EXTENSIONS OF SIMPLE MODULES FOR THE UNIVERSAL CHEVALLEY GROUPS AND ITS PARABOLIC SUBGROUPS

SAMY EL BADAWY YEHIA

A Thesis submitted for the degree of Doctor of Philosophy in Mathematics

January, 1982

Mathematics Institute
University of Warwick
COVENTRY CV4 7AL

ACKNOWLEDGEMENTS

Most of all, I would like to express my gratitude to my supervisor, Dr. S. Donkin, who has provided support, encouragement, and much more besides during the preparation of this thesis.

I wish to thank Professor R.W. Carter for many illuminating discussions.

Grateful thanks to my family who kept with me through the ups and downs of the last years of research study, and apologies for the lost weekends and summer holidays.

I wish also to acknowledge the financial support of the Egyptian Government.

Finally, thanks to Terri Moss for the elegant typing.

CONTENTS

INTRODUCTION

The modular representation theory of Chevalley groups is still in a tentative stage (see the introduction of [8]). As far as this topic is concerned, we know: the indexing set for simple modules, the linkage principle, the strong linkage principle, the blocks, etc.

In this thesis two problems have been solved. Both of them deal with the extension of a simple module by another simple one. The first problem deals with the extension group between simple G-modules when G is the universal Chevalley group, and describes this group for type A_2. The second one investigates the blocks of the parabolic subgroups of the universal Chevalley groups which is highly related to the extension problem.

The wide open problem of describing the modular representations of Chevalley groups, and the solution of the above mentioned problems recall to mind the Hindu fable of the blind men and the elephant as written by J.G. Saxe, however see the paragraph below.

It was six men of Indostan to learning much inclined,
Who went to see the Elephant (though all of them were blind)
That each by observation might satisfy his mind.

The First approached the Elephant, and happening to fall
Against his broad and sturdy side, at once began to bawl:
"God bless me! but the Elephant is very like a wall!"

The Second, feeling of the tusk, cried: "Ho! what have we here
So very round and smooth and sharp? To me 'tis very clear
This wonder of an Elephant is very like a spear!"

The Third approached the animal, and happening to take
The squirming trunk within his hands, thus boldly up and spake:
"I see," quoth he, "the Elephant is very like a snake!"

The Fourth reached out an eager hand, and felt about the knee.
"What most this mighty beast is like is mighty plain", quoth he;
"'Tis very clear the Elephant is very like a tree!"

The Fifth, who chanced to touch the ear, said: "E'en the blindest man
Can tell what this resembles most; deny the fact who can,
This marvel of an Elephant is very like a fan!"

The Sixth no sooner had begun about the beast to grope,
Then, seizing on the swinging tail that fell within his scope,
"I see," quoth he, "the Elephant is very like a rope!"

And so these men of Indostan disputed loud and long,
Each in his own opinion exceeding stiff and strong,
Though each was partly in the right, and all were in the wrong!

One way to study the representation theory of a group is to get hold of the simple modules. The modular representations of the Chevalley groups (and its parabolic subgroups) are not necessarily completely reducible, so the extension problem appears naturally. The natural question is, if V is a module (with two composition factors say), when is it completely reducible? Conversely, given two simple modules L_1, L_2 what modules V may be constructed with L_1, L_2 as its composition factors, and when do these extensions split? Another important aspect of the extension problem is Anderson's conjucture (conjucture 7.2 of [4]), which may be very strongly connected with Lusztig's conjuncture on the character of the simple modules (problem IV of [33]).

This thesis consists of five chapters. Since we cannot put a sharp line between the blocks and the extensions, the first chapter is meant to be a preliminary for both our problems, and also it presents the necessary background.

The second chapter deals with the extension group in general (when G is the universal Chevalley group), and puts some relations between the extension functor and Jantzen's translation functor.

In the third and fourth chapters we investigate this functor when G is of type A_2. In the third one we determine the functor $\mathrm{Ext}^1_{U_1}$ between simple U_1-modules, where U_1 is the restricted enveloping algebra of the Lie algebra of G. The extensions $\mathrm{Ext}^1_G(L(\mu),L(\lambda))$ i.e. between simples, have been determined in the fourth chapter.

Finally, in the fifth chapter we determine the blocks of the parabolic subgroups of the universal Chevalley groups.

Throughout this thesis, the notations dim and $\otimes$ are abbreviations for $\dim_K$ and $\otimes_K$ respectively. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the natural, integral, rational, real, and the complex numbers respectively. Modules for the affine algebraic groups will always mean the rational ones defined in Section 1.1. A submodule or a direct summand may mean isomorphic to a submodule or to a direct summand. Finally, the end of the proofs (if any), definitions, examples, etc., will be marked thus: $\square$.

CHAPTER 1 - PRELIMINARIES

In this chapter we review the rather well known results required for the later chapters, and also build up the necessary notation. These preliminaries come under three titles, Hopf algebras, induced modules, and Chevalley groups. The third section of this chapter is devoted to type A_2 to see what these look like in this special case.

1.1. Hopf Algebras and Induced Modules.

Let $(G,K[G])$ be an affine algebraic group over an algebraically closed field K of characteristic $p > 0$, and co-ordinate ring $K[G] = A$. A Hopf algebra structure in A can be defined as follows. We identify $A \otimes A$ with the coordinate ring of the product $G \times G$ i.e. by defining $(f \otimes g)(x,y) = f(x)g(y)$ for every $f,g \in A$ and $x,y \in G$. The comultiplication $\mu:A \to A \otimes A$ is defined by $\mu(f)(x,y) = f(xy)$. The augmentation $\varepsilon:A \to K$ is defined by $\varepsilon(f) = f(1)$, where $1 \in G$ is the identity element. The antipode $S:A \to A$ is defined by $S(f)(x) = f(x^{-1})$. Clearly they satisfy the conditions:

$$(H1)\ (\mu \otimes 1_A)\mu = (1_A \otimes \mu)\mu,$$

$$(H2)\ (\varepsilon \otimes 1_A)\mu = (1_A \otimes \varepsilon)\mu = 1_A,$$

$$(H3)\ m(1_A \otimes S)\mu = m(S \otimes 1_A)\mu = \varepsilon 1.$$

In the above $m:A \otimes A \to A$ is the K-linear map sending $f \otimes g$ to fg for every $f,g \in A$, 1_A is the identity map on A, and 1 is the unit element of A.

The dual space A^* is made into an associative K-algebra. The multiplication, $*$ say, is defined by:

CHAPTER 1 - PRELIMINARIES

In this chapter we review the rather well known results required for the later chapters, and also build up the necessary notation. These preliminaries come under three titles, Hopf algebras, induced modules, and Chevalley groups. The third section of this chapter is devoted to type A_2 to see what these look like in this special case.

1.1. Hopf Algebras and Induced Modules.

Let $(G, K[G])$ be an affine algebraic group over an algebraically closed field K of characteristic $p > 0$, and co-ordinate ring $K[G] = A$. A Hopf algebra structure in A can be defined as follows. We identify $A \otimes A$ with the coordinate ring of the product $G \times G$ i.e. by defining $(f \otimes g)(x,y) = f(x)g(y)$ for every $f,g \in A$ and $x,y \in G$. The comultiplication $\mu : A \to A \otimes A$ is defined by $\mu(f)(x,y) = f(xy)$. The augmentation $\varepsilon : A \to K$ is defined by $\varepsilon(f) = f(1)$, where $1 \in G$ is the identity element. The antipode $S : A \to A$ is defined by $S(f)(x) = f(x^{-1})$. Clearly they satisfy the conditions:

$$(H1)\ (\mu \otimes 1_A)\mu = (1_A \otimes \mu)\mu,$$

$$(H2)\ (\varepsilon \otimes 1_A)\mu = (1_A \otimes \varepsilon)\mu = 1_A,$$

$$(H3)\ m(1_A \otimes S)\mu = m(S \otimes 1_A)\mu = \varepsilon 1.$$

In the above $m : A \otimes A \to A$ is the K-linear map sending $f \otimes g$ to fg for every $f,g \in A$, 1_A is the identity map on A, and 1 is the unit element of A.

The dual space A^* is made into an associative K-algebra. The multiplication, $*$ say, is defined by:

for every $\gamma_1, \gamma_2 \in A^*$ and $f \in A$ put

$$(\gamma_1 * \gamma_2)(f) = (\gamma_1 \otimes \gamma_2)\mu(f) = \sum_i \gamma_1(f_i)\gamma_2(f_i'),$$

where $\mu(f) = \sum_i f_i \otimes f_i'$. Clearly the augmentation ε is its unit element.

The Lie algebra of G, Lie (G), is defined by

$$\mathrm{Lie}(G) = \{\gamma \in K[G]^*/\gamma(fg) = \gamma(f)\varepsilon(g) + \varepsilon(f)\gamma(g) \text{ for every } f,g \in K[G]\}.$$

It is direct to show that

$$[\gamma_1,\gamma_2] = \gamma_1 * \gamma_2 - \gamma_2 * \gamma_1 \in \mathrm{Lie}\ (G)$$

for every $\gamma_1,\gamma_2 \in$ Lie (G).

A (left) rational G-module V is a module for the group algebra KG such that:

(R1) KGv is finite dimensional for every $v \in V$.

(R2) If $\{v_i | i \in I\}$ is some (hence any) basis of V.

Then the maps $\{f_{ij}: G \to K/i,j \in I\}$ defined by

$$gv_i = \sum_{j \in I} f_{ji}(g)v_j,$$

$i \in I$, are all in K[G].

The Lie algebra Lie(G) becomes a rational G-module via the adjoint action i.e. the action

$$\mathrm{ad}(x)\gamma = \varepsilon_x * \gamma * \varepsilon_{x^{-1}}$$

for every $x \in G$, and $\gamma \in$ Lie (G). The evaluation at $x \in G$

$$\varepsilon_x : K[G] \to K$$

is defined by

$$\varepsilon_x(f) = f(x)$$

for every $f \in K[G]$.

An action of G on K[G] (denoted by c_x) is defined by:

$$(c_x f)(y) = f(x^{-1}yx)$$

for every $x,y \in G$ and $f \in K[G]$. The proof of the following lemma is straightforward.

1.1.1. Lemma.

$(ad(x)\gamma)(f) = \gamma\ (c_{x^{-1}}f)$ *for every* $x \in G$, $\gamma \in K[G]^*$, *and* $f \in K[G]$. □

Let H be a closed subgroup of G. The inclusion $H \to G$ induces a Lie algebra monomorphism $\mathrm{Lie}(H) \to \mathrm{Lie}(G)$. Sometimes Lie(H) is identified with a sub-Lie algebra of Lie(G).

For a G-module V (resp. an R-module V, where R is a ring), the socle is its maximal completely reducible submodule, and it will be denoted by $\mathrm{soc}_G\, V$ (resp. $\mathrm{soc}_R\, V$). We put $\overset{G}{\simeq}$ (resp. $\overset{R}{\simeq}$) between two modules to mean that they are isomorphic as G-modules (resp. R-modules).

A (right) A-comodule is a pair (V,τ), where V is a K-space, and $\tau : V \to V \otimes A$ a K-map such that:

(M1) $(\tau \otimes 1_A)\tau = (1_V \otimes \mu)\tau$,

(M2) $(1_V \otimes \varepsilon)\tau = 1_V$,

where 1_A, 1_V are the identity maps in A, V respectively.

Remark (1).

Suppose A is a coalgebra i.e. a vector space with μ and ε as before satisfying (H1), (H2). Remark (2) on p. 140 of [18] shows that, there is a functor from the category of right A-comodules to the category of left (unital) A^*-modules. This functor associates to each A-comodule (V,τ) the A^*-module ${}_{A^*}V$ which has the same underlying space and A^* acts by

$$\gamma v = \sum_i \gamma(a_i)v_i$$

for every $\gamma \in A^*$, $v \in V$, where $\tau(v) = \sum_i v_i \otimes a_i$.

Conversely, if A is finite dimensional, an inverse functor exists. This functor associates the comodule (V,τ) to each A^*-module ${}_{A^*}V$, the map $\tau: V \to V \otimes A$ is defined as follows: let $\{a_i\}$, $\{a_i^*\}$ be a basis and its dual of A, A^* respectively, define for every $v \in V$

$$\tau(v) = \sum_i a_i^* v \otimes a_i.$$

It now follows directly that τ satisfies (M1), and

$$\sum_i \varepsilon(a_i)a_i^* = \varepsilon$$

guarantees (M2).

Thus, when A is finite dimensional, we obtain an inclusion preserving equivalence of categories between the category of right A-comodules and that of left A^*-modules.

Remark (2).

Using the two sets $\{v_i\}$, $\{f_{ij}\}$ defined in (R2) above, we can easily show that; there is an equivalence of categories between the category of right K[G]-comodules and that of rational G-modules. The last category will be denoted by $\mathcal{M}_G$.

For a more detailed discussion of the functors in the previous two remarks e.g. when A is an infinite dimensional space, we refer to §3.1 of [1] and Appendix II of [37].

Our final notation in this section is that of the induced modules.

Let A be a K-space and $V \in \mathcal{M}_G$. Define the rational G-module $(A) \otimes V$ as follows: the underlying space is $A \otimes V$, and G acts by

$$g(a \otimes v) = a \otimes gv$$

for each $g \in G$, $a \in A$, and $v \in V$. See (1.2f) of [18]. The parentheses are to indicate that A is a "dummy" i.e. intervenes only as a K-space.

The coordinate ring K[G] is made into a G-bimodule via the right and left translations i.e. put

$$(x.f)(y) = f(yx), \text{ and } (f.x)(y) = f(xy)$$

for every $f \in K[G]$, and $x,y \in G$.

Let H be a closed subgroup of G, and $V \in \mathcal{M}_H$. Define

$$V|^G = \{\sum_i v_i \otimes f_i \in V \otimes K[G] / \sum_i hv_i \otimes f_i = \sum_i v_i \otimes f_i.h \text{ for every } h \in H\}.$$

It is easy to show that, for every $g \in G$

$$\sum_i v_i \otimes f_i \in V|^G \longrightarrow \sum_i v_i \otimes g.f_i \in V|^G .$$

So $V|^G$ becomes a G-module with action defined by

$$g(\sum_i v_i \otimes f_i) = \sum_i v_i \otimes g.f_i .$$

It is a rational G-module since it is a submodule of $(V) \otimes K[G]$. The induced module $V|^G$ will be denoted by $\mathrm{Ind}_H^G(V)$.

1.2. Chevalley Groups.

Let L be a complex, finite dimensional, and simple Lie algebra. Let H be a Cartan subalgebra, Φ be the root system, and let $\Delta = \{\alpha_1,\ldots,\alpha_\ell\}$ be a set of simple roots which determine the positive (resp. negative) roots Φ^+ (resp. Φ^-).

The Killing form will be denoted by $(\ , \ )$. Thus we get for each $\gamma \in H^*$ a unique $H'_\gamma \in H$ such that

$$(H'_\gamma , H) = \gamma(H)$$

for every $H \in H$. We put

$$H_i = \frac{2H'_{\alpha_i}}{(H'_{\alpha_i}, H'_{\alpha_i})}$$

for every $i = 1,\ldots,\ell$. The set

$$\{X_\alpha , H_i / \alpha \in \Phi, \ i = 1,\ldots,\ell\}$$

will be a Chevalley basis of L. We can define a symmetric, bilinear, and positive definite form on H^* (call it $(\ , \ )$ also), by putting

$$(\gamma , \delta) = (H'_\gamma, H'_\delta)$$

for every γ, $\delta \in H^*$. See §1 of [34].

The (abstract) Weyl group associated to Φ will be denoted by W. This is the group of automorphisms of Φ generated by the reflections $\{\omega_\alpha / \alpha \in \Delta\}$. The reflection ω_α acts on $\beta \in \Phi$ by

$$\omega_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha,$$

where

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$$

is the dual root associated to $\alpha \in \Phi$.

The integral (resp. dominant) weights will be denoted by X (resp. X^+). These are the two sets

$$X = \{\lambda \in H^*/(\lambda, \alpha^\vee) \in \mathbb{Z} \text{ for every } \alpha \in \Delta\},$$

$$X^+ = \{\lambda \in X/(\lambda, \alpha^\vee) \geq 0 \text{ for every } \alpha \in \Delta\}.$$

The fundamental dominant weights $\{\lambda_i / i = 1,\ldots,\ell\}$ (or $\{\lambda_\alpha / \alpha \in \Delta\}$) are the weights satisfying $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$ for every $i,j = 1,\ldots,\ell$. These weights form a $\mathbb{Z}$-basis of X. There is a partial ordering in X, namely $\lambda \leq \mu$ iff $\mu-\lambda$ is a sum of positive (hence simple) roots or $\mu = \lambda$. The root lattice will be denoted by $X_r = \mathbb{Z}\Phi$.

The dual root system $\Phi^\vee = \{\alpha^\vee / \alpha \in \Phi\}$ is also irreducible as Φ(§10.4 of [22]). The root $\alpha_0 \in \Phi$ such that $\alpha_0^\vee$ is the unique maximal root (with respect to the above partial ordering) of $\Phi^\vee$ is called the highest short root of Φ. The Coxeter number h is defined to be

$$h = 1 + (\rho, \alpha_0^\vee),$$

where

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{\alpha \in \Delta} \lambda_\alpha,$$

see §13.3 of [22].

For each dominant weight λ, the unique (up to an isomorphism) finite dimensional simple L-module with highest weight λ will be denoted by V_λ (§21.2 of [22]).

Passing to an algebraically closed field K of positive characteristic p, we have two important structures. The universal Chevalley group G, and the hyperalgebra U_K.

Let $\{X_\alpha, H_i / \alpha \in \Phi,\ i = 1,\ldots,\ell\}$ be a Chevalley basis of L. Let V be a finite dimensional L-module, $V_{\mathbb{Z}}$ be an admissible lattice, and $V_K = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. For $\alpha \in \Phi$ and $t \in K$, let $x_\alpha(t)$ be the automorphism of V_K:

$$x_\alpha(t)(v \otimes k) = \sum_{t=0}^{\infty} t^n \left(\frac{X_\alpha^n}{n!} v \otimes k\right),$$

$v \in V_{\mathbb{Z}}$, $k \in K$. The above sum is finite since X_α^n acts as zero for n sufficiently large. Define:

$$\chi_\alpha = \{x_\alpha(t) / t \in K\},$$

the root subgroup, and

$$G = \langle \chi_\alpha / \alpha \in \psi \rangle.$$

A group G so constructed will be called a Chevalley group.

A weight $\lambda \in X$ is called a weight of V if

$$V^\lambda = \{v \in V / Hv = \lambda(H)v \text{ for every } H \in H\} \neq 0.$$

If the $\mathbb{Z}$-span of the weights of V is the whole X, our G will be called the universal Chevalley group (of type Φ). Except where stated to the contrary G will always denote this universal Chevalley group.

The universal Chevalley group G is a connected, almost simple, semisimple, simply connected algebraic group by Theorem 6 of [34], and §27.5 of [33]. It is also uniquely determined (up to an affine algebraic group isomorphism) by Φ (§32.1 of [23]).

For each $\alpha \in \Phi$, there is a unique homomorphism $\phi_\alpha : SL(2,K) \to G$, where SL(2,K) is the special linear group of order 2×2 over K (§3.2 of [6]). This ϕ_α sends $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$) to $x_\alpha(t)$ (resp. $x_{-\alpha}(t)$) for every $t \in K$. Let $h_\alpha(t)$ be the image of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $t \in K^*$. The maximal torus of G generated by $\{h_\alpha(t)/\alpha \in \Delta, t \in K^*\}$ will be denoted by T. For the equivalence between the definition of T here and that of Steinberg in [34] we refer to Lemma 6.1.1 of [7], however see Theorem 6 of [34].

Each element $h \in T$ has a unique expression as a product

$$h = \prod_{\alpha \in \Delta} h_\alpha(t_\alpha)$$

where $t_\alpha \in K^*$, see corollary on p. 44 of [34]. So we get an isomorphism of abelian groups $X \to X(T)$, where X(T) is the character group of T. This isomorphism is the one that associates to each $\lambda \in X$ the character $\tilde{\lambda}$, where

$$\tilde{\lambda}\Big(\prod_{\alpha \in \Delta} h_\alpha(t_\alpha)\Big) = \prod_{\alpha \in \Delta} t_\alpha^{(\lambda, \alpha^\vee)}.$$

Therefore X may be identified with X(T), and we have a natural action of W on X(T). The "dot" action of W on X(T) is defined by

$$\omega.\lambda = \omega(\lambda + \rho) - \rho$$

for every $\omega \in W$, and $\lambda \in X(T)$.

Now we define the hyperalgebra U_K. Let $U_{\mathbb{Z}}$ be a Kostant $\mathbb{Z}$-form of the universal enveloping algebra $U_{\mathbb{C}}$ of L i.e. the $\mathbb{Z}$-subalgebra of $U_{\mathbb{C}}$ generated by all $\frac{X_\alpha^r}{r!}$ for every $\alpha \in \Phi$ and every integer $r \geq 0$ (by convention X_α^0 is the unit element). The hyperalgebra

$$U_K = U_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$$

has a K-basis deduced from the usual $\mathbb{Z}$-basis of $U_{\mathbb{Z}}$. This is the set of all elements of the form

$$\prod_{\alpha\in\Phi^+} X_{-\alpha,a_\alpha} \prod_{i=1}^{\ell} H_{i,b_i} \prod_{\alpha\in\Phi^+} X_{\alpha,c_\alpha},$$

for some ordering of Φ^+ fixed in the multiplications of the $X_{\alpha,n}$'s, and

$$X_{\alpha,n} = \frac{X_\alpha^n}{n!} \otimes 1, \quad H_{i,b} = \frac{H_i(H_i-1)\ldots(H_i-b+1)}{b!} \otimes 1.$$

Also all the a_α's, b_i's, and c_α's are non-negative integers. Hence $X_{\alpha,o}$ is the unit element of U_K.

As in the remarks of §1.1, there is an inclusion preserving equivalence of categories between the (locally finite) U_K-modules and the rational G-modules (see §1 of [16] and §5.5, §6.1, §9.1 of [12]). This is realized as follows.

Let G_a be the affine algebraic group $(K^+, K[X])$, where $X:K \to K$ is defined by $X(t) = t$ for every $t \in K$. For every $\alpha \in \Phi$, define the affine algebraic group homomorphism

$$\psi_\alpha : G_a \to G$$

by $\psi_\alpha(t) = x_\alpha(t)$ for every $t \in K$ (Lemma 34 of [34]). This ψ_α gives the K-algebras homomorphism

$$\psi_\alpha^* : K[G] \to K[X]$$

defined by $\psi_\alpha^*(f)(t) = f(x_\alpha(t))$ for every $f \in K[G]$ and $t \in K$. Now for each non-negative integer r, define

$$\xi_{\alpha,r} : K[G] \to K$$

by

$$\psi_\alpha^*(f) = \sum_{r=0}^{\infty} \xi_{\alpha,r}(f) X^r$$

for every $f \in K[G]$. Given $\alpha \in \Phi$ and $f \in K[G]$, we can see that the $\xi_{\alpha,r}(f)$'s are almost all zeros.

There is a monomorphism of K-algebras

$$\psi : U_K \to K[G]^*$$

satisfying $\psi(X_{\alpha,r}) = \xi_{\alpha,r}$ for every $\alpha \in \Phi$ and $r \geq 0$ (§6.5 of [12]).

Any rational G-module V may be regarded as a locally finite U_K-module by defining:

$$uv_i = \sum_j \psi(u)(f_{ji}) v_j,$$

for any $u \in U_K$, and $\{v_i\}$, $\{f_{ij}\}$ are as in (R2).

Conversely, any locally finite U_K-module V may be considered as a rational G-module (see also §6.8 and §9.2 of [12]). The generators $x_\alpha(t)$'s act by

$$x_\alpha(t)v = \sum_{r=0}^{\infty} t^r X_{\alpha,r} v$$

for every $v \in V$. Thus if V is a G-module, a submodule will mean either a U_K or a G-submodule of V .

The integral group ring of the integral weights X (not to be confused with X in $(K^+, K[X])$ above) will be $\mathbb{Z}[X]$. Its $\mathbb{Z}$-basis is

$$\{e(\lambda)/\lambda \in X\} ,$$

and the multiplication is $e(\lambda)e(\mu) = e(\lambda+\mu)$. The Weyl group W acts naturally on $\mathbb{Z}[X]$ by $w(e(\lambda)) = e(w(\lambda))$ for every $\omega \in W$ and $\lambda \in X$.

For a finite dimensional G-module V the formal character, ch. V, is defined by

$$\text{ch. } V = \sum_{\mu \in X} m_V(\mu) e(\mu) ,$$

where $m_V(\mu)$ is the dimension of the weight subspace

$$V^\mu = \{v \in V/tv = \mu(t)v \quad \text{for every} \quad t \in T\} .$$

The Weyl module associated to $\lambda \in X^+$ will be denoted by $V(\lambda)$. Recall $V(\lambda)$ is defined by reduction mod p of the minimal $U_{\mathbb{Z}}$-lattice $U_{\mathbb{Z}} v_\lambda$ in the irreducible L-module V_λ i.e. v_λ is a maximal vector in V_λ .

Since $V(\lambda)$ is of particular interest here, we shall state in the next theorem the most important results for it.

1.2.1. <u>Theorem</u>.

(i) *The dimension of the Weyl module is given by:*

$$\dim V(\lambda) = \dim_{\mathbb{C}} V_\lambda = \prod_{\alpha\in\Phi^+} (\lambda+\rho,\alpha^\vee) / \prod_{\alpha\in\Phi^+} (\rho,\alpha^\vee) .$$

The formal character, ch. (λ) , *is given by:*

$$\text{ch.}\ (\lambda) = \mathbb{A}(\lambda+\rho)/\mathbb{A}(\rho) ,$$

where for $\mu \in X$

$$\mathbb{A}(\mu) = \sum_{\omega\in W} \varepsilon(\omega) e(\omega(\mu)) ,$$

and $\varepsilon(\omega) = \pm 1$ *is the sign of* ω . *See* §24.3 *of* [22].

(ii) *For* $\lambda \in X$, *let* $\chi(\lambda) = \mathbb{A}(\lambda+\rho)/\mathbb{A}(\rho)$, *and let* $\omega \in W$ *be the unique element of* W *such that* $\omega(\lambda-\rho) \in X^+$. *Then we have:*

$$\chi(\lambda) = \begin{cases} 0 & \text{if } \omega.\lambda \notin X^+ , \\ \varepsilon(\omega)\ \text{ch.}(\omega.\lambda) & \text{if } \omega.\lambda \in X^+ . \end{cases}$$

See §7.2 *of* [24], *and Lemma* 8 *of* [30].

(iii) *Let*

$$\theta = \sum_{\mu \in X} m_\mu e(\mu) \in \mathbb{Z}[X]^W$$

i.e. W *invariant, and let* $\lambda \in X^+$. *Then we have:*

$$\theta\chi(\lambda) = \sum_{\mu \in X} m_\mu \chi(\lambda+\mu) ,$$

see also §7.2 *of* [24], *and Lemma* 8 *of* [30].

(iv) *The weight* λ *is the unique highest weight of* $V(\lambda)$ *and its multiplicity is one. The value of the partition function at* $\nu \in X$, $p(\nu)$, *is defined to be the number of ways* ν *can be written as a nonnegative sum of positive roots. The multiplicity of a weight* μ *of* $V(\lambda)$, $m_\lambda(\mu)$, *is given by* (§24.2 *of* [22]) :

$$m_\lambda(\mu) = \sum_{\omega \in W} \varepsilon(\omega) p(\omega.\lambda-\mu) .$$

(v) *The* G-*module* $V(\lambda)$ *is generated by a maximal vector of weight* λ *and is indecomposable. It has a unique maximal submodule* $M(\lambda)$. *The unique top composition factor modules:*

$$\{L(\lambda) = V(\lambda)/M(\lambda) / \lambda \in X^+\}$$

form a complete set of simple (irreducible) G-*modules* (§2.1 *of* [24]).

(vi) *If* $L(\mu)$ *is a composition factor of* $V(\lambda)$ *.* *Then there exist* $\mu_1,\ldots,\mu_n \in X^+-\rho$ *such that* $\mu = \mu_1\uparrow\mu_2\uparrow\ldots\uparrow\mu_n = \lambda$*.* *Two weight* ν,χ *are denoted* $\nu\uparrow\chi$ *iff there exist an integer* $n \in \mathbb{N}$ *and* $\alpha \in \Phi^+$ *such that* $(\chi+\rho,\alpha^\vee) \geq np$ *and* $\nu = \omega_\alpha.\chi+np\alpha$ *.* *This is the strong linkage principle, see Corollary 3 of* [2]. □

Our final technical definition in this section is the affine Weyl group W_a . For more information we refer to §3 of [8], §1 of [35], and §1.2 of [3].

For $\alpha \in \Phi$ and $n \in \mathbb{Z}$, we let $\omega_{\alpha,n}$ denote the affine reflection in $X \otimes_{\mathbb{Z}} \mathbb{R}$ given by

$$\omega_{\alpha,n}.\lambda = \omega_\alpha.\lambda+np\alpha$$

for each $\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R}$. The action of W on $X \otimes_{\mathbb{Z}} \mathbb{R}$ is deduced from its action on X . We may consider $X \subseteq X \otimes_{\mathbb{Z}} \mathbb{R}$ via the injection $\lambda \mapsto \lambda\otimes 1$.

The $\omega_{\alpha,n}$'s are the reflections in the affine hyperplanes

$$L_{\alpha,n} = \{\lambda \in X\otimes_{\mathbb{Z}}\mathbb{R}/(\lambda+\rho,\alpha^\vee) = np\} .$$

The affine Weyl group W_a is defined to be the group of isometries of $X \otimes_{\mathbb{Z}} \mathbb{R}$ generated by all $\omega_{\alpha,n}$ for every $\alpha \in \Phi$ and $n \in \mathbb{Z}$. Let $\omega \in W_a$, so there exist $\omega' \in W$ and $\alpha \in \mathbb{Z}\Phi$ such that $\omega.\lambda = \omega'.\lambda+p\alpha$ for every $\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R}$. These ω',α associated to $\omega \in W_a$ may not be unique.

The connected components of $X \otimes_{\mathbb{Z}} \mathbb{R} - \bigcup_{\alpha,n} L_{\alpha,n}$ are called alcoves. The alcove

$$A_o = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} / 0 < (\lambda+\rho,\alpha^\vee) < p \text{ for every } \alpha \in \Phi^+\}$$

is called the fundamental alcove. The closure of A_o is

$$\bar{A}_o = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} / 0 \leq (\lambda+\rho,\alpha^\vee) \leq p \text{ for every } \alpha \in \Phi^+\}.$$

Any alcove A is a subset of $X \otimes_{\mathbb{Z}} \mathbb{R}$ of the form $\omega.A_o$, where $\omega \in W_a$ is uniquely determined by A. The closure of A is therefore $\bar{A} = \omega.\bar{A}_o$.

Finally, since we are interested only in the integral weights, when we say the weights in an alcove we mean the (discrete) integral weights.

1.3. Type A_2 Case.

In type A_2, $L = sl(3,\mathbb{C})$ the set of all 3×3 matrices over $\mathbb{C}$ with trace zero. The rank $\ell = 2$, $|\Phi| = 6$, $\Delta = \{\alpha,\beta\}$, $\Phi^+ = \{\alpha,\beta,\alpha+\beta\}$, and $(\alpha,\alpha) = 2$ for every $\alpha \in \Phi$. So we may identify Φ with $\Phi^\vee$. Our two fundamental dominant weights are $\{\lambda_\alpha,\lambda_\beta\}$, and if $\lambda = r\lambda_\alpha + s\lambda_\beta \in X$ we write $\lambda = (r,s)$. Thus $\alpha = (2,-1)$, $\beta = (-1,2)$, and $\alpha_o = \rho = (1,1)$. The Weyl group $W = \{1,\omega_\alpha,\omega_\beta,\omega_\alpha\omega_\beta,\omega_\beta\omega_\alpha,\omega_o\}$ with $\omega_\alpha^2 = \omega_\beta^2 = \omega_o^2 = 1$, and $\omega_o = \omega_\alpha\omega_\beta\omega_\alpha = \omega_\beta\omega_\alpha\omega_\beta$ (ω_o will always denote the longest element in W, not only in this type). The action of W is given by:

$$\omega_\alpha(r,s) = (-r,r+s) \quad , \quad \omega_\beta(r,s) = (r+s,-s),$$
$$\omega_\alpha\omega_\beta(r,s) = (-r-s,r) \quad , \omega_\beta\omega_\alpha(r,s) = (s,-r-s),$$
$$\omega_o(r,s) = (-s,-r) \quad .$$

The Coxeter number $h = 3$.

The universal Chevalley group is $G = SL(3,K)$, the set of all 3×3 matrices over K of determinant one (p. 27 of [34]). Let $\lambda = (r,s) \in X^+$, hence $\dim V(\lambda) = \frac{(r+1)(s+1)(r+s+2)}{2}$.

The decomposition of $X \otimes_{\mathbb{Z}} \mathbb{R}$ into alcoves is shown in the following diagram.

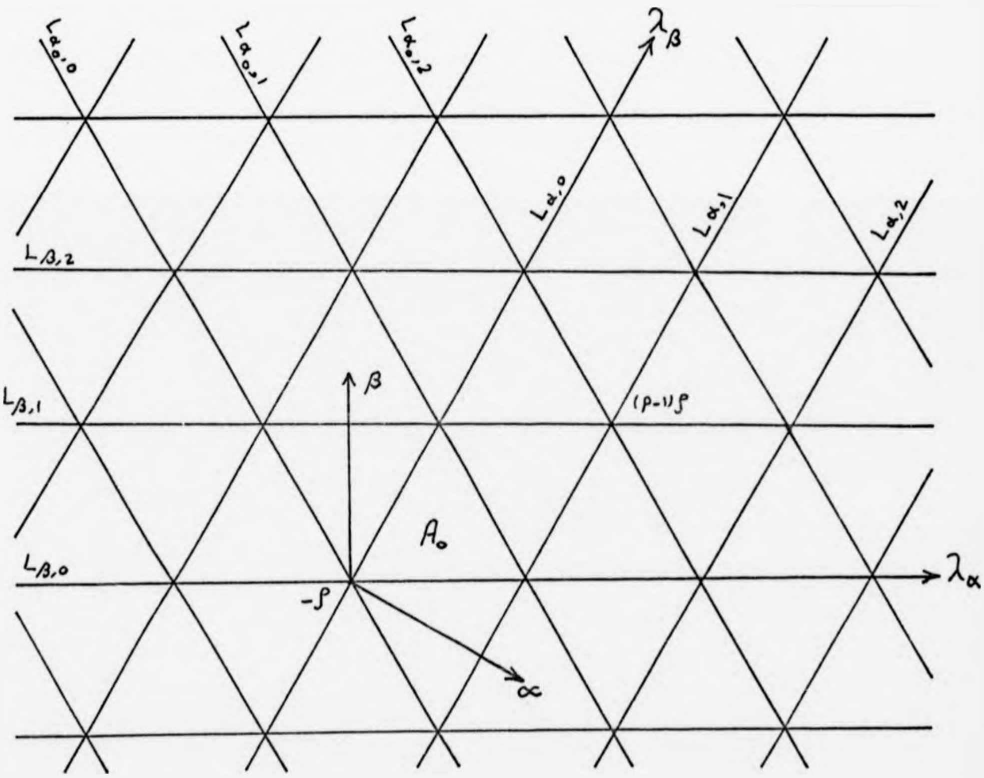

Figure 1.

To get the multiplicity $m_\lambda(\mu)$ of the weight μ of $V(\lambda)$, we apply Theorem 1.2.1(iv). The following diagram shows these multiplicities. In this diagram we assume $\lambda = (r,s) \in X^+$ with $r \geq s$ (the case $r \leq s$ is nearly the same).

The numbers in the figure are the multiplicities of the weights on the corresponding line, see §2 on p. 1562 of [5].

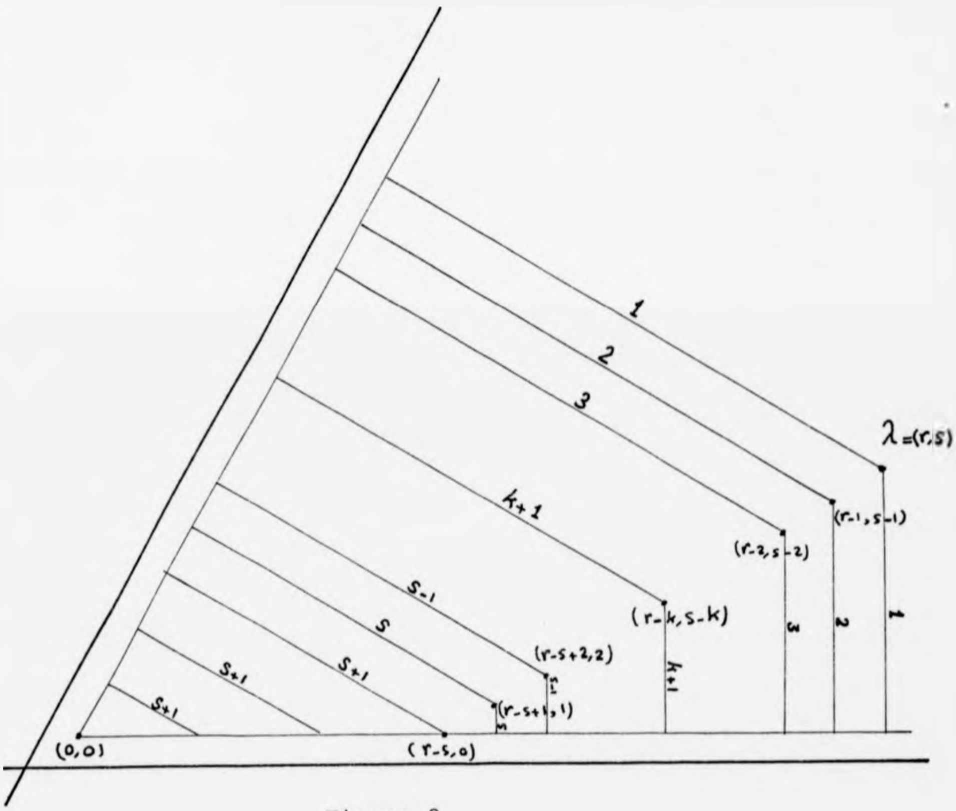

Figure 2.

Let $0 \neq v \in V(\lambda)^{\lambda}$ be a maximal vector. Thus by (v) of Theorem 1.2.1 $V(\lambda) = KGv$, and hence $V(\lambda) = U_K v$. The vector $(\prod_{\alpha\in\Phi^+} X_{-\alpha,a_\alpha})v$ is either zero or of weight $\lambda - a_\alpha\alpha - a_\beta\beta - a_{\alpha_0}\alpha_0$, see Proposition 5.13 of [6]. Let $\mu \in X$ be a weight of $V(\lambda)$. The vector v is maximal and generates $V(\lambda)$ as a U_K-module, therefore the set of all vectors of the form $(\prod_{\alpha\in\Phi^+} X_{-\alpha,a_\alpha})v$ such that $\lambda - a_\alpha\alpha - a_\beta\beta - a_{\alpha_0}\alpha_0 = \mu$ generates the weight subspace $V(\lambda)^{\mu}$. The condition of all a_α's to be nonnegative integers guarantees this set of generators to be finite. In order to get a basis for this weight subspace out of these generators, we have to reject some elements. These rejected elements depend on the chosen ordering in the multiplication of the $X_{-\alpha,a_\alpha}$'s, however see p. 218 of [9].

1.3.1. Example.

We consider the weight subspace $V(0,3)^{(0,0)}$ which is one dimensional. There are two solutions for $(0,3)-a_\alpha\alpha-a_\beta\beta-a_{\alpha_0}\alpha_0 = (0,0)$. The first one is $a_\alpha = 0$, $a_\beta = a_{\alpha_0} = 1$. The second is $a_\alpha = 1$, $a_\beta = 2$, $a_{\alpha_0} = 0$. For the ordering $X_{-\alpha,a_\alpha} X_{-\alpha_0,a_{\alpha_0}} X_{-\beta,a_\beta}$ (this corresponds to the lexicographic order of [9]), both the solutions work and we get a set of generators consists of two elements: $\{(X_{-\alpha_0,1}\ X_{-\beta,1})v,\ (X_{-\alpha,1}\ X_{-\beta,2})v\}$. Each element of the previous set is a basis. On the other hand if we consider the ordering $X_{-\beta,a_\beta}\ X_{-\alpha_0,a_{\alpha_0}}\ X_{-\alpha,a_\alpha}$, we find $(X_{-\beta,2}\ X_{-\alpha,1})v$ is zero (since $(-2,4)$ is not a weight of $V(0,3)$), and $\{(X_{-\beta,1}X_{-\alpha_0,1})v\}$ is a basis of $V(0,3)^{(0,0)}$. □

If it happened that for some weight μ of $V(\lambda)$, the number of solutions for $\lambda-a_\alpha\alpha-a_\beta\beta-a_{\alpha_0}\alpha_0 = \mu$ is equal to $m_\lambda(\mu)$. Then any ordering will do it, and the resulting set of generators is a basis of $V(\lambda)^\mu$ for any such ordering. We shall see that this is the case we meet in Section 3.2.

CHAPTER 2 - EXTENSIONS OF MODULES

In this chapter we investigate our main object, the (bi)functor Ext^1. The first section may be well known to many readers ((1.5f) of [18] and §2.1 of [20]), however the approach in this section uses more algebraic group techniques.

In some results of the last section p is assumed to be large enough ($p \geq 3(h-1)$), and in others λ is assumed to be small enough ($\lambda \in A_0$). These restrictions are lifted when $G = SL(3,K)$ in the next chapters. We hope these results may eventually be proved without these restrictions.

2.1. Rationally Injective Modules.

In this section G will be an affine algebraic group. We want to prove that any rational G-module may be embedded into an injective one.

2.1.1. Lemma.

Let I *be a rational* G-*module. Then* I *is an injective module iff the contravariant functor* $\mathrm{Hom}_G(-,I)$ *is exact.*

Proof.

Let $0 \to A \xrightarrow{j} B \xrightarrow{\pi} C \to 0$ be any short exact sequence in M_G. Applying the functor $\mathrm{Hom}_G(-,I)$, we get the exact sequence

$$0 \to \mathrm{Hom}_G(C,I) \xrightarrow{\bar{\pi}} \mathrm{Hom}_G(B,I) \xrightarrow{\bar{j}} \mathrm{Hom}_G(A,I),$$

where $\bar{j}(\phi) = \phi \circ j$, $\bar{\pi}(\psi) = \psi \circ \pi$ for every $\phi \in \mathrm{Hom}_G(B,I)$, $\psi \in \mathrm{Hom}_G(C,I)$

We can easily see that $\bar{j}$ is an epimorphism iff I is an injective module (for any short exact sequence like above). $\square$

2.1.2. Lemma.

The functor $Hom_G(-,I)$ *is exact iff it is exact on finite dimensional* G-*modules.*

Proof.

Suppose $Hom_G(-,I)$ is exact on finite dimensional G-modules. We want to prove I is injective.

Suppose

$$\begin{array}{ccccc} 0 & \to & U & \xrightarrow{j} & V \\ & & {\scriptstyle\theta}\downarrow & & \\ & & I & & \end{array}$$

is any G-diagram with exact row. Let V_1, V_2 be two submodules of V, and α_1,α_2: $V_1,V_2 \to I$ be two G-homomorphisms. Define a partial ordering on the pairs $(V_1,\alpha_1) << (V_2,\alpha_2)$ iff $V_1 \subseteq V_2$ and $\alpha_2|_{V_1} = \alpha_1$. Let

$$S = \{(V',\alpha)/(j(U),\ \theta o j^{-1}) << (V',\alpha),\ \ \alpha o j = \theta\}.$$

This S is not empty since $(j(U),\ \theta o j^{-1}) \in S$.

Let $S' = \{(A_i,\alpha_i)/i \in I\} \subseteq S$ be a non-empty totally ordered subset. Put $A = \bigcup_{i \in I} A_i$, and $\alpha:A \to I$ defined by $\alpha|_{A_i} = \alpha_i$ for every $i \in I$. Clearly (A,α) is an upper bound for S', and hence S is inductively ordered. Zorn's lemma completes the proof. Note that the fact that V is locally finite is used to show the maximal element we have by Zorn's lemma is V itself.

The other way is obvious. □

2.1.3. Lemma.

Let $\{I_\alpha/\alpha \in \Lambda\}$ *be a set of injectives in* M_G. *Then* $\sum_{\alpha\in\Lambda}^{\oplus} I_\alpha$ *is also injective.*

Proof.

It is known when Λ is finite (Theorem 5.7.3 of [14]). Suppose Λ is infinite.

Let

$$\begin{array}{ccccc} 0 & \to & U & \to & V \\ & & \theta\big\downarrow & & \\ & & \sum_{\alpha\in\Lambda}^{\oplus} I_\alpha & & \end{array}$$

be any G-diagram with exact row. Thus it is enough to consider the case when U,V are finite dimensional. So there exists $\{\alpha_1,\ldots,\alpha_n\} \subseteq \Lambda$ such that $\theta(U) \subseteq \sum_{i=1}^{n\oplus} I_{\alpha_i}$. □

2.1.4. Proposition.

K[G] *is an injective (left)* G-*module via the right translation.*

Proof.

It is enough to prove the functor $\mathrm{Hom}_G(-,K[G])$ is exact on finite dimensional modules. So it is enough to prove: $V \in M_G$ and finite dimensional implies $\dim \mathrm{Hom}_G(V,K[G]) = \dim V$.

Pick $\{v_1,\ldots,v_n\}$ a basis of V. Let $\{f_{ij}/i,j=1,\ldots,n\}$ be the corresponding coordinate functions (see (R2) before).

Define for $j = 1,\dots,n$, $\theta_j : V \to K[G]$ by $\theta_j(v_i) = f_{ji}$ for every $i = 1,\dots,n$ (and extend by linearity).

Claim $\{\theta_j / j = 1,\dots,n\}$ is a basis of $\mathrm{Hom}_G(V,K[G])$. From the definition of G-modules we have $x(yv_i) = (xy)v_i$ for every $x,y \in G$ and $i = 1,\dots,n$ which implies all these θ_j's are G-homomorphisms. Also $f_{jm}(1) = \delta_{jm}$ implies they are linearly independent, where $1 \in G$ is the identity element. Finally it is straightforward to prove: for every $\theta \in \mathrm{Hom}_G(V,K[G])$ we have $\theta = \sum_{j=1}^{n} k_j\theta_j$, where $\theta(v_j)(1) = k_j$. □

2.1.5. <u>Proposition</u>.

Every rational G*-module* V *is isomorphic to a submodule of a direct sum of copies of* K[G].

<u>Proof</u>.

The module $(V) \otimes K[G]$ (see Section 1.1) is isomorphic to $\sum_{i \in I}^{\oplus} K[G]$. In this direct sum I is the indexing set for $\{v_i / i \in I\}$, a basis of V, see p. 144 of [18]. Let $\{f_{ij}\}$ be the corresponding coordinate functions. It is clear that for every $i \in I$ the set $\{j \in I / f_{ji} \neq 0\}$ is finite. Define the map $\phi : V \to (V) \otimes K[G]$ by $\phi(v_i) = \sum_j v_j \otimes f_{ji}$. It follows directly that ϕ is a G-modules monomorphism. □

2.2. <u>The Functor Ext^1</u>.

Throughout this section G will be our universal Chevalley group except in the next definition and the following remarks where G may be an affine algebraic group.

2.2.1. Definition.

Let $M \in \mathcal{M}_G$ with injective resolution

$$0 \to M \xrightarrow{\xi} I_o \xrightarrow{d_o} I_1 \xrightarrow{d_1} I_2 \to \dots .$$

For every $V \in \mathcal{M}_G$, this resolution gives the complex

$$0 \to \mathrm{Hom}_G(V,M) \xrightarrow{\bar{\xi}} \mathrm{Hom}_G(V,I_o) \xrightarrow{\underline{d}_o} \mathrm{Hom}_G(V,I_1) \xrightarrow{\underline{d}_1} \dots ,$$

where $\underline{d}_n(\phi) = d_n \circ \phi$ for every $\phi \in \mathrm{Hom}_G(V,I_n)$. Define

$$\mathrm{Ext}_G^o(V,M) = \ker \underline{d}_o,$$

$$\mathrm{Ext}_G^n(V,M) = \frac{\ker \underline{d}_n}{\mathrm{Im}\, \underline{d}_{n-1}} \quad , n = 1,2,\dots .$$

Therefore we obtain the (bi)functors $\mathrm{Ext}_G^n(-,-)$, $n = 0,1,\dots$ from $\mathcal{M}_G \times \mathcal{M}_G$ to the category of K-spaces. □

Remark (1).

(i) When R is a ring, the functors Ext_R^n may be defined dually using projective resolution. We don't use it here since there is no guarantee for the existence of enough projectives in our category.

(ii) These functors are independent of the particular injective resolution used (Corollary 2.2 on page 90 of [19]).

(iii) For every V, $I \in \mathcal{M}_G$ with I injective we have:

$$\mathrm{Ext}_G^n(V,I) = 0.$$

(iv) For every $A,B,C \in \mathcal{M}_G$ with B finite dimensional we have:

$$\mathrm{Ext}_G^n (A \otimes B, C) \cong \mathrm{Ext}_G^n (A, B^* \otimes C).$$

(v) Given a short exact sequence in $\mathcal{M}_G$

$$0 \to A \to B \to C \to 0,$$

and $V \in \mathcal{M}_G$. We obtain the following long exact sequences:

$$0 \to \mathrm{Hom}_G(V,A) \to \mathrm{Hom}_G(V,B) \to \mathrm{Hom}_G(V,C) \to \mathrm{Ext}_G^1(V,A) \to \cdots$$

$$0 \to \mathrm{Hom}_G(C,V) \to \mathrm{Hom}_G(B,V) \to \mathrm{Hom}_G(A,V) \to \mathrm{Ext}_G^1(C,V) \to \cdots .$$

See Chapter IV, Section 7 of [19].

<u>Remark (2)</u>.

For each $V \in \mathcal{M}_G$ and $n \geq 0$, the Hochshild cohomology, $H^n(G,V)$, is defined to be

$$H^n(G,V) = \mathrm{Ext}_G^n(K,V),$$

where K is the one dimensional trivial G-module.

Let N be a closed and normal subgroup of G, and suppose $V \in \mathcal{M}_G$. The Hochshild cohomology $H^n(N,V)$ is made into a G-module as follows. Let

$$0 \to V \to I \to M \to 0$$

be a short exact sequence of G-modules with I injective (hence $I|_N$ is also injective by Theorem 4.3 and Proposition 2.1 of [11]). So we obtain the exact sequence

$$0 \to V^N \to I^N \xrightarrow{\alpha} M^N \to H^1(N,V) \to 0,$$

where $A^N \cong \mathrm{Hom}_N(K,A)$ (for every $A \in \mathcal{M}_G$) is a G-submodule of A. The isomorphism (of K-spaces) $H^1(N,V) \cong \frac{M^N}{\mathrm{Im}\ \alpha}$ gives a G-module structure to $H^n(N,V)$ when $n = 1$. We proceed inductively using the dimension shift ($H^m(N,V) \cong H^{m-1}(N,M)$, $m > 1$) to obtain such structure for every $n > 1$. So if $A,B \in \mathcal{M}_G$ such that A is finite dimensional, $\mathrm{Ext}_N^n(A,B)$ has a G-module structure with N acts trivially. Also we have:

$$H^n(N,A \otimes B) \overset{G}{\cong} H^n(N,A) \otimes B$$

for every $A,B \in \mathcal{M}_G$ such that $B|_N$ is trivial.

Remark (3).

For every $A,B \in \mathcal{M}_G$ we have:

$$\mathrm{Ext}_G^1(A,B) \cong E_G(A,B),$$

where $E_G(A,B)$ is the space of the equivalence classes of the extensions $0 \to B \to M \to A \to 0$ and $M \in \mathcal{M}_G$ (Theorem 2.4 on p. 91 of [19]). So $\mathrm{Ext}_G^1(A,B) \neq 0$ iff there exists such an extension which does not split. Thus the name "extension" comes.

Recall that for $\lambda \in X^+$, $M(\lambda)$ is the unique maximal submodule of the Weyl module $V(\lambda)$. Let $\lambda^* = -\omega_0(\lambda)$, where $\omega_0 \in W$ is its longest element.

2.2.2. Proposition.

Let $\lambda,\mu \in X^+$. *Then we have:*

$$\mathrm{Ext}_G^1(L(\mu),L(\lambda)) \cong \begin{cases} \mathrm{Hom}_G(M(\lambda^*),L(\mu^*)) & \textit{if } \lambda \geq \mu, \\ \mathrm{Hom}_G(M(\mu),L(\lambda)) & \textit{if } \lambda \leq \mu, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof.

The first two cases are in 3.10 of [13]. The last one comes from: a necessary condition for $L(\lambda)$ to be a composition factor of $V(\mu)$ is that $\lambda \leq \mu$. □

For a positive integer n define U_n to be the subalgebra of U_K generated by

$$\{X_{\alpha,a}/\alpha \in \Phi,\ 1 \leq a < p^n\}.$$

It is a finite dimensional algebra of dimension

$$\dim U_n = p^{n \dim_{\mathbb{C}} L}.$$

By Proposition 2.1 of [25], the following set

$$\{\prod_{\alpha\in\Phi^+} X_{-\alpha,a_\alpha} \prod_{i=1}^{\ell} H_{i,b_i} \prod_{\alpha\in\Phi^+} X_{\alpha,c_\alpha}/0 \leq a_\alpha, b_i, c_\alpha < p^n\}$$

is a basis of U_n. Moreover U_n is a symmetric algebra (the corollary for Theorem 1 of [26]).

Define the set

$$X_n = \{\lambda \in X^+/(\lambda,\alpha^\vee) < p^n \text{ for every } \alpha \in \Delta\}.$$

Thus

$$\{L(\lambda)/\lambda \in X_n\}$$

is a full set of simple U_n-modules (§2.2 of [25]).

For a positive integer n let F^n be the $n^{\underline{th}}$ Frobenius morphism. For every $V \in M_G$, let V^{F^n} be the G-module resulting from "twisting" the action of G on V by F^n. So if $A \in M_G$ such that U_n acts trivially on it, $A = A'^{F^n}$ for some $A' \in M_G$. Conversely, U_n acts trivially on V^{F^n} for every $V \in M_G$. For a fuller discussion of the above results we refer the reader to Section 2 of [32].

The monomorphism $\psi: U_K \to K[G]^*$ (see p. 11) sends U_n to a subalgebra of $K[G]^*$ isomorphic to $(K[G]/_{M}[p^n])^*$, where $M^{[p^n]}$ is the ideal of K[G] generated by $\{f^{p^n}/f(1) = 0\}$(§9.1 of [12], and §1 of [16]). Thus by Remark (1) of Section 1.1, we have an inclusion preserving equivalence of categories between the category of U_n-modules and that of $K[G]/M^{[p^n]}$ -comodules.

Recall that the adjoint action of G on $K[G]^*$ is defined by $ad(x)\gamma = \varepsilon_x * \gamma * \varepsilon_{x^{-1}}$ for every $x \in G$ and $\gamma \in K[G]^*$. This adjoint action (via ψ) makes U_K a G-module with U_n a submodule (§2.1 of [32]). Let A, B $\in M_G$, using

$$x(ua) = (ad(x)u)(xa)$$

for every $x \in G$, $u \in U_n$, and $a \in A$, we can define a G-module structure in $Hom_{U_n}(A,B)$ by

$$(x.f)(a) = x(f(x^{-1}a))$$

for every $x \in G$, $f \in Hom_{U_n}(A,B)$, and $a \in A$. This G-module structure gives another one to $Ext^1_{U_n}(A,B)$. Note that $I|_{U_n}$ is an injective U_n-module for each injective module $I \in M_G$.

Beginning with two G-modules, we can define the extension space between them as $K[G]/M^{[p^n]}$-comodules, and give it a G-module structure as well, see Remark (2) of Section 2.2. This extension space is isomorphic to the U_n-extension space between the same G-modules, and this isomorphism is as G-modules. See Section 1 of [17] for more details.

So Corollary 2 of [17] using our notation reads.

2.2.3 Proposition.

Suppose $\lambda, \mu \in X^+, \lambda = \lambda_o + p\lambda', \mu = \mu_o + p\mu'$, *where* $\lambda_o, \mu_o \in X_1$ *and* $\lambda', \mu' \in X^+$. *Then we have:*

$$\mathrm{Ext}^1_G(L(\mu),L(\lambda)) = \begin{cases} \mathrm{Ext}^1_G(L(\mu'),L(\lambda')) \oplus (\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda)))^G & \textit{if} \quad \mu_o = \lambda_o, \\ (\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda)))^G & \textit{if} \quad \mu_o \neq \lambda_o. \end{cases} \square$$

2.3. The Translation and The Extension Functors.

Let λ be a weight in the closure of the fundamental alcove ($\lambda \in \bar{A}_o$). A G-module V is said to belong to λ if all its composition factors have highest weights belonging to $W_a.\lambda$. Proposition 2.3 of [27] together with the strong linkage principle show that any indecomposable G-module belongs to some $\lambda \in \bar{A}_o$. The category of finite dimensional G-modules belonging to $\lambda \in \bar{A}_o$ will be denoted by C_λ.

In the definition of the translation functor T^μ_λ we follow §3 of [31]. We can decompose any finite dimensional G-module M into a direct sum $M = \sum_{i=1}^{n} {}^{\oplus} M_i$ of indecomposables. For every $\lambda \in \bar{A}_o$ set $p_\lambda(M)$ equal to the sum of M_i belonging to λ. In fact $p_\lambda(M)$ is the biggest submodule of M belonging to λ and is independent of the particular choice of the decomposition. Also we have:

$$M = \sum_{\lambda \in \bar{A}_o} p_\lambda(M).$$

For a weight ν we define ν_* to be the unique dominant weight in $\{\omega(\nu)/\omega \in W\}$, see §13.2 of [22].

For every $\lambda, \mu \in \bar{A}_o$ define the translation functor:

$$T_\lambda^\mu : C_\lambda \to C_\mu$$

by $T_\lambda^\mu V = p_\mu(V \otimes L(\mu-\lambda)_*)$ for every $V \in C_\lambda$.

The functor T_λ^μ is exact, and T_λ^μ, T_μ^λ are adjoint functors. So for every $A \in C_\lambda$ and $B \in C_\mu$ we have:

$$\mathrm{Hom}_G(T_\lambda^\mu A, B) \simeq \mathrm{Hom}_G(A, T_\mu^\lambda B),$$

$$\mathrm{Hom}_G(T_\mu^\lambda B, A) \simeq \mathrm{Hom}_G(B, T_\lambda^\mu A),$$

see §2 of [3].

Let $S \subseteq W_a$ be the set of reflections with respect to the walls of A_o. For $\lambda \in \bar{A}_o$ define

$$S_\lambda = \{\omega \in S / \omega.\lambda = \lambda\}.$$

For $\omega \in W_a$ define

$$\tau(\omega) = \{\omega' \in S / \omega\omega'.\lambda < \omega.\lambda\},$$

where λ is an arbitrary element of A_o, see §3 of [31].

2.3.1. <u>Lemma (§3, Lemma, [31])</u>.

Let $\lambda \in A_o$, $\mu \in \bar{A}_o$, and $\omega \in W_a$ be such that $\omega.\lambda \in X^+$. Then we have:

$$T_\lambda^\mu V(\omega.\lambda) \simeq \begin{cases} V(\omega.\mu) & \text{if } \omega.\mu \in X^+, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

2.3.2. Proposition (§3,Proposition, [31]).

Let $\lambda \in A_o$, $\mu \in \bar{A}_o$, *and* $\omega \in W_a$ *be such that* $\omega.\lambda \in X^+$. *Then we have:*

$$T_\lambda^\mu\, L(\omega.\lambda) \cong \begin{cases} L(\omega.\mu) & \text{if } S_\mu \cap \tau(\omega) \text{ is empty,} \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

For λ, $\mu \in X^+$, $[V(\lambda):L(\mu)]$ denotes the multiplicity of $L(\mu)$ as a composition factor in a Jorden-Hölder series of G-modules for $V(\lambda)$.

2.3.3. Theorem (§3, Theorem, [31]).

Let $\lambda \in A_o$, *and* $\omega,\omega' \in W_a$ *be such that* $\omega.\lambda, \omega'.\lambda \in X^+$. *Then for all* $\mu \in \bar{A}_o$ *such that* $\tau(\omega') \cap S_\mu = \phi$ *we have:*

$$[V(\omega.\lambda):L(\omega'.\lambda)] = \begin{cases} [V(\omega.\mu):L(\omega'.\mu)] & \text{if } \omega.\mu \in X^+, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

2.3.4. Proposition.

Let $\lambda \in A_o$, *and* $\omega,\omega' \in W_a$ *be such that* $\omega.\lambda, \omega'.\lambda \in X^+$. *Then for every* $\mu \in A_o$ *we have:*

$$\mathrm{Ext}_G^1(L(\omega.\lambda),L(\omega'.\lambda)) \cong \mathrm{Ext}_G^1(L(\omega.\mu),L(\omega'.\mu)).$$

Proof.

The weights λ, $\mu \in A_o$ and $\omega.\lambda,\omega'.\lambda \in X^+$, so $S_\mu = \phi$ and $\omega.\mu,\omega'.\mu \in X^+$. Applying T_λ^μ to the short exact sequence

$$0 \to M(\omega.\lambda) \to V(\omega.\lambda) \to L(\omega.\lambda) \to 0,$$

we get the short exact sequence

$$0 \to T_\lambda^\mu M(\omega.\lambda) \to V(\omega.\mu) \to L(\omega.\mu) \to 0.$$

Hence $\quad T_\lambda^\mu M(\omega.\lambda) \cong M(\omega.\mu).$

We have three cases.

<u>Case (i)</u> : $\omega.\lambda \geq \omega'.\lambda.$

Proposition 2.2.2 gives

$$\mathrm{Ext}_G^1(L(\omega.\lambda),L(\omega'.\lambda)) \cong \mathrm{Hom}_G(M(\omega.\lambda),L(\omega'.\lambda)).$$

Suppose its dimension is $n > 0$. Therefore (see 3.11(b) of [13]) there exists a submodule M of $M(\omega.\lambda)$ such that $M(\omega.\lambda)/M \cong \overset{n}{\Sigma}{}^{\oplus} L(\omega'.\lambda)$. Applying T_λ^μ to the short exact sequence

$$0 \to M \to M(\omega.\lambda) \to \overset{n}{\Sigma}{}^{\oplus} L(\omega'.\lambda) \to 0,$$

we obtain $n \leq \dim \mathrm{Ext}_G^1 (L(\omega.\mu), L(\omega'.\mu))$.

<u>Case (ii)</u>: $\omega.\lambda \leq \omega'.\lambda.$

Proposition 2.2.2 gives

$$\mathrm{Ext}_G^1(L(\omega.\lambda), L(\omega'.\lambda)) \cong \mathrm{Hom}_G(M((\omega'.\lambda)^*), L((\omega.\lambda)^*)).$$

The weight $\lambda^* \in A_o$ and $(\omega'.\lambda)^* = \omega_1'.\lambda^*$ for some $\omega_1' \in W_a$.

Thus

$$T_{\lambda^*}^{\mu^*} M((\omega'.\lambda)^*) \cong M((\omega'.\mu)^*),$$

$$T_{\lambda^*}^{\mu^*} L((\omega.\lambda)^*) \cong L((\omega.\mu)^*).$$

So as in case (i) we obtain:

$$\dim \mathrm{Ext}^1_G(L(\omega.\lambda), L(\omega'.\lambda)) \leq \dim \mathrm{Ext}^1_G(L(\omega.\mu), L(\omega'.\mu)).$$

Case (iii): $\omega.\lambda, \omega'.\lambda$ are not comparable.

This gives $\mathrm{Ext}^1_G(L(\omega.\lambda), L(\omega'.\lambda)) = 0$.

Hence in general we obtain:

$$\dim \mathrm{Ext}^1_G(L(\omega.\lambda), L(\omega'.\lambda)) \leq \dim \mathrm{Ext}^1_G (L(\omega.\mu), L(\omega'.\mu)).$$

Beginning with $\omega.\mu$, $\omega'.\mu$, we get the reverse inequality. □

2.3.5. Definition.

Let G be an affine algebraic group with $\{L_\alpha / \alpha \in \Lambda\}$ a full set of simple rational G-modules. For each $\alpha \in \Lambda$ choose I_α an injective cover for L_α. We say that α_1 and α_2 are adjacent if either $\mathrm{Hom}_G(I_{\alpha_1}, I_{\alpha_2}) \neq 0$ or $\mathrm{Hom}_G(I_{\alpha_2}, I_{\alpha_1}) \neq 0$ or both. The equivalence classes of Λ of the equivalence relation generated by adjacency are called the blocks. The simple module L_α is said to belong to a block if α is in this block. □

2.3.6. Definition.

A finite dimensional rational G-module V is said to belong to a block if all its composition factors belong to that block. A rational G-module is in a block if all its finite dimensional submodules are in that block. □

2.3.7. Proposition.

Let G be an affine algebraic group. Then we have:

(i) *Each indecomposable* $V \in M_G$ *belongs to one block.*

(ii) *For* $V \in M_G$, *let* V_α *be its unique maximal submodule which lies in the block* $\beta(\alpha)$ $(\alpha \in \beta(\alpha))$. *Then*

$$V = \sum_{\alpha \in \Lambda}^{\oplus} V_\alpha .$$

Proof.

See (1.6b), (1.6c) of [18]. □

2.3.8. Corollary.

Two indices α, δ *are in the same block iff there exist* $\alpha = \alpha_1, \ldots, \alpha_n = \delta$ *such that either* $\mathrm{Ext}_G^1(L_{\alpha_i}, L_{\alpha_{i+1}}) \neq 0$ *or* $\mathrm{Ext}_G^1(L_{\alpha_{i+1}}, L_{\alpha_i}) \neq 0$ *or both,* $i = 1, \ldots, n-1$.

Proof.

Suppose α, δ are in the same block. To prove the necessity of the condition we may assume L_α is a composition factor of I_δ. By (R1) in Section 1.1, we can find M a finite dimensional submodule of I_δ such that L_α is a composition factor of it.

Suppose

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = M,$$

a composition series for M. The order of L_α in such a series is defined to be i such that $L_\alpha \cong V_i/V_{i-1}$. We choose a composition series for M such that this order is minimal and call it $n(M,\alpha)$.

We argue by induction on $n(M,\alpha)$. If $n(M,\alpha) = 1$, we have: $L_\alpha \cong L_\delta$ and the corollary follows. Suppose $n(M,\alpha) > 1$, and the corollary is true for each $L_{\alpha'}$ a composition factor of M such that its $n(M,\alpha') < n(M,\alpha)$. Let

$$V_{n(M,\alpha)-1}/V_{n(M,\alpha)-2} \cong L_{\alpha_1}$$

for some $\alpha_1 \in \Lambda$. By the minimality of $n(M,\alpha)$ we have: either $\mathrm{Ext}^1_G(L_\alpha, L_{\alpha_1}) \neq 0$ or $\mathrm{Ext}^1_G(L_{\alpha_1}, L_\alpha) \neq 0$ or both.

The sufficiency of the condition is a direct consequence of 2.3.7(i). □

The following proposition plays an important role in the next two chapters as well as in the two results just next to it, but first we need a definition.

2.3.9. <u>Definition</u>.

Two weights λ,μ are said to be linked if there exists $\omega \in W$ (the Weyl group) for which $\omega.\lambda \equiv \mu \pmod{pX}$. □

2.3.10. <u>Proposition</u>.

Let $\lambda, \mu \in X_1$. *Then we have:*

(i) *Each G-composition factor of* $\mathrm{Ext}^1_{U_1}(L(\mu), L(\lambda))$ *is of the form* $L(\xi)^F$ *such that* $\mu + p\xi \le 2(p-1)\rho + \omega_0(\lambda)$.

(ii) *A necessary condition for* $L(\xi)^F$ *to be a G-composition factor of* $\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda))$ *is that* $\mu + p\xi = \omega.\lambda$ *for some* $\omega \in W_a$.

(iii) *A necessary condition for* $\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda))$ *to be non-zero is that* μ *is linked to* λ.

Proof.

(i) Let $St = L((p-1)\rho)$ be the first Steiberg module, and let $\hat{\lambda} = (p-1)\rho + \omega_o(\lambda)$. Thus $L(\lambda)$ is contained exactly once as a U_1-submodule of $L(\hat{\lambda}) \otimes St$ (p. 41 of [24]) i.e. $L(\lambda)$ is the $L(\lambda)$-isotypic component of $\mathrm{soc}_{U_1} L(\hat{\lambda}) \otimes St$. Hence $L(\hat{\lambda}) \otimes St$ contains $L(\lambda)$ exactly once as a G-submodule (§2.2 of [32]). Therefore we get the short exact sequence of G-modules:

$$0 \to L(\lambda) \to L(\hat{\lambda}) \otimes St \to L(\hat{\lambda}) \otimes St/L(\lambda) \to 0.$$

It is well known that the restriction $St|_{U_1}$ is projective (§5.5 of [24]). Thus $L(\hat{\lambda}) \otimes St$ is projective as a U_1-module (§8.1 of [24]), and hence it is injective by the symmetry of U_1 ([26]). So we have the exact sequence (of G-modules):

$$0 \to \mathrm{Hom}_{U_1}(L(\mu),L(\lambda)) \to \mathrm{Hom}_{U_1}(L(\mu),L(\hat{\lambda})\otimes St) \to \mathrm{Hom}_{U_1}(L(\mu),L(\hat{\lambda})\otimes St/L(\lambda))$$

$$\to \mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda)) \to 0.$$

To complete the proof of Part (i), we need the following lemma together with the fact that any weight of $L(\hat{\lambda}) \otimes St$ is $\leq 2(p-1)\rho + \omega_o(\lambda)$.

2.3.11. Lemma.

Let V *be a finite dimensional* G-*module, and let* $\mu \in X_1$. *Then all* G-*composition factors of* $\mathrm{Hom}_{U_1}(L(\mu),V)$ *are of the form* $L(\xi)^F$ *such that* $L(\mu + p\xi)$ *is a* G-*composition factor of* V.

Proof.

We argue by induction on the composition length of V.

Suppose V is simple i.e. $V \simeq L(\lambda + p\xi)$ for some $\lambda \in X_1$ and $\xi \in X^+$. By Steinberg's tensor product theorem and §2.3 of [32] we have:

$$\mathrm{Hom}_{U_1}(L(\mu),V) \overset{G}{\simeq} \mathrm{Hom}_{U_1}(L(\mu),L(\lambda)) \otimes L(\xi)^F.$$

Hence $\mathrm{Hom}_{U_1}(L(\mu),V) \neq 0$ iff $\mu = \lambda$, and in this case $\mathrm{Hom}_{U_1}(L(\mu),L(\mu)) \overset{G}{\simeq} K$.

Now suppose the composition length of V is greater than one. Pick $L \in \mathcal{M}_G$ a simple submodule of V. The short exact sequence of G-modules

$$0 \to L \to V \to V/L \to 0,$$

gives the exact sequence (of G-modules)

$$0 \to \mathrm{Hom}_{U_1}(L(\mu),L) \to \mathrm{Hom}_{U_1}(L(\mu),V) \to \mathrm{Hom}_{U_1}(L(\mu),V/L).$$

The composition factors of the middle term are composition factors of the outer two terms. □

Proof of 2.3.10 (Cont.)

(ii) Let M be the block component of $L(\hat{\lambda}) \otimes St$ containing $L(\lambda)$, see Proposition 2.3.7(ii). Thus

$$L(\hat{\lambda}) \otimes St = M \oplus M'$$

for some $M' \in \mathcal{M}_G$. The simple module $L(\nu)$ is a G-composition factor of M implies $\nu = \omega.\lambda$ for some $\omega \in W_a$, see [16] and the introduction of Chapter 3. Also M is an injective U_1-module. Replacing $L(\hat{\lambda}) \otimes St$ by M in the first part we get our result.

(iii) It is a direct consequence of (ii). □

2.3.12. Proposition.

Suppose $p \nmid f = |X/X_r|$. *Then we have:*

$$\mathrm{Ext}^1_{U_1}(L(\lambda),L(\lambda)) = 0$$

for every $\lambda \in A_o$.

Proof.

Claim 1. For every ξ, $\xi' \in X^+$ we have:

$$\mathrm{Ext}^1_G(L(\xi)^F,L(\xi')^F) \cong \mathrm{Ext}^1_G(L(\xi),L(\xi')).$$

By Proposition 2.2.3, it is enough to prove

$$\mathrm{Ext}^1_{U_1}(L(0),L(0)) = 0.$$

Section 6.10 of [12] states that $\mathrm{Ext}^1_{U_1}(L(0),L(0)) = 0$ unless Φ is of type $C_\ell (\ell \geq 1)$ and $p = 2$. This is not our case, thus we are done. For the values of f for different types of Φ, we refer to §13.1 of [22], or to p. 99 of [7].

<u>Claim 2</u>. Suppose $\lambda \in A_o$ and $\xi \in X^+$ such that

$$\mathrm{Ext}^1_G(L(\lambda + p\xi), L(\lambda)) \neq 0$$

then $\xi \in X_r$.

The two weights $\lambda + p\xi$ and λ are comparable, hence $p\xi \in X_r$. Let $\bar{\xi} = \xi + X_r \in X/X_r$. So $p\bar{\xi} = \bar{0}$ implies p is an exponent of $\bar{\xi}$. By assumption $p \nmid f = |X/X_r|$, therefore $\bar{\xi} = \bar{0}$ and $\xi \in X_r$. This proves Claim 2.

Now let $\lambda \in A_o$ and $\xi \in X^+$. Hence by Proposition 2.2.3 we have:

$$\mathrm{Ext}^1_G(L(\lambda+p\xi), L(\lambda)) \cong \mathrm{Ext}^1_G(L(\xi), L(0)) \oplus \mathrm{Hom}_G(L(\xi)^F, \mathrm{Ext}^1_{U_1}(L(\lambda), L(\lambda))).$$

Using 2.3.10(i), it is enough to prove

$$\mathrm{Ext}^1_G(L(\lambda+p\xi), L(\lambda)) \neq 0 \Rightarrow \mathrm{Ext}^1_G(L(\lambda+p\xi), L(\lambda)) \cong \mathrm{Ext}^1_G(L(\xi), L(0)).$$

Since $\xi \in X_r$, there exists $\omega \in W_a$ such that $\omega.\lambda = \lambda + p\xi$. Applying 2.3.4 (in that proposition we put our λ and ω above, and $\mu = 0$, $\omega' = 1$) we get:

$$\mathrm{Ext}^1_G(L(\lambda+p\xi), L(\lambda)) \cong \mathrm{Ext}^1_G(L(p\xi), L(0)).$$

Claim 1 completes the proof. $\square$

2.3.13. <u>Proposition</u>.

Suppose $p \geq 3(h-1)$. *Then* $\mathrm{Ext}^1_{U_1}(L(\mu), L(\lambda))$ *is completely reducible as a G-module for every* $\lambda, \mu \in X_1$.

Proof.

For every $\xi \in A_o$, we have: $V(\xi) \cong L(\xi)$ (§4.1 of [24]). Thus by 2.2.2 and Claim 1 in the proof of the last proposition it is enough to prove: $L(\xi)^F$ is a composition factor of $\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda))$ implies $\xi \in A_o$.

Suppose $L(\xi)^F$ is a composition factor of $\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda))$. So $\mu + p\xi \le 2(p-1)\rho + \omega_o(\lambda)$, this gives

$$(2(p-1)\rho + \omega_o(\lambda) - \mu - p\xi,\ \alpha_o^\vee) \ge 0.$$

Therefore

$$p(\xi,\alpha_o^\vee) \le 2p(\rho,\alpha_o^\vee) - 2(\rho,\alpha_o^\vee) + (\omega_o(\lambda),\alpha_o^\vee) - (\mu,\alpha_o^\vee).$$

Hence

$$(\xi,\alpha_o^\vee) < 2(\rho,\alpha_o^\vee) \Rightarrow (\xi+\rho,\alpha_o^\vee) < 3(h-1) \le p. \quad \Box$$

CHAPTER 3 - THE FUNCTOR $\mathrm{Ext}^1_{U_1}$ FOR TYPE A_2

A finite dimensional G-module V is said to have a Weyl filtration if there exists a series of submodules

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V,$$

such that

$$V_i/V_{i-1} \cong V(\lambda_i)$$

for some $\lambda_i \in X^+$, and $i = 1,\ldots,n$.

Define

$$B = \langle T, \chi_{-\alpha} / \alpha \in \Phi^+ \rangle.$$

This B is a Borel subgroup of G. For each $\lambda \in X$, let K_λ be the one dimensional B-module on which T acts by weight λ, and the unipotent radical of B acts trivially. Let $Y(\lambda)$ denote the induced module

$$\mathrm{Ind}_B^G(K_\lambda) = Y(\lambda).$$

It is well known that for each $\lambda \in X^+$

$$Y(\lambda) \cong V(\lambda^*)^*,$$

and $V(\lambda)$, $Y(\lambda)$ have the same character (and composition factors). See p. 55 of [2], §1 of [29], and Proposition 2 on p. 684 of [35].

A finite dimensional G-module V is said to have a good filtration if it has a series like above with each quotient isomorphic to some induced module $Y(\lambda_i)$. By taking the duals, we can see that a finite dimensional G-module has a Weyl filtration iff its dual has a good filtration (Corollary 2.6 of [36]).

Beside the above mentioned filtrations we introduce the blocks of the universal Chevalley groups as well. For each $\lambda \in X^+$, we let $\beta(\lambda)$ to be the block containing λ. Define $r(\lambda)$ to be the nonnegative integer such that

$$\lambda + \rho \in p^{r(\lambda)}X \backslash p^{r(\lambda)+1} X.$$

Then we have ([16]):

$$\beta(\lambda) = (W.\lambda + p^{r(\lambda)+1} \mathbb{Z}\Phi) \cap X^+.$$

For a small p (p = 3 say) and when G = SL(3,K), it is easy to identify the block of each weight which is not so far from the origin. This can be done by calculating the $r(\lambda)$ above, and determining which weights are inside the alcoves (each alcove contains only one weight, see Fig. 1), and which are on the boundaries.

3.1. <u>Weyl Filtration For $Q(\lambda)$</u>.

In this section we assume that G = SL(3,K).

We know that $\{L(\lambda)/\lambda \in X_1\}$ is a full set of simple U_1-modules. We want to show for each $\lambda \in X_1$ (and each p) there exists a rational G-module $Q(\lambda)$ with Weyl filtration such that the restriction $Q(\lambda)|_{U_1}$ is isomorphic to $Q(1,\lambda)$, the principal indecomposable U_1-module associated to $L(\lambda)$.

The restricted region X_1 is divided into two alcoves and three boundaries. They are the lower alcove A_0, the upper alcove, and the boundaries: $\{(r,s) \in X^+/r + s = p - 2\}$ (this boundary separates the above two alcoves), $\{(r,p-1)/0 \leq r \leq p-1\}$, and $\{(p-1,s)/0 \leq s \leq p-1\}$.

3.1.1. Lemma.

Let $\lambda \in A_o$, or be on one of the above boundaries. Then $V(\lambda) \cong L(\lambda)$.

Proof.

There is no dominant weight $\mu < \lambda$, and μ is strongly linked to λ, however see §4.1 of [24]. □

3.1.2. Lemma.

For $p = 2$ *or* 3, *the tensor product* $L(\lambda) \otimes \mathrm{St}$ *has a Weyl filtration for every* $\lambda \in X_1$.

Proof.

Unless $p = 3$ and $\lambda = (1,1)$, we have $V(\lambda) \cong L(\lambda)$. Thus, apart from this case, our tensor product has a Weyl filtration by Theorem 4.2 of [36] (this theorem states that in type A_ℓ, the module $V(\lambda) \otimes V(\mu)$ has such filtration for every $\lambda, \mu \in X^+$).

When $p = 3$, direct calculations show that

$$\mathrm{ch}.L(1,1) = e(1,1) + e(2,-1) + e(1,-2) + e(-1,-1) + e(-2,1) + e(-1,2) + e(0,0).$$

So all the weights of $L(1,1)|_B \otimes K_{(2,2)}$ are dominant. Hence $\mathrm{Ind}_B^G(L(1,1)|_B \otimes K_{(2,2)})$ has a good filtration by Corollary 2.8 of [36]. But by Proposition 1.5 of [11] we have:

$$\mathrm{Ind}_B^G(L(1,1)|_B \otimes K_{(2,2)}) \cong L(1,1) \otimes Y(2,2).$$

Thus

$$(L(1,1) \otimes Y(2,2))^* \cong L(1,1) \otimes V(2,2)$$

has a Weyl filtration. □

3.1.3. Lemma.

Let $p = 2$ *or* 3, *and for* $\lambda \in X_1$ *let* $\hat{\lambda} = (p-1)\rho + w_o(\lambda)$. *Then there is a rational* G-*module* $Q(\lambda)$ *occurring precisely once as a* G-*summand of* $L(\hat{\lambda}) \otimes St$, *and such that the restriction* $Q(\lambda)|_{U_1}$ *is isomorphic to the* U_1-*module* $Q(1,\lambda)$, *the principal indecomposable module associated to* $L(\lambda)$.

Proof.

Theorem 1.2.1(iii) and an easy calculation in some Weyl modules give us the coefficients $a_{\lambda\tau}$ such that

$$\text{ch. } L(\hat{\lambda}) \text{ ch. St} = \sum_{\tau \in X^+} a_{\lambda\tau} \text{ ch.} L(\tau).$$

Knowing the composition factors of $L(\hat{\lambda}) \otimes St$ allows us to decompose it into block components (see the introduction of this chapter, and see also Proposition 2 on p. 684 of [35]). So, if M is the block component of $L(\hat{\lambda}) \otimes St$ which contains $L(\lambda)$, we can easily calculate its dimension.

Now M is an injective U_1-module containing $L(\lambda)$ exactly once as a U_1-submodule (and as a G-submodule). Therefore $Q(1,\lambda)$ is a U_1-submodule of M. The dimension of $Q(1,\lambda)$ can be calculated using §5.2 and §5.4 of [24]. We find that $\dim Q(1,\lambda) = \dim M$ in all cases except when $p = 3$ and $\lambda = (0,0)$. This proves the lemma except in that case, which will be proved in what follows.

The block decomposition of $St \otimes St$, when $p = 3$, is

$$St \otimes St = M \oplus L(5,2) \oplus L(2,5) \oplus 3\ St.$$

Also we have:

$$\mathrm{Hom}_G(L(1,1), St \otimes St) \simeq \mathrm{Hom}_G(L(1,1)\otimes St, St) \simeq \mathrm{Hom}_G(Q(1,1) \oplus St, St) \simeq K.$$

This gives that $L(1,1)$ is contained exactly once as a G-submodule of $St \otimes St$, hence as a G-submodule of M. Therefore

$$Q(1,(0,0)) \oplus Q(1,(1,1)) \subseteq M|_{U_1}.$$

From p. 26 of [24] we have:

$$\dim Q(1,(0,0)) = 6\times 2\times 3^3, \text{ and } \dim Q(1,(1,1)) = 3\times 2\times 3^3.$$

It follows that

$$Q(1,(0,0)) \oplus Q(1,(1,1)) \overset{U_1}{\simeq} M.$$

Let P be the G-indecomposable component of M containing $L(0,0)$. So it is enough to prove P does not contain $L(1,1)$. We have:

$$St/L(1,0) \otimes L(1,2),$$

where "/" means "divides" i.e. St is a direct summand of $L(1,0) \otimes L(1,2)$. Hence

$$St \otimes St/L(1,0) \otimes L(1,2) \otimes St \simeq L(1,0) \otimes Q(0,1) \oplus V,$$

for some $V \in M_G$. Also we have:

$$\mathrm{Hom}_G(L(0,0), L(1,0)\otimes L(1,2)\otimes St) \simeq \mathrm{Hom}_G(L(0,1), L(1,2)\otimes St) \simeq K,$$

$$\mathrm{Hom}_G(L(0,0), L(1,0) \otimes Q(0,1)) \simeq K.$$

Hence

$$L(0,0) \not\subseteq V, \text{ and } P/L(1,0) \otimes Q(0,1).$$

So it is enough to prove $L(1,1) \not\subseteq L(1,0) \otimes Q(0,1)$.

We have:

$$\mathrm{Hom}_{U_1}(L(1,1), L(1,0) \otimes Q(0,1)) \simeq \mathrm{Hom}_{U_1}(L(1,1) \otimes L(0,1), Q(0,1)).$$

So the next lemma (which is true for any type of Φ) together with $L(1,1) \otimes L(0,1) \simeq L(1,2) \oplus L(2,0)$ completes the proof. $\square$

3.1.4. Lemma.

Let A *be a finite dimensional* U_1*-module. Then* $\dim \mathrm{Hom}_{U_1}(A, Q(1,\lambda))$ *is equal to the multiplicity of* $L(\lambda)$ *as a* U_1*-composition factor of* A $(\lambda \in X_1)$.

Proof.

Let

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

be a short exact sequence of U_1-modules. Thus we get the short exact sequence

$$0 \to \mathrm{Hom}_{U_1}(V_3, Q(1,\lambda)) \to \mathrm{Hom}_{U_1}(V_2, Q(1,\lambda)) \to \mathrm{Hom}_{U_1}(V_1, Q(1,\lambda)) \to 0.$$

We then argue by induction on the U_1-composition length of A. $\square$

3.1.5. Proposition.

For every $\lambda \in X_1$ *there exists a rational* G*-module* $Q(\lambda)$ *such that* $Q(\lambda)|_{U_1} \simeq Q(1,\lambda)$, *moreover* $Q(\lambda)$ *has a Weyl filtration.*

Proof.

When $p \geq 2(h-1) = 4$, see §5.6 of [32].

When $p = 2$ or 3, the module $Q(\lambda)$ is a direct summand of $L(\hat{\lambda}) \otimes St$. The latter has such filtration by Lemma 3.1.2. Thus $Q(\lambda)$ also has one by §5.2(3) of [32]. □

3.1.6. Corollary.

The socle

$$\mathrm{Soc}_G\, V(2(p-1)\rho + \omega_o(\lambda)) = L(\lambda)$$

for every $\lambda \in X_1$.

Proof.

We claim that $Q(\lambda)$ has $2(p-1)\rho + \omega_o(\lambda)$ as its unique highest weight. By Lemma 3.2 of [15], the weight $\lambda - 2(p-1)\rho$ is the unique minimal weight of $Q(\lambda)$, also μ is a weight of $Q(\lambda)$ implies $\omega_o(\mu)$ is a weight of it. Hence $2(p-1)\rho + \omega_o(\lambda)$ is the unique highest weight of $Q(\lambda)$.

The next argument is due to J. Jantzen ([32]). Let $0 \neq v \in Q(\lambda)$ be of weight $2(p-1)\rho + \omega_o(\lambda)$. Hence from §5.2(2) of [32] we have:

$$KGv \simeq V(2(p-1)\rho + \omega_o(\lambda)).$$

Therefore

$$0 \neq \mathrm{Soc}_G KGv \simeq \mathrm{Soc}_G V(2(p-1)\rho + \omega_o(\lambda)) \subseteq \mathrm{Soc}_G Q(\lambda) \simeq L(\lambda). \quad \square$$

3.2. Some Weyl Modules of Small Weights.

One step on the way to get $\mathrm{Ext}^1_G(L(\mu), L(\lambda))$ when $G = SL(3,K)$ is to determine $\mathrm{Ext}^1_{U_1}$ between simple U_1-modules. To determine this $\mathrm{Ext}^1_{U_1}$, we need some $\mathrm{Ext}^1_G(L(\tau), L(\lambda))$ when $\lambda \in X_1$, and τ rather small. This will be done in this section.

When we put $\overset{G}{\longleftrightarrow}$ between two G-modules, we mean that they have the same composition factors.

Throughout this section we assume that $G = SL(3,K)$.

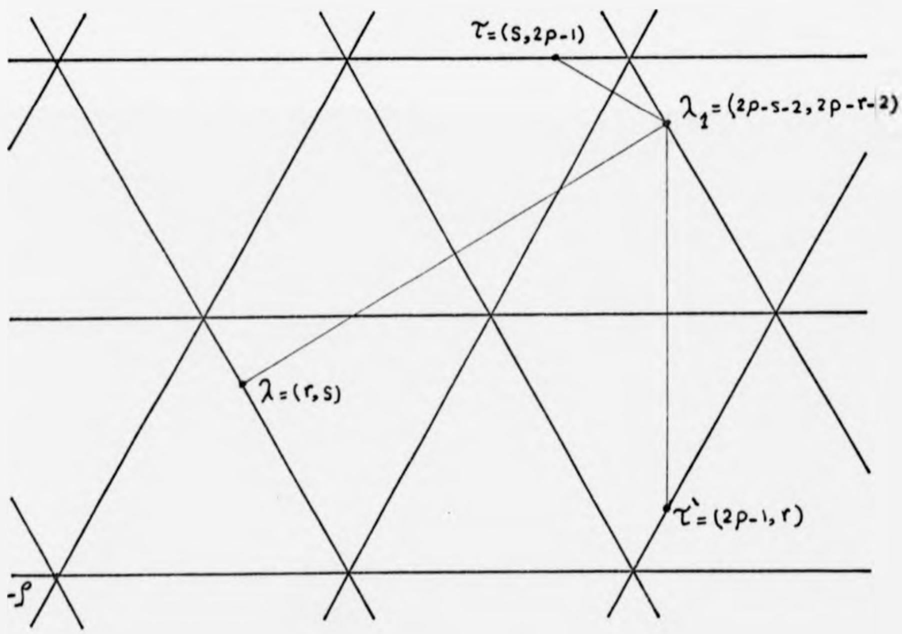

Figure 3.

3.2.1. Lemma.

Suppose the notation of Figure 3. For $p \neq 3$ *we have* :

(i) $\mathrm{Soc}_G V(\mu) \simeq L(\lambda)$, $\mu \in \{\lambda, \tau, \tau', \lambda_1\}$.

(ii) $V(\mu) \overset{G}{\longleftrightarrow} L(\mu) \oplus L(\lambda)$, $\mu \in \{\tau, \tau'\}$.

(iii) $V(\lambda_1) \overset{G}{\longleftrightarrow} L(\lambda_1) \oplus L(\tau) \oplus L(\tau') \oplus L(\lambda)$.

Proof.

(i) The socle $soc_G\, V(\lambda_1) \simeq L(\lambda)$ by Corollary 3.1.6. The Weyl module $V(\lambda_1)$ contains a copy of both $V(\tau)$ and $V(\tau')$, see p. 177 of [32]. Finally $V(\lambda)$ is simple.

(ii) It is a direct consequence of dimensional calculations

(iii) This part follows from (i), (ii) above and dimensional calculations. □

When p = 3, the only change in the above lemma is that $[V(\lambda_1) : L(\lambda)] = 2$. This is the only case for which V(1,1) is no longer simple. The dimension of L(1,1) is seven (not eight as for all other cases i.e. $p \neq 3$). This smaller dimension affects the multiplicity of the composition factor above.

3.2.2. Lemma.

Suppose the notation of Fig. 3. For $p \neq 3$ *we have:*

(i) $\mathrm{Ext}^1_G(L(\mu),L(\lambda)) = 0$, $\mu \in \{\lambda,\lambda_1\}$.

(ii) $\mathrm{Ext}^1_G(L(\mu),L(\lambda)) \simeq K$, $\mu \in \{\tau,\tau'\}$.

Proof.

(i) It is obvious from Proposition 2.2.2 that $\mathrm{Ext}^1_G(L(\lambda),L(\lambda) = 0$, however see §4.1 of [25].

If $\mathrm{Ext}^1_G(L(\lambda_1),L(\lambda)) \neq 0$, the simple module $L(\lambda)$ will appear as a "next to the top" composition factor of $V(\lambda_1)$ leaving a different one in the bottom, this is a contradiction to Lemma 3.2.1(i).

(ii) It is a direct application of Proposition 2.2.2 and the last lemma. □

3.2.3. Lemma.

Suppose the notation of Fig. 3. For $p = 3$ *we have:*

(i) $\mathrm{Ext}^1_G(L(\lambda), L(\lambda)) = 0$

(ii) $\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \cong K$, $\mu \in \{\tau, \tau', \lambda_1\}$.

Proof.

We prove the case when $\mu = \lambda_1 = (2p-s-2, 2p-r-2)$. The upper alcove contains only one weight $(1,1)$, and $(0,0)$ is its image in the lower one. Hence we have:

$$\mathrm{Ext}^1_G(L(1,1), L(0,0)) \cong K.$$

Also we have (see Proposition 2.2.3):

$$\mathrm{Ext}^1_G(L(2p-s-2, 2p-r-2), L(r,s)) \cong$$

$$\cong \mathrm{Ext}^1_G(L(p-s-2, p-r-2) \otimes L(1,1)^F, L(r,s)) \cong$$

$$\cong \mathrm{Ext}^1_G(L(1,1), L(0,0)) \oplus (\mathrm{Ext}^1_{U_1}(L(2p-s-2, 2p-r-2), L(r,s)))^G.$$

It follows that

$$1 \le \dim \mathrm{Ext}^1_G(L(\lambda_1), L(\lambda)) \le 2.$$

This dimension cannot be two, otherwise $L(\lambda)$ will appear twice just next to the top composition factor of $V(\lambda_1)$ leaving a different one in the bottom, this is a contradiction to Lemma 3.2.1(i). $\square$

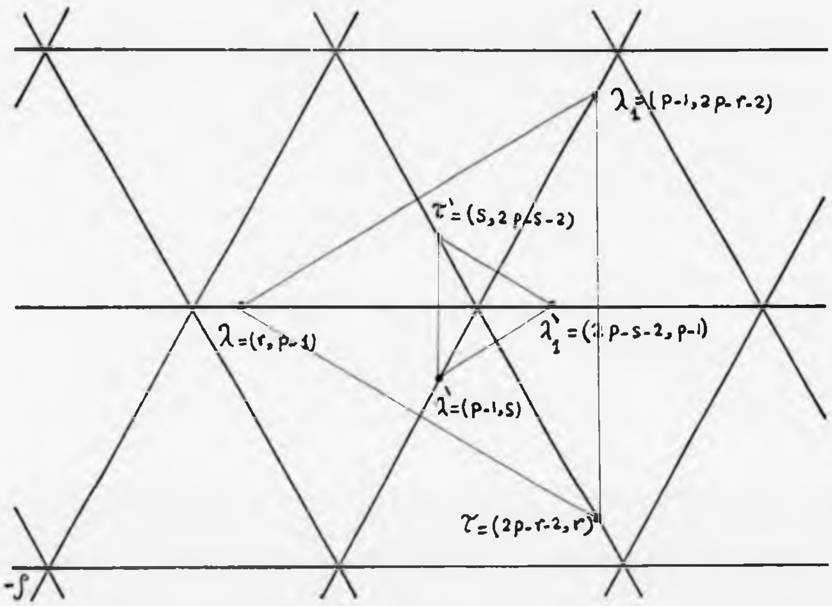

Figure 4.

3.2.4. Lemma.

Suppose the notation of Fig. 4. We have:

(i) $\mathrm{soc}_G\, V(\mu) \simeq L(\lambda)$, $\mu \in \{\lambda, \tau, \lambda_1\}$.

(ii) $\mathrm{soc}_G\, V(\mu) \simeq L(\lambda')$, $\mu \in \{\lambda', \tau', \lambda_1'\}$.

(iii) $V(\mu) \simeq L(\mu)$, $\mu \in \{\lambda, \lambda'\}$.

(iv) $V(\tau) \xleftrightarrow{G} L(\tau) \oplus L(\lambda)$.

(v) $V(\tau') \xleftrightarrow{G} L(\tau') \oplus L(\lambda')$.

(vi) $V(\lambda_1) \xleftrightarrow{G} L(\lambda_1) \oplus L(\tau) \oplus L(\lambda)$.

(vii) $V(\lambda_1') \xleftrightarrow{G} L(\lambda_1') \oplus L(\tau') \oplus L(\lambda')$.

Proof.

Imitate the proof of Lemma 3.2.1. $\square$

3.2.5. Lemma.

Suppose the notation of Fig. 4. We have:

(i) $\mathrm{Ext}^1_G(L(\mu),L(\lambda)) = 0, \quad \mu \in \{\lambda,\lambda_1\}.$

(ii) $\mathrm{Ext}^1_G(L(\mu),L(\lambda')) = 0, \quad \mu \in \{\lambda',\lambda'_1\}.$

(iii) $\mathrm{Ext}^1_G(L(\tau),L(\lambda)) \cong K.$

(iv) $\mathrm{Ext}^1_G(L(\tau'),L(\lambda')) \cong K.$ $\square$

3.2.6. Lemma.

Suppose μ,λ *are dominant weights satisfying the following conditions:*

(i) $\mu < \lambda$,

(ii) μ , λ *are in the closure of two adjacent alcoves,*

(iii) μ , λ *are mirror images in the hyperplane between these two alcoves. Then* $\mathrm{Hom}_G(V(\mu),V(\lambda)) \neq 0$.

Proof.

This is a special case of the theorem of [10], however see the figure on page 238 of [9] . $\square$

Now our next step is to investigate the weights inside the two alcoves. For p = 2, these two alcoves (and all the other ones) are empty in the sense that they don't contain any integral weight.

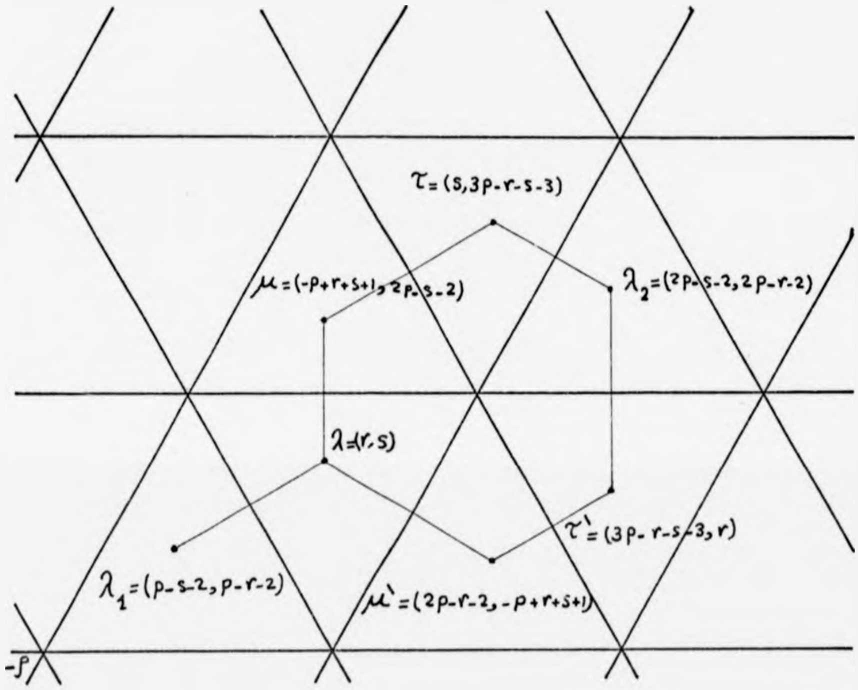

Figure 5.

3.2.7. Lemma.

Suppose the notation of Fig. 5. We have:

(i) $V(\lambda_1) \simeq L(\lambda_1)$.

(ii) $V(\lambda) \overset{G}{\longleftrightarrow} L(\lambda) \oplus L(\lambda_1)$.

(iii) $V(\nu) \overset{G}{\longleftrightarrow} L(\nu) \oplus L(\lambda)$, $\nu \in \{\mu, \mu'\}$.

Proof.

(i), (ii) See §4.3 of [24].

(iii) By Lemma 3.2.6 we have:

$$\mathrm{Hom}_G(V(\lambda), V(\nu)) \neq 0$$

for every $\nu \in \{\mu, \mu'\}$. Thus $L(\lambda)$ is a composition factor of $V(\nu)$. Counting the dimensions completes the proof. □

3.2.8. Lemma.

Suppose the notation of Fig. 5. For $p \neq 3$ *we have:*

(i) $\mathrm{soc}_G\, V(\lambda) \simeq L(\lambda_1)$.

(ii) $\mathrm{soc}_G\, V(\nu) \simeq L(\lambda)$, $\nu \in \{\mu, \mu', \tau, \tau', \lambda_2\}$.

(iii) $V(\nu) \overset{G}{\longleftrightarrow} L(\nu) \oplus L(\mu) \oplus L(\mu') \oplus L(\lambda_1) \oplus L(\lambda)$, $\nu \in \{\tau, \tau'\}$.

(iv) $V(\lambda_2) \overset{G}{\longleftrightarrow} L(\lambda_2) \oplus L(\tau) \oplus L(\tau') \oplus L(\mu) \oplus L(\mu') \oplus L(\lambda_1) \oplus L(\lambda)$.

Proof.

(i) This part follows directly from Lemma 3.2.7(ii).

(ii) The socle $\mathrm{soc}_G\, V(\lambda_2) \simeq L(\lambda)$ by Corollary 3.1.6. The Weyl module $V(\lambda_2)$ contains a copy of both $V(\tau)$ and $V(\tau')$, see p. 177 of [32]. Hence $\mathrm{soc}_G\, V(\lambda_2) \simeq \mathrm{soc}_G\, V(\tau) \simeq \mathrm{soc}_G\, V(\tau') \simeq L(\lambda)$. It follows from Lemma 3.2.7(iii) that $\mathrm{soc}_G\, V(\mu) = \mathrm{soc}_G\, V(\mu') \simeq L(\lambda)$.

(iii) We prove the case when $\nu = \tau$ (the case $\nu = \tau'$ can be done similarly). From the figure on p. 238 of [9] we have:

$$\mathrm{Hom}_G(V(\mu), V(\tau)) \neq 0, \text{ and } \mathrm{Hom}_G(V(\mu'), V(\tau)) \neq 0.$$

So both $L(\mu)$ and $L(\mu')$ are composition factors of $V(\tau)$. The weight $(s, 2p-r-s-3)$ is in the upper alcove and $(2p-s-2, p-r-2)$ is its image in the hyperplane $L_{\alpha,1}$. Hence Lemma 3.2.7(iii), and Proposition 2.2.2 give us

$$\mathrm{Ext}^1_G(L(\tau), L(\lambda_1)) \simeq \mathrm{Ext}^1_G(L(s,2p-r-s-3) \otimes L(0,1)^F, L(p-s-2,p-r-2)) \simeq$$

$$\simeq \mathrm{Ext}^1_G(L(s,2p-r-s-3), L(2p-s-2,p-r-2)) \simeq K.$$

From the above we get, all $L(\tau)$, $L(\mu)$, $L(\mu')$, $L(\lambda_1)$, $L(\lambda)$ are composition factors of $V(\tau)$. The dimensions of these modules force the multiplicities to be one.

(iv) This part follows from (ii), (iii) above and dimensional calculations. $\square$

As in Lemma 3.2.1, when $p = 3$, the only change in Lemma 3.2.8 is that $[V(\lambda_2) : L(\lambda_1)] = 2$. This does not affect the following Lemma.

3.2.9. Lemma.

Suppose the notation of Fig. 5. We have:

(i) $\mathrm{Ext}^1_G(L(\nu),L(\lambda)) = 0$, $\nu \in \{\lambda,\tau,\tau',\lambda_2\}$.

(ii) $\mathrm{Ext}^1_G(L(\nu),L(\lambda)) = K$, $\nu \in \{\lambda_1,\mu,\mu'\}$.

Proof.

(i) When $\nu \in \{\lambda,\tau,\tau',\lambda_2\}$, and $\nu \neq \lambda$, the simple module $L(\lambda)$ is the unique bottom composition factor of $V(\nu)$, with other composition factors between it and the top one $L(\nu)$. Hence $[V(\nu):L(\lambda)] = 1$ forces $L(\lambda)$ to never appear as a *next to the top* composition factor. Thus $\mathrm{Ext}^1_G(L(\nu),L(\lambda)) = 0$. When $\nu = \lambda$, the extension $\mathrm{Ext}^1_G(L(\lambda),L(\lambda))$ is zero from §4.1 of [25].

(ii) It is a direct application of Proposition 2.2.2, and Lemma 3.2.7. $\square$

We come now to our final case i.e. $\lambda \in A_0$. The weights in which we are interested ($\mu + p\xi \in W_a.\lambda$, $\mu + p\xi \leq 2(p-1)\rho + \omega_0(\lambda)$) are shown in the following figure.

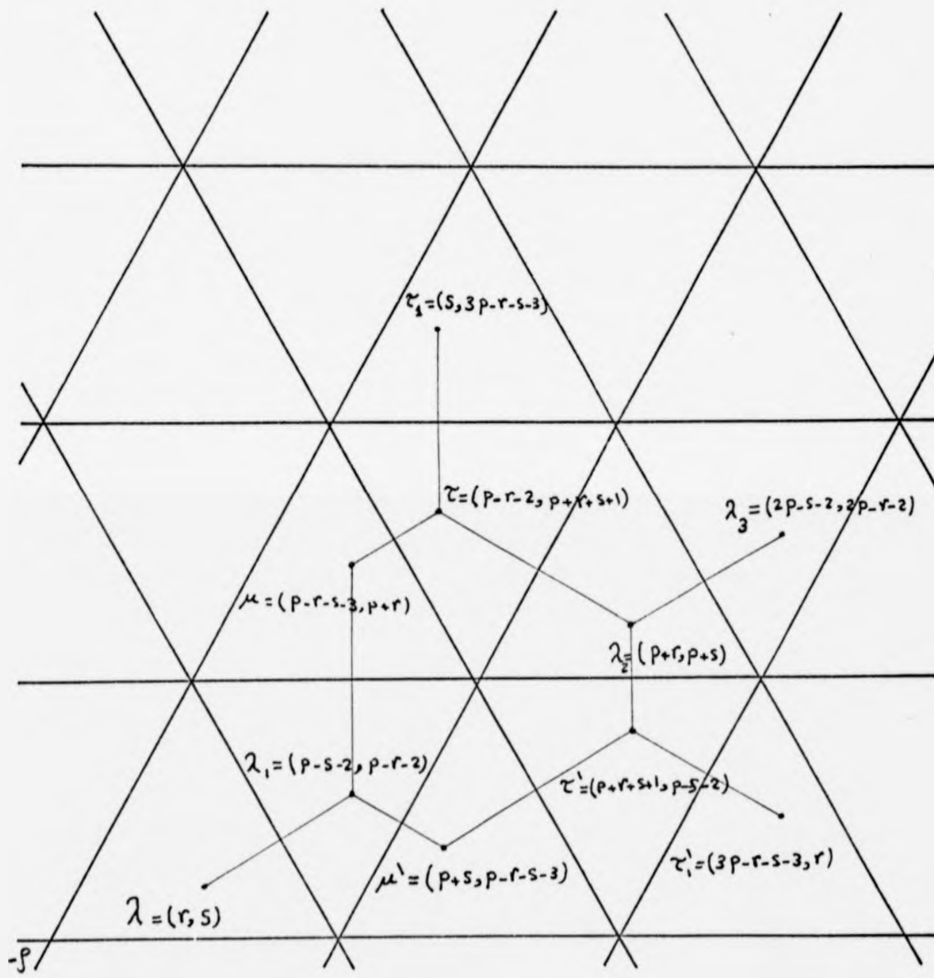

Figure 6.

3.2.10. Lemma.

Suppose the notation of Fig. 6. For $p \neq 3$ *we have:*

(i) $\mathrm{soc}_G V(\nu) = L(\lambda)$, $\nu \in \{\tau_1, \tau_1', \lambda_3\}$.

(ii) $[V(\nu):L(\lambda)] = 1$, $\nu \in \{\tau_1, \tau_1', \lambda_3\}$.

Proof.

(i) See the proof of Lemma 3.2.8(ii).

(ii) The weight $\lambda_3 = \omega.\lambda$ for some $\omega \in W_a$. Let μ be a dominant weight on the boundary between the upper and the lower alcoves . $(\mu \in \bar{A}_o)$. Then $\omega.\mu$ is on the hyperplane $L_{\alpha_o,3}$ (ω is our one above such that $\lambda_3 = \omega.\lambda$). Thus by Lemma 3.2.1(iii) and Theorem 2.3.3 we have:

$$[V(\lambda_3) : L(\lambda)] = [V(\omega.\mu) : L(\mu)] = 1,$$

note that all the conditions of Theorem 2.3.3 are satisfied. Similarly we can prove:

$$[V(\tau_1) : L(\lambda))] = [V(\tau_1') : L(\lambda)] = 1. \qquad \square$$

By a similar way we prove.

3.2.11. Lemma.

Suppose the notation of Fig. 6. For p = 3 *we have:*

(i) $\operatorname{soc}_G V(\nu) \cong L(\lambda)$, $\nu \in \{\tau_1, \tau_1', \lambda_3\}$.

(ii) $[V(\nu) : L(\lambda)] = 1$, $\nu = \{\tau_1, \tau_1'\}$.

(iii) $[V(\lambda_3) : L(\lambda)] = 2. \qquad \square$

3.2.12. Lemma.

Suppose the notation of Fig. 6. For $p \neq 3$ *we have:*

(i) $\operatorname{Ext}^1_G(L(\nu), L(\lambda)) = 0$, $\nu \in \{\lambda, \mu, \mu', \tau_1, \tau_1', \lambda_3, \lambda_2\}$.

(ii) $\operatorname{Ext}^1_G(L(\nu), L(\lambda)) \cong K$, $\nu \in \{\lambda_1, \tau, \tau'\}$.

Proof.

(i) When $\nu \in \{\lambda,\mu,\mu',\tau_1,\tau_1',\lambda_3,\lambda_2\}$, and $\nu \neq \lambda_2$, the extension $\mathrm{Ext}^1_G(L(\nu),L(\lambda)) = 0$ may be proved using exactly the same argument of Lemma 3.2.9(i).

Now we prove the case $\nu = \lambda_2$.

Suppose $\mathrm{Ext}^1_G(L(\lambda_2),L(\lambda)) \neq 0$. Then there exists a submodule $M \subseteq M(\lambda_2) \subseteq V(\lambda_2)$ such that $M(\lambda_2)/M \simeq L(\lambda)$, moreover

$$M \overset{G}{\longleftrightarrow} L(\tau) \oplus L(\tau') \oplus L(\mu) \oplus L(\mu') \oplus L(\lambda_1).$$

Let

$$0 \neq \varphi \in \mathrm{Hom}_G(V(\lambda_2),\ V(\lambda_3)),$$

thus

$$\ker \phi = M,$$

this is because $\mathrm{soc}_G\, V(\lambda_3) \simeq L(\lambda)$.

Suppose $V(\lambda_3) = KGw$ (resp. $V(\lambda_2) = KGv$) for some $0 \neq w \in V(\lambda_3)^{\lambda_3}$ (resp. $0 \neq v \in V(\lambda_2)^{\lambda_2}$). Applying Section 1.3 we obtain:

$$\{(X_{-\alpha,i}\ X_{-\alpha_0,p-r-s-2-i}\ X_{-\beta,i})w / 0 \le i \le p-r-s-2\},$$

a basis of $V(\lambda_3)^{\lambda_2}$.

$$\{(X_{-\alpha,r+1+i}\ X_{-\alpha_0,p-r-s-2-i}\ X_{-\beta,i})w / 0 \le i \le p-r-s-2\},$$

a basis of $V(\lambda_3)^{\tau}$.

$$\{(X_{-\alpha,r+1})v\},$$

a basis of $V(\lambda_2)^{\tau}$.

Now for our ϕ above we have:

$$\phi(v) = \sum_{i=0}^{p-r-s-2} k_i(X_{-\alpha,i}\, X_{-\alpha_0,p-r-s-2-i}\, X_{-\beta,i})w \neq 0,$$

for some $k_i \in K$ not all zeros. Let $0 \neq v' \in M$ be a vector of weight τ. So $\phi(v') = 0$, and

$$v' = k(X_{-\alpha,r+1})v$$

for some $0 \neq k \in K$. Therefore (we consider ϕ as a U_K-homomorphism)

$$\phi(v') = k(X_{-\alpha,r+1})\phi(v) = \sum_{i=0}^{p-r-s-2} kk_i(X_{-\alpha,r+1}X_{-\alpha,i}X_{-\alpha_0,p-r-s-2-i}X_{-\beta,i})w =$$

$$= \sum_{i=0}^{p-r-s-2} \binom{r+1+i}{i} kk_i(X_{-\alpha,r+1+i}\, X_{-\alpha_0,p-r-s-2-i}\, X_{-\beta,i})w = 0 \quad .$$

This implies all the k_i's must be zeros; which is a contradiction. Note that we have used

$$X_{-\alpha,a}\, X_{-\alpha,b} = \binom{a+b}{a} X_{-\alpha,a+b} \neq 0$$

for every nonnegative integers a,b such that $a+b < p$ (§2 of [25]).

(ii) When $\nu = \lambda_1$, see Lemma 3.2.7(ii). When $\nu = \tau$ or τ', the extension $\mathrm{Ext}^1_G(L(\nu),L(\lambda)) \simeq K$ may be shown by arguing as in the proof of Lemma 3.2.8(iii). □

To make our exposure complete, we have to borrow a result from the next section. This result is: when $p = 3$, the extension $\mathrm{Ext}^1_{U_1}(L(1,1),L(0,0))$ is isomorphic to

$L(0,0) \oplus L(0,1)^F \oplus (1,0)^F$. It will be used to prove $\mathrm{Ext}^1_G(L(4,4), L(0,0)) = 0$ in the next lemma. In Proposition 3.3.2 we shall need to know some $\mathrm{Ext}^1_G(L(\mu+p\xi), L(\lambda))$ to obtain $\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda))$ $(\mu,\lambda \in X_1)$, but actually the case $\mu + p\xi = (4,4)$ and $\lambda = (0,0)$ (when $p = 3$) is not required to obtain $\mathrm{Ext}^1_{U_1}(L(1,1),L(0,0))$.

3.2.13. Lemma.

Suppose the notation of Fig. 6. For $p = 3$ *we have:*

(i) $\mathrm{Ext}^1_G(L(\nu),L(\lambda)) = 0$, $\nu \in \{\lambda,\mu,\mu',\tau_1,\tau_1',\lambda_3\}$.

(ii) $\mathrm{Ext}^1_G(L(\nu),L(\lambda)) \simeq K$, $\nu \in \{\lambda_1,\lambda_2,\tau,\tau'\}$.

Proof.

(i) When $\nu \in \{\lambda,\mu,\mu'\}$, see Lemma 3.2.12(i). When $\nu = \tau_1 = (0,6)$ (or $\nu = \tau_1' = (6,0)$), we use Proposition 2.2.3:

$$\mathrm{Ext}^1_G(L(0,6),L(0,0)) \simeq$$

$$\simeq \mathrm{Ext}^1_G(L(0,2),L(0,0)) \oplus \mathrm{Hom}_G(L(0,2)^F,\mathrm{Ext}^1_{U_1}(L(0,0),L(0,0))) = 0$$

When $\nu = \lambda_3 = (4,4)$, using Proposition 2.2.3 we have:

$$\mathrm{Ext}^1_G(L(4,4),L(0,0)) \simeq (\mathrm{Ext}^1_{U_1}(L(1,1),L(0,0)) \otimes L(1,1)^F)^G \simeq$$

$$\simeq \mathrm{Hom}_G(L(1,1)^F,L(0,0) \oplus L(1,0)^F \oplus L(0,1)^F) = 0.$$

(ii) When $\nu \in \{\lambda_1,\tau,\tau'\}$, see Lemma 3.2.12(ii). When $\nu = \lambda_2 = (3,3)$, we have:

$$\mathrm{Ext}^1_G(L(3,3),L(0,0)) \simeq$$

$$\simeq \mathrm{Ext}^1_G(L(1,1),L(0,0)) \oplus \mathrm{Hom}_G(L(1,1)^F,\mathrm{Ext}^1_{U_1}(L(0,0),L(0,0))) \simeq K. \quad \square$$

Putting all the above together, we get our final result in this section.

3.2.14. Proposition.

$\lambda \in X_1$	$\mu+p\xi \in (W_a.\lambda)\cap X^+$, $\mu+p\xi \le 2(p-1)\rho+\omega_o\lambda$	$\mathrm{Ext}^1_G(L(\mu+p\xi),L(\lambda)) \simeq$
$\lambda = (r,s) \in A_o$	(r,s)	0
	$(p-s-2,p-r-2)$	K
	$(s,p-r-s-3) + p(1,0)$	0
	$(p-r-s-3,r) + p(0,1)$	0
	$(r+s+1,p-s-2) + p(1,0)$	K
	$(p-r-2,r+s+1) + p(0,1)$	K
	$(r,s) + p(1,1)$	0 (K when $p = 3$)
	$(p-s-2,p-r-2) + p(1,1)$	0
	$(p-r-s-3,r) + p(2,0)$	0
	$(s,p-r-s-3) + p(0,2)$	0
$\lambda = (r,s)$ in the upper alcove	(r,s)	0
	$(p-s-2,p-r-2)$	K
	$(p-r-2,-p+r+s+1)+p(1,0)$	K
	$(-p+r+s+1,p-s-2) + p(0,1)$	K
	$(p-s-2,p-r-2) + p(1,1)$	0
	$(2p-r-s-3,r) + p(1,0)$	0
	$(s,2p-r-s-3) + p(0,1)$	0

$(r,s) \in X^+$, $r+s = p-2$	(r,s)	0
	$(p-1,r) + p(1,0)$	K
	$(s,p-1) + p(0,1)$	K
	$(p-s-2,p-r-2) + p(1,1)$	0 (K when $p = 3$)
$(r,p-1)$, $0 \leq r < p-1$	$(r,p-1)$	0
	$(p-r-2,r) + p(1,0)$	K
	$(p-1,p-r-2) + p(0,1)$	0
$(p-1,s)$, $0 \leq s < p-1$	$(p-1,s)$	0
	$(p-s-2,p-1) + p(1,0)$	0
	$(s,p-s-2) + p(0,1)$	K
$(p-1,p-1)$	$(p-1,p-1)$	0

3.3. $\mathrm{Ext}^1_{U_1}(-,-)$ Between Simple U_1-modules of Type A_2.

Let V be a finite dimensional G-module. For a non-negative integer n, let $nL(\lambda) = L(\lambda) \oplus \ldots \oplus L(\lambda)$ n-times ($\lambda \in X^+$). The U_1-socle of V

$$\mathrm{soc}_{U_1} V \overset{U_1}{\simeq} \sum_{\lambda \in X_1}^{\oplus} n_\lambda(V) L(\lambda)$$

is a G-submodule of V. Moreover, the $L(\lambda)$-isotypic component of $\mathrm{soc}_{U_1} V$ ("$n_\lambda(V) \cdot L(\lambda)$") is also a G-submodule of V. The homomorphism

$$L(\lambda) \otimes \mathrm{Hom}_{U_1}(L(\lambda),V) \longrightarrow V$$

defined by $v \otimes f \longmapsto f(v)$ for every $v \in L(\lambda)$ and $f \in \mathrm{Hom}_{U_1}(L(\lambda),V)$ is a G-module isomorphism between $L(\lambda) \otimes \mathrm{Hom}_{U_1}(L(\lambda),V)$ and

"$n_\lambda(V)\cdot L(\lambda)$". Lemma 2.3.11 shows that any G-composition factor of "$n_\lambda(V)\cdot L(\lambda)$" is of the form $L(\lambda+p\xi)$ for some $\xi \in X^+$. For more information see §2.2 and §2.3 of [32].

Throughout the rest of this section we assume that $G = SL(3,K)$.

3.3.1. Lemma.

When $p = 3$, *the simple module* $L(1,1)^F$ *is not a composition factor of* $\mathrm{Ext}^1_{U_1}(L(0,0),L(1,1))$.

Proof.

For $\lambda \in X_1$, the short exact sequence of G-modules

$$0 \to L(\lambda) \to Q(\lambda) \to Q(\lambda)/L(\lambda) \to 0$$

gives the exact sequence of G-modules ($\mu \in X_1$)

$$0 \to \mathrm{Hom}_{U_1}(L(\mu),L(\lambda)) \to \mathrm{Hom}_{U_1}(L(\mu),Q(\lambda)) \to \mathrm{Hom}_{U_1}(L(\mu),Q(\lambda)/L(\lambda)) \to$$

$$\to \mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda)) \to 0.$$

Thus $L(\xi)^F$ is a composition factor of $\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda))$ implies $L(\mu+p\xi)$ is a G-composition factor of the $L(\mu)$-isotypic component of $\mathrm{soc}_{U_1} Q(\lambda)/L(\lambda)$ i.e. $n_\mu(Q(\lambda)/L(\lambda))\cdot L(\mu)$. Let $U(\mu,\lambda)$ be the G-submodule of $Q(\lambda)$ such that

$$n_\mu(Q(\lambda)/L(\lambda))\cdot L(\mu) \overset{G}{\cong} U(\mu,\lambda)/L(\lambda).$$

Now suppose $L(1,1)^F$ is a composition factor of $\mathrm{Ext}^1_{U_1}(L(0,0),L(1,1))$. So $L(3,3)$ is a composition factor of $U((0,0),(1,1))/L(1,1)$. Hence there exists a non-zero vector $v \in U((0,0),(1,1)) \subseteq Q(1,1)$ of weight $(3,3)$. The weight $(3,3)$ is the unique maximal weight of

$Q(1,1)$. Thus by Proposition 3.1.5 and (2) of §5.2 of [32] we obtain:

$$V(3,3) = KGv \subseteq U((0,0),(1,1)) \subseteq Q(1,1).$$

Hence

$$V(3,3)/L(1,1) \subseteq n_{(0,0)}(Q(1,1)/L(1,1))\cdot L(0,0).$$

Finally $L(1,4)$ is a composition factor of $V(3,3)$ implies $L(1,1) \otimes L(0,1)^F$ is a composition factor of $n_{(0,0)}(Q(1,1)/L(1,1))\cdot L(0,0)$, this is a contradiction (all the composition factors of $n_{(0,0)}(Q(1,1)/L(1,1))\cdot L(0,0)$ must be of the form $L(\xi)^F$). □

Recall two weights μ and λ are linked if there exists $\omega \in W$ such that $\omega.\lambda \equiv \mu \pmod{pX}$. In X_1, this "linkage" relation forms equivalence classes. The class containing λ is

$$a_\lambda = \{\mu \in X_1 \mid \mu \text{ is linked to } \lambda\}.$$

When $\lambda, \mu \in X_1$, the extension $\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda))$ is not zero implies $\mu \in a_\lambda$. The following table shows a_λ for different positions of $\lambda \in X_1$.

λ	a_λ
$(r,s) \in A_0$	$\{(r,s),(p-s-2,p-r-2),(s,p-r-s-3),(p-r-s-3,r),$ $(r+s+1,p-s-2),\ (p-r-2,r+s+1)\}$
(r,s) in the upper alcove	$\{(r,s),(p-s-2,p-r-2),(p-r-2,-p+r+s+1),$ $(-p+r+s+1,p-s-2),(2p-r-s-3,r),(s,2p-r-s-3)\}$
$(r,s)\in X^+, r+s=p-2$	$\{(r,s),(p-1,r),(s,p-1)\}$
$(r,p-1), 0\le r<p-1$	$\{(r,p-1),(p-r-2,r),(p-1,p-r-2)\}$
$(p-1,s), 0\le s<p-1$	$\{(p-1,s),(p-s-2,p-1),\ (s,p-s-2)\}$
$(p-1,p-1)$	$\{(p-1,p-1)\}$

Remark.

When $p = 2$, the first two a_λ's of the above table don't contain any weight. When $p = 3$, the six weights of each of the first two a_λ's collapse into two weights only.

The following figure relates the weights of a_λ to the weights mentioned in the last section.

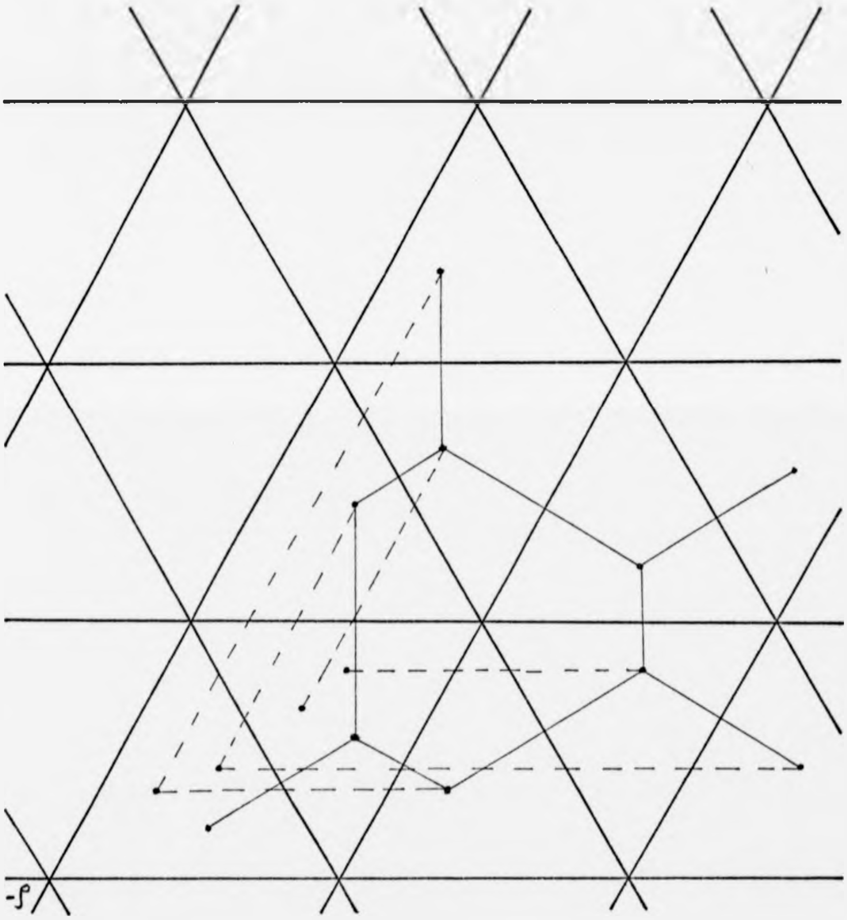

Figure 7.

We conclude this chapter with the next proposition and its two corollaries. These two corollaries lift the restrictions in Proposition 2.3.12, and Proposition 2.3.13 respectively when $G = SL(3,K)$.

3.3.2. <u>Proposition.</u>

For every $\lambda \in X_1$, and $\mu \in a_\lambda$. The extension $\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda))$ is given by the following table.

$\lambda \in X_1$	$\mu \in a_\lambda$	$\xi \in X^+$ such that $\mu+p\xi \le 2(p-1)\rho+w_o(\lambda)$, $\mu+p\xi \in W_a\cdot\lambda$	$\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda)) \overset{G}{\cong}$
$(r,s)\in A_o$	(r,s)	$(0,0)$, $(1,1)$ (*)	0
	$(p-s-2,p-r-2)$	$(0,0)$, $(1,1)$ (*)	$K(K\oplus L(1,0)^F\oplus L(0,1)^F, p=3)$
	$(s,p-r-s-3)$	$(1,0)$, $(0,2)$ (*)	0
	$(p-r-s-3,r)$	$(0,1)$, $(2,0)$ (*)	0
	$(r+s+1,p-s-2)$	$(1,0)$ (*)	$L(1,0)^F(K\oplus L(1,0)^F\oplus L(0,1)^F, p=3)$
	$(p-r-2,r+s+1)$	$(0,1)$ (*)	$L(0,1)^F(K\oplus L(1,0)^F\oplus L(0,1)^F, p=3)$
		(*) $p \neq 3$	
(r,s) in the upper alcove	(r,s)	$(0,0)$ $((0,0),(1,0),(0,1),p=3)$	0
	$(p-s-2,p-r-2)$	$(0,0)$, $(1,1)$ (*)	$K(K\oplus L(1,0)^F\oplus L(0,1)^F, p=3)$
	$(p-r-2,-p+r+s+1)$	$(1,0)$ (*)	$L(1,0)^F(K\oplus L(1,0)^F\oplus L(0,1)^F, p=3)$
	$(-p+r+s+1,p-s-2)$	$(0,1)$ (*)	$L(0,1)^F(K\oplus L(1,0)^F\oplus L(0,1)^F, p=3)$
	$(2p-r-s-3,r)$	$(1,0)$ (*)	0
	$(s,2p-r-s-3)$	$(0,1)$ (*)	0
		(*) $p \neq 3$	
$(r,s)\in X^+$, $r+s=p-2$	(r,s)	$(0,0)$, $(1,1)$	0
	$(p-1,r)$	$(1,0)$	$L(1,0)^F$
	$(s,p-1)$	$(0,1)$	$L(0,1)^F$

$(r,p-1)$, $0\le r<p-1$	$(r,p-1)$ $(p-r-2,r)$ $(p-1,p-r-2)$	$(0,0)$ $(1,0)$ $(0,1)$	0 $L(1,0)^F$ 0
$(p-1,s)$, $0\le s<p-1$	$(p-1,s)$ $(p-s-2,p-1)$ $(s,p-s-2)$	$(0,0)$ $(1,0)$ $(0,1)$	0 0 $L(0,1)^F$
$(p-1,p-1)$	$(p-1,p-1)$	$(0,0)$	0

Proof.

Suppose first that p is not equal to three. For any two weights in the third column (ξ,ξ' say) we have:

$$\mathrm{Ext}^1_G(L(\xi)^F,L(\xi')^F) \simeq \mathrm{Ext}^1_G(L(\xi),L(\xi')) = 0.$$

These $L(\xi)^F$'s are the only possibilities for the composition factors of $\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda))$. Hence it is completely reducible as a G-module. Therefore

$$\dim \mathrm{Hom}_G(L(\xi)^F,\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda))) = [\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda)):L(\xi)^F].$$

Case by case we determine the multiplicities of these composition factors using Proposition 3.2.14 and Proposition 2.2.3.

Now suppose that p is equal to three. The cases that need attention are:

$$\mathrm{Ext}^1_{U_1}(L(0,0),L(1,1)) \simeq \mathrm{Ext}^1_{U_1}(L(1,1),L(0,0)),$$
$$\mathrm{Ext}^1_{U_1}(L(0,1),L(0,1)), \text{ and } \mathrm{Ext}^1_{U_1}(L(1,0),L(1,0)).$$

The last two cases are similar.

By Lemma 3.3.1, the simple module $L(1,1)^F$ is not a composition factor of $\mathrm{Ext}^1_{U_1}(L(0,0),L(1,1))$. Hence $L(0,0)$, $L(1,0)^F$, and $L(0,1)^F$ are the only possible composition factors of $\mathrm{Ext}^1_{U_1}(L(0,0),L(1,1))$. Thus it is completely reducible. Proposition 3.2.14 determines the multiplicities of these composition factors.

The possible composition factors of $\mathrm{Ext}^1_{U_1}(L(0,1),L(0,1))$ are $L(0,0)$ and $L(1,1)^F$. The zero extension $\mathrm{Ext}^1_G(L(0,1),L(0,1)) = 0$ shows that $L(0,0)$ cannot be in the G-socle of $\mathrm{Ext}^1_{U_1}(L(0,1),L(0,1))$. Also $\mathrm{Ext}^1_G(L(3,4), L(0,1)) \cong K$ guarantees that $L(1,1)^F$ cannot be in its G-socle either. □

3.3.3. Corollary.

For every $\lambda \in X_1$, *we have:*

$$\mathrm{Ext}^1_{U_1}(L(\lambda),L(\lambda)) = 0. \qquad \square$$

3.3.4. Corollary.

The G-module $\mathrm{Ext}^1_{U_1}(L(\mu),L(\lambda))$ *is completely reducible for every* $\mu, \lambda \in X_1$. □

CHAPTER 4 - THE EXTENSION $\mathrm{Ext}_G^1(L(\mu),L(\lambda))$ FOR TYPE A_2

Throughout this chapter we assume that $G = SL(3,K)$.

We want to know the dimension of $\mathrm{Ext}_G^1(L(\mu),L(\lambda))$ for every μ, $\lambda \in X^+$. Also, given $\lambda \in X^+$, we want to know for which dominant weights μ the above dimension is nonzero?

4.1. The Socles of $L(1,0) \otimes L(\lambda)$, $L(0,1) \otimes L(\lambda)$.

4.1.1. Proposition.

Suppose $\lambda,\mu \in X^+$, $\lambda = \lambda_0 + p\lambda'$, $\mu = \mu_0 + p\mu'$, *where* $\lambda_0,\mu_0 \in X_1$ *and* λ', $\mu' \in X^+$. *Moreover suppose* $\lambda_0 \neq \mu_0$. *Then we have:*

(i) *A necessary condition for* $\mathrm{Ext}_G^1(L(\mu),L(\lambda))$ *to be nonzero is that* $\mu_0 \in a_{\lambda_0}$.

(ii) $\mathrm{Ext}_G^1(L(\mu),L(\lambda))$ *is zero outside the following table.*

λ_0	μ_0	$\mathrm{Ext}_G^1(L(\mu),L(\lambda)) \simeq$
$(r,s) \in A_0$	$(p-s-2,p-r-2)$	$\mathrm{Hom}_G(L(\mu'),L(\lambda'))$ (*)
	$(r+s+1,p-s-2)$	$\mathrm{Hom}_G(L(\mu'),L(1,0)\otimes L(\lambda'))$ (*)
	$(p-r-2,r+s+1)$	$\mathrm{Hom}_G(L(\mu'),L(0,1)\otimes L(\lambda'))$ (*)
(r,s) in the upper alcove	$(p-s-2,p-r-2)$	$\mathrm{Hom}_G(L(\mu'),L(\lambda'))$ (*)
	$(p-r-2,-p+r+s+1)$	$\mathrm{Hom}_G(L(\mu'),L(1,0)\otimes L(\lambda'))$ (*)
	$(-p+r+s+1,p-s-2)$	$\mathrm{Hom}_G(L(\mu'),L(0,1)\otimes L(\lambda'))$ (*)
$(r,s) \in X^+$, $r+s=p-2$	$(p-1,r)$	$\mathrm{Hom}_G(L(\mu'),L(1,0)\otimes L(\lambda'))$
	$(s,p-1)$	$\mathrm{Hom}_G(L(\mu'),L(0,1)\otimes L(\lambda'))$

$(r,p-1)$, $0 \le r < p-1$	$(p-r-2,r)$	$\mathrm{Hom}_G(L(\mu'),L(1,0)\otimes L(\lambda'))$
$(p-1,s)$, $0 \le s < p-1$	$(s,p-s-2)$	$\mathrm{Hom}_G(L(\mu'),L(0,1)\otimes L(\lambda'))$

(*) $\mathrm{Hom}_G(L(\mu'),L(\lambda')\oplus L(1,0)\otimes L(\lambda')\oplus L(0,1)\otimes L(\lambda'))$ *when* $p = 3$.

Proof.

From Proposition 2.2.3, and the trivial action of U_1 on $L(\mu')^F$, $L(\lambda')^F$ we have:

$$\mathrm{Ext}^1_G(L(\mu),L(\lambda)) \cong \mathrm{Hom}_G(L(\mu')^F,\mathrm{Ext}^1_{U_1}(L(\mu_o),L(\lambda_o))\otimes L(\lambda')^F).$$

Hence we use Proposition 3.3.2 and

$$\mathrm{Hom}_G(A,B) \cong \mathrm{Hom}_G(A^F,B^F)$$

for every $A, B \in M_G$ to prove our proposition. □

The last proposition makes the socle of $L(1,0)\otimes L(\lambda)$ and that of $L(0,1)\otimes L(\lambda)$, $\lambda \in X^+$, of particular interest to us. So the rest of this section is devoted for it.

For each p we have: $V(1,0) \cong L(1,0)$ and $V(0,1) \cong L(0,1)$. Thus

$$ch.L(1,0) = e(1,0) + e(0,-1) + e(-1,1),$$
$$ch.L(0,1) = e(0,1) + e(-1,0) + e(1,-1).$$

By Theorem 1.2.1(iii), for every $\lambda = (r,s) \in X_1$ we have:

$$\chi(1,0)\chi(r,s) = \chi(r+1,s) + \chi(r,s-1) + \chi(r-1,s+1),$$
$$\chi(0,1)\chi(r,s) = \chi(r,s+1) + \chi(r-1,s) + \chi(r+1,s-1).$$

So the character of $L(1,0) \otimes L(r,s)$ (resp. $L(0,1) \otimes L(r,s)$) depends on the relative positions of the weights in Fig. 8 (resp. Fig. 9) with respect to the two alcoves and the three boundaries. Moreover, each χ in the above two equations is either zero or equal to the character of the corresponding Weyl module.

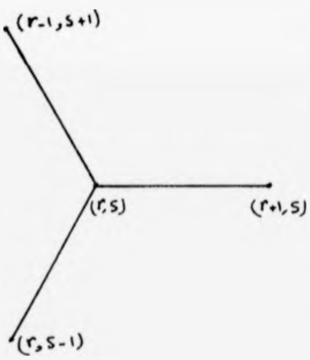

Figure 8.

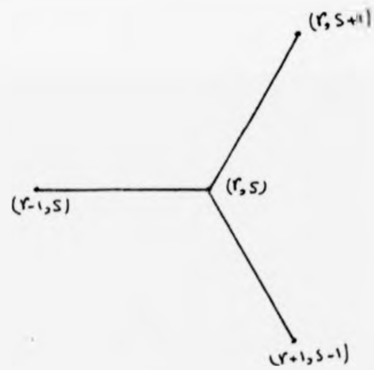

Figure 9.

The different cases for $\lambda \in X_1$ we have to consider separately when calculating the socle of $L(1,0) \otimes L(\lambda)$ (resp. $L(0,1) \otimes L(\lambda)$) are illustrated in Fig. 10 (resp. Fig. 11).

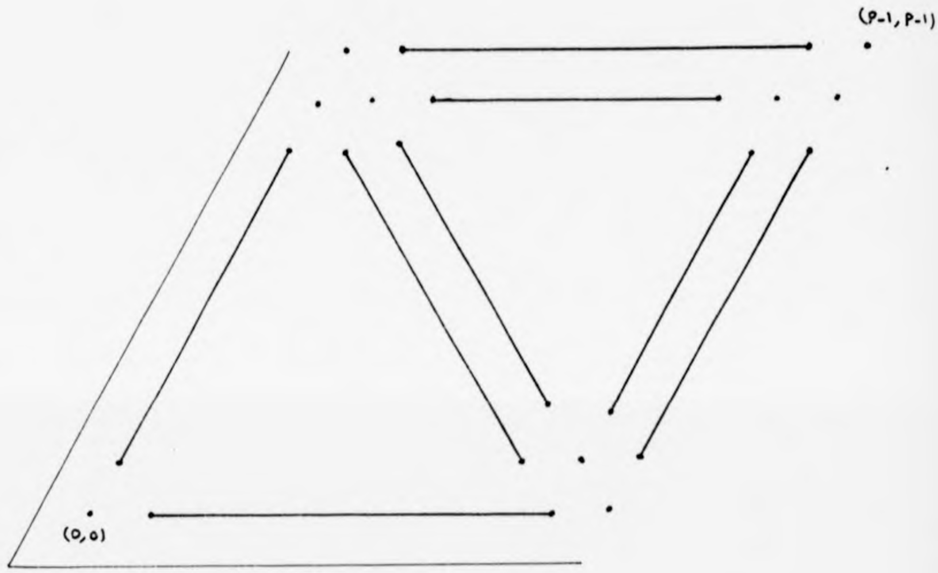

Figure 10.

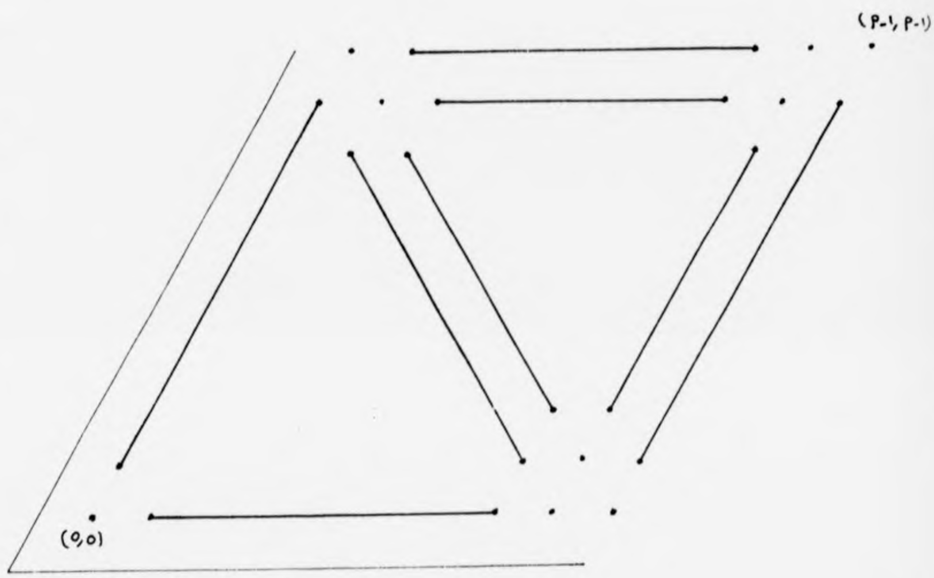

Figure 11.

4.1.2. Proposition.

Let $\lambda \in X_1$. *Then* $\mathrm{soc}_G\, L(1,0) \otimes L(\lambda) = \mathrm{soc}_{U_1} L(1,0) \otimes L(\lambda)$, *and it is given by the following table.*

	$\lambda = (r,s) \in X_1$	$\mathrm{soc}_G\, L(1,0) \otimes L(\lambda)$
(*),(2),(3)	$(0,0)$	$L(1,0)$
(*)	$(0,s), 1 \le s \le p-3$	$L(1,s) \oplus L(0,s-1)$
(3)	$(0,p-2)$	$L(0,p-3)$
	$(r,s), r+s=p-2, 1\le r\le p-3$	$L(r,s-1) \oplus L(r-1,s+1)$
(*), (3)	$(r,0),\ 1 \le r \le p-2$	$L(r+1,0) \oplus L(r-1,1)$
(*)	(r,s) deep inside A_0	$L(r+1,s) \oplus L(r,s-1) \oplus L(r-1,s+1)$
(*),(2),(3)	$(0,p-1)$	$L(1,p-1) \oplus L(0,p-2)$
(3)	$(r,p-1), 1 \le r \le p-2$	$L(r+1,p-1) \oplus L(r,p-2)$
(2),(3)	$(p-1,p-1)$	$L(p-1,p-2)$
(*),(3)	$(1,p-2)$	$L(2,p-2) \oplus L(0,p-1)$
(*)	$(r,p-2), 2 \le r \le p-3$	$L(r+1,p-2) \oplus L(r,p-3) \oplus L(r-1,p-1)$
(*)	$(p-2,p-2)$	$L(p-1,p-2) \oplus L(p-2,p-3) \oplus L(p-3,p-1)$
(3)	$(p-1,p-2)$	$L(p-2,p-1) \oplus L(p-1,p-3)$
(*)	$(r,s), r+s=p-1, 2\le r\le p-3$	$L(r+1,s) \oplus L(r-1,s+1)$
(*)	$(p-2,1)$	$L(p-1,1) \oplus L(p-3,2)$
(2),(3)	$(p-1,0)$	$L(p-2,1)$
	$(p-1,s), 1 \le s \le p-3$	$L(p-2,s+1) \oplus L(p-1,s-1)$
(*)	$(p-2,s), 2 \le s \le p-3$	$L(p-1,s) \oplus L(p-2,s-1) \oplus L(p-3,s+1)$
(*)	(r,s) deep in the upper alcove	$L(r+1,s) \oplus L(r,s-1) \oplus L(r-1,s+1)$

(*) *The socle is the whole module.* (2) *Valid when* $p = 2$.

(3) *Valid when* $p = 3$.

Proof.

We shall prove the case when $\lambda = (p-1,p-2)$. All the other cases are either similar or easier.

Direct calculations show that:

$$ch.L(1,0)\,ch.L(p-1,p-2)=ch.L(p,p-2)+2ch.L(p-2,p-1)+ch.L(p-1,p-3), \quad (1)$$

$$ch.L(0,1)\,ch.L(p,p-2) = ch.L(p,p-1) + ch.L(p+1,p-3), \quad (2)$$

$$ch.L(0,1)\,ch.L(p-2,p-1)=ch.L(p-3,p-1)+2ch.L(p-1,p-2)+ch.L(p-2,p). \quad (3)$$

So by (2) we have:

$$\mathrm{Hom}_G(L(p,p-2),L(1,0)\otimes L(p-1,p-2)) \cong \mathrm{Hom}_G(L(0,1)\otimes L(p,p-2),L(p-1,p-2)) = 0.$$

Thus

$$L(p,p-2) \not\subseteq L(1,0) \otimes L(p-1,p-2).$$

The simple module

$$L(p-1,p-3) \subsetneq \mathrm{soc}_G\, L(1,0) \otimes L(p-1,p-2),$$

this is because

$$\mathrm{Ext}^1_G(L(\tau),L(p-1,p-3)) = \mathrm{Ext}^1_G(L(p-1,p-3),L(\tau)) = 0$$

for every $\tau \in X^+$ such that $L(\tau)$ is a composition factor of $L(1,0) \otimes L(p-1,p-2)$ (see Proposition 2.2.2).

Hence

$$[\mathrm{soc}_G L(1,0) \otimes L(p-1,p-2) : L(p-2,p-1))] = 1 \text{ or } 2.$$

To complete the proof, we have to show that the above multiplicity is one.

Suppose that multiplicity is two. By that assumption we obtain:

$$\dim \mathrm{Hom}_G(L(p-2,p-1), L(1,0) \otimes L(p-1,p-2)) =$$

$$= \dim \mathrm{Hom}_G(L(0,1) \otimes L(p-2,p-1), L(p-1,p-2)) = 2.$$

Thus we can find $V \subseteq L(0,1) \otimes L(p-2,p-1)$ such that

$$L(0,1) \otimes L(p-2,p-1)/V \simeq 2L(p-1,p-2).$$

Equation (3) gives

$$V \overset{G}{\longleftrightarrow} L(p-3,p-1) \oplus L(p-2,p),$$

and

$$\mathrm{Ext}^1_G(L(p-3,p-1), L(p-2,p)) = \mathrm{Ext}^1_G(L(p-2,p), L(p-3,p-1)) = 0$$

gives

$$V \simeq L(p-3,p-1) \oplus L(p-2,p).$$

Hence

$$\mathrm{Hom}_G(L(p-2,p),\ L(0,1) \otimes L(p-2,p-1) \simeq$$

$$\simeq \mathrm{Hom}_G(L(1,0) \otimes L(p-2,p), L(p-2,p-1)) \neq 0.$$

The last nonzero homomorphism is a contradiction to

$$\mathrm{ch}.L(1,0)\mathrm{ch}.L(p-2,p) = \mathrm{ch}.L(p-1,p) + \mathrm{ch}.L(p-3,p+1).$$

From all above we conclude that

$$\mathrm{soc}_G L(1,0) \otimes L(p-1,p-2) \simeq L(p-2,p-1) \oplus L(p-1,p-3).$$

Now we prove the G and the U_1 socles are equal. Let $V \in \mathcal{M}_G$ and suppose

$$\mathrm{soc}_{U_1} V \overset{U_1}{\simeq} \sum_{\mu \in X_1} n_\mu(V) L(\mu).$$

From the introduction of Section 3.3 we know that the $L(\mu)$-isotypic component of $\mathrm{soc}_{U_1} V$ ("$n_\mu(V) \cdot L(\mu)$") is a G-submodule of V.

This adds a necessary condition on "$n_\mu(V) \cdot L(\mu)$" to be nonzero. This condition is that $L(\mu+p\xi)$ is a composition factor of V for some $\xi \in X^+$. Suppose for each $\mu \in X_1$ there is at most one $\xi \in X^+$ such that $[V:L(\mu+p\xi)] \neq 0$. Then the $L(\mu)$-isotypic component of $\mathrm{soc}_{U_1} V$ is completely reducible as a G-module (§4.1 of [25]). Therefore $\mathrm{soc}_{U_1} V$ is completely reducible as a G-module and is equal to $\mathrm{soc}_G V$. This is our case. This completes the proof of the proposition when $\lambda = (p-1,p-2)$.

We should mention that, when $\lambda = (p-1,p-1)$, we use Lemma 3.1.3. □

Similarly we get.

4.1.3. Proposition.

Let $\lambda \in X_1$. *Then* $\mathrm{soc}_G\, L(0,1) \otimes L(\lambda) = \mathrm{soc}_{U_1} L(0,1) \otimes L(\lambda)$, *and it is given by the following table.*

	$\lambda = (r,s) \in X_1$	$\mathrm{soc}_G\, L(0,1) \otimes L(\lambda)$
(*),(2),(3)	(0,0)	L(0,1)
(*),(3)	(0,s), $1 \le s \le p-2$	L(0,s+1) $\oplus$ L(1,s-1)
(*)	(r,0), $1 \le r \le p-3$	L(r,1) $\oplus$ L(r-1,0)
(3)	(p-2,0)	L(p-3,0)
	(r,s),r+s=p-2,$1 \le r \le p-3$	L(r-1,s) $\oplus$ L(r+1,s-1)
(*)	(r,s) deep inside A_o	L(r,s+1) $\oplus$ L(r-1,s) $\oplus$ L(r+1,s-1)
(2),(3)	(0,p-1)	L(1,p-2)
	(r,p-1),$1 \le r \le p-3$	L(r-1,p-1) $\oplus$ L(r+1,p-2)
(3)	(p-2,p-1)	L(p-3,p-1) $\oplus$ L(p-1,p-2)
(2),(3)	(p-1,p-1)	L(p-2,p-1)
(*)	(1,p-2)	L(1,p-1) $\oplus$ L(2,p-3)
(*)	(r,p-2),$2 \le r \le p-3$	L(r,p-1) $\oplus$ L(r-1,p-2) $\oplus$ L(r+1,p-
(*)	(p-2,p-2)	L(p-2,p-1)$\oplus$L(p-3,p-2)$\oplus$L(p-1,p-3)
(*)	(r,s),r+s=p-1,$2 \le r \le p-3$	L(r,s+1) $\oplus$ L(r+1,s-1)
(*),(3)	(p-2,1)	L(p-2,2) $\oplus$ L(p-1,0)
(*),(2),(3)	(p-1,0)	L(p-1,1) $\oplus$ L(p-2,0)
(3)	(p-1,s),$1 \le s \le p-2$	L(p-1,s+1) $\oplus$ L(p-2,s)
(*)	(p-2,s),$2 \le s \le p-3$	L(p-2,s+1) $\oplus$ L(p-3,s) $\oplus$ L(p-1,s-
(*)	(r,s) deep inside the upper alcove	L(r,s+1) $\oplus$ L(r-1,s) $\oplus$ L(r+1,s-1) $\square$

4.1.4. Corollary.

Suppose $\lambda \in X^+$, $\lambda = \lambda_o + p\lambda'$, *where* $\lambda_o \in X_1$ *and* $\lambda' \in X^+$. *Then we have:*

$$\mathrm{soc}_G\, L(1,0) \otimes L(\lambda) \simeq (\mathrm{soc}_G\, L(1,0) \otimes L(\lambda_o)) \otimes L(\lambda')^F,$$

$$\mathrm{soc}_G\, L(0,1) \otimes L(\lambda) \simeq (\mathrm{soc}_G\, L(0,1) \otimes L(\lambda_o)) \otimes L(\lambda')^F.$$

Proof.

It is straightforward to prove

$$\mathrm{soc}_G\, L(1,0) \otimes L(\lambda) \simeq \mathrm{soc}_G(\mathrm{soc}_{U_1} L(1,0) \otimes L(\lambda_o)) \otimes L(\lambda')^F.$$

From Proposition 4.1.2 we have:

$$\mathrm{soc}_{U_1}\, L(1,0) \otimes L(\lambda_o) = \mathrm{soc}_G\, L(1,0) \otimes L(\lambda_o) \overset{G}{\simeq} \sum_{\mu\in X_1}^{\oplus} n_{(1,0)}(\mu,\lambda_o)L(\mu)$$

for some $n_{(1,0)}(\mu,\lambda_o) = 0$ or 1. Thus by Steinberg's tensor product theorem, the module

$$(\mathrm{soc}_G\, L(1,0) \otimes L(\lambda_o)) \otimes L(\lambda')^F$$

is completely reducible as a G-module. □

4.2. The Extension $\mathrm{Ext}^1_G(L(\mu),L(\lambda))$ For Type A_2.

4.2.1. Definition.

For each $\lambda \in X^+$. Define

$$A(\lambda) = \{\mu \in X^+/\mathrm{Ext}^1_G(L(\mu),L(\lambda)) \neq 0\}. \quad \square$$

4.2.2. Proposition.

$$A(0,0) = \{p^n(p-2,p-2),p^n(p+1,p-2),p^n(p-2,p+1)/n=0,1,..\}\backslash\{(0,0)\}.$$

Proof.

We put "$\backslash\{(0,0)\}$" to exclude $(0,0)$ when $p = 2$.

Suppose $\mu \in A(0,0)$. Let

$$\mu = \sum_{i=0}^{r} p^i \mu_i$$

be a p-adic expansion of μ i.e. $\mu_i \in X_1$ for every $i = 0,1,\ldots,r$. We have two cases.

<u>Case (i)</u> : $\mu_0 \neq (0,0)$.

Let $\mu = \mu_0 + p\mu'$, thus by Proposition 2.2.3 we obtain:

$$\mathrm{Ext}^1_G(L(\mu),L(0,0)) \cong \mathrm{Hom}_G(L(\mu')^F,\mathrm{Ext}^1_{U_1}(L(\mu_0),L(0,0))) \neq 0.$$

This implies

$$\mu' \in \{(0,0),\ (1,0),(0,1)\},$$

and hence

$$\mu \in \{(p-2,p-2),(p+1,p-2),(p-2,p+1)\}\backslash\{(0,0)\}.$$

<u>Case (ii)</u>: $\mu_0 = (0,0)$.

Let m be the least integer such that $\mu_m \neq 0 (0 < m \leq r)$.

Thus

$$\mu = p^m \left(\sum_{i=0}^{r-m} p^i \mu_{m+i}\right).$$

Using Claim 1 in the proof of Proposition 2.3.12 and then replacing μ by $\sum_{i=0}^{r-m} p^i \mu_{m+i}$ in Case (i) we obtain:

$$\mu \in \{p^m(p-2,p-2),p^m(p+1,p-2),p^m(p-2,p+1)\}\backslash\{(0,0)\}.$$

Conversely, suppose

$$\mu \in \{p^n(p-2,p-2),p^n(p+1,p-2),p^n(p-2,p+1)/n=0,1,..\}\backslash\{(0,0)\}.$$

Put $\lambda = (0,0)$ in Lemma 3.2.12 (Lemma 3.2.2 when $p = 2$, and Lemma 3.2.13 when $p = 3$), we obtain:

$$\mathrm{Ext}^1_G(L(\nu), L(0,0)) \simeq K,$$

for every

$$\nu \in \{(p-2,p-2),(p+1,p-2),(p-2,p+1)\}\backslash\{(0,0)\}.$$

We then use, as before, Claim 1 in the proof of Proposition 2.3.12. □

We are now ready to introduce and prove our final result.

4.2.3. Theorem.

Suppose $\lambda \in X^+$, $\lambda = \lambda_o + p\lambda'$, $\lambda' = \lambda'_o + p\lambda''$, *where* $\lambda_o, \lambda'_o \in X_1$ *and* $\lambda', \lambda'' \in X^+$. *Moreover suppose* $\lambda_o = (r,s)$ *and* $\lambda'_o = (r',s')$. *Then we have:*

(i) $\dim \mathrm{Ext}^1_G(L(\mu), L(\lambda)) \leq 1$ *for every* $\mu \in X^+$.

(ii) $A(\lambda) = (A + p\lambda') \cup (\lambda_o + pA(\lambda'))$, *where*

$A = \{\eta_1, \eta_2 + p\nu_2, \eta_3 + p\nu_3 / K \subseteq \mathrm{Ext}^1_{U_1}(L(\eta_1), L(\lambda_o))$,

$L(1,0)^F \subseteq \mathrm{Ext}^1_{U_1}(L(\eta_2), L(\lambda_o)), L(0,1)^F \subseteq \mathrm{Ext}^1_{U_1}(L(\eta_3), L(\lambda_o))$,

$L(\lambda'_o + \nu_2) \subseteq \mathrm{soc}_G L(1,0) \otimes L(\lambda'_o), L(\lambda'_o + \nu_3) \subseteq \mathrm{soc}_G L(0,1) \otimes L(\lambda'_o)\}$.

Explicitly A *is given by the following table.*

λ_o	λ_o'	A
$\lambda_o=(r,s)\in A_o$	$(0,0)$ (3)	$\{\eta_1,\eta_2+p(1,0),\eta_3+p(0,1)\}$
Let:	$(0,s'),1\le s'\le p-3$	$\{\eta_1,\eta_2+p(1,0),\eta_2+p(0,-1),\eta_3+p(0,1),$
$\eta_1=(p-s-2,p-r-2),$		$\eta_3+p(1,-1)\}$
$\eta_2=(r+s+1,p-s-2),$	$(r',0),1\le r'\le p-3$	$\{\eta_1,\eta_2+p(1,0),\eta_2+p(-1,1),\eta_3+p(0,1),$
$\eta_3=(p-r-2,r+s+1).$		$\eta_3+p(-1,0)\}$
or	$(0,p-2)$ (3)	$\{\eta_1,\eta_2+p(0,-1),\eta_3+p(0,1),\eta_3+p(1,-1)\}$
$\lambda=(r,s)$ in the	$(p-2,0)$ (3)	$\{\eta_1,\eta_2+p(1,0),\eta_2+p(-1,1),\eta_3+p(-1,0)\}$
upper alcove.	$(r',s'),r'+s'=p-2,$ $1\le r'\le p-3$	$\{\eta_1,\eta_2+p(0,-1),\eta_2+p(-1,1),\eta_3+p(-1,0),$
Let:		$\eta_3+p(1,-1)\}$
$\eta_1=(p-s-2,p-r-2),$	$(0,p-1)$ (3)	$\{\eta_1,\eta_2+p(1,0),\eta_2+p(0,-1),\eta_3+p(1,-1)\}$
$\eta_2=(p-r-2,-p+r+s+1),$	$(p-1,0)$ (3)	$\{\eta_1,\eta_2+p(-1,1),\eta_3+p(0,1),\eta_3+p(-1,0)\}$
$\eta_3=(-p+r+s+1,p-s-2)$	$(r',s'),r'+s'=p-1,$ (3) $1\le r'\le p-2$	$\{\eta_1,\eta_2+p(1,0),\eta_2+p(-1,1),\eta_3+p(0,1),$
		$\eta_3+p(1,-1)\}$
	$(r',p-1),1\le r'\le p-2$ (3)	$\{\eta_1,\eta_2+p(1,0),\eta_2+p(0,-1),\eta_3+p(-1,0),$
		$\eta_3+p(1,-1)\}$
	$(p-1,s'),1\le s'\le p-2$ (3)	$\{\eta_1,\eta_2+p(0,-1),\eta_2+p(-1,1),\eta_3+p(0,1),$
		$\eta_3+p(-1,0)\}$
	$(p-1,p-1)$ (3)	$\{\eta_1,\eta_2+p(0,-1),\eta_3+p(-1,0)\}$
	(r',s') otherwise	$\{\eta_1,\eta_2+p(1,0),\eta_2+p(0,-1),\eta_2+p(-1,1),$
		$\eta_3+p(0,1),\eta_3+p(-1,0),\eta_3+p(1,-1)\}$
$(r,s),r+s=p-2$	$(0,0)$ (2),(3)	$\{\eta_1+p(1,0),\eta_2+p(0,1)\}$
Let:	$(0,s'),1\le s'\le p-3$	$\{\eta_1+p(1,0),\eta_1+p(0,-1),\eta_2+p(0,1),$
$\eta_1=(p-1,r),$		$\eta_2+p(1,-1)\}$
$\eta_2=(s,p-1).$	$(r',0),1\le r'\le p-3$	$\{\eta_1+p(1,0),\eta_1+p(-1,1),\eta_2+p(0,1),$
		$\eta_2+p(-1,0)\}$

	$(0,p-2)$	(3)	$\{\eta_1+p(0,-1),\eta_2+p(0,1),\eta_2+p(1,-1)\}$
	$(p-2,0)$	(3)	$\{\eta_1+p(1,0),\eta_1+p(-1,1),\eta_2+p(-1,0)\}$
	$(r',s'),r'+s'=p-2,$ $1\le r'\le p-3$		$\{\eta_1+p(0,-1),\eta_1+p(-1,1),\eta_2+p(-1,0),$ $\eta_2+p(1,-1)\}$
	$(0,p-1)$	(2),(3)	$\{\eta_1+p(1,0),\eta_1+p(0,-1),\eta_2+p(1,-1)\}$
	$(p-1,0)$	(2),(3)	$\{\eta_1+p(-1,1),\eta_2+p(0,1),\eta_2+p(-1,0)\}$
	$(r',s'),r'+s'=p-1,$ $1\le r'\le p-2$	(3)	$\{\eta_1+p(1,0),\eta_1+p(-1,1),\eta_2+p(0,1),$ $\eta_2+p(1,-1)\}$
	$(r',p-1),1\le r'\le p-2$	(3)	$\{\eta_1+p(1,0),\eta_1+p(0,-1),\eta_2+p(-1,0),$ $\eta_2+p(1,-1)\}$
	$(p-1,s'),1\le s'\le p-2$	(3)	$\{\eta_1+p(-1,1),\eta_1+p(0,-1),\eta_2+p(0,1),$ $\eta_2+p(-1,0)\}$
	$(p-1,p-1)$	(2),(3)	$\{\eta_1+p(0,-1),\eta_2+p(-1,0)\}$
	(r',s') otherwise		$\{\eta_1+p(1,0),\eta_1+p(0,-1),\eta_1+(-1,1),$ $\eta_2+p(0,1),\eta_2+p(-1,0),\eta_2+p(1,-1)\}$
$(r,p-1),0\le r\le p-2$	$(0,0)$	(2),(3)	$\{\eta+p(1,0)\}$
Let:			
	$(0,s'),1\le s'\le p-3$		
$\eta=(p-r-2,r)$	$(0,p-1)$	(2),(3)	$\{\eta+p(1,0),\eta+p(0,-1)\}$
	$(r',p-1),1\le r'\le p-2$	(3)	
	$(0,p-2)$	(3)	$\{\eta+p(0,-1)$
	$(p-1,p-1)$	(2),(3)	
	$(r',s'),r'+s'=p-2,$ $1\le r'\le p-3$		
	$(p-1,p-2)$	(3)	$\{\eta+p(0,-1),\eta+p(-1,1)\}$
	$(p-1,s'),1\le s'\le p-3$		
	$(r',0),1\le r'\le p-2$	(3)	
	$(1,p-2)$	(3)	
	$(r',s'),r'+s'=p-1,$ $2\le r'\le p-3$		$\{\eta+p(1,0),\eta+p(-1,1)\}$
	$(p-2,1)$	(3)	

	$(p-1,0)$	(2),(3)	$\{\eta+p(-1,1)\}$
	(r',s) otherwise		$\{\eta+p(1,0),\eta+p(0,-1),\eta+p(-1,1)\}$
$(p-1,s), 0\le s\le p-2$	$(0,0)$	(2),(3)	$\{\eta+p(0,1)\}$
Let:			
	$(0,s'), 1\le s'\le p-2$	(3)	
$\eta=(s,p-s-2)$			
	$(1,p-2)$	(3)	
	$(r',s'), r'+s'=p-1,$ $2\le r'\le p-3$		$\{\eta+p(0,1),\eta+p(1,-1)\}$
	$(p-2,1)$	(3)	
	$(r',0), 1\le r'\le p-3$		
	$(p-1,0)$	(2),(3)	$\{\eta+p(0,1),\eta+p(-1,0)\}$
	$(p-1,s'), 1\le s'\le p-2$	(3)	
	$(p-2,0)$	(3)	
			$\{\eta+p(-1,0)\}$
	$(p-1,p-1)$	(2),(3)	
	$(r',s'), r'+s'=p-2,$ $1\le r'\le p-3$		
	$(r',p-1), 1\le r'\le p-3$		$\{\eta+p(-1,0),\eta+p(1,-1)\}$
	$(p-2,p-1)$	(3)	
	$(0,p-1)$	(2),(3)	$\{\eta+p(1,-1)\}$
	(r',s') otherwise		$\{\eta+p(0,1),\eta+p(-1,0),\eta+p(1,-1)\}$
$(p-1,p-1)$	(r',s')	(2),(3)	empty

(2) *Applied when* $p = 2$.

(3) *Applied when* $p = 3$.

Proof.

(i) Let $\mu = \mu_o + p\mu' \in X^+$. When $\mu_o = \lambda_o$, we go down to μ',λ' using

$$\mathrm{Ext}^1_G(L(\mu),L(\lambda)) \cong \mathrm{Ext}^1_G(L(\mu'),L(\lambda'))$$

and so on. So we may assume that $\mu_o \neq \lambda_o$.

Proposition 4.1.1 and Corollary 4.1.4 show that

$\mathrm{Ext}^1_G(L(\mu),L(\lambda))$ is isomorphic to $\mathrm{Hom}_G(L(\mu'),L(\lambda'))$ or to

$\mathrm{Hom}_G(L(\mu'),(\mathrm{soc}_G\ L(1,0) \otimes L(\lambda_o')) \otimes L(\lambda'')^F)$ or to

$\mathrm{Hom}_G(L(\mu'),(\mathrm{soc}_G\ L(0,1) \otimes L(\lambda_o')) \otimes L(\lambda'')^F)$,

and when $p = 3$ it may be isomorphic to the direct sum of the three Hom_G's above. All the following G-modules

$$L(\lambda'),(\mathrm{soc}_G L(1,0) \otimes L(\lambda_o')) \otimes L(\lambda'')^F,(\mathrm{soc}_G\ L(0,1)\otimes L(\lambda_o'))\otimes L(\lambda'')^F$$

are completely reducible, and when $p = 3$ no two of them share a composition factor. Also each composition factor of any of the three modules above occurs exactly once by Propositions 4.1.2, and 4.1.3. So dim $\mathrm{Ext}^1_G(L(\mu),L(\lambda))$ is at most one.

(ii) When $\mu_o = \lambda_o$, the extension $\mathrm{Ext}^1_G(L(\mu),L(\lambda))$ is not zero iff $\mu' \in A(\lambda')$. When $\mu_o \neq \lambda_o$, suppose

$$\mathrm{Ext}^1_G(L(\mu),L(\lambda)) \cong \mathrm{Hom}_G(L(\mu'),L(1,0) \otimes L(\lambda')).$$

Thus it is nonzero iff $\mu' = \lambda_o' + \nu_2 + p\lambda''$ for some $\nu_2 \in X$ such that $L(\lambda_o' + \nu_2) \subseteq \mathrm{soc}_G L(1,0) \otimes L(\lambda_o')$, and $\mu_o = \eta_2$ above. □

CHAPTER 5 - THE BLOCKS OF THE PARABOLIC SUBGROUPS

The parabolic subgroups of G containing the Borel subgroup

$$B = \langle T, \chi_{-\alpha} / \alpha \in \Phi^{+} \rangle$$

are in one to one correspondence with the set of subsets of the simple roots Δ. For each subset $\Omega \subseteq \Delta$, the group

$$P_{\Omega} = \langle B, \chi_{\alpha} / \alpha \in \Omega \rangle$$

is the associated parabolic subgroup.

The parabolic subgroup P_{Ω} has a Levi decomposition

$$P_{\Omega} = U^{\Omega} G_{\Omega},$$

where

$$U^{\Omega} = \langle \chi_{-\alpha} / \alpha \in \Phi^{+} \backslash \mathbb{Z}\Omega \rangle$$

is the unipotent radical of P_{Ω}, and

$$G_{\Omega} = \langle T, \chi_{\alpha}, \chi_{-\alpha} / \alpha \in \Omega \rangle$$

is the Levi factor of P_{Ω}.

Recall the roots of an affine algebraic group with respect to a maximal torus are the nonzero weights of the adjoint representation of its Lie algebra (see Section 1.1 for the definition of the adjoint representation). The subroot system $\Phi_{\Omega} = \Phi \cap \mathbb{Z}\Omega$ (resp. $\Phi_{\Omega} \cup (\Phi^{-} \backslash \mathbb{Z}\Omega)$) are the roots of G_{Ω} (resp. P_{Ω}) with respect to T.

In fact Φ_{Ω} is an abstract root system (with respect to the Euclidean subspace $\mathbb{R}\,\Phi_{\Omega}$), and

$$W_{\Omega} = \langle \omega_{\alpha} / \alpha \in \Omega \rangle \subseteq W$$

is its associated abstract Weyl group. Moreover Ω and $\Phi_\Omega^+ = \Phi^+ \cap \Phi_\Omega$ are simple and positive roots in Φ_Ω respectively.

For a fuller discussion of all the above results we refer the reader to §2.5, §8.5 of [7], and §30.1, §30.2 of [23]. All the notation above will be used in what follows.

5.1. The Blocks of The Derived Subgroup G'_Ω .

5.1.1. Lemma.

Let $G'_\Omega = (G_\Omega, G_\Omega)$ *be the derived subgroup of* G_Ω, *and let* $Z(G_\Omega)$ *be the centre of* G_Ω. *Then we have:*

(i) $G_\Omega = Z(G_\Omega)\, G'_\Omega$, *and* $Z(G_\Omega) \cap G'_\Omega$ *is finite.*

(ii) G'_Ω *is a connected semisimple algebraic group, and*

$$G'_\Omega = \langle X_\alpha, X_{-\alpha} / \alpha \in \Omega\rangle$$

Proof.

The group G_Ω is reductive (§30.2 of [23]). Thus all the lemma, except $G'_\Omega = \langle X_\alpha, X_{-\alpha}/\alpha \in \Omega\rangle$, follows from §27.5 of [23].

Recall the relations (Lemma 20 on p. 29 of [34]):

(i) $x_\alpha(u)x_\alpha(u') = x_\alpha(u+u')$ for every $\alpha \in \Phi$ and $u,u' \in K$.

(ii) $h_\beta(t)x_\alpha(u)h_\beta(t)^{-1} = x_\alpha(t^{(\alpha,\beta^\vee)}u)$ for every $\alpha,\beta \in \Phi$, $t \in K^*$, and $u \in K$.

Let $H = \langle X_\alpha, X_{-\alpha}/\alpha \in \Omega\rangle$. Clearly H is a normal subgroup of G_Ω. The epimorphism $T \to G_\Omega/H$ defined by $t \mapsto tH$ for every $t \in T$ shows that G_Ω/H is an abelian group. Thus $G'_\Omega \subseteq H$.

From the relations (i), (ii) above, the commutator

$$(x_\alpha(u), h_\beta(t)) = x_\alpha(u - t^{(\alpha,\beta^\vee)}u)$$

for every $\alpha, \beta \in \Phi$, $t \in K^*$, and $u \in K$. So $H \subseteq G'_\Omega$. □

Remark.

For every $\alpha \in \Phi_\Omega$, $u \in K$, and $t \in K^*$ we have:

$$x_\alpha(u) \in G'_\Omega \text{ , and } h_\alpha(t) \in G'_\Omega.$$

This is because there exists $\omega \in W_\Omega$ such that $\omega(\alpha) \in \Omega$, and also

$$\omega_\beta(1)x_\alpha(u)\omega_\beta(1)^{-1} = x_{\omega_\beta(\alpha)}(\pm u)$$

for every $\beta \in \Omega$, where $\omega_\beta(1) = x_\beta(1)x_{-\beta}(-1)x_\beta(1)$, see Lemma 20 of [34].

5.1.2. Lemma.

$T_\Omega = \langle h_\alpha(t)/\alpha \in \Omega,\ t \in K^*\rangle$ *is a maximal torus in* G'_Ω .

Proof.

It is enough to prove T_Ω is of maximal dimension.

By definition, the rank of G'_Ω is the dimension of any of its maximal tori. Moreover, G'_Ω is semisimple, hence

$$\text{rank } G'_\Omega = \text{rank } \Phi(G'_\Omega,T')$$

the root system associated to (G'_Ω,T'), where T' is any maximal torus (§27.1 of [23]). The roots of G_Ω and of G'_Ω are in one to one correspondence (p. 164 of [23]), and Φ_Ω is a root system of (G_Ω,T). Hence

$$\text{rank } G'_\Omega = |\Omega| \ .$$

The inclusion $T_\Omega \subseteq T$ implies any element of T_Ω has a unique expression as a product $\prod_{\alpha\in\Omega} h_\alpha(t_\alpha)$, $t_\alpha \in K^*$ (Corollary on p. 44 of [34]). Hence

$$T_\Omega \cong K^* \times \ldots \times K^* \quad |\Omega|\text{-times.} \quad \square$$

5.1.3. Definition.

(i) $X_\Omega = \{\lambda \in X(T)/(\lambda,\alpha^\vee) \geq 0$ for every $\alpha \in \Omega\}$.

(ii) $X_{\Omega,0} = \{\lambda \in X(T)/(\lambda,\alpha^\vee) = 0$ for every $\alpha \in \Omega\}$.

(iii) $X'_\Omega = \{\lambda \in X(T)/(\lambda,\beta^\vee) = 0$ for every $\beta \in \Delta\backslash\Omega\}$.

(iv) $X'^{+}_\Omega = X'_\Omega \cap X_\Omega$. □

For every $\lambda_\Omega \in X_\Omega$, there is a unique expression

$$\lambda_\Omega = \lambda'_\Omega + \lambda_{\Omega,0},$$

with $\lambda'_\Omega \in X'^{+}_\Omega$, and $\lambda_{\Omega,0} \in X_{\Omega,0}$.

The inclusion

$$i_\Omega : T_\Omega \to T$$

gives the epimorphism

$$i^*_\Omega : X(T) \to X(T_\Omega).$$

Since $\mathbb{R}$ is a torsion free $\mathbf{Z}$-module, the last epimorphism gives another one (call it i^*_Ω also)

$$i^*_\Omega : X(T) \otimes_{\mathbf{Z}} \mathbb{R} \to X(T_\Omega) \otimes_{\mathbf{Z}} \mathbb{R}.$$

It follows directly that:

$$\ker i^*_\Omega = X_{\Omega,0} \otimes_{\mathbf{Z}} \mathbb{R},$$

$$|\Phi_\Omega| = |\{i^*_\Omega(\alpha)/\alpha \in \Phi_\Omega\}|.$$

Note that $\Phi_\Omega \cap X_{\Omega,0}$ is empty.

The proof of the following lemma is an imitation of the proof of the analogous result for G which is originally due to J.A. Green.

5.1.4. Lemma.

$\Phi_\Omega{}' = i^*_\Omega(\Phi_\Omega)$ is the root system associated to (G'_Ω, T_Ω).

Proof.

For each $\alpha \in \Phi_\Omega$, define affine algebraic group homomorphism

$$\phi^\Omega_\alpha : G_a \to G'_\Omega$$

by $u \longmapsto x_\alpha(u)$ for every $u \in G_a$. This ϕ^Ω_α gives the K-algebra homomorphism

$$\phi^{\Omega *}_\alpha : K[G'_\Omega] \to K[X],$$

and $\phi^{\Omega *}_\alpha$ in turn gives

$$\xi^\Omega_{\alpha,r} \in K[G'_\Omega]^*, \ r = 0,1,\dots .$$

These are defined as follows, for every $f \in K[G'_\Omega]$ put

$$\phi^{\Omega *}_\alpha(f) = \sum_{r=0}^{\infty} \xi^\Omega_{\alpha,r}(f) X^r .$$

See Section 1.2 for the analogous ϕ_α for G, see also Section 1.1 for the notation which we will use next. We prove the lemma in two steps.

Step 1. We prove $\xi^\Omega_{\alpha,1} \in \mathrm{Lie}(G'_\Omega)$ for every $\alpha \in \Phi_\Omega$.

Let $f,g \in K[G'_\Omega]$ and $\alpha \in \Phi_\Omega$. Hence

$$\phi^{\Omega *}_\alpha(fg) = \phi^{\Omega *}_\alpha(f)\ \phi^{\Omega *}_\alpha(g),$$

expanding each side in a polynomial in X, and equating the coefficient of X^r $(r = 0,1,\dots)$ we obtain:

$$\xi^{\Omega}_{\alpha,r}(fg) = \sum_{i=0}^{r} \xi^{\Omega}_{\alpha,i}(f) \ \xi^{\Omega}_{\alpha,r-i}(g). \tag{1}$$

Let $u \in K$, $\alpha \in \Phi_{\Omega}$, and $f \in K[G'_{\Omega}]$. Thus we have:

$$\phi^{\Omega *}_{\alpha}(f)(u) = f(x_{\alpha}(u)) = \sum_{r=0}^{\infty} \xi^{\Omega}_{\alpha,r}(f)u^{r} .$$

Putting $u = 0$ we get:

$$f(1) = \xi^{\Omega}_{\alpha,0}(f) = \varepsilon_{\Omega}(f),$$

where ε_{Ω} is the augmentation of the Hopf algebra $K[G'_{\Omega}]$.

Hence

$$\xi^{\Omega}_{\alpha,0} = \varepsilon_{\Omega}. \tag{2}$$

From (1), (2) and for each $f,g \in K[G'_{\Omega}]$ we have:

$$\xi^{\Omega}_{\alpha,1}(fg) = \xi^{\Omega}_{\alpha,1}(f)\varepsilon_{\Omega}(g) + \varepsilon_{\Omega}(f) \ \xi^{\Omega}_{\alpha,1}(g).$$

This shows

$$\xi^{\Omega}_{\alpha,1} \in \mathrm{Lie}(G'_{\Omega}).$$

Step 2. We prove $\mathrm{ad}(t)\xi^{\Omega}_{\alpha,1} = \alpha(t)\xi^{\Omega}_{\alpha,1}$ for every $t \in T_{\Omega}$ i.e. $\xi^{\Omega}_{\alpha,1}$ is of weight $\alpha|_{T_{\Omega}} = i^{*}_{\Omega}(\alpha)$.

Let $t \in T_{\Omega}$ and $\alpha \in \Phi_{\Omega}$, then by Lemma 20 of [34] we have for every $u \in K$:

$$tx_{\alpha}(u)t^{-1} = x_{\alpha}(\alpha(t)u) .$$

Thus

$$\phi^{\Omega *}_{\alpha}(f)(\alpha(t)u) = f(x_{\alpha}(\alpha(t)u)) = f(tx_{\alpha}(u)t^{-1}) = (c_{t^{-1}}f)(x_{\alpha}(u)) =$$

$$= \phi^{\Omega *}_{\alpha}(c_{t^{-1}}f)(u).$$

Hence

$$\sum_{r=0}^{\infty} \xi^{\Omega}_{\alpha,r}(f)(\alpha(t)u)^{r} = \sum_{r=0}^{\infty} \xi^{\Omega}_{\alpha,r}(c_{t^{-1}}f)(u^{r}).$$

Lemma 1.1.1 gives

$$(\mathrm{ad}(t)\xi^{\Omega}_{\alpha,1})(f) = \xi^{\Omega}_{\alpha,1}(c_{t^{-1}}f).$$

Therefore

$$\xi^{\Omega}_{\alpha,1}(f)(\alpha(t)u) = \xi^{\Omega}_{\alpha,1}(c_{t^{-1}}f)(u) = (\mathrm{ad}(t)\xi^{\Omega}_{\alpha,1})(f)(u),$$

which gives

$$\mathrm{ad}(t)\xi^{\Omega}_{\alpha,1} = \alpha(t)\xi^{\Omega}_{\alpha,1} . \qquad \square$$

For each $\alpha, \beta \in \Omega$, define

$$(i^{*}_{\Omega}(\alpha),\ i^{*}_{\Omega}(\beta)) = (\alpha,\beta).$$

Since $\Omega' = i^{*}_{\Omega}(\Omega)$ generates Φ'_{Ω}, and Φ'_{Ω} in turn generates $X(T_{\Omega}) \otimes_{\mathbb{Z}} \mathbb{R}$ (§27.1 of [23]), this form can be extended to a bilinear symmetric form on $X(T_{\Omega}) \otimes_{\mathbb{Z}} \mathbb{R}$.

5.1.5. Lemma.

(i) *For every* $\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ *and* $\alpha \in \Omega$ *we have:*

$$(\lambda,\alpha^{\vee}) = (i^{*}_{\Omega}(\lambda), i^{*}_{\Omega}(\alpha)^{\vee}).$$

(ii) *The map* i^{*}_{Ω} *induces an isomorphism of root systems between* Φ_{Ω} *and* Φ'_{Ω}. *Moreover* $\Omega' = i^{*}_{\Omega}(\Omega)$ *and* $\Phi'^{+}_{\Omega} = i^{*}_{\Omega}(\Phi^{+}_{\Omega})$ *are simple roots and its corresponding positive ones in* Φ'_{Ω} *respectively.*

Proof.

(i) We may omit the "v".

For every $\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ we have:

$$i_\Omega^*(\lambda) = \sum_{\beta\in\Omega} m_\beta \, i_\Omega^*(\beta)$$

for some $m_\beta \in \mathbb{R}$.

Let $\lambda' = \sum_{\beta\in\Omega} m_\beta \beta$. Thus $\lambda \equiv \lambda' \mod X_{\Omega,0} \otimes_{\mathbb{Z}} \mathbb{R}$, and hence

$(\lambda,\alpha) = (\lambda',\alpha)$ for every $\alpha \in \Omega$.

Therefore

$$(\lambda,\alpha) = (\lambda',\alpha) = \sum_{\beta\in\Omega} m_\beta(\beta,\alpha) = \sum_{\beta\in\Omega} m_\beta(i_\Omega^*(\beta),i_\Omega^*(\alpha)) = (i_\Omega^*(\lambda),i_\Omega^*(\alpha)).$$

(ii) Clearly the above bilinear form on $X(T_\Omega) \otimes_{\mathbb{Z}} \mathbb{R}$ is positive definite, and i_Ω^* induces an Euclidean space isomorphism between the subspace of $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by Φ_Ω and $X(T_\Omega) \otimes_{\mathbb{Z}} \mathbb{R}$. □

This paragraph is the only part of this chapter in which we assume G to be a connected semisimple affine algebraic group. We follow §31.1 of [23] in defining G to be simply connected. The symbols T and N(T) will denote a maximal torus and its normalizer in G respectively. We let $W(G,T) = N(T)/T$, and $\Phi(G,T)$ to be the associated Weyl group and root system respectively. The Weyl group $W(G,T)$ has a natural action on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Let (,) be a bilinear, symmetric, positive definite, and $W(G,T)$ invariant form on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Choose Δ a simple system in $\Phi(G,T)$. An element $\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is called an *integral weight* if $(\lambda,\alpha^\vee) \in \mathbb{Z}$ for every $\alpha \in \Delta$. The group G is called *simply connected* if the set of all integral weights in $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is equal to $X(T)$. Alternatively, the semisimple group G is simply connected iff the set of the fundamental dominant weights $\{\lambda_\alpha / \alpha \in \Delta\}$ is in $X(T)$. Recall the fundamental dominant weights are the weights λ_α's $(\alpha \in \Delta)$ satisfying $(\lambda_\alpha, \beta^\vee) = \delta_{\alpha\beta}$ for every $\beta \in \Delta$, and they form a $\mathbb{Z}$-basis for the lattice of integral weights.

Now we return to the original notation for G i.e. G is our universal Chevally group.

5.1.6. Proposition.

G'_Ω *is simply connected.*

Proof.

There is a bilinear, symmetric, and positive definite form on $X(T_\Omega) \otimes_{\mathbb{Z}} \mathbb{R}$ (see Lemma 5.1.5). The abstract Weyl group associated to the abstract root system Φ'_Ω has an action on $X(T_\Omega) \otimes_{\mathbb{Z}} \mathbb{R}$. The elementary reflections of this abstract Weyl group

$$\{\omega_{i^*_\Omega(\alpha)} / \alpha \in \Omega\}$$

act on $\lambda' \in X(T_\Omega) \otimes_{\mathbb{Z}} \mathbb{R}$ by

$$\omega_{i^*_\Omega(\alpha)}(\lambda') = \lambda' - (\lambda', i^*_\Omega(\alpha)^\vee) i^*_\Omega(\alpha).$$

The Weyl group associated to (G'_Ω, T_Ω) is isomorphic to the abstract Weyl group associated to Φ'_Ω which is in turn isomorphic to

$$W_\Omega = \langle \omega_\alpha / \alpha \in \Omega \rangle \subseteq W.$$

See §27.1 and p. 164 of [23], §2.5 of [7], and Lemma 22 of [34]. Moreover we have:

$$i^*_\Omega(\omega_\alpha(\lambda)) = \omega_{i^*_\Omega(\alpha)}(i^*_\Omega(\lambda))$$

for every $\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Thus we may identify W_Ω with the Weyl group associated to (G'_Ω, T_Ω), and we have:

$$i^*_\Omega(\omega(\lambda)) = \omega(i^*_\Omega(\lambda))$$

for every $\omega \in W_\Omega$, and $\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

We can easily see that $\{i^*_\Omega(\lambda_\alpha)/\alpha \in \Omega\}$ is the set of the fundamental dominant weights in which we are interested (λ_α is the fundamental dominant weight of $X(T)$ associated to $\alpha \in \Delta$). Finally

$$\lambda \in X(T) \text{ implies } i^*_\Omega(\lambda) \in X(T_\Omega). \quad \square$$

The dot action of the Weyl group W_Ω needs more care to handle. Let

$$\rho_\Omega = \frac{1}{2} \sum_{\alpha \in \Phi^+_\Omega} \alpha .$$

Although ρ_Ω need not be in $X(T)$ (e.g. type A_2), we have:

$$\omega(\lambda+\rho_\Omega) - \rho_\Omega \in X(T)$$

for each $\omega \in W_\Omega$ and $\lambda \in X(T)$. This is because

$$(\rho_\Omega, \alpha) = (\omega_\alpha(\rho_\Omega), \omega_\alpha(\alpha)) = (\rho_\Omega - \alpha, -\alpha)$$

for each $\alpha \in \Omega$, which gives

$$(\rho_\Omega, \alpha^\vee) = 1$$

for every $\alpha \in \Omega$. Thus

$$\rho \equiv \rho_\Omega \mod X_{\Omega,0} \otimes_{\mathbb{Z}} \mathbb{R},$$

and hence

$$\omega.\lambda = \omega(\lambda + \rho_\Omega) - \rho_\Omega, \text{ for every } \omega \in W_\Omega.$$

From above we obtain:

$$i^*_\Omega(\rho_\Omega) = \frac{1}{2} \sum_{\alpha' \in \Phi'^+_\Omega} \alpha', \text{ and } i^*_\Omega(\omega.\lambda) = \omega.(i^*_\Omega(\lambda))$$

for every $\omega \in W_\Omega$ and $\lambda \in X(T)$.

5.1.7. Definition.

For a weight $\lambda' \in X(T_\Omega)$, not equal to $-i^*_\Omega(\rho_\Omega)$, we define $r(\lambda')$ to be the nonnegative integer satisfying

$$\lambda' + i^*_\Omega(\rho_\Omega) \in p^{r(\lambda')}X(T_\Omega)\backslash p^{r(\lambda')+1}X(T_\Omega).$$

For each weight $\lambda \in X(T)$ such that $i^*_\Omega(\lambda) \neq -i^*_\Omega(\rho_\Omega)$, we define $r_\Omega(\lambda)$ to be

$$r_\Omega(\lambda) = r(i^*_\Omega(\lambda)). \quad \square$$

5.1.8. Theorem, [16].

For every $\lambda' \in X(T_\Omega)^+$, *let* $\beta(\lambda')$ *be the* G'_Ω-*block containing* λ'. *Then we have:*

$$\beta(\lambda') = (W_\Omega.\lambda' + p^{r(\lambda')+1}\,\mathbb{Z}\Phi'_\Omega) \cap X(T_\Omega)^+. \quad \square$$

5.2. Representations and Blocks of The Levi Factor G_Ω.

The levi factor G_Ω may not be semisimple, so to determine its blocks we have to find out the indexing set for its simple modules. This will be done in the following proposition and its corollary.

5.2.1. Proposition.

(i) *Each simple P_Ω-module is a module of highest weight with respect to the partial ordering involving Φ^+_Ω only. That module is nonzero iff its highest weight is in X_Ω. Let $L_\Omega(\lambda_\Omega), \lambda_\Omega \in X_\Omega$, denote such module. Then $\{L_\Omega(\lambda_\Omega)/\lambda_\Omega \in X_\Omega\}$ is a full set of simple P_Ω-modules.*

(ii) dim $L_\Omega(\lambda_{\Omega,0}) = 1$ *for every* $\lambda_{\Omega,0} \in X_{\Omega,0}$.

(iii) *For each $\lambda_\Omega \in X_\Omega$, let $\lambda_\Omega = \lambda'_\Omega + \lambda_{\Omega,0}$ be its unique decomposition with $\lambda'_\Omega \in X'_\Omega, \lambda_{\Omega,0} \in X_{\Omega,0}$. Then:*

$$L_\Omega(\lambda_\Omega) \overset{P_\Omega}{\simeq} L_\Omega(\lambda'_\Omega) \otimes L(\lambda_{\Omega,0}).$$

Proof.

It is an imitation of the proof of the analogous result for G, e.g. Theorem 39 of [34], §31.4 of [23], and §5.7 of [6]. □

5.2.2. Corollary.

The restrictions $\{L_\Omega(\lambda_\Omega)|_{G_\Omega}/\lambda_\Omega \in X_\Omega\}$ is a full set of simple G_Ω-modules.

Proof.

The unipotent radical U^Ω is a normal subgroup of P_Ω. Thus $L_\Omega(\lambda_\Omega)|_{U^\Omega}$ is completely reducible (as a U^Ω-module) by Clifford's theorem (Theorem 49.2 of [14]). But U^Ω is a unipotent group, then $L_\Omega(\lambda_\Omega)|_{U^\Omega}$ is trivial (§17.5 of [23]). Hence $L_\Omega(\lambda_\Omega)|_{G_\Omega}$ is a simple G_Ω-module.

Conversely, any simple G_Ω-module can be made into a simple P_Ω-module by letting U^Ω act trivially. □

The following lemma may be well known to most readers.

5.2.3. Lemma.

Let V *be a finite dimensional* G_Ω*-module, and let* $\omega \in W_\Omega$. *Suppose* $\lambda \in X(T)$ *is a weight of* V, *then* $\omega(\lambda)$ *is also a weight of* V *with the same multiplicity.*

Proof.

It is enough to prove it when $\omega = \omega_\alpha$, $\alpha \in \Omega$.

For each $\alpha \in \Omega$, $t \in K^*$, let $\omega_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$, then $h_\alpha(t) = \omega_\alpha(t)\omega_\alpha(1)^{-1}$ and $\omega_\alpha(1)h_\beta(t)\omega_\alpha(1)^{-1} = h_{\omega_\alpha(\beta)}(t)$, for each $\beta \in \Omega$ (Lemma 20 of [34]).

We want to prove $\omega_\alpha(1)^{-1}\, V^\lambda = V^{\omega_\alpha(\lambda)}$. Let $0 \neq v \in V^\lambda$, and β any element of Ω, $t \in K^*$. Then:

$$h_\beta(t)(\omega_\alpha(1)^{-1}v) = \omega_\alpha(1)^{-1}(\omega_\alpha(1)h_\beta(t)\omega_\alpha(1)^{-1}v) = \omega_\alpha(1)^{-1}(h_{\omega_\alpha(\beta)}(t)v) =$$

$$= \omega_\alpha(1)^{-1}\,(t^{(\omega_\alpha(\lambda),\beta^\vee)}v) = \omega_\alpha(\lambda)(h_\beta(t))(\omega_\alpha(1)^{-1}v).$$

Hence $\omega_\alpha(1)^{-1}v \in V^{\omega_\alpha(\lambda)}$, so $\omega_\alpha(1)^{-1}\, V^\lambda \subseteq V^{\omega_\alpha(\lambda)}$.

Similarly $\omega_\alpha(1)^{-1}\, V^{\omega_\alpha(\lambda)} \subseteq V^\lambda$. □

5.2.4. Lemma.

Let $\omega_{\Omega,0} \in W_\Omega$ *be its longest element. Then we have:*

$$L_\Omega(\lambda_\Omega)^* \overset{G_\Omega}{\cong} L_\Omega(-\omega_{\Omega,0}(\lambda_\Omega))$$

for every $\lambda_\Omega \in X_\Omega$.

Proof.

By Lemma 5.2.3, the weight $\omega_{\Omega,0}(\lambda_\Omega)$ is the unique minimal weight of $L_\Omega(\lambda_\Omega)$. So $-\omega_{\Omega,0}(\lambda_\Omega)$ is the unique maximal weight of $L_\Omega(\lambda_\Omega)^*$ (since μ_Ω is a weight of $L_\Omega(\lambda_\Omega)$ iff $-\mu_\Omega$ is a weight of $L_\Omega(\lambda_\Omega)^*$ with the same multiplicity). By induction on the composition length of $L_\Omega(\lambda_\Omega)^*$, we can easily find that $L_\Omega(-\omega_{\Omega,0}(\lambda_\Omega))$ is a composition factor of $L_\Omega(\lambda_\Omega)^*$. Finally the isomorphism

$$L_\Omega(\lambda_\Omega)^{**} \overset{G_\Omega}{\cong} L_\Omega(\lambda_\Omega)$$

completes the proof. □

5.2.5. Proposition.

Suppose $\lambda_\Omega, \mu_\Omega \in X_\Omega$, $\lambda_\Omega = \lambda'_\Omega + \lambda_{\Omega,0}$, $\mu_\Omega = \mu'_\Omega + \mu_{\Omega,0}$, *where* λ'_Ω, $\mu'_\Omega \in X'_\Omega$ *and* $\lambda_{\Omega,0}, \mu_{\Omega,0} \in X_{\Omega,0}$. *Then we have:*

$$\mathrm{Ext}^1_{G_\Omega}(L_\Omega(\mu_\Omega), L_\Omega(\lambda_\Omega)) \cong \mathrm{Hom}_{G_\Omega/G'_\Omega}(L_\Omega(\mu_{\Omega,0}-\lambda_{\Omega,0}), \mathrm{Ext}^1_{G'_\Omega}(L_\Omega(\mu'_\Omega), L_\Omega(\lambda'_\Omega))).$$

Proof.

Claim any G_Ω/G'_Ω-module is completely reducible. It is enough to prove that if U is a rational G_Ω-module on which G'_Ω acts trivially, then U is completely reducible. Let $Z = Z(G_\Omega)^0$ be the connected component of the centre of G_Ω, hence $G_\Omega = ZG'_\Omega$. For every $\eta \in X(Z)$, where $X(Z)$ is the character group of Z, let

$$U^{(\eta)} = \{u \in U/zu = \eta(z)u \text{ for every } z \in Z\}.$$

Therefore

$$U = \sum_{\eta \in X(Z)}^{\oplus} U^{(\eta)}$$

is a G_Ω-decomposition of U. So it is enough to prove $U^{(\eta)}$ is G_Ω-completely reducible. If $\{v_i/i \in I\}$ is a basis of $U^{(\eta)}$, and $g = zg'$ where $g \in G_\Omega$, $z \in Z$, and $g' \in G'_\Omega$. Then $gv_i = \eta(z)v_i$ for every $i \in I$. This proves the claim.

Now let V be a rational G_Ω-module. The Hochschild-Serre five term exact sequence (§4.6(b) of [12]) reads:

$$0 \to H^1(G_\Omega/G'_\Omega, V^{G'_\Omega}) \to H^1(G_\Omega, V) \to H^1(G'_\Omega, V)^{G_\Omega/G'_\Omega} \to H^2(G_\Omega/G'_\Omega, V^{G'_\Omega}) \to H^2(G_\Omega, V).$$

By the claim

$$H^1(G_\Omega/G'_\Omega, V^{G'_\Omega}) = H^2(G_\Omega/G'_\Omega, V^{G'_\Omega}) = 0.$$

Thus we obtain:

$$H^1(G_\Omega, V) \cong H^1(G'_\Omega, V)^{G_\Omega/G'_\Omega}.$$

Put $V = L_\Omega(\mu_\Omega)^* \otimes L_\Omega(\lambda_\Omega)$. Using the fact that both $L_\Omega(\mu_{\Omega,0})$ and $L_\Omega(\lambda_{\Omega,0})$ are trivial G'_Ω-modules together with the previous Lemma and Proposition 5.2.1(iii), we get our result (see Remark (2) of §2.2). □

5.2.6. Lemma.

Suppose $\lambda_\Omega \in X_\Omega$, $\lambda_\Omega = \lambda'_\Omega + \lambda_{\Omega,0}$, *where* $\lambda'_\Omega \in X'_\Omega$ *and* $\lambda_{\Omega,0} \in X_{\Omega,0}$. *Then we have:*

$$L_\Omega(\lambda_\Omega) \overset{G'_\Omega}{\cong} L_\Omega(\lambda'_\Omega) \overset{G'_\Omega}{\cong} L(i^*_\Omega(\lambda_\Omega)),$$

where $L(\mu')$ *is the simple* G'_Ω*-module with highest weight* $\mu' \in X(T_\Omega)^+$.

Proof.

The first isomorphism is a direct consequence of Proposition 5.2.1(iii) and the fact that $L_\Omega(\lambda_{\Omega,0})$ is isomorphic to the trivial G'_Ω-module i.e. K.

The two subgroups

$$B_\Omega = B \cap G_\Omega, \quad B'_\Omega = B \cap G'_\Omega$$

are Borel subgroups of G_Ω and G'_Ω respectively. For every $\lambda \in X(T)$, let $K_{\Omega,\lambda}$ be the one dimensional B_Ω-module on which T acts by weight λ and the unipotent radical of B_Ω acts trivially. Let

$$Y_\Omega(\lambda) = \mathrm{Ind}_{B_\Omega}^{G_\Omega}(K_{\Omega,\lambda}).$$

Theorem 2.9 of [36] states

$$Y_\Omega(\lambda)|_{G'_\Omega} \overset{G'_\Omega}{\simeq} \mathrm{Ind}_{B'_\Omega}^{G'_\Omega}(K_{\Omega,\lambda}|_{B'_\Omega}). \qquad (*)$$

Claim $Y_\Omega(\lambda_\Omega)$ is nonzero iff $\lambda_\Omega \in X_\Omega$, and for $\lambda_\Omega \in X_\Omega$, $Y_\Omega(\lambda_\Omega)$ has a simple socle $L_\Omega(\lambda_\Omega)$.

As in the semisimple case we can show that $L_\Omega(\lambda_\Omega)$, as a B_Ω-module, is generated by any nonzero vector of weight λ_Ω. Let $\mu_\Omega \in X_\Omega$. Then

$$\mathrm{Hom}_{G_\Omega}(L_\Omega(\mu_\Omega), Y_\Omega(\lambda_\Omega)) \simeq \mathrm{Hom}_{B_\Omega}(L_\Omega(\mu_\Omega)|_{B_\Omega}, K_{\Omega,\lambda_\Omega}),$$

see Proposition 1.4 of [11]. So if

$$0 \neq \theta \in \mathrm{Hom}_{B_\Omega}(L_\Omega(\mu_\Omega)|_{B_\Omega}, K_{\Omega,\lambda_\Omega})$$

and $0 \neq v \in L_\Omega(\mu_\Omega)^{\mu_\Omega}$, we have:

$$\theta(KB_\Omega v) = KB_\Omega\theta(v) \neq 0.$$

Hence $\theta(v) \neq 0$, and so $\theta(v) \in K_{\Omega,\lambda_\Omega}$ is a vector of weight λ_Ω, thus v is of weight λ_Ω. This shows this homomorphism is zero when $\mu_\Omega \neq \lambda_\Omega$. Let

$$U = \sum_{\mu \neq \lambda_\Omega} L_\Omega(\lambda_\Omega)^\mu.$$

For any $\theta \in \mathrm{Hom}_{B_\Omega}(L_\Omega(\lambda_\Omega), K_{\Omega,\lambda_\Omega})$, we have $\theta(U) = 0$. So

$$\mathrm{Hom}_{B_\Omega}(L_\Omega(\lambda_\Omega), K_{\Omega,\lambda_\Omega}) \cong \mathrm{Hom}_{B_\Omega}(L_\Omega(\lambda_\Omega)/U, K_{\Omega,\lambda_\Omega}) \cong K.$$

This completes the proof of the claim.

It is well known that (p. 55 of [2]):

$$\mathrm{soc}_{G'_\Omega} \mathrm{Ind}_{B'_\Omega}^{G'_\Omega}(K_{\Omega,\lambda_\Omega}\big|_{B'_\Omega}) \cong L(i^*_\Omega(\lambda_\Omega)).$$

By Clifford's theorem, the restriction $L_\Omega(\lambda_\Omega)\big|_{G'_\Omega}$ is completely reducible, and from (*) we have:

$$L_\Omega(\lambda_\Omega)\big|_{G'_\Omega} \subseteq \mathrm{Ind}_{B'_\Omega}^{G'_\Omega}(K_{\Omega,\lambda_\Omega}\big|_{B'_\Omega}). \quad \square$$

5.2.7. Corollary.

Suppose the notation of Proposition 5.2.5 and Lemma 5.2.6. Then we have:

$$\mathrm{Ext}^1_{G'_\Omega}(L_\Omega(\mu_\Omega), L_\Omega(\lambda_\Omega)) \overset{K}{\cong} \mathrm{Ext}^1_{G'_\Omega}(L_\Omega(\mu'_\Omega), L_\Omega(\lambda'_\Omega)) \overset{K}{\cong} \mathrm{Ext}^1_{G'_\Omega}(L(i^*_\Omega(\mu_\Omega)), L(i^*_\Omega(\lambda_\Omega))). \quad \square$$

5.2.8. Lemma.

Let λ,μ belong to the same coset of $\mathbf{Z}\Phi_\Omega$ in $X(T)$, and suppose $i^*_\Omega(\lambda) = i^*_\Omega(\mu)$. Then $\lambda = \mu$.

Proof.

We can put

$$\lambda = \mu + \sum_{\alpha\in\Omega} n_\alpha \alpha$$

for some $n_\alpha \in \mathbf{Z}$. Thus

$$\sum_{\alpha\in\Omega} n_\alpha \, i^*_\Omega(\alpha) = 0 \text{ implies } n_\alpha = 0$$

for every $\alpha \in \Omega$, this is because $\{i^*_\Omega(\alpha)/\alpha \in \Omega\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Phi'_\Omega$. □

5.2.9. Proposition.

For $\lambda_\Omega \in X_\Omega$, let $\beta_\Omega(\lambda_\Omega)$ be the block of G_Ω containing λ_Ω. Then we have:

$$\beta_\Omega(\lambda_\Omega) = (W_\Omega.\lambda_\Omega + p^{r_\Omega(\lambda_\Omega)+1}\,\mathbb{Z}\Phi_\Omega) \cap X_\Omega .$$

Proof.

Suppose $\mu_\Omega \in \beta_\Omega(\lambda_\Omega)$. Hence there exist $\mu_\Omega = \lambda_0, \lambda_1, \ldots, \lambda_n = \lambda_\Omega$ all are in X_Ω such that either

$$\mathrm{Ext}^1_{G_\Omega}(L_\Omega(\lambda_i), L_\Omega(\lambda_{i+1})) \neq 0 \text{ or } \mathrm{Ext}^1_{G_\Omega}(L_\Omega(\lambda_{i+1}), L_\Omega(\lambda_i)) \neq 0$$

or both for every $i = 0,\ldots,n-1$. We argue by induction on n. So we may assume $n = 1$, also we can suppose

$$\mathrm{Ext}^1_{G_\Omega}(L_\Omega(\mu_\Omega), L_\Omega(\lambda_\Omega)) \neq 0.$$

The above nonzero extension gives us; that there exists an indecomposable G_Ω-module V such that $L_\Omega(\mu_\Omega)$ and $L_\Omega(\lambda_\Omega)$ are its only composition factors. For each $\mu \in X(T)$, let $\bar{\mu} = \mu + \mathbb{Z}\Phi_\Omega$. Let $V^{(\bar{\mu})}$ be the sum of all weight subspaces of V of weights in $\bar{\mu}$. Thus

$$V = \sum^{\oplus}_{\mu\in X(T)} V^{(\bar{\mu})} .$$

It follows directly from §27.2 of [23] that

$$X_\alpha \, V^{(\bar{\mu})} = V^{(\bar{\mu})}$$

for every $\alpha \in \Phi_\Omega$. Hence $V^{(\bar{\mu})}$ is a G_Ω-submodule of V, which gives

$$V = V^{(\bar{\mu})}$$

for some $\mu \in X(T)$, and all the weights of V are in the same coset of $\mathbf{Z}\Phi_\Omega$ in $X(T)$.

By Proposition 5.2.5 we have:

$$\mathrm{Ext}^1_{G'_\Omega}(L(i^*_\Omega(\mu_\Omega)), L(i^*_\Omega(\lambda_\Omega))) \neq 0.$$

Therefore

$$i^*_\Omega(\mu_\Omega) \in (W_\Omega . i^*_\Omega(\lambda_\Omega) + p^{r_\Omega(\lambda_\Omega)+1}\mathbf{Z}\Phi'_\Omega) \cap X(T_\Omega)^+.$$

Hence

$$\mu_\Omega + \nu \in (W_\Omega . \lambda_\Omega + p^{r_\Omega(\lambda_\Omega)+1}\mathbf{Z}\Phi_\Omega) \cap X_\Omega$$

for some $\nu \in X_{\Omega,0}$.

The previous two paragraphs show that; all μ_Ω, $\mu_\Omega + \nu, \lambda_\Omega$ are in the same coset of $\mathbf{Z}\Phi_\Omega$ in $X(T)$. Lemma 5.2.8 forces ν to be zero.

Conversely, suppose

$$\mu_\Omega \in (W_\Omega . \lambda_\Omega + p^{r_\Omega(\lambda_\Omega)+1}\mathbf{Z}\Phi_\Omega) \cap X_\Omega.$$

Hence

$$i^*_\Omega(\mu_\Omega) \in (W_\Omega . i^*_\Omega(\lambda_\Omega) + p^{r_\Omega(\lambda_\Omega)+1}\mathbf{Z}\Phi'_\Omega) \cap X(T_\Omega)^+.$$

Therefore, there exist $i^*_\Omega(\mu_\Omega) = \lambda'_0, \ldots, \lambda'_n = i^*_\Omega(\lambda_\Omega)$ all are in $X(T_\Omega)^+$ such that either

$$\mathrm{Ext}^1_{G'_\Omega}(L(\lambda'_i), L(\lambda'_{i+1})) \neq 0 \text{ or } \mathrm{Ext}^1_{G'_\Omega}(L(\lambda'_{i+1}), L(\lambda'_i)) \neq 0$$

or both for every $i = 0,\ldots,n-1$. Choose $\lambda_{\Omega,i} \in X_\Omega$ such that

$$i_\Omega^*(\lambda_{\Omega,i}) = \lambda_i', \ i = 0,\ldots,n.$$

Claim there exists $\nu \in X_{\Omega,0}$ such that

$$\mu_\Omega + \nu \in \beta_\Omega(\lambda_\Omega).$$

We argue by induction on n above. So we may assume $n = 1$, and also we can suppose

$$\mathrm{Ext}^1_{G_\Omega'}(L(i_\Omega^*(\mu_\Omega)),\ L(i_\Omega^*(\lambda_\Omega))) \neq 0.$$

Hence the G_Ω/G_Ω'-module ($\mu_\Omega = \mu_\Omega' + \mu_{\Omega,0}, \lambda_\Omega = \lambda_\Omega' + \lambda_{\Omega,0}$ as usual)

$$\mathrm{Ext}^1_{G_\Omega'}(L_\Omega(\mu_\Omega'), L_\Omega(\lambda_\Omega')) \neq 0,$$

see Corollary 5.2.7. Choose $\nu' \in X_{\Omega,0}$ such that

$$L_\Omega(\nu') \subseteq \mathrm{soc}_{G_\Omega/G_\Omega'} \mathrm{Ext}^1_{G_\Omega'}(L_\Omega(\mu_\Omega'), L_\Omega(\lambda_\Omega')).$$

Thus, Proposition 5.2.5 gives

$$\mathrm{Ext}^1_{G_\Omega}(L_\Omega(\mu_\Omega' + \nu' + \lambda_{\Omega,0}), L_\Omega(\lambda_\Omega)) \simeq \mathrm{Hom}_{G_\Omega/G_\Omega'}(L_\Omega(\nu'), \mathrm{Ext}^1_{G_\Omega'}(L_\Omega(\mu_\Omega'), L_\Omega(\lambda_\Omega'))) \neq 0.$$

Put $\nu = \nu' + \lambda_{\Omega,0} - \mu_{\Omega,0}$. So

$$\mu_\Omega + \nu \in \beta_\Omega(\lambda_\Omega), \qquad (*)$$

this proves our claim.

By assumption, μ_Ω and λ_Ω are in the same coset of $\mathbb{Z}\Phi_\Omega$ in $X(T)$, and from (*) above $\mu_\Omega + \nu$ and λ_Ω are in that coset. Thus $\nu = 0$ and $\mu_\Omega \in \beta_\Omega(\lambda_\Omega)$. □

or both for every $i = 0,\dots,n-1$. Choose $\lambda_{\Omega,i} \in X_\Omega$ such that

$$i_\Omega^*(\lambda_{\Omega,i}) = \lambda_i', \ i = 0,\dots,n.$$

Claim there exists $\nu \in X_{\Omega,0}$ such that

$$\mu_\Omega + \nu \in \beta_\Omega(\lambda_\Omega).$$

We argue by induction on n above. So we may assume $n = 1$, and also we can suppose

$$\mathrm{Ext}^1_{G_\Omega'}(L(i_\Omega^*(\mu_\Omega)), L(i_\Omega^*(\lambda_\Omega))) \neq 0.$$

Hence the G_Ω/G_Ω'-module ($\mu_\Omega = \mu_\Omega' + \mu_{\Omega,0}, \lambda_\Omega = \lambda_\Omega' + \lambda_{\Omega,0}$ as usual)

$$\mathrm{Ext}^1_{G_\Omega'}(L_\Omega(\mu_\Omega'), L_\Omega(\lambda_\Omega')) \neq 0,$$

see Corollary 5.2.7. Choose $\nu' \in X_{\Omega,0}$ such that

$$L_\Omega(\nu') \subseteq \mathrm{soc}_{G_\Omega/G_\Omega'} \mathrm{Ext}^1_{G_\Omega'}(L_\Omega(\mu_\Omega'), L_\Omega(\lambda_\Omega')).$$

Thus, Proposition 5.2.5 gives

$$\mathrm{Ext}^1_{G_\Omega}(L_\Omega(\mu_\Omega' + \nu' + \lambda_{\Omega,0}), L_\Omega(\lambda_\Omega)) \simeq \mathrm{Hom}_{G_\Omega/G_\Omega'}(L_\Omega(\nu'), \mathrm{Ext}^1_{G_\Omega'}(L_\Omega(\mu_\Omega'), L_\Omega(\lambda_\Omega'))) \neq 0.$$

Put $\nu = \nu' + \lambda_{\Omega,0} - \mu_{\Omega,0}$. So

$$\mu_\Omega + \nu \in \beta_\Omega(\lambda_\Omega), \qquad (*)$$

this proves our claim.

By assumption, μ_Ω and λ_Ω are in the same coset of $\mathbb{Z}\Phi_\Omega$ in $X(T)$, and from (*) above $\mu_\Omega + \nu$ and λ_Ω are in that coset. Thus $\nu = 0$ and $\mu_\Omega \in \beta_\Omega(\lambda_\Omega)$. □

5.3. The Blocks of the Parabolic Subgroup P_Ω.

For every $\lambda_\Omega \epsilon X_\Omega$, let $I_\Omega(\lambda_\Omega)$ (resp. $J_\Omega(\lambda_\Omega)$) be the injective cover of $L_\Omega(\lambda_\Omega)$ as a P_Ω-module (resp. G_Ω-module). The injective G_Ω-module $J_\Omega(\lambda_\Omega)$ can be made into a P_Ω-module by letting U^Ω act trivially. So the tensor product identity (Proposition 1.5 of [11]) gives

$$\mathrm{Ind}_{G_\Omega}^{P_\Omega}(J_\Omega(\lambda_\Omega)) \overset{P_\Omega}{\simeq} J_\Omega(\lambda_\Omega) \otimes \mathrm{Ind}_{G_\Omega}^{P_\Omega}(K).$$

Moreover, this induced module is injective and has a simple P_Ω-socle $L_\Omega(\lambda_\Omega)$ (p. 4 and Proposition 1.4 of [11]). Thus

$$I_\Omega(\lambda_\Omega) \overset{P_\Omega}{\simeq} J_\Omega(\lambda_\Omega) \otimes \mathrm{Ind}_{G_\Omega}^{P_\Omega}(K).$$

Directly from the definition of the induced module, we can identify

$$\mathrm{Ind}_{G_\Omega}^{P_\Omega}(K) = \{f \epsilon K[P_\Omega]/f.g = f \text{ for every } g \epsilon G_\Omega\}.$$

The coordinate ring $K[U^\Omega]$ can be made into a G_Ω-module by the action

$$(c_g f)(u) = f(g^{-1}ug)$$

for every $g \epsilon G_\Omega$, $f \epsilon K[U^\Omega]$, and $u \epsilon U^\Omega$. This action is well defined since U^Ω is a normal subgroup of P_Ω. The module $\mathrm{Ind}_{G_\Omega}^{P_\Omega}(K)$ is a G_Ω-module via the right translation. So we have:

5.3.1. Proposition.

$$K[U^\Omega] \overset{G_\Omega}{\simeq} \mathrm{Ind}_{G_\Omega}^{P_\Omega}(K).$$

Proof.

The map

$$\psi\colon G_\Omega \times U^\Omega \to P_\Omega$$

defined by $\psi(g,u) = gu$ for every $g \in G_\Omega$ and $u \in U^\Omega$ is an isomorphism of varieties (p. 185 of [23]). This gives the isomorphism of K-algebras

$$\psi^*\colon K[P_\Omega] \to K[G_\Omega] \otimes K[U^\Omega]$$

defined by $\psi^*(f)(g,u) = f(gu)$ for every $f \in K[P_\Omega]$, $g \in G_\Omega$, and $u \in U^\Omega$.

Let $1_{G_\Omega} \in K[G_\Omega]$ be the map $1_{G_\Omega}\colon G_\Omega \to K$ sending $g \longmapsto 1 \in K$ for every $g \in G_\Omega$. Thus $K1_{G_\Omega}$ is a subalgebra of $K[G_\Omega]$, and considering the action by conjugation ("c_g") defined above we have:

$$K1_{G_\Omega} \otimes K[U^\Omega] \overset{G_\Omega}{\cong} K[U^\Omega].$$

Claim 1. $\psi^*(\mathrm{Ind}_{G_\Omega}^{P_\Omega}(K)) = K1_{G_\Omega} \otimes K[U^\Omega]$.

Let $f \in \mathrm{Ind}_{G_\Omega}^{P_\Omega}(K)$. Hence we have:

$$\psi^*(f) = 1_{G_\Omega} \otimes f|_{U^\Omega} \in K1_{G_\Omega} \otimes K[U^\Omega].$$

Conversely, let $1_{G_\Omega} \otimes \eta \in K1_{G_\Omega} \otimes K[U^\Omega]$. Suppose $f \in K[P_\Omega]$ be such that $\psi^*(f) = 1_{G_\Omega} \otimes \eta$. We can easily see that $f \in \mathrm{Ind}_{G_\Omega}^{P_\Omega}(K)$.

Claim 2. $\psi^*\colon \mathrm{Ind}_{G_\Omega}^{P_\Omega}(K) \to K1_{G_\Omega} \otimes K[U^\Omega]$

is a G_Ω-modules isomorphism.

Indeed, let $f \in \mathrm{Ind}_{G_\Omega}^{P_\Omega}(K)$, g and $g' \in G_\Omega$, $u \in U^\Omega$, and suppose $\psi^*(f) = 1_{G_\Omega} \otimes \eta$. Hence

$\psi^*(g.f)(g',u)=f(g^{-1}ug)=(1_{G_\Omega}\otimes\eta)(g^{-1}g'g,g^{-1}ug)=(c_g(1_{G_\Omega}\otimes\eta))(g',u)$. □

From now on, we let M_Ω to be the kernel of the augmentation map for the Hopf algebra $K[U^\Omega]$ i.e.

$$M_\Omega = \{f \in K[U^\Omega]/f(1) = 0\},$$

where $1 \in U^\Omega$ is the identity element.

We can easily see that M_Ω is a G_Ω-submodule of $K[U^\Omega]$ (via the action "c_g" above), M_Ω is a maximal ideal in $K[U^\Omega]$, and

$$K[U^\Omega] \overset{G_\Omega}{\cong} M_\Omega \oplus K.$$

Krull's intersection theorem gives

$$\bigcap_{r=1}^{\infty} M_\Omega^r = 0.$$

5.3.2. Lemma.

$$\text{ch. } M_\Omega/M_\Omega^2 = \sum_{\lambda\in \Phi^+\setminus\Phi_\Omega^+} e(\lambda).$$

Proof.

Considering the adjoint action, the weights of Lie (U^Ω) as a G_Ω-module with respect to T are $\Phi^-\setminus\Phi_\Omega^-$ (the first paragraph of §30.2 of [23]). So the weights of $(\text{Lie}(U^\Omega))^*$ are $\Phi^+\setminus\Phi_\Omega^+$.

The map

$$\theta:\text{Lie }(U^\Omega) \to (M_\Omega/M_\Omega^2)^*$$

defined by $\theta(\gamma)(m+M_\Omega^2) = \gamma(m)$ for every $\gamma \in$ Lie (U^Ω), and $m \in M_\Omega$ is a G_Ω-modules isomorphism (we use Lemma 1.1). □

5.3.3. Corollary.

The weight $\lambda \in X(T)$ *is a weight of* $M_\Omega^r/M_\Omega^{r+1}$, $r \geq 1$, *iff* $\lambda = \sum_{j=1}^{r} \alpha_j$ *for some* $\alpha_j \in \Phi^+\setminus\Phi_\Omega^+$. *The multiplicity of* λ *is equal* *to the number of different ways λ can be written as such a sum.*

Proof. (S. Donkin).

Let $m = \dim M_\Omega/M_\Omega^2$, and for every $r \geq 1$, let $f(m,r)$ denote the dimension of the space of homogenous polynomials of degree r in m variables. Let

$$R = K[U^\Omega]_{M_\Omega}$$

be the localization of $K[U^\Omega]$ at M_Ω.

Now R is a regular local ring by Theorem A of §5.3 of [23]. Then

$$\dim M_\Omega R/(M_\Omega R)^2 = m,$$

so that

$$\dim (M_\Omega R)^r/(M_\Omega R)^{r+1} = f(m,r).$$

The natural map

$$K[U^\Omega] \to R$$

induces an isomorphism

$$M_\Omega^r/M_\Omega^{r+1} \to (M_\Omega R)^r/(M_\Omega R)^{r+1}.$$

Thus

$$\dim M_\Omega^r/M_\Omega^{r+1} = f(m,r). \quad \square$$

5.3.4. Lemma.

Let V *be a* P_Ω*-module. Then* $L_\Omega(\lambda_\Omega)$, $\lambda_\Omega \in X_\Omega$, *is a* P_Ω*-composition factor of* V *iff it is a* G_Ω*-composition factor.*

Proof.

Suppose $L_\Omega(\lambda_\Omega)$ is a G_Ω-composition factor of V. So there exists two finite dimensional G_Ω-submodules of V, $V_1 \subseteq V_2$, such that $V_2/V_1 \simeq L_\Omega(\lambda_\Omega)$. Hence we may assume V to be finite dimensiona

We argue by induction on the P_Ω-composition length of V, $\ell(V)$. If $\ell(V) = 1$ nothing to prove. So suppose $\ell(V) > 1$. Let

$$V^{U^\Omega} = \{v \in V/uv = v \text{ for every } u \in U^\Omega\},$$

it is nonzero by §17.5 of [23]. If $V^{U^\Omega} = V$, then U^Ω acts trivially on V and the lemma follows. If $V^{U^\Omega} \subsetneq V$, so V^{U^Ω} and V/V^{U^Ω} both are P_Ω-modules with P_Ω-composition length less than $\ell(V)$, moreover $L_\Omega(\lambda_\Omega)$ is a G_Ω-composition factor of one of them or of both. □

Remark.

By the last lemma, if V is a rational P_Ω-module, when we say a composition factor of it, we mean either a P_Ω or a G_Ω one.

5.3.5. Lemma.

Suppose $L_\Omega(\mu_\Omega)$ *is a composition factor of* $I_\Omega(\lambda_\Omega)$. *Then it is either a composition factor of* $J_\Omega(\lambda_\Omega)$ *or of* $L_\Omega(\lambda'_\Omega) \otimes M^r_\Omega/M^{r+1}_\Omega$, *where* $L_\Omega(\lambda'_\Omega)$ *is a composition of* $J_\Omega(\lambda_\Omega)$, *and* $r \geq 1$.

Proof.

We have:

$$I_\Omega(\lambda_\Omega) \overset{G_\Omega}{\simeq} J_\Omega(\lambda_\Omega) \otimes K[U^\Omega] \overset{G_\Omega}{\simeq} J_\Omega(\lambda_\Omega) \otimes M_\Omega \oplus J_\Omega(\lambda_\Omega).$$

Hence $L_\Omega(\mu_\Omega)$ is a composition factor of $J_\Omega(\lambda_\Omega)$ and the lemma follows, or $L_\Omega(\mu_\Omega)$ is a composition factor of $J_\Omega(\lambda_\Omega) \otimes M_\Omega$ and the lemma follows using Krull's intersection theorem. □

5.3.6. Corollary.

For $\lambda_\Omega \in X_\Omega$, *let* $\beta_\Omega(\lambda_\Omega)$ *and* $\beta_{P_\Omega}(\lambda_\Omega)$ *be the* G_Ω *and* P_Ω *blocks containing* λ_Ω *respectively. Then we have:*

$$\beta_\Omega(\lambda_\Omega) \subseteq \beta_{P_\Omega}(\lambda_\Omega).$$

Proof.

This follows directly from the lemma. □

5.3.7. Lemma.

Let V *be a finite dimensional rational* G_Ω*-module. Suppose* $\lambda_\Omega \in X_\Omega$ *be such that* $\lambda_\Omega + \phi \in X_\Omega$ *for every* ϕ *a weight of* V. *Then* $Y_\Omega(\lambda_\Omega) \otimes V$ *has a filtration by the* $Y_\Omega(\lambda_\Omega + \phi)$*'s, and each occurs multiplicity of* ϕ *times.*

Proof.

See the proof of Corollary 2.8 of [36], and note that

$$Y_\Omega(\lambda_\Omega + \phi) = \mathrm{Ind}_{B_\Omega}^{G_\Omega}(K_{\Omega,\lambda_\Omega+\phi}) \neq 0$$

for every ϕ a weight of V. □

5.3.8. Lemma.

Suppose $\Omega \subsetneqq \Delta$. *Then* $\mathbf{Z}\Phi = \mathbf{Z}(\Phi\backslash\Phi_\Omega)$.

Proof.

Let $\Omega' = \Delta\backslash\{\alpha\}$ for some $\alpha \in \Delta\backslash\Omega$. Thus $\Omega \subseteq \Omega' \subsetneqq \Delta$. Hence $\Phi\backslash\Phi_{\Omega'} \subseteq \Phi\backslash\Phi_\Omega$. So we may assume $\Omega = \Delta\backslash\{\alpha\}$ for some $\alpha \in \Delta$. We fix this $\alpha \in \Delta$, $\alpha \notin \Omega$.

It is enough to prove: $\Omega \subseteq \mathbf{Z}(\Phi\backslash\Phi_\Omega)$. Then it is enough to show that, for every $\beta \in \Omega$, there exists $\Delta(\alpha,\beta) \subseteq \Delta$ such that $\alpha + \beta + \sum_{\gamma\in\Delta(\alpha,\beta)} \gamma \in \Phi$ and $\alpha + \sum_{\gamma\in\Delta(\alpha,\beta)} \gamma \in \Phi$. For each $\beta \in \Omega$, let $\Delta(\alpha,\beta)$ to be the set of simple roots joining α and β in the Dynkin diagram of the corresponding Δ (it may be empty), see §3.6. of [7]. □

By the previous lemma, for each $\phi \in \mathbf{Z}\Phi$, we can find θ_1 a weight of $M_\Omega^{r_1}/M_\Omega^{r_1+1}$, and θ_2 a weight of $M_\Omega^{r_2}/M_\Omega^{r_2+1}$ for some $r_1, r_2 \geq 0$, such that $\phi = \theta_1 - \theta_2$. We put $r_1 = 0$ (resp. $r_2 = 0$) if $\theta_1 = 0$ (resp. $\theta_2 = 0$), and as before we assume $\Omega \subsetneq \Delta$.

5.3.9. Definition.

For every $\phi \in \mathbf{Z}\Phi$, choose the θ_1, θ_2 above such that $r_1 + r_2$ is minimal. Put $n_\phi = r_1 + r_2$. □

5.3.10. Proposition.

Let $\phi \in \mathbf{Z}\Phi$, *and let* $\lambda_\Omega \in X_\Omega$ *be big enough with respect to* ϕ ($(\lambda_\Omega, \alpha^\vee) > 6n_\phi$ *for every* $\alpha \in \Omega$ *say*), *moreover suppose* $\Omega \subsetneq \Delta$. *Then we have:*

$$\lambda_\Omega + \phi \in \beta_{P_\Omega}(\lambda_\Omega).$$

Proof.

The module

$$I_\Omega(\lambda_\Omega) \overset{G_\Omega}{\cong} J_\Omega(\lambda_\Omega) \otimes M_\Omega \oplus J_\Omega(\lambda_\Omega),$$

and also we have

$$\mathrm{soc}_{G_\Omega} J_\Omega(\lambda_\Omega) \cong \mathrm{soc}_{G_\Omega} Y_\Omega(\lambda_\Omega) \cong L_\Omega(\lambda_\Omega).$$

Then $J_\Omega(\lambda_\Omega)$ contains a copy of $Y_\Omega(\lambda_\Omega)$, and hence for every $r \geq 1$, all the composition factors of $Y_\Omega(\lambda_\Omega) \otimes M_\Omega^r/M_\Omega^{r+1}$ are composition factors of $I_\Omega(\lambda_\Omega)$.

Let $\phi = \theta_1 - \theta_2$ as before.

Step 1. We prove $\lambda_\Omega + \theta_1 \in \beta_{P_\Omega}(\lambda_\Omega)$.

Let θ be any weight of $M_\Omega^{r_1}/M_\Omega^{r_1+1}$. Thus $(\lambda_\Omega + \theta, \alpha^\vee) > 0$ for every $\alpha \in \Omega$. By Lemma 5.3.7, the module $Y_\Omega(\lambda_\Omega + \theta_1)$ is a section of

$Y_\Omega(\lambda_\Omega) \otimes M_\Omega^{r_1}/M_\Omega^{r_1+1}$, therefore $L_\Omega(\lambda_\Omega + \theta_1)$ is a composition factor of $I_\Omega(\lambda_\Omega)$.

Step 2. We prove $\lambda_\Omega + \theta_1 - \theta_2 \in \beta_{P_\Omega}(\lambda_\Omega)$.

Let $\mu = \lambda_\Omega + \theta_1 - \theta_2$, and let θ be any weight of $M_\Omega^{r_2}/M_\Omega^{r_2+1}$. So $(\mu + \theta, \alpha^\vee) > 0$ for every $\alpha \in \Omega$. By the last step, $\mu + \theta_2 \in \beta_{P_\Omega}(\mu)$ i.e. $\lambda_\Omega + \theta_1 \in \beta_{P_\Omega}(\lambda_\Omega + \theta_1 - \theta_2)$. □

5.3.11. Theorem.

Suppose $\lambda_\Omega \in X_\Omega$, and $\Omega \subsetneq \Delta$. Then we have:

$$\beta_{P_\Omega}(\lambda_\Omega) = (\lambda_\Omega + \mathbb{Z}\Phi) \cap X_\Omega.$$

Proof.

Let $2\rho_\Omega = \sum_{\alpha \in \Phi_\Omega^+} \alpha$. From before we know $(2\rho_\Omega, \alpha^\vee) = 2$ for every $\alpha \in \Omega$.

Now let $\phi \in \mathbb{Z}\Phi$ be an arbitrary element such that $\lambda_\Omega + \phi \in X_\Omega$. By Proposition 5.3.10, we can choose $m \in \mathbb{N}$ big enough such that

$$\lambda_\Omega + mp^{r_\Omega(\lambda_\Omega)+r_\Omega(\lambda_\Omega+\phi)+1} 2\rho_\Omega + \phi \in \beta_{P_\Omega}(\lambda_\Omega + mp^{r_\Omega(\lambda_\Omega)+r_\Omega(\lambda_\Omega+\phi)+1} 2\rho_\Omega).$$

Also we have (by Proposition 5.2.9, and Corollary 5.3.6):

$$\lambda_\Omega + \phi + mp^{r_\Omega(\lambda_\Omega)+r_\Omega(\lambda_\Omega+\phi)+1} 2\rho_\Omega \in \beta_\Omega(\lambda_\Omega + \phi) \subseteq \beta_{P_\Omega}(\lambda_\Omega + \phi),$$

$$\lambda_\Omega + mp^{r_\Omega(\lambda_\Omega)+r_\Omega(\lambda_\Omega+\phi)+1} 2\rho_\Omega \in \beta_\Omega(\lambda_\Omega) \subseteq \beta_{P_\Omega}(\lambda_\Omega).$$

Hence

$$\lambda_\Omega + \phi \in \beta_{P_\Omega}(\lambda_\Omega).$$

Conversely, let $\mu_\Omega \in \beta_{P_\Omega}(\lambda_\Omega)$. Thus we may assume $L_\Omega(\mu_\Omega)$ is a composition factor of $I_\Omega(\lambda_\Omega)$. Lemma 5.3.5, Corollary 5.3.3, and

Proposition 5.2.9 give

$$\mu_\Omega \in (\lambda_\Omega + \mathbf{Z}\Phi) \cap X_\Omega. \quad \square$$

REFERENCES

[1] E. Abe; *Hopf Algebras*, Cambridge University Press, Cambridge, 1977.

[2] H.H. Anderson; *The strong linkage principle*, J. Reine Angew. Math. 315 (1980), 53-59.

[3] H.H. Anderson; *On the structure of the cohomology of line bundles on* $G/_B$, J. Algebra 71 (1981), 245-258.

[4] H.H. Anderson; *Line bundles on flag manifolds*, To appear.

[5] J.-P. Antoine, D. Speiser; *Characters of irreducible representations of the simple groups II. Application to the classical groups*, J. Mathematical Physics 5 (1964), 1560-1572.

[6] A. Borel; *Properties and linear representations of Chevalley groups*, In Lecture Notes in Mathematics, 131, Springer, 1970.

[7] R.W. Carter; *Simple Groups of Lie Type*, Wiley, London, New York, 1972.

[8] R.W. Carter; *The relation between characteristic o representations and characteristic p representations of finite groups of Lie type*, The Santa Cruz Conference on Finite Groups, Proceeding of Symposia in Pure Mathematics, A.M.S. 37 (1980), 301-311.

[9] R.W. Carter, G. Lusztig; *On the modular representations of the general linear and symmetric groups*, Math. Z. 136 (1974), 193-242.

[10] R.W. Carter, M.T.J. Payne; *On homomorphisms between Weyl and Specht modules*, Math. Proc. Camb. Phil. Soc. 87 (1980), 419-425.

[11] E. Cline, B. Parshall, L. Scott; *Induced modules and affine quotients*, Math. Ann. 230 (1977), 1-14.

[12] E. Cline, B. Parshall, L. Scott; *Cohomology, hyperalgebras, and representations*, J. Algebra 63 (1980), 98-123.

[13] E. Cline, B. Parshall, L. Scott, W. van der Kallen; *Rational and generic cohomology*, Invensions Math. 39 (1977), 143-169.

[14] C.W. Curtis, I. Reiner; *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York, 1962.

[15] S. Donkin; *On a question of Verma*, J. London Math. Soc. 21 (1980), 445-455.

[16] S. Donkin; *The blocks of a semisimple algebraic group*, J. Algebra 67 (1980), 36-53.

[17] S. Donkin; *On Ext^1 for semisimple groups and infinitesimal subgroups*, Preprint, University of Warwick, Math. Institute, 1981.

[18] J.A. Green; *Locally finite representations*, J. Algebra 41 (1976), 137-171.

[19] P.J. Hilton, U. Stammbach; *A Course in Homological Algebra*, Graduate Texts in Mathematics, 4, Springer, 1970.

[20] G. Hochschild; *Cohomology of algebraic linear groups*, J. Math. 5 (1961), 492-519.

[21] J.E. Humphreys; *Modular representations of classical Lie algebras and semisimple groups*, J. Algebra 19 (1971), 51-79.

[22] J.E. Humphreys; *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, 9, Springer, 1972.

[23] J.E. Humphreys; *Linear Algebraic Groups*, Graduate Texts in Mathematics, 21, Springer, 1975.

[24] J.E. Humphreys; *Ordinary and Modular Representations of Chevalley Groups*, Lecture Notes in Mathematics, 528, Springer, 1976.

[25] J.E. Humphreys; *On the hyperalgebra of a semisimple algebraic group*, Contribution to Algebra: A Collection of Papers Dedicated to Ellic Kolchin. Academic Press, New York, 1977.

[26] J.E. Humphreys; *Symmetry for finite dimensional Hopf algebras*, Proc. Amer. Math. Soc. 68 (1978), 143-146.

[27] J.E. Humphreys, J.C. Jantzen; *Blocks and indecomposable modules for semisimple algebraic groups*, J. Algebra 54 (1978), 494-503.

[28] J.C. Jantzen; *Zur Charakterformel gewisser Darstellungen halbeinfacher Gruppen und Lie-Algebren*, Math. Z. 40 (1974), 127-149.

[29] J.C. Jantzen; *Darstellungen halbeinfacher Gruppen und kontravariante Formen*, J. Reine Angew. Math. 290 (1977), 117-141.

[30] J.C. Jantzen; *Über das Dekompositionsverhalten gewisser modularer Darstellungen halbeinfacher Gruppen und ihrer Lie-Algebren*, J. Algebra 49 (1977), 441-469.

[31] J.C. Jantzan; *Weyl modules for groups of Lie type*, In Proc. of London Math. Soc., Durham 1978.

[32] J.C. Jantzen; *Darstellungen halbeinfacher Gruppen und ihrer Frobenius-Kerne*, J. Reine Angew. Math. 317 (1980), 157-199.

[33] G. Lusztig; *Some problems in representation theory of finite Chevalley groups*, The Santa Cruz Conference on Finite Groups, Proceeding of Symposia in Pure Mathematics, A.M.S. 37 (1980), 313-317.

[34] R. Steinberg; *Lectures on Chevalley Groups*, Yale Univ., Math. Dept., 1968.

[35] D.-N. Verma; *The role of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras*, Proc. Summer School on Group Representations, Budapest, 1971, 653-705.

[36] Jian-pan Wang; *Sheaf cohomology on G/B and tensor product of Weyl modules*, Preprint, East China Normal University, Dept. of Math., 1981.

[37] H. Yanagihara; *Theory of Hopf Algebras Attached to Group Schemes*, Lecture Notes in Mathematics, 614, Springer, 1977.

GLOSSARY OF NOTATION

$(G, K[G])$	1	$\leq$	7
$\gamma_1 * \gamma_2$	2	$X_r = \mathbb{Z}\Phi$	7
$\mathrm{Lie}(G)$	2	$\Phi^\vee$	7
$\mathrm{ad}(x)\gamma$	2	h	7
$c_x f$	3	α_o	7
$\mathrm{soc}_G\ V$	3	ρ	8
$\mathrm{soc}_R\ V$	3	V_λ	8
$\underline{\underline{G}}$	3	$x_\alpha(t)$	8
$\underline{\underline{R}}$	3	χ_α	8
$\mathcal{M}_G$	5	G	8
$(A) \otimes V$	5	V^λ	8
$\mathrm{Ind}_H^G(V)$	6	$h_\alpha(t)$	9
$\mathcal{L}$	6	T	9
$\mathcal{H}$	6	$X(T)$	9
Φ	6	$\omega.\lambda$	10
Δ	6	$X_{\alpha,n}$	10
Φ^+	6	$H_{i,b}$	10
Φ^-	6	U_K	10
$\{X_\alpha, H_i\}$	6	G_a	11
$(\ ,\ )$	6	ψ_α	11
W	7	$\xi_{\alpha,r}$	11
ω_α	7	$\psi : U_k \rightarrow K[G]^*$	11
$\alpha^\vee$	7	$\mathbb{Z}[X]$	12
X	7	$\{e(\lambda)/\lambda \in X\}$	12
X^+	7	$\mathrm{ch}.V$	11
$\{\lambda_i\}$ or $\{\lambda_\alpha\}$	7	$V(\lambda)$	13