COMBINATORIAL DYNAMICS ON THE INTERVAL AND A GENERALIZATION OF SHARKOVSKIĬ'S THEOREM

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DECLARATION

This thesis is submitted to the University of Warwick in fulfilment of the requirements of the degree of Doctor of Philosophy. No part of the thesis has been submitted in support of an application for another degree or qualification of this or any other institution of learning. Chapter 1 and chapter 2 appeared in the following papers in which my own work was that of full pro-rata contribution.

1. J. Bobok, M. Kuchta, *Invariant measures for maps of the interval that do not have points of some period*, Ergod. Th. & Dynam. Sys. **14** (1994), 9–21.
2. J. Bobok, M. Kuchta, *X-minimal patterns and a Generalization of Sharkovskii's Theorem*, Preprint (submitted to Trans. Amer. Math. Soc.).

A. Blokh obtained results in [Bl] similar to some of those in our chapter 2. He used a completely different technique and worked independently at the same time.

M. Kuchta 8. August, 1994

SUMMARY

We study the discrete one dimensional dynamical systems given by continuous functions mapping a closed real interval into itself and the law of coexistence of periodic orbits for such systems.

In chapter 1 we study invariant measures for a continuous function which maps a real interval into itself. We show that the ratio of the measures of the two subintervals into which it is divided by a fixed point is constrained by the the set of periods of periodic points. As a consequence of this we get new information about the law of coexistence of periodic orbits.

In chapter 2 we study the law of coexistence of different types of periodic orbits more closely. Based on the idea from chapter 1 we introduce the term eccentricity of a periodic orbit and study the coexistence law between periodic orbits with different eccentricities. We also characterize those periodic orbits with a given eccentricity that are simplest from the point of view of the coexistence law.

We obtain a generalization of Sharkovskii's Theorem where the notion of period of periodic orbit is replaced by the notion of eccentricity of periodic orbit.

Chapter 2 is independent of chapter 1 but uses ideas that originated in the work covered by chapter 1.

With love
to Eva and Alicka

A false balance is abomination to the Lord: but a just weight is his delight.
[Proverbs 11:1]

The appearance of the wheels and their work was like unto the colour of a beryl: and they four had one likeness: and their appearance and their work was as it were a wheel in the middle of a wheel.
[Ezekiel 1:16]

INTRODUCTION

In the middle of the 1970's a boom started in the investigation of one dimensional dynamical systems. There were complex reasons for this. One reason is that one dimensional dynamical systems are good as models of many problems in physics, biology, mechanics, electricity, etc. One dimensional dynamical systems are especially good in connection with computers. They are usually much cheaper to calculate and the second big advantage is the possibility of an easily understandable output in two dimensions. One can even make an output of the whole one-parameter family using one axis for the space and another for the parameter (see Fig.1).

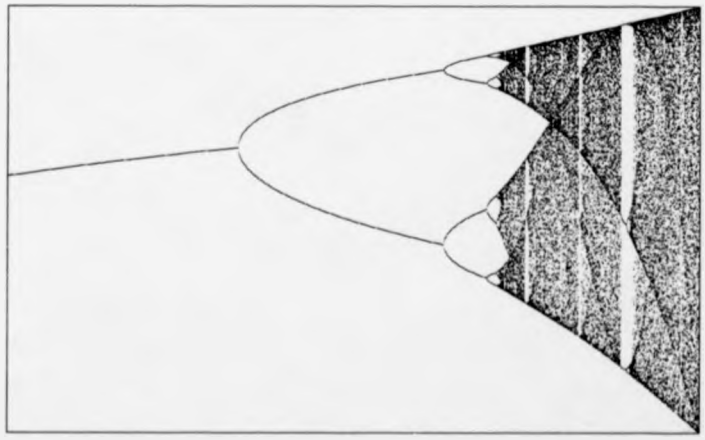

FIGURE 1. The output for the function $f_a(x) = ax(1-x)$. The x-axis is used for parameter $a \in [2.5, 4]$ and the y-axis is used for space $[0, 1]$. After 10,000 initial iterates of a point 0.33 we printed out 100 successive iterates. We can observe for which values of the parameters there is an attracting periodic point and also the period doubling phenomena.

A second reason for the popularity of one dimensional dynamical systems was the whole situation in the theory of dynamical systems. Most of the problems

considered at that time as important had either been solved or had turned out to be very difficult. Therefore attention turned towards the simplest systems. They still can have very complex behavior and solving possibly easier problems here can be useful for understanding more general systems.

Also there developed the idea of reducing the dimension of mathematical models which made it possible in many cases to pass from a many dimensional model to a one dimensional model. An example of the successful use of this procedure was the reduction of a model of atmospheric behavior to the Lorenz family of flows in three dimensions and then to a class of maps in one dimension. This procedure also made it possible to explain period doubling phenomena for many dimensional systems through the Feigenbaum universality theory for one dimensional systems.

In the theory of one dimensional systems the periodic orbits (cycles) play an important role. The first remarkable result is Sharkovskii's Theorem. It describes all the possible sets of periods of all cycles of a continuous map of a real interval into itself. This theorem leads naturally to the idea that there is a "coexistence law" for the cycles of some "type". In Sharkovskii's Theorem the type is given by the period of a cycle. But the period itself gives us very little information about the cycle. The full information tells us about the way the points of a cycle are mapped. When ordering the points of a cycle $p_1 < p_2 < \cdots < p_n$, one gets a cyclic permutation σ corresponding to it, such that each point p_i is mapped to $p_{\sigma(i)}$. It turns out that this permutation (which is the same as "ordered pattern" of a cycle) is probably the best way to classify cycles in order to study their coexistence law. For these patterns nice properties of the coexistence law (also called the "forcing relation") have been proved. But still the exact structure of the forcing relation is not known. In order to be able to get some more information one must group patterns into the bigger sets and study the forcing relation on these sets. For example in Sharkovskii's Theorem the patterns with the same period are in one set.

The theory developed around the ideas of pattern and forcing relation deals mainly with combinatorial objects: permutations, graphs etc. That is why L. Alsedà, J. Llibre and M. Misiurewicz in [ALM2] call it Combinatorial Dynamics. As the first half of our title suggests we will use this theory extensively. Instead of period we will take different information from a pattern to define a new "type" of pattern. We will consider the position of a fixed point whose existence is implied by a pattern, or more precisely the ratio of the number of points of the cycle on each side of such fixed point. Patterns with the same ratio will be said to be of the same *type*. As the second half of our title suggests, using these types we will be able to get better information about the structure of the forcing relation which will result in a generalization of Sharkovskii's Theorem.

BACKGROUND

By $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ we denote the sets of real, rational, integer and positive integer numbers respectively. By $\mathrm{conv}(X)$, $\mathrm{int}(X)$, $\overline{X}$ we denote the convex hull, interior and closure of the set X respectively. We will put sets inside $\{,\}$ brackets and for us a set is a collection of elements without multiple membership. An ordered collection of elements with possibly multiple membership will be called a sequence and we put it in $\langle,\rangle$. We will use $[,]$ and $(,)$ to denote closed and open intervals respectively.

By $C(X,X)$ we denote the set of continuous functions that map a topological space X into itself. For a function f and a nonnegative integer $i \in \mathbb{Z}$ we define ith iteration f^i where $f^0(x) = x$ and $f^{i+1}(x) = f(f^i(x))$. We will often map whole sets and so we will use the convention $f(X) = \{f(x);\ x \in X\}$. This allows us to define iteration for negative integers $i \in \mathbb{Z}$ as $f^{-1}(x) = \{y;\ f(y) = x\}$ and $f^{i-1}(x) = f^{-1}(f^i(x))$.

For a function $f : X \to X$ we define the orbit of a point $x \in X$ as $\mathrm{orb}_f(x) = \{f^i(x);\ i = 0, 1, 2, \ldots\}$. Often we omit f and write simply $\mathrm{orb}(x)$. There are two possibilities for $\mathrm{orb}(x)$. Either it is finite or infinite. In the first case the sequence $\langle f^i(x)\rangle_{i=0}^{\infty}$ must be periodic from some point on. It contains "periodic" points.

We say that a point $x \in X$ is periodic if there is $n \in \mathbb{N}$ such that $f^n(x) = x$. The least such n is called the period of x and we use $\mathrm{per}(x)$ to denote it. The set of all periodic points of a function f will be $\mathrm{Per}(f)$. A periodic point with period 1 will be called a fixed point and $\mathrm{Fix}(f)$ will be the set of all fixed points of the function f. The orbit of a periodic point will be called a periodic orbit.

Now if we look at a periodic orbit we can see that the important thing is only how the function f maps the points of this orbit. We can forget everything else and we still have a periodic orbit. So we can have the following definition.

Let $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}$ and $\varphi : P \to P$. Then (P, φ) is called a *periodic orbit* (or *cycle*) if φ is a cyclic permutation of P. We will usually omit φ and simply say that P is a cycle. The *period* of a cycle P is $\mathrm{per}(P) = n$.

For a cycle $(\{p_1, \ldots, p_n\}, \varphi)$ we will usually use one of the following labelings. A spatial labeling where $p_1 < p_2 < \cdots < p_n$ and a dynamical labeling where $\varphi(p_i) = p_{i+1}$ for $i = 1, \ldots, n-1$ and $\varphi(p_n) = p_1$

Now if we look at the cycle under a different scale or if we turn the paper upside down (reverse the orientation) that does not make our cycle really different even though it looks different. Hence in order to be able to distinguish "really different" cycles we will group similar ones.

Two periodic orbits $(P, \varphi), (Q, \psi)$ are *equivalent* if there exists a homeomorphism $h : \mathrm{conv}(P) \to \mathrm{conv}(Q)$ such that $h(P) = Q$ and $\psi \circ h|_P = h \circ \varphi|_P$. It is easy to see that this relation is an equivalence relation. An equivalence class of this relation will be called a *pattern*. (If we permit only an order preserving homeomorphism h then we have an *oriented pattern*.) If A is a pattern and $(P, \varphi) \in A$ we say that the cycle P has pattern A (or P is a representative of A) and we will use the symbol $[P]$ to denote the pattern A. The period of the pattern A is $\mathrm{per}(A) = \mathrm{per}(P)$.

We consider the space $C(I, I)$ of all continuous maps $f : I \to I$, where I is a closed real interval. A function $f \in C(I, I)$ has a cycle (P, φ) if $f|_P = \varphi$. We shall say that f *exhibits* the pattern $[P]$ and we can define the forcing relation between patterns.

Definition. A pattern A forces a pattern B if all maps in $C(I, I)$ exhibiting A exhibit also B.

We have the following information about the forcing relation

Theorem. ([B], [ALM2]) *The forcing relation is a partial order.*

Now we can state Sharkovskii's Theorem. First we introduce the Sharkovskii ordering:

$$3 \succ 5 \succ 7 \succ \cdots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \cdots \succ 2^2 \cdot 3 \succ 2^2 \cdot 5 \succ 2^2 \cdot 7 \succ \ldots$$
$$\succ 2^\infty \succ \cdots \succ 2^3 \succ 2^2 \succ 2 \succ 1.$$

Sharkovskii's Theorem. ([S2], [St])

(i) *A pattern with period m forces some pattern with period n for any $n \in \mathbb{N}$ such that $m \succ n$.*

(ii) *For any $m \in \mathbb{N} \cup \{2^\infty\}$ there is a continuous map $f : I \to I$ such that it has a cycle of period $n \in \mathbb{N}$ if and only if $m = n$ or $m \succ n$.*

The beauty of this theorem is that its first part allows us to obtain complex information about the set of periodic orbits from information about only one periodic orbit. Its second part gives a full characterization of the possible sets of periodic orbits as far as their period is concerned.

If we have a periodic orbit then we usually have more information about it than only its period. Taking this extra information into account we can get a better result than Sharkovskii's Theorem. Also we do not have to take a periodic orbit as our

starting point. A natural generalization of periodic orbit is an invariant measure. In chapter 1 we will use this in order to improve the first part of Sharkovskii's Theorem. In chapter 2 we will use the idea from chapter 1 for a new definition of the "type" of periodic orbits. Using this new type we will be able to prove a generalization of Sharkovskii's Theorem.

CHAPTER 1

INVARIANT MEASURES, FIXED POINTS AND PERIODIC ORBITS

1.0 Preliminaries

Consider a continuous function which maps a closed interval of a real line into itself and an invariant measure on this interval. There is a close connection between invariant measures and periodic orbits. We can construct any invariant measure of a map from the periodic orbits. So one can naturally assume that if we reduce the set of periodic orbits then it restricts the possibilities for invariant measures. On the other hand if we have information about an invariant measure then we can get information about the set of periodic orbits.

Now what information about an invariant measure should we take into consideration? First of all our function must have a fixed point. Let us assume for simplicity that it has a unique fixed point. Then this divides our interval into two parts and we can measure these two parts using the invariant measure. We will use the ratio of the measures of the two intervals given by our unique fixed point. From this ratio we will be able to deduce some information about the set of periods of periodic points for our function.

Let μ be a finite measure on the interval $I = [a, b]$ and $f \in C(I, I)$. For simplicity from now if we say "measure" then we in fact mean "finite measure" and if we measure some set then we assume that it is measurable.

The measure of a set $S \subset I$ will be denoted by $\mu(S)$ and for $x \in I$ let $\|x\|_\mu$ denote $\mu([a, x])$. We will say that $f \in C(I, I)$ preserves the measure μ (or that μ is an invariant measure for f) if $\mu(f^{-1}(S)) = \mu(S)$ for any measurable $S \subset I$. We will denote the set of all functions $f \in C(I, I)$ preserving a measure μ by $C_\mu(I, I)$ and the set of all measures preserved by a function $f \in C(I, I)$ by $\mathcal{M}(f)$.

Our argument will make heavy use of the following simple inequality.

$$(*) \qquad \text{If } \mu \in \mathcal{M}(f) \text{ then } \mu(f(S)) \geq \mu(S) \text{ for any } S \subset I.$$

In particular if $\mu(\{p\}) > 0$ then p must be a periodic point and $\mu(\{x\}) = \mu(\{p\})$ for any $x \in \text{orb}(p)$.

We say that a set $S \subset I$ is f-invariant if $f(S) \subset S$. The measure $\mu \in \mathcal{M}(f)$ is called ergodic if for any f-invariant set $S \subset I$ either $\mu(S) = 0$ or $\mu(S) = \mu(I)$. We denote the set of all f-invariant ergodic measures by $\mathcal{E}(f)$.

The support of the measure μ, denoted by $\mathrm{supp}(\mu)$, is the smallest closed set $S \subset I$ such that $\mu(S) = \mu(I)$.

If $\mu \in \mathcal{M}(f)$ then $f(\mathrm{supp}(\mu)) = \mathrm{supp}(\mu)$ and if μ is ergodic then either $\mathrm{supp}(\mu) = \mathrm{orb}(p)$ for some $p \in \mathrm{Per}(f)$ or $\mathrm{supp}(\mu)$ is a perfect set. We have the following ergodic decomposition (see e.g. [P]).

THEOREM 1.0.1. *Let $f \in C(I, I)$ and $\mu \in \mathcal{M}(f)$. Then there is a measure m on the set $\mathcal{E}(f)$ such that $\mu(S) = \int_{\mathcal{E}(f)} \lambda(S)\, dm(\lambda)$ for any measurable set S.*

Now we will put a restriction on the set of periodic orbits of the map f and we will study the properties of the possible set $\mathcal{M}(f)$. The restriction we make is the restriction of the set $\mathrm{Per}(f)$. We will work with the set $C_k(I, I) \subset C(I, I)$ of all functions that do not have any periodic point of period $2k + 1$ ($k > 0$). We also define $C_{\mu,k}(I, I) = C_\mu(I, I) \cap C_k(I, I)$.

First we will start with the simplest case.

1.1 Maps with only one fixed point

As the title suggests, throughout this section we will assume that the function under consideration has only one fixed point.

DEFINITION. Let $f \in C(I, I)$ ($I = [a, b]$), $p \in \mathrm{Fix}(f)$ and $z \in [a, p)$. Let

$$\begin{aligned}
&y_0(z) = z \\
&x_i(z) = \inf\{x \in [p, b];\ f(x) = y_{i-1}(z)\} && \text{for } i \geq 1 \\
&y_i(z) = \sup\{x \in [a, p];\ f(x) = x_i(z)\} && \text{for } i \geq 1 \\
&x_i(z) = \sup\{f(x);\ x \in [y_i(z), p]\} && \text{for } i \leq 0 \\
&y_i(z) = \inf\{f(x);\ x \in [p, x_{i+1}(z)]\} && \text{for } i < 0.
\end{aligned}$$

(See Fig. 1.1 which gives an idea of what these sequence mean.)

Note that sometimes $x_i(z), y_i(z)$ are not well defined. But from the continuity of f we obtain the following two lemmas. We omit their proofs.

LEMMA 1.1.1. *Let $f \in C(I, I)$, $\mathrm{Fix}(f) = \{p\}$ and $z \in [a, p)$. If $x_1(z), y_1(z)$ exist and $y_1(z) > y_0(z)$ then $x_i(z), y_i(z)$ exist for all $i \in \mathbf{Z}$ and*

$$\cdots \leq y_{-2}(z) \leq y_{-1}(z) \leq y_0(z) < y_1(z) < y_2(z) < \cdots < p$$
$$p < \cdots < x_2(z) < x_1(z) \leq x_0(z) \leq x_{-1}(z) \leq x_{-2}(z) \leq \cdots$$

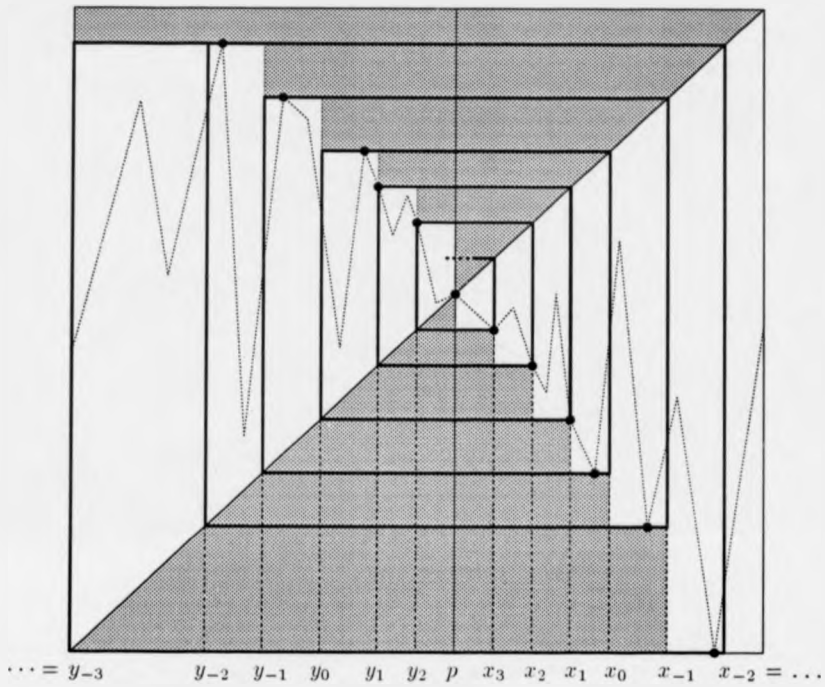

FIGURE 1.1. Function f (dotted) and sequences $\langle y_i \rangle$, $\langle x_i \rangle$. The filled area is forbidden for a function f with a unique fixed point.

LEMMA 1.1.2. *Let $f \in C(I, I)$, $\mathrm{Fix}(f) = \{p\}$, $x \in [a, p)$ and $y_1(x) > y_0(x)$. If $x^* \in [y_0(x), y_1(x)]$ then $y_i(x^*) \in [y_i(x), y_{i+1}(x)]$ and $x_i(x^*) \in [x_{i+1}(x), x_i(x)]$ for all $i \in \mathbf{Z}$.*

THEOREM 1.1.3. *Let μ be a measure on $I = [0, 1]$, $f \in C_{\mu,k}(I, I)$ and $\mathrm{Fix}(f) = \{p\}$. Then either $\mu(\{p\}) = \mu(I)$ or $\frac{\mu([0,p))}{\mu((p,1])} \in (\frac{k}{k+1}, \frac{k+1}{k})$.*

PROOF. From Theorem 1.0.1 it follows that it suffices to prove this for ergodic measures. So let μ be ergodic.

If $\mu(\{p\}) > 0$ then $\mu(\{p\}) = \mu(I)$ because μ is ergodic. So we can assume that $\mu(\{p\}) = 0$. Because p is the only fixed point of f we have $f(x) > x$ for any $x < p$. Hence $\bigcap_{i=0}^{\infty} f^i(S) = \{p\}$ for any f-invariant closed set $S \subset [0, p]$. Therefore $\mathrm{supp}(\mu) \not\subset [0, p]$ and so $\mu((p, 1]) > 0$. We will show only that $\frac{\mu([0,p))}{\mu((p,1])} < \frac{k+1}{k}$ since second inequality can be shown similarly.

If $f(x) \geq p$ for all $x \in \mathrm{supp}(\mu) \cap [0, p]$ then clearly $\frac{\mu([0,p))}{\mu((p,1])} \leq 1$ and we are done. Hence we can assume that there is $y^* \in \mathrm{supp}(\mu) \cap [0, p)$ such that $f(y^*) < p$.

Moreover if μ is not atomic then we can choose $y^* > 0$ such that for any $z_1 < y^*$ and $z_2 > y^*$

$$\mu((z_1, y^*)) > 0 \quad \text{and} \quad \mu((y^*, z_2)) > 0. \tag{1}$$

Because $f(\mathrm{supp}(\mu)) = \mathrm{supp}(\mu)$ we can choose y^* such that $x_1(y^*)$ exists. If $y_1(y^*) \leq y_0(y^*)$ (or $y_1(y^*)$ does not exist) then $[y_0(y^*), x_1(y^*)]$ is an invariant interval and from (1) we obtain that y^* is a periodic point. So $\mathrm{orb}(y^*) \subset [y_0(y^*), x_1(y^*)]$. But $f^{-1}(y^*) \cap [y_0(y^*), x_1(y^*)] = \{x_1(y^*)\}$ and $f^{-1}(x_1(y^*)) \cap [y_0(y^*), x_1(y^*)] = \emptyset$. Hence $y_1(y^*) > y_0(y^*)$ and we can use Lemma 1.1.1. Set

$$\begin{aligned} a_1 &= \lim_{i\to\infty} y_i(y^*) & b_1 &= \lim_{i\to\infty} x_i(y^*) \\ a_2 &= \lim_{i\to-\infty} y_i(y^*) & b_2 &= \lim_{i\to-\infty} x_i(y^*). \end{aligned} \tag{2}$$

Because $[a_1, b_1]$, $[a_2, b_2]$ are invariant by (1) and Lemma 1.1.1

$$\mu([a_1, b_1]) = 0 \quad \text{and} \quad \mu(I \setminus [a_2, b_2]) = 0. \tag{3}$$

Now we may take $z \in [y_0(y^*), y_1(y^*))$ (we can use Lemma 1.1.2). We will prove that for every $i \in \mathbf{Z}$

$$\inf\{f(x);\ x \in [y_i(z), p]\} > y_{i+k}(z).$$

Assume the contrary. Then $y_i(z) \neq y_{i+k}(z)$ and so $f([y_i(z), y_{i+k+1}(z)]) \supset [y_{i+k}(z), x_i(z)]$. But $f([y_n(z), x_m(z)]) \supset [y_{m-1}(z), x_n(z)]$ for any $m, n \in \mathbf{Z}$ and so $f^{2k+1}([y_i(z), y_{i+k+j}(z)]) \supset [y_i(z), y_{i+k+j}(z)]$. Hence there is $x \in [y_i(z), y_{i+k+j}(z)]$ such that $f^{2k+1}(x) = x$. So x is a periodic point with odd period smaller then or equal to $2k+1$. But because $p \notin [y_i(z), y_{i+k+j}(z)]$ we have that period x is greater than 1 and from Sharkovskii's Theorem the function f has a periodic point with period $2k+1$ — a contradiction.

Therefore we have that for $i \in \mathbf{Z}$

$$f([y_i(z), x_{i+k+1}(z)]) \subset [y_{i+k}(z), x_i(z)]$$

and then

$$0 \leq \mu([y_{i+k}(z), x_i(z)]) - \mu([y_i(z), x_{i+k+1}(z)])$$

or equivalently

$$0 \leq \|x_i(z)\|_\mu - \|y_{i+k}(z)\|_\mu + \mu(\{y_{i+k}(z)\}) - \|x_{i+k+1}(z)\|_\mu + \|y_i(z)\|_\mu - \mu(\{y_i(z)\}). \tag{4}$$

Moreover from Lemma 1.1.2 and (2)

$$\begin{aligned} a_1 &= \lim_{i\to\infty} y_i(z) & b_1 &= \lim_{i\to\infty} x_i(z) \\ a_2 &= \lim_{i\to-\infty} y_i(z) & b_2 &= \lim_{i\to-\infty} x_i(z). \end{aligned} \tag{5}$$

Because $y_k(x)$ is continuous and $f(y^*) < p$ we can choose $x^* > y^*$ such that $\inf\{f(x);\ x \in [y^*, p]\} \geq y_k(x^*)$ and $\sup\{f(x);\ x \in [y^*, p]\} = x_0(x^*)$. Hence we have that $f([y^*, x_{k+1}(x^*)]) \subset [y_k(x^*), x_0(x^*)]$ and $f([y_{-k}(x^*), x_1(x^*)]) \subset [y^*, x_{-k}(x^*)]$. Hence from (1)

(6)
$$0 < \varepsilon \leq \|x_{-k}(x^*)\|_\mu - \|y^*\|_\mu + \mu(\{y^*\}) - \|x_1(x^*)\|_\mu + \|y_{-k}(x^*)\|_\mu - \mu(\{y_{-k}(x^*)\})$$
$$0 \leq \|x_0(x^*)\|_\mu - \|y_k(x^*)\|_\mu + \mu(\{y_k(x^*)\}) - \|x_{k+1}(x^*)\|_\mu + \|y^*\|_\mu - \mu(\{y^*\})$$

where $\varepsilon = \mu([y^*, x^*))$ (see (1)).

Now by summing (4) for $z = x^*$ and $i \in \{-n, -n+1, \ldots, n\}$ except for the cases $i \in \{-k, 0\}$ when inequalities (4) are replaced by (6) we get

$$0 < \varepsilon \leq \sum_{i=-n}^{-n+k} \|x_i(x^*)\|_\mu + \sum_{i=-n}^{-n+k-1} (\|y_i(x^*)\|_\mu - \mu(\{y_i(x^*)\})) -$$
$$- \sum_{i=n+1}^{n+k+1} \|x_i(x^*)\|_\mu - \sum_{i=n+1}^{n+k} (\|y_i(x^*)\|_\mu - \mu(\{y_i(x^*)\})).$$

Finally we will use (5) and take the limit as $n \to \infty$. But we have to distinguish two cases.

1. $a_2 < y_i(x^*)$ and $x_i(x^*) < b_2$ for any $i \in \mathbf{Z}$. Then $\{a_2, b_2\}$ forms a periodic orbit. Hence $\mu(\{a_2\}) = \mu(\{b_2\}) = 0$ and after taking the limit we get

$$0 < \varepsilon \leq (k+1)\mu([0, b_2)) + k\mu([0, a_2]) - (k+1)\mu([0, b_1]) - k\mu([0, a_1)).$$

2. There is $n \in \mathbf{N}$ such that for all $i > n$ we have $y_{-i}(x^*) = a_2$ and $x_{-i}(x^*) = b_2$. After taking the limit we get

$$0 < \varepsilon \leq (k+1)\mu([0, b_2]) + 0 - (k+1)\mu([0, b_1]) - k\mu([0, a_1)).$$

Hence using (3) in both cases we get $\frac{\mu([0,p))}{\mu([p,1))} < \frac{k+1}{k}$. ■

Unfortunately a continuous function has in general several fixed points. We will now show that we can use Theorem 1.1.3 at least for the class of functions which have a dense set of periodic points. We need some more information for this.

We say that $f \in C(I, I)$ is transitive if there is a point $x \in I$ such that $\mathrm{orb}_f(x)$ is dense in I.

From [BaM] we have the following information about a function with a dense set of periodic points.

LEMMA 1.1.4. ([BaM]) *Let $f \in C(I, I)$ have a dense set of periodic points. Then one of the following conditions is true*

1. *f is transitive.*

2. *$f(x) = x$ for all $x \in I$.*
3. *There are $a, b \in I$ such that $a < b$, $f|_{[a,b]}$ is transitive and $f(a) = a$ or $f(b) = b$.*
4. *There are $a, b \in I$ such that $f|_{[a,b]}$ is transitive (here $a = b$ is possible), $f([0, a]) = [b, 1]$ and $f([b, 1]) = [0, a]$.*

For transitive functions we have

LEMMA 1.1.5. (Lemma 3.3 from [BC]) *Let $f \in C(I, I)$ be a transitive function that does not have a periodic point of period 3. Then it has a unique fixed point and this is not an endpoint of I.*

Now we can easily prove

LEMMA 1.1.6. *Let $f \in C_k(I, I)$ and* $\mathrm{Per}(f)$ *be dense in I. Then either $f(x) = x$ for all $x \in I$ or f has a unique fixed point.*

PROOF. We use Lemma 1.1.4.

In case 1 from Sharkovskiĭ's Theorem and Lemma 1.1.5 we have that f has a unique fixed point.

In case 2 there is nothing to prove.

In case 3 using Lemma 1.1.5 we obtain that f has a periodic point with period 3 - a contradiction. So this case is not possible.

Finally in case 4 we get that there is no fixed point outside the interval $[a, b]$ and using Lemma 1.1.5 we get that $f|_{[a,b]}$ has a unique fixed point. ■

So a restriction on the set of periodic orbits of a map gives a restriction on invariant measures. Now we will give examples that will show that this restriction is the best possible.

If we consider the set of invariant measures then there are two extremal types of measures: nonatomic measures with full support, for example Lebesgue measure, and purely atomic measures, for example on a single periodic orbit. We will give examples for both cases.

LEMMA 1.1.7. *Let $k \in \mathbb{N}$ and λ be Lebesgue measure on $I = [0, 1]$. Then for every $p \in (\frac{k}{2k+1}, \frac{k+1}{2k+1})$ there is a map $f \in C_{\lambda,k}(I, I)$ such that* $\mathrm{Fix}(f) = \{p\}$.

PROOF. We can assume that $p \geq \frac{1}{2}$ (case $p \leq \frac{1}{2}$ is similar). If $p = \frac{1}{2}$ then we can simply put $f(x) = 1 - x$ and we are done. So assume that $p > \frac{1}{2}$. Define the function f such that $f(0) = \frac{k}{2k+1}$, $f(2p - 1) = p$, $f(p - \frac{k}{2k+1}) = 1$, $f(p) = p$, $f(\frac{k+1}{2k+1}) = \frac{k}{2k+1}$, $f(1) = 0$ and let f be linear between these points (see Fig. 1.2).

Clearly f is Lebesgue measure preserving and has a unique fixed point p. Moreover if f has a periodic point with period $2k + 1$ then there is $p_0 \in I$ such that both

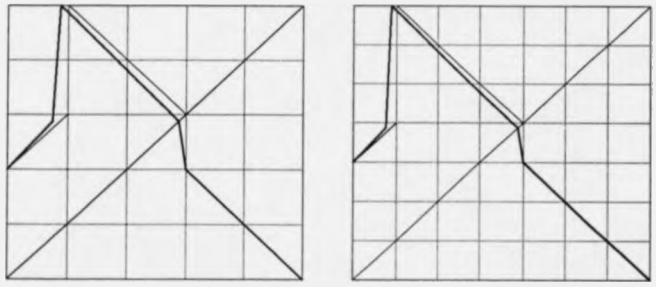

FIGURE 1.2. Functions $f \in C_{\Lambda,2}(I,I)$ and $f \in C_{\Lambda,3}(I,I)$.

p_0 and $f(p_0)$ are on the same side of a fixed point p. Hence it suffices to show that there is no periodic point with period $2k+1$ in the interval $[0, 2p-1]$.

For every $x \in [0, 2p-1]$ we have $f(x) \geq x + \frac{k}{2k+1}$. Moreover for $x \in [\frac{1}{2k+1}, p]$ we have $f(x) < 1 - x + \frac{1}{2k+1}$ and for $x \in [p, 1]$ we have $f(x) \geq 1 - x$. Hence for every $x \in [0, 2p-1]$ we obtain $f^{2k+1}(x) > x$. ■

DEFINITION. Let $m, n \in \mathbb{N}$ and $m \geq n$. Define $f_{m,n} \in C(I,I)$ such that $I = [1, m+n]$,

$$\begin{aligned} f_{m,n}(i) &= n + i && \text{for} \quad 0 < i \leq m - n, \\ f_{m,n}(x_i) &= 2m + 1 - i && \text{for} \quad m - n < i \leq m, \\ f_{m,n}(x_i) &= m + n + 1 - i && \text{for} \quad m < i \leq m + n \end{aligned}$$

and $f_{m,n}$ is linear between these points. If $m \leq n$ then $f_{m,n}$ is defined symmetrically (see Fig. 1.3).

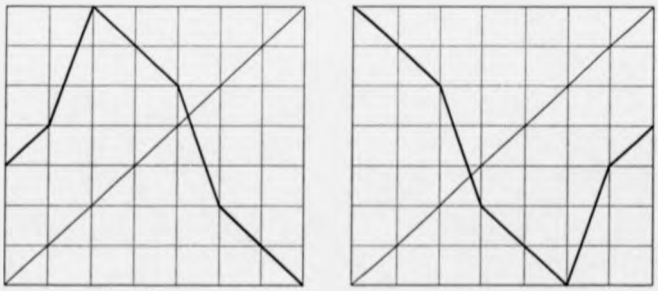

FIGURE 1.3. Functions $f_{5,3}$ and $f_{3,5}$.

LEMMA 1.1.8. *Let $m, n, k \in \mathbb{N}$ and m, n coprime. If $\frac{m}{n} \in (\frac{k}{k+1}, \frac{k+1}{k})$ then $f_{m,n} \in C_k(I,I)$.*

PROOF. We can assume that $\frac{m}{n} \geq 1$. If $m = n$ then we have $f^2_{m,n}(x) = x$ for all $x \in I$ and we are done. Hence we can assume that $m > n$.

It is easy to see that function $f_{m,n}$ is unimodal and so we can use kneading theory (see e.g. [MT] or Appendix). If $f_{m,n}$ has a periodic orbit of a period $2k+1$ then it has the Štefan orbit of period $2k+1$ (see e.g. [ALM2],[S2], [St]). Hence the kneading invariant $\underline{S} = RLR^{2(k-1)}C$ of the Štefan orbit of period $2k+1$ has to be compared with the kneading invariant $\underline{K}$ of $f_{m,n}$. If function $f_{m,n}$ has a periodic orbit of a period $2k+1$ then $\underline{K} \geq \underline{S}$.

It can be shown that $\underline{K} = R\underline{B}_0\underline{B}_1 \ldots \underline{B}_{m-2}C$ where $\underline{B}_i = L$ if $in = l \pmod m$ with $0 \leq l < m-n$ and $\underline{B}_i = R^2$ if $in = l \pmod m$ with $m-n < l < m$. Therefore $\underline{K} = RLR^{2s}L \ldots C$ if $m-n > 1$ or $\underline{K} = RLR^{2m-4}C$ if $m-n = 1$.

Thus, $\underline{K} \geq \underline{S}$ if and only if $s \leq k-1$ (if $m-n > 1$) or $m-2 \leq k-1$ (if $m-n = 1$). Notice that $\pmod m$ adding n is the same as subtracing $m-n$, so s is the largest number such that $s(m-n) < n$. Hence if $m-n > 1$ then $\underline{K} > \underline{S}$ if and only if $\frac{n}{m-n} < k$ and if $m-n = 1$ then $\underline{K} > \underline{S}$ if and only if $\frac{n}{m-n} \leq k$. But in both cases we have $\frac{n}{m} \leq \frac{k}{k+1}$. This completes the proof. ■

1.2 Maps with several fixed points

Now we will try to get some information even in the case when our map has several fixed points. Before proceeding we need to have more information about the invariant measure μ. More precisely, we want to know more about the set $\mathrm{supp}(\mu)$ in order to be able to make a decomposition of the measure μ. The notion of the center of a map is important here.

DEFINITION. Let $f : S \to S$. The point $x \in S$ is nonwandering if for any neighborhood U of x in S there is an $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. We denote by $\Omega(f)$ the set of nonwandering points of f. Define the center of f to be $C(f) \subset S$, the maximal closed invariant set such that $\Omega(f|_{C(f)}) = C(f)$.

We have

LEMMA 1.2.1. (Sharkovskii [S1]) *Let $f \in C(I,I)$. Then $C(f) = \overline{\mathrm{Per}(f)}$.*

The following result holds for any compact metric space.

LEMMA 1.2.2. *Let μ be a measure on I and $f \in C_\mu(I,I)$. Then* $\mathrm{supp}(\mu) \subset C(f)$.

Now we will define a new invariant measure which is connected with our invariant measure μ and a particular fixed point q.

DEFINITION. Let $f \in C(I,I)$ and $q \in \mathrm{Fix}(f)$. Let $C_q = \overline{S_q}$ where $S_q = \{p \in \mathrm{Per}(f);\ q \in \mathrm{conv}(\mathrm{orb}(p))\}$.

LEMMA 1.2.3. *Let $f \in C(I, I)$ and $q \in \mathrm{Fix}(f)$. Then $f(C_q) = C_q$.*

PROOF. Clearly $f(S_q) = S_q$ and from continuity $f(C_q) = C_q$. ■

LEMMA 1.2.4. *Let $f \in C(I, I)$. Then $C(f) = \bigcup_{q \in \mathrm{Fix}(f)} C_q$.*

PROOF. Let $a \in C(f)$ and $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence of periodic points such that $\lim_{n\to\infty} p_n = a$. We can assume that $f(p_n) \geq p_n$ for all $n \geq 1$. Let $q_n = \inf(\mathrm{Fix}(f) \cap [p_n, 1])$. Because $\mathrm{Fix}(f)$ is closed $q_n \in \mathrm{Fix}(f)$ and it is easy to see that $q_n \in \mathrm{conv}(\mathrm{orb}(p_n))$. But for $i \neq j$ either $q_i = q_j$ or $[p_i, q_i] \cap [p_j, q_j] = \emptyset$. Hence there is a sequence $\langle n(i) \rangle_{i=1}^{\infty}$ such that either $q_{n(i)} = q_{n(i+1)}$ or $\lim_{i\to\infty} q_{n(i)} = a$. In the first case $q_{n(1)} \in \mathrm{conv}(\mathrm{orb}(p_{n(i)}))$ and so $a = \lim_{i\to\infty} p_{n(i)} \in C_{q_{n(1)}}$. In the second case $a \in \mathrm{Fix}(f)$ and so $a \in C_a$. ■

DEFINITION. Let μ be a measure on I and let $S \subset I$. Define the measure μ_S by $\mu_S(A) = \mu(A \cap S)$ for any $A \subset I$.

LEMMA 1.2.5. *Let μ be a measure on I and $f \in C_\mu(I, I)$. Then $\mu_{C_q} \in \mathcal{M}(f)$ for any $q \in \mathrm{Fix}(f)$. (Note that μ_{C_q} can be zero measure.)*

PROOF. Let $S \subset I$. We have that $\mu_{C_q}(S) = \mu(S \cap C_q) = \mu(f^{-1}(S \cap C_q))$. But $f^{-1}(S \cap C_q) = ((I \setminus C_q) \cap f^{-1}(S \cap C_q)) \cup (C_q \cap f^{-1}(S))$. Hence it suffices to show that $\mu((I \setminus C_q) \cap f^{-1}(C_q)) = 0$. But $f(f^{-1}(C_q) \cup C_q) \subset C_q$ and so using $(*)$ we are done. ■

So we have a system of new invariant measures μ_{C_q} given by a measure μ. We will derive more information about the measure μ_{C_q}.

LEMMA 1.2.6. *Let $f \in C_k(I, I)$. Then for any $q \in \mathrm{Fix}(f)$ and $p \in \mathrm{Per}(f)$ such that $q \in \mathrm{conv}(\mathrm{orb}(p))$ we have $(p - q)(p - f(p)) \geq 0$.*

PROOF. See e.g. (9) in [St]. ■

LEMMA 1.2.7. *Let $f \in C_k(I, I)$. Then $C_q \cap \mathrm{Fix}(f) = \{q\}$ for any $q \in \mathrm{Fix}(f)$.*

PROOF. Assume that there is a $q^* \in \mathrm{Fix}(f) \cap C_q$ such that $q^* < q$ (the other case is similar). Then there is a sequence $\langle p_n \rangle_{n=1}^{\infty}$ of periodic points such that $q \in \mathrm{conv}(\mathrm{orb}(p_n))$ and $\lim_{n\to\infty} p_n = q^*$. If we can choose a subsequence $\langle p_{n(i)} \rangle_{i=1}^{\infty}$ such that $p_{n(i)} < q^*$ for all $i \geq 1$ then from Lemma 1.2.6 (using point q) and continuity of f there are $i, j \in \mathbb{N}$ such that $f^j(p_{n(i)}) \in (q^*, q)$ and $f^{j+1}(p_{n(i)}) > f^i(p_{n(i)})$. But then again from Lemma 1.2.6 (using point q^*) we obtain that f has a 3-cycle which is a contradiction. Therefore we can assume that $p_n > q^*$ for all $n \geq 1$. Moreover again from Lemma 1.2.6 we have $\min(\mathrm{orb}(p_n)) > q^*$ and so we can assume that $\min(\mathrm{orb}(p_n)) = p_n$ for all $n \geq 1$.

Let $p_n^1 \in \mathrm{orb}(p_n)$ such that $f(p_n^1) = p_n$. Then $p_n^1 > q$ and we can assume that $\lim_{n\to\infty} p_n^1 = q_1$. We have $f(q_1) = q^*$ and so $q_1 > q$.

Let $p_n^2 = \max(\mathrm{orb}(p_n))$. We can assume that $\lim_{n\to\infty} p_n^2 = q_2$ and because $p_n^2 \geq p_n^1$ then $q_2 \geq q_1$.

Let $p_n^3 \in \mathrm{orb}(p_n)$ such that $f(p_n^3) = p_n^2$. We can assume that $\lim_{n\to\infty} p_n^3 = q_3$ and from Lemma 1.2.6 we have $q^* < p_n^3 < q$. Moreover $f(q_3) = q_2 \geq q_1 > q$ and so $q^* < q_3 < q$. Hence $f([q^*, q_3]) \cap f([q_3, q_1]) \supset [q^*, q_1]$ and therefore f has 3-cycle which is a contradiction. ■

DEFINITION. Let $S \subset I$ such that $I = \mathrm{conv}(S)$ and let $\varphi : S \to S$ be a continuous function (pair (S, φ) can be considered as a generalized periodic orbit). Define function $f_S \in C(I, I)$ such that $f_S|_S = \varphi$ and for any interval $J \subset I$ such that $J \cap S = \emptyset$ we have that $f_S|_J$ is linear.

THEOREM 1.2.8. *Let $f \in C(I, I)$, $S \subset I$ be closed such that $f(S) \subset S$. Take $\varphi = f|_S$ and a function f_S. Then for any $p \in \mathrm{Per}(f_S)$ there is $p^* \in \mathrm{Per}(f)$ such that $f_S|_{\mathrm{orb}_{f_S}(p)} \circ h = h \circ f|_{\mathrm{orb}_f(p^*)}$ where $h : \mathrm{orb}_f(p^*) \to \mathrm{orb}_{f_S}(p)$ is an order preserving bijection.*

REMARK. We can also say that $\mathrm{orb}_f(p^*)$ and $\mathrm{orb}_{f_S}(p)$ have the same oriented pattern.

The Theorem 1.2.8 is in fact a generalization of Theorem 2.6.13 from [ALM2] which is one of the key tools in the theory of forcing relation. Therefore it deserves more attention and we will prove it in the section 1.3.

THEOREM 1.2.9. *Let μ be a measure on $I = [0, 1]$ and $f \in C_{\mu,k}(I, I)$. Then for any $q \in \mathrm{Fix}(f)$ either $\mu_{C_q}(\{q\}) = \mu_{C_q}(I)$ or $\frac{\mu_{C_q}([0,q))}{\mu_{C_q}((q,1])} \in (\frac{k}{k+1}, \frac{k+1}{k})$.*

PROOF. Take a pair $(C_q, f|_{C_q})$ and a function f_{C_q}. By Lemma 1.2.6 and Lemma 1.2.7 the function f_{C_q} has a unique fixed point q and, by Theorem 1.2.8, no cycle of period $2k+1$. Moreover $\mathrm{supp}(\mu_{C_q}) \subset C_q$ and $f_{C_q}|_{C_q} = f|_{C_q}$. So $f_{C_q} \in C_{\mu_{C_q},k}(I, I)$. Hence we can apply Theorem 1.1.3 to the function f_{C_q}. Finally it suffices to realize that $\mu_{C_q}([0, q)) = \mu_{C_q}([\min C_q, q))$ and $\mu_{C_q}((q, 1]) = \mu_{C_q}((q, \max C_q])$. ■

1.3 Proof of Theorem 1.2.8

DEFINITION. Let $f \in C(I, I)$ and $x < y \in I$. If $f(x) < f(y)$ then $\mathrm{sign}_f([x, y]) = 1$, if $f(x) > f(y)$ then $\mathrm{sign}_f([x, y]) = -1$ and if $f(x) = f(y)$ then $\mathrm{sign}_f([x, y]) = 0$.

LEMMA 1.3.1. *Let $f \in C(I, I)$ and $a, b, c, d \in I$ such that $\mathrm{conv}(\{f(a), f(b)\}) \supset [c, d]$. Then there are $a^*, b^* \in [a, b]$ such that $f([a^*, b^*]) = [c, d]$, $f(\{a^*, b^*\}) = \{c, d\}$ and $\mathrm{sign}([a^*, b^*]) = \mathrm{sign}([a, b])$.*

PROOF. If $f(a) > f(b)$ then $a^* = \sup\{x \in [a, b]: f(x) = d\}$ and $b^* = \inf\{x \in [a^*, b]; f(x) = a\}$. The second case is similar. ■

THEOREM 1.2.8. *Let $f \in C(I, I)$. $S \subset I$ be closed such that $f(S) \subset S$. Take $\varphi = f|_S$ and a function f_S. Then for any $p \in \mathrm{Per}(f_S)$ there is $p^* \in \mathrm{Per}(f)$ such that $f_S|_{\mathrm{orb}_{f_S}(p)} \circ h = h \circ f|_{\mathrm{orb}_f(p^*)}$ where $h : \mathrm{orb}_f(p^*) - \mathrm{orb}_{f_S}(p)$ is an order preserving bijection.*

PROOF. We can assume that $\mathrm{orb}(p) \cap S = \emptyset$ and $\mathrm{per}(p) > 1$ (the other cases are trivial). Let $f_S^i(p^*) \in J_i$ for $0 \le i \le \mathrm{per}(p) - 1$ where J_i is the biggest closed interval such that $\mathrm{int}(J_i) \cap S = \emptyset$. Hence $f(J_i) \supset J_{i+1}$ (all indices will be taken mod $\mathrm{per}(p)$). There are two possibilities.

1. $\langle J_i \rangle_{i=0}^{\mathrm{per}(p)-1} \ne \langle J_{i+n} \rangle_{i=0}^{\mathrm{per}(p)-1}$ for any $0 < n < \mathrm{per}(p)$.
2. $\langle J_i \rangle_{i=0}^{\mathrm{per}(p)-1} = \langle J_{i+n} \rangle_{i=0}^{\mathrm{per}(p)-1}$ for some $0 < n < \mathrm{per}(p)$.

Case 1. Let $\{I_i\}_{i=1}^k = \{J_i\}_{i=0}^{\mathrm{per}(p)-1}$ such that $I_i \ne I_j$ for $i \ne j$, and choose $p_i \in \{1, 2, \ldots, k\}$ such that $f_S^i(p) \in I_{p_i}$. Let $\alpha_1\alpha_2 = (a_1, \ldots, a_n, b_1, \ldots, b_m)$ where $\alpha_1 = (a_1, \ldots, a_n)$ and $\alpha_2 = (b_1, \ldots, b_m)$. Set $\alpha_i^1 = (p_i)$ for $0 \le i < \mathrm{per}(p)$, $\alpha_i^j = \alpha_i^{j-1}\alpha_{i+1}^{j-1}$ and $I_{\alpha_i^1} = I_{p_i}$. Let $I_{\alpha_i^j} \subset I_{\alpha_i^{j-1}}$ such that $f_S(I_{\alpha_i^j}) = I_{\alpha_{i+1}^{j-1}}$ (clearly $f_S(I_{\alpha_i^{j-1}}) \supset I_{\alpha_{i+1}^{j-1}}$). We have $f_S^i(p) \in I_{\alpha_i^{\mathrm{per}(p)}}$ and so $f_S^i(p) \in \mathrm{int}(I_{\alpha_i^{\mathrm{per}(p)}})$ because the endpoints of $I_{\alpha_i^{\mathrm{per}(p)}}$ are mapped by $f_S^{\mathrm{per}(p)}$ into the set S.

Let $K_{\alpha_i^1} = I_{p_i}$ for $1 \le i \le k$. By Lemma 1.3.1 we can choose $K_{\alpha_i^j} = [a_i^j, b_i^j]$ such that

(i) $K_{\alpha_i^j} \subset K_{\alpha_i^{j-1}}$

(ii) $f(K_{\alpha_i^j}) = K_{\alpha_{i+1}^{j-1}}$ and $\mathrm{conv}(\{f(a_i^j), f(b_i^j)\}) = K_{\alpha_{i+1}^{j-1}}$

(iii) $\mathrm{sign}_{f_S}(I_{\alpha_i^j}) = \mathrm{sign}_f(K_{\alpha_i^j})$ and

(iv) the order of $\{K_{\alpha_i^j}\}_{i=0}^{\mathrm{per}(p)-1}$ is the same as the order of $\{I_{\alpha_i^j}\}_{i=0}^{\mathrm{per}(p)-1}$.

We have $\mathrm{int}(K_{\alpha_i^j}) \ne \emptyset$ and because $\mathrm{int}(I_i) \cap \mathrm{int}(I_j) = \emptyset$ for $1 \le i \ne j \le k$ then $\mathrm{int}(I_{\alpha_i^n}) \cap \mathrm{int}(I_{\alpha_j^n}) = \emptyset$ and $\mathrm{int}(K_{\alpha_i^n}) \cap \mathrm{int}(K_{\alpha_j^n}) = \emptyset$ for $\alpha_i^n \ne \alpha_j^n$. But the 2^jth letter of $\alpha_i^{\mathrm{per}(p)}$ is p_{i+j} and so we have $\alpha_i^{\mathrm{per}(p)} \ne \alpha_j^{\mathrm{per}(p)}$ for $0 \le i \ne j < \mathrm{per}(p)$. Hence $K_{\alpha_{i+1}^{\mathrm{per}(p)}} \subset f(K_{\alpha_i^{\mathrm{per}(p)}})$ and $\mathrm{int}(K_{\alpha_i^{\mathrm{per}(p)}}) \cap \mathrm{int}(K_{\alpha_j^{\mathrm{per}(p)}}) = \emptyset$ for $0 \le i \ne j < \mathrm{per}(p)$. So there is $p^* \in \mathrm{Per}(f)$ such that $f^i(p^*) \in K_{\alpha_i^{\mathrm{per}(p)}}$ and $f^{\mathrm{per}(p)}(p^*) = p^*$.

If $\mathrm{per}(p^*) = \mathrm{per}(p)$ then the order of $\{f^i(p^*)\}_{i=0}^{\mathrm{per}(p)-1}$ is the same as the order of $\{K_{\alpha_i^{\mathrm{per}(p)}}\}_{i=0}^{\mathrm{per}(p)-1}$ which is the same as the order of $\{I_{\alpha_i^{\mathrm{per}(p)}}\}_{i=0}^{\mathrm{per}(p)-1}$ which is the same as the order of $\{f_S^i(p)\}_{i=0}^{\mathrm{per}(p)-1}$ and we are done.

If $\mathrm{per}(p^*) \ne \mathrm{per}(p)$ then $f^i(p^*) \notin \mathrm{int}(K_{\alpha_i^{\mathrm{per}(p)}})$ and so $p^* \in S$. But then $f^i(p^*) = f_S^i(p^*) \in I_{\alpha_i^1}$ and so $f_S^i(p^*) \in I_{\alpha_i^{\mathrm{per}(p)}}$. Because $f_S^{\mathrm{per}(p)}|_{I_{\alpha_i^{\mathrm{per}(p)}}}$ is linear and $f_S^{\mathrm{per}(p^*)}(p) \ne p$ then $f_S^{\mathrm{per}(p)}|_{I_{\alpha_i^{\mathrm{per}(p)}}}$ has at least two fixed points p, p^* and so it is the identity. Hence we have $I_{\alpha_i^{\mathrm{per}(p)}} = I_{\alpha_i^1} = I_{p_i}$. Let the endpoints of I_{p_i} be a_i, b_i. We can assume that $p \in [a_0, p^*]$. Then $f_S^i([a_0, p]) \cap f_S^j([a_0, p]) = \emptyset$ for

$0 \leq i \neq j < \mathrm{per}(p)$ and $f_S^{\mathrm{per}(p)}(a_0) = a_0$. So $\mathrm{orb}_{f_S}(a_0)$ is the same as $\mathrm{orb}_{f_S}(p)$. But $a_0 \in S$ and so $\mathrm{orb}_{f_S}(a_0) = \mathrm{orb}_f(a_0)$. Hence we can set $p^* = a_0$.

Case 2. There exists an $0 < n < \mathrm{per}(p)$ such that $f_S^n(J_i) \supset J_i$ and we can take the minimal one. Let $J_i^* \subset J_i$ be such that $f_S^j(J_i^*) \subset J_{i+j}$ for $0 < j < n$ and $f_S^n(J_i^*) = J_i$. Then we see that $f_S^n|_{J_i^*}$ is linear and moreover if $f_S^j(p) \in J_i$ then $f_S^j(p) \in J_i^*$. But $\mathrm{orb}(p) \cap J_i$ forms a periodic orbit of the function f_S^n and because $f_S^n|_{J_i^*}$ is linear the period of this orbit is 1 or 2. Hence we obtain $n = \frac{\mathrm{per}(p)}{2}$ and the slope of $f_S^n|_{J_i^*}$ is -1. But $f_S^n(J_i^*) = J_i$ and so $J_i^* = J_i$. Hence the endpoints of J_i form a periodic orbit which is the same as $\mathrm{orb}(p)$ and this orbit is a subset of S. The end of the proof is the same as in case 1. ■

CHAPTER II

ECCENTRICITY AND A GENERALIZATION OF SHARKOVSKIĬ'S THEOREM

2.0 Preliminaries

Consider Theorem 1.1.3 once again. We will call $\frac{\mu([0,p])}{\mu([p,1])}$ the eccentricity of measure μ around the fixed point p. We can reformulate Theorem 1.1.3 to say that a measure μ with eccentricity at least $\frac{k+1}{k}$ forces a periodic orbit with period $2k+1$.

But we know that any periodic orbit with period $2k+1$ forces a Štefan orbit with period $2k+1$ (see eg. [ALM2]) and the measure given by a Štefan orbit with period $2k+1$ has eccentricity $\frac{k+1}{k}$. Hence we have a statement that an invariant measure with eccentricity at least $\frac{k+1}{k}$ forces an invariant measure with eccentricity $\frac{k+1}{k}$. The natural hypothesis is that a measure with eccentricity $a \geq 1$ forces a measure with eccentricity b for any $b \in [1, a]$. Of course this is trivial if we take a linear combination of our measure and a measure concentrated on a fixed point p but is it true for ergodic measures?

Periodic orbits can be considered as a special class of ergodic measures and we will prove this hypothesis for them. Or to be more precise we will prove it for patterns. We will define our notion of "type" of a pattern. Let $(P = \{p_1, \ldots, p_n\}, \varphi)$ be a cycle with spatial labeling. If

$$(*) \qquad (p_i - \varphi(p_i)) \cdot (p_{i+1} - \varphi(p_{i+1})) < 0$$

then any continuous function with cycle P has a fixed point in the open interval (p_i, p_{i+1}). On the other hand if $(*)$ is not true, then there is a function with the cycle P that does not have a fixed point in the interval (p_i, p_{i+1}). Hence we can make the following

DEFINITION. A cycle (P, φ) has *eccentricity* $r \in \mathbb{Q}$ if for any map $f \in C(I, I)$ with the cycle P there is a fixed point $c \in \mathrm{Fix}(f)$ such that $\frac{\#\{x \in P;\ x \leq c\}}{\#\{x \in P;\ x \geq c\}} = r$. Note that a cycle $(h(P), h^{-1} \circ \varphi \circ h)$ where $h(x) = -x$ has eccentricity $\frac{1}{r}$ and so we define the eccentricity of a pattern $[P]$ as the eccentricity of a representative whose eccentricity is not smaller than one.

REMARK. Note that a pattern (or cycle) can have more than one eccentricity (see Fig. 2.1).

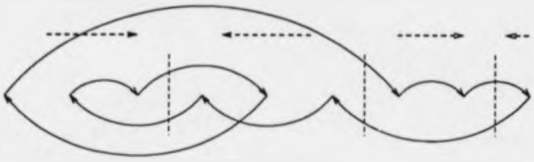

FIGURE 2.1. An example of a cycle P with eccentricities $\frac{1}{2}$, $\frac{2}{1}$ and $\frac{8}{1}$. The pattern $[P]$ has eccentricities $\frac{2}{1}$ and $\frac{8}{1}$.

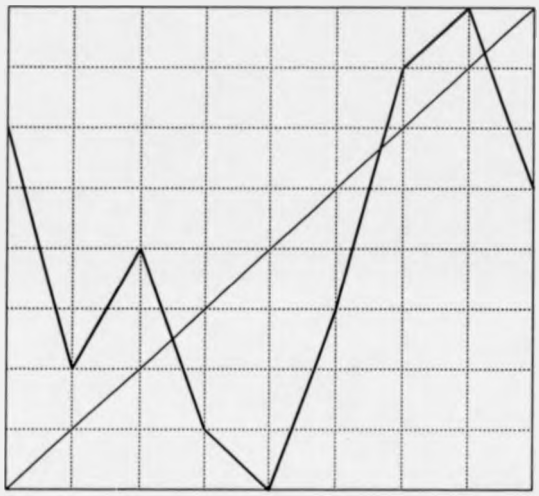

FIGURE 2.2. The graph of the function f_P.

We will fix an $r \in \mathbb{Q}$ and consider the set of all patterns with eccentricity r. We can look at the forcing relation restricted to this set. Some of the patterns may not force any other pattern from this set. These will be called *X-minimal patterns* with eccentricity r (*X-minimal r-patterns*) and for their representatives we shall use term *X-minimal cycles* (*X-minimal r-cycles*).

Note that a pattern with eccentricities $r, q \in \mathbb{Q}$ can theoretically be an X-minimal r-pattern but not an X-minimal q-pattern. (Later we will show that in fact this case is not possible).

The structure of this chapter is as follows. In section 2.1 we study the forcing relation between patterns with different eccentricity. Section 2.2 is devoted to the characterization of X-minimal patterns. In section 2.3 we prove the existence of X-minimal patterns and we give a simple algorithm for constructing all X-minimal

patterns. Finally in section 2.4 we define a new notion of the "type" of a pattern and using it we prove our generalization of Sharkovskii's Theorem.

We shall use some standard notions and techniques from combinatorial dynamics. The most important is the notion of P-linear map.

DEFINITION. Let (P, φ) be a periodic orbit and $I = \mathrm{conv}(P)$. Then $f_P \in C(I, I)$ such that $f_P|_P = \varphi$ and $f_P|_J$ is linear for any interval $J \subset I$ such that $J \cap P = \emptyset$. The function f_P is the piecewise linear function given by the cycle P and sometimes it is called the connect-the-dot map (see Fig. 2.2).

Very often we will use the following basic fact.

LEMMA 2.0.1. (Theorem 2.6.13 from [ALM2]) *Let (P, φ) be a cycle. If f_P exhibits pattern B then $[P]$ forces B.*

REMARK. Note that this lemma is only a restricted form of the Theorem 1.2.8.

We say that an interval J P-covers an interval L if $L \subset f_P(J)$. We will denote this by $J \xrightarrow{P} L$. A sequence of closed intervals $\mathcal{A} = \langle I_k \rangle_{k=1}^{m}$ is called P-cyclic if $I_1 \xrightarrow{P} I_2 \xrightarrow{P} \dots \xrightarrow{P} I_m \xrightarrow{P} I_1$. Note that a P-cyclic sequence is in fact a cycle of intervals and therefore we will consider two P-cyclic sequences equal if they form the same cycle and have the same length. This will allow us to start a cyclic sequence wherever we want by simply rotating it. The P-cyclic sequences and cycles of the function f_P are in close relation. Namely we have

LEMMA 2.0.2. ([B2], [BGMY], Lemma 1.2.7 from [ALM2]) *Let P be a periodic orbit and $\mathcal{A} = \langle I_k \rangle_{k=0}^{m-1}$ be P-cyclic. Then there is a periodic point $x \in \mathrm{Per}(f_P)$ such that $f_P^k(x) \in I_k$ for $k = 0, \dots, m-1$ and $f_P^m(x) = x$. The period of the cycle given by the point x and the function f_P divides m.*

We will say that a cycle obtained from a P-cyclic sequence $\mathcal{A}$ using the Lemma 2.0.2 is *contained* in the $\mathcal{A}$.

We will use the following simple notation for connecting two sequences. If $\mathcal{A} = \langle a_1, \dots, a_n \rangle$ and $\mathcal{B} = \langle b_1, \dots, b_m \rangle$ then $\mathcal{A} + \mathcal{B} = \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$.

Finally if $A, B \subset \mathbb{R}$ then we say that $A < B$ if $A \neq B$ and $a \leq b$ for any $a \in A$ and $b \in B$. If $x \in \mathbb{R}$ then we say that $x < A$ ($x > A$) if $\{x\} < A$ ($\{x\} > A$).

2.1 Unipatterns and forcing between patterns with different eccentricity

A cycle P with unique eccentricity will be called a *unicycle*[1]. We shall denote the eccentricity of a unicycle P by $E(P)$. A unicycle P with $E(P) = r$ will be

[1] Unfortunately there is no bicycle. However according to A. Manning there may be a little comfort for cyclists. Actually the object we study (a periodic orbit and a fixed point "in" it) consists of two cycles so it is a bicycle

called an *r-unicycle*. Similarly we shall use the terms *unipattern*, $E([P])$ and *r-unipattern*. Note that if a cycle P is not a unicycle then the pattern $[P]$ has at least two different eccentricities. Therefore a representative of a unipattern is a unicycle.

We will show that an X-minimal pattern must be a unipattern.

LEMMA 2.1.1. *Suppose the cycle P is not a unicycle. Then f_P has an r-unicycle for any positive $r \in \mathbb{Q}$.*

PROOF. Because P is not a unicycle the function f_P has at least two fixed points (in fact at least three). If $r = 1$ then a fixed point gives an r-cycle. So we can assume that $r > 1$ (the case $r < 1$ is similar).

Let $z_1 < z_2$ be two rightmost fixed points of f_P. Hence $f_P(x) > x$ for $x \in (z_1, z_2)$ and $f_P(x) < x$ for $x > z_2$. Let $a \in (z_1, z_2)$ be such that $f_P(a) \geq f_P(x)$ for any $x \in (z_1, z_2)$. Clearly $f_P(a) > z_2$ (otherwise the interval $[z_1, z_2]$ would be f_P-invariant which is impossible because it contains a point from P). Now let $J_1 = [z_1, a]$ and $J_2 = [z_2, f_P(a)]$. We have that $J_1 \xrightarrow{P} J_1$ and $J_1 \xrightarrow{P} J_2$. Moreover $J_2 \xrightarrow{P} J_1$ (otherwise interval $[z_1, f_P(a)]$ would be f_P-invariant) (see Fig. 2.3). Hence for any $m \geq n$

$$\overbrace{\langle J_1, \ldots, J_1 \rangle}^{m-n \text{ times } J_1} + \overbrace{\langle J_1, J_2, \ldots, J_1, J_2 \rangle}^{n \text{ times } J_1, J_2}$$

is a P-cyclic sequence. Using Lemma 2.0.2 we obtain a cycle Q with period $m+n$. So it does not contain any fixed point. But $Q \subset [z_1, f_P(a)]$ and $(z_1, f_P(a)) \cap \mathrm{Fix}(f_P) = \{z_2\}$ and so Q is a unicycle. Finally because $J_1 < z_2 < J_2$ we have that Q is an $\frac{m}{n}$-unicycle and we can choose $\frac{m}{n} = r$. ■

Hence we have the straightforward

COROLLARY 2.1.2. *An X-minimal pattern is a unipattern.*

PROOF. If an r-pattern A is not a unipattern then its representative P is not a unicycle. By Lemma 2.1.1 function f_P exhibits an r-unipattern B and by Lemma 2.0.1 pattern A forces B. But $A \neq B$ (one is a unipattern and the other is not) and so A is not an X-minimal r-pattern. ■

Now we would like to find all patterns forced by a unipattern $[P]$.

First note that f_P has a unique fixed point and therefore every cycle it has is a unicycle. So a unipattern can force only unipatterns. Later we will often use this fact without mentioning it.

One possible way to find patterns forced by a unipattern $[P]$ is to find all P-cyclic sequences and using Lemma 2.0.1 and 2.0.2 we can get some of the patterns forced by $[P]$. But in general if we have a P-cyclic sequence then we have no information about the eccentricity of a pattern forced by this sequence,. Fortunately, for some special P-cyclic sequences we can get this information.

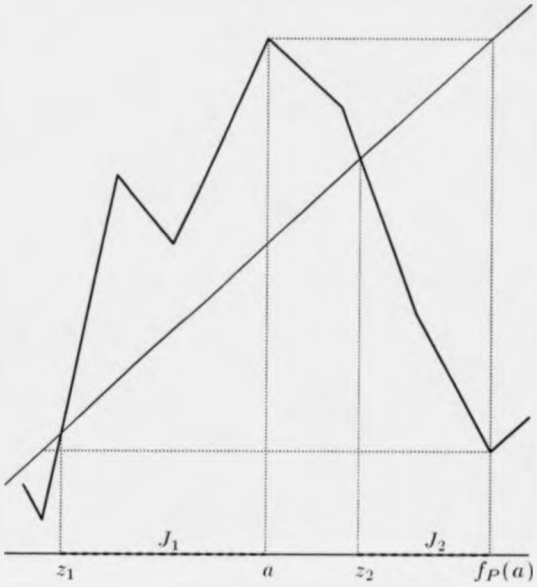

FIGURE 2.3

DEFINITION. Let P be a unicycle and $\mathrm{Fix}(f_P) = \{c\}$. A P-cyclic sequence $\mathcal{A} = \langle I_i \rangle_{i=1}^{a}$ will be called *separated* if $c \notin \mathrm{int}(I)$ for any $I \in \mathcal{A}$. The eccentricity of a separated P-cyclic sequence $\mathcal{A}$ will be $E(\mathcal{A}) = \frac{\#\{i; I_i \leq c\}}{\#\{i; I_i \geq c\}}$.

LEMMA 2.1.3. *Let P be a unicycle, $c \in \mathrm{Fix}(f_P)$ and $\mathcal{A}$ be a separated P-cyclic sequence. Then f_P has an $E(\mathcal{A})$-unicycle contained in the P-cyclic sequence $\mathcal{A}$.*

PROOF. Assume that $E(\mathcal{A}) \geq 1$ (the case $E(\mathcal{A}) \leq 1$ is similar).

If there is $I \in \mathcal{A}$ such that $c \notin I$ then the cycle Q that we get from the sequence $\mathcal{A}$ by using Lemma 2.0.2 clearly has eccentricity $E(\mathcal{A})$.

Assume that $c \in I$ for all $I \in \mathcal{A}$. If $E(\mathcal{A}) = 1$ then the fixed point c gives such a cycle. So we can assume that $E(\mathcal{A}) > 1$. Then $\mathcal{A} = \langle \ldots, I, J, K \ldots \rangle$ where $I, J < c$ and $K > c$ ($I \neq \{c\}$ because $E(\mathcal{A}, c) \neq 1$). Now there are two possibilities.

1. $f_P(x) \geq c$ for any $x \in J$ or
2. there is an $a \in J$ such that $f_P(a) = c$ and $a \neq c$.

In case 1 we have that $J \subset I$. So there is a point $a \in I$ such that $f_P(a) = \inf J$ and a point $b \in I$ such that $b \neq c$ and $f_P(b) = c$. Hence we can replace interval I by $I^* = \mathrm{conv}(\{a, b\})$ and we have again a separated P-cyclic sequence $\mathcal{B}$.

In case 2 let $b \in J$ such that $f_P(b) = \sup\{f_P(x); x \in J\}$. Now we can replace interval J by $J^* = \mathrm{conv}(\{a, b\})$ to get a separated P-cyclic sequence $\mathcal{B}$.

In both cases we obtain a new separated P-cyclic sequence $\mathcal{B}$ with eccentricity $E(\mathcal{A})$. But now there is an interval in $\mathcal{B}$ (I^* or J^*) that does not contain the point c. Hence using the argument above there is a $E(\mathcal{A})$-unicycle Q in f_P. ■

Now the question is how can we tell whether we have picked up all possible P-cyclic sequences that can give us some information about patterns forced by $[P]$. We will show that it is enough to examine those P-cyclic sequences that have their elements only from a set of intervals given by the cycle P.

DEFINITION. For a unicycle (P, φ) let $\mathfrak{P}$ be the partition of the interval $I = \mathrm{conv}(P)$ into intervals with endpoints in $P \cup \mathrm{Fix}(f_P)$.

In particular if $P = \{p_1, p_2, \ldots, p_{k(m+n)}\}$ is a unicycle with spatial labeling where $k, m, n \in \mathbb{N}$, m, n are coprime, $E(P) = \frac{m}{n}$ and $\mathrm{Fix}(f_P) = \{c\}$, the partition $\mathfrak{P} = \{J_i\}_{i=1}^{k(m+n)}$ where

$$\begin{aligned} J_i &= [p_i, p_{i+1}] && \text{for } i < km \\ J_{km} &= [p_{km}, c] \\ J_{km+1} &= [c, p_{km+1}] \\ J_i &= [p_{i-1}, p_i] && \text{for } i > km+1. \end{aligned}$$

A P-cyclic sequence $\mathcal{A} = \langle I_i \rangle_{i=1}^{a}$ such that each $I_i \in \mathfrak{P}$ will be called a P-loop. (Note that any P-loop is separated.)

Now we can prove some kind of converse of Lemma 2.0.2.

LEMMA 2.1.4. *Let P be a unicycle and $Q = \{q_1, \ldots, q_m\}$ be a cycle of f_P with* $\mathrm{per}(Q) = m > 1$. *Then there is a unique P-loop $\mathcal{A} = \langle I_i \rangle_{i=1}^{m}$ such that $f_P^{i-1}(q_1) \in I_i$ for $1 \le i \le m$.*

PROOF. If $Q \neq P$ then for any q_i there is a unique interval $I_i \in \mathfrak{P}$ such that $q_i \in I_i$. Moreover because f_P is linear on any interval $I \in \mathfrak{P}$ and $q_i \in \mathrm{int}(I_i)$ we have that $I_i \xrightarrow{P} I_{i+1}$ and so the sequence $\mathcal{A} = \langle I_i \rangle_{i=1}^{m}$ is a P-loop.

So assume that $Q = P = \{p_1, \ldots, p_n\}$ with spatial labeling. Then there is a unique interval $I_1 \in \mathfrak{P}$ such that $p_1 \in I_1$. Assume that there is given interval $I_j \in \mathfrak{P}$ such that $f_P^{j-1}(p_1) \in I_j$. There are at most two intervals $I \in \mathfrak{P}$ such that $f_P^j(p_1) \in I$ but only one of them satisfies condition $I_j \xrightarrow{P} I$ (because f_P is linear on I_j and $f_P^{j-1}(p_1)$ is an endpoint of I_j). Hence there is a unique $I_{j+1} \in \mathfrak{P}$ such that $f_P^j(p_1) \in I_{j+1}$ and $I_j \xrightarrow{P} I_{j+1}$. Therefore there is a unique P-loop of a length $\mathrm{per}(P)$ containing the cycle P too. ■

REMARK. Of course a cycle Q can be contained in more than one P-loop. But any P-loop containing Q is only a repetition of a unique P-loop $\mathcal{A}$ that has length equal $\mathrm{per}(Q)$.

DEFINITION. We shall denote the P-loop containing cycle P by $\mathcal{A}_P$.

We will say that P-loop $\mathcal{A}$ is simple if there are no two nonempty P-loops $\mathcal{B}, \mathcal{C}$ such that $\mathcal{A} = \mathcal{B} + \mathcal{C}$.

LEMMA 2.1.5. *A P-loop containing some interval more than once is not simple.*

PROOF. After rotating we can write our P-loop as $\overbrace{\langle I, \ldots, K \rangle}^{\mathcal{B}} + \overbrace{\langle I, \ldots, L \rangle}^{\mathcal{C}}$ and both $\mathcal{B}$ and $\mathcal{C}$ are nonempty P-loops. ■

Now we will look at a unicycle P and its loop $\mathcal{A}_P$. There are basically two possibilities. Either $\mathcal{A}_P$ is simple or not. The next lemma shows the importance of simple $\mathcal{A}_P$.

LEMMA 2.1.6. *Let P be a unicycle with* $\mathrm{per}(P) > 2$ *and simple loop $\mathcal{A}_P$. Then for each P-loop $\mathcal{A}$ there is a unique cycle contained in $\mathcal{A}$.*

PROOF. Assume the contrary. Let $\mathcal{A} = \langle I_i \rangle_{i=1}^m$ be a P-loop and let $x < y \in \mathrm{Per}(f_P)$ be such that $f_P^{i-1}(x), f_P^{i-1}(y) \in I_i$ for $1 \leq i \leq m$ and $f_P^m(x) = x$ and $f_P^m(y) = y$. Hence $f_P^m|_{[x,y]}$ is linear and therefore the identity. We will take the smallest possible $a \in \mathbb{N}$ such that $f_P^a|_{[x,y]}$ is the identity and that means that x, y have either period a or period $\frac{a}{2}$.

Take the points $x^*, y^* \in I_1$ such that $x^* \leq x < y \leq y^*$, $f_P^n(x^*), f_P^n(y^*) \in P \cup \mathrm{Fix}(f_P)$ for some $n \in \mathbb{N}$, $f_P^n|_{[x^*,y^*]}$ is linear and $(x^*, y^*) \cap (P \cup \mathrm{Fix}(f_P)) = \emptyset$.

So we have that $f_P^n|_{[x^*,y^*]}$ is the identity and therefore $x^*, y^* \in P \cup \mathrm{Fix}(f_P)$. Hence $f_P^i([x^*, y^*]) \in \mathfrak{P}$ for any $i \geq 0$ and if $\{x^*, y^*\} \cap \mathrm{Fix}(f_P) \neq \emptyset$ then $\mathrm{per}(P) \leq 2$. Therefore $x^*, y^* \in P$. Moreover a is the smallest possible number such that $f_P^a|_{[x^*,y^*]}$ is the identity and therefore $\mathrm{per}(P) = a$.

Take a sequence $\mathcal{A}^* = \langle f_P^i([x^*, y^*]) \rangle_{i=1}^a$. Clearly $\mathcal{A}^*$ is a P-loop and moreover it contains the cycle P. Therefore from Lemma 2.1.4 we have $\mathcal{A}^* = \mathcal{A}_P$. But it is easy to see that $\mathcal{A}^*$ is not simple (it contains the interval $[x^*, y^*]$ twice - once covering the point x^* and second time covering the point y^*) – a contradiction. ■

Now we will investigate the forcing relation between patterns with different eccentricities.

LEMMA 2.1.7. *Let P be a unicycle with $E(P) \geq 1$ and $\mathcal{A}_P$ be not simple. Then f_P has a unicycle Q such that* $\mathrm{per}(Q) < \mathrm{per}(P)$ *and $E(Q) \geq E(P)$.*

PROOF. Because $\mathcal{A}_P$ is not simple there are P-loops $\mathcal{B}, \mathcal{C}$ such that $\mathcal{A}_P = \mathcal{B} + \mathcal{C}$. But either $E(\mathcal{B}) \geq E(\mathcal{A}_P)$ or $E(\mathcal{C}) \geq E(\mathcal{A}_P)$ and they are both shorter than loop $\mathcal{A}_P$. Hence we are finished by Lemmas 2.1.3 and 2.0.2. ■

LEMMA 2.1.8. *A unipattern A forces some unipattern B such that $E(A) \leq E(B)$, $\mathrm{per}(A) \geq \mathrm{per}(B)$ and Q, representative of B, has a simple loop $\mathcal{A}_Q$.*

PROOF. If P, a representative of A, has a simple loop then $B = A$. If not then by Lemma 2.1.7 the pattern A forces a unipattern A^* such that $E(A) \leq E(A^*)$, $\mathrm{per}(A) > \mathrm{per}(A^*)$. Because $\mathrm{per}(A)$ is finite after repeating this finitely many times we must get our unipattern B. ■

LEMMA 2.1.9. *Let A be an r-unipattern and let its representative P have a simple loop $\mathcal{A}_P$. Then the pattern A forces some q-unipattern for each $q \in \mathbb{Q}$ such that $r \geq q \geq 1$.*

PROOF. We may assume that $E(P) = r > 1$ (the case $r = 1$ is trivial) and $\mathrm{per}(P) = k(m+n)$ where $\frac{m}{n} = r$ (m, n are coprime).

Because $\mathcal{A}_P$ is simple it contains every interval from the partition $\mathfrak{P}$. So we may assume that the loop $\mathcal{A}_P$ starts with interval J_{km}. Moreover $J_{km} \xrightarrow{P} J_{km+1}$ and $J_{km+1} \xrightarrow{P} J_{km}$. Hence

$$B = \langle \overbrace{J_{km}, J_{km+1}, \ldots, J_{km}, J_{km+1}}^{a \text{ times } J_{km}, J_{km+1}} \rangle + \overbrace{\mathcal{A}_P + \cdots + \mathcal{A}_P}^{b \text{ times } \mathcal{A}_P}$$

is a P-loop with eccentricity $\frac{a+bm}{a+bn}$. If $q = \frac{r}{s}$ then we can choose $a = ms - rn$ and $b = r - s$. So $\frac{a+bm}{a+bn} = q$. Hence from Lemma 2.1.3 we get that f_P has a q-unicycle and by Lemma 2.0.1 the pattern A forces a q-unipattern. ■

Now we can easily get the final statement of this section.

THEOREM 2.1.10. *Let $r, q \in \mathbb{Q}$ satisfy $r \geq q \geq 1$. Then any r-pattern forces a q-unipattern.*

PROOF. Let A be an r-pattern. If A is not a unipattern then, by Lemma 2.1.1 and Lemma 2.0.1, A forces a q-unipattern. If A is a unipattern then, by Lemma 2.1.8, it forces a unipattern B such that $E(B) \geq r$ and P a representative of B has a simple loop $\mathcal{A}_P$. By Lemma 2.1.9 the pattern B forces a q-unipattern but because forcing relation is transitive we have that A forces a q-unipattern. ■

2.2 X-minimal patterns

First we recall the

DEFINITION. An r-pattern is *X-minimal* if it does not force any other r-pattern.

Now we would like to find all X-minimal r-patterns. We already have some information about such patterns. More precisely we have

LEMMA 2.2.1. *An X-minimal pattern is a unipattern and its representative P has a simple loop $\mathcal{A}_P$.*

PROOF. If $[P]$ is an X-minimal pattern then by Corollary 2.1.2 it is a unipattern. If it does not have a simple loop $\mathcal{A}_P$ then by Lemma 2.1.8 it forces a unipattern $[Q]$ with a simple loop $\mathcal{A}_Q$ and $E(Q) > E(P)$. So $[P] \neq [Q]$ and by Lemma 2.1.9 the pattern $[Q]$ forces a pattern with eccentricity $E(P)$. Finally because the forcing relation is antisymmetrical we have that $[P]$ is not X-minimal - a contradiction. ■

DEFINITION. Let P be a unicycle and $c \in \mathrm{Fix}(f_P)$. A sequence $Q = \langle q_i \rangle_{i=0}^{a}$ will be called a *P-semicycle* if

$$\begin{aligned} q_i &\in P \\ f_P(q_{i-1}) &= q_i \qquad \text{for} \quad 1 \leq i \leq a \\ q_0 &\neq q_a \\ q_0 &\in \mathrm{conv}\{q_a, c\}. \end{aligned}$$

Eccentricity of the P-semicycle Q will be $E(Q) = \frac{\#\{i>0:q_i \leq c\}}{\#\{i>0:q_i > c\}}$. (See Fig. 2.4.)

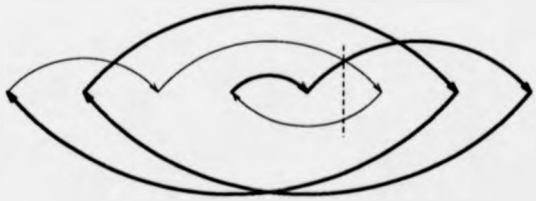

FIGURE 2.4. A cycle P with a semicycle Q (thick lines). $E(Q) = \frac{3}{2}$.

LEMMA 2.2.2. *Let P be a unicycle with a P-semicycle Q. Then f_P has a $E(Q)$-cycle R such that* $\mathrm{per}(R) \neq \mathrm{per}(P)$.

PROOF. Let $Q = \langle q_i \rangle_{i=0}^{a}$ and $I_i = \mathrm{conv}\{q_i, c\}$. Clearly $I_0 \xrightarrow{P} I_1 \xrightarrow{P} \ldots \xrightarrow{P} I_a$ and $I_0 \subset I_a$. Therefore $\langle I_i \rangle_{i=1}^{a}$ is a separated P-cyclic sequence with eccentricity $E(Q)$. By Lemma 2.1.3 the function f_P has an $E(Q)$-cycle R. Moreover a is not divisible by $\mathrm{per}(P)$ and therefore $\mathrm{per}(R) \neq \mathrm{per}(P)$. ■

DEFINITION. Let P be an $\frac{m}{n}$-unicycle where $m \geq n \in \mathbf{N}$ are coprime and $c \in \mathrm{Fix}(f_P)$. Define the coding $K_P : P \to \mathbf{Z}$ by

$$\begin{aligned} K_P(p_1) &= 0 \\ K_P(f_P(p_i)) &= K_P(p_i) + n \qquad \text{for} \quad p_i < c \\ K_P(f_P(p_i)) &= K_P(p_i) - m \qquad \text{for} \quad p_i > c. \end{aligned}$$

We say that P has monotone code if and only if either $\mathrm{per}(P) = 1$ or $E(P) > 1$ and for any $p, q \in P$ such that $p \neq q$ and $q \in \mathrm{conv}(\{p, c\})$ we have $K_P(q) > K_P(p)$ (see Fig. 2.5).

If P has monotone code we also say that the pattern $[P]$ has monotone code.

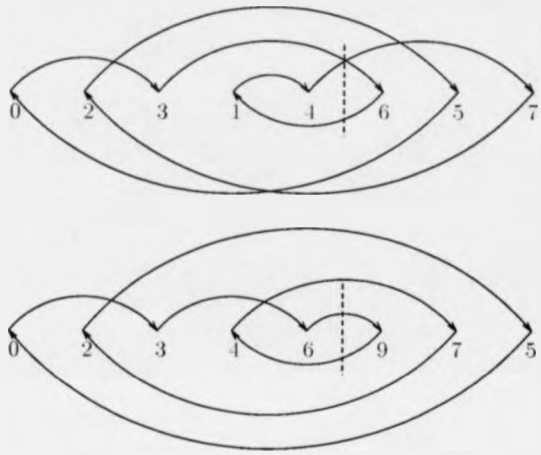

FIGURE 2.5. An example of a cycle without (top) and with (bottom) monotone code.

LEMMA 2.2.3. *An X-minimal pattern has monotone code.*

PROOF. Let A be an X-minimal $\frac{m}{n}$-pattern (m, n are coprime) and an $\frac{m}{n}$-cycle P be a representative of A. From Lemma 2.1.1 we have that P is a unicycle.

Assume that P does not have monotone code. Then there are two different $p, q \in P$ such that $q \in \mathrm{conv}(\{p, c\})$ and $K_P(q) \leq K_P(p)$. Set $Q = \langle q_j \rangle_{j=0}^{a}$ where $q_0 = q$, $q_{j+1} = f_P(q_j)$ and $q_a = p$. Clearly Q is a semicycle and we can estimate $E(Q)$. From definition of K_P we have that

$$K_P(q_a) = K_P(q_0) + n\#\{j;\ 0 \leq j < a,\ q_j < c\} - m\#\{j;\ 0 \leq j < a,\ q_j > c\}.$$

Hence $E(Q) = \frac{\#\{j;\ 0 \leq j < a,\ q_j < c\}}{\#\{j;\ 0 \leq j < a,\ q_j > c\}} \geq \frac{m}{n}$. Using Lemma 2.2.2 and Lemma 2.0.1 we get that A forces a pattern $B \neq A$ such that $E(B) \geq \frac{m}{n}$ and from Theorem 2.1.10 we have that B forces an $\frac{m}{n}$-pattern C. But $A \neq C$ because the forcing relation is antisymmetric and so A is not an X-minimal $\frac{m}{n}$-pattern - a contradiction. ■

So we have proved that an X-minimal pattern is a unipattern with monotone code. Now we are going to get more information about a unicycle with monotone code.

Let $(P = \{p_1, p_2, \ldots, p_{k(m+n)}\}, \varphi)$ be an $\frac{m}{n}$-unicycle with spatial labeling, monotone code ($m \geq n$ are coprime) and $c \in \mathrm{Fix}(f_P)$. From the monotonicity of code we immediately have that $\varphi(p_i) < c$ for $i > km$.

Hence we can define a new cycle (P^*, ψ) where $P^* = \{p_i\}_{i=1}^{km}$ and

$$\begin{aligned} &\text{if} \quad \varphi(p_i) \in P^* \qquad \text{then} \quad \psi(p_i) = \varphi(p_i) \\ &\text{if} \quad \varphi(p_i) \notin P^* \qquad \text{then} \quad \psi(p_i) = \varphi^2(p_i). \end{aligned}$$

So we can have

DEFINITION. Let $C_P = \langle c_i \rangle_{i=1}^{kn}$ where $c_i \in \{0, 1\}$ be a code given to the cycle P in the following way

$$\begin{aligned} c_i = 0 \qquad &\text{if} \quad \psi^i(p_1) = \varphi(\psi^{i-1}(p_1)) \\ c_i = 1 \qquad &\text{if} \quad \psi^i(p_1) = \varphi^2(\psi^{i-1}(p_1)). \end{aligned}$$

From the monotonicity of code K_P it can be seen that the code C_P can be also obtained from the cycle (P^*, ψ) if we start in the point p_1 and following cycle we write 0 if we move right and 1 if left (see Fig. 2.6).

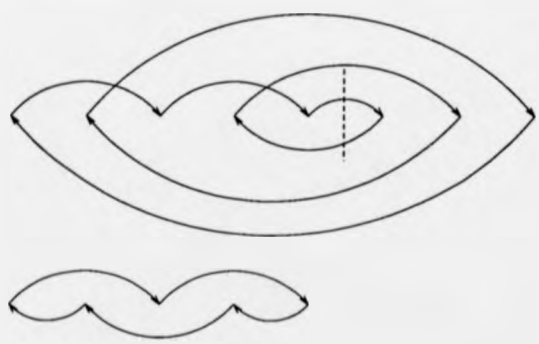

FIGURE 2.6. An example of a cycle (P, φ) (top) and (P^*, ψ) (bottom) with $C_P = \langle 0, 0, 1, 1, 1 \rangle$.

Note that C_P contains kn units and $k(m-n)$ zeros. Moreover

$$\begin{aligned} &\text{if} \quad c_i = 0 \qquad \text{then} \quad K_P(\psi^i(p_1)) = K_P(\psi^{i-1}(p_1)) + n \\ &\text{if} \quad c_i = 1 \qquad \text{then} \quad K_P(\psi^i(p_1)) = K_P(\psi^{i-1}(p_1)) - m + n. \end{aligned}$$

Hence we have the following connection between K_P and C_P:

$$K_P(\psi^i(p_1)) = in - m \sum_{j=1}^{i} c_j$$

LEMMA 2.2.4. *Let $m > n$ be coprime and P be an $\frac{m}{n}$-unicycle with monotone code. Then* $\mathrm{per}(P) = m + n$.

PROOF. Assume that $P = \{p_1, \ldots, p_{k(m+n)}\}$ with spatial labeling and $k > 1$. We will study the code C_P.

Let i_j be such that $c_{i_j} = 1$ and $\sum_{i=1}^{i_j} c_i = j$ (i_j is the place of jth unit in the sequence C_P).

Because $k > 1$ we have $\psi^{i_n}(p_1) \neq p_1$ and from monotonicity of the code we have $K_P(\psi^{i_n}(p_1)) > 0$. But $K_P(\psi^{i_n}(p_1)) = ni_n - mn$ and so we have that $i_n > m$.

Moreover from monotonicity of code we have that no two points from P^* can have the same value of the code K_P. If there is a part $C^* = \langle c_i \rangle_{i=j+1}^{j+m}$ of the code C_P such that $\sum_{i=j+1}^{j+m} c_i = n$ then $K_P(\psi^{j+m}(p_1)) = K_P(\psi^j(p_1)) + (m-n)n + n(n-m) = K_P(\psi^j(p_1))$. But $\psi^{j+m}(p_1) \neq \psi^j(p_1)$ ($k > 1$) and so we have a contradiction with the monotonicity of code. So there must not be part of a sequence C_P of length m which contains $m - n$ times 0 and n times 1. Hence we get that $i_n - i_1 \geq m$ (otherwise the sequence $\langle c_i \rangle_{i=i_n-m+1}^{i_n}$ contains $m - n$ times 0 and n times 1).

Therefore $i_1 < i_{n+1} - m + 1$ and using sequence $\langle c_i \rangle_{i=i_{n+1}-m+1}^{i_{n+1}}$ as above we obtain $i_{n+1} - i_2 \geq m$. Inductively for all $j \leq (k-1)n$

$$i_{n+j} - i_{1+j} \geq m.$$

We have $c_1 = 0$ because $K_P(\psi(p_1)) \geq 0$ (monotonicity) and so $1 < i_1 < i_2 < \cdots < i_{kn-1} < i_{kn} \leq kn$. Using the inequalities above we obtain

$$km \geq 1 + \sum_{j=1}^{k} (i_{jn} - i_{(j-1)n+1}) \geq 1 + \sum_{j=1}^{k} m = 1 + km$$

which is a contradiction. ■

LEMMA 2.2.5. *Let P be a unicycle which is not X-minimal. Then f_P has a unicycle R such that that* $\mathrm{per}(R) < \mathrm{per}(P)$ *and* $E(R) \geq E(P)$.

PROOF. If $\mathcal{A}_P$ is not simple then by Lemma 2.1.7 function f_P has a unicycle Q with $\mathrm{per}(Q) < \mathrm{per}(P)$ and $E(Q) \geq E(P)$. By Lemma 2.1.8 we have that f_P has a unicycle R such that $\mathrm{per}(R) \leq \mathrm{per}(Q)$ and $E(R) \geq E(Q)$ and so we are done.

Assume that $\mathcal{A}_P$ is simple. If $\mathrm{per}(P) = 2$ then cycle given by a fixed point is our cycle R. So we can assume that $\mathrm{per}(P) > 2$ (if $\mathrm{per}(P) = 1$ then P is X-minimal).

Because P is not X-minimal the function f_P contains a cycle $Q \neq P$ with $E(Q) = E(P)$. Let $\mathcal{A}$ be the unique P-loop containing cycle Q (Lemma 2.1.6). Because $\mathcal{A}_P$ is simple and $P \neq Q$, using Lemma 2.1.6 we have that $\mathcal{A} \neq \mathcal{A}_P + \cdots + \mathcal{A}_P$. Hence $\mathcal{A}$ can be written as the sum of two P-loops $\mathcal{B} + \mathcal{C}$ (the loop $\mathcal{C}$ may be empty) such that $\mathcal{B}$ is a simple P-loop, $E(\mathcal{B}) \geq E(\mathcal{A}_P)$ and $\mathcal{B} \neq \mathcal{A}_P$.

If the length of $\mathcal{B}$ is smaller then per(P) then cycle R given by the P-loop $\mathcal{B}$ (Lemma 2.0.2) is the one we are looking for (see Lemma 2.1.3).

So the length of $\mathcal{B}$ is at least per(P) and from Lemma 2.1.5 we have that it must be equal per(P). Hence both $\mathcal{A}_P$ and $\mathcal{B}$ contain all intervals from $\mathfrak{P}$. Because they are different there are intervals $I, J, K \in \mathfrak{P}$ such that $J \neq K$ and

$$\begin{aligned} \mathcal{A}_P &= \langle \ldots, I, J, \ldots \rangle \\ \mathcal{B} &= \langle \ldots, I, K, \ldots \rangle. \end{aligned}$$

Hence after suitable rotation we can write

$$\begin{aligned} \mathcal{A}_P &= \overbrace{\langle J, \ldots, L \rangle}^{\mathcal{D}} + \overbrace{\langle K, \ldots, I \rangle}^{\mathcal{E}} \\ \mathcal{B} &= \langle K, \ldots, I \rangle. \end{aligned}$$

Note that $\mathcal{D}$ is nonempty and the loop $\mathcal{E}$ is P-cyclic.

If $E(\mathcal{E}) \geq E(\mathcal{A}_P)$ then the loop $\mathcal{E}$ gives us a cycle R with period smaller than per(P) and we are done (Lemmas 2.0.2 and 2.1.3).

Otherwise $\mathcal{D} + \mathcal{B}$ is a P-cyclic loop with $E(\mathcal{D} + \mathcal{B}) > E(\mathcal{A}_P)$. So it can be written as a sum of two P-loops such that one of them is a simple P-loop $\mathcal{F}$ such that $E(\mathcal{F}) > E(\mathcal{A}_P)$. This loop has length smaller then per(P) (all simple P-loops with length have eccentricity $E(\mathcal{A}_P)$) and so it will give us a cycle R with period smaller than per(P) (Lemmas 2.0.2 and 2.1.3). ■

Let P be an $\frac{m}{n}$-unicycle with a simple loop $\mathcal{A}_P = \langle I_i \rangle_{i=1}^{k(m+n)}$. We have $f_P^{i-1}(p_1) \in I_i$. So we may define a map $\pi : P \to \mathfrak{P}$ such that $\pi(f_P^{i-1}(p_1)) = I_i$. Because $\mathcal{A}_P$ is simple π is a bijection. Moreover if $\pi(p_j) = I_i$ then $p_j \in I_i$. Hence

$$\pi(p_i) = J_i \qquad \text{for} \quad 1 \leq i \leq k(m+n).$$

We recall that $\mathfrak{P} = \{J_i\}_{i=1}^{k(m+n)}$ with spatial labeling (see the definition of $\mathfrak{P}$) (see Fig. 2.7).

FIGURE 2.7. The arrows show on which intervals points of the cycle are mapped by π.

We know that $\pi(x) \xrightarrow{P} \pi(f_P(x))$ and using the bijection π we can naturally define a coding $K : \mathfrak{P} \to \mathbb{Z}$ similar to the coding $K_P : P \to \mathbb{Z}$ where

$$K(J_i) = K(\pi(p_i)) = K_P(p_i).$$

Now we shall estimate how the code of the intervals in the partition $\mathfrak{P}$ depends on the P-covering property for these intervals.

LEMMA 2.2.6. *Let P be an $\frac{m}{n}$-unicycle with monotone code, $\mathfrak{P} = \{J_i\}_{i=1}^{m+n}$ with spatial labeling and $J_i, I \in \mathfrak{P}$ such that $J_i \xrightarrow{P} I$. If $i \leq m$ then $K(I) \geq K(J_i) + n$ and if $i > m$ then $K(I) \geq K(J_i) - m$.*

PROOF. If $i = m$ then $J_i = [p_m, c]$ and so $I \subset [c, f_P(p_m)]$. From the monotonicity of the code we have $K(I) \geq K(\pi(f_P(p_m))) = K(J_i) + n$.

If $i < m$ then $J_i = [p_i, p_{i+1}]$. But f_P is linear on J_i and so if $J_i \xrightarrow{P} J_j$ then $p_j \in \operatorname{conv}\{f_P(p_i), f_P(p_{i+1})\}$. From the monotonicity of the code we have that $K_P(p_j) \geq \min\{K_P(f_P(p_i)), K_P(f_P(p_{i+1}))\}$. But $K_P(f_P(p_i)) < K_P(f_P(p_{i+1}))$ and so $K_P(p_j) \geq K_P(f_P(p_i)) = K_P(p_i) + n$. Hence $K(I) \geq K(J_i) + n$.

Now let $i > m$. From monotonicity of code we have that $f_P(p_{j+1}) < f_P(p_j) < c$ for all $j > m$. Hence $f_P([c, p_i]) = [f_P(p_i), c]$ and so $K(I) \geq K_P(f_P(p_i)) = K(J_i) - m$. ■

Finally we are ready to prove

THEOREM 2.2.7. *Let P be a periodic orbit. Then P is X-minimal if and only if it is a unicycle with monotone code.*

PROOF. The necessity of these conditions is proved in Lemmas 2.1.1 and 2.2.3. Now we will show that they are sufficient too.

Let P be an $\frac{m}{n}$-unicycle with monotone code. If P is not X-minimal then by Lemma 2.2.5 there is cycle Q of f_P such that $\operatorname{per}(Q) < \operatorname{per}(P)$ and $E(Q) \geq E(P)$. By Lemma 2.1.4 there is a P-loop $\mathcal{A} = \langle I_i \rangle_{i=1}^{a+b}$ such that $E(\mathcal{A}) = E(Q) = \frac{a}{b}$ where $a = \#\{i; I_i < c\}$ and $b = \#\{i; I_i > c\}$. Finally by Lemma 2.2.6 we have that

$$K(I_1) = K(I_1) + \sum_{i=1}^{a} n_i - \sum_{i=1}^{b} m_i$$

where $n_i \geq n$ and $m_i \leq m$. Therefore $\frac{a}{b} \leq \frac{m}{n}$. But from Lemma 2.2.4 we have that $a + b < m + n$ and hence $\frac{a}{b} < \frac{m}{n}$ which contradicts $E(Q) \geq E(P)$. ■

From this Theorem we have immediately

COROLLARY 2.2.8. *A pattern is X-minimal if and only if it is a unipattern with monotone code.*

Although it is very easy to check if a pattern is a unipattern with monotone code it is still not a "look and see" (geometrical) characterization. We have at least some easy necessary geometrical condition.

LEMMA 2.2.9. *Let P be a representative of an X-minimal pattern with $E(P) > 1$. Then P is unicycle and for any $p, q \in P$ and* $\mathrm{Fix}(f_P) = \{c\}$ *we have*

$$\begin{array}{llllll} \textit{if} & p < q < c & \textit{and} & f_P(p), f_P(q) < c & \textit{then} & f_P(p) < f_P(q) \\ \textit{if} & p < q < c & \textit{and} & f_P(p), f_P(q) > c & \textit{then} & f_P(p) > f_P(q) \\ \textit{if} & p > q > c & \textit{then} & f_P(p), f_P(q) < c & \textit{and} & f_P(p) < f_P(q). \end{array}$$

PROOF. This follows easily from Theorem 2.2.7 and the monotonicity of the code. ■

Unfortunately these conditions are not sufficient (see Fig. 2.8) and we do not know if there exists a good geometrical characterization at all.

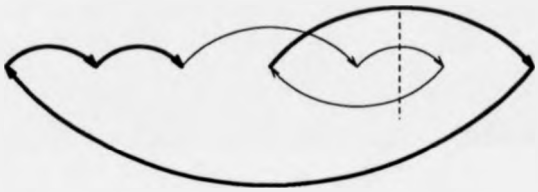

FIGURE 2.8. A cycle P satisfying conditions from Lemma 2.2.9 which is not X-minimal. It can be easy checked from code K_P or by finding a semicycle with eccentricity $\frac{4}{1}$ (thick lines).

2.3 Existence of X-minimal orbits

In the previous section we gave a characterization of X-minimal orbits. However, if we have a function $f \in C(I, I)$ with a periodic orbit with eccentricity r it is still not clear whether this map has an X-minimal r-cycle. This is because the set of all patterns with given eccentricity is infinite and so theoretically there may exist a sequence of r-patterns such that each one forces the next one and none of them forces an X-minimal r-pattern. In this section we will show that this is impossible.

LEMMA 2.3.1. *An r-unipattern A forces an X-minimal q-pattern B such that $q \geq r$ and* $\mathrm{per}(A) \geq \mathrm{per}(B)$.

PROOF. If A is an X-minimal r-pattern take $B = A$ and we are done.

If A is not an X-minimal pattern then by Lemmas 2.2.5 and 2.0.1 we have that A forces a unipattern A^* such that $E(A^*) \geq E(A)$ and $\mathrm{per}(A^*) < \mathrm{per}(A)$. Applying this finitely many times we must get an X-minimal pattern B. ■

Now we investigate X-minimal patterns more closely. Let P be an X-minimal cycle with eccentricity $\frac{m}{n}$. We would like to know what cycles does it force. If we

think a little about Lemma 2.2.1 and the way we proved Lemma 2.2.5 we can see that among all the cycles forced by P only those with period lower than per(P) are important. Any other cycle forced by P can be obtained by a "gluing" some of these cycles together. Because P is X-minimal we have that eccentricities of these cycles depending on the period are bounded above by $\frac{\left[\frac{im}{n}\right]}{i}$ for $i = 1, \ldots, n$ where $[x]$ denotes the integer part of x. We now consider those that give us a maximal possible eccentricity with a minimal possible period.

DEFINITION. Let $m, n \in \mathbb{N}$, $m > n$ coprime. The fraction $\frac{\left[\frac{im}{n}\right]}{i}$ is called an $\frac{m}{n}$*-extremal fraction* if $1 \leq i \leq n$ and $\frac{\left[\frac{im}{n}\right]}{i} > \frac{\left[\frac{jm}{n}\right]}{j}$ for all $j \in \{1, \ldots, i-1\}$.

REMARK. Note that $\left[\frac{im}{n}\right]$ and i are coprime for an $\frac{m}{n}$-extremal fraction $\frac{\left[\frac{im}{n}\right]}{i}$.

LEMMA 2.3.2. *Let $\frac{p_{j-1}}{q_{j-1}} \leq \frac{p}{q} \leq \frac{p_j}{q_j}$ where $\frac{p_{j-1}}{q_{j-1}} < \frac{p_j}{q_j}$ are consecutive $\frac{m}{n}$-extremal fractions. There are nonnegative integer numbers b, c such that $p = bp_{j-1} + cp_j$ and $q = bq_{j-1} + cq_j$.*

PROOF. We will use Farey series (see e.g. [HW] or Appendix). We show that $\frac{p_{j-1}}{q_{j-1}}, \frac{p_j}{q_j}$ are consecutive terms of the Farey series of order q_j. If not then there is a term $\frac{p^*}{q^*}$ such that $\frac{p_{j-1}}{q_{j-1}} < \frac{p^*}{q^*} < \frac{p_j}{q_j}$ and $\frac{p^*}{q^*}, \frac{p_j}{q_j}$ are consecutive terms of the Farey series of order q_j. So we have that $q^* \leq q_j$.

If $q^* < q_j$ then $\frac{\left[\frac{q^* m}{n}\right]}{q^*}$ is $\frac{m}{n}$-extremal fraction - a contradiction with assumption that $\frac{p_{j-1}}{q_{j-1}}, \frac{p_j}{q_j}$ are consecutive $\frac{m}{n}$-extremal fractions.

If $q^* = q_j$ then from Theorem 28 [HW] we have $p_j q^* - p^* q_j = 1$. But this is possible only if $q_j = 1$ which is contradiction with definition of $\frac{m}{n}$-extremal fractions.

Hence we have that $\frac{p_{j-1}}{q_{j-1}}, \frac{p_j}{q_j}$ are consecutive terms of the Farey series of order q_j and our lemma follows from 3.3. First proof of Theorems 28 and 29. [HW] ■

LEMMA 2.3.3. *Let P be an X-minimal $\frac{m}{n}$-unicycle ($m > n$ coprime) and $\frac{p}{q}$ be an $\frac{m}{n}$-extremal fraction. Then f_P has a $\frac{p}{q}$-unicycle Q with* per$(Q) = p + q$.

PROOF. If $\frac{p}{q} = \frac{m}{n}$ then we can set $Q = P$. So we can assume that $\frac{p}{q} < \frac{m}{n}$ and let $c \in \mathrm{Fix}(f_P)$.

We will show that there is a P-semicycle with eccentricity $\frac{p}{q}$. We will define a code $C = \langle c_i \rangle_{i=1}^{(m+n)q}$ where

$$c_i = 0 \quad \text{if} \quad f_P^{i-1}(p_1) < c$$
$$c_i = 1 \quad \text{if} \quad f_P^{i-1}(p_1) > c.$$

Note that $c_1 = 0$ and if $c_i = 1$ then $c_{i-1} = 0$ and $c_{i+1} = 0$ (if $i + 1 \leq (m+n)q$).

There is a close connection between C and K_P:

$$K_P(f_P^i(p_1) = in - (m+n)\sum_{j=1}^{i} c_j.$$

We will show that there is a piece of code $C^* = (c_i)_{i=j+1}^{j+p+q}$ such that $c_{j+1} = 0$, $c_{j+p+q} = 1$ and $\sum_{i=j+1}^{j+p+q} c_i = q$.

Assume to the contrary that there is no such sequence C^*.

We will use very similar technique as in the proof of Lemma 2.2.4. Let i_j be such that $c_{i_j} = 1$ and $\sum_{i=1}^{i_j} c_i = j$ (i_j is the position of the jth unit in C).

Note that $K_P(p) \geq 0$ for all $p \in P$ because P has monotone code. If $i_q \leq p+q$ then $K_P(f_P^{i_q}(p_1) < 0$ (because $\frac{p}{q} < \frac{m}{n}$) – a contradiction. Hence $i_q > p+q$.

If $i_q - i_1 < p+q$ then $(c_i)_{i=i_q-(p+q)}^{i_q}$ would be our sequence C^*. Hence $i_q - i_1 \geq p+q$ and because $c_{i_q+1} = 0$ we have that $i_{q+1} > i_q + 1$ and so $i_{q+1} - (i_1+1) > p+q$. Again because there is no sequence C^* we have that $i_{q+1} - i_{1+1} \geq p+q$. Repeating this argument we obtain

$$i_{j+q} - i_{j+1} \geq p+q$$

for $1 \leq j \leq (n-1)q$. Now using these inequalities and the fact that

$$1 < i_1 < i_q < i_q + 1 < i_{q+1} < \cdots < i_{jq} < i_{jq} + 1 < i_{jq+1} < \cdots < i_{nq} \leq (m+n)q$$

we get the inequality

$$(m+n)q \geq \sum_{j=1}^{n}(1 + i_{jq} - i_{(j-1)q+1}) \geq n + n(p+q).$$

From here we have that $\frac{m}{n} \geq \frac{p+1}{q}$ which contradicts the assumption that $\frac{p}{q}$ is $\frac{m}{n}$-extremal fraction. Hence we have proved the existence of a sequence C^*.

Now we will show that sequence $A = (f_P^i(p_1))_{i=j+1}^{j+p+q+1}$ connected with C^* is a P-semicycle.

Because $c_{j+1} = 0$ and $c_{j+p+q} = 1$ we have that both $f_P^{j+1}(p_1), f_P^{j+p+q+1}(p_1) < c$. Moreover

$$K_P(f_P^{j+p+q+1}(p_1)) = K_P(f_P^{j+1}(p_1)) + pn - qm < K_P(f_P^{j+1}(p_1))$$

and from monotonicity of code we have that $f_P^{j+p+q+1}(p_1) < f_P^{j+1}(p_1) < c$. Therefore A is a P-semicycle. Clearly its eccentricity is $\frac{p}{q}$ and so using Lemma 2.2.2 the function f_P has a $\frac{p}{q}$-unicycle Q. Finally because p, q are coprime we have that $\mathrm{per}(Q) = p+q$. ■

LEMMA 2.3.4. *Suppose m, n are coprime, p, q are coprime and $\frac{m}{n} \geq \frac{p}{q} \geq 1$. Then an X-minimal $\frac{m}{n}$-unipattern forces some $\frac{p}{q}$-unipattern with period $p+q$.*

PROOF. Let A be an X-minimal $\frac{m}{n}$-unipattern.

If $\frac{p}{q} \leq \frac{p_i}{q_i} < \frac{m}{n}$ where $\frac{p_i}{q_i}$ is an $\frac{m}{n}$-extremal fraction then from Lemmas 2.3.1 and 2.3.3 the pattern A forces an X-minimal $\frac{m^*}{n^*}$-pattern A^* such that $\frac{p_i}{q_i} \leq \frac{m^*}{n^*}$ and $\mathrm{per}(A^*) < \mathrm{per}(A)$. But forcing relation is transitive and so it is enough to prove that A^* forces some $\frac{p}{q}$-unipattern with period $p+q$. We can repeat this reduction and because we decrease the period we must stop after finitely many steps.

Hence we can assume that $\frac{p_{a-1}}{q_{a-1}} < \frac{p}{q} \leq \frac{p_a}{q_a}$ where $\frac{p_{a-1}}{q_{a-1}}, \frac{p_a}{q_a}$ are two biggest $\frac{m}{n}$-extremal fractions.

From Lemma 2.3.2 we have that $p = bp_{a-1} + cp_a$ and $q = bq_{a-1} + cq_a$ for some nonnegative integers b, c. It is clear that $\frac{p_a}{q_a} = \frac{m}{n}$ and $c > 0$ because $\frac{p_{a-1}}{q_{a-1}} < \frac{p}{q}$.

Let P be a representative of A. By Lemma 2.3.3 the function f_P has $\frac{p_{a-1}}{q_{a-1}}$-unicycle P_{a-1} with period $p_{a-1} + q_{a-1}$ and by Lemma 2.1.4 there is a P-loop $\mathcal{A}_{a-1}$ of length $p_{a-1} + q_{a-1}$ and eccentricity $\frac{p_{a-1}}{q_{a-1}}$ connected with this cycle.

Loop $\mathcal{A}_P$ has length $p_a + q_a$ and eccentricity $\frac{p_a}{q_a}$. Moreover because P is X-minimal by Lemma 2.2.1 the loop $\mathcal{A}_P$ is simple and therefore it contains all the intervals from $\mathfrak{P}$.

Hence we can connect c times loop $\mathcal{A}_P$ and b times loop $\mathcal{A}_{a-1}$ into one single P-loop with length $p+q$ and eccentricity $\frac{p}{q}$. This P-loop gives us a cycle Q (Lemma 2.1.3) such that $\mathrm{per}(Q) = p+q$ (p, q are coprime). Finally let $B = [Q]$ and apply Lemma 2.0.1. ■

THEOREM 2.3.5. *Any r-pattern forces an X-minimal r-pattern.*

PROOF. By Theorem 2.1.10 an r-pattern forces an r-unipattern and by Lemma 2.3.1 an r-unipattern forces an X-minimal q-pattern for some $q \geq r$. By Lemma 2.3.4 this pattern forces an r-unipattern with minimal possible period. Because there are only finitely many r-patterns with this period after repeating this procedure finitely many times we must get an X-minimal r-pattern. ■

We shall end this part by an easy algorithm to construct all X-minimal patterns. Let us consider a cycle P with an eccentricity $\frac{m}{n}$. We have defined a code $K_P : P \to \mathbf{Z}$. Clearly different orbits have different codes. Moreover if we have a code $K_P : P \to \mathbf{Z}$ of an X-minimal $\frac{m}{n}$-cycle then we can easily reconstruct the function φ of the cycle (P, φ) from the given code using the following simple algorithm (assume that $P = \{1, \ldots, m+n\}$).

ALGORITHM 1. *If $K_P(i) - K_P(j) = m$ or $K_P(j) - K_P(i) = n$ then $\varphi(i) = j$.*

We have also defined a code C_P connected with a given X-minimal cycle P. Again if we have a code $C_P = \langle c_1, \ldots, c_m \rangle$ connected with a given X-minimal $\frac{m}{n}$-

cycle P then we can easily reconstruct the code $K_P : P \to \mathbb{Z}$ and hence the cycle (P, φ).

ALGORITHM 2. *The function $K_P : P \to \mathbb{Z}$ is*

- *increasing on the set $\{1, \ldots, m\}$ with values from $\left\{\sum_{j=1}^{k}(n - c_j m);\ k = 1, \ldots, m\right\}$*
- *decreasing on the set $\{m+1, \ldots, m+n\}$ with values from $\left\{m + \sum_{j=1}^{k}(n - c_j m);\ c_k = 1\right\}$.*

Now we take some code C^* which is only a rotation of the code C_P. The function K^* obtained from the code C^* using Algorithm 2 is nothing but the function K_P shifted by some negative multiple of n. Therefore if we apply Algorithm 1 to the function K^* we again obtain the cycle (P, φ).

Hence we can get any X-minimal $\frac{m}{n}$-cycle by choosing a sequence $C \in C(m, n)$ where $C(m, n) = \{\{c_i\}_{i=1}^{m} \in \{0, 1\}^m;\ \sum_{i=1}^{m} c_i = n\}$, then using Algorithm 2 we get a code K^* and finally by Algorithm 1 we get an X-minimal cycle (see Fig. 2.9).

Note that you get different patterns if and only if you start from $C_1, C_2 \in C(m, n)$ such that C_1 is not a rotation of C_2. So we have the following simple

COROLLARY 2.3.6. (Proposition 4.4 from [BM]) *There are $\frac{m!}{m \cdot n!(m-n)!}$ different X-minimal $\frac{m}{n}$-patterns (m, n coprime).*

REMARK. According to this Corollary there is a unique X-minimal $\frac{n+1}{n}$-pattern. This is of course the pattern of the Stefan cycle because this is the only pattern that does not force any other pattern with period $2n + 1$ (see eg. [ALM2]) and clearly any pattern with period $2n + 1$ has eccentricity at least $\frac{n+1}{n}$.

2.4 A Generalization of Sharkovskii's Theorem

If we look at back at Lemma 2.2.4 and Theorems 2.2.7, 2.3.5 and 2.1.10 then we can see that they in fact give a generalization of a part of Sharkovskii's Theorem for odd periods. Indeed a pattern with a period $2k + 1$ has eccentricity at least $\frac{k+1}{k}$, by Lemma 2.2.4 it forces a pattern with eccentricity $\frac{k+2}{k+1}$ which by Theorems 2.2.7, 2.3.5 and 2.1.10 forces a pattern with period $2k + 3$.

The part of Sharkovskii's Theorem concerning even periods is somehow hidden in eccentricity equal to 1. So in order to get a full generalization we need to define a better type of patterns that will make a finer division of the set of all patterns with eccentricity 1.

Let look a bit closer at a periodic orbit (P, φ) with $\mathrm{per}(P) > 1$ and $E(P) = 1$. It is clear that $\mathrm{per}(P)$ is even. So $P = P_1 \cup P_2$ such that (P_i, φ^2) is a periodic orbit with period $\frac{\mathrm{per}(P)}{2}$ for $i = 1, 2$.

$C_P = \langle 0,0,1,0,1,0,1 \rangle$

$C_P = \langle 0,0,1,0,0,1,1 \rangle$

$C_P = \langle 0,0,0,1,1,0,1 \rangle$

$C_P = \langle 0,0,0,1,0,1,1 \rangle$

$C_P = \langle 0,0,0,0,1,1,1 \rangle$

FIGURE 2.9. The list of all X-minimal $\frac{7}{3}$-patterns.

We shall say that $[(P,\varphi)]$ is a $(2,r)$-pattern if $E([(P,\varphi)]) = 1$, $\mathrm{per}(P) > 1$ and $E([(P_i,\varphi^2)]) = r$ for some $i \in \{1,2\}$. Inductively we say that $[(P,\varphi)]$ is a $(2^k,r)$-pattern for $k > 1$ if $E([(P,\varphi)]) = 1$ and $[(P_i,\varphi^2)]$ is a $(2^{k-1},r)$-pattern for some $i \in \{1,2\}$. Finally we say that an r-pattern is a $(1,r)$-pattern. (See Fig. 2.10.)

We define a space $\mathcal{N} = \{(2^k,a): k \in \mathbb{N} \cup \{0\}, a \in \mathbb{R} \cup \{\infty\}, a \geq 1\} \cup \{(2^\infty,1)\}$ and a total ordering relation on $\mathcal{N}$ such that

$$(2^{k-1},a) > (2^{k-1},b) > (2^k,a) > (2^\infty,1) > (2^k,1) > (2^{k-1},1)$$

for any $a,b \in \mathbb{R} \cup \{\infty\}$ such that $a > b > 1$ and $k \in \mathbb{N}$.

For any $(a,b) \in \mathcal{N}$ we define sets

$$\mathcal{N}(a,b) = \{(c,d) \in \mathcal{N}: (a,b) \geq (c,d),\ c \in \mathbb{N},\ d \in \mathbb{Q}\}$$
$$\mathcal{N}_o(a,b) = \{(c,d) \in \mathcal{N}: (a,b) > (c,d),\ c \in \mathbb{N},\ d \in \mathbb{Q}\}.$$

A cycle (P, φ) with eccentricities $\frac{1}{11}$, $\frac{1}{5}$, $\frac{1}{1}$, $\frac{3}{1}$ and $\frac{5}{1}$:

(P, φ) is a 1-cycle so we look at (P_1, φ^2) and (P_2, φ^2).

The cycle (P_1, φ^2) has eccentricities $\frac{1}{2}$, $\frac{1}{1}$ and $\frac{2}{1}$:

The cycle (P_2, φ^2) has eccentricities $\frac{1}{5}$, $\frac{1}{2}$ and $\frac{2}{1}$:

(P_1, φ^2) is a 1-cycle so we look at $(P_{1,1}, \varphi^4)$ and $(P_{1,2}, \varphi^4)$.

The cycle $(P_{1,1}, \varphi^4)$ has eccentricity $\frac{2}{1}$:

The cycle $(P_{1,2}, \varphi^4)$ has eccentricity $\frac{2}{1}$:

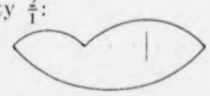

FIGURE 2.10. The pattern $[(P, \varphi)]$ has types $(1, \frac{11}{1})$, $(1, \frac{5}{1})$, $(1, \frac{3}{1})$, $(1, \frac{1}{1})$, $(2, \frac{5}{1})$, $(2, \frac{2}{1})$, $(2, \frac{1}{1})$ and $(4, \frac{2}{1})$.

Now we may state

THEOREM 2.4.1 (GENERALIZED SHARKOVSKIĬ'S THEOREM).

(i) *Any (a, b)-pattern forces some (c, d)-pattern for any $(c, d) \in \mathcal{N}(a, b)$.*

(ii) *For any $(a, b) \in \mathcal{N}$ there is a function $f \in C(I, I)$ such that f exhibits some (c, d)-pattern if and only if $(c, d) \in \mathcal{N}(a, b)$.*

(iii) *For any $(a, b) \in \mathcal{N}$ there is a function $f \in C(I, I)$ such that f exhibits some (c, d)-pattern if and only if $(c, d) \in \mathcal{N}_\omega(a, b)$.*

(iv) *A $(2^k, \frac{m}{n})$-pattern with $\frac{m}{n} > 1$ (m, n coprime) forces a pattern with period $2^k(m + n)$. A $(2^k, 1)$-pattern forces a pattern with period 2^k.*

PROOF. This will follow from Lemmas 2.4.9, 2.4.10, 2.4.15, 2.4.16 and Claim 2.4.17. ■

In order to prove the theorem above we need the notion of block structure.

Let (P, φ) be a cycle of period n and $B = [(\{1, 2, \ldots, m\}, \psi)]$ be a pattern of period m. Let $P = \{p_1, p_2, \ldots, p_n\}$ have the spatial labeling. We say that (P, φ) has a block structure over B if $n = sm$, $P = P_1 \cup P_2 \cup \cdots \cup P_m$ with $P_i = \{p_{(i-1)s+1}, \ldots, p_{(i-1)s+s}\}$ for all $i = 1, 2, \ldots, m$ and $\varphi(P_i) = P_{\psi(i)}$. Each of the sets P_i will be called a block of P. In other words, we could consider each block as a "fat" point and P as a "fat" cycle with pattern B (see Fig. 2.11).

FIGURE 2.11. A pattern which has a block structure over a pattern with period 4.

Assume that P has a block structure over B and (Q, ψ) is a cycle with pattern B. Then we also say that P has a block structure over (Q, ψ). If P has a block structure over B (respectively over Q) we also say that pattern $[P]$ has a block structure over B (respectively over Q).

Note that if (P, φ) has a block structure over a pattern of period m, then (P_i, φ^m) is a cycle of period $\frac{\mathrm{per}(P)}{m}$ for all i.

If a cycle (pattern) has a block structure over a pattern with period 2 then we say that it has a division.

We already have enough information about patterns of type $(1, r)$ but we have no information about forcing relation for patterns of type $(2^k, r)$ where $k \in \mathbb{N}$. So take a $(2^k, r)$-pattern and let (P, φ) be its representative. Because $k \geq 1$ our pattern is also a $(2, q)$-pattern (either $q = 1$ or $q = r$). Hence $\mathrm{per}(P) = 2n$ where $n \in \mathbb{N}$. There are two possibilities. Either P has a division or not. In the case when P does not have a division we can use the following lemma.

LEMMA 2.4.2. (Proposition 3.4 from [LMPY]) *Let A be a pattern with* $\mathrm{per}(A) = 2n$ *that does not have a division (so $n > 1$). Then if n is odd, the pattern A forces a pattern with period n. If n is even it forces a pattern with period $n + 1$.*

Hence we have following simple

COROLLARY 2.4.3. *A pattern with period greater than 1 which does not have a division forces a $(1, q)$-pattern for some $q > 1$.*

PROOF. This is straightforward from Lemma 2.4.2. ∎

Now we will look closely at the patterns that have a division. Let (P, φ) be a representative of such a pattern. Obviously P is a unicycle, $E(P) = 1$ and $\mathrm{per}(P) > 1$. We can look at the two cycles (P_1, φ^2) and (P_2, φ^2). If we have information about the types of patterns forced by $[P_1]$ and $[P_2]$ we can deduce information about the patterns forced by $[P]$. More precisely we have

LEMMA 2.4.4. *Let (P, φ) be a representative of a pattern with a division and $P_1 \subset P$ such that (P_1, φ^2) is a cycle. If $[P_1]$ forces an (a, b)-pattern A then $[P]$ forces a $(2a, b)$-pattern with division and period $2 \cdot \mathrm{per}(A)$.*

PROOF. Suppose $[P_1]$ forces an (a, b)-pattern A. Consider the function f_P^2 and the interval $I = \mathrm{conv}(P_1)$. Because P has a division we have $f_P(P_1) = P_2$, $f_P(P_2) = P_1$ and $\mathrm{conv}(P_1) \cap \mathrm{conv}(P_2) = \emptyset$. Hence $f_P^2|_I \in C(I, I)$. But $f_P^2|_I$ exhibits the pattern $[P_1]$ (it has the cycle P_1) and therefore it has a cycle Q_1 which is a representative of the pattern A. Let $f_P(Q_1) = Q_2$ and $Q = Q_1 \cup Q_2$. Clearly $(Q, f_P|_Q)$ is a cycle. We have $Q_2 \subset \mathrm{conv}(P_2)$ and therefore Q has a division, $E(Q) = 1$ and $\mathrm{per}(Q) > 1$. Hence $[Q]$ is a $(2a, b)$-pattern with a division. Clearly $\mathrm{per}(Q) = 2\,\mathrm{per}(A)$. ■

LEMMA 2.4.5. *A $(1, r)$-pattern with $r > 1$ forces a $(2, q)$-pattern with a division for each $q \geq 1$.*

PROOF. Let A be a $(1, r)$-pattern with $r > 1$. There is a $k \in \mathbf{N}$ such that $r > \frac{k+1}{k}$. By Theorems 2.1.10 and 2.3.5 A forces an X-minimal $(1, \frac{k+1}{k})$-pattern (which must be a Štefan pattern). So we may assume that A is a Štefan pattern and $(P = \{p_1, \ldots, p_{2k+1}\}, \varphi)$ is its representative. We have

$$\begin{aligned}
\varphi(p_1) &= p_{k+1} \\
\varphi(p_i) &= p_{2k+3-i} && \text{for} \quad i = 2, \ldots, k+1 \\
\varphi(p_i) &= p_{2k+2-i} && \text{for} \quad i = k+2, \ldots, 2k+1
\end{aligned}$$

and $p_{k+1} < c < p_{k+2}$ for $c \in \mathrm{Fix}(f_P)$. So $\mathfrak{P} = \{J_i\}_{i=1}^{2k+1}$ and

$$\begin{aligned}
J_1 &\xrightarrow{P} J_j && \text{for} \quad j = k+1, \ldots, 2k+1 \\
J_j &\xrightarrow{P} J_{2k+3-j} && \text{for} \quad j = 2, \ldots, k+1 \\
J_j &\xrightarrow{P} J_{2k+2-j} && \text{for} \quad j = k+2, \ldots, 2k+1 \\
J_{k+2} &\xrightarrow{P} J_{k+1}.
\end{aligned}$$

Note that only J_1 and J_{k+2} P-cover more than one interval. Hence

$$\mathcal{A} = \overbrace{\langle J_{k+2}, J_{k+1}, \ldots, J_{k+2}, J_{k+1} \rangle}^{s+1 \text{ times } J_{k+2}, J_{k+1}} + \langle J_{k+2}, J_k, J_{k+3}, J_{k-1}, \ldots, J_{2k+1}, J_1 \rangle$$

is a P-loop of length $2(k+s+1)$. Because the interval J_1 is only once in the loop $\mathcal{A}$ and $\mathcal{A} \neq \mathcal{A}_P$ the cycle Q given by the P-loop $\mathcal{A}$ has period $2(k+s+1)$. We can write $Q = \{q_1, q_2, \ldots, q_{2(k+s+1)}\}$ with spatial labeling. By alternating structure of $\mathcal{A}$, whenever $f_P(q_i) = q_j$ we have $c \in \operatorname{conv}(\{q_i, q_j\})$ and so Q has a division. Moreover, because f_P is monotone on the interval $[p_2, p_{2k+1}]$ we have that

$$\begin{aligned} f_P(q_1) &= q_{k+s+2} \\ f_P(q_i) &= q_{2(k+s+2)-i} && \text{for} \quad i = 2, \ldots, k+s+1 \\ f_P(q_i) &= q_{2(k+s+1)+1-i} && \text{for} \quad i = k+s+2, \ldots, 2(k+s+1). \end{aligned}$$

Take $Q_1 = \{q_1, \ldots, q_{k+s+1}\}$. Then $(Q_1, f_P^2|_{Q_1})$ is a cycle and

$$\begin{aligned} f_P^2(q_1) &= q_{k+s+1} \\ f_P^2(q_i) &= q_{i-1} && \text{for} \quad i = 2, \ldots, k+s+1. \end{aligned}$$

Hence the cycle Q_1 is a $\frac{k+s}{1}$-unicycle. Because we can choose s arbitrarily large we are done by Theorem 2.1.10 and Lemma 2.4.4. ■

In order to be able to use Lemma 2.4.5 effectively we must use patterns with a special structure.

DEFINITION. Let A be a $(2^k, r)$-pattern and (P_1^0, φ) be a representative of A. From the definition of $(2^k, r)$-pattern we see that for $1 \leq j \leq k$ there are sets P_1^j, P_2^j such that $P_1^{j-1} = P_1^j \cup P_2^j$, $\varphi^{2^{j-1}}(P_1^j) = P_2^j$ and (P_1^j, φ^{2^j}) is a $(2^{k-j}, r)$-cycle. The sequence of sets $\langle P_1^j \rangle_{j=0}^k$ will be called $(2^k, r)$-*determining*. Moreover if the cycle (P_1^j, φ^{2^j}) has a division for all $j < k$ then the sequence $\langle P_1^j \rangle_{j=0}^k$ will be called *splitting*. In this case we say that pattern A has a splitting $(2^k, r)$-determining sequence (see Fig. 2.12).

LEMMA 2.4.6. *An (a, b)-pattern with an (a, b)-determining sequence which is not splitting forces a (c, d)-pattern A for some $c < a$ and $d > 1$ such that any (c, e)-determining sequence of A is splitting.*

PROOF. Let (P_1^0, φ) be a representative of our (a, b)-pattern and $\langle P_1^j \rangle_{j=0}^k$ be an (a, b)-determining sequence which is not splitting. Take the smallest $j < k$ such that the cycle (P_1^j, φ^{2^j}) does not have a division. Using Corollary 2.4.3 and repeatedly Lemma 2.4.4 we have that our pattern forces a (c, d)-pattern such that $c = 2^j < a$ and $d > 1$. We can repeat the same procedure for new (c, d)-pattern and after finitely many steps we must get a pattern A with splitting determining sequences. ■

LEMMA 2.4.7. *A $(2^k, r)$-pattern with $r > 1$ and a splitting $(2^k, r)$-determining sequence forces a $(2^k, q)$-pattern for any $q \in \mathbb{Q}$ such that $r \geq q \geq 1$.*

PROOF. This follows easily from Theorem 2.1.10 and Lemma 2.4.4. ■

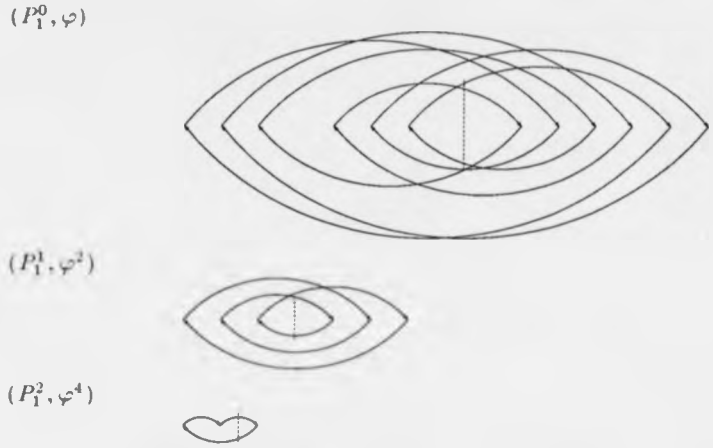

FIGURE 2.12. A $(2^2, \frac{2}{1})$-pattern with a splitting $(2^2, \frac{2}{1})$-determining sequence $\langle P_1^j \rangle_{j=0}^2$.

LEMMA 2.4.8. *A $(2^k, r)$-pattern with $r > 1$ and a splitting $(2^k, r)$-determining sequence forces a $(2^{k+1}, q)$-pattern for any $1 \leq q \in \mathbb{Q}$.*

PROOF. This follows easily from Lemma 2.4.5 and Lemma 2.4.4. ■

Now we are quite ready to prove two parts from Theorem 2.4.1.

LEMMA 2.4.9. *An (a, b)-pattern forces a (c, d)-pattern for any $(c, d) \in \mathcal{X}(a, b)$.*

PROOF. From Lemma 2.4.6 we have that an (a, b)-pattern A forces an (a^*, b^*)-pattern B with splitting (a^*, b^*)-determining sequence for some $(a^*, b^*) \geq (a, b)$. If $b^* > 1$ then using Lemmas 2.4.7 and 2.4.8 inductively we have that the pattern B forces a (c, d)-pattern for any $(c, d) \leq (a^*, b^*)$ such that $c \geq a^*$.

Now if $(c, d) \leq (a^*, b^*)$ and $c < a^*$ then $d = 1$. But from the definition it is clear that an (a^*, b^*)-pattern is a $(c, 1)$-pattern for any $(c, 1) \leq (a^*, b^*)$ such that $c < a^*$.

So we have that the pattern B forces a (c, d)-pattern for each $(c, d) \in \mathcal{X}(a^*, b^*)$. But A forces B and $(a^*, b^*) \geq (a, b)$. Hence we are done because the forcing relation is transitive. ■

LEMMA 2.4.10. *A $(2^k, \frac{m}{n})$-pattern with $\frac{m}{n} > 1$ (m, n coprime) forces a $(2^k, \frac{m}{n})$-pattern with period $2^k(m+n)$. A $(2^k, 1)$-pattern forces a $(2^k, 1)$-pattern with period 2^k.*

PROOF. Using Lemmas 2.4.6 and 2.4.5 as in the proof of Lemma 2.4.9 we get that a $(2^k, \frac{m}{n})$-pattern forces a $(2^k, \frac{m}{n})$-pattern A with a splitting $(2^k, \frac{m}{n})$-determining

sequence. Let (P_1^0, φ) be a representative of A and $\langle P_1^j \rangle_{j=0}^k$ be the splitting $(2^k, \frac{m}{n})$-determining sequence. So we have that $[P_1^k]$ is an $\frac{m}{n}$-pattern. Using Theorems 2.3.5 and 2.2.7 together with Lemma 2.2.4 we have that $[P_1^k]$ forces a $\frac{m}{n}$-pattern with period $m+n$ or period 1 if $\frac{m}{n} = 1$. Finally repeatedly applying Lemma 2.4.4 we are done. ■

So we have only two parts of Theorem 2.4.1 left to prove. As you may already have guessed a knowledge of X-minimal (a, b)-patterns will be very useful for proving them. So first

DEFINITION. An (a, b)-pattern which does not force any other (a, b)-pattern will be called an X-minimal (a, b)-pattern.

We will try to prove that some patterns are X-minimal (a, b)-patterns. For this we need to define a special type of block structure.

Let (P, φ) be a cycle and A, B be patterns. We say that (P, φ) is an A-extension of B if P has a block structure over B, φ is monotone on each block of P except at most one and, with the notation from the definition of block structure, we have that (P_i, φ^m) has pattern A for some $i \in \{1, 2, \ldots, m\}$ (in fact this does not depend on i). As above, if P is an A-extension of B and (Q, ψ) is a cycle with pattern B, then we say that P is an A-extension of (Q, ψ). We also say that $[P]$ is an A-extension of B (respectively of Q) if P is an A-extension of B (respectively of Q).

We define two special types of A-extension. An A-extension where $\mathrm{per}(A) = 2$ will be called a 2-extension. An A-extension where A is an X-minimal r-pattern will be called an r-extension.

A cycle will be called simple if it can be obtained from a cycle of period 1 by making 2-extensions k times and then one r-extension for some $k \in \mathbf{N}$ and $r \in \mathbf{Q}$. A pattern of a simple cycle will be called a simple pattern.

Note that a simple pattern obtained from a cycle of period 1 by making 2-extensions k times and then one r-extension will be a $(2^k, r)$-pattern. Moreover if A is a simple $(2^k, r)$-pattern and B is a simple pattern of period 2^s then an A-extension of B will be a simple $(2^{k+s}, r)$-pattern (see Fig. 2.13).

LEMMA 2.4.11. (Proposition 2.10.6 from [ALM2]) *Let A, B, C, D be patterns such that C is an A-extension of B and C forces D. Then either B forces D or D is an A^*-extension of B for some pattern A^* forced by A. If $C \neq D$ then in the last case $A^* \neq A$.*

LEMMA 2.4.12. (Lemma 2.11.4 from [ALM2]) *Let C, B, D be patterns such that C is a 2-extension of B and C forces D. Then either $C = D$ or B forces D.*

The next lemma is only a slight modification of Lemma 2.11.5 from [ALM2].

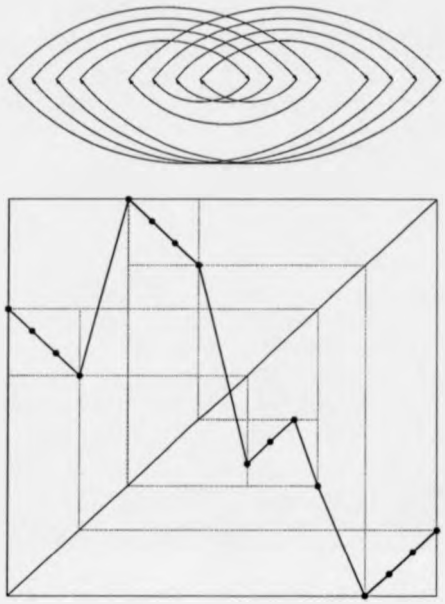

FIGURE 2.13. An example of a simple $(4, \frac{3}{1})$-cycle P and the function f_P.

LEMMA 2.4.13. *Let C be a simple pattern of period 2^k, $k \geq 0$. If C forces a pattern D where $D \neq C$ then D is a simple pattern of period 2^i, $i < k$.*

PROOF. We use induction. For $k = 0$ this is obvious. Assume that we know it for simple patterns of period 2^{k-1}. If C is a simple pattern of period 2^k, then C is a 2-extension of a simple pattern of period 2^{k-1}. If C forces D then we are done by Lemma 2.4.12 and the induction hypothesis. ■

THEOREM 2.4.14. *A simple pattern is X-minimal.*

PROOF. Let C be a simple $(2^k, r)$-pattern. Then there is an X-minimal r-pattern A and a simple pattern B of a period 2^k such that C is an A-extension of B. If $r = 1$ then we are done by Lemma 2.4.13. Assume now that $r > 1$ and D is a $(2^k, r)$-pattern such that C forces D. Then by Lemma 2.4.11 and 2.4.13 we have that D is an A^*-extension of B where A forces A^*. But because D is a $(2^k, r)$-pattern we have that $E(A^*) = r$. Finally because A is an X-minimal r-pattern we have that $A = A^*$ and from Lemma 2.4.11 we have that $D = C$. ■

Now we are ready to prove the remaining two parts from Theorem 2.4.1.

LEMMA 2.4.15. *Let $(a,b) \in \mathcal{N}(1,\infty)$ and P be a representative of a simple (a,b)-pattern. The function f_P exhibits some (c,d)-pattern if and only if $(c,d) \in \mathcal{N}(a,b)$.*

PROOF. From Lemmas 2.0.1 and 2.4.9 we have that f_P exhibits a (c,d)-pattern for any $(c,d) \in \mathcal{N}(a,b)$. Moreover exactly like in the proof of Theorem 2.4.14 we can prove that f_P does not exhibit a (c,d)-pattern if $(c,d) \notin \mathcal{N}(a,b)$ (see also Lemma 2.4.18). ■

LEMMA 2.4.16. *For any $(a,b) \in \mathcal{N}$ there is a function $f \in C(I,I)$ such that f exhibits some (c,d)-pattern if and only if $(c,d) \in \mathcal{N}_o(a,b)$.*

PROOF. Clearly there is a sequence of patterns $\langle A_i \rangle_{i=1}^{\infty}$ such that A_i is a simple (a_i,b_i)-pattern, $(a,b) > (a_{i+1},b_{i+1}) > (a_i,b_i)$ and $(a,b) = \sup\{(a_i,b_i)\colon i \geq 1\}$.

Let P_i be a representative of A_i such that $\mathrm{conv}(P_i) = [x_i,y_i] \subset (0,1)$ and $y_i < x_{i+1}$ for all $i \in \mathbb{N}$.

Define a function $f \in C(I,I)$ where $I = [0,1]$. Let $f(0) = 0$, $f(1) = 1$, $f|_{[x_i,y_i]} = f_{P_i}$ for $i \geq 1$ and $f|_J$ be linear for any interval $J \subset I$ such that $J \cap [x_i,y_i] = \emptyset$ for each $i \geq 1$ (see Fig. 2.14).

It is easy to see that outside of the intervals $[x_i,y_i]$ the function f has only fixed points. Hence if f has a (c,d)-cycle P then there is $i \geq 1$ such that $P \subset [x_i,y_i]$. So f_{P_i} exhibits $[P]$ and from Lemma 2.4.15 we have that $(c,d) \in \mathcal{N}(a_i,b_i) \subset \mathcal{N}_o(a,b)$. Moreover for any $(c,d) \in \mathcal{N}_o(a,b)$ there is an $i \geq 1$ such that $(a,b) > (a_i,b_i) \geq (c,d)$. By Lemma 2.4.15 we have that f_{P_i} exhibits a (c,d)-pattern and therefore f exhibits a (c,d)-pattern too. ■

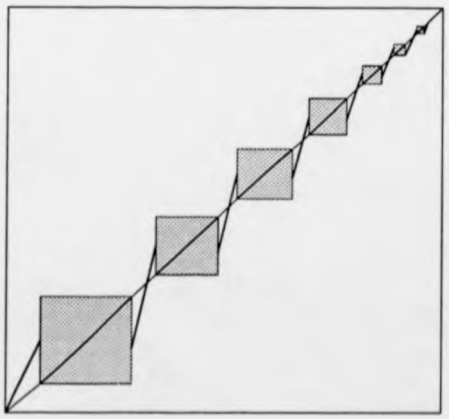

FIGURE 2.14. A graph of a function f. Inside the filled squares are the functions f_{P_i}.

Finally it suffices to realize

CLAIM 2.4.17. *If* $(a,b) \in \mathcal{N} \setminus \mathcal{N}(1,\infty)$ *then* $\mathcal{N}(a,b) = \mathcal{N}_o(a,b)$.

Now let us study X-minimal (a,b)-patterns more closely. The first natural question is to characterize all X-minimal (a,b)-patterns. Unfortunately it is not true, as one might expect, that the only X-minimal patterns are the simple ones (this idea seems to be natural for someone who knows the characterization of "primary" patterns – see [ALM2]) (see Fig. 2.15).

A cycle P

The function f_P and an important part of the function f_P^2 (dotted)

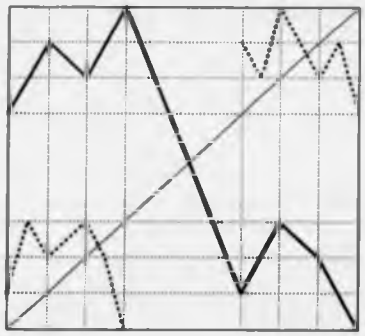

FIGURE 2.15. An example of an X-minimal $(2,\frac{3}{1})$-pattern which is not simple.

We can still get some more information about X-minimal (a,b)-patterns. First question is what types of patterns does an X-minimal (a,b)-pattern force? Can an X-minimal (a,b)-pattern which is not simple force a (c,d)-pattern for $(c,d) \notin \mathcal{N}(a,b)$?

LEMMA 2.4.18. *An X-minimal* (a,b)*-pattern forces a* (c,d)*-pattern if and only if* $(c,d) \in \mathcal{N}(a,b)$.

PROOF. The "if" part follows from Lemma 2.4.9. Assume that an X-minimal (a,b)-pattern A forces a (c,d)-pattern for some $(c,d) > (a,b)$. Clearly A has only

finite set of types (less than per(A)) and $\mathcal{N}(c,d) \setminus \mathcal{N}(a,b)$ has infinitely many elements. Using this fact, Lemma 2.4.9 and the fact that the forcing relation is transitive we have that A forces an (e,f)-pattern B such that $(e,f) > (a,b)$ and $A \neq B$. From Lemma 2.4.9 we have that B forces an (a,b)-pattern C. So A forces C and because the forcing relation is antisymmetrical we have that $A \neq C$ - a contradiction. ■

LEMMA 2.4.19. *An X-minimal $(2^k, r)$-pattern has a block structure over a simple $(2^k, 1)$-pattern.*

PROOF. It suffices to realize that if a $(2^k, r)$-pattern has not a block structure over a simple $(2^k, 1)$-pattern then either it has a type $(2^n, s)$ for $n < k$ and $s > 1$ or there is a $(2^k, q)$-determining sequence that is not splitting (note that q does not have to be equal to r). Because we have an X-minimal $(2^k, r)$-pattern using Lemma 2.4.18 we get that the first case is not possible. In the second case by Lemma 2.4.6 we have that our pattern forces an (a,b)-pattern for some $(a,b) > (2^k, r)$ - again a contradiction with Lemma 2.4.18. ■

LEMMA 2.4.20. *An X-minimal $(2^k, \frac{m}{n})$-pattern has period $2^k(m+n)$ if $m > n$ and period 2^k if $m = n$ (m, n coprime).*

PROOF. This follows immediately from Lemma 2.4.10. ■

So immediately we have following simple

COROLLARY 2.4.21. *An X-minimal $(2^k, 1)$-pattern is a simple $(2^k, 1)$-pattern.*

PROOF. This follows straight from Lemmas 2.4.19 and 2.4.20. ■

Now let A be an X-minimal $(2^k, r)$-pattern and (P, φ) be a representative of A. From Lemma 2.4.19 we have that P consists of 2^k blocks P_i. From Lemmas 2.1.1, 2.4.4 and 2.4.18 we have that (P_i, φ^{2^k}) is a unicycle and $E(P_i) \leq r$. Of course the pattern $[P_i]$ must not force a pattern with eccentricity greater than r. So if $E(P_i) = r$ then $[P_i]$ is an X-minimal r-pattern (see Lemma 2.2.5).

Now assume that A is an X-minimal $(2^k, \frac{n+1}{n})$-pattern. From Lemma 2.4.20 we have that the cycle (P_i, φ^{2^k}) has period $2n+1$. But the minimal possible eccentricity of a pattern with this period is $\frac{n+1}{n}$. Hence in this case for every i pattern $[(P_i, \varphi^{2^k})]$ must be an X-minimal $\frac{n+1}{n}$-pattern. But there is only one X-minimal $\frac{n+1}{n}$-pattern and that is exactly the pattern of the Ŝtefans cycle of the period $2n+1$.

In this special case we can prove even more. We need the following

LEMMA 2.4.22. ([B], Theorem 2.11.1 from [ALM2]) *Let A be a pattern with* per$(A) = 2^k(2n+1)$ *and B be the pattern of the Ŝtefan cycle of period $2n+1$. If A is not a B-extension of a simple $(2^k, 1)$-pattern then A forces another pattern with period $2^k(2n+1)$.*

Now we can prove

LEMMA 2.4.23. *An X-minimal* $(2^k, \frac{n+1}{n})$*-pattern is simple.*

PROOF. By Lemma 2.4.20 we have that an X-minimal $(2^k, \frac{n+1}{n})$-pattern A has period $2^k(2n+1)$. If it is not simple then by Lemma 2.4.22 it forces another pattern B with the same period. It is easy to see that if (a, b) is a maximal (in the sense of ordering on X) type of the pattern B then $(a, b) \geq (2^k, \frac{n+1}{n})$. Hence by Lemma 2.4.9 we have that B forces a $(2^k, \frac{n+1}{n})$-pattern. So the pattern A is not an X-minimal $(2^k, \frac{n+1}{n})$-pattern – a contradiction. ■

We will end this section by a conjecture.

CONJECTURE. *If A is an X-minimal $(2^k, r)$-pattern and (P, φ) is a representative of A such that (P_i, φ^{2^k}) is an r-cycle for every block P_i then A is simple.*

POSSIBLE FUTURE DIRECTIONS

In mathematics solving one problem usually leads to many new ones. Here we will give some of the possible directions of development in this area.

The first idea is to generalize these results. As we mentioned at the beginning of chapter 2 it is possible to study the forcing relation for invariant measures. We can define "pattern" for measures and the forcing relation exactly as for patterns of periodic orbits.

The second idea it to study other properties of our X-minimal patterns. The topological entropy given by a pattern is the first thing we would like to know (see e.g. [BM]).

The next problem is the structure of the forcing relation restricted to the set of X-minimal patterns. In the case when we use period to determine the type of a pattern the structure of the forcing relation restricted to the set of primary patterns with odd period is known. It is a simple linear structure. But if we have two X-minimal patterns with eccentricity greater than 1 we are not able to easily recognize whether one forces another.

Then there is the problem of the maximal eccentricity forced by a pattern and there may be some effective algorithm to decide this.

Finally we can start to use ideas similar to eccentricity for functions and patterns on different spaces.

APPENDIX

Here we give some basic information about theories used in the paper. This will not cover them. It will only show the main ideas from these theories and should help reader to understand more without reading other papers.

Kneading Theory

For details look at [MT]. A simple account can be also found in [D].

Let $I = [a, b]$ and $f \in C(I, I)$. The function f is called unimodal if there is $c \in I$ such that $f|_{[a,c]}$ is increasing and $f|_{[c,b]}$ is decreasing. The point c is called the critical point of the function f. For a point $x \in I$ define the itinerary $S(x) = \langle s_i \rangle_{i=0}^{\infty}$ where

$$\begin{aligned} s_i &= L \quad \text{if} \quad f^i(x) < c \\ s_i &= C \quad \text{if} \quad f^i(x) = c \\ s_i &= R \quad \text{if} \quad f^i(x) > c \end{aligned}$$

The kneading sequence of the function f is $K(f) = S(f(c))$.

Now we define an ordering on itineraries. Let $s = \langle s_i \rangle_{i=0}^{\infty}$ and $t = \langle t_i \rangle_{i=0}^{\infty}$ be two different itineraries. So, for some n, $s_i = t_i$ for $0 \leq i < n$ and $s_n \neq t_n$. Let $\tau_j(s)$ be the number of R's in $\langle s_i \rangle_{i=0}^{j}$ and $L < C < R$. Now $s < t$

$$\begin{aligned} &\text{if} \quad \tau_{n-1}(s) \quad \text{is even and} \quad s_n < t_n \\ &\text{or} \quad \tau_{n-1}(s) \quad \text{is odd and} \quad s_n > t_n. \end{aligned}$$

If $s = \langle s_i \rangle_{i=0}^{n}$ is an itinerary of a point in I under some unimodal function f and $\langle s_i \rangle_{i=k}^{n} \leq K(f)$ for all $k = 0, 1, \ldots$ then there is a point $x \in I$ that has itinerary s under the function f.

So we can use this for deciding whether a unimodal cycle forces another unimodal cycle (the cycle P is unimodal if f_P is). If we want to check whether the cycle P forces a cycle Q then we have to compare the kneading invariant $K(f_P)$ with itineraries given by the cycle Q. But from all these itineraries $K(f_Q)$ is the biggest one and so we have to compare only $K(f_P)$ and $K(f_Q)$. If $K(f_P) \geq K(f_Q)$ then $[P]$ forces $[Q]$.

Farey series

For details look at [HW].

The Farey series of order n is the ascending sequence of irreducible fractions whose denominators do not exceed n.

There are two basic theorems that express characteristic properties of Farey series.

THEOREM 28 from [HW]. *If $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive terms of the Farey series of order n then*

$$kh' - hk' = 1.$$

THEOREM 29 from [HW]. *If $\frac{h}{k}$, $\frac{h''}{k''}$ and $\frac{h'}{k'}$ are three successive terms of the Farey series of order n then*

$$\frac{h''}{k''} = \frac{h + h'}{k + k'}.$$

Moreover from Theorem 28 we have

$$kh'' - hk'' = 1, \qquad k''h' - h''k' = 1$$

and solving these equations for h'' and k'' we obtain

$$h''(kh' - hk') = h + h', \qquad k''(kh' - hk') = k + k'.$$

Hence $h'' = h + h'$ and $k'' = k + k'$ which then easily leads to the statement used in Lemma 2.3.2.

BIBLIOGRAPHY

[ALM1] L. Alsedà, J. Llibre, M. Misiurewicz, *Periodic orbits of maps of Y*, Trans. Amer. Math. Soc. **313** (1989), 475–538.

[ALM2] L. Alsedà, J. Llibre, M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, Adv. Ser. Nonlinear Dynam. 5, World Sci., Singapore, 1993.

[ALS] L. Alsedà, J. Llibre, R. Serra, *Minimal periodic orbits for continuous maps of the interval*, Trans. Amer. Math. Soc. **286** (1984), 595–627.

[B] S. Baldwin, *Generalizations of a theorem of Sharkovskii on orbits of continuous real-valued functions*, Discrete Math. **67** (1987), 111–127.

[BaM] M. Barge, J. Martin, *Dense periodicity on the interval*, Proc. Amer. Math. Soc. **94** (1985), 731–735.

[Be1] C. Bernhardt, *Simple permutations with order a power of two*, Ergodic Theory Dynamical Systems **4** (1984), 179–186.

[Be2] C. Bernhardt, *The ordering on permutations induced by continuous maps of the real line*, Ergodic Theory Dynamical Systems **7** (1987), 155–160.

[BE] N. P. Bhatia, W. O. Egerland, *A refinement of Sarkovskii's Theorem*, Proc. Amer. Math. Soc. **102** (1988), 965–972.

[B1] L. Block, *Simple periodic orbits of mappings of the interval*, Trans. Amer. Math. Soc. **254** (1979), 391–398.

[B2] L. Block, *Periodic orbits of continuous mappings of the circle*, Trans. Amer. Math. Soc. **260** (1980), 553–562.

[BCo] L. Block, W. A. Coppel, *Stratification of continuous maps of an interval*, Trans. Amer. Math. Soc. **297** (1986), 587–604.

[BC] L. Block, E. M. Coven, *Topological conjugacy and transitivity for a class of piecewise monotone maps of the interval*, Trans. Amer. Math. Soc. **300** (1987), 297–306.

[BGMY] L. Block, J. Guckenheimer, M. Misiurewicz, L. S. Young, *Periodic points and topological entropy of one dimensional maps*, Global theory of dynamical systems, Lecture Notes in Math., vol. 819, Springer, Berlin, 1980, pp. 18–34.

[BH] L. Block, D. Hart, *Stratification of the space of unimodal interval maps*, Ergodic Theory Dynamical Systems **3** (1983), 533–539.

[Bl] A. Blokh, *Rotation numbers, twists and a Sharkovskii-Misiurewicz-type ordering for patterns on the interval*, Ergodic Theory Dynamical Systems (to appear).

[BM] A. Blokh, M. Misiurewicz, *Entropy of Twist Interval maps*, MSRI Preprint No. 041-94, Mathematical Sciences Research Institute, Berkeley, 1994.

[BK1] J. Bobok, M. Kuchta, *Invariant measures for maps of the interval that do not have points of some period*, Ergodic Theory Dynamical Systems **14** (1994), 9–21.

[BK2] J. Bobok, M. Kuchta, *N-minimal patterns and a Generalization of Sharkovskii's Theorem*, preprint (submitted to Trans. Amer. Math. Soc.) (1994).

[BF] R. Bowen, J. Franks, *The periodic points of maps of the disk and the interval*, Topology **15** (1976), 337–342.

[Bu] U. Burkart, *Interval mapping graphs and periodic points of continuous functions*, Combin. Theory Ser. B **32** (1982), 57–68.

[Ca] L. A. V. Carvallho, *On an extension of Sarkovskii's order*, J. Math. Anal. Appl. **138** (1989), 52–58.

[CE] P. Collet, J.-P. Eckmann, *Iterated maps on the interval as dynamical systems*, Progr. Phys., vol. 1, Birkhäuser, Boston, 1980.

[C] W. A. Coppel, *Sarkovskii-minimal orbits*, Math. Proc. Cambridge Philos. Soc. **93** (1983), 397–408.

[DGP] B. Derrida, A. Gervois, Y. Pomeau, *Universal metric properties of bifurcations of endomorphisms*, J. Phys. A **12** (1979), 269–296.

[D] R. L. Devaney, *An introduction to chaotic dynamical systems*, Second edition, Addison-Wesley, Reading, MA, 1989.

[F1] V. V. Fedorenko, *Canonical periodic trajectories of one-dimensional dynamical systems*, (in Russian), Approximative and qualitative methods of theory of differential-functional equations, Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1983, pp. 106–109.

[F2] V. V. Fedorenko, *Properties of cyclic permutations, employed in one-dimensional dynamical systems*, (in Russian), Differential-functional equations and their applications, Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1985, pp. 80–86.

[F3] V. V. Fedorenko, *Partially ordered set of types of periodic trajectories of one-dimensional dynamical systems*, (in Russian), Dynamical systems and differential-difference equations, Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1986, pp. 90–97.

[F4] V. V. Fedorenko, *Simple one-dimensional dynamical systems*, (in Russian), Differential-functional equations and their applications to nonlinear boundary problems, Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1987, pp. 94–96.

[Fe1] M. J. Feigenbaum, *Quantitative universality for a class of non-linear transformations*, J. Statist. Phys. **19** (1978), 25–52.

[Fe2] M. J. Feigenbaum, *The universal metric properties of nonlinear transformations*, J. Statist. Phys. **21** (1979), 669–706.

[G] J. Guckenheimer, *On the bifurcation of maps of the interval*, Invent. Math. **39** (1977), 165–178.

[HW] G.H. Hardy, E.M. Wright, *An introduction to the theory of numbers*, Fourth edition, Oxford Univ. Press, Oxford, 1960.

[H] C.-W. Ho, *On Block's condition for simple periodic orbits of functions on an interval*, Trans. Amer. Math. Soc. **281** (1984), 827–832.

[HM] C.-W. Ho, C. Morris, *A graph theoretical proof of Sharkovsky's theorem on the periodic points of continuous functions*, Pacific J. Math. **96** (1981), 361–370.

[J] L. Jonker, *Periodic orbits and kneading invariants*, Proc. London Math. Soc. **39** (1979), 428–450.

[LMPY] T.-Y. Li, M. Misiurewicz, G. Pianigiani, J. A. Yorke, *No division implies chaos*, Trans. Amer. Math. Soc. **273** (1982), 191–199.

[LY] T. Li, J. Yorke, *Period three implies chaos*, Amer. Math. Monthly **82** (1975), 985–992.

[MT] J. Milnor, W. Thurston, *On iterated maps of the interval*, Dynamical systems, Lecture Notes in Math., vol. 1342, Springer, Berlin, 1988, pp. 465–563.

[MN] M. Misiurewicz, Z. Nitecki, *Combinatorial patterns for maps of the interval*, Mem. Amer. Math. Soc. **94** (1991), 156.

[MS] M. Misiurewicz, J. Smítal, *Smooth chaotic maps with zero topological entropy*, Ergodic Theory Dynamical Systems **8** (1988), 421–424.

[P] R. R. Phelps, *Lectures on Choquet's Theorem*, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1966.

[S1] A. N. Sharkovskiĭ, *Fixed points and the center of a continuous mapping of the line into itself*, (Ukrainian. Russian and English summaries), Dopovidi Akad. Nauk Ukraïn. RSR **7** (1964), 865–868.

[S2] A. N. Sharkovskiĭ, *Co-existence of cycles of a continuous mapping of the line into itself*, (Russian. English summary), Ukrainian Mat. Zh. **16** (1964), 61–71.

[S3] A. N. Sharkovskiĭ, *On cycles and structure of a continuous map*, (in Russian), Ukrainian Mat. Zh. **17** (1965), 104–111.

[Sn1] L'. Snoha, *Minimal periodic orbits of mappings of an interval*, Zeszyty Nauk. WSP w Bydgoszczy, Problemy matemat. **7** (1985), 146–149.

[Sn2] L'. Snoha, *On functions having periodic orbits which are not potentially minimal*, Teubner-Texte zur Math., Ergodic theory and related topics II, vol. 94, Teubner, Leipzig, 1987, pp. 185–189.

[Sn3] L'. Snoha, *Characterization of potentially minimal periodic orbits of continuous mappings of an interval*, Acta Math. Univ. Comenian. **52-53** (1987), 111–124.

[St] P. Stefan, *A theorem of Sarkovskiĭ on the existence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Phys. **54** (1977), 237–248.

[Str] P. D. Jr. Straffin, *Periodic points of continuous functions*, Math. Mag. **51** (1978), 99–105.

Publications of the author

[1] M. Kuchta, Jaroslav Smítal, *Two-point scrambled set implies chaos*, European Conference on Iteration Theory, (Caldes de Malavella, 1987), World Sci. Publishing, Teaneck, NJ, 1989, pp. 427–430.

[2] M. Kuchta, *Characterization of chaos for continuous maps of the circle*, Comment. Math. Univ. Carolin. **31** (1990), 383–390.

[3] T. Gedeon, M. Kuchta, *Shadowing property of continuous maps*, Proc. Amer. Math. Soc. **115** (1992), 271–281.

[4] M. Kuchta, *Shadowing property of continuous maps with zero topological entropy*, Proc. Amer. Math. Soc. **119** (1993), 641–648.

[5] J. Bobok, M. Kuchta, *Topological entropy and variation for transitive maps points of some period*, Math. Slovaca **43** (1993), 345–355.

[6] J. Bobok, M. Kuchta, *Register shifts versus transitive f-cycles for piecewise monotone maps*, Real Anal. Exchange (to appear).

[7] J. Bobok, M. Kuchta, *Invariant measures for maps of the interval that do not have points of some period*, Ergodic Theory Dynamical Systems **14** (1994), 9–21.

[8] J. Bobok, M. Kuchta, *N-minimal patterns and a Generalization of Sharkovskiĭ's Theorem*, preprint (submitted to Trans. Amer. Math. Soc.) (1994).

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