Type III Diffeomorphisms of Manifolds.

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Declaration.

The contents of Chapter II are contained in a paper with Klaus Schmidt as co-author, which will be submitted for publication.

The nature of the joint work is the following. The construction given in Chapter II, Section §4 arose from some suggestions given by Klaus Schmidt to this author to improve an earlier weaker result. Most of the estimates, as they appear here, were hammered out together.

Summary.

This thesis is about a classification of ergodic diffeomorphisms of manifolds introduced by Krieger [13]. We start with the definition which states that if f denotes a non-singular ergodic transformation of a probability space (X,S,μ) which admits no σ-finite invariant measure equivalent to the given measure, then f is of type III. We then look at a finer division of the type III class of transformations by specifying the ratio set of f, which gives information about the values that the Radon-Nikodym derivative of f^n (the $n^{\underline{th}}$ iterate of f) takes on sets of positive measure. The ratio set is an invariant of weak equivalence, hence of conjugation by diffeomorphisms.

We examine some relationships between the differentiable and metric structures of diffeomorphisms of manifolds, starting with some known results on the circle [7, 8, 11].

The first chapter introduces most of the necessary definitions, background theorems, and notation.

The second chapter extends results of Herman and Katznelson by giving a construction for C^2 diffeomorphisms of the circle which are of type III_1. The rotation numbers of these diffeomorphisms form a set of measure 1 in $[0,1]$; Herman's theorem shows this is not possible for C^3 diffeomorphisms.

In Chapter 3 we discuss the topology of type III_1 diffeomorphisms in the space of C^∞ diffeomorphisms of T^n. We prove that type III_1 diffeomorphisms form a dense G_δ in certain closed subspaces of $\mathrm{Diff}^\infty(T^n)$.

The fourth chapter deals with the more pathological cases: type III_λ and type III_0 diffeomorphisms. The main result of this chapter is a theorem which proves that every smooth paracompact manifold of dimension ≥ 3 admits a smooth type III_0 diffeomorphism. This uses techniques from Herman [7], and is based on a result about ergodic smooth real line extensions of diffeomorphisms proved earlier in the chapter. Type III_λ diffeomorphisms are shown to be easy to construct from the results on type III_0 and type III_1 diffeomorphisms.

Introduction.

Let f be a non-singular automorphism of a Lebesgue space (X,S,μ). The problem of describing the conditions under which μ is equivalent to an f-invariant measure has been the subject of much study [2,3,4,6, 11,15]. Ornstein was the first, in 1960, to construct a non-singular transformation of a measure space which admits no σ-finite invariant measure equivalent to the given measure [15]. Herman constructed examples of diffeomorphisms of the circle which are $C^{1+\varepsilon}$, and later some smooth maps of every paracompact manifold of dimension greater than or equal to three, which do not admit σ-finite invariant measures equivalent to Lebesgue measure [7]. Katznelson also constructed some C^{∞} examples on the circle.

We develop ideas from [7] and [14] in this thesis to study the problem in more detail. After introducing some definitions and notation in Chapter I, we first recall the relationship between rotation numbers of diffeomorphisms of the circle and conjugacy to the rotation diffeomorphism. Herman's Theorem proves the existence of a set $A \subset [0,1]$ of full Lebesgue measure such that every C^3 diffeomorphism of the circle whose rotation number lies in A admits a unique finite invariant measure equivalent to Lebesgue measure. In this chapter we also introduce the concept of the ratio set and describe briefly Krieger's classification of non-singular transformations given in [13].

Chapter II consists of a construction which proves that for every $\alpha \in$, there exists a C^2 diffeomorphism of the circle with rotation number α which admits no σ-finite invariant measure equivalent to Lebesgue measure. Some topological results concerning the size of the set of type III_1 diffeomorphisms of the circle are included, extending a result of Herman [8].

The third chapter deals with type III_1 diffeomorphisms of higher dimensional manifolds. Herman has shown in [7] the existence of a C^∞ type III_1 diffeomorphism on every smooth paracompact manifold of dimension ≥ 3 . We study the topology of the set of such diffeomorphisms. The results concerning the rotation number of a diffeomorphism of the circle generalise only to a limited extent to higher dimensions; the rotation number of a diffeomorphism of a higher dimensional torus is not always well-defined if we use the definition given in Chapter I.

We prove, in a general setting, that C^r type III_1 diffeomorphisms form a G_δ on any compact manifold. The task is then to determine which manifolds and which spaces of diffeomorphisms admit dense sets of type III_1 diffeomorphisms. We give several methods for constructing C^∞ type III_1 diffeomorphisms on T^n , starting with Katznelson's construction on T^1 . The existence of a residual set of type III_1 diffeomorphisms can be shown in various closed subspaces of $\mathrm{Diff}^{\infty}(T^n)$.

The fourth chapter deals with more pathological types of diffeomorphisms which do not admit any invariant measure equivalent to Lebesgue measure. In particular we study type III_0 and type III_λ diffeomorphisms. Sections §10 and §11 deal with extending smooth type III_0 diffeomorphisms of the circle to the n-torus.

Section §12 contains a useful result on smooth ergodic real line extensions. We see in Chapter II that for certain irrational numbers $\alpha \in [0,1]$, *every* C^1 cocycle for R_α is a coboundary. A similar result holds for C^∞ cocycles; the proof uses the same Fourier series techniques of II.5.7. Therefore, it is not obvious that a type III diffeomorphism of T^1 has an ergodic real line extension, since we need a smooth cocycle which is not a coboundary. In this section we prove that if X is a locally compact manifold, f an ergodic diffeomorphism of X with a recurrent cocycle which is not a coboundary but a C^∞ limit of coboundaries, then there exists a dense G_δ of ergodic real line extensions of f . It is a trivial consequence of Herman's Theorem that $\log Df$ is always a C^∞ limit of coboundaries for ergodic $f \in \mathrm{Diff}^{\infty}(T^1)$. By our construction of type III_0 diffeomorphisms of T^n this condition still holds, so we can apply the theorem. Hence we can extend our type III diffeomorphisms to $T^n \times \mathbb{R}^p$, and now using methods similar to those of Herman we can find type III_0 diffeomorphisms of arbitrary paracompact manifolds.

The chapters are labelled with Roman numerals, and the sections are numbered continuously. A reference to II.5.3 means Chapter II, Lemma 5.3 (in Section §5), for example.

CHAPTER I.

Notation and Definitions.

§1. Diffeomorphisms of T^n .

Let $\mathrm{Diff}_+^r(T^n)$ denote the group of orientation preserving C^r diffeomorphisms of the n-torus isotopic to the identity, $T^n = \mathbb{R}^n/\mathbb{Z}^n$ where $n \geq 1$ and $1 \leq r \leq \infty$. If we lift T^n to $\mathbb{R}^n$ we can also lift $f \in \mathrm{Diff}_+^r(T^n)$ to $\tilde{f}:\mathbb{R}^n \to \mathbb{R}^n$, where $\tilde{f} = \mathrm{Id} + \phi$, and ϕ is $\mathbb{Z}^n$-periodic. We define the set

$$D^r(T^n) = \{\tilde{f}:\mathbb{R}^n \to \mathbb{R}^n \mid \tilde{f} \text{ is a diffeomorphism}, \tilde{f} = \mathrm{Id} + \phi,\ \phi \text{ is } \mathbb{Z}^n\text{-periodic}\} .$$

We define the C^r norm for $1 \leq r < \infty$ by:

$$||f||_r = ||f||_0 + ||Df||_0 + \dots + ||D^r f||_0 \quad ,$$

and if $r = \infty$, we define the metric:

$$\|f-g\|_\infty = \sum_{r=0}^{\infty} 2^{-r} \frac{||f-g||_r}{1+||f-g||_r} .$$

Then $D^r(T^n)$ is a topological group with centre $C = \{R_p(x) = x+p \mid p \in \mathbb{Z}^n\}$. Clearly we have $\mathrm{Diff}_+^r(T^n) \cong \dfrac{D^r(T^n)}{C}$ for each $1 \leq r \leq +\infty$. We will usually write $f \in D^r(T^n)$ for both the map f and its lifting, unless confusion arises.

The rotation number of $f \in D^r(T^1)$, denoted $\rho(f)$, is defined to be the uniform limit of the sequence $\{\dfrac{f^k - Id}{k}\}$, $k \in \mathbb{N}$. We normalise $\rho(f)$ so that $0 \leq \rho(f)(=\rho(\tilde{f})) < 1$. It is well-known that ρ is invariant under conjugation (i.e., $\rho(f) = \rho(h^{-1} \circ f \circ h)$ for every $f \in D^r(T^1)$ and every $h \in D^0(T^1))$, and that $\rho : D^r(T^1) \to \mathbb{R}/\mathbb{Z}$ is continuous with respect to the C^r topology. Furthermore, $f \in D^r(T^1)$ has periodic points if and only if $\rho(f)$ is rational. A full discussion of the rotation number and its properties is given in [8]. We write $F_\alpha^r = \{f \in D^r(T^1) \mid \rho(f) = \alpha\}$, and denote $F^r(T^1) = D^r(T^1) - \mathrm{int}\, \rho^{-1}(\mathbb{Q})$, (where int denotes interior).

Consider $\alpha \in \mathbb{R}-\mathbb{Q}/\mathbb{Z}$, and recall the continued fraction expansion of α given by:

$$\alpha = [a_1, a_2, \ldots, \ldots]$$

$$= \frac{1|}{a_1} + \frac{1|}{a_2} + \ldots \quad .$$

The $n^{\underline{th}}$ convergent of α is denoted by p_n/q_n, where p_n and q_n satisfy the well-known recursion formulae: $p_0 = 0$, $p_1 = q_0 = 1$, $q_1 = a_1$, $p_n = a_n p_{n-1} + p_{n-2}$, and $q_n = a_n q_{n-1} + q_{n-2}$, for $n \geq 2$. We can now define the set A for which Herman's Theorem is true.

Definition 1.1.

Let $\alpha \in \mathbb{R}-\mathbb{Q}/\mathbb{Z}$, $\alpha = [a_1,\dots,a_n,\dots]$. We say that α satisfies condition $\mathcal{A}$ if for every $\varepsilon > 0$, there exists $B > 0$ such that

$$\prod_{\substack{(1+a_i)(1+a_{i+1}) > B \\ 1\leq i\leq n}} (1+a_i)(1+a_{i+1}) = O(q_n^{\varepsilon}) \quad \text{as} \quad n \to \infty .$$

We now define:

$$A = \{\alpha \in [0,1] - \mathbb{Q} \mid \alpha \text{ satisfies condition } \mathcal{A}\} ;$$

it is shown in [8] that $m(A) = 1$. We say that α is of constant type if $\sup_{n\geq 1} a_n < \infty$.

We define the set $\Omega = \{\alpha \in A \mid \alpha \text{ is not of constant type}\}$.

We state Herman's Theorem since it will be referred to throughout the thesis.

Theorem 1.2.

If $f \in D^r(T^1)$, $3 \leq r \leq \infty$ and $\rho(f) \in A$, then f is C^{r-2} conjugate to R_α . If $r = \infty$, then f is C^∞ conjugate to R_α .

§2. Ergodic Theory Preliminaries.

Let $(X,\mathcal{S},\mu)$ denote a Borel space where μ is a probability measure on $(X,\mathcal{S})$. Let f denote a non-singular ergodic transformation of $(X,\mathcal{S},\mu)$, i.e. every f-invariant set $B \in \mathcal{S}$ satisfies either $\mu(B) = 0$ or $\mu(B) = 1$. We define the set $\mathrm{Aut}(X,\mathcal{S},\mu) = \{T:(X,\mathcal{S})\circlearrowleft$ such that T is a non-singular Borel automorphism of $(X,\mathcal{S})\}$,

and let

$$O_f(x) = \{f^n x : n \in \mathbb{Z}\} .$$

The full group of f is defined by

$$[f] = \{V \in \mathrm{Aut}(X,\mathcal{S},\mu) : Vx \in O_f(x) \text{ for every } x \in X\} .$$

Definition 2.1.

Two transformations $f,g \in \mathrm{Aut}(X,\mathcal{S},\mu)$ are weakly equivalent if there exists a measurable invertible map $\psi : X \to X$ with $\psi_*^{-1}\mu \sim \mu$ and $\psi(O_f(x)) = O_g(\psi x)$ for μ-a.e. $x \in X$.

We now introduce an invariant of weak equivalence.

Definition 2.2.

Let $f \in \mathrm{Aut}(X, \mathfrak{S}, \mu)$ be an invertible, ergodic transformation. A non-negative real number t is said to lie in the ratio set of f, $r^*(f)$, if for every Borel set $B \in \mathfrak{S}$ with $\mu(B) > 0$, and for every $\varepsilon > 0$,

$$\mu\Big(\bigcup_{n \in Z} \Big(B \cap f^n B \cap \Big\{x \in X : \Big| \frac{d\mu f^{-n}}{d\mu}(x) - t \Big| < \varepsilon\Big\}\Big)\Big) > 0 .$$

Here $\frac{d\mu f^{-n}}{d\mu}$ denotes the Radon-Nikodym derivative of $f^n_* \mu$ with respect to μ. We set $r(f) = r^*(f) \setminus \{0\}$. One can show that $r(f)$ is a closed subgroup of the multiplicative group of positive real numbers $\mathbb{R}^+$, and that f admits a σ-finite invariant measure if and only if $r^*(f) = \{1\}$ [13]. If f has no σ-finite invariant measure equivalent to μ, there are three possibilities:

(1) $r^*(f) = \{t \in R : t \geq 0\}$, in which case f is said to be of type III_1;

(2) $r^*(f) = \{0\} \cup \{\lambda^n : n \in \mathbb{Z}\}$ for $0 < \lambda < 1$; in this case f is said to be of type III_λ; or,

(3) $r^*(f) = \{0, 1\}$. Then f is of type III_0.

The ratio set is actually an example of the set of essential values for a particular cocycle for f . We shall briefly introduce these more general concepts from the study of non-singular group actions on measure spaces. For the purposes of this thesis, we will give the definitions in the differentiable context; for the most general definitions, we refer the reader to [16].

Let $(X,\mathfrak{S},\mu)$ denote a C^∞ , compact manifold with smooth probability measure μ . Let $f \in \mathrm{Diff}^\infty(X)$ be μ-ergodic (it is always non-singular) and let H be a locally compact second countable abelian group. The action $(n,x) \mapsto f^n(x)$ of $\mathbb{Z}$ on X is clearly non-singular since for every $n \in \mathbb{Z}$, $x \mapsto f^n x$ is a Borel automorphism of X which leaves μ quasi-invariant.

<u>Definition 2.3</u>.

A Borel map $a:\mathbb{Z} \times X \to H$ is called a <u>cocycle</u> for the $\mathbb{Z}$-action of f on X if the following condition holds:

For every $n,m \in \mathbb{Z}$ and for every $x \in X$, we have

$$a(n,f^m x) - a(n+m,x) + a(m,x) = 0 \ .$$

A cocycle $a:\mathbb{Z} \times X \to H$ is called a <u>coboundary</u> if there exists a Borel map $b:X \to H$ with $a(n,x) = b(f^n x) - b(x) \quad n \in \mathbb{Z}$, for μ-a.e.

$x \in X$. Two cocycles a_1 and a_2 are said to be <u>cohomologous</u> if their difference is a coboundary.

The following defines a cohomology invariant which generalises the concept of the ratio set.

<u>Definition 2.4</u>.

Let $(X,\mathfrak{S},\mu)$ be as above, f an automorphism which acts non-singularly and ergodically on $(X,\mathfrak{S},\mu)$ and let $a: \mathbb{Z} \times X \to H$ be a cocycle. An element $\alpha \in \bar{H} = H \cup \{\infty\}$ is called an <u>essential value</u> of a if, for every Borel set $B \in \mathfrak{S}$ with $\mu(B) > 0$ and for every neighbourhood $N(\alpha)$ of α in $\bar{H}$,

$$\mu(B \cap f^{-n} B \cap \{x: a(n,x) \in N(\alpha)\}) > 0$$

for some $n \in \mathbb{Z}$. The set of essential values is denoted by $\bar{E}(a)$, and we put $E(a) = \bar{E}(a) \cap H$. We will state a few well-known properties of $\bar{E}(a)$.

(1) $\bar{E}(a)$ is a non-empty closed subset of $\bar{H}$;

(2) $E(a)$ is a closed subgroup of H;

(3) $\bar{E}(a) = \{0\}$ if and only if a is a coboundary.

(4) $\bar{E}(a_1) = \bar{E}(a_2)$ whenever a_1 and a_2 are cohomologous.

CHAPTER II.

Rotation Numbers of Type III_1 Diffeomorphisms of T^1.

§3. Introduction.

In [11], Katznelson constructs ergodic C^∞ diffeomorphisms of the circle which do not admit any σ-finite measure equivalent to Lebesgue measure. The rotation numbers of these diffeomorphisms are Liouville numbers of a very special form and are, in particular, contained in a subset of the circle of Lebesgue measure zero. Subsequently Herman [7] proved that the set of C^∞ diffeomorphisms of the circle, which are of type III_1 form a dense G_δ in F^∞. Herman's Theorem implies, however, that the rotation numbers of all these type III_1 diffeomorphisms lie in the complement of the set A defined in Definition 1.1. Herman's Theorem also implies that the situation is basically unchanged if one looks at C^3-diffeomorphisms of the circle. His methods suggest that C^2-diffeomorphisms might exhibit a different behavior, and this is the starting point for this chapter. Using a refinement of Katznelson's construction, we obtain, for a.e. $\alpha \in [0,1)$, a C^2-diffeomorphism of the circle of type III_1 with rotation number α. The main result of this chapter is the following:

Theorem 3.1.

For every $\alpha \in \Omega$, there exists a diffeomorphism $f \in D^2(T^1)$ with $\rho(f) = \alpha$, which is of type III_1.

Theorem 3.1 will be a consequence of a slightly stronger assertion (Proposition 4.4.), which in turn has an interesting topological implication. In Section 5 we prove that, for every $\alpha \in A$, the set of type III_1-diffeomorphisms is a dense G_δ in $F^2_\alpha(T^1)$.

In [8, XI], Herman posed the following problem:
If α is a number of constant type, and if $2 \leq r < +\infty$, does there exist $f \in F^r_\alpha$ such that f is not C^{r-1} conjugate to R_α? The answer to this question is still not known since Ω does not contain numbers of constant type.

Theorem 3.1 also sheds some light on a related problem. If α is of constant type and if $\phi \in C^1(T^1)$ with $\int_{T^1} \phi(x)dx = 0$, then there exists $\psi \in L^2(T^1,m)$ satisfying $\psi - \psi o R_\alpha = \phi$ m-a.e. We have proved that for each $\alpha \in \Omega$ there exist uncountably many $f \in F^2$, and for each f a function $\phi \in C^1(T^1)$ with $\int_{T^1} \phi(x)dm = 0$ which is such that the equation $\psi - \psi o f = \phi$ has no m-measurable solution.

Later in the thesis we see that if $\mathcal{B} = \overline{\{\phi \in C^1(T^1) | \phi = \psi - \psi o f\}}$, (where the closure is taken with respect to the C^1 topology) and if there exists a single ϕ in $\mathcal{B}$ with the above property then there is a dense G_δ in $\mathcal{B}$ with the same property.

§4. The Proof of Theorem 3.1.

Let $f \in D^2(T^1)$ with $\alpha = \rho(f) \in T^1 \setminus Q$, and let $\alpha = [a_1, a_2, \ldots]$ be the continued fraction expansion of $\rho(f)$. For every $n \geq 0$, $P_n(f)$ denotes the partition of T^1 given by the points $\{f^j(0): 0 \leq j \leq q_n - 1\}$. f sends each interval in $P_n(f)$ onto some other interval in the same partition, with the exception of two subintervals. Furthermore, if $I \in P_n(f)$, and if $J \subset I$ is one of the subintervals in the partition of I defined by $P_{n+1}(f)$, then f^{q_n} sends J onto one of its two neighbours in I, except possibly when J lies at one of the two ends of I. A much more detailed discussion can be found in [11]. We now turn to the proof of Theorem 3.1 and start with a measure theoretic proposition closely related to [11,II, Theorem 1.1].

Proposition 4.1.

Let $f \in D^2(T^1)$ with $\alpha = \rho(f) \in T^1 \setminus Q$, and let, for every $n \geq 1$, $P_n(f)$ be the partition of T^1 described above. Suppose that the following condition holds for infinitely many n: for every $I \in P_n(f)$ there exists a Borel set $C \subset I$ and a positive integer j_o with

$$f^{j_o}(C) \subset I \tag{4.1}$$

$$m(C) \geq 10^{-4} m(I) \tag{4.2}$$

and such that

$$10^{-3} \leq |\log Df^{j_0}(t)| \leq 1 \tag{4.3}$$

for every $t \in \mathbf{C}$. Then f admits no σ-finite invariant measure equivalent to m.

Proof.

If f admits a σ-finite invariant measure m' equivalent to m, we have $dm'(t) = g(t)dm(t)$ for some Borel function $g > 0$ on T^1 which satisfies $g \circ f(t) \cdot Df(t) = g(t)$ for m a.e. $t \in T^1$. There exists a $c > 0$ such that the set $E = \{t \in T^1 : c \leq g(t) \leq 1.001c\}$ has positive measure. Let t_0 be a point of density of E. By definition, we have

$$\lim_{\substack{n \to \infty \\ t_0 \in I \in P_n(f)}} \frac{m(I \cap E)}{m(I)} = 1 \quad .$$

Choose n large enough so that, for $t_0 \in I \in P_n(f)$,

$$m(I \cap E) > (1-10^{-6})m(I) \quad ,$$

and such that n is one of the numbers satisfying (4.1) - (4.3).

From (4.2) and (4.3) we get

$$\frac{1}{3}m(C) \leq m(f^{j_o}(C)) \leq 3m(C) \quad ,$$

$$m(f^{j_o}(C)) \geq 3^{-1}\ 10^{-4}m(I) \quad ,$$

$$m(C\cap E) \geq m(C) - m(I\setminus E) = m(C) - (m(I)-m(I\cap E))$$

$$> m(C) - 10^{-6}m(I) \geq (1-10^{-2})m(C) \ ,$$

and

$$m(f^{j_o}(C) \cap E) \geq m(f^{j_o}(C))-m(I\setminus E) > m(f^{j_o}(C))-10^{-6}m(I)$$

$$> m(f^{j_o}(C)) - 3\cdot 10^{-2}m(f^{j_o}(C))$$

$$= (1-3\cdot 10^{-2})m(f^{j_o}(C)) \ .$$

This implies

$$m(f^{-j_o}(f^{j_o}(C) \cap E)) > 3^{-1}m(f^{j_o}(C) \cap E)$$

$$> 3^{-1}\cdot(1-3\cdot 10^{-2})m(f^{j_o}(C))$$

$$> 3^{-2}\cdot(1-3\cdot 10^{-2})m(C) \ ,$$

and hence

$$m(C \cap E \cap f^{-j_0}(E)) \geq m(C \cap f^{-j_0}(E)) - m(C \setminus E)$$

$$> 3^{-2}(1-3\cdot 10^{-2})m(C)-(m(C)-m(C\cap E))$$

$$> 3^{-2}(1-3\cdot 10^{-2})m(C)-10^{-2}m(C)$$

$$> 0.09m(C) > 0 .$$

Put $B = C \cap E \cap f^{-j_0}(E)$. $m(B) > 0$, and we get

$$10^{-3} \leq |\log Df^{j_0}(t)|$$

for every $t \in B$, from (4.3). On the other hand,

$$Df^{j_0}(t) = \frac{g\circ f^{j_0}(t)}{g(t)} ,$$

and hence

$$|\log Df^{j_0}(t)| = |\log g\circ f^{j_0}(t) - \log g(t)|$$

$$\leq \log 1.001 < 10^{-3} ,$$

since both t and $f^{j_0}(t)$ lie in E whenever $t \in B$. This contradiction proves the non-existence of an invariant measure $m' \sim m$.

Remark 4.2.

Any reader familiar with the notion of the ratio set $r^*(f)$ (cf. I, §2) will realize that we have just proved that $r^*(f) \neq \{1\}$. A closer look at the proof of Proposition 4.1. shows that $r(f) \neq \{1\}$, so that the diffeomorphism f must either be of type III_λ for some λ with $1 > \lambda \geq \frac{1}{e}$, or of type III_1. If $f: T^1 \to T^1$ or $f: T^1 \to \mathbb{R}$ is a C^r-map, we denote by $||f||_r$ its C^r-norm.

Proposition 4.3.

Let $\alpha \in \Omega$ (cf. I, 1.1), and let $0 < \delta, \eta < 1$ and $N \geq 1$ be fixed. There exists a diffeomorphism $f \in D^\infty(T^1)$ and integers j_0, M such that $M > N$ and $1 \leq j_0 < q_{M+1}$ which satisfy

$$\rho(f) = \alpha, \tag{4.4}$$

$$P_n(f) = P_n(R_\alpha) \quad \text{for} \quad 0 \leq n \leq M, \tag{4.5}$$

$$||f - R_\alpha||_2 < \eta, \tag{4.6}$$

$$||\log Df||_0 \leq 10^{-1}.\delta.n.d_{-1} \quad , \tag{4.7}$$

and for every $I \in P_M(f)$ there exists an interval $C \subset I$ such that $f^{j_o}(C) \subset I$, $m(C) \geq 2.10^{-4}\ m(I)$, and

$$|\log Df^{j_o}(t)| \geq 2.10^{-3}\delta \text{ for every } t \in C\,. \tag{4.8}$$

Furthermore we have

$$||\log Df^{j_o}||_0 \leq 0.5\delta \quad . \tag{4.9}$$

Proof.

Let $\alpha = [a_1, a_2, \ldots]$ be the continued fraction expansion of $\alpha \in A$, and let, for every $n \geq 1$, $d_n = ||q_n\alpha||$, where $||\cdot||$ denotes the distance from the nearest integer. For every $n \geq 1$, the partition $P_n(R_\alpha)$ consists of q_n intervals; q_{n-2} of these have length $d_{n-1} + d_n$, and $a_n q_{n-1}$ have length d_{n-1} . The partition $P_{n+1}(R_\alpha)$ divides each interval $I \in P_n(R_\alpha)$ into either a_{n+1} or $a_{n+1} + 1$ subintervals, all of which have length d_n , except for one interval at one of the ends of I , which will have length $d_n + d_{n+1}$. From this discussion we conclude that, for every $n \geq 1$,

$$1/2q_n \leq d_{n-1} \leq 1/q_n \ . \tag{4.10}$$

We now fix $M > N$ with

$$a_{M+1} \geq 10^3 \ \eta^{-1},$$

where η, N are the numbers appearing in the statement of this proposition. (2.10) implies

$$d_M \leq q_{M+1}^{-1} \leq a_{M+1}^{-1} q_M^{-1} \leq 2.10^{-3} \eta \ d_{M-1},$$

so that the longer intervals in $P_M(R_\alpha)$, which have length $d_{M-1}+d_M$, differ from the shorter ones (of length d_{M-1}) by at most $2.10^{-3} d_{M-1}$. Let $\psi_1 : T^1 \to \mathbb{R}$ be the function

$$\psi_1(t) = \begin{cases} \frac{1}{2}\left(1-\cos\left(\frac{64}{15}\pi \, d_{M-1}^{-1} \cdot \left(t - \frac{d_{M-1}}{64}\right)\right)\right) & \text{for } \frac{d_{M-1}}{64} \leq t \leq \frac{31 d_{M-1}}{64} \\ \\ 0 & \text{otherwise.} \end{cases}$$

We choose and fix a C^∞-function $\psi_2 : T^1 \to R$ such that $\psi_2(t) \geq 0$ for every $t \in T^1$, $\int \psi_2 \, dm = 1$, $\psi_2(t) = \psi_2(1-t)$, and $\psi_2(t) = 0$ for $10^{-3} d_{M-1} \leq t \leq 1-10^{-3} . d_{M-1}$. The function $\psi_+ : T^1 \to R$ will denote the convolution $\psi_1 * \psi_2$, and we set $\psi_-(t) = \psi_+(1-t)$. Let now, for every $I \in P_M(R_\alpha)$, $\gamma(I)$ denote the mid-point of the interval I, and put

$$\phi_+(t) = \sum_{I \varepsilon P_M(R_\alpha)} \psi_+(t-\gamma(I)) \ ,$$

$$\phi_-(t) = -\sum_{I \varepsilon P_M(R_\alpha)} \psi_-(t-\gamma(I)) \ .$$

Clearly ϕ_+ and ϕ_- are C^∞-functions, and they are easily seen to satisfy the following conditions.

$$||\phi_+||_0 = ||\phi_-||_0 \leq 1 \ , \tag{4.11}$$

$$||D\phi_+||_0 = ||D\phi_-||_0 \leq \frac{32}{15} \pi d^{-1}_{M-1} < 7 \ d^{-1}_{M-1} \ , \tag{4.12}$$

$$||D^2\phi_+||_0 = ||D^2\phi_-||_0 \leq \frac{1}{2} \left(\frac{64}{15} \pi d^{-1}_{M-1}\right)^2 < 90 \ d^{-2}_{M-1} \ , \tag{4.13}$$

and

$$\phi_+(\ell\alpha) = \phi_-(\ell\alpha) = 0 \quad \text{for} \quad 0 \leq \ell \leq q_M-1 \ . \tag{4.14}$$

We conclude that

$$||\phi_+||_2 = ||\phi_-||_2 < 100 d^{-2}_{M-1} \ .$$

For every $I \ \varepsilon \ P_M(R_\alpha)$, we have

$$D\phi_+(t) \geq 10^{-1} d_{M-1}^{-1} \text{ for every } t \in J_1(I) = [\gamma(I)+0.018 d_{M-1}, \gamma(I) + 0.247 d_{M-1}] \tag{4.15}$$

and

$$D\phi_-(t) \leq -10^{-1} d_{M-1}^{-1} \text{ for every } t \in J_2(I) = [\gamma(I)-0.247 d_{M-1}, \gamma(I) - 0.018 d_{M-1}] \ . \tag{4.16}$$

Consider now, for every $c \in [-1,1]$,

$$\phi_c(t) = \begin{cases} \phi_+(t) + (1-c)\ \phi_-(t) & \text{for } 0 \leq c \leq 1 , \\ \phi_-(t) + (1+c)\ \phi_+(t) & \text{for } -1 \leq c \leq 0 , \end{cases}$$

and put

$$f_c(t) = t+\alpha + 10^{-2}.\delta.\eta.d_{M-1}^{2}. \quad \phi_c(t) \ . \tag{4.17}$$

Since the functions ϕ_+ and ϕ_- have disjoint supports, it is clear that the relations (4.11) - (4.14) hold with ϕ_c , $c \in [-1,1]$, replacing ϕ_+ or ϕ_- . In particular we note that f_c can be considered as a diffeomorphism of T^1 , and that $||f_c - R_\alpha||_2 < \eta$

for every $c \in [-1,1]$. In our next step we fix the value of c. An easy estimate shows that $\rho(f_{+1})$ and $\rho(f_{-1})$ lie on different sides of α. Since the map $c \to f_c$ is continuous from $[-1,1]$ to $D^{\infty}(T^1)$, there exists a $c_0 \in [-1,1]$ with $\rho(f_{c_0}) = \alpha$, and we put $f = f_{c_0}$. So far we have proved that f satisfies (4.4) and (4.6). (4.5) follows from (4.14) and (4.17), and (4.7) is clear from the inequalities:

$$
\begin{aligned}
&0.95x \leq \log(x+1) \leq x \quad \text{for } 0 \leq x \leq 0.1 \\
\text{and} \qquad & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.18) \\
&1.06x \leq \log(x+1) \leq x \quad \text{for } -0.1 \leq x \leq 0 ,
\end{aligned}
$$

and from (4.12) and (4.17). In order to prove (4.8) and (4.9), we set $\ell_0 = \mathrm{Int}\left(\frac{5}{\eta}\right)$ and

$$j_0 = \ell_0 q_M \quad \text{, where Int denotes the integral part.} \qquad (4.19)$$

Then (4.12), (4.17) and (4.18) yield

$$
\begin{aligned}
\| \log Df^{j_0} \|_0 \leq j_0 \cdot \| \log Df \|_0 &\leq 1.06 . j_0 . 10^{-2} \delta\eta \, d_{M-1}^{2} \quad \| D\phi_c \|_0 \\
&\leq 1.06 \cdot 7 \cdot 10^{-2} . \delta\eta \, d_{M-1} \cdot j_0 \\
&\leq 10^{-1} \, \delta\eta \, d_{M-1} . 5\eta^{-1} q_M \\
&\leq 0.5 \, \delta ,
\end{aligned}
$$

by (4.10). Having proved (4.9), we consider (4.8) and assume first that $c_o \geq 0$. Put

$$J_1'(I) = [\gamma(I) + 0.132\, d_{M-1}, \gamma(I) + 0.133\, d_{M-1}] \quad .$$

The inequalities

$$|f^k(t) - R_\alpha^k(t)| \leq k.10^{-2}\ \delta n\ d_{M-1}^2 ||\phi_{c_o}||_o$$

$$\leq \ell_o q_M \cdot 10^{-2}\ \delta n\ d_{M-1}^2$$

$$\leq 5.10^{-2}\ d_{M-1} ,$$

where $0 \leq k \leq j_o$ and

$$|f^{j_o}(t) - t| \leq \ell_o\ R_\alpha^{q_M}(t) - t| + \ell_o q_M\ \delta n\ d_{M-1}^2 . 10^{-2}$$

$$\leq \ell_o(d_M + \delta n\ d_{M-1} . 10^{-2})$$

$$\leq 10^{-2}\ d_{M-1}$$

imply the following: for every $I \in P_M(R_\alpha)$ and for every k with $0 \leq k \leq j_o$ there exists an interval $\tilde{I} \in P_M(R_\alpha)$ with

$$f^k(J_1'(I)) \subset J_1(\tilde{I}) \ . \tag{4.20}$$

In the particular case $k = j_o$ we get $\tilde{I} = I$. For later reference we give here a more precise estimate: if $I \in P_M(R_\alpha)$, one has

$$f^{j_o}(J_1'(I)) \subset [\gamma(I) + 0.07\ d_{M-1}, \gamma(I) + 0.2\ d_{M-1}] \ . \tag{4.21}$$

From (4.7) and (4.18) we have

$$0.95 \cdot 10^{-2} \delta n d_{M-1} d_{M-1} \sum_{k=0}^{j_o-1} D\phi_{c_o}(f^k(t))$$

$$\leq \sum_{k=0}^{j_o-1} \log\ (1 + 10^{-2}\ \delta n d_{M-1} d_{M-1} D\phi_{c_o}(f^k(t)))$$

$$= \sum_{k=0}^{j_o-1} \log\ Df\ (f^k(t)) = \log\ Df^{j_o}(t) \ ,$$

and we can apply (4.15) and (4.20) to get

$$0.002375\delta \leq 0.95 \cdot 10^{-3}\ \delta n\ j_o d_{M-1} \leq \log\ Df^{j_o}(t)$$

for every $t \in C = J_1'(I)$, $I \in P_M(R_\alpha)$. This proves (4.8) under the assumption that $c_o \geq 0$. If $c_o < 0$, one uses $J_2(I)$, defined in (4.16), chooses $J_2'(I) = [\gamma(I) - 0.133d_{M-1}, \gamma(I) - 0.132d_{M-1}]$, and obtains

$$-0.0025\delta \geq \log Df^{j_o}(t)$$

for every $t \in C = J_2'(I)$, $I \in P_M(R_\alpha) = P_M(f)$. Again we have verified (4.8). The expression (4.21) is now replaced by

$$f^{j_o}(J_2'(I)) \subset [\gamma(I) - 0.2\, d_{M-1}, \gamma - 0.07\, d_{M-1}] \quad . \tag{4.22}$$

The proof of Proposition 4.3 is complete.

Proposition 4.4.

Let $\alpha \in \Omega$ and $0 < \varepsilon < 1$ be fixed. There exists a diffeomorphism $f_o \in D^2(T^1)$ satisfying the following conditions.

$$\rho(f_o) = \alpha , \tag{4.23}$$

$$||f_o - R_\alpha||_2 < \varepsilon , \tag{4.24}$$

$$f_o \text{ is of type } III_1 . \tag{4.25}$$

Proof.

Using an induction argument we shall construct a sequence $(f_n) \subset D^\infty(T^1)$ which converges in $D^2(T^1)$ to a limit f_o, and f_o will satisfy (4.23) - (4.25). For this construction we choose and fix a sequence $(\delta_n : n \geq 1)$ of real numbers satisfying

$$0 < \delta_n \leq 1 \quad \text{for every} \quad n \geq 1 \, ,$$
$$\delta_{2n} = 1 \quad \text{for every} \quad n \geq 1 \, , \tag{4.26}$$

and

$$\text{the set } \{\delta_n : n \geq 1 \text{ is dense in } [0,1] \, . \tag{4.27}$$

The sequence $(f_n) \subset D^\infty(T^1)$ will be obtained through repeated applications of Proposition 4.3: Given $\delta_\ell, \eta_\ell, N_\ell$ we use Proposition 4.3 to define a function $f = \tilde{f}_{\ell+1} \in D^\infty(T^1)$ satisfying (4.4) - (4.9), and we put $j_o = j_{\ell+1}$, $M = M_{\ell+1}$, and $\mathfrak{C} = \mathfrak{C}_{\ell+1}(I) \subset I$, for every $I \in P_{M_{\ell+1}}(\tilde{f}_{\ell+1})$. $N_{\ell+1}$ will then be chosen depending on $M_{\ell+1}$ and $\tilde{f}_{\ell+1}$. To start the process, let $f_1 = R_\alpha$, $M_1 = 0, N_1 = 1$, $\eta_1 = \delta_1 \cdot \varepsilon \cdot 2^{-1} 10^{-4} a_1^{-1}$, and apply Proposition 4.3 with $\delta = \delta_1$, $\eta = \eta_1$, $N = N_1$ to get $f = \tilde{f}_2 = f_2$, $M = M_2$, $j_o = j_2$, and

$\tilde{C}_2(I) = C_2(I) \subset I$ for every $I \in P_{M_2}(f_2)$, satisfying (4.4) - (4.9).

Suppose now that we have constructed $f_1,\ldots,f_\ell$, $\eta_1,\ldots,\eta_{\ell-1}$, $M_1,\ldots,M_\ell$, $N_1,\ldots,N_{\ell-1}$. By Herman's Theorem (cf. I, §1) there exists $h_\ell \in \mathrm{Diff}^\infty(T^1)$ with $f_\ell = h_\ell^{-1} R_\alpha h_\ell$. Choose $N_\ell \geq M_\ell$ such that

$$\max_{I \in P_{N_\ell}(\tilde{f}_\ell)} \sup_{t_1,t_2 \in I} |Dh_\ell^{-1}(t_1) - Dh_\ell^{-1}(t_2)| < 3/5||h_\ell||_1 \ , \tag{4.28}$$

and

$$\max_{I \in P_{N_\ell}(\tilde{f}_\ell)} \sup_{t_1,t_2 \in I} |\log Dh_\ell^{-1}(t_1) - \log Dh_\ell^{-1}(t_2)| < 10^{-4}\delta_{\ell+1}. \tag{4.29}$$

We put $h_1 = \mathrm{id}$, define

$$\eta_\ell = (\min_{1 \leq n \leq \ell} \delta_n . (1+||h_n||_3 + ||h_n^{-1}||_3)^{-2}).\varepsilon.2^{-\ell}10^{-5}.q^{-1}_{M_\ell+1} \tag{4.30}$$

and apply Proposition 4.3 with $\eta = \eta_\ell$, $\delta = \delta_\ell$, $N = N_\ell$ to obtain $f = \tilde{f}_{\ell+1}$, $j_0 = j_{\ell+1}$, $M = M_{\ell+1}$ and $\mathbf{C} = \tilde{\mathbf{C}}_{\ell+1}(I) \subset I$, $I \in P_{M_{\ell+1}}(\tilde{f}_{\ell+1})$ satisfying (4.4) - (4.9). The general inequality

$$||h^{-1}R_\alpha h - h^{-1}g\,h||_2 \leq 10||R_\alpha - g||_2.(1+||h||_3 + ||h^{-1}||_3)^2 \ , \tag{4.31}$$

$h \in \mathrm{Diff}^3(T^1)$, $g \in D^2(T^1)$, implies that

$$f_{\ell+1} = h_\ell^{-1} \tilde{f}_{\ell+1} h_\ell$$

satisfies

$$||f_\ell - f_{\ell+1}||_2 \leq (\min_{1 \leq n \leq \ell} \delta_n)\varepsilon . 2^{-\ell} 10^{-4} q_{M_{\ell+1}}^{-1} . \tag{4.32}$$

For every $I \in P_{M_{\ell+1}}(f_{\ell+1})$, put

$$C_{\ell+1}(I) = h_\ell^{-1} (\tilde{C}_{\ell+1} (h_\ell(I))) . \tag{4.33}$$

From (4.28) one proves easily

$$m(C_{\ell+1}(I)) \geq 10^{-4} m(I) \tag{4.34}$$

for every $I \in P_{M_{\ell+1}}(f_{\ell+1})$.

This induction procedure allows us to define a sequence $(f_\ell : \ell \geq 1)$ in $D^\infty(T^1)$, which satisfies (4.28) - (4.34) for every $\ell \geq 1$. (4.32) implies in particular that (f_ℓ) is a Cauchy sequence in $D^2(T^1)$, and we define

$$f_0 = \lim_\ell f_\ell \tag{4.35}$$

in $D^2(T^1)$. It is now necessary to go through a series of estimates in order to prove that f_o satisfies the required conditions. By construction (4.23) and (4.24) are obvious, but (4.25) is somewhat more difficult to prove. We first observe that, for every $k \leq \ell$,

$$P_{M_k}(f_k) = P_{M_k}(f_\ell) ,$$

so that

$$P_{M_k}(f_k) = P_{M_k}(f_o) \tag{4.36}$$

for every $k \geq 1$. Our next aim is to show that

$$f_o^{j_k} (\mathbb{C}_k(I)) \subset I \tag{4.37}$$

for every $I \in P_{M_k}(f_o)$ and for every $k \geq 2$. Statements (4.21) and (4.22) imply that, for every $k \geq 2$, and for every $I \in P_{M_k}(\tilde{f}_k)$, the interval $\tilde{f}_k^{j_k} (\tilde{\mathbb{C}}_k(I))$ has distance $> 10^{-2} d_{M_k - 1}$ from the endpoints of I. Now (4.33) shows that the interval $f_k^{j_k} (\mathbb{C}_k(I))$ has distance $> 10^{-2} d_{M_k-1} \cdot ||Dh_k||_o^{-1}$ from the endpoints of I, for every $I \in P_{M_k}(f_k)$. We will prove (4.37) if we can show that

$$||f_o^{j_k} - f_k^{j_k}||_o \leq 10^{-2}\, d_{M_k-1}||Dh_k||_o^{-1} \tag{4.38}$$

for every $k \geq 2$. (4.38) will follow from

$$||f_p^{j_k} - f_k^{j_k}||_o \leq 10^{-2}\, d_{M_k-1}\cdot||Dh_k||_o^{-1} \tag{4.39}$$

for every $k \leq p$. To prove (4.39) we first note that (4.7) and (4.30) imply, for every $j = 1,\dots,q_{M_\ell-1}$, $\ell \geq 1$,

$$\begin{aligned} ||\log D\tilde{f}^j_{\ell+1}||_o &\leq j||\log D\tilde{f}_{\ell+1}||_o \\ &\leq j.10^{-1}\, \delta_\ell\, \eta_\ell\, d_{M_\ell-1} \\ &\leq 2^{-\ell}10^{-6}\, q^{-1}_{M_\ell+1}\, (\inf_{1\leq n\leq\ell}(1+||h_n||_3 + ||h_n^{-1}||_3)^{-2}) \\ &< 10^{-6} \quad . \end{aligned}$$

Hence $||\log D\tilde{f}^j_{\ell+1}||_o > 2^{-1}\, ||1 - D\tilde{f}^j_{\ell+1}||_o$ and

$$
\begin{aligned}
||f^j_\ell - f^j_{\ell+1}||_0 &= ||h^{-1}_\ell \, R^j_\alpha \, h_\ell - h^{-1}_\ell \, \tilde{f}^j_{\ell+1} \, h_\ell||_0 \\
&\leq ||Dh^{-1}_\ell||_0 \cdot ||R^j_\alpha - \tilde{f}^j_{\ell+1}||_0 \\
&\leq ||Dh^{-1}_\ell||_0 \cdot ||DR^j_\alpha - D\tilde{f}^j_{\ell+1}||_0 \\
&= ||Dh^{-1}_\ell||_0 \cdot ||1 - D\tilde{f}^j_{\ell+1}||_0 \qquad (4.40)\\
&< 2||Dh^{-1}_\ell||_0 \cdot ||\log D\tilde{f}^j_{\ell+1}||_0 \\
&< 10^{-5} . 2^{-\ell} d_{M_\ell} . (\min_{1\leq n\leq \ell} (1 + ||h_n||_3 + ||h^{-1}_n||_3)^{-1}) \quad .
\end{aligned}
$$

During the last estimate we have used the fact that $\rho(\tilde{f}^j_{\ell+1}) = \rho(R^j_\alpha) = j\alpha$.

Adding up the inequalities (4.40) we get, for every $k < p$,

$$
\begin{aligned}
||f^{j_k}_k - f^{j_k}_p||_0 &\leq \sum_{\ell=k}^{p-1} ||f^{j_k}_\ell - f^{j_k}_{\ell+1}||_0 \\
&\leq 10^{-5} \, d_{M_k} . \left(\min_{1\leq n\leq k} (1 + ||h_n||_3 + ||h^{-1}_n||_3)^{-1} \right. \\
&\leq 10^{-5} \, d_{M_k} . ||Dh_k||^{-1}_0 \quad ,
\end{aligned}
$$

which proves (4.39). We now intend to prove the inequality

$$\delta_k . 10^{-3} \leq |\log Df_o^{j_k}(t)| \leq \delta_k . 0.6 \tag{4.41}$$

for every $k \geq 2$, every $t \in \mathbf{C}_k(I)$, and every $I \in P_{M_k}(f_o)$.
To verify (4.41), we use induction. Our hypothesis is that, for every $\ell = 2,\ldots,p$, and for every $k \leq \ell$,

$$\delta_k(2.10^{-3}-10^{-4}(2-2^{1-\ell})) \leq |\log Df_\ell^{j_k}(t)| \leq \delta_k(0.5+10^{-4}(2-2^{1-\ell})) \tag{4.42}$$

To prove that (4.42) holds for $\ell = p+1$, we proceed exactly as in (4.40):

$$||\log Df_p^{j_k} - \log Df_{p+1}^{j_k}||_o$$

$$= ||\log D(h_p^{-1} R_\alpha^{j_k} h_p) - \log D(h_p^{-1} \tilde{f}_{p+1}^{j_k} h_p)||_o$$

$$\leq ||\log Dh_p^{-1} (R_\alpha^{j_k} h_p) - \log Dh_p^{-1} (\tilde{f}_{p+1}^{j_k} h_p)||_o$$

$$+ ||\log D\tilde{f}_{p+1}^{j_k}||_o$$

$$\leq ||D^2 h_p^{-1}||_o . ||Dh_p||_o . ||R_\alpha^{j_k} - \tilde{f}_{p+1}^{j_k}||_o + j_k . ||\log D\tilde{f}_{p+1}||_o$$

$$\leq \|D^2h_p^{-1}\|_o \cdot \|Dh_p\|_o \cdot \|1 - D\tilde{f}_{p+1}^{j_k}\|_o + j_k\|\log D\tilde{f}_{p+1}\|_o$$

$$\leq \quad j_k \cdot \|\log D\tilde{f}_{p+1}\|_o \cdot (2\|D^2h_p^{-1}\|_o \cdot \|Dh_p\|_o + 1)$$

$$\leq \quad j_k \cdot 10^{-1}\, \delta_p\, \eta_p\, d_{M_p-1} \cdot (2\|D^2h_p^{-1}\|_o \cdot \|Dh_p\|_o + 1)$$

$$\leq \quad j_k \cdot \delta_p \cdot d_{M_p-1} \cdot (2\|D^2h_p^{-1}\|_o\, \|Dh_p\|_o + 1)\,.$$

$$(\min_{1\leq n\leq p} \delta_n \cdot (1 + \|h_n\|_3 + \|h_n^{-1}\|_3)^{-2}).\varepsilon.2^{-p}.10^{-6}.q_{M_p+1}^{-1}$$

$$\leq \quad \delta_k \cdot 2^{-p} \cdot 10^{-6}$$

for every $k = 1,\ldots,p$, since $j_k \leq q_{M_{k+1}} \leq q_{M_{p+1}}$. Adding up these inequalities as before we see that (4.42) holds for $\ell = p+1$ and $k < \ell$. Turning now to $k = p+1$ we have

$$\log Df_{p+1}^{j_{p+1}}(t) = \log(D(h_p^{-1}\, \tilde{f}_{p+1}^{j_{p+1}} h_p))(t)$$

$$= \log Dh_p^{-1}(\tilde{f}_{p+1}^{j_{p+1}}(h_p(t))) + \log Dh_p(t)$$

$$+ \log D\tilde{f}_{p+1}^{j_{p+1}}(h_p(t))$$

$$= \log Dh_p^{-1}(\tilde{f}_{p+1}^{j_{p+1}}(h_p(t)) - \log Dh_p^{-1}(h_p(t))$$

$$+ \log D\tilde{f}_{p+1}^{j_{p+1}}(h_p(t)) \quad .$$

Since $t \in \tilde{C}_{p+1}(I)$ implies $\tilde{f}_{p+1}^{j_{p+1}}(t) \in I$ for every $I \in P_{M_{p+1}}(\tilde{f}_{p+1})$ (cf. 4.8), we can apply (4.29) and (4.33) to get

$$|\log Df_{p+1}^{j_{p+1}}(t) - \log D\tilde{f}_{p+1}^{j_{p+1}}(t)| < 10^{-4} \delta_{p+1}$$

for every $t \in C_{p+1}(I)$, $I \in P_{M_{p+1}}(f_{p+1})$. We have that (4.8) and (4.9) imply (4.42) for $k = p+1$, $\ell = p+1$, so that (4.41) follows by continuity.

Finally we see that (4.34), (4.36), (4.37) and (4.41), together with (4.26), allow us to apply Proposition 4.1 and Remark 4.2 now shows that f_0 is either of type III_1 or of type III_λ for $\lambda < 1$. To prove that f_0 is indeed of type III_1 , we use an indirect argument and assume that f_0 is of type III_λ . By (4.27), the set

$$S = \{n \geq 1: - \log \lambda/2 \leq \delta_n \leq - \log \lambda\}$$

is infinite. A routine calculation, involving the Lebesgue density theorem, (4.34), (4.36), (4.37) and (4.41) (for $k \in S$) now implies that $r(f_o)$ must contain an element γ satisfying

$$1 < e^{-10^{-3} \log \lambda/2} \leq \gamma \leq e^{-0.6 \log \lambda} < \lambda^{-1} ,$$

which is absurd. This contradiction proves that f_o satisfies (4.25), and the proposition is completely proved.

Remark 4.5.

Proposition 4.4 implies Theorem 3.1.

§5. Topological Properties of the set of Type III_1 - Diffeomorphisms.

Motivated by [8,II.4], we define, for $r,s \geq 0$, $\alpha \in T^1$,

$$O^{r,s} = \{g^{-1}R_\alpha g \in F^r_\alpha(T^1) : g \in D^s(T^1)\} ,$$

$$O^{r,ac}_\alpha = \{g^{-1}R_\alpha g \in F^r_\alpha(T^1) : g \text{ is absolutely continuous with respect to Lebesgue measure } m\} ,$$

$$O^r_\alpha = \{ g^{-1}R_\alpha g : g \in D^r(T^1)\} ,$$

and

$$F^{r,m} = \{f \in F^r_\alpha(T^1) : f \text{ admits a } \sigma\text{-finite invariant measure equivalent to } m\} .$$

In a more general form Herman's Theorem states that, for every $\alpha \in A$, and for every $r \geq 3$,

$$O^{r,r-2}_\alpha = O^{r,ac}_\alpha = F^{r,m}_\alpha = F^r_\alpha .$$

Furthermore Herman has shown that O_α^r is meagre in F_α^r, for every $\alpha \in T^1\backslash Q$, and for every r with $1 \leq r < \infty$. For $\alpha \in T^1\backslash Q$, α not of constant type, a stronger assertion holds: $O_\alpha^{r,r-1}$ is meagre in $\overline{O_\alpha^r}$, where the bar denotes closure in the C^r-topology (cf. [8,XI.4]).

In this section we shall indicate some consequences of Proposition 4.5 concerning the 'size' of $O_\alpha^{2,ac}$ and $F_\alpha^{2,m}$ in F_α^2, which will improve Herman's result in this special case. Our main assertion in this section is the following:

Proposition 5.1.

For all $\alpha \in \Omega$, the set

$$G_\alpha = \{f \in F_\alpha^2 : f \text{ is of type } III_1\} \tag{5.1}$$

is a dense G_δ in F_α^2.

Corollary 5.2.

$F_\alpha^{2,m}$ and $O_\alpha^{2,ac}$ are meagre subsets of F_α^2.

We shall prove Proposition 5.1 by applying the lemmas below.

Lemma 5.3.

For $\alpha \in A$, $0 \leq r \leq \infty$, $\overline{O^r_\alpha} = F^r_\alpha$.

Proof.

This is a trivial consequence of Herman's Theorem since F^∞_α is dense in F^r_α for every $r = 0,\ldots,\infty$.

Lemma 5.4.

Let $f \in F^r_\alpha$, $r \geq 2$ and assume that f is of type III_1. Then $h^{-1} \circ f \circ h$ is of type III_1 for every $h \in D^0(T^1)$ which is absolutely continuous with respect to m .

Proof.

The ratio set is an invariant of weak equivalence and hence of conjugacy by non-singular automorphisms.

Lemma 5.5.

The set G_α in (5.1) is a G_δ in F^2_α , for every $\alpha \in T^1 \setminus Q$.

Proof.

By [8, Ch.V], the set $G = \{f \in F^2 : f$ is of type $III_1\}$ is a dense G_δ in $F^2 = D^2(T^1)\setminus\text{int}\ \rho^{-1}(\mathbb{Q})$, where int denotes interior. Hence $G_\alpha = F_\alpha^2 \cap G$ is a G_δ in $F_\alpha^2 = F^2 \cap \rho^{-1}(\alpha)$. Note that G_α may be empty!

Proof of Proposition 5.1.

Lemma 5.5 shows that it suffices to prove that G_α is dense in F_α^2. Suppose there exists an open set U in F_α^2 such that $U \cap G_\alpha = \phi$. By Lemma 5.3, there exists $g \in U$ such that $g = h^{-1} \circ R_\alpha \circ h$, where $h \in D^2(T^1)$. Now we define the map $H: F_\alpha^2 \to F_\alpha^2$ by $H(f) = h \circ f \circ h^{-1}$. We have $H(g) = R_\alpha$, and it is easy to see that H is a homeomorphism with respect to the C^2 topology. Thus $H(U)$ is open in F_α^2 and contains R_α. By Proposition 4.5, $H(U)$ contains a type III_1 diffeomorphism so by Lemma 5.4, U does as well. This contradiction proves the proposition.

Corollary 5.6.

For every $\alpha \in \Omega$ there exists $f \in F_\alpha^2$ which is not C^1 conjugate to R_α.

Using the notation of Proposition 5.1 we see that for every $\alpha \in A$, $f \in G_\alpha$ implies that the equation $\log Df = \psi - \psi \circ f$ has no m-measurable solution. We compare this with a known result for the linearised equation, where α is of constant type implies the existence of an L^2 solution for every C^1 cocycle. We include the result and a proof for completeness. If α is of constant type, it is not yet known if Corollary 5.6 is true.

Proposition 5.7.

If $\alpha \in \mathbb{R}\backslash\mathbb{Q}/\mathbb{Z}$ is of constant type and if $\phi \in C^1(T^1)$ with $\int_{T^1} \phi(x)dm = 0$, then there exists $\psi \in L^2(T^1,m)$ satisfying $\psi - \psi \circ R_\alpha = \phi$ m-a.e.

Proof.

We write $\phi(x) = \sum_{k=-\infty}^{\infty} \hat{\phi}(k)e^{2\pi ikx}$, where $\hat{\phi}(k)$, $k \in \mathbb{Z}$ are the Fourier coefficients of ϕ. Solving the equation $\psi - \psi \circ R_\alpha = \phi$ then reduces to showing that $\sum_{k\neq 0} \left|\frac{\hat{\phi}(k)}{1-e^{2\pi ik\alpha}}\right|^2 < +\infty$ (for an L^2 solution). Since α is of constant type there exists a constant $C > 0$ such that for every $k \in Z$, $||k\alpha|| > \frac{C}{k}$, and therefore a constant C' such that $|1-e^{2\pi ik\alpha}| > \frac{C'}{k}$.

Now $\sum_{k\neq 0} \left|\frac{\hat{\phi}(k)}{1-e^{2\pi ik\alpha}}\right|^2 \leq \sum_{k\neq 0} \left|\frac{k}{C'}\right|^2 \cdot \left|\hat{\phi}(k)\right|^2$.

Since $\phi \in C^1(T^1)$, we have $\sum_{-\infty}^{\infty} |k|^2 \cdot |\hat{\phi}(k)|^2 < +\infty$, and the result follows.

CHAPTER III.

Type III_1-Diffeomorphisms of Higher Dimensional Tori.

§6. Introduction.

orientation preserving
Every ergodic diffeomorphism of the circle can be extended to an ergodic flow on T^2 by taking the suspension flow. Once an ergodic flow is obtained, a standard result gives the existence of a set of $t \in \mathbb{R}$ for which the diffeomorphism obtained by fixing the flow at time t is ergodic and preserves the measure theoretic properties of the original diffeomorphism. We use this technique, as suggested by Herman in [7], to obtain some results on the topology of type III_1 diffeomorphisms on T^n.

Another method for obtaining ergodic flows and diffeomorphisms on T^2 is to look at ergodic circle extensions of the dynamical system given by:

$$\begin{aligned} F: T^n \times \mathbb{R} &\to T^n \times \mathbb{R} \\ (x,y) &\mapsto \left(fx, y + \log \frac{dmf^{-1}}{dm}(x)\right) \end{aligned} .$$

Motivated by a paper of Jones and Parry [10], we show that "most" C^∞ functions $\phi: T^n \times \mathbb{R} \to T^1$ give ergodic extensions for each F as defined

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Motivated by a paper of Jones and Parry [10], we show that "most" C^∞ functions $\phi: T^n \times \mathbb{R} \to T^1$ give ergodic extensions for each F as defined above. Thus we obtain results about smooth type III_1 diffeomorphisms of the skew product type.

Using conjugacy we obtain topological results for a larger class in $D^r(T^n)$.

We begin Chapter III with a lemma proving that type III_1 diffeomorphisms always form a G_δ . Thus to prove the existence of a residual set of such diffeomorphisms, the question is reduced to showing the existence of a dense set.

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§7. Type III_1 Diffeomorphisms Are a G_δ.

Let $(X,\mathcal{S},\mu)$ be a compact manifold, with μ a C^∞ probability measure on X. Consider the space

$$L_0^2(X,\mu) = \{h \in L^2(X,\mu) \mid \int_X h\,d\mu = 0\}$$

and let $\{h_i\}_{i\in N}$ denote a countable dense set in $L_0^2(X,\mu)$. We prove the following lemma similar to [7,V].

Lemma 7.1.

Let $g: X \to X$ be an invertible transformation which preserves μ. Then g is μ-ergodic if and only if

$$\inf_{m\geq 1} \left| \frac{1}{m} \sum_{j=1}^{m-1} h_i \circ g^j \right|_2 = 0 \quad \text{for all}$$

$i\in N$ and where $|\quad|_2$ denotes the L^2-norm.

Proof.

($\Rightarrow$) Assume g is μ-ergodic. Then by Von Neumann's mean ergodic theorem, for all $h \in L^2(X,\mu)$,

$$\lim_{m\to\infty} \left| \frac{1}{m} \sum_{j=0}^{m-1} h \circ g^j - \int_X h d\mu \right|_2 = 0 \quad .$$

This implies in particular, for every $i \in N$,

$$\lim_{m\to\infty} \left| \frac{1}{m} \sum_{j=1}^{m-1} h_i \circ g^j \right|_2 = 0 \quad .$$

(<=) Assume that

$$\inf_{m\geq 1} \left| \frac{1}{m} \sum_{j=1}^{m-1} h_i \circ g^j \right|_2 = 0 \quad \text{for every} \quad i \in N \quad .$$

Suppose there exists $f \in L_0^2(X,\mu)$ such that $f \circ g = f$ μ-a.e. $x \in X$. Then we have for every $i \in N$, $\int_X h_i . f d\mu = \int_X h_i . (f \circ g^{-1}) d\mu$ (since g preserves μ) $= \int_X (h_i \circ g) . f d\mu = \int_X (h_i \circ g^j) . f d\mu$ for every $j \in Z$, $= \frac{1}{m} \sum_{j=0}^{m-1} \int_X h_i \circ g^j . f d\mu$ (since the integral is independent of j). Now by Holder's inequality,

$$\frac{1}{m} \sum_{j=0}^{m-1} \int_X h_i \circ g^j . f d\mu \leq \left| \frac{1}{m} \sum_{j=0}^{m-1} h_i \circ g^j \right|_2 . | f |_2$$

for every m , so $\int_X h_i . f d\mu = 0$ for every i . Clearly $\int_X h_i . f d\mu = 0 \; \forall \; i \in N$ implies that $f \equiv$ constant μ-a.e. and hence that g is μ-ergodic.

We use Lemma 7.1 to prove the following theorem.

Theorem 7.2.

Let $\mathrm{Diff}^{\infty}(X)$ denote the space of C^{∞} diffeomorphisms of X. With the C^{∞} topology on $\mathrm{Diff}^{\infty}(X)$, the set of type III_1 diffeomorphisms is a G_{δ}.

Proof.

By definition and [16], $f \in \mathrm{Diff}^{\infty}(X)$ is of type III_1 if the map

$$\begin{aligned} F: X\times\mathbb{R} &\to X\times\mathbb{R} \\ (x,y) &\mapsto \left(fx, y + \log\frac{d\mu f^{-1}}{d\mu}(x)\right) \end{aligned} \qquad \text{given by:}$$

is $\nu = e^{-y}d\mu\otimes dy$ - ergodic. (Note that F preserves ν).

If we let $C_n = X\times(-n,n)$ denote the cylinder without boundary, and let ν_n denote $\nu|_{C_n}\cdot|\nu(C_n)|^{-1}$, we can consider the map $F_{C_n} = F_n$, the map induced by F on C_n. Then F_n preserves the probability measure ν_n and we see that F is ν-ergodic if and only that g is μ-ergodic.

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Let $(h_{n,i})_{i\epsilon N}$ be a countable dense subset of $L_0^2(C_n,\nu_n)$; i.e. $\int_{C_n} h_{n,i}d\nu_n = 0$ for every $i \epsilon N$, for fixed $n \epsilon N$.

For fixed values of $i,n,$ and k we define:

$$\mathcal{B}_{k,i,n} = \{f\epsilon \mathrm{Diff}^{\infty}(X) \Big| \inf_{m\geq 1} \Big| \frac{1}{m} \sum_{j=0}^{m-1} h_{n,i} \circ F_n^j \Big|_2 < \frac{1}{k}\} .$$

We claim that for each triplet (k,i,n) the set $B_{k,i,n}$ is open in the C^{∞} topology. This is because the map

$$f \to \Big| \frac{1}{m} \sum_{j=0}^{m-1} h_{n,i} \circ F_n^j \Big|_2$$

is continuous with respect to the C^{∞} topology on the domain and the L^2 topology on the range, for fixed m,n,i . Since the infimum of continuous maps is upper semicontinuous, it follows that $B_{k,i,n}$ is open.

We now consider:

$$\mathbb{B} = \bigcap_n \bigcap_i \bigcap_k B_{k,i,n} .$$

Clearly $\mathcal{B}$ is a G_δ and $f \in \mathcal{B}$ implies $\forall\, n \in N$, $\forall\, i \in N$, and $\forall\, k \in N$,

$$\inf_{m \geq 1} \left| \frac{1}{m} \sum_{j=0}^{m-1} h_{n,i} \circ F_n^j \right|_2 < \frac{1}{k} \quad ,$$

so F_n is ν_n-ergodic for every $n \in N$, hence F is ν-ergodic.

Note that, a priori, this G_δ may be empty. If $X = T^1$, from Katznelson's construction it follows that the set of type III_1 diffeomorphisms in $F^\infty = D^\infty(T^1)\text{-int}\, \rho^{-1}(Q)$ is a dense G_δ. This result is proved using different methods in [7].

§8. Cartesian Product Diffeomorphisms.

We will show that type III_1 diffeomorphisms form a dense G_δ in certain closed subspaces of the space of diffeomorphisms of higher dimensional tori.

Let $PR^r(T^n)$ denote the set of C^r diffeomorphisms of the form:

$$f \in PR^r(T^n) \Rightarrow f(x_1,x_2,\ldots,x_n) = (f_1(x_1),\ldots,f_n(x_n)) \quad ,$$

where $f_i \in D^r(T^1)$. Clearly $PR^r(T^n) \subset D^r(T^n)$ and for each $n \geq 1$, $0 \leq r \leq \infty$, $PR^r(T^n)$ is a closed subgroup of $D^r(T^n)$ (with respect to composition).

Herman has shown in [8,XIII] that in general one cannot define the rotation number of $f \in D^r(T^n)$ for $n \geq 2$. In $PR^r(T^n)$, however, there is a natural extension of the rotation number. We define, for $f \in PR^r(T^n)$, $\rho(f) = \alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n) \in \mathbb{R}^n$ where $\alpha_i = \rho(f_i)$ for $1 \leq i \leq n$.

We write $PR^r_\alpha(T^n) = \{f \in PR^r(T^n) \mid \rho(f) = \alpha \in \mathbb{R}^n\}$,

and

$$FR^r(T^n) = \{f \in PR^r(T^n) \mid f_i \in F^r(T^1) \ , \ 1 \leq i \leq n\} \quad .$$

Recall that $F^r(T^1) = D^r(T^1) \backslash \mathrm{int}\, \rho^{-1}(\mathbb{Q})$.

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Recall that $F^r(T^1) = D^r(T^1) \setminus \operatorname{int} \rho^{-1}(\mathbb{Q})$.

To extend our results to higher dimensions we need to recall some elementary facts about ergodic flows on compact manifolds.

Definition 8.1.

A C^r flow on a manifold X is a C^r map $f: X \times \mathbb{R} \to X$ such that if we denote $f_t(x) = f(x,t)$, $\forall x \in X$, $t \in \mathbb{R}$, then

(i) $f_{s+t}(x) = f_s \circ f_t(x) \quad \forall s,t \in \mathbb{R}$,

(ii) $f_0 = Id_X$. Then it follows that

(iii) $f_{-t} = f_t^{-1}$.

A flow on $(X, \mathcal{S}, \mu)$ is a μ-ergodic if whenever $f_t(A) = A$ for some $A \in \mathcal{S}$ and for every $t \in \mathbb{R}$, then either $\mu(A) = 0$ or $\mu(A^c) = 0$.

Equivalently, f_t is μ-ergodic if whenever $\phi \circ f_t = \phi$ for ϕ in $L^\infty(X,\mu)$, then ϕ is a constant μ-almost everywhere.

Definition 8.2.

A non-singular ergodic flow f_t on $(X, \mathcal{S}, \mu)$ is of type III if it admits no σ-finite invariant measure equivalent to μ.

We say f_t is of type III_1 if the flow

$$S: X \times \mathbb{R} \times \mathbb{R} \to X \times \mathbb{R} \quad \text{given by}$$

$$(x,z,t) \to \left(f_t(x),\ z + \log \frac{d\mu f_t^{-1}}{d\mu}(x)\right) \quad \text{is}$$

$\nu = e^{-z} d\mu \otimes dz$ - ergodic.

If f is any diffeomorphism of a compact manifold of dimension $n \geq 1$, there exists a canonical method for obtaining a flow on an $(n+1)$-dimensional manifold, associated with f, called the suspension flow of f. We define F_t = {suspension flow of f on $X \times /\sim$, where $(x,y) \sim (f^{-1}x, y+n)$ $n \in \mathbb{Z}$} ; for details of this standard construction we refer the reader to [17]. The following proposition is an easy consequence of the construction.

Proposition 8.3.

If $f \in \mathrm{Diff}^r(X)$ and f is μ-ergodic, then F_t is $\mu \otimes m$-ergodic. It is then a trivial corollary of this proposition that the suspension flow of a type III_1 diffeomorphism is also of type III_1.

We use the following well-known result [14] to obtain higher dimensional diffeomorphisms from flows.

Lemma 8.4.

Let $(Y,\mathcal{G},\nu)$ be a Lebesgue space, ν a positive σ-finite measure and G_t a flow on Y preserving ν and ν-ergodic. If G_t has no orbit of full ν-measure, then there exists a set $\mathcal{C} \subset [0,1]$, $m(\mathcal{C}) = 1$, such that for all $t_o \in \mathcal{C}$, $G_{t_o} \in \mathrm{Aut}(Y,\nu)$ is ν-ergodic.

Theorem 8.5.

For every $n \geq 1$ there exists a set $B_n \subset \mathbb{R}^n - \mathbb{Q}^n/\mathbb{Z}^n$, $m(B_n) = 1$ such that for every $\alpha \in B_n$ there is a residual set of type III_1 C^2 diffeomorphisms in $FR^2_\alpha(T^n)$.

Proof.

We use induction on n. For $n = 1$, let $B_1 = A$ (cf.I 1.1) and apply II.5.1. Now assume the theorem is true for $n = k$; we show this implies the theorem for $n = k+1$ as well. If $g \in FR^2_\alpha(T^k)$ is of type III_1, then the following map is $\nu = e^{-z}dm\otimes dz$ -ergodic.

$$S_g : T^k \times \mathbb{R} \to T^k \times \mathbb{R}$$

$$(x,z) \mapsto (gx, z + \log \frac{dmg^{-1}}{dm}(x)) .$$

Taking the suspension flow of S_g , one obtains a $\nu' = \nu \otimes m$-ergodic flow on $T^k \times T^1 \times \mathbb{R}$. We apply Lemma 8.4 to obtain a set $A_g \subset [0,1]$, $m(A_g) = 1$ such that for every $t_o \in A_g$, S_{g,t_o} is a ν'-ergodic diffeomorphism of $T^{k+1} \times \mathbb{R}$. The map S_{g,t_o} is of the form:

$$T^K \times T^1 \times \mathbb{R} \to T^k \times T^1 \times \mathbb{R}$$

$$(x,y,z) \to (gx, y+t_o, z+\log \frac{dmg^{-1}}{dm}(x)) \quad .$$

Now consider a countable dense subset $\{P_i\}_{i \in N}$ of $G_\alpha(T^k)$, (cf. II.5.1), then for every $g \in \{P_i\}_{i \in N}$ we obtain a set $A_i \subset [0,1]$, with $m(A_i) = 1$. Using the definition of set A from I.1.1, we let $\tilde{A} = \bigcap_{i \in N} (A_i \cap A)$.

Putting $B_{k+1} = B_k \times \tilde{A}$ implies that $m(B_{k+1}) = 1$.

If $\beta = (\beta_1, \ldots, \beta_{k+1}) \in B_{k+1}$ then there exists $f \in FR^2_\beta(T^{k+1})$ of the form $f(x_1, \ldots, x_{k+1}) = (g(x_1, \ldots, x_k), x_{k+1} + \beta_{k+1})$ where $g \in \{P_i\}_{i \in N} \subset FR^2_{(\beta_1, \ldots, \beta_k)}(T^k)$, and both g and f are of type III_1.

By Theorem 7.2 we have a G_δ of type III_1 diffeomorphisms in $FR^2_\beta(T^{k+1})$.

To show that the set

$$\{f \in FR^2_\beta(T^{k+1}) \mid f \text{ is of type } III_1\} \quad \text{is dense,}$$

we recall that the ratio set is an invariant of absolutely continuous conjugacy. We note also that for every $\alpha \in \tilde{A}$, the set of functions in $F^2_\alpha(T^1)$ which are conjugate to R_α in an absolutely continuous way are dense in F^2_α (because F^∞_α is dense in F^2_α and Herman's Theorem applies). Therefore, since the product of dense sets is dense in $FR^2_{(\beta_1,\ldots,\beta_k)}(T^k) \times F^2_\alpha(T^1)$, the theorem is proved.

For every $n \geq 1$ and $r \geq 1$, we define

$G^r(T^n) = \{g^{-1} \circ f \circ g \mid g \in D^r(T^n),\ f \in FR^r(T^n)\}$. Taking the closure of this set with respect to C^r topology gives a set denoted $\overline{G^r(T^n)}$. We first prove a corollary to the proof of Theorem 8.5.

Corollary 8.6.

It is generic in $FR^\infty(T^n)$ that $f \in FR^\infty(T^n)$ is of type III_1.

Proof.

We use induction. The case $n = 1$ is true from [11]. By methods used in 8.5 we can obtain the result on T^n, $n \geq 2$.

Corollary 8.7.

It is generic in $\overline{G^r(T^n)}$, $n \geq 1$, $r \geq 2$, that $f \in \overline{G^r(T^n)}$ is of type III_1.

Proof.

Given any element $f \in \overline{G^r(T^n)}$, f is arbitrarily close in the C^r topology to a map of the form $\tilde{g} = g^{-1} \circ \tilde{f} \circ g$, $g \in D^r(T^n)$, $\tilde{f} \in FR^r(T^n)$, and $\tilde{f}$ is of type III_1 . By invariance under conjugacy, $\tilde{g}$ is of type III_1, so density is proved.

§9. Skew products of type III_1.

In this section we will first present some results about smooth circle extensions of discrete dynamical systems. These results are similar to results of Jones and Parry [10] for the continuous case. A slight modification of the first proposition gives the main result of this section.

Theorem 9.1.

Given $(X,\mathcal{S},\mu)$ a connected compact manifold and $f \in \mathrm{Diff}^\infty(X)$, f μ-ergodic and of type III_1. It is generic in $C^\infty(X,T^1)$ (with the C^∞ topology) that the skew product extension given by:

$$X \times T^1 \to X \times T^1$$

$$(x,z) \mapsto (fx, z\cdot\psi x) \quad \text{for} \quad \psi \in C^\infty(X,T^1)$$

is of type III_1.

Recalling the definitions given in Chapter 1, Section §2, we see that every C^∞ map from X to T^1 determines a multiplicative C^∞ $\mathbb{Z}$-cocycle as follows. Given any C^∞ (Borel) map $\psi:X \to T^1$ we define a multiplicative cocycle for the $\mathbb{Z}$-action of $f \in \mathrm{Diff}^\infty(X)$ on X by:

$$a(n,x) = \begin{cases} \prod_{k=0}^{n-1} \psi(f^k x) & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \\ -a(-n,f^n x) & \text{if } n < -1 . \end{cases}$$

The set $C^\infty(X,T^1) = \{\psi:X \to T^1 | \psi \text{ is } C^\infty\}$ forms a group under pointwise multiplication; it is a complete topological group with respect to the C^∞ topology.

Consider also the set $G = \{\phi:X \to T^1 | \phi$ is a Borel map and $\phi \circ f/\phi = h$ μ-a.e. for some $h \in C^\infty(X,T^1)\}$. We identify two functions in G if and only if they are equal μ-a.e. G is a group under pointwise multiplication.

We define the map $\eta:G \to C^\infty(X,T^1)$ by $\phi \to \phi \circ f/\phi$. We see that η is a homomorphism and that $\ker \eta$ = constant maps $\cong T^1$. If we now define a metric on G by: for every $\phi_1,\phi_2 \in G$,

$$\delta(\phi_1,\phi_2) = \int_X |\phi_1-\phi_2|d\mu + ||\eta\phi_1-\eta\phi_2||_\infty \quad ,$$

then we see that $\mathcal{G}$ is complete and separable with respect to δ, and that η is a continuous group homomorphism.

The smooth skew product uniquely determined by a C^∞ cocycle is the following:

$$S\psi : X \times T^1 \to X \times T^1$$

$$(x,z) \to (fx, z\cdot a(1,x))$$

$$= (fx, z\cdot\psi(x)) \quad \text{for } \psi \in C^\infty(X,T^1) \ .$$

The next theorem is proved in [16].

<u>Theorem 9.2</u>.

The skew product $S\psi$ is $\mu\otimes m$-ergodic if and only if $E(a) = T^1$. (Recall the definition of $E(a)$ given in I,§2.)

This helps us to prove the following.

Proposition 9.3.

Given (X,μ) as in 9.1 and $f \in \mathrm{Diff}^\infty(X)$, f μ-ergodic, the set $\{\psi\in C^\infty(X,T^1) \mid S\psi$ is $\mu\otimes m$-ergodic$\}$ is a residual set in $C^\infty(X,T^1)$ with respect to the C^∞ topology.

Proof.

By using I.2.3 and Theorem 9.2, it is clear that $S\psi$ is $\mu\otimes m$-ergodic if and only if for every $k \geq 1$, and for every $\phi \in G$, $\psi^k \neq \phi\circ f/\phi$ a.e. (Because if $E(a) \neq T^1$, then $\overline{E}(a) = \{0\}$ or $\overline{E}(a) = \{\omega_i\}_{i=1}^k = k^{\underline{th}}$ roots of unity.) Therefore it suffices to show that ηG is of the first category in $C^\infty(X,T^1)$; i.e. it can be written as the countable union of nowhere dense sets. Suppose that ηG is of the second category in $C^\infty(X,T^1)$; then the closure of ηG has non-empty interior in $C^\infty(X,T^1)$.

By the Open Mapping Theorem (*), η is continuous implies that $\eta: G \to C^\infty(X,T^1)$ is open. Then $\eta G = C^\infty(X,T^1)$ because the only subgroup of $C^\infty(X,T^1)$ which is both open and closed (and non-empty) is itself. However $\eta G \neq C^\infty(X,T^1)$ because there exists $\beta\in(0,1)$ such that $e^{2\pi i\beta}$ is not a coboundary. (Just choose β not in the L^∞

spectrum of f or use Lemma 8.4.) This contradiction proves the proposition.

(*) Version of the Open Mapping Theorem used: Let E and D be complete, metrisable, separable groups. The homomorphism ξ of E into D is open if it satisfies: (i) the graph of ξ is a closed subset of $E \times D$; (ii) the closure of $\xi(V)$ is a neighbourhood of Id_D whenever V is a neighbourhood of Id_E in E . (See [12].) In Proposition 9.3, (i) is satisfied by the continuity of η , and it can be shown that (ii) holds if ηG is non-meagre.

Remark 9.4.

Let $f \in Diff^{\infty}(X)$ be of type III_1 ; then $F: X \times \mathbb{R} \to X \times \mathbb{R}$ given by $(x,z) \to (fx, z + \log \frac{d\mu f^{-1}}{d\mu}(x))$ is $\nu = e^{-z} d\mu \otimes dz$ - ergodic. A smooth cocycle for F is given by any C^{∞} map $\psi: X \times \mathbb{R} \to T^1$ and defined as before. In particular, the subset of $C^{\infty}(X \times \mathbb{R}, T^1)$ defined by

$$H = \{\psi \in C^{\infty}(X \times \mathbb{R}, T^1) \mid \psi(x,z_1) = \psi(x,z_2) \ , \ \forall \ x \in X \ , \ z_1, z_2 \in \mathbb{R}\}$$

describes the set of C^{∞} maps which do not depend on the $\mathbb{R}$-coordinate. It is not difficult to see that H forms a topological group under pointwise multiplication with respect to the C^{∞} topology, and that

$H \equiv C^\infty(X,T^1)$. By defining $G' = \{\phi : X \times \mathbb{R} \to T^1 | \phi$ is Borel and $\phi \circ F/\phi = h$ a.e. for some $h \in H\}$, η' such that $\eta'(\phi) = \phi \circ F/\phi$, $\phi \in G'$, and

$$\delta'(\phi_1,\phi_2) = \int_{X\times\mathbb{R}} \frac{|\phi_1-\phi_2|}{1+|\phi_1-\phi_2|} d\mu \otimes m + ||\eta'\phi_1 - \eta'\phi_2||_\infty ,$$

we can use 9.3 to prove Theorem 9. as a simple corollary.

Proof of Theorem 9.1.

It is generic in $C^\infty(X,T^1)$ that the skew product $S_\psi : X\times T^1 \to X\times T^1$ defined by $(x,y) \to (fx, y\cdot\psi(x))$, (where f is as above), is of type III_1.

Proof.

By 9.3 and 9.4 we see that $\eta' G'$ is meagre in H , so the set

$$\{\psi \in C^\infty(X,T^1) | (fx, y\cdot\psi(x), z+\log\frac{d\mu f^{-1}}{d\mu}(x)) \text{ is } \mu\otimes m\text{-ergodic}\} \text{ is}$$

residual in $C^\infty(X,T^1)$ for each f of type III_1.

We define the set

$$SP^r(T^n) = cl\{f\varepsilon D^r(T^n) | f(x_1,\ldots,x_n)$$

$$= (f_1(x_1), x_2+\Psi_1(x_1), x_3+\Psi_2(x_1,x_2),\ldots,x_n+\Psi_{n-1}(x_1,\ldots,x_{n-1}))$$

where $f_1 \varepsilon F^r(T^1)$ and $\Psi_i \varepsilon C^r(T^i,T^1)\}$,

where cl denotes closure taken with respect to the C^r topology; then as another corollary of 9.3 we obtain the following result.

Corollary 9.6.

For $r \geq 1$, $n \geq 1$, it is generic in $SP^r(T^n)$ that $f \varepsilon SP^r(T^n)$ is of type III_1 .

Proof.

By Theorem 7.2, it suffices to show the set $\{f\varepsilon SP^r(T^n) | f$ is of type $III_1\}$ is dense. Since $SP^r(T^n) \cong F^r(T^1) \times C^r(T^1,T^1)\times\ldots\times C^r(T^{n-1},T^1)$ we can prove the result by induction. For $n = 1$ it is true. Assume there exists a dense set in $SP^r(T^k)$ for which $f \varepsilon SP^r(T^k)$ is of type III_1. We then can find, by 9.3, a dense (residual) set in $C^\infty(T^k,T^1)$ for which $(f(x), x_{k+1}\psi(x))$ is of type III_1, where $x = (x_1,\ldots,x_k) \varepsilon T^k$ and $\psi \varepsilon C^\infty(T^k,T^1)$. Using the product topology we are done.

Corollay 9.7.

For $r \geq 1$, $n \geq 1$ it is generic in the set

$$GP^r(T^n) = cl\{g^{-1} \circ f \circ g \mid g \in D^r(T^n) , \; f \in SP^r(T^n)\}$$

that $h \in GP^r(T^n)$ is of type III_1.

CHAPTER IV.

SMOOTH TYPE III_0 DIFFEOMORPHISMS.

§10. Introduction.

In [7], Herman proved that every paracompact manifold of dimension ≥ 3 admits a C^∞ type III_1 diffeomorphism. Since a type III_0 diffeomorphism, f, has uncountably many ergodic components of the skew product with $a(n,x) = \log\frac{d\mu f^{-n}}{d\mu}(x)$, we need a different method to extend smooth type III_0 diffeomorphisms from the circle (where we know from [11] such an f exists) to $T^n \times \mathbb{R}^m$ for $m,n \geq 1$. Once we obtain type III_0 diffeomorphisms on $T^n \times \mathbb{R}^m$, we can apply Herman's methods with slight modifications to obtain the existence of a smooth type III_0 diffeomorphism on every paracompact manifold.

We begin the chapter with some results about the topology of type III_0 diffeomorphisms in certain spaces of smooth diffeomorphisms of T^n, for $n \geq 1$.

In section §12 we prove the existence of smooth, ergodic real line extensions ~~for~~ the $\mathbb{Z}$-action of a type III_0 diffeomorphism on a

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We begin the chapter with some results about the topology of type III_0 diffeomorphisms in certain spaces of smooth diffeomorphisms of T^n, for $n \geq 1$.

In section §12 we prove the existence of smooth, ergodic real line extensions for the $\mathbb{Z}$-action of a type III_0 diffeomorphism on a manifold X. Special thanks are due to Klaus Schmidt and Ralf Spatzier for helpful discussions on the construction given in Theorem 12.7. It is not difficult to show that most of these ergodic extensions give a smooth type III_0 diffeomorphism of $X \times \mathbb{R}$.

In §13 and §14 we add some results about type III_λ, diffeomorphisms of manifolds $0 < \lambda < 1$, for completeness.

§11. Type III_0 Diffeomorphisms of T^n .

Recall from III.8 that

$FR^\infty(T^n) = \{f \varepsilon D^\infty(T^n) | f = (f_1, f_2, \ldots, f_n), f_i \varepsilon F^\infty(T^1)\}$. To obtain a type III_0 diffeomorphism of the torus in higher dimensions, the construction is similar to the one in III.8.5 but has the added complication of a non-trivial ergodic decomposition of $(x,y) \to (fx, y + \log\frac{d\mu f^{-1}}{d\mu}(x))$.

Theorem 11.1.

For every $n \geq 1$, the set $\theta^n = \{f \varepsilon FR^\infty(T^n) | f$ is of type $III_0\}$ is dense in $FR^\infty(T^n)$ in the C^∞ topology.

Proof.

Let $n = 1$. The theorem is true by [11]. Assume it is true for $n = k$. We claim there exists a measurable set $C \subset [0,1]$, $m(C) = 1$ such that for every $t \varepsilon C$, the map $(f, R_t): T^{k+1} \to T^{k+1}$ is of type III_0 . We prove this claim as follows. Consider the ergodic decomposition of the skew product defined by

$$F: T^k \times \mathbb{R} \to T^k \times \mathbb{R}$$

$$(x,z) \to (fx, z + \log\frac{d\mu f^{-1}}{d\mu}(x)) \quad x \varepsilon T^k , z \varepsilon \mathbb{R} ,$$

which preserves the measure $\nu = e^{-z} d\mu \otimes dz$. By [16] there exists a Borel probability space $(Y, \mathcal{T}, \rho)$ and σ-finite measures q_y on $(T^k \times \mathbb{R}, \mathcal{S}_k, \nu)$ such that:

(i) $y \to q_y(B)$ is Borel for every $B \in \mathcal{S}_k$

(ii) $\nu(B) = \int_{T^k \times R} q_y(B) d\rho(y)$

(iii) Every $q_y, y \in Y$ is invariant and ergodic under F, and q_y and $q_{y'}$ are mutually singular when $y \neq y'$.

(iv) Let $\mathcal{Z} = \{B \in \mathcal{S}_k : F(B) = B\}$. For every $B \in \mathcal{Z}$, put $B_y = \{y \in Y : q_y(B) > 0\}$. Then $\mathcal{Z}_Y = \{B_Y : B \in \mathcal{Z}\}$ is equal to $\mathcal{T}$ mod sets of ρ-measure zero.

For each $y \in Y$, the map $F_t : T^k \times \mathbb{R} \times T^1 \to T^k \times \mathbb{R} \times T^1$ defined by $(x,z,w) \to (fx, z + \log \frac{d\mu f^{-1}}{d\mu}(x), w+t)$ is $q_y \otimes m$-ergodic for m-a.e. $t \in [0,1]$. This can be shown by taking the suspension flow of F and applying Lemma 8.4for each $y \in Y$. If we can prove that the set $Q = \{(y,t) \in (Y \times I, \mathcal{T} \times \mathcal{J}, \rho \times m) : F_t$ is $q_y \otimes m$ ergodic$\}$ is $\rho \otimes m$ measurable, then by Fubini's Theorem, since $\rho \otimes m(Q) = 1$ there exists a set $C \subset [0,1]$, $m(C) = 1$ such that if $t \in C$ for ρ-a.e. $y \in Y$, F_t is $q_y \otimes m$

ergodic. By the uniqueness of ergodic decomposition, this implies that F_t is of type III_0.

Now given any $g \in FR^{\infty}(T^{k+1})$, g is of the form: $g(x_1,\dots,x_k,x_{k+1}) = (g_1(x_1),\dots g_{k+1}(x_{k+1}))$, where $g_i \in F^{\infty}(T^1)$. By hypothesis θ^k is dense in $FR^{\infty}(T^k)$, so for every $\varepsilon > 0$ there exists $\tilde{g} \in \theta^k$ such that

$$||(\tilde{g}(x_1,\dots,x_k),g_{k+1}(x_{k+1})) - g(x_1,\dots,x_{k+1})||_{\infty} < \varepsilon/2 .$$

By continuity of $\rho(g)$, and recalling from I.1.1. that $m(A) = 1$ (A is defined in I.1.1), we can find $h \in F^{\infty}(T^1)$ such that:

(i) $||h-g_{k+1}||_{\infty} < \varepsilon/2^k$, and

(ii) $\rho(h) \in C_{\tilde{g}} \cap A$, where $C_{\tilde{g}}$ is the set obtained above for which $t \in C_{\tilde{g}} \Rightarrow (\tilde{g},R_t)$ is of type III_0. Since $m(A) = m(C_{\tilde{g}}) = 1$, we have $m(A \cap C_{\tilde{g}}) = 1$.

By Herman's theorem, $h = k^{-1} \circ R_{t_o} \circ k$ for some $t_o \in C_{\tilde{g}}$ and $k \in D^{\infty}(T^1)$. By invariance under conjugation of the ratio set, the map $(\tilde{g},R_{t_o})$ is of type III_0 if and only if $(\tilde{g},h)$ is of type III_0.

ergodic. By the uniqueness of ergodic decomposition, this implies that F_t is of type III_0.

Now given any $g \in FR^{\infty}(T^{k+1})$, g is of the form: $g(x_1,\ldots,x_k,x_{k+1}) = (g_1(x_1),\ldots g_{k+1}(x_{k+1}))$, where $g_i \in F^{\infty}(T^1)$. By hypothesis θ^k is dense in $FR^{\infty}(T^k)$, so for every $\varepsilon > 0$ there exists $\tilde{g} \in \theta^k$ such that

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(i) $||h-g_{k+1}||_{\infty} < \varepsilon/2^k$, and

(ii) $\rho(h) \in C_{\tilde{g}} \cap A$, where $C_{\tilde{g}}$ is the set obtained above for which $t \in C_{\tilde{g}} \Rightarrow (\tilde{g},R_t)$ is of type III_0. Since $m(A) = m(C_{\tilde{g}}) = 1$, we have $m(A \cap C_{\tilde{g}}) = 1$.

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Thus we have shown that given any $g \in FR^{\infty}(T^{k+1})$, we can find a map of the form:

$$\bar{h}: FR^{\infty}(T^{k+1}) \to FR^{\infty}(T^{k+1})$$

such that $\quad (x_1,\ldots,x_k,x_{k+1}) \to (\tilde{g}(x_1,\ldots,x_k), h(x_{k+1}))$,

for some $\tilde{g} \in FR^{\infty}(T^k)$ of type III_0 , and some h satisfying (i) and (ii). Furthermore we have $||\bar{h}-g||_{\infty} < \varepsilon$.

The proof is done if we can show that the set Q defined above is $\rho \otimes m$-measurable. We prove this in the lemma that follows.

<u>Lemma 11.2</u>.

The set

$$Q_{\rho\otimes m} = \{(y,t) \in (Y\times I, \mathcal{J}\times\mathcal{L}, \rho\otimes m) : F_t \text{ is } q_y\otimes m\text{-ergodic}\}$$

is measurable.

<u>Proof</u>.

Let $X = T^k \times \mathbb{R}$, and denote by X_n the [illegible] manifold

$X_n = T^k \times (-n,n)$. The skew product F defined in 11.1 is always conservative, so we can induce on X_n . Let F_n denote the induced transformation on X_n , and we write F_{nt} for the map

$$F_{nt} : X_n \times T^1 \to X_n \times T^1$$

$$(x,z,w) \to (F_n(x,z), w+t) \qquad \text{for every}$$

$x \in T^k$, $z \in (-n,n)$, $w \in T^1$. We define, for every $y \in Y$ and $n \geq 1$, a normalised measure on X_n equivalent to the induced measure obtained from q_y restricted to X_n . Call these measures $p_y^{(n)}$; then $p_y^{(n)}(X_n) = 1$ for $n \geq 1$. Clearly we have F_t is $q_y \otimes m$-ergodic if and only if F_{nt} is $p_y^{(n)} \otimes m$-ergodic for every $n \geq 1$. To show Q is measurable, we show that $Q_n = \{(y,t) \in Y \times I | F_{nt}$ is $p_y^{(n)} \otimes m$-ergodic$\}$ is $\rho \otimes m$-measurable for each $n \in N$.

To show Q_n is measurable, we use Lemma 7.1 from Chapter III in the following form.

Lemma 7.1'.

With the above notation, F_{nt} is $p_y^{(n)} \otimes m$-ergodic if and only if

$$\inf_{m \geq 1} \left| \frac{1}{m} \sum_{j=0}^{m-1} h_i \circ F_{nt}^j - \int_{X_n \times I} h_i \, dp_y^{(n)} \otimes m \right|_{L^2(X_n \times I, p_y \otimes m)} = 0 \quad ,$$

where $\| \quad \|_{L^2(X_n\, I, p_y^{(n)} \otimes m)}$ denotes the relevant L^2 norm and $h_i \in \{h_k\}_{k\in N}$ and $\{h_k\}_{k\in N}$ is a countable dense sequence of Borel (L^2) functions on $X_n \times I$, (hence measurable for $p_y^{(n)} \otimes m$, for every $y \in Y$).

Since the infimum of measurable functions is measurable, and since the countable intersection of measurable sets is measurable, it suffices to show that for each fixed m,i, and $n \in N$, the map

$$(x,z,w,y,t) \to \Big\| \frac{1}{m} \sum_{j=0}^{m-1} h_i \circ F^y_{nt}\,(x,z,w)$$

$$- \int_{X_n \times I} h_i \, dp_y^{(n)} \otimes m \Big\|_{L^2(X_n \times I, p_y^{(n)} \otimes m)}$$

is measurable, for every $x \in X_n$, $z \in (-n,n)$, $w \in T^1$, $y \in Y$, and $t \in [0,1]$.

Using the definition of Lebesgue integral and elementary facts about measurable functions, it is not difficult to see that the map:

$$\Phi_{(m,i,n)} : X \times (-n,n) \times T^1 \times Y \times I \to \mathbb{R} \qquad \text{given by:}$$

where $| \quad |_{L^2(X_n \, I, p_y^{(n)} \otimes m)}$ denotes the relevant L^2 norm and $h_i \in \{h_k\}_{k \in N}$ and $\{h_k\}_{k \in N}$ is a countable dense sequence of Borel (L^2) functions on $X_n \times I$, (hence measurable for $p_y^{(n)} \otimes m$, for every $y \in Y$).

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$$(x,z,w,y,t) \to \Big| \frac{1}{m} \sum_{j=0}^{m-1} h_i \circ F^y_{nt} (x,z,w)$$

$$- \int_{X_n \times I} h_i \, dp_y^{(n)} \otimes m \Big|_{L^2(X_n \times I, p_y^{(n)} \otimes m)}$$

is measurable, for every $x \in X_n$, $z \in (-n,n)$, $w \in T^1$, $y \in Y$, and $t \in [0,1]$.

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$$(x,z,w,y,t) \to \left| \frac{1}{m}\sum_{j=0}^{m-1} h_i \circ F^j_{nt}(x,z,\ ) - \int_{X_n \ T^1} h_i \, dp_y^{(n)} \otimes m \right|_{L^2(X_n \times I)}$$

is Borel for each fixed triplet (m,i,n) and hence the infimum map, denoted $\Phi_{i,n}$ is Borel also.

Thus the set $\overline{Q} = \bigcap_{n\varepsilon N} \bigcap_{i\varepsilon N} \phi^{-1}_{i,n}(0)$

is a measurable set in $X \times \mathbb{R} \times T^1 \times Y \times I$, and by Fubini's Theorem we have that the projection on $Y \times I$ of $\overline{Q}$ is measurable in $Y \times I$. Now we conclude by observing that

$$\Pi_{Y\times I}(\overline{Q}) = \{(y,t) \in Y\times I \,|\, F_t \text{ is } \rho\otimes m\text{-ergodic}\}$$

$$= Q \text{ , so the lemma is proved.}$$

§12. Type III_0 Diffeomorphisms of $T^1 \times R$

We begin with a result which is analogous to Proposition 9.3(Ch. III), but is weaker due to the non-compactness of $\mathbb{R}$.

Proposition 12.1.

Let $(X,\mathcal{S},\mu)$ be a smooth compact manifold with μ a C^∞ probability measure on X . Let $f \in \mathrm{Diff}^\infty(X)$ be an ergodic diffeomorphism. Suppose there exists an element which is not a coboundary in the set:

$$\mathcal{C} = \overline{\{\phi \in C^\infty(X,\mathbb{R}) \mid \phi = \eta - \eta \circ f \quad \text{for some Borel map} \quad \eta : X \to \mathbb{R}}\ ,$$

(where the closure is taken with respect to the C^∞ topology). Then the set of coboundaries is meagre in $\mathcal{C}$.

Proof.

The proof is similar to that of III.9.3. We consider $\mathcal{C}$ as a complete topological group under pointwise addition and with respect to the C^∞ topology. If we let

$$E = \{\phi : X \to \mathbb{R} \mid \phi \text{ is a Borel map and } \phi - \phi \circ f = h \text{ a.e. for some } h \in \mathcal{C}\}\ ,$$

and identify two functions in E if and only if they are equal μ-a.e., then we see that E is a group under pointwise addition.

We define the map: $L:E \to \mathcal{C}$ by setting $L(\phi) = \phi - \phi \circ f$. We see that L is a group homomorphism and $\ker L$ = constant maps $\cong \mathbb{R}$. We now define a metric on E by:

for all $\phi_1, \phi_2 \in E$,

$$\delta_{\mathbb{R}}(\phi_1, \phi_2) = \int_X \frac{|\phi_1 - \phi_2|}{1 + |\phi_1 - \phi_2|} d\mu + ||L\phi_1 - L\phi_2||_\infty \ ;$$

then we see that E is complete and separable with respect to $\delta_{\mathbb{R}}$, and that L is a continuous group homomorphism.

Using the Open Mapping Theorem, and the assumption that there exists $\psi \in \mathcal{C}$ such that $\psi \notin$ image L, the proposition is proved.

Remark 12.2.

Under the assumption that there exists at least one element which is not a coboundary in $\mathcal{C}$, we will prove that there is in fact a dense G_δ of elements in $\mathcal{C}$ which contain all of $\mathbb{R}$ in their essential range; i.e. there is a dense G_δ in $\mathcal{C}$, call it $\mathcal{E}_c$, such that if $\phi \in \mathcal{E}_c$, then the skew product given by:

$F : X \times \mathbb{R} \to X \times \mathbb{R}$, where

$$(x,y) \to (fx, y+\phi x) \quad \text{is} \quad \mu m\text{-ergodic.}$$

We begin with an easy lemma, whose proof is similar to [16, 9.6].
We denote $E(a_\phi) = E(\phi)$.

Lemma 12.3.

Let $(X,\mathcal{S},\mu)$ and f be as in 12.1. The set
$\mathcal{E} = \{\phi \in C^\infty(X,\mathbb{R}) \mid F : X \to \mathbb{R}$ defined in Remark 12.2 is μm-ergodic$\}$
is a G_δ .

Proof.

By the most general form of III.9.2, $\mathcal{E} = \{\phi \in C^\infty(X,R) \mid E(\phi) = \mathbb{R}\}$.
Let S_0 denote a countable, dense subalgebra for X . Let $\{\lambda_i\}_{i\in N}$
be a dense sequence in $\mathbb{R}$. Fix any element $B \in \mathcal{S}_0$ and any number
$\beta \in \{\lambda_i\}_{i\in N}$. Claim that for every fixed $\delta > 0$ and $\varepsilon > 0$ the set

$$U(B,\beta,\varepsilon,\delta) = \{\phi \in C^\infty(X,\mathbb{R}) \mid \sup_{V\in[f]} \mu(B \cap V^{-1}B \cap$$

$\{x : a_\phi(V,x) \in (\beta-\varepsilon, \beta+\varepsilon)\}) > \delta\}$ is open in $C^\infty(X,\mathbb{R})$ with the C^∞
topology. (Proof of claim. Clearly, for fixed n , the map
$\phi \to \sum_{i=0}^{n-1} \phi \circ f^i$ is continuous in $C^\infty(X,\mathbb{R})$ with respect to the C^∞

topology. We recall that $Vx = f^{n(x)}x$ for some $n \in Z$ and for every $x \in X$. By [16,2.6], $a_\varphi(V,x) = a_\phi(n,x) = \sum_{i=0}^{n-1} \phi \circ f^i(x)$. Therefore by continuity, we can find $\delta > 0$ small enough s.t. $||\phi-\tilde{\phi}||_\infty < \delta$ implies that $\mu\{x: |a_\varphi(V,x)| < \varepsilon\} = \mu\{x: |a_{\tilde{\varphi}}(V,x)| < \varepsilon\}$.)

Then $\bigcap_{B \in S_0} \bigcap_m U(B,\beta,\frac{1}{m},\frac{\mu(B)}{4})$

$= \{\phi \in C^\infty(X,\mathbb{R}) \mid \beta \in E(\phi)\}$, and finally we have

$$\mathcal{E} = \bigcap_i \bigcap_B \bigcap_m U(B,\lambda_i,\frac{1}{m},\frac{\mu(B)}{4})$$

$= \{\phi \in C^\infty(X,\mathbb{R}) \mid E(\phi) = \mathbb{R}\}$, which proves the proposition.

Remark 12.4.

Since $\mathcal{C} \subset C^\infty(X,\mathbb{R})$ is a closed subgroup, the set $\mathcal{E}_\mathcal{C} = \{\phi \in \mathcal{C} \mid E(\phi) = \mathbb{R}\}$ is also a G_δ . It is easy to see that the set $\mathcal{E}_\mathcal{C}$ is dense in $\mathcal{C}$ if it is not empty. For if there exists an element $\psi \in \mathcal{E}_\mathcal{C}$, then by adding a suitable coboundary to ψ we can find another element of $\mathcal{E}_\mathcal{C}$ arbitrarily close in the C^∞ topology to any $\phi \in \mathcal{C}$.

In the next lemma we will prove that if there exists an element $\psi \in \mathcal{C}$ such that $E(\psi) = \{n\lambda\}_{n \in \mathbb{Z}}$ for some $0 < \lambda$, then there exists an element $\tilde{\psi} \in \mathcal{E}_\mathcal{C}$.

Lemma 12.5.

Let $(X,\mathcal{S},\mu)$ and f be as in 12.1. Suppose there exists an element $\psi \in \mathcal{C}$ such that $E(\psi) = \{n\lambda\}_{n\in Z}$ for some $0 < \lambda$. Then $\mathcal{E}_{\mathcal{C}}$ is a dense G_δ in $\mathcal{C}$.

Proof.

Since $\lambda \in E(\psi)$, we have for all $B \in \mathcal{S}$, $\mu(B) > 0$ and all $\varepsilon > 0$,

$$\sup_{V\in[f]} \mu(B \cap V^{-1}B \cap \{x: |a_\psi(V,x)-\lambda| < \varepsilon\}) > \frac{\mu(B)}{2} .$$

We can choose an irrational scalar $c \in R$, $0 < c < 1$ such that $\beta = c\lambda$, and λ and β are rationally independent. Then for all $x \in \{x: |a_\psi(V,x)-\lambda| < \varepsilon\}$, we have

$$a_{c\psi}(V,x) = \sum_{i=0}^{n-1} c\psi o f^i(x) = c \sum_{i=0}^{n-1} \psi o f^i(x) ,$$

for some $n \in \mathbb{Z}$, so $|a_{c\psi}(V,x) - \beta| < \varepsilon$ as well. Let $\tilde{\psi} = c\psi$. Thus $\beta \in E(\tilde{\psi})$. By adding suitable coboundaries to $\tilde{\psi}$, we see that the set $\{\phi \in \mathcal{C} | E(\phi) \ni \{n\beta\}_{n\in\mathbb{Z}}\}$ is dense in $\mathcal{C}$, and the proof of Lemma 12.3 shows us that it is in fact a G_δ . Similarly, we have that $U_\lambda = \{\phi \varepsilon \mathcal{C} \mid E(\phi) \ni \{n\lambda\}_{n\varepsilon\mathbb{Z}}\}$ is a dense G_δ . Then $U_\lambda \cap U_\beta$ is also a dense G_δ in $\mathcal{C}$, and since the set of essential values for $\phi \varepsilon \mathcal{C}$ forms a closed additive subgroup of $\mathbb{R}$, $\lambda \varepsilon E(\phi)$ and $\beta \varepsilon E(\phi)$ imply that $E(\phi) = \mathbb{R}$ since λ and β are rationally independent. Therefore the proposition is proved.

The only remaining possibility is that every element $\phi \varepsilon \mathcal{C}$ which is not a coboundary satisfies $\overline{E}(\phi) = \{0,\infty\}$. In Theorem 12.7 we will show that even under this assumption we can construct elements in $\mathcal{E}_{\mathcal{C}}$. This theorem is sufficient to ensure that type III_0 diffeomorphisms of compact manifolds have ergodic real line extensions, which is what is needed to extend type III_0 diffeomorphisms to arbitrary manifolds.

We first prove a proposition which is necessary for the construction in Theorem 12.7.

Proposition 12.6.

Let $(X,\mathcal{S},\mu)$ be a smooth paracompact manifold with μ a smooth σ-finite measure on X . Let $f \varepsilon \mathrm{Diff}^\infty(X)$ be a μ-ergodic diffeomorphism. We denote by $\mathcal{S}_0$ a countable dense subalgebra of $\mathcal{S}$, and $\mathfrak{C}$ is as in 12.1. Suppose there exists $\phi \varepsilon \mathfrak{C}$ such that ϕ is recurrent and $\overline{E}(\phi) = \{0,\infty\}$. Then there exists a set $\Psi \subset \mathfrak{C}$, such that Ψ is a dense G_δ in $\mathfrak{C}$ with respect to the C^∞ topology and every element $\psi \varepsilon \Psi$ satisfies the following condition:

For every $\varepsilon > 0$, for every $M \in \mathbb{R}^+$, and for every $B \in \mathfrak{S}_0$, there exists $\phi \in \mathfrak{C}$ with $||\phi||_\infty \leq 1$ and $\overline{E}(\phi) = \{0,\infty\}$, and $V \in [f]$ such that $\mu(B \cap V^{-1}B \cap \{x: |a_\psi(V,x)|<\varepsilon\} \cap \{x: |a_\phi(V,x)| > M\}) \geq \frac{\mu(B)}{2}$.

Proof.

We choose a countable dense set $\{\phi_i\}_{i\in\mathbb{N}}$ in the unit ball of $\mathfrak{C}$, where for every $i \in \mathbb{N}$, $\overline{E}(\phi_i) = \{0,\infty\}$ and ϕ_i is recurrent. We choose a countable dense set in the full group of f , denoted $\{V_k\}_{k\in\mathbb{N}}$. Let $M \in \mathbb{N}$ denote a positive integer. We define the set:

$$\begin{aligned}&\Lambda(B,M,\varepsilon,\ell,j,k,i)\\ &= \{\psi\in\mathfrak{C} \mid \mu(B\cap V_k^{-1}B \cap \{x: |a_\psi(V_k,x)| < \varepsilon\}) \cap \\ &\qquad \{x| \; a_{\frac{1}{j}\phi_i}(V_k,x)| > M\}) \\ &\qquad\quad > (1 - \tfrac{1}{\ell})\mu(B)\cdot 2^{-1} \; .\end{aligned}$$

By the continuity of ψ , and using techniques from 12.3 we can show that for each fixed $(B,M,\varepsilon,\ell,j,k,i)$ the set $\Lambda(B,M,\varepsilon,\ell,j,k,i)$ is open in $\mathfrak{C}$ with respect to the C^∞ topology.

We also claim that $\Gamma(B,M,\varepsilon,\ell,j) = \bigcup_k \bigcup_i \Lambda(B,M,\varepsilon,\ell,j,k,i)$ is open and dense in $\mathfrak{C}$. Clearly it is open, and it is dense because each $\Gamma(B,M,\varepsilon,\ell,j)$

contains the coboundaries. To show this, fix $\varepsilon_o, M_o, B_o, j_o$ and ℓ_o. Suppose $\psi \in \mathfrak{C}$ is a coboundary. Then choose any $\phi_o \in \mathfrak{C}$ which satisfies $\overline{E}(\phi_o) = \{0,\infty\}$ and $\frac{1}{j_o}||\phi_o||_\infty \leq 1$. Since ψ is a coboundary we write

$\psi = \eta - \eta \circ f$ where η is a Borel function on X, and we find a set $D_o \subset B_o$ such that $|\eta(x) - \eta(y)| < \varepsilon_o/4$ for all $x, y \in D_o$. Since $\infty \in \overline{E}(\phi_o)$, we can find an integer p such that $\mu(D_o \cap f^{-p} D_o \cap \{x | a_{\frac{1}{j_o}\phi_o}(p,x)| > M_o\}) > 0$. Using the exhaustion argument

method of [16, 9.4], we can find an element $V_k \in [f]$ such that

$$\mu(B \cap V_k^{-1} B \cap \{x : |a_\psi(V_k, x)| < \varepsilon_o\} \cap$$

$$\{x : |a_{\frac{1}{j_o}\phi_o}(V_k, x)| > M_o\}) > (1 - \frac{1}{\ell_o})\mu(B_o)2^{-1} .$$

This proves that $\psi \in \Gamma(B_o, M_o, \varepsilon_o, \ell_o, j_1)$.

We now define $\Psi = \bigcap_{B \in S_o} \bigcap_M \bigcap_\varepsilon \bigcap_\ell \bigcap_j \Gamma(B, M, \varepsilon, \ell, j)$, (where

$\varepsilon \in \{\varepsilon_r\}_{r \in \mathbb{N}}$ is a countable set such that $\varepsilon_r \leq \frac{1}{r}$).

Clearly Ψ is a dense G_δ, and it remains to show that $\psi \in \Psi$ satisfies the hypotheses of the proposition. If $\psi \in \Psi$, then for every $\varepsilon_r > 0$, for every $M \in \mathbb{N}^+$, for every $B \in S_o$, and for every $j, \ell \in \mathbb{N}^+$ there exists $\phi_i \in C$, $\overline{E}(\phi_i) = \{0,\infty\}$, ϕ_i is recurrent, and there exists $V_k \in [f]$ satisfying:

contains the coboundaries. To show this, fix $\varepsilon_0, M_0, B_0, j_0$ and ℓ_0. Suppose $\psi \in \mathcal{C}$ is a coboundary. Then choose any $\phi_0 \in \mathcal{C}$ which satisfies $E(\phi_0) = \{0,\infty\}$ and $\frac{1}{j_0}||\phi_0||_\infty \leq 1$. Since ψ is a coboundary we write $\psi = \eta - \eta \circ f$ where η is a Borel function on X, and we find a set $D_0 \subset B_0$ such that $|\eta(x)-\eta(y)| < \varepsilon_0/4$ for all $x,y \in D_0$. Since $\infty \in E(\phi_0)$, we can find an integer p such that
$\mu(D_0 \cap f^{-p} D_0 \cap \{x | a_{\frac{1}{j_0}\phi_0}(p,x)| > M_0\}) > 0$. Using the exhaustion argument method of [15, 9.4], we can find an element $V_k \in [f]$ such that

$$\mu(B \cap V_k^{-1} B \cap \{x: |a_\psi(V_k,x)| < \varepsilon_0\} \cap$$

$$\{x: |a_{\frac{1}{j_0}\phi_0}(V_k,x)| > M_0\}) > (1-\frac{1}{\ell_0})\mu(B_0)2^{-1} .$$

This proves that $\psi \in \Gamma(B_0, M_0, \varepsilon_0, \ell_0, j_1)$.

We now define $\Psi = \bigcap_{B \in S_0} \bigcap_M \bigcap_\varepsilon \bigcap_\ell \bigcap_j \Gamma(B,M,\varepsilon,\ell,j)$, (where $\varepsilon \in \{\varepsilon_r\}_{r \in \mathbb{N}}$ is a countable set such that $\varepsilon_r \leq \frac{1}{r}$).

Clearly Ψ is a dense G_δ, and it remains to show that $\psi \in \Psi$ satisfies the hypotheses of the proposition. If $\psi \in \Psi$, then for every $\varepsilon_r > 0$, for every $M \in \mathbb{N}^+$, for every $B \in S_0$, and for every $j,\ell \in \mathbb{N}^+$ there exists $\phi_j \in C$, $E(\phi_j) = \{0,\infty\}$, ϕ_j is recurrent, and there exists $V_k \in [f]$ satisfying:

$\mu(B \cap V_k^{-1}B \cap \{x: |a_\psi(V_k,x)| < \varepsilon\} \cap$

$\{x: |a_{\frac{1}{j}\phi_i}(V_k,x)| > M\}) > (1 - \frac{1}{\ell})\mu(B)\cdot 2^{-1}$.

This concludes the proof.

We make Ψ into a complete metric space by defining a metric on Ψ given by:

$$D_\infty(\phi_1,\phi_2) = ||\phi_1-\phi_2||_\infty + d_\infty(\phi_1,\phi_2) \quad ,$$

where $d_\infty(\phi_1,\phi_2)$ is defined in the following way.

Let $d(\phi,A) = \inf\limits_{\psi\varepsilon A}||\phi-\psi||_\infty$ for any set $A \subset \mathfrak{E}$. We index the countable set of sets $\Gamma(B,\varepsilon,M,j,k)$ by $s \varepsilon \mathbb{N}$, say. Then we define

$$d_s(\phi_1,\phi_2) = \left|\frac{1}{d(\phi_1,\Gamma_s{}^c)} - \frac{1}{d(\phi_2,\Gamma_s{}^c)}\right| , \quad \text{where}$$

Γ_s^c denotes the complement of the open set $\Gamma_s = \Gamma_s(B,M,\varepsilon,\ell,j)$.

Finally we let $d_\infty(\phi_1,\phi_2) = \sum\limits_{s=1}^{\infty} 2^{-s} \dfrac{d_s(\phi_1,\phi_2)}{1+d_s(\phi_1,\phi_2)}$.

An easy calculation shows that D_∞ is a metric on Ψ , and that Ψ is complete with respect to D_∞ .

We are now ready for the main theorem of this section.

Theorem 12.7.

Let (X,S,μ) and $f \in \mathrm{Diff}^\infty(X)$ be as in 12.6. Suppose there exists an element $\phi \in \mathfrak{C}$ which is recurrent and not a coboundary. Then $\mathcal{E}_{\mathfrak{C}}$ is a dense G_δ in $\mathfrak{C}$ with the C^∞ topology.

Proof.

By 12.3-12.5 it suffices to assume that every element of $\mathfrak{C}$ which is not a coboundary satisfies $\overline{E}(\phi) = \{0,\infty\}$.

Let $\mathfrak{S}_0$ denote a countable, dense subalgebra of X . We fix an element $B \in \mathfrak{S}_0$, $\mu(B) > 0$, and we choose and fix any $\varepsilon > 0$.

We will construct $\psi \in C$ and $V \in [f]$ such that
$\mu(B \cap V^{-1}B \cap \{x: |a_\psi(V,x)-1| < \varepsilon\}) \geq \mu(B)/2$.
Then, using the notation and methods of Lemmas 12.3-12.5 we see that $U(B,1,\varepsilon,\frac{\mu(B)}{2})$ is open, dense, and non-empty in $\mathfrak{C}$ (in the C^∞ topology) for each $B \in \mathfrak{S}_0$ and $\varepsilon > 0$, and therefore the theorem is proved.

A. Setting Up the Construction.

Let Ψ be defined as in 12.6. We start the induction process by defining $\phi_0 = 0$, $\bar{B}_0 = B$, $M_0 = 1$, and $\varepsilon_0 = \varepsilon/2$. Since $\phi_0 \in \Psi$ we apply 12.6 to obtain p_1 and ϕ_1 satisfying:

$$\mu(\tilde{B}_0 \cap f^{-p_1}\tilde{B}_0 \cap \{x:|a_{\phi_0}(p_1,x)|<\varepsilon_0\} \cap \{x:|a_{\phi_1}(p_1,x)|>M_0\}) > 0. \quad (12.1)$$

Since the set Ψ is dense in $\mathfrak{C}$, we can perturb ϕ_1 slightly if necessary so that $\phi_1 \in \Psi$, and (12.1) still holds. We choose

$$B_1 \subseteq \tilde{B}_0 \cap f^{-p_1}\tilde{B}_0 \cap \{x:|a_{\phi_0}(p_1,x)|<\varepsilon_0\} \cap \{x:|a_{\phi_1}(p_1,x)| > M_0\}$$

such that $B_1 \cap f^{p_1}B_1 = \emptyset$. Choose $c_1 \leq 1$ satisfying

$$\mu(\tilde{B}_0 \cap f^{-p_1}\tilde{B}_0 \cap \{x:|a_{\phi_0}(p_1,x)|<\varepsilon_0\} \cap \{x:|a_{c_1\phi_1}(p_1,x)-1|<\varepsilon_0\}) > 0 \quad .$$

We define $V_1 \in [f]$ by

$$V_1(x) = \begin{cases} f^{p_1}x & \text{if } x \in B_1 \\ f^{-p_1}x & \text{if } x \in f^{p_1}B_1 \\ x & \text{otherwise,} \end{cases} \quad \text{and let}$$

$\tilde{B}_1 = \tilde{B}_0 \setminus (B_1 \cap f^{p_1}B_1)$. We define $\zeta_1 = c_1\phi_1$.

B. The $j^{\underline{th}}$ Stage.

We will define inductively:

$\phi_j \in \mathfrak{C}, ||\phi_j||_\infty \leq 1$, $c_j \in \mathbb{R}^+$, $s_j \in \mathbb{N}^+$, $M_j \in \mathbb{R}^+$, $\varepsilon_j > 0$, $B_j \subset B$, $\tilde{B}_j \subset B$, $p_j \in \mathbb{N}$, $\zeta_j \in \Psi$ and $V_j \in [f]$ satisfying:

$(1)_j \quad \zeta_j = \sum_{\ell=1}^{j} c_\ell \phi_\ell \in \Psi$,

$(2)_j \quad \mu(\tilde{B}_{j-1} \cap f^{-p_j}\tilde{B}_{j-1} \cap \{x: |a_{c_j\phi_j}(p_j,x)-1| < \varepsilon_j\} \cap$

$\{x: |a_{\zeta_{j-1}}(p_j,x)| < \varepsilon_j\}) > 0$,

$(3)_j \quad D_\infty(\zeta_{j-1},\zeta_j) < \varepsilon_j$,

$(4)_j \quad c_j || \sum_{\ell=0}^{p_\ell - 1} \phi_\ell ||_\infty < \varepsilon(2^{j+2\ell})^{-1}$ for $0 \leq \ell \leq j-1$,

$(5)_j \quad p_j > p_{j-1}$, $\varepsilon_j = \varepsilon(2^{j+1})^{-1}$, $M_j \geq M_{j-1}$,

$(6)_j \quad B_j \subset \tilde{B}_{j-1} \cap f^{-p_j}\tilde{B}_{j-1} \cap \{x: |a_{c_j\phi_j}(p_j,x)-1| < \varepsilon_j\}$,

$\mu(B_j) > 0$, and $B_j \cap f^{p_j}B_j = \emptyset$. We define $\tilde{B}_j = \tilde{B}_{j-1} \setminus (B_j \cup f^{p_j}B_j)$.

$(7)_j$

$$V_j(x) = \begin{cases} V_k(x) & \text{if } x \in B_k , \quad k \leq j-1 \\ f^{p_j}(x) & \text{if } x \in B_j \\ f^{-p_j}(x) & \text{if } x \in f^{p_j}B_j \\ x & \text{otherwise.} \end{cases}$$

C. <u>The induction step.</u>

Assume we are at the $j^{\underline{th}}$ stage. First we choose $s_{j+1} \in \mathbb{N}^+$ large enough so that

$(2)_j$ $\mu(\tilde{B}_{j-1} \cap f^{-p_j}\tilde{B}_{j-1} \cap \{x: |a_{c_j\phi_j}(p_j,x)-1| < \varepsilon_j\} \cap$

$\{x: |a_{\zeta_{j-1}}(p_j,x)| < \varepsilon_j\}) > 0$,

$(3)_j$ $D_\infty(\zeta_{j-1},\zeta_j) < \varepsilon_j$,

$(4)_j$ $c_j\|\sum_{\ell=0}^{p_\ell-1} \phi_{\ell}\|_\infty < \varepsilon(2^{j+2\ell})^{-1}$ for $0 \le \ell \le j-1$,

$(5)_j$ $p_j > p_{j-1}$, $\varepsilon_j = \varepsilon(2^{j+1})^{-1}$, $M_j \geq M_{j-1}$,

$(6)_j$ $B_j \subset \tilde{B}_{j-1} \cap f^{-p_j}\tilde{B}_{j-1} \cap \{x: |a_{c_j\phi_j}(p_j,x)-1| < \varepsilon_j\}$,

$\mu(B_j) > 0$, and $B_j \cap f^{p_j}B_j = \emptyset$. We define $\tilde{B}_j = \tilde{B}_{j-1} \setminus (B_j \cup f^{p_j}B_j)$.

$(7)_j$

$$V_j(x) = \begin{cases} V_k(x) & \text{if } x \in B_k , \; k \le j-1 \\ f^{p_j}(x) & \text{if } x \in B_j \\ f^{-p_j}(x) & \text{if } x \in f^{p_j}B_j \\ x & \text{otherwise.} \end{cases}$$

C. The induction step.

Assume we are at the $j^{\underline{th}}$ stage. First we choose $s_{j+1} \in \mathbb{N}^+$ large enough so that

$\sum_{s=s_{j+1}}^{\infty} 2^{-s} < \varepsilon(2^{j+3})^{-1}$. Then we define

$\gamma_{s_{j+1}} = \gamma_{j+1} = \min\limits_{s \leq s_{j+1}} d(\zeta_j, \Gamma_s^c)$. Since $\zeta_j \in \Psi$, $\gamma_{j+1} > 0$.

Now we choose $M_{j+1} \geq \max(M_j, \varepsilon_j^{-1} \gamma_{j+1}^{-2} S_{j+1} 2^{4j} p_j)$.

Using $(1)_j$ and Proposition 12.6 we can find $p_{j+1} > p_j$ and $\phi \in \mathfrak{t}$ such that $||\phi||_\infty \leq 1$, and for $\varepsilon_{j+1} = \varepsilon_j \cdot 2^{-1}$ we have

$$\mu(\tilde{B}_j \cap f^{-p_{j+1}} \tilde{B}_j \cap \{x: |a_{\zeta_j}(p_{j+1},x)| < \varepsilon_{j+1}\} \cap \{x: |a_\phi(p_{j+1},x)| > M_{j+1}\}) > 0 \tag{12.2}$$

Let $\phi_{j+1} = \phi$. We choose $c_{j+1} < \frac{1}{M_{j+1}}$ such that

$$\mu(\tilde{B}_j \cap f^{-p_{j+1}} \tilde{B}_j \cap \{x: |a_{\zeta_j}(p_{j+1},x)| < \varepsilon_{j+1}\} \cap$$

$$\{x: |a_{c_{j+1}\phi_{j+1}}(p_{j+1},x) - 1| < \varepsilon_{j+1}\}) > 0 \quad . \tag{12.3}$$

We define the set $B_{j+1} \subsetneq (\tilde{B}_j \cap f^{-p_{j+1}} \tilde{B}_j \cap \{x: |a_{\zeta_j}(p_{j+1})x)| < \varepsilon_{j+1}\} \cap$

$\{x: |a_{c_{j+1}\phi_{j+1}}(p_{j+1},x) - 1| < \varepsilon_{j+1}\}$ such that

$\mu(B_{j+1}) > 0$ and $B_{j+1} \cap f^{p_{j+1}} B_{j+1} = \emptyset$.

We can assume that $\zeta_j + c_{j+1}\phi_{j+1} \in \Psi$, because if not we could have chosen $\tilde{c}_{j+1}$ or $\tilde{\phi}_{j+1}$ arbitrarily close by such that (12.2) still holds and $\zeta_j + \tilde{c}_{j+1}\tilde{\phi}_{j+1} \in \Psi$ (since Ψ is dense in $\mathfrak{C}$). Define $\zeta_{j+1} = \zeta_j + c_{j+1}\phi_{j+1} = \sum_{i=1}^{j+1} c_i\phi_i$.

We must now check to see if $(1)_{j+1}$ through $(7)_{j+1}$ hold.

For $(7)_{j+1}$, we just define:

$$V_{j+1}(x) = \begin{cases} V_k(x) & \text{if } x \in B_k, \; k \leq j \\ f^{p_{j+1}}(x) & \text{if } x \in B_{j+1} \\ f^{-p_{j+1}}(x) & \text{if } x \in f^{p_{j+1}}B_{j+1} \\ x & \text{otherwise.} \end{cases}$$

By our construction, $(1)_{j+1}$, $(2)_{j+1}$, $(5)_{j+1}$, and $(6)_{j+1}$ obviously are satisfied.

To check $(3)_{j+1}$ we proceed as follows

$$D_\infty(\zeta_j,\zeta_{j+1}) = ||\zeta_j-\zeta_{j+1}||_\infty + \sum_{s=1}^{\infty} 2^{-s} \frac{d_s(\zeta_j,\zeta_{j+1})}{1+d_s(\zeta_j,\zeta_{j+1})}$$

$$= ||\zeta_j-\zeta_{j+1}||_\infty + \sum_{s=1}^{s_{j+1}} 2^{-s} \frac{d_s(\zeta_j,\zeta_{j+1})}{1+d_s(\zeta_j,\zeta_{j+1})} + \sum_{s=s_{j+1}+1}^{\infty} 2^{-s} \frac{d_s(\zeta_j,\zeta_{j+1})}{1+d_s(\zeta_j,\zeta_{j+1})}$$

$$\leq c_{j+1}||\phi_{j+1}||_\infty + \sum_{s=1}^{s_{j+1}} 2^{-s} \frac{d_s(\zeta_j,\zeta_{j+1})}{1+d_s(\zeta_j,\zeta_{j+1})} + \sum_{s=s_{j+1}+1}^{\infty} 2^{-s}$$

$$\leq c_{j+1}||\phi_{j+1}||_\infty + \sum_{s=1}^{s_{j+1}} 2^{-s} \frac{d_s(\zeta_j,\zeta_{j+1})}{1+d_s(\zeta_j,\zeta_{j+1})} + \varepsilon(2^{j+3})^{-1} ,$$

by our choice of s_{j+1} .

We have that

$$d_s(\zeta_j,\zeta_{j+1}) = \left| \frac{1}{d(\zeta_j,\Gamma_s^c)} - \frac{1}{d(\zeta_{j+1},\Gamma_s^c)} \right| \leq \left| \frac{1}{d(\zeta_j,\Gamma_s^c)} - \frac{1}{d(\zeta_j,\Gamma_s^c) + ||\zeta_j-\zeta_{j+1}||_\infty} \right| ,$$

since $\left|d(\zeta_j,\Gamma_s^c) - d(\zeta_{j+1},\Gamma_s^c)\right| \leq ||\zeta_j-\zeta_{j+1}||_\infty$ for each s , so the denominators can vary by at most $||\zeta_j-\zeta_{j+1}||_\infty$, and now we have

$$||\zeta_j-\zeta_{j+1}||_\infty \leq c_{j+1}||\phi_{j+1}||_\infty \leq c_{j+1}\cdot 1 \leq \varepsilon_j\cdot\gamma_{j+1}^2\cdot 2^{-4j}\cdot p_j^{-1}\cdot s_{j+1}^{-1}$$

$$\leq \varepsilon\cdot\gamma_{j+1}^2\cdot 2^{-5j}\cdot p_j^{-1}\cdot s_{j+1}^{-1} ,$$

and recalling our choice of $\gamma_{j+1} > 0$, we have

$$D_\infty(\zeta_j,\zeta_{j+1}) \leq \varepsilon\cdot 2^{-5j} + \sum_{s=1}^{s_{j+1}} \frac{\gamma_{j+1}^2\cdot\varepsilon\cdot 2^{-(j+6)}s_{j+1}^{-1}}{\gamma_{j+1}^2} \Bigg/ 1+ \frac{\gamma_{j+1}^2\cdot\varepsilon\cdot 2^{-(j+6)}s_{j+1}^{-1}}{\gamma_{j+1}^2}$$

$$+ \varepsilon\cdot 2^{-(j+3)}$$

$$\leq c_{j+1}\|\phi_{j+1}\|_\infty + \sum_{s=1}^{s_{j+1}} 2^{-s}\frac{d_s(\zeta_j,\zeta_{j+1})}{1+d_s(\zeta_j,\zeta_{j+1})} + \varepsilon(2^{j+3})^{-1} \quad ,$$

by our choice of s_{j+1} .

We have that

$$d_s(\zeta_j,\zeta_{j+1}) = \left| \frac{1}{d(\zeta_j,\Gamma_s^c)} - \frac{1}{d(\zeta_{j+1},\Gamma_s^c)} \right| \leq \left| \frac{1}{d(\zeta_j,\Gamma_s^c)} - \frac{1}{d(\zeta_j,\Gamma_s^c) + \|\zeta_j-\zeta_{j+1}\|_\infty} \right| \quad ,$$

since $\left|d(\zeta_j,\Gamma_s^c) - d(\zeta_{j+1},\Gamma_s^c)\right| \leq \|\zeta_j-\zeta_{j+1}\|_\infty$ for each s , so the denominators can vary by at most $\|\zeta_j-\zeta_{j+1}\|_\infty$, and now we have

$$\|\zeta_j-\zeta_{j+1}\|_\infty \leq c_{j+1}\|\phi_{j+1}\|_\infty \leq c_{j+1}\cdot 1 \leq \varepsilon_j\cdot\gamma_{j+1}^2\cdot 2^{-4j}\cdot p_j^{-1}\cdot s_{j+1}^{-1}$$

$$\leq \varepsilon\cdot\gamma_{j+1}^2\cdot 2^{-5j}\cdot p_j^{-1}\cdot s_{j+1}^{-1} \quad ,$$

and recalling our choice of $\gamma_{j+1} > 0$, we have

$$D_\infty(\zeta_j,\zeta_{j+1}) \leq \varepsilon\cdot 2^{-5j} + \sum_{s=1}^{s_{j+1}} \frac{\gamma_{j+1}^2\cdot\varepsilon\cdot 2^{-(j+6)}s_{j+1}^{-1}}{\gamma_{j+1}^2} \Bigg/ 1+\frac{\gamma_{j+1}^2\cdot\varepsilon\cdot 2^{-(j+6)}s_{j+1}^{-1}}{\gamma_{j+1}^2}$$

$$+ \varepsilon\cdot 2^{-(j-3)}$$

$$\leq \varepsilon\cdot 2^{-5j} + s_{j+1}(\varepsilon\cdot 2^{-(j+6)} s_{j+1}^{-1}) + \varepsilon\cdot 2^{-(j+3)}$$

$$\leq \varepsilon\cdot 2^{-5j} + \varepsilon\cdot 2^{-(j+6)} + \varepsilon\cdot 2^{-(j+3)}$$

$$< \varepsilon\cdot 2^{-(j+3)} = \varepsilon_{j+1} \quad .$$

We check $(4)_{j+1}$ as follows. Clearly

$$c_{j+1}\left\|\sum_{i=0}^{p_\ell -1}\phi_{j+1}\circ f^i\right\|_\infty \leq c_{j+1}\cdot p_j\cdot\|\phi_{j+1}\|_\infty \quad \text{since} \quad 0 \leq \ell \leq j \quad ,$$

$$\leq \varepsilon_j\cdot\gamma_{j+1}^2\cdot s_{j+1}^{-1}\cdot p_j^{-1}\cdot 2^{-4j}\cdot p_j\cdot\|\phi_{j+1}\|_\infty$$

$$\leq \varepsilon\cdot 2^{-(j+1)}\cdot 2^{-(4j)}$$

$$\leq \varepsilon\cdot 2^{-(j+1+2\ell)} \quad .$$

D. Taking the limit.

We let $\psi = \sum_{i=1}^{\infty} c_i\phi_i$. Since $(3)_j$ holds for all $j \geq 1$, ψ is the limit of a Cauchy sequence in Ψ . More precisely, given any $\delta > 0$ (we might as well assume $0 < \delta < \varepsilon$) , we choose $j_0 \in \mathbb{N}^+$ such that $\sum_{j=j_0}^{\infty} 2^{-j} < \frac{\delta}{\varepsilon}$. Then for any $m > n \geq j_0$ we have

$$D_\infty(\zeta_n,\zeta_m) \leq \sum_{k=0}^{m-n-1} D\,(\zeta_{n+k},\zeta_{n+k+1})$$

$$\leq \sum_{k=0}^{m-n-1} \varepsilon_{n+k+1}\ , \quad \text{by} \quad (3)_j\ .$$

Now $\displaystyle\sum_{k=0}^{m-n-1} \varepsilon_{n+k+1} \leq \varepsilon \sum_{k=0}^{m-n-1} 2^{-(n+k+1)}$

$$\leq \varepsilon \sum_{k=j_0}^{\infty} 2^{-k} < \delta\ ,$$

so $D_\infty(\zeta_n,\zeta_m) < \delta$.

Therefore $\psi \varepsilon \Psi$, since Ψ is complete. We can apply 12.6 to ψ and continue the induction process. Then, using an exhaustion argument, we obtain a sequence of sets B_i such that $B_i \cap B_j = B_i \cap f^{p_j} B_j = B_j \cap f^{p_i} B_i = B_i \cap f^{p_i} B_i = B_j \cap f^{p_j} B_j = \emptyset$, for all $i \neq j$, and such that $\mu(\bigcup_{i \in \mathbb{N}} (B_i \cup f^{p_i} B_i)) > \frac{\mu(B)}{2}$. We also obtain $V \varepsilon [f]$ such that

$$V(x) = \begin{cases} V_k(x) & \text{if } x \varepsilon B_k \cup f^{p_k} B_k \\ x & \text{otherwise.} \end{cases} \tag{12.4}$$

For all $x \varepsilon B_j$, we have $V(x) = f^{p_j}(x)$, and this implies for $x \varepsilon B_j$,

$$|a_\psi(V,x)| = |a_\psi(p_j,x)|$$

$$\leq |a_{\zeta_{j-1}}(p_j,x)| + |a_{c_j\phi_j}(p_j,x)| + |\sum_{k=0}^{p_j-1} (\sum_{i=j+1}^{\infty} c_i\phi_i) \circ f^k(x)|$$

$$\leq \varepsilon_j + 1 + \varepsilon_j + |\sum_{i=j+1}^{\infty} c_i (\sum_{k=0}^{p_j-1} \phi_i \circ f^k(x)|$$

by $(2)_j$, and now by $(4)_j$,

$$\leq \varepsilon_j + 1 + \varepsilon_j + \sum_{i=j+1}^{\infty} \varepsilon (2^{i+2(i-1)})^{-1}$$

$$\leq \varepsilon_j + 1 + \varepsilon_j + \varepsilon/8 \leq 1 + (5/8)\varepsilon \quad .$$

From the above and (12.4), an easy calculation shows that $\mu(B \cap V^{-1}B \cap \{x: |a_\psi(V,x) - 1| < \varepsilon\}) \geq \mu(B)/2$, and we are done.

We should point out that 12.3 is true for non-compact X , and the hypotheses on (X,S,μ) in 12.6 are sufficient for 12.7 to be true. We have proved the existence of a dense G_δ of $\mathfrak{C}$ whose elements give ergodic extensions for f ; we now need to see which of these skew products have the same ratio sets as f . We will give a necessary and sufficient condition, but first we will recall some easily proved facts.

<u>Lemma 12.8</u>.

Let (X,S,μ) be as in 12.6. and let $f \in \mathrm{Diff}^\infty(X)$ be any ergodic

$$
\begin{aligned}
|a_\psi(V,x)| &= |a_\psi(p_j,x)| \\
&\le |a_{\zeta_{j-1}}(p_j,x)| + |a_{c_j\phi_j}(p_j,x)| + \left|\sum_{k=0}^{p_j-1} \left(\sum_{i=j+1}^{\infty} c_i\phi_i\right) \circ f^k(x)\right| \\
&\le \varepsilon_j + 1 + \varepsilon_j + \left|\sum_{i=j+1}^{\infty} c_i\left(\sum_{k=0}^{p_j-1} \phi_i \circ f^k(x)\right)\right|
\end{aligned}
$$

by $(2)_j$, and now by $(4)_j$,

$$
\le \varepsilon_j + 1 + \varepsilon_j + \sum_{i=j+1}^{\infty} \varepsilon\left(2^{i+2(i-1)}\right)^{-1}
$$

$$
\le \varepsilon_j + 1 + \varepsilon_j + \varepsilon/8 \le 1 + (5/8)\varepsilon \quad .
$$

From the above and (12.4), an easy calculation shows that $\mu(B \cap V^{-1}B \cap \{x : |a_\psi(V,x) - 1| < \varepsilon\}) \ge \mu(B)/2$, and we are done.

We should point out that 12.3 is true for non-compact X , and the hypotheses on (X,S,μ) in 12.6 are sufficient for 12.7 to be true. We have proved the existence of a dense G_δ of $\mathfrak{C}$ whose elements give ergodic extensions for f ; we now need to see which of these skew products have the same ratio sets as f . We will give a necessary and sufficient condition, but first we will recall some easily proved facts.

<u>Lemma 12.8</u>.

Let (X,S,μ) be as in 12.6. and let $f \in \mathrm{Diff}^\infty(X)$ be any ergodic diffeomorphism of X. If we consider the skew product F_ϕ defined by:

$$F_\phi : X\times\mathbb{R} \to X\times\mathbb{R}$$

$$(x,y) \mapsto (fx, y+\phi x) \quad \text{where} \quad \phi\varepsilon\mathcal{C} , \quad \text{then} \quad r^*(F_\phi) \subset r^*(f) .$$

Proof.

Let $F = F_\phi$. By definition,

$r^*(F) = \overline{E}(\log \frac{d\mu\mathrm{m}F^{-1}}{d\mu\mathrm{m}})$ where $\frac{d\mu\mathrm{m}F^{-1}}{d\mu\mathrm{m}}(x,y)$ denotes the Radon-Nikodym derivative of the measure $\mu\mathrm{m}(F^{-1})$ with respect to $\mu\mathrm{m}$ at the point $(x,y) \in X\times\mathbb{R}$. This implies

$$\frac{d\mu\mathrm{m}F^{-1}}{d\mu\mathrm{m}}(x,y) = \det DF(x,y)$$

$$= \det\begin{vmatrix} df(x) & 0 \\ d\phi(x) & 1 \end{vmatrix}$$

$$= df(x) = \frac{d\mu f^{-1}}{d\mu}(x) .$$ From this we see that

$\lambda\varepsilon r^*(F) \Rightarrow \lambda\varepsilon r^*(f)$.

Remark 12.9.

Given two cocycles on X, $\phi_1,\phi_2 \in C^\infty(X\times\mathbb{Z},\mathbb{R})$, we can define a cocycle $(\phi_1,\phi_2):\mathbb{Z}\times X \to \mathbb{R}^2$ by $(\phi_1,\phi_2)(n,x) = (\phi_1(n,x),\phi_2(n,x)) \; \forall \; n \in \mathbb{Z}$, $x \in X$. We compactify $\mathbb{R}^2$ by adding lines of the form (α,∞), $(\alpha,-\infty)$, (∞,α), $(-\infty,\alpha)$ for all $\alpha \in \mathbb{R}$, plus four points at $(-\infty,\infty)$, (∞,∞) ,

$(\infty,-\infty)$, $(-\infty,-\infty)$. Then $(\lambda,\beta) \in \overline{E}(\phi_1,\phi_2)$ means that for every $B \in \mathcal{S}$, $\mu(B) > 0$ and for every $\varepsilon > 0$, there exists $n \in \mathbb{Z}$ such that $\mu(B \cap f^{-n}B \cap \{x: |(\phi_1(n,x),\phi_2(n,x))-(\lambda,\beta)| < \varepsilon\}) > 0$, or equivalently, $\mu(B \cap f^{-n}B \cap \{x: |\phi_1(n,x)-\lambda| < \varepsilon\} \cap \{x: |\phi_2(n,x)-\beta| < \varepsilon\}) > 0$.

It is clear that $(\lambda,\beta) \in \overline{E}(\phi_1,\phi_2)$ implies that $\lambda \in \overline{E}(\phi_1)$ and $\beta \in \overline{E}(\phi_2)$, but the converse is not necessarily true. We give an example of the usefulness of considering two cocycles together in the next proposition.

Proposition 12.10.

With (X,S,μ) an m-dimensional manifold and f as in 12.6, we assume further that f is of type III_0 and that the map F_ϕ defined in 12.8 is $\mu\otimes m$-ergodic. Then $(0,\infty) \in \overline{E}(\phi,\log \frac{d\mu f^{-1}}{d\mu})$ if and only if F_ϕ is of type III_0 .

Proof.

(=>) Assume that $(0,\infty) \in \overline{E}(\phi,\log \frac{d\mu f^{-1}}{d\mu})$. By 12.8 it suffices to show that $\infty \in r^*(F_\phi)$. Let $\mathcal{C} \in \mathcal{S} \times \mathcal{J} \subset X \times \mathbb{R}$ be such that $\mu\otimes m(\mathcal{C}) > 0$. Choose t_0 to be a point of density of $\mathcal{C}$. Then there exists an m+1-dimensional cube $R \subset X \times I$ of volume $\delta > 0$, centred at $t_0 = (t_1,t_2)$ such that $\mu\otimes m(R \cap \mathcal{C}) > .99\delta$. By setting $B = \pi_X(R \cap \mathcal{C})$, we see that $\mu(B) > 0$. Since $(0,\infty) \in \overline{E}(\phi, \log \frac{d\mu f^{-1}}{d\mu})$, there exists $n \in \mathbb{Z}$ such that

$$\mu(B \cap f^{-n}B \cap \{x: |\sum_{i=0}^{n-1} \phi \circ f^{i}(x)| < \delta^{\frac{1}{m+1}}\} \cap \{x: |\log \frac{d\mu f^{n}}{d\mu}(x)| > M\}) > 0 \quad .$$

This implies that

$$\mu \otimes m((R \cap \hat{C}) \cap \{(x,y) \in R \cap \hat{C} | (f^{-n}x, y - \sum_{i=0}^{n-1} \phi \circ f^{i}(x)) \in R \cap \hat{C}\} \cap$$

$$\{(x,y) | \ |\log \frac{d\mu f^{n}}{d\mu}(x)| > M\}) > 0 \ .$$

Therefore $\infty \in r^{*}(F)$.

($\Leftarrow$) Suppose that F is of type III_0 , i.e. $r^{*}(F) = \{0,\infty\}$. Then for every set $\hat{C} \subset S \times \mathcal{S}$, $\mu \otimes m(\hat{C}) > 0$, and for every $M \in \mathbb{R}^{+}$, we can find an integer n such that

$$\mu \otimes m(\hat{C} \cap F^{-n}\hat{C} \cap \{(x,y): |\log \frac{d\mu \otimes m F^{n}}{d\mu \otimes m}(x,y)| > M\}) > 0 \quad .$$

Since $\log \frac{d\mu \otimes m F^{n}}{d\mu \otimes m}(x,y) = \log \frac{d\mu f^{n}}{d\mu}(x)$,

and since $F^{-n}C = \{(f^{-n}x, y - \sum_{i=0}^{n-1} \phi \circ f^{i}(x)), \ (x,y) \in \hat{C}\}$,

then for any $B \in S$, $\mu(B) > 0$, we just choose $\hat{C} = B \times (-\varepsilon/2, \varepsilon/2)$.

Then clearly there exists $n \in \mathbb{Z}$ such that

$$\mu(B \cap f^{-n}B \cap \{x: |\sum_{i=0}^{n-1} \phi o f^{i}(x)| < \varepsilon\} \cap \{x: |\log\frac{d\mu f^{-n}}{d\mu}(x)| > M\}) > 0 \ .$$

This implies that $(0,\infty) \in \overline{E}(\phi, \log\frac{d\mu f^{-1}}{d\mu})$.

Finally we prove the existence of a dense G_δ of elements in $\mathfrak{C}$ which satisfy the hypotheses of 12.10. We assume X and f are as in 12.6.

<u>Proposition 12.11.</u>

Let $\psi: X \to \mathbb{R}$ be a fixed (C^∞) cocycle for f with $\overline{E}(\psi) = \{0,\infty\}$. Then the set $\mathfrak{A} = \{\phi \in \mathfrak{C} \mid [0,\infty) \in \overline{E}(\phi,\psi)\}$ is a dense G_δ in $\mathfrak{C}$.

<u>Proof.</u>

Let $\mathfrak{S}_0$ be a countable dense subalgebra for X . Choose any $B \in \mathfrak{S}_0$, and fix $M \in \mathbb{R}^+$. Since $\infty \in \overline{E}(\psi)$, there exists $V \in [f]$ such that $\mu(B \cap V^{-1}B \cap \{x: |a_\psi(V,x)| > M\}) > \frac{\mu(B)}{2}$. If we define the set

$$\mathcal{A}(B,M,\varepsilon) = \{\phi \in \mathfrak{C} \mid \sup_{V \in [f]} \mu(B \cap V^{-1}B \cap \{x: |a_\psi(V,x)| > M\} \cap$$

using the same argument as in Lemma 12.3 we see that it is open for fixed $B,M,$ and ε . Now

$$\mathfrak{A} = \bigcap_{B\in S_0} \bigcap_{M\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \mathcal{A}(B,M,\tfrac{1}{m}) = \{\phi\in C \mid (0,\infty) \in \overline{E}(\phi,\psi)\} \quad .$$

Clearly this set is a G_δ . To show that it is dense, we observe that the coboundaries are dense in $\mathfrak{C}$ and obviously lie in $\mathfrak{A}$.

Theorem 12.12.

With X and f as in 12.6, suppose further that f is of type III_0. Then the set $\mathfrak{C}_0 = \{\phi\in\mathfrak{C} \mid (x,y) \mapsto (fx,y+\phi x)$ is of type $III_0\}$ is a dense G_δ in $\mathfrak{C}$.

Proof.

By 12.7, we have that $\mathcal{E}_{\mathfrak{C}}$ is a dense G_δ . By 12.10 and 12.11, we have $\mathfrak{A}$ is a dense G_δ . Then $\mathcal{E}_{\mathfrak{C}} \cap \mathfrak{A}$ is a dense G_δ of $\mathfrak{C}$ and $\mathfrak{C}_0 = \mathcal{E}_{\mathfrak{C}} \cap \mathfrak{A}$.

Corollary 12.13.

There are uncountably many C^∞ type III_0 diffeomorphism on $T^n \times \mathbb{R}^p$, for every $n \geq 1$, $p \geq 0$.

Proof.

For $n = 1$, $p = 0$, we use Katznelson's construction. By 11.1, the result is true for all $n \geq 1$ when $p = 0$. By repeated applications of 12.3-12.12 and an induction argument, the corollary is proved.

§13. Type III_λ Diffeomorphisms of $T^n \times \mathbb{R}^p$, $0 < \lambda < 1$.

In this section we will examine type III_λ diffeomorphisms and show that all the results from §12 hold true for type III_λ , with $0 < \lambda < 1$. In some sense, type III_λ transformations are better behaved than III_0 ; there is only one type III_λ ergodic transformation, up to weak equivalence, for each $0 < \lambda < 1$. We will state the analogous theorems to those in §12 and mention the necessary modifications of the proofs.

Theorem 13.1.

The set $O^n_\lambda = \{f \in FR^\infty(T^n) | f$ is of type $III_\lambda\}$ is dense in $F^\infty(T^n)$ for every $0 < \lambda < 1$.

Proof.

The proof is the same as 11.1 and 11.2; we take the suspension flow of the skew product

$$F: T^k \times \mathbb{R} \to T^k \times \mathbb{R}$$

$$(x,z) \mapsto \left(fx, z+\log \frac{d_\mu f^{-1}}{d\mu}(x)\right) ,$$ and obtain type III_λ diffeomorphisms of $T^{k+1} \times \mathbb{R}$ once we have proved that $Q_\lambda = \{(y,t) \in (Y\times I, \mathfrak{I}\times\mathfrak{J}, \rho\times m): \ F_t$ is $q_y \times m$-ergodic$\}$ is measurable. (Here, $(Y,\mathfrak{I},\rho)$ corresponds to the ergodic decomposition of a type III_λ

transformation. We refer to §11 and [16] for details.)

Theorem 13.2.

Let (X,S,μ) and f be as in 12.6. Suppose further that f is of type III_λ, $0 < \lambda < 1$. Then the set

$$\mathfrak{C}_\lambda = \{\phi \in \mathfrak{C} \mid (x,y) \to (fx, y+\phi x) \text{ is of type } III_\lambda\}$$

is a dense G_δ in $\mathfrak{C}$.

Proof.

Use Lemma 12.7 and an obvious modification of 12.11 and 12.12.

Corollary 13.3.

For every λ, $0 < \lambda < 1$, there are uncountably many C^∞ type III_λ diffeomorphisms of $T^n \times \mathbb{R}^p$, for every $n \geq 1$, $p \geq 0$.

Proof.

The same as 12.13.

§14. Type III_λ, $0 \leq \lambda \leq 1$ Diffeomorphisms of Arbitrary Manifolds.

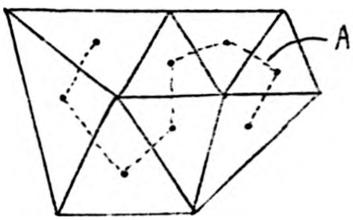
Herman proved in [7] that every connected paracompact manifold of dimension ≥ 3 has a C^∞ type III_1 diffeomorphism on it. He gave a nice method for extending results on $T^2 \times \mathbb{R}^{m-2}$ to any connected paracompact m-dimensional manifold for $m \geq 3$. We will outline the method here for completeness, also including some explanations and modifications for our particular circumstances.

Lemma 14.1.

Let X be an m-dimensional C^∞ paracompact connected manifold and μ a C^∞ measure on X. Then there exists an open set $V \subset X$, diffeomorphic to $\mathbb{R}^m$ and satisfying $\mu(X-V) = 0$.

Proof.

Let T be a C^∞ triangulation of X, and let T' denote the dual complex of T, i.e. every m-simplex of T corresponds to a vertex of T', and if two m-simplices of T meet in an $(m-1)$-simplex, this corresponds to a dual edge in T' formed by joining interior points in the touching m-simplices. Let C denote the one-skeleton of T', (so C consists of dual vertices and dual edges of T'). Let A be a maximal tree for C; then by definition A is a one-dimensional simplicial complex which contains every dual vertex of T' and is contractible. Each vertex of A corresponds to an m-simplex of T, so we can consider the interior of the union of m-simplices of T traced out by A. We therefore have an m-dimensional open set V which is PL isomorphic to $\mathbb{R}^m$ and therefore diffeomorphic to $\mathbb{R}^m$. Since $X-V$ forms part of the $(m-1)$-skeleton of T, it clearly has Lebesgue measure zero.



Lemma 14.2.

If $m \geq 3$, there exists an open set U of $\mathbb{R}^m$ diffeomorphic to $T^2 \times \mathbb{R}^{m-2}$ such that $\mu(\mathbb{R}^m - U) = 0$.

Proof.

We write $\mathbb{R}^m = \mathbb{R}^3 \times \mathbb{R}^{m-3}$ and then it suffices to prove the theorem for $m = 3$. We write $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$. Then by removing $\{0,0\} \times \mathbb{R}$ from $\mathbb{R}^2 \times \mathbb{R}$ we have $(\mathbb{R}^2 - \{0,0\}) \times \mathbb{R} \equiv T^1 \times \mathbb{R}^2$.
Now $T^1 \times \mathbb{R}^2 - \{T^1 \times (0,0)\} \equiv T^2 \times \mathbb{R}$.

Lemma 14.3.

There exists a C^∞ type III_0 (III_λ, $0 < \lambda < 1$) flow on $T^2 \times \mathbb{R}^p$ for every $p \in \mathbb{N}$.

Proof.

We apply 12.13 (13.3) to obtain a C^∞ type III_0 $(III_\lambda,\ 0 < \lambda \leq 1)$ diffeomorphism of $T^1 \times \mathbb{P}^p$, then we take the suspension flow.

Lemma 14.4.

Let U be an open set of $\mathbb{R}^m$, and let f_t be a C^∞ flow of type III_0 $(III_\lambda,\ 0 < \lambda \leq 1)$ on U . Let χ be the infinitesimal generator of f_t , i.e. χ is defined by:

$\frac{\partial f_t}{\partial t}(x)\Big|_{t=0} = \chi \circ f_t(x)$. Let $\phi \in C^\infty(U,\mathbb{R})$, $\phi > 0$, be defined such that the vector field $\phi\chi$ is globally integrable and defines a flow g_t . Then the flow g_t is weakly equivalent to f_t .

Proof.

The flow f_t satisfies the differential equation: $\frac{\partial f_t}{\partial t} = \chi \circ f_t$, $f_0(x) = x$, and $df_t(x) = \det(Df_t(x))$ satisfies:

$$\frac{\partial}{\partial t}\log(df_t) = \operatorname{div}(\chi)\circ f_t\ ,\quad \log(df_0(x)) = 0\ ,$$

where $\operatorname{div}\chi = \sum_{i=1}^{m} \frac{\partial \chi_i}{\partial x_i}$. It follows that g_t satisfies:

Proof.

By 14.1 and 14.2 we have an open set $U \subset X$ of full μ-measure and such that μ is diffeomorphic to $T^2 \times \mathbb{R}^{m-2}$. Let f_t be a type III_0 (III_λ, $0 < \lambda \leq 1$) flow on U with infinitesimal generator χ; such as flow exists by 14.3. Let $\phi \in C^\infty(X,\mathbb{R})$ be such that $\phi > 0$ on U, $\phi = 0$ on $X-U$, and such that the vector field

$$Y(x) = \begin{cases} \phi(x)\chi(x) , & \text{if } x \in U , \\ 0 , & \text{if } x \in X-U \end{cases}$$

is C^∞ on X and globally integrable, thus defining a flow, $\tilde{f}_t$, on X. The flow $\tilde{f}_t$ is of type III_0 (III_λ) by 14.4.

Corollary 14.6.

There exist uncountably many C^∞ type III_0 (III_λ, $0 < \lambda \leq 1$) diffeomorphisms on every connected, paracompact C^∞ manifold of dimension ≥ 3.

Proof.

Apply 14.5, 8.4, and the technique used in 11.1.

Proof.

We apply 12.13 (13.3) to obtain a C^∞ type III_0 (III_λ, $0 < \lambda \leq 1$) diffeomorphism of $T^1 \times \mathbb{R}^p$, then we take the suspension flow.

Lemma 14.4.

Let U be an open set of $\mathbb{R}^m$, and let f_t be a C^∞ flow of type III_0 (III_λ, $0 < \lambda \leq 1$) on U. Let χ be the infinitesimal generator of f_t, i.e. χ is defined by:

$\frac{\partial f_t}{\partial t}(x)\Big|_{t=0} = \chi \circ f_t(x)$. Let $\phi \in C^\infty(U,\mathbb{R})$, $\phi > 0$,

be defined such that the vector field $\phi\chi$ is globally integrable and defines a flow g_t. Then the flow g_t is weakly equivalent to f_t.

Proof.

The flow f_t satisfies the differential equation: $\frac{\partial f_t}{\partial t} = \chi \circ f_t$, $f_0(x) = x$, and $df_t(x) = \det(Df_t(x))$ satisfies:

$$\frac{\partial}{\partial t}\log(df_t) = \mathrm{div}(\chi)\circ f_t\ , \quad \log(df_0(x)) = 0\ ,$$

where $\mathrm{div}\ \chi = \sum_{i=1}^{m} \frac{\partial \chi_i}{\partial x_i}$. It follows that g_t satisfies:

$$\frac{\partial g_t}{\partial t} = \phi\cdot\chi^\circ g_t \ , \quad g_o = Id \ , \quad \text{and}$$

$$\frac{\partial}{\partial t}\log(dg_t) = div(\phi\cdot\chi)^\circ g_t = (\phi\cdot div\ \chi)^\circ g_t + \frac{\partial}{\partial t}\log(\phi^\circ g_t) \ .$$

We can associate to f_t and g_t flows F_t and G_t respectively, on $U \times \mathbb{R}$ which satisfy the following differential equations:

$$\frac{\partial}{\partial t}F_t\Big|_{t=0} = (\chi, div(\chi)) \ , \quad F_o = Id \ ;$$

$$\frac{\partial}{\partial t}G_t\Big|_{t=0} = (\phi\cdot\chi, div(\phi\cdot\chi)) \ , \quad G_o = Id \ .$$

Let $\tilde{G}_t$ denote the flow on $U \times \mathbb{R}$ associated to the vector field $(\phi\cdot\chi, \phi\cdot(div\ \chi))$. Clearly $\tilde{G}_t$ is weakly equivalent to F_t because it has the same orbits as F_t , hence the same ergodic decomposition. Now we define the map $h: U \times \mathbb{R} \to U \times \mathbb{R}$, where $h(x,y) = (x, y+\log\phi(x))$. It is easy to see that $G_t = h\circ\tilde{G}_t{}^\circ h^{-1}$, and the lemma is proved.

<u>Theorem 14.5</u>.

There exists a C^∞ type III_o $(III_\lambda, 0 < \lambda \leq 1)$ flow on every paracompact, connected manifold X of dimension $m \geq 3$.

Proof.

By 14.1 and 14.2 we have an open set $U \subset X$ of full μ-measure and such that μ is diffeomorphic to $T^2 \times \mathbb{R}^{m-2}$. Let f_t be a type III_0 (III_λ, $0 < \lambda \leq 1$) flow on U with infinitesimal generator χ; such as flow exists by 14.3. Let $\phi \in C^\infty(X,\mathbb{R})$ be such that $\phi > 0$ on U, $\phi = 0$ on $X-U$, and such that the vector field

$$Y(x) = \begin{cases} \phi(x)\chi(x), & \text{if } x \in U, \\ 0, & \text{if } x \in X-U \end{cases}$$

is C^∞ on X and globally integrable, thus defining a flow, $\tilde{f}_t$, on X. The flow $\tilde{f}_t$ is of type III_0 (III_λ) by 14.4.

Corollary 14.6.

There exist uncountably many C^∞ type III_0 (III_λ, $0 < \lambda \leq 1$) diffeomorphisms on every connected, paracompact C^∞ manifold of dimension ≥ 3.

Proof.

Apply 14.5, 8.4, and the technique used in 11.1.

$$\frac{\partial g_t}{\partial t} = \phi\cdot\chi\circ g_t \ , \quad g_o = \mathrm{Id} \quad , \quad \text{and}$$

$$\frac{\partial}{\partial t}\log(dg_t) = \mathrm{div}(\phi\cdot\chi)\circ g_t = (\phi\cdot\mathrm{div}\ \chi)\circ g_t + \frac{\partial}{\partial t}\log(\phi\circ g_t) \quad .$$

We can associate to f_t and g_t flows F_t and G_t respectively, on $U \times \mathbb{R}$ which satisfy the following differential equations:

$$\frac{\partial}{\partial t}F_t\Big|_{t=0} = (\chi,\mathrm{div}(\chi)) \ , \quad F_o = \mathrm{Id} \quad ;$$

$$\frac{\partial}{\partial t}G_t\Big|_{t=0} = (\phi\cdot\chi,\mathrm{div}(\phi\cdot\chi)) \ , \quad G_o = \mathrm{Id} \quad .$$

Let $\bar{G}_t$ denote the flow on $U \times \mathbb{R}$ associated to the vector field $(\phi\cdot\chi,\phi\cdot(\mathrm{div}\ \chi))$. Clearly $\bar{G}_t$ is weakly equivalent to F_t because it has the same orbits as F_t , hence the same ergodic decomposition. Now we define the map $h: U \times \mathbb{R} \to U \times \mathbb{R}$, where $h(x,y) = (x,y+\log\phi(x))$. It is easy to see that $G_t = h\circ\bar{G}_t\circ h^{-1}$, and the lemma is proved.

<u>Theorem 14.5</u>.

There exists a C^∞ type III_0 (III_λ, $0 < \lambda \leq 1$) flow on every paracompact, connected manifold X of dimension $m \geq 3$.

Proof.

By 14.1 and 14.2 we have an open set $U \subset X$ of full μ-measure and such that μ is diffeomorphic to $T^2 \times \mathbb{R}^{m-2}$. Let f_t be a type III_0 $(III_\lambda,\ 0 < \lambda \leq 1)$ flow on U with infinitesimal generator χ ; such as flow exists by 14.3. Let $\phi \in C^\infty(X,\mathbb{R})$ be such that $\phi > 0$ on U, $\phi = 0$ on $X-U$, and such that the vector field

$$Y(x) = \begin{cases} \phi(x)\chi(x) , & \text{if } x \in U , \\ 0 , & \text{if } x \in X-U \end{cases}$$

is C^∞ on X and globally integrable, thus defining a flow, $\tilde{f}_t$, on X. The flow $\tilde{f}_t$ is of type III_0 (III_λ) by 14.4.

Corollary 14.6.

There exist uncountably many C^∞ type III_0 $(III_\lambda,\ 0 < \lambda \leq 1)$ diffeomorphisms on every connected, paracompact C^∞ manifold of dimension ≥ 3 .

Proof.

Apply 14.5, 8.4, and the technique used in 11.1.

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