ON NOETHERIAN RINGS OF FINITE GLOBAL DIMENSION

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CONTENTS

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DECLARATION.

The results contained in Chapters 2, 3 and 4 were obtained during a period of joint work with my supervisor, Dr. C.R. Hajarnavis and with Dr. K.A. Brown. Some of these results form part of a joint paper to appear in the Proceedings of the London Mathematical Society [13]. For the most part, no one person has been responsible for the final versions of the results which appear in the above paper and have subsequently been included in this thesis. However, the results of Section 2.2 and their applications in Section 4.1 are due to K.A. Brown, as are the examples in Section 4.4. The examples in Section 6.2 are also based on suggestions by K.A. Brown.

PREFACE

A commutative Noetherian local ring of finite global dimension is a regular local ring and the structure of such a ring is well described in many texts (see §1.9). It is perhaps surprising that very little seems to be known about the corresponding non-commutative rings and it is our aim in this thesis to examine some of the ways in which the commutative theory can, and cannot be extended to a right Noetherian local ring of finite right global dimension.

Chapter 1 contains basic definitions and, in Chapter 2, results are obtained on the projective and injective dimensions of modules over right Noetherian local rings.

Commutative regular local rings are domains and we begin Chapter 3 by considering the question of when a right Noetherian local ring R of finite right global dimension is a prime ring. An example of Stafford's shows that R is not necessarily prime; however, by examining the Nilpotent radical of R, we are able to extend results of Ramras [57] and Walker [75] and show that R is indeed prime when certain prime ideals are localisable. In Chapter 4 we consider the lattice of prime ideals of R when R is an AR-ring and provide theorems which generalise some of the basic results from the theory of commutative regular local rings. Section 4.3 contains examples which not only illustrate points arising in Chapters 3 and 4 but also show that some of the techniques which are mainstays of the commutative theory, fail dramatically in a non-commutative setting.

In Chapter 5, we generalise the concepts of regular sequences and Cohen Macaulay rings enabling us to prove, in §5.2 and Chapter 6, that a right Noetherian local ring of finite right global dimension, which is integral over its centre, is a prime ring and exhibits many properties similar to those enjoyed by commutative regular local rings. Examples are provided which show that some alternative approaches are not applicable to the rings considered in these chapters.

Regular local rings are Gorenstein rings and, as such, are the subject of an elegant structure theorem due to Bass [5]. In Chapter 7, we generalise his theorem to the situation of rings integral over their centres. Each chapter begins with a summary of the results contained in that chapter.

NOTATION

	Page.
$ann_R(M)$, $ann_M(S)$: annihilators	2
AR-ring: ring whose ideals satisfy the Artin Rees property	15
cl.Kdim(R): classical Krull dimension of R	11
$C(I)$, $C_R(I)$: regular elements modulo I	5
C-Macaulay ring:	78
DVR: Discrete valuation ring	28
$E_R(M)$: Injective hull of M_R	23
G.U.: Going Up (for prime ideals)	17
$G_C(I,M)$: C-grade of I on M	72
gl.dim(R): global dimension of R	21,22
$id_R(M)$: injective dimension of M_R	23
INC: Incomparability (for prime ideals)	17
$I_R(P)$: Indecomposable Injective R-module corresponding to prime ideal P	111,119
J(R): Jacobson radical of R	5
K.dim(R): Krull dimension of R	11
L.O.: Lying Over (for prime ideals)	17
$\ell(I)$: left annihilator of I	2
N(R): Nilpotent radical of R	4
PI ring: ring satisfying a polynomial identity	12
PID: Principal Ideal Domain	28
$pd_R(M)$: projective dimension of M_R	21
QF-ring: Quasi Frobenius ring	25
$\rho(M)$: reduced rank of M	6
r(I): right annihilator of I	2
R_P, R_S, R_ρ: localisations of R	14,20
$Soc_R(M)$: Socle of M_R	2
$Sing_R(M)$: Singular submodule of M_R	3

	Page
UFD: Unique Factorisation domain	29
$u.\dim_R(M)$: uniform dimension of M_R	1
$Z(R)$: centre of R	1
$\mathcal{Z}_R(M)$: set of zero divisors in R on M	2

$\square$ denotes the end of a proof.

CHAPTER 1

PRELIMINARIES

We begin by introducing some of the background theory which will be required later on. All rings are assumed to be associative with an identity element and, unless otherwise stated, are not assumed to be commutative. With the exception of §7.3, subrings are assumed to have the same identity element as the ring. The *centre* of a ring R will be denoted Z(R) and subrings of the centre are called *central subrings*. One-sided ideals will be referred to specifically as right ideals or left ideals and the term ideal will mean a two-sided ideal. Italics will be used to denote ideals of a central subring. Some knowledge of semi-simple Artinian rings, modules and elementary homological algebra is assumed.

§1.1. Rings and Modules.

Unless further indication is given, modules will be assumed to be right modules and definitions below are made accordingly. Corresponding definitions for left modules will be taken as given. Let R denote a ring and M a right R-module. The notation M_R will occasionally be used for the sake of brevity.

A submodule U of M_R is said to be *uniform* if every pair of non-zero submodules of U have non-zero intersection. A submodule E of M_R is *essential* (*in* M) if E has non-trivial intersection with every non-zero submodule of M. The module M_R is said to be *finite dimensional* if it contains no infinite direct sums of submodules and, in this case, M contains an essential submodule which is a finite direct sum of uniform submodules.

Lemma 1.1.1.

If M is a finite dimensional right R-module then there exists an integer $n \geq 0$ such that

(i) A submodule E of M is essential $\Longleftrightarrow$

E contains a direct sum of n uniform submodules.

(ii) Every direct sum of uniform submodules has at most n terms.

Proof: [28, Theorem 1.07] □

This lemma enables us to define the (right) *uniform dimension* of a (right) R-module M as the maximum number of uniform submodules occurring in an essential direct sum. It will be denoted by $u.\dim_R(M)$.

A right R-module M is said to be *irreducible* if

(i) $MR \neq 0$

(ii) The only submodules of M are 0 and M.

The *socle* of an R-module M is the sum of all the irreducible submodules of M and is denoted by $\mathrm{Soc}(M_R)$. If M contains no irreducible submodules, its socle is zero.

When considering the ring R as a module over itself, the obvious definitions may be made for uniform, essential and irreducible right (and left) ideals. The term *minimal* will also be used to describe irreducible (right) ideals.

For a subset S of a ring R and submodule N of M_R, we write $\mathrm{ann}_M(S)$ and $\mathrm{ann}_R(N)$ for the *annihilator of* S *in* M and *of* N *in* R respectively, where

$$\mathrm{ann}_M(S) = \{m \in M \mid mS = 0\} \qquad \mathrm{ann}_R(N) = \{r \in R \mid Nr = 0\}$$

Considering the subset S as a submodule of each of ${}_RR$ and R_R, we define the *left annihilator of* S, $\ell(S)$ and *right annihilator of* S, $r(S)$ as follows

$$\ell(S) = \{r \in R \mid rS = 0\}$$

$$r(S) = \{r \in R \mid Sr = 0\}$$

An element $c \in R$ is *right regular* if $r(c) = 0$, *left regular* if $\ell(c) = 0$ and *regular* if $r(c) = 0 = \ell(c)$. A *domain* is a ring in which every non-zero element is regular.

A right (left) ideal I of a ring R is a *right (left) annihilator* if $I = r(X)$ (or $I = \ell(X)$) for some subset X of R.

Let R be a ring and M a right R-module. An element x of R is a *zero divisor on* M if $mx = 0$ for some $0 \neq m \in M$. The set of zero divisors on M is denoted by $\mathcal{Z}_R(M)$, thus

$$\mathcal{Z}_R(M) = \{r \in R \mid mr = 0 \text{ for some } 0 \neq m \in M\}.$$

The *singular submodule* of a right R-module M will be denoted by $\mathrm{Sing}_R(M)$ and is defined by

$$\mathrm{Sing}_R(M) = \{m \in M \mid \mathrm{ann}_R(m) \text{ is an essential right ideal of } R\}$$

A ring R is right *non-singular* if, when viewed as a right R-module, the singular submodule of R is zero.

<u>Notation</u>: In the literature, a common notation for the singular submodule of a ring R is Z(R) however we reserve this notation for the centre of the ring. Throughout the later chapters we shall be concerned with the centres of various rings, whilst the singular submodule is only required in Chapter 3 so no confusion over notation should arise.

Let R be a ring. An R-module M is said to be *Noetherian* if it satisfies any of the following equivalent conditions,

(i) M satisfies the maximum condition on R-submodules.

(ii) M satisfies the ascending chain condition (ACC) on R-submodules.

(iii) Every R-submodule of M is finitely generated.

M is said to be *Artinian* if either of the following equivalent conditions hold:

(i) M satisfies the minimum condition on R-submodules.

(ii) M satisfies the descending chain condition (DCC) on R-submodules.

The ring R is said to be *right (left) Noetherian* if R is Noetherian as a right (left) R-module.

A *Noetherian ring* is one which is both right and left Noetherian. Corresponding definitions may be made for right Artinian, left Artinian and Artinian rings.

A ring R satisfying the following conditions is said to be *right Goldie*.

(i) R is finite dimensional as a right R-module.

(ii) R satisfies ACC on right annihilators.

A module over an arbitrary ring R is *cyclic* if it can be generated over R by a single element. A right ideal of R is *right principal* if it is a cyclic R-module and a domain in which every right ideal is principal is called a *right principal ideal domain*, (right PID).

§1.2. Primes and Radicals.

A right, left or two-sided ideal I of a ring R is said to be *proper* if (i) $I \neq 0$ and (ii) $I \neq R$.

A ring R is *simple* if (i) $R^2 \neq 0$ and (ii) R contains no proper two-sided ideals.

An ideal P of R is a *prime ideal* if given ideals A, B of R such that $AB \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$. It is well known that the following are equivalent:

(i) P is a prime ideal of R.

(ii) $IJ \subseteq P$ for right ideals I, J of $R \Longrightarrow I \subseteq P$ or $J \subseteq P$.

(iii) $aRb \subseteq P$ for $a, b \in R \Longrightarrow a \in P$ or $b \in P$.

A ring is a *prime ring* if 0 is a prime ideal of the ring.

Let I be a right, left or two-sided ideal of a ring R. I is said to be *nilpotent* if for some integer $n \geq 2$, $I^n = 0$. The sum of all nilpotent ideals of R is called the *Nilpotent radical* of R and will be denoted N(R), or simply N. The Nilpotent radical is known to contain all the nilpotent one-sided ideals of the ring. In general, N(R) will not itself be nilpotent but by a result of Levitski's [31, Theorem 1.45], the Nilpotent Radical of a right Noetherian ring is nilpotent.

If the Nilpotent radical of a ring R is zero, then R is said to be a *semiprime ring*. An ideal I of R is a *semiprime ideal* if R/I is a semiprime ring. The following are equivalent for an ideal I of a ring R:

(i) I is a semiprime ideal.

(ii) $A^n \subseteq I$ for A an ideal of $R \Longrightarrow A \subseteq I$

(iii) $aRa \subseteq I$ for $a \in R \Longrightarrow a \in I$.

An ideal P of a ring R is said to be *right (left) primitive* if there exists an irreducible right (left) R-module M such that $P = \mathrm{ann}_R(M)$. It is easy to see that the following implications hold for an ideal of a ring R,

$$\left\{\begin{matrix}\text{prime ideals}\\ \text{of R}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{primitive ideals}\\ \text{of R}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{maximal ideals}\\ \text{of R}\end{matrix}\right\}$$

For an arbitrary ring, these inclusions are strict.

The *Jacobson radical* of a ring R will be denoted by J(R) and is defined by any of the following:

$$J(R) = \bigcap \{M \mid M \text{ is a maximal right ideal of } R\}$$
$$= \bigcap \{P \mid P \text{ is a right primitive ideal of } R\}$$
$$= \{x \in R \mid (1-xr) \text{ is a unit in } R \text{ for all } r \in R\}$$

Left-handed versions of the above may also be used to define J(R). The following property of the Jacobson radical is well known.

<u>Lemma 1.2.1</u>. (<u>Nakayama's Lemma</u>)

Let R be a ring with Jacobson radical J. If M is a finitely generated right R-module such that MJ = M, then M = 0.

<u>Proof</u>:[54, Lemma 7.2.4]. □

Let R be a ring and let J denote the Jacobson radical of R, then R is said to be

(i) *semilocal* if R/J is a semisimple Artinian ring.

(ii) *local* if R/J is simple Artinian.

(iii) *scalar local* if R/J is a division ring.

For a commutative ring, local and scalar local are the same but for a non-commutative ring R with R/J(R) Artinian, we have

R is local $\Longleftrightarrow$ J(R) is the unique maximal ideal of R

R is scalar local $\Longleftrightarrow$ J(R) is the unique maximal right ideal of R

In the above situation, R is semilocal when J(R) is *semimaximal* i.e. intersection of finitely many maximal ideals of R.

§1.3. <u>Quotient Rings and Reduced Rank</u>.

For an ideal I of a ring R, we define

$$C(I) = \{c \in R \mid (c + I) \text{ is regular element of } R/I\}$$

thus C(0) is the set of regular elements of R.

Let S denote a multiplicatively closed set of elements of a ring R. We say that R satisfies the *right Ore condition with respect to* S (right Ore w.r.t. S) if given elements $a, c \in R$ with $c \in S$, there exist elements $a_1, c_1 \in R$, $c_1 \in S$ such that $ac_1 = ca_1$.

Similarly, R is said to satisfy the *left Ore condition with respect to* S (left Ore w.r.t. S) if given elements $a, c \in R$, $c \in S$, there exist elements $a_1, c_1 \in R$, $c_1 \in S$ such that $c_1 a = a_1 c$.

A ring is said to be a *quotient ring* if every regular element is a unit of the ring.

Let Q be a ring with 1, R a subring of Q then we say that Q is a *right (left) quotient ring of* R if

(i) regular elements of R are units in Q

(ii) every element of Q may be expressed as ac^{-1} $(c^{-1}a)$ where $a, c \in R$, $c \in C(0)$.

Such a ring Q is a quotient ring as defined above. It is straightforward to show that a ring R has a right (left) quotient ring Q if and only if R satisfies the right (left) Ore condition w.r.t. C(0) and, in this case, Q is unique up to isomorphism. If the ring R has both left and right quotient rings then they are isomorphic and we may refer to "the quotient ring of R".

One of the major theorems on the existence of quotient rings is due to Goldie.

<u>Theorem 1.3.1</u>. (<u>Goldie's Theorem</u>)

A ring R has a semisimple Artinian right quotient ring Q if and only if

(i) R is a right Goldie ring

(ii) R is semiprime.

Further, Q is simple Artinian if and only if R is prime.

<u>Proof</u>: [28, Theorems 1.37, 1.38]. □

<u>Reduced Rank</u>.

The uniform dimension defined in §1.1 has the disadvantage of not being additive over short exact sequences. Goldie's reduced rank, which we now define, overcomes this difficulty, for details see [27] or [20].

Let M be a finitely generated module over a right Noetherian ring R and let N denote the Nilpotent radical of R. The *reduced rank* $\rho(M)$ may be defined as follows:

Firstly suppose $MN = 0$, then set

$$S(M) = \{m \in M \mid mc = 0, \text{ for some } c \in C(N)\},$$

This will be the singular submodule of M by the hypothesis on R and M. Define

$$\rho(M) = \text{u.dim } (M/S(M))$$

In the general case, N is nilpotent so we may choose a series of submodules

$$M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_i \supseteq M_{i+1} \supseteq \dots \supseteq M_t = 0 \quad \text{such that}$$

$$M_iN \subseteq M_{i+1} \text{ for } 0 \le i \le t \quad \text{and then define}$$

$$\rho(M) = \sum_{i=0}^{t-1} \rho(M_i/M_{i+1}).$$

$\rho(M)$ is independent of the series chosen.

The following lemma describes the important properties of the reduced rank. In particular notice that additivity is preserved over short exact sequences.

<u>Lemma 1.3.2.</u>

Let R be a right Noetherian ring and M a finitely generated R-module, then

(i) if N is a submodule of M,

$$\rho(M) = \rho(M/N) + \rho(N)$$

(ii) $\rho(M) = 0$ if and only if each element $m \in M$ satisfies $mc = 0$ for some $c \in C(N)$.

<u>Proof</u>: (i) [28, Theorem 1.22]

(ii) This follows from the definition. □

The reduced rank may be used in the proofs of the next two results from the theory of quotient rings.

<u>Lemma 1.3.3.</u>

Let E be an essential right ideal of a semiprime right Noetherian ring R, then E contains a regular element of R.

<u>Proof</u>: Since E is essential, $\rho(E) = \rho(R)$ and hence $\rho(R/E) = 0$ by Lemma 1.3.2(i). It follows that there exists $c \in C(N) = C(0)$ such that $1.c \in E$ i.e. E contains a regular element. □

Lemma 1.3.4.

Let R be a right Noetherian ring. Suppose a,c are elements of R, $c \in C(0)$ then there exists elements $a_1, c_1 \in R$ with $c_1 \in C(N)$ such that $ac_1 = ca_1$.

Proof: Since c is regular, $R \cong cR$ as R-modules thus $\rho(R) = \rho(cR)$ and $\rho(R/cR) = 0$ by Lemma 1.3.2(i). Hence there exists $c_1 \in C(N)$ such that $ac_1 \in cR$ and the result follows. □

If R is a right Noetherian ring in which $C(0) = C(N)$ then the above Lemma shows that R satisfies the right Ore condition w.r.t. $C(0)$ and hence has a right quotient ring Q. In this case, we can say more about the structure of the quotient ring Q as the following theorem of Small indicates.

Theorem 1.3.5. (Small's Theorem)

Let R be a right Noetherian ring, then R has a right Artinian right quotient ring if and only if $C(0) = C(N)$.

Proof: [28, Theorem 2.7]. □

§1.4. Prime Ideals in Noetherian Rings.

We begin with a definition. An ideal P of a ring R is a *maximal annihilator prime* of an R-module M if

(i) $P = \mathrm{ann}_R(N)$ for a non-zero submodule N of M

(ii) for any submodule $0 \neq N' \subseteq N$, $\mathrm{ann}_R(N') = \mathrm{ann}_R(N)$.

It is easy to see that such an ideal is necessarily prime. In the commutative case, such primes are called maximal associated primes but this term has a number of interpretations in non-commutative ring theory so we avoid its use.

Proposition 1.4.1.

If C is a central subring of the right Noetherian ring R and M is a finitely generated R-module then

(i) there exists only finitely many maximal annihilator primes of M as a C-module and each is the annihilator of a non-zero element of M.

(ii) if S is a subring of C consisting of zero-divisors on M, then there exists a non-zero element $m \in M$ such that $mS = 0$.

Proof: c.f. [40, Theorem 82]

(i) C satisfies ACC on annihilator ideals of M_C since it is a subring of a right Noetherian ring. Consider the set of annihilators in C of non-zero elements of M. Each annihilator lies in a maximal one which will be a maximal annihilator prime.

Let $\{p_i = \mathrm{ann}_C(m_i) \mid i \in I\}$ denote these primes.

then

$$\mathcal{Z}_C(M) = \bigcup_{i \in I} p_i$$

As an R-submodule of the right Noetherian R-module M, $\sum_{i \in I} m_i R$ is finitely generated by $m_1, \ldots, m_n$, say.

We claim that $p_1, \ldots, p_n$ are the only maximal annihilator primes.

For, if $p = \mathrm{ann}_C(m)$ is another such prime then, since $m = m_1 r_1 + \ldots + m_n r_n$ for some elements $r_i \in R$.

$$p \supseteq p_1 \cap \ldots \cap p_r$$

hence $p \supseteq p_j$ for some j, $1 \le j \le n$.

However p_j was chosen maximal so $p = p_j$ and the claim is proved.

(ii) Suppose $S \subseteq \mathcal{Z}_C(M) = \bigcup_{i=1}^{n} p_i$

then $S \subseteq p_j$ for some $j, 1 \le j \le n$ by [40, Theorem 81]

so $S \subseteq \mathrm{ann}_C(m_j)$ and $m_j S = 0$, $0 \ne m_j \in M$. □

The above proposition is familiar in a commutative setting and, even then, it is false if either of the Noetherian or the finitely generated hypotheses are removed [40, p.56].

We note the following result from commutative ring theory which will be used implicitly in many proofs.

Lemma 1.4.2.

Let C be a commutative ring and $\mathfrak{a}$ a non-zero ideal of C. Suppose $\{p_i, 1 \le i \le n\}$ is a finite set of prime ideals of C such that $\mathfrak{a} \not\subseteq \bigcup_{i=1}^{n} p_i$

then

(i) there exists an element $c \in \mathfrak{a}$ such that $c \notin p_i$ for $1 \le i \le n$

(ii) if M is a C-module such that $\mathcal{Z}_C(M) = \bigcup_{i=1}^{n} p_i$, then $\mathfrak{a}$ contains a non-zero divisor on M.

Proof: Follows from [49, Chapter 2, Proposition 5]. □

We now consider some properties enjoyed by prime ideals of right Noetherian rings. A *minimal prime* is one which does not contain any other primes of the ring.

Lemma 1.4.3.

Let R be a right Noetherian ring. Then

(i) Every ideal of R contains a product of prime ideals

(ii) Every prime ideal contains a minimal prime

(iii) If I is a right ideal of R and $I \subseteq \bigcup_{i=1}^{n} P_i$, a finite union of prime ideals P_i, $1 \le i \le n$ then $I \subseteq P_j$ for some $1 \le j \le n$.

Proof: (i) Easily seen by taking a maximal counterexample.

(ii) Follows from (i) and definition.

(iii) Assume false and that $P_i \not\subseteq P_j$ for $1 \le i, j \le n$.

For each j, there exists $x_j \in I$ such that $x_j \notin P_j$ and

$$IP_1 \ldots P_{j-1} P_{j+1} \ldots P_n \, x_j \not\subseteq P_j.$$

Choose an element t_j in the LHS with $t_j \notin P_j$ and set $t = t_1 + \ldots + t_n$.

Then $t \in I \subseteq \bigcup_{j=1}^{n} P_j$ so $t \in P_k$ for some k. However, $t_j \in P_k$ for all $j \ne k$ and $t = t_1 + \ldots + t_n$ so $t_k \in P_k$. Contradiction. □

Proposition 1.4.4.

Let R be a right Noetherian ring and let N denote the Nilpotent radical of R then

(i) N is the intersection of all prime ideals

(ii) $N = \bigcap_{i=1}^{n} P_i$ where $\{P_1, \ldots, P_n\}$ is the finite set of minimal primes of R

(iii) $C(N) = \bigcap_{i=1}^{n} C(P_i)$

(iv) A prime ideal P of R is minimal $\Longleftrightarrow$ $P \cap C(N) = \emptyset$.

Proof: (i) (ii) (iv) [28, 2.17]

(iii) [28, 2.14]. □

A prime ideal P of a ring R is said to contain a *chain of primes of length* n, if there exist n distinct prime ideals of R, $P_1,\ldots,P_n$, such that

$$P \supsetneq P_1 \supsetneq P_2 \supsetneq \cdots \supsetneq P_n$$

The prime ideal P has *rank* n if it contains a chain of primes of length n but none longer. If P contains chains of arbitrary length, P is said to have rank = ∞. *The rank of an ideal* I is the infimum of the ranks of those primes containing I.

For a commutative ring, the (krull) dimension is defined to be the sup. of all ranks of its prime ideals, [40, p. 32]. We shall be concerned with two dimensions which generalise this concept to an arbitrary ring R. The first is the *classical Krull dimension*, denoted cl.K.dim(R), which is again defined using prime ideals but may take any ordinal value, see [29, p. 48] for details.

The second is the *Krull dimension* introduced by Rentschler and Gabriel [60]. This is defined inductively and applies to modules as well as rings. For a right R-module M, the Krull dimension will be denoted Kdim(M). The definitions and properties of both these dimensions may be found in [29] however, in the next propositions, we give those properties that will be required in later chapters. It should be noted that the Krull dimension of an arbitrary ring need not exist, see [29, Example 10.1].

Proposition 1.4.5.

Let R be a ring and M a non-zero right R-module then

(i) Kdim(M) = 0 ⟺ M is Artinian

(ii) cl.K.dim(R)=0 ⟺ All prime ideals of R are maximal.

(iii) If the Krull dimension of R is defined then R has a classical Krull dimension and cl.K.dim(R)≤Kdim(R).

(iv) If R is right Noetherian, then R has a Krull dimension.

Proof: (i) This follows from the definition [29, p.5].

(ii) This also follows from the definition [29, p. 48].

(iii) [29, Proposition 7.9].

(iv) [29, Proposition 1.3]. □

For a commutative Noetherian ring , R, the classical Krull dimension, cl.Kdim(R) and the Krull dimension, Kdim(R), coincide but for non-commutative rings, the inequality (iii) of the above proposition is, in general, strict, see [29, Example 10.10]. There is, however, an important class of non-commutative rings for which equality holds:

We say that a ring R is *right (left) fully bounded* if, in any prime factor ring of R, every essential right (left) ideal contains a non-zero two-sided ideal. R is a *right FBN (left FBN) ring* if R is right (left) fully bounded and right (left) Noetherian.

Clearly all commutative Noetherian rings are fully bounded. Other examples include rings satisfying a polynomial identity *(P.I. rings)* [1] and as we show in Lemma 1.6.3 certain rings integral over a central subring.

Proposition 1.4.6.

If R is a right FBN ring then

cl.Kdim(R) = Kdim(R).

Proof: [29, Theorem 8.12]. □

§1.5. Localisation and the AR Property.

Let T be a multiplicatively closed subset of elements of a ring R, then T is a *right (left) divisor set.* if

(i) $0 \notin T$, $1 \in T$

(ii) R satisfies the right (left) Ore condition w.r.t. T.

If T is a right divisor set of regular elements of a ring R, we may form the *partial right quotient ring* R_T where

$$R_T = \{rt^{-1} | r \in R, t \in T\}$$

R_T is uniquely defined (for proof see [54, Theorem 10.2.12] and has the following properties.

Lemma 1.5.1.

(i) R is a subring of R_T and elements of T are units in R_T.

(ii) If $r_1t_1^{-1},\ldots,r_nt_n^{-1}$ are elements of R_T then there exist elements $s_1,\ldots,s_n \in R$ and $t \in T$ such that

$$r_it_i^{-1} = s_it^{-1} \text{ for } 1 \le i \le n.$$

(iii) If I is a right ideal of R_T then $I = (I \cap R)R_T$.

(iv) If R is right Noetherian, so is R_T.

Proof: [54, Lemma 10.2.13]. □

We wish to apply the above procedure in situations where the elements of T are not regular and we also need to consider modules. Let R be a ring, M a right R-module and let T denote a right divisor set in R.

Set

$$M^* = \{m \in M | mt = 0 \text{ for some } t \in T\}.$$

Lemma 1.5.2.

(i) If $m_1,\ldots,m_n \in M^*$ then there exists $t \in T$ such that $m_it = 0$ for $1 \le i \le n$.

(ii) M^* is an R-submodule of M.

Proof: [54, Lemma 11.2.11] □

When considering the action of T on R, we can say more:

Lemma 1.5.3.

For a ring R with right divisor set T, let

$$R^* = \{r \in R | rt = 0 \text{ for some } t \in T\}$$

Then

(i) R^* is a two-sided ideal of R.

(ii) If R^* is finitely generated as a left ideal, then there exists $t \in T$ such that $R^*t = 0$.

If further, R is prime, then $R^* = 0$.

(iii) Suppose R is right Noetherian and let $\bar{R} = R/R^*$ then $\bar{T}$ is a right divisor set of regular elements of $\bar{R}$.

Proof: [54, Lemma 11.2.12]. □

If R, T satisfy the hypotheses of Lemma 1.5.3(iii) then we define the *(right) localisation of* R *at* T denoted R_T, to be the partial right quotient ring $\bar{R}_{\bar{T}}$, i.e.

$$R_T = \{\bar{r}\bar{t}^{-1} | \bar{r} \in \bar{R},\ \bar{t} \in \bar{T}\}.$$

Similarly we define the *localisation of* $M \not\subset T$, M_T by

$$M_T = \{\bar{m}\bar{t}^{-1} | \bar{m} \in \bar{M} = M/M^*,\ \bar{t} \in \bar{T}\}.$$

A semiprime ideal S of a right Noetherian ring R is said to be *right localisable* if R satisfies the right Ore condition w.r.t. $C(S)$. In this case $C(S)$ is a right divisor set but need not consist of regular elements. As above, we may form the partial right quotient ring $\bar{R}_{C(\bar{S})}$. The ring thus formed is the *(right) localisation of* R *at* S and will be denoted by R_S.

There is a natural map $\phi: R \to R_S$ which will be injective if the elements of $C(S)$ are regular.

The terminology is explained by the following result.

<u>Proposition 1.5.4.</u>

Let R be a right Noetherian ring and P a right localisable semiprime ideal of R. Then R_P is a right Noetherian semilocal ring with $J(R_P) = PR_P$. If P is prime then R_P is a local ring.

<u>Proof</u>: It is easy to show that R_P/PR_P is isomorphic to the right quotient ring of R/P and is therefore semisimple Artinian by Theorem 1.3.1. If $\bar{p}\bar{c}^{-1} \in PR_P$ with $c \in C(P)$ then $(1-\bar{p}\bar{c}^{-1}) = (\bar{c}-\bar{p})\bar{c}^{-1}$ is a unit in R_P. Thus $PR_P = J(R_P)$. The prime case also follows from Theorem 1.3.1. □

Corresponding definitions may be made for left localisability and partial left quotient rings and if an ideal is both left and right localisable then the two partial quotient rings are isomorphic.

It is not true that every prime ideal of a Noetherian ring is right localisable - see the proof of Example 6.2.2. In fact, if it were true, the results of Chapters 3 and 4 would be sufficient to describe all the rings discussed in this thesis. We now consider some classes of rings in which localisable prime ideals exist.

An ideal I of a right Noetherian ring R is said to possess the *right* AR *property* if, given an right ideal E of R, there exists an integer $n \geq 1$ such that

$$E \cap I^n \subseteq EI.$$

The *left* AR *property* is similarly defined and a Noetherian ring is said to be an AR *ring* if all its ideals have the right and left AR properties. Our interest in AR rings stems from the following result of P.F. Smith.

Proposition 1.5.5.

Every prime ideal of a Noetherian AR ring is localisable.

Proof: [69, Proposition 3.4.] □

An ideal I of a right Noetherian ring R is said to be *polycentral* if it has a centralising set of generators (c.s.g.) i.e. if there exist elements $x_1,\ldots,x_n \in I$ such that

(i) $I = \sum_{i=1}^{n} x_i R$

(ii) $x_1 \in Z(R)$ and $x_j + \sum_{i=1}^{j-1} x_i R \in Z(R/\sum_{i=1}^{j-1} x_i R)$ for $2 \leq j \leq n$.

The ring R is said to be a *polycentral ring* if all its ideals are polycentral.

Proposition 1.5.6. (Nouaze-Gabriel [52]).

In a right Noetherian ring, polycentral ideals have the right AR property.

Proof: [54, Theorem 11.2.8]. □

Examples of polycentral Noetherian rings, and hence AR rings are

(a) Commutative Noetherian rings [2, Chapter 10].

(b) Integral group rings of finitely generated nilpotent groups [54, Corollary 11.3.12].

(c) Universal enveloping algebras of finite dimensional nilpotent Lie algebras over fields [45, Theorem 4.2].

The group ring kG, where k is a field of characteristic zero and G a polycyclic-by-finite group, is polycentral if and only if G is finite-by-nilpotent [54, Corollary 11.3.12]. This result, together with the corresponding one for

prime characteristic, combines with the work of Roseblade and Smith [63] to provide examples of Noetherian group rings which are AR rings but not polycentral rings, see [13,§4]. Finally notice that the localisation of an AR ring is also an AR ring.

In [46], Muller introduces a class of rings with certain localisable semiprime ideals which includes the classes of polycentral and AR rings. Following him, we say that a set of prime ideals $\{P_1,\ldots,P_n\}$ of a Noetherian ring R is a *clan* if

(i) The semiprime ideal $S = \bigcap_{i=1}^{n} P_i$ is right and left localisable.

(ii) The ideal SR_S of R_S has the right and left AR property.

(iii) No proper subset of $\{P_1,\ldots,P_n\}$ satisfies the properties (i) and (ii).

If $\{P = P_1,P_2,\ldots,P_n\}$ is a clan, we say that P *belongs to a clan* and we call the semiprime ideal $S = \bigcap_{i=1}^{n} P_i$ the *chieftain* of P.

<u>Theorem 1.5.7.</u>

A prime ideal of a Noetherian ring belongs to at most one clan.

<u>Proof</u>: [46, Theorem 5] □

A Noetherian ring R is said to have *enough clans* if every prime ideal of R belongs to a clan. It is clear from Proposition 1.5.5 that Noetherian AR rings have enough clans. Muller [46] provides further examples of rings which have enough clans but are not AR rings. In particular, certain FBN rings have enough clans, see Proposition 1.6.6 and [47]. At present, no example is known of a local Noetherian ring which does not have enough clans.

§1.6. <u>Rings Integral over a Central Subring</u>.

Let R be a ring and C a subring of the centre of R such that $1 \in C$. An element $x \in R$ is said to be *integral over* C if it satisfies a monic polynomial whose coefficients lie in C i.e. there exists an integer $n \geq 1$ and elements $a_0,a_1,\ldots,a_{n-1} \in C$ such that

$$x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0.$$

The ring R is *integral over* C if every element of R is integral over C. It is easy to show that if I is a proper ideal of a ring R which is integral over a central subring C, then the factor ring $C/(C \cap I)$ is a central subring of R/I and that R/I is integral over $C/(C \cap I)$.

The class of rings integral over a central subring includes those rings finitely generated as modules over their centres, though the two classes are distinct. Blair [10] provides examples to show that, unlike the finitely generated case, a right Noetherian ring integral over its centre need not be left Noetherian and the centre need not be Noetherian. [10, Examples 3.2, 3.3]

We are interested in the relationship between the prime ideals and the Krull dimension of the ring and the central subring over which it is integral. We begin with some terminology familiar in commutative ring theory [40, p. 28].

Let S be a central subring of a ring T. For a prime ideal P of T, $P' = P \cap S$ is a prime ideal of S and P is said to *lie over* P'. We say that the pair (T,S) satisfies:

(i) <u>Lying Over</u> (L.O.) if given a prime ideal P' of S, there exists a prime ideal of T lying over P'.

(ii) <u>Going Up</u> (G.U.) if given prime ideals $P_0 \subseteq P_1$ of S and a prime ideal Q_0 of T lying over P_0, there exists a prime ideal Q_1 of T such that

$$Q_0 \subseteq Q_1 \text{ and } Q_1 \cap S = P_1$$

(iii) <u>Incomparability</u> (INC) if given prime ideals P, Q of T such that

$$P \cap S = Q \cap S \text{ and } P \subseteq Q \text{ then } P = Q.$$

<u>Proposition 1.6.1.</u>

Let R be a ring integral over a central subring C then

(i) The pair (R,C) satisfies LO and GU

(ii) If R/P is a right Goldie ring for every prime P of R, then the pair (R,C) satisfies INC.

In particular, (R,C) satisfies INC if any of the following hold:

(a) R has right Krull dimension

(b) R is right Noetherian

(c) R satisfies a polynomial identity.

Proof: (i) Straightforward adaptations of the commutative cases see [40, Theorems 41 and 44]

(ii) [10, Proposition 1.5]. For the particular cases, (a) follows from [29, Corollary 3.4] and Posner's Theorem [55] gives the result in case (c). □

Proposition 1.6.2.

Let S be a central subring of a ring T

(i) If the pair (T,S) satisfies INC and P is a prime ideal of T then

$$\text{rank } P \leq \text{rank } (P \cap S)$$

(ii) If the pair (T,S) satisfies GU and Q is a prime ideal of S of finite rank n then there exists a prime ideal P of T lying over Q such that

$$\text{rank } P \geq n = \text{rank } (P \cap S)$$

(iii) If the pair (T,S) satisfies both GU and INC then given a prime ideal Q of S, there exists a prime ideal P of T lying over Q such that

$$\text{rank } P = \text{rank } (P \cap S) = \text{rank } Q.$$

Proof: (i) and (ii) follow easily from the definitions as in [40, Theorems 45, 46] (iii) follows directly from (i) and (ii), see [40, Theorem 46] □

Remark: In Proposition 1.6.2(i) the inequality may be strict even when T is integral over S [40, Example 25, p. 43]. Furthermore, Proposition 1.6.2(iii) does not imply that rank P = rank (P ∩ S) for all primes P of T. We consider rings for which equality holds in Chapters 5 and 6.

In their paper [16] Chamarie and Hudry show that a number of important properties of commutative rings are also enjoyed by rings integral over a central subring. In particular they prove the following

Proposition 1.6.3.

Let R be a ring integral over a central subring C. If the right Krull dimension of R is defined then

(i) R is right and left fully bounded

(ii) The classical Krull dimension of C is also defined and

$$\mathrm{Kdim}(R) = \mathrm{cl.K.dim}(R) = \mathrm{cl.K.dim}(C)$$

(iii) If P is a prime ideal of R and $p = P \cap C$ then

$$\mathrm{Kdim}(R/P) = \mathrm{Kdim}(R/pR) = \mathrm{cl.K.dim}(C/p)$$

Proof: (i) [16, Corollary 1.4]

(ii) and (iii) [16, Lemma 1.7, Corollary 1.8] □

As one might expect there is a close connection between the Jacobson radicals of the two rings.

Proposition 1.6.4.

Let R be a ring integral over a central subring C then $J(R) \cap C = J(C)$.

If R is right Noetherian then

(i) $\bigcap_{i=1}^{\infty} J^n(R) = 0$

(ii) R is semilocal $\Longleftrightarrow$ C is semilocal

(iii) If R is semilocal then J(R) satisfies the right AR property.

Proof: Hoechsmann [32] proves that $J(R) \cap C = J(C)$. For (i) see [10, Theorem 2.4] and for (ii) and (iii) see [16, Lemma 2.1] □

In the context of the above, we note the following result which will be of technical significance in Chapter 5.

Lemma 1.6.5.

(i) In a right FBN ring R, right primitive ideals are maximal

(ii) If R is a right FBN ring which satisfies GU and INC with respect to a central subring C, then

$$J(C) \subseteq J(R).$$

Proof: (i) If P is a right primitive ideal of R then there exists a maximal right ideal K of R/P which contains no non-zero two-sided ideals of R/P. Since R is right FBN, K is not essential and there exists a minimal right ideal I of R/P such that $I \oplus K = R/P$.

Thus R/P has a non-zero essential socle and is therefore right Artinian.

It follows that P is maximal.

(ii) Let m be a maximal ideal of C and M a prime ideal of R lying over m, then, by INC, M is also a maximal ideal.

Now $J(C) = \bigcap\{m \mid m \text{ is a maximal ideal of } C\}$

$= \bigcap\{M \cap C \mid M \text{ is a maximal ideal of } R\}$

$\subseteq \bigcap\{M \mid M \text{ is a maximal ideal of } R\}$

Since $J(R) = \bigcap\{P \mid P \text{ is right primitive ideal of } R\}$, the result follows from (i). □

We close this section by considering localisation and note that Muller's results on rings with enough clans require the ring to be Noetherian on both sides. In the situation of rings integral over a central subring, Chamarie and Hudry use a similar approach to localisation which requires only one-sided conditions. An immediate corollary to Proposition 1.6.6 is that Noetherian rings integral over a central subring have enough clans, see also [47, Proposition 9].

<u>Proposition 1.6.6.</u>

Let R be a right Noetherian ring integral over a central subring C. Let p be a prime ideal of C and $\{P_1,\ldots,P_n\}$ the finite non-empty set of prime ideals of R lying over p, so $P_i \cap C = p$ for $1 \le i \le n$. Let $S = \bigcap_{i=1}^{n} P_i$ then

(i) R satisfies the right and left Ore conditions with respect to C(S)

(ii) The partial right quotient rings R_S may be formed by central localisation, i.e.

$R_S = R_p = R \otimes_C C_p$ where C_p is obtained by inverting elements of $C \backslash p$

(iii) R_p is a semilocal right Noetherian ring integral over C_p whose Jacobson radical $J(R_p) = SR_p$ satisfies the right AR property.

<u>Proof</u>: [16, Proposition 2.2]. □

§1.7. <u>Homological Algebra</u>

In this section we introduce some terminology from homological algebra. For a detailed treatment of this subject, we refer the reader to [64], [14] or

[51]. Some knowledge of homomorphisms, exact sequences and functors is assumed.

<u>Lemma 1.7.1.</u>

The following conditions on a right R-module P are equivalent:

(i) P is a direct summand of a free R-module

(ii) $\mathrm{Hom}_R(P,-)$ is an exact functor

(iii) Every exact sequence $0 \to A \to B \to P \to 0$ of right R-modules splits.

(iv) For each diagram of R-modules

$$\begin{tikzcd} & P \arrow[dl, dashed, "\gamma"'] \arrow[d, "\alpha"] & \\ B \arrow[r, "\beta"'] & C \arrow[r] & 0 \end{tikzcd}$$

with β an epimorphism, there exists a homomorphism γ making the diagram commute i.e. such that $\beta\gamma = \alpha$.

<u>Proof</u>: [64, Theorems 3.8, 3.10, 3.11] □

An R-module P is *projective* if it satisfies any of the conditions (i) - (iv) of Lemma 1.7.1.

Let M be any (right) R-module. A *projective resolution of* M is an exact sequence

$$\cdots \to P_{n+1} \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

in which each P_i is a projective (right) R-module. The projective resolution has *length* n if $P_i = 0$ for $i \geq n+1$. The module M_R has *projective dimension* n if there exists a projective resolution of M of length n but none shorter. If no projective resolution of finite length exists, M has infinite projective dimension. We denote the projective dimension of the right R-module M by $\mathrm{pd}_R(M)$. If M is also a left R-module, we shall use the notation $\mathrm{rt.pd}_R$ and left pd_R to distinguish between the projective dimensions of M_R and ${}_RM$.

The *right global dimension* of a ring R, denoted rt.gl.dim(R), is defined by

$$\mathrm{rt.gl.dim}(R) = \sup.\ \{\mathrm{pd}_R(M) \mid M_R \text{ a right R-module}\}$$

Similarly

$$\text{left gl.dim}(R) = \sup.\ \{\text{left pd}_R(M) \mid {}_RM \text{ a left R-module}\}$$

It follows from the Artin-Wedderburn Theorem [64, Theorem 4.3] that

$$\text{rt.gl.dim}(R) = 0 \Longleftrightarrow R \text{ is semisimple Artinian} \Longleftrightarrow \text{left gl.dim}(R) = 0.$$

A ring is said to be *right (left) hereditary* if it has right (left) global dimension equal to one. It is not true that a right hereditary ring is left hereditary [64, p. 73], indeed Jategoankar [35] provides examples of left Noetherian rings whose left and right global dimensions are finite but differ by an arbitrary integer. Auslander showed that this cannot occur if the ring is Noetherian on both sides:

Theorem 1.7.2.

If R is a Noetherian ring then rt.gl.dim(R) = left gl.dim(R). This value is the *global dimension* of R, denoted gl.dim(R).

Proof: [64, Corollary 9.20]. □

Lemma 1.7.3.

Let R be a right Noetherian ring and let T be a right divisor set of R then rt.gl.dim $(R_T) \leq$ rt.gl.dim(R).

Proof: We may clearly assume that rt.gl. dim(R) $< \infty$. If T consists of regular elements, this result is proved in [54, Lemma 10.3.14]. The general case follows from [72, Proposition 15.7 and Corollary 13.2] and [14, Exer. 10 p.123] □

The dual concept to projective modules is that of *injective modules* which may be defined by any of the equivalent conditions in the following Lemma.

Lemma 1.7.4.

The following are equivalent for a right R-module E.

(i) $\mathrm{Hom}_R(-,E)$ is an exact functor

(ii) Every exact sequence $0 \to E \to A \to B \to 0$ of right R-module splits

(iii) For each diagram of R-modules

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[dl, dashed, "\gamma"] \\ & E & \end{tikzcd}$$

with α a monomorphism, there exists a homomorphism γ making the diagram commute i.e. such that $\gamma\alpha = \beta$.

Proof: [64, Theorems 3.13, 3.18] □

The existence of projective modules is guaranteed by Lemma 1.7.1(i). It can be shown that every R-module embeds in an injective R-module [67, Theorem 2.11]. The "smallest" injective right R-module containing a given right R-module M is called the *injective hull of* M and is denoted $E_R(M)$ or simply $E(M)$. The module M is essential in $E_R(M)$ and if N is a right R-module containing $E_R(M)$, then M is not an essential submodule of N_R. The injective hull is unique up to isomorphism [67, Theorem 2.21] so "smallest" is well defined.

An *injective resolution* of a right R-module M is an exact sequence

$$0 \to M \xrightarrow{\varepsilon} E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \to \dots \to E_n \xrightarrow{d_n} E_{n+1} \to \dots$$

where each E_i is an injective right R-module and the d_i's are R-homomorphisms.

A particular injective resolution of M may be formed in which

$$E_0 = E_R(M)$$
$$E_{i+1} = E_R(E_i/d_{i-1}(E_{i-1})) \text{ for } i \geq 1.$$

Such an injective resolution is called the *minimal resolution of* M and is unique up to isomorphism [51, §3.7, Theorem 19].

The above injective resolution has *length* n if $E_i = 0$ for $i \geq n+1$ and the module M has *injective dimension* n if there exists an injective resolution of length n but none shorter. If no injective resolution of finite length exists, M is said to have infinite injective dimension. We use $id_R(M)$ to denote the injective dimension of M_R.

A ring R is called *Gorenstein* if R_R has finite injective dimension and is (right) *self-injective* if R_R is injective.

Throughout Chapters 2 and 7 we shall be concerned with the properties of the derived functors $\mathrm{Ext}^n_R(-,-)$ and $\mathrm{Tor}^n_R(-,-)$. We shall assume some familiarity with the homological algebra associated with these functors, the details of which may be found in any of the standard texts, such as those given above - see also §2.3. The following four results will be required in later chapters.

Proposition 1.7.5.

The following are equivalent for a right R-module M,

(i) $pd_R(M) \le n$.

(ii) $Ext_R^k(M,N) = 0$ for all right R-modules N, all $k \ge n+1$

(iii) $Ext_R^{n+1}(M,N) = 0$ for all right R-modules N.

If R is right Noetherian, the above are equivalent to

(iv) $Tor_R^k(M,L) = 0$ for all left R-modules L, all $k \ge n+1$

(v) $Tor_R^{n+1}(M,L) = 0$ for all left R-modules L.

Proof: (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) [64, Theorem 9.6]

(i) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) [64, Theorem 9.19 and Ex. 9.13] □

Proposition 1.7.6.

The following are equivalent for a right R-module N,

(i) $id_R(N) \le n$

(ii) $Ext_R^k(M,N) = 0$ for all right R-modules M, all $k \ge n+1$.

(iii) $Ext_R^{n+1}(M,N) = 0$ for all right R-modules M.

Proof: [64, Theorem 9.10]. □

Proposition 1.7.7. (Long Exact Sequence of Ext).

Let M be a right R-module and suppose $0 \to A \to B \to C \to 0$ is a short exact sequence of right R-modules, then the following are (infinitely) long exact sequences,

(i) $0 \to Hom_R(M,A) \to Hom_R(M,B) \to Hom_R(M,C) \to Ext_k^1(M,A) \to \dots$

$\dots \to Ext_R^n(M,B) \to Ext_R^n(M,C) \to Ext_R^{n+1}(M,A) \to \dots$

(ii) $0 \to Hom_R(C,M) \to Hom_R(B,M) \to Hom_R(A,M) \to Ext_R^1(C,M) \to \dots$

$\dots \to Ext_R^n(B,M) \to Ext_R^n(A,M) \to Ext_R^{n+1}(C,M) \to \dots$

Proof: [64, Theorems 6.13, 6.14, 7.3 and 7.5]. □

Proposition 1.7.8. (Long Exact Sequence of Tor).

Let M be a right R-module and suppose $0 \to A \to B \to C \to 0$ is a short exact sequence of left R-modules, then the following are(infinitely) long exact sequence

$$\ldots \operatorname{Tor}_R^n(M,B) \to \operatorname{Tor}_R^n(M,C) \to \operatorname{Tor}_R^{n-1}(M,A) \to \ldots$$

$$\ldots \to \operatorname{Tor}_R^1(M,C) \to M \otimes A \to M \otimes B \to M \otimes C \to 0$$

A similar result may be obtained in the other variable of $\operatorname{Tor}_R^n(-,-)$.

Proof: [64, Theorems 6.12 and 8.3]. □

§1.8. The Socle of a QF ring.

A *Quasi Frobenius* or QF *ring* is a Noetherian ring which satisfies any of the equivalent conditions (i) - (v) of the following theorem.

Theorem I.8.1.

For a Noetherian ring R, the following are equivalent:

(i) R is right self-injective

(ii) R is left self-injective

(iii) Every projective R-module is injective

(iv) R is right and left Artinian and right and left self-injective.

(v) For any right ideal K and left ideal L,

$$r(\ell(K)) = K \quad \text{and} \quad \ell(r(L)) = L.$$

Proof: [33, pp. 80-82]. □

Unlike arbitrary Noetherian rings, QF rings have the property that their left and right socles coincide, and we may therefore refer to the socle of a QF ring.

Lemma 1.8.2.

Let R be a QF ring and let E, E' denote the right and left socles of R, then $E = E'$.

Proof: R is right Artinian so E is the unique minimal essential right ideal of R and Sing (R) = $\ell(E)$. By [28, 1.35] $\ell(E)$ is therefore nilpotent and hence is contained in the Jacobson radical, J i.e. $\ell(E) \subseteq J$. From 1.8.1(v)

$$r(J) \subseteq r(\ell(E)) = E.$$

Now $E' \subseteq r(J)$

So $E' \subseteq E$.

Similarly $E \subseteq \ell(J) \subseteq E'$ and the result follows. □

Notice that a QF ring R is semilocal and associated to each of its maximal ideals, there is a unique irreducible right R-module. In the remainder of this section, we investigate the occurrences of each irreducible in the socle of R and establish some notation which will be used in the proof of Corollary 7.3.2. This corollary is used to relate injective indecomposable modules over certain rings to prime ideals of the ring in order to provide a structure theorem for particular Gorenstein rings.

An element e of any ring is an *idempotent* if $e \neq 0$ and $e^2 = e$. An idempotent e is *primitive* if it cannot be expressed as the sum of two non-zero idempotents f,g such that $fg = gf = 0$.

If a uniform (right) ideal is generated by an idempotent e, it is easy to see that e is a primitive idempotent.

<u>Proposition 1.8.3.</u>

Let R be a QF ring with socle E, then there exists a decomposition of 1 into primitive idempotents with the following properties:

(i) $1 = e_{11} + e_{12} + . + e_{1k_1} + e_{21} + \dots e_{2k_2} + \dots + e_{n1} + \dots + e_{nk_n}$

where e_{ij} are primitive idempotents $1 \le i \le n$, $1 \le j \le k_i$

and $e_{ij}R \cong e_{st}R \iff i = s$.

(ii) Each $e_{ij}R$ is a uniform right ideal containing a unique minimal right ideal $e_{ij}E$.

(iii) Writing $f_i = e_{i1}$ so that $f_iR \cong e_{ij}R \quad 1 \le j \le k_i$

then $\{f_iR,\ 1 \le i \le n\}$ is a full set of non-isomorphic right ideals generated by a primitive idempotent.

(iv) There exists a permutation σ of $(1,\dots,n)$ such that

$$\frac{f_iR}{f_iJ} \cong f_{\sigma(i)}E \quad \text{where J is the Jacobson radical of R.}$$

<u>Proof</u>: Write E as a direct sum of minimal right ideals $K_{ij}, 1 \le i \le n,\ 1 \le j \le k_i$ choosing the notation such that

(a) $K_{ij} \cong K_{st} \Longleftrightarrow i = s$

(b) $\{K_{11},\ldots,K_{n1}\}$ form a full set of non-isomorphic minimal right ideals.

Then $E = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} K_{ij}$.

R is self-injective and E is essential in R so from [67, Propositions 2.22 and 2.23],

$$R = E_R(E) = \bigoplus_{i=1}^{n} \bigoplus_{i=1}^{k_i} E_R(K_{ij}).$$

Since each K_{ij} is irreducible, $E_R(K_{ij})$ is a uniform R-module with unique minimal submodule K_{ij}. Hence

$$E_R(K_{ij}) \cong E_R(K_{st}) \Longleftrightarrow K_{ij} = K_{st} \Longleftrightarrow i = s.$$

Choosing idempotents $e_{ij} \in R$ such that $e_{ij}R = E_R(K_{ij})$ yields (i) and (ii).

(iii) Let $f_i = e_{i1}$. Then $\{f_iR,\ 1 \le i \le n\}$ is a set of non-isomorphic right ideals each generated by a primitive idempotent.

Suppose e is another primitive idempotent and let $I = eR$. Since R is Artinian, I contains a minimal right ideal K which is unique since e is primitive.

By assumption, $K = K_{ij}$ for some $1 \le i \le n$, $1 \le j \le k_i$ and since I is an essential extension of K_{ij}, it follows that

$$I \cong E_R(K_{ij}) \cong f_iR$$

Thus $\{f_iR,\ 1 \le i \le n\}$ is a full set of non-isomorphic right ideals generated by a primitive idempotent.

(iv) This follows from [48, pp.8-9]. □

<u>Proposition 1.8.4.</u>

Let R be a QF ring with maximal ideals $M_1,\ldots,M_n$. Let S_i denote the unique irreducible right R-module occurring as a composition factor of R/M_i and let E be the socle of R.

Then the number of copies of S_i in E = length of the composition series of R/M_i

$$= u.\dim_{R/M_i}(R/M_i)$$

<u>Proof</u>: Using the notation of Proposition 1.8.3. there exists a permutation σ of $\{1,\ldots,n\}$, such that

$$\frac{f_iR}{f_iJ} \cong f_{\sigma(i)}E \quad \text{where } J = J(R) = \bigcap_{i=1}^{n} M_i$$

so

$$R/J \cong \underbrace{f_{\sigma(1)}E \oplus \dots \oplus f_{\sigma(1)}E}_{k_1 \text{ copies}} \oplus \dots \oplus \underbrace{f_{\sigma(n)}E \oplus \dots \oplus f_{\sigma(n)}E}_{k_n \text{ copies}}$$

But for each $1 \le i \le n$, there exists a unique integer $\pi(i) \in \{1,\dots,n\}$ such that $f_{\sigma(i)}E \cong S_{\pi(i)}$, hence

$$R/J \cong \underbrace{S_{\pi(1)} \oplus \dots \oplus S_{\pi(1)}}_{k_1 \text{ copies}} \oplus \dots \oplus \underbrace{S_{\pi(n)} \oplus \dots \oplus S_{\pi(n)}}_{k_n \text{ copies}}.$$

However, from the Artin Wedderburn Theorem,

$$R/J \cong \underbrace{S_1 \oplus \dots \oplus S_1}_{\ell_1 \text{ copies}} \dots \oplus \underbrace{S_n \oplus \dots \oplus S_n}_{\ell_n \text{ copies}},$$

where ℓ_i = length of composition series of R/M_i

$= \text{u.dim}_{R/M_i}(R/M_i)$

By the uniqueness of composition factors (up to ordering and isomorphism)

$$k_i = \ell_{\pi^{-1}(i)}$$

i.e. the number of copies of S_i in $E = \ell_i = \text{u.dim}_{R/M_i}(R/M_i)$ □

§1.9. Commutative Rings

The following concepts from commutative ring theory will be referred to in the text. The details of these may be found in [39] and [40].

A *Discrete Valuation Ring* (DVR) is a commutative ring R which is a PID with a unique non-zero prime ideal and for any $a,b \in R$, either $a \in bR$ or $b \in aR$. A commutative domain R is a *Krull domain* if the following are satisfied.

(i) R_p is a DVR for each $p \in P$; $R = \bigcap_{p \in P} R_p$

(ii) Any non-zero element of R lies in only finitely many primes $p \in P$.

where P is the set of rank 1 prime ideals of R. A Noetherian Krull domain

all of whose non-zero prime ideals are maximal is called a *Dedekind domain.*

If R is a commutative domain with quotient field K, then R is said to be *integrally closed* if all elements of K which are integral over R lie in R. A Noetherian domain is integrally closed if and only if it is a Krull domain.

An *irreducible element* of a commutative domain is a non-unit which cannot be expressed as the product of two non-units. A *Unique Factorisation Domain* (UFD) is a commutative domain in which every non-zero element is expressible as a unique product of irreducible elements and units. Further characterisations of UFDs may be found in [40, p.132]. We note that a UFD is a Krull domain but that an integrally closed Noetherian domain is not necessarily a UFD, [40,p. 82 and p.132].

The ordered sequence of elements $x_1,\ldots,x_n$ of a commutative ring R is a *regular sequence of length* n if

(i) $\sum_{i=1}^{n} x_iR \neq R$

(ii) For each $1 \leq j \leq n$, $x_j \notin Z_R(R/\sum_{i=1}^{j-1} x_iR)$.

If R is Noetherian, the *grade* of an ideal is the length of a maximal regular sequence of elements of the ideal. That this is well defined is proved in [40, Chapter 3] and in §5.1, where a generalisation of grade is discussed. A commutative Noetherian ring is *Cohen Macaulay* if grade and rank coincide for every maximal ideal of the ring.

For a commutative Noetherian local ring with unique maximal ideal m, the R-module m/m^2 is a vector space over the residue field $k = R/m$. If its vector space dimension d equals the (classical) Krull dimension of R, the ring is said to be *regular local of dimension* d.

The following results provide a detailed description of the structure of regular local rings.

<u>Theorem 1.9.1</u>. (Serre)

A commutative Noetherian local ring R is regular local if and only if it has finite global dimension. Moreover, in this case, the dimension of R as a

regular local ring equals the global dimension of R.

Proof: [39, p. 183-4]. □

Theorem 1.9.2. (Auslander and Buchsbaum)

A regular local ring is a UFD.

Proof:[40, §4.2] □

In the remaining chapters, we shall study some of the ways in which the theory of regular local rings does, and does not, extend to right Noetherian local rings of finite global dimension. In particular, we shall consider the following properties of a commutative regular local ring R of dimension d with unique maximal ideal m,

(R1) R is a domain

(R2) $pd_R(R/m) = gl.dimR = \text{rank } m = d$.

(R3) m is generated by a regular sequence of length d.

(R4) R is integrally closed

(R5) Prime ideals of grade 1 have rank 1.

(R6) R is Cohen Macaulay.

Notice that (R1) and (R4) follow from Theorem 1.9.2, (R2) is proved by Theorem 1.9.1 and (R3) $\Rightarrow$ (R6) $\Longrightarrow$ (R5).

CHAPTER 2

MODULES AND HOMOLOGICAL ALGEBRA

Let S denote a commutative regular local ring of dimension n. In this chapter we obtain results on the projective dimensions of modules over a right Noetherian local ring of finite right global dimension, which are related to the properties of modules over the ring S. We begin in §2.1 by considering the projective dimension of irreducible modules.

It is well known [50, §9.4, Theorem 27] that an S-module M has projective dimension n if and only if M has a non-zero socle. In §2.2 we examine this result in a non-commutative setting, the main results being Lemma 2.2.1 and Proposition 2.2.4 from which we deduce a generalisation of the familiar result that an ideal of S generated by a regular sequence of length t has projective dimension t-1. [50, §9.3, Theorem 20].

Recall that if R is a ring and M,N are right R-modules, the abelian groups $\mathrm{Ext}_R^n(M,N)$, $n \geq 0$, do not have a natural R-module structure unless R is commutative. In §2.3 we define an R-module action for any ring R and obtain the following result,

Let R be a Noetherian ring, M a finitely generated R-module and x a central element of J(R) such that the induced map $x^*:\mathrm{Ext}_R^n(M,R) \to \mathrm{Ext}_R^n(M,R)$ is surjective, then

$$\mathrm{Ext}_R^n(M,R) = 0.$$

If R is commutative, this result is a simple consequence of Nakayama's lemma. It is of technical importance in proving that the injective dimension of certain local Noetherian rings is determined by the Ext modules of their irreducible modules.

§2.1. Modules of finite projective dimension.

Theorem 2.1.1(i) is a generalisation of [8, Proposition 1.1] on the proof of which ours is based, and of [12, §3, No.2, Prop. 5, Cor. 2] while parts (ii) and (iii) are due to Boratynski, [11].

Theorem 2.1.1.

(i) Let R be a Noetherian ring and let A be a finitely generated R-module

module of finite projective dimension then

$$pd_R(A) = n \iff \text{(a) } \mathrm{Tor}_R^{n+i}(A,X) = 0 \text{ for all irreducible left R-modules } X \text{ and all } i \geq 1 \text{ and}$$

(b) there exists an irreducible left R-module Y such that $\mathrm{Tor}_R^n(A,Y) \neq 0$.

(ii) Let R be a right Noetherian ring, let M be a finitely generated R-module and let n be a non-negative integer. Then

$$pd_R(M) \leq n \iff \mathrm{Ext}_R^{n+1}(M,S) = 0$$

for all finitely generated $R/J(R)$-modules S.

(iii) If in (ii), R is semilocal, then

$$pd_R(M) \leq n \iff \mathrm{Ext}_R^{n+1}(M,S) = 0$$

for all irreducible R-modules S.

<u>Proof</u>. (i) Suppose $pd_R(A) = n$. Then (a) is clear. By [14, Chapter VI, Exercises 3 and 6], there exists a left ideal I of R such that $\mathrm{Tor}_R^n(A,R/I) \neq 0$. Suppose that I is chosen maximal such that $\mathrm{Tor}_R^n(A,R/I) \neq 0$. If I is a maximal left ideal, R/I is the required irreducible module. Otherwise, let $\{I_\lambda : \lambda \in \Lambda\}$ be the proper left ideals strictly containing I, and put $T = \bigcap_{\lambda\in\Lambda} I_\lambda$.

We claim that $I \subsetneq T$. For otherwise there exists a short exact sequence

$$0 \longrightarrow R/I \longrightarrow \prod_{\lambda\in\Lambda} R/I_\lambda \longrightarrow X \longrightarrow 0,$$

yielding an exact sequence

$$\mathrm{Tor}_R^{n+1}(A,X) \longrightarrow \mathrm{Tor}_R^n(A,R/I) \longrightarrow \mathrm{Tor}_R^n(A, \prod_\lambda R/I_\lambda),$$

by Proposition 1.7.8. Now $\mathrm{Tor}_R^{n+1}(A,X) = 0$, since $pd_R(A) = n$. Moreover,

$$\mathrm{Tor}_R^n(A, \prod_{\lambda\in\Lambda} R/I_\lambda) \cong \prod_{\lambda\in\Lambda} \mathrm{Tor}_R^n(A,R/I_\lambda) = 0,$$

by [9, Prop. 1.2] and the maximality of I. Hence $\mathrm{Tor}_R^n(A,R/I) = 0$, a contradiction.

Since $T \neq I$, T/I is an irreducible left R-module, by construction. Consider the short exact sequence

$$0 \longrightarrow T/I \xrightarrow{j} R/I \longrightarrow R/T \longrightarrow 0,$$

where j is the inclusion map. This induces an exact sequence

$$\mathrm{Tor}_R^n(A,T/I) \xrightarrow{j^*} \mathrm{Tor}_R^n(A,R/I) \longrightarrow \mathrm{Tor}_R^n(A,R/T),$$

where the right hand term is zero, by choice of I. Thus j^* is onto, and so $\mathrm{Tor}_R^n(A,T/I) \neq 0$. This completes the proof.

Suppose now that (a) and (b) hold. Then (b) implies that $\mathrm{pd}_R(A) \geq n$. If $\mathrm{pd}_R(A) > n$, the argument of the first part of the proof can be repeated to yield a contradiction to (a). Thus,

$$\mathrm{pd}_R(A) = n.$$

(ii) This is proved in the proof of [11, Lemma 3].

(iii) If $X = \sum_{i=1}^{n} \oplus S_i$ is a sum of irreducible modules S_i, then

$$\mathrm{Ext}_R^t(M,X) = \sum_{i=1}^{n} \oplus \mathrm{Ext}_R^t(M,S_i),$$

by [64, Theorem 7.11], so this is a special case of (ii). □

<u>Corollary 2.1.2</u>.

Let R be a right Noetherian local ring

(i) If R is left Noetherian, and gl.dim.(R) = $n < \infty$, then

$$\mathrm{Tor}_R^n(R/J(R), R/J(R)) \neq 0.$$

(ii) If J(R) has the right AR property, then

$$\mathrm{Ext}_R^n(R/J(R), R/J(R)) \neq 0 \text{ if and only if rt.gl.dim.}(R) \geq n$$

<u>Proof</u>: (i) This follows from Theorem 2.1.1 and its left-handed analogue

(ii) Follows from [11, Corollary] and Theorem 2.1.1(ii). □

Example 2.1.3 is due to Fields [23] and shows that both parts of the above corollary are false for arbitrary right Noetherian local rings. This example exhibits a number of properties of interest later in this chapter.

<u>Example 2.1.3</u>. (Fields [23])

There exists a right Noetherian scalar local domain S with Jacobson radical J such that

(i) S has right global dimension 2

(ii) $pd_S\,(S/J) = 1$

(iii) S contains a non-maximal prime ideal P such that $pd_S(S/P) = 2$.

(iv) J does not satisfy the right AR property.

<u>Proof</u>: Let F be a field of rational functions in countably many indeterminates over an arbitrary field k. Let $R = F[[x]]$, the formal power series ring, then there exists an injective endomorphism σ of R such that $\sigma(R) \subseteq F$, see [34]. Form $S = R[[x_1,x_2;\sigma]]$, the twisted power series ring in two commuting indeterminates with multiplication defined by $rx_i = x_i\sigma(r)$ for $i = 1,2$, $r \in R$. We claim that this ring S has the required properties. If $T = R[[x_1;\sigma]]$ then T is a right PID by [34]. If $t \in T$ then $t = x_1^n\left(\sum_{i\geq 0} x_1^i\, r_i\right)$ for some $n \geq 0$, $r_i \in R$ and $t.x_2 = x_2x_1^nu$ where u is a unit in T. A version of the Hilbert basis theorem [54, Theorem 10.2.6] may be used to prove that $T[x_2;\sigma]$ is right Noetherian. That S is also right Noetherian follows as in the commutative case. For each $n \geq 1$, $0 \neq \sigma(x^n) \in F$ so there exists $g_n \in F$ such that $g_n^{-1} = \sigma(x^n)$ and hence $x_i = x^nx_ig_n$ for $i = 1,2$. It follows that xS is a maximal ideal and, since 1-x is a unit in R, it is the Jacobson radical J of S. S is therefore a scalar local domain.

The ideal $I = x_1S$ is essential as a right ideal and, for each $n \geq 1$, $I \subseteq (xS)^n = x^nS$. Thus $I \cap J^n = I$ for every n and so J cannot satisfy the right AR property otherwise we have a contradiction to Nakayama's Lemma. Let $P = x_1S + x_2S$. It is clear that P is a non-maximal prime ideal and, using [40, Theorem B, p.124], $pd_S(S/P) = 2$. Finally S has right global dimension 2 since $T = S/x_2S$ is a right PID by above and the hypotheses of Small change of rings theorem [68] are satisfied, see [34]. □

In connection with Corollary 2.1.2(ii), note that at present no example is known of a Noetherian ring whose Jacobson radical fails to have the AR property. Both parts of the corollary are familiar results in the commutative case - see, for example, [50, §9.3, Theorems 17 and 18]. The following corollary is also well known for commutative rings, indeed a stronger result can be proved in this case, see [40, Theorem 214].

Corollary 2.1.4.

Let R be a right Noetherian local ring of finite right global dimension n. If M is a non-zero finitely generated R-module, $id_R(M) = n$.

Proof: Let S be the irreducible R-module. Since rt.gl. dim.(R) = n, Theorem 2.1.1(iii) implies the existence of a module A such that $Ext_R^n(A,S) \neq 0$. Since M is finitely generated, it has a submodule T with $M/T \cong S$. There is thus an exact sequence

$$Ext_R^n(A,M) \longrightarrow Ext_R^n(A,S) \longrightarrow Ext_R^{n+1}(A,T).$$

Because rt. gl. dim.(R) = n, the right hand term is zero. It follows that $Ext_R^n(A,M) \neq 0$, so $id_R(M) \geq n$. The reverse inequality is clear, since rt.gl.dim(R)=n. □

The following consequence of Theorem 2.1.1(i) will be required in the proof of Theorem 5.2.7.

Corollary 2.1.5

Let R be a right Noetherian local ring and M a finitely generated right R-module of finite projective dimension k. If N is a left R-module with an irreducible submodule T, then

$$Tor_R^k(M,N) \neq 0.$$

Proof: We have the exact sequence of left R-modules

$$0 \longrightarrow T \xrightarrow{j} N \longrightarrow N/T \longrightarrow 0$$

where j is the inclusion map. This induces the exact sequence

$$Tor_R^{k+1}(M,N/T) \to Tor_R^k(M,T) \xrightarrow{j^*} Tor_R^k(M,N)$$

Now $pd_R(M) = k$ so the left hand term is zero by Proposition 1.7.5(v) and thus j^* is injective. All irreducible left R-modules are isomorphic, so

$$Tor_R^k(M,T) \neq 0 \text{ by Theorem 2.1.1(i)}$$

This embeds in $Tor_R^k(M,N)$ as j^* is injective hence $Tor_R^k(M,N) \neq 0$. □

§2.2. Modules with non-zero Socles.

Lemma 2.2.1.

Let R be a right Noetherian local ring of finite right global dimension.

Suppose that R is either left Noetherian, or J(R) has the right AR property. Then, if M is an R-module with a non-zero socle,

$$pd_R(M) = \text{rt.gl.dim.}(R).$$

Proof: By Corollary 2.1.2, rt.gl.dim.(R) = $pd_R(X)$ where X is the unique irreducible R-module. Now [40, Theorem B, p.124] applied to the sequence

$$0 \rightarrowtail X \rightarrow M \rightarrow M/X \rightarrow 0$$

yields the desired conclusion. □

Our aim now is to obtain a result which corresponds to the reverse implication and, in the commutative case, reduces to the fact that if the projective dimension of an R-module M equals the global dimension of R then M has a non-zero socle.

The following observation (of B. Hartley) will be needed in the proof of Lemma 2.2.3.

Lemma 2.2.2.

Suppose that the ideal I of a right Noetherian ring R has the right AR property and let M be a finitely generated R-module with a submodule N. Then there exists an integer $t \geq 1$ such that

$$MI^t \cap N \subseteq NI.$$

Proof: The result is evident when M is a cyclic module. For the general case, let X be a submodule chosen maximal with respect to $X \cap N = NI$, then $\bar{N} = \frac{N+X}{X}$ is essential in $\bar{M} = M/X$.

Suppose $\bar{M} = M_1 + \ldots + M_n$ where each M_i is cyclic. Applying the cyclic result to each of the essential submodules $N_i = \bar{N} \cap M_i$ yields integers t_i, $1 \leq i \leq n$, such that $M_i I^{t_i} = 0$ and the result follows. □

Lemma 2.2.3.

Let R be a right Noetherian ring, and let I be an ideal of R with the right AR property. Let M be a finitely generated R-module, and let t and n be integers, $n \geq 0$, $t \geq 1$. Then there exists an integer $s \geq 0$ such that any homomorphism $\psi: M \to M$ for which $\psi(M) \subseteq MI^s$ can be lifted to a homomorphism ϕ of the nth term P_n of a given finitely generated projective resolution of M in such a way that

$\phi(P_n) \subseteq P_n I^t$.

<u>Proof</u>: Let $0 \to K_n \to P_n \xrightarrow{d_n} P_{n-1} \to \dots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$

be the given resolution of M by finitely generated projective modules P_i. We argue by induction on n.

(i) $n = 0$: In this case we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & K_0 \arrow[r] & P_0 \arrow[r] \arrow[d, dashed, "\phi"] & M \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & K_0 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd} .$$

If $\psi(M) \subseteq MI^s$, then clearly ϕ can be defined so that $\phi(P_0) \subseteq P_0 I^s$, so that putting $s = t$ gives the required result.

(ii) $n > 0$: Suppose the result is true for all finitely generated projective resolutions of M of length less than n. Let $\psi_i : P_i \to P_i$ be a lifting of ψ to P_i, $0 \le i < n$. Suppose first that, writing $\psi_{n-1}(P_{n-1}) = E_{n-1}$,

$$E_{n-1} \cap d_n(P_n) \subseteq d_n(P_n) I^t. \qquad (*)$$

Consider the diagram

$$\begin{tikzcd} P_n \arrow[r, "d_n"] & P_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\psi_{n-1}"] & P_{n-2} \arrow[d, "\psi_{n-2}"] \\ P_n \arrow[r, "d_n"'] & P_{n-1} \arrow[r, "d_{n-1}"'] & P_{n-2} \end{tikzcd}$$

where, if $n = 1$, $P_{n-2} = M$ and $\psi_{n-2} = \psi$. By the commutativity of the right hand square,

$$\psi_{n-1} d_n(P_n) \subseteq \ker d_{n-1} = \operatorname{im} d_n,$$

so that

$$\psi_{n-1} d_n(P_n) \subseteq E_{n-1} \cap d_n(P_n) \subseteq d_n(P_n) I^t,$$

by (*). It follows that a lifting $\psi_n : P_n \to P_n$ can be found making the above diagram commute, and such that $\psi_n(P_n) \subseteq P_n I^t$.

To ensure that (*) holds, note that there exists $u \geq 1$ such that

$$P_{n-1} I^u \cap d_n(P_n) \subseteq d_n(P_n)I^t,$$

since I, and so I^t, have the right AR property. Thus if ζ is chosen such that there exists a lifting $\psi_{n-1} : P_{n-1} \to P_{n-1}$ of ψ, with

$$E_{n-1} = \psi_{n-1}(P_{n-1}) \subseteq P_{n-1}I^u,$$

then we have shown that ψ can be lifted so that $\psi_n(P_n) \subseteq P_n I^t$. Therefore, by induction, there exists an integer s with the required property. □

<u>Proposition 2.2.4</u>.

Let R be a right Noetherian ring whose Jacobson radical J has the right AR property. Let M be a finitely generated R-module, and suppose that there exists $k \geq 0$ such that, for all $s \geq 1$, there exists an embedding θ_s of M into MJ^s, such that $pd_R(M/\theta_s(M)) \leq k$. If V is a finitely generated right R/J-module, then $Ext_R^k(M,V) = 0$. Moreover, $pd_R(M) \leq k - 1$.

<u>Proof</u>: Note that by Theorem 2.1.1(ii), the two conclusions of the proposition are equivalent, so it is enough to show that $pd_R(M) \leq k - 1$. Suppose for a contradiction that this is false, so that, by Theorem 2.1.1(ii) again, there exists $n \geq k$ and a finitely generated right R/J-module V, such that $Ext_R^n(M,V) \neq 0$.

Fix a resolution of M by finitely generated projective modules with n^{th} term P_n. By the previous lemma (with $t = 1$) and the given hypotheses, there exists a monomorphism $\psi:M \to M$ such that, putting $X = M/\psi(M)$, $pd_R(X) \leq k$, and ψ can be lifted to a map $\psi_n:P_n \to P_n$ so that $\psi_n(P_n) \subseteq P_nJ$. There is thus an exact sequence

$$0 \longrightarrow M \xrightarrow{\psi} M \longrightarrow X \longrightarrow 0,$$

yielding, by Proposition 1.7.7 an exact sequence

$$Ext_R^n(M,V) \xrightarrow{\psi^*} Ext_R^n(M,V) \longrightarrow Ext_R^{n+1}(X,V).$$

Now $n \geq k$, and $pd_R(X) \leq k$, so that $Ext_R^{n+1}(X,V) = 0$, and ψ^* is onto. But ψ^* is induced from a map $\psi_n:P_n \to P_n$ such that $\psi_n(P_n) \subseteq P_nJ$ and $VJ = 0$, so from the definition of ψ^*, it follows that ψ^* is the zero map. Hence $Ext_R^n(M,V) = 0$, a

contradiction. This completes the proof. □

The following corollary is an immediate consequence of the above proposition.

Corollary 2.2.5

Let R be a right Noetherian ring of finite right global dimension n, and let M be a finitely generated right R-module such that M embeds in MJ. Suppose further that the Jacobson Radical J has the right AR property. Then $pd_R(M) \leq n-1$.

This corollary is the complementary result to Lemma 2.2.1 that we have been aiming for. Its connection with the commutative result [50, §9.4, Theorem 27] is demonstrated by:

Corollary 2.2.6

Let R be a right Noetherian local ring integral over a central subring C and of finite right global dimension n. If M is a non-zero finitely generated R-module, then

$$pd_R(M) = n \iff M \text{ has a non-zero socle.}$$

Proof: $\Longleftarrow$ This follows from Proposition 1.6.4(iii) and Lemma 2.2.1

$\Longrightarrow$ Let J denote the Jacobson radical of R and let $m = J \cap C$, the unique maximal ideal of C. Suppose there exists an element $x \in m$ such that x is regular on M, then $\theta: M \to M$ where $\theta: m \mapsto mx$ defines an embedding of M into MJ, and so $pd_R(M) \leq n-1$ by Corollary 2.2.5. This contradiction implies that $m \subseteq \mathfrak{Z}(M)$. Thus by Proposition 1.4.1(ii), there exists $0 \neq a \in M$ such that $am = 0$. Now R/mR is right Artinian by Proposition 1.6.3(iii) so aR is a non-zero Artinian submodule of M and the result follows. □

We conclude this section with three applications of Proposition 2.2.4 and Corollary 2.2.5. The first two applications are concerned with prime ideals and enable inductive procedures to be used to show that for certain rings, the rank of a prime ideal is bounded by the global dimension of the ring, see §4.1 and §6.1. The final application is concerned with the projective dimension of an ideal generated by a regular sequence.

Corollary 2.2.7.

Let R be a right Noetherian ring of finite right global dimension n. Suppose that the Jacobson radical J has the right AR property and let P be a prime ideal of R.

If $J \not\subseteq P$ then $pd_R(R/P) \leq n-1$.

Proof: Since $J \not\subseteq P$, it follows from Lemma 1.3.3 that there exists $c \in C(P) \cap J$. It follows that R/P embeds as a right R-module in (J+P)/P so the result follows from Corollary 2.2.5. □

Notice that the above corollary will apply when, in addition to the other properties, R is semilocal and P is a non-maximal prime ideal.

Although the assumption that the Jacobson radical has the right AR property is made for each of the previous five results (implicit in the hypotheses of Corollary 2.2.6), it is only used in the proof of Lemma 2.2.3. It is perhaps surprising that these results are false without this assumption. This is shown by the non-maximal prime ideal P of the ring S in Example 2.1.3 for which $pd_S(S/P) = 2 =$ rt.gl.dim S.

Hence Proposition 2.2.4 and its corollaries are not true without the assumption that J has the right AR property and so this hypothesis is not superfluous for Lemma 2.2.3.

Returning to applications of Corollary 2.2.5, we have

Corollary 2.2.8.

Let R be a right Noetherian semilocal ring of finite right global dimension n whose Jacobson Radical J has the right AR property. Let $\{P_1 \ldots P_m\}$ be a set of non-maximal prime ideals of R such that

(i) the semiprime ideal $S = \bigcap_{i=1}^{m} P_i$ is right localisable.

(ii) $J(R_S)$ has the right AR property

then

$$\text{rt.gl.dim.}(R_S) \leq n-1.$$

Proof: By Corollary 2.2.7, $pd_R(R/P_i) \leq n-1$ for $1 \leq i \leq m$ and since R_S is flat,

$$pd_{R_S}(R_S/P_iR_S) \le n-1 \quad \text{for } 1 \le i \le m.$$

It follows that $pd_{R_S}(R_S/J(R_S)) \le n-1$ and the proof is completed by applying [11, Theorem]. □

<u>Corollary 2.2.9</u>.

Let R be a right Noetherian ring whose Jacobson radical J has the right AR property. Let $x_1,\ldots,x_t \in J$ be such that $x_1 \in C(0)$, and, for $2 \le i \le t$,

$$\sum_{j=1}^{i-1} x_jR = \{r \in R \mid x_ir \in \sum_{j=1}^{i-1} x_jR\}.$$

Put $I = \sum_{i=1}^{t} x_iR$. Then $pd_R(R/I) = t$.

<u>Proof</u>: It is easily shown that $pd_R(R/I) \le t$, by applying [40, Theorem B, p.124] to short exact sequences of the form

$$0 \longrightarrow \sum_{i=1}^{j} x_iR \Big/ \sum_{i=1}^{j-1} x_iR \longrightarrow R \Big/ \sum_{i=1}^{j-1} x_iR \longrightarrow R \Big/ \sum_{i=1}^{j} x_iR \longrightarrow 0,$$

where $1 \le j \le t$, and noting that the left hand module in the above sequence is isomorphic to $R/\sum_{i=1}^{j-1} x_iR$. Let $pd_R(R/I) = n \le t$. We aim to show that $n = t$. The proof is by induction on t, the case $t = 0$ being trivial. Suppose that $t \ge 1$, and that $pd_R(R/Y) = t - 1$, where $Y = \sum_{i=1}^{t-1} x_iR$.

For each $s \ge 1$, there exists an embedding of R/Y in $(J^s + Y)/Y$ afforded by the map $\psi^s : r + Y \mapsto x_t^s r + Y$. Put $M = R/Y$ and consider $M/\psi^s(M)$. For $0 \le m < s$ it is easily checked that

$$I = \{r \in R \mid x_t^m r \in Y + x_t^{m+1}R\}.$$

Hence $M/\psi^s(M)$ contains a chain of s submodules, yielding s factor modules, each isomorphic to R/I. It follows from [40, Theorem B, p. 124] that

$$pd_R\,(M/\psi^s(M)) \le n.$$

Therefore, by Proposition 2.2.4

$$t - 1 = pd_R(M) \le n - 1,$$

and so $n \geq t$. Hence $n = t$, as required. □

When R is a commutative ring, the above result reduces to the well known fact that an ideal of R, generated by an R-sequence of length t, has projective dimension $t - 1$, [50, §9.3, Theorem 20]. Once again the Proposition is false if the hypothesis that J has the right AR property is removed, as can be seen by considering the sequence $\{x_1, x_2, x\}$ of elements in the Jacobson radical of the ring S in Example 2.1.3. More pathological examples still are provided by the local PIDs constructed by Jategaonkar in [35].

§2.3. A Module structure on Ext.

Our aim is to construct an R-module action on the abelian groups $\mathrm{Ext}^n_R(-,-)$ when R is non-commutative, in such a way that for suitable modules M,N, $\mathrm{Ext}^n_R(M,N)$ is finitely generated.

We begin by recalling the definition of $\mathrm{Ext}^n_R(M,N)$ for right R-modules M,N via a projective resolution of M.

$$P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \dots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

By definition [64, p.132] $\mathrm{Ext}^n_R(M,N) = \dfrac{\mathrm{Ker}\, d^*_{n+i}}{\mathrm{Im}\, d^*_n}$

where $d^*_i : \mathrm{Hom}_R(P_{i-1},N) \to \mathrm{Hom}_R(P_i,N)$

$$f \mapsto d^*_i f.$$

such that $d^*_i f(x) = f(d_i(x))$ for $x \in P_i$.

Suppose that N is a right and left R-module, then for each $n \geq 0$, $\mathrm{Hom}_R(P_n,N)$ is a left R-module as follows:

for $f \in \mathrm{Hom}_R(P_n,N)$ $r \in R$, $x \in P_n$

define

$$(rf)(x) = r(f(x)).$$

It is easy to show that this defines a left R-module action and that the maps d^*_n become R-homomorphisms under this action. It follows that for right R-modules M,N such that ${}_RN$ is also a left R-module,

$$\mathrm{Ext}^n_R(M,N) \text{ is a left R-module } n \geq 0$$

Lemma 2.3.1.

If, in the above situation, R is Noetherian, M_R is finitely generated and ${}_RN$ is finitely generated then $\mathrm{Ext}^n_R(M,N)$ is a finitely generated left R-module for $n \geq 0$.

Proof: Since R is Noetherian, the P_n of the projective resolution of M are all finitely generated. $\mathrm{Ext}^n_R(M,N)$ is a factor of a submodule of $\mathrm{Hom}_R(P_n,N)$ so it is enough to show that for any finitely generated projective right R-module P, $\mathrm{Hom}_R(P,N)$ is a finitely generated left R-module under the action described above.

Now there exists a finitely generated free right R-module F such that $F \to P \to 0$ is exact and hence Hom (P,N) is a summand of Hom(F,N). We may therefore assume that that P is a free module and $P \cong R^{(m)}$ for some $m \geq 1$.

Then, as left R-modules,

$$ {}_R\mathrm{Hom}_R(P,N) = \bigoplus_{i=1}^{m} \mathrm{Hom}_R(R_i,N) \qquad R_i \cong R $$

$$ \cong \underbrace{{}_RN \oplus \ldots \oplus {}_RN}_{m \text{ copies}} $$

${}_RN$ is finitely generated so the result follows. □

Lemma 2.3.1 has the following useful corollary which, in the commutative case, is an easy consequence of Nakayama's lemma.

Corollary 2.3.2.

Let I be an ideal of a Noetherian ring R and let x be a central element of the Jacobson radical J of R. Suppose M is a finitely generated right R module and let $\mu: M \to M$ denote the map defined by right multiplication by x i.e. $\mu(m) = mx$. If the induced map μ^* is such that for some $n \geq 1$

$$ \mathrm{Ext}^n_R(M,R/I) \xrightarrow{\mu^*} \mathrm{Ext}^n_R(M,R/I) \to 0 \text{ is exact,} $$

then $\mathrm{Ext}^n_R(M,R/I) = 0$.

Proof: Let P_n denote the nth term of a projective resolution of M then, by definition,

$$\mathrm{Ext}^n(M,R/I) = \frac{\mathrm{Ker}\ d^*_{n+1}}{\mathrm{Im}\ d^*_n}$$

where

$$d^*_i : \mathrm{Hom}(P_{i-1},R/I) \to \mathrm{Hom}(P_i,R/I) \text{ is induced from } d_i : P_i \to P_{i-1}$$

Let $f \in \mathrm{Ker}\ d^*_{n+1}$. By exactness, there exists $g \in \mathrm{Ker}\ d^*_{n+1}$ and $h \in \mathrm{Im}\ d^*_n$ such that

$$\mu^* g = f + h$$

Thus, for $a \in P_n$, $\mu^* g(a) = f(a) + h(a)$

but $\mu^* g(a) = g(ax)$

$= g(a).x$

$= x.g(a)$ since $x \in Z(R)$

$= (xg)(a)$

So $f = x.g - h \in x.\mathrm{Ker}\ d^*_{n+1} + \mathrm{Im}\ d^*_n$.

It follows that

$$\mathrm{Ext}^n_R(M,R/I) \subseteq x.\mathrm{Ext}^n_R(M,R/I)$$

and hence

$$\mathrm{Ext}^n_R(M,R/I) \subseteq J.\mathrm{Ext}^n_R(M,R/I).$$

Now by Lemma 2.3.1, $\mathrm{Ext}^n_R(M,R/I)$ is finitely generated as a left R-module so Nakayama's lemma completes the proof. □

As an application of the above corollary, we obtain a generalisation of a result due to Auslander, see [5, Proposition 2.7], which will be used in Chapter 5.

<u>Proposition 2.3.3</u>.

Let R be a local Noetherian ring with a local central subring C whose maximal ideal m is such that R/mR is Artinian.

Let k be a positive integer, then

$$\mathrm{id}_R R < k \iff \mathrm{Ext}^n_R(S,R) = 0 \quad \text{for } n \geq k$$

and for all irreducible right R-modules S.

<u>Proof</u>: $\Longrightarrow$ Proposition 1.7.6.

$\Longleftarrow$ Suppose, for a contradiction, that $id_R R \geq k$, then by [43, Lemma 1], there exists a right ideal I of R such that

$$\mathrm{Ext}_R^k(R/I,R) \neq 0.$$

Let K be a right ideal chosen maximal with respect to the property that $\mathrm{Ext}_R^\ell(R/K,R) \neq 0$ for some $\ell \geq k$, and assume $\mathrm{Ext}_R^j(R/K,R) \neq 0$. By hypothesis, R/K is not an irreducible R-module and $K \neq R$.

Suppose $m \subseteq \mathcal{Z}(R/K)$ then by 1.4.1(ii) there exists $0 \neq \bar{r} \in R/K$ such that $\bar{r}m = 0$.

R/mR is Artinian so R/K contains an irreducible R-module K'/K. The short exact sequence

$$0 \to K'/K \to R/K \to R/K' \to 0$$

gives rise to the following exact sequence, by Proposition 1.7.7.

$$\mathrm{Ext}_R^j(R/K',R) \xrightarrow{\phi} \mathrm{Ext}_R^j(R/K,R) \longrightarrow \mathrm{Ext}_R^j(K'/K,R).$$

The third term is zero since K'/K is irreducible, so ϕ is an epimorphism. By the choice of K, $\mathrm{Ext}_R^j(R/K,R) \neq 0$ so

$$\mathrm{Ext}_R^j(R/K',R) \neq 0.$$

This contradicts the maximality of K.

Therefore $m \not\subseteq \mathcal{Z}(R/K)$ and by Lemma 1.4.2, there exists $x \in m \subseteq J(R)$ such that x is regular on R/K.

Let μ denote right multiplication on R/K by the element x then the following sequence is exact,

$$0 \to R/K \xrightarrow{\mu} R/K \to R/xR+K \to 0.$$

Apply Proposition 1.7.7 to give the exact sequence

$$\mathrm{Ext}_R^j(R/K,R) \xrightarrow{\mu^*} \mathrm{Ext}_R^j(R/K,R) \to \mathrm{Ext}_R^{j+1}(R/xR+K,R)$$

By the choice of K, the third term is zero and the induced map μ^* is an epimorphism. Thus,

$$\mathrm{Ext}_R^j(R/K,R) = 0 \text{ by Corollary 2.3.2.}$$

This contradicts our assumption on K so $id_R R < k$. $\square$

The above proposition applies to local FBN rings which satisfy GU and INC over a central subring, so, in particular, it applies to Noetherian local rings integral over a central subring.

We do not know whether the injective dimension of an arbitrary local Noetherian ring R is determined by the Ext modules of the irreducible R-modules. One-sided Noetherian assumptions are not sufficient as may be seen from Example 2.1.3. Recall that this example provides a right Noetherian scalar local ring S of right global dimension 2, all of whose irreducible S-modules have right projective dimension 1, being isomorphic to S/J(S).

Thus, from Proposition 1.7.5,

$$\mathrm{Ext}^2_S(K,S) = 0 \text{ for all irreducible S modules K}$$

but

$$\mathrm{id}_S S = 2 \text{ by Corollary 2.1.4.}$$

CHAPTER 3

LOCAL NOETHERIAN RINGS

Let R denote a right Noetherian local ring of finite right global dimension. In this chapter we consider the question: Is R necessarily a prime ring?

If R is right Artinian, it is an easy consequence of Lemma 2.2.1 that R is simple and therefore a matrix ring over a local domain. Ramras [57] considerably generalised this, obtaining the same conclusion when R is assumed to have a right Artinian quotient ring. He conjectured that the above question has an affirmative answer; however a recent example of Stafford's has shown that this is not the case. We give his example in 3.1.6. It should be noted that this example is neither left Noetherian nor does the Jacobson radical have the right AR-property. At present, there is no known counterexample to the above question which has either of these additional properties.

Walker [75] has shown that R is prime in the case when R is scalar local and is right non-singular. This leads us to consider the Nilpotent radical N of R and the main result of §3.1 shows that R/N is a matrix ring over a local domain and that $\rho(N) = 0$. The results of Ramras and Walker follow immediately. In §3.2 we use this result to show that R is a prime ring in situations where certain prime ideals of R are localisable.

§3.1. The Nilpotent Radical.

Lemma 3.1.1.

Let R be a right Noetherian local ring. Then there exists an idempotent $e \in R$ such that every finitely generated projective R-module is a finite direct sum of copies of eR.

Proof: Contained in first paragraph of the proof of [57, Theorem 4]. □

The module eR of the above lemma is called the *unique minimal projective R-module*.

Lemma 3.1.2 illustrates the use of the reduced rank $\rho(M)$ of a module M.

Lemma 3.1.2.

Let R be a right Noetherian ring. Then for a finitely generated R-module M

of finite projective dimension there exists an integer $k \geq 0$ such that

$$\rho(M) = k\rho(P) \text{ where } P \text{ is the unique minimal projective } R\text{-module.}$$

Proof: The proof is by induction on the projective dimension of M, the case when M is projective being clear from Lemma 3.1.1. Suppose $pd_R(M) = t > 0$ and that the result is known for all modules of projective dimension less than t. There exists a short exact sequence

$$0 \to K \to F \to M \to 0$$

where F is a finitely generated free R-module and K is finitely generated with projective dimension t-1. By induction, there exist integers $k_1, k_2 \geq 0$ such that $\rho(K) = k_1\rho(P)$ $\rho(F) = k_2\rho(P)$.

By the additivity of the reduced rank across short exact sequences

$$\rho(M) = (k_2 - k_1)\rho(P) \text{ as required.} \qquad \square$$

Theorem 3.1.3.

Let R be a right Noetherian local ring of finite right global dimension. Let N be the Nilpotent radical of R. Then R/N is a complete matrix ring over a local domain and $\rho(N) = 0$.

In particular N is a prime ideal. If, in addition, N is a finitely generated left ideal then $Nc = 0$ for some $c \in C(N)$.

Proof: Let P denote the unique minimal projective R-module, then from Lemma 3.1.1.

$$R = e_1R \oplus e_2R \oplus \ldots \oplus e_nR, \text{ where } e_i = e_i^2 \text{ and } e_i R \cong P \text{ for } 1 \leq i \leq n.$$

We claim that e_iR/e_iN is uniform, $1 \leq i \leq n$.

Let $e = e_1$ and choose $x \in eR$ so that $xR+eN/eN$ is a non-zero uniform module.

Then

$$\rho(eR/xR) + \rho(xR) = \rho(eR).$$

Lemma 3.1.2 implies that either $\rho(xR) = 0$ or $\rho(eR/xR) = 0$. If $\rho(xR) = 0$, then $xc = 0$ for some $c \in C(N)$ contradicting our choice of x outside N.

Hence $\rho(eR/xR) = 0$, so there exists $d \in C(N)$ such that $ed \in xR$. It follows that $xR + eN/eN$ is an essential uniform submodule of eR/eN, thus e_1R/e_1N is uniform.

Let D be the ring $\mathrm{End}_{R/N}(e_1R/e_1N)$. Since R/N is semiprime right Noetherian, D is an integral domain. Now $e_1R \cong e_iR$ for $1 \leq i \leq n$.

Hence $\frac{e_1R}{e_1N} \cong \frac{e_iR}{e_iN}$ $1 \le i \le n$.

Since $R/N \cong \frac{e_1R}{e_1N} \oplus \ldots \oplus \frac{e_nR}{e_nN}$, it follows that

$$R/N \cong M_n(D).$$

Clearly D is a local so R/N is a complete matrix ring over a local domain. In particular N is a prime ideal.

Finally, from the short exact sequence

$$0 \to N \to R \to R/N \to 0$$

it follows that $\rho(N) = 0$.

Suppose now that N is a finitely generated left ideal. Since the right Ore condition is satisfied with respect to the set $C(N)$ in the ring R/N by Theorem 1.3.1, the hypothesis that N is finitely generated ensures the existence of $c_i \in C(N)$ such that $N^i c_i \in N^{i+1}$ for $1 \le i \le t$ where $N^{t+1} = 0 \ne N^t$.

Thus, setting $c = c_1c_2\ldots c_t$ yields the final conclusion. □

Notice that if $R/J(R) \cong M_m(D)$ where D is a division ring then the integer n occurring in the proof of the above result necessarily divides m.

From Theorem 3.1.3 we can immediately deduce the result of Ramras [57] and a generalisation of Walker's Theorem [75, Theorem 2.9]

<u>Corollary 3.1.4.</u>

Let R be a right Noetherian local ring of finite right global dimension. If either

(a) R is non-singular (i.e. $\mathrm{Sing}_R(R) = 0$) or

(b) R has a right Artinian right quotient ring,

then R is prime.

Moreover, if R is scalar local, then R is a domain.

<u>Proof</u>: For $c \in C(N)$, $cR + N$ is an essential right ideal and by assumption (a) or (b), has zero left annihilator in R. If $N^{k+1} = 0$ and $0 \ne n \in N^k$, then by Theorem 3.1.3, there exists $c \in C(N)$ such that

$$n(cR + N) = 0.$$

Thus $N = 0$ and the result follows from Theorem 3.1.3. □

For Noetherian rings, essentially the same result may be stated in terms of annihilator primes, see §1.1 . Note that if R is right Noetherian and is either right non-singular or has a right Artinian right quotient ring, then every annihilator prime is minimal, [28, 2.14].

Corollary 3.1.5.

If R is a Noetherian local ring of finite global dimension, all of whose right annihilator primes are minimal, then $R \cong M_n(D)$ for some $n \geq 1$ and a local domain D.

Moreover, if $R/J(R) \cong M_m(F)$ where F is a division ring, then n divides m.

Proof: By [57,Theorem 4], it is enough to show that R is semiprime. If $N = N(R) \neq 0$, Theorem 3.1.3 guarantees the existence of $c \in C(N)$ such that $Nc = 0$,

Since N is prime, any right annihilator prime containing r(N) contains an element of C(N) so cannot be minimal. Therefore $N = 0$. □

The following example, due to J.T. Stafford [71] shows that a right Noetherian local ring of finite right global dimension is not necessarily a prime ring. This answers a question of Ramras [57].

Example 3.1.6 (Stafford)

There exists a local right Noetherian ring R with right global dimension 2 which is not a prime ring.

Proof: By considering an appropriate factor ring of Jategoankar's principal right ideal domains [35, Theorem 4.6], we obtain a ring R with Jacobson radical J which has as its lattice of right ideals the unique chain

$$R \supsetneq J \supsetneq J^2 \supsetneq \cdots \supsetneq Q = \bigcap_{n=1}^{\infty} J^n \supsetneq 0.$$

Further, $QJ = 0$ but $JQ = Q$.

We claim that R has the required properties. Clearly every right ideal of R is cyclic and is a two-sided ideal, so R is right Noetherian and is scalar local. To show that R has right global dimension equal to 2 it suffices to show that $pd_R(R/I) \leq 2$ for all right ideals I of R by [64, Theorem 9.14]. Thus it is enough to consider the cyclic R-modules R, $N = R/Q$ and $M_n = R/J^n$, for $1 \leq n < \infty$.

Let $J^n = a_nR$ then since $J^nQ = Q$, $r(a_n) = 0$ i.e. a_n a right regular. So $J^n \cong R$ as right R-modules. Thus from the short exact sequence

$$0 \to J^n \to R \to M_n \to 0.$$

we have $pd_R(M_n) \leq 1$. Since R has no direct summands, this implies that $pd_R(M_n) \cong 1$. Finally $Q = qR \cong R/r(q) = R/J$ so from the short exact sequence

$$0 \to Q \to R \to R/Q \to 0,$$

it follows that $pd_R(R/Q) = 2$. Thus R has right global dimension equal to 2. □

Notice that the above ring R has right Krull dimension 1 and left global dimension > 2. For, if left gl.dim (R) ≤ 2, the exact sequence

$$0 \to \ell(q) \to R \xrightarrow{q} R \to R/Rq \to 0$$

gives $\ell(q)$ as left projective and therefore free. However $\ell(q) = Q$ has non-zero annihilator which gives a contradiction.

Ramras has shown [57] that a right Noetherian local ring S is prime if S has right and left global dimensions ≤ 2. In [13, Section 5] we have shown that the ring S is also prime when S has right Krull dimension ≤ 1 and S is either left Noetherian or J(S) has the right AR property. Example 3.1.6 therefore fits neatly into discussions of primeness in low dimensions.

§3.2. <u>Rings with localisable Prime Ideals</u>.

<u>Proposition 3.2.1</u>.

Let R be a Noetherian ring of finite global dimension. If the Nilpotent radical N is non-zero, then there are no right localisable prime ideals of R minimal over r(N).

<u>Proof</u>: Suppose that the result is false. Let P be a right localisable prime ideal minimal over r(N). Then $I = \{r \in R \mid rc = 0, \text{ for some } c \in C(P)\}$ is an ideal of R and there exists $d \in C(P)$ such that $Id = 0$. Now $d \notin r(N)$ since $d \in C(P)$ and $r(N) \subseteq P$. It follows that $N \nsubseteq I$ and the ring R/I has non-zero nilpotent radical. Localising at P we obtain a local ring R_P of finite global

dimension as in Lemma 1.7.3. However R_P is a (partial) right quotient ring of R/I. So by [28, Corollary 2.6] R_P has non-zero nilpotent radical. Since P is minimal over r(N) it follows that $J(R_P)$ has non-zero left annihilator in R_P. Thus R_P has non-zero right socle. Therefore by Lemma 2.2.1 R_P must be simple Artinian. This is a contradiction. Therefore the result is proved. □

For practical purposes, the chief usefulness of this result occurs in situations where *all* the prime ideals are localisable. Thus, for example, from [57, Theorem 4] and Proposition 3.2.1, we deduce

Corollary 3.2.2.

Let R be a local Noetherian ring all of whose prime ideals are right localisable, and suppose that R has finite global dimension. Then R is isomorphic to $M_n(D)$, for some local Noetherian domain D. If R is scalar local, then n = 1 and R is a domain.

Globalising this result yields

Corollary 3.2.3

Let R be a Noetherian ring of finite global dimension all of whose prime ideals are right localisable. Then R is a direct sum of finitely many prime rings, each having finite global dimension.

Proof: By Proposition 3.2.1, R is a semi-prime ring. Let $P_1,\ldots,P_n$ be the minimal primes of R, and define a homomorphism

$$\psi: R \to \sum_{i=1}^{n} \oplus R/P_i \text{ by } \quad x \mapsto \sum_{i=1}^{n} (x + P_i).$$

Since $\bigcap_{i=1}^{n} P_i = 0$, ψ is a ring monomorphism. We claim that ψ is onto. Fix j, $1 \le j \le n$. If $P_j + (\bigcap_{\substack{i=1\\ i\neq j}}^{n} P_i) \neq R$, there exists a maximal ideal M of R containing both P_j and P_t, for some $t \neq j$. Moreover

$$I = \{r \in R \mid rc = 0, \text{ for some } c \in \mathcal{C}(M)\}$$

is an ideal of M contained in $P_j \cap P_t$. It follows that R_M contains at least 2 minimal prime ideals, contradicting Corollary 3.2.2. Hence $P_j + (\bigcap_{\substack{i=1 \\ i \neq j}}^{n} P_i) = R$, for all $j = 1,\ldots,n$. It follows easily that ψ is onto.

Thus, $R \cong \sum_{i=1}^{n} \oplus R/P_i$; and, clearly, R/P_i has finite global dimension for $i = 1,\ldots,n$. □

It follows from Proposition 1.5.5 that AR rings form a large class of rings to which the above results apply. Examples of such rings are given in Section 1.

The hereditary Artinian ring $R = \begin{pmatrix} Q & Q \\ 0 & Q \end{pmatrix}$ illustrates the fact that the global result Corollary 3.2.3 becomes false if the hypothesis that all prime ideals are localisable is replaced by the weaker assumption that every prime ideal of R belongs to a clan. Notice that this ring R is also finitely generated as a module over its centre.

CHAPTER 4

RINGS OF FINITE GLOBAL DIMENSION

In this chapter we consider the structure of Noetherian rings of finite global dimension in which certain prime ideals are localisable. We have shown in Corollary 3.2.2 that if R is a local Noetherian AR ring of finite global dimension, {0} is a prime ideal of R. One is led to ask what other aspects of the prime ideal structure of commutative regular local rings extend to AR rings. In particular, property (R2) of §1.9 states that the rank of the Jacobson radical of such a commutative ring S is equal to the global dimension of S. The main result of §4.1 shows that if R is a local Noetherian ring of finite global dimension having enough clans, then the rank of the Jacobson radical is at least bounded by gl.dim (R). However, this inequality can be strict even when R is an AR ring, as we show in Example 4.1.3.

Our aim in §4.2 is to obtain non-commutative analogues of the following properties from §1.9:

Let S be a commutative regular local ring then

(R4) S is integrally closed

(R5) If P is a prime ideal of S having grade 1 then P has rank 1.

In order to generalise (R5) we introduce a concept which corresponds to "grade 1" but may be applied in arbitrary Noetherian rings. This agrees with the natural definition of grade available in rings with sufficiently large centres, for example polycentral rings, PI rings and rings integral over central subrings.

Many proofs in the theory of commutative regular local rings rely on the inductive technique of factor ring by certain regular elements to obtain a ring of lower dimension. This technique has successfully been generalised by Walker [75]. In §4.3, we outline two examples of K.A. Brown which show that this inductive technique fails dramatically for general polycentral rings. These examples illustrate a number of other important differences between the results of the commutative theory and the general case.

§4.1. Rank of Maximal Ideals.

Recall that by Corollary 2.2.7, in a right Noetherian semilocal ring of finite right global dimension n, whose Jacobson radical satisfies the right AR property, any non-maximal prime ideal P has projective dimension $\leq n-1$.

Theorem 4.1.1.

Let R be a Noetherian ring of global dimension n and suppose that R has enough clans. If M is a maximal ideal of R then rank (M) $\leq n$

Proof

We may clearly assume that $n < \infty$. The proof is by induction on n, the case $n = 0$ being clear. By localising at the chieftain of M, we may assume that R is semilocal and that its Jacobson radical J has the right AR property.

Let P be a prime ideal of R such that

$P \subseteq M$ and rank $(M/P) = 1$.

Let $\{P = P_1, \ldots, P_t\}$ denote the clan of P with chieftain S. Since every prime ideal of R belongs to at most one clan by Theorem 1.5.7, none of the primes P_i, $1 \leq i \leq t$, are maximal. The hypotheses of Corollary 2.2.8 are therefore satisfied and hence

$$\text{gl.dim } (R_S) \leq n-1$$

By induction, it follows that rank (P) = rank $(P_S) \leq n-1$ and the proof is complete. □

In particular, the above result applies to AR rings so we have

Corollary 4.1.2.

Let R be a Noetherian AR ring. If M is a maximal ideal of R, then rank (M) $\leq$ gl.dim (R).

In Example 4.1.3, we show that the inequality obtained above can be strict, even when the ring R is local and has finite global dimension. There is, however, some evidence to suggest that, for such a ring R,

$$\text{K.dim}(R) = \text{gl.dim } (R) \qquad (\dagger)$$

This problem is discussed in §5 of [13] where (†) is shown to be true when either of the terms involved is at most one. In Theorem 6.1.2 we shall show that (†) is satisfied when R is a right Noetherian local ring integral over a central subring.

In fact, we know of no right Noetherian local ring of finite right global dimension, whose Jacobson radical has the right AR property, for which (†) is false.

The following example provides Noetherian local AR rings of arbitrary finite global dimension whose Jacobson radicals have rank 1. They are constructed from certain group rings and we refer the reader to [13] and [54] for details.

Example 4.1.3. (K.A. Brown)

For every positive integer n, there exists a scalar local Noetherian AR ring R_n of global dimension n whose Jacobson radical is the unique non-zero prime.

Proof:

If $n = 1$, the integers localised at a prime ideal will suffice, so fix $n > 1$.

Let k be a field of positive characteristic. Take a poly - (infinite cyclic) group G which is torsion free and contains an abelian normal subgroup A of rank n with basis $\{x_1,\ldots,x_n\}$, such that G/A is poly - (infinite cyclic). As shown in [13, Example 7.2], G and A can be chosen such that in the group ring kG, the ideal $I = \sum_{i=1}^{n} (x_i - 1)kG$ has the following properties:

(i) kG/I is a domain

(ii) I is a rank 1 prime ideal

(iii) I has the AR property and is localisable

(iv) for each $1 \le t \le n$, the ideal $\sum_{i=1}^{t} (x_i-1)kA$ is a prime ideal of kA.

[These are satisfied when A is a plinth for G and G is orbitally sound (see [13] and [62])].

Since I is assumed to be localisable, we may form the ring $R_n = kG_I$ and we claim that this ring has the required properties.

Certainly R_n is Noetherian by [54, Corollary 10.2.8] and since G satisfies the hypotheses of [54, Theorem 13.1.11], kG and R_n are domains. kG/I is also a domain so R_n is scalar local. The Jacobson radical J of R_n has the AR property since I does, and J is centrally generated by $\{(x_1-1),\ldots,(x_n-1)\}$. We claim that these generators satisfy the conditions of Corollary 2.2.9.

To see this suppose that $1 \le t \le n$ and that

$$(x_t-1)\gamma = \sum_{i=1}^{t-1} (x_i-1)\beta_i \quad \text{for some } \gamma, \beta_i \in R_n .$$

There exist elements $r, c, \delta_i \in kG$, $1 \le i \le t-1$, such that $\gamma = rc^{-1}$ and $\beta_i = \delta_i c^{-1}$ for $1 \le i \le t-1$,

so that

$$(x_t-1)r = \sum_{i=1}^{t-1} (x_i-1)\delta_i \qquad (*)$$

Let $\{s_\lambda : \lambda \in \Lambda\}$ be a transversal to A in G and write

$r = \sum_\lambda r_\lambda s_\lambda$, $\delta_i = \sum_\lambda \delta_{i\lambda} s_\lambda$, $1 \le i \le t-1$, where $r_\lambda, \delta_{i\lambda} \in kA$ then for each $\lambda \in \Lambda$, we deduce from (*) that

$$(x_t-1)r_\lambda = \sum_{i=1}^{t-1} (x_i-1)\delta_{i\lambda}$$

But from (iv) $\sum_{i=1}^{t-1} (x_i-1)kA$ is a prime ideal of kA not containing (x_t-1).

It follows that $r \in \sum_{i=1}^{t-1} (x_i-1)kG$

and hence $\gamma \in \sum_{i=1}^{t-1} (x_i-1)R_n$ as required.

It follows from Corollary 2.2.9 that R_n/J has projective dimension n so by [11, Theorem], R_n has global dimension n.

Since I has rank 1, rank (J) is also 1 i.e. J is the unique non-zero prime ideal of R_n. Since every prime ideal of R_n has the AR property, R_n is easily seen to be an AR ring. □

§4.2. <u>Rings with Enough Invertible Ideals</u>.

We begin with some definitions. Let I be an ideal of a ring R which is an order in a right quotient ring Q. Set $I^* = \{q \in Q \mid qI \subseteq R\}$, and $I^+ = \{q \in Q \mid Iq \subseteq R\}$. Then I is said to be *invertible*, if

$$II^+ = I^*I = R$$

and in this case it is clear that $I^+ = I^*$. We write $I^* = I^{-1}$. The right Noetherian ring R is said to have *enough invertible ideals* if every non-zero ideal

of R contains an invertible ideal. Since an ideal which is generated by a regular central element is visibly invertible, prime polycentral rings and prime Noetherian PI rings each have enough invertibles (Use [65, Theorem 2] in the latter case). Further examples are provided by Asano orders (see e.g. [30]) and prime rings integral over a central subring [16, Lemma 1.1].

Since an invertible ideal is projective by the dual basis lemma [64, Lemma 4.15], the following result applies to invertible ideals.

Lemma 4.2.1.

Let I be an ideal of the local, prime right Noetherian ring R. If I is projective as a right R-module then there exists $a \in C(0)$ such that $I = aR$.

Proof: I is essential as a right ideal so the right uniform dimension of I equals that of R. Thus I is a free right ideal. □

Proposition 4.2.2.

Let R be a Noetherian ring of finite global dimension all of whose primes are right localisable, such that PR_P has the right AR property in R_P for all primes P of R. If I is an invertible ideal of R and P/I is a right annihilator prime ideal of R/I then P has rank 1.

Proof: Suppose P/I is a right annihilator in R/I of X/I where one may take $X = \{r \in R | rP \subseteq I\}$. Thus X is an ideal of R with $I \subseteq X$.

If $IR_P = XR_P$ then, since X is finitely generated as a left ideal, there exists $d \in C(P)$ such that $Xd \subseteq I$, so $d \in P \cap C(P)$, which is impossible. Thus $IR_P \subsetneq XR_P$ and XR_P/IR_P is in the right socle of R_P/IR_P, since $PR_P = J(R_P)$.

By Lemma 2.2.1, $\text{gl.dim}(R_P) = \text{pd.}(R_P/IR_P) \leq 1 + \text{pd}(IR_P)$. Since I is invertible and so projective, $\text{gl.dim}(R_P) \leq 1$. By Theorem 4.1.1, $\text{rank}(P) = \text{rank}(PR_P) \leq 1$.

If rank $P = 0$, PR_P is nilpotent so there exists $c \in C(P)$ and $n \geq 1$ such that $P^n c = 0$. Since P contains an invertible ideal, this cannot happen and so rank $P = 1$. □

Corollary 4.2.3.

Let R be a Noetherian AR ring of finite global dimension and let I be an invertible ideal of R. Then R/I has an Artinian quotient ring.

Proof: By [21, Proposition 2.1], R/I has primary decomposition and, by the above proposition, all the associated primes of R/I are minimal. It follows from the remark at the end of §2 of [21] that R/I has an Artinian quotient ring. □

In particular, Proposition 4.2.2 and its corollary apply when $I = xR$ is an ideal of a Noetherian polycentral ring with x central and regular. They, thus incorporate the result (R5) of the introduction i.e.. that a commutative regular local ring over a prime ideal of grade 1 has rank 1.

We now consider property (R4) and begin by proving a result which in the commutative context, yields the well known fact that a regular local ring is an intersection of Dedekind domains and so in particular is integrally closed.

Let R be a prime Noetherian ring with enough invertible ideals. Let Q be the quotient ring of R. Define

$$S = \{q \in Q \mid Bq \subseteq R \text{ for some } 0 \neq B \text{ ideal of } R\}$$

Then, as shown at the beginning of §2 of [30] S is a simple Noetherian ring. Moreover, being a direct limit of inverses of invertible ideals, S is a flat extension of R. It follows that if R is a ring of finite global dimension, then so is S.

Recall that by Corollary 3.2.3 and Proposition 1.5.5, a Noetherian AR ring of finite global dimension is a direct sum of prime rings with the same properties.

We now have

Theorem 4.2.4.

Let R be a Noetherian, prime AR ring of finite global dimension. Suppose that R has enough invertible ideals. Then

$$R = \bigcap_{P \in \mathcal{P}} R_P \cap S \qquad (*)$$

where $\mathcal{P}$ is the set of rank one prime ideals of R and S is the ring defined above. Further

(i) R_P is a hereditary ring for all primes $P \in \mathcal{P}$ and

(ii) S is a simple, Noetherian ring of finite global dimension.

Proof: Let Q be the quotient ring of R. Each R_P and S are subrings of Q.

Clearly $R \subseteq \bigcap_{P \in \mathcal{P}} R_P \cap S$. Suppose now that $q \in \bigcap_{P \in \mathcal{P}} R_P \cap S$.

Then there exists a non-zero ideal B of R such that $Bq \subseteq R$. Clearly, by hypothesis, we may assume that B is invertible. Let $X = \{r \in R | qr \in R\}$. Then $X \cap C(P) \neq \emptyset$ for all rank one primes P of R. Let $P_1, \ldots, P_n$ be the prime ideals of R minimal over B. Then by [20, Theorem 4.6] $P_1, \ldots, P_n$ are rank one primes so $X \cap C(P_i) \neq \emptyset$ for $i = 1,2,\ldots,n$. Therefore, by [21, Lemma 2.8] $X \cap \bigcap_{i=1}^{n} C(P_i) \neq \emptyset$. Thus by Small's theorem (Theorem 1.3.5) and Corollary 4.2.3, $X \cap C(B) \neq \emptyset$. Choose $d \in X \cap C(B)$. Then $qd \in R$ so that $Bqd \subseteq B$. Hence, since $d \in C(B)$ it follows that $Bq \subseteq B$. Therefore

$$q \in Rq = (B^{-1}B)q = B^{-1}(Bq) \subseteq B^{-1}B = R.$$

Then $\bigcap_{P \in \mathcal{P}} R_P \cap S \subseteq R$ as required.

S is a simple, Noetherian ring of finite global dimension as noted above. Thus it remains to prove that R_P is a hereditary ring for $P \in \mathcal{P}$. Let I be an invertible ideal contained in P. By Lemma 4.2.1, IR_P is a principal right ideal and hence $pd_{R_P}(R_P/IR_P) = 1$. Since P has rank 1, PR_P is the unique non-zero prime ideal of R_P. It follows that R_P/IR_P is Artinian. Since R_P has finite global dimension, Lemma 2.2.1 shows that gl.dim. $R_P = pd_{R_P}(R_P/IR_P)$. Thus R_P is a hereditary ring. □

Any simple Noetherian ring of finite global dimension will satisfy the hypotheses of Theorem 4.2.4 so the ring S need not be hereditary. Even when R is not simple, S may still exist and not be hereditary. This may be seen by a modification to the example in [30, §4].

When R is a commutative ring, S is simply the quotient ring of R so Theorem 4.2.4 yields the conclusion that R is integrally closed, c.f. (R4) of §1.9.

Specialising in another direction, recall that a prime Noetherian hereditary ring in which every non-zero ideal is invertible is called a *Dedekind prime ring*. As shown in [30, §1 and Lemma 2.1], a Noetherian prime ring is a Dedekind prime ring if and only if it is a hereditary AR ring. Thus Theorem 4.2.4 applies to

Dedekind prime rings yielding a result of Kuzmanovich [42].

We state two further special cases.

Corollary 4.2.5.

(i) Let R be a Noetherian, prime, polycentral ring of finite global dimension. Then

$$R = \bigcap_{P \in \mathcal{P}} R_P \cap S$$

where $\mathcal{P}$ and S are as in the theorem above.

(ii) Let R be a Noetherian prime AR ring satisfying a polynomial identity. Then if R has finite global dimension,

$$R = \bigcap_{P \in \mathcal{P}} R_P \quad \text{where } \mathcal{P} \text{ is the set of rank one prime ideals of } R.$$

Proof: (i) Clearly a Noetherian, prime, polycentral ring has enough invertible ideals.

(ii) By [65, Theorem 2] every non-zero ideal of a prime PI ring contains a central element. Thus such a ring contains enough invertible ideals. Every element of the quotient ring Q of R is expressible as ac^{-1} where $a \in R$ and c is a central regular element [65, Corollary 1]. Hence S = Q and thus in this case (*) takes the form

$$R = \bigcap_{P \in \mathcal{P}} R_P. \qquad \square$$

As mentioned above, simple Noetherian rings of finite global dimension satisfy the hypotheses of Theorem 4.2.4, so the ring S clearly cannot be omitted from (*) in general. Nevertheless, in view of Corollary 4.2.5 (ii) it is perhaps not immediately obvious that its presence is necessary even when R is a local Noetherian polycentral ring. This fact is illustrated by Example 4.3.2.

Corollary 4.2.6.

Let R be a Noetherian PI-ring of finite global dimension, all of whose ideals have the AR property. Then the centre of R is a direct sum of n integrally closed domains, where n is the number of minimal prime ideals of R.

Proof: We may assume that R is prime by Corollary 3.2.3. By Corollary 4.2.5(ii), we have

$$Z(R) = \bigcap_{P \in \mathcal{P}} Z(R_P),$$

where $\mathcal{P}$ is the set of rank one primes of R. If $P \in \mathcal{P}$, then R_P is a prime hereditary Noetherian P.I. ring by Theorem 4.2.4(i). It follows from the main result of [61] that $Z(R_P)$ is a Dedekind domain for all $P \in \mathcal{P}$. Thus $Z(R_P)$ is integrally closed, for all P, and hence Z(R) is integrally closed. □

If R is as in Corollary 4.2.6, its centre Z may fail to be Noetherian, even if R is also local. For, in [38], a scalar local Noetherian PI ring R of global dimension 2 is constructed, such that Z is not Noetherian. Since the Jacobson radical of a fully bounded Noetherian ring has the AR property [36, Corollary 3.6], the fact that this ring is an AR-ring follows from

Lemma 4.2.7.

Let R be a Noetherian local ring of global dimension 2, such that the Jacobson radical J has the AR property. Then every rank one prime ideal P of R is right (and left) principal, and R is an AR-ring.

Proof: By [57, Proposition 7], R is a prime ring. Let P be a prime ideal of R such that rank (J/P) = 1. If $P \neq 0$, then P is right and left projective, by Corollary 2.2.7. Since R is prime, Lemma 3.1.1 implies that P is right and left free. Thus P is an invertible ideal, and so has the AR property, by [30, Lemma 2.1] Moreover, rank (P) = 1, by [20, Theorem 4.6]. Hence, every prime ideal of R has the AR property, and it follows easily that R is an AR-ring. □

Notice that the previous two results imply, in particular, that *the centre of a local Noetherian PI ring of global dimension at most two is an integrally closed domain.*

§4.3. Two Examples of Polycentral Rings.

In [75], Walker introduced the class of right Noetherian *regular local* rings, defined as follows: An ideal I of a ring R is said to have a *regular normalising*

set of generators (r.s.g) if there exist elements $x_1, \ldots, x_n \in I$, $n \geq 1$, such that

(i) $I = \sum_{i=1}^{n} x_i R$

(ii) $x_1 \in C(0)$

(iii) $\sum_{i=1}^{j} x_i R = \sum_{i=1}^{j} R x_i$ for $1 \leq j \leq n$ and

(iv) $x_j \in C(\sum_{i=1}^{j-1} x_i R)$ for $2 \leq j \leq n$

A right Noetherian local ring R is said to be *regular local* if and only if J(R) has an r.s.g. Such a regular local ring has right global dimension and right Krull dimension equal to the number of elements in the r.s.g of J(R) [75, Theorem 2.7 - hence the description "regular local" as in the commutative case, see §1.9.

Many naturally occurring local polycentral rings, notably the localisations of certain group algebras and enveloping algebras, are regular local [75] and [70]. However, Example 4.3.1 shows that not all polycentral Noetherian local rings are regular and hence the results in this chapter could not have been deduced from those of Walker.

Both examples in this section are due to K.A. Brown and are constructed from certain group rings. We refer to [54] or [13] for details.

<u>Example 4.3.1.</u>

There exists a scalar local Noetherian polycentral ring R of global dimension 3 which is not regular local. Moreover, R is finitely generated as a module over its centre.

<u>Proof</u>: Let G be the group

$$G = \langle x,y \mid x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2} \rangle$$

This group is discussed in [54, Lemma 13.3.3]; we note in particular the following facts proved there: G is torsion free and contains a normal free Abelian subroup $A = \langle x^2, y^2, (xy)^2 \rangle$ of rank 3 with G/A the direct product of two groups of order 2. In particular, G is polycyclic.

Let k be the field of two elements and set $S = kG$. Then S is Noetherian by [54, Corollary 10.2.8]. Now S is polycentral [54, Corollary 11.3.12(ii)] so the augmentation ideal I of S is localisable [69, Theorem 2.2].

Let $R = S_I$ so that R is a scalar local polycentral Noetherian ring. Note that S is prime by [54, Theorem 4.2.10] so that $C_S(I) \leq C_S(0)$ and S embeds in R by [54, Lemma 11.2.12]. By [9, §7.4] and [54, Theorem 10.3.6], the global dimension of S is 3 so by [54, Lemma 10.3.14] gl.dim (R) ≤ 3.

The elements (x^2-1), (y^2-1), $((xy)^2-1)$ of J(R) are easily seen to satisfy the hypotheses of Corollary 2.2.9 and since J(R) has the AR property, it follows that gl.dim (R) ≥ 3.

Now suppose that R is regular local and let $\{x_1, x_2, x_3\}$ be a regular normalising set of generators of J(R); (this set has 3 elements by [75, Theorem 2.7]).

By [75, Lemma 2.6], $P = x_1R + x_2R$ is a prime ideal so that $Q = P \cap S$ is a prime ideal of S.

As shown in [13] using the results of [62], $Q \subseteq \omega A$ where ωA denotes the ideal of S generated by the augmentation ideal of A. We shall obtain a contradiction by showing that $I/\omega A$ is cyclic.

Let $x_3 = ac^{-1}$, $a \in S$, $c \in C_S(I)$ and let $\alpha \in I \backslash \omega A$. Then there exist $\beta \in kG$, $\delta \in \omega A$ and $d \in C_S(I)$ such that

$$\alpha - a\beta d^{-1} = \delta d^{-1}$$

that is

$$\alpha d - a\beta = \delta \in \omega A.$$

Since k has characteristic 2 and $|G/A| = 4$, the ring $S/\omega A$ is local by [54, Lemma 3.1.6]. Thus there exists $z \in S$ such that $(dz - 1) \in \omega A$.

Hence

$$\alpha - a\beta z = \delta z - \alpha(dz-1) \in \omega A, \quad \text{and}$$

$$I/\omega A = \frac{aS+\omega A}{\omega A} \quad \text{where } a \in S.$$

Since $I = \omega G$, this is clearly false and so we conclude that R is not regular local.

To prove that R is finitely generated over its centre, note first that S is finitely generated over its centre Z by [53, Lemma 7], where $Z \subseteq kA$ by [54, Lemma 4.11.] Straightforward arguments now show that $R = S_B$, where $B = Z \cap C_S(I)$, so that R is also finitely generated over its centre. □

Consider a hereditary Noetherian local ring R. Then K.dim(R) = 1 by [18, Corollary 2.3] and, if R is not prime, it follows from Theorem 3.1.3 that R has a non-zero socle. This contradicts Lemma 2.2.1 so R is therefore prime. Applying Lemma 4.2.1 we have that a hereditary Noetherian local ring is regular. We have been unable to determine whether there exist polycentral Noetherian local rings of global dimension 2 which are not regular.

The next example should be considered in the context of the identity

$$R = \bigcap_{P \in \mathcal{P}} R_P \cap S$$

obtained for polycentral rings in Corollary 4.2.5(i).

<u>Example 4.3.2.</u>

There exists a scalar local polycentral Noetherian ring R such that

(i) gl.dim R = K.dim R = 3

(ii) every saturated chain of primes of R has length 3

(iii) R has a unique rank one prime.

<u>Proof</u>: Let G be the group

$$G = \langle x,y,z \mid [x,y] = z^2,\ z^{-1}xz = x^{-1}, z^{-1}yz = y^{-1}\rangle$$

It may be checked that G is a torsion free polycyclic group containing a normal nilpotent subgroup $N = \langle x,y\rangle$ of index 2. Writing Cent (A) and $\Delta(A)$ for the centre and FC-subgroup [54, Chapter 4] of a group A, we have

$$\text{Cent}(G) = \text{Cent}(N) = \Delta(G) = \langle z^2\rangle$$

which is torsion free abelian.

Let k be the field of two elements and set S = kG. As in Example 4.3.1, S is a prime polycentral Noetherian ring whose augmentation ideal I is localisable. Let $R = S_I$, then R is a scalar local prime polycentral Noetherian ring. Employing

similar arguments to those of Example 4.3.1 and applying Corollary 2.2.9 to the elements (z^2-1), $(x-1)$, $(y-1)$ shows that R has global dimension 3.

Now let P be a rank one prime of R. Then since kG is polycentral,

$$0 \neq P \cap Z(kG) = P \cap k\langle z^2\rangle,$$

and since $P \cap kG \subseteq I$, and $I \cap k\langle z^2\rangle = (z^2-1)\, k\langle z^2\rangle$, we deduce that

$$0 \neq P \cap (z^2-1)\, k\langle z^2\rangle.$$

Since $P \cap k\langle z^2\rangle$ is a prime ideal of $k\langle z^2\rangle$, it follows that

$$(z^2-1)\, R \subseteq P.$$

Now $G/\langle z^2\rangle$ has no non-trivial finite normal subgroups so by [54, Theorem 4.2.10] the group ring $k(G/\langle z^2\rangle)$ is prime.

In view of the isomorphism between $kG/(z^2-1)kG$ and $k(G/\langle z^2\rangle)$, this means that $(z^2-1)kG$ is a prime ideal. Hence $(z^2-1)R$ is prime and since P has rank one, we deduce that $P = (z^2-1)R$. □

As discussed in [13], there are a number of important aspects of the structure of the ring R constructed above which show marked differences from the commutative situation.

Firstly one may easily show that $(z-1) \notin P$ whilst $(z-1)^2 = z^2-1 \in P$ so R/P is not a domain. Since R is scalar local, it follows from [75, Lemma 2.6] that, as in Example 4.3.1, J(R) cannot have a regular normalising set of generators.

Recall that, in the study of commutative regular local rings, the technique of factoring by an element lying in the Jacobson radical but not in its square to yield a regular local ring of smaller dimension [40, Theorem 161] is an essential ingredient of many inductive arguments. However, if R is the ring of Example 4.3.2, central elements of the radical are given by $J(R) \cap Z(R) = (z^2-1)Z(R)$ and since $z^2-1 = (z-1)^2 \in J(R)^2$, *there are no non-zero central elements in* $J(R)\backslash J(R)^2$.

As this might lead one to suspect, factoring the ring R by αR, $0 \neq \alpha \in Z(R)$ yields a ring of infinite global dimension. However, much more than this is true and it is shown in [13] that *the only factor rings of* R *having finite global*

dimension are 0,R *and* R/J(R).

Thus, the technique of induction, based on the passage to a factor ring of smaller global dimension is simply not available in the general non-commutative context.

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CHAPTER 5

CENTRALLY MACAULAY RINGS

We have already seen from Example 4.3.2 that the commutative technique based on factoring by regular elements cannot be employed when examining the structure of Noetherian local AR-rings of finite global dimension. Walker's generalisation of this technique [75] fails even for rings with "large centres", as shown by Example 4.3.1, although some major results have been obtained for rings finitely generated as modules over their centres, notably by Ramras [56] and Vasconcelos [74]. Our aim in the next two chapters, is to extend their results to the case of rings integral over a central subring and to provide structure theorems comparable with those obtained in Chapter 4.

The importance of regular sequences and Cohen Macaulay rings in commutative ring theory is well known and it is these concepts we consider in §5.1. By providing a particular interpretation of regular sequences on modules, we introduce the concepts of *C-grade* and *C-Macaulay* for rings integral over a central subring C. C-Macaulay, or *centrally Macaulay* rings are shown to be well behaved under central localisation and factoring by regular central elements. An inductive procedure is then used in §5.2 to show that certain Noetherian local rings of finite global dimension are prime centrally Macaulay rings. These rings exhibit many properties enjoyed by Cohen Macaulay rings and in §5.3 we examine some of the ways in which the two concepts are related for rings with a Cohen Macaulay centre. It should be noted that in Example 6.2.3 we provide a ring finitely generated over its centre which is centrally Macaulay but whose centre is not Cohen Macaulay. This ring is a prime Noetherian local ring of global dimension 5 and answers a question posed by Vasconcelos [74].

§5.1. C-Sequences and Grade

Throughout this section, C will denote a central subring of a ring R and N_R will denote a right R-module. Italics will be used to denote ideals of C.

If I is an ideal of R (or C) we define a *C-sequence (of length n) in I on* N_R to be an ordered sequence of n elements $x_1,\ldots,x_n$ of $I \cap C$ such that

(i) $N(\sum_{i=1}^{n} x_iR) \neq N$

(ii) $x_1 \notin \mathcal{Z}(N)$ and for $1 \leq i \leq n-1$,

$x_{i+1} \notin \mathcal{Z}(N/N(\sum_{j=1}^{i} x_jR))$

Notation

We shall adopt the conventions of commutative ring theory and omit references to the module N_R and the ideal I in the cases where the module is R_R and where the ideal is clear from the context. We write $N(x_1,\ldots,x_n)$ for the submodule $N(\sum_{i=1}^{n} x_iR)$ of N_R.

Notice that the *"regularity condition"* (ii) of the definition may be stated in the form: x_1 is a non-zero divisor on N_R and for $1 \leq i \leq n-1$, x_{i+1} is a non-zero divisor on $N/N(x_1,\ldots,x_i)$.

Lemma 5.1.1.

Let $x_1,\ldots,x_n$ be a C-sequence in the ring R on N_R then the sequence obtained by interchanging x_i and x_{i+1} is a C-sequence on N_R if and only if

$$x_i \notin \mathcal{Z}(N/N(x_1,\ldots,x_{i-1},x_{i+1})$$

Proof: This follows inductively as in [40, Theorems 117, 118] since x_i and x_{i+1} are central elements of R. □

This interchanging of elements of a C-sequence can always be effected when the elements of the sequence lie in the Jacobson radical of the ring and the module concerned is finitely generated enabling the use of Nakayama's Lemma. Thus we have:

Corollary 5.1.2.

Let R be a right Noetherian ring, C a central subring and N_R a finitely generated R-module. Suppose the elements $x_1,\ldots,x_n$ form a C-sequence on N_R and

lie in the Jacobson radical J of R, then any permutation of $x_1,\ldots,x_n$ is also a C-sequence in J on N_R.

Proof: Follows from Lemma 5.1.1 and Lemma 1.2.1. For details see [40, Theorem 119] □

Corollary 5.1.3.

If R is right Noetherian, then any C-sequence (on R) in the Jacobson radical of R consists of regular elements. □

A C-sequence $x_1,\ldots,x_n$ in ideal I on N_R is *maximal*, if $I \cap C$ consists of zero divisors of $N/N(x_1,\ldots,x_n)$.

Lemma 5.1.4.

Suppose R is right Noetherian, N_R is finitely generated and I is an ideal of R or C, then

(i) maximal C-sequences in I on N exist and have only finitely many terms

(ii) If $N(IR) \neq N$, then any C-sequence in I on N can be extended to a maximal C-sequence in I on N.

Proof: (i) Suppose $x_1,x_2\ldots$ is a C-sequence in I on N, and let $I_j = \sum_{i=1}^{j} x_iR$. The "regularity condition" ensures that $I_j \subsetneq I_{j+1}$, for otherwise x_{j+1} is a zero divisor on N/NI_j, thus the chain of ideals $I_1 \subsetneq I_2 \subsetneq \ldots$ is strictly ascending. Since R is right Noetherian, any such chain of ideals must stop and (i) follows.

(ii) Let $x_1,\ldots,x_n$ denote a C-sequence in I on N. If this is not maximal then

$$(I \cap C) \not\subseteq \mathcal{Z}(N/N(x_1,\ldots,x_n))$$

From Proposition 1.4.1, $I \cap C$ is not contained in any of the finitely many maximal annihilator primes (in C) of $N/N(x_1,\ldots,x_n)$ and so by Lemma 1.4.2, there exists $x_{n+1} \in I \cap C$ such that x_{n+1} is a non-zero divisor on $N/N(x_1\ldots x_n)$. By assumption, $N(IR) \neq N$, so $N(x_1\ldots x_{n+1}) \neq N$ and $x_1\ldots x_{n+1}$ is also C-sequence in I on N_R. Repeating this process we obtain a maximal C-sequence. □

The following proposition shows that all such maximal C-sequences have the same length. It is a familiar result from commutative ring theory. However we give the proof in full because of its importance to the development of the theory of grade.

Proposition 5.1.5.

Let R be right Noetherian and I an ideal of R or C. If N_R is finitely generated and $N(IR) \neq N$ then any two maximal C-sequences in I on N_R have the same length.

Proof: (See also [40, Theorem 121]). We use induction on the length of the C-sequence. Firstly suppose that x and y are two C-sequences in I on N of length 1 with x a maximal C-sequence. For this case, it suffices to show that $(I \cap C) \subseteq \mathcal{Z}(N/Ny)$, for then y is also a maximal C-sequence. By assumption on x,

$$\mathcal{a} = I \cap C \subseteq \mathcal{Z}(N/Nx)$$

so there exists $u \in N$, $u \notin Nx$ such that

$$u\mathcal{a} \subseteq Nx.$$

In particular, $uy = vx$ for some $v \in N$. We claim that $v \notin Ny$, and $v\mathcal{a} \subseteq Ny$. Suppose not, then $v = u'y$ for some $u' \in N$ and $uy = vx = u'yx = u'xy$ but $y \notin \mathcal{Z}(N)$ so $u = u'x \in Nx$.

This contradicts the assumption that $u \notin Nx$, so $v \notin Ny$.

Now $vx\mathcal{a} = uy\mathcal{a} \subseteq Nxy = Nyx$ so $v\mathcal{a} \subseteq Ny$ since $x \notin \mathcal{Z}(N)$.

This proves the claim and hence $\mathcal{a} = (I \cap C)$ consists of zero divisors on N/Ny. Thus y is a maximal C-sequence in I on N.

To deal with the general case let $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ denote two C-sequences in I on N_R such that $x_1,\ldots,x_n$ is maximal. We will show that $y_1,\ldots,y_n$ is also maximal and to this end, we assume that the result is true for sequences of length less than n. Let $U_1 = V_1 = N$ and for $2 \leq i \leq n$, let

$$U_i = N/N(x_1,\ldots,x_{i-1}) \text{ and } V_i = N/N(y_1,\ldots,y_{i-1}).$$

At each stage of the C-sequences $x_i \notin \mathcal{Z}(U_i)$ and $y_i \notin \mathcal{Z}(V_i)$ so for $1 \le i \le n$

$$\mathfrak{a} \not\subseteq \mathcal{Z}(U_i) \text{ and } \mathfrak{a} \not\subseteq \mathcal{Z}(V_i) \text{ where } \mathfrak{a} = I \cap C.$$

Set $T = U_1 \oplus \ldots \oplus U_n \oplus V_1 \ldots \oplus V_n$. Then $\mathfrak{a}$ cannot consist entirely of zero divisors of the C-module T. Now T is finitely generated as an R-module so it follows from Propositions 1.4.1 and 1.4.2 that there exist an element $z \in \mathfrak{a} = I \cap C$ such that $z \notin \mathcal{Z}(T)$. In particular, $z \notin \mathcal{Z}(U_n)$ so z is a C-sequence in I on U_n. By assumption x_n is a maximal C-sequence in I on U_n so by the n=1 case, z is also a maximal C-sequence. Since $z \notin \mathcal{Z}(U_i)$ for $1 \le i \le n$, we may apply Lemma 1.5.1 n times to give $z, x_1,\ldots,x_{n-1}$ as a C-sequence in I on N which is maximal. Repeating this procedure for the modules V_i, $1 \le i \le n$ on which z is also a non-zero divisor, yields the C-sequence $z, y_1\ldots y_{n-1}$ in I on N which is not necessarily maximal.

Now $x_1,\ldots,x_{n-1}$ and $y_1,\ldots,y_{n-1}$ are C-sequences in I on the R-module N/Nz, the first of which is maximal. It follows by induction that $y_1\ldots y_{n-1}$ is also maximal and hence $z, y_1,\ldots,y_{n-1}$ is a maximal C-sequence on N.

Finally we use Lemma 5.1.1 to show that $y_1\ldots y_{n-1}, z$ is also a maximal C-sequence on N and then the n=1 case to give both z and y_n as maximal C-sequences on the R-module $N/N(y_1\ldots y_{n-1})$. Thus $y_1,\ldots,y_n$ is a maximal C-sequence in I on N and the proof is complete. □

We are now in a position to define grade in the same way as the commutative case.

Let R be a right Noetherian ring with a central subring C and let I be an ideal of R or C. For a finitely generated R-module N_R, the length of any maximal C-sequence in I on N_R is called the *C-grade of* I *on* N_R and is denoted by $G_C(I,N)$.

<u>Notation:</u>

For the majority of applications, it will be clear which central subring is being considered, so we shall merely refer to *grade* and use the notation G(I,N).

When considering C-sequences on R_R, we shall refer to the *grade of* I.

Remarks:

(1) Notice that an annihilator prime P in R of an ideal $I = xR$ where x is regular and central will have grade 1 by the above definition. This concept of grade 1 therefore agrees with that used in §4.2.

(2) Under certain extra hypotheses, this C-sequence definition of grade is equivalent to the homological definition introduced by Rees [59]. In Chapter 7, we demonstrate this equivalence and prove results which require both definitions. For this chapter, however, it is convenient to use the above definition and to work with C-sequences.

Let R be a right Noetherian ring with a central subring C. Suppose $x_1,\ldots,x_n$ is a C-sequence in a prime ideal P of R. Let $\bar{P} = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_k$ be a chain of prime ideals in $\bar{R} = R/x_1R$. In the lifting of this chain to R, P_k is not a minimal prime since it contains the regular element x_1. The chain of primes in P may therefore be extended by at least one. It is straightforward to deduce that the grade of P is bounded by its rank, c.f. [40, Theorems 131, 132].

That grade and rank need not coincide is easily seen by considering the ideal of $K[x,y]/(x^2,xy)$ generated by x and y. We aim to show that for certain algebras equality of grade and rank for maximal ideals ensures equality for all prime ideals. We begin with some lemmas enabling a reduction to the local situation.

Lemma 5.1.6.

Let C be a central subring of a right Noetherian ring R and let N be a finitely generated right R-module. Suppose that T is a multiplicatively closed subset of C such that $0 \notin T$, $1 \in T$.

If $x_1,\ldots,x_n$ is a C-sequence in R on N such that

$N_T(\sum_{i=1}^{n} \bar{x}_i R_T) \neq N_T$ where N_T is the localisation of N at T

then

$\bar{x}_1,\ldots,\bar{x}_n$ is a C_T-sequence in R_T on N_T, (where bars denote images in R_T)

<u>Proof:</u> R clearly satisfies the right Ore condition w.r.t. T so T is a right divisor set.

Let $N^* = \{u \in N | ut = 0 \text{ for some } t \in T\}$

$R^* = \{r \in R | rt = 0 \text{ for some } t \in T\}$

then, by definition in §1.5, $N_T = (N/N^*)_T$

Suppose $x_1 \in R^*$, then $x_1 t = 0$ for some $t \in T$ and hence $Nx_1 t = 0$. But $x_1 \notin \mathcal{Z}(N)$ so $Nt = 0$ and hence $N = N^*$. This contradicts the assumption that $N_T(\sum_{i=1}^{n} \bar{x}_i R_T) \neq N_T$. Hence $x_1 \notin R^*$ and similarly $x_2,\ldots,x_n \notin R^*$ so the images $\bar{x}_1,\ldots,\bar{x}_n$ are non-zero elements of R_T.

It remains to show that these elements form a C_T-sequence on N_T i.e. for $1 \leq i \leq n$, $\bar{x}_i$ is a non-zero divisor on the R_T-module $N_T/N_T(\sum_{j=1}^{i-1} \bar{x}_j R_T)$.

Suppose there exist elements $\bar{u}\bar{t}^{-1}$, $\bar{u}_j \bar{t}_j^{-1} \in N_T$ with $\bar{u}\bar{t}^{-1} \neq 0$ and such that

$$\bar{u}\bar{t}^{-1}\bar{x}_i = \sum_{j=1}^{i-1} \bar{u}_j \bar{t}^{-1} \bar{x}_j \qquad (*)$$

Multiplying (*) by a suitable element of T yields an equation in N,

$$ux_i c = \sum_{j=1}^{i-1} u_j x_j c_j \qquad (**) \text{ where } c, c_1,\ldots,c_{i-1} \in T$$

Since $x_1,\ldots,x_i$ form a C-sequence on N, (**) implies that

$$uc \in N(\sum_{j=1}^{i-1} x_j R).$$

It follows that $\bar{u}\bar{t}^{-1} \in N_T(\sum_{j=1}^{i-1} \bar{x}_j R_T)$ and hence $\bar{x}_i$ is a non-zero divisor as required. □

Corollary 5.1.7.

Let R be a right Noetherian ring with a central subring C such that the pair (R,C) satisfies L.O. Then for an ideal I of R such that $I \cap C$ is contained in a prime ideal p of C,

$$G_C(I,R) \leq G_{C_p}(IR_p,R_p)$$

Proof: Let $x_1,\ldots,x_n$ denote a C-sequence in I on R. If $R_p \neq \sum_{i=1}^{n} \bar{x}_i R_p$, the hypotheses of Lemma 5.1.6 are satisfied so the images $\bar{x}_1,\ldots,\bar{x}_n$ form a C_p-sequence of length n on R_p. It is therefore enough to show that $(I \cap C)R_p \neq R_p$ or that $pR_p \neq R_p$.

Suppose that the latter is false and let P be a prime ideal of R lying over p. Then $PR_p \supseteq pR_p = R_p$ so there exist elements $p \in P$ and $c \in C\backslash p$ such that $\bar{p}\bar{c}^{-1} = 1$. Hence there exists $t \in R^* = \{r \in R | rd = 0 \text{ for some } d \in C\backslash p\}$ such that $c = p + t$.

But $C_C(p) \subseteq C_R(P)$ so $R^* \subseteq P$ and hence $c \in P$. Contradiction. Therefore $pR_p \neq R_p$ and the result follows from Lemma 5.1.6. □

Proposition 5.1.8.

Let R be a right Noetherian ring with central subring C such that the pair (R,C) satisfies L.O. If a is a proper ideal of C then there exists a maximal ideal m of C containing a such that

$$G_C(a,R) = G_{C_m}(a_m,R_m)$$

Proof: Let $x_1,\ldots,x_n$ denote a maximal C-sequence in a on R and let $K = \sum_{i=1}^{n} x_i R$. Then $a \subseteq \mathcal{Z}(R/K)$.

By Proposition 1.4.1(ii), there exists $r \in R$, $r \notin K$ such that $ra \subseteq K$. (*)

Let $b = \{b \in C \mid rb \in K\}$ then $a \subseteq b \triangleleft C$ and $b \neq C$ since $r \notin K$. Choose a maximal ideal m of C containing ideal b and therefore a.

By Corollary 5.1.7, $G_{C_m}(a_m, R_m) \geq n$ so it remains to prove the maximality of $\bar{x}_1, \ldots, \bar{x}_n$ as a C_m-sequence in a_m i.e. that

$$a_m \subseteq \mathcal{Z}(R_m/KR_m)$$

Let $\bar{r}$ denote the image of r in R_m then $\bar{r}a_m \subseteq KR_m$ from (*).

If $\bar{r} \in KR_m$, then there exists $c \in C \backslash m$ with $rc \in K$ but then $c \in b$, by definition, and this contradicts $b \subseteq m$. The C_m-sequence $\bar{x}_1, \ldots, \bar{x}_n$ is therefore maximal and the proof is complete. □

<u>Proposition 5.1.9.</u>

Let R be a right Noetherian ring with central subring C such that the pair (R,C) satisfies L.O. and let N be a finitely generated right R-module. If a is an ideal of C then there exists a prime ideal P of R lying over a prime ideal p of C such that (i) $a \subseteq p = P \cap C$

(ii) $G(a,N) = G(P,N) = G(p,N)$

<u>Proof</u>: Let $x_1, \ldots, x_n$ be a maximal C-sequence in a on N_R and set $K = \sum_{i=1}^{n} x_i R$.

Then $a \subseteq \mathcal{Z}(N/NK)$ so applying Proposition 1.4.1(i) we may enlarge a to a maximal annihilator prime p of the C-module N/NK.

Let $U = \mathrm{ann}_{N/NK}(p)$. Since p is central in R, U is a right R-module and, by Proposition 1.4.1(ii), U_R is non-zero. Let P denote a maximal annihilator prime in R of U_R, so $P = \mathrm{ann}_R(V)$ for some non-zero R-submodule V of U_R.

Now $Vp \subseteq Up = 0$ so $p \subseteq P \cap C$. The prime ideal $P \cap C$ of C is an annihilator of a non-zero submodule of N/NK so by the maximality of p, $P \cap C = p$.

Thus P lies over p and, since $x_1, \ldots, x_n$ is a maximal C-sequence in P on N, the proof is complete. □

Lemma 5.1.10.

Let C be a central subring of a right Noetherian ring R and let a be an ideal of C. If $x \in C$ such that $b = a + xC \subseteq J(R)$, the Jacobson radical of R, then, for a finitely generated R-module N,

$$G_C(b,N) \leq 1 + G_C(a,N)$$

Proof: Suppose $G_C(a,N) = n$ and let $x_1,\ldots,x_n$ be a maximal C-sequence in a on N, then $a \subseteq \mathcal{Z}(N/N(x_1,\ldots,x_n))$.
By considering the module $N/N(x_1,\ldots,x_n)$, we may reduce to the case $G_C(a,N) = 0$ and it then suffices to show that $G_C(b,N) \leq 1$.

If $b \subseteq \mathcal{Z}(N)$ we are done, so suppose not, in which case,

$b = a + xC \not\subseteq p_1 \cup \ldots \cup p_k$ where p_i, $1 \leq i \leq k$, are the maximal annihilator primes of the C-module N.
By [40, Theorem 124] in the ring C, there exists $a \in a$ such that

$$y = a + x \notin p_1 \cup \ldots \cup p_k.$$

Now $a + yC = a + xC = b$ and $y \notin \mathcal{Z}(N)$ so it is enough to show that $b \subseteq \mathcal{Z}(N/Ny)$ for then y is a maximal C-sequence in b on N and $G(b,N) = 1$.

Clearly $y \in \mathcal{Z}(N/Ny)$ so it remains to show that $a \subseteq \mathcal{Z}(N/Ny)$. Let $U = \mathrm{ann}_N(a)$ then, as in Proposition 5.1.9, U is a non-zero finitely generated R-submodule of N by Proposition 1.4.1(i). Suppose $U \subseteq Ny$ and let $0 \neq u \in U$. Then there exists $v \in N$ such that $u = vy$ and thus $vya = ua = 0$. However $y \notin \mathcal{Z}(N)$ so $va = 0$ i.e. $v \in U$ and hence $U = Uy$. This is a contradiction to Nakayama's Lemma since $y \in b \subseteq J(R)$. Hence $U \not\subseteq Ny$ and $a \subseteq \mathcal{Z}(N/Ny)$ as required. □

Proposition 5.1.11.

Let R be a right Noetherian ring with a local central subring C whose maximal ideal m lies in the Jacobson radical of R. Suppose further that the pair (R,C) satisfies L.O.

If N is a finitely generated R-module and a an ideal of C such that $G(a,N) \lneq G(m,N)$, then there exist prime ideals P of R and p of C such that

(i) $a \subseteq p = P \cap C$

(ii) $G(P,N) = G(p,N) = 1 + G(a,N)$

<u>Proof</u>: Let $x_1,\ldots,x_k$ be a maximal C-sequence in a on N and let $K = \sum_{i=1}^{k} x_i R$. Since $G(m,N) > k$, there exists $y \in m$ such that $y \notin \mathcal{Z}(N/NK)$ and then $G(a + yC,N) \geq k+1$. It follows from Lemma 5.1.10 that $G(a + yC,N) = k+1$. The proof is completed by applying Proposition 5.1.9 to the ideal $a + yC$. □

Having adapted the commutative theory of grade to right Noetherian rings with central subrings, we are able to make the following definition which generalises that of a commutive Cohen Macaulay ring.

A right Noetherian ring R is said to be *centrally Macaulay* if it contains a central subring C such that, for all maximal ideals M of R,

C-grade of M on R = rank M

i.e. $G_C(M,R)$ = rank M.

<u>Notation</u>

In order to be specific about the central subring concerned, we shall use the term *C-Macaulay* where C indicates the central subring and is not a reference to Cohen. We shall refer to commutative Cohen-Macaulay rings by their full title so no confusion should arise.

If the central subring C is local with unique maximal ideal m, all maximal ideals of R have the same C-grade on R, namely $G_C(m,R)$. This value is said to be the *C-grade of R*.

We are now in a position to prove the main results of this section. Theorem 5.1.12 is stated in its full generality to illustrate the importance of the INC condition on primes to the development of the theory. In applications, we shall only be concerned with right Noetherian rings integral over a central

subring, where this property holds by Proposition 1.6.1, so, in later chapters, we will refer to Corollary 5.1.13 rather than the theorem.

Theorem 5.1.12.

Let R be a right Noetherian, right fully bounded ring with a central subring C such that the pair (R,C) satisfies GU. Suppose that R is C-Macaulay, then

Grade and rank coincide for all primes of R and C if and only if the pair (R,C) satisfies INC on primes.

Corollary 5.1.13.

Let R be a right Noetherian ring integral over a central subring C such that R is C-Macaulay then

(i) $G_C(P,R)$ = rank P for all primes P of R

(ii) $G_C(p,R)$ = rank p for all primes p of C

(iii) rank P = rank$(P \cap C)$ for all primes P of R.

Before proving the theorem and its corollary, we note that commutative Cohen-Macaulay rings are characterised by the property that grade and rank coincide for all primes. It is important also to note that GU and INC are not sufficient conditions to ensure that rank P = rank $P \cap C$ even for commutative integral extensions, see [40, Ex. 25, p.43]

Proof of 5.1.12:

$\Longrightarrow$ The assumptions on grade and rank imply that

$$\text{rank } P = G_C(P,R) = G_C(P \cap C,R) = \text{rank } (P \cap C) \quad (*)$$

Suppose that P and Q are prime ideals of R such that $P \subsetneq Q$ and $P \cap C = Q \cap C$, then from (*), rank P = rank Q. This contradicts $P \subsetneq Q$ so the pair (R,C) satisfies the INC condition on primes.

$\Longleftarrow$ Let m be a maximal ideal of C and M a prime ideal of R lying over m, then M is a maximal ideal by INC.

Now $G(m,R) = G(M,R) =$ rank M by the C-Macaulay assumption on R and hence all the prime ideals of R lying over m are maximal and have the same rank.

It follows from GU and INC that

$$\text{rank } m = \text{rank } M \text{ for all } M \text{ lying over } m,$$

and hence $G(m,R) = \text{rank } m$, for all maximal ideals m of C.

For prime ideals of C, it is therefore enough to consider the non-maximal primes.

Let p be such a prime ideal, then by Proposition 5.1.8 there exists a maximal ideal m of C such that

$$G_C(p,R) = G_{C_m}(p_m,R_m).$$

Further rank p = rank p_m.

We may therefore reduce to the case where C is local with unique maximal ideal m. Notice that central localisation preserve the hypotheses on R and by Lemma 1.6.5, $J(C) = m \subseteq J(R)$.

Suppose the result is false and amongst those primes of C whose rank exceeds their grade, choose one of maximal rank. Let p denote this prime then $p \subsetneq m$ and

$$G(p,R) \subsetneq \text{rank } p \subsetneq \text{rank } m = G(m,R).$$

Applying Proposition 5.1.11 to the prime ideal p yields a prime ideal q of C such that $p \subsetneq q$

$$G(q,R) = G(p,R) + 1.$$

Now

$$\text{rank } q \supsetneq \text{rank } p \supsetneq G(p,R) = G(q,R) - 1$$

so $$\text{rank } q \supsetneq G(q,R).$$

This contradicts the choice of p and therefore

$$G(p,R) = \text{rank } p \quad \text{for all primes } p \text{ of } C.$$

To deal with prime ideals of R, we may again assume C to be local and choose a counterexample of maximal rank. Denote this prime ideal of R by P then

$$G(P,R) \lneq \text{rank } P \quad \text{and} \quad P \cap C \lneq m.$$

Applying Proposition 5.1.11 to $P \cap C$ and using INC, there exists a prime ideal $Q \gneq P$ such that

$$G(Q,R) = 1 + G(P \cap C,R) = 1 + G(P,R)$$

This gives a contradiction as before and completes the proof. □

One of the most important properties of centrally Macaulay rings satisfying Theorem 5.1.12 is that they also satisfy the following generalisation of the Classical Unmixedness Theorem [40, Theorem 137].

<u>Theorem 5.1.14</u>. (A Generalised Unmixedness Theorem).

Let R be a right FBN ring with central subring C such that

(a) the pair (R,C) satisfy GU and INC

(b) R is C-Macaulay

Let $x_1,\ldots,x_t$ denote a C-sequence on R and set $I = \sum_{i=1}^{t} x_i R$ then

(i) Annihilator primes (in R) of I are minimal over I.

(ii) R/I has a right Artinian right quotient ring.

<u>Proof</u>: (i) Observe that if P is an annihilator prime in R on R/I then

$$P \cap C \subseteq \mathcal{Z}(R/I) \text{ and } P \cap C \supseteq \sum_{i=1}^{t} x_i C.$$

so

$$G(P,R) = G(P \cap C,R) = t = \text{rank } P \text{ by Theorem 5.1.12}$$

Further all prime ideals Q containing I contain a C-sequence of length t so $G(Q,R) \geq t$. Thus from Theorem 5.1.12, rank $Q \geq t$ for all primes containing I. It follows that all annihilator primes of I are minimal over I and all have rank t.

(ii) Suppose that P is a prime ideal of R minimal over I. We claim that $P \cap C$ is a prime ideal of C minimal over $I \cap C$. For if q is another prime ideal of C, such that $I \cap C \subseteq q \subsetneq P \cap C$ and q is minimal over $I \cap C$, then by GU and INC, there exist prime ideals Q and P' of R with Q lying over q and P' lying over $P \cap C$ such that

$$I \subseteq Q \subsetneq P'$$

By Theorem 5.1.12, rank P' = rank $(P' \cap C)$ = rank $(P \cap C)$.
P is minimal over I so rank $(P \cap C) = G(P,R) = t$ and thus rank $Q \subsetneq$ rank $P' = t$.
However $t = G(I,R) = G(Q,R)$ since q is minimal over $I \cap C$, and

$$t = G(Q,R) = \text{rank } Q \subsetneq t. \quad \text{Contradiction .}$$

This proves the claim that $P \cap C$ is minimal over $I \cap C$ for P a prime ideal of R minimal over I.

If q is a prime ideal minimal over $I \cap C$ then there exists a prime ideal Q of R lying over q which is minimal over I (otherwise we get a contradiction to INC). Thus passage between R and C preserves minimality of prime ideals over I and $I \cap C$ respectively.

(ii) Let $P_1,\ldots,P_k$ be the prime ideals of R minimal over I then, by above, $\{p_i \mid p_i = P_i \cap C,\ 1 \le i \le k\}$ is the set of prime ideals of C minimal over $I \cap C$.

Let $T = \bigcap_{i=1}^{k} C_R(P_i) \cap C = \bigcap_{i=1}^{k} C_C(p_i)$.

Since every annihilator prime is minimal over I, $T \subseteq C_R(I)$.
Now $0 \neq C/I \cap C \longleftrightarrow R/I$ and R/I satisfies GU and INC on primes over $C/I \cap C$.
These properties are preserved by localisation w.r.t. T.

The only prime ideals of C which survive in $(C/I \cap C)_T$ are those minimal over $I \cap C$ and hence every prime ideal of $(C/I \cap C)_T$ is maximal and by INC, the prime ideals of $(R/I)_T$ are also maximal. Since $(R/I)_T$ is right FBN it follows that $(R/I)_T$ is right Artinian and is the required right quotient ring of R/I. □

§5.2. Properties and Examples of Centrally Macaulay Rings

In this section we restrict our attention to rings which are integral over a central subring.

Proposition 5.2.1.

Let R be a right Noetherian ring integral over a central subring C. If R is C-Macaulay then for any prime ideal p of C, the localisation R_p is C_p-Macaulay.

Proof: By Corollary 5.1.13, $G_C(p,R) = \text{rank}\ (p)$ so from Corollary 5.1.7, we have

$$G_{C_p}(p_p,R_p) \geq G_C(p,R) = \text{rank}\ (p) = \text{rank}\ (p_p)$$

Now R_p is integral over the central subring C_p by Proposition 1.6.6(iii) and since C_p is local with unique maximal ideal p_p, $M \cap C_p = p_p$ for every maximal ideal M of R_p. It follows that rank M = rank p_p and hence

$$\text{rank } M \leq G_{C_p}(p_p,R_p) = G_{C_p}(M,R_p). \quad (*)$$

Since the C_p-grade of M is bounded by its rank, we have equality in (*) and hence R_p is C_p-Macaulay. □

Proposition 5.2.2.

Let R be a right Noetherian ring integral over its centre Z. Let x be a regular central element which is not a unit.

(i) If R is Z-Macaulay then R/xR is Z/xZ-Macaulay

(ii) If $x \in J(R)$ and R/xR is Z/xZ-Macaulay then R is Z-Macaulay

Further if, in case (ii), Z is local then grade R = 1 + grade R/xR.

Proof: (i) First notice that $xR \cap Z = xZ$.

For, if $a = xt \in xR \cap Z$ then $x(rt - tr) = rxt - xtr = 0$ for all $r \in R$, but $x \in C_R(0)$ so $rt = tr$ and $t \in Z$.

It follows that Z/xZ is a central subring of R/xR and that R/xR is integral over this central subring.

Let $\bar{m}$ be a maximal ideal of Z/xZ then $\bar{m} = m/xZ$ for some maximal ideal m of Z.

Let $x, x_1, \ldots, x_n$ be a maximal Z-sequence in m on R then $\bar{x}_1, \ldots, \bar{x}_n$ is a maximal Z/xZ-sequence in $\bar{m}$ on R/xR (bars denoting images in R/xR) thus

$$G_Z(m,R) = G_{Z/xZ}(\bar{m}, R/xR) + 1.$$

By hypotheses, $G_Z(m,R) = \text{rank } m$ and, since $\text{rank } m \geq 1 + \text{rank } \bar{m}$, we have

$$G_{Z/xZ}(\bar{m}, R/xR) \geq \text{rank } \bar{m}.$$

However, rank bounds grade so it follows that

$$G_{Z/xZ}(\bar{m}, R/xR) = \text{rank } \bar{m} \text{ for all maximal ideals } \bar{m} \text{ of } Z/xZ.$$

The pair $(R/xR, Z/xZ)$ satisfies GU and INC by Proposition 1.6.1 so $G(\bar{M}, R/xR) = \text{rank } \bar{M}$ for all maximal ideals $\bar{M}$ of R/xR. Hence R/xR is Z/xZ-Macaulay.

(ii) We note that $x \in J(R) \cap Z = J(Z)$ so x lies in every maximal ideal of Z and hence of R.

Let M be a maximal ideal of R and write $\bar{M} = M/xR$ then by hypothesis,

$$\text{rank } \bar{M} = G_{Z/xZ}(\bar{M}, R/xR).$$

As above, x starts a Z-sequence in M on R so

$$G_Z(M,R) = 1 + G_{Z/xZ}(\bar{M}, R/xR).$$

Since $x \in C_R(0)$, $\text{rank } M = 1 + \text{rank } \bar{M}$ by [20, Theorem 4.6] thus

$$G_Z(M,R) = \text{rank } M.$$

This holds for every maximal ideal of R so R is Z-Macaulay.

If Z is local with unique maximal ideal m of rank n then

$$n = \text{rank } M \text{ for all maximal ideals } M \text{ of } R \text{ and grade } R = n.$$

Clearly grade R/xR = rank $m/xZ = n-1$ = grade $R - 1$. □

<u>Proposition 5.2.3</u>.

Let R be a Noetherian local ring integral over a central subring C. If R has finite injective dimension then

$$\mathrm{id}_R R = \text{C-grade of } R.$$

<u>Proof</u>: Let m denote the unique maximal ideal of C and let $x_1,\ldots,x_n$ be a maximal C-sequence in m on R.

Thus

$$\text{C-grade of } R = G(m,R) = n.$$

If $\mathrm{id}_R R = t \lneq n$, it follows from [4, Theorem 2.2] that there exists an integer $s \leq t$ such that

$$\mathrm{id}_{\bar{R}}\bar{R} = 0 \text{ where } \bar{R} = R/\sum_{i=1}^{s} x_i R$$

But then $\bar{R}$ is a Noetherian self-injective ring so is QF. In particular, $\bar{R}$ is Artinian and $m \subseteq \mathcal{Z}(\bar{R})$.

However $x_{s+1} \in m$ is regular on $\bar{R}$ by assumption, so we have a contradiction and hence $\mathrm{id}_R R \geq n$.

Suppose now that $\mathrm{id}_R R = k \gneq n$.

Let $A = R/\sum_{i=1}^{n} x_i R$.

By the maximality of the C-sequence $x_1,\ldots,x_n$, $m \subseteq \mathcal{Z}(A)$ and there exists $0 \neq a \in A$ such that $am = 0$ by Proposition 1.4.1(ii).

Now $mR \cap C = m$ and R/mR is integral over C/m so by Proposition 1.6.3, R/mR is Artinian.

Thus A contains a non-zero irreducible right R-module K.

Applying Corollary 2.2.9 to $A = R/\sum_{i=1}^{n} x_i R$ gives $\mathrm{pd}_R A = n$, so from the assumption that $k \gneq n$,

$$\mathrm{Ext}_R^k(A,R) = 0 \text{ by Proposition 1.7.5.}$$

Furthermore, Proposition 1.7.6, gives

$$\mathrm{Ext}_R^{k+1}(A/K,R) = 0 \text{ since } \mathrm{id}_R R = k.$$

From the exact sequence of Ext modules, Proposition 1.7.7,

$$\mathrm{Ext}_R^{k}(K,R) = 0.$$

Since R is local, every irreducible right R-module is isomorphic to K, thus

$$\mathrm{Ext}_R^{k}(L,R) = 0 \text{ for all irreducible right R-modules } L.$$

Hence, by Proposition 2.3.3,

$$\mathrm{id}_R R < k.$$

This contradiction gives $\mathrm{id}_R R = n$ as required. □

<u>Corollary 5.2.4.</u>

If R is a Noetherian local ring integral over its centre Z and of finite injective dimension n then

R is Z-Macaulay of grade n.

<u>Proof</u>: Use induction on $n = \mathrm{id}_R R$.

If $n = 0$, R is self injective so is a QF-ring.

Then R is Artinian and

$$\text{rank } J(R) = G(m,R) = 0$$

where $m = J(R) \cap Z$, the unique maximal ideal of Z.

If $n > 0$ then $\mathrm{id}_R R = G(m,R) = n$ by Proposition 5.2.3. Now there exists $0 \neq x \in m$, x regular on R so by [4, Theorem 2.2] $\mathrm{id}_{\bar{R}} \bar{R} \leq n-1$ where $\bar{R} = R/xR$. Since $xR \cap Z = xZ$, $\bar{R}$ is a Noetherian local ring integral over the central subring Z/xZ.

By Proposition 5.2.3, $\mathrm{id}_{\bar{R}} \bar{R} = G_{Z/xZ}(\bar{m},\bar{R})$ where $\bar{m} = m/xZ$.

Now x starts a Z-sequence in m on R, so

$$G_{Z/xZ}(\bar{m},\bar{R}) = G_Z(m,R) - 1$$

and hence $\mathrm{id}_{\bar{R}} \bar{R} = n-1$.

Thus by induction $\bar{R}$ is $Z(\bar{R})$-Macaulay of grade n-1

i.e. $\quad$ rank $J(\bar{R}) = n-1$

But since R is local, $J(\bar{R}) = J(R)/xR$, so

$$\text{rank } J(R) = 1 + \text{rank } J(\bar{R}) = n = G(m,R).$$

Hence R is Z-Macaulay of grade n. □

Corollary 5.2.5.

Let R be a Noetherian local ring integral over its centre Z. Then

$$\text{gl.dim } R = n < \infty \Rightarrow \text{id}_R R = n$$
$$\Rightarrow R \text{ is Z-Macaulay of grade } n.$$

Corollary 5.2.6.

A Noetherian local ring of finite global dimension which is integral over its centre is a prime centrally Macaulay ring.

Proof: The ring is centrally Macaulay by Corollary 5.2.5 so by applying Corollary 5.1.14 to a trivial regular sequence, it has an Artinian quotient ring. Primeness follows from [57, Theorem 4]. □

Corollary 5.2.5 is a natural generalisation of the observations made by Bass [4]. Although we prove a stronger version of 5.2.6 in Chapter 6, it serves at this stage to explain our interest in centrally Macaulay rings and with Corollary 5.2.5, provides examples of centrally Macaulay rings. Further examples are given in the next section where we discuss the relationship between the centrally Macaulay ring and the central subring concerned.

We conclude this section by demonstrating that centrally Macaulay rings satisfy natural generalisations of two further properties enjoyed by commutative Cohen Macaulay rings. These have been noted under stronger hypotheses by Northcott [51, Chapter 6], Ramras [56] and Vasconcelos [74].

Theorem 5.2.7.

Let R be a Noetherian local ring integral over a central subring C whose unique maximal ideal is denoted by m.

If N is a finitely generated right R-module of finite R-projective dimension then

$$pd_R(N) + G_C(m,N) = G_C(m,R).$$

<u>Proof</u>: Suppose $G(m,R) = n$ and $G(m,N) = d$.

Let $y_1,\ldots,y_d$ be a maximal C-sequence in m on the R-module N and $x_1,\ldots,x_n$ a maximal C-sequence in m on R.

Writing

$$R' = R/\sum_{i=1}^{n} x_i R \quad \text{and} \quad N' = N/N(y_1\cdots y_d)$$

then

$$G(m,R') = 0 \text{ and } G(m,N') = 0.$$

Consider the R-module N'. It follows from [40, Theorem B p.124] that

$$pd_R(N') \le pd_R(N) + d.$$

The reverse inequality is obtained by modifying the proof of Corollary 2.2.9, the inductive step being given as follows

Let $Y = N(y_1,\ldots,y_{d-1})$ and $N^* = N/Y$

then, for each $s \ge 1$, there exists a monomorphism ψ_s where

$$\psi_s : N^* \longrightarrow N^*$$
$$u \longmapsto uy_d^s$$

Noting that for $0 \le t \le s$, $\dfrac{Ny_d^t + Y}{Ny_d^{t+1}+Y} \cong N'$

we have

$$pd_R(N^*/\psi_s(N^*)) \le pd_R N'.$$

Proposition 2.2.4 may now be applied, as in Corollary 2.2.9, to give

$$pd_R(N') \ge pd_R(N) + d.$$

It is therefore enough to show that

$$pd_R(N') = G(m,R) = n$$

Now $G(m,N') = 0$ so it follows from Proposition 1.4.1(ii) that there exists $0 \ne u \in N'$ such that $um = 0$ and since R/mR is Artinian, N' contains an irreducible right R-module.

When viewed as a left R-module, R' is finitely generated and has left projective dimension n by [40, Theorem B, p.124]. Applying the left handed version of Corollary 2.1.5 gives

$$\mathrm{Tor}_R^n(N',R') \neq 0.$$

It follows from Proposition 1.7.5 that

$$\mathrm{pd}_R(N') \geq n \qquad (*)$$

Similarly $G(m,R') \neq 0$ so by Proposition 1.4.1(ii) there exists $0 \neq r \in R'$ such that $rm = 0$.

Since m consists of central elements, $mr = 0$ and hence the left R-module R' contains an irreducible left R-module.

It follows from Corollary 2.1.5 that

$$\mathrm{Tor}_R^k(N',R') \neq 0 \quad \text{where } k = \mathrm{pd}_R(N')$$

Hence $\mathrm{pd}_R(N') = k \leq \text{left } \mathrm{pd}_R(R') = n$ by Proposition 1.7.5. This inequality, together with (*) above, completes the proof. □

The second property we consider is the *saturated chain condition* (or *equidimensionality*).

A chain of prime ideals is said to be *saturated* if no further prime can be inserted and a ring satisfies the *saturated chain condition* if any two saturated chains of prime ideals between two fixed primes have the same length.

We define the *little rank* of a prime ideal P to be the length of the shortest saturated chain of prime-ideals between P and a minimal prime ideal contained in P.

Clearly the little rank of a prime ideal is bounded above by its rank. For a centrally Macaulay ring, we show that the two ranks coincide and are equal to the grade of the prime ideal. The saturated chain condition therefore holds for such rings.

Lemma 5.2.8.

Let R be a right Noetherian ring integral over a central subring C and let p denote a prime ideal of C, then

$$G(p,R) \leq \text{little rank } p$$

Proof: Suppose the result is false so that

$$G(p,R) \gneq \text{little rank } p.$$

By Corollary 5.1.7, $G(PR_p,R_p) \geq G(P,R)$ for any prime ideal P of R lying over p.

Now little rank p = little rank p_p, hence

$$G(p_p,R_p) \gneq \text{little rank } p_p.$$

We may therefore reduce to the case where C is local with maximal ideal p such that

$$G(p,R) \gneq \text{little rank } p = t.$$

Using induction, we may assume that the result of the lemma holds for all prime ideals of C whose little rank is less than t.

Let q be a prime ideal of C which is the top of saturated chain of primes in p, so that little rank q = t-1 and rank $(p/q) = 1$.

By induction, $G(q,R) \leq$ little rank q = t-1

so $\quad G(q,R) \lneq G(p,R)$.

Applying Proposition 5.1.11 to the ideal q of C yields a prime ideal q' of C such that

$$q \subsetneq q'$$

and $\quad G(q',R) = G(q,R) + 1 \leq t$

But $\quad q \subsetneq q' \subseteq p$ and rank $(p/q) = 1$

so $\quad q' = p$ and $G(p,R) = t$. Contradiction.

Hence little rank $p \geq G(p,R)$ as required. □

Corollary 5.2.9.

Let R be a right Noetherian ring integral over a central subring C such that R is a C-Macaulay ring, then rank and little rank coincide for all prime ideals of R and C. Both R and C satisfy the saturated chain condition.

Proof: Let p denote a prime ideal of C then by Lemma 5.2.8 we have

$$G(p,R) \leq \text{little rank } p \leq \text{rank } p$$

But R is C-Macaulay so grade and rank coincide thus

$$\text{little rank } p = \text{rank } p.$$

For a prime ideal P of R, intersecting a saturated chain of prime ideals contained in P with C yields

$$\text{little rank } (P \cap C) \leq \text{little rank } (P) \text{ by INC.}$$

Hence

$$\text{rank } (P \cap C) = \text{little rank } (P \cap C) \leq \text{little rank } P \leq \text{rank } P.$$

It follows from Corollary 5.1.13(iii) that

$$\text{little rank } P = \text{rank } P.$$

The saturated chain conditions follow easily from the equality of rank and little rank. □

§5.3. Rings with a Cohen Macaulay Centre.

We begin by considering the lifting of regular sequences.

Lemma 5.3.1.

Let R be a right FBN ring with central subring C such that

(a) the pair (R,C) satisfies GU and INC

(b) R is C-Macaulay

Suppose $x_1,\ldots,x_t$ are elements of C lying in the Jacobson radical J of R which form a regular sequence on the C-module C, then $x_1,\ldots,x_t$ is a C-sequence in J on R.

<u>Proof</u>: Suppose not, then there exists $k, 0 \le k \lneq t$ such that $\{x_1,\ldots,x_k\}$ forms a C-sequence on R but x_{k+1} is a zero divisor on $R/\sum_{i=1}^{k} x_i R$

Set $I = \sum_{i=1}^{k} x_i R$, then $[x_{k+1} + I]$ is a zero-divisor in the ring R/I and, being central, lies in an annihilator prime ideal Q of I.

I is generated by a C-sequence of length k so by Theorem 5.1.14(i), every annihilator prime of I is minimal over I and has grade k.

It follows from Theorem 5.1.12 that rank Q = k = rank $(Q \cap C)$. Now $Q \cap C \supseteq I \cap C \supseteq \sum_{i=1}^{k} x_i C$ and by [40, Theorem 132] prime ideals of C minimal over $\sum_{i=1}^{k} x_i C$ have rank $\ge k$. It follows that $Q \cap C$ therefore consists of zero divisors on $C/\sum_{i=1}^{k} x_i C$ [40, Theorem 84]. However $x_{k+1} \in Q \cap C$ is regular on $C/\sum_{i=1}^{k} x_i C$ by the assumption on $\{x_1,\ldots,x_t\}$. This contradiction completes the proof. □

It follows from the above result that if R is a Noetherian ring integral over a commutative local Cohen-Macaulay ring Z and R is Z-Macaulay then

grade Z = Z-grade R = rank J(R) = rank m

where m is the unique maximal ideal of Z.

In the non-local situation we have the following:

<u>Proposition 5.3.2.</u>

Let R be a Noetherian ring with central subring C and suppose that R is a finitely generated C-module of finite projective dimension. Then any two of the following imply the third.

(a) R is C-Macaulay

(b) C is Cohen Macaulay

(c) R is C-projective.

Proof: (a) + (b) $\Rightarrow$ (c).

R is C-Macaulay so if M is a maximal ideal of R,

$$G(M,R) = \text{rank } M = \text{rank } (M \cap C)$$

Let $m = M \cap C$, then C_m is a local Cohen-Macaulay ring [40, Theorem 139] and by Proposition 5.2.1, R_m is C_m-Macaulay.

By the remarks above, C_m-grade (R_m) = grade (C_m).

Now R_m is finitely generated as a C_m-module so

$$C_m\text{-grade } (R_m) + \text{pd}_{C_m} R_m = \text{grade } (C_m) \text{ by [40, Theorem 173]}$$

hence $\text{pd}_{C_m} R_m = 0$.

This argument may be repeated for each maximal ideal of C. It follows from [50, §9.2, Theorem 11] that

$$\text{pd}_C R = 0 \text{ as required.}$$

(b) + (c) $\Rightarrow$ (a).

Let M be a maximal ideal of R and let $m = M \cap C$. Localisation at m gives R_m as a finitely generated projective module over the local Cohen Macaulay ring C_m so by [40, Theorem 173]

$$G_{C_m}(m_m, R_m) = \text{grade } C_m = \text{rank } m_m.$$

However rank m_m = rank MR_m = rank M, and

$G_{C_m}(m_m, R_m) = G_C(m,R) = G_C(M,R)$ by Proposition 5.1.8.

Hence $G_C(M,R) = \text{rank } M$.

This holds for all maximal ideals of R so R is C-Macaulay.

(a) + (c) $\Rightarrow$ (b).

Let m be a maximal ideal of C and localise at m. R_m is a projective C_m-module so from [40, Theorem 173]

$$G_{C_m}(m_m, C_m) = G_{C_m}(m_m, R_m)$$

However, R_m is C_m-Macaulay so $G_{C_m}(m_m.R_m) = \text{rank } m_m$. It follows that every localisation of C at a maximal ideal is Cohen Macaulay and therefore C is Cohen Macaulay, [40, Theorem 140]. □

The above result can be compared with those obtained by Ramras [56] for algebras finitely generated as modules over regular local rings. For such algebras, the C-projective dimension of R is finite and C is Cohen Macaulay.

Consider now a local Noetherian ring R finitely generated as a module over its centre Z and suppose that R has finite global dimension. In view of Corollary 5.2.5, one is led to ask whether Z is a Cohen Macaulay ring. In his paper, Vasconcelos [74] asks whether Z is necessarily a regular local ring, and therefore Cohen Macaulay. In Example 6.2.3. we provide a ring R as above whose centre Z is not a Cohen Macaulay ring and we then discuss the relationship between the global dimensions of R and Z.

CHAPTER 6

RINGS INTEGRAL OVER A CENTRAL SUBRING

Section 6.1 is concerned with the proof of the following structure theorem.

Theorem

Let R be a right Noetherian local ring of finite right global dimension n which is integral over its centre Z, then

(i) R is a prime Z-Macaulay ring

(ii) The Jacobson Radical J of R contains a maximal Z-sequence of length n.

(iii) Rank $J = G_Z(J,R) = pd_R\ R/J = rt.gl.dim\ R = n$

and $K.dim\ R = cl.K.dim\ Z = n$

(iv) $R = \bigcap_{p \in P} R_p$ where P is the set of rank 1 prime ideal of Z and each R_p is hereditary.

(v) The centre Z is a Krull domain and, if Z is Noetherian, Z is integrally closed and has Krull dimension n.

The above theorem generalises that of Vasconcelos [74] and should be compared with theorems obtained in Chapter 4 for AR rings and with the structure of regular local rings, §1.9.

In §6.2 we show by means of two examples that rings satisfying the above theorem need not be AR-rings or finitely generated as modules over their centres. Therefore the above theorem applies to a class of rings not covered by existing results.

Let S denote a Noetherian local ring of finite global dimension which is finitely generated as a module over a central subring. Vasconcelos asks whether the centre of S is necessarily regular local. In Example 6.2.3, we provide such a ring S in which the centre is not Cohen Macaulay and is therefore far from regular local.

In his paper, Vasconcelos shows that the ring S is a maximal order in the sense of [3]. We have been unable to obtain a comparable result when the ring

is integral over its centre, as in the above theorem. Some remarks on this problem follow the proof of Theorem 6.1.8.

§6.1. A Structure Theorem

The theorem is proved in a number of stages, each of independent interest. The first step is to obtain a result on the ranks of maximal ideals similar to that of Theorem 4.1.1.

Theorem 6.1.1.

Let R be a right Noetherian ring integral over a central subring C and of finite right global dimension n, then rank $M \leq n$ for all maximal ideals M of R.

Proof: The proof proceeds by induction, the case $n = 0$ being clear.

Let P be a prime ideal of R such that $P \subseteq M$ and rank $(M/P) = 1$. Let $p = P \cap C$ and let $\{P = P_1, P_2, \ldots, P_t\}$ denote the prime ideals of R minimal over pR. By GU and INC, none of these primes are maximal and by Proposition 1.6.6, R is localisable w.r.t. their intersection. Corollary 2.2.8 may now be applied and the proof completed as in Theorem 4.1.1. □

If, in the above result, R is also assumed to be left Noetherian, Theorem 6.1.1 is a simple corollary to Theorem 4.1.1 by [47, Proposition 9]

The following theorem is a generalisation of Corollaries 5.2.5 and 5.2.6 to one sided Noetherian rings. Unlike these corollaries, the proof does not use the injective dimension, indeed we do not know whether a one-sided version of Corollary 5.2.4 is true.

Theorem 6.1.2.

Let R be a right Noetherian local ring integral over a central subring C and of finite right global dimension n, then

(i) A maximal C-sequence in the Jacobson Radical J of R has length n.

(ii) R is prime C-Macaulay ring of grade n

(iii) rank J = K.dim R = cl.K.dim C = rank m = n where $m = J \cap C$.

Note:

We do not assume that C has Krull dimension (or is Noetherian) however in this case K.dim C = n as well [29, Corollary 8.14] (or Propositions 1.4.5, 1.4.6)

Proof: Let $x_1,\ldots,x_t$ be a maximal C-sequence in J on R. By the remarks above Lemma 5.1.6, the rank of J bounds its grade, therefore we have

$$t \le \text{rank } J \le n \qquad \text{from Theorem 6.1.1.}$$

We require t = n.

Let $I = \sum_{i=1}^{t} x_i R$, then by the maximality of the C-sequence,

$$m \subseteq \mathcal{Z}(R/I) \quad \text{where } m = J \cap C.$$

Now $m \subseteq mR \cap C \subseteq J \cap C = m$ so C/m embeds in R/mR as a central subring and R/mR is integral over C/m.

$$C/m \text{ is a field so cl.K.dim } (C/m) = 0.$$

Hence, by Proposition 1.6.3, K.dim$(R/mR) = 0$ and R/mR is right Artinian, by Proposition 1.4.5(i).

It follows from Proposition 1.4.1 that R/I has a non-zero socle.

J has the right AR property so Lemma 2.2.1 applies and $pd_R(R/I) = \text{rt.gl.dim } R$. However, $pd_R(R/I) = t$ by Corollary 2.2.9 so n = t as required.

(ii) It follows from (i) and Theorem 6.1.1 that

$$G_C(J,R) = n \le \text{rank } J \le \text{rt.gl.dim } R = n$$

hence R is C-Macaulay of grade n.

Primeness follows from the Generalised Unmixedness Theorem (5.1.14) applied to a trivial C-sequence and a result of Ramras [57, Theorem 4].

(iii) Proposition 1.6.3. □

Proposition 6.1.3.

Let R be a right Noetherian ring integral over a local central subring C with maximal ideal m. Suppose R has non-zero finite right global dimension n such that

rank m = rt.gl.dim R = n > 0 then

the localisation R_p of R at a rank 1 prime p of C is right hereditary.

Proof: Use induction on n = rt.gl.dim. R.

If n = 1 the result is trivial, for then R_p is the localisation of a right hereditary ring and is not semisimple Artinian since rank p = 1.

Suppose the result holds for rings with right global dimension less than n and let rt.gl.dim R = n > 1.

Then if p is a rank 1 prime, $p \subsetneq m$. Let q be a prime ideal of C such that

$$p \subseteq q \subsetneq m \text{ and rank } m/q = 1.$$

If Q is a prime ideal of R lying over q then Q is not a maximal ideal and hence

$$pd_R\,(R/Q) \leq n-1 \text{ by Corollary 2.2.7.}$$

As in the proof of Theorem 6.1.2, this holds for all prime ideals lying over q and is preserved by localisation at q, thus

$$pd_{R_q}\,(R_q/QR_q) \leq n-1 \text{ for all primes ideals Q lying over } q$$

and

$$pd_{R_q}\,(R_q/J(R_q)) \leq n-1\,.$$

It follows from [11, Theorem] that rt.gl.dim $R_q \leq n-1$.

R_q is a right Noetherian local ring integral over the local central subring C_q so by Proposition 1.6.3,

$$\text{K.dim } R_q = \text{cl.K.dim } C_q = \text{rank } q_q = \text{rank } q.$$

Every prime ideal of R_q lying over q_q is a maximal ideal of R_q and therefore has rank bounded by rt.gl.dim (R_q) by Theorem 6.1.1. Thus

$$n-1 = \text{rank } q_q \leq \text{rt.gl.dim } R_q \leq n-1,$$

and rank q_q = rt.gl.dim R_q.

R_q now satisfies the hypotheses of the theorem so by induction, localisations of R_q at rank 1 primes of C_q are hereditary.

In particular, p_q is a rank 1 prime ideal of C_q since localisation preserves rank of primes contained in q.

Therefore $(R_q)_{p_q}$ is hereditary. This is merely the localisation of R at the rank 1 prime ideal p of C so the result follows. □

Lemma 6.1.4.

Let Z be the centre of a prime ring R and let p be a prime ideal of Z, then the centre of R_p is Z_p.

Proof: Suppose $ac^{-1} \in Z(R_p)$ where $a \in R$, $c \in Z\backslash p$

then $ac^{-1}r - rac^{-1} = 0$ for all $r \in R$

i.e. $(ar - ra)c^{-1} = 0$ for all $r \in R$.

Since R is prime, $ar = ra$ for all $r \in R$, and hence $ac^{-1} \in Z_p$. The result follows. □

Corollary 6.1.5.

Let R be a prime right Noetherian ring integral over its centre Z and of finite non-zero right global dimension.

If the centre is local with maximal ideal m such that

rank m = rt.gl.dim.R

then

Z_p is a DVR for each rank one prime ideal p of Z.

Proof: From Proposition 6.1.3, R_p is hereditary for a rank 1 prime ideal p of Z. Lemma 6.1.4 gives the centre of R_p as Z_p and since R_p is also prime, Z_p is a Krull domain,[6, Theorem 7.1.]. Z_p is also a local ring whose maximal ideal p_p has rank one so Z_p is a DVR. □

The next result should be compared with Theorem 4.2.4.

Theorem 6.1.6.

Let R be a right Noetherian local ring integral over a central subring C and of finite right global dimension then

$$R = \bigcap_{p \in P} R_p \text{ where } P = \{\text{rank 1 prime ideals of } C\}$$

<u>Proof</u>: R is prime by Theorem 6.1.2 so $R \subseteq \bigcap_{p \in P} R_p$.

Let $q \in \bigcap_{p \in P} R_p$ and set $X = \{r \in R \mid rq \in R\}$.

Clearly $X \cap C_C(p) \neq \emptyset$ for all $p \in P$ so

$$X \cap C_R(P) \neq \emptyset$$

where P is a prime ideal of R lying over a rank one prime ideal of C.

Hence $X \cap C_R(P) \neq \emptyset$ for all rank 1 prime ideals P of R.

Let c be a non-zero element of C then $c \in X \cap C_R(0)$ and is a C-sequence on R. Hence R/cR has a right Artinian right quotient ring by the Generalised Unmixedness Theorem 5.1.14.

The Principal Ideal Theorem [20, Theorem 4.6.] shows that all primes minimal over cR have rank 1, so

$$X \cap C_R(Q) \neq \emptyset \text{ for all prime ideals } Q \text{ of } R \text{ minimal over } cR$$

Therefore $X \cap C_R(N(R/cR)) \neq \emptyset$ by Proposition 1.4.4(iii) and

$$X \cap C_R(cR) \neq \emptyset \text{ by Small's Theorem 1.3.5.}$$

Let $d \in X \cap C_R(cR)$ then $dq \in R$ and $dcq = cdq \in cR$, so $cq \in cR$. Thus $q \in R$ and $\bigcap_{p \in P} R_p \subseteq R$ as required. □

<u>Corollary 6.1.7</u>.

If a right Noetherian local ring R of finite right global dimension is integral over its centre Z then the centre is a Krull domain.

<u>Proof</u>: We have to show that Z is the locally finite intersection of the localisation at its rank 1 prime ideals and that each localisation is a DVR. By Theorem 6.1.2, R is prime and has right Artinian right quotient ring Q. From Theorem 6.1.6, we have

$Z = Z(R) = R \cap Z(Q) = \bigcap_{p \in P} R_p \cap Z(Q)$, with P as in Theorem 6.1.6

$$= \bigcap_{p \in P} Z(R_p)$$

$$= \bigcap_{p \in P} Z_p \quad \text{by Lemma 6.1.4.}$$

Let s denote a non-zero element of the centre then there exist only finitely many prime ideals of R minimal over sR and these have rank 1 by [20, Theorem 4.6]. Intersecting these prime ideals with Z yields the finitely many rank 1 prime ideals of Z containing s.

Thus Z is the locally finite intersection of its localisation at rank 1 prime ideals. Each Z_p is a DVR by Corollary 6.1.5. □

We are now able to prove the main theorem mentioned in the introduction which we restate for convenience.

<u>Theorem 6.1.8.</u>

Let R be a right Noetherian local ring of finite right global dimension n which is integral over its centre Z. Then

(i) R is a prime Z-Macaulay ring

(ii) The Jacobson Radical J of R contains a maximal Z-sequence of length n

(iii) Rank $J = G(J,R) = pd_R\, R/J$ = rt.gl.dim.R = K.dim.R.

(iv) $R = \bigcap_{p \in P} R_p$ where P is the set of rank 1 prime ideals of the centre Z and each R_p is hereditary.

(v) The centre Z is a Krull domain with cl.K.dim Z = n. If Z is Noetherian, then K.dim.Z = n and Z is integrally closed.

<u>Proof</u>:

(i) and (ii) are proved in Theorem 6.1.2.

(iii) follows from Theorem 6.1.2 and [11, Theorem]

(iv) proved in Proposition 6.1.3 and Theorem 6.1.6

(v) The first part is Corollary 6.1.7 and Proposition 1.6.3. The final

observation follows from [29, Theorem 8.14] and the definition of integrally closed. □

The above theorem should be compared with the properties (R1)-(R6) of a commutative regular local ring S described in §1.9. The property (R3) that the maximal ideal J(S) is generated by a regular sequence cannot be extended to the situation of rings integral over a central subring as shown by Example 4.3.1.

Some Remarks on Maximal Orders.

Let C be a Krull domain which is also a central subring of a ring R and let K denote the field of fractions of C. Then R is said to be a *C-order* (in the sense of Fossum [25]) if

(i) R is a subring of a finite dimensional central simple (f.d.c.s) K-algebra

(ii) R is integral over C.

If, in the above definition, R is a finitely generated C-module, then R is an order in the sense of Auslander and Goldman [3]. An equivalence relation may be defined on orders in the same f.d.c.s. K-algebra and a *maximal order* is one which contains every order equivalent to it, see [3] or [25].

When R is a prime PI ring, the f.d.c.s algebra is simply the quotient ring of R since this may be formed by central localisation [65, Corollary1]. We therefore have the following:

Lemma 6.1.9.

Let R be a right Noetherian local ring of finite global dimension and let Z denote the centre of R.

(i) If R is a finitely generated Z-module then R is an order in its quotient ring Q (in the sense of Auslander and Goldman).

(ii) If R is integral over Z and is a PI ring then R is an order in its quotient ring Q (in the sense of Fossum).

Proof: In both cases, the existence of Q and the fact that Z is a Krull domain is given by Theorem 6.1.8. □

In the context of (ii), it follows from a result of Schelter [66, Corollary 2, p. 252] that a prime PI ring R whose centre is a Krull domain, is integral over Z. He provides an example in which R is not a finitely generated Z-module however this example does not have finite global dimension. Schelters result is used by Chamarie and Maury [17] to show that R embeds in a maximal order R' in the sense of Fossum and that R' is a maximal order equivalent to R in the sense of Asano,(see [30] for the definition of an Asano order).

Let R denote a right Noetherian local ring of finite global dimension which is integral over its centre Z. If R is assumed to be a PI ring then R is a Z-order in its quotient ring as in Lemma 6.1.9(ii).

From [25, Proposition 1.3], we have that

R is a maximal order $\Longleftrightarrow$ (i) $R = \bigcap_{p \in P} R_p$ where P is the set of rank 1 primes of Z

(ii) Each R_p is a maximal Z_p-order.

Theorem 6.1.6 shows that (i) is satisfied however we have only been able to show that each R_p is a hereditary Z_p-order, (Theorem 6.1.8)and hence that R is a *tame order*, [25, p.325]. In order to prove that R is a maximal order, it is sufficient to consider the localisations of R at rank one primes of the centre. In this direction, Chamarie [15] has shown that, with the above notation, the following are equivalent.

(i) R is a maximal order

(ii) R_p is a maximal Z_p-order for all $p \in P$

(iii) R_p is local for all $p \in P$

(iv) R satisfies the right Ore condition w.r.t. C(P) for every rank 1 prime P of R

(v) For each $p \in P$, there exists a unique prime ideal P of R lying over p.

We have been unable to show that any of the above conditions hold when the ring R is not a finitely generated Z-module and it remains an open problem as to whether a Noetherian local ring of finite global dimension which is integral over its centre and a PI ring is necessarily a maximal order. In conclusion, we observe that any counterexample to the above question must have a non-Noetherian centre by Formanek's result [24] and Vasconcelos' Theorem for the finitely generated case.

§6.2. Examples and Counterexamples

It has already been shown in Example 4.3.1 that Walkers technique of using regular normalising sets of generators to reduce the global dimension of a ring cannot be applied to rings finitely generated as modules over their centres. We now provide examples which show that two other approaches to the problem of the structure of rings of finite global dimension will also fail in the context of rings integral over a central subring. The following example relates to the results of Vasconcelos [74]

Example 6.2.1.

For every integer $n \geq 1$, there exists a scalar local Noetherian domain of finite global dimension n which is integral over its centre but not finitely generated as a module over its centre.

Proof: Let D be a division ring with centre T which satisfies the following properties

(i) D is an infinite dimensional vector space over T, i.e. $\dim_T D = \infty$.

(ii) For every finite set of elements $a_1,\ldots,a_k$ of D, there exists a sub-division ring $F \subseteq D$ containing $a_1,\ldots,a_k$ such that

(a) T is the centre of F

(b) $\dim_T F < \infty$.

The existence of such rings was proved by Koethe [41], see also [44].

Take $n \geq 1$ and let $S = D[X_1,\ldots,X_n]$ where $X_i, 1 \leq i \leq n$ are commuting indeterminates.

An element $s \in S$ is a polynomial in the indeterminates, let $\{f_1,\ldots,f_t\}$ denote its coefficients in D.

Let F be the subdivision ring of D containing $\{f_1,\ldots,f_t\}$ which is finite dimensional over T (using property (ii) of D), then

$s \in F[X_1,\ldots,X_n]$ and s is integral over the centre Z of S, namely $T[X_1,\ldots,X_n]$. It follows that S is integral over its centre Z. Using the Hilbert Basis Theorem [54, Theorem 10.2.6], S is Noetherian and by [22] has global dimension n.

Let M denote the maximal ideal of S generated by $X_1,\ldots,X_n$. M is the unique prime ideal of S lying over $m = \sum_{i=1}^{n} X_i Z$.

The localisation S_m is therefore local, indeed it is scalar local since $S/M \cong D$.

Let R denote S_m. R is thus a scalar local Noetherian ring integral over its centre Z_m. From Lemma 1.7.3, gl.dim (R) $\leq$ gl.dim(S) = n.

The unique maximal ideal J of R is generated by the elements $X_1,\ldots,X_n$ which satisfy the hypotheses of Corollary 2.2.9 so $pd_R(R/J) = n$ and hence gl.dim.R = n by [11, Theorem]. R is therefore a domain by the scalar local version of Theorem 6.1.8. The hypotheses on D guarantee that R is not finitely generated as a module over its centre. □

Our second example shows that the techniques used in Chapters 3 and 4 for AR-rings cannot be applied to rings integral over their centres or indeed to finitely generated algebras. Expanding an example of Ramras [58] we construct a finitely generated algebra of finite global dimension which is not an AR ring. Notice that, in view of Lemma 4.2.7, the global dimension of such an algebra must be at least three.

Example 6.2.2.

There exists a scalar local Noetherian ring finitely generated over its centre and of finite global dimension 3 which is not an AR-ring.

Proof: Let S be a commutative regular local ring of dimension 3 in which 2 is a unit. For example, let $S = \mathbb{Q}[[X,Y,Z]]$, the formal power series ring over the rational numbers in three commuting indeterminates.

Construct the skew polynomial ring $T = S[A,B;\sigma]$ where $\sigma(A) = -A$ and $\sigma(s) = s$ for $s \in S$.

Set $R = T/\langle A^2-(X+Z^2), B^2-Y\rangle$

Then $R = S\bar{A} + S\bar{B} + S\bar{A}\bar{B} + S$ and R is a finitely generated free S-module.

Clearly $S \subseteq Z(R)$. Let $\tau = \alpha\bar{A} + \beta\bar{B} + \gamma\bar{A}\bar{B} + \delta$ be an element of Z(R), then

$$\bar{A}\tau = \tau\bar{A}$$

i.e. $$\alpha(X+Z^2) + \beta\bar{A}\bar{B} + \gamma(X+Z^2)\bar{B} + \delta\bar{A}$$
$$= \alpha(X+Z^2) - \beta\bar{A}\bar{B} - \gamma(X+Z^2)\bar{B} + \delta\bar{A}.$$

hence $\beta = 0$ and $\gamma = 0$.

Similarly, from $\bar{B}\tau = \tau\bar{B}$, we deduce $\alpha = 0$.

Hence the centre of R is S.

The elements $\bar{Z},\bar{B},\bar{A}$ generate the unique maximal ideal J of R and satisfy the hypotheses of Corollary 2.2.9 so $pd_R(R/J) = 3$.

It follows from [11, Theorem] that R is a scalar local Noetherian domain of global dimension 3. Further, it is a maximal order [74, Theorem 4.3] and a finitely generated free module over its centre S.

Now let $P = \bar{X}R + \bar{B}R + (\bar{A}-\bar{Z})R$, then P is an ideal of R generated by the normalising set of generators $X,\bar{B},\bar{A}-Z$.

$$R/P \cong S[A,B;\sigma]/\langle \bar{X},\bar{B},\bar{A}-\bar{Z}, A^2 - (X+Z^2), B^2-Y\rangle \cong \mathbb{Q}[[Z]]$$

Thus R/P is a domain and $C(P) = R\backslash P$.

In order to show that R is not an AR ring, it suffices to show that P fails to satisfy the right Ore condition wrt C(P).

To this end, let r,s be elements of R with $s \in C(P)$ such that

$$(\bar{A}+Z)r = \bar{B}s \qquad (\bar{A} + Z \in C(P))$$

Now $(\bar{A}+Z) \in C(\bar{B}R)$ so $r \in \bar{B}R$.

Let $r = \bar{B}t$ then $(\bar{A}+Z)\bar{B}t = \bar{B}s$ (*)

However $(\bar{A}+Z)\bar{B} = -\bar{B}\bar{A} + Z\bar{B} = \bar{B}(Z-\bar{A})$, so (*) yields

$$\bar{B}((Z-\bar{A})t-s) = 0.$$

Hence $s = (Z-\bar{A})t$ since R is a domain.

But then $s \in P$ giving a contradiction to $s \in C(P)$.

It follows that R is not an AR ring. □

Our third example answers a question of Vasconcelos [74] and relates to the discussions of Section 5.3.

<u>Example 6.2.3.</u>

There exists a scalar local Noetherian ring of finite global dimension which is finitely generated as a module over its centre and such that the centre is not Cohen Macaulay.

<u>Proof</u>: Let $k = \{0,1\}$, the field of two elements and set $S = k[X_1,X_2,X_3,X_4]$ the ring of polynomials in four commuting indeterminates over k.

Define an automorphism $\sigma: S \to S$ by

$$\begin{aligned} \sigma:\ & X_1 \mapsto X_1 + X_2 \\ & X_2 \mapsto X_2 + X_3 \\ & X_3 \mapsto X_3 + X_4 \\ & X_4 \mapsto X_4 \end{aligned}$$

and extend by linearity. Note that $\sigma^4 = 1$.

Form the skew polynomial ring $T = S[Y;\sigma]$, then T is a Noetherian ring of global dimension 5 by [22] and[54, Theorem 10.2.6]

The maximal ideal $M = \langle X_1, X_2, X_3, X_4, Y\rangle$ of T has a centralising set of generators namely $\{X_4, X_3, X_2, X_1, Y\}$ in this order. By [69], we may localise T at M and form the local ring $R = T_M$, in fact R is scalar local since $T/M \cong k$.

Now gl.dim (R) ≤ 5 by Lemma 1.7.3.

However, the generators of the maximal ideal MR_M satisfy the hypothesis of Corollary2.2.9 so R has global dimension 5.

Let $U = k[X_1, X_2, X_3, X_4, Y^4]$ and $m = M \cap U$.

Then U_m is clearly a commutative subring of R and R is a free U_m-module of rank 4. R is therefore a P.I. ring.

Let $S_0 = \{s \in S \mid \sigma(s) = s\}$.

The centre C of $T = S[Y;\sigma]$ is easily seen to be $S_0[Y^4]$ and it follows that the centre Z of R is given by $Z = C_{(M\cap C)}$.

By adapting the proof of [53, Lemma 7], S_0 is Noetherian and hence so is the centre Z of R.

The ring S_0 has already been calculated, see [26] or [7]. As shown in [26], S_0 is not a Cohen Macaulay ring. It follows from [7,Theorem 5] or the remarks on p. 88 of [26] that $S_{0(M\cap S_0)}$ is not Cohen Macaulay.

Now consider $Z = C_{M\cap C}$. If Z is a Cohen Macaulay ring, then so is Z/Y^4Z [40, Theorem 141].

However $Z/Y^4Z \cong S_{0(M\cap S_0)}$ which has been shown not to be Cohen Macaulay.

Finally, since R is a prime PI ring whose centre Z is Noetherian R is a finitely generated Z-module by [24, Theorem 2]. □

In contrast to the above example, a consequence of the next result is that the centre Z of a local Noetherian ring R of finite global dimension is a regular local ring when R is a free Z-module of finite rank. Note that by Example 6.2.1, rings satisfying the following proposition need not be finitely generated over their centres.

Proposition 6.2.4.

Let R be a Noetherian ring integral over a local central subring C and of finite global dimension n. If R is a projective C-module then C is a regular local ring of dimension $\leq n$.

Proof: Let m denote the unique maximal ideal of C and let $k = C/m$ be the residue field.

R is semilocal with finitely many maximal ideals $\{M_1 \ldots M_t\}$ lying over m. The Jacobson radical J of R has the AR-property, Proposition 1.6.4, so by [11, Theorem] $pd_R(R/J) = n$.

Now $R/J \cong R/M_1 \oplus \ldots \oplus R/M_t$ as R-modules so there exists s, $1 \leq s \leq t$ such that $pd_R\, R/M_s = n$ by [64, Exercise 9.6].

Set $M = M_s$, then $M \cap C = m$ and k embeds as a central subring in R/M.

R/M is therefore a vector space (possibly infinite dimensional) over k and R/M is isomorphic as a C-module to a direct sum of copies of k. It follows from [64, Exercise 9.6] that, $pd_C(R/M) = pd_C(k)$.

Now choose a projective resolution of R/M as an R-module

$$0 \to P_n \to P_{n-1} \to \ldots \to P_o \to R/M \to 0.$$

Each P_i is projective as an R-module but R is C-projective so each P_i is also C-projective. Further P_n is non-zero since $pd_R(R/M) = n$ thus we have a C-projective resolution of R/M of length n and

$$pd_C(R/M) = pd_C(k) \leq n.$$

Hence gl.dim $C \leq n$ and by Serre's Theorem 1.9.1, C is a regular local ring of dimension $\leq n$. □

Corollary 6.2.5.

If R is a local Noetherian ring finitely generated over a central subring Z and of finite global dimension n then

Z is regular local ring of dimension n $\Longleftrightarrow$ R is Z-free.

Proof:

$\Longrightarrow$ By Theorem 6.1.2, grade of R = n = grade of Z so [40, Theorem 173] gives the result.

$\Longleftarrow$ Z is a Cohen Macaulay ring and by Theorem 6.1.2, R is a Z-Macaulay ring. Result now follows from Proposition 5.3.3. □

The literature contains many examples which show that the inequality of Proposition 6.2.4 can be strict. We give some references to these and other related examples. To this end, let R denote a Noetherian ring finitely generated as a module over its centre Z.

In 6.2.3 we have provided an example where R and Z are local domains, R has global dimension 5 but Z has infinite global dimension. An easier example with gl.dim R = 2 is given in [38] however the centre here is Gorenstein and hence Cohen Macaulay.

For any $n \geq 1$, V.A. Jategoankar [37] and Tarsy [73] provide semilocal Noetherian rings, finitely generated as modules over discrete valuation rings, whose global dimensions are n. These rings are not local and not prime so the hypothesis that R is local is not superfluous in any of Corollaries 5.2.5, 5.2.6 or Theorems 6.1.2, 6.1.8.

Finally, an unpublished example of A.W. Chatters [19] provides a prime scalar local Noetherian ring which is finitely generated as a module over a principal ideal domain but does not have finite global dimension. This ring has Krull dimension 1 and every ideal has a centralising set of generators.

CHAPTER 7

GORENSTEIN RINGS

It is well known that Noetherian rings of finite global dimension also have finite injective dimension and are therefore Gorenstein rings. Like regular local rings, commutative Gorenstein rings are the subject of an elegant structure theorem which connects their homological properties with their prime ideals. The theorem we consider in this chapter is Bass' Fundamental Theorem [5]:

Theorem (Bass)

Let S be a commutative Noetherian Gorenstein ring of injective dimension n and let

$$0 \to S \to E_0 \to E_1 \to \dots \to E_n \to 0$$

denote the minimal injective resolution of S. Then, for $0 \le k \le n$,

$$E_k \cong \bigoplus_{\text{rank } p=k} E_S(S/p) \quad \text{where } p \text{ a prime ideal of S.}$$

Before attempting to generalise this theorem to a non-commutative Noetherian ring, it is instructive to consider what properties are required of the ring in order that a comparable result may be achieved.

In the theorem, each injective hull $E_S(S/p)$ is *indecomposable* i.e. has no non-zero direct summands. This is not necessarily the case for the R-injective hull of R/P where P is a prime ideal of a right Noetherian ring R, although every injective module over such a ring may be expressed as a direct sum of indecomposable injectives [67, Theorem 4.4]. The correspondence between prime ideals and isomorphism classes of indecomposable injectives was provided by Gabriel and has been shown to be 1-1 if and only if the ring is right FBN, see [29, Chapter 8]. We outline the Gabriel correspondence in §7.2 and, for a right FBN ring R we let $I_R(P)$ denote a representative of the unique isomorphism class corresponding to prime ideal P. Observe that in the above theorem, $I_S(p) = E_S(S/p)$.

An examination of Bass' paper [5] reveals that the properties of Cohen Macaulay rings play a significant role in the proof of the theorem. In Chapter 5, we provided generalisations of these properties for Noetherian rings integral over central subrings. Such rings are also FBN, so they are natural candidates for a generalisation of Bass' Theorem.

We shall prove the following theorem.

<u>Theorem</u>

Let R be a Noetherian local ring integral over its centre and of finite injective dimension n. Suppose that

$$0 \to R \to E_0 \to E_1 \to \dots \to E_n \to 0$$

is the minimal injective resolution of R, then for $0 \le k \le n$,

$$E_k = \bigoplus_{\text{rank } P=k} I_R(P)^{d_P}$$

where (i) P is a prime ideal of R

(ii) $I_R(P)$ is a representative of the isomorphism class of indecomposable injectives corresponding to P

(iii) d_P = uniform dimension $_{R/P}$ R/P.

Note that if R is commutative, $I_R(P) = E_R(R/P)$ and $d_P = 1$ so the above theorem reduces to that of Bass.

Our approach to the proof of the theorem is to use the properties of grade in centrally Macaulay rings to reduce the problem to considering only maximal ideals. This will be sufficient to show that $I_R(P)$ can only occur as a summand of the k-th term of the minimal injective resolution, where k = rank P. In order to count the number of occurrences of $I_R(P)$ in this term, it is necessary to view the abelian group $\mathrm{Ext}^k_R(R/J,R)$ as a module over the ring $\mathrm{Hom}_R(R/J,R/J)$ where J is the Jacobson radical of R.

In §7.1 we describe the module constructions necessary to give a homological description of grade. This was first introduced by Rees [59]. In §7.2, we show

that $I_R(P)$ can only occur as a summand of the appropriate term in the minimal injective resolution and the proof is completed in §7.3 where isomorphisms are constructed in order to count the number of occurrences of each $I_R(P)$.

7.1. <u>A Homological Description of Grade.</u>

Let R be any ring, I an ideal of R and let $A = \mathrm{Hom}_R(R/I, R/I)$. A is a ring under addition and composition of maps (denoted by fog).

Let M be any right R-module and define a right A-module action on $\mathrm{Hom}_R(R/I,M)$ as follows:

For $f \in \mathrm{Hom}_R(R/I,M)$, $\theta \in A$, $x \in R/I$, let

$$\begin{aligned} \mathrm{Hom}_R(R/I,M) \times A &\longrightarrow \mathrm{Hom}_R(R/I,M) \\ (f,\theta) &\longmapsto f^\theta \end{aligned}$$

where $f^\theta : R/I \to M$ is given by $x \mapsto f(\theta(x))$.

It is clear that $((f,\theta),\phi) = (f,(\theta\phi))$

for $f^{\theta\phi}(x) = f((\theta\phi)(x)) = f^\theta(\phi(x)) = (f^\theta)^\phi(x)$.

Now suppose that N is another right R-module and that $d:M \to N$ is an R-homomorphism. Let $d^*:\mathrm{Hom}_R(R/I,M) \to \mathrm{Hom}_R(R/I,N)$ be the induced map, i.e. $[d^*f](x) = dof(x)$. We claim that d^* is an A-homomorphism.

For, if $\theta \in A$, $f \in \mathrm{Hom}(R/I,M)$ and $x \in R/I$,

then $(d^*f)^\theta : x \longmapsto d^*f(\theta(x)) = dof(\theta(x))$

and $d^*(f^\theta) : x \longmapsto dof^\theta(x) = dof(\theta(x))$.

Since d,f and θ are all R-homomorphisms, it is clear that the A-action commutes with that of d^* so d^* is an A-homomorphism as claimed.

Now let $0 \to M \xrightarrow{\varepsilon} N_0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \to \dots$ be a complex of R-modules.

Applying the functor $\mathrm{Hom}_R(R/I,-)$ yields

$$0 \to \mathrm{Hom}_R(R/I,M) \xrightarrow{\varepsilon^*} \mathrm{Hom}_R(R/I,N_0) \xrightarrow{d_0^*} \mathrm{Hom}_R(R/I,N_1) \xrightarrow{d_1^*} \dots$$

This is a complex of A-modules and A-homomorphisms so by the definition of $\mathrm{Ext}_R^n(R/I,M) = \dfrac{\mathrm{Ker}\ d_n^*}{\mathrm{Im}\ d_{n-1}^*}$, $\mathrm{Ext}_R^n\ (R/I,M)$ is also an A-module.

Further, if $0 \to L \to M \to N \to 0$ is a short exact sequence of R-modules, then the long exact sequence of Ext,

$$\ldots \to \mathrm{Ext}_R^n\ (R/I,M) \to \mathrm{Ext}_R^n\ (R/I,N) \to \mathrm{Ext}_R^{n+1}(R/I,L) \to \ldots$$

is an exact sequence of A-modules. This may be seen by examining the homomorphisms in a proof of the existence of such a long exact sequence, for example see [64, Chapters 6,7].

<u>Lemma 7.1.1.</u>

Let R be a ring and M,N right R-modules. Let x be a central element of R such that $Nx = 0$ and x is a non-zero divisor on M,

then $\mathrm{Hom}_R(N,M) = 0$.

<u>Proof</u>: See Kaplansky [40, Appendix 3-1, p.100]. □

Northcott observed that Rees' reduction of the Ext and Tor functors required only that the factoring elements were central. We require isomorphisms as A-modules, not just as abelian groups. Thus, in our notation, we have

<u>Proposition 7.1.2.</u>

Let R be a ring with central subring C. Let M be a right R-module and let $x_1,\ldots,x_n$ be a C-sequence in R on M,

then (i) If N is another right R-module such that $N(x_1,\ldots,x_n) = 0$, then

$\mathrm{Ext}_R^n\ (N,M) \cong \mathrm{Hom}_R\ (N,\ M/M(x_1,\ldots,x_n))$ as abelian groups.

(ii) If $x_1,\ldots,x_n$ lie in a proper ideal I of R, then

$\mathrm{Ext}_R^n\ (R/I,M) \cong_A \mathrm{Hom}_R\ (R/I,\ M/M(x_1,\ldots,x_n))$

where the isomorphism is as A-modules, $A = \mathrm{Hom}_R(R/I,R/I)$.

Proof:(i) The proof is by induction, the case $n = 0$ being clear from [64, Theorem 7.2].

Let μ denote right multiplication by x_1 on M. Since x_1 is central in R, μ is an R-homomorphism and the regularity of x_1 on M gives rise to the short exact sequence

$$0 \to M \xrightarrow{\mu} M \to M/Mx_1 \to 0.$$

Applying Proposition 1.7.7 yields the long exact sequence of Ext

$$\ldots \to \mathrm{Ext}_R^{n-1}(N,M) \to \mathrm{Ext}_R^{n-1}(N,M/Mx_1) \to \mathrm{Ext}_R^{n}(N,M) \xrightarrow{\mu^*} \mathrm{Ext}_R^{n}(N,M) \to \ldots \quad (*)$$

Now $Nx_1 = 0$ so μ^* is the zero map.

By induction, $\mathrm{Ext}_R^{n-1}(N,M) \cong \mathrm{Hom}_R(N,M/M(x_1,\ldots,x_{n-1}))$. (**)

From Lemma 7.1.1, with $x = x_n$, the RHS of (**) is zero.

Thus from (*),

$$\mathrm{Ext}_R^{n-1}(N,M/Mx_1) \cong \mathrm{Ext}_R^{n}(N,M).$$

Applying the inductive hypotheses to the C-sequence $(x_2,\ldots,x_n)$ on the module M/Mx_1 completes the proof of (i).

(ii) If N is replaced by R/I in the above proof, then the isomorphism as abelian groups is achieved. From the remarks above, all the maps are A-homomorphisms and hence the final isomorphism is as A-modules as well as abelian groups. □

Lemma 7.1.3.

Let R be a right Noetherian ring integral over a local central subring C. Suppose $x_1,\ldots,x_n$ are elements of the Jacobson radical J of R forming a C-sequence on the finitely generated right R-module N then

$$\mathrm{Hom}_R\,(R/J,\ N/N(x_1,\ldots,x_n)) \neq 0 \iff x_1,\ldots,x_n \text{ is a maximal C-sequence in J on N.}$$

Proof: $\Longrightarrow$ Let $f: R/J \to N/N(x_1,\ldots,x_n)$ be a non-zero R-homomorphism and let $u \in N$, $u \notin N(x_1,\ldots,x_n)$ such that $f(1 + J) = \bar{u} = u + N(x_1,\ldots,x_n)$.

Then $\bar{u}.J = f(1 + J).J = f(J) \subseteq N(x_1,\ldots,x_n)$

so $\quad J \subseteq \mathcal{Z}(N/N(x_1,\ldots,x_n))$.

Therefore J consists of zero divisors of $N/N(x_1,\ldots,x_n)$ and hence the C-sequence is maximal.

$\Longleftarrow$: Since $x_1,\ldots,x_n$ is a maximal C-sequence in J on N,

$$m = J \cap C \subseteq \mathcal{Z}(N/N(x_1,\ldots,x_n)) .$$

By Proposition 1.4.1(ii), there exists $u \in N$, $u \notin N(x_1,\ldots,x_n)$ such that $u.m \subseteq N(x_1,\ldots,x_n)$.

Now m is the unique maximal ideal of C and $mR \cap C = m$ so R/mR is integral over the central subring C/m which is a field.

Thus R/mR is right Artinian by Proposition 1.6.3, and there exists an integer $k \geq 1$ such that $J^k \subseteq mR$.

Therefore $u.J^k \subseteq u.mR \subseteq N(x_1,\ldots,x_n)$

Choose k such that $u.J^k \subseteq N(x_1,\ldots,x_n)$ but

$$u.J^{k-1} \not\subseteq N(x_1,\ldots,x_n),$$

and let $v \in uJ^{k-1}$ such that $v \notin N(x_1,\ldots,x_n)$.

Define $f: R/J \to N/N(x_1\ldots x_n)$

$$1 + J \mapsto v + N(x_1\ldots x_n)$$

then f is a non-zero element of $\mathrm{Hom}_R\,(R/J, N/N(x_1,\ldots,x_n))$ and the proof is complete. $\square$

Corollary 7.1.4.

Let R be a right Noetherian ring integral over a local central subring C. Let N be a finitely generated R-module and let J denote the Jacobson radical of R, then

$G_C(J,N) = n \iff$ n is the least integer k such that

$$\mathrm{Ext}_R^k\,(R/J,N) \neq 0.$$

<u>Proof</u>: $\Longrightarrow$: Let $x_1,\ldots,x_n$ denote a maximal C-sequence in J on N, then by Lemma 7.1.3,

$$\mathrm{Hom}_R\,(R/J,\ N/N(x_1,\ldots,x_n)) \neq 0.$$

Using the isomorphism of Proposition 7.1.2(ii), we have

$$\mathrm{Ext}_R^n\,(R/J,N) \neq 0.$$

If $\mathrm{Ext}_R^k\,(R/J,N) \neq 0$ for $k < n$, then by Proposition 7.1.2(ii),

$$\mathrm{Hom}_R(R/J,\ N/N(x_1,\ldots,x_k)) \neq 0.$$

But then $x_1,\ldots,x_k$ would be a maximal C-sequence in J on N by Lemma 7.1.3 which contradicts the assumption that $G_C(J,N) = n$. Thus $\mathrm{Ext}_R^k(R/J,N) = 0$ for $k < n$.

$\Longleftarrow$: Suppose $G_c(J,N) = t$ and let $x_1,\ldots,x_t$ be a maximal C-sequence in J on N. By Proposition 7.1.2(ii) and Lemma 7.1.3,

$$\mathrm{Ext}_R^t\,(R/J,N) \cong \mathrm{Hom}_R\,(R/J,\ N/N(x_1,\ldots x_t)) \neq 0$$

and $\quad \mathrm{Ext}_R^k\,(R/J,N) = 0$ for $k < t$.

Thus $t \geq n$ by hypothesis.

Now $\mathrm{Ext}_R^n\,(R/J,N) \cong \mathrm{Hom}_R\,(R/J,\ N/N(x_1,\ldots,x_n)) \neq 0$ so by Lemma 7.1.3, $x_1,\ldots,x_n$ is a maximal C-sequence in J on N and hence $t = n$ by Proposition 5.1.5. □

Results similar to those in 7.1.2 - 7.1.4 have been obtained by other authors under different hypotheses. In particular, versions may be found in papers by Northcott, Ramras and Vasconcelos.

7.2. <u>Indecomposable Injectives in the Minimal Resolution</u>

We begin by showing that localisation at a prime ideal of the central subring preserves the required properties of the ring.

<u>Proposition 7.2.1</u>.

Let R be a Noetherian ring integral over a local central subring C such that

$G_C(J,N) = n \iff n$ is the least integer k such that

$$\mathrm{Ext}_R^k\,(R/J,N) \neq 0.$$

<u>Proof</u>: $\Longrightarrow$: Let $x_1,\ldots,x_n$ denote a maximal C-sequence in J on N, then by Lemma 7.1.3,

$$\mathrm{Hom}_R\,(R/J,\ N/N(x_1,\ldots,x_n)) \neq 0.$$

Using the isomorphism of Proposition 7.1.2(ii), we have

$$\mathrm{Ext}_R^n\,(R/J,N) \neq 0.$$

If $\mathrm{Ext}_R^k\,(R/J,N) \neq 0$ for $k < n$, then by Proposition 7.1.2(ii),

$$\mathrm{Hom}_R(R/J,\ N/N(x_1,\ldots,x_k)) \neq 0.$$

But then $x_1,\ldots,x_k$ would be a maximal C-sequence in J on N by Lemma 7.1.3 which contradicts the assumption that $G_C(J,N) = n$. Thus $\mathrm{Ext}_R^k(R/J,N) = 0$ for $k < n$.

$\Longleftarrow$: Suppose $G_C(J,N) = t$ and let $x_1,\ldots,x_t$ be a maximal C-sequence in J on N. By Proposition 7.1.2(ii) and Lemma 7.1.3,

$$\mathrm{Ext}_R^t\,(R/J,N) \cong \mathrm{Hom}_R\,(R/J,\ N/N(x_1,\ldots,x_t)) \neq 0$$

and $\quad \mathrm{Ext}_R^k\,(R/J,N) = 0$ for $k < t$.

Thus $t \geq n$ by hypothesis.

Now $\mathrm{Ext}_R^n\,(R/J,N) \cong \mathrm{Hom}_R\,(R/J,\ N/N(x_1,\ldots,x_n)) \neq 0$ so by Lemma 7.1.3, $x_1,\ldots,x_n$ is a maximal C-sequence in J on N and hence $t = n$ by Proposition 5.1.5. □

Results similar to those in 7.1.2 - 7.1.4 have been obtained by other authors under different hypotheses. In particular, versions may be found in papers by Northcott, Ramras and Vasconcelos.

7.2. <u>Indecomposable Injectives in the Minimal Resolution</u>

We begin by showing that localisation at a prime ideal of the central subring preserves the required properties of the ring.

<u>Proposition 7.2.1</u>.

Let R be a Noetherian ring integral over a local central subring C such that

R has finite injective dimension.

If $id_R R$ = grade R = K.dim R

then $id_{R_p} R_p$ = grade R_p = K.dim R_p for all prime ideals p of C.

Furthermore R and R_p are centrally Macaulay rings.

<u>Proof</u>:

Since C is local with maximal ideal m, say,

rank m = K.dim R = rank J where J = J(R).

R is therefore C-Macaulay and by Proposition 5.2.1, R_p is C_p-Macaulay.

The result holds for the prime ideal m. By working down a chain of primes contained in m, it is enough to show that the result holds for a prime ideal $p \subsetneq m$ with rank $(m/p) = 1$.

Let $id_R R = n$, then $id_{R_p} R_p \leq n$ by [4, Corollary 1.4]

Now $G_{C_p}(p_p, R_p)$ = rank p_p = n-1, so there exist elements $x_1, \ldots, x_{n-1} \in C_p$ forming a maximal C_p-sequence on R_p.

As in the proof of Proposition 5.2.3, if $id_{R_p} R_p = t \lneq n-1$,

then factoring by $x_1, \ldots, x_k$ for some $k \leq t \lneq n-1$ we have $id_{R_p} R_p / \sum_{i=1}^{k} x_i R_p = 0$ by [4, Theorem 2.2].

Thus $R_p / \sum_{i=1}^{k} x_i R_p$ is a QF ring whose Jacobson radical contains a regular element, x_{k+1}. This contradiction (to Theorem 1.8.1) shows that $id_{R_p} R_p \geq n-1$. Now let S be an irreducible right R_p-module and suppose that $Ext^n_{R_p}(S, R_p) \neq 0$. Let L be a finitely generated R-module such that $L \otimes R_p \cong S$.

Write $\phi: L \to S$ for the natural map and let $K = L/\mathrm{Ker}\,\phi$. If $m \subseteq \mathfrak{Z}(K)$ then by Proposition 1.4.1(ii) there exists $k \in K$, $k \notin \mathrm{Ker}\,\phi$ such that $k.m \subseteq \mathrm{Ker}\,\phi$. Now there exists $d \in m \cap C_C(p)$ so $kd \subseteq \mathrm{Ker}\,\phi$ i.e. there exists $c \in C_C(p)$ such

that $kdc = 0$. But then $k \in \operatorname{Ker} \phi$ which is a contradiction. We therefore have $x \in m$ which is a non-zero divisor on K and writing μ for right multiplication by x, we have the short exact sequence

$$0 \to K \xrightarrow{\mu} K \to K/Kx \to 0.$$

Applying Proposition 1.7.7, gives rise to the exact sequence

$$\ldots \to \operatorname{Ext}_R^n(K,R) \xrightarrow{\mu^*} \operatorname{Ext}_R^n(K,R) \to \operatorname{Ext}_R^{n+1}(K/Kx,R) \to \ldots$$

Since $\mathrm{id}_R R = n$, the third term is zero, from Proposition 1.7.6, so μ^* is an epimorphism and by Corollary 2.3.2, $\operatorname{Ext}_R^n(K,R)$ is also zero.

It follows that $\operatorname{Ext}_{R_p}^n(S,R_p) = 0$ which contradicts the assumption on S.

Therefore $\operatorname{Ext}_{R_p}^n(X,R_p) = 0$ for all irreducible right R_p-modules X and by Proposition 2.3.3, $\mathrm{id}_{R_p} R_p < n$.

We now have $\mathrm{id}_{R_p} R_p = n-1 = \text{grade } R_p = \text{K.dim } R_p$ as required. □

The *assassinator* of a uniform right R-module U is denoted $\mathrm{ass}_R U$ and defined by,

$$\mathrm{ass}_R U = \{r \in R \mid Vr = 0 \text{ for some non-zero submodule } V \text{ of } U\}.$$

If R is right Noetherian, $\mathrm{ass}_R U$ is a prime ideal of R [29, Theorem 8.3] and furthermore, $E_R(U)$ is an indecomposable injective R-module having assassinator equal to that of U_R. The Gabriel correspondence between isomorphism classes of indecomposable injectives and prime ideals of R is given by the mapping $\{I\} \mapsto \mathrm{ass}_R I$. In [29, Theorem 8.6] this mapping is shown to be an isomorphism when the ring R is right FBN. In this case, given a prime ideal P of R, we write $I_R(P)$ for an indecomposable injective module having assassinator P. Then $I_R(P)$ is unique up to isomorphism and $I_R(P) = E_R(U)$ for some uniform right ideal U of R/P.

Lemma 7.2.2.

Let R be a right Noetherian ring integral over a central subring C and let P be a prime ideal of R. If $p = P \cap C$ then

(i) $I_R(P)_p$ is a non-zero indecomposable injective R_p-module corresponding to PR_p.

(ii) If $I_R(P)$ is a direct summand of the R-injective R-module E then $I_R(P)_p$ is an R_p-direct summand of the injective R_p-module E_p.

Proof: Let $I_R(P) = E_R(U)$ where $U = K/P$ a uniform right ideal of R/P and let $0 \neq u = [k+P] \in U$. If $u.c = 0$ for some $c \in C_C(p)$, then $kc \in P$. This implies that $k \in P$ since $C_C(p) \subseteq C_R(P)$ and this gives a contradiction to $u \neq 0$. Thus elements of $C_C(p)$ are regular on U and U_p is non-zero. Now U is an essential submodule of its injective hull $E_R(U) = I_R(P)$ and it is easily shown that U_p is essential in $I_R(P)_p$. Consider the following diagram of R_p-modules,

$$\begin{tikzcd} 0 \arrow[r] & U_p \arrow[r, "i"] \arrow[d, "j"'] & I_R(P)_p \arrow[dl, "\phi"] \\ & E_{R_p}(U_p) & \end{tikzcd}$$

Since $I_R(P)_p$ is R_p-injective by [4, Lemma 1.2b], it follows from [67, Proposition 2.1.8] that ϕ is an embedding and hence $I_R(P)_p$ is a non-zero R_p-direct summand of $E_{R_p}(U_p)$. Now U_p is isomorphic to a uniform right ideal of R_p/PR_p so its R_p-injective hull is indecomposable. ϕ is therefore an isomorphism and the result follows.

(ii) This is a simple consequence of (i) and [4, Lemma 1.2b] □

Proposition 7.2.3.

Let R be a Noetherian ring integral over a local central subring C and let P denote a prime ideal of R. Suppose that the localisation of R at $p = P \cap C$

is of finite injective dimension k. Then, if

$$0 \to R \to E_0 \to E_1 \to \dots$$

is an injective resolution of R, $I_R(P)$ is not a direct summand of E_i for $i > k$.

<u>Proof</u>: Suppose $I_R(P) \oplus L \cong_R E_i$ for some $i > k$ and some right R-module L. Then by Lemma 7.2.2(ii), there exists an R_p-module K such that

$$I_R(P)_p \oplus K \cong_{R_p} (E_i)_p$$

Now $I_R(P)_p \neq 0$ by Lemma 7.2.2(i) so $(E_i)_p \neq 0$.

However, $0 \to R_p \to (E_0)_p \to (E_1)_p \to \dots$ is a minimal injective resolution of R_p by [4, Corollary 1.3] so by the hypothesis on the injective dimension of R_p, $(E_i)_p = 0$. This is a contradiction and therefore $I_R(P)$ cannot be a summand of E_i for $i > k$. □

<u>Lemma 7.2.4.</u>

Let R be a semilocal FBN ring and let M denote a maximal ideal of R and J the Jacobson radical. Suppose E is an injective R-module containing a submodule isomorphic to $I_R(M)$, then

$$\mathrm{Hom}_R(R/J, E) \neq 0.$$

<u>Proof</u>: We may assume that $I_R(M) \subseteq E$.

Let $I_R(M) = E_R(U)$ where U is a uniform right ideal of R/M. U_R is therefore an irreducible module and there exist an epimorphism ϕ and monomorphisms i_1 and i_2 such that

$$\phi: R/J \to U \quad \text{and} \quad 0 \neq U \xrightarrow{i_1} E_R(U) \xrightarrow{i_2} E.$$

Writing $f = i_2 \circ i_1 \circ \phi$ gives a non-zero homomorphism $f: R/J \to E$ as required. □

Proposition 7.2.5.

Let R be a Noetherian ring of finite injective dimension n and let

$$0 \to R \to E_0 \to E_1 \to \dots \to E_n \to 0$$

denote a minimal injective resolution of R. If J is the Jacobson radical of R and $A = \mathrm{Hom}_R(R/J, R/J)$ then

$$\mathrm{Ext}^i_R\,(R/J,R) \cong_A \mathrm{Hom}\,(R/J,E_i) \text{ for each } 0 \le i \le n$$

Proof: Let S_i be the submodule of E_i annihilated by J, then $S_i = \mathrm{Soc}\,(E_i)$. By definition of the minimal injective resolution,

$$E_i = E_R(d_{i-1}(E_{i-1})).$$

Now $\mathrm{Soc}\,(E(M)) = \mathrm{Soc}\,(M)$ for all R-modules by [67, Proposition 3.16], so $S_i = \mathrm{Soc}\,(d_{i-1}(E_{i-1}))$. Therefore we have $d_iS_i \subseteq d_id_{i-1}(E_{i-1}) = 0$ for each i. Apply the functor $\mathrm{Hom}_R\,(R/J,-)$ to the minimal resolution above to give

$$0 \to \mathrm{Hom}_R\,(R/J,E_0) \xrightarrow{d_0^*} \mathrm{Hom}_R\,(R/J,E_1) \xrightarrow{d_1^*} \dots \qquad (*)$$

But $\mathrm{Hom}_R\,(R/J,E_i) = \mathrm{Hom}\,(R/J,S_i)$ so (*) is a complex in which the connecting homomorphisms d_i^* are all zero maps. By definition,

$$\mathrm{Ext}^i_R\,(R/J,R) = \frac{\mathrm{Ker}\,d_i^*}{\mathrm{Im}\,d_{i-1}^*}$$

so $\mathrm{Ext}^i_R\,(R/J,R) \cong \mathrm{Hom}\,(R/J,E_i)$ for each i. All the maps d_i^* are A-homomorphisms so from the discussions in §7.1, the above isomorphism of abelian groups is also an isomorphism of A-modules. □

We are now in a position to prove the first part of our main theorem.

Theorem 7.2.6 (Structure Theorem Version 1)

Let R be a Noetherian ring integral over a local central subring C and of finite injective dimension such that

$$\mathrm{id}_R R = \text{C-grade } R = \text{K.dim } R.$$

Let P be a prime ideal of R of rank k. Then the unique indecomposable injective R-module corresponding to P can only occur as a summand in the kth term of a minimal injective resolution of R.

Proof: Suppose that $id_R R = n$ and let

$$0 \to R \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \dots \to E_n \to 0$$

denote a minimal injective resolution of R. If $p = P \cap C$ then from Proposition 7.2.1 we have $id_{R_p} R_p = K.\dim R_p = k$ and so $I_R(P)$ cannot occur as a summand of E_t for $t > k$ by Proposition 7.2.3. Using Lemma 7.2.2(ii), we may localise at p and reduce the problem to the case where P is a maximal ideal of a semilocal ring R such that

$$\text{rank } P = id_R R = k.$$

Let $J = J(R)$ then $G_C(J,R) = k$ since R is C-Macaulay by Proposition 7.2.1. It follows from Corollary 7.1.4 that

$$\mathrm{Ext}^i_R(R/J,R) = 0 \text{ for all } i < k.$$

However, if $I_R(P)$ is a summand of E_i for some i, then by Lemma 7.2.4. $\mathrm{Hom}_R(R/J,E_i) \neq 0$ But then $\mathrm{Ext}^i_R(R/J,R) \neq 0$ by Proposition 7.2.5 which, by above, can only occur if $i \geq k$. It follows that $I_R(P)$ can only occur as a summand of E_k where $k = \text{rank } P$. □

7.3. Occurrences of $I_R(P)$

In order to count the number of times each indecomposable injective R-module occurs in the minimal resolution of R, it is necessary to distinguish those modules associated to different maximal ideals. We do this by considering the uniform dimension of certain modules over subrings of the ring $A = \mathrm{Hom}(R/J,R/J)$ where J denotes the Jacobson Radical of R. These subrings arise naturally in the

theory of semisimple Artinian rings. Indeed, the first result of the section is essentially a result on minimal right ideals of semisimple Artinian rings. The need to describe the module action and to count the uniform dimensions dictates the use of the more complex notation which we now describe.

Let R be a semilocal ring with Jacobson Radical J and maximal ideals $M_1, M_2, \ldots, M_n$. Recall that there exist elements $e_1, \ldots, e_n$ of R such that for $1 \leq i \leq n$

(i) $\bar{e}_i$ is a central idempotent in $\bar{R} = R/J$

(ii) $R/J \cong \bar{e}_1 \bar{R} \oplus \bar{e}_2 \bar{R} \oplus \ldots \oplus \bar{e}_n R$

(iii) $\bar{e}_i \bar{R} \cong R/M_i$.

Let K_i be the unique irreducible R-module occurring in a composition series of R/M_i then $\{K_1, \ldots, K_n\}$ is a full set of non-isomorphic irreducible R-modules. Write $D_i = \mathrm{Hom}_R(R/M_i, R/M_i)$.

This ring may be considered as a subring of $A = \mathrm{Hom}_R(R/J, R/J)$ using (ii) and (iii) above, i.e.

for $\theta \in D_i$ define $\bar{\theta} : R/J \to R/J$

$$r+J \mapsto \theta(e_i r + M_i).$$

The identity element of D_i is the map λ_i where

$$\lambda_i : R/J \to R/J$$
$$r+J \mapsto e_i r + J$$

Notice that the subrings D_i of A have different identity elements from that of A. In the following Lemma the D_i-module action is that induced from the A-module action.

<u>Lemma 7.3.1.</u>

Let R be a semilocal ring and N an irreducible R-module isomorphic to K_i (in the above notation) then

(i) $\mathrm{Hom}_R\,(R/J,N)$ is trivial as a D_j-module, $i \neq j$

(ii) $\mathrm{Hom}_R\,(R/J,N)$ is a uniform D_i-module.

<u>Proof</u>: (i) Without loss of generality, we may assume that $N = K_i \subseteq \bar{e}_i\bar{R}$.

Then $Ne_j = 0$ for $i \neq j$ and $ue_i = u$ for all $u \in N$.

If $f \in \mathrm{Hom}_R\,(R/J,N)$ then $f\circ\lambda_j\,(1 + J) = f(e_j + J) = f(1 + J)e_j \in Ne_j = 0$.

Thus $f \circ \lambda_j$ is the zero map and, since λ_j is the identity element of D_j,

$$\mathrm{Hom}_R\,(R/J,N).D_j = 0.$$

(ii) Suppose that f,g are non-zero maps from $R/J \to N$,

then they are surjective since N is irreducible.

Let $0 \neq u \in N$ then there exist $r_1,r_2 \in R$ such that

$$f(r_1 + J) = g(r_2 + J) = u \neq 0.$$

Define $\theta_1,\theta_2 \in \mathrm{Hom}_R\,(R/J,R/J)$ by left multiplication by the elements r_1e_i and r_2e_i respectively. Then $\theta_1,\theta_2 \in D_i$ and

$$f^{\theta_1}(1+J) = f(r_1e_i+J) = f(r_1+J)e_i = u.e_i = u$$

$$g^{\theta_2}(1+J) = g(r_2e_i+J) = g(r_2+J)e_i = u.e_i = u.$$

Thus $0 \neq f^{\theta_1} = g^{\theta_2} \in fD_i \cap gD_i$ and $\mathrm{Hom}_R(R/J,N)$ is a uniform D_i-module. □

<u>Corollary 7.3.2.</u>

Let R be a semilocal Noetherian ring with maximal ideals $M_1,\ldots,M_n$ and let X be an ideal of R such that R/X is a QF ring.

Then

$$u.\dim_{D_j} \mathrm{Hom}_R\,(R/J,R/X) = u.\dim_{R/M_j} R/M_j \text{ for } 1 \leq j \leq n.$$

<u>Proof</u>: Let $S = \mathrm{Soc}\,(R/X)$, then by Proposition 1.8.4,

$$S \cong K_1^{\ell_1} \oplus \ldots \oplus K_n^{\ell_n}$$

where K_i is the unique irreducible R/M_i-module and $\ell_i = u.\dim_{R/M_i} R/M_i$.

So

$$\mathrm{Hom}\,(R/J,S) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_R(R/J,K_i)^{\ell_i}$$

Equating uniform dimensions of each side as D_j-modules we have

$$u.\dim_{D_j}(L.H.S) = \ell_j \times u.\dim_{D_j} \mathrm{Hom}_R\,(R/J,K_j)$$

$$= \ell_j \quad \text{by Lemma 7.3.1.}$$

Since $\mathrm{Hom}_R\,(R/J,R/X) = \mathrm{Hom}_R\,(R/J,S)$, the result follows. □

<u>Corollary 7.3.3.</u>

Let R be a semilocal FBN ring with maximal ideals $M_1,\ldots,M_n$. Let E be an injective R-module containing k copies of $I_R(M_i)$, the unique indecomposable injective module corresponding to M_i, then

$$u.\dim_{D_i} \mathrm{Hom}_R\,(R/J,E) = k.$$

<u>Proof</u>: By definition, $I_R(M_i) \cong E_R(K_i)$, the R-injective hull of the unique minimal right ideal of R/M_i.

Now $\mathrm{Hom}_R\,(R/J,E) = \mathrm{Hom}_R(R/J, \mathrm{Soc}(E))$, however

$$\mathrm{Soc}(E) = \mathrm{Soc}(E_R(K_i)^k) \oplus T$$

$$= K_i^{\,k} \oplus T$$

where T is a direct sum of irreducible R-modules not isomorphic to K_i.

$$\text{Thus } \mathrm{Hom}_R\,(R/J,E) = \mathrm{Hom}_R\,(R/J,K_i)^k \oplus \mathrm{Hom}_R\,(R/J,T) \qquad (*)$$

Consider each side of (*) as $D_i = \mathrm{Hom}_R\,(R/M_i,R/M_i)$ modules and equating uniform dimensions as before, then

$$u.\dim_{D_i}(L.H.S) = k \times u.\dim_{D_i} \mathrm{Hom}_R\,(R/J,K_i) + u.\dim_{D_i} \mathrm{Hom}_R\,(R/J,T)$$

$$= k \qquad \text{by Lemma 7.3.1.}$$

Hence $u.\dim_{D_i} \mathrm{Hom}\,(R/J,E) = k$ as required. □

where K_i is the unique irreducible R/M_i-module and $\ell_i = u.\dim_{R/M_i} R/M_i$.

So

$$\mathrm{Hom}\,(R/J,S) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_R(R/J,K_i)^{\ell_i}$$

Equating uniform dimensions of each side as D_j-modules we have

$$u.\dim_{D_j}(L.H.S) = \ell_j \times u.\dim_{D_j} \mathrm{Hom}_R\,(R/J,K_j)$$

$$= \ell_j \quad \text{by Lemma 7.3.1.}$$

Since $\mathrm{Hom}_R\,(R/J,R/X) = \mathrm{Hom}_R\,(R/J,S)$, the result follows. □

<u>Corollary 7.3.3.</u>

Let R be a semilocal FBN ring with maximal ideals $M_1,\ldots,M_n$. Let E be an injective R-module containing k copies of $I_R(M_i)$, the unique indecomposable injective module corresponding to M_i, then

$$u.\dim_{D_i} \mathrm{Hom}_R\,(R/J,E) = k.$$

<u>Proof</u>: By definition, $I_R(M_i) \cong E_R(K_i)$, the R-injective hull of the unique minimal right ideal of R/M_i.

Now $\mathrm{Hom}_R\,(R/J,E) = \mathrm{Hom}_R(R/J,\ \mathrm{Soc}(E))$, however

$$\mathrm{Soc}(E) = \mathrm{Soc}(E_R(K_i)^k) \oplus T$$

$$= K_i^{\,k} \oplus T$$

where T is a direct sum of irreducible R-modules not isomorphic to K_i.

$$\text{Thus } \mathrm{Hom}_R\,(R/J,E) = \mathrm{Hom}_R\,(R/J,K_i)^k \oplus \mathrm{Hom}_R\,(R/J,T) \qquad (*)$$

Consider each side of (*) as $D_i = \mathrm{Hom}_R\,(R/M_i,R/M_i)$ modules and equating uniform dimensions as before, then

$$u.\dim_{D_i}(L.H.S) = k \times u.\dim_{D_i} \mathrm{Hom}_R\,(R/J,K_i) + u.\dim_{D_i} \mathrm{Hom}_R\,(R/J,T)$$

$$= k \qquad \text{by Lemma 7.3.1.}$$

Hence $u.\dim_{D_i} \mathrm{Hom}\,(R/J,E) = k$ as required. □

We are now able to complete the proof of the structure theorem.

Theorem 7.3.4. (Structure Theorem)

Let R be a Noetherian ring integral over a local central subring C and of finite injective dimension n such that

$$\mathrm{id}_R R = \text{C-grade } R = \text{K.dim } R = n \qquad \text{(A)}$$

Let $0 \to R \to E_0 \to E_1 \to \ldots \to E_n \to 0$ denote a minimal injective resolution of R, then for each $0 \le k \le n$

$$E_k \cong \bigoplus_{\text{rank } P=k} I_R(P)^{d_P}$$

where (i) $I_R(P)$ is the unique indecomposable injective R-module associated to the prime ideal P under the Gabriel correspondence.

(ii) $d_P = \text{u.dim}_{R/P} R/P$.

Proof: From the first version of the theorem, 7.2.6, $I_R(P)$ only occurs as a direct summand in E_k where k = rank P. Localisation at $p = P \cap C$ preserves the hypotheses on the ring by Proposition 7.2.1 and the minimality of the injective resolution by [4, Corollary 1.3]. Further, it follows from Lemma 7.2.2 that the number of occurrences of $I_{R_p}(PR_p)$ in $(E_k)_p$ equals the number of occurrences of $I_R(P)$ in E_k. Finally notice that

$$\text{u.dim}_{R/P} R/P = \text{u.dim}_{R_p/PR_p} R_p/PR_p = d_P$$

We may therefore assume that R is semilocal, $\mathrm{id}_R R = k$ and P is a maximal ideal with rank k. It is enough to show that $I_R(P)$ occurs d_P times in E_k.
Let $A = \mathrm{Hom}_R(R/J, R/J)$ where $J = J(R)$, then

$$\mathrm{Ext}^k_R(R/J,R) \cong_A \mathrm{Hom}_R(R/J,E_k) \text{ by Proposition 7.2.5.}$$

R has C-grade equal to $id_R R = k$, by Proposition 7.2.1, so there exist elements $x_1,\ldots,x_k$ in J forming a C-sequence R. Set $X = \sum_{i=1}^{k} x_i R$. By Proposition 7.1.2(ii)

$$\mathrm{Ext}_R^k(R/J,R) \cong_A \mathrm{Hom}_R\,(R/J,R/X).$$

Applying[4, Theorem 2.2] k times, we have that R/X is a QF ring so Corollary 7.3.2 is applicable and hence, writing $D = \mathrm{Hom}_R\,(R/P,R/P)$,

$$\mathrm{u.dim}_D\,\mathrm{Hom}_R(R/J,R/X) = \mathrm{u.dim}_{R/P}\,R/P = d_P \qquad (*)$$

However, by Corollary 7.3.3, the number of copies of $I_R(P)$ in E_k equals the D-uniform dimension of $\mathrm{Hom}_R\,(R/J,E_k)$ and

$$\mathrm{Hom}_R\,(R/J,E_k) \cong_A \mathrm{Ext}_R^k\,(R/J,R) \cong_A \mathrm{Hom}_R\,(R/J,R/X) \qquad (**)$$

Comparing (*) and (**) we have $\mathrm{u.dim}_D\,\mathrm{Hom}_R\,(R/J,E_k) = d_P$ so $I_R(P)$ occurs d_P times in E_k and the proof is complete. □

Finally we observe that the hypothesis (A) of the above theorem is satisfied when R is local and either

(i) R has finite global dimension or

(ii) C is the centre of R.

The first of these follows from Theorem 6.1.2 whilst the second is a consequence of Corollary 5.2.4.

R has C-grade equal to $id_R R = k$, by Proposition 7.2.1, so there exist elements $x_1,\ldots,x_k$ in J forming a C-sequence R. Set $X = \sum_{i=1}^{k} x_i R$. By Proposition 7.1.2(ii)

$$\mathrm{Ext}_R^k(R/J,R) \cong_A \mathrm{Hom}_R\,(R/J,R/X).$$

Applying[4, Theorem 2.2] k times, we have that R/X is a QF ring so Corollary 7.3.2 is applicable and hence, writing $D = \mathrm{Hom}_R\,(R/P,R/P)$,

$$\mathrm{u.dim}_D\,\mathrm{Hom}_R(R/J,R/X) = \mathrm{u.dim}_{R/P}\,R/P = d_P \qquad (*)$$

However, by Corollary 7.3.3, the number of copies of $I_R(P)$ in E_k equals the D-uniform dimension of $\mathrm{Hom}_R\,(R/J,E_k)$ and

$$\mathrm{Hom}_R\,(R/J,E_k) \cong_A \mathrm{Ext}_R^k\,(R/J,R) \cong_A \mathrm{Hom}_R\,(R/J,R/X) \qquad (**)$$

Comparing (*) and (**) we have $\mathrm{u.dim}_D\,\mathrm{Hom}_R\,(R/J,E_k) = d_P$ so $I_R(P)$ occurs d_P times in E_k and the proof is complete. □

Finally we observe that the hypothesis (A) of the above theorem is satisfied when R is local and either

(i) R has finite global dimension or

(ii) C is the centre of R.

The first of these follows from Theorem 6.1.2 whilst the second is a consequence of Corollary 5.2.4.

REFERENCES

1. S.A. Amitsur, Prime rings having polynomial identities with arbitrary coefficients, Proc. London Math. Soc 17 (1967), 470-486.
2. M.F. Atiyah and I.G. Macdonald, An introduction to commutative algebra, Addison and Wesley, 1969.
3. M. Auslander and O. Goldman, Maximal Orders, Trans. Amer. Math. Soc. 97 (1960), 1-24.
4. H. Bass, Injective dimension in Noetherian rings, Trans. Amer. Math. Soc. 102 (1962), 18-29.
5. H. Bass, On the ubiquity of Gorenstein rings, Math. Z. 82 (1963), 8-28.
6. G.M. Bergman and P.M. Cohn, The centres of 2-firs and hereditary rings, Proc. London Math. Soc. 23 (1971), 83-98.
7. M-J. Bertin, Anneaux d'invariants d'anneaux de polynomes en caractérisque p, C.R. Acad. Sci. Paris, 264 (1967) Ser A, 653-656.
8. S.M. Bhatwadekar, On the global dimension of some filtered algebras, J. London Math. Soc. 13 (1976), 239-248.
9. R. Bieri, Homological dimension of discrete groups, Queen Mary College Mathematical Notes, 1976.
10. W.D. Blair, Right Noetherian rings integral over their centres, J. Algebra 27 (1973), 187-198.
11. M. Boratyński, A change of rings theorem and the Artin Rees property, Proc. Amer. Math. Soc. 53(1975), 307-310.
12. N. Bourbaki, Eléments de mathématique; Algèbre commutative, Hermann, Paris, 1967.
13. K.A. Brown, C.R. Hajarnavis and A.B. MacEacharn, Noetherian rings of finite global dimension, to appear in Proc. London Math. Soc.
14. H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, N.J. 1956.

15. M. Chamarie, Ordres maximaux et R-ordres maximaux, C.R. Acad. Sci. Paris, 285, (1977) Ser. A, 989-991.

16. M. Chamarie and A. Hudry, Anneaux noethériens à droite entiers sur un sous-anneau de leur centre, Comm. in Algebra 6 (1978), 203-222.

17. M. Chamarie and G. Maury, Un théorème de l'ideal à gauche principal dans les anneaux premiers, noethériens à identite polynomiale dont le centre est un domaine de Krull, C.R. Acad. Sci. Paris, 286 (1978) Ser.A, 609-611.

18. A.W. Chatters, The restricted minimum condition in Noetherian hereditary rings, J. London Math. Soc 4 (1971-72), 83-87.

19. A.W. Chatters, unpublished note.

20. A.W. Chatters, A.W. Goldie, C.R. Hajarnavis and T.H. Lenegan, Reduced rank in Noetherian rings, to appear in J. of Algebra.

21. A.W. Chatters and C.R. Hajarnavis, Non-commutative rings with primary decomposition, Quart. J. Math. Oxford (2) 22 (1971), 73-83.

22. K.W. Fields, On the global dimension of skew polynomial rings, J. Algebra 13 (1969), 1-4.

23. K.W. Fields, On the global dimension of residue rings, Pac. J. Math. 32 (1970), 345-349.

24. E. Formanek, Noetherian PI rings, Comm. in Algebra 1 (1974), 79-86.

25. R.M. Fossum, Maximal orders over Krull domains, J. Algebra 10 (1968), 321-332.

26. R.M. Fossum, The divisor class group of a Krull domain, Springer-Verlag, Berlin, 1973.

27. A.W. Goldie, Torsion-free modules and rings, J. Algebra 1 (1964), 268-287.

28. A.W. Goldie, The structure of Noetherian rings, in Lectures on Rings and Modules, Lecture Notes in Math. No. 246 Springer-Verlag, Berlin, 1972.

29. R. Gordon and J.C. Robson, Krull dimension, Mem. Amer. Math. Soc. 133 (1973).

30. C.R. Hajarnavis and T.H. Lenegan, Localisation in Asano orders, J. Algebra 21 (1972), 441-449.

31. I.N. Herstein, Non-Commutative Rings, Carus Math. Monograph No. 15, Math. Assoc. Amer., 1968.

32. K. Hoechsmann, Lifting ideals in non-commutative integral extensions, Canad. Math. Bull. 13 (1970), 129-130.

33. J.P. Jans, Rings and Homology, Holt, Reinhart and Winston, 1964.

34. A.V. Jategaonkar, Left principal ideal domains, J. Algebra 8 (1968), 148-155.

35. A.V. Jategaonkar, A. counter-example in ring theory and homological algebra, J. Algebra 12 (1969), 418-440.

36. A.V. Jategaonkar, Jacobson's conjecture and modules over fully bounded Noetherian rings, J. of Algebra 30 (1974), 103-121.

37. V.A. Jategaonkar, Global dimension of tiled orders over commutative Noetherian domains, Trans. Amer. Math. Soc. 190 (1974), 357-374.

38. C.U. Jensen and S. Jøndrup, Centres and fixed-point rings of Artinian rings, Math. Z. 130 (1972), 189-197.

39. I. Kaplansky, Fields and Rings, University of Chicago Press, 1969.

40. I. Kaplansky, Commutative Rings, Allyn and Bacon, Boston, 1970.

41. G. Koethe, Schiefkörper unendlichen ranges über dem Zentrum, Math. Annallen 105 (1931) 15-39.

42. J. Kuzmanovich, Localisations of Dedekind prime rings, J. Algebra 21 (1972), 378-393.

43. E. Matlis, Applications of Duality, Proc. Amer. Math. Soc 10 (1959), 659-661.

44. G. Maury, Exemples et compléments en théorie des ordres maximaux au sens de K. Asano, C.R. Acad. Sci. Paris, 284 (1977) Ser.A 839-841.

45. J.C. McConnell, The intersection theorem for a class of non-commutative Noetherian rings, Proc. London Math. Soc. 17 (1967), 487-498.

46. B.J. Muller, Localisation in non-commutative Noetherian rings, Can. J. Math. 28 (1976), 600-610.

47. B.J. Muller, Localisation in Fully Bounded Noetherian rings, Pac. J. Math. 67 (1976), 233-245.

48. T. Nakayama, On Frobeniusean Algebras II, Annals of Math. 42 (1941), 1-21.

49. D.G. Northcott, Rings, Modules and Multiplicities, Cambridge University Press, 1968.

50. D.G. Northcott, An Introduction to Homological Algebra, Cambridge University Press, 1972.

51. D.G. Northcott, A First course in Homological Algebra, Cambridge University Press, 1973.

52. Y. Nouaze and P. Gabriel, Idéaux premiers de l'algèbre enveloppante d'une algèbre de Lie nilpotente, J. Algebra 6 (1967), 77-99.

53. D.S. Passman, Radicals of group rings, Comm. in Algebra 2 (1974), 295-305.

54. D.S. Passman, The Algebraic Structure of Group Rings, Interscience 1977.

55. E.C. Posner, Prime rings satisfying a polynomial identity, Proc. Amer. Math. Soc. 11 (1960), 180-183.

56. M. Ramras, Maximal orders over Regular local rings of dimension 2. Trans. Amer. Math. Soc 142 (1969), 457-479.

57. M. Ramras, Orders with finite global dimension, Pac. J. Math. 50 (1974), 583-587.

58. M. Ramras, Maximal orders over regular local rings, Trans. Amer. Math. Soc. 155 (1971), 345-352.

59. D. Rees, The grade of an ideal or module, Proc. Camb. Phil. Soc. 53 (1957), 28-42.

60. R. Rentschler and P. Gabriel, Sur la dimension des anneaux et ensembles ordonnés, C.R. Acad. Sci. Paris, 265 (1967) Ser. A, 712-715.

61. J.C. Robson and L.W. Small, Hereditary prime PI rings are classical hereditary orders, J. London Math. Soc. 8 (1974), 499-503.

62. J.E. Roseblade, Prime ideals in group rings of polycyclic groups, Proc. London Math. Soc. 36 (1978), 385-447.

63. J.E. Roseblade and P.F. Smith, A note on the Artin-Rees property of certain polycyclic group algebras, to appear.

64. J.J. Rotman, Notes on Homological Algebra, van Nostrand, 1970.

65. L. Rowen, Some results on the centre of a ring with polynomial identity, Bull. Amer. Math. Soc. 79 (1973), 219-223.

66. W. Schelter, Integral extensions of rings satisfying a polynomial identity, J. Algebra 40 (1976), 245-257.

67. D.W. Sharpe and P. Vámos, Injective Modules, Cambridge University Press, 1972.

68. L.W. Small, A change of rings theorem, Proc. Amer. Math. Soc. 19 (1968), 661-666.

69. P.F. Smith, Localisation and the AR property, Proc. London Math. Soc. 22 (1971), 39-68.

70. P.F. Smith, On non-commutative regular local rings, Glasgow Math. J. 17 (1976), 98-102.

71. J.T. Stafford, On a conjecture of Ramras, unpublished note.

72. B. Stenström, Rings and Modules of Quotients, Lecture Notes in Math. No. 237, Springer-Verlag, 1971.

73. R.B. Tarsy, Global dimension of orders, Trans. Amer. Math. Soc 151 (1970), 335-340.

74. W.V. Vasconcelos, On quasi-local regular algebras, Symposia Mathematica Vol XI (1973).

75. R. Walker, Local rings and normalising sets of elements, Proc. London Math. Soc. 24 (1972), 27-45.