# MATHEMATICAL MODELS IN CAPITAL INVESTMENT APPRAISAL

by

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## DECLARATION

I declare that this thesis describes the author's original work except where otherwise stated. The control variable technique applied to the case of percentiles was the subject of a joint paper with Dr. R. Ashford accepted for publication on Journal of Operational Research Society.

# ABSTRACT

The large and complex problem of project appraisal is considered here from a financial point of view, by representing an investment project by a stream of cash flows. Two elements affecting a project are focussed on here: the tax system and uncertainty.

A review is made of the most common models for the selection of projects in a deterministic situation. Interdependency between projects and ongoing activities of a firm is recognized, and a special emphasis is given to the tax system as a source of interdependency. Different ways of dealing with uncertainty in the context of capital investment are reviewed, and stochastic simulation is acknowledged as the relevant technique to use in this study. Variance reduction techniques are presented as an efficient tool for stochastic simulation in particular the use of control variables in the calculation of the mean and also, a relative innovation, in the calculation of a percentile.

Several groups of projects are chosen in a deterministic way, with different underlying philosophies. Two simplified tax systems are considered - the old one and the new one, retaining two of the main characteristics of the UK tax system, both pre and post 1984. Uncertainty is introduced in each of these groups using a model.

The results show that taxation and uncertainty considerably reduce the expected net present value of the groups of projects, the reduction being greater with the old tax system than with the new one. The new tax system overall seems to generate higher net present values with no higher risk than the old tax system. The difference in ranking in the groups when uncertainty is introduced suggests that the benefits from a deterministic mathematical programming model are diminished.

# CHAPTER I. INTRODUCTION.

## 1.1. CAPITAL INVESTMENT APPRAISAL: THE PROBLEM.

The appraisal of capital investment decisions, taken in its more general form, is a complex activity that can be approached from different points of view. An investment can be examined from its social aspects and its economic consequences. If an investment has been implemented, its results can be seen as part of a larger capital budgeting exercise. Financial appraisal is one way to look at an investment and it is this perspective that is going to be studied. Capital investment appraisal is here considered as the financial evaluation of decisions involving capital investments. Even when reduced to a financial point of view, capital investment appraisal is a large problem with several approaches necessary to handle the different issues.

The selection of investment projects and the treatment of corporate tax affecting the capital expenditure decisions of firms are examples of important practical problems. Practicioners, however, have not yet favoured recent academic methods of solution. Risk is another important element to take into consideration when dealing with the appraisal of capital investments. In project selection, explicit handling of tax and of risk are not usually considered at the same time. The association of the three is too complicated to be handled by the most common models. However studying the impact of corporate tax on project selection, even if in a deterministic environment, and investigating the effect of uncertainty on a group of selected projects, are potentially important issues addressed in this research.

Financial costs and benefits are going to be associated with cash flow rather than with accounting profit in this study. The accounting profit provides a measure of performance over a certain period of the life of a firm, or project, relative to which the costs are apportioned. In investment appraisal one is interested in the entire life of the project, not in an arbitrary accounting period. Cash is also necessary to make the payments of bills, interest and dividends. These two reasons support the use of cash flows, although they are difficult to obtain with accuracy in a direct way, in all other but the simplest situations.

It is important to identify the relevant items in cash flow calculations. All cash inflows and outflows which change because of the undertaking of a project are relevant to the project. It is usual to take into account the sales revenue and costs of goods sold, the changes in current assets and liabilities, the acquisition and disposal of assets and tax based cash flows. Depreciation is an accounting convention not affecting cash flows and thus is not considered in capital investment appraisal.

Inflation is another point to take into consideration. It can be introduced directly onto the different items used to calculate the project's cash flows which respond in different ways to the effect of inflation. The values used to produce the cash flows in nominal terms, that is to say, in inflated prices, should reflect those differences.

Once the necessary relevant factors to obtain the cash flows during the life of a project have been identified, the problem now is how to combine them in a useful manner as they arise in different time periods. The present value of a certain amount of money X, obtainable at some future time is less than X. This statement is reasonable, because if one had the amount X now, it could be invested and start earning interest. A common way of taking this into account is to multiply X by a factor less than 1, related to the interest rate or rate of return which is the reward that investors demand for accepting delayed payment. The interest rate is also known as the discount rate.

The present value approach is not the only one used to obtain a summary measure of a project's cash flows. Other approaches, some of them also involving the discount rate, are used and even favoured by firms. Some of these will be reviewed later.

The evaluation of a capital investment project is often separated from other activities of a firm, although it is accepted that the relevant way to consider the project's cash flows is as a marginal change in the global cash stream of the firm. This is due to adding the project to the existing activities. Resources limitations restrict the number of projects that can be accepted because of the competition among projects themselves and the existing activities of the firm for fixed amounts of materials, equipment and labour. Most situations with restrictions are reasonably well approximated with a linear programming model.

Another source of interdependency among projects is the amount of available capital. This means that it is not correct to evaluate a project in isolation from the firm's other projects and ongoing activities. Weingartner (1963) was a pioneer in the use of linear programming for capital rationing. Other mathematical programming models were developed in more complex practical situations taking into account the investment, financing and dividend options facing a firm.

The impact of taxation has long been recognised on the appraisal of individual projects, both on the cash flow profile and on the discount rate applied to the after-tax cash flows. The tax system is normally used not only to raise revenue to pay for government spending but also to stimulate particular activities which are considered of common interest in both the public and private sector.

Usually, encouraging investment in industry with its social and economic consequences, is one aim of certain aspects of the tax system. This is is the case in the UK tax system which treats capital expenditure in a distinctive way For tax purposes special capital allowances, sometimes known as writing-down allowances, substitute for the deduction from taxable profits of the depreciation of fixed assets, as the latter is completely disallowed. For example, before the 1984 Finance Act there was a first year allowance of 100 per cent on plant, machinery and equipment. The effect was that qualifying capital expenditure was treated, for tax purposes, in the year of acquisition, as an expense. This has changed and there is now a 25 per cent annual allowance on the declining balance basis. There have been other considerable changes in the tax system since 1983. These changes can be summed up by saying that the UK tax system has moved from a high tax, high allowance system to a low tax, low allowance system.

It is nowadays quite common practice to consider the effect of taxation on projects on an individual basis. More recent work, however, has shown that corporate tax can generate interdependencies between a firm's ongoing activities and a project (Buckley, 1975), and also among otherwise independent projects (Burns and Grundy, 1979).

Berry and Dyson (1979) have shown that the most simple tax system can also create such interdependencies. The simplified tax system they have considered retained one important aspect of the pre-1984 UK tax system: the 100% first year capital allowance. Berry and Dyson developed a mathematical programming model to solve this problem. The model was then extended to cope with a more complex tax system closer to the actual UK tax system.

## 1.2. UNCERTAINTY AND CAPITAL INVESTMENT APPRAISAL.

There is risk in any investment. Future cash flows are usually not known with certainty. Future revenues depend on an uncertain demand for the final product and future costs depend on uncertain activity levels and factor market conditions. The existing approaches for dealing with these uncertainties differ both in technique and in the perception of those whose risk is involved. Managers and individual investors have access to different type of information about firms and look at risk from different points of view.

Individual investors usually take their decisions about investments using only information which has been made public. Management also have access to other information within a firm, allowing a different type of analysis of investment projects.

It is current practice for an investor, acting in a rational way, to diversify his investments. It is known that diversification reduces the variability of the total return, see for example, Brealey and Myers (1981), but it does not completely eliminate risk. The variability of a portfolio of investments is measured by the standard deviation of the portfolio. It is possible to remove specific or unsystematic risk with diversification. Specific risk is particular to certain areas of activity and can affect an individual company and perhaps its immediate competitors. This risk is caused by factors independent of the market as a whole and is based on covariance with the market portfolio. Diversified portfolios, because they are exposed to variations caused by their connection with the market, are affected by another type of risk which diversification cannot eliminate. This is the systematic or market risk. Total risk is obtained by adding up the systematic and the specific risk. The total risk is therefore irrelevant for individual investors because by diversifying they can remove specific risk. Individual investors can hold a wider portfolio of investments than firms which are bound by specific areas of work where they can develop their activities. It is the total risk which is important to management. In what follows it is the management perspective that is considered.

A common way of obtaining the risk of a project is through the calculation of the variance of the project cash flow. An alternative procedure is to regard the project's contribution to the variance of the firm's cash flow as important. In this case managerial judgement about risk involves a judgement about the riskiness of the entire portfolio of the firm's activities and consequently about the relationship of any individual investment project with the firm's total risk.

Despite the difficulties inherent in obtaining a project's cash flows, they constitute a requirement for most of the performance measures used in capital investment appraisal. In this context, Hertz (1964) proposed risk analysis as an approach to tackle uncertainty. Probability distributions are assigned to factors affecting the various elements that are aggregated to form the project cash flow. In order to apply the risk analysis process it is necessary to build up the probability distributions of the components of the cash stream. This may be a difficult task because it is not easy to obtain plausible values for the distributions based on information from management. The job is even more difficult if some interdependency among components is introduced. The advantage of the approach is that a clearer picture of the uncertainty is established to be used in the analysis. This approach models separately the relevant elements forming a project cash flow. A different approach is to construct directly a model of the cash stream in each time period. The model can include interdependencies between projects and among time periods and can allow for the attempt to make market information as realistic as possible and at the same time to be simple enough to be useful in the calculations.

In relation to the financial perspective of the large and complex problem of capital investment appraisal a few points have been stressed:

_ the use of a project's cash flows and some of the difficulties associated with obtaining them;

_ the interdependencies among projects and the firm ongoing activities, in particular those specific to the tax system;

_ the uncertainty of future cash flows.

These aspects of the problem are usually tackled separately. After obtaining the cash flows the proposed projects can be ranked by means of one or more summary measures which aggregate this information. The decision to accept or reject the projects can be based on the ordering obtained. The use of a deterministic model necessary to take account of the interdependencies may give an answer that is far removed from a more realistic situation where uncertainty is important.

The object of stochastic programming is to deal at one and the same time with interdependency and uncertainty or risk. Some stochastic programming models to treat risk and interaction simultaneously can be seen for example in Kall, 1976. Finance was an area where stochastic programming models were applied. These models are difficult to solve unless the problem is small or a special case is considered (Bereanu,1980). Thus, although the theoretical instruments exist to solve the problem with interdependency and risk, they seem to be of no general use in practice.

The methods and techniques that are effectively used in capital investment appraisal can only give insight into particular aspects of the problem, which may lead to decisions which are not optimal in overall terms.

This research is concerned with the effect of uncertainty on the groups of projects accepted by a firm. The portfolio of projects is generated by taking into consideration the tax induced interdependencies among a larger set of initial projects. Stochastic simulation is used to study the combined effect of tax and uncertainty on a firm's ongoing activities and additional projects.

The UK underwent a considerable change in its corporation tax system in 1984. A comparison will therefore be made of two simplified tax systems: that of the old and new tax system. The old tax system includes a key characteristic of the pre-1984 UK tax system, the first year 100% capital allowance and a 52% tax rate; the new tax system includes a 25% reduction in balance for capital allowances and a 35% tax rate. The no tax situation is also considered.

Uncertainty is introduced through a stochastic model to generate the cash flows for a firm's ongoing activities and for its projects. Using methods of stochastic simulation the three tax situations - no tax, old tax and new tax systems, are then compared and some conclusions drawn. The results of the stochastic simulation are also compared with the results obtained when uncertainty is not present.

Risk is an important element in capital investment. Risk is going to be measured here by the probability that the value of a firm and additional projects will fall below a certain critical value. In a real problem this critical value could be a value below which the firm could become bankrupt, need to reduce dividends, or experience other liquidity problems. Again the three tax situations are compared.

## 1.3. A BRIEF DESCRIPTION OF THE CHAPTERS.

Given an initial set of proposed projects whose cash flows are supposedly known, the problem that confronts the decision-maker is the acceptance or rejection of all or part of them. Chapter II reviews several methods of selecting capital investment projects in a deterministic situation. Firstly, the most common summary measures obtained from the projects' cash flows are introduced as well as the decision criteria which firms actually use for setting up new projects. Some mathematical programming methods which handle interdependencies created by several causes are then discussed. Mathematical programming theory can also be used to obtain a framework within which the most used rules of thumb in capital investment can be analysed.

Models for dealing with uncertainty in capital investment problems are considered in chapter III. Firstly, modelling risk for individual projects is discussed. Secondly, models for introducing uncertainty in a portfolio of projects are described and the model to be used in these calculations is presented. This model can be applied both in a deterministic and in a stochastic simulation, depending on some basic elements of the model being taken as fixed constants or as random variables.

In chapter IV, simulation in the context of capital investment is examined. A brief reference to the use of deterministic and stochastic simulations is made The importance of stochastic analysis in capital investment is stressed and the sampling procedures are outlined. The basic problem chosen for this research was taken from Weingartner (1963). As capital investment appraisal is a sensitive and important area in the life of a firm, it has not been easy to obtain suitable data for the purposes of this research. Thus an academic problem, already used in other studies, was adopted as a basis for the numerical work and is described here. In order to increase the accuracy of the estimators of the mean and percentile, variance reduction techniques (VRT) are considered. Chapter V presents several VRT applied to these estimators. The VRT are compared in the context of the capital investment problem and one of them, judged to be the most efficient, has then been chosen for use in subsequent calculations.

Chapter VI contains the detailed description of how the previously chosen models were used to make the selection of projects and to carry out the investigation of the behaviour of several sets of projects under alternative tax regimes both in deterministic and stochastic environments. This chapter also contains the results of the research, the main conclusions of which relate to:

____ the efficiency of various variance reduction techniques in the context of capital investment appraisal;

____ the impact of the old and new tax system on the value and riskiness of projects;

____ the use of deterministic procedures to select projects in uncertain conditions;

____ the value of portfolios of projects assuming deterministic and stochastic environments.

Finally, chapter VII gives the main points of the work and its conclusions, and presents some ideas for future research.

## CHAPTER II. SELECTION OF CAPITAL INVESTMENT PROJECTS IN A DETERMINISTIC SITUATION.

The financial appraisal of capital investments usually involves the calculation of a summary measure of a stream of cash flows. Two of the problems associated with the development of such a measure arise from the fact that the cash flows occur at different points in time and from the inherent uncertainty in future cash flows. The measures available for reducing the cash stream associated with a capital investment to a single value, in order to deal with the time dimension, range from ignoring the problem to using a discount technique. The uncertainty problem is considered in chapter III.

The measures to be considered in section II.1 are: the payback period, the accounting rate of return, the net present value, the profitability index, the internal rate of return and the fixed interest equivalent.

Section II.2 presents some decision criteria which firms actually use for setting up new projects. It is well known that a good capital budgeting system does not make accept-reject decisions on individual projects. It allows for capital rationing, simultaneous investment and financing decisions, and interdependencies among projects created by the tax system. Although not in wide use, mathematical programming models can handle these complexities of capital project selection. Section II.3 gives a quick survey of models of this type which have been used and presents one of these models in some detail.

The mathematical programming theory offers the possibility of interpreting some of the most common rules for project selection and this is reviewed in section II.4.

## II.1. SUMMARY MEASURES OF A STREAM OF CASH FLOWS.

Let $C_0$, $C_1$, ..., $C_i$, .., $C_n$ be the cash stream representing a capital project with n years of life, where $C_i$ is the cash flow in year i. It is assumed that $C_i$ occurs at the end of year i. $C_0$, or even the first few values of $C_i$, will represent the cash outflow at the beginning of the project life and as such will be negative. Other $C_i$ can also be negative, possibly representing some substitution of equipment or, if i=n, a lagged tax payment or some kind of cleaning up operation.

Payback Period.

The payback period of a project is the number of years it takes before the cumulative forecast net cash flow equals the initial investment. A payback rule involves comparing the calculated payback period with some predetermined target period A calculated figure less than the target one indicates that the project should be accepted. If a number of projects are being ranked, the most acceptable will be the one which has the shortest payback period.

Payback is an ad hoc rule. It does not use all the available information, as it ignores the cash flows outside the payback period. It ignores the order in which cash flows come within the payback period as it does not consider the time value of money for cash flows within that period. It gives no indication of how to set the target payback period.

The discounted payback rule uses discounted cash flows before the calculation of the payback period. It is a little better than the undiscounted payback, but yet does not answer the other two criticisms. Nevertheless the payback rule is in common use in combination with other summary measures. Its continued use in practice, despite its major faults, may perhaps be attributed to its being a rough screening device which gives some indication, at an early stage, of whether the project is likely to be acceptable. It may also be a reflection of the management's perception of the quality of the available cash flow data or of the costs of data collection. Finally, it may be perceived as a simple approach to dealing with the uncertainty of future cash flows. A short payback period may provide some assurance that acceptance of the project is unlikely to have serious consequences for the firm.

Accounting Rate of Return.

The accounting rate of return (ARR) is another non-discounting method of project appraisal and is based on accounting profit rather than cash flow. The ARR is essentially a ratio and can be computed in many ways differing only in the definitions of the accounting numbers involved. The numerator is the average profit of the project after depreciation and taxes, while the denominator is the average book value of the investment. A decision rule is based on some predetermined target value. A project should be accepted if its calculated ARR is greater than the target value.

This summary measure has a number of faults. It uses accounting numbers instead of cash flows; it does not consider the time value of money; it deals in ratios and therefore says nothing about the size of the projects, it does not say how to set the target value. ARR is probably a worse rule than the payback one.

Net Present Value.

The net present value (NPV) is a summary measure of project appraisal based on discounted cash flows. It incorporates the time value of money using a discount factor which is related to the firm's relevant interest rate in order to bring all future cash flows back to the present decision date. In the absence of interdependencies a firm should accept all opportunities with a positive NPV and reject those with a negative NPV.

The general formula for the net present value is:

$$NPV = C_0 + C_1/(1+r) + C_2/(1+r)^2 + \ldots + C_n/(1+r)^n$$

where r is the interest rate.

A positive NPV means that the project is yielding higher returns than can be obtained by simply lending at the rate of return r. This interpretation suggests that r is a minimum acceptable rate of return. The rate of return is also referred to as the discount rate, the hurdle rate or the opportunity cost of capital.

NPV is a measure whose use is increasing and is much favoured in finance textbooks. It is cash flow based and takes all cash flows into account as well as the time value of money. Furthermore, with an appropriate discount rate the NPV of a project is exactly the same as the increase in shareholder wealth.

A similar measure to NPV which uses the same discount rate but assesses the value of a project at its termination is the net terminal value (NTV). Its expression is:

$$NTV = C_0(1+r)^n + C_1(1+r)^{n-1} + \ldots + C_i(1+r)^{n-i} + \ldots + C_n$$

and hence,

$$NTV = NPV(1+r)^n$$

The net terminal value is the surplus available at the end of the project after repaying the investment and assuming that the money borrowed, or surpluses invested during the life of the project, were both made at an interest rate of r. A decision rule to accept any project with a positive NTV would lead to the same decision as the NPV decision rule.

Profitability Index.

The profitability index (PI), or the benefit/cost ratio as it is sometimes called, is the present value (PV) of forecasted future cash flows divided by the initial investment:

$$PI = (PV\ cash\ flows) / (Initial\ investment)$$

The profitability index decision rule is to accept all projects with an index greater than 1. The PI leads to exactly the same decisions as the NPV because when PI>1, the present value is greater than the initial investment so the NPV must be positive. However, the PI can be misleading when there is a need to choose between two mutually exclusive investments because the order of magnitude of their NPV can be very different. This problem can be dealt with looking at the PI on the incremental investment. The PI very closely resembles the NPV and in some cases can even be the more useful rule. But for most purposes it is safer to work with the net present values which add up, rather than with profitability indexes that do not.

Internal Rate of Return.

The internal rate of return (IRR) of a project may be defined as the discount rate at which the present value of all future cash flows, both positive and negative, is equal to the investment cost of the project. Hence, it is the discount rate which makes NPV=0. This means that to find the IRR of an investment project lasting n years the following equation must be solved for IRR:

$$C_0 + C_1/(1+IRR) + C_2/(1+IRR)^2 + \dots + C_n/(1+IRR)^n = 0$$

The solution method is usually of the iterative type.

The decision rule for capital budgeting on the basis of the internal rate of return is to accept an investment project if the opportunity cost of capital is less than the IRR. The IRR is a profitability measure which depends solely on the amount and timing of the project cash flows. It can be interpreted as the highest rate of interest at which the company could afford to finance the project.

There are some problems with the use of the IRR. If there is more than one change in the sign of the cash flows $C_i$, $i=0, \dots, n$ there can be different rates of return. There can be as many changes in this rate as there are changes in the sign of $C_i$. There are also cases in which no IRR exists.

When there is a need to choose from among mutually exclusive projects, the IRR identifies the good projects, but as it ranks them differently from the NPV, a project which does not have the highest NPV can be chosen. The IRR rule can be salvaged in these cases by looking at the internal rate of the return on the incremental flows. The IRR gives also a ranking different from the one obtained with the NPV to projects which offer different patterns of cash flows over the time period under consideration. This situation is represented in the figure 2.1. For values of interest rate below P, $NPV_A > NPV_B$ although $IRR_B > IRR_A$.

The IRR rule requires the comparison of the project's IRR with the opportunity cost of capital. But short term interest rates may be different from long term rates. In these cases there is no simple criterion for evaluating the IRR of a project.

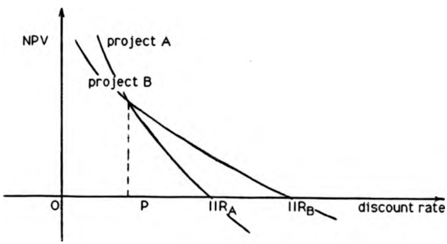


Figure 2.1

Although IRR is a popular measure with practitioners it is unfavourably compared with NPV in textbooks. Several reformulations of IRR have been designed to remove the problems with this measure while retaining its essential characteristics.

The internal rate of return assumes in its calculation that any surplus funds generated by the project have opportunity costs for the project which are equal to the IRR. But cash flows should be discounted at the market determined opportunity costs. A better assumption therefore, would be to assume that surplus funds can be invested and capital raised at the discount rate used in an NPV calculation. This leads to the fixed interest equivalent.

Fixed Interest Equivalent.

The fixed interest equivalent (FIE) rate of return is an alternative interest rate measure which can be obtained, using these assumptions, by computing the NTV of the project and calculating the interest rate required to yield a similar terminal value if the funds were invested in a fixed interest investment, e.g. Dyson and Berry, 1984.

Net terminal value of an equivalent fixed interest investment

at k% = TV(investment) - TV(cost of investment)

Assuming $C_0, C_n < 0$ and $C_i > 0, i=1, \ldots, n-1$ :

$$NTV(FIE_k) = -C_0(1 + k)^n - C_n + C_0(1+r)^n + C_n$$

where r is the discount rate.

The rate of interest required for the two terminal values to be equal is obtained by solving for k in the equation:

$$-C_0(1+k)^n + C_0(1+r)^n = C_0(1+r)^n + C_1(1+r)^{n-1} + \dots + C_{n-1}(1+r) + C_n$$

Weston and Brigham (1979) state that if the pattern of investment rates is known, then one should calculate the NTV and an IRR* (obtained equating NTV to zero ), because they are more accurate measures of project profitability than the NPV and IRR.

## II.2. CAPITAL BUDGETING TECHNIQUES.

Capital budgeting has been defined as the art of finding assets and the science of developing models to evaluate their worth (Pike, 1983). As such it is a much wider subject than can be treated here, as this study is confined to reviewing the evaluation techniques applied by firms of a reasonable size. It is known that most companies use a number of different criteria for project selection (Brealey & Myers, 1981). This can be seen in table 2.1 based on the results of Pike (1983), which were obtained from 150 usable replies from questionnaires sent to the 208 largest (measured in terms of market capitalisation values) UK manufacturing and retailing companies.

Table 2.1 Percentage of firms using combinations of investment criteria

| Payback | ARR | IRR | NPV | % of firm |
|---|---|---|---|---|
| x | x | x | x | 10 |
| x | x | x | | 10 |
| x | x | | x | 6 |
| x | | x | x | 10 |
| | x | x | x | 1 |
| x | x | | | 11 |
| x | | x | | 12 |
| x | | | x | 6 |
| | x | x | | 3 |
| | x | | x | 1 |
| | | x | x | 4 |
| x | | | | 11 |
| | x | | | 8 |
| | | x | | 4 |
| | | | x | 3 |

Source: Pike, 1983 - Table 6

Pike also found that payback and IRR were the most favoured techniques with a great advantage to payback. The NPV came only in fourth place in the actual use of firms contacted in the survey. It seems, though, that the discount methods have been gaining support, especially with the largest firms.

Two different interpretations have been proposed for the use of these combinations of criteria. They can be summarised in the following way. Firstly, no single appraisal technique is simple enough to be clearly understood by managers and at the same time complex enough to express the most important relationships holding in the real world. Secondly, managers tend to select the technique that presents the project in the best possible light relative to the way their performance is evaluated and rewarded.

The rate of return r used in discount methods is not determined in a straightforward manner. It is frequently assumed to have three components: a real rate of interest, the expected level of inflation and a risk premium allowing for the riskiness of the projects. This last aspect will be considered in more detail in chapter III.

As the aim of the cost of capital is to offer a minimum acceptable return for all projects undertaken by the firm, another way of obtaining r is to consider a weighted average cost of capital (WACC).

Although many types of long-term capital exist, the main broad types are debt and equity. Equity capital represents the funds provided by the owners of the firm. The dividends which are payments made to the owners of equity capital vary amongst other things, according to how well the firm is doing. Firstly, the firm has to meet its obligations to the providers of debt capital then afterwards dividends can be paid to equity holders, subject to the firm having sufficient income and cash resources from which to make this payment. The equity capital itself is rarely repaid until the firm is liquidated. The returns paid to debt holders, which is the interest, are usually fixed by contract, and must be paid regardless of the size of the firm's income or cash resources. Furthermore, much debt capital is redeemable, that is, the capital must be repaid by a specified date or dates. Failure to pay interest or repay capital on due dates often results in the control of the firm passing to the debt holders.

As debt and equity carry different levels of risk related both to interest and dividend payments, and also to capital repayment, they often have different costs. The WACC incorporates both the cost of equity capital, $k_E$ and the cost of debt capital, $k_D$, weighting $k_E$ and $k_D$ according to market values of the source of capital.

$$WACC = (E/V)\,k_E + (D/V)\,k_D$$

with $V = E + D$ being the total market value of debt and equity, and therefore the market value of the firm. Market values are used in the WACC expression because they are both current and observable. Furthermore, they measure the amount sacrificed by investors in a company as a result of their continuing to own equity or debt which could otherwise be sold at its market value.

The expression to calculate WACC is not controversial, it can be seen merely as a definition. If capital structure is to affect value, it must do so by operating either on expected earnings or on the cost of capital or on both. Because interest is tax deductible, gearing ( or financial leverage) generally increases expected earnings, at least as long as the firm does not use so much gearing that liquidation seriously threatens its continued existence. The effect of gearing on the cost of capital is much less clear and is the subject of much controversy in finance.

The capital budgeting techniques considered here evaluate a project in isolation although it has long been recognised that the cash stream for the project influences, and is influenced by the existing activities of the firm and other projects also under consideration. Mathematical programming presents an approach to modelling the interdependencies between the projects and the firm and is the subject of the following section.

## II.3. MATHEMATICAL PROGRAMMING MODELS.

The selection of investment projects in a situation of capital rationing is a classic application of mathematical programming (MP). In 1963 Weingartner presented a solution, in terms of an MP model, to the following problem: a firm is confronted with several possible independent investment projects and a fixed capital budget. For each project the net present value of its cash flows is computed. The objective is then to select from the available projects, which require outlays in several time periods, those which lead to the highest present value for the firm, given the fixed capital budget. Weingartner suggested an MP model to solve the problem. One form of Weingartner's model, reduced to its essentials, is as follows:

$$\text{maximize} \quad \sum_{j=1}^{n} \sum_{t=0}^{T} [a_{tj}/(1+r)^t]\, x_j$$

subject to

$$- \sum_{j=1}^{n} a_{tj}\, x_j \le c_t \,, \quad t = 0,1,\ldots,T \tag{2.1}$$

$$0 \le x_j \le 1 \,, \quad j = 1,\ldots,n$$

where

- $r$ is a fixed discount rate (the 'cost of capital');
- $a_{tj}$ is the net cash flow, which may be negative, obtained from a unit of project $j$ in period $t$;

$x_j$ is the fraction of project j accepted (not more than one of a project may be accepted);

$c_t$ is the fixed amount of cash available at t;

T is the last period in which capital rationing occurs and

n is the number of projects currently under consideration.

Weingartner's model on capital budgeting under capital rationing led to some controversy, with the nature of the objective function being specially questioned.

Baumol and Quandt (1965), were in favour of formulating the objective function as the sum, over all time periods, of the utility of the cash withdrawn for owner consumption in each time period t. One of their criticisms of Weingartner's model was that, with the existence of capital rationing, the objective function could not be specified until the solution was found, because r should itself be internally determined. Baumol and Quandt's criticism was dismissed both in the situation of soft (Carleton,1969) and hard capital rationing (Elton,1970). Myers' suggestion (1972) that the formulation of the objective function in terms of utility and in terms of net present value were equivalent led Bhaskar (1976) to contest the equivalence and to present another formulation in which the objective function is the present value of the future stream of dividends.

The model (2.1) does not permit funds attached to one period to be used in another period, that is, funds not used in a period disappear completely. Bhaskar (1974) extended the model to allow for the possibility of lending funds from one period to another at a rate of interest k. He suggested treating the lending operation as an additional investment.

If the cost of capital is greater than the lending rate ($r > k$), any money lent has a negative NPV, even so it may happen that the firm should undertake this 'project'. The situation of accepting a project with a negative NPV is more general than in a lending case. There are circumstances in capital rationing in which it is beneficial for the firm to undertake projects with a negative NPV.

In essence, borrowing is similar in nature to lending, and can be modelled in very much the same way. The real problem when borrowing is incorporated into the model is caused by the relationship between the level of borrowing and the cost of capital. The acknowledgement of the existence of debt capital raises the question of the correct way of calculating the NPV of projects. Bhaskar (1974) bases his model on the Modigliani-Miller

(MM) theory which states that the firm's overall cost of capital, which is the weighted average cost of equity and debt, is a constant, the value of which is invariant to the level of gearing in the firm. So, the objective function of the model with borrowing consists of just the net present value of the projects considered.

The full model with the two different types of financial transactions, lending and borrowing, is then the follows:

$$\text{maximise} \quad \sum_{j=1}^{n} \sum_{t=1}^{T} [a_{tj}/(1+r)^t]\, x_j + [(1+k)(1+r)^{-1} - 1] \sum_{t=0}^{T} y_t/(1+r)^t$$

( NPV of projects + NPV of funds lent )

subject to

$$\sum_{j=1}^{n} a_{0j}x_j + y_0 - z_0 = c_0 \qquad (2.2)$$

$$\sum_{j=1}^{n} a_{tj}x_j + y_t - (1+k)y_{t-1} - z_t + \sum_{i=0}^{t-1} e_{ti}\, z_i = c_t\,, \quad t=1, 2, \ldots, T$$

$$z_t \leq S_t\,, \qquad t = 0, 1, \ldots, T$$

$$0 \leq x_j \leq 1\,, \qquad j = 1, 2, \ldots, n$$

$$y_t \geq 0,\; z_t \geq 0, \qquad t = 0, 1, \ldots, T$$

where $r$, $a_{tj}$, $x_j$, $c_t$, $T$ and $n$ have the same meaning as in (2.1) and

$y_t$ is the amount of money lent from period $t$ to period $t+1$,

$k$ is the lending rate ($k < r$);

$z_t$ is the amount borrowed in period t;

$e_{ti}$ is the repayment and interest payable in period t associated with one pound borrowed in period i;

$S_t$ is the maximum amount of long term debt that can be used in period t.

Two easy extentions of model (2.2) can be constructed ( Bhaskar, 1974) to include increasing marginal costs of borrowing and competitive borrowing.

Weingartner's model for capital budgeting has led to several variants which try to incorporate the actual situation facing the firm. In order to achieve this more constraints were considered and slightly different objective functions were used. The cost of trying to embody the most important relationships holding in the real world is the increased complexity of the model and the possibility of diminishing its usefulness in practice. A more sophisticated model is more difficult to be understood and accepted by the non-technical user. The overall problem of capital budgeting is so vast that in fact what is achieved with the several models proposed is a situation of sub-optimisation. This is what happens with the models developed by Chambers (1967, 1971), Hamilton and Moses (1972), Myers and Pogue (1977) and Bhaskar (1978). These models jointly consider the problems of investment, financing and dividend options facing the firm and their implications for capital budgeting.

The Myers and Pogue model is consistent with the main results of modern finance theory. In particular, it is based on the following two propositions:

1. The risk characteristics of a capital investment opportunity can be evaluated independently of the risk characteristics of the firm's existing assets or other opportunities.

2. The Modigliani-Miller result that the total market value of the firm is equal to its unlevered value plus the present value of the taxes saved due to debt financing.

The firm is therefore assumed to choose a combination of investment and financing options which maximises the total market value of the firm, specified according to these two propositions. The major constraints are a debt limit (specified as a function of the value and risk characteristics of the firm's assets and new investment) and a requirement that planned sources and uses of funds are equal. In addition there are other habitual constraints in this type of model, like, for example, constraints on liquidity, dividend policy and investment choices.

In the models reviewed and referred to, the capital investment appraisals are based on after-tax cash flows. Explicit corporation taxation and other features of the tax system are omitted from those models. Although Bhaskar (1978) recognises that tax may affect financing decisions quite independently of the investment decision, it appears that Berry and Dyson (1979) were the first to apply mathematical programming to the problem of selection among a set of potential projects while taking into account the true tax situation of the firm.

Usually, the tax benefits of any capital allowances generated by a project are treated by one of two distinct standard procedures:

1. The benefits can be had immediately the project is begun, whether or not the project itself being appraised generates sufficient profits to make this possible; of course, in this case, the firm to which the project is incremental is, implicitly or

explicitly, assumed to be in a profit situation which allows this treatment of capital allowances.

2. The project is treated as something separate from the firm to which it is incremental. In this case the tax benefits of any capital allowances can be taken only when the profits generated by the project under appraisal make this possible.

Neither of these approaches attempt to place the project in the context of the firm's actual tax-paying situation. Berry and Dyson demonstrate using an example, that the most simple of the tax systems creates interdependencies between projects on an after-tax basis where they did not exist in the pre-tax situation. They use a mathematical programming model as a selection procedure.

When Berry and Dyson constructed their model there was a 100 per cent first year allowance on investments on plant, machinery and equipment. This was a key characteristic of the UK tax system. A simplified tax system (pre-1984) including only a corporate tax rate t (52%), a 100 per cent capital allowance on investment and with a zero time lag between tax becoming payable and date of payment, can be translated into a mixed integer programming (MIP) formulation. The Berry and Dyson's complete model for this situation is as follows:

$$\text{maximise } P = \sum_{j=1}^{N} x_j \, NPV_j - t \sum_{i=1}^{M} z_i d_i \qquad (2.3.a)$$

subject to

$$\sum_{j=1}^{N} x_j(k_{j1} + c_{j1}) + y_1 + u_1 - z_1 = 0 \qquad (2.3.b)$$

$$\sum_{j=1}^{N} x_j(k_{ji} + c_{ji}) + y_i - u_{i-1} + u_i - z_i = 0, \quad i=2,\ldots,M \qquad (2.3.c)$$

$$u_i \geq 0, \quad z_i \geq 0 \quad , \quad i=1,\ldots,M$$

$$0 \leq x_j \leq 1 \quad \text{and integer}, \quad j=1,\ldots,N \qquad (2.3.d)$$

where

- $N$ is the number of projects under consideration,
- $M$ is the total number of time periods;
- $x_j$ =0, if project j is rejected; =1, if project j is accepted;
- $NPV_j$ is the pre-tax discounted net present value of project j;
- $y_i$ is the ongoing cash flow in year i;
- $d_i$ is the discount factor relevant to year i;
- $k_{ji}$ is the capital expenditure for project j, in year i;
- $c_{ji}$ is the pre-tax net revenue cash flow for project j, in year i;
- $u_i$ is the total unrelieved balance of capital allowances up

to and including year $i$;

$z_i$ is the total taxable income in year $i$, after allowances.

There is only one type of constraint in this model, other than the restrictions on the variables. In year i the available allowances

$$\sum_{j=1}^{N} x_j k_{ji} \quad \text{and the taxable income} \quad \sum_{j=1}^{N} x_j c_{ji} + y_i$$

are such that their sum can be negative, zero or positive, and it can be represented by the difference of two non-negative variables $z_i$ and $u_i$ to give

$$\sum_{j=1}^{N} x_j k_{ji} + \sum_{j=1}^{N} x_j c_{ji} + y_i = z_i - u_i$$

from which (2.3.b) follows immediately. From the second year up to the planning horizon under consideration, the constraint will differ in form from the year 1 constraint only in that an unrelieved balance from the previous year will have to be taken into account. It should be noted that the $k_{ji}$ are negative or zero.

The objective function involves the maximisation of the after tax NPV:

$$\text{maximise} \left[ \sum_{j=1}^{N} x_j \, NPV_j - t \sum_{i=1}^{M} z_i d_i + \sum_{i=1}^{M} y_i d_i \right]$$

The three terms within brackets are: the sum of the discounted pre-tax NPV's of the chosen projects, the discounted sum of the tax payments and a discounted value for the ongoing firm cash flows $y_t$. The last term is a constant and can be discarded in the optimisation.

As there are no capital rationing constraints the discount rates are not different from those generated by the market. The model could include internally generated capital constraints, the soft capital rationing case; if hard capital rationing was the case, the model would have to be reformulated.

The system of taxing corporate profit in the UK is considerably more complex than the simple system considered. The MIP model can be extended to include other peculiarities of the tax system (Berry and Dyson, 1979; Berry, 1981; Ashford, Berry and Dyson, 1987). Berry and Dyson have established that the nature of the UK tax system supports the case for the use of mathematical programming in the selection of capital projects.

Pointon (1982) presents additional sets of constraints which can be of assistance in attempting to incorporate tax complexities into a mathematical programming model of capital budgeting. As Pointon notes, not all the constraints should be programmed for a particular firm on every occasion. If it is obvious that the firm has taxable profits from other existing projects which are so large as to absorb the capital allowances on new projects being considered, then the appropriate constraints can be omitted. The degree of realism required will clearly be a matter of judgement for the individual model maker

Cooper and Franks (1983) describe and compare various mechanisms for exploiting the tax losses of a firm. These include both financial and real assets transactions. They use a choice model which is converted to an equivalent linear program. Considering the properties of this model and its dual, Cooper and Franks show that the 'effective' tax rate for the firm with tax losses is less than the full tax rate and is endogeneous to the firm's current and future set of real assets and financial transactions. As a result, they conclude that the value of any asset can only be calculated simultaneously with the firm's optimal choice of both real assets and financial transactions. This, of course, has implications for real asset decisions and the evaluation of such financial market transactions as leasing.

A lease contract is an example of a simultaneous investment and financing instrument. The contractual nature of a financial lease repayment schedule means that the firm is undertaking a form of debt financing while at the same time, acquiring an asset which will alter the future cash revenue patterns of the organisation. One possible method of evaluating a lease in practice is to incorporate the lease as a project into an MP model of the firm in which all investment and financing decisions are considered simultaneously. But the MP models provide more than a mere computational tool for lease evaluation: they provide a generalised framework in which analytical expressions for the value of a lease may be derived.

Myers, Dill and Bautista (MDB, 1976) derived, from an MP model, a formula for evaluating financial lease contracts. In their model the objective was the maximisation of the equilibrium market value of the firm taking into consideration the interactions between the decision to lease and the use of other financing instruments by the lessee and the lessor. The analysis applies the adjusted present value methodology developed by

Myers (1974). In the MDB's derivation the debt capacity displaced by a lease is conceived as a present value and this must be estimated simultaneuosly with the value of the lease.

With a derivation similar to that used by MDB (1976), Ashton (1978) presented a general solution to the lease valuation problem which directly relates the value of the lease to the initial set of assumptions which are made about capital markets. Thus, in his approach, the value of a lease is a direct logical consequence of, and is consistent with, the approach adopted for the valuation of the project and debt cash flows, as well as the specification of debt capacity restrictions. It is not necessary to assume that the lease replaces debt, but rather the analysis assumes that any lease decision is made so that the subsequent rearrangement of equity and debt financing is optimal. The general analytical expression for the value of a lease is obtained by using Kuhn-Tucker optimality conditions on a constrained optimisation problem.

Important and interesting as these MP models may be, they have not obtained generalised acceptance. The next section gives a brief summary of some analysis on the results of the conventional capital budgeting techniques and the LP models.

## II.4. LINKS BETWEEN RULES OF THUMB AND MP MODELS.

The advantage of LP models over conventional discounting for the selection of capital investment projects, from the point of view of financial theory is well established, when taking into account investment and financing decisions. But this advantage depends on the complexity of the model employed. The LP model formulation and the NPV rule therefore, are completely equivalent (Weingartner, 1963) when there is a situation of unlimited borrowing and lending at the same rate, which is the simplest possible model. At the other extreme (Hamilton and Moses,1973), with a very complex model, involving interdependencies between project opportunities, the conventional discounting methods can be very misleading. The problem seems to be the case of middle ground complexity, and the point at issue, the ability to judge how good, or bad are the rules of thumb when compared with the results of the LP models.

Based on an MP model, Myers (1974) presented a framework in which the interactions of corporate investment and financing decisions can be analysed. Within this Myers developed a more general and flexible capital budgeting rule that allows for the tax benefits of debt; incorporates easily the impact of dividend policy, if relevant; and provides a natural basis for analysis of the lease vs buy or lease vs borrow decision. The new rule of thumb is to accept a project if its adjusted present value (APV) is positive, that is, accept project if APV = base-case NPV + present value of financing side effects > 0 . The base-case NPV is the usual project's NPV calculated assuming all equity financing and perfect capital markets. The financing side effects can involve interest tax shields, special financing (sometimes special financing opportunities are tied to project acceptance) and issue costs.

Ashton and Atkins (1979) have shown that for certain types of models, in particular for those including constraints on debt capacity, the use of simple rules of thumb is a good approximation. These rules can be investigated by means of the approximations that they imply to the dual of the MP model.

From the summary measures revised in section II.1 the NPV is the chosen one for comparing the effect of several groups of projects on the firm. These groups are obtained considering just the interdependencies among projects created by the tax system. Berry and Dyson's MIP model is used to generate those groups. A detailed description of the application of the model is presented later.

The models considered in this chapter exclude an important element associated with most of the real problems: the uncertainty of future values. Models for dealing with uncertainty in capital investment is the subject of next chapter.

# CHAPTER III. MODELS FOR DEALING WITH UNCERTAINTY IN CAPITAL INVESTMENT PROBLEMS.

## III.1. INTRODUCTION.

The amount of risk is often an important issue in the evaluation of proposed capital investments. In this chapter risk is going to be considered under two different perspectives. First, some existing models for the evaluation of uncertain cash streams are discussed, then, ways of modelling the uncertainty in cash flow streams are presented.

Let $X_t$ be the net cash flow in period t for a certain project, then the net present value NPV is given by

$$NPV = \sum_{t=0}^{n} X_t /(1 + r)^t \qquad (3.1)$$

where r is the cost of capital and n is the life of the project. Usually, the $X_t$ will be calculated from forecasts of other variables ( sales levels, unit costs, etc). Let these variables be $Y_k$, $k=1, \ldots, m$ , then

$$X_t = \Phi(Y_1, \ldots, Y_m), \quad t=1, \ldots, n \qquad (3.2)$$

The $Y_k$ are, in most situations, random variables and so are the $X_t$. The life n of a project is also, in many cases, a random variable.

The NPV therefore, is a random variable whose probability distribution is of interest.

There are several factors which influence the uncertainty of the net present value:

* The degree of detail in the definition of the Y variables and the correlations among them, in each time period.

* The stream of cash flows $X_t$, t = 1, ..., n is a time series, many of the Y variables are themselves time series (prices, sales, costs, for example). Most time series present autocorrelated patterns. Although the process underlying the time series formed by the $X_t$ is often difficult to identify, the autocorrelated behaviour can have a significant impact on the distribution of the NPV.

* The uncertainty about project life.

The physical and economic life of a project is usually not known with certainty at the time the project is evaluated. It has been found (R. Bey, 1981) that by incorporating the uncertainty associated with the life of projects into a capital investment decision analysis a significant impact resulted on the risk and return characteristics, measured by the mean and variance of the

NPV of individual capital investment projects.

* Strategy decisions related to capital investments.

Many capital investments are more complicated than simply an investment now and returns in the future. They can involve a strategy, that is a sequence of decisions, rather than a single accept/reject decision. Since the strategy alternatives can have an important effect upon project profitability, the investment analysis should include them in some way. A particularly important decision is whether or not to abandon an investment. A project may be abandoned even if it still has useful life remaining and is generating positive cash flows if the abandonment value (salvage value, tax effects, etc) is greater than the expected discounted future cash flows. The abandonment decision depends upon uncertain cash flows and it has been found (Robichek and van Horne, 1967) that failure to include this option can have a substantial impact on project profitability and the distribution of NPV.

When looking directly at the sources of uncertainty in the NPV, trying to assess them and incorporating them in the $X_t$ values, the r to use in (3.1) should be the risk free rate. Alternatively an adjustment for risk can be made in terms of the cost of capital using a risk adjusted discount rate.

Several types of models have been proposed for evaluation of project profitability under uncertainty. In this chapter section III.2 contains a brief description of some of those models, with some comments about the extent to which each one is able to incorporate the above mentioned factors which can affect the uncertainty of NPV. From a different point of view, in section III.3 some models are presented for introducing uncertainty into a portfolio of projects. In particular, a model is presented which has been constructed with the same objective and which will be used to obtain the results of the next chapters.

## III.2. BRIEF DESCRIPTION OF SEVERAL TYPES OF MODELS.

### III.2.1. The certainty model.

This model is not directly designed to evaluate uncertainty, but it is often used for this purpose by means of sensitivity analysis. In the certainty model for project evaluation one number is given for each factor which can affect the cash flow of a project, including the project life, and the NPV is calculated from the cash flows obtained, using a discount factor reflecting the firm's cost of capital. Some degree of uncertainty can be introduced into the analysis. That is, for the uncertain variables, optimistic and pessimistic values are considered and the effect on the NPV is calculated. This analysis is sometimes sufficient if under optimistic estimates the project is not profitable, leading to a reject decision, or if under pessimistic estimates the project is highly profitable, leading to a clear acceptance of the project. This model is useful as a first step in risk evaluation and it can be used to determine which variables affect profitability and hence, which factors must be included in further information gathering and modelling efforts.

Nevertheless, as the model does not formally include uncertainty, only crude ideas of risk can be obtained from sensitivity analysis. The NPV values obtained are also subjected to bias which can arise from several sources: the use of the most likely estimate for the factors instead of the expected value (Hertz, 1964); the uncertainty about project life or about the timing of any cash flow (Solomon, 1966); and the inability of the certainty model to incorporate strategy and abandonment decisions (Robichek and van Horne, 1967). Even if these biases are small and can be ignored, they can only be determined in magnitude and direction by analysis beyond the certainty model.

III.2.2. The decision tree approach.

This model starts by developing a decision tree for the investment, a standard approach in decision making under uncertainty (Brealey and Myers, 1981). The tree is first analysed in the backward direction, and the optimum decisions determined subjected to some criteria such as maximise NPV. Then, the tree is analysed again in a forward direction to determine all possible NPV's and their probabilities, for the strategy selected as optimal. Finally, the NPV's are classified in a frequency distribution and statistics such as the variance can be calculated.

Once the decision tree has been analysed to find the optimum strategy, and the non-optimal decisions eliminated, the tree becomes a probability tree and the second part of the analysis, the determination of the distribution of NPV, can be done either by the enumeration of all branches of the tree or by simulation methods.

The decision tree method incorporates and optimises over the strategic or abandonment decisions; it also considers the interrelationships between variables; and if the problem is simple, the tree is a visual display that is easy to understand. The method can be very cumbersome to use in real life situations where the investments have a life span of several years and many uncertain variables, it is also difficult to include the time series dependencies into a decision tree.

III.2.3. The analytical approach.

The analytical approach was first suggested by Hillier (1963) to determine the distribution of NPV when the net cash flow in time period t, $X_t$, is expressed as a sum of independent cash flows emanating from different sources. Later, Hillier has extended the methodology to deal with situations where there are dependencies in the model and Wagle (1967) extended it still further, to deal with situations where the cash flows are calculated from estimates of other variables.

A generalised description of Hillier's model is the following (Wagle). Consider an investment with a life of n years, n being a random variable. The cash flow in period t, $X_t$, is made up of independent cash flows, $Y_{it}$ with a finite mean $\mu_{it}$ and variance $\sigma_{it}^2$, i=1, ..., m . Then:

$$X_t = \sum_{i=1}^{m} Y_{it} \tag{3.3}$$

$$E[X_t] = \sum_{i=1}^{m} \mu_{it} = \Omega_t \tag{3.4}$$

$$Var[X_t] = \sum_{i=1}^{m} \sigma_{it}^2 + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} Cov[Y_{it}, Y_{jt}] \tag{3.5}$$

If the cash flows continue over n periods, with a cost of capital r, then the net present value of this investment is defined as

$$P_n(r) = \sum_{t=0}^{n} X_t/(1+r)^t \tag{3.6}$$

Its expected value and variance are given by:

$$E[P_n(r)] = \sum_{t=0}^{n} \Omega_t/(1+r)^t \tag{3.7}$$

$$Var[P_n(r)] = \sum_{t=0}^{n} Var[X_t]/(1+r)^{2t} + 2 \sum_{t=0}^{n-1} \sum_{t'=t+1}^{n} Cov[X_t, X_{t'}]/(1+r)^{t+t'} \tag{3.8}$$

Also,

$$E[P_n^2(r)] = Var[P_n(r)] + \{E[P_n(r)]\}^2 \tag{3.9}$$

For a certain value of r the equations (3.7), (3.8), (3.9) give the conditional mean, variance and second moment about the origin of the net present value for a given n. Usually, n is a discrete random variable. When the discrete probability distribution for the project life is known, the overall expectation, E[P(r)] and variance, Var [P(r)] of NPV can be easily determined.

These two parameters of the distribution of P(r) provide a basis for evaluating a measure of risk in the proposal. The model in its general form, involving the estimates of all covariances between various cash flows, is much too cumbersome. Hillier has suggested three assumptions, which seem reasonable in many situations; they greatly reduce the number of estimates needed and thus greatly increase the practical applicability of the model. These assumptions are:

1. The correlation $\rho_{ij}$, between cash flows from sources i and j, $Y_i$ and $Y_j$ respectively, is the same for all periods t.

2. The correlation of cash flows from source j between periods t and t+1 is $\rho_j$ and this is the same for all periods. Furthermore, the cash flow from source j in period t+2 and other subsequent periods depends only upon the correlation between adjacent periods. Cash flows from a given source therefore, have a one

period autocorrelation relationship. This implies that the correlation

$$cor(Y_{jt}, Y_{jt'}) = \rho_j^{t'-t} \qquad \text{for } t'>t \tag{3.10}$$

3. The correlation between different cash flows in different periods is given as a function of the correlations determined in 1 and 2 above:

$$cor(Y_{it}, Y_{jt'}) = cor(Y_{it}, Y_{jt}).cor(Y_{jt}, Y_{jt'}) = \rho_{ij} \cdot \rho_j^{t'-t} \quad \text{for } t'>t \tag{3.11}$$

The covariance between $Y_{it}$ and $Y_{jt'}$ is thus:

$$Cov[Y_{it}, Y_{jt'}] = \rho_{ij} \cdot \rho_j^{t'-t} \cdot \sqrt{Var[Y_{it}].Var[Y_{jt'}]} \tag{3.12}$$

The mean and variance of P(r) provide a basis for evaluating and comparing prospective investments. Furthermore, certain weak probability statements can be made using the Tchebyscheff's inequality which states:

$$Prob\{\, P(r) - E[P(r)] \geq k \sqrt{Var[P(r)]}\,\} \leq 1/k^2 \tag{3.13}$$

Should P(r) be normally distributed, more precise probabilistic statements could be made. Hillier has considered some of the conditions under which P(r) will be normally distributed in great detail. The most important conditions, as summarised by Wagle, are:

a) If $X_1, X_2, \ldots, X_n$ have a multivariate normal distribution, then P(r) being a linear function of the X's would itself be normally distributed.

b) Since P(r) is the sum of a number of random variables it follows by the Central Limit theorem that, under certain conditions, P(r) is asymptotically normally distributed. The net cash flows may also themselves be explicitly, or implicitly sums of a number of variates and in some cases it may therefore, be reasonable to assume that they are normally distributed.

An essential theoretical difficulty, pointed out by Hillier, is that P(r) is a weighted sum of random variables, in which the weights are the discounting factors. The effect of this is that the shape of the distribution of P(r) may be dominated by the early cash flows, especially at a high discount rate.

Hull (1977) has identified three main reasons why NPV might not be approximately normal in a given situation. These are:

1. The investor might have options (e.g. abandonment or expansion) open to him at some stage during the life of the project.

2. There might be non-linearities in the cash flow model which could be caused either by the presence of variables such as

'growth rate' and 'life of project', or by conditions within the model itself.

3. There might be insufficient uncertain variables. Hull has found that if a cash flow model is sufficiently 'well behaved' for the Hillier-Wagle analytic approach to be applicable, then the distribution of NPV will be approximately normal providing that there are a sufficient number of uncertain variables in the model.

If the Hillier model is considered in a simplified form (cash flows treated as from one source, for example) it is easy to use. From estimates of means, variances and correlations the probability distribution of NPV can be crudely estimated with a few minutes work on a pocket calculator. Hence, a simplified version can be useful as a first-order measure of risk. The model can be also easily generalized into a portfolio type analysis, as Hillier has done. It is difficult, though, to incorporate the accounting relationships which often involve non-linearities and discontinuities. The model requires unconditional variances and their estimation is also a difficult task. The model only produces a mean and a variance. The assumption of normality is required for evaluating probabilities, and this assumption may not be reasonable in some situations. Finally, the Hillier model is subject to potential biases due to non-linearities and the failure to include decisions (in particular, the abandonment decision) in the analysis.

III.2.4. The simulation approach.

The use of simulation in the evaluation of investments projects, or risk analysis, was first suggested by Hertz (1964). Basically, it involves the definition of the variables $Y_{it}$ in the expression of the cash flow in time period t, and the estimation of the probability distributions for the unknown variables, including project life, if it is uncertain. Then, samples are drawn from the distribution of each variable, for each year, and for the life of the project; the cash flow is determined for each year using the sampled values; and the NPV is calculated using the sampled value for project life. This process is repeated a large number of times to obtain a frequency distribution for NPV. From this a mean NPV is estimated as well as other statistics.

This approach is very flexible. The model of cash flows can easily include non-linearities, discontinuities, limits, uncertainty about project life and also, though not so easily, correlation effects between variables in a given period and autocorrelations between periods. The computer programme can be written so that at each period a decision is made about continuing or abandoning the project, conditional on the circumstances at the time. Similarly, strategy decisions, such as expansion, can be incorporated into the model. It should be noted that the simulation approach can only evaluate strategy alternatives, not optimise to find the best

The use of risk analysis requires that the entire probability distributions for the uncertain variables have to be estimated. In addition, all the relationships between the variables must be clearly specified, including correlations and autocorrelations. These two types of dependencies are difficult to estimate and they are often ignored in simulation models, the variables being then assumed as independent.

Eilon and Fowkes (1973) show that the omission of the interdependencies may lead to significant errors in the observed distributions of the selected appraisal criteria. They introduce various forms of discriminate sampling which is a compromise between the two extremes of independent and conditional sampling. In the particular example they considered, the results obtained from conditional sampling were closely reproduced by a discriminate scheme which only involved the figures for the range and the most likely value of each variable.

In risk analysis the cash flows used to obtain the probability distribution of the NPV are discounted using a risk free rate. A different type of procedure uses rates of return adjusted for risk. The next section reviews some models of this kind.

### III.2.5 Finance theory models.

Another possibility in considering the uncertainty is to represent a future cash flow by its expected value and discount it at a risk adjusted discount rate. In this case, the present value, PV, of an uncertain cash flow, X, occurring one period ahead, is given by

$$PV = EV[X]/(1+r) \tag{3.14}$$

where EV is the expected value operator and r is the risk adjusted discount rate.

#### Certainty Equivalent

Another possibility is to find the smallest certain cash flow amount, the certainty equivalent, which one would accept in exchange for the uncertain cash flow X. Then,

$$PV = CE[X] / (1+r_f) \tag{3.15}$$

where CE is the certainty equivalent operator and $r_f$ is the risk free rate. Now, the two expressions for PV are equivalent.

$$PV = EV[X] / (1+r) = CE[X] / (1+r_f) \tag{3.16}$$

As long as one considers only one future period the two expressions are exactly the same.

The risk adjusted discount rate formula can be easily extended to a multi-period situation if an appropriate discount rate is used in each period. If it is assumed, as is usual, that risk increases at a steady rate as one looks further into the future, a constant risk adjusted rate is used.

The certainty equivalent can also be easily extended to the multi-period case.

$$PV = \sum_{t=1}^{T} CE_t / (1+r_f)^t = \sum_{t=1}^{T} a_t EV_t / (1+r_f)^t \tag{3.17}$$

where $a_t$ is the ratio of the certainty equivalent of a cash flow $X_t$, $CE_t$, to its expected value, $EV_t$.

The equivalence of the certainty equivalent and risk adjusted discount rate formulae implies that the value of $a_t$ decreases over time at a constant rate. This is what happens when a constant risk adjusted discount rate r is used to value the cash flow for each period (Brealey and Myers, 1981). The CE operator is usually defined in terms of a utility function (Haley and Schall, 1979). So, this approach for accommodating risk presents practical difficulties in the formulation of probability density functions and the formulation of a utility function. The latter, in particular, raises the question of whose utility function: should it be the manager's or the shareholder's ? Dyson and Berry (1985) address this point. When acting in the interest of the shareholders management must accept investments that increase share value and reject those that do not. So, the ideal would be for management to find a firm trading on the stock market which is a duplicate of the project being considered. Then a comparison could be made of the cost of undertaking the project and the value the stock market would place on the project if it were undertaken. The project should be accepted if the market value is greater than the cost. As an exact replica of the project is not easy to find, the alternative is to discover and apply a valuation mechanism which generates share prices. Current financial theory offers a few models that can be used to this purpose, e.g., the time state preference model (TSP), the capital asset pricing model (CAPM), the arbitrage pricing theory (APT) and the option pricing theory (OPT) which are now discussed

The TSP, detailed in Haley and Schall (1979), provides a framework to tackle uncertainty about the future. The CAPM, the APT and the OPT are models to establish the price of future uncertain cash flows that can be used on their own or in combination with the TSP. Only CAPM is going to be later reviewed in some detail. The APT was developed by Ross (1976) as an alternative to the CAPM. It has suffered several simplifications and extensions introduced by Huberman (1982) and Ingersoll (1984), among others. Berry and Dyson (1985) present simple examples of application of both CAPM and APT to investment decisions.

The OPT is a continuous time model developed in 1973 by Black and Scholes. The detailed theory concerning this model can be seen, for example, in Haley and Schall (1979). Banz and Miller (1978) have presented the estimates of state prices that can be used for pricing all ordinary capital assets once their payoffs relative to the market portfolio have been specified.

Time State Preference Model.

In the TSP model the uncertainty about the value of a cash flow at a future point in time, t, is a function of the length of time between now and t, and of the state at time t. Each state is a particular sequence of events occurring from the present to time t, where the state is defined. For each t the states are exhaustive and mutually exclusive. All the other characteristics of the model - return from assets, utility functions and probabilities - are based on the definition of states.

The TSP approach can be summarised as follows. Assets provide returns, for each period, depending on the state that occurs. The consequences of acquiring an asset are evaluated by determining the expected value of the utility of the returns provided by the asset in each period. This approach can be adapted to capital budgeting under uncertainty assuming the following:

1. The possible future outcomes for the economy as a whole, in the next period, can be partitioned into n mutually exclusive and exhaustive states, classified as 'boom', 'depression' and any other finer gradations also judged adequate.

2. The decision maker somehow knows the expected value of the payoffs $X_j$ of the investment project under analysis in each of

the considered states.

3. There are known current prices $v_j$ of a set of securities that pay £1 if state j occurs and zero otherwise. The future consequences then, in the next period of the investment opportunity, are obtained by C calculated as

$$C = \sum_{j=1}^{n} v_j E[X_j]$$

If C exceeds any initial outlays required, the project is worth undertaking; otherwise, the decision is negative. The valuation expression C is now a weighted sum across both time and states, rather than merely time alone as in the NPV and certainty equivalent cases. The prices $v_j$ can be obtained through the use of the different finance models CAPM, APT and OPT.

The Capital Asset Pricing Model.

The CAPM is a model of risk and return in a well functioning capital market. In the CAPM the equilibrium rates of return on all risky assets are a function of their covariance with the market portfolio. As shareholders are able to hold diversified portfolios of shares the relevant risk is the non diversifiable risk, that is, the extent to which returns on a share change within the market portfolio. The model is expressed by the equation

$$r_i = r_f + (r_m - r_f)\,\sigma_{im} / \sigma_m^2 \qquad (3.18)$$

where

$r_i$ is the expected rate of return on asset i;

$r_f$ is the risk free rate;

$r_m$ is the expected rate of return of the market portfolio;

$\sigma_{im}$ is the covariance between asset i and the market portfolio;

$\sigma_m^2$ is the variance of the market portfolio

This equation, the derivation of which can be seen in a finance textbook (Haley and Schall, 1979 for example), expresses a linear relationship and its graphical representation is known as the market line. The required rate of return on any asset i, $r_i$, is equal to the risk free rate of return plus a risk premium. This risk premium is the price of risk, $(r_m - r_f)$, multiplied by the quantity of risk which is often called beta, $\beta_i$:

$$\beta_i = \sigma_{im} / \sigma_m^2 \qquad (3.19)$$

The risk free asset has a beta of zero and the market portfolio has a beta of one. Beta is a measure of the share's risk relative to that of the market.

Equation (3.18) can be written as:

$$r_i = r_f + \beta_i (r_m - r_f) \qquad (3.20)$$

The CAPM looks at rates of return and prices over one period at a time. To calculate $r_i$ through CAPM it is necessary to have the values of $r_f$, $\beta_i$ and the price of risk. Dyson and Berry (1985) present the usual way to obtain these values.

With CAPM used as a rule of thumb, an investment project under consideration must offer at least the calculated $r_i$ if it is not to reduce the company's share price. This implies that the CAPM formula for calculating the expected rate of return:

$$r = r_f + \beta (r_m - r_f) \qquad (3.21)$$

can be used in the standard discounted cash flow formula to obtain

$$NPV = \sum_{t=0}^{n} X_t \,/\, (1+r)^t = \sum_{t=0}^{n} X_t \,/\, [1+r_f+\beta(r_m-r_f)]^t \qquad (3.22)$$

On applying (3.22) the same discount rate is used in all time periods; in particular, this assumes, among other things (Fama, 1977), that the beta will be constant over the project's entire life-span.

The CAPM, being a pricing mechanism, can also be used in a certainty equivalent form:

$$PV(X) = \{ FV(X) - \lambda \, \mathrm{Cov}[X, r_m] \} / (1 + r_f) = CE(X)/(1 + r_f) \qquad (3.23)$$

where

- $X$ is the uncertain future cash flow;
- $r_m$ is the uncertain future return on the market portfolio;
- PV is the present value;
- FV is the forecast value;
- CE is the certainty equivalent operator and

$$\lambda = (r_m - r_f) / \sigma_m^2$$

where $\sigma_m^2$ is the variance of the market return.

The derivation of equation (3.23) can be found among others in Brealey and Myers (1981). With (3.23) the certainty equivalent is obtained by measuring the risk of the cash flow by a covariance which expresses the extent to which the cash flow moves in line with the market portfolio.

Until now, risk has been considered in relation to the evaluation of uncertain cash flows. A different point of view is that of modelling the uncertainty into the cash flows. This is the subject of the following section.

## III.3. MODELS FOR INTRODUCING UNCERTAINTY IN A PORTFOLIO OF PROJECTS.

Several models, whether or not they are based on finance theory results, have been used to introduce uncertainty on the cash flows of the projects considered in a certain period of time. They have been used in simulation studies to investigate decision rules of capital investment.

Sundem (1975) has evaluated six capital budgeting models in a simulated environment, the models considered were the mean variance (MV) portfolio, the MV model with a diagonal simplification, the variability of returns, a chance constrained model, the net present value and the payback period. He applied the technique to thirty different investments adapted from Weingartner (1967). Sundem's analysis does not consider the dynamic effects of numerous investment alternatives arising period by period, with all the extra uncertainty that this creates in the decision to invest now rather than later. He presumed that all the thirty projects would begin in time period one. A time-state preference model was used to provide market values for each proposed capital budgeting project and a set of parameters of that model was fixed.

With the same type of approach Whitaker (1984) has compared the payback period, the net present value, the profitability index and the internal rate of return. He derives a method of simulating investment alternatives which may exhibit dependent cash flows. Then, the simulation model is used to evaluate the effectiveness of the major decision criteria in the selection of investments from a sample of investments occurring in a continual stream of alternatives when capital is rationed.

In his method of simulating investment alternatives a single investment is viewed as a series of cash flows generated by some project which is characterised by the following parameters: duration of the investment (m), the initial outlay at time t ($C_t$) and the cash flow in the $j^{th}$ period of the investment ($E_j$, $j=t+1$, $t+2$, ..., $t+m$). Now the approach is to select randomly, for each project, m and $C_t$ from a given frequency distribution. With m and $C_t$ known at time t, each $E_j$ is given by $E_j = Ag_j$ where A is a constant and $g_j$ is a random variable. The constant A is calculated so that $E_j$ will on average generate an investment that yields an internal rate of return of 100r%. With a standardised $g_j$, such that

$E[g_j] = 1$, $E_j$ is given by

$$E_j = r\, C_t g_j \,/\, (1-(1+r)^{-m})$$

Dependence on the generated cash flows is introduced through $g_j$ expressed as a Markov process, stationary about an expected value of 1. A sequence of values is obtained from

$$g_j = a g_{j-1} + (1-a) + e_j\,, \qquad 0 \le a \le 1$$

where the $e_j$ are independent sample values from a given distribution with zero mean and a variance of $\sigma_j^2$. The constant a is the autocorrelation of this autoregression series of which it is known that the expected return is not affected by the dependence but the variance becomes larger as a approaches 1. When the $e_j$ are normally distributed, from $N(0, \sigma_j^2)$, the $g_j$ may be generated by sampling from a normal distrbution

$$g_j \sim N(a g_{j-1} + (1-a), \sigma_j^2)$$

The variances of cash flow forecasts generally increase with time, so Whitaker's assumption about these variances takes the form:

$$\sigma_p^2 = \omega\, \sigma_{p-1}^2 \qquad \text{with} \qquad \omega \ge 1\,,\ p = 1,2,\ldots,m$$

or

$$\sigma_p^2 = \omega^{p-1}\,\sigma^2$$

where $\sigma^2$ is the initial forecasted variance.

Then Whitaker uses this model to create a simulated environment where numerous investment opportunities are presented for selection, period after period, so that each of the decision criteria can be applied over a given time period in order to determine their relative effectiveness in generating returns.

A different model was used in this study to introduce uncertainty in the cash flows considered. The model of the cash flow in period t, for project i, is formed by two distinct parts: one specific to the project and the second representing a systematic effect influencing all projects and the firm's ongoing activities. The model is more direct and simple than Whitaker's. But even so, it creates dependencies between time periods for the cash flows associated with a project, and also dependencies among projects in the same time period.

In time period t ($t>1$) the specific part of the model, is composed of two terms. The first one is a basic cash flow for that period, $X_{it}$, which is a random variable with a known mean, $\mu_{it}$ and variance, $\sigma_{it}^2$. The variance is assumed to be proportional to the mean: $\sigma_{it}^2 = C\,\mu_{it}$ , where C is a constant. Other forms have been considered in the context of other types of problems for example inventory and stock control problems; in particular,

$$\sigma_{it}^2 = C\,\mu_{it}^p \quad \text{and} \quad \sigma_{it}^2 = A\,\mu_{it} + B\,\mu_{it}^2$$

where p, A, B and C are constants. Hull (1977) has also used the function form $\sigma_{it}^2 = C\,\mu_{it}^2$ when generating cash flows, although he does not mention if there is some empirical evidence to recommend it.

In principle, any distribution dependent on those two parameters, mean and variance, could be used to generate the $X_{it}$, but it seems natural to use the normal distribution because usually the cash flow of a project in time period t is a sum of random variables. Hull (1977) used triangular, uniform and ◺-shaped distributions to generate cash flows in five case studies and Wagle (1967) refers the use of a beta distribution for this sort of problem.

The second term of the specific part of the model is a kind of correction: the generated cash flow in the previous time period being above or below its expected value is supposed to directly influence the cash flow one period ahead. If its coefficient is negative, the model tries to restore the amount lost if the value is less than the mean or to counterbalance the growth effect if the value is greater. If the coefficient is positive, the response will be different: a value higher than the mean is considered in an optimistic way and the actual cash flow value is increased; a value lower than the mean is seen in a pessimistic way and the cash flow is diminished accordingly. This part of the model therefore, takes the following form:

$$X_{it} + a_{it} ( P_{i,t-1} - \mu_{i,t-1} ), \qquad t=2, 3, \ldots, T$$

where $P_{i,t-1}$ is the actual cash flow for project i in time period t-1.

The second part of the model corresponding to the systematic effect, $S_t$, is considered to have the form:

$$S_1 = E_1$$

$$S_t = d_t S_{t-1} + E_t \;, \quad t=2, 3, \ldots, T \qquad (3.24)$$

where $E_t \sim N(0, \sigma_t^2)$, $\sigma_t^2 = p^{t-1}\sigma_1^2$ and $d_t$ is a constant. In this way the systematic effect will only increase the uncertainty of the cash flow values. If $\mu_t$ was instead the mean of $E_t$, there would also be a trend effect.

The model for the cash flows with dependencies then takes the form:

$$P_{i1} = X_{i1} + b_{i1}S_1 \tag{3.25}$$

$$P_{it} = X_{it} + a_{it}\,(\,P_{i,t-1} - \mu_{i,t-1}\,) + b_{it}S_t\,, \quad t=2,3,\ldots,T\,;\ i=1,2,\ldots,N$$

where capital letters are used to represent random variables and the $a_{it}$ and $b_{it}$ are constants. Also, $X_{it} \sim N\,(\mu_{it}, \sigma_{it}^2\,)$ with $\sigma_{it}^2 = C\,\mu_{it}$ and the systematic effect is given by (3.24). The way in which the constants and initial values were chosen to generate cash flows with this model is detailed in the next chapter.

# CHAPTER IV. SIMULATION IN THE CONTEXT OF CAPITAL INVESTMENT.

## IV.1. INTRODUCTION.

Simulation is a powerful technique which has been used within the context of capital investment since the facilities of computation allowed the easy and fast execution of repetitive calculations. Simulation may be used in a deterministic way, using recursive models to do the necessary calculations, or in a stochastic way by introducing uncertainty into the situation being modelled.

Deterministic simulation can be used to produce income statements, sources and uses of funds, and balance sheets to help in financial planning. When the corresponding model equations are known, it is easy to study the effect of changes in some of the elements allowing a sensitivity analysis and the study of the performance of some project under different scenarios. With the growing use of small computers this type of simulation seems to be increasingly used.

Sensitivity analysis allows us to consider the effect of changing one variable at a time. By looking at a project under alternative scenarios, one can consider the effect of a limited number of plausible combinations of variables. Stochastic simulation is a technique for considering all possible combinations. It permits the introduction of uncertainty in a situation through the use of random variables. This technique has been applied to help decision making in practical situations, under the name of risk analysis, and has also been used to create environments where different rules and approaches used in capital investment can be compared.

Section IV.2 gives some justification for the use of stochastic analysis in capital investment studies. The use of stochastic simulation brings the need for two different types of exercise: the sampling procedure and the analysis of the results. This is the subject of section IV.3, while in IV.4 a small example is presented of an application of stochastic simulation to study the effect of uncertainty on two small projects. Finally, in section IV.5, the details are given of the problem and model chosen for the study of the simultaneous effect of tax and uncertainty on a portfolio of projects.

## IV.2. STOCHASTIC ANALYSIS AND CAPITAL INVESTMENT.

An important element in the analysis of a capital investment project is uncertainty which is usually dealt with in different ways as reviewed in chapter III. In a stochastic analysis the uncertainty is introduced in the situation of interest by appropriate models, an essential part of which are random variables with known distributions. In many circumstances these variables are interdependent, and their relationships are also introduced in the simulation. Giving values to those variables permits the calculation of the quantity of interest, say X, which is, in fact, a random variable. The repeated application of this procedure gives a sequence (sample) of values for X. This sample is then used to obtain some statistics of interest and, eventually, to reach some more general conclusions about the quantity under study, X, which is usually the NPV, in the case of a capital investment study

Risk analysis, as stochastic analysis is also known, can be used to mimic a real life situation about which some data is known, or could be gathered. The distribution of the random variables that affect the NPV should be sufficiently realistic. It is usually difficult to obtain these distributions and even more difficult to obtain the interrelationships among the variables. The interdependencies are often excluded from the models under the justification that they are accounted for implicitly in the values attributed to the variables. Eilon and Fowkes (1973) demonstrated that such exclusions may lead to significant errors in the observed distributions of the NPV. They consider the problem of dependence in some detail and suggest various forms of discriminate sampling, where the range of possible values of the dependent variable is restricted in some way according to the value sampled for the independent variable. It should be noted that a scheme of discriminate sampling can bring problems of consistency if the unconditional distribution of the dependent variable has been assessed in advance. Hull (1977) suggested a procedure for dealing with dependence in risk simulation taking into account a numerical estimate of the extent of the dependence. Hull showed that if $X_2$ is dependent on $X_1$ in a risk simulation, conditional distributions can be chosen for $X_2$, providing the unconditional distributions for $X_1$ and $X_2$ are suitable transformations of the normal distributions.

Another way of using stochastic analysis is to make reasonable assumptions about the factors that influence the quantity of interest and decide on the distributions that fulfill those assumptions. A model can be used to create the interdependencies. This was the way Sundem (1975) and Whitaker (1984) used simulation in their investigation into the decision rules of capital investment.

As reviewed in chapter II, linear programming can be used to model the interdependency that the tax system creates within a group of projects and with the ongoing activities of a firm, in a deterministic situation. Stochastic programming, which is a natural extension of linear programming to deal with the introduction of uncertainty, is not usually practicable. Stochastic analysis is a viable procedure to study the influence of tax and uncertainty on the NPV of capital investment projects. In this study a model is used to generate the cash flows for each project of a portfolio of investments during a certain number of time periods. With these cash flows, net cash flows for all time periods can be obtained using a simplified tax system which retains the main characteristics of the UK tax system. Interdependencies are therefore introduced directly by the model used, which explicitly allows for the relationships between the projects, and by the tax system. Before going into details of the problem and the model chosen for the study, a review is made of some aspects of stochastic simulation which are going to be used.

## IV.3. STOCHASTIC SIMULATION.

Stochastic simulation is a typical technique of performing sampling experiments with a model of the situation under study. It is usual to distinguish the independent variables, which are not determined by the model, and the variables that depend on the model, in particular intermediate variables and output variables. Besides the variables of the model, there are the parameters which are constant quantities which influence the dependent variables. The connections between variables and parameters are described by relationships which are usually translated by mathematical expressions. Different values for the independent variables and the parameters, and different relationships between them, may give different values for the dependent variables. Statistical methods can be used to analyse the output values. In stochastic simulation therefore, once the model is chosen, two different types of exercise need to be considered: the sampling procedure and the analysis of the results. The next two sections deal with these problems.

IV.3.1. The sampling procedures.

In any simulation experiment there is a need for a source of random numbers uniformly distributed in [0,1]. When using a digital computer, it is common to use the computer itself to generate the random numbers. In fact, they are pseudo-random numbers which behave, to a reasonable extent, as numbers following a uniform distribution in [0,1]. Several relatively sophisticated procedures have been proposed for testing whether a sequence of numbers constitutes a sample of random numbers or not. To generate sequences of pseudo-random numbers several methods exist, of which the most popular are the congruential methods. The congruential method used in this study to generate a sequence of uniform pseudo- random numbers in [0,1], $\{u_n\}$, is well known. It mainly obtains the sequence $\{u_n\}$ from $u_{n+1} = f(u_n)$, given an initial value $u_0$, the seed, and an adequate form for function f.

Several authors, Knuth (1969) and Naylor (1971) for example, present details about the generation of random numbers and a description of the statistical tests used in connection with them. They also present generators for stochastic variables, continuous and discrete, with different distributions. In this case the generation of values from normal distributions with given parameters is of interest. They are obtained through the polar method, as described by Knuth (1969).

As in stochastic simulation the sequences of generated random values are controllable they can be repeated when the same values of constants and seeds are used. This can be an advantage over truly random sequences which cannot be exactly reproduced.

The use of common random numbers (CRN) attempts to improve the efficiency of response difference estimation by comparing alternatives under the same conditions. CRN can be effective for systems whose responses are piecewise monotonic transformations of input variables. Wright and Ramsay (1979) point out that for complex systems the CRN effectiveness and synchronization impact may be counter-productive, because one cannot be certain that common random input streams will yield pairs of positively correlated situations.

The next section considers some methods of analysing the stream of values obtained in the simulation.

### IV.3.2. The analysis of results.

Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with a finite population mean $\mu$ and a finite population variance $\sigma^2$. The variables $X_1, X_2, \ldots, X_n$ can be seen as n independent observations of a random variable X such that $E(X) = \mu$ and $Var(X) = \sigma^2$. The sample can be used to obtain some statistics, four of which are shown below: mean, variance, percentile and quantile. The mean and percentile are more detailed because they are going to be used in the simulation study.

The sample mean is an unbiased point estimator for $\mu$:

$$\bar{X}_n = \sum_{j=1}^{n} X_j / n \tag{4.1}$$

An estimator $\hat{T}$ of a parameter T is unbiased if $E(\hat{T}) = T$.

The sample mean is itself a random variable and without some more information there is no way of evaluating how close $\bar{X}_n$ is to $\mu$. The usual way of assessing the accuracy of $\bar{X}_n$ as an estimator of $\mu$ is to construct a confidence interval for $\mu$. The construction of such a confidence interval needs an estimate of Var $(\bar{X}_n)$. Since the $X_i$'s are independent,

$$\mathrm{Var}\ (\bar{X}_n) = \sigma^2/n$$

and an unbiased estimator of Var $(\bar{X}_n)$ is given by

$$\widehat{\mathrm{Var}}\ (\bar{X}_n) = s_n^2/n \tag{4.2}$$

where $s_n^2$, the sample variance, given by :

$$s_n^2 = \sum_{j=1}^{n} [\, X_j - \bar{X}_n \,]^2 \,/\, (n-1) \tag{4.3}$$

is an unbiased estimator for $\sigma^2$.

Working with the sample mean is appealing because it is asymptotically normal and an estimate of its variance can be easily derived from the same n observations. Using the central limit theorem the distribution of the statistic $\sqrt{n}\ (\bar{X}_n - \mu)/s$ converges to $N(0,1)$ provided that the $\{X_i\}$ obeys some regularity conditions. Approximately, $\bar{X}_n$ has a normal distribution with mean $\mu$ and variance $\sigma^2/n$. If the $X_i$'s have normal distribution, $\bar{X}_n$ has the exact distribution $N\ (\mu, \sigma^2/n\ )$ and

$$(X_n - \mu)\ /\ \sqrt{(s_n^2/n)}$$

has the t distribution with $n-1$ degrees of freedom. In this case an exact confidence interval (c.i.) for $\mu$ has extremes

$$\overline{X}_n - t_{n-1,1-\alpha/2}\sqrt{(s_n^2/n)} \quad \text{and} \quad \overline{X}_n + t_{n-1,1-\alpha/2}\sqrt{(s_n^2/n)} \tag{4.4}$$

$t_{n-1,1-\alpha/2}$ being the $1-\alpha/2$ quantile for the t-distribution with $n-1$ degrees of freedom. The coverage for this c.i. is $1-\alpha$, that is, if one constructs a very large number of $100(1-\alpha)\%$ c.i., each based on n observations, the proportion of the c.i.'s that cover $\mu$ will be $1-\alpha$.

When the variables $X_i$ are not normal, which is the usual case in practice, it can be shown that, as n increases, the distribution of $(\overline{X}_n - \mu)/\sqrt{(s_n^2/n)}$ converges to that of $(\overline{X}_n - \mu)/\sqrt{(\sigma_n^2/n)}$ which is $N(0,1)$. It is therefore common practice to treat $(\overline{X}_n - \mu)/\sqrt{(s_n^2/n)}$ as a t variate and compute an approximate c.i. for $\mu$ using (4.4). The actual coverage of this approximate c.i. is less than $1-\alpha$, but will be close to $1-\alpha$ if the sample size n is sufficiently large.

Another important characteristic of a c.i. is its precision. Of two different $100(1-\alpha)\%$ c.i. for $\mu$, the smaller of the c.i. would be favoured since it gives a more precise idea of the exact value of $\mu$.

The mean is a measure of the central tendency of the distribution of X and is one of the statistics most used. Other characteristics of the distribution of X that are often of interest are the variance $\sigma^2$, the percentiles $p_c$ and the quantiles $x_p$.

An unbiased point estimator of $\sigma^2$ is the sample variance $s_n{}^2$ which was given in (4.3). The sample variance $s_n{}^2$ is asymptotically normal ( Fishman, 1973) with mean $\sigma^2$ and variance given by

$$Var(s_n{}^2) = (\mu_4 / \sigma^4 - (n-3)/(n-1))\, \sigma^4 / n = \sigma^4 (2/(n-1) + \gamma_2/n)$$

where

$$\gamma_2 = \mu_4 / \sigma^3 - 3 \qquad \text{with} \qquad \mu_4 = E[(X - E(X))^4]$$

If X is normal, $\gamma_2 = 0$ and $(n-1)s_n{}^2 / \sigma^2$ has the chi-square distribution with n-1 degrees of freedom. If X is not normal, to ignore the fourth moment would underestimate the width of the confidence interval for $\sigma^2$. The problem can be solved in practice by replication, as explained in Fishman (1973).

The percentile $p_c$ is the probability that the variable X is less than or equal to the constant C:

$$p_c = P\{X \leq C\} \tag{4.5}$$

The usual estimator of $p_c$ is $\hat{p}_c$, the sample value of an indicator function $\delta_C(X)$ defined the following way:

$$\delta_C(X) = \begin{cases} 1 & \text{if } X \leq C \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$P(\delta_C(X_i) = 1) = p_c, \quad P(\delta_C(X_i) = 0) = 1 - p_c$$

and $E(\delta_C(X_i)) = p_c$.

Then,

$$\hat{p}_c = \sum_{i=1}^{n} \delta_C(X_i) / n \tag{4.6}$$

As

$$E(\hat{p}_c) = p_c$$

$\hat{p}_c$ is an unbiased estimator of $p_c$.

The variance of $\hat{p}_c$ can be easily calculated by:

$$\mathrm{Var}(\hat{p}_c) = p_c(1-p_c) / n \tag{4.7}$$

An unbiased estimator of $\mathrm{Var}(\hat{p}_c)$ is given by $\underline{\mathrm{Var}}(\hat{p}_c)$:

$$\underline{\mathrm{Var}}(\hat{p}_c) = \hat{p}_c(1-\hat{p}_c) / (n-1)$$

as can be shown without difficulty.

As $\hat{p}_c$ is being calculated as a mean of the sample values of the indicator function $\delta_C(X)$, a confidence interval can be obtained of the type given in (4.4), using $\underline{Var}\ (\hat{p}_c)$ as the sample mean variance.

The number $x_p$ is said to be the p-quantile, $0<p<1$, of the distribution of X if $P\,[\,X < x_p\,] \leq p$ and $P\,[\,X > x_p\,] \leq 1-p$.

If the distribution function of X, F(X), is continuous and strictly increasing, then $x_p = F^{-1}(p)$. A point estimator for the p-quantile is the sample p-quantile $\hat{x}_p$ which is given by

$$\hat{x}_p = X\,(\,G(np+1)\,)$$

where G(k) is the greatest integer less than or equal to k and X(j), the jth order statistic, is the jth smallest of the $X_j$'s for $j=1,2,\ldots,n$. A confidence interval can also be constructed for $x_p$ using the order statistics (Conover,1980). The difficulty with these estimation methods is that they require a large amount of computer storage and computing time to sort the sample if the sample size n is large, and it must be large in order to obtain meaningful results. Several alternatives leading to more efficient methods of obtaining point and interval estimations of the quantiles are available. This is particularly important when estimating extreme quantiles.

## IV.4. STOCHASTIC SIMULATION: A SMALL EXAMPLE.

As was seen in previous sections, uncertainty is an important enough element to be introduced in capital investment analysis. In some special cases, the analytical approach, first suggested by Hillier, can be used to study the problem mathematically. This approach does not deal easily with interdependence between projects, and the situations of interdependence are very important in real life. Stochastic simulation therefore, allowing for interdependencies, assumes an important role in the study of more realistic cases.

Even in a very simple situation uncertainty can change the ranking of projects as is exemplified in the following. Consider two projects A and B whose cash flows are first assumed to be deterministic and are given in table 4.1:

| Project | Time period 1 | 2 | 3 | 4 | NPV obtained with a 5% interest rate |
|---|---|---|---|---|---|
| A | -100 | 80 | 100 | 100 | 153.28 |
| B | — | -90 | 100 | 150 | 134.56 |

Table 4.1. Cash flows and net present values for projects A and B.

Project A having a higher NPV is preferable to B. Now let uncertainty be introduced under the form of normal distributions for the cash flows in each time period, which is reasonable as was seen in section III.3. Consider the means of the normal distributions as given by the values of table 4.1, which leads to expected NPV for A and B equal to the NPV's presented in that table. These expected NPV's are calculated as indicated by expression (3.7). Let the variance which is the second parameter needed to completely define the normal distribution for each variable, be given as in table 4.2.

| | Time period | | | |
|---|---|---|---|---|
| Project | 1 | 2 | 3 | 4 |
| A | — | 6400 | 10000 | 10000 |
| B | — | 8100 | 10000 | 22500 |

Table 4.2. Variances for the cash flows of A and B in each time period.

The initial investment for A, in time period 1, the present time, is considered to be correct and thus has a zero variance. All other cash flows are supposed to have a variance equal to the square of the respective means. Now, values are generated from the normal distributions to have the uncertain cash flows of A and B in time periods 2, 3 and 4. With them the NPV's of A and B can be obtained. Of course, these NPV's are now random variables that approximately follow a normal distribution. A sample can be obtained for each of these random NPV's and its mean can be calculated. Several batches of 100 values were generated and the estimated expected NPV's were obtained. The results are presented in table 4.3, columns A and B1.

| Batch | A | B1 | B2 |
|---|---|---|---|
| 1 | 167.78 | 148.57 | 114.70 |
| 2 | 146.48 | 138.61 | 141.96 |
| 3 | 154.88 | 105.55 | 132.14 |
| 4 | 154.60 | 136.38 | 135.01 |
| 5 | 181.30 | 147.21 | 100.03 |
| 6 | 140.62 | 164.77 | 147.72 |
| 7 | 156.29 | 125.60 | 134.82 |
| 8 | 151.67 | 116.68 | 137.43 |
| 9 | 178.17 | 165.90 | 109.15 |
| 10 | 155.13 | 132.44 | 126.47 |
| 11 | 165.01 | 138.72 | 119.52 |
| 12 | 165.23 | 157.67 | 118.35 |
| 13 | 150.54 | 106.11 | 142.16 |
| 14 | 155.01 | 140.62 | 136.67 |
| 15 | 152.42 | 155.80 | 134.93 |
| 16 | 161.37 | 159.14 | 126.60 |
| 17 | 134.98 | 176.81 | 161.13 |
| 18 | 162.39 | 99.65 | 123.68 |
| 19 | 151.40 | 133.43 | 137.48 |
| 20 | 163.15 | 179.19 | 123.11 |

Table 4.3. NPV values for projects A and B.

Some interdependency can be introduced in this simple example in the following way: if a value for project A's cash flow is below its expected value, the value of project's B cash flow is above its expected value in a standardised way. The values for the batches obtained when this interdependency is introduced are given in table 4.3; only the B values are different and are under B2. The NPV values of this table exemplify well that the sample mean is a random variable. Comparing these results with the NPV's of table 4.1, the effect of uncertainty, and uncertainty associated with some interdependency can be seen. For batches 6,15,17 and 20 of A and B1, and 6 and 17 of A and B2 the results show that project A is no longer the best of the two. Uncertainty can therefore greatly influence a project.

In this research uncertainty was introduced in a portfolio of projects through the model detailed next section.

## IV.5. THE PROBLEM AND MODEL CHOSEN.

Due to difficulties in obtaining current data on capital investment projects for the study it seemed natural to choose those portfolios from a problem which has already been used in different contexts. The basic problem then, from which several portfolios of investment projects were obtained was taken from Weingartner (1965). This problem was also used by Sundem (1975) in his evaluation of capital budgeting models.

Table 4.5. Cash flows for the 30 investments projects and the firm ongoing activities.

| Project Number | cash flow in year 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -100 | 20 | 20 | 20 | 19 | 19 | 18 | 16 | 14 | 11 | 6 | -8 | | | | | | | | | |
| 2 | -100 | 20 | 18 | 18 | 18 | 18 | 14 | 14 | 14 | 14 | 14 | 10 | 10 | 10 | 10 | 10 | 6 | 6 | 6 | 6 | 6 |
| 3 | -100 | 15 | 15 | 15 | 15 | 15 | 13 | 13 | 13 | 13 | 13 | 11 | 11 | 11 | 11 | 11 | 9 | 9 | 9 | 9 | 9 |
| 4 | -100 | 20 | 6 | 11 | 7 | 16 | 5 | 14 | 18 | 3 | 20 | 2 | 22 | 8 | 10 | 18 | 6 | 9 | 14 | 24 | |
| 5 | -100 | -60 | -60 | 80 | 74 | 66 | 56 | 44 | 30 | 14 | | | | | | | | | | | |
| 6 | -200 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 119.77 |
| 7 | | | | | | | -80 | 20 | 20 | 20 | 19 | 17 | 14 | 10 | 6 | 2 | | | | | |
| 8 | | | | | | | | | -60 | -30 | -10 | 45 | 34 | 25 | 16 | 12 | 8 | -20 | 21 | 16 | 32.47 |
| 9 | | | -120 | 25 | 25 | 30 | 35 | 30 | 25 | 20 | 15 | 10 | 5 | | | | | | | | |
| 10 | | -100 | 18 | 17 | 15 | 12 | 8 | -10 | 18 | 17 | 15 | 12 | 8 | -10 | 18 | 17 | 15 | 12 | 8 | | |
| 11 | -150 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | | | | | | | | | |
| 12 | -100 | 20 | 18 | 16 | 14 | 12 | 10 | 4 | -20 | 20 | 18 | 16 | 14 | 12 | 10 | 4 | | | | | |
| 13 | -150 | -75 | -75 | 60 | 60 | 55 | 50 | 44 | 38 | 36 | 35 | 34 | 33 | 30 | 25 | 17 | 9 | | | | |
| 14 | -50 | -100 | -175 | 50 | 55 | 60 | 65 | 60 | 50 | 40 | 30 | 20 | 10 | -25 | 50 | 41 | 35 | 25 | 15 | 5 | |
| 15 | -100 | -150 | -100 | 10 | 20 | 30 | 40 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 287.45 |
| 16 | | | | | | | | -95 | -60 | 47 | 42 | 37 | 31 | 24 | 18 | 13 | 9 | 6 | 4 | 3 | |
| 17 | | | | | | | | | | -175 | 50 | 45 | 35 | 25 | 10 | -60 | 45 | 35 | 25 | 10 | |
| 18 | -250 | 45 | 45 | 40 | 30 | 25 | 20 | 15 | 10 | -40 | 40 | 32 | 25 | 19 | 14 | 10 | 7 | 5 | | | |
| 19 | -75 | -75 | -40 | 40 | 40 | 40 | 35 | 35 | 30 | 25 | 15 | 5 | | | | | | | | | |
| 20 | -180 | 20 | 12 | 16 | 13 | 11 | 19 | 17 | 12 | 15 | 19 | 13 | 14 | 17 | 20 | 14 | 11 | 15 | 17 | 12 | |
| 21 | | | | | | | | | | | | -80 | 18 | 16 | 14 | 12 | 10 | 4 | 16 | 14 | 15.45 |
| 22 | | -85 | 20 | 20 | 16 | 15 | 13 | 10 | 7 | 3 | | | | | | | | | | | |
| 23 | | -270 | -100 | 125 | 115 | 105 | 80 | 60 | 35 | 25 | 15 | 10 | | | | | | | | | |
| 24 | | | | | | | | | | -40 | 15 | 13 | 9 | 7 | 5 | 2 | | | | | |
| 25 | | | | | | | | -50 | 10 | 10 | 9 | 7 | 4 | -14 | 9 | 9 | 8 | 6 | 3 | -16 | 18.5 |
| 26 | | -200 | 60 | 40 | 30 | 15 | -25 | -25 | 50 | 40 | 30 | 20 | 10 | | | | | | | | |
| 27 | | | | | | | | | | | -70 | 15 | 13 | 11 | 10 | 9 | 7 | 6 | 4 | 3 | 2 |
| 28 | | -355 | 60 | 70 | 80 | 70 | 55 | 40 | 25 | 15 | 5 | | | | | | | | | | |
| 29 | -275 | 40 | 45 | 45 | 40 | 35 | 30 | 25 | 20 | 15 | -75 | 35 | 30 | 25 | 20 | 15 | 10 | 5 | | | |
| 30 | -140 | 20 | 20 | 18 | 16 | 14 | 11 | 8 | -25 | 18 | 18 | 16 | 13 | 10 | 6 | -25 | 16 | 16 | 14 | 11 | 14.2 |
| 31 | 400 | 360 | 320 | 280 | 240 | 200 | 160 | 120 | 80 | 40 | 0 | -40 | -80 | -120 | -160 | -200 | -240 | -280 | -320 | -360 | -400 |

The cash flows associated with thirty hypothetical investment projects over a period of 21 years and the cash flows for the firm's ongoing activities, considered as project 31, are given in table 4.5. These cash flows are used to define the coefficients of a mixed integer programming model to obtain portfolios of investment projects. Uncertainty is then introduced in these groups of projects in the way described as follows.

The procedure considered here to generate values of the cash flows of N projects during T time periods, uses the model presented in section III.3. The model is repeated below for convenience.

The cash flow for project i in time period t, $P_{it}$ is given by:

$$P_{i1} = X_{i1} + b_{i1}S_1$$

$$P_{it} = X_{it} + a_{it}(P_{i,t-1} - \mu_{i,t-1}) + b_{it}S_t , \quad t=2,3,\dots,T; \; i=1,2,\dots,N$$

where $X_{it} \sim N(\mu_{it},\sigma_{it}^2)$, $\sigma_{it}^2 = c\mu_{it}$ ; c, $a_{it}$ and $b_{it}$ are constants and the systematic effect $S_t$ is given by:

$$S_1 = E_1$$

$$S_t = d_t S_{t-1} + E_t , \quad t=2,3,\dots,T$$

where $E_t \sim N(0,\sigma_t^2)$, $\sigma_t^2 = p^{t-1}\sigma_1^2$ and $d_t$ and p are constants.

For simplicity, in the experiments with the model the constants were taken as:

$$a_{it} = a, \quad b_{it} = b, \quad d_t = d$$

The problem now is to attribute sensible figures to the constants in the model. The positive value 0.5 was assigned to constant a to model the influence of $P_{i,t-1}$ on $P_{it}$ trying to be neither too optimistic, when that value is higher than its mean, $\mu_{i,t-1}$, nor too pessimistic, when it is lower than $\mu_{i,t-1}$. The value of b was taken as equal to 0.5 to smooth the influence of the high variability introduced by the systematic effect; and c was made equal to 0.5, supposing that the values of $X_{it}$ are reasonably accurate. The values of 10, 1.1 and 1 were chosen to $\sigma_1{}^2$, p and d respectively, to begin with a rather high variance and proceed with a moderate increase on the variance of the systematic effect. Reasonable values therefore, seem to be the following:

$$a = 0.5 \ , \ b = 0.5 \ , \ c = 0.5 \ ,$$

$$d = 1.0 \ , \ p = 1.1 \ \text{and} \ \sigma_1{}^2 = 10.0 \ .$$

The $\mu_{it}$ were chosen to be the values of cash flows of some projects given in table 4.5. The $X_{it}$ and $E_t$ are generated as values from a normal distribution of known parameters through the polar method. The values of the cash flows are obtained from the model above. Then, for the chosen set of projects, using the generated cash flow values, the net cash flows can be obtained considering a simplified tax system. The tax system, as was reviewed in chapter II, is by itself a factor of interdependency among projects, thus affecting the net cash flows in each period. A group of projects chosen in a deterministic situation using the peculiarities of the tax system in the best possible way, by a linear programming model for example, can behave very differently when uncertainty is introduced. Because of the uncertainty, the benefits of tax allowances, for example can not be so accurately used. Finally, from the net cash flows, a value of the NPV can be calculated. The procedure is repeated in order to obtain a sequence of NPV values which constitute the object of the analysis.

The values that constitute the sequence $(X_i)$ of the NPV's of a group of projects and firm ongoing activities are generated independently (different random numbers are used to obtain each NPV) and the value of n was fixed as equal to 250. Although all of the four descriptors of the probabilistic behaviour of the NPV sequence could be interesting, only two of them were considered in the study in detail. They are the sample mean and a percentile.

The c.i. for the mean given by (4.4) is approximated since the $\{NPV_i\}$ will not be normal although one would expect a good approximation for the c.i. because n is not small. Also, the c.i. for the percentile will be approximated. As the half width of the c.i. is related to the variance of the statistic, one would like to have an estimator with a small variance. The next chapter studies some methods of variance reduction applied to the calculation of the mean and the percentile to choose a method to be used in the subsequent analysis.

# CHAPTER V. APPLICATION OF VARIANCE REDUCTION TECHNIQUES TO THE SIMULATION OF A CAPITAL INVESTMENT PROBLEM.

## V.1. INTRODUCTION.

The accuracy of an estimated response of a simulated system may be measured by the standard deviation of the mean of the estimator. This accuracy can be increased either by taking a larger sample, using the original sampling procedure, or by using a variance reduction technique (VRT).

Some VRT's change the original sampling process completely, as is the case in importance sampling. Other VRT's modify the sampling process in a subtle way as happens with antithetic variates and common random numbers. Some VRT's use the same original sampling procedure but after the sampling, use a more sophisticated estimator than the one associated with the crude simulator, as for example, in stratification after sampling and control variates.

The objective of this study is to find an efficient way of measuring the accuracy of several estimators related to the net present value of a group of projects. The purpose of this chapter is to describe and compare some VRT's in the simulation of a group of capital investment projects.

When such an efficient technique is found the sample obtained through its application is used to study other aspects of the capital investment problem. Hence, a VRT that completely disturbs the sampling procedure is not considered here because the output of the new sampling process cannot be used to investigate other aspects of the problem in hand, including the dynamic behaviour of the system under study. This is the case in importance sampling where the original process is replaced by another one (Clark, 1961, Kleijnen, 1974) So, although importance sampling has been used as a VRT, for example in the simulation of periodic queueing systems (Moy, 1971), it is not referred to in this study.

Selective sampling, a VRT technique devised by Brenner (1963) and criticised by Kleijnen (1974) among others, as being a biased procedure is also not going to be considered. In the case of the NPV of a group of projects it does not also seem practical to use descriptive sampling, a technique proposed by Saliby (1980), which is in line with Brenner's method. In descriptive sampling the sample values are deterministically selected, but the sequence is randomised. This technique involves the inverse transformation of the distribution function of the random variables in use, which is not always readily available.

After the definition of the problem a brief description of several VRT's is presented, applied to the mean and followed by the results obtained for the case under study. The techniques explicitly considered are : stratified sampling, antithetic variate sampling and regression sampling (or control variate technique).

Afterwards, the application of the techniques is extended to the other statistic considered. Finally, based on the results obtained, a technique is chosen to be used for the rest of the study.

## V.2. THE PROBLEM.

The problem is the estimation of $\theta=E[X]$ where $\theta$ is any quantity that can be expressed as the expected value of some random variable X. Usually the method of estimating $\theta$ is to generate n independent observations $x_1, x_2, \ldots, x_n$ and to consider

$$T = \sum_{i=1}^{n} x_i / n = \bar{x}_n$$

Then $E[T] = \theta$ and $var[T] = var[X]/n$. The variance of T will in general not be known, but it can be estimated by

$$s_n^2 = \sum_{i=1}^{n} (x_i - T)^2/(n-1)$$

The estimated value gives some idea of how well T represents $\theta$. This method gives convergence to $\theta$ of the order of $\sqrt{n}$ and a great number of observations may be needed to obtain an acceptable precision in the estimate of $\theta$. Variance reduction techniques are used to reduce the amount of sampling for a fixed precision or to increase the accuracy in the case of a fixed size sample. As the VRT's are going to be applied to a sample of fixed size they are going to be compared by the decrease in the standard deviation, since the relative change of the standard deviation is equal to that of the length of the confidence interval.

## V.3. APPLICATION OF VRT TO THE ESTIMATION OF A MEAN.

Consider now the problem of estimating the expected net present value of the generated net cash flows in a group of projects associated with a firm's ongoing activities. Three different VRT are going to be applied to this statistic: stratified sampling, antithetic variate sampling and control variate technique.

V.3.1. Brief description of the methods used.

V.3.1.1. Stratified sampling.

In stratified sampling besides the variable of interest X, another variable Y is measured for each of n observations. Variable Y is called the stratification variable and it serves to classify each sampled value to one of K mutually exclusive and exhaustive classes or strata, $S_k$, with mean $\mu_k$, $k=1,2, \ldots ,K$.

Denote by $p_k$ the probability that a particular value of Y, y, belongs to class $S_k$ :

$$p_k = \text{Prob}(y \in S_k)$$

Let $n_1, n_2, \ldots, n_K$ , adding up to n, be the specified number of observations to be drawn from each stratum

$$n = \sum_{k=1}^{K} n_k$$

The population mean $\mu$ can be estimated by the stratified estimator $\overline{x}_{ST}$:

$$\overline{x}_{ST} = \sum_{k=1}^{K} p_k \overline{x}_k$$

where $\bar{x}_k$ is the usual sample mean of the k'th stratum. It can be proved that the stratified estimator is an unbiased estimator of $\mu$. The variance of $\bar{x}_{ST}$ is given by

$$\mathrm{var}[\bar{x}_{ST}] = \sum_{k=1}^{K} p_k^2 \,\mathrm{var}[\bar{x}_k] = \sum_{k=1}^{K} p_k^2 \sigma_k^2/n_k$$

where

$$\sigma_k^2 = E[(X-\mu_k)^2 \mid X \in S_k]$$

Obviously, $\sigma_k^2$ can be estimated by $s_k^2$ :

$$s_k^2 = \sum_{i=1}^{n_k} (x_{ki} - \bar{x}_k)^2/(n_k - 1)$$

An unbiased estimator of $\mathrm{var}[\bar{x}_{ST}]$ is given in :

$$s_{ST}^2 = \sum_{k=1}^{K} p_k^2 \; s_k^2/n_k$$

The variance of $\bar{x}_{ST}$ depends on the choice of $n_k$, the number of observations in stratum k. Choosing $n_k = p_k n$ yields

$$\mathrm{var}[\bar{x}_{ST}] = \sum_{k=1}^{K} p_k \,\sigma_k^2/n$$

It has been proved ( Tocher, 1963 and Cochran, 1966 ) that

$$\mathrm{var}[\bar{x}_n] = \mathrm{var}[\bar{x}_{ST}] + \sum_{k=1}^{K} p_k (\mu_k - \mu)^2/n$$

So the stratification is useful if the $\mu_k$'s are not all equal to $\mu$. The means $\mu_k$ differ if the variable of interest X depends on the stratification variable Y, hence if there is a Y strongly correlated with X the stratification is efficient. A confidence interval to the mean can be obtained in the usual way (Cochran, 1966) assuming that $\bar{x}_{ST}$ is normally distributed and $s^2_{ST}$ is well determined. If there are not enough observations in the k'th stratum, the central limit theorem may not apply and $\bar{x}_k$ may not be well determined. In any case the confidence limits for the stratified estimator hold only approximately (Kleijnen, 1974).

Another way of using stratification, is to apply it after sampling (Kleijnen, 1974). After obtaining a sample of n observations in the usual way, that is without stratification, the observations are classified to the adequate strata. The estimator of the mean, $\bar{x}_{SA}$ is obtained in a similar way to $\bar{x}_{ST}$, with the difference that now the number of observations per stratum is not fixed. Cochran (1966) has shown that stratification after sampling is almost as good as proportionate sampling.

The main problem with stratification after sampling is that it can be more difficult to determine a confidence interval to the mean than with stratified sampling.

In the case of the NPV of a group of projects the stratification variable chosen was the sum of the cash flows of all time periods considered, which is strongly correlated with the NPV, and the comparison value was the sum of the expected cash flows, C. The cash flows were normally distributed with known mean and variance.

Only two strata were considered : $S_1$, when the sum of the cash flows is less than C and $S_2$, otherwise. Then, as the distribution of that sum is symmetric, $p_1=p_2=0.5$ .

The expressions used to do the calculations were:

$$\overline{NPV}_{SA} = p_1 \sum_{k=1}^{n_1} NPV_{1k}/n_1 + p_2 \sum_{k=1}^{n_2} NPV_{2k}/n_2$$

$$var[\ \overline{NPV}_{SA}\ ] = p_1^2\ s_1^2/n_1 + p_2^2\ s_2^2/n_2$$

where $NPV_{ik}$ is the NPV of the k'th observation in stratum i and $s^2_i$ is the estimated variance of the NPV in stratum i.

V.3.1.2. Antithetic variate sampling

This technique was first proposed by Hammersley and Morton (1956) in the context of Monte Carlo integration. The method is based on the fact that the mean of two negatively correlated observations from the same population give a better estimate for the population mean than the mean of two independent observations.

Let $x_1$ and $x_2$ be two values of the response of a system with mean response $\mu$. An estimate of $\mu$ is given by x:

$$x = (x_1+x_2)/2$$

with variance $\mathrm{var}(x) = \mathrm{var}(x_1)/4+\mathrm{var}(x_2)/4+\mathrm{cov}(x_1,x_2)/2$.

Hence , $\mathrm{var}(x) < \mathrm{var}(x_1)/4+\mathrm{var}(x_2)/4$ , if $\mathrm{cov}(x_1,x_2)$ is negative, that is, if $x_1$ and $x_2$ are negatively correlated.

The problem is then to obtain negatively correlated values. This can be achieved using random numbers r and 1-r to generate $x_1$ and $x_2$, if these values are obtained through a monotonic function.

Tocher (1963) suggested that the use of complementary random streams in general simulation problems would also produce negatively correlated outputs. The designation 'antithetics' then became associated with the use of complementary random streams.

In this case $x_1$ and $x_2$ are produced by a complicated function represented by the computer programme used to generate them. It is usually very difficult, if not impossible, to show analytically that antithetic variates lead to a negative correlation between $x_1$ and $x_2$. In most cases it can only be found empirically that the values are negatively correlated.

In certain situations, as Zeigler (1979) points out, the technique can even increase the variance (positive covariance between $x_1$ and $x_2$). It seems advisable on applying this technique, to do an initial small sample test comparing the estimated variances of the parent and averaged model realisations. Only if the results in this test yield a significant variance reduction should it be used in the full scale simulation experimentation.

Despite its weaknesses the antithetic variate technique is attractive since in most cases it gives worthwhile variance reductions and it is easy to apply with only a little extra programming and running time. Also the statistical analysis of the results is not difficult.

Let $z_j$ be the average of each antithetic pair generated, this is,

$$z_j = (x_j + x_j^a)/2\ , \quad j=1,\dots,n/2\ ;\ n \text{ even}$$

where $x_j^a$ is the antithetic value associated with $x_j$. All $x_j$ use new sequences of random numbers, so they are independent; hence, all $z_j$ are independent. Then,

$$\bar{x}_n = \sum_{k=1}^{n} x_k/n = \sum_{j=1}^{n/2} z_j/(n/2) = \bar{z}_{n/2}$$

$$\mathrm{var}(\bar{x}_n) = \mathrm{var}(\bar{z}_{n/2}) = \sum_{j=1}^{n/2} (z_j - \bar{z}_{n/2})^2/(n/2-1)$$

and Z has smaller variance than X.

The case of the NPV of a group of projects seems to be a very good one for the application of this technique. A value of NPV, $NPV_j$, is generated with a stream of uniform random numbers $R=(r_1, r_2, \dots, r_s)$ ; with the complementary stream $(1-R)= (1-r_1,\dots,1-r_s)$ another value of NPV , $NPV_j^a$ , is obtained. Then,

$$NPV^j = (NPV_j + NPV_j^a)\ /2\ , \qquad j=1,2,\dots,n/2$$

The same number of uniform random numbers is used in NPV and $NPV^a$ , and the complementary values are used in precisely the same calculations. This may not happen in the simulation of more complex situations where there is no guarantee of a perfect synchronization of the complementary values as there is in this case.

V.3.1.3. Control variate technique.

The control variate technique or regression sampling, is considered a most promising VRT in the context of its application to general simulation. This VRT uses control variables to obtain the statistical estimator of interest.

Let X be a random variable whose expected value m is to be estimated. A random variable Y is a control variable if its expectation is known and if it is correlated with X. The control variable can be used to construct an unbiased estimator for m which has smaller variance than X. For any constant a, X(a) given by :

$$X(a) = X - a(Y-\mu_Y)$$

is an unbiased estimator of m and

$$var[X(a)] = var[X] - 2a cov[X,Y] + a^2 var[Y] . \qquad (5.1)$$

Then, $\mathrm{var}[X(a)] < \mathrm{var}[X]$ if $2a\,\mathrm{cov}[X,Y] > a^2\mathrm{var}[Y]$

This inequality can be used to establish limits for the control coefficient a. Still better,the value of a, $a_{opt}$, which minimises $\mathrm{var}[X(a)]$ can be obtained by differentiating (5.1) and solving for a:

$$a_{opt} = \mathrm{cov}[X,Y]/\mathrm{var}[Y] = \rho(X,Y) \quad \sigma(X)/\sigma(Y)$$

where $\rho(X,Y)$ is the correlation coefficient between X and Y.

Substituting in (5.1) yields the minimum variance

$$\mathrm{var}[X(a_{opt})] = [1 - \rho^2(X,Y)]\ \mathrm{var}[X]$$

The more correlated Y is with the variable X therefore, the greater the reduction in variance.

The control variate technique can be extended to the case of more than one control variable :

$$X(a_1, \quad ,a_k) = X - \sum_{k=1}^{K} a_k\,(Y_k - \mu_k)$$

with $\mu_k = E[Y_k]$.

Using a more condensed notation

$$X(\mathbf{a}) = X - \mathbf{a}'[\mathbf{Y} - \mu_Y]$$

where $\mathbf{a}$ is now a column vector of constant coefficients, $a_k$, $\mathbf{Y}$ a column vector of K control variables and $\boldsymbol{\mu}_Y$ the expectation of Y.

$X(a)$ is also an unbiased estimator of m.

The vector $\mathbf{a}_{opt}$ which minimises the variance of $X(a)$ is given by (Anderson, 1958) :

$$a_{opt} = \boldsymbol{\Sigma}_Y^{-1} \boldsymbol{\sigma}_{XY}$$

where $\boldsymbol{\Sigma}_Y$ is the covariance matrix of Y and $\boldsymbol{\sigma}_{XY}$ is a K-dimensional vector whose components are the covariances between X and $Y_k$'s.

The minimum variance is (Anderson, 1958):

$$\mathrm{var}[X(a_{opt})] = [1-R^2_{XY}]\, \mathrm{var}[X]$$

where

$$R_{XY}{}^2 = \sigma'_{XY}\ \boldsymbol{\Sigma}_Y^{-1}\ \boldsymbol{\sigma}_{XY} / \mathrm{var}[X]$$

is the square of the multiple correlation coefficient between X and Y.

The quantity $1-R^2_{XY}$, called the minimum variance ratio, is the factor by which the variance of X can be reduced if the optimum coefficient vector $\mathbf{a}_{opt}$ is known.

It should be noted that although the theoretical expressions for $\mathbf{a}_{opt}$ and $\mathrm{var}[X(a_{opt})]$ are known, they cannot usually be used in practice because $\boldsymbol{\sigma}_{XY}$ and $\boldsymbol{\Sigma}_Y^{-1}$ are unknown.

So, the practical application of control variables requires the finding of variables which are highly correlated with the variables of interest and the estimation of the optimum coefficient value, $a_{opt}$.

The control variables can be chosen directly as a function of the basic uniform random values used in the simulation, in this case being system independent. There is evidence, as Kleijnen (1974) states, that better results are achieved with control variables that are system dependent, that is, the variates $y_k$ are defined directly in terms of the model being simulated. These control variables are sometimes called concomitant control variables. Usually there are many control variables available and they do not require much extra computer time for their calculation.

The standard estimator of $\mathbf{a}_{opt}$ is the sample equivalent of its theoretical expression :

$$\hat{\mathbf{a}}_{opt} = \hat{\boldsymbol{\Sigma}}_Y^{-1}\, \hat{\boldsymbol{\sigma}}_{XY}$$

where $\hat{\boldsymbol{\sigma}}_{XY}$ and $\hat{\boldsymbol{\Sigma}}_Y$ are the sample covariance vector and the sample covariance matrix whose elements are given by

$$(\hat{\sigma}_{XY})_i = \sum_{j=1}^{n} (x_j - \bar{x})(y_{ij} - \bar{y}_i)/(n-1)$$

and

$$(\hat{\Sigma}_Y)_{ik} = \sum_{j=1}^{n} (y_{ij} - \bar{y}_i)(y_{kj} - \bar{y}_k)/(n-1)$$

where $y_{ij}$ is the j'th element of $y_i$ and $\bar{y}_i$ is the mean of $y_{ij}$ for $j=1,\ldots,n$. Substituting $\hat{a}_{opt}$ in X(a) yields :

$$x_j(\hat{a}_{opt}) = x_j - \hat{a}_{opt}'(y - \mu_Y)$$

and

$$\bar{x}(\hat{a}_{opt}) = \sum_{j=1}^{n} x_j(\hat{a}_{opt})/n$$

Now $\bar{x}(\hat{a}_{opt})$ is not in general an unbiased estimator of m. Also, as Lavenberg and Welch (1981) point out, the t-distribution with (n-1) degrees of freedom cannot be used to generate a confidence interval. Lavenberg, Moeller and Welch (1982) recommend another method which is based on theory assuming that the vector $(X, Y_1, \ldots, Y_K)$ has a multivariate normal distribution. In many simulations X and the control variables are such that the multivariate normal assumption seems a reasonable one. Then the conditional distribution of X given Y=y is univariate normal with expectation $E[X \mid Y=y] = m - a'(y-\mu_Y)$ where $a = \Sigma_Y^{-1}\sigma_{XY}$ and

$$var[X \mid Y=y] = \sigma_X^2(1-R^2_{XY})$$

which is the minimum variance of $X(a)$.

Hence, under the multivariate normal assumption, and conditional on $Y_j=y_j$, $j=1,\ldots,n$, $\hat{\sigma}^2(X(a))$ can be obtained using standard regression techniques (Lavenberg and Welch, 1981).

Let $\quad \mathbf{X} = M'b + \varepsilon$

$$X' = (X_1,\ldots,X_N), \quad b = (m, a)' \quad \text{and}$$

$$M' = \begin{bmatrix} 1 & y_{11}-\mu_1 & \cdots & y_{K1}-\mu \\ 1 & y_{1n}-\mu_1 & \cdots & y_{Kn}-\mu_K \end{bmatrix}$$

where $\varepsilon$ is a vector of independent normally distributed random variables with mean zero and common variance $\sigma^2$ given by $\sigma_X^2(1-R^2_{XY})$.

Let $\hat{m}$ and $\hat{a}$ be the least squares estimators of m and a. Then

$$var[\hat{m}] = s_{11}\,\sigma^2$$

where $s_{11}$ is the upper leftmost element of $(MM')^{-1}$.

From regression theory an unbiased estimator of $\sigma^2$ is

$$\hat{\sigma}^2 = [\sum_{j=1}^{n} X_j^2 - \sum_{j=1}^{n} (\hat{m} - \hat{a}(y_j - \mu_Y))^2 ]/(n-K-1)$$

When $a_{opt}$ has to be estimated the full potential of the variance reduction of this VRT is reduced by the factor (n-2)/(n-K-2) (Lavenberg and Welch, 1981). In the applications therefore, it is important to use a small number of control variables selected from the full set of possible control variables

In the case of estimating the NPV of a group of projects one can identify control variables by looking for variables correlated with total NPV and whose mean can be obtained easily. One possible set of control variables can be constructed by using the sums of independent variables closely related to the NPV of each project. For example, the sum of all cash flows of all projects in all time periods can be used as a control variable. Several variables were considered and it was found that one variable that worked well was variable V, given below. Using the notation of the stochastic model of section IV.3, where $X_{it}$ are independent normal variables with parameters $\mu_{it}$ and $\sigma_{it}^2$, and $E_t$ are independent normal variables with parameters 0 and $\sigma_t^2$, the control variable V was defined in the following way:

$$V = \sum_{i=1}^{N} \sum_{t=1}^{T} \gamma_t X_{it} + \sum_{t=1}^{T} d_t E_t$$

where

$$d_t = \gamma_t b_t + c_{t+1} \gamma_{t+1} b_{t+1} + \dots + c_{t+1} c_{t+2} \dots c_T \gamma_T b_T$$

with

$$b_t = \sum_{i=1}^{N} b_{it}$$

Appendix 5.1 presents a detailed way of obtaining V with this form. Associating the twenty one time periods in several groups, more than one control variable similar to V can be constructed. For example, three control variables VC(1), VC(2), VC(3) can be obtained considering three groups of seven time periods.Thus,

$$VC(1) = \sum_{i=1}^{N} \sum_{t=1}^{7} \gamma_t X_{it} + \sum_{t=1}^{7} d_t E_t$$

$$VC(2) = \sum_{i=1}^{N} \sum_{t=8}^{14} \gamma_t X_{it} + \sum_{t=8}^{14} d_t E_t$$

$$VC(3) = \sum_{i=1}^{N} \sum_{t=15}^{21} \gamma_t X_{it} + \sum_{t=15}^{21} d_t E_t$$

The calculations were done with one, three, seven and twenty one control variables obtained in the way above described. The results are given in V.3.2.

V.3.1.4 Combination of antithetic and control variate techniques.

After some experimentation with the VRT previously described it became apparent that a possible combination of the two techniques with the best performances, antithetic and control variates, could bring even better results.

Consider a pair of antithetic values for the variable of interest, $x_i$ and $x_i^a$ and define $x_i^A$ as their average :

$$x_i^A = (x_i + x_i^a)/2$$

Similarly for the control variable, consider the pair $y_i$ and $y_i^a$ and define $y_i^A$ : $\qquad y_i^A = (y_i + y_i^a)/2$

Then, using a notation similar to that of V.3.1.3 define $X^A(b)$ :

$$X^A(b) = X^A - b\,(Y^A - \mu_Y)$$

where $X^A$ and $Y^A$ are variables whose values are the averages $x_i^A$ and $y_i^A$ respectively, and b a constant coefficient; $X^A(b)$ is an unbiased estimator of the expected value m.

Under the multivariate normal assumption and conditional on

$Y^A = y^A$, $\hat{\sigma}^2[X^A(b)]$ can be obtained using standard regression techniques as in the case of V.3.1.3.

V.3.2. Some numerical results.

The techniques of section V.3.1 were applied to the estimation of the mean NPV of a group of fifteen projects associated with the firm's ongoing activities.

The results in the table 5.1 represent the percentage improvement on the standard deviation when considered without VRT, $SD_1$ and with the indicated VRT, $SD_2$. In each column therefore the value presented is

$$[(SD_1-SD_2)/SD_1]\times 100$$

Each batch of 250 evaluations begins with a different value for the seed of the uniform random number generator, $IS_i$, $i=1, \ldots, 20$. The values $IS_i$ are the same for batch i in all the VRT applied. With the antithetic technique each batch has 125 direct evaluations and the corresponding 125 complementary evaluations.

Table 5.1 shows a considerable difference in efficiency between stratification after sampling using as a stratification variable the sum of the cash flows of all time periods, with the sum of all expected cash flows as the comparison value, and the other VRT considered. The antithetic technique compares well with the case of one control variable but is worse than the control variable technique when the number of control variables increases. Joining the antithetic and control variable techniques (with one control variable) gave poorer results than each of the techniques considered separately. The joining of the two techniques with more than one control variable was not pursued. As can be concluded from table 5.1 the control variable technique with more than one control variable gives consistently better results than the other VRT considered in this study.

The cases of seven and twenty one control variables gave very similar results, so the VRT chosen for the study of the mean of the generated NPV was the regression sampling with seven variables. This method gave a good percentage improvement on the standard deviation. There is now the question of measuring the robustness of the confidence interval.

Let the parameter $\mu$ be estimated in m batches of n evaluations using k control variables. Consider $\alpha_0 \geq 1/m$ as an adequate coverage at an $\alpha_0$ level, where

$$t_{n-k-1}^{\alpha_0/2} = a_0$$

and

$$a_0 = \min\{ a : \exists\ \mu \in \bigcap_{i=1}^{m} [\hat{\mu}_i - a\,\hat{\sigma}_i\,,\ \hat{\mu}_i + a\,\hat{\sigma}_i] \}$$

with $\hat{\mu}_i$ being the estimated value of the parameter $\mu$ in batch i and $\hat{\sigma}_i$ being the standard deviation of the values of $\mu_i$.

Table 5.1 Percentage improvement on the standard deviation with the VRT indicated.

| Batch number | stratification after sampling | antithetic technique | antithetic & 1control variable | 1 control variable | 3 control var. | 7 control var. | 21control var. |
|---|---|---|---|---|---|---|---|
| 1 | 41.8 | 85.4 | 64.0 | 83.9 | 86.8 | 87.8 | 87.8 |
| 2 | 35.1 | 85.2 | 74.1 | 81.7 | 85.3 | 85.7 | 85.7 |
| 3 | 37.5 | 80.6 | 58.0 | 80.4 | 83.9 | 85.7 | 85.5 |
| 4 | 37.9 | 76.9 | 56.0 | 81.0 | 85.7 | 87.4 | 86.9 |
| 5 | 35.9 | 83.1 | 76.5 | 80.2 | 83.9 | 85.0 | 85.4 |
| 6 | 36.7 | 81.6 | 58.4 | 81.6 | 84.3 | 86.2 | 86.0 |
| 7 | 35.0 | 81.1 | 62.8 | 82.0 | 85.2 | 86.7 | 86.9 |
| 8 | 36.2 | 83.4 | 62.3 | 83.1 | 86.1 | 87.4 | 86.4 |
| 9 | 38.9 | 77.8 | 70.8 | 82.6 | 85.5 | 87.0 | 86.8 |
| 10 | 40.9 | 80.1 | 55.0 | 83.6 | 86.8 | 87.2 | 87.0 |
| 11 | 40.1 | 81.4 | 60.8 | 82.2 | 85.8 | 86.9 | 86.8 |
| 12 | 37.6 | 82.6 | 66.2 | 83.3 | 86.4 | 86.8 | 86.5 |
| 13 | 37.4 | 84.0 | 66.2 | 81.3 | 85.4 | 86.2 | 86.3 |
| 14 | 37.8 | 83.1 | 61.5 | 82.5 | 85.7 | 86.9 | 87.0 |
| 15 | 39.8 | 85.3 | 71.8 | 82.5 | 86.0 | 86.7 | 86.9 |
| 16 | 36.3 | 79.2 | 54.1 | 81.1 | 84.0 | 85.4 | 85.7 |
| 17 | 41.9 | 79.5 | 54.6 | 83.1 | 86.1 | 87.6 | 87.4 |
| 18 | 36.7 | 88.1 | 73.8 | 80.8 | 85.6 | 86.1 | 86.3 |
| 19 | 38.6 | 82.1 | 58.9 | 83.9 | 86.9 | 87.3 | 87.3 |
| 20 | 37.5 | 83.3 | 73.0 | 82.4 | 85.3 | 86.3 | 86.3 |

In each batch, following Lavenberg and Welch (1981), a confidence interval for the mean can be obtained with the estimated values of the mean and its variance from the regression and the adequate percentile of the t-distribution with ( n-k-1) degrees of freedom.

An easy way of verifying the robustness of the interval estimation is as follows. Using the results of the twenty batches as many confidence intervals can be obtained for the mean of NPV. Considering these intervals there is adequate coverage if their intersection is non empty. Table 5.2 shows the 90% confidence intervals obtained for twenty batches of two hundred and fifty evaluations in the determination of the expected NPV using seven control variables.

As the intersection of the twenty confidence intervals is non-empty the conclusion is that regression sampling with the seven control variables defined as explained above is a robust method of estimating the mean of NPV.

Table 5.2 . The 90% confidence intervals obtained for the mean.

$$a = t_{242}^{0.05} = 1.645$$

| Batch number | mean $\hat{\mu}_i$ | standard deviation $\hat{\sigma}_i$ | $a\hat{\sigma}_i$ | $\hat{\mu}_i - a\hat{\sigma}_i$ | $\hat{\mu}_i + a\hat{\sigma}_i$ |
|---|---|---|---|---|---|
| 1 | 516.73 | 3.7545 | 6.17615 | 510.554 | 522.906 |
| 2 | 514.89 | 4.6889 | 7.71324 | 507.177 | 522.603 |
| 3 | 524.31 | 4.2107 | 6.92660 | 517.383 | 531.237 |
| 4 | 517.63 | 4.0047 | 6.58773 | 511.042 | 524.218 |
| 5 | 512.71 | 4.9445 | 8.13370 | 504.576 | 520.844 |
| 6 | 513.10 | 4.3150 | 7.09818 | 506.002 | 520.198 |
| 7 | 522.94 | 3.9438 | 6.48755 | 516.452 | 529.428 |
| 8 | 519.23 | 3.9318 | 6.46781 | 512.762 | 525.698 |
| 9 | 523.22 | 3.8941 | 6.40579 | 516.814 | 529.626 |
| 10 | 519.39 | 4.0090 | 6.59481 | 512.795 | 525.985 |
| 11 | 516.79 | 4.2039 | 6.91542 | 509.875 | 523.705 |
| 12 | 519.38 | 4.1799 | 6.87594 | 512.504 | 526.256 |
| 13 | 520.32 | 4.1226 | 6.78168 | 513.538 | 527.102 |
| 14 | 516.51 | 4.0827 | 6.71604 | 509.794 | 523.226 |
| 15 | 515.89 | 4.2708 | 7.02547 | 508.865 | 522.915 |
| 16 | 526.74 | 4.0083 | 6.59365 | 520.146 | 533.334 |
| 17 | 520.55 | 3.6756 | 6.04636 | 514.504 | 526.596 |
| 18 | 519.45 | 4.5998 | 7.56667 | 511.883 | 527.017 |
| 19 | 513.89 | 4.0283 | 6.62655 | 507.263 | 520.517 |
| 20 | 513.70 | 4.5088 | 7.41698 | 506.283 | 521.117 |

## V.4 APPLICATION OF VRT TO THE ESTIMATION OF A PERCENTILE.

Now, the statistic to be considered is a percentile p defined as

$$p = \mathrm{Prob}(Y \leq K) \qquad (5.2)$$

where K is some given critical value of the random variable of interest Y.

The usual estimator of p is $\hat{p}$, the sample value of an indicator function $\delta_K(Y)$ as defined in IV.3.2. The aim is to use VRT to obtain an estimated value of p with lower variance than that of $\hat{p}$.

From the three VRTs used in the case of the mean value only the control and antithetic variates are going to be considered in the estimation of p. Stratification does not make sense in this case because it estimates the relevant quantity in several strata and then combines the results. When Y is the net present value and the stratification variable is the sum of cash flows, p is certainly very different in each stratum.

V.4.1. Theoretical developments

V.4.1.1. Antithetic variate technique.

Let $y_i$ and $y_i^a$ be the ith pair of antithetic values for the NPV, with $i=1,2,\ldots,n$. One possible way of applying the antithetic variate technique to the estimation of p is to average the unbiased estimators obtained from the sequences of $y_i$ and $y_i^a$, $i=1,2,\ldots,n$. Thus,

$$\hat{p} = \sum_{i=1}^{n} \delta_K(y_i)/n \quad , \quad \hat{p}^a = \sum_{i=1}^{n} \delta_K(y_i^a)/n$$

and

$$\hat{\hat{p}} = (\hat{p} + \hat{p}^a)/2$$

It is not possible to obtain an expression for the variance of $\hat{\hat{p}}$, because $\hat{p}$ and $\hat{p}^a$ are not independent. The variance of $\hat{\hat{p}}$ has to be estimated empirically.

Another VRT technique that can be applied to the estimation of p is the control variate technique.

V.4.1.2. Control variate technique.

Let X be a control variable which is correlated with the response variable Y and whose distribution is known. Consider a function f(X) with zero expected value and use it to construct an estimator $p_c$ of p of the form: $p_c = \delta_K(Y) - f(X)$

where $\delta_K(Y)$ is the indicator function:

$$\delta_K(Y) = \begin{cases} 1 & \text{for } Y \leq K \\ 0 & \text{for } Y > K \end{cases}$$

Then $p_c$ is an unbiased estimator of the percentile p. The problem is now the construction of f(X) such that

$$\text{var}[\, p_c \,] = \text{var}[\, \delta_K(Y) - f(X) \,] < \text{var}[\, \delta_K(Y) \,] = p(1-p)$$

The function f(X) can be constructed as it is suggested in Ashford and Guedes (1986), in the following way. Define the indicator function $\delta_Q(X)$ such that

$$\delta_Q(X) = \begin{cases} 1 & \text{for } X \le Q \\ 0 & \text{for } X > Q \end{cases} \tag{5.3}$$

with

$$q = \text{Prob}\,[X \le Q] \tag{5.4}$$

The parameter Q is not fixed for the moment. Its value is going to be determined by satisfying some conditions specified below. Now let

$$f(X) = \beta(\,\delta_Q(X) - q)$$

where $\beta$ is a constant parameter. Thus $E[f(X)] = 0$ and

$$p_c = \delta_K(Y) - \beta\,(\,\delta_Q(X) - q)$$

Then p can be estimated by $\hat{p}_c$, the sample estimator of $p_c$:

$$\hat{p}_c = \sum_{i=1}^{n} \delta_K(y_i)/n - \sum_{i=1}^{n} \beta(\,\delta_Q(x_i) - q)/n$$

$$= \sum_{i=1}^{n} [\,\delta_K(y_i) - \beta\,\delta_Q(x_i)]/n + \beta q$$

$$= \sum_{i=1}^{n} \Delta_i/n + \beta q$$

where $\Delta_i$ is the value of $\Delta$ realised in the ith trial, with $\Delta = \delta_K(Y) - \beta\delta_Q(X)$. The variance of $\hat{p}_c$ may be estimated by the usual sample estimator $\hat{\sigma}_c^2$:

$$\hat{\sigma}_c^2 = \sum_{i=1}^{n} (\Delta_i - \bar{\Delta})^2/[n(n-1)] \tag{5.5}$$

where

$$\bar{\Delta} = \sum_{i=1}^{n} \Delta_i/n$$

The values of the parameters Q (and hence q) and $\beta$ are determined to minimise var($\hat{p}_c$) and at the same time ensuring that its value is less than var($\hat{p}$). The expressions for the optimal $\beta$ and q are derived in Ashford and Guedes (1986) but for the sake of completeness are also expressed and developed below. The estimator $p_c$ of p is then obtained through the following expression:

$$p_c = \delta_K(Y) - \beta(\delta_Q(X) - q) = \delta_K(Y) - \beta\delta_Q(X) + \beta q \tag{5.6}$$

where $\delta_K(Y)$ and $\delta_Q(X)$ are indicator functions and q is given by (5.4).

For convenience the control variable X is scaled so that it has zero mean and unit variance, and let $F_X(x)$ be its known distribution function, which is assumed to be continuous.

Defining J(q) as in (5.7):

$$J(q) = \mathrm{Prob}\{ Y \le K \text{ and } X \le Q \} \tag{5.7}$$

and expressing the right hand side of (5.7) in terms of the conditional probability yields (5.8):

$$\mathrm{Prob}\{Y \le K \text{ and } X \le Q\} = \mathrm{Prob}\{Y \le K \mid X \le Q\}.\mathrm{Prob}\{X \le Q\} \tag{5.8}$$

As X is a continuous variable and by definition $F_X(x) = \mathrm{Prob}\{X \le x\}$, also considering the meaning of q, J(q) can be expressed as in (5.9):

$$J(q) = \int_{F_X(\xi)\le q} \mathrm{Prob}\{Y \le K \mid X \le \xi\}.dF_X(\xi) \tag{5.9}$$

Now, as

$$\Delta = \delta_K(Y) - \beta\,\delta_Q(X)$$

comes that

$$\mathrm{Prob}\{\Delta = -\beta\} = q - J(q)$$

$$\mathrm{Prob}\{\Delta = 1-\beta\} = J(q)$$

$$\mathrm{Prob}\{\Delta = 1\} = p - J(q)$$

Then the expression (5.10) for the variance of $\Delta$ can be developed as below:

$$E(\Delta) = -\beta(q - J(q)) + (1-\beta)J(q) + p - J(q) = p - \beta q$$

$$E(\Delta^2) = \beta^2(q - J(q)) + (1-\beta)^2 J(q) + p - J(q) = \beta^2 q - 2\beta J(q) + p$$

and $$\mathrm{var}(\Delta) = p(1-p) - 2\beta[J(q) - pq] + \beta^2 q(1-q). \tag{5.10}$$

Since $\mathrm{var}(p_c) = \mathrm{var}(\Delta)/n$, minimising $\mathrm{var}(\Delta)$ with respect to $\beta$

and q yields a minimum of var($p_c$) with respect to the same parameters. Then,

$$\partial \, \mathrm{var}(\Delta)/\partial \, \beta = -2J(q) + 2pq + 2\beta q(1-q)$$

This will be zero whenever

$$\beta = (J(q) - pq)/[q(1-q)] \qquad (5.11)$$

On the other hand,

$$\partial \, \mathrm{var}(\Delta)/\partial \, q = -2\beta[J'(q) - p] + \beta^2(1-2q)$$

which will be zero whenever

$$J'(q) - \beta(0.5 - q) - p = 0 \qquad (5.12)$$

Also,

$$\partial^2 \mathrm{var}(\Delta)/\partial\beta^2 = 2q(1-q)$$

$$\partial^2 \mathrm{var}(\Delta)/\partial\beta\partial q = -2J'(q) + 2p + 2\beta(1-2q)$$

$$\partial^2 \mathrm{var}(\Delta)/\partial q^2 = -2\beta J''(q) - 2\beta^2$$

The Hessian of var($\Delta$) with respect to $\beta$ and q at its stationary values given by (5.11) and (5.12) is

$$\begin{bmatrix} 2q(1-q) & \beta(1-2q) \\ \beta(1-2q) & -2\beta[J''(q) + \beta] \end{bmatrix} \qquad (5.13)$$

Since $q(1-q)>o$ the matrix in (5.13) will be positive definite if its determinant is positive:

$$-4\beta q(1-q)[J'(q) + \beta] - \beta^2(1-2q)^2 > 0$$

that is, if

$$\beta\{-4q(1-q)J''(q) - \beta\} > 0 \qquad (5.14)$$

This can be guaranteed if J(q) is expressed in a special way. In practice J(q) and J'(q) must be estimated and this can be done using a single variable linear model of regression analysis. In common with the usual control variate technique suppose that

$$Y = \alpha_0 + \alpha_1 X + \varepsilon \qquad (5.15)$$

where Y is the vector of observations, X is the vector of the independent variables, $\alpha_0$ and $\alpha_1$ are the parameters to be estimated and $\varepsilon$ is the the vector of errors. If $F_e(x)$ denotes the distribution function of $\varepsilon$, J(q) can be written as

$$J(q) = \int_{F_X(\xi)\le q} F_e(K - \alpha_0 - \alpha_1\xi)dF_X(\xi)$$

$$= \int_{\xi \le F_X^{-1}(q)} F_e(K - \alpha_0 - \alpha_1\xi)F'_X(\xi)d\xi \qquad (5.16)$$

Then, as from (5.4) $F_X^{-1}(q) = Q$ ,

$$dJ(q)/dq = F_e(K - \alpha_0 - \alpha_1 F_X^{-1}(q)).F'_X(Q)/F'_X(Q)$$

because

$$q = F_X(Q)$$

and

$$dQ/dq = (dq/dQ)^{-1} = 1/F'_X(Q)$$

Thus,

$$J'(q) = F_e(K - \alpha_0 - \alpha_1 F_X^{-1}(q))$$

Also,

$$d^2J(q)/dq^2 = -\alpha_1 F'_e(K - \alpha_0 - \alpha_1 F_X^{-1}(q))/F'_X(F_X^{-1}(q))$$

Let G be the standardised distribution of the error $\varepsilon$, i. e.

$$G(\xi) = F_e(\xi/\sigma_e)$$

where $\sigma_e^2$ is the variance of $\varepsilon$. Without loss of generality the control variable is assumed to be positively correlated with the response variable, i.e. $\alpha_1 > 0$. Let

$$\alpha_x = [\max_{\xi} F'_X(\xi)]^{-1}$$

Since, by hypothesis, X is continuously distributed, $0 < \alpha_X < \infty$. Then,

$$J''(q) \le -\sigma_e^{-1}\alpha_1\alpha_X G'(K - \alpha_0 - \alpha_1 F_X^{-1}(q)) \qquad (5.17)$$

When X is normally distributed with unit variance, $\alpha_x = \sqrt{2\pi}$.

Considering the error $\varepsilon$ normally distributed, if

$$r = \text{Prob}\,(Y \le K \mid X = F_X^{-1}(q)\,) = \Phi(K - \alpha_0 - \alpha_1 F_X^{-1}(q))$$

and using the result of appendix 5.2,

$$\Phi'(K - \alpha_0 - \alpha_1 F'_X(q)) \ge 4r(1-r)/\sqrt{2\pi}$$

thus $\quad -\Phi'(K - \alpha_0 - \alpha_1 F'_X(q)) \le -4r(1-r)/\sqrt{2\pi} \le -1/\sqrt{2\pi}$

Then,

$$J''(q) \le \alpha_1/\sigma_e \, . \, \sqrt{2\pi} \, . \, (-1/\sqrt{2\pi}) = -\alpha_1/\sigma_e \tag{5.18}$$

as $J(q) \le q$ it is true that $J(q) - pq \le q(1-p)$ and thus

$$\beta \le (1-p)/(1-q) \tag{5.19}$$

It follows from (5.14) with (5.18) and (5.19) that

$$4q(1-q)\beta J''(q) + \beta^2 \le -4q(1-q)[(1-p)/(1-q)](\alpha_1/\sigma_e) + (1-p)^2/(1-q)^2$$

$$= [-4q(1-q)^2(1-p)\alpha_1 + (1-p)^2\sigma_e] \,/\, [(1-q)^2\sigma_e] \tag{5.20}$$

If $\alpha_1 > 0$, and for $\sigma_e$ sufficiently small, the expression in the numerator of (5.20) is negative. Hence the inequality (5.14) is verified and the Hessian of $var(\Delta)$ is positive definite. Then the values of $\beta$ and $q$ determined respectively by (5.11) and (5.12) minimise $var(p_c)$.

In practice $p$, $J(q)$ and $J'(q)$ must be estimated, and a root of equation (5.12) has to be determined numerically. Let $\hat{p}$, $\widehat{J}(q)$ and $\widehat{J}'(q)$ be the estimated values of $p$, $J(q)$ and $J'(q)$ respectively. $\widehat{J}(q)$ and $\widehat{J}'(q)$ are obtained as described in appendix 5.3. A technique to obtain $\hat{p}_c$ as an estimator of $p$ is as follows (Ashford and Guedes, 1986):

Step 1: Set $\hat{p}_c = \hat{p}$, $q = \hat{p}_c$ and $s^2 = \hat{p}(1-\hat{p})/n$

Step 2: Set $\beta = [\hat{J}(q) - \hat{p}_c q] / [q(1-q)]$

Step 3: Compute $\hat{p}_c = \Sigma_i \Delta_i/n + \beta q$ and $\hat{\sigma}_c^2 = \Sigma_i (\Delta_i - \bar{\Delta})^2 / [n(n-1)]$

If $\hat{\sigma}_c^2 > s^2$ then stop. Otherwise set $s^2 = \hat{\sigma}_c^2$.

Step 4: Solve $\hat{J}(q) - \beta(0.5-q) = \hat{p}_c$ for q. Go to step 2.

If the procedure stops on the first iteration through step 3, then this approach using the chosen control variable will not work and it is likely that the estimate of J(q) is too inaccurate. In such circumstances or when the residuals do not exhibit any convenient distribution, then some variance reduction may be achieved by simply setting $q = \hat{p}$ and $\beta = [\bar{J}(\hat{p}) - \hat{p}^2] / [\hat{p}(1-\hat{p})]$ where $\bar{J}(\hat{p})$ is the sample value of the probability that both Y and X are not greater than their respective critical values. The value of $\hat{p}_c$ can then be obtained as in step 3. It should be noted that the procedure is dependent on the chosen control variable, so other control variables can be tried if desired. Some results obtained with this algorithm are presented in V.4.2.

V.4.1.3. Combination of antithetic and control variate techniques.

In the calculation of a percentile there is also a possibility of combining the antithetic and control variate techniques.

Let $y_i$ and $y_i^a$, $x_i$ and $x_i^a$ be pairs of antithetic values obtained respectively for the variable of interest, Y and a control variable, X with known distribution and correlated with Y. The estimator $\hat{p}_c$ of p defined in V.4.1.2 can be calculated with the direct and antithetic values. Therefore, let

$$\hat{p}_c = \sum_{i=1}^{n} \Delta_i/n + \beta q$$

with

$$\Delta_i = \delta_K(y_i) - \beta\delta_Q(x_i) \qquad \text{and} \qquad q = \text{Prob}(X \leq Q)$$

and

$$\hat{p}_c{}^a = \sum_{i=1}^{n} \Delta_i{}^a/n + \beta$$

with

$$\Delta_i{}^a = \delta_K(y_i{}^a) - \beta\delta_Q(x_i{}^a)$$

Then, the estimator $\hat{\hat{p}}_c$ obtained combining the two techniques is given by

$$\hat{\hat{p}}_c = (\hat{p}_c + \hat{p}_c{}^a)/2$$

The variance of $\hat{\hat{p}}_c$, similarly to that of $\hat{\hat{p}}$, has to be estimated empirically.

### V.4.2. Some numerical results

The methods described and developed in V.4.1 were applied to the same problem used in V.3.2 to calculate a percentile. The variable of interest was the NPV of a group of fifteen capital projects associated with the firm's ongoing activities. As in the case of the control variables used for variance reduction in estimating the mean, to construct control variables for variance reduction in estimating a percentile, one is interested in finding a variable which is well correlated with the NPV being estimated, and for which there exists a known mean and variance. Several control variables can be constructed. Varible V, defined in V.3.1.3, was found to work well for reducing variance on estimates of a percentile.

The control variate technique gives an estimated variance for each estimated value of $p_c$. This is a great advantage when compared with the antithetic technique for which the variance of the estimator of p has to be calculated empirically. Table 5.3 gives the results of twenty batches of 125 evaluations of $\hat{\hat{p}}$ using the antithetic technique (column 2) and of $\hat{\hat{p}}_c$ using the method of V.4.1.3 (column 3). This corresponds to 250 evaluations of the adequate indicator functions.

Table 5.3. Values of a percentile obtained for K = 591.05

| 1<br>Batch number | 2<br>$\hat{\hat{p}}$ | 3<br>$\hat{\hat{p}}_c$ | 4<br>$\hat{p}_c$ | 5<br>$var(\hat{p}_c).10^{-3}$ | 6<br>$\hat{p}$ | 7<br>$var(\hat{p}).10^{-3}$ |
|---|---|---|---|---|---|---|
| 1 | .524 | .5217 | .5341 | .22078 | .532 | 0.9999 |
| 2 | .532 | .5536 | .5282 | .32645 | .492 | 1.0038 |
| 3 | .528 | .5304 | .5161 | .20105 | .520 | 1.0024 |
| 4 | .524 | .5465 | .5157 | .30480 | .512 | 1.0034 |
| 5 | .520 | .5029 | .4828 | .32768 | .516 | 1.0030 |
| 6 | .528 | .5161 | .5416 | .15833 | .560 | 0.9896 |
| 7 | .524 | .5170 | .5206 | .26531 | .516 | 1.0030 |
| 8 | .512 | .5148 | .5146 | .32285 | .512 | 1.0034 |
| 9 | .512 | .5082 | .5183 | .24774 | .508 | 1.0038 |
| 10 | .524 | .5235 | .5305 | .22649 | .456 | 0.9962 |
| 11 | .516 | .5190 | .5209 | .23318 | .540 | 0.9976 |
| 12 | .512 | .5234 | .5117 | .25449 | .488 | 1.0034 |
| 13 | .524 | .5302 | .5429 | .19174 | .548 | 0.9948 |
| 14 | .524 | .4994 | .5235 | .15421 | .548 | 0.9948 |
| 15 | .520 | .5120 | .5230 | .15653 | .560 | 0.9896 |
| 16 | .528 | .5030 | .5101 | .30480 | .468 | 0.9999 |
| 17 | .520 | .5147 | .5313 | .23781 | .556 | 0.9914 |
| 18 | .520 | .5133 | .5139 | .29433 | .552 | 0.9932 |
| 19 | .512 | .5332 | .5232 | .25862 | .468 | 0.9999 |
| 20 | .520 | .5221 | .5556 | .14297 | .524 | 1.0017 |

Column 4 gives $\hat{p}_c$, estimator of p using the control variate technique, obtained with 250 evaluations and column 5 gives $var(\hat{p}_c)$ for each batch. Columns 6 and 7 give $\hat{p}$ and $var(\hat{p})$ obtained also with 250 evaluations; $\hat{p}$ is the usual estimator of p without VRT indicated in IV.3.2. The standard deviation of $\hat{\hat{p}}$, $\hat{\hat{p}}_c$, $\hat{p}_c$ and $\hat{p}$ estimated from the twenty batches are $5.96.10^{-3}$, $1.37.10^{-2}$, $1.50.10^{-2}$ and $3.18.10^{-2}$, respectively.

These results give the clear advantage of the antithetic method, with a smaller dispersion of the values of p estimated by that method. However, as the variance needs to be obtained empirically, that method is much less convenient than the control variate technique which for each estimated value of p gives an estimated value for its variance. The method chosen therefore, to estimate a percentile is the control variate technique

That method was applied to three critical values of NPV chosen so that the corresponding percentiles would be around 10%, 50% and 90%. The calculations were done using 20 batches of 250 evaluations. Appendix 5.4 gives the results obtained in detail. Table 5.4 presents the percentage improvement on the standard deviation calculated without VRT, SD1 and with the control variate technique, SD2. In each column the value presented is 100(SD1 - SD2)/SD1.

The procedure used to estimate $\hat{p}_c$ and its variance permits the study of the robustness of the method. A confidence interval at an $\alpha_0$ level can be constructed for the value of the percentile obtained in each batch and using a method similar to the one used in V.3.2 some conclusions can be taken about the coverage at that level.

Table 5.4. Percentage improvement on the standard deviation of the estimated value of p for critical value K.

| Batch number | K=-150 | K=585 | K=1150 |
|---|---|---|---|
| 1 | 63.5 | 53.8 | 43.6 |
| 2 | 49.3 | 49.5 | 57.7 |
| 3 | 44.2 | 55.8 | 61.4 |
| 4 | 48.6 | 50.2 | 26.8 |
| 5 | 47.2 | 53.0 | 38.5 |
| 6 | 43.8 | 67.1 | 52.3 |
| 7 | 45.5 | 63.1 | 33.8 |
| 8 | 77.0 | 55.1 | 61.8 |
| 9 | 53.6 | 54.0 | 43.0 |
| 10 | 26.2 | 57.5 | 52.2 |
| 11 | 28.0 | 51.4 | 70.9 |
| 12 | 50.4 | 60.5 | 60.0 |
| 13 | 54.6 | 58.9 | 58.9 |
| 14 | 45.6 | 61.2 | 56.0 |
| 15 | 60.0 | 60.4 | 58.0 |
| 16 | 36.6 | 47.6 | 63.8 |
| 17 | 39.5 | 57.2 | 62.2 |
| 18 | 62.9 | 59.3 | 52.7 |
| 19 | 53.1 | 47.7 | 68.6 |
| 20 | 69.7 | 69.2 | 48.3 |

In table 5.5 $a_i = \hat{\mu}_i - c\,\hat{\sigma}_i$, $b_i = \hat{\mu}_i + c\,\hat{\sigma}_i$, $\hat{\mu}_i$ and $\hat{\sigma}_i$ being respectively the expected value and the standard deviation estimated for $\hat{p}_c$ in batch i using the control variate technique as developed in V.4.1.2. The value of c is the 90% point of the t-distribution with 249 degrees of freedom.

The method works well for the three types of percentiles as can be seen from table 5.5.

Table 5.5. Confidence intervals at a 90% level obtained in the calculation of the percentiles.

| Batch number | K=150 $a_1$ | K=150 $b_1$ | K=585 $a_1$ | K=585 $b_1$ | K=1150 $a_1$ | K=1150 $b_1$ |
|---|---|---|---|---|---|---|
| 1 | .0874 | .1150 | .5068 | .5641 | .9057 | .9400 |
| 2 | .0719 | .1091 | .5066 | .5693 | .8923 | .9220 |
| 3 | .0716 | .1064 | .4885 | .5434 | .9008 | .9301 |
| 4 | .0691 | .1106 | .4879 | .5497 | .9223 | .9612 |
| 5 | .0884 | .1270 | .4614 | .5197 | .8992 | .9377 |
| 6 | .0876 | .1287 | .5187 | .5593 | .9065 | .9408 |
| 7 | .0724 | .1100 | .5037 | .5495 | .9173 | .9551 |
| 8 | .0760 | .0925 | .4828 | .5385 | .8979 | .9242 |
| 9 | .0772 | .1117 | .4752 | .5323 | .9071 | .9428 |
| 10 | .0838 | .1324 | .4914 | .5440 | .8847 | .9239 |
| 11 | .0832 | .1338 | .4797 | .5400 | .9040 | .9268 |
| 12 | .0778 | .1126 | .5069 | .5559 | .8961 | .9264 |
| 13 | .0760 | .1036 | .5097 | .5605 | .9017 | .9301 |
| 14 | .0827 | .1240 | .4937 | .5417 | .8970 | .9280 |
| 15 | .0737 | .1051 | .5077 | .5566 | .8996 | .9285 |
| 16 | .0710 | .1060 | .4704 | .5354 | .9048 | .9286 |
| 17 | .0803 | .1237 | .4992 | .5520 | .8945 | .9211 |
| 18 | .0896 | .1178 | .4957 | .5461 | .8948 | .9259 |
| 19 | .0884 | .1240 | .4855 | .5502 | .9009 | .9234 |
| 20 | .0820 | .1054 | .5328 | .5710 | .9058 | .9398 |

# CHAPTER VI. SELECTING CAPITAL INVESTMENT PROJECTS IN A DETERMINISTIC WORLD AND IN A STOCHASTIC WORLD.

In chapters II, III and IV, several models which can be used in capital investment appraisal and some methods for calculating the required values were discussed. A model was also introduced for generating interdependent cash flows (sections III.3 and IV.5), and efficient methods to calculate the expected NPV and a percentile were proposed (sections V.3 and V.4).

The objective of this chapter is now to study the effect of taxation, and the combined effect of taxation and uncertainty, on a portfolio of capital investments. In particular to investigate possible alterations in the ranking of several portfolios of projects due to these effects.

The investment problem used as a basis for the experimentation was presented in section IV.5. In section VI.1 some groups of projects are selected under two tax systems through the use of the mixed integer programming model (MILP model) proposed by Berry and Dyson (1979), which was given in chapter II.

In section VI.2 uncertainty is introduced and applied to the portfolios of previously generated projects. Its effect is investigated in terms of the change in the expected net present value associated with each portfolio considered, and in terms of a percentile of the distribution of the NPV. Some conclusions are presented in section VI.4.

## VI.1. SELECTION OF PROJECTS WITH A DETERMINISTIC MODEL UNDER TWO SIMPLIFIED TAX SYSTEMS.

To study the effect of taxation, two tax systems are considered. Both of them are simplifications although they retain the key features of the U.K. tax system: one pre-1984, the other one post-1986. The first tax system, hereafter designated by 'old tax system', is the same as that used by Berry and Dyson (1979); it includes a corporate tax rate of 52%, a 100% capital allowance on investment and it considers a zero time lag between tax becoming payable and the date of payment. The second tax system, hereafter designated by 'new tax system', includes a corporate tax rate of 35%, a 25% writing down capital allowance and considers a zero time lag between tax becoming payable and date of payment.

Five groups of projects were generated with the objective of taking into account both tax systems and of obtaining different behaviours when uncertainty is introduced.

Group 1 came as a result of using the old tax system and the

MILP model given by equations 2.3.a to 2.3.d. The coefficients for the model were either directly available from table 4.5 or easily calculated from the information from the table. An interest rate of 6% (risk and inflation free) was used to calculate the pre-tax discounted net present value of project j, $NPV_j$, and to obtain the discount factor $d_i$, $i=1,\ldots,21$ relevant to year i, which after being multiplied by t=52% form the coefficients of variables $z_i$, the total taxable income in year i, after allowances.

A similar type of MILP model for the new tax system, with the same interest rate, was used to obtain group 2. In this case the coefficients of the variables $x_i$ in the constraints have to be calculated taking into account the 25% writing down capital allowances. Some of the coefficients of the objective function are also changed because the tax rate is now 35%.

Group 3 is formed by a certain number of individually evaluated projects with positive NPV. The NPV is calculated with an interest rate of 6% and under the new tax system. Two sets of projects were considered within this group: set 1 with the 15 projects with the highest NPV's; set 2 , with all the projects with positive NPV, 21 in number.

The remaining two groups were generated with the idea of, somehow, anticipating the effect of uncertainty on the NPV. Group 4 is the result of considering a different way of using the allowances. On the MILP model that generated group 2, an upper bound on the variables $u_i$ was introduced to cause a smoothing effect on the unused allowances. Variable $u_i$ represents the total unrelieved balance of capital allowances up to and including year i. It was hoped to control, to some extent, the variability of the NPV by imposing a narrower range of values for the $u_i$. Another approach was used on group 5 where the projects chosen are such that the probability of the group's NPV being greater than a certain target value, $p_0$, is maximised.

Let M be the total expected NPV and $S^2$ the total variance of the chosen projects. Then,

$$M = \sum_{j=1}^{30} p_j x_j \qquad \text{and} \qquad S^2 = \sum_{i=1}^{30} \sum_{j=1}^{30} c_{ij} x_i x_j$$

where $p_j$ is the expected NPV of project j;

$c_{ij}$, $i \neq j$ is the covariance between the net present value of projects i and j;

$c_{ii}$ is the variance of the net present value of project i;

$x_j = 1$ if project j is chosen, $= 0$ otherwise.

Supposing that $NPV_{group5} \sim N(M, S^2)$ the problem can take the form:

to maximise

$$\text{Prob}\,(\,NPV_{group5} \geq p_0\,) = 1 - \Phi(\,(p_0 - M)\,/\,S\,) \qquad (6.1.a)$$

subject to

$$\sum_{j=1}^{30} p_j\, x_j = M$$

$$\sum_{i=1}^{30} \sum_{j=1}^{30} c_{ij}\, x_i\, x_j = S^2 \qquad (6.1.b)$$

$$x_i \in \{0,1\}\ , \quad i=1,\ldots,30$$

where $\Phi$ is the standard normal distribution function. As $\Phi$ is a monotonic increasing function this problem is equivalent to

$$\text{minimise} \quad (\,p_0 - M\,)\,/\,S \qquad (6.2)$$

subject to the constraints (6.1.b).

As detailed in appendix 6.1, problem 6.2 can be transformed into a mixed integer linear programming problem the solution of which gave the set of projects forming group 5. This transformation was developed in close association with Dr. Robert Ashford. The MILP models used to obtain groups 1, 2, 4 and 5 were solved with SCICONIC in an IBM 4381 computer. Table 6.2 presents the five groups of projects forming the portfolio of investment projects, chosen in the way previously described.

Table 6.1. Groups of projects used to study the effect of taxation and uncertainty.

| Project number | Group 1 | Group 2 | Group 3 Set 1 | Group 3 Set 2 | Group 4 | Group 5 |
|---|---|---|---|---|---|---|
| 1 | √ | √ | √ | √ | √ | √ |
| 2 | √ | √ | √ | √ | √ | √ |
| 3 | √ | √ | √ | √ | √ | √ |
| 4 | √ | √ | √ | √ | √ | √ |
| 5 | √ | √ | √ | √ | √ | √ |
| 6 | √ | √ | √ | √ | √ | √ |
| 7 | √ | √ | √ | √ | √ | √ |
| 8 | √ | √ | √ | √ | √ | √ |
| 9 | √ | √ | √ | √ | √ | √ |
| 10 | √ | √ | √ | √ | √ | √ |
| 11 | | | | √ | | √ |
| 12 | | √ | | √ | √ | √ |
| 13 | √ | √ | √ | √ | √ | √ |
| 14 | √ | √ | √ | √ | √ | √ |
| 15 | √ | √ | √ | √ | √ | √ |
| 16 | √ | √ | √ | √ | √ | √ |
| 17 | √ | | | | √ | |
| 18 | | | | | | |
| 19 | | | | √ | | √ |
| 20 | | | | | √ | |
| 21 | √ | √ | | √ | √ | |
| 22 | | | | √ | | √ |
| 23 | √ | √ | √ | √ | √ | √ |
| 24 | √ | | | √ | | |
| 25 | √ | | | | √ | √ |
| 26 | | | | | | |
| 27 | √ | | | | √ | |
| 28 | | | | | | |
| 29 | | | | | | |
| 30 | | | | | | |
| no of projects | 20 | 17 | 15 | 21 | 20 | 20 |

The groups were obtained as follows:

Group 1 _ MILP model for the old tax system;

Group 2 _ MILP model for the new tax system;

Group 3 _ Projects with the highest NPV values:

set 1 _ the best 15 projects;

set 2 _ the 21 projects with positive NPV;

Group 4 _ MILP model for the new tax system with an upper bound on the unused capital allowance;

Group 5 _ Maximising the probability that the total NPV exceed a target value $p_0$=910.

## VI.2. ANALYSIS OF THE PERFORMANCE OF SEVERAL SETS OF PROJECTS IN A DETERMINISTIC WORLD.

Now the portfolios presented in table 6.1 are compared under three different tax situations: the old tax, the new tax and no tax systems. The NPV values for these three situations and for all groups and sets studied are presented in table 6.2. The calculations were done on an IBM machine. The NPV values were calculated with an interest rate of 6%. There is ample evidence in table 6.2 that tax considerably reduces the NPV value. The reduction is much higher with the old tax system which has a tax rate of 52%, than with the new tax system where the tax rate is only 35%. This reduction is of about 60% in the old tax system and of about 45% in the new tax system. The combined effect of spreading in time of the cash flows and of tax leads to a percentage decrease in the NPV value which is higher than the tax rate. The very different way of treating capital allowances is one factor influencing the amount reduced which favours the old tax system. The big decrease however, in corporate tax rate is no doubt the main cause of the considerable difference between the NPV values.

Table 6.2. Summary of results for the deterministic values of the NPV calculated with an interest rate of 6%.

| | Old tax system | New tax system | No tax |
|---|---|---|---|
| Group 1 | 769.558 (1) | 1027.42 (2) | 1893.08 (4) |
| Group 2 | 743.011 (3) | 1037.14 (1) | 1911.61 (3) |
| Group 3: Set 1 | 730.448 (4) | 1027.02 (3) | 1891.11 (5) |
| Set 2 | 717.684 (5) | 1024.12 (4) | 1928.35 (1) |
| Group 4 | 754.592 (2) | 1019.08 (5) | 1880.24 (6) |
| Group 5 | 705.542 (6) | 1015.53 (6) | 1917.75 (2) |

Inside the parenthesis is the ranking of the set in terms of the NPV.

The groups were obtained as follows:

Group 1 _ MILP model for the old tax system;

Group 2 _ MILP model for the new tax system;

Group 3 _ Projects with the highest NPV values:

set 1 _ the best 15 projects;

set 2 _ the 21 projects with positive NPV;

Group 4 _ MILP model for the new tax system with an upper bound on the unused capital allowance;

Group 5 _ Maximising the probability that the total NPV exceed a target value $p_0=910$.

Group 1 is the best performer under the old tax system and group 2 is the best performer under the new tax system, as one would expect because of the way they are generated. Thus the benefit of incorporating the tax rules into the selection process can be seen. It is however less important under the new tax system than under the old one. Ignoring tax in the evaluation, and selecting all projects with positive NPV leads to group 3, set 2. Under the old tax system group1 is 7% better and under the new tax system group 2 is 1.3% better. In the next section the behaviour of these groups is investigated when uncertainty is introduced.

## VI.3. ANALYSIS OF THE PERFORMANCE OF SEVERAL SETS OF PROJECTS IN A STOCHASTIC WORLD.

Uncertainty is introduced through the use of model (3.37) presented in chapter III. The model is applied to generate the cash flow $P_{it}$ of project i in time period t in the way described in section IV.3.1. The sets in each group are going to be compared in terms of their expected NPV and in terms of a percentile taken as a measure of risk. To calculate these statistics the control variable technique of variance reduction was used in the way detailed in chapter V.

Table 6.3 presents the results for the expected values of the NPV obtained from 250 evaluations of this summary measure with the stochastic cash flows. The interest rate used was 6%, as in the deterministic situation. The three tax situations previously considered, the old tax, the new tax and no tax systems, were also examined. Again, as one would expect, the introduction of taxation greatly reduces the NPV value, and much more so with the old tax system than with the new tax system. Comparing the results of tables 6.2 and 6.3 shows that there is a clear decrease in the NPV values under the tax regimes, but no change in the no tax situation. The values of the groups of projects are systematically lower under uncertainty and taxation so that any values obtained deterministically are biased. This may occur because the system of tax allowances ensures that the high revenues incur a proportionaly greater tax liability than low ones and also because the benefits of the tax system cannot be so accurately exploited, when stochastic values are present. It is also an important result given that in practice deterministic evaluation is common. The reduction due to uncertainty is 18% for group 1 under the old tax system, and 6.7% for group 2 under the new tax system.

Uncertainty also changes the rankings of the groups of projects. Group 1 is now only ranked third under the old tax system despite that system being used in its generation. Under the new tax system group 2 (generated under that system) is edged into second place by Group 3 set 1. Group 3 set 1 is now

Table 6.3. Summary of results for the stochastic values of the NPV calculated with an interest rate of 6%.

| | Old tax system | New tax system | No tax |
|---|---|---|---|
| Group 1 | 630.852 (3) | 928.034 (5) | 1893.34 (4) |
| Group 2 | 652.327 (2) | 967.706 (2) | 1912.58 (3) |
| Group 3: Set 1 | 655.509 (1) | 968.514 (1) | 1892.05 (5) |
| Set 2 | 619.510 (4) | 948.493 (6) | 1928.36 (1) |
| Group 4 | 612.045 (5) | 915.462 (6) | 1880.15 (6) |
| Group 5 | 611.417 (6) | 943.065 (4) | 1917.95 (2) |

Inside the parenthesis is the ranking of the set in terms of the NPV.

The groups were obtained as follows:

Group 1 _ MILP model for the old tax system;

Group 2 _ MILP model for the new tax system;

Group 3 _ Projects with the highest NPV values:

set 1 _ the best 15 projects;

set 2 _ the 21 projects with positive NPV;

Group 4 _ MILP model for the new tax system with an upper bound on the unused capital allowance;

Group 5 _ Maximising the probability that the total NPV exceed a target value $p_0=910$.

ranked 1 under both tax systems despite the fact that the selection criterion (best 15 projects) is somewhat arbitrary.

Group 3 set 2 (all projects with a positive NPV) does not perform well under tax and uncertainty. The findings on rankings are thus inconclusive but the preeminence of selection by deterministic mathematical programming incorporating taxation is lost when the projects are placed in an uncertain world.

To assess the riskiness of a set of projects it can be important to know the probability of falling below a certain fixed value, K, which represents a threshold of financial trouble for the firm, in real situations. In the situation of this research, a value $K_\alpha$ for K was fixed so that there was an $\alpha$% probability of the best set of projects in each of the three tax situations being less than $K_\alpha$. The values of $\alpha$ were taken to be 1, 5, 10, 15 and 20 and the $K_\alpha$ values, given in table 6.4, were estimated from normal distributions with parameters estimated during the expected values calculations.

The percentiles obtained are given in table 6.5. The best performers within both tax systems, in terms of the means, continue to be the best performers in terms of the percentiles: group 3 - set 1 and group 2. Group 5, which was constructed controlling the variability in some way, behaved well with the new tax system but less well with the old tax system. The other group for which it was hoped to control the variability, group 4, behaved rather badly. In the no tax situation there is a change in the ranking which is similar when considering the percentiles compared to the deterministic and stochastic mean results.

Table 6.4. $K_\alpha$ values used in the percentile calculations.

| $\alpha$ | Old tax system | New tax system | No tax |
|---|---|---|---|
| 1% | -1247.973 | -1362.765 | -1499.631 |
| 5% | -707.139 | -701.322 | -533.140 |
| 10% | -418.784 | -348.662 | -17.839 |
| 15% | -224.167 | -110.644 | 329.951 |
| 20% | -69.552 | 78.451 | 606.253 |

Table 6.5. Summary of results for the percentile values .

| | Old tax system | New tax system | No tax |
|---|---|---|---|
| Group 1 | .0414 (5) | .0336 (5) | .0157 (3) |
| Group 2 | .0302 (2) | .0303 (2) | .0088 (1) |
| Group 3: set 1 | .0087 (1) | .0088 (1) | .0088 (1) |
| set 2 | .0316 (4) | .0313 (4) | .0161 (5) |
| Group 4 | .0503 (6) | .0384 (6) | .0231 (6) |
| Group 5 | .0309 (3) | .0304 (3) | .0157 (3) |
| | | | |
| Group 1 | .1035 (6) | .1046 (5) | .0605 (4) |
| Group 2 | .0698 (2) | .0643 (2) | .0483 (2) |
| Group 3: set 1 | .0551 (1) | 0514 (1) | .0308 (1) |
| set 2 | .0958 (4) | 0922 (4) | .0604 (3) |
| Group 4 | .1051 (5) | .1056 (6) | .0724 (6) |
| Group 5 | .0926 (3) | .0829 (3) | .0618 (5) |
| | | | |
| Group 1 | .1407 (5) | .1437 (5) | .1152 (4) |
| Group 2 | .1141 (2) | 1063 (2) | .0854 (2) |
| Group 3: set 1 | .0995 (1) | .0961 (1) | .0743 (1) |
| set 2 | .1348 (3) | 1291 (4) | .1210 (6) |
| Group 4 | .1489 (6) | .1659 (6) | .1142 (3) |
| Group 5 | 1378 (4) | .1238 (3) | .1161 (5) |
| | | | |
| Group 1 | .1717 (3) | .1772 (3) | .1718 (4) |
| Group 2 | .1540 (2) | .1603 (2) | .1213 (2) |
| Group 3: set 1 | .1230 (1) | .1233 (1) | .1148 (1) |
| set 2 | .1866 (5) | .1898 (6) | .1663 (3) |
| Group 4 | .1849 (4) | .1846 (4) | .1872 (6) |
| Group 5 | .1900 (6) | 1889 (5) | 1758 (5) |
| | | | |
| Group 1 | .2115 (3) | .2418 (5) | .2061 (3) |
| Group 2 | .1835 (2) | .1809 (2) | 1837 (2) |
| Group 3: set 1 | .1587 (1) | .1671 (1) | .1710 (1) |
| set 2 | .2234 (5) | .2273 (4) | 2205 (6) |
| Group 4 | .2309 (6) | .2509 (6) | 2167 (4) |
| Group 5 | .2198 (4) | .2267 (3) | 2174 (5) |

Note: The ranking of the set in terms of the percentiles is inside the parenthesis

## VI.4. CONCLUSIONS.

It has long been recognised that taxation can create interdependencies among capital investment projects and, for some time, mathematical programming has been suggested as an adequate tool to help with the decision of obtaining the best group of projects in a deterministic context.

The UK 1984 Finance Act has radically changed the way of treating the capital allowances for plant and machinery. The two simplified tax systems considered in the study contain the main characteristics relating to the use of capital allowances. The application of both simplified tax systems to five groups of projects revealed that the NPV values obtained with the new tax system were consistently much better than with the old tax system.

One of the difficulties of the problem of choosing a portfolio of capital investments projects is that a decision has to be taken now for activities that are developed along some future period of time and their outcomes are uncertain. By investigating the effect of introducing uncertainty in the selected portfolios, it became evident that under the tax regimes uncertainty significantly reduces the value of the expected NPV which means that deterministic valuations are biased. This reduction is greater with the old tax system than with the new tax system but is significant. This is an important result for the practice of capital investment appraisal. Taxation and uncertainty have the important effect of considerably diminishing the expected NPV, and they also seem to alter the ordering of the groups considered. Under taxation alone the benefit of a mathematical programming selection procedure are evident. When uncertainty is introduced however this benefit becomes questionable.

Taking a percentile as a measure of risk , the new tax system and the old tax system in the cases considered seem to perform in a rather similar way. The no tax situation seems to behave a little better in terms of such a measure of risk. Overall, it can be concluded that, with the uncertainty introduced with the proposed model, the new tax system generates higher NPV with no higher risk than the old tax system.

# CHAPTER VII. CONCLUSIONS.

Capital investment appraisal was here considered as the financial evaluation of projects involving capital investments. Financial costs and benefits were expressed through cash flows and the present value approach was used to obtain a summary measure of a project's cash flows.

Resource limitations are obvious causes of interdependency among projects but other not so evident sources, such as taxation, can also create interdependency among otherwise independent projects and between a firm's ongoing activities and a project.

The amount of risk is often an important issue in the evaluation of proposed capital investments. Risk is the result of several types of uncertainty that affect investment projects. Different type of information about firms and different degree of facility for diversification are also factors leading managers and individual investors to look at risk from different points of view. It was the management perspective that was considered here.

Dealing simultaneously with the problems of interdependency and risk associated with proposed projects of a firm is a difficult problem. On assessing a group of proposed projects competing with the ongoing activities of the firm for limited resources, the question is how to choose which projects to implement taking into account the interdependency and risk associated with them.

A comprehensive review was made, in previous chapters, of methods using different philosophies and techniques to deal with the problem of project selection. A common way of tackling this problem, which may not give the best answers, is to consider interdependency and uncertainty one at a time. This research has explored different methodologies for selecting and evaluating capital projects, giving special emphasis to the effect of both taxation and uncertainty on groups of projects.

Selection of projects.

The basic data from which five groups of projects were chosen was that previously used by Weingartner. The basic idea to create these groups was that they should be different enough to behave in a distinct way when considered in a deterministic and in a stochastic environment. Different approaches were then considered to generate these groups. A mixed integer linear programming model where taxation is the only source of interdependency was used to generate two groups of projects. Two simplified tax systems, the old one and the new one, retaining two of the main characteristics (tax rate and allowances) of the UK tax system, both pre and post 1984, were respectively used to obtain groups 1 and 2. Choosing projects, individually evaluated, with positive NPV and ranking them by their NPV is a frequent selection procedure. Group 3 displays this method of selection: set 1 contains a certain number of projects with the highest NPVs and set 2 contains all the projects with positive NPV. The remaining two groups were generated with the idea of, somehow, anticipating the effect of uncertainty on the NPV. Two different procedures were applied. To obtain group 4 a mixed integer linear programming model was used where the restrictions were not only the taxation system (the simplified post-1984 system) but also an upper bound on the unused allowances to cause a more equitable use of the reductions through time. Another procedure was used to obtain group 5: the projects were chosen such that the probability of the group's NPV being greater than a certain target value is maximised. This was the only situation where uncertainty was introduced directly in the selection model.

The selected groups were then studied under three different tax regimes (the two tax systems and no tax situation), in both a deterministic and a stochastic environment.

Evaluation of groups.

All the selected groups were evaluated under deterministic conditions. For each group of projects cash flows were generated through a basic model, constructed for this research, which allows for some interdependency between cash flows of the same project in different time periods and between the cash flows of the projects implemented in a certain time period. The NPV for all groups under the three tax situations were then calculated.

The evaluation of all groups was also done under stochastic conditions. Simulation was the technique chosen to study the groups of projects in a stochastic environment. Uncertainty was introduced in the generated cash flows through the use of random variables, with known probability distributions, in the basic model. The NPV of each group of projects is now a random variable for which a sample of values can be obtained.

As stochastic simulation is a technique of performing sampling experiments with a model, the analysis of the results generated is done by the usual statistical procedures. The accuracy of a statistic, measured by the standard deviation of the mean of the estimator, may be increased by using a variance reduction technique. Some VRTs were applied in the simulation of a group of capital investment projects giving improved results in the calculation of the expected values of NPV. It was found that the control variable technique was a robust and efficient method of estimating the mean NPV.

The VRTs applied to the estimation of the mean, though not very common in the context of capital investment problems, are known techniques used in other simulation situations. The extension of the control variable technique to the estimation of percentiles is a new development in the use of this VRT.

Experimentation.

The calculations were done with the different groups of selected projects under the three different tax situations, in both deterministic and stochastic conditions. The main conclusions which can be taken from the results obtained are the following:

— the NPVs achieved with the new tax system were consistently much better than with the old tax system;

— the expected NPV of any group of projects is systematically and significantly lower than its deterministic valuation; this is due to the combined effect of uncertainty and taxation so that any values obtained deterministically are biased;

— deterministic optimal methods of selection seem less attractive under uncertain as opposed to a deterministic world.

Future developments.

A few more questions directly related to the study can be thought of as points for further research. One of them is to investigate the effect of different values of the constants of the uncertainty model.

If uncertainty is introduced by a different model would the conclusions related to the ordering of the sets still be the same ? In other words, how sensitive are those conclusions to the way uncertainty is modelled ?

Although inflation is certainly not as important in the late eighties as it was ten years ago, it would perhaps be interesting to investigate its effect over the performance of the portfolios

Another open question is the adequacy of simpler selection procedures when compared with a more sophisticated stochastic programming approach.

Further work in the area beyond that of the thesis seems of limited appeal if it is done only on a theoretical basis. At this point it is important to have some data from real investment situations to help determine what future paths are worth exploring.

APPENDIX 5.1. Obtaining the expression of control variable V.

Considering the stochastic model for generating the cash flows and using the same notation, the variable V is defined with the following expression:

$$V = \sum_{i=1}^{N} \sum_{t=1}^{T} \gamma_t X_{it} + \sum_{i=1}^{N} \sum_{t=1}^{T} \gamma_t b_{it} S_t$$

where $X_{it}$ are independent normal variables with parameters $\mu_{it}$ and $\sigma_{it}^2$, $\gamma_t$ are the discount factors, $S_t$ are variables representing the systematic effect of the model and are such that

$$S_1 = E_1$$

$$S_t = c_t S_{t-1} + E_t, \quad t=1,2,\ldots,T$$

where $E_t$ are independent normal variables with parameters 0 and $\sigma_t^2$. The coefficients $b_{it}$ are the same as in the model. Each variable $S_k$ can be expressed in terms of $E_t$, $t=1,2,\ldots,k$ in this way:

$$S_2 = c_2 S_1 + E_2 = c_2 E_1 + E_2$$

$$S_3 = c_3 S_2 + E_3 = c_3 c_2 E_1 + c_3 E_2 + E_3$$

$$\vdots$$

$$S_k = c_k S_{k-1} + E_k = c_k c_{k-1} \cdots c_2 E_1 + c_k c_{k-1} \cdots c_3 E_2 + \ldots + c_k E_{k-1} + E_k$$

Also,

$$\sum_{i=1}^{N}\sum_{t=1}^{T} \gamma_t b_{it} S_t = \sum_{t=1}^{T} \gamma_t \left( \sum_{i=1}^{N} b_{it} \right) S_t = \sum_{t=1}^{T} \gamma_t b_t S_t$$

with $\quad b_t = \sum_{i=1}^{N} b_{it}.$

Then,

$$\sum_{t=1}^{T} \gamma_t b_t S_t = \gamma_1 b_1 E_1 + \gamma_2 b_2 (c_2 E_1 + E_2) + \gamma_3 b_3 (c_3 c_2 E_1 + c_3 E_2 + E_3) +$$

$$+ \ldots + \gamma_T b_T (c_T c_{T-1} \ldots c_2 E_1 + c_T c_{T-1} \ldots c_3 E_2 + \ldots + c_T E_{T-1} + E_T)$$

$$= (\gamma_1 b_1 + \gamma_2 b_2 c_2 + \gamma_3 b_3 c_3 c_2 + \ldots + \gamma_T b_T c_T c_{T-1} \ldots c_2) E_1$$

$$+ (\gamma_2 b_2 + \gamma_3 b_3 c_3 + \ldots + \gamma_T b_T c_T c_{T-1} \ldots c_3) E_2$$

$$+ \ldots + (\gamma_T b_T + \gamma_{T-1} b_{T-1} c_T) E_{T-1} + \gamma_T b_T E_T$$

hence,

$$\sum_{i=1}^{N}\sum_{t=1}^{T} \gamma_t b_{it} S_t = \sum_{t=1}^{T} d_t E_t$$

where

$$d_t = \gamma_t b_t + \gamma_{t+1} b_{t+1} c_t c_{t+1} + \ldots + \gamma_T b_T c_t c_{t+1} \ldots c_{T-1} c_T$$

APPENDIX 5.2. Obtaining a lower bound for the standard normal density function.

A lower bound for the standard normal density function can be obtained through the study of maxima and minima of a function F(x) defined by the following expression:

$$F(x) = \sqrt{2\pi}.\Phi'(x) - 4\Phi(x)[1 - \Phi(x)]$$

where $\Phi(x)$ is the standard normal distribution function.

By definition

$$\Phi(x) = \int_{-\infty}^{x} \exp(-t^2/2) / \sqrt{2\pi}.dt$$

and

$$\Phi'(x) = \exp(-x^2/2) / \sqrt{2\pi}$$

Hence, as $\Phi'(0) = 1/\sqrt{2\pi}$ and $\Phi(0) = 0.5$, $F(0) = 0$.

The first derivative of F(x) has the following expression:

$$F'(x) = \sqrt{2\pi}\ \Phi''(x) - 4[\Phi'(x) - 2\Phi(x)\Phi'(x)]$$

with

$$\Phi''(x) = -x\exp(-x^2/2) / \sqrt{2\pi}$$

and the stationary points of F(x) are obtained as roots of

$$F'(x) = 0,$$

that is,

$$-x\exp(-x^2/2) - 4\exp(-x^2/2)/\sqrt{2\pi} +$$

$$+ 8 \left( \int_{-\infty}^{x} \exp(-t^2/2)/\sqrt{2\pi}.dt \right) . \exp(-x^2/2)/\sqrt{2\pi} = 0$$

$$\exp(-x^2/2) \, [ -x - 4/\sqrt{2\pi} + 8/\sqrt{2\pi} \int_{-\infty}^{x} \exp(-x^2/2) / \sqrt{2\pi} \; dt ] = 0$$

The equation

$$\int_{-\infty}^{x} \exp(-t^2/2) / \sqrt{2\pi} \, dt - (\sqrt{2\pi}/8).x - 0.5 = 0$$

has three real roots $x_1$ (<0), $x_2$ (=0) and $x_3$ (>0). The values of $x_1$ and $x_3$ can be calculated numerically. It is not difficult to find their values and determine the sign of $F''(x)$ to conclude that $x_1$ and $x_2$ are maxima for F(x). Also, $x_2$ is a minimum for F(x), because

$$F''(x) = \sqrt{2\pi} \, \Phi'''(x) + 4\Phi''(x) \, [2\Phi(x) - 1] + 8[\Phi'(x)]^2$$

with

$$\Phi'''(x) = \exp(-x^2/2) \, (x^2-1)/\sqrt{2\pi} \quad \text{and} \quad F''(0) > 0.$$

Then,

$$\sqrt{2\pi} \, \Phi'(x) - 4 \, \Phi(x)(1 - \Phi(x)) \geq 0$$

and

$$\Phi'(x) \geq 4/\sqrt{2\pi} . \Phi(x)(1 - \Phi(x))$$

which gives the required lower bound for the standard normal density function.

Appendix 5.3. Calculation of $\hat{J}(q)$ and $\hat{J}'(q)$.

The value of J(q) is defined through (5.16), that is :

$$J(q) = \int_{x \le F_X^{-1}(q)} F_e(K - \alpha_0 - \alpha_1 x) dF_X(x)$$

and is approximated by $\hat{J}(q)$ calculated as described below.

The values of $\alpha_0$ and $\alpha_1$ are obtained using a regression program. Let $x_i$, i=1,...,n be the sample values for the control variable X, and suppose that $t=F_X^{-1}(q)$ is such that $x_m < t < x_{m+1}$. Then, the value of $\hat{J}(q)$ can be obtained from the sample values in the following way:

$$J(q) = \int_{x \le t} F_e(K - \alpha_0 - \alpha_1 x) dF_X(x)$$

$$= \int_{-\infty}^{x_1} F_e(K - \alpha_0 - \alpha_1 x) dF_X(x) + \sum_{j=1}^{m-1} \int_{x_j}^{x_{j+1}} F_e(K - \alpha_0 - \alpha_1 x) dF_X(x) +$$

$$+ \int_{x_m}^{t} F_e(K - \alpha_0 - \alpha_1 x) dF_X(x)$$

Let $I_j$ be defined as

$$I_j = \int_{x_j}^{x_{j+1}} F_e(K - \alpha_0 - \alpha_1 x) dF_X(x) = \int_{x_j}^{x_{j+1}} F_e(K - \alpha_0 - \alpha_1 x) F'_X(x) dx \,, \qquad j=1, \ldots, m-1$$

Supposing that X follows a normal distribution with expected value $\mu_X$ and variance $\sigma_X^2$ and approximating the distribution function of the residuals, $F_e$, by a piecewise linear function, a sample value for $I_j$ is given by $\hat{I}_j$:

$$\hat{I}_j = \int_{x_j}^{x_{j+1}} (A_j + B_j).(1/\sigma_X\sqrt{2\pi}).\exp(-((x-\mu_X)/\sigma_X)^2/2)dx$$

where

$$A_j = (\hat{F}_e(y_{j+1}) - \hat{F}_e(y_j))/(x_{j+1} - x_j)$$

and

$$B_j = (x_{j+1}\hat{F}_e(y_j) - x_j\hat{F}_e(y_{j+1}))/(x_{j+1} - x_j)$$

with

$$y_j = K - \hat{\alpha}_0 - \hat{\alpha}_1 x_j$$

and $\hat{F}_e(y)$ being the sample distribution function of the residuals.

Calculating the values of the integral in the expression of $\hat{I}_j$ comes

$$\hat{I}_j = (\sigma_X/\sqrt{2\pi}).(\exp(-z_j^2/2) - \exp(-z_{j+1}^2/2) + \mu_X(\Phi(z_{j+1})-\Phi(z_j))$$

with $z_j=(x_j-\mu_X)/\sigma_X$ and where $\Phi$ is the distribution function for the standard normal variable.

Let

$$I_m = \int_{x_m}^{t} F_e(K - \alpha_0 - \alpha_1 x)dF_X(x)$$

Then $\hat{I}_m$ can be obtained in a similar way to $\hat{I}_j$, $j=1,\dots,m-1$.

Now, noting that when $\hat{\alpha}_1>0$, $\hat{F}_e(K - \hat{\alpha}_0 - \hat{\alpha}_1 x) \approx 1$,

$$\int_{-\infty}^{x_1} F_e(K - \alpha_0 - \alpha_1 x)dF_X(x) \approx \int_{-\infty}^{x_1} dF_X(x) = F_X(x_1)$$

Thus,

$$\hat{J}(q) = F_X(x_1) + \sum_{j=1}^{m} I_j$$

The value of $J'(q)$ is needed to solve equation (5.12):

$$J'(q) - \beta(0.5 - q) - p = 0$$

for q.

This is done within an iterative process where $\beta$ and p are given by approximated values $\hat{\beta}$ and $\hat{p}_c$.

By the definition of $J(q)$ it is easy to conclude that

$$J'(q) = F_e(K - \alpha_0 - \alpha_1 F_X^{-1}(q))$$

Then, a sample value for $J'(q)$ is given by

$$\widehat{J'}(q) = F_e(K - \hat{\alpha}_0 - \hat{\alpha}_1 \widehat{F_X}^{-1}(q))$$

where $\hat{F}_X$ is the sample distribution function of the control variable and the residuals are supposed to follow a normal distribution with zero expected value and variance obtained through the regression program.

Appendix 5.4. Results of the calculations with the percentiles.

| Batch no | K=-150 $\hat{p}$ | $\sigma(\hat{p})10^{-2}$ | $\hat{p}_c$ | $\sigma(\hat{p}_c)10^{-2}$ | K= 585 $\hat{p}$ | $\sigma(\hat{p})10^{-2}$ | $\hat{p}_c$ | $\sigma(\hat{p}_c)10^{-2}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | .104 | 1.934 | .101 | 0.704 | .528 | 3.164 | .536 | 1.463 |
| 2 | .096 | 1.867 | .091 | 0.947 | .488 | 3.168 | .538 | 1.600 |
| 3 | .068 | 1.595 | .089 | 0.890 | .512 | 3.168 | .516 | 1.400 |
| 4 | .120 | 2.059 | .090 | 1.059 | .508 | 3.168 | .519 | 1.578 |
| 5 | .096 | 1.867 | .108 | 0.986 | .504 | 3.168 | .491 | 1.489 |
| 6 | .096 | 1.867 | .108 | 1.048 | .560 | 3.146 | .539 | 1.036 |
| 7 | .084 | 1.758 | .091 | 0.959 | .512 | 3.168 | .527 | 1.170 |
| 8 | .092 | 1.832 | .084 | 0.421 | .504 | 3.168 | .511 | 1.421 |
| 9 | .100 | 1.901 | .094 | 0.881 | .500 | 3.169 | .504 | 1.456 |
| 10 | .076 | 1.679 | .108 | 1.240 | .456 | 3.156 | .518 | 1.340 |
| 11 | .088 | 1.795 | .108 | 1.292 | .536 | 3.160 | .510 | 1.537 |
| 12 | .088 | 1.795 | .095 | 0.890 | .476 | 3.165 | .531 | 1.250 |
| 13 | .064 | 1.551 | .090 | 0.704 | .544 | 3.156 | .535 | 1.296 |
| 14 | .104 | 1.934 | .103 | 1.053 | .544 | 3.156 | .518 | 1.225 |
| 15 | .112 | 1.999 | .089 | 0.799 | .556 | 3.149 | .532 | 1.246 |
| 16 | .052 | 1.407 | .088 | 0.892 | .464 | 3.160 | .503 | 1.656 |
| 17 | .092 | 1.832 | .102 | 1.108 | .556 | 3.149 | .526 | 1.348 |
| 18 | .104 | 1.934 | .104 | 0.718 | .536 | 3.160 | .521 | 1.286 |
| 19 | .104 | 1.934 | .106 | 0.908 | .464 | 3.160 | .518 | 1.652 |
| 20 | .108 | 1.967 | .094 | 0.597 | .512 | 3.168 | .552 | 0.976 |

| Batch no | K=1150 $\hat{p}$ | $\sigma(\hat{p})10^{-2}$ | $\hat{p}_c$ | $\sigma(\hat{p}_c)10^{-2}$ |
|---|---|---|---|---|
| 1 | .936 | 1.551 | .923 | .8753 |
| 2 | .912 | 1.795 | .907 | .7597 |
| 3 | .896 | 1.934 | .915 | .7475 |
| 4 | .952 | 1.355 | .942 | .9919 |
| 5 | .932 | 1.595 | .918 | .9818 |
| 6 | .908 | 1.832 | .924 | .8744 |
| 7 | .944 | 1.457 | .936 | .9641 |
| 8 | .916 | 1.758 | .911 | .6723 |
| 9 | .932 | 1.595 | .925 | .9090 |
| 10 | .876 | 2.089 | .904 | .9983 |
| 11 | .888 | 1.999 | .915 | .5809 |
| 12 | .896 | 1.934 | .911 | .7732 |
| 13 | .916 | 1.758 | .916 | .7223 |
| 14 | .912 | 1.795 | .912 | .7899 |
| 15 | .916 | 1.758 | .914 | .7384 |
| 16 | .924 | 1.679 | .917 | .6082 |
| 17 | .912 | 1.795 | .908 | .6789 |
| 18 | .924 | 1.679 | .910 | .7943 |
| 19 | .908 | 1.832 | .912 | .5754 |
| 20 | .924 | 1.679 | .923 | .8681 |

Appendix 6.1. The MILP model to generate group 5.

1. The model.

Consider the problem:

$$\text{minimise} \quad ( p_0 - M ) / S$$

subject to

$$\sum_{j=1}^{30} p_j x_j = M$$

$$\sum_{i=1}^{30} \sum_{j=1}^{30} c_{ij} x_i x_j = S^2 \qquad (6.1.1)$$

$$x_i \in \{ 0,1 \}, \; i=1,\dots,30$$

where

$p_0$ is a given constant;

$p_j$ is the expected net present value of project j;

$c_{ij}$, $i \neq j$ is the covariance between the net present value of project i and j;

$c_{ii}$ is the variance of the net present value of project i.

This nonlinear integer mathematical programming model can be converted into a mixed integer linear programming model in the following way. The nonlinearities of the objective function and the second constraint are going to be transformed in order to obtain separability.

Nonlinearities of the type $x_i x_j$ can be substituted by the following formulation. Let $w_{ij}=x_i x_j$, then the constraints:

$$w_{ij} - x_i \leq 0$$

$$w_{ij} - x_j \leq 0$$

$$- w_{ij} + x_i + x_j \leq 1$$

$$w_{ij} \geq 0$$

are equivalent to $x_i x_j$ with $x_i$, $x_j \in ( 0,1 )$, as can easily be checked.

Now,

$$x_i = 0 \text{ or } x_j = 0 \text{ or both } \rightarrow w_{ij}=0 \quad ( x_i x_j = 0 )$$

$$x_i = 1 \text{ and } x_j = 1 \quad \rightarrow w_{ij} = 1 \quad ( x_i x_j = 1 )$$

Also,

$$w_{ij}=0 \rightarrow x_i + x_j \leq 1 \rightarrow x_i = 0 \text{ or } x_j = 0 \text{ or both } \rightarrow x_i x_j = 0$$

and

$$w_{ij}=1 \rightarrow x_i \geq 1, x_j \geq 1 \text{ and } x_i + x_j \leq 2 \rightarrow x_i = x_j = 1 \rightarrow x_i x_j = 1$$

The other nonlinearities are going to be treated in a different way, using a suitable piecewise linear approximation. Let

$$y = ( p_0 - M ) / S .$$

Then, problem (6.1.1) can be written as:

$$\text{minimise } y$$

$$\text{subject to} \quad yS + \sum_{j=1}^{30} p_j x_j = p_0$$

$$S^2 - \sum_{i=1}^{30} \sum_{j=1}^{30} c_{ij} x_i x_j = 0 \qquad (6.1.2)$$

$$x_i \in \{ 0,1 \} , i=1,\dots,30$$

As the minimisation of y is equivalent to the minimisation of $y^2$ for $y > 0$, two cases are considered according to $p_0$ being greater than or less than M.

<u>Case 1:</u> $p_0 > M \quad ( y > 0 )$

The product yS can be written as $y^2 \sqrt{(S^2 / y^2)}$.

Define $z = S^2 / y^2$ and let $\{ z^{(k)} \}$ with $k=1,\dots, K$, be a suitable grid of values for z.

$$y^2 \sqrt{(S^2 / y^2)} = y^2 \sqrt{z} \approx \sum_{k=1}^{K} \sqrt{z^{(k)}} ( \lambda_k y^2 )$$

with

$$\sum_{k=1}^{K} \lambda_k = 1 \quad \text{and} \quad \lambda_k \geq 0$$

Also,

$$S^2 = y^2 \; S^2 / y^2 = y^2 z \approx \sum_{k=1}^{K} z^{(k)} ( \lambda_k y^2 )$$

Let $\mu_k = \lambda_k y^2$, then $\mu_k \geq 0$ and

$$\sum_{k=1}^{K} \mu_k = y^2$$

Hence, problem (6.1.1) can be written as:

$$\text{minimise} \qquad \sum_{k=1}^{K} \mu_k$$

subject to

$$\sum_{k=1}^{K} \sqrt{z^{(k)}\mu_k} + \sum_{j=1}^{30} p_j x_j = p_0$$

$$\sum_{k=1}^{K} z^{(k)} \mu_k - \sum_{i=1}^{30} \sum_{j=1}^{30} c_{ij} w_{ij} = 0 \qquad (6.1.3)$$

$$\sum_{k=1}^{K} z^{(k)} \mu_k \geq \varepsilon$$

$$\begin{aligned} & w_{ij} - x_i \leq 0 \\ & w_{ij} - x_j \leq 0 \\ & -w_{ij} + x_i + x_j \leq 1 \qquad\qquad i,j = 1, \ldots, 30 \\ & w_{ij} \geq 0 \\ & x_j \in \{0,1\} \\ & 0 \leq w_{ij} \leq 1 \\ & \mu_k \geq 0 \quad , k = 1, \ldots, K \end{aligned}$$

where $\varepsilon$ is a small positive constant.

<u>Case 2</u>: $p_0 < M$

Define $y = -(p_0 - M)/S \quad (> 0)$

Following a procedure similar to that used in case 1, problem (6.1.1) can be written as:

$$\text{maximise} \quad \sum_{k=1}^{K} \mu_k$$

subject to

$$\sum_{k=1}^{K} \sqrt{z^{(k)}}\mu_k - \sum_{j=1}^{30} p_j x_j = -p_0$$

$$\sum_{k=1}^{K} z^{(k)} \mu_k - \sum_{i=1}^{30} \sum_{j=1}^{30} c_{ij} w_{ij} = 0 \qquad (6.1.4)$$

$$\sum_{k=1}^{K} z^{(k)} \mu_k \geq \varepsilon$$

$$\begin{aligned} & w_{ij} - x_i \leq 0 \\ & w_{ij} - x_j \leq 0 \\ & -w_{ij} + x_i + x_j \leq 1 \qquad\qquad i,j = 1, \ldots, 30 \\ & w_{ij} \geq 0 \\ & x_j \in \{0,1\} \\ & 0 \leq w_{ij} \leq 1 \\ & \mu_k \geq 0 \quad, k = 1, \ldots, K \end{aligned}$$

where $\varepsilon$ is a small positive constant.

2. The coefficients of the model.

Let $P_{it}$, the cash flow for project i in time period t, be given by the model introduced in chapter III, section 3.1:

$$P_{11} = X_{11} + bS_1 \tag{6.1.5}$$

$$P_{1t} = X_{1t} + a( P_{1,t-1} - \mu_{1,t-1} ) + bS_t, \quad t = 2, \ldots, T$$

where

$$S_1 = E_1$$

$$S_t = dS_{t-1} + E_t, \quad t=2, \ldots, T$$

$$X_{1t} \sim N ( \mu_{1t}, \sigma_{1t}^2 ) \qquad \text{with} \qquad \sigma_{1t}^2 = c\mu_{1t}$$

and $\quad E_t \sim N ( 0, \sigma_t^2 ) \qquad \text{with} \qquad \sigma_t^2 = p^{t-1}\sigma_1^2.$

For project 1, with the model (6.1.5), is easy to calculate the expected value in each period:

$$E ( P_{1t} ) = \mu_{1t}, \quad t = 1, 2, \ldots, T$$

The variable $P_{1t}$ can be expressed in terms of the independent variables $X_{1j}$ and $S_j$, $j = 1, \ldots, t$.

$$P_{12} = X_{12} + a( P_{11} - \mu_{11} ) + bS_2$$

$$= X_{12} + a( X_{11} + bS_1 - \mu_{11} ) + bS_2$$

$$P_{13} = X_{13} + a [ X_{12} + a( X_{11} + bS_1 - \mu_{11} ) + bS_2 - \mu_{12} ] + bS_3$$

$$= X_{13} + a( X_{12} + bS_2 - \mu_{12} ) + a^2( X_{11} + bS_1 - \mu_{11} ) + bS_3$$

$$P_{it} = X_{it} + \sum_{k=1}^{t-1} a^{t-k} ( X_{ik} + bS_k - \mu_{ik} ) + bS_t$$

The variance of $P_{it}$ can be calculated by:

$$Var(P_{it}) = Var(X_{it}) + \sum_{k=1}^{t-1} [a^{2(t-k)} Var( X_{ik}) + a^{2(t-k)} b^2 Var( S_k)]$$

$$+ b^2 Var(S_t)$$

Now,

$$S_1 = dE_1$$

$$S_2 = dS_1 + E_2 = dE_1 + E_2$$

$$S_3 = dS_2 + E_3 = d^2E_1 + dE_2 + E_3$$

$$\vdots$$

$$S_t = d^{t-1}E_1 + d^{t-2}E_2 + \ldots + dE_{t-1} + E_t = \sum_{k=1}^{t} d^{t-k}E_k$$

Thus, $E(S_t) = 0$ and

$$S_t^2 = \sum_{k=1}^{t} d^{2(t-k)} E_k^2 + 2 \sum_{k=1}^{t-1} \sum_{k'=k+1}^{t} d^{2t-k-k'} E_k E_{k'}$$

Also, with $i < j$:

$$S_i S_j = ( \sum_{k=1}^{i} d^{i-k} E_k ) ( \sum_{p=1}^{j} d^{j-k} E_p )$$

$$= \sum_{k=1}^{i} d^{2(i-k)} E_k^2 + 2 \sum_{k=1}^{i-1} \sum_{k'=k+1}^{j} d^{i+j-k-k'} E_k E_{k'}$$

As the variables $E_k$ and $E_{k'}$, with $k=k'$, are independent

$$E( S_t^2 )= \sum_{k=1}^{t} d^{2(t-k)} \sigma_k^2$$

$$E( S_i S_j ) = \sum_{k=1}^{i} d^{2(i-k)} \sigma_k^2 \ , \ i < j$$

and

$$Var( S_t ) = E( S_t^2) - [ E( S_t) ]^2 = \sum_{k=1}^{t} d^{2(t-k)} \sigma_k$$

Then,

$$Var(P_{it}) = \sigma_{it}^2 + \sum_{k=1}^{t-1} [ a^{2(t-k)}\sigma_{ik}^2 + a^{2(t-k)}b^2 \sum_{j=1}^{k} d^{2(t-k)} \sigma_k^2 ]$$

$$+ b^2 \sum_{k=1}^{t} d^{2(t-k)} \sigma_k^2$$

and

$$E ( P_{it}^2 ) = Var (P_{it} ) + ( \mu_{it} )^2 \qquad (6.1.6)$$

The net present value for project i, $NPV_i$, is given by

$$NPV_i = \sum_{t=1}^{T} f_t P_i$$

where $f_t$ is the discount factor in time period i. Then, $p_i$, the expected net present value of project i, is given by:

$$p_i = E(NPV_i) = \sum_{t=1}^{T} f_t\, E(P_{it}) = \sum_{t=1}^{T} f_t\, \mu_{it}$$

The coefficient $c_{ii}$, the variance of the net present value of project i, is:

$$c_{ii} = Var(NPV_i) = E(NPV_i^2) - \left(\sum_{t=1}^{T} f_t\, \mu_{it}\right)^2$$

Now,

$$NPV_i^2 = \left(\sum_{t=1}^{T} f_t P_{it}\right)^2 = \sum_{t=1}^{T} f_t^2\, P_{it}^2 + 2 \sum_{t=1}^{T-1} \sum_{t'=t+1}^{T} f_t f_{t'}\, P_{it} P_{it'}$$

and

$$E(NPV_i^2) = \sum_{t}^{T} f_t^2\, E(P_{it}^2) + 2 \sum_{t=1}^{T-1} \sum_{t'=t+1}^{T} f_t f_{t'}\, E(P_{it} P_{it'}) \qquad (6.17)$$

From the expression of $P_{it}P_{it'}$, $t < t'$ is easy to obtain:

$$
\begin{aligned}
E(P_{it}P_{it'}) = {} & \mu_{it}\mu_{it'} \\
& + E\left\{ \left[ \sum_{k=1}^{t-1} a^{t-k}(X_{ik} + b\,S_k - \mu_{ik}) \right] \left[ \sum_{p=1}^{t'-1} a^{t'-p}(X_{ip} + b\,S_p - \mu_{ip}) \right] \right\} \\
& + E\left( b\,S_t \sum_{p=1}^{t'-1} a^{t'-p}\, b\,S_p \right) \\
& + E\left( b\,S_{t'} \sum_{k=1}^{t-1} a^{t-k}\, b\,S_k \right) \\
& + E(b^2 S_t^2)
\end{aligned}
$$

$$= \mu_{it}\mu_{it'} + \sum_{k=1}^{t-1} a^{t+t'-2k}\sigma_{ik}^2$$

$$+ \sum_{k=1}^{t-1} a^{t+t'-2k} b^2 \left( \sum_{j=1}^{k} d^{2(k-j)} \sigma_j^2 \right)$$

$$+ 2 \sum_{k=1}^{t-1} \sum_{p=k+1}^{t'-1} a^{t+t'-k-p} b^2 \left( \sum_{j=1}^{k} d^{2(k-j)} \sigma_j^2 \right)$$

$$+ \sum_{k=1}^{t-1} a^{t-k} b^2 \left( \sum_{j=1}^{k} d^{2(k-j)} \sigma_j^2 \right)$$

$$+ b^2 \sum_{k=1}^{t} d^{2(t-k)} \sigma_k^2$$

$$+ \sum_{p=1}^{t-1} a^{t'-p} b^2 \left( \sum_{j=1}^{p} d^{2(p-j)} \sigma_j^2 \right)$$

$$+ \sum_{p=t}^{t'-1} a^{t'-p} b^2 \sum_{j=1}^{p} d^{2(p-j)} \sigma_j^2 \qquad (6.1.8)$$

Using (6.1.6) and (6.1.8) in (6.1.7) yields the expression to calculate $E(NPV_i^2)$. The coefficient $c_{ij}$, the covariance between the net present value of projects i and j, is given by:

$$c_{ij} = Cov(NPV_i, NPV_j) = E(NPV_i\, NPV_j) - E(NPV_i)\, E(NPV_j) \qquad (6.1.9)$$

Now, using the expression for $NPV_i$:

$$E(NPV_i NPV_j) = \sum_{t=1}^{T} f_t^2 E(P_{it} P_{jt}) + 2 \sum_{t=1}^{T-1} \sum_{t'=t+1}^{T} f_t f_{t'} E(P_{it'} P_{jt'})$$

also,

$$E(P_{it} P_{jt}) = \mu_{it}\mu_{jt}$$

$$+ \sum_{k=1}^{t-1} a^{2(t-k)} b^2 \left( \sum_{j=1}^{k} d^{2(k-j)} \sigma_j^2 \right)$$

$$+ 2 \sum_{k=1}^{t-2} \sum_{k'=k+1}^{t-1} a^{2t-k-k'} b^2 \left( \sum_{j=1}^{k} d^{2(k-j)} \sigma_j^2 \right)$$

$$+ 2 \sum_{k=1}^{t-1} a^{t-k} b^2 \left( \sum_{j=1}^{k} d^{2(k-j)} \sigma_j^2 \right)$$

$$+ b^2 \sum_{k=1}^{t} d^{2(t-k)} \sigma_k^2$$

and, for $t < t'$,

$$E(P_{it} P_{jt'}) = \mu_{it}\mu_{jt'} +$$

$$+ \sum_{k=1}^{t-1} a^{t+t'-2k} b^2 \left( \sum_{j=1}^{k} d^{2(k-j)} \sigma_j^2 \right) +$$

$$+ 2 \sum_{k=1}^{t-1} \sum_{p=k+1}^{t'-1} a^{t+t'-k-p} b^2 \left( \sum_{j=1}^{k} d^{2(k-j)} \sigma_j^2 \right)$$

$$+ \sum_{k=1}^{t-1} a^{t-k} b^2 \left( \sum_{j=1}^{k} d^{2(k-j)} \sigma_j^2 \right)$$

$$+ \; b^2 \sum_{k=1}^{t} d^{2(t-k)} \sigma_k{}^2$$

$$+ \sum_{p=1}^{t-1} a^{t^{\cdot}-p} b^2 \left( \sum_{j=1}^{p} d^{2(p-j)} \sigma_j{}^2 \right)$$

$$+ \sum_{p=t}^{t^{\cdot}-1} a^{t^{\cdot}-p} b^2 \sum_{j=1}^{p} d^{2(p-j)} \sigma_j{}^2$$

Then, the value of $c_{ij}$ can be calculated substituting in (6.1.9) the adequate expressions.

Taking into account that z is defined as $S^2/y^2$ and that $y = (p_0 - M)/S$ a grid of points was obtained for z. These points are the $z^{(k)}$ of the model.

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