec_Y$ satisfies (23) and (24) (with Y for X and G for G), it is clear that $\prec$ satisfies (23) and (24) as well (with X for X and G for G).

$$\alpha_X(x,y) = 0 \quad \text{otherwise.}$$

In particular, if X is a group and $\mathfrak{L}$ is a local system of subgroups of X such that for each $Y \in \mathfrak{L}$ there is a binary relation $\prec_Y$ on Y, then Lemma 5.3.1 says that

there is a binary relation $\prec$ on X such that, for every finite subset $\{(x_1,y_1), \ldots, (x_n,y_n)\}$ of X x X there is $Y \in \mathfrak{L}$ such that $x_i, y_i \in Y$ and $x_i \prec y_i$ if and only if $x_i \prec_Y y_i$ for $0 \leq i \leq n$. (26)

Proposition 5.3.3. Let G be a group acting on a group X.

(a) If $\mathfrak{L}$ is a local system of subgroups of X such that for each $Y \in \mathfrak{L}$ there is a G-invariant series $\mathscr{S}_Y$ in Y on whose factors the action induced by G is trivial, then there is a G-invariant series in X with the same property.

(b) If $\mathfrak{L}_1$ is a local system of subgroups of G such that for all $H \in \mathfrak{L}_1$ there is an H-invariant series $\mathscr{S}_H$ in X on whose factors the action induced by H is trivial, then there exists a G-invariant series in X on whose factors the action induced by G is trivial.

Proof (a) For each $Y \in \mathfrak{L}$ the binary relation $\prec_Y$ on Y determined by $\mathscr{S}_Y$ satisfies (23) and (24) (with Y and G for X and G respectively). By Remark 5.3.2 there is a binary relation $\prec$ on X satisfying (26) (with $\mathfrak{L}$ for $\mathfrak{L}$ and X for X). Then, since for each $Y \in \mathfrak{L}$ the binary relation $\prec_Y$ satisfies (23) and (24) (with Y for X and G for G), it is clear that $\prec$ satisfies (23) and (24) as well (with X for X and G for G).

Consequently, as shown in the beginning of the section, the series in X associated to $\prec$ satisfies the required conditions.

(b) For each $H \in \mathfrak{L}_1$ the binary relation $\prec_H$ on X determined by $\mathcal{S}_H$ satisfies (23) and (24) (with X for X and H for G). By considering $\mathfrak{L}' = \{X_H | X_H = X \text{ for all } H \in \mathfrak{L}_1\}$ as a local system of subgroups of X and associating to each X_H the binary relation $\prec_H$, by Remark 5.3.2 it follows that there is a binary relation $\prec$ on X satisfying (26) (with $\mathfrak{L}'$ for $\mathfrak{L}$ and X for X). Then, since for each $H \in \mathfrak{L}_1$ the binary relation $\prec_H$ satisfies (23) and (24) (with X for X and H for G), it is clear that $\prec$ satisfies (23) and (24) as well (with X for X and G for G). Consequently the series in X associated to $\prec$ satisfies the required conditions.

$\square$

We are now ready to prove

Theorem 5.3.4. Let G and G_1 be groups, $H \triangleleft G$, and suppose that $G \in \mathcal{A}$. Let $\pi: G \to G_1$ be a projectivity such that H^π is core-free in G_1. Then there exist a G-invariant series $\mathcal{S}$ in $H^{\pi,G}$ and a G_1-invariant series $\mathcal{S}_1$ in $(H^\pi)^{G_1}$ whose factors are cyclic and if, in addition, H^π is quasinormal in G_1, then G induces the identity on the factors of $\mathcal{S}$ and G_1 induces the identity on the factors of $\mathcal{S}_1$.

Proof. As we have already pointed out in 5.1, as a result of Theorem 5.1.1, we may assume that H^π is quasinormal in G_1. Let $\mathcal{F}$ be the set of finitely generated subgroups of G. If $F \in \mathcal{F}$ set $\mathcal{F}_F = \{ E \in \mathcal{F} \mid E \geq F \}$. By

Theorem 5.1.2 $F/C_F(H^{\pi,<H,F>}/H_{\pi,<H,F>})$ and $F^{\pi}/C_{F^{\pi}}((H^{\pi})^{<H,F>^{\pi}}/(H^{\pi})_{<H,F>^{\pi}})$ are periodic and therefore finite (use the projective invariance of $\mathcal{A}$ for the finiteness of the latter) by the hypothesis on G. In particular H^{π} has a finite number of conjugates in $<H,F>^{\pi}$; then, considering that $|(H^{\pi})^{<H,F>^{\pi}} : H| < \infty$ (Lemma 1.2.9 (i)), it follows that $(H^{\pi})^{<H,F>^{\pi}}/(H^{\pi})_{<H,F>^{\pi}}$, and hence also $H^{\pi,<H,F>}/H_{\pi,<H,F>}$ are finite groups. Again Theorem 5.1.2 implies that there exists an integer n_F such that

$$(H^{\pi})^{<H,F>^{\pi}}/(H^{\pi})_{<H,F>^{\pi}} \leq Z_{n_F}(<H,F>^{\pi}/(H^{\pi})_{<H,F>^{\pi}}) \tag{27}$$

and

$$H^{\pi,<H,F>}/H_{\pi,<H,F>} \leq Z_{n_F}(<H,F>/H_{\pi,<H,F>}). \tag{28}$$

Let now $X \in \mathcal{F}$, $Y \in \mathcal{F}_X$. Set

$$\gamma_0 = H^{\pi,<H,Y>}, \quad \gamma_i = [H^{\pi,<H,Y>}, \underbrace{X,\dots,X}_{i\ \text{times}}] \quad \text{for all } 1 \leq i \in \mathbb{N}$$

and

$$\delta_0 = (H^{\pi})^{<H,Y>^{\pi}}, \quad \delta_i = [(H^{\pi})^{<H,Y>^{\pi}}, \underbrace{X^{\pi},\dots,X^{\pi}}_{i\ \text{times}}] \quad \text{for all } 1 \leq i \in \mathbb{N}.$$

Then (27) and (28) show that, if $Z \in \mathcal{F}_Y$,

$$\gamma_{n_Z} \leq H_{\pi,<H,Z>} \quad \text{and} \quad \delta_{n_Z} \leq (H^{\pi})_{<H,Z>^{\pi}} .$$

Thus, since $\bigcap_{Z \in \mathcal{F}_Y} H^{\pi}_{<H,Z>^{\pi}} = 1$, we obtain

$$\bigcap_{i \in \mathbb{N}} \gamma_i = 1 \quad , \quad \bigcap_{i \in \mathbb{N}} \delta_i = 1$$

Therefore $\{\gamma_i\}_{i \in \mathbb{N}}$ is an X-invariant series in $H^{\pi,<H,Y>}$ on whose factors X acts trivially. Similarly $\{\delta_i\}_{i \in \mathbb{N}}$ is an X^{π}-invariant series in $(H^{\pi})^{<H,Y>^{\pi}}$ on whose factors X^{π} acts trivially. As Y varies in $\mathcal{F}_X$, the groups $H^{\pi,<H,Y>}$ form a local system of X-invariant subgroups of $H^{\pi,G}$ and the groups $(H^{\pi})^{\langle H,Y\rangle^{\pi}}$ form a local system of X^{π}-invariant subgroups of $(H^{\pi})^{G_1}$. Therefore, by Proposition 5.3.3 (a) there exist an X-invariant series in $H^{\pi,G}$ and an X^{π}-invariant series in $(H^{\pi})^{G_1}$ on whose factors X and X^{π} respectively act trivially. Finally, as X varies in $\mathcal{F}$, the groups X and X^{π} form local systems of G and G_1 respectively. Applying Proposition 5.3.3 (b) it follow that there exist a G-invariant series in $H^{\pi,G}$ and a G_1invariant series in $(H^{\pi})^{G_1}$, on whose factors G and G_1 respectively induce the identity.

$\square$

References

1. E. Artin, Geometric Algebra. New York: Interscience Publishers, Inc. 1957.

2. R. Baer, Crossed isomorphisms. Amer. Journal of Math. 66,341-404 (1944).

3. G. Birkhoff, Lattice Theory. Amer. Math. Soc. Colloquium Publications 25, New York, 1949.

4. G. Busetto, Proprietà di immersione dei sottogruppi quasinormali periodici. Le Matematiche 34, 243-249 (1979).

5. F. Gross, p-subgroups of core-free quasinormal subgroups. Rocky Mountain J. Math. 1, 541-550 (1971).

6. ———, p-subgroups of core-free quasinormal subgroups II. Rocky Mountain J. Math. 5, 349-359 (1975).

7. ———, Infinite permutable subgroups. Rocky Mountain J. Math. 12, 333-343 (1982).

8. B. Huppert, Endliche Gruppen I, Springer, Berlin-Heidelberg-New York (1967).

9. H. Kurzweil, Endliche Gruppen, Springer, Berlin-Heidelberg-New York (1977).

10. J. Lennox, A note on quasinormal subgroups of finitely generated groups, J. London Math. Soc. 24, 127-128 (1981).

11. R. Maier-P. Schmid, The embedding of quasinormal subgroups in finite groups. Math. Zeitschrift 131, 269-272 (1973).

12. F. Menegazzo, Normal subgroups and projectivities of finite groups. Rend. Sem. Mat. Univ. Padova 59, 11-15 (1978).

13. ———, Immagini di sottogruppi normali nelle proiettività tra gruppi. Conferenza-Convegno di Teoria dei gruppi, Povo (TN), Italy (1979).

14. F. Napolitani, G. Zacher, Über das Verhalten der Normalteiler unter Projektivitäten. Math. Zeitschrift 183, 371-380 (1983).

15. E. Rips, Lattice isomorphism of groups, subgroups of finite index and normality. Manuscript.

16. D. J. S. Robinson, Finiteness conditions and generalized soluble groups, Erg. der Math. 62, Springer-Verlag, Berlin (1972).

17. E. Schenkman, On the norm of a group. Illinois J. Math. 4, 150-152 (1960).

18. R. Schmidt, Modulare Untergruppen endlicher Gruppen. Illinois J. Math. 13, 358-377 (1969).

19. ————, Normal subgroups and lattice isomorphisms of finite groups. Proc. London Math. Soc. 30, 287-300 (1975).

20. W. R. Scott, Group Theory, Prentice Hall (1964).

21. S. E. Stonehewer, Permutable subgroups of infinite groups. Math. Zeitschrift 125, 1-16 (1972).

22. ————, Modular subgroups of infinite groups. Symposia Mathematica 17, 207-214 (1976).

23. M. Suzuki, On the lattice of subgroups of finite groups. Trans. Amer. Math. Soc. 70, 345-371 (1951).

24. ————, Structure of a group and the structure of its lattice of subgroups. Erg. der Math. 10, Springer-Verlag,Berlin (1956).

25. B. V. Yakovlev, Lattice isomorphisms of solvable groups. Alg. i Log. 9, 349-369 (1970).

26. G. Zacher, Una caratterizzazione reticolare della finitezza del l'indice di un sottogruppo in un gruppo. Rend. Acc. Naz. Lincei 69, 317-323 (1980).

27. ————, Sulle immagini dei sottogruppi normali nelle proiettività. Rend. Sem. Mat. Univ. Padova. 67, 39-74 (1982).

28. G. Zappa, Sulla risolubilità dei gruppi finiti in isomorfismo reticolare con un gruppo risolubile. Giornale di Mat. Battaglini 80, 213-255 (1951).