Some subalgebras of the Schur algebra

by

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SUMMARY

In this thesis we study some subalgebras of the Schur algebra for the general linear group $GL_n(k)$, particularly the Schur algebra $S(B^+)$ for the Borel subgroup B^+ of $GL_n(k)$.

In many ways it is easier to work in $S(B^+)$ than in the more complicated algebra $S(GL_n(k))$. Using the properties of $S(B^+)$ we give a new treatment of the Weyl modules for $GL_n(k)$. We then construct 2-step minimal projective resolutions of the irreducible $S(B^+)$-modules and from these we obtain very easily 2-step projective resolutions of the Weyl modules for $GL_n(k)$.

We study the Cartan invariants of $S(B^+)$ and show that under certain conditions they satisfy an interesting identity.

For particular cases of the field k and of the integer n we prove several results on minimal projective resolutions of the irreducible $S(B^+)$-modules.

The methods we use are combinatorial and do not involve algebraic group theory.

CONTENTS

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DECLARATION

The work in this thesis is original as far as I am aware, except when explicitly stated to the contrary.

0. INTRODUCTION

Let k be an infinite field and let n and r be positive integers.

Suppose that E is an n-dimensional k-vector space where $G = GL_n(k)$ acts naturally. Then, the r-fold tensor product $E^{\otimes r} = E \otimes \dots \otimes E$ ($\otimes$ denotes $\otimes_k$) can be made into a left kG-module by the rule

$$g(x_1 \otimes \dots \otimes x_r) = gx_1 \otimes \dots \otimes gx_r; \quad \text{all } g \in G,\ x_1,\dots,x_r \in E.$$

Let

$$T_r : kG \to \mathrm{End}_k(E^{\otimes r})$$

be the representation afforded by $E^{\otimes r}$ (regarded as kG-module). The image of T_r, i.e., $T_r(kG)$ is a subalgebra of $\mathrm{End}_k(E^{\otimes r})$.

Definition: For each subgroup H of G the subalgebra $T_r(kH)$ of $T_r(kG)$ will be called the *Schur algebra* for H, n, r and k and denoted $S_k(n, r; H)$, or simply $S(H)$ if no confusion regarding n, r and k arises.

In his dissertation [S], I. Schur introduced a k-algebra, denoted $S_k(n,r)$ in [G1], and used it to study the polynomial representations of the complex general linear group $GL_n(\mathbb{C})$.

The Schur algebra $S(G) = S_k(n,r; G)$, defined above, may be identified with $S_k(n,r)$. In fact, in [G2; p.5] it is proved that there is a k-algebra isomorphism

$$(0.1) \qquad \Xi : S_k(n,r) \longrightarrow S(G)$$

which takes the basis element $\xi_{i,j}$ of $S_k(n,r)$ (defined in [G1; p. 21]) to the basis element $\xi_{i,j}$ of $S(G)$ (defined in §2).

Let H be any subgroup of G. The Schur algebra $S(H)$ is a powerful tool in the study of polynomial representations of H. It is a classical fact (cf. [G1; (2.4d)]) that there is an equivalence between the category mod $S(G)$, of all $S(G)$-modules which are finite dimensional over k, and the category of polynomial representations of G which are homogeneous of degree r. It is easy to see that this equivalence of categories still holds if we replace G by H.

This thesis is mainly devoted to the study of the Schur algebra $S(B^+)$ for the Borel subgroup B^+ of G (B^+ consists of all upper triangular matrices in G) and its applications to $S(G)$. Our methods are combinatorial and we shall not use algebraic group theory.

Our interest in $S(B^+)$ arose from our attempts to construct projective resolutions of K_λ, the Weyl module for G with highest weight λ. In recent years it has been proved by several authors (cf. e.g. [D], [AB2], [P]) that $S(G)$ has finite global dimension. This led to the problem of constructing projective resolutions of K_λ. An answer to this problem was given in [AB1] in the case when $n = 2$, and in [A] and [Z] when the field k has characteristic zero. We use the properties of $S(B^+)$ to give a new treatment of the Weyl modules K_λ, and to obtain some results on projective resolutions of these modules.

The study of $S(B^+)$ in itself proved to be interesting, and in particular the analysis of an identity involving its Cartan invariants (see (0.5)).

We begin in Chapter 1 by introducing some basic material which will be used in the following chapters. Sections 1 and 2 contain notation and elementary results. In §3 we use the method of [G2; §3] to determine bases of the Schur algebras, $S(G_J^+)$ and $S(L_J)$, for the standard parabolic subgroups G_J^+ of G and its Levi factors L_J. In §4 and §5 we define weight spaces and contravariant duals, and prove some results which will be very useful in the next chapter. We think Theorem (5.6) may be known, but we cannot find any reference for it. We also remark that a result similar to (4.8) is known from the theory of algebraic groups (cf. e.g. [St; theor. 39]).

In the first section of Chapter 2 we determine full sets of pairwise non-isomorphic irreducible, and projective indecomposable, $S(B^+)$-modules. These are indexed by the elements of $\Lambda(n,r)$ (see p.1.1 and (7.12) for the definitions of $\Lambda(n,r)$ and $\Lambda^+(n,r)$). From now on let k_λ and $V_\lambda = S(B^+)\xi_\lambda$ denote, respectively, the irreducible and projective indecomposable $S(B^+)$-modules associated with $\lambda = (\lambda_1,...,\lambda_n) \in \Lambda(n,r)$.

In §7 we define, for each $\lambda \in \Lambda^+(n,r)$, the Weyl module K_λ associated with λ, by

$$K_\lambda = S(G) \otimes_{S(B^+)} k_\lambda.$$

This definition is equivalent to the classical one given in [CL]. In fact, in [G1; pp. 64, 65] it is proved that the Weyl module for G associated with λ (as defined in [CL]) is the contravariant dual of the rational G-module $\mathrm{Ind}_{B^-}^{G} k_\lambda^-$, where k_λ^- is the irreducible B^--module associated with λ. It can be seen (cf. [G2; p.25]) that $\mathrm{Ind}_{B^-}^{G} k_\lambda^-$ is equivalent, via $T_r : kG \to S(G)$, to the $S(G)$-module $M_\lambda = \mathrm{Hom}_{S(B^-)}(S(G), k_\lambda^-)$. In (7.14) we prove that M_λ is the contravariant dual of K_λ. This proves the equivalence of the definitions.

We use the properties of $S(B^+)$ to give an alternative proof of some of the results in [CL] about Weyl modules. In particular, we prove that these are cyclic modules containing a unique maximal submodule, and that the quotients by these submodules give a full set of pairwise non-isomorphic irreducible $S(G)$-modules.

In §8 we study the modules $K_{\lambda,J} = S(G_J^+) \otimes_{S(B^+)} k_\lambda$. Let $J = \{1,\dots,n\}\backslash\{m_1,\dots,m_s\}$, for integers $m_0,m_1,\dots,m_s$ satisfying $0 = m_0 < m_1 < \dots < m_s = n$. Write $n_a = m_a - m_{a-1}$, and for each $\lambda = (\lambda_1,\dots,\lambda_n) \in \Lambda(n,r)$ satisfying

(0.2) $$\lambda_{m_{a-1}+1} \geq \dots \geq \lambda_{m_a}; \quad \text{for } a = 1,\dots,s,$$

define $\lambda(a) = (\lambda_{m_{a-1}+1},\dots,\lambda_{m_a})$. Then we prove that $K_{\lambda,J}$ is isomorphic as $S(L_J)$-module to $K_{\lambda(1)} \otimes \dots \otimes K_{\lambda(s)}$ ($\otimes$ means $\otimes_k$), where $K_{\lambda(a)}$ is the Weyl module for $S(GL_{n_a}(k))$ associated with $\lambda(a)$. It is quite simple to show that $K_{\lambda,J}$ is zero if λ does not satisfy (0.2).

Chapter 3 is dedicated to the construction of a 2-step minimal projective resolution of k_λ in mod $S(B^+)$. In §9 we determine a minimal set of $S(B^+)$-generators of the radical of $V_\lambda = S(B^+)\xi_\lambda$. This is not too hard, since V_λ has a very well behaved k-basis. From this result it is easy to construct the 2-step minimal projective resolution of k_λ

$$\bigoplus_{1 \leq \nu \leq n-1} \ \bigoplus_{1 \leq p^{d_\nu} \leq \lambda_{\nu+1}} V_{\lambda(\nu,p^{d_\nu})} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0,$$

where $\lambda \in \Lambda(n,r)$ and char $k = p$ (≥ 0) (for notation see §9).

Now we only need to apply the right exact functor $S(G) \otimes_{S(B^+)} \cdot : \text{mod } S(B^+) \to S(G)$ to the sequence above, and we obtain the 2-step projective resolution of the Weyl module K_λ

$$\text{(0.3)} \qquad \coprod_{1 \le \nu \le n-1} \; \coprod_{1 \le p^{d_\nu} \le \lambda_{\nu+1}} S(G)\xi_{\lambda(\nu, p^{d_\nu})} \xrightarrow{\psi_1} S(G)\xi_\lambda \xrightarrow{\psi_0} K_\lambda \to 0,$$

where $\lambda \in \Lambda^+(n,r)$ and char $k = p$ (≥ 0).

In [ABW] there is given (as part of the construction of a standard basis of K_λ) a 2-step projective resolution of K_λ ($\lambda \in \Lambda^+(n,r)$). This is done using symmetric, exterior and divided power algebra theory. But since in the work cited it is not assumed that k is a field (more general rings are allowed) the resolution obtained

$$\coprod_{1 \le \nu \le n-1} \; \coprod_{1 \le m \le \lambda_{\nu+1}} S(G)\xi_{\lambda(\nu,m)} \to S(G)\xi_\lambda \to K_\lambda \to 0$$

is less economical (for the case that k is a field of characteristic p) than (0.3).

Chapter 4 deals with the Cartan invariants

$$c_{\lambda\alpha} = \dim_k \operatorname{Hom}_{S(B^+)}(V_\alpha, V_\lambda), \quad \text{all } \alpha, \lambda \in \Lambda(n,r)$$

of $S(B^+)$. As is expected from the algebraic group theory of B^+, we show that $c_{\lambda\alpha} \ne 0$ iff $\lambda \trianglelefteq \alpha$, i.e., iff

$$\text{(0.4)} \qquad \alpha = A_1^{m_1} \cdots A_{n-1}^{m_{n-1}} \lambda = (\lambda_1 + m_1, \lambda_2 + m_2 - m_1, \ldots, \lambda_n - m_{n-1}),$$

for non-negative integers $m_1, \ldots, m_{n-1}$.

If this condition holds, we have two cases to consider. First suppose that the integers m_ν in (0.4) satisfy $m_\nu \le \lambda_{\nu+1}$, for $\nu = 1, \ldots, n-1$. Then $c_{\lambda\alpha}$ may be expressed in terms of the integers $n(m_1, \ldots, m_{n-1})$ (cf. (11.9)) which depend only on $m_1, \ldots, m_{n-1}$.

We then determine a generating function for $n(m_1,\ldots,m_{n-1})$, which allows us to prove that the following identity holds

$$(0.5)\qquad \sum_{\omega \in P(n)} \varepsilon(\omega)\, c_{\omega(\lambda)\alpha} = \delta_{\lambda,\alpha}\,,$$

where $P(n)$ is the symmetric group on $\{1,\ldots,n\}$, $\varepsilon(\omega)$ is the sign of the permutation ω, $\omega(\lambda) = (\lambda_1 + \omega(1)-1,\ldots,\lambda_n + \omega(n)-n)$, and $\delta_{\lambda,\alpha}$ is the Kronecker delta.

Now suppose that $m_\nu > \lambda_{\nu+1}$, for some $\nu \in \{1,\ldots,n-1\}$. Then the expression which describes $c_{\lambda\alpha}$ is much more complicated, and in this case we are not able to prove (0.5). Nevertheless, we show that the relation (0.5) holds for any α and λ in $\Lambda(n,r)$, provided $n \le 3$.

In Chapter 5 we return to the construction of minimal projective resolutions of k_λ, for any $\lambda \in \Lambda(n,r)$. In [G2] it is proved that $S(B^+)$ is a quasi-hereditary algebra. Therefore it has finite global dimension (cf. [CPS]), and minimal projective resolutions of k_λ are finite. In §13 we determine these resolutions in the case when the field k has characteristic zero and $n \le 3$. These are formally very similar to the resolutions obtained in [A] and [Z] for the Weyl modules K_λ ($\lambda \in \Lambda^+(n,r)$). Section 14 deals with the case when k has positive characteristic p and $n = 2$. Let $\lambda = (\lambda_1,\lambda_2) \in \Lambda(2,r)$ and suppose that $p^d \le \lambda_2 < p^{d+1}$ (some $d \ge 0$). Then we prove that

$$(0.6)\qquad \bigoplus_{m=1}^{d} \left(V_{\lambda(1,p^m)} \oplus V_{\lambda(1,1+p^m)} \oplus V_{\lambda(1,p+p^m)} \oplus \cdots \oplus V_{\lambda(1,p^{m-1}+p^m)}\right)$$
$$\xrightarrow{\varphi_2} \bigoplus_{m=0}^{d} V_{\lambda(1,p^m)} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0$$

are the first three terms of a minimal projective resolution of k_λ. Note that if $\operatorname{char} k = 0$ we have shown (cf. (13.1)) that

$$(0.7) \qquad 0 \to V_{\lambda(1,1)} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0$$

is a minimal projective resolution of k_λ. This illustrates how the difficulty in the construction of these sequences increases when we go from a field of characteristic zero to a field of positive characteristic. We should remark that the major obstacle with which we were confronted in our attempts to give a complete solution of this problem is the complicated rule for the multiplication of two basis elements of $S(G)$.

We conclude Chapter 5 by applying the functor $S(G) \otimes_{S(B^+)} \cdot$ to the sequences (0.6) and (0.7) and obtain similar exact sequences for the Weyl module K_λ $(\lambda \in \Lambda^+(n,r))$. This is justified by a recent theorem of D.J. Woodcock (cf. (15.1)).

Finally in Chapter 6 we study the Schur algebra $S(U^+)$ for the unipotent subgroup U^+ of B^+. We determine a k-basis of $S(U^+)$ which, unlike the basis of $S(G_J^+)$ determined in §3, is not a subset of the basis $\{\xi_{i,j} \mid (i,j) \in \Omega\}$ of $S(G)$ (cf. (2.2)). Then we prove that $S(U^+)$ is a local ring. We end this chapter by studying the natural epimorphism

$$\begin{array}{ccc} S(T) \otimes S(U^+) & \longrightarrow & S(G) \\ \xi \otimes \eta & \mapsto & \xi\eta \end{array}$$

determined by the decomposition $B^+ = TU^+$ of B^+ as the semidirect product of the group T (of all diagonal matrices in G) and U^+.

1. SCHUR ALGEBRAS

§1. Notation and basic definitions

k is an infinite field of any characteristic, n and r are positive integers which will be fixed throughout and $G = GL_n(k)$ denotes the general linear group of degree n over k.

If s is any positive integer, we write $\underline{s}$ for the set $\{1,\dots,s\}$.

$I = I(n,r) = \{i = (i_1,\dots,i_r) \mid i_\rho \in \underline{n}$ for all $\rho \in \underline{r}\}$, will also be regarded as the set of all functions $i : \underline{r} \to \underline{n}$ $(i_\rho = i(\rho)$, for all $\rho \in \underline{r})$, and

$$\Lambda = \Lambda(n,r) = \{\lambda = (\lambda_1,\dots,\lambda_n) \mid \lambda_\nu \in \mathbb{Z}, \lambda_\nu \geq 0\ (\nu \in \underline{n}), \sum_{\nu \in \underline{n}} \lambda_\nu = r\}$$

is the set of all unordered partitions of r into n parts (zero parts being allowed).

(1.1) Definition: $\lambda \in \Lambda$ is the *weight* of $i \in I$ (and we write $i \in \lambda$) if $\lambda_\nu = \#\{\rho \in \underline{r} \mid i_\rho = \nu\}$, for all $\nu \in \underline{n}$.

$P = P(r)$ denotes the symmetric group on $\underline{r}$. It acts on the right of $I(n,r)$ by

(1.2) $$i\pi = (i_{\pi(1)}, \dots, i_{\pi(r)}), \text{ all } i \in I,\ \pi \in P.$$

P also acts on the right of $I \times I$ by

$$(i,j)\pi = (i\pi, j\pi), \text{ all } i,j \in I,\ \pi \in P.$$

We write $i \sim j$ if i and j are in the same P-orbit of I and similarly $(i,j) \sim (i',j')$ means that (i,j) and (i',j') are in the same P-orbit of $I \times I$.

(1.3) Remark: Note that $i \sim j$ iff i and j have the same weight, so we may think of $\Lambda(n,r)$ as the set of all P-orbits in $I(n,r)$.

We will now introduce some pre-orderings on $I(n,r)$.

(1.4) If $m_0, m_1, \dots, m_s$ are integers satisfying $0 = m_0 < m_1 < \dots < m_{s-1} < m_s = n$, define $J = \underline{n} \setminus \{m_1, \dots, m_s\}$ $(s \geq 1)$.

Clearly $\underline{n} = \dot{\bigcup}_{a \in \underline{s}} N_a$, where $N_a = \{m_{a-1} + 1, \dots, m_a\}$ $(a \in \underline{s})$.

For $\mu, \nu \in \underline{n}$ say $\mu \underset{J}{=} \nu$ if μ and ν are in the same set N_a, for some $a \in \underline{s}$.

(1.5) Definition: For $\mu, \nu \in \underline{n}$, $\mu \underset{J}{\leq} \nu$ means that $\mu \leq \nu$ or $\mu \underset{J}{=} \nu$.

We may extend these concepts to $I(n,r)$ as follows

(1.6) Definition: Let $i,j \in I(n,r)$. Then we say

(i) $\quad i \underset{J}{=} j$ if $i_\rho \underset{J}{=} j_\rho$, all $\rho \in \underline{r}$;

(ii) $\quad i \underset{J}{\leq} j$ if $i_\rho \underset{J}{\leq} j_\rho$, all $\rho \in \underline{r}$.

(1.7) Remarks: (i) The relation $\underset{J}{\leq}$ is reflexive and transitive on I. Also $i \underset{J}{\leq} j$ and $j \underset{J}{\leq} i$ iff $i \underset{J}{=} j$ (but not necessarily $i = j$). Hence $\underset{J}{\leq}$ is a pre-ordering on I.

(ii) For any $i,j \in I$ we have that $i \underset{J}{\leq} j$ implies $i\pi \underset{J}{\leq} j\pi$, for any $\pi \in P$. So if $i \underset{J}{\leq} j$ and $(i,j) \sim (h,\ell)$ (some $h, \ell \in I$) then $h \underset{J}{\leq} \ell$.

A similar result holds if we use $=_J$ instead of $\leq_J$.

We shall now pay special attention to the case when $J = \emptyset$, i.e., $s = n$ and $N_a = \{a\}$ for all $a \in \underline{n}$.

If $\mu, \nu \in \underline{n}$, $\mu \leq_{\emptyset} \nu$ means $\mu \leq \nu$ (in the usual sense). Thus, if $i, j \in I$ we have $i \leq_{\emptyset} j$ iff $i_\rho \leq j_\rho$, all $\rho \in \underline{r}$. We shall write $\leq$ for $\leq_{\emptyset}$ and $i < j$ will mean $i \leq j$ but $i \neq j$.

As $i \leq j$ and $j \leq i$ implies $i = j$, we have in this case a partial order on I (it coincides with the partial order defined in [G2; p.11]).

(1.8) Lemma: Let $i \in I$ and $\pi \in P$. Then $i\pi \leq i$ iff $i\pi = i$.

Proof: One "if" is obvious. Now suppose $i\pi \leq i$ but $i\pi \neq i$, i.e., $i_{\pi(\rho)} \leq i_\rho$, all $\rho \in \underline{r}$, and $i_{\pi(\tau)} < i_\tau$, for some $\tau \in \underline{r}$. Then

$$\sum_{\rho \in \underline{r}} i_\rho > \sum_{\rho \in \underline{r}} i_{\pi(\rho)} = \sum_{\rho \in \underline{r}} i_\rho ,$$

a contradiction. So $i\pi \leq i$ implies $i\pi = i$. $\square$

Now we will introduce a partial order $\trianglelefteq$ on $\Lambda(n,r)$, usually called the *dominance order* (cf. [JK; (1.4.6)]).

(1.9) Definition: If $\alpha, \beta \in \Lambda(n,r)$ we say that $\alpha \trianglelefteq \beta$ if $\sum_{\nu=1}^{\mu} \alpha_\nu \leq \sum_{\nu=1}^{\mu} \beta_\nu$, for all $\mu \in \underline{n}$.

(1.10) Lemma: If $i, j \in I$ have weights α and β, respectively, then $i \le j$ implies $\beta \trianglelefteq \alpha$.

Proof: Suppose $i \le j$. Then $i_\rho \le j_\rho$ for all $\rho \in \underline{r}$, which implies that, for any $\mu \in \underline{n}$, $\{\rho \in \underline{r} \mid j_\rho \le \mu\} \subseteq \{\rho \in \underline{r} \mid i_\rho \le \mu\}$. Hence

$$\sum_{\nu=1}^{\mu} \beta_\nu = \# \{\rho \in \underline{r} \mid j_\rho \le \mu\} \le \# \{\rho \in \underline{r} \mid i_\rho \le \mu\} = \sum_{\nu=1}^{\mu} \alpha_\nu, \text{ i.e., } \beta \trianglelefteq \alpha. \qquad \square$$

We now define some notation involving λ-tableaux. Essentially this will be the same as in [G1].

Let λ be any element of $\Lambda(n,r)$.

The *diagram* of λ is the set

$$[\lambda] = \{(\mu,\nu) \in \mathbf{Z} \times \mathbf{Z} \mid \mu \ge 1 \text{ and } 1 \le \nu \le \lambda_\mu\}$$

and any map from $[\lambda]$ to a set is called a λ-*tableau*. We shall choose a bijective map $T^\lambda : [\lambda] \to \underline{r}$ and call it the *basic* λ-*tableau*. If $T^\lambda((\mu,\nu)) = a_{\mu\nu}$ $((\mu,\nu) \in [\lambda])$ we shall write

$$\text{(1.11)} \qquad T^\lambda = \begin{array}{|llll|} \hline a_{11} & a_{12} & \cdots & a_{1\lambda_1} \\ \hline a_{21} & a_{22} & \cdots & a_{2\lambda_2} \\ \hline & & \cdots & \\ \hline a_{n1} & a_{n2} & \cdots & a_{n\lambda_n} \\ \hline \end{array}$$

Associated with T^λ we have the subgroup of P consisting of all those $\pi \in P$ which preserve the rows (resp. columns) of (1.11). This is called the *row stabilizer* (resp. *column stabilizer*) of T^λ.

Now let $i \in I(n,r)$. Since i may be regarded as a map from $\underline{r}$ to $\underline{n}$ we may consider the λ-tableau iT^λ. We shall denote it by T_i^λ and write

$$T_i^\lambda = \begin{array}{|llll|} \hline i_{a_{11}} & i_{a_{12}} & \cdots & i_{a_{1\lambda_1}} \\ \hline i_{a_{21}} & i_{a_{22}} & \cdots & i_{a_{2\lambda_2}} \\ \hline & & \cdots & \\ \hline i_{a_{n1}} & i_{a_{n2}} & \cdots & i_{a_{n\lambda_n}} \\ \hline \end{array}$$

A final remark on notation. If V, V' are k-vector spaces we shall write $V \otimes V'$ for $V \otimes_k V'$.

§2. The Schur algebras $S_k(n,r;\ H)$

Let E be an n-dimensional k-vector space with basis $\{e_1, \ldots, e_n\}$ where G acts naturally, i.e.,

$$ge_\nu = \sum_{\mu \in \underline{n}} g_{\mu\nu} e_\mu, \quad \text{all } g \in G,\ \nu \in \underline{n}.$$

The r-fold tensor product $E^{\otimes r} = E \otimes \ldots \otimes E$ (r factors) has k-basis

$$\{e_i = e_{i_1} \otimes \ldots \otimes e_{i_r} \mid i \in I(n,r)\}$$

and it can be made into a left kG-module by the rule

$$ge_i = ge_{i_1} \otimes \ldots \otimes ge_{i_r}, \text{ all } g \in G, \ i \in I.$$

Using (1.2) we may also define a right P-action on $E^{\otimes r}$, which commutes with that of G, by

$$e_i\pi = e_{i\pi}, \text{ all } \pi \in P, \ i \in I.$$

Let

$$T_r : kG \to \mathrm{End}_k(E^{\otimes r})$$

be the representation afforded by $E^{\otimes r}$ regarded as kG-module. Then the image of T_r, i.e., $T_r(kG)$, is a subalgebra of $\mathrm{End}_k(E^{\otimes r})$.

If we consider any subgroup H of G, then $T_r(kH)$ will be a subalgebra of $T_r(kG)$ and we make the

(2.1) Definition: Let H be any subgroup of G. Then the algebra $T_r(kH)$ will be called the *Schur algebra* for H, n, r and k and will be denoted by $S_k(n, r; H)$ (or simply $S(H)$ if no confusion relative to n,r and k arises).

It is well known (see e.g. [G1; (2.6c)]) that $S(G)$ is the algebra $\mathrm{End}_{kP}(E^{\otimes r})$, consisting of all kP-endomorphisms of $E^{\otimes r}$ (regarded as right kP-module).

In order to obtain a basis for $S(G)$ consider, for each $(i,j) \in I \times I$, the element $\xi_{i,j}$ of $\mathrm{End}_k(E^{\otimes r})$ whose matrix, $\left(A_{h,\ell}(i,j)\right)_{h,\ell \in I \times I}$, relative to the basis $\{e_m \mid m \in I\}$, has

$$A_{h,\ell}(i,j) = \begin{cases} 1 & \text{if } (h,\ell) \sim (i,j) \\ 0 & \text{if } (h,\ell) \not\sim (i,j), \end{cases} \quad (h,\ell) \in I \times I.$$

Then $\xi_{i,j} \in \mathrm{End}_{kP}(E^{\otimes r}) = S(G)$ and it is clear that $\xi_{i,j} = \xi_{h,\ell}$ iff $(i,j) \sim (h,\ell)$. Hence to obtain distinct elements $\xi_{i,j}$ we should take a transversal Ω of the set of all P-orbits of $I \times I$. Once we have done this we get the result

(2.2) Theorem: (Schur) (cf. [G2; (2.2)]) $\{\xi_{i,j} \mid (i,j) \in \Omega\}$ is a k-basis for $S(G)$.

The next proposition will tell how to express $T_r(g)$ as a linear combination of the elements of this basis.

(2.3) Proposition: [G2; (3.1)]. For any $g = (g_{\mu\nu})_{\mu,\nu \in \underline{n}}$ in G there holds

$$T_r(g) = \sum_{(i,j) \in \Omega} g_{i,j}\, \xi_{i,j}\,.$$

where $g_{i,j}$ means $g_{i_1j_1}\, g_{i_2j_2} \cdots g_{i_rj_r}$.

A formula for the multiplication of two basis elements $\xi_{i,j}$ and $\xi_{h,\ell}$ of $S(G)$ is due to Schur (see [S; p. 20] or [G1; (2.3b)]) and it says

$$\xi_{i,j}\,\xi_{h,\ell} = \sum_{(p,q) \in \Omega} (z(i,j,h,\ell,p,q)\cdot 1_k)\xi_{p,q}\,. \tag{2.4}$$

where $z(i,j,h,\ell,p,q) = \# \{s \in I(n,r) \mid (i,j) \sim (p,s)$ and $(h,\ell) \sim (s,q)\}$, for any $i,j,h,\ell \in I(n,r)\}$.

The following lemma is an easy consequence of this rule

(2.5) Lemma: [G1; (2.3c)]. For any $i,j,h,\ell \in I$ there holds

(i) $\xi_{i,j}\,\xi_{h,\ell} = 0$, unless $j \sim h$

(ii) $\xi_{i,i}\,\xi_{i,j} = \xi_{i,j}\,\xi_{j,j} = \xi_{i,j}$

(iii) $\xi_{i,i}{}^2 = \xi_{i,i}$ and $\xi_{i,i}\,\xi_{j,j} = 0$ if $i \neq j$.

Let $i,j \in I(n,r)$ and supose i has weight λ. Then $\xi_{i,i} = \xi_{j,j}$ iff $(i,i) \sim (j,j)$ iff $i \sim j$, i.e., iff j has weight λ. So from now on we shall write ξ_λ for $\xi_{i,i}$.

Using (2.3) it is easy to see that $T_\ell(id) = \sum_{\lambda \in \Lambda} \xi_\lambda$. Also form (2.5)(iii) we know that $\xi_\lambda^2 = \xi_\lambda$ and $\xi_\lambda\,\xi_\alpha = 0$ if $\lambda \neq \alpha$ $(\alpha, \lambda \in \Lambda)$. Thus, since $1_{S(G)} = T_\ell(id)$, we have that

(2.6) $$1_{S(G)} = \sum_{\lambda \in \Lambda} \xi_\lambda$$

is an orthogonal idempotent decomposition of $1_{S(G)}$.

Calculations using rule (2.4) turn out to be very long and complicated, so we shall use a new version of this formula, given by J.A. Green in [G2], which is more convenient for our work. We state it now.

For $i,j,\ell \in I$, let P_i denote the stabilizer of i in P, i.e., $P_i = \{\pi \in P \mid i\pi = i\}$, and write $P_{i,j} = P_i \cap P_j$, $P_{i,j,\ell} = P_i \cap P_j \cap P_\ell$. Then, if $[P_{i,\ell} : P_{i,\ell,j}]$ denotes the index of $P_{i,\ell,j}$ in $P_{i,\ell}$, we have the

(2.7) Theorem: [G2; (2.6)]. For any $i,j,\ell \in I(n,r)$ there holds

$$\xi_{i,j}\,\xi_{j,\ell} = \sum_{\delta} ([P_{i\delta,\ell} : P_{i\delta,\ell,j}]1_k)\,\xi_{i\delta,\ell}\,,$$

where the sum is over a transversal $\{\delta\}$ of the set of all double cosets $P_{i,j}\,\delta\,P_{j,\ell}$ in P_j.

Remarks: (i) It is assumed that $\delta = 1$ is a member of the transversal.

(ii) The elements $\xi_{i\delta,\ell}$ considered above may not be all distinct.

§3. Bases for $S(G_J)$ and $S(L_J)$

In this paragraph we will apply the method used in [G2; pp. 11, 13] to determine k-bases for $S(G_J)$ and $S(L_J)$, where G_J is any standard parabolic subgroup of G and L_J is its Levi factor. We start with some notation.

B^+ (resp. B^-) denotes the *Borel subgroup* of G, consisting of all upper (resp. lower) triangular matrices in G. T is the group of all diagonal matrices in G and U^+ (resp. U^-) is the group of all unipotent matrices in B^+ (resp. B^-).

For each $\mu, \nu \in \underline{n}$, $\mu \neq \nu$, let $\varepsilon_{\mu\nu}$ be the element of $\mathbf{Z}^n$ with 1 in position μ, -1 in position ν, and zeros elsewhere. These are called the *roots* (of G) and $\Delta = \{e_{\mu,\mu+1} \mid \mu \in \underline{n-1}\}$ is the set of *simple roots.*

$U_{\mu\nu} = U_{\varepsilon_{\mu\nu}}$ is the *root subgroup* associated with the root $\varepsilon_{\mu\nu}$ ($\mu,\nu \in \underline{n}$, $\mu \neq \nu$), i.e., $U_{\mu\nu} = \{u_{\mu\nu}(t) \mid t \in k\}$, where $u_{\mu\nu}(t)$ is the element of G with 1's in the main diagonal, t in position (μ,ν) and zeros elsewhere. It is well known that $U^+ = \langle u_{\mu,\mu+1}(t) \mid \mu \in \underline{n-1},\ t \in k\rangle$.

For any subset J of $\underline{n-1}$ we will consider the *standard parabolic subgroups* of G,

$G_J^+ = \langle B^+, x_\mu \mid \mu \in J\rangle$ and $G_J^- = \langle B^-, x_\mu \mid \mu \in J\rangle$, where, for any $\mu \in \underline{n-1}$

$$(3.1) \qquad x_\mu = \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ 0 & & & & & & \ddots\ 1 \end{pmatrix} \begin{array}{l} \\ \\ \\ (\text{row } \mu) \\ (\text{row } \mu+1) \\ \\ \\ \end{array}$$

Finally we write $L_J = \langle T, U_{\mu\nu} \mid e_{\mu\nu} \in \Phi_J\rangle$ and

$$U_J^+ = \prod_{\substack{e_{\mu\nu} \notin \Phi_J \\ \mu<\nu}} U_{\mu\nu}, \qquad U_J^- = \prod_{\substack{e_{\mu\nu} \notin \Phi_J \\ \mu>\nu}} U_{\mu\nu},$$

where $\Phi_J = \{e_{\mu\nu} \mid \mu, \nu \in \underline{n}\} \cap (\bigoplus_{\mu \in J} \mathbb{Z}\, e_{\mu,\mu+1})$.

Suppose $J = \underline{n} \setminus \{m_1,\ldots,m_s\}$, for integers $0 = m_0 < m_1 < \ldots < m_{s-1} < m_s = n$ $(s \geq 1)$. We are in the situation of (1.4) and as we did there we define $N_a = \{m_{a-1}+1,\ldots, m_a\}$, for each $a \in \underline{s}$. Then a typical element, $g = (g_{\mu\nu})_{\mu,\nu \in \underline{n}}$, of G_J^+ has the form

$$g = \begin{array}{r} (\text{row } m_1) \\ (\text{row } m_2) \\ \\ (\text{row } m_{s-1}) \end{array} \begin{pmatrix} \boxed{} & & * \\ & \ddots & \\ 0 & & \boxed{} \end{pmatrix}$$

i.e., $g_{\mu\nu} = 0$, unless $\mu \leq \nu$ or μ and ν are in the same set N_a, for some $a \in \underline{s}$. Thus (cf. (1.5))

$$(3.2) \qquad G_J^+ = \{g \in G \mid g_{\mu\nu} = 0 \text{ unless } \mu \underset{J}{\leq} \nu, \text{ for all } \mu, \nu \in \underline{n}\},$$

and for any $i,j \in I(n,r)$ and for any $g \in G_J^+$ there holds

$$(3.3) \qquad g_{i,j} = g_{i_1j_1} \cdots g_{i_rj_r} = 0, \text{ unless } i \underset{J}{\leq} j.$$

So from (2.3) we have

$$(3.4) \qquad T_r(g) = \sum_{(i,j) \in \Omega} g_{i,j}\, \xi_{i,j} = \sum_{(i,j) \in \Omega,\, i \underset{J}{\leq} j} g_{i,j}\, \xi_{i,j}.$$

This means that $S(G_J^+) = T_r(kG_J^+)$ is contained in the k-span of $D = \{\xi_{i,j} \mid (i,j) \in \Omega, i \underset{J}{\leq} j\}$. Being a subset of a basis of $S(G)$, D is linearly independent so, if we show that D is contained in $S(G_J^+)$ we have proved the

(3.5) Proposition: $S(G_J^+)$ has k-basis $\{\xi_{i,j} \mid (i,j) \in \Omega,\ i \underset{J}{\leq} j\}$.

Proof: In this proof we write $M = \{(\mu,\nu) \in \underline{n} \times \underline{n} \mid \mu \underset{J}{\leq} \nu\}$.

Suppose $S(G_J^+)$ is a proper subset of the k-span of D. Then there are elements

$b_{i,j} \in k$, not all zero, such that

$$(3.6) \qquad \sum_{(i,j)\in\Omega,\ i \underset{J}{\leq} j} b_{i,j}\, g_{i,j} = 0, \quad \text{for all } g \in G_J^+ .$$

Consider in the polynomial ring $k[x_{\mu\nu} \mid (\mu,\nu) \in M]$ on the indeterminates $x_{\mu,\nu}$ $((\mu,\nu) \in M)$, the polynomials

$$(3.7) \qquad b(x) = \sum_{(i,j)\in\Omega,\ i \underset{J}{\leq} j} b_{i,j}\, x_{i,j} \quad \text{and} \quad c(x) = \prod_{a \in \underline{s}} \det(x_{\mu\nu})_{\mu,\nu \in N_a}.$$

Then (3.6) says that [1] $b((g_{\mu\nu})_{(\mu,\nu) \in M}) = 0$, for all values $g_{\mu\nu} \in k$ that satisfy $c((g_{\mu\nu})_{\mu,\nu \in M}) \neq 0$. At this point we may use the

Principle of irrelevance of algebraic inequalities (cf. e.g. [C; p. 140]).

Let $f, g, h \in k[x_1,\dots,x_m]$, $h \neq 0$ (where k is an infinite field) and suppose that $f(\alpha) = g(\alpha)$ for all $\alpha = (\alpha_1,\dots,\alpha_m)$ for which $h(\alpha) \neq 0$. Then $f = g$.

And we have that $b(x) = 0$. But the monomials $x_{i,j} = x_{i_1 j_1} \cdots x_{i_r j_r}$, all $(i,j) \in \Omega$, $i \underset{J}{\leq} j$, are all distinct and so linearly independent elements of $k[x_{\mu\nu} \mid (\mu,\nu) \in M]$. Hence $b(x) = 0$ implies $b_{i,j} = 0$, for all $(i,j) \in \Omega$, $i \underset{J}{\leq} j$. This contradicts our hypothesis and proves (3.5). □

Applying the same process to

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[1] By $b((g_{\mu\nu}))$ and $c((g_{\mu\nu}))$ we mean the element of k obtained by replacing the indeterminate $x_{\mu\nu}$ in (3.7) by $g_{\mu\nu}$, for all $(\mu,\nu) \in M$.

$$G_J^- = \{g \in G \mid g_{\mu\nu} = 0 \text{ unless } \nu \underset{J}{\leq} \mu, \text{ for all } \mu, \nu \in \underline{n}\}$$

and

$$L_J = \{g \in G \mid g_{\mu\nu} = 0 \text{ unless } \mu \underset{J}{=} \nu, \text{ for all } \mu, \nu \in \underline{n}\},$$

we obtain

(3.8) Proposition: $S(G_J^-)$ and $S(L_J)$ have k-bases

$$\{\xi_{i,j} \mid (i,j) \in \Omega,\ j \underset{J}{\leq} i\} \qquad \text{and} \qquad \{\xi_{i,j} \mid (i,j) \in \Omega,\ i \underset{J}{=} j\},$$

respectively.

§ 4 Weight spaces

Let H be a subgroup of G containing T and let $V \in \operatorname{mod} S(H)$.

We know that, for all $\lambda \in \Lambda$, $\xi_\lambda \in S(H)$ (since $S(T) \subseteq S(H)$ and, taking $J = \emptyset$ in (3.8), we get that $\{\xi_\lambda \mid \lambda \in \Lambda\}$ is a k-basis of $S(T)$). Hence there is the orthogonal idempotent decomposition

$$1 = \sum_{\lambda \in \Lambda} \xi_\lambda$$

of 1 in $S(H)$ (cf. (2.6)), which yields the decomposition of V

$$V = \bigoplus_{\lambda \in \Lambda} \xi_\lambda V \tag{4.1}$$

as a direct sum of subspaces.

(4.2) Definition: For each $\lambda \in \Lambda$, $V^\lambda = \xi_\lambda V$ is called the λ-*weight space* of V. We say that λ is a *weight* of V if $\dim_k V^\lambda > 0$.

It is well known (cf. [G1; (3.2)]) that this definition coincides with the usual definition of weight space when we regard V as a rational T-module and identify λ with the multiplicative character $T \to k$ given by $g \mapsto g_{11}^{\lambda_1} \cdots g_{nn}^{\lambda_n}$ (all $g \in T$).

The next proposition is an easy consequence of the definition of weight space and of the fact that ξ_λ is idempotent.

(4.3) Proposition: [G1; (3.3b)] Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence in mod $S(H)$. Then the naturally induced sequence of k-spaces $0 \to V_1{}^\lambda \to V_2{}^\lambda \to V_3{}^\lambda \to 0$ is exact, for any $\lambda \in \Lambda$.

Before we proceed we need to define some notation. For any $\lambda \in \Lambda(n,r)$ we choose a basic λ-tableau T^λ and define $\ell(\lambda) \in I(n,r)$ by the λ-tableau

(4.4)
$$T^\lambda_{\ell(\lambda)} = \begin{array}{|l|l} \hline 1\ 1\ \ldots\ \ldots\ 1 & \text{(row 1)} \\ \hline 2\ 2\ \ldots\ \ldots\ \ldots\ 2 & \text{(row 2)} \\ \hline \ldots\ \ldots & \\ \hline n\ n\ \ldots\ \ldots\ \ldots\ n & \text{(row n)} \\ \hline \end{array}$$

If $\mu, \nu \in \underline{n}$ and $\mu \neq \nu$ define, for each non-negative integer $m \le \lambda_\nu$, the element

$\ell(\mu, \nu, m, \lambda)$ of $I(n,r)$ by the λ-tableau

$$(4.5) \quad T^{\lambda}_{\ell(\mu,\nu,m,\lambda)} = \begin{array}{|l|l}
\hline
1\ 1\ \cdots\cdots\ 1 & \text{(row 1)} \\
\hline
2\ 2\ \cdots\cdots\ \ 2 & \text{(row 2)} \\
\hline
\cdots\cdots & \\
\hline
\underbrace{\mu \ldots \mu}_{m}\ \nu\ \cdots\cdots\ \nu & \text{(row } \nu) \\
\hline
\cdots\cdots & \\
\hline
n\ n\ \cdots\cdots\ n & \text{(row } n) \\
\hline
\end{array}$$

i.e. $T^{\lambda}_{\ell(\mu,\nu,m,\lambda)}$ is obtained from $T^{\lambda}_{\ell(\lambda)}$ by substituting the first m ν's in row ν by μ's and keeping all other entries unchanged.

In this section we write $\ell(m, \lambda)$ for $\ell(\mu, \nu, m, \lambda)$ if no confusion relative to μ and ν arises.

(4.6) Proposition: [G2; (5.8)] Let $\mu, \nu \in \underline{n}$ and suppose that $\mu < \nu$ and $\lambda_\mu < \lambda_\nu$. Then

$$\xi_\lambda = \sum_{m=\lambda_\nu-\lambda_\mu}^{\lambda_\nu} z_m \, \xi_{\ell(\lambda),\, \ell(m,\lambda)} \, \xi_{\ell(m,\lambda),\, \ell(\lambda)} \,,$$

where z_m are integers independent of char k.

(4.7) Lemma: Suppose $\mu, \nu \in \underline{n}$, $\mu \neq \nu$ and let $u_{\mu\nu}(t)$ be the element of G with 1's in the main diagonal, t in position (μ, ν), and zeros elsewhere ($t \in k$). Then

$$T_r(u_{\mu\nu}(t)) = \sum_{m=0}^{r} t^m \, \Gamma_{\mu\nu}^{(m)} ,$$

where

$$\Gamma_{\mu\nu}^{(m)} = \sum_{\lambda} \xi_{\ell(m,\lambda),\,\ell(\lambda)} ,$$

sum over all weights $\lambda \in \Lambda$ such that $m \leq \lambda_\nu$.

Proof: Write g for $u_{\mu\nu}(t)$. Then, from (2.3), we have that $T_r(g) = \sum_{(i,j) \in \Omega} g_{i,j} \, \xi_{i,j}$. But $g_{i,j} = 0$ unless $(i_\rho, j_\rho) \in \{(1,1), (2,2), \ldots, (n,n), (\mu, \nu)\}$, all $\rho \in \underline{r}$. If this last condition holds and if m is the number of $\rho \in \underline{r}$ such that $(i_\rho, j_\rho) = (\mu, \nu)$, then $g_{i,j} = t^m$ and $(i,j) \sim (\ell(m,\lambda), \ell(\lambda))$, for some λ with $m \leq \lambda_\nu$.

Now consider any $\lambda \in \Lambda$ with $m \leq \lambda_\nu$. Clearly $g_{\ell(m,\lambda),\,\ell(\lambda)} = t^m$. So the proof will be complete when we show that $(\ell(m,\lambda), \ell(\lambda)) \not\sim (\ell(m',\alpha), \ell(\alpha))$ if $m \neq m'$ or $\lambda \neq \alpha$ ($\alpha \in \Lambda$, $m' \leq \alpha_\nu$). But this is immediate, since $\ell(\lambda)$ and $\ell(\alpha)$ (if $\lambda \neq \alpha$) or $\ell(m,\lambda)$ and $\ell(m',\alpha)$ (if $\lambda = \alpha$) have different weights, so they are not in the same P-orbit of I. □

(4.8) Proposition: Let J be any subset of $\underline{n-1}$ and let H be one of the groups G_J^+, G_J^- or L_J defined in §3. Let $V \in \mathrm{mod}\, S(H)$ and suppose there is $v \in V$ such that

(i) $v \neq 0$ and $\xi_\lambda v = v$, for some $\lambda \in \Lambda$;

(ii) there are $\mu, \nu \in \underline{n}$ such that $\mu < \nu$, $\mu \underset{J}{=} \nu$ and $T_r(u_{\mu\nu}(t))v = v$, for all $t \in k$.

Then $\lambda_\nu \leq \lambda_\mu$.

Proof: Suppose v, λ, and μ, ν satisfy (i) and (ii) above, and let m be any non-negative integer such that $m \leq \lambda_\nu$. Then, as $\mu \underset{j}{=} \nu$, $\ell(m, \lambda) \underset{j}{=} \ell(\lambda)$ and the elements $\xi_{\ell(m,\lambda),\ell(\lambda)}$ and $\xi_{\ell(\lambda),\ell(m,\lambda)}$ are in $S(H)$. Also, as $\xi_\lambda v = v$, we have

$$\Gamma^{(m)}_{\mu\nu} v = \Gamma^{(m)}_{\mu\nu}\xi_\lambda v = \sum_\alpha \xi_{\ell(m,\alpha),\ell(\alpha)}\xi_\lambda v$$

sum over all weights $\alpha \in \Lambda$ such that $m \leq \alpha_\nu$.

But $\xi_{\ell(m,\alpha),\ell(\alpha)}\xi_\lambda = 0$ or $\xi_{\ell(m,\lambda),\ell(\lambda)}$, according as $\alpha \neq \lambda$ or $\alpha = \lambda$, and so

$$\Gamma^{(m)}_{\mu\nu} v = \begin{cases} \xi_{\ell(m,\lambda),\ell(\lambda)} v\,; & \text{if } m \leq \lambda_\nu \\ 0\,; & \text{if } m > \lambda_\nu. \end{cases}$$

Hence, by lemma (4.7), we have $T_r(u_{\mu\nu}(t))v = \sum_{m=0}^{\lambda_\nu} t^m \xi_{\ell(m,\lambda),\ell(\lambda)} v$, for all $t \in k$.

Note that $\ell(0, \lambda) = \ell(\lambda)$ and so $\xi_{\ell(0,\lambda),\ell(\lambda)} = \xi_\lambda$. Therefore $T_r(u_{\mu\nu}(t))v = v$ iff

$$\xi_\lambda v + \sum_{m=1}^{\lambda_\nu} t^m \xi_{\ell(m,\lambda),\ell(\lambda)} v = v \quad \text{iff}$$

(4.9)
$$\sum_{m=1}^{\lambda_\nu} t^m \xi_{\ell(m,\lambda),\ell(\lambda)} v = 0, \quad \text{all } t \in k.$$

Since k is an infinite field we may choose $t_1, \ldots, t_{\lambda_\nu} \in k$ such that $\det(t_a{}^b)_{a,b \in \underline{\lambda_\nu}} \neq 0$. So (4.9) implies

(4.10)
$$\xi_{\ell(m,\lambda),\ell(\lambda)} v = 0, \quad \text{for all } m \in \underline{\lambda_\nu}.$$

Suppose $\lambda_\mu < \lambda_\nu$.

From (4.6) we know that there are integers z_m such that

$$\xi_\lambda = \sum_{m=\lambda_\nu-\lambda_\mu}^{\lambda_\nu} z_m \, \xi_{\ell(\lambda),\, \ell(m,\lambda)} \xi_{\ell(m,\lambda),\, \ell(\lambda)}.$$ Hence

$$v = \xi_\lambda v = \sum_{m=\lambda_\nu-\lambda_\mu}^{\lambda_\nu} z_m \, \xi_{\ell(\lambda),\, \ell(m,\lambda)} \xi_{\ell(m,\lambda),\, \ell(\lambda)} v = 0 \quad \text{(by (4.10))}.$$

This contradicts the assumption of $v \neq 0$. So $\lambda_\nu \leq \lambda_\mu$. □

§5. Contravariant duals

We start this section with a result for a very general class of k-algebras and then we apply it to Schur algebras.

Let S be a finite dimensional k-algebra equipped with an involutory anti-automorphism $^\circ : S \to S$. Let R be a subalgebra of S and write $^\circ R$ for its image by $^\circ$, i.e., $^\circ R = {}^\circ(R)$ (similarly $^\circ\xi$ denotes $^\circ(\xi)$, for any $\xi \in S$).

If $V \in \mathrm{mod}\, R$, its dual, $V^* = \mathrm{Hom}_k(V,k)$, can be made into a left $^\circ R$-module by

(5.1) $$(\xi\,\theta)v = \theta({}^\circ\xi v), \quad \theta \in V^*, \ \xi \in {}^\circ R, \ v \in V.$$

(5.2) Definition: For each $V \in \mathrm{mod}\, R$, the $^\circ R$-module V^*, defined above, will be called the *contravariant dual* of V (relative to $^\circ$) and will be denoted V°.

(5.3) Remark: It is not difficult to see that the natural isomorphism $V \to (V^*)^*$, of finite dimensional k-spaces, is an R-isomorphism $V \to (V^\circ)^\circ$.

Let $V \in \mathrm{mod}\, R$ and $W \in \mathrm{mod}\, {}^\circ R$ be given. A k-bilinear form $(\ ,\ ) : W \times V \to k$

is called *contravariant* (in ${}^*R$) if it satisfies $(\xi w, v) = (w, {}^*\xi v)$, for all $\xi \in {}^*R$, $w \in W$, $v \in V$. It is well known that such a non-singular form exists iff W and V^* are isomorphic ${}^*R$-modules (the isomorphism $\gamma : W \to V^*$ being given by $\gamma(w)(v) = (w,v)$).

Now let Q be another subalgebra of S such that $R \le Q$. Then ${}^*R \le {}^*Q$ and, Q and ${}^*Q$ may be regarded as (R, Q)- and $({}^*Q, {}^*R)$-bimodules, respectively.

Consider the right exact functor

(5.4) $$F = {}^*Q \otimes_{{}^*R} \cdot : \text{mod } {}^*R \to \text{mod } {}^*Q$$

and the left exact functor[2]

(5.5) $$F' = \text{Hom}_R(Q, \cdot) : \text{mod } R \to \text{mod } Q.$$

(5.6) Theorem: With the notation above, there is a ${}^*Q$-isomorphism

$$F(V^*) \cong (F'(V))^* ,$$

natural in $V \in \text{mod } R$.

Proof: It is enough to describe, for each $V \in \text{mod } R$, a non-singular bilinear form $\Phi_V : F(V^*) \times F'(V) \to k$, which is contravariant in ${}^*Q$ and is natural in mod R.

Let $(\, ,\,)_V : V^* \times V \to k$ be the k-bilinear contravariant non-singular form defined by

[2] If $V \in \text{mod } R$, Q acts on the left of $\text{Hom}_R(Q,V)$ by $(\xi u)(\eta) = u(\eta\xi)$, $u \in F'(V)$, $\xi, \eta \in Q$.

$$(\theta, v)_V = \theta(v), \ \theta \in V^*, v \in V;$$

(the contravariant property comes form (5.1)). For each $u \in F'(V) = \mathrm{Hom}_R(Q,V)$, we may define a k-bilinear map $h'_u : {}^*Q \times V^* \to k$ by $h'_u(\eta, \theta) = (\theta, u({}^*\eta))_V$ (all $\eta \in {}^*Q$, $\theta \in V^*$). Since $(,)_V$ is contravariant and u is an R-map, we have

$$h'_u(\eta\xi, \theta) = (\theta, u({}^*\xi\, {}^*\eta))_V =$$
$$= (\theta, {}^*\xi\, u({}^*\eta))_V = (\xi\theta, u({}^*\eta))_V = h'_u(\eta, \xi\theta)$$

(for any $\eta \in {}^*Q$, $\xi \in {}^*R$, $\theta \in V^*$) which proves that h'_u is ${}^*R$-balanced. Hence we may define a k-linear map $h_u : {}^*Q \otimes_{{}^*R} V^* \to k$ by $h_u(\eta \otimes \theta) = (\theta, u({}^*\eta))_V$, and the k-bilinear form $\Phi_V : F(V^*) \times F'(V) \to k$ by

(5.7) $\quad \Phi_V(\eta \otimes \theta, u) = h_u(\eta \otimes \theta) = (\theta, u({}^*\eta))_V$, all $\theta \in V^*$, $\eta \in {}^*Q$, $u \in F'(V)$.

To prove that Φ_V is contravariant, take θ, η, u as above and any $\xi \in {}^*Q$. Then the left ${}^*Q$-action on $F(V^*)$ gives $\xi(\eta \otimes \theta) = \xi\eta \otimes \theta$. So $\Phi_V(\xi(\eta \otimes \theta), u) = (\theta, u({}^*(\xi\eta)))_V = (\theta, u({}^*\eta{}^*\xi))_V$. But the left action of Q on $F'(V)$ gives $({}^*\xi u)({}^*\eta) = u({}^*\eta{}^*\xi)$. So $\Phi_V(\xi(\eta \otimes \theta), u) = (\theta, ({}^*\xi u)({}^*\eta))_V = \Phi_V(\eta \otimes \theta, {}^*\xi u)$.

The next step is to prove that Φ_V is non-singular.

Consider the k-spaces $X = {}^*Q \otimes V^*$ and $Y = \mathrm{Hom}_k(Q,V)$. Clearly these have the same dimension (viz. dim Q dim V). Define a k-bilinear form $\tilde{\Phi}_V : X \times Y \to k$, using the same formula as for Φ_V, i.e.,

$$\tilde{\Phi}_V (\eta \otimes \theta, u) = (\theta, u({}^*\eta))_V \text{, all } u \in Y, \eta \in {}^*Q, \theta \in V^*.$$

The right kernel of $\tilde{\Phi}_V$ is the set of all $u \in Y$ such that $\tilde{\Phi}_V(x, u) = 0$, for all $x \in X$, or equivalently, $\tilde{\Phi}_V(\eta \otimes \theta, u) = 0$, for all $\eta \in {}^*Q$, $\theta \in V^*$. As $\tilde{\Phi}_V(\eta \otimes \theta, u) = (\theta, u({}^*\eta))_V$ and $(,)_V$ is non-singular we have that $u \in$ right ker $\tilde{\Phi}_V$ iff $u({}^*\eta) = 0$, for all $\eta \in {}^*Q$, i.e., iff $u = 0$. Hence $\tilde{\Phi}_V$ is non-singular since its right kernel is trivial and $\dim X = \dim Y$.

From the definition of tensor product, we know that $F(V^*) = {}^*Q \otimes_{{}^*R} V^* = X/M$, where M is the subspace of X k-spanned by $\{\eta\xi \otimes \theta - \eta \otimes \xi\theta \mid \eta \in {}^*Q,\ \xi \in {}^*R,\ \theta \in V^*\}$. Let $M^\perp = \{u \in Y \mid \tilde{\Phi}_V(x,u) = 0,$ for all $x \in M\}$. It is clear that there is a non-singular k-bilinear form $\hat{\Phi}_V : X/M \times M^\perp \to k$, given by $\hat{\Phi}_V(x + M, u) = \tilde{\Phi}_V(x,u)$, all $x \in X$, $u \in M^\perp$. So if we prove that $M^\perp = F'(V)$, we have that $\Phi_V = \hat{\Phi}_V$ is non-singular. So let $u \in Y$. Then $u \in M^\perp$ iff, for all $\eta \in {}^*Q$, $\xi \in {}^*R$, $\theta \in V^*$, there holds

$$\tilde{\Phi}_V(\eta\xi \otimes \theta, u) = \tilde{\Phi}_V(\eta \otimes \xi\theta, u), \text{ i.e., } (\theta, u({}^\circ\xi{}^\circ\eta))_V = (\xi\theta, u({}^\circ\eta))_V$$

which means

$$\theta(u({}^\circ\xi{}^\circ\eta)) = (\xi\theta)(u({}^\circ\eta)) \text{ i.e. } \theta(u({}^\circ\xi{}^\circ\eta)) = \theta({}^\circ\xi u({}^\circ\eta)).$$

But this is equivalent to $u({}^\circ\xi{}^\circ\eta) = {}^\circ\xi u({}^\circ\eta)$, for all $\eta \in {}^*Q$, $\xi \in {}^\circ R$, i.e., $u \in \mathrm{Hom}_R(Q,V)$. Hence $M^\perp = F'(V)$.

The proof of the theorem will be complete when we show that Φ_V is natural in $V \in \mathrm{mod}\ R$. This amounts to the condition that for any $V, V' \in \mathrm{mod}\ R$, and for all $f \in \mathrm{Hom}_R(V, V')$

$$\Phi_V(\eta \otimes \tau f, u) = \Phi_{V'}(\eta \otimes \tau, fu)$$

i.e. $(\tau f, u(\eta^\circ))_V = (\tau, fu({}^\circ\eta))_{V'}$, for all $\eta \in {}^\circ Q$, $\tau \in V'^*$ and $u \in F'(V)$, which is trivially true. $\square$

Returning to the Schur algebra $S(G)$ we may define a k-linear automorphism $^\circ : S(G) \to S(G)$, by

(5.8) $$^{*}\xi_{i,j} = \xi_{j,i}, \text{ all } (i,j) \in \Omega.$$

This is in fact an involutary anti-automorphism of $S(G)$ (cf. [G1; p. 32]) and so we are in the conditions referred to above.

For any subset J of $\underline{n-1}$ consider the Schur algebras $S(G_J^+)$ and $S(G_J^-)$. It is clear from its definition that this anti-automorphism carries the basis

$\{\xi_{i,j} \mid (i,j) \in \Omega, i \underset{J}{\leq} j\}$ of $S(G_J^+)$, into the basis $\{\xi_{i,j} \mid (i,j) \in \Omega, j \underset{J}{\leq} i\}$ of $S(G_J^-)$, and vice-versa, hence

(5.9) $$^{*}S(G_J^-) = S(G_J^+).$$

So if we consider any $V \in \text{mod } S(G_J^-)$ (resp. $V' \in \text{mod } S(G_J^+)$) its dual, V^*, is in $\text{mod } S(G_J^+)$ (resp. $V'^* \in \text{mod } S(G_J^-)$).

Also if J' is another subset of $\underline{n-1}$, such that $J' \subseteq J$, we may use (5.6) with $R = S(G_{J'}^+)$ and $Q = S(G_J^+)$ or $R = S(G_{J'}^-)$ and $Q = S(G_J^-)$.

2. THE MODULES $K_{\lambda,J}$

§6 The Schur algebra $S(B^+)$

We shall now give special attention to the Schur algebra $S(B^+) = S_k(n,r; B^+)$ for the Borel subgroup B^+ of G.

Using the notation of §3, $B^+ = G^+_\emptyset$. So if $\Omega' = \{(i,j)) \in \Omega \mid i \leq j\}$ we get from (3.5) that

(6.1) $S(B^+)$ has k-basis $\{\xi_{i,j} \mid (i,j) \in \Omega'\}$.

This result is not new, it can be found in [G2] where it is also proved that

(6.2) rad $S(B^+)$ has k-basis $\{\xi_{i,j} \mid (i,j) \in \Omega', i \neq j\}$.

For each $\lambda \in \Lambda(n,r)$ consider the left ideal

$$V_\lambda = S(B^+)\xi_\lambda .$$

As $S(B^+) = \bigoplus_{(i,j) \in \Omega'} k\xi_{i,j}$, V_λ is k-spanned by all $\xi_{i,j}\,\xi_\lambda$, $(i,j) \in \Omega'$. But from (2.5) we know that $\xi_{i,j}\,\xi_\lambda$ is $\xi_{i,j}$ or 0, according as j has weight λ or not.

Thus, $V_\lambda = \bigoplus_{(i,j) \in \Omega',\, j \in \lambda} k\xi_{i,j}$, i.e.,

(6.3) V_λ has k-basis $\{\xi_{i,j} \mid (i,j) \in \Omega',\ j \in \lambda\}$ [3]

[3] In §9 we shall give another description of this basis involving row-semistandard tableaux and the element $\ell(\lambda)$ defined in (4.4).

Now consider the k-algebra $\xi_\lambda S(B^+)\xi_\lambda = \xi_\lambda V_\lambda$. It is spanned by $\xi_\lambda \xi_{i,j}$, for all $(i,j) \in \Omega'$ such that $j \in \lambda$. Once more, we have $\xi_\lambda \xi_{i,j} = 0$, unless i has weight λ and if so, there is $\pi \in P$ such that $i = j\pi$. But then we have $j\pi = i \leq j$, which implies $i = j$ (cf (1.8)), and so $\xi_{i,j} = \xi_{j,j} = \xi_\lambda$. Hence

$$\xi_\lambda S(B^+)\xi_\lambda = k\xi_\lambda$$

is a local ring and ξ_λ is a primitive idempotent of $S(B^+)$. Putting this together with (2.6) and using that $1_{S(B^+)} = 1_{S(G)}$, we have proved that

(6.4)
$$1_{S(B^+)} = \sum_{\lambda \in \Lambda} \xi_\lambda$$

is a primitive orthogonal idempotent decomposition of $1_{S(B^+)}$, and

$$S(B^+) = \bigoplus_{\lambda \in \Lambda} V_\lambda$$

is a direct sum decomposition of $S(B^+)$ into projective indecomposable $S(B^+)$-modules.

As an immediate consequence of this result we have that, for any $\lambda \in \Lambda$, V_λ has a unique maximal submodule, viz. $\operatorname{rad} V_\lambda = (\operatorname{rad} S(B^+))\xi_\lambda$, and so $V_\lambda/\operatorname{rad} V_\lambda$ is an irreducible $S(B^+)$-module.

Using the same argument as for (6.3) we have, as a consequence of (6.2), that

(6.5) $\operatorname{rad} V_\lambda$ has k-basis $\{\xi_{i,j} \mid (i,j) \in \Omega', i \neq j, j \in \lambda\}$.

Therefore $V_\lambda/\operatorname{rad} V_\lambda = k(\xi_\lambda + \operatorname{rad} V_\lambda)$ is a one-dimensional vector space and it is clear that

$$V_\lambda/\mathrm{rad}\,V_\lambda \underset{S(B^+)}{\cong} V_\alpha/\mathrm{rad}\,V_\alpha \text{ iff } \alpha = \lambda \ (\alpha \in \Lambda).$$

This together with (6.4) gives that

$$\left\{V_\lambda/\mathrm{rad}\,V_\lambda \mid \lambda \in \Lambda(n,r)\right\} \text{ and } \left\{V_\lambda \mid \lambda \in \Lambda(n,r)\right\}$$

are full sets of pairwise non-isomorphic irreducible and projective indecomposable $S(B^+)$-modules, respectively.

In order to give a better characterization of these modules we define, for each $\lambda \in \Lambda$, the k-linear maps $\chi_\lambda : kB^+ \to k$ and $\aleph_\lambda : S(B^+) \to k$ by

(6.6) $\chi_\lambda(b) = b_{11}^{\lambda_1} \cdots b_{nn}^{\lambda_n}$, all $b \in B^+$, and

$$\aleph_\lambda(\xi_{i,j}) = \begin{cases} 1, & \text{if } i = j \text{ has weight } \lambda \\ 0, & \text{otherwise} \end{cases}, \text{ all } (i,j) \in \Omega',$$

respectively.

It is easy to see that χ_λ is a k-algebra map and that $\chi_\lambda(b) = \aleph_\lambda(T_r(b))$, for all $b \in B^+$. Thus $\aleph_\lambda$ is also a k-algebra map and we make the

(6.7) Definition: For each $\lambda \in \Lambda$, k_λ is the field k regarded either as a rational B^+-module affording the representation χ_λ or as an $S(B^+)$-module affording the representation $\aleph_\lambda$.

It is clear from the definitions that if

(6.8) $\qquad \aleph'_\lambda : V_\lambda \to k_\lambda$ is the restriction of $\aleph_\lambda$ to V_λ

then $\aleph'_\lambda$ is an $S(B^+)$-epimorphism with $\ker \aleph'_\lambda = \mathrm{rad}\, V_\lambda$. Thus $V_\lambda/\mathrm{rad}\, V_\lambda \underset{S(B^+)}{\cong} k_\lambda$.

As a summary of the main results of this section we have,

(6.9) Theorem: (i) $1 = \sum_{\lambda \in \Lambda(n,r)} \xi_\lambda$, is a primitive orthogonal idempotent decomposition of 1 in $S(B^+)$.

(ii) $\{k_\lambda \mid \lambda \in \Lambda(n,r)\}$ is a full set of pairwise non-isomorphic irreducible $S(B^+)$-modules.

(iii) $\{V_\lambda \mid \lambda \in \Lambda(n,r)\}$ is a full set of pairwise non-isomorphic projective indecomposable $S(B^+)$-modules.

(6.10) Remark: A result parallel to (6.9) can be obtained if we consider the Schur algebra $S(B^-)$. In this case, for each $\lambda \in \Lambda$, k_λ^- will denote the one-dimensional $S(B^-)$-module (or one-dimensional rational B^--module) affording the representation $\aleph_\lambda^- : S(B^-) \to k$ (resp. $\chi_\lambda^- : B^- \to k$), defined by

$$\aleph_\lambda^-(\xi_{i,j}) = \begin{cases} 1, & \text{if } i = j \text{ has weight } \lambda \\ 0, & \text{otherwise} \end{cases} \qquad ; \text{ all } (i,j) \in \Omega \text{ such that } j \leq i$$

(resp. $\chi_\lambda^-(b) = b_{11}^{\lambda_1} \cdots b_{nn}^{\lambda_n}$, all $b \in B^-$).

§7. Weyl modules

In [CL] R. Carter and G. Lusztig define, for each dominant weight λ, a $GL_n(k)$-module K_λ (there denoted $\bar{V}_\lambda$) and call it the Weyl module for $GL_n(k)$ associated with λ. Working with the universal enveloping algebra of the Lie algebra $gl(n)$, they prove that these are cyclic modules containing a unique maximal submodule and that the quotients by these give a full set of pairwise non-isomorphic polynomial irreducible $GL_n(k)$-modules. In particular if char $k = 0$ Weyl modules are themselves irreducible. A k-basis for K_λ, indexed by standard tableaux, is also produced in the work cited.

The same results were later obtained in [G1] within the framework of Schur algebras. Using a result of G. James [J, (26.4)] it is there proved that, in fact, K_λ may also be characterized as the contravariant dual of the induced module $\mathrm{Ind}_{B^-}^{G} k_\lambda^-$ (for any dominant weight λ).

Here we give an alternative definition of Weyl modules and we show how some of the results referred to above can be easily obtained from the properties of $S(B^+)$.

Take $J = \underline{n-1}$ and $J' = \emptyset$. Then $G_J^+ = G$, $G_{J'}^+ = B^+$, $G_{J'}^- = B^-$ and we may apply the results of §5 to $S(G)$, $S(B^+)$ and $S(B^-)$.

We have from (5.9) that ${}^\circ S(B^-) = S(B^+)$. Also ${}^\circ S(G) = S(G)$. Thus taking $Q = S(G)$ and $R = S(B^-)$ in (5.4) and (5.5) we get, $F(V^\circ) = S(G) \otimes_{S(B^+)} V^\circ$ and $F'(V) = \mathrm{Hom}_{S(B^-)}(S(G), V)$, and, by (5.6),

(7.1) there is an $S(G)$-isomorphism

$$S(G) \otimes_{S(B^+)} V^\circ \cong (\mathrm{Hom}_{S(B^-)}(S(G), V))^\circ,$$

for any $V \in \mathrm{mod}\ S(B^-)$.

For any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$ consider the irreducible $S(B^+)$-module k_λ and define

$$K_\lambda = S(G) \otimes_{S(B^+)} k_\lambda. \tag{7.2}$$

It is then clear that

(7.3) Lemma: $K_\lambda = S(G)\omega_\lambda$, where $\omega_\lambda = 1_{S(G)} \otimes 1_{k_\lambda}$. Hence K_λ is a cyclic $S(G)$-module.

In [G2; p. 14] it is proved that $S(G)$ has the decomposition

$$S(G) = S(B^+)S(B^-). \tag{7.4}$$

We now apply this result to K_λ.

From the action of $S(B^+)$ on k_λ (cf. (6.6) and (6.7)) there holds

$\xi_{i,j}\,\omega_\lambda = \xi_{i,j} \otimes 1_{k_\lambda} = 1_{S(G)} \otimes \xi_{i,j}\,1_{k_\lambda} = \omega_\lambda$ if $\xi_{i,j} = \xi_\lambda$, and zero otherwise (all $i \leq j$).

Thus $S(B^+)\omega_\lambda = \sum_{(i,j) \in \Omega'} k\xi_{i,j}\,\omega_\lambda = k\omega_\lambda$ and using (7.4) we get

$$K_\lambda = S(G)\omega_\lambda = S(B^-)\,S(B^+)\omega_\lambda = S(B^-)\omega_\lambda. \tag{7.5}$$

But $S(B^-)$ has k-basis $\{\xi_{i,j} \mid (j,i) \in \Omega'\}$. Hence by (7.5),

(7.6) $$K_\lambda = \sum_{(j,i)\in\Omega'} k\xi_{i,j}\omega_\lambda = \sum_{(j,i)\in\Omega',\, j\in\lambda} k\xi_{i,j}\omega_\lambda\,,$$

since $\omega_\lambda = \xi_\lambda\omega_\lambda$, and so $\xi_{i,j}\omega_\lambda = \xi_{i,j}\,\xi_\lambda\omega_\lambda = 0$, unless j has weight λ.

(7.7) Lemma: (i) $K_\lambda{}^\lambda = k\omega_\lambda$. Thus $\dim_k(K_\lambda{}^\lambda) \le 1$ and it is zero iff $K_\lambda = 0$.

(ii) If $\alpha \in \Lambda$ is a weight of K_λ then $\alpha \trianglelefteq \lambda$.

Proof: (ii) From (7.6) we have that, for any $\alpha \in \Lambda$,

$$\xi_\alpha K_\lambda = \sum_{(j,i)\in\Omega',\, j\in\lambda} k\xi_\alpha\,\xi_{i,j}\omega_\lambda = \sum_{(j,i)\in\Omega',\, j\in\lambda,\, i\in\alpha} k\xi_{i,j}\omega_\lambda.$$

So $\xi_\alpha\, K_\lambda \neq 0$ implies that there are $i,j \in I$ such that $i \in \alpha, j \in \lambda$ and $j \le i$. But then, by (1.10), $\alpha \trianglelefteq \lambda$.

(i) Consider now $\alpha = \lambda$. Then $\xi_\lambda\, K_\lambda = \sum_{(j,i)\in\Omega',\, i,\, j\in\lambda} k\xi_{i,j}\omega_\lambda$. But from (1.8) we know that if $i,j \in \lambda$ and $j \le i$ then $i = j$. Thus $\xi_\lambda K_\lambda = k\xi_\lambda\omega_\lambda = k\omega_\lambda$. □

It is just natural to ask under which conditions is $K_\lambda \neq 0$? The next proposition answers this question.

(7.8) Proposition: Let $\lambda = (\lambda_1,\dots,\lambda_n) \in \Lambda(n,r)$. Then $K_\lambda \neq 0$ iff $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n$.

Proof: Suppose first $K_\lambda \neq 0$. Then $\omega_\lambda \neq 0$ and $\xi_\lambda\,\omega_\lambda = \omega_\lambda$. If we prove that

$T_r(u_{\mu,\mu+1}(t))\omega_\lambda = \omega_\lambda$ (all $\mu \in \underline{n-1}$, $t \in k$), condition (ii) of (4.8) is satisfied (with $\nu = \mu + 1$ and any $\mu \in \underline{n-1}$). hence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Fix $\mu \in \underline{n-1}$ and write $\ell(m,\alpha) = \ell(\mu, \mu+1, m, \alpha)$ (cf. (4.4) and (4.5)). Then, from (4.7) we have $T_r(u_{\mu,\mu+1}(t)) = \sum_{m=0}^{r} t^m \Gamma^{(m)}_{\mu,\mu+1}$ where $\Gamma^{(m)}_{\mu,\mu+1} = \sum_\alpha \xi_{\ell(m,\alpha),\ell(\alpha)}$ (sum over all weights $\alpha \in \Lambda$ such that $m \leq \alpha_{\mu+1}$).

Note that if $m = 0$, $\alpha_{\mu+1} \geq 0$ for all $\alpha \in \Lambda$, so $\Gamma^{(0)}_{\mu,\mu+1} = 1_{S(G)}$. On the other hand if $m > 0$, $\ell(m,\alpha) < \ell(\alpha)$ (since $\mu < \mu + 1$) and so $\xi_{\ell(m,\alpha),\ell(\alpha)}\omega_\lambda = 0$, for all α. Thus, for any $t \in k$, we have

$$\text{(7.9)} \quad T_r(u_{\mu,\mu+1}(t))\omega_\lambda = \Gamma^{(0)}_{\mu,\mu+1}\,\omega_\lambda + \sum_{m=1}^{r} t^m \Gamma^{(m)}_{\mu,\mu+1}\,\omega_\lambda = \Gamma^{0}_{\mu,\mu+1}\,\omega_\lambda = \omega_\lambda.$$

As this holds for any $\mu \in \underline{n-1}$, we get the required result.

Now suppose $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and consider the contravariant dual $(k_\lambda^-)^*$ of the irreducible $S(B^-)$-module k_λ^-. Then $(k_\lambda^-)^*$ is a one-dimensional $S(B^+)$-module and for any $\theta \in (k_\lambda^-)^*$, $c \in k_\lambda^-$ and $(i,j) \in \Omega'$ there holds

$$(\xi_{i,j}\theta)(c) = \theta(\xi_{j,i}\, c) = \theta(c) \quad \text{if } \xi_{i,j} = \xi_\lambda, \text{ and zero otherwise.}$$

Therefore $(k_\lambda^-)^*$ affords the representation κ_λ and $(k_\lambda^-)^* \underset{S(B^+)}{\cong} k_\lambda$. Thus, from (7.1), we have

$$\text{(7.10)} \quad K_\lambda = S(G) \otimes_{S(B^+)} k_\lambda \underset{S(G)}{\cong} (\mathrm{Hom}_{S(B^-)}(S(G), k_\lambda^-))^* .$$

It is a classical fact that if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ then $\mathrm{Hom}_{S(B^-)}(S(G), k_\lambda^-) \neq 0$ (cf. e.g. [G1, p. 64] or [G2, p. 25]), so (7.8) follows. □

(7.11) Remark: Note that, since $U^+ = \langle u_{\mu,\mu+1}(t) \mid \mu \in \underline{n-1},\ t \in k\rangle$, (7.9) implies that $T_r(u)\omega_\lambda = \omega_\lambda$, for all $u \in U^+$.

(7.12) Definition: $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$ is called *dominant* if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

We write $\Lambda^+ = \Lambda^+(n,r) = \{\lambda \in \Lambda(n,r) \mid \lambda \text{ is dominant}\}$.

(7.13) Definition: Let $\lambda \in \Lambda^+(n,r)$. Then K_λ will be called the *Weyl module* for S(G) associated with λ.

Similarly, $M_\lambda = \mathrm{Hom}_{S(B^-)}(S(G), k_\lambda^-)$ will be called the *Schur module* for S(G) associated with λ.

(7.14) Corollary: Let $\lambda \in \Lambda^+(n,r)$. Then $K_\lambda = M_\lambda^*$.

Proof: (cf. (7.10)).

We use now a familiar argument to prove the

(7.15) Lemma: If $\lambda \in \Lambda^+$ then K_λ has a unique maximal S(G)-submodule.

Proof: Let V be a proper submodule of K_λ. It cannot contain ω_λ, since $S(G)\omega_\lambda = K_\lambda$, so $V^\lambda = V \cap K_\lambda{}^\lambda = V \cap k\,\omega_\lambda = 0$. Let $X = \sum_V V$ (sum over all proper $S(G)$-submodules of K_λ). Then

$$X^\lambda = \xi_\lambda \sum_V V = \sum_V \xi_\lambda V = \sum_V V^\lambda = 0.$$

Hence X is a proper submodule of K_λ and it is clearly its unique maximal submodule $\square$

(7.16) Lemma: For each $\lambda \in \Lambda^+(n,r)$ define $F_\lambda = K_\lambda/ \operatorname{rad} K_\lambda$. Then $\{F_\lambda \mid \lambda \in \Lambda^+\}$ is a full set of pairwise non-isomorphic irreducible $S(G)$-modules.

Proof: Let $\lambda \in \Lambda^+$. We know from (7.15) that K_λ has a unique maximal submodule, which must then be $\operatorname{rad} K_\lambda$. Thus, $F_\lambda = K_\lambda/\operatorname{rad} K_\lambda$ is irreducible and it is $S(G)$-generated by $\bar\omega_\lambda = \omega_\lambda + \operatorname{rad} K_\lambda$ ($\neq 0$ since $\omega_\lambda \notin \operatorname{rad} K_\lambda$).

From the definition of F_λ and from (4.3) we know that there is a short exact sequence of k-spaces $0 \to (\operatorname{rad} K_\lambda)^\alpha \to K_\lambda{}^\alpha \to F_\lambda{}^\alpha \to 0$, for any $\alpha \in \Lambda$. Thus by (7.7), we have

(7.17) (i) $F_\lambda{}^\lambda = k\bar\omega_\lambda$ and $\dim F_\lambda{}^\lambda = 1$;

(ii) If $F_\lambda{}^\alpha \neq 0$, for some $\alpha \in \Lambda$, then $\alpha \trianglelefteq \lambda$.

As an immediate consequence of (7.17) we have

$$F_\alpha \underset{S(G)}{\not\cong} F_\lambda \text{ if } \alpha \neq \lambda \quad (\alpha, \lambda \in \Lambda^+).$$

Now let V be any irreducible $S(G)$-module and suppose that

(7.18) $\quad \mathrm{Hom}_{S(G)}(K_\lambda, V) \neq 0$, for some $\lambda \in \Lambda^+$.

Then, if $0 \neq \theta \in \mathrm{Hom}_{S(G)}(K_\lambda, V)$, we have, $V \underset{S(G)}{\cong} K_\lambda/\ker\theta$ and $\ker\theta$ is a maximal submodule of K_λ, i.e., $\ker\theta = \mathrm{rad}\, K_\lambda$ and $V \underset{S(G)}{\cong} K_\lambda/\mathrm{rad}\, K_\lambda = F_\lambda$.

Thus in order to finish the proof of (7.16) we only need to prove that (7.18) holds. For this we shall use the

Adjoint Isomorphism Theorem: (cf. e.g. [R, (2.11)]). Given rings R and S, let A be a left R-module, B be an (S,R)-bimodule and C be a left S-module. Then there is an isomorphism of groups

$$\tau : \mathrm{Hom}_S(B \otimes_R A, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C)).$$

Regarded as an $S(B^+)$-module V has some irreducible submodule. This has to be isomorphic to k_λ, for some $\lambda \in \Lambda$, which implies $\mathrm{Hom}_{S(B^+)}(k_\lambda, V) \neq 0$. Now if in the Adjoint Isomorphism theorem we take $R = S(B^+)$, $S = B = S(G)$, $A = k_\lambda$ and $C = V$, we get an isomorphism of groups

$$\tau : \mathrm{Hom}_{S(G)}(S(G) \otimes_{S(B^+)} k_\lambda, V) \cong \mathrm{Hom}_{S(B^+)}(k_\lambda, \mathrm{Hom}_{S(G)}(S(G), V)).$$

But $\mathrm{Hom}_{S(G)}(S(G), V) \cong V$ as an $S(B^+)$-module. Thus

$$\mathrm{Hom}_{S(G)}(K_\lambda, V) \cong \mathrm{Hom}_{S(B^+)}(k_\lambda, V).$$

Since $\mathrm{Hom}_{S(B^+)}(k_\lambda, V) \neq 0$, we must have $\mathrm{Hom}_{S(G)}(K_\lambda, V) \neq 0$ and $\lambda \in \Lambda^+$ (since $K_\lambda \neq 0$ iff $\lambda \in \Lambda^+$). Hence (7.18). □

In the next theorem we summarise the main results of this section, but before we need a definition.

(7.19) Definition: Let H be a subgroup of G containing T. We say that an S(H)-module V has *highest weight* λ $(\lambda \in \Lambda)$ if λ is a weight of V and $\alpha \lhd \lambda$, for all other weights α of V.

(7.20) Theorem: (cf. [G1; §5] and [CL; §3]). For $\lambda \in \Lambda^+(n,r)$ there holds

(i) The Weyl module K_λ is a cyclic S(G)-module generated by $\omega_\lambda = 1_{S(G)} \otimes 1_{k_\lambda}$;

(ii) K_λ has highest weight λ, $K_\lambda{}^\lambda = k\omega_\lambda$ and $T_r(u)\omega_\lambda = \omega_\lambda$, for all $u \in U^+$;

(iii) K_λ is the contravariant dual of the Schur module M_λ;

(iv) K_λ has a unique maximal submodule, rad K_λ, and $\{F_\alpha = K_\alpha/\mathrm{rad}\, K_\alpha \mid \alpha \in \Lambda^+(n,r)\}$ is a full set of pairwise non-isomorphic irreducible S(G)-modules;

(v) F_λ has highest weight λ and $\dim_k F_\lambda{}^\lambda = 1$.

§8. $K_{\lambda,J}$ and the Schur algebra $S(L_J)$

Consider any standard parabolic subgroup G_J^+ of G. In §6 and §7 we studied the $S(G_J^+)$-modules $S(G_J^+) \otimes_{S(B^+)} k_\lambda$, in the two extreme cases of $J = \emptyset$ and

$J = \underline{n-1}$ respectively. We are now interested in the intermediate cases.

As in §3, let $J = \underline{n} \setminus \{m_1,\dots,m_s\}$, where $m_0, m_1, \dots, m_s$ are integers satisfying $0 = m_0 < m_1 < \dots < m_{s-1} < m_s = n$. Let $N_a = \{m_{a-1}+1,\dots,m_a\}$ $(a \in \underline{s})$, and define, for each $\lambda \in \Lambda$, the $S(G_J^+)$-module

$$K_{\lambda,J} = S(G_J^+) \otimes_{S(B^+)} k_\lambda .$$

Note that, in particular, $K_{\lambda,\emptyset} \underset{S(B^+)}{\cong} k_\lambda$ and $K_{\lambda,\underline{n-1}} = K_\lambda$.

It is clear that $K_{\lambda,J} = S(G_J^+)\ \omega_\lambda$, where $\omega_\lambda = 1_{S(G)} \otimes 1_{k_\lambda}$. Also, as in §7 (cf. (7.11)), we have

(8.1) $\xi_\lambda\, \omega_\lambda = \omega_\lambda$ and $T_r(u)\omega_\lambda = \omega_\lambda$, for all $u \in U^+$.

So applying (4.8) to $K_{\lambda,J}$ we get the following.

(8.2) Lemma: Let $\lambda \in \Lambda(n,r)$. Then $K_{\lambda,J} = 0$, unless $\lambda_{m_{a-1}+1} \geq \lambda_{m_{a-1}+2} \geq \dots \geq \lambda_{m_a}$, for all $a \in \underline{s}$.

Proof: Suppose $K_{\lambda,J} \neq 0$. Then $\omega_\lambda \neq 0$.

In (4.8) take $H = G_J^+$, $V = K_{\lambda,J}$, $v = \omega_\lambda$, and $(\mu,\nu) = (\mu, \mu+1)$, where $\mu \underset{J}{\sim} \mu+1$. Then the hypotheses of (4.8) are satisfied. Thus, $\lambda_{\mu+1} \leq \lambda_\mu$, for all $\mu \in \underline{n-1}$ such that $m_{a-1}+1 \leq \mu \leq m_a - 1$ (some $a \in \underline{s}$). □

Notation: $\Lambda_J^+ = \Lambda_J^+(n,r) = \{\lambda \in \Lambda(n,r) \mid \lambda_{m_{a-1}} + 1 \geq \lambda_{m_{a-1}} + 2 \geq \ldots \geq \lambda_{m_a}$, for all $a \in \underline{s}\}$.

Consider the subgroups U_J^+ and L_J of G_J^+, defined in §3.Then G_J^+ has the Levi decomposition $G_J^+ = L_J U_J^+$, and so $S(G_J^+) = S(L_J)\, S(U_J^+)$.

As U_J^+ is a subgroup of U^+, (8.1) implies $T_r(u)\omega_\lambda = \omega_\lambda$, all $u \in U_J^+$. Thus

$$K_{\lambda,J} = S(G_J^+)\omega_\lambda = S(L_J)\, S(U_J^+)\omega_\lambda = S(L_J)\omega_\lambda,$$

and, in order to understand $K_{\lambda,J}$, we need to study the Schur algebra $S(L_J)$.

L_J consists of all matrices of the form

$$g = \begin{pmatrix} g^{(1)} & 0 & .. & 0 \\ 0 & g^{(2)} & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & & g^{(s)} \end{pmatrix}$$

where, for each $a \in \underline{s}$, the matrix $g^{(a)} = (g_{\mu\nu})_{\mu,\nu \in N_a}$ is non-singular; in other words, L_J consists of all $g \in G$ such that $g_{\mu\nu} = 0$ for all $(\mu,\nu) \in \underline{n} \times \underline{n}$ such that $\mu \underset{J}{\neq} \nu$.

For convenience of notation write $G_a = GL_{n_a}(k)$, where $n_a = m_a - m_{a-1} = \# N_a$.

L_J is isomorphic to $G_1 \times \ldots \times G_s$ (external direct product) and so, we should be able to obtain $S(L_J)$ from the Schur algebras for G_a ($a \in \underline{s}$). To do this we shall use coalgebra theory. We start with some standard results which can be found in [G1; pp. 4-6, 18-20].

Let H be any group, and let k^H denote the k-algebra of all maps $f : H \to k$ (addition and multiplication in k^H being defined pointwise).

We identify $k^H \otimes k^H$ with a k-subspace of $k^{H \times H}$, via the k-monomorphism $k^H \otimes k^H \to k^{H \times H}$, which takes $f \otimes f'$ to the map $f'' : H \times H \to k$, defined by $f''(h, h') = f(h)\, f'(h')$, for all $f, f' \in k^H$, $h, h' \in H$.

Let $\Delta_H : k^H \to k^{H \times H}$, and $\varepsilon_H : k^H \to k$, be the k-algebra maps defined by

$$\Delta_H(f)(h,h') = f(hh'), \text{ and } \varepsilon_H(f) = f(1_H), \text{ all } f \in k^H, h, h' \in H.$$

Then, the set $\mathcal{F}(k^H) = \{f \in k^H \mid \Delta_H(f) \in k^H \otimes k^H\}$ is a k-bialgebra: it is a subalgebra of k^H and the comultiplication and counit maps are the restrictions of Δ_H and ε_H, respectively, to $\mathcal{F}(k^H)$.

Now make $H = G$.

For each $\mu, \nu \in \underline{n}$, define the *coordinate map* $c_{\mu\nu} \in k^G$, by

$$c_{\mu\nu}(g) = g_{\mu\nu}, \text{ all } g \in G.$$

Let $A(G) = k[c_{\mu\nu} \mid \mu, \nu \in \underline{n}]$ be the k-subalgebra of k^G generated by the $c_{\mu\nu}$ ($\mu, \nu \in \underline{n}$). As the field k is infinite, the $c_{\mu\nu}$ are algebraically independent over k. Hence $A(G)$ may be regarded as the algebra of all polynomials over k in the indeterminates $c_{\mu\nu}$ ($\mu, \nu \in \underline{n}$).

For each $q \geq 0$, let $A_q(G)$ denote the k-subspace of $A(G)$ consisting of all those elements in $A(G)$ which, considered as polynomials in the $c_{\mu\nu}$'s, are homogenous of degree q. Then

$$A(G) = \bigoplus_{q \geq 0} A_q(G).$$

It is clear that, for each $q \geq 1$,

(8.3) $A_q(G)$ has k-basis $\{c_{i,j} = c_{i_1 j_1} \dots c_{i_q j_q} \mid (i,j) \in \Omega_q\}$,

where Ω_q is a transversal of the set of all P_q-orbits of $I(n,q) \times I(n,q)$.

Also, by the definition of Δ_G,

$$\Delta_G(c_{\mu\nu}) = \sum_{\tau \in \underline{n}} c_{\mu\tau} \otimes c_{\tau\nu}, \quad \mu, \nu \in \underline{n}.$$

As Δ_G is a k-algebra map this gives,

$$\Delta_G(c_{i,j}) = \sum_{h \in I(n,q)} c_{i,h} \otimes c_{h,j}, \text{ all } i,j \in I(n,q);\ q \geq 1.$$

Similarly $\varepsilon_G(c_{\mu\nu}) = \delta_{\mu\nu}$, and $\varepsilon_G(c_{i,j}) = \delta_{i,j} = \delta_{i_1 j_1} \dots \delta_{i_q j_q}$ ($\mu, \nu \in \underline{n}$, $i,j \in I(n,q)$).

This shows that $A(G)$ is a sub-bialgebra of $\mathcal{F}(k^G)$, and that $A_q(G)$ is a subcoalgebra of $A(G)$. Thus $A_q(G)^* = \mathrm{Hom}_k(A_q(G),k)$ is a k-algebra.

The algebra $S_k(n,q)$ introduced by I. Schur in [S] coincides with $A_q(G)^*$ (cf. [G1; pp. 18–21]). Thus, as we mentioned in the introduction

(8.4) $A_q(G)^*$ and $S_k(n,q;G)$ will be identified, via the k-algebra isomorphism $\Xi : A_q(G)^* \to S_k(n,q;G)$, defined in (0.1).

Note that if Ω_q is as in (8.3) then $\{\xi_{i,j} \mid (i,j) \in \Omega_q\}$ is the basis of $A_q(G)^*$ dual to the basis $\{c_{i,j} \mid (i,j) \in \Omega_q\}$ of $A_q(G)$.

Now consider the subgroup L_J of G.

For each $c \in A(G)$, denote the restriction of c to L_J by $\bar{c}$. Let $A(L_J) = \{\bar{c} \mid c \in A(G)\}$. Then $A(L_J)$ is a subalgebra of k^{L_J}, and it is clearly generated by those $\bar{c}_{\mu\nu}$ which satisfy $\bar{c}_{\mu\nu} \neq 0$ $(\mu, \nu \in \underline{n})$.

Note that, for any $g \in L_J$, we have $\bar{c}_{\mu\nu}(g) = g_{\mu\nu} = 0$, unless $\mu \underset{J}{=} \nu$. Hence $\bar{c}_{\mu\nu} = 0$ if $\mu \underset{J}{\neq} \nu$. Now, using an argument similar to that in the proof of (3.5), we can show that

(8.5) Lemma: The $\bar{c}_{\mu,\nu}$ $(\mu, \nu \in \underline{n},\ \mu \underset{J}{=} \nu)$ are algebraically independent over k.

Therefore, we can identify $A(L_J)$ with $k[\bar{c}_{\mu\nu} \mid \mu, \nu \in \underline{n}, \mu \underset{J}{=} \nu]$, the algebra of all polynomials over k in the indeterminates $\bar{c}_{\mu\nu}$ $(\mu, \nu \in \underline{n}, \mu \underset{J}{=} \nu)$.

Let $\mu, \nu \in \underline{n}$, $\mu \underset{J}{=} \nu$, and consider $\bar{c}_{\mu\nu}$. From the definition of the k-algebra map Δ_{L_J}, we have

$$\Delta_{L_J}(\bar{c}_{\mu\nu})\,(g,g') = \bar{c}_{\mu\nu}(gg') = (gg')_{\mu\nu} =$$

$$= \sum_{\tau \in \underline{n}} g_{\mu\tau}\, g'_{\tau\nu} = \Big(\sum_{\tau \in \underline{n},\, \tau \underset{J}{=} \mu} \bar{c}_{\mu\tau} \otimes \bar{c}_{\tau\nu}\Big)(g,g'), \text{ all } g, g' \in L_J.$$

Hence

$$\text{(8.6)} \quad \Delta_{L_J}(\bar{c}_{\mu\nu}) = \sum_{\tau \underset{J}{=} \mu (\underset{J}{=} \nu)} \bar{c}_{\mu\tau} \otimes \bar{c}_{\tau\nu} \in A(L_J) \otimes A(L_J),$$

and $A(L_J)$ is a sub-bialgebra of $\mathcal{F}(k^{L_J})$.

Notice that as Δ_{L_J} is a k-algebra map then, for each $q \geq 1$,

$$\Delta_{L_J}(\bar{c}_{i,j}) = \sum_{\substack{h \in I(n,q) \\ h \underset{J}{=} i}} \bar{c}_{i,h} \otimes \bar{c}_{h,j}, \text{ all } i,j \in I(n,q)\,,\ i \underset{J}{=} j$$

(here $h \underset{J}{=} i$ means $h_\rho \underset{J}{=} i_\rho$, all $\rho \in \underline{q}$).

Therefore, $A_q(L_J) = \sum_{\substack{i,j \in I(n,q) \\ i \underset{J}{=} j}} k\, c_{i,j}$ $(q \geq 1)$, is a subcoalgebra of $A(L_J)$.

Now let us return to the groups $G_a = GL_{n_a}(k)$ $(a \in \underline{s})$. Everything we have said about G applies, in particular, to G_a. So we may consider the bialgebras $A(G_a)$. For each $\mu, \nu \in \underline{n_a}$, we also denote by $c_{\mu\nu}$ the coordinate map in k^{G_a} given by, $c_{\mu\nu}(g) = g_{\mu\nu}$ (all $g \in G_a, a \in \underline{s}$).

The tensor product $A(G_1) \otimes \ldots \otimes A(G_s)$ is a k-bialgebra, with counit and comultiplication maps defined by

$$\varepsilon_\otimes = \varepsilon_{G_1} \otimes \ldots \otimes \varepsilon_{G_s} \quad \text{and} \quad \Delta_\otimes = \mathfrak{t}(\Delta_{G_1} \otimes \ldots \otimes \Delta_{G_s}),$$

where $t : \bigotimes_{a \in s} (A(G_a) \otimes A(G_a)) \to (\bigotimes_{a \in s} A(G_a)) \otimes (\bigotimes_{a \in s} A(G_a))$ is the "twisting" map,

i.e., $t(\bigotimes_{a \in s} (c_a \otimes c'_a)) = (\bigotimes_{a \in s} c_a) \otimes (\bigotimes_{a \in s} c'_a)$, for all $c_a, c'_a \in A(G_a)$ (cf. [Sw; p.49]).

(8.7) Lemma: The k-bialgebras $A(L_J)$ and $\bigotimes_{a \in s} A(G_a)$ are isomorphic.

Proof: As $\underline{n} = \dot{\bigcup}_{a \in s} N_a$, we may define a map $\theta : \underline{n} \to \underline{n}$, by

$$\theta(\mu) = \mu - m_{a-1}, \text{ all } \mu \in N_a,\ a \in \underline{s}.$$

(8.8) Note that the restriction of θ to N_a gives a bijection between N_a and $\underline{n}_a$.

Now let $\psi : A(L_J) \to \bigotimes_{a \in s} A(G_a)$ be the k-algebra map defined by

(8.9) $\psi(c_{\mu\nu}) = 1 \otimes \ldots \otimes \underset{(a)}{c_{\theta(\mu)\,\theta(\nu)}} \otimes \ldots \otimes 1$, if $\mu, \nu \in N_a$.

We claim that ψ is a bialgebra isomorphism. To prove this we need to show that

(i) $\Delta_{\otimes} \psi = (\psi \otimes \psi)\Delta_{L_J}$, and $\varepsilon_{\otimes} \psi = \varepsilon_{L_J}$;

(ii) ψ is bijective.

As $\Delta_{\otimes}$, ψ and Δ_{L_J} are k-algebra maps, we have $\Delta_{\otimes} \psi = (\psi \otimes \psi)\Delta_{L_J}$ iff $\Delta_{\otimes} \psi(c_{\mu\nu}) = (\psi \otimes \psi)\Delta_{L_J}(c_{\mu\nu})$ $(\mu, \nu \in N_a, a \in \underline{s})$. So consider $\mu, \nu \in N_a$. By (8.6) and (8.8),

$$(\hat{\psi}\otimes\hat{\psi})\Delta_{L_J}(\bar{c}_{\mu\nu}) = \hat{\psi}\otimes\hat{\psi}\,(\sum_{\mu \underset{J}{=} \tau (\underset{J}{=} \nu)} \bar{c}_{\mu\tau}\otimes\bar{c}_{\tau\nu}) =$$

$$= \hat{\psi}\otimes\hat{\psi}\,(\sum_{\tau\in N_a} \bar{c}_{\mu\tau}\otimes\bar{c}_{\tau\nu}) = \sum_{\tau\in N_a} (1\otimes\ldots\otimes \underset{(a)}{c_{\theta(\mu)\,\theta(\tau)}}\otimes\ldots\otimes 1)\otimes$$

$$\otimes\ (1\otimes\ldots\otimes \underset{(a)}{c_{\theta(\tau)\theta(\nu)}}\otimes\ldots\otimes 1) =$$

$$= \sum_{\sigma\in \underline{n_a}} (1\otimes\ldots\otimes c_{\theta(\mu)\,\sigma}\otimes\ldots\otimes 1)\otimes(1\otimes\ldots\otimes c_{\sigma\,\theta(\nu)}\otimes\ldots\otimes 1) =$$

$$= \Delta_{\otimes}\,\hat{\psi}\,(\bar{c}_{\mu\nu}).$$

The proof of $\varepsilon_{\otimes}\,\hat{\psi} = \varepsilon_{L_J}$ is similar. Hence (i).

Now to prove (ii) we consider, for each $a \in \underline{s}$, the k-algebra map $f_a : A(G_a) \to A(L_J)$, given by, $f_a(c_{\mu\nu}) = \bar{c}_{m_{a-1}+\mu,\, m_{a-1}+\nu}$, for all $\mu, \nu \in \underline{n_a}$.

Also, let $f : \bigotimes_{a\in\underline{s}} A(G_a) \to A(L_J)$, be the k-algebra map defined by $f(c_1\otimes\ldots\otimes c_s) = f_1(c_1)\ldots f_s(c_s)$, for all $c_a \in A(G_a)$, $a\in\underline{s}$. Clearly $f = \hat{\psi}^{-1}$. Hence $\hat{\psi}$ is bijective. □

Let $R(J) = \{d = (d_1,\ldots,d_s) \in \mathbb{Z}^s \mid d_a \geq 0\ (a \in \underline{s});\ \sum_{a\in\underline{s}} d_a = r\}$, and define

$$A_{R(J)} = \bigoplus_{d\in R(J)}\ \bigotimes_{a\in\underline{s}} A_{d_a}(G_a).$$

Consider any $d \in R(J)$, and let $D_a = \{d_1 + \ldots + d_{a-1} + \mu \mid \mu \in \underline{d_a}\}$ $(a \in \underline{s},\ d_0 = 0)$. As

$r = \sum_{a \in s} d_a$, we have $\underline{r} = \dot{\bigcup}_{a \in \underline{s}} D_a$.

Suppose $i(a), j(a) \in I(n_a, d_a)$ $(a \in \underline{s})$. Then we have the following diagram

$$\begin{array}{ccccccc} D_a & \longrightarrow & \underline{d_a} & \xrightarrow{i(a)} & \underline{n_a} & \longrightarrow & N_a \\ d_1 + \ldots + d_{a-1} + \mu & \longmapsto & \mu & \longmapsto & i(a)_\mu & \longrightarrow & m_{a-1} + i(a)_\mu, \end{array}$$

and similar for $j(a)$. Thus, we may define $\bar{i}\bar{j} \in I(n,r)$ as follows

(8.10) $\bar{i}_\rho = m_{a-1} + i(a)_\mu$; $\quad \bar{j}_\rho = m_{a-1} + j(a)_\mu$, if $\rho = d_1 + \ldots + d_{a-1} + \mu \in D_a$.

It is then clear that

(8.11) (i) $\bar{i}_\rho \underset{J}{=} \bar{j}_\rho$, all $\rho \in \underline{r}$. Hence $\bar{i} \underset{J}{=} \bar{j}$.

(ii) $\{\rho \in \underline{r} \mid \bar{i}_\rho \in N_a\} = D_a$

(iii) $$\prod_{\rho \in D_a} c_{\theta(\bar{i}_\rho)\,\theta(\bar{j}_\rho)} = \prod_{\mu \in \underline{d_a}} c_{i(a)_\mu\, j(a)_\mu} = c_{i(a), j(a)} \in A_{d_a}(G_a).$$

(8.12) Theorem: With the notation above there is a coalgebra isomorphism

$$\tilde{\psi} : A_r(L_J) \to A_{R(J)}$$

satisfying $\tilde{\psi}(\bar{c}_{\bar{i}\bar{j}}) = \bigotimes_{a \in \underline{s}} c_{i(a),j(a)}$, all $i(a), j(a) \in I(n_a, d_a)$, $d \in R(J)$.

Proof: By definition, $A_r(L_J) = \sum_{\substack{i,j \in I(n,r) \\ i \underset{J}{=} j}} k\, \bar{c}_{i,j}$. So, consider any $i,j \in I(n,r)$ such that

$i \underset{J}{=} j$, and define

$$R_a(i) = \{\rho \in \underline{r} \mid i_\rho \in N_a\} = \{\rho \in \underline{r} \mid j_\rho \in N_a\}, \text{ all } a \in \underline{s}.$$

Then, $\underline{r} = \dot{\bigcup}_{a \in \underline{s}} R_a(i)$ (since $\underline{n} = \dot{\bigcup}_{a \in \underline{s}} N_a$). Also, if $r_a(i) = \# R_a(i)$ we have

(8.13) $(r_1(i),\ldots,r_s(i)) \in R(J)$.

Let $\bar{\psi}$ be as in (8.9). Then

$$\bar{\psi}(\bar{c}_{i,j}) = \prod_{\rho \in \underline{r}} \psi(\bar{c}_{i_\rho j_\rho}) = \bigotimes_{a \in \underline{s}} \prod_{\rho \in R_a(i)} c_{\theta(i_\rho)\theta(j_\rho)}.$$

But, $\prod_{\rho \in R_a(i)} c_{\theta(i_\rho),\theta(j_\rho)} \in A_{r_a(i)}(G_a)$. Hence, by (8.13), $\bar{\psi}(\bar{c}_{i,j}) \in \bigotimes_{a \in \underline{s}} A_{r_a(i)}(G_a) \subseteq A_{R(J)}$.

Therefore, $\bar{\psi}(A_r(L_J)) \subseteq A_{R(J)}$.

Now, consider any $d \in R(J)$, and let $i(a), j(a) \in I(n_a, d_a)$ $(a \in \underline{s})$. Then, if $\hat{i}, \hat{j}$ are as in (8.10), $\bar{c}_{\hat{i}\hat{j}} \in A_r(L_J)$, and by (8.11)(ii) and (iii),

(8.14) $\bar{\psi}(\bar{c}_{\hat{i}\hat{j}}) = \bigotimes_{a \in \underline{s}} \prod_{\rho \in R_a(\hat{i})} c_{\theta(\hat{i}_\rho)\theta(\hat{j}_\rho)} = \bigotimes_{a \in \underline{s}} \prod_{\rho \in D_a} c_{\theta(\hat{i}_\rho)\theta(\hat{j}_\rho)} = \bigotimes_{a \in \underline{s}} c_{i(a),j(a)}.$

Since $A_{R(J)}$ is k-spanned by $\{ \bigotimes_{a \in \underline{s}} c_{i(a),j(a)} \mid i(a), j(a) \in I(n_a, d_a)\ (a \in \underline{s}),\ d \in R(J)\}$, (8.14) shows that $\bar{\psi}(A_r(L_J)) = A_{R(J)}$. Thus, we define $\bar{\psi} : A_r(L_J) \to A_{R(J)}$ to be the restriction of $\tilde{\psi}$ to $A_r(L_J)$ and this ends the proof of the theorem. □

It is now easy to obtain a description of $S(L_J)$ in terms of $S(n_a, d_a; G_a)$ $(a \in \underline{s},$ $d \in R(J))$.

Since the dual of a coalgebra map is an algebra map (cf. [Sw; [1.4.1]), by (8.12), there is a k-algebra isomorphism

$$\tilde{\psi}^* : A^*_{R(J)} \to A_r(L_J)^*.$$ [4]

But (cf. (8.4)),

$$A^*_{R(J)} = (\bigoplus_{d \in R(J)} \bigotimes_{a \in \underline{s}} A_{d_a}(G_a))^* \underset{k\text{-alg}}{\cong} \coprod_{d \in R(J)} \bigotimes_{a \in \underline{s}} A_{d_a}(G_a)^* \underset{k\text{-alg}}{\cong} \coprod_{d \in R(J)} \bigotimes_{a \in \underline{s}} S(n_a, d_a; G_a).$$

So, if we write

$$S_{R(J)} = \coprod_{d \in R(J)} \bigotimes_{a \in \underline{s}} S(n_a, d_a; G_a),$$

we have just seen that the algebras $A_r(L_J)^*$ and $S_{R(J)}$ are isomorphic. Now we have the following

(8.15) Theorem: There is a k-algebra isomorphism $\psi : S_{R(J)} \to S(L_J)$.

Proof: Let $\varphi : A_r(G) \to A_r(L_J)$ be "restriction to L_J". Clearly φ is a coalgebra epimorphism. Thus, we have the short exact sequence

[4] If V, V' are k-vector spaces and $f \in \mathrm{Hom}_k(V,V')$, $f^* \in \mathrm{Hom}_k(V'^*, V^*)$ denotes the map defined by, $f^*(\theta') = \theta' f$, all $\theta' \in V'^*$.

$$0 \longrightarrow \ker\varphi \xrightarrow{\text{inc}} A_r(G) \xrightarrow{\varphi} A_r(L_J) \to 0.$$

Taking duals (and since all k-spaces involved are finite dimensional) we obtain the short exact sequence

$$0 \longrightarrow A_r(L_J)^* \xrightarrow{\varphi^*} A_r(G)^* \xrightarrow{\text{inc}^*} (\ker\varphi)^* \longrightarrow 0.$$

Therefore,

$$S_{R(J)} \underset{k\text{-alg}}{\cong} A_r(L_J)^* \underset{k\text{-alg}}{\cong} \operatorname{Im}\varphi^* = \ker \text{inc}^*.$$

But $\ker\varphi$ is k-spanned by $c_{i,j}$, for all $i,j \in I(n,r)$ such that $i \underset{J}{\neq} j$ (cf. (8.3) and (8.5)).

Thus,

$$\ker \text{inc}^* = \{\xi \in A_r(G)^* \mid \xi(c_{i,j}) = 0, \text{ for all } i,j \in I(n,r) \text{ such that } i \underset{J}{\neq} j\}$$

$$\underset{k\text{-alg}}{\cong} \bigoplus_{\substack{(i,j) \in \Omega \\ i \underset{J}{=} j}} k\,\xi_{i,j} = S(L_J).$$

Hence $S_{R(J)} \underset{k\text{-alg}}{\cong} S(L_J)$, and we define the isomorphism $\psi : S_{R(J)} \to S(L_J)$ so that the diagram below commutes.

$$\begin{tikzcd}
A^*_{R(J)} \arrow[r, "\tilde{\psi}^*"] & A_r(L_J)^* \arrow[r, "\varphi^*"] & S(L_J) \\
S_{R(J)} \arrow[u, "\eta"] \arrow[urr, "\psi"'] & &
\end{tikzcd}$$

.

where η is the natural isomorphism

$$(\bigoplus_{d \in R(J)} \bigotimes_{a \in s} A_{d_a}(G_a))^* \cong \coprod_{d \in R(J)} \bigotimes_{a \in s} A_{d_a}(G_a)^* \cong \coprod_{d \in R(J)} \bigotimes_{a \in s} S(n_a, d_a; G_a). \ \square$$

For each $d \in R(J)$, let $\iota_d : \bigotimes_{a \in s} S(n_a, d_a; G_a) \to S_{R(J)}$, and $\pi_d : S_{R(J)} \to \bigotimes_{a \in s} S(n_a, d_a; G_a)$ be the natural injection and projection, respectively.

Let $i(a), j(a) \in I(n_a, d_a)$ $(a \in s)$. By (8.12), $\tilde{\psi}\,\varphi(c_{\bar{i}\bar{j}}) = \bigotimes_{a \in s} c_{i(a),j(a)}$. Thus, as $\xi_{h,\ell}$ is the basis element of $A_r(G)^*$ dual to the basis element $c_{h,\ell}$ of $A_r(G)$ (all $h, \ell \in I(n,r)$) and a similar relation exists between $\xi_{i(a),j(a)}$ and $c_{i(a),j(a)}$ $(a \in s)$, we have

$$\textbf{(8.16)} \quad \psi\,\iota_d(\bigotimes_{a \in s} \xi_{i(a),j(a)}) = \varphi^* \psi^* \eta\,\iota_d(\bigotimes_{a \in s} \xi_{i(a),j(a)}) = \xi_{\bar{i}\bar{j}}.$$

(8.17) Remarks: (i) As $\bigotimes_{a \in s} S(n_a, d_a; G_a)$ is k-spanned by all $\bigotimes_{a \in s} \xi_{i(a),j(a)}$, $i(a), j(a) \in I(n_a, d_a)$ $(a \in s)$, $S(L_J)$ is k-spanned by $\{\psi\,\iota_d(\bigotimes_{a \in s} \xi_{i(a),j(a)}) = \xi_{\bar{i}\bar{j}} \mid i(a), j(a) \in I(n_a, d_a)$ $(a \in s)$, $d \in R(J)\}$. Hence, for each $i,j \in I(n,r)$ satisfying $i \underset{J}{=} j$, there is some $\xi_{\bar{i}\bar{j}} = \psi\,\iota_d(\bigotimes_{a \in s} \xi_{i(a),j(a)})$ such that $\xi_{i,j} = \xi_{\bar{i}\bar{j}}$.

(ii) Recall that, $\bar{i}$ and $\bar{j}$ are determined by $i(a), j(a) \in I(n_a, d_a)$ $(a \in s)$ as follows

$$\bar{i}_\rho = m_{a-1} + i(a)_\mu; \quad \bar{j}_\rho = m_{a-1} + j(a)_\mu,$$

if $\rho = d_1 + \ldots + d_{a-1} + \mu$ $\quad (\mu \in \underline{d_a},\ d_0 = 0)$.

Hence, $\bar{i} \le \bar{j}$ (resp. $\bar{i} = \bar{j}$) iff $i(a) \le j(a)$ (resp. $i(a) = j(a)$), for all $a \in \underline{s}$.

(iii) Suppose $\bar{j}$ has weight $\alpha \in \Lambda(n,r)$, and $j(a)$ has weight $\alpha(a) \in \Lambda(n_a,d_a)$ ($a \in \underline{s}$). Then α and $\alpha(a)$ are related by

$$\alpha(a)_\nu = \alpha_{m_{a-1}+\nu}, \text{ all } \nu \in \underline{n_a}.$$

It is now time to return to the study of the module $K_{\lambda,J}$.

Let $\lambda \in \Lambda_J^+$. As $S(L_J)$ is a subalgebra of $S(G_J^+)$, we may regard $K_{\lambda,J}$ as an $S(L_J)$-module (by restriction).

For each $a \in \underline{s}$, let $r_a(\lambda) = \lambda_{m_{a-1}+1} + \ldots + \lambda_{m_a}$, and define $\lambda(a) \in \Lambda(n_a, r_a(\lambda))$ by

$$\lambda(a)_\nu = \lambda_{m_{a-1}+\nu}, \text{ all } \nu \in \underline{n_a}.$$

Note that $r(\lambda) = (r_1(\lambda),\ldots,r_s(\lambda)) \in R(J)$. Also, since $\lambda \in \Lambda_J^+$, $\lambda(a)_1 = \lambda_{m_{a-1}+1} \ge \lambda_{m_{a-1}+2} = \lambda(a)_2 \ge \ldots \ge \lambda_{m_a} = \lambda(a)_{n_a}$. Hence $\lambda(a) \in \Lambda^+(n_a, r_a(\lambda))$, all $a \in \underline{s}$.

Let B_a^+ denote the subgroup of G_a consisting of all upper triangular matrices in G_a. Consider the irreducible $S(n_a, r_a(\lambda); B_a^+)$-module $k_{\lambda(a)}$ affording the representation $\kappa_{\lambda(a)}$ (cf. (6.7) and (6.9)(ii)).

From (7.8), we know that the $S(n_a, r_a(\lambda); G_a)$-module

$$K_{\lambda(a)} = S(n_a, r_a(\lambda); G_a) \otimes_{S(n_a,r_a(\lambda);B_a^+)} k_{\lambda(a)}$$

is non-zero, since $\lambda(a) \in \Lambda^+(n_a, r_a(\lambda))$, for all $a \in \underline{s}$. Therefore, if we consider the k-vector space $\bigotimes_{a \in \underline{s}} K_{\lambda(a)}$, we have

(8.18) $\bigotimes_{a \in \underline{s}} K_{\lambda(a)} \neq 0$, for all $\lambda \in \Lambda_J^+$.

As each $K_{\lambda(a)}$ is an $S(n_a, r_a(\lambda); G_a)$-module, $\bigotimes_{a \in \underline{s}} K_{\lambda(a)}$ may be regarded as a $\bigotimes_{a \in \underline{s}} S(n_a, r_a(\lambda); G_a)$-module by

$$(\bigotimes_{a \in \underline{s}} \xi_a)(\bigotimes_{a \in \underline{s}} (\zeta_a \otimes 1_k)) = \bigotimes_{a \in \underline{s}} (\xi_a \zeta_a \otimes 1_k), \text{ all } \xi_a, \zeta_a \in S(n_a, r_a(\lambda); G_a) \quad (a \in \underline{s}).$$

But, since we have the k-algebra epimorphism

$$S(L_J) \xrightarrow{\psi^{-1}} S_{R(J)} \xrightarrow{\pi_{r(\lambda)}} \bigotimes_{a \in \underline{s}} S(n_a, r_a(\lambda); G_a)$$

(where ψ is the isomorphism defined in (8.15) and $\pi_{r(\lambda)}$ is the natural projection) we may also regard $\bigotimes_{a \in \underline{s}} K_{\lambda(a)}$ as an $S(L_J)$-module via $\pi_{r(\lambda)} \psi^{-1}$.

It is our aim to prove that, under these conditions, we have the following result.

(8.19) Theorem: Let $\lambda \in \Lambda_J^+$. Then $K_{\lambda,J}$ and $\bigotimes_{a \in \underline{s}} K_{\lambda(a)}$ are isomorphic $S(L_J)$-modules.

As an easy consequence of (8.19) we have the corollary.

(8.20) Corollary: Let $\lambda \in \Lambda(n,r)$. Then $K_{\lambda,J} \neq 0$ iff $\lambda \in \Lambda_J^+$.

Proof: By (8.2), (8.18) and (8.19), the corollary follows. □

(8.21) Remark: Let $\lambda \in \Lambda(n,r)$ and let J be any proper subset of $\underline{n-1}$. Then we know from §7 that $(k_\lambda^-)^o \underset{S(B^+)}{\cong} k_\lambda$. Also, by (5.9), ${}^o S(G_J^-) = S(G_J^+)$ and ${}^oS(B^-) = S(B^+)$. Hence by (5.6), $(\mathrm{Hom}_{S(B^-)}(S(G_J^-), k_\lambda^-))^o \underset{S(G_J^+)}{\cong} S(G_J^+) \otimes_{S(B^+)} k_\lambda = K_{\lambda,J}$. Thus, from (8.20) we obtain

$$\mathrm{Hom}_{S(B^-)}(S(G_J^-), k_\lambda^-) \neq 0 \text{ iff } \lambda \in \Lambda_J^+.$$

Note that in the case when $J = \underline{n-1}$, we have used the fact that $\mathrm{Hom}_{S(B^-)}(S(G), k_\lambda^-) \neq 0$ to prove that $K_\lambda \neq 0$, for all $\lambda \in \Lambda^+$ (cf. proof of (7.8)).

Proof of (8.19) Let $\lambda \in \Lambda_J^+$. Define $B_J^+ = B^+ \cap L_J$.

Then $S(B_J^+)$ is a subalgebra both of $S(B^+)$ and of $S(L_J)$, and we may consider the $S(L_J)$-module.

$$S(L_J) \otimes_{S(B_J^+)} k_\lambda$$

(here k_λ being regarded as the restriction of k_λ to $S(B_J^+)$).

Now the proof of (8.19) follows from the next two lemmas. □

(8.22) Lemma: Let $\lambda \in \Lambda_J^+$. Then $\bigotimes_{a \in s} K_{\lambda(a)}$ and $S(L_J) \otimes_{S(B_J^+)} k_\lambda$ are isomorphic $S(L_J)$-modules.

(8.23) Lemma: If $\lambda \in \Lambda_J^+$, the $S(L_J)$-modules $K_{\lambda,J}$ and $S(L_J) \otimes_{S(B_J^+)} k_\lambda$ are isomorphic.

Proof of (8.22) Let $\lambda \in \Lambda^+(n,r)$. In this proof we write

$$S(G_a^+) = S(n_a, r_a(\lambda); G_a) \text{ and } S(B_a^+) = S(n_a, r_a(\lambda); B_a^+) \quad (a \in s).$$

As $S(L_J)$ is k-spanned by $\{\xi_{i,j} \mid i,j \in I(n,r), i \underset{J}{-} j\}$, $S(L_J) \otimes_{S(B_J^+)} k_\lambda$ is k-spanned by $\{\xi_{i,j} \otimes 1_k \mid i,j \in I(n,r), i \underset{J}{-} j\}$.

But, if $j \notin \lambda$ then $\xi_{i,j} \otimes 1_k = \xi_{i,j} \otimes \xi_\lambda 1_k = \xi_{i,j}\, \xi_\lambda \otimes 1_k = 0$. Hence

(8.24) $S(L_J) \otimes_{S(B_J^+)} k_\lambda$ is k-spanned by $\{\xi_{i,j} \otimes 1_k \mid (i,j) \in I(n,r), i \underset{J}{-} j$ and $j \in \lambda\}$.

Now consider the Schur algebra $S(B_J^+)$. By an argument similar to that used in the proof of (3.5), we can show that

(8.25) $S(B_J^+)$ has k-basis $\{\xi_{i,j} \mid (i,j) \in \Omega, i \underset{J}{-} j$ and $i \leq j\}$.

For each $a \in s$, write $\omega_{\lambda(a)} = 1_{S(G_a)} \otimes 1_k \in K_{\lambda(a)}$. Then we may define a k-linear map, $\theta_1 : S(L_J) \otimes_{S(B_J^+)} k_\lambda \to \bigotimes_{a \in s} K_{\lambda(a)}$, by

$$\theta_1(\xi \otimes 1_k) = \pi_{r(\lambda)}\, \psi^{-1}(\xi)\big(\bigotimes_{a \in s} \omega_{\lambda(a)}\big), \text{ all } \xi \in S(L_J)$$

(recall that $\pi_{r(\lambda)}\,\psi^{-1}(\xi) \in \bigotimes_{a \in s} S(G_a)$ and ψ is as in (8.15)).

To prove that θ_1 is well defined, consider any basis element $\xi_{i,j}$ of $S(B_J^+)$ (i.e., $(i,j) \in \Omega$, $i \underset{J}{=} j$ and $i \le j$), and any $\xi \in S(L_J)$.

If $j \notin \lambda$, then $\xi\xi_{i,j} \otimes 1_k = \xi \otimes \xi_{i,j}\, 1_k = 0$. So suppose that $j \in \lambda$.

By (8.17)(i), $\xi_{i,j} = \xi_{\bar{i}\bar{j}} = \psi\, \iota_d(\bigotimes_{a \in s} \xi_{i(a),j(a)})$, for some $d \in R(J)$ and $i(a), j(a) \in I(n_a,d_a)$ $(a \in s)$. But, by (8.17)(ii) and (iii), $i(a) \le j(a)$, and $j(a)$ has weight $\lambda(a) \in \Lambda(n_a, r_a(\lambda))$, for all $a \in s$. Hence $d = r(\lambda)$, and $\xi_{i(a),j(a)} \in S(B_a^+)$ $(a \in s)$. Also $\xi_{i(a),j(a)}\,\omega_{\lambda(a)} = \xi_{i(a),j(a)} \otimes 1_k = 1_{S(G_a)} \otimes \xi_{i(a),j(a)}\, 1_k = \kappa_{\lambda(a)}(\xi_{i(a),j(a)})\omega_{\lambda(a)}$.

Therefore

$$\theta_1(\xi\,\xi_{i,j} \otimes 1_k) = \pi_{r(\lambda)}\,\psi^{-1}(\xi)\,\pi_{r(\lambda)}\,\psi^{-1}(\xi_{i,j})\,(\bigotimes_{a \in s} \omega_{\lambda(a)}) =$$

$$= \pi_{r(\lambda)}\,\psi^{-1}(\xi)\,\pi_{r(\lambda)}\,\psi^{-1}\,\psi\,\iota_{r(\lambda)}\,(\bigotimes_{a \in s} \xi_{i(a),j(a)})\,(\bigotimes_{a \in s} \omega_{\lambda(a)}) =$$

$$= \pi_{r(\lambda)}\,\psi^{-1}(\xi)\,(\bigotimes_{a \in s} \xi_{i(a),j(a)}\,\omega_{\lambda(a)}) =$$

$$= \kappa_{\lambda(1)}(\xi_{i(1),j(1)}) \cdots \kappa_{\lambda(s)}(\xi_{i(s),j(s)})\,\theta_1(\xi \otimes 1_k) =$$

$$= \begin{cases} \theta_1(\xi \otimes 1_k)\,; & \text{if } i(a) = j(a), \text{ for all } a \in s \\ 0\,; & \text{otherwise.} \end{cases}$$

But, by (8.17)(ii), $i(a) = j(a)$ (all $a \in s$) iff $\bar{i} = \bar{j}$, i.e., iff $i = j$. Hence $\theta_1(\xi\xi_{i,j} \otimes 1_k)$ if $i = j \in \lambda$, and zero otherwise.

On the other hand, $\theta_1(\xi \otimes \xi_{i,j}\, 1_k) =$

$$= \kappa_\lambda(\xi_{i,j})\,\theta_1(\xi \otimes 1_k) = \begin{cases} 1; & \text{if } i = j \in \lambda \\ 0; & \text{otherwise.} \end{cases}$$

Hence $\theta_1(\xi\xi_{i,j} \otimes 1_k) = \theta_1(\xi \otimes \xi_{i,j}\, 1_k)$.

Thus θ_1 is well defined. Also, since $\pi_{r(\lambda)}\,\psi^{-1}$ is a k-algebra map, θ_1 is an $S(L_J)$-map.

Now, to prove that θ_1 is bijective, we consider the k-map

$$\theta_2 : \bigotimes_{a \in \underline{s}} K_{\lambda(a)} \to S(L_J) \otimes_{S(B_J^+)} k_\lambda, \text{ given by}$$

$$\theta_2\Big(\bigotimes_{a \in \underline{s}} (\xi_a \otimes 1_k)\Big) = \Big(\psi\, \iota_{r(\lambda)}\Big(\bigotimes_{a \in \underline{s}} \xi_a\Big)\Big) \otimes 1_k, \text{ all } \xi_a \in S(G_a), a \in \underline{s}.$$

In a similar way to that used for θ_1, we can show that for any $i(a), j(a) \in I(n_a, r_a(\lambda))$ such that $i(a) \le j(a)$, and for any $\xi_a \in S(G_a)$ $(a \in \underline{s})$, there holds

(8.26) $$\theta_2\Big(\bigotimes_{a \in \underline{s}} (\xi_a\, \xi_{i(a),j(a)} \otimes 1_k)\Big) = \theta_2\Big(\bigotimes_{a \in \underline{s}} (\xi_a \otimes \xi_{i(a),j(a)}\, 1_k)\Big) =$$

$$= \begin{cases} \Big(\psi\, \iota_{r(\lambda)}\Big(\bigotimes_{a \in \underline{s}} \xi_a\Big)\Big) \otimes 1_k; & \text{if } i(a) = j(a) \in \lambda(a), \text{ all } a \in \underline{s} \\ 0 & ; \text{ otherwise.} \end{cases}$$

As $S(B_a^+)$ is k-spanned by $\{\xi_{i(a),j(a)} \mid i(a), j(a) \in I(n_a, r_a(\lambda)),\ i(a) \le j(a)\}$, by (8.26), θ_2 is well defined.

Now using (8.24) and the fact that if $i, j \in I(n,r)$ satisfy $i \underset{J}{=} j$ and $j \in \lambda$ then

$\xi_{i,j} = \psi\, \iota_{r(\lambda)} (\bigotimes_{a \in s} \xi_{i(a),j(a)})$ for some $i(a), j(a) \in I(n_a, r_a(\lambda))$ $(a \in s)$, it is easy to see

that $\theta_1^{-1} = \theta_2$ □

Proof of (8.23): In this proof we write $\Omega_J = \{(i,j) \in \Omega \mid i \underset{J}{=} j\}$.

As G_J^+ is the semidirect product $L_J U_J^+$, each $g \in G_J^+$ may be written in a unique way $g = \ell u$, for some $\ell \in L_J$, $u \in U_J^+$. So we may define a k-algebra map $d : k\, G_J^+ \to k\, L_J$ by, $d(g) = \ell$ (the multiplicative property of d comes from the fact that U_J^+ is a normal subgroup of G_J^+). So we have the following diagram

$$\begin{array}{ccc} k\, G_J^+ & \xrightarrow{d} & k\, L_J \\ T_r \downarrow & & \downarrow T_r \\ S(G_J) & \underset{\delta}{\dashrightarrow} & S(L_J) \end{array}$$

and we would like to define $\delta : S(G_J^+) \to S(L_J)$ so that the diagram commutes.

For this we only need to prove that, for any $\gamma \in kG_J^+$, $T_r(\gamma) = 0$ implies $T_r(d(\gamma)) = 0$.

Consider any $\ell \in L_J$, $u \in U_J^+$, $(i,j) \in \Omega_J$, $\rho \in \underline{r}$. Then $(i_\rho, j_\rho) \in N_a \times N_a$ (some $a \in \underline{s}$) and we have

$$(\ell\, u)_{i_\rho j_\rho} = \sum_{\mu \in \underline{n}} \ell_{i_\rho \mu}\, u_{\mu\, j_\rho}.$$

But $\ell_{i_\rho \mu} = 0$ unless $\mu \in N_a$, in which case $u_{\mu j_\rho} = 0$ or 1, according as $\mu \neq j_\rho$ or

$\mu = j_\rho$. So $(\ell u)_{i_\rho j_\rho} = \ell_{i_\rho j_\rho}$ for all $\rho \in \underline{r}$, which implies

(8.27) $(\ell u)_{i,j} = \ell_{i,j}$, all $(i,j) \in \Omega_J$.

Now, let γ be any element of kG_J^+. Then $\gamma = \sum_{\ell u} a_{\ell u}\, \ell u$, $(a_{\ell u} \in k)$ sum over a finite number of elements $\ell \in L_J$ and $u \in U_J^+$, and

$$T_r(\gamma) = \sum_{\ell u} a_{\ell u}\, T_r(\ell u) = \sum_{\ell u} a_{\ell u} \Big(\sum_{\substack{(i,j) \in \Omega \\ i \leq_J j}} (\ell u)_{i,j}\, \xi_{i,j} \Big)$$

$$= \sum_{\substack{(i,j) \in \Omega \\ i \leq_J j}} \Big(\sum_{\ell u} a_{\ell u}\, (\ell u)_{i,j} \Big) \xi_{i,j}.$$

As $\{\xi_{i,j} \mid (i,j) \in \Omega, i \leq_J j\}$ is a k-basis of G_J^+, $T_r(\gamma) = 0$ implies $\sum_{\ell u} a_{\ell u} (\ell u)_{i,j} = 0$, for all $(i,j) \in \Omega, i \leq_J j$. In particular we have

$$\sum_{\ell u} a_{\ell u}\, (\ell u)_{i,j} = 0, \text{ for all } (i,j) \in \Omega_J.$$

But from (8.27), we know this is the same as

(8.28) $\sum_{\ell u} a_{\ell u}\, \ell_{i,j} = 0$, for all $(i,j) \in \Omega_J$.

Thus, $T_r(d(\gamma)) = \sum_{\ell u} a_{\ell u}\, T_r(\ell) =$

$$= \sum_{\ell u} a_{\ell u} \sum_{(i,j) \in \Omega_J} \ell_{i,j}\, \xi_{i,j} = \sum_{(i,j) \in \Omega_J} \Big(\sum_{\ell u} a_{\ell u}\, \ell_{i,j} \Big) \xi_{i,j} = 0, \text{ by } (8.28).$$

So $T_r(\gamma) = 0$ implies $T_r(d(\gamma)) = 0$, for all $\gamma \in kG_J^+$.

Now define a k-linear map

$$\eta_1 : K_{\lambda,J} \longrightarrow S(L_J) \otimes_{S(B_J^+)} k_\lambda$$

by

$$\eta_1(\xi \otimes 1_k) = \delta(\xi) \otimes 1_k, \text{ all } \xi \in S(G_J^+).$$

To prove this is well defined we need to show that for any $b \in B^+$, and any $\xi \in S(G_J^+)$, there holds

$$\eta_1(\xi\, T_r(b) \otimes 1_k) = \eta_1(\xi \otimes T_r(b)1_k).$$

For this note that

(i) $d(b) \in B_J^+$, so $T_r(d(b)) \in S(B_J^+)$;

(ii) $\aleph_\lambda(T_r(d(b))) = \aleph_\lambda(T_r(b))$.

Hence

$\eta_1(\xi\, T_r(b) \otimes 1_k) = \delta(\xi\, T_r(b)) \otimes 1_k =$ (since δ is a k-algebra map)

$= \delta(\xi)\, \delta(T_r(b)) \otimes 1_k = \delta(\xi)\, T_r(d(b)) \otimes 1_k =$

$\delta(\xi) \otimes T_r(d(b))1_k = \aleph_\lambda\,(T_r(b))\, \delta(\xi) \otimes 1_k =$

$= \eta_1(\xi \otimes T_r(b)1_k)$.

On the other hand it is easy to see that we may define an $S(L_J)$-map

$$\eta_2 : S(L_J) \otimes_{S(B_J^+)} k_\lambda \longrightarrow K_{\lambda,J}$$

by

$$\eta_2(\xi \otimes 1) = \xi \otimes 1, \text{ all } \xi \in S(L_J).$$

Since U_J^+ acts trivially on k_λ and the restriction of δ to $S(L_J)$ is the identity map on $S(L_J)$ we have $\eta_2 = \eta_1^{-1}$, hence the lemma. □

3. 2-STEP PROJECTIVE RESOLUTIONS

§9. The radical of V_λ

The notation introduced in this chapter will be in force hereafter.

Recall from §4 that for each $\alpha \in \Lambda(n,r)$ we choose a basic α-tableau T^α and define $\ell(\alpha) \in I(n,r)$ by

$$T^\alpha_{\ell(\alpha)} = \begin{array}{|cccc|l} \hline 1 & 1 & \ldots & 1 & \text{(row 1)} \\ \hline 2 & 2 & \ldots & 2 & \text{(row 2)} \\ \hline & & \ldots & & \\ \hline n & n & \ldots & n & \text{(row n)} \\ \hline \end{array}$$

Clearly $\ell(\alpha)$ has weight α and the stabilizer, $P_{\ell(\alpha)}$, of $\ell(\alpha)$ in P coincices with the row stabilizer of T^α.

(9.1) Definition: Let $i \in I$. We say that the α-tableau T^α_i is *row-semistandard* if the entries in each row of T^α_i are weakly increasing ($\leq$) from left to right.

Let $I(\alpha) = \{i \in I \mid i \leq \ell(\alpha)$ and T^α_i is row-semistandard$\}$.

We use $\lambda = (\lambda_1,\ldots,\lambda_n)$ to denote an arbitrarily chosen element of Λ with basic λ-tableau

$$\text{(9.2)}\qquad T^{\lambda} = \begin{array}{|l|}\hline a_{11}\ a_{12}\ \cdots\ a_{1\lambda_1} \\ \hline a_{21}\ a_{22}\ \cdots\ a_{2\lambda_2} \\ \hline \cdots \\ \hline a_{n1}\ a_{n2}\ \cdots\ a_{n\lambda_n} \\ \hline \end{array}$$

and we write $\ell = \ell(\lambda)$, if no confusion arises.

We are interested in describing the basis (6.3), of $V_\lambda = S(B^+)\xi_\lambda$, in terms of λ-tableaux. For that we need a small lemma

(9.3) Lemma: Suppose $i \in I$, $i \le \ell$ and T_i^λ is not row-semistandard. Then there is $i' \in I$ such that $i' \le \ell$, $T_{i'}^\lambda$ is row-semistandard and $(i, \ell) \sim (i', \ell)$.

Proof: Suppose i is in the conditions of the lemma. Then there is $\pi \in P_\ell$ such that $T_{i\pi}^\lambda$ is row-semistandard (since P_ℓ equals the row-stabilizer of T^λ). As $i\pi \le \ell\pi = \ell$ and $(i\pi, \ell) = (i\pi, \ell\pi) \sim (i, \ell)$, we make $i' = i\pi$. □

(9.4) Proposition: V_λ and rad V_λ have k-bases

$$X_1 = \{\xi_{i,\ell} \mid i \in I(\lambda)\} \text{ and } X_2 = \{\xi_{i,\ell} \mid i \in I(\lambda) \text{ and } i \ne \ell\},$$

respectively.

Proof: As P_ℓ coincides with the row stabilizer of T^λ, the elements of $I(\lambda)$ are all distinct and so linearly independent. Thus, the result follows from (6.3) and (6.5), once we have proved that if $(i,j) \in \Omega'$ and $j \in \lambda$, then there is $i' \in I$ such that $i' \le \ell$, $T_{i'}^\lambda$ is row-semistandard and $(i', \ell) \sim (i,j)$. But this is clear from (1.3) and (9.3). □

Our next step is to determine a set of $S(B^+)$-generators of rad V_λ.

For each $\nu \in \underline{n-1}$, and each non-negative integer m, define $A_\nu^m : \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$A_\nu^m(z_1,\dots,z_n) = (z_1,\dots,z_\nu + m,\ z_{\nu+1} - m,\dots,z_n),^{5} \text{ all } (z_1,\dots,z_n) \in \mathbb{Z}^n.$$

If $m \le \lambda_{\nu+1}$ then $A_\nu^m\lambda \in \Lambda(n,r)$, and we choose the basic $A_\nu^m\lambda$-tableau to be

$$T^{A_\nu^m\lambda} = \begin{array}{|l|l}
\hline
a_{11} \ \cdots \ a_{1\lambda_1} & \text{(row 1)} \\
\hline
\quad \cdots & \\
\hline
a_{\nu 1} \ \cdots \ a_{\nu\lambda_\nu}\ a_{\nu+1,1} \ \cdots \ a_{\nu+1,m} & \text{(row } \nu) \\
\hline
a_{\nu+1,m+1} \ \cdots \ a_{\nu+1,\lambda_{\nu+1}} & \text{(row } \nu+1) \\
\hline
\quad \cdots & \\
\hline
a_{n1} \ \cdots \ a_{n\lambda_n} & \text{(row } n). \\
\hline
\end{array}$$

Thus

$$\text{(9.5)} \qquad T^{\lambda}_{\ell(A_\nu^m\lambda)} = \begin{array}{|l|l}
\hline
1 \ \cdots \ 1 & \text{(row 1)} \\
\hline
\quad \cdots & \\
\hline
\nu \ \cdots \ \nu & \text{(row } \nu) \\
\hline
\underbrace{\nu \ \cdots \ \nu}_{m}\ \nu+1 \ \cdots \ \nu+1 & \text{(row } \nu+1) \\
\hline
\quad \cdots & \\
\hline
n \ \cdots \ n & \text{(row } n). \\
\hline
\end{array}$$

---

[5] This map, A_ν^m, is a *raising operator*, as defined in [M; p. 8].

To simplify notation we write $\lambda(\nu, m) = A_\nu^m\lambda$, and $\ell(\nu, m) = \ell(A_\nu^m\lambda)$. Also if $m > \lambda_{\nu+1}$ we make the convention that $\xi_{\ell(\nu,m),\ell} = \xi_{\ell(\nu,m),\ell(\nu,m)} = 0$, and $V_{\lambda(\nu,m)} = 0$.

(9.6) Remarks: Let $\nu \in \underline{n-1}$ and $0 \le m \le \lambda_{\nu+1}$. Then

(i) $\ell(\nu, m)$ is the element $\ell(\nu, \nu+1, m, \lambda)$, defined in (4.5), and $\ell(\nu,0) = \ell$;

(ii) $\ell(\nu, m)$ has weight $\lambda(\nu, m)$;

(iii) $T^\lambda_{\ell(\nu, m)}$ is row-semistandard and if $m \ge 1$ then $\ell(\nu, m) < \ell$. Hence $\xi_{\ell(\nu,m),\ell} \in \text{rad } V_\lambda$, for all $m \ge 1$.

(9.7) Lemma: $X = \{\xi_{\ell(\nu,\, m_\nu),\ell} \mid \nu \in \underline{n-1},\ 1 \le m_\nu \le \lambda_{\nu+1}\}$ is a set of $S(B^+)$-generators of rad V_λ.

Proof: Let M be the $S(B^+)$-module generated by X. It is our aim to prove that $M = \text{rad } V_\lambda$.

By (9.6)(iii), it is clear that $M \subseteq \text{rad } V_\lambda$. To prove the equality we will show that all the elements of the basis X_2 of rad V_λ (defined in (9.4)) are in M.

Suppose $i \in I$ satisfies (9.8) below

(9.8) $\quad i < \ell$ and T^λ_i is row-semistandard.

Then there is $\rho \in \underline{r}$ such that $i_\rho < \ell_\rho$. Suppose this situation occurs for the first time in row $\nu + 1$ of T^λ, where $\nu \in \underline{n-1}$ (notice that this can never occur in row 1 of T^λ, since $i_{a_{1\mu}} = \ell_{a_{1\mu}} = 1$, for all $1 \le \mu \le \lambda_1$). Then

$$T_i^\lambda = \begin{array}{ll} \boxed{1 \ \ \ldots \ \ 1} & \text{(row 1)} \\ \quad\ldots & \\ \boxed{\nu \ \ \ldots \ \ \nu} & \text{(row } \nu) \\ \boxed{\underbrace{\tau \,..\, \tau}_{m} \ \mu \ \ldots} & \text{(row } \nu{+}1) \\ \quad\ldots & \end{array}$$

where $1 \le \tau < \mu \le \nu + 1$ and $1 \le m \le \lambda_{\nu+1}$. As $\tau \le \nu$ and $i \le \ell$, we have $i \le \ell(\nu, m)$. Thus $\xi_{i,\ell(\nu,m)} \in S(B^+)$. But $\xi_{\ell(\nu,m),\ell} \in X$ and so $\xi_{i,\ell(\nu,m)}\, \xi_{\ell(\nu,m),\ell} \in M$. We now analyse this product.

$$\xi_{i,\ell(\nu,m)}\, \xi_{\ell(\nu,m),\ell} = \sum_{\delta \in D} a_\delta\, \xi_{i\delta,\ell}\,,$$

where δ runs over a transversal D of the set of all double cosets $P_{i,\ell(\nu,m)}\, \delta\, P_{\ell(\nu,m),\ell}$ in $P_{\ell(\nu,m)}$, and $a_\delta = [P_{i\delta,\ell} : P_{i\delta,\ell,\ell(\nu,m)}]$ (and $1 \in D$).

Suppose first that $\delta \in D$ and $\xi_{i\delta,\ell} = \xi_{i,\ell}$. Then there is $\pi \in P_\ell$ such that $i\delta = i\pi$, and so $\delta = \sigma\pi$, for some $\sigma \in P_i$. As $\delta \in P_{\ell(\nu,m)}$, we have $\ell(\nu,m)\sigma\pi = \ell(\nu,m)$. Hence $\ell(\nu,m)\sigma = \ell(\nu,m)\pi^{-1}$. But $\pi^{-1} \in P_\ell$. Thus

$$T^\lambda_{\ell(\nu,m)\pi^{-1}} = \begin{array}{ll} \boxed{1 \ \ \ldots \ \ 1} & \text{(row 1)} \\ \quad\ldots & \\ \boxed{\nu \ \ \ldots \ \ \nu} & \text{(row } \nu) \\ \boxed{\nu\ \nu+1 \ \ \nu\ \nu+1 \ \ldots\ \nu} & \text{(row } \nu + 1) \\ \quad\ldots & \\ \boxed{n \ \ \ldots \ \ n} & \text{(row } n) \end{array}$$

i.e., $T^{\lambda}_{\ell(\nu,m)\pi^{-1}}$ is obtained from $T^{\lambda}_{\ell(\nu,m)}$ by permuting the elements of row $\nu+1$ amongst themselves. On the other hand as $\sigma \in P_i$ and $\tau < \mu$, there are no $\nu+1$'s in the first m-entries of row $\nu+1$ of $T^{\lambda}_{\ell(\nu,m)\sigma}$. Hence, $\ell(\nu,m)\pi^{-1} = \ell(\nu,m)\sigma$ implies $\ell(\nu,m)\pi^{-1} = \ell(\nu,m) = \ell(\nu,m)\sigma$, i.e., $\sigma \in P_{i,\ell}(\nu,m)$, $\pi \in P_{\ell(\nu,m),\ell}$ and $P_{i,\ell(\nu,m)}\delta P_{\ell(\nu,m),\ell} = P_{i,\ell(\nu,m)} P_{\ell(\nu,m),\ell}$. Therefore $\delta = 1$ and $\xi_{i,\ell}$ has coefficient $a_1 = [P_{i,\ell} : P_{i,\ell,\ell(\nu,m)}]$. But since $\tau < \mu$, we have $P_{i,\ell} = P_{i,\ell,\ell(\nu,m)}$. Thus $a_1 = 1$. There are two possibilities now:

(i) If $\tau = \nu$ we have $D = \{1\}$, and so $\xi_{i,\ell} = \xi_{i,\ell(\nu,m)}\xi_{\ell(\nu,m),\ell} \in M$, as desired.

(ii) Suppose now $\tau < \nu$.

For each $j \in I(n,r)$ define $\beta(j) = (\beta_1(j),\dots,\beta_n(j))$, where $\beta_\mu(j)$ is the sum of the entries in row μ of T^{λ}_j, and order these vectors lexicographically.

Let $\delta \in D\backslash\{1\}$. Then $T^{\lambda}_{i\delta}$ is obtained from T^{λ}_i by exchanging some of the τ's in row $\nu+1$ with ν's in row ν, and keeping fixed all other entries. As $\tau < \nu$, we will then have $\beta(i\delta) < \beta(i)$. If $T^{\lambda}_{i\delta}$ is not row-semistandard there is $\pi \in P_\ell$ such that $T^{\lambda}_{i\delta\pi}$ is row-semistandard and $\xi_{i\delta\pi,\ell} = \xi_{i\delta,\ell}$. Also, as $\pi \in P_\ell$ and $i \le \ell(\nu,m) < \ell$, we have $\beta(i\delta\pi) = \beta(i\delta) < \beta(i)$, and $i\delta\pi \le \ell(\nu,m)\delta\pi = \ell(\nu,m)\pi < \ell\pi = \ell$.

So we have proved that,

(9.9) If $i \in I$ satisfies (9.8), there exist a subset I' of $I(n,r)$, $\eta \in M$ and integers a_j ($j \in I'$) such that

(i) $\xi_{i,\ell} = \eta + \sum_{j \in I'} a_j \xi_{j,\ell}$;

(ii) j satisfies (9.8), all $j \in I'$;

(iii) $\beta(j) < \beta(i)$, all $j \in I'$.

If I' is the empty set $\xi_{i,\ell} \in M$, as required. Otherwise we apply (9.9) to each $j \in I'$. As the set $\{\beta(j) \mid j \in I(n,r)\}$ is finite, the process must stop.

Hence $X_2 = \{\xi_{i,\ell} \mid i$ satisfies (9.8)$\} \subseteq M$, and the lemma follows. □

As we are interested in a minimal set of generators of rad V_λ, we need to make some more calculations.

Consider $\nu \in \underline{n-1}$, and integers q,m satisfying $1 \le q \le m \le \lambda_{\nu+1}$. We have tableaux

$$T^{\lambda}_{t(\nu,m)} = \begin{array}{l} \cdots \\ \boxed{\nu \ \cdots \ \nu} \\ \boxed{\underbrace{\nu\cdots\nu}_{m}\ \nu+1\ \cdots\ \nu+1} \\ \cdots \end{array} \qquad \begin{array}{l} \\ (\text{row } \nu) \\ (\text{row } \nu+1) \\ \end{array}$$

$$T^{\lambda}_{t(\nu,q)} = \begin{array}{l} \cdots \\ \boxed{\nu \ \cdots \ \nu} \\ \boxed{\underbrace{\nu\cdots\nu}_{q}\ \nu+1\ \cdots\ \nu+1} \\ \cdots \end{array} \qquad \begin{array}{l} \\ (\text{row } \nu) \\ (\text{row } \nu+1) \\ \end{array} .$$

It is not difficult to see that $\quad P_{t(\nu,q)} = P_{t(\nu,m),\,t(\nu,q)}\,P_{t(\nu,q),\ell} \quad$ and

$[P_{t(\nu,m),\ell} : P_{t(\nu,m),\ell,\,t(\nu,q)}] = \dfrac{m!}{q!\,(m-q)!} = \dbinom{m}{q}$. Thus, by (2.7),

(9.10) $$\xi_{\ell(\nu,m),\ \ell(\nu,q)}\,\xi_{\ell(\nu,q),\ell} = \binom{m}{q}\,\xi_{\ell(\nu,m),\ell}.$$

Note that $q \le m$ implies $\ell(\nu,m) \le \ell(\nu,q)$ and so $\xi_{\ell(\nu,m),\ \ell(\nu,q)} \in S(B^+)$.

Lets consider first the case when char $k = 0$. Then, taking $q = 1$ in (9.10),

$$\xi_{\ell(\nu,m),\ell} = \frac{1}{m}\xi_{\ell(\nu,m),\ \ell(\nu,1)}\,\xi_{\ell(\nu,1),\ell},\ \text{all}\ \ 1 \le m \le \lambda_{\nu+1},\ \nu \in \underline{n-1}.$$

This together with (9.7) give

(9.11) If char $k = 0$, rad V_λ is $S(B^+)$-generated by $\{\xi_{\ell(\nu,1),\ell} \mid \nu \in \underline{n-1}\}$.

Now suppose char $k = p \ne 0$. We will use the following lemma.

(9.12) Lemma: [J; (22.4)]. Assume that a,b are positive integers and

$$a = a_0 + a_1 p + \ldots + a_t p^t \qquad (0 \le a_\mu < p,\ a_\mu \in \mathbb{Z})$$

$$b = b_0 + b_1 p + \ldots + b_t p^t \qquad (0 \le b_\mu < p,\ b_\mu \in \mathbb{Z}).$$

Then $\binom{a}{b} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_t}{b_t}$ (mod p). In particular, p divides $\binom{a}{b}$ iff $a_\mu < b_\mu$, for some μ.

Let $p^d \le m < p^{d+1}$, i.e., $m = m_0 + m_1 p + \ldots + m_d p^d$, where $0 \le m_\mu < p$, $m_\mu \in \mathbb{Z}$ ($\mu \in \underline{d}$) and $m_d \ne 0$.

Then, from (9.12), we know that $p \nmid \binom{m}{p^d}$ and by (9.10),

$$\xi_{\ell(\nu,m),\ell} = \frac{1}{\binom{m}{p^d}}\, \xi_{\ell(\nu,m),\,\ell(\nu,p^d)}\ \xi_{\ell(\nu,p^d),\ell}.$$

Thus, similarly to the previous case, we get

(9.13) If $\operatorname{char} k = p$, $\operatorname{rad} V_\lambda$ is $S(B^+)$-generated by $\{\xi_{\ell(\nu,p^{d_\nu}),\ell} \mid \nu \in \underline{n-1},\ 1 \le p^{d_\nu} \le \lambda_{\nu+1}\}$.

It is our aim to prove that these sets are in fact minimal sets of generators of $\operatorname{rad} V_\lambda$. For this we need to define a grading of $S(B^+)$.

Let $i,j \in I$ have weights α and β, respectively, and suppose that $i \le j$. By (1.10), $\beta \trianglelefteq \alpha$. Thus, there are non-negative integers $m_1,\dots,m_{n-1}$ such that

$$\alpha = \beta + \sum_{\mu \in \underline{n-1}} m_\mu\, e_{\mu,\mu+1} = A_1^{m_1} A_2^{m_2} \cdots A_{n-1}^{m_{n-1}} \beta,$$

where $$e_{\mu,\mu+1} = (0,\dots,0,\underset{(\mu)}{1},\underset{(\mu+1)}{-1},0,\dots,0) \in \mathbb{Z}^n \quad (\mu \in \underline{n-1}).$$

Hence, $\alpha - \beta \in \Psi = \{\sum_{\mu \in \underline{n-1}} z_\mu e_{\mu,\mu+1} \mid z_\mu \in \mathbb{Z},\ z_\mu \ge 0\ (\mu \in \underline{n-1})\}$ (i.e. $\alpha - \beta$ is a sum of positive roots). In these conditions we say that $\xi_{i,j}$ has *degree* $d(\xi_{i,j})$, where

$$d(\xi_{i,j}) = \alpha - \beta\,.$$

For each $\sigma \in \Psi$, let $S(B^+)_\sigma$ be the k-subspace of $S(B^+)$ spanned by all $\xi_{i,j}$ $(i \le j)$ of degree σ. Then

(9.14) $$S(B^+) = \bigoplus_{\sigma \in \Psi} S(B^+)_\sigma$$

is a grading of $S(B^+)$.

In fact, suppose that $i, j, h, f \in I$ have weights α, β, β', γ, respectively, and that $i \le j$, and $h \le f$. Then $\xi_{i,j}\,\xi_{h,f} = 0$, unless $\beta = \beta'$. If this last condition holds, there is $\pi \in P$ such that $j\pi = h$, and so

$$\xi_{i,j}\,\xi_{h,f} = \xi_{i\pi,h}\,\xi_{h,f} = \sum_{\delta} a_\delta\, \xi_{i\pi\delta,f}\,.$$

where the sum is over a subset $\{\delta\}$ of P_h, and a_δ are non-negative integers. Since $i\pi\delta$ has weight α, we have

$$d(\xi_{i\pi\delta,f}) = \alpha - \gamma = (\alpha - \beta) + (\beta - \gamma) = d(\xi_{i,j}) + d(\xi_{h,f}),$$

for all δ. Hence

(9.15) $$\xi_{i,j}\,\xi_{h,f} \in S(B^+)_{d(\xi_{i,j}) + d(\xi_{h,f})}\,.$$

It follows now easily that $S(B^+)_\sigma\, S(B^+)_{\sigma'} \subseteq S(B^+)_{\sigma+\sigma'}$, for all $\sigma, \sigma' \in \Psi$. Hence (9.14) is a grading of $S(B^+)$.

(9.16) Proposition: Let char $k = p$ (≥ 0). Then

$$\Upsilon = \{\xi_{\ell(v,p^{d_v}),\ell} \mid v \in \underline{n-1},\ 1 \le p^{d_v} \le \lambda_{v+1}\}$$

is a minimal set of $S(B^+)$-generators of rad V_λ.

Proof: By (9.11) and (9.13), we know that Y generates $\operatorname{rad} V_\lambda$. Thus, to prove the proposition, we only need to show that if $Y' \subseteq Y$ and $S(B^+)Y' = \operatorname{rad} V_\lambda$ then $Y' = Y$. Suppose this does not happen, i.e., there is Y' satisfying

$$Y' \subsetneq Y \text{ and } S(B^+)Y' = \operatorname{rad} V_\lambda.$$

Then, there are some $\nu \in \underline{n-1}$, and some non-negative integer d such that

$$1 \le p^d \le \lambda_{\nu+1} \text{ and } \xi_{\ell(\nu,p^d),\ell} \in Y \setminus Y'.$$

As $\xi_{\ell(\nu,p^d),\ell} \in \operatorname{rad} V_\lambda$, there are $\eta_1,\dots,\eta_q \in S(B^+)$, and $\xi_{\ell(\nu_1,p^{d_1}),\ell},\dots,\xi_{\ell(\nu_q,p^{d_q}),\ell} \in Y'$ such that

$$\xi_{\ell(\nu,p^d),\ell} = \sum_{s=1}^{q} \eta_s \, \xi_{\ell(\nu_s,p^{d_s}),\ell}.$$

Write $\quad \eta_s = \sum_{(i,j) \in \Omega'} a_{i,j}^{(s)} \xi_{i,j}, \quad (a_{i,j}^{(s)} \in k)$. Then

$$\xi_{\ell(\nu,p^d),\ell} = \sum_{s=1}^{q} \sum_{(i,j) \in \Omega'} a_{i,j}^{(s)} \xi_{i,j} \, \xi_{\ell(\nu_s,p^{d_s}),\ell} .$$

But, since distinct $\xi_{f,h}$'s are linearly independent, this implies that there are $s \in \underline{q}$ and $i \le \ell(\nu_s, p^{d_s})$ satisfying

$$(9.17) \quad \xi_{i,\ell(\nu_s,p^{d_s})} \, \xi_{\ell(\nu_s,p^{d_s}),\ell} = \sum_{\delta \in D} a_\delta \, \xi_{i\delta,\ell} = a \, \xi_{\ell(\nu,p^d),\ell} + \sum_{\delta \in D \setminus D'} a_\delta \, \xi_{i\delta,\ell} .$$

where:

(1) D is a transversal of the set of double cosets $P_{i,\ell(\nu_s,p^{d_s})}\,\delta\,P_{\ell(\nu_s,p^{d_s}),\ell}$ in $P_{\ell(\nu_s,p^{d_s})}$;

(2) $D' = \{\delta \in D \mid \xi_{i\delta,\ell} = \xi_{\ell(\nu,p^{d}),\ell}\}$ and $a_\delta = [P_{i\delta,\ell} : P_{i\delta,\ell,\,\ell(\nu_s,p^{d_s})}]$, all $\delta \in D$;

(3) $a = \sum_{\delta \in D'} a_\delta$ satisfies $a \not\equiv 0 \pmod p$.

Write $d(\xi_{i,\ell(\nu_s,p^{d_s})}) = \sum_{\mu \in \underline{n-1}} m_\mu\, e_{\mu,\mu+1}$ $(m_\mu \in \mathbb{Z},\ m_\mu \geq 0)$.

Then, (9.17) and (9.15) imply $d(\xi_{\ell(\nu,p^{d}),\ell}) = d(\xi_{i,\ell(\nu_s,p^{d_s})}) + d(\xi_{\ell(\nu_s,p^{d_s}),\ell})$, i.e.,

$$p^{d}\, e_{\nu,\nu+1} = \sum_{\mu \in \underline{n-1}} m_\mu\, e_{\mu,\mu+1} + p^{d_s}\, e_{\nu_s,\nu_s+1}$$

and, since the vectors $e_{\mu,\mu+1}$ $(\mu \in \underline{n-1})$ are linearly independent over $\mathbb{R}$, this implies

(9.18) $\qquad \nu_s = \nu,\quad m_\nu + p^{d_s} = p^{d},\quad$ and $\quad m_\mu = 0$ if $\mu \neq \nu$.

(i) Suppose first $p = 0$.

Then, from (9.18), we have $p^{d} = p^{d_s} = 1$ and $m_\nu = 0$. Thus,

$$\xi_{\ell(\nu,p^{d}),\ell} = \xi_{\ell(\nu,1),\ell} = \xi_{\ell(\nu_s,p^{d_s}),\ell} \in Y',$$

which contradicts our hypothesis.

(ii) Suppose now that $p \neq 0$.

If $p^{d} = p^{d_s}$ we get the same contradiction as in (i).

Thus, let $p^{d} > p^{d_s}$ and consider any $\delta \in D'$. As $\nu_s = \nu$, we have

(a) $i \leq \ell(\nu,p^{d_s})$ implies $i\delta \leq \ell(\nu,p^{d_s})$ (since $\delta \in P_{\ell(\nu,p^{d_s})}$);

(b) $\quad \xi_{i\delta,\ell} = \xi_{\ell(\nu,p^{d_*}),\ell}$ implies $i\delta = \ell(\nu,p^{d})\pi$, for some $\pi \in P_\ell$.

Hence

$$
T^{\lambda}_{i\delta} = \begin{array}{ll}
\boxed{1 \ \ldots \ 1} & \text{(row 1)} \\
\quad \ldots & \\
\boxed{\nu \ \ldots \ \nu} & \text{(row } \nu\text{)} \\
\boxed{\underbrace{\nu \ldots \nu}_{p^{d_*}} \ \underbrace{\nu+1 \ \nu \ldots \nu}_{m_\nu \ \nu\text{'s and } (\lambda_{\nu+1}-p^{d}) \ \nu+1\text{'s}} \ \nu+1} & \text{(row } \nu+1\text{)} \\
\quad \ldots & \\
\boxed{n \ \ldots \ n} & \text{(row n)}
\end{array}
$$

(i.e. $i\delta\tau = \ell(\nu,p^{d})$, for some $\tau \in P_{\ell(\nu,p^{d_*}),\ell)}$, and

$$a_\delta = [\, P_{i\delta,\ell} : P_{i\delta,\ell,\ell(\nu,p^{d_*})}] = \binom{p^{d}}{p^{d_*}} \equiv 0 \pmod p.$$

Therefore, $a = \sum_{\delta \in D'} a_\delta \equiv 0 \pmod p$, which gives a contradiction.

Thus Y is a minimal set of generators of rad V_λ. $\square$

§10. A 2-step minimal projective resolution of k_λ and its applications to Weyl modules

Now, that we have defined a minimal set of generators of rad V_λ, it is easy to determine a 2-step minimal projective resolution of k_λ, i.e., an exact sequence in mod $S(B^+)$

$$P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} k_\lambda \longrightarrow 0$$

where P_0 and P_1 are projective $S(B^+)$-modules and $\ker \varphi_\mu \subseteq \text{rad } P_\mu$, $\mu = 0,1$.

We know from §6 that $V_\alpha = S(B^+)\xi_{\alpha\alpha}$ $(\alpha \in A)$ is a projective $S(B^+)$-module. Also, by (6.8), there is an $S(B^+)$-epimorphism $\varkappa'_\lambda : V_\lambda \to k_\lambda$ (defined by $\varkappa'_\lambda(\xi_{i,\ell}) = 1$ or 0, according as $i = \ell$, or $i < \ell$ $(i \leq \ell)$) with $\ker \varkappa'_\lambda = \text{rad } V_\lambda$. So, we make

(10.1) $$P_0 = V_\lambda \text{ and } \varphi_0 = \varkappa'_\lambda.$$

Now, suppose that char $k = 0$, and define $\varphi_1 : \bigoplus_{\nu \in \underline{n-1}} V_{\lambda(\nu,1)} \to V_\lambda$, by

$$\varphi_1 \Big(\sum_{\nu \in \underline{n-1}} \eta_\nu \Big) = \sum_{\nu \in \underline{n-1}} \eta_\nu \, \xi_{\ell(\nu,1),\ell}, \text{ all } \eta_\nu \in V_{\lambda(\nu,1)}.$$

Then, φ_1 is an $S(B^+)$-map and, since $\xi_{\ell(\nu,1),\ell} \in \text{rad } V_\lambda$,

$\text{Im } \varphi_1 = \varphi_1 \big(\bigoplus_{\nu \in \underline{n-1}} V_{\lambda(\nu,1)} \big) \subseteq \text{rad } V_\lambda$. Thus, if we prove that $\xi_{\ell(\nu,1),\ell} \in \text{Im } \varphi_1$ $(\nu \in \underline{n-1})$, by (9.16), we will have $\text{Im } \varphi_1 = \text{rad } V_\lambda$. But this is easy, since $\xi_{\lambda(\nu,1)} \in V_{\lambda(\nu,1)}$, and

$$\varphi_1(\xi_{\lambda(\nu,1)}) = \xi_{\lambda(\nu,1)} \, \xi_{\ell(\nu,1),\ell} = \xi_{\ell(\nu,1),\ell}.$$

Hence

$$\bigoplus_{\nu \in \underline{n-1}} V_{\lambda(\nu,1)} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \longrightarrow 0$$

is an exact sequence in mod $S(B^+)$.

Similarly, if char $k = p$, we obtain the exact sequence

$$\bigoplus_{\nu \in \underline{n-1}} \bigoplus_{1 \le p^{d_\nu} \le \lambda_{\nu+1}} V_{\lambda(\nu,p^{d_\nu})} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \longrightarrow 0$$

where φ_1 is defined by

$$(10.2)\quad \varphi_1 \Big(\sum_{\nu \in \underline{n-1}} \sum_{1 \le p^{d_\nu} \le \lambda_{\nu+1}} \eta_{(\nu,p^{d_\nu})} \Big) = \sum_{\nu \in \underline{n-1}} \sum_{1 \le p^{d_\nu} \le \lambda_{\nu+1}} \eta_{(\nu,p^{d_\nu})} \, \xi_{\ell(\nu,p^{d_\nu}),\ell},$$

all $\eta_{(\nu,p^{d_\nu})} \in V_{\lambda(\nu,p^{d_\nu})}$.

Now, we know that $\ker \varphi_0 = \operatorname{rad} V_\lambda$. Thus, to prove that the 2-step projective resolution of k_λ, defined above, is minimal it is sufficient to show that

$$\ker \varphi_1 \subseteq \bigoplus_{\nu \in \underline{n-1}} \bigoplus_{1 \le p^{d_\nu} \le \lambda_{\nu+1}} \operatorname{rad} V_{\lambda(\nu,p^{d_\nu})} \quad \Big(= \operatorname{rad} \Big(\bigoplus_{\nu \in \underline{n-1}} \bigoplus_{1 \le p^{d_\nu} \le \lambda_{\nu+1}} V_{\lambda(\nu,p^{d_\nu})} \Big)\Big).$$

Suppose this is not true, i.e., there are $\eta_{(\nu,p^{d_\nu})} \in V_{\lambda(\nu,p^{d_\nu})}$ such that

$$\varphi_1 \Big(\sum_{\nu \in \underline{n-1}} \sum_{1 \le p^{d_\nu} \le \lambda_{\nu+1}} \eta_{(\nu,p^{d_\nu})} \Big) = 0 \text{ and } \eta_{(\mu,p^d)} \notin \operatorname{rad} V_{\lambda(\mu,p^d)},$$

for some $\mu \in \underline{n-1}$, and some p^d such that $1 \le p^d \le \lambda_{\mu+1}$.

Write $C = \{(\nu,p^{d_\nu}) \mid \nu \in \underline{n-1},\ 1 \le p^{d_\nu} \le \lambda_{\nu+1},\ (\nu, p^{d_\nu}) \ne (\mu, p^d)\}$.

Then $\varphi_1 \Big(\sum_{\nu \in \underline{n-1}} \sum_{1 \le p^{d_\nu} \le \lambda_{\nu+1}} \eta_{(\nu,p^{d_\nu})} \Big) = 0$ iff $\eta_{(\mu,p^d)} \, \xi_{\ell(\mu,p^d),\ell} = - \sum_{c \in C} \eta_c \, \xi_{\ell(c),\ell}$.

But as $\eta_{(\mu,p^d)} \notin \operatorname{rad} V_{\lambda(\mu,p^d)}$, we have

$$\eta_{(\mu,p^d)} = a_1\, \xi_{\ell(\mu,p^d),\ell(\mu,p^d)} + \sum_{i < \ell(\mu,p^d)} a_i\, \xi_{i,\ell(\mu,p^d)}\,.$$

where $a_i \in k$ and $a_1 \neq 0$. Thus

$$(10.3) \qquad a_1\, \xi_{(\mu,p^d),\ell} + \sum_{i < \ell(\mu,p^d)} a_i\, \xi_{i,\ell(\mu,p^d)}\, \xi_{\ell(\mu,p^d),\ell} = - \sum_{c \in C} \eta_c\, \xi_{\ell(c),\ell}.$$

But, since $i < \ell(\mu,p^d)$ implies $i\delta < \ell(\mu,p^d)$ (all $\delta \in P_{\ell(\mu,p^d)}$), the coefficient of $\xi_{\ell(\mu,p^d),\ell}$ on the left side of (10.3) is a_1 ($\neq 0$). On the other hand, we know from (9.16) that, this coefficient on the right side of (10.3) is zero.

This yields a contradiction, and so

$$\ker \varphi_1 \subseteq \operatorname{rad}\Big(\bigoplus_{\nu \in \underline{n-1}} \; \bigoplus_{1 \le p^{d_\nu} \le \lambda_{\nu+1}} V_{\lambda(\nu,p^{d_\nu})} \Big).$$

Hence we proved the

(10.4) Theorem: Suppose char $k = p$ (≥ 0). Then the sequence below is a 2-step minimal projective resolution of k_λ

$$\bigoplus_{\nu \in \underline{n-1}} \; \bigoplus_{1 \le p^{d_\nu} \le \lambda_{\nu+1}} V_{\lambda(\nu,p^{d_\nu})} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \longrightarrow 0\,,$$

where φ_0 and φ_1 are the maps defined in (10.1) and (10.2), respectively.

It is now easy to use this result to obtain 2-step projective resolutions of $K_{\lambda,J}$ in mod $S(G_J^\dagger)$. Unfortunatley these resolutions are not necessarily minimal.

Let J be any subset of $\underline{n-1}$ and suppose that $\lambda \in \Lambda_J^+$.

By applying the right exact functor

$$F_J = S(G_J^\dagger) \otimes_{S(B^+)} \cdot : \text{mod } S(B^+) \to \text{mod } S(G_J^\dagger)$$

to the sequence in (10.4), we obtain the exact sequence

$$\coprod_{\nu \in \underline{n-1}} \ \coprod_{1 \le p^{d_\nu} \le \lambda_{\nu+1}} F_J\left(V_{\lambda(\nu, p^{d_\nu})}\right) \xrightarrow{F_J(\varphi_1)} F_J(V_\lambda) \xrightarrow{F_J(\varphi_0)} F_J(k_\lambda) \to 0.$$

But $F_J(k_\lambda) = S(G_J^\dagger) \otimes_{S(B^+)} k_\lambda$ is $K_{\lambda,J}$.

Also, for each $\alpha \in \Lambda$, the map

(10.5) $$f_{\alpha,J} : S(G_J^\dagger) \otimes_{S(B^+)} V_\alpha \to S(G_J^\dagger)\xi_\alpha$$

defined by

$$f_{\alpha,J}(\xi \otimes \eta) = \xi\eta, \text{ all } \xi \in S(G_J^\dagger),\ \eta \in V_\alpha,$$

is an $S(G_J^\dagger)$-isomorphism. Thus, from (10.4), we get

(10.6) Corollary: Suppose char $k = p\ (\geq 0)$ and let J be any subset of $\underline{n-1}$.

Then, if $\lambda \in \Lambda_J^+(n,r)$ the sequence below is a 2-step projective resolution of $K_{\lambda,J}$

$$\coprod_{\nu \in \underline{n-1}} \coprod_{1 \leq p^{\ell_\nu} \leq \lambda_{\nu+1}} S(G_J^+)\xi_{\lambda(\nu,p^{\ell_\nu})} \xrightarrow{\Psi_1} S(G_J^+)\xi_\lambda \xrightarrow{\Psi_0} K_{\lambda,J} \to 0,$$

where $\Psi_0 = F_J(\varphi_0) f_{\lambda,J}^{-1}$ and $\Psi_1 = f_{\lambda,J}\, F_J(\varphi_1) \left(\coprod_{\nu \in \underline{n-1}} \coprod_{1 \leq p^{\ell_\nu} \leq \lambda_{\nu+1}} f^{-1}_{\lambda(\nu,p^{\ell_\nu}),J} \right)$.

Considering the particular case of $J = \underline{n-1}$, we have

(10.7) Corollary: Suppose char $k = p\ (\geq 0)$. Then, if $\lambda \in \Lambda^+(n,r)$ the sequence below is a 2-step projective resolution of the Weyl module K_λ

$$\coprod_{\nu \in \underline{n-1}} \coprod_{1 \leq p^{\ell_\nu} \leq \lambda_{\nu+1}} S(G)\xi_{\lambda(\nu,p^{\ell_\nu})} \xrightarrow{\Psi_1} S(G)\xi_\lambda \xrightarrow{\Psi_0} K_\lambda \to 0,$$

where $\Psi_0 = F_{\underline{n-1}}(\varphi_0) f^{-1}_{\lambda,\underline{n-1}}$ and

$\Psi_1 = f_{\lambda,\underline{n-1}}\ F_{\underline{n-1}}(\varphi_1) \left(\coprod_{\nu \in \underline{n-1}} \coprod_{1 \leq p^{\ell_\nu} \leq \lambda_{\nu+1}} f^{-1}_{\lambda(\nu,p^{\ell_\nu}),\underline{n-1}} \right)$.

4. $S(B^+)$ REVISITED

In this chapter we will look in more detail at the Schur algebra $S(B^+)$, in particular at its Cartan invariants.

§11. The spaces $\mathrm{Hom}_{S(B^+)}(V_\alpha, V_\lambda)$

We recall that λ is a fixed element of $\Lambda(n,r)$, T^λ is the basic tableau (9.2) and $\ell = \ell(\lambda)$.

It was proved in (9.4) that V_λ has k-basis $\{\xi_{i,\ell} \mid i \in I(\lambda)\}$, which implies the following.

(11.1) Lemma: $\dim_k V_\lambda = \prod_{\mu \in \underline{n}} \binom{\lambda_\mu + \mu - 1}{\mu - 1}$

Proof: As $\dim_k V_\lambda = \# I(\lambda)$ = number of λ-tableaux of the form

$$T_i^\lambda = \begin{array}{|l|l} \hline 1 \;\dots\; 1 & \text{(row 1)} \\ \hline 1 \dots 1\, 2 \dots 2 & \text{(row 2)} \\ \hline \dots & \\ \hline 1 .. 1\, 2 .. 2 \;\dots\dots\; \mu \dots \mu & (\text{row } \mu) \\ \hline \dots & \\ \hline 1 .. 1\, 2 .. 2 \;\dots\; n .. n & (\text{row } n) \\ \hline \end{array}$$

we have that $\dim_k V_\lambda = p_{\lambda_1} \cdots p_{\lambda_n}$, where, for each $\mu \in \underline{n}$, p_{λ_μ} = number of distinct sequences of integers

$$\underbrace{1 \ldots 1}_{a_1} \; \underbrace{2 \ldots 2}_{a_2} \; \underbrace{\mu \ldots \mu}_{a_\mu}, \qquad a_\nu \geq 0 \; (\nu \in \underline{\mu}), \; \sum_{\nu \in \underline{\mu}} a_\nu = \lambda_\mu,$$

i.e., $p_{\lambda_\mu} = \binom{\lambda_\mu + \mu - 1}{\mu - 1}$ □

Now let α be any element of $\Lambda(n,r)$ and consider the k-space

$$(V_\alpha, V_\lambda)_{S(B^+)} = \mathrm{Hom}_{S(B^+)}(V_\alpha, V_\lambda).$$

As $V_\alpha = S(B^+)\xi_\alpha$ and $V_\lambda = S(B^+)\xi_\lambda$ there is a k-isomorphism

(11.2) $$(V_\alpha, V_\lambda)_{S(B^+)} \cong \xi_\alpha S(B^+)\xi_\lambda = (V_\lambda)^\alpha .$$

(11.3) Lemma: Let $\alpha \in \Lambda(n,r)$. Then the following statements are equivalent.

(i) $(V_\alpha, V_\lambda)_{S(B^+)} \neq 0$

(ii) $\lambda \trianglelefteq \alpha$

(iii) $\alpha = A_1^{m_1} \ldots A_{n-1}^{m_{n-1}}\lambda$, for non-negative integers $m_1, \ldots, m_{n-1}$.[6]

Proof: (ii) and (iii) above are obviously equivalent. Now let $\alpha \in \Lambda(n,r)$ and consider $\xi_\alpha S(B^+)\xi_\lambda$.

As $S(B^+)\xi_\lambda = \bigoplus_{i \in I(\lambda)} k\,\xi_{i,\ell}$, there holds

$$\xi_\alpha S(B^+)\xi_\lambda = \bigoplus_{\substack{i \in I(\lambda) \\ i \in \alpha}} k\,\xi_{i,\ell} .$$

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[6] Recall that $A_1^{m_1} \ldots A_{n-1}^{m_{n-1}}\lambda = (\lambda_1 + m_1, \lambda_2 + m_2 - m_1, \ldots, \lambda_n - m_{n-1})$.

Therefore $\xi_\alpha\, S(B^+)\xi_\lambda \neq 0$ iff there is $i \in I(\lambda)$ with weight α. If such i exists, then $i \le \ell$ and, by (1.10), $\lambda \trianglelefteq \alpha$.

Conversely if $\lambda \trianglelefteq \alpha$ let i be the element of $I(n,r)$ whose λ-tableau T_i^λ has the first α_1 entries equal to 1, the next α_2 entries equal to 2, ... Then $i \in \alpha$ and since $\alpha_1 \ge \lambda_1$, $\alpha_1 + \alpha_2 \ge \lambda_1 + \lambda_2, \ldots$, $i \in I(\lambda)$. Hence $\lambda \trianglelefteq \alpha$ implies $\xi_\alpha\, S(B^+)\xi_\lambda \neq 0$.

Now the result follows from (11.2). □

It follows from the fact that $\dim_k k_\alpha = 1$ (all $\alpha \in \Lambda(n,r)$) that k is a splitting field for $S(B^+)$. So (cf. [CR; (54.16)]) the *Cartan invariants* $c_{\lambda\alpha}$ of $S(B^+)$ may be defined by

$$c_{\lambda\alpha} = \dim_k (V_\alpha, V_\lambda)_{S(B^+)} = \dim_k (V_\lambda)^\alpha.$$

Recall that $(V_\lambda)^\lambda = \xi_\lambda\, S(B^+)\xi_\lambda = k\xi_\lambda$ (cf. §6). Also, by the previous lemma, $\dim_k (V_\alpha, V_\lambda)_{S(B^+)} \neq 0$ iff $\lambda \trianglelefteq \alpha$. Thus we have the following

(11.4) Theorem: The Cartan invariants $c_{\lambda\alpha}$ of $S(B^+)$ satisfy (i) and (ii) below.

(i) $c_{\lambda\alpha} \neq 0$ iff $\lambda \trianglelefteq \alpha$.

(ii) $c_{\lambda\lambda} = 1$.

If we arrange the elements of $\Lambda(n,r)$ in some total order $\underline{\prec}$ such that $\lambda \trianglelefteq \alpha$ implies $\lambda \underline{\prec} \alpha$, and use this total order to arrange the rows and columns of the Cartan matrix C of $S(B^+)$ then, by (11.4), C takes the unitriangular form

$$C = \begin{array}{cc} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \cdots \; c_{\lambda\alpha} \; \cdots & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} & \text{(row } \lambda) \\ \text{(column } \alpha) & \end{array}$$

Now let $\alpha \in \Lambda(n,r)$ and suppose that $\lambda \trianglelefteq \alpha$, i.e., $\alpha = A_1^{m_1} \ldots A_{n-1}^{m_{n-1}}\lambda$, for non-negative integers $m_1,\ldots,m_{n-1}$.

As $(V_\alpha, V_\lambda)_{S(B^+)} \underset{k}{\cong} \xi_\alpha S(B^+)\xi_\lambda = \bigoplus_{\substack{i \in I(\lambda) \\ i \in \alpha}} k\,\xi_{i,\ell}$, it is easy to see that

(11.5) $(V_\alpha, V_\lambda)_{S(B^+)}$ has k-basis $\{\cdot\, \xi_{i,\ell} \mid i \in I(\lambda),\ i \in \alpha\}$, where, for each $i \in I(\lambda)$ satisfying $i \in \alpha$, $\cdot\, \xi_{i,\ell}$ is the element of $(V_\alpha, V_\lambda)_{S(B^+)}$ defined by

$$\cdot\, \xi_{i,\ell}(\xi) = \xi\, \xi_{i,\ell}\,, \text{ for all } \xi \in V_\alpha.$$

Therefore $\dim_k (V_\alpha, V_\lambda)_{S(B^+)} = \#\{i \in I(\lambda) \mid i \in \alpha\}$ = number of tableaux of the type

(11.6)

$$T_i^\lambda = \begin{array}{ll}
\boxed{1 \;\ldots\; 1} & \text{(row 1)} \\
\boxed{\underbrace{1 \ldots 1}_{b_{11}} \; \underbrace{2 \;\ldots\; 2}_{b_{12}}} & \text{(row 2)} \\
\quad \cdots\cdots & \\
\boxed{\underbrace{1 \ldots 1}_{b_{\mu-1,1}} \; \underbrace{2 \ldots 2}_{b_{\mu-1,2}} \; \ldots\ldots\ldots \; \underbrace{\mu \ldots \mu}_{b_{\mu-1,\mu}}} & \text{(row } \mu) \\
\quad \cdots\cdots & \\
\boxed{\underbrace{1 \,..\, 1}_{b_{n-1,1}} \; \underbrace{2 \,..\, 2}_{b_{n-1,2}} \; \ldots \; \underbrace{n-1 \,..\, n-1}_{b_{n-1,n-1}} \; \underbrace{n \ldots n}_{\alpha_n}} & \text{(row n)}
\end{array}$$

where

$$b_{\mu\nu} \geq 0 \; (\mu \in \underline{n-1},\ \nu \in \underline{\mu+1}); \sum_{\mu \in \underline{n-1}} b_{\mu 1} = \alpha_1 - \lambda_1; \sum_{\mu = \nu-1}^{n-1} b_{\mu\nu} = \alpha_\nu,\ \nu = 2,\dots,n-1.$$

This tableau determines a matrix $b = (b_{\mu\nu})_{\mu,\nu \in \underline{n-1}}$ whose entries $b_{\mu\nu}$ satisfy

(11.7) (i) $b_{\mu\nu} \in \mathbb{Z}$; $b_{\mu\nu} \geq 0$; and if $\nu > \mu + 1$ then $b_{\mu\nu} = 0$ (all $\mu, \nu \in \underline{n-1}$).

(ii) $\displaystyle\sum_{\mu \in \underline{n-1}} b_{\mu 1} = \alpha_1 - \lambda_1$; $\quad \displaystyle\sum_{\mu \in \underline{n-1}} b_{\mu\nu} = \alpha_\nu,\ \nu = 2,\dots,n-1.$

(iii) $\displaystyle\sum_{\nu \in \underline{n-1}} b_{\mu\nu} = \lambda_{\mu+1},\ \mu = 1,\dots,n-2;$ $\quad \displaystyle\sum_{\nu \in \underline{n-1}} b_{n-1,\nu} = \lambda_n - \alpha_n.$

Conversely, given a matrix, $b = (b_{\mu\nu})_{\mu,\nu \in \underline{n-1}}$, satisfying (11.7) it determines a tableau T_i^λ of the type (11.6), by the rule: T_i^λ is row semistandard, all the entries in row 1 of T_i^λ are equal to 1, and $b_{\mu\nu}$ is the number of ν's in row $\mu+1$ of T_i^λ, for all $\mu, \nu \in \underline{n-1}$.

Thus we have a bijective correspondence, $T_i^\lambda \longleftrightarrow (b_{\mu\nu})_{\mu,\nu \in \underline{n-1}}$, between the sets $\{T_i^\lambda \mid i \in I(\lambda)$ and $i \in \alpha\}$ and $\mathscr{B}(\alpha,\lambda) = \{b = (b_{\mu\nu})_{\mu,\nu \in \underline{n-1}} \mid b_{\mu\nu}$ satisfies (11.7) for all $\mu, \nu \in \underline{n-1}\}$.

This proves the following.

(11.8) Lemma: With the notation above, we have

$$c_{\lambda\alpha} = \dim_k(V_\alpha, V_\lambda)_{S(B^+)} = \#\mathscr{B}(\alpha,\lambda).$$

The remainder of this section will be dedicated to the study of the Cartan invariants $c_{\lambda\alpha}$, in the case when $\alpha = A_1^{m_1} \ldots A_{n-1}^{m_{n-1}}\lambda$, for non-negative integers $m_1,\ldots,m_{n-1}$ satisfying

$$m_\nu \leq \lambda_{\nu+1}\,, \text{ all } \nu \in \underline{n-1}.$$

The case $m_\nu > \lambda_{\nu+1}$, for some $\nu \in \underline{n-1}$, will be studied in §12.

(11.9) Definition: Given integers $m_1,\ldots,m_s$ $(s \geq 1)$ let $\mathcal{D}(m_1,\ldots,m_s)$ be the set of all matrices, $d = (d_{\mu\nu})_{\mu,\nu\in\underline{s}}$, whose entries satisfy

$$(11.10)\quad \begin{cases} d_{\mu\nu} \in \mathbb{Z}\,;\ d_{\mu\nu} \geq 0\,;\ d_{\mu\nu} = 0 \text{ if } \nu > \mu+1,\ (\mu,\nu \in \underline{s}) \\ \sum_{\mu\in\underline{s}} d_{\mu\nu} = m_\nu \qquad ;\ \nu \in \underline{s} \\ d_{\nu,\nu+1} = \sum_{\tau=1}^{\nu} (d_{\nu+1,\tau} + \ldots + d_{s\tau})\,;\ \nu \in \underline{s-1}. \end{cases}$$

Define $n(m_1,\ldots,m_s) = \# \mathcal{D}(m_1,\ldots,m_s)$.

Note that if $m_\mu < 0$, for some $\mu \in \underline{s}$, then $\mathcal{D}(m_1,\ldots,m_s) = \emptyset$ and $n(m_1,\ldots,m_s) = 0$.

(11.11) Proposition: Let $\alpha = A_1^{m_1} \ldots A_{n-1}^{m_{n-1}}\lambda \in \Lambda(n,r)$, where $m_1,\ldots,m_{n-1}$ are non-negative integers satisfying $m_\nu \leq \lambda_{\nu+1}$, for all $\nu \in \underline{n-1}$. Then

$$c_{\lambda\alpha} = n(m_1,\ldots,m_{n-1}).$$

Proof: Let α satisfy the conditions above. Since we know from (11.8) that $c_{\lambda,\alpha} = \# \mathcal{B}(\alpha,\lambda)$, to prove the proposition we only need to show that $\# \mathcal{B}(\alpha,\lambda) = \# \mathcal{D}(m_1,\ldots,m_{n-1})$. For simplicity we shall write $s = \underline{n-1}$.

As $m_\nu \le \lambda_{\nu+1}$ ($\nu \in \underline{s}$), we may define non-negative integrs $q_0, q_1,\ldots,q_s$ as follows

$$q_\nu = \lambda_{\nu+1} - m_\nu, \text{ for all } \nu \in \underline{s-1}, \text{ and } q_0 = q_s = 0.$$

Then $\alpha_{\nu+1} = \lambda_{\nu+1} + m_{\nu+1} - m_\nu = m_{\nu+1} + q_\nu$ ($\nu \in \underline{s-1}$) and the set of equations (11.7)(ii) and (iii) can be rewritten

$$\begin{cases} \sum\limits_{\mu \in \underline{s}} b_{\mu\nu} = m_\nu + q_{\nu-1}\,; & \nu \in \underline{s} \\ \sum\limits_{\nu \in \underline{s}} b_{\mu\nu} = m_\mu + q_\mu\,; & \mu \in \underline{s} \end{cases} .$$

So (11.7) is equivalent to the set of equations

$$\textbf{(11.12)} \qquad \begin{cases} b_{\mu\nu} \in \mathbb{Z};\ b_{\mu\nu} \ge 0;\ b_{\mu\nu} = 0 \text{ if } \nu > \mu + 1 \ (\text{all } \mu, \nu \in \underline{s}) \\ \sum\limits_{\mu \in \underline{s}} b_{\mu\nu} = m_\nu + q_{\nu-1};\ \nu \in \underline{s} \\ b_{\nu,\nu+1} = q_\nu + \sum\limits_{\tau=1}^{\nu} (b_{\nu+1,\tau} + \ldots + b_{s\tau});\ \nu \in \underline{s-1} \end{cases} .$$

Hence we have the following new expression for $\mathcal{B}(\alpha,\lambda)$

(11.13) $\mathcal{B}(\alpha,\lambda) = \{b = (b_{\mu\nu})_{\mu,\nu \in \underline{s}} \mid b_{\mu\nu}$ satisfies (11.12), all $\mu, \nu \in \underline{s}\}$.

Now for each $b \in \mathcal{B}(\alpha,\lambda)$, define $\theta(b) \in \mathcal{D}(m_1,\ldots,m_s)$, by

$$\theta(b)_{\mu\nu} = \begin{cases} b_{\mu\nu} & , \quad \text{if } \nu \neq \mu+1 \\ b_{\mu,\mu+1} - q_\mu & , \quad \text{if } \nu = \mu+1 \text{ ; all } \mu, \nu \in \underline{s}. \end{cases}$$

Since $q_\nu \geq 0$ $(\nu = 0,\ldots,s)$, it is clear that the map $\theta : \mathcal{B}(\alpha,\lambda) \to \mathcal{D}(m_1,\ldots,m_s)$, which takes $b \in \mathcal{B}(\alpha,\lambda)$ to $\theta(b) \in \mathcal{D}(m_1,\ldots,m_s)$, is a bijection. Hence $\#\mathcal{B}(\alpha,\lambda) = \# \mathcal{D}(m_1,\ldots,m_s)$. □

This proposition shows that the integers $n(m_1,\ldots,m_{s-1})$ have an important role in our work.

In some cases they are very easy to calculate. For example let $n = 3$, and let m_1, m_2 be any non-negative integers. Then

$$\mathcal{D}(m_1,m_2) = \# \left\{ d = \begin{pmatrix} d_{11} & d_{21} \\ d_{21} & d_{22} \end{pmatrix} \middle| \begin{array}{l} d_{\mu\nu} \in \mathbb{Z},\ d_{\mu\nu} \geq 0 \ (\mu, \nu = 1,2); \\ d_{11} + d_{21} = m_1 \ ; \ d_{21} + d_{22} = m_2 \end{array} \right\} .$$

Now it is easy to see that $d \in \mathcal{D}(m_1,m_2)$ iff $d_{11} = m_1 - d_{21}$; $d_{22} = m_2 - d_{21}$; $d_{21} \in \mathbb{Z}$ and $0 \leq d_{21} \leq \min(m_1,m_2)$. Therefore, $n(m_1,m_2) = \#\mathcal{D}(m_1,m_2) = \min(m_1,m_2) + 1$, and we have the corollary.

(11.14) Corollary: Let $\alpha, \lambda \in \Lambda(3,r)$ and suppose that $\alpha = A_1^{m_1} A_2^{m_2} \lambda$, for non-negative integers m_1, m_2 satisfying $m_1 \leq \lambda_2$. Then

$$c_{\lambda\alpha} = \min(m_1, m_2) + 1.$$

In general, $n(m_1,\dots,m_s)$ can not be expressed in such a nice way. What we will do now is to determine a generating function for these integers, which enable us to establish some relations amongst the $c_{\lambda\alpha}$.

Let s be any positive integer. Take s indeterminates $x_1,\dots,x_s$ and define the formal series

$$Q(x_1,\dots,x_s) = \sum_{m_1,\dots,m_s \geq 0} n(m_1,\dots,m_s)\, x_1^{m_1} \dots x_s^{m_s}$$

(11.15) Proposition: With the notation above, we have

$$Q(x_1,\dots,x_s) = \frac{1}{P(x_1,\dots,x_s)},$$

where $P(x_1,\dots,x_s) = \prod_{1 \leq \nu < \mu \leq s+1} (1 - x_\nu x_{\nu+1} \dots x_{\mu-1}).$

Proof: Let $P'(x_1,\dots,x_s) = \dfrac{1}{P(x_1,\dots,x_s)}$. As,

$(1 - x_\nu x_{\nu+1} \dots x_{\mu-1})^{-1} = \sum_{h_{\mu-1,\nu} > 0} (x_\nu x_{\nu+1} \dots x_{\mu-1})^{h_{\mu-1,\nu}}$, we have

$$P'(x_1,\dots,x_s) = \prod_{1 \leq \nu < \mu \leq s+1} \Big[\sum_{h_{\mu-1,\nu} \geq 0} (x_\nu x_{\nu+1} \dots x_{\mu-1})^{h_{\mu-1,\nu}} \Big]$$

$$= \sum_{\substack{h_{\mu-1,\nu} \geq 0 \\ 1 \leq \nu < \mu \leq s+1}} x_1^{h_{11}+\dots+h_{s1}} \dots x_\nu^{\sum_{\tau=1}^{\nu} (h_{\nu\tau} + \dots + h_{s\tau})} \dots x_s^{h_{s1}+\dots+h_{ss}}.$$

Thus, for any non-negative integers $m_1,\dots,m_s$, the coefficient of $x_1^{m_1}\dots x_s^{m_s}$ in $P'(x_1,\dots,x_s)$ equals the number of matrices, $h = (h_{\mu\nu})_{\mu,\nu\in \underline{s}}$, whose entries satisfy

$$(11.16)\qquad \begin{cases} h_{\mu\nu} \in \mathbf{Z}\,;\ h_{\mu\nu} \geq 0 \text{ and } h_{\mu\nu} = 0 \text{ if } \nu > \mu\ (\mu,\nu \in \underline{s}); \\ \sum_{\tau=1}^{\nu} (h_{\nu\tau} + \dots + h_{s\tau}) = m_\nu\,,\ \nu \in \underline{s}\,. \end{cases}$$

Let $\mathcal{H}(m_1,\dots,m_s)$ be the set of all these matrices, i.e.,
$\mathcal{H}(m_1,\dots,m_s) = \{h = (h_{\mu\nu})_{\mu,\nu\in\underline{s}} \mid h_{\mu\nu}$ satisfies (11.16), all $\mu,\nu \in \underline{s}\}$.

We can define a map, $\hat{\theta} : \mathcal{H}(m_1,\dots,m_s) \to \mathcal{D}(m_1,\dots,m_s)$, by

$$\hat{\theta}(h)_{\mu\nu} = \begin{cases} h_{\mu\nu} & ;\ \text{if } \nu \neq \mu+1 \\ \sum_{\tau=1}^{\mu} (h_{\mu+1,\tau} + \dots + h_{s\tau})\,; & \text{if } \nu = \mu+1\,, \end{cases} \quad \text{all } \mu,\nu \in \underline{s},\ h \in \mathcal{H}(m_1,\dots,m_s)\,.$$

In fact, if $h \in \mathcal{H}(m_1,\dots,m_s)$ we have that

$$\hat{\theta}(h)_{\mu,\mu+1} = \sum_{\tau=1}^{\mu} (h_{\mu+1,\tau} + \dots + h_{s\tau}) = \sum_{\tau=1}^{\mu} (\hat{\theta}(h)_{\mu+1,\tau} + \dots + \hat{\theta}(h)_{s\tau}),$$

for all $\mu \in \underline{s-1}$.

Also, $\displaystyle\sum_{\mu\in\underline{s}} \hat{\theta}(h)_{\mu\nu} = \hat{\theta}(h)_{\nu-1,\nu} + \sum_{\mu=\nu}^{s} \hat{\theta}(h)_{\mu\nu} =$

$$= \sum_{\tau=1}^{\nu-1} (h_{\nu\tau} + \dots + h_{s\tau}) + \sum_{\mu=\nu}^{s} h_{\mu\nu} = \sum_{\tau=1}^{\nu} (h_{\nu\tau} + \dots + h_{s\tau}) = m_\nu.$$ Hence,

$\hat{\theta}(h) \in \mathcal{D}(m_1,\dots,m_s)$.

It is easy to see that $\hat{\theta}$ is a bijection. Thus, $\# \mathcal{H}(m_1,\ldots,m_s) = \# \mathcal{D}(m_1,\ldots,m_s) = n(m_1,\ldots,m_s)$, i.e., the coefficient of $x_1^{m_1} \ldots x_s^{m_s}$ in $P'(x_1,\ldots,x_s)$ is $n(m_1,\ldots,m_s)$. Hence $P'(x_1,\ldots,x_s) = Q(x_1,\ldots,x_s)$. □

(11.17) Definition: For each $\omega \in P(n)$, define $\omega(\lambda) \in \mathbb{Z}^n$ by

$$\omega(\lambda) = (\lambda_1 + \omega(1) - 1,\ \lambda_2 + \omega(2) - 2,\ldots,\lambda_n + \omega(n) - n).$$

(11.18) Remarks: For any $\omega \in P(n)$, we have:

(i) Let $\delta = (n-1,\ n-2,\ldots,\ 1,0) \in \mathbb{Z}^n$. Then $\omega(\lambda) = \lambda + \delta - (\delta_{\omega(1)},\ldots,\delta_{\omega(n)})$ $(= \lambda + \delta - \omega^{-1}\delta$ in the notation of [M] (cf. [M; p. 8])).

(ii) For each $\nu \in \underline{n-1}$, let $a_\nu(\omega)$ be the non-negative integer given by

$a_\nu(\omega) = \omega(1) + \omega(2) + \ldots + \omega(\nu) - (1 + \ldots + \nu)$. Then, $\omega(\lambda) = A_1^{a_1(\omega)} \ldots A_{n-1}^{a_{n-1}(\omega)}\lambda$.

Conventions: Here we generalize the convention made in §9 as follows: if $m_1,\ldots,m_{n-1}$ are non-negative integers and $A_1^{m_1} \ldots A_{n-1}^{m_{n-1}}\alpha \notin \Lambda(n,r)$, then $V_{A_1^{m_1} \ldots A_{n-1}^{m_{n-1}}\alpha} = 0$ and $\xi_{i,\ell(A_1^{m_1} \ldots A_{n-1}^{m_{n-1}}\alpha)} = \xi_{\ell(A_1^{m_1} \ldots A_{n-1}^{m_{n-1}}\alpha),i} = 0$, for all $\alpha \in \Lambda(n,r)$, $i \in I(n,r)$.

We will also write $c_{A_1^{m_1} \ldots A_{n-1}^{m_{n-1}}\alpha\,\beta} = \dim_k(V_\beta, V_{A_1^{m_1} \ldots A_{n-1}^{m_{n-1}}\alpha})_{S(B^+)} = 0$ (all $\beta \in \Lambda(n,r)$).

We can now prove the main result of this section

(11.19) Theorem: Let $\alpha = A_1^{m_1} \ldots A_{n-1}^{m_{n-1}} \lambda \in \Lambda(n,r)$, for non-negative integers $m_1,\ldots,m_{n-1}$ satisfying $m_\nu \le \lambda_{\nu+1}$ ($\nu \in \underline{n-1}$). Then the Cartan invariants of $S(B^+)$ satisfy the identity

$$\sum_{\omega \in P(n)} \varepsilon(\omega) c_{\omega(\lambda)\alpha} = \delta_{\lambda,\alpha}$$

(where $\varepsilon(\omega)$ is the sign of the permutation ω, and $\delta_{\lambda,\alpha} = 1$ or 0, according as $\lambda = \alpha$ or $\lambda \ne \alpha$).

Proof: If $n = 1$ the theorem is obvious. So suppose that $n \ge 2$.

Let $\omega \in P(n)$ and write $a_\nu(\omega) = \sum_{\mu=1}^{\nu} (\omega(\mu)-\mu)$, for all $\nu \in \underline{n}$. Then

$\omega(\lambda) = A_1^{a_1(\omega)} \ldots A_{n-1}^{a_{n-1}(\omega)} \lambda.$

Suppose in the first place that $\omega(\lambda) \notin \Lambda(n,r)$. Then, there is some $\nu \in \underline{n-1}$ such that $\lambda_{\nu+1} + a_{\nu+1}(\omega) - a_\nu(\omega) < 0$. But then, since $m_\nu \le \lambda_{\nu+1}$, we have

$$m_\nu - a_\nu(\omega) \le \lambda_{\nu+1} - a_\nu(\omega) < -a_{\nu+1}(\omega) \le 0.$$

Hence $c_{\omega(\lambda)\alpha} = n(m_1 - a_1(\omega),\ldots,m_{n-1} - a_{n-1}(\omega)) = 0$ (recall that $n(b_1,\ldots,b_s) = 0$ if $b_\nu < 0$, for some $\nu \in \underline{s}$.)

Now suppose that $\omega(\lambda) \in \Lambda(n,r)$. There are two possibilities:

(i) $m_\nu - a_\nu(\omega) < 0$, for some $\nu \in \underline{n-1}$

(ii) $m_\nu - a_\nu(\omega) \ge 0$, for all $\nu \in \underline{n-1}$.

In the first case we have $n(m_1 - a_1(\omega),\ldots,m_{n-1} - a_{n-1}(\omega)) = 0$. Also $\omega(\lambda) \ntrianglelefteq \alpha$.

So, by (11.4)(i), $c_{\omega(\lambda)\alpha} = 0$. Thus $c_{\omega(\lambda)\alpha} = n(m_1 - a_1(\omega),\ldots,m_{n-1} - a_{n-1}(\omega))$.

Now consider the case (ii).

We have $\alpha = A_1^{m_1} \ldots A_{n-1}^{m_{n-1}} 0 = A_1^{m_1 - a_1(\omega)} \ldots A_{n-1}^{m_{n-1} - a_{n-1}(\omega)} \omega(\lambda)$.

Also, $\omega(\lambda)_{\nu+1} - (m_\nu - a_\nu(\omega)) = \lambda_{\nu+1} + a_{\nu+1}(\omega) - a_\nu(\omega) - (m_\nu - a_\nu(\omega)) =$ $\lambda_{\nu+1} - m_\nu + a_{\nu+1}(\omega)$. Since $a_{\nu+1}(\omega) \geq 0$, this implies

$$\omega(\lambda)_{\nu+1} - (m_\nu - a_\nu(\omega)) \geq \lambda_{\nu+1} - m_\nu \geq 0, \text{ all } \nu \in \underline{n-1}.$$

Therefore, α and $\omega(\lambda)$ satisfy the hypothesis of (11.11), and so $c_{\omega(\lambda)\alpha} = n(m_1 - a_1(\omega),\ldots,m_{n-1} - a_{n-1}(\omega))$.

Thus, in any of these cases $c_{\omega(\lambda)\alpha} = n(m_1 - a_1(\omega),\ldots,m_{n-1} - a_{n-1}(\omega))$, for all $\omega \in P(n)$, and we have

$$\sum_{\omega \in P(n)} \varepsilon(\omega) c_{\omega(\lambda)\alpha} = \sum_{\omega \in P(n)} \varepsilon(\omega)\, n(m_1 - a_1(\omega),\ldots,m_{n-1} - a_{n-1}(\omega)).$$

Now the theorem follows from the lemma (11.20) below. □

(11.20) Lemma: Let s be a positive integer. For each $\omega \in P(s+1)$ let $a_\nu(\omega) = \sum_{\mu=1}^{\nu} (\omega(\mu) - \mu)$, $\nu \in \underline{s}$. Then, for any non-negative integers $m_1,\ldots,m_s$, there holds

$$\sum_{\omega \in P(s+1)} \varepsilon(\omega)\, n(m_1 - a_1(\omega),\ldots,m_s - a_s(\omega)) = \begin{cases} 1; & \text{if } m_\nu = 0, \text{ all } \nu \in \underline{s} \\ 0; & \text{if } m_\nu \neq 0, \text{ some } \nu \in \underline{s}. \end{cases}$$

Proof: Let $X_1,\ldots,X_{s+1}$ be $s+1$ independent variables and consider the ring of Laurent polynomials $\mathbb{Z}[X_1^{\pm1},\ldots,X_{s+1}^{\pm1}]$. In [M; p. 26 (proof of (3.4''))] it is proved that in this ring there holds

$$\text{(11.21)} \quad \sum_{\omega \in P(s+1)} \varepsilon(\omega)\, X_1^{\omega(1)-1}\, X_2^{\omega(2)-2} \cdots X_{s+1}^{\omega(s+1)-(s+1)} = \prod_{1 \le \nu < \mu \le s+1} (1 - X_\nu X_\mu^{-1}).$$

Now consider the polynomial ring $\mathbb{Z}[x_1,\ldots,x_s]$ in the independent variables $x_1,\ldots,x_s$, and let $f : \mathbb{Z}[x_1,\ldots,x_s] \to \mathbb{Z}[X_1^{\pm1},\ldots,X_{s+1}^{\pm1}]$ be the ring homomorphism defined by,

$$f(x_1^{b_1}\cdots x_s^{b_s}) = X_1^{b_1}\, X_2^{b_2-b_1} \cdots X_s^{b_s-b_{s-1}} X_{s+1}^{-b_s}, \text{ all monomials } x_1^{b_1}\cdots x_s^{b_s} \in \mathbb{Z}[x_1,\ldots,x_s].$$

Note that $f(x_\nu x_{\nu+1} \cdots x_\mu) = X_\nu X_{\mu+1}^{-1}$, all $1 \le \nu < \mu \le s$.

Suppose that

$$P(x_1,\ldots,x_s) = \prod_{1 \le \nu < \mu \le s+1} (1 - x_\nu x_{\nu+1} \cdots x_{\mu-1}) = \sum_{b_1,\ldots,b_s \ge 0} p(b_1,\ldots,b_s)\, x_1^{b_1} \cdots x_s^{b_s}.$$

Then $f\Big(\prod_{1 \le \nu < \mu \le s+1} (1 - x_\nu x_{\nu+1} \cdots x_{\mu-1})\Big) = \sum_{b_1,\ldots,b_s \ge 0} p(b_1,\ldots,b_s)\, f(x_1^{b_1} \cdots x_s^{b_s})$,

i.e.,

$$\prod_{1 \le \nu < \mu \le s+1} (1 - X_\nu X_\mu^{-1}) = \sum_{b_1,\ldots,b_s \ge 0} p(b_1,\ldots,b_s)\, X_1^{b_1} X_2^{b_2-b_1} \cdots X_{s+1}^{-b_s}.$$

Hence, by (11.21),

$$\sum_{b_1,\ldots,b_s \ge 0} p(b_1,\ldots,b_s) X_1^{b_1} X_2^{b_2-b_1} \cdots X_{s+1}^{-b_s} = \sum_{\omega \in P(s+1)} \varepsilon(\omega) X_1^{\omega(1)-1}\, X_2^{\omega(2)-2} \cdots X_{s+1}^{\omega(s+1)-(s+1)}.$$

This implies that

$$p(b_1,\dots,b_s) = \begin{cases} \varepsilon(\omega); & \text{if } (b_1,\dots,b_s) = (a_1(\omega),\dots,a_s(\omega)) \\ 0 \; ; & \text{if } (b_1,\dots,b_s) \neq (a_1(\omega),\dots,a_s(\omega)) . \end{cases}$$

Therefore

$$P(x_1,\dots,x_s) = \prod_{1 \le \nu < \mu \le s+1} (1 - x_\nu x_{\nu+1} \cdots x_{\mu-1}) = \sum_{\omega \in P(s+1)} \varepsilon(\omega) x_1^{a_1(\omega)} \cdots x_s^{a_s(\omega)} .$$

Now let $Q(x_1,\dots,x_s) = \sum_{q_1,\dots,q_s \ge 0} n(q_1,\dots,q_s) x_1^{q_1} \cdots x_s^{q_s}$. By (11.15), $P(x_1,\dots,x_s)\, Q(x_1,\dots,x_s) = 1$. Hence

$$\sum_{\omega \in P(s+1)} \sum_{q_1,\dots,q_s \ge 0} \varepsilon(\omega)\, n(q_1,\dots,q_s)\, x_1^{q_1 + a_1(\omega)} \cdots x_s^{q_s + a_s(\omega)} = 1.$$

The coefficient of $x_1^{m_1} \cdots x_s^{m_s}$ on the left side of this equality is

$$\sum_{\omega \in P(s+1)} \varepsilon(\omega)\, n(m_1 - a_1(\omega),\dots,m_s - a_s(\omega)).$$

On the other hand, this coefficient on the right side of the equality is 1 if $m_1 = \dots = m_s = 0$, and it is zero otherwise. Hence

$$\sum_{\omega \in P(s+1)} \varepsilon(\omega)\, n(m_1 - a_1(\omega),\dots,m_s - a_s(\omega)) = \begin{cases} 1; & \text{if } m_1 = \dots = m_s = 0 \\ 0; & \text{if } m_\nu \neq 0, \text{ some } \nu \in \underline{s}. \end{cases}$$

This completes the proof of the lemma. $\square$

§12. Some more results on $c_{\lambda\alpha}$

In this section we proceed with the study of the Cartan invariants $c_{\lambda\alpha}$ of $S(B^+)$. We use the same notation as in §11.

In (11.11) we proved that $c_{\lambda\alpha} = n(m_1,\dots,m_{n-1})$ if $\alpha = A_1^{m_1}\dots A_{n-1}^{m_{n-1}}\lambda$, for non-negative integers $m_1,\dots,m_{n-1}$ satisfying $m_\nu \le \lambda_{\nu+1}$ ($\nu \in \underline{n-1}$). In the general case we have a weaker result.

(12.1) Proposition: Let $\alpha = A_1^{m_1}\dots A_{n-1}^{m_{n-1}}\lambda \in \Lambda(n,r)$, where $m_1,\dots,m_{n-1}$ are non-negative integers. Then

$$c_{\lambda\alpha} \le n(m_1,\dots,m_{n-1}).$$

Proof: Write $s = \underline{n-1}$ and define integers $q_0,\dots,q_s$ as follows

$$q_\nu = \lambda_{\nu+1} - m_\nu, \text{ for all } \nu \in \underline{s-1};\ q_0 = q_s = 0.$$

Note that, since we are not assuming that $m_\nu \le \lambda_{\nu+1}$, q_ν may be a negative integer.

It is easy to see that, as in the proof of (11.11), $\mathscr{B}(\alpha,\lambda)$ has the expression

(12.2) $\qquad \mathscr{B}(\alpha,\lambda) = \{b = (b_{\mu\nu})_{\mu,\nu\in\underline{s}} \mid b_{\mu\nu} \text{ satisfy (12.3), all } \mu,\nu \in \underline{s}\}$

where

(12.3) (i) $b_{\mu\nu} \in \mathbb{Z}$; $b_{\mu\nu} \ge 0$, and $b_{\mu\nu} = 0$ if $\nu > \mu + 1$ (all $\mu, \nu \in \underline{s}$).

(ii) $\sum_{\mu \in \underline{s}} b_{\mu\nu} = m_\nu + q_{\nu-1},\ \nu \in \underline{s}\,;$

(iii) $b_{\nu,\nu+1} = q_\nu + \sum_{\tau=1}^{\nu} (b_{\nu+1,\tau} + \ldots + b_{s\tau});\ \nu \in \underline{s-1}.$

Thus, we may define an injective map $\theta : \mathcal{B}(\alpha,\lambda) \to \mathcal{D}(m_1,\ldots,m_s)$, by

$$\theta(b)_{\mu\nu} = \begin{cases} b_{\mu\nu}, & \text{if } \nu \neq \mu+1 \\ b_{\mu,\mu+1} - q_\mu, & \text{if } \nu = \mu+1\,; \end{cases} \qquad \text{all } \mu,\nu \in \underline{s},\ b \in \mathcal{B}(\alpha,\lambda).$$

But, since q_ν may be negative, θ may not be surjective. In fact we have

$$\operatorname{Im}\theta = \{d = (d_{\mu\nu}) \in \mathcal{D}(m_1,\ldots,m_s) \mid d_{\mu,\mu+1} \geq -q_\mu,\ \text{all } \mu \in \underline{s-1}\}.$$

Therefore, $\#\mathcal{B}(\alpha,\lambda) \leq \#\mathcal{D}(m_1,\ldots,m_s) = n(m_1,\ldots,m_s)$, and by (11.8),

$$c_{\lambda\alpha} = \#\mathcal{B}(\alpha,\lambda) \leq n(m_1,\ldots,m_s). \quad \square$$

(12.4) Remark: Note that if α and λ are as above, from the proof of (12.1), we have

$$c_{\lambda\alpha} = n(m_1,\ldots,m_{n-1}) - \#\{d \in \mathcal{D}(m_1,\ldots,m_{n-1}) \mid d_{\mu,\mu+1} < m_\mu - \lambda_{\mu+1},\ \text{some } \mu \in \underline{n-1}\}.$$

We shall now describe $c_{\lambda\alpha}$ in the case when $n = 3$. Recall from §11 that $c_{\lambda\alpha} = 0$, unless $\lambda \trianglelefteq \alpha$. Also $n(m_1,m_2) = \min(m_1,m_2) + 1$ if $m_1,m_2 \geq 0$, and it is zero otherwise.

(12.5) Theorem: Let $\lambda, \alpha \in A(3,r)$ and suppose that $\lambda \trianglelefteq \alpha$, i.e., $\alpha = A_1^{m_1} A_2^{m_2} \lambda$, for non-negative integers m_1, m_2. Then

$$c_{\lambda\alpha} = \begin{cases} \min(m_1, m_2) + 1 & , \text{ if } m_1 \le \lambda_2 \\ \min(\lambda_2, \lambda_2 + m_2 - m_1) + 1, & \text{if } m_1 \ge \lambda_2. \end{cases}$$

Proof: By (11.14), $c_{\lambda\alpha} = \min(m_1, m_2) + 1$ if $m_1 \le \lambda_2$.

Now suppose that $m_1 > \lambda_2$ and write $q = \lambda_2 - m_1$ (< 0).

From (12.2), we know that

$$\mathcal{B}(\alpha,\lambda) = \left\{ b = \begin{pmatrix} b_{11} & b_{21} + q \\ b_{21} & b_{22} \end{pmatrix} \middle| \begin{array}{l} b_{\mu\nu} \in \mathbb{Z},\ b_{\mu\nu} \ge 0\ (\mu, \nu = 1,2),\ b_{21} + q \ge 0, \\ b_{11} + b_{21} = m_1 ;\ b_{21} + b_{22} = m_2 . \end{array} \right\}$$

So, we may define $\bar{\theta} : \mathcal{B}(\alpha,\lambda) \to \mathcal{D}(m_1 + q, m_2 + q)$, by

$$\bar{\theta}(b)_{\mu\nu} = \begin{cases} b_{\mu\nu} & , \text{ if } (\mu,\nu) \ne (2,1) \\ b_{21} + q, & \text{if } (\mu,\nu) = (2,1) ; \end{cases} \qquad \mu, \nu = 1,2;\ b \in \mathcal{B}(\alpha,\lambda).$$

Clearly $\bar{\theta}$ is injective. Also, since $q \le 0$, we may define, for each $d \in \mathcal{D}(m_1 + q, m_2 + q)$, $b(d) \in \mathcal{B}(\alpha,\lambda)$, by

$$b(d)_{\mu\nu} = \begin{cases} d_{\mu\nu}, & \text{if } (\mu,\nu) \ne (2,1) \\ d_{21} - q, & \text{if } (\mu,\nu) = (2,1) ; \end{cases} \qquad \mu, \nu = 1,2.$$

Then $\bar{\theta}(b(d)) = d$. Hence $\bar{\theta}$ is surjective. Therefore, $\#\mathcal{B}(\alpha,\lambda) = \#\mathcal{D}(m_1+q, m_2+q) = n(m_1+q, m_2+q) = \min(m_1+q, m_2+q) + 1$.

But, $m_1+q = \lambda_2$ and $m_2+q = \lambda_2+m_2-m_1$. Hence, $c_{\lambda\alpha} = \min(\lambda_2, \lambda_2+m_2-m_1)+1$. □

We now generalize theorem (11.19) in the case $n = 3$.

(12.6) Theorem: Let $\alpha, \lambda \in \Lambda(3,r)$. Then we have

$$\sum_{\omega \in P(3)} \varepsilon(\omega)\, c_{\omega(\lambda)\alpha} = \delta_{\lambda,\alpha}.$$

Proof: If $\lambda \not\trianglelefteq \alpha$, then $\omega(\lambda) \not\trianglelefteq \alpha$ (since $\lambda \trianglelefteq \omega(\lambda)$) and so $c_{\omega(\lambda)\alpha} = 0$, for all $\omega \in P(3)$. Thus $\sum_{\omega \in P(3)} \varepsilon(\omega)\, c_{\omega(\lambda)\alpha} = 0$.

Now suppose that $\lambda \trianglelefteq \alpha$, i.e., $\alpha = A_1^{m_1} A_2^{m_2}\lambda$, for non-negative integers m_1, m_2.

If $m_1 \leq \lambda_2$ the theorem follows from (11.19).

Now consider the case $m_1 > \lambda_2$, and write $q = \lambda_2 - m_1$ (< 0).

Let $\omega \in P(3)$. Once more we define $a_\nu(\omega) = \sum_{\mu=1}^{\nu} (\omega(\mu) - \mu)$ ($\nu = 1,2,3$), so that $\omega(\lambda) = A_1^{a_1(\omega)} A_2^{a_2(\omega)}\lambda$. Calculating $a_\nu(\omega)$, for all $\omega \in P(3)$ ($\nu = 1,2$), we obtain

$$\text{(12.7)} \quad \sum_{\omega \in P(3)} \varepsilon(\omega)\, c_{\omega(\lambda)\alpha} = c_{\lambda\alpha} - c_{A_1\lambda,\alpha} - c_{A_2\lambda,\alpha} + c_{A_1^2 A_2\lambda,\alpha} + c_{A_1 A_2^2\lambda,\alpha} - c_{A_1^2 A_2^2\lambda,\alpha}.$$

Suppose that $\omega(\lambda) \trianglelefteq \alpha$, for all $\omega \in P(3)$. Then $\omega(\lambda)_2 = \lambda_2 + a_2(\omega) - a_1(\omega)$. Also $m_1 - a_1(\omega) > \lambda_2 - a_1(\omega)$. Thus

$$m_1 - a_1(\omega) \geq \omega(\lambda)_2 \quad \text{if} \quad a_2(\omega) \leq 1$$

and, by (12.5), $c_{\omega(\lambda)\alpha} = \min\,(\omega(\lambda)_2,\ \omega(\lambda)_2 + m_2 - a_2(\omega) - m_1 + a_1(\omega)) + 1 = \min\,(m_1 - a_1(\omega),\ m_2 - a_2(\omega)) + a_2(\omega) + q + 1$. Hence (since $m_1 - a_1(\omega) \geq 0$ and $m_2 - a_2(\omega) \geq 0$)

$$c_{\omega(\lambda)\alpha} = n(m_1 - a_1(\omega),\ m_2 - a_2(\omega) + a_2(\omega) + q \quad \text{if} \quad a_2(\omega) \leq 1. \tag{12.8}$$

Now suppose that $a_2(\omega) = 2$, i.e., $\omega(\lambda) = A_1 A_2^2\lambda$ or $\omega(\lambda) = A_1^2 A_2^2\lambda$. We have two cases to consider

(i) $m_1 \geq \lambda_2 + 2$. Then $m_1 - a_1(\omega) \geq \omega(\lambda)_2$, and $c_{\omega(\lambda)\alpha}$ is given by (12.8), for all $\omega \in P(3)$. Therefore, by (11.20) and (12.7),

$$\sum_{\omega \in P(3)} \varepsilon(\omega)\, c_{\omega(\lambda)\alpha} = \sum_{\omega \in P(3)} \varepsilon(\omega)\, n(m_1 - a_1(\omega),\ m_2 - a_2(\omega)) + \sum_{\omega \in P(3)} \varepsilon(\omega)\, (a_2(\omega) + q) = 0$$

(since $\sum_{\omega \in P(3)} \varepsilon(\omega)\, a_2(\omega) = -1 + 1 + 2 - 2 = 0$).

(ii) $m_1 < \lambda_2 + 2$. Then $m_1 - 1 < (A_1 A_2^2\lambda)_2$ and $m_1 - 2 < (A_1^2 A_2^2\lambda)_2$. Hence

$c_{A_1 A_2^2\lambda,\alpha} = n(m_1 - 1,\ m_2 - 2)$, and $c_{A_1^2 A_2^2\lambda,\alpha} = n(m_1 - 2,\ m_2 - 2)$.

Thus,

$$\sum_{\omega \in P(3)} \varepsilon(\omega)\, c_{\omega(\lambda)\alpha} = \sum_{\omega \in P(3)} \varepsilon(\omega)\, n(m_1 - a_1(\omega),\ m_2 - a_2(\omega)) +$$

$$\sum_{\omega \in P_3,\ a_2(\omega) \neq 2} \varepsilon(\omega)\, (a_2(\omega) + q) = 0 - (1 + q) + (1 + q) = 0.$$

This ends the proof of the theorem in the case when $\omega(\lambda) \trianglelefteq \alpha$, for all $\omega \in P(3)$. The proof in the other cases is similar. $\square$

(12.9) Remark: In (13.4) we construct a minimal projective resolution of k_λ

$$0 \to V_{A_1^2A_2^2\lambda} \xrightarrow{\varphi_3} V_{A_1^2A_2\lambda} \oplus V_{A_1A_2^2\lambda} \xrightarrow{\varphi_2} V_{A_1\lambda} \oplus V_{A_2\lambda} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0$$

when $\operatorname{char} k = 0$ and $\lambda \in \Lambda(3,r)$.

So for any $\alpha \in \Lambda(3,r)$, we obtain a short exact sequence of k-spaces

$$0 \to (V_{A_1^2A_2^2\lambda})^\alpha \to (V_{A_1^2A_2\lambda})^\alpha \oplus (V_{A_1A_2^2\lambda})^\alpha \to (V_{A_1\lambda})^\alpha \oplus (V_{A_2\lambda})^\alpha \to (V_\lambda)^\alpha \to (k_\lambda)^\alpha \to 0$$

(since $V^\alpha = \xi_\alpha V$ and ξ_α is an idempotent).

This implies that

$$\dim_k(k_\lambda)^\alpha = \dim_k (V_\lambda)^\alpha - \dim_k (V_{A_1\lambda})^\alpha - \dim_k (V_{A_2\lambda})^\alpha + \dim_k (V_{A_1^2A_2\lambda})^\alpha$$

$$+ \dim_k (V_{A_1A_2^2\lambda})^\alpha - \dim_k (V_{A_1^2A_2^2\lambda})^\alpha.$$

Or equivalently

(12.10) $$\delta_{\lambda,\alpha} = \dim_k(k_\lambda)^\alpha = \sum_{\omega \in P(3)} \varepsilon(\omega)c_{\omega(\lambda)\alpha} \quad \text{if } \operatorname{char} k = 0.$$

But, by (12.5), $c_{\omega(\lambda)\alpha}$ depends only on $\omega(\lambda)$ and α, and not on the field k. In fact, the equality $\sum_{\omega \in P(3)} \varepsilon(\omega)c_{\omega(\lambda)\alpha} = \delta_{\lambda,\alpha}$ may be rewritten in terms of the integers $n(m_1, m_2)$, which do not depend on k. So, from (12.10), we obtain an alternative proof of the theorem (12.6) (for any field k).

Theorems (11.19) and (12.6) lead us to conjecture the following

(12.11) Conjecture: For any $\alpha, \lambda \in \Lambda(n,r)$ there holds

$$\sum_{\omega \in P(n)} \varepsilon(\omega)\, c_{\omega(\lambda)\alpha} = \delta_{\lambda,\alpha}.$$

(12.12) Remarks: (i) Note that the conjecture is obvious if $n \leq 2$. Also, by (11.4), it holds for any $\alpha \in \Lambda(n,r)$ such that $\lambda \not\trianglerighteq \alpha$.

(ii) To support (12.11) we have, in addition to theorems (11.19) and (12.6), many examples in the case when $n = 4$.

(iii) Consider the ring $\mathbb{Z}[x_1,\ldots,x_n]$ of the polynomials in the independent variables $x_1,\ldots,x_n$ with coefficients in $\mathbb{Z}$. We remark here the analogy between (12.11) and the Jacobi-Trudi identity

$$s_\lambda(x_1,\ldots,x_n) = \sum_{\omega \in P(n)} \varepsilon(\omega)\, h_{\omega(\lambda)}(x_1,\ldots,x_n), \quad \text{all } \lambda \in \Lambda^+(n,r)$$

which expresses the Schur function $s_\lambda(x_1,\ldots,x_n)$, corresponding to λ, in terms of the complete symmetric functions $h_{\omega(\lambda)_\mu}(x_1,\ldots,x_n)$ (cf. [M; pg. 14, (3.1), (3.4')]).

Let $m = (m_1,\ldots,m_s)$, $q = (q_1,\ldots,q_s)$, where $m_1,\ldots,m_s$, $q_1,\ldots,q_s$ are non-negative integers and $s \geq 1$. Define

$$\tilde{\mathcal{B}}(m,q) = \{b = (b_{\mu\nu})_{\mu,\nu \in \underline{s}} \mid b_{\mu\nu} \text{ satisfy (12.13), all } \mu, \nu \in \underline{s}\}$$

and

$$\tilde{n}(m,q) = \# \tilde{\mathcal{B}}(m,q),$$

where

(12.13) (i) $b_{\mu\nu} \in \mathbb{Z}$; $b_{\mu\nu} \geq 0$ and $b_{\mu\nu} = 0$ if $\nu > \mu + 1$ $(\mu, \nu \in \underline{s})$.

(ii) $\sum_{\mu \in \underline{s}} b_{\mu\nu} = m_\nu \quad (\nu \in \underline{s})$, and $\sum_{\nu \in \underline{s}} b_{\mu\nu} = q_\mu \quad (\mu \in \underline{s})$.

Then, by (11.8),

$$c_{\lambda\alpha} = \hat{n}((\alpha_1-\lambda_1, \alpha_2,\ldots,\alpha_{n-1}), (\lambda_2, \lambda_3,\ldots,\lambda_{n-1}, \lambda_n-\alpha_n)),$$

for all $\alpha \in \Lambda(n,r)$ such that $\lambda \trianglelefteq \alpha$.

To end this section, we determine a generating function for the integers $\hat{n}(m,q)$. Take 2s indeterminantes $x_1,\ldots,x_s$, $y_1,\ldots,y_s$ $(s \geq 1)$, and define the series

$$\hat{Q}(x,y) = \hat{Q}(x_1,\ldots,x_s, y_1,\ldots,y_s) =$$

$$= \sum_{\substack{m_1,\ldots,m_s \geq 0 \\ q_1,\ldots,q_s \geq 0}} \hat{n}(m,q)\ x_1^{m_1} \ldots x_s^{m_s} y_1^{q_1} \ldots y_s^{q_s}\ .$$

(12.14) Lemma: With the notation above, we have

$$\hat{Q}(x,y) = \frac{1}{\hat{P}(x,y)},$$

where $\hat{P}(x,y) = \prod_{1 \leq \nu \leq \mu \leq s+1} (1 - x_\nu y_{\mu-1})$ (here $y_0 = x_{s+1} = 0$).

Proof: As $(1 - x_\nu y_{\mu-1})^{-1} = \sum_{b_{\mu-1,\nu} \geq 0} (x_\nu y_{\mu-1})^{b_{\mu-1,\nu}}$, we have

$$\frac{1}{\hat{P}(x,y)} = \sum_{\substack{b_{\mu,\nu}\ge 0\\ \nu<\mu+1;\ \mu,\nu\in s}} x_1^{\sum_{\tau=1}^{s} b_{\tau 1}} \cdots x_\nu^{\sum_{\tau=\nu-1}^{s} b_{\tau\nu}} \cdots x_s^{b_{s-1s}+b_{ss}}\, y_1^{b_{11}+b_{12}} \cdots y_\mu^{\sum_{\tau=1}^{\mu+1} b_{\mu\tau}} \cdots y_s^{\sum_{\tau=1}^{s} b_{s\tau}}.$$

Therefore, the coefficient of $x_1^{m_1} \ldots x_s^{m_s}\ y_1^{q_1} \ldots y_s^{q_s}$ in $\dfrac{1}{\hat{P}(x,y)}$ is

$\hat{n}((m_1,\ldots,m_s), (q_1,\ldots,q_s))$, i.e., $\dfrac{1}{\hat{P}(x,y)} = \hat{Q}(x,y)$. □

5. ON MINIMAL PROJECTIVE RESOLUTIONS OF k_λ

In Chapter 3 we produced 2-step minimal projective resolutions of k_λ, for any $\lambda \in \Lambda(n,r)$. This led us to consider the problem of constructing minimal projective resolutions of k_λ.

It is known that $S(B^+)$ has finite global dimension (cf. [G2]). Therefore minimal projective resolutions of k_λ are finite and, by (10.4), they depend on the characteristic p of k.

We now look at this problem for some particular cases of n and p.

§13. The case $n \le 3$ and char $k = 0$

In §13 we assume that k has characteristic zero.

Suppose first that $n = 1$. Then $\Lambda(1,r)$ has only one element, (r), and $k_{(r)} = V_{(r)}$ is a projective module.

Now suppose that $n = 2$ and let $\lambda \in \Lambda(2,r)$. By (10.4), there is the 2-step minimal projective resolution of k_λ[7]

$$V_{A_1\lambda} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0,$$

where $\operatorname{Im} \varphi_1 = \operatorname{rad} V_\lambda$.

But, from (9.4) and (11.1), we know that

$$\dim \operatorname{rad} V_\lambda = \dim V_\lambda - 1 = \lambda_2 = \dim V_{A_1\lambda}.$$

Hence, $\dim \ker \varphi_1 = \dim V_{A_1\lambda} - \dim \operatorname{rad} V_\lambda = 0$, and we have the following

[7] Recall that $\lambda(1,1) = A_1\lambda$ and $\ell = \ell(\lambda)$.

(13.1) Theorem: Let char $k = 0$ and $\lambda \in \Lambda(2,r)$. Then

$$0 \to V_{A_1\lambda} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0,$$

where φ_0 and φ_1 are as in (10.4), is a minimal projective resolution of k_λ.

It is now convenient to introduce a matrix notation for $S(B^+)$-maps.

Let $\alpha^{(1)},\dots,\alpha^{(s)}, \beta^{(1)},\dots,\beta^{(q)} \in \Lambda(n,r)$, and consider the matrix $F = (\eta_{a,b})_{a\in \underline{s}, b\in \underline{q}}$, where $\eta_{a,b} \in V_{\beta^{(b)}}$, all $a \in \underline{s}$, $b \in \underline{q}$.

Then we identify F with the $S(B^+)$-map $\varphi : \bigoplus_{a\in\underline{s}} V_{\alpha^{(a)}} \to \bigoplus_{b\in\underline{q}} V_{\beta^{(b)}}$, given by

$$\varphi(\sum_{a\in\underline{s}} \eta_a) = \sum_{a\in\underline{s}, b\in\underline{q}} \eta_a \eta_{a,b}, \text{ all } \eta_a \in V_{\alpha^{(a)}},\ a \in \underline{s}.$$

Suppose now that $n = 3$ and that $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda(3,r)$. Let

$$T^\lambda = \begin{array}{|lll|}\hline a_{11} & \cdots & a_{1\lambda_1} \\ \hline a_{21} & \cdots & a_{2\lambda_2} \\ \hline a_{31} & \cdots & a_{3\lambda_3} \\ \hline \end{array}$$

be the chosen basic λ-tableau, and define $h, j \in \Lambda(3,r)$, by the λ-tableaux

(13.2) $$T^\lambda_h = \begin{array}{|lllll|}\hline 1 & 1 & 1 & \dots & 1 \\ \hline 1 & 2 & 2 & \dots & 2 \\ \hline 1 & 3 & 3 & \dots & 3 \\ \hline \end{array} \quad ; \quad T^\lambda_j = \begin{array}{|lllll|}\hline 1 & 1 & 1 & \dots & 1 \\ \hline 2 & 2 & 2 & \dots & 2 \\ \hline 2 & 1 & 3 & \dots & 3 \\ \hline \end{array}$$

Let F_1, F_2, F_3 be the matrices defined as follows

(13.3) $F_1 = \begin{bmatrix} \xi_{\ell(A_1\lambda),\,\ell} \\ \xi_{\ell(A_2\lambda),\,\ell} \end{bmatrix}$;

$$F_2 = \begin{bmatrix} \xi_{\ell(A_1^2A_2\lambda),\ell(A_1\lambda)} + a\,\xi_{h,\ell(A_1\lambda)} & b\,\xi_{\ell(A_1^2A_2\lambda),\ell(A_2\lambda)} \\ -2\,\xi_{\ell(A_1A_2^2\lambda),\ell(A_1\lambda)} & \xi_{\ell(A_1A_2^2\lambda),\ell(A_2\lambda)} - \xi_{j,\ell(A_2\lambda)} \end{bmatrix};$$

$$F_3 = \begin{bmatrix} c\,\xi_{\ell(A_1^2A_2^2\lambda),\,\ell(A_1^2A_2\lambda)} & \xi_{\ell(A_1^2A_2^2\lambda),\,\ell(A_1A_2^2\lambda)} \end{bmatrix};$$

where $\begin{cases} a = 2, \quad b = -2, \quad c = 1; & \text{if } \lambda_2 \neq 1 \\ a = 0, \quad b = -1, \quad c = 2; & \text{if } \lambda_2 = 1. \end{cases}$

Then we have the following result.

(13.4) Theorem: Suppose that char $k = 0$ and that $\lambda \in \Lambda(3,r)$. Then the sequence below is a minimal projective resolution of k_λ

$$0 \to V_{A_1^2A_2^2\lambda} \xrightarrow{\varphi_3} V_{A_1^2A_2\lambda} \oplus V_{A_1A_2^2\lambda} \xrightarrow{\varphi_2} V_{A_1\lambda} \oplus V_{A_2\lambda} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0,$$

where $\varphi_0 = \aleph'_\lambda$ (cf. (10.1)), and $\varphi_1, \varphi_2, \varphi_3$ are defined by the matrices F_1, F_2, F_3 above.

(13.5) Remarks: (i) Note that $h = \ell(A_1^2A_2\lambda)\,(a_{22}\,a_{31})$ and $j = \ell(A_1A_2^2\lambda)\,(a_{21}\,a_{32})$.[8]

[8] For any $a, a' \in \underline{r}$, $(a\,a')$ denotes the transposition in P which interchanges a and a'.

(ii) According to the convention made in Chapter 4, some of the entries $\xi_{\ell(A_1^{m_1}A_2^{m_2}\lambda),\,\ell(A_1^{q_1}A_2^{q_2}\lambda)}$, $\xi_{h,\ell(A_1\lambda)}$, $\xi_{j,\ell(A_2\lambda)}$ of the matrices F_1, F_2, F_3 may be zero (when $\lambda_2 = 0$ or $\lambda_3 \leq 1$).

A similar remark applies to the modules $V_{A_1^{m_1}A_2^{m_2}\lambda}$.

Proof of (13.4) To simplify notation, in this proof we write $\ell(A_1^{m_1}A_2^{m_2})$ for $\ell(A_1^{m_1}A_2^{m_2}\lambda)$, and $p_{i',j',h'}$ for $[P_{i',h'} : P_{i',j',h'}]$ $(i', j', h' \in I(3,r))$.

Suppose $\lambda_2, \lambda_3 \geq 2$.

We have the λ-tableaux

(13.6) $T_\ell^\lambda = \begin{array}{|l|} \hline 1\ 1\ 1\ \ldots\ 1 \\ \hline 2\ 2\ 2\ \ldots\ \ 2 \\ \hline 3\ 3\ 3\ \ldots\ 3 \\ \hline \end{array}$; $\quad T_{\ell(A_1)}^\lambda = \begin{array}{|l|} \hline 1\ 1\ 1\ \ldots\ 1 \\ \hline 1\ 2\ 2\ \ldots\ \ 2 \\ \hline 3\ 3\ 3\ \ldots\ 3 \\ \hline \end{array}$;

$T_{\ell(A_2)}^\lambda = \begin{array}{|l|} \hline 1\ 1\ 1\ \ldots\ 1 \\ \hline 2\ 2\ 2\ \ldots\ \ 2 \\ \hline 2\ 3\ 3\ \ldots\ 3 \\ \hline \end{array}$; $\quad T_{\ell(A_1A_2^2)}^\lambda = \begin{array}{|l|} \hline 1\ 1\ 1\ \ldots\ 1 \\ \hline 1\ 2\ 2\ \ldots\ \ 2 \\ \hline 2\ 2\ 3\ \ldots\ 3 \\ \hline \end{array}$;

$T_{\ell(A_1^2A_2)}^\lambda = \begin{array}{|l|} \hline 1\ 1\ 1\ \ldots\ 1 \\ \hline 1\ 1\ 2\ \ldots\ \ 2 \\ \hline 2\ 3\ 3\ \ldots\ 3 \\ \hline \end{array}$; $\quad T_{\ell(A_1^2A_2^2)}^\lambda = \begin{array}{|l|} \hline 1\ 1\ 1\ \ldots\ 1 \\ \hline 1\ 1\ 2\ \ldots\ \ 2 \\ \hline 2\ 2\ 3\ \ldots\ 3 \\ \hline \end{array}$.

It is clear that the $S(B^+)$-map φ_1, defined by the matrix F_1, is the map defined in (10.2). So, by (10.4),

$$V_{A_1\lambda} \oplus V_{A_2\lambda} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0$$

is a 2-step minimal projective resolution of k_λ. We now explain how to obtain the matrix F_2.

By (11.5), the k-spaces $(V_{A_1^2A_2\lambda}, V_{A_1\lambda})_{S(B^+)}$, $(V_{A_1^2A_2\lambda}, V_{A_2\lambda})_{S(B^+)}$,

$(V_{A_1A_2^2\lambda}, V_{A_1\lambda})_{S(B^+)}$, $(V_{A_1A_2^2\lambda}, V_{A_2\lambda})_{S(B^+)}$, have k-bases

$$\{\cdot\,\xi_{\ell(A_1^2A_2),\,\ell(A_1)}, \cdot\,\xi_{h,\ell(A_1)}\}; \quad \{\cdot\,\xi_{\ell(A_1^2A_2),\,\ell(A_2)}\};$$

$$\{\cdot\,\xi_{\ell(A_1A_2^2),\,\ell(A_1)}\}; \quad \{\cdot\,\xi_{\ell(A_1A_2^2),\,\ell(A_2)}, \cdot\,\xi_{j,\ell(A_2)}\},$$

respectively.

Thus, $\varphi_2 \in (V_{A_1^2A_2\lambda} \oplus V_{A_1A_2^2\lambda}, V_{A_1\lambda} \oplus V_{A_2\lambda})_{S(B^+)}$, iff it is defined by a matrix of the type

$$F_2 = \begin{bmatrix} a_1\ \xi_{\ell(A_1^2A_2),\,\ell(A_1)} + a_2\ \xi_{h,\ell(A_1)} & a_3\ \xi_{\ell(A_1^2A_2),\,\ell(A_2)} \\ a_4\ \xi_{\ell(A_1A_2^2),\,\ell(A_1)} & a_5\ \xi_{\ell(A_1A_2^2),\,\ell(A_2)} + a_6\ \xi_{j,\ell(A_2)} \end{bmatrix},$$

$a_\mu \in k$, $\mu = 1,\dots,6$.

It is clear that $\varphi_1 \varphi_2 = 0$ iff $F_2 F_1 = 0$. So our next step is to determine those $a_\mu \in k$ ($\mu = 1,\dots,6$) for which $F_2 F_1 = 0$.

From the structure of the λ-tableaux (13.2) and (13.6), it is not hard to see that

$$P_{\ell(A_1)} = P_{\ell(A_1^2A_2),\ell(A_1)}\, P_{\ell(A_1),\ell} = P_{h,\ell(A_1)}\, P_{\ell(A_1),\ell}, \qquad \text{and}$$

$$P_{\ell(A_1^2A_2),\ell(A_1),\ell} = \frac{\lambda_1!\,(\lambda_2-2)!\,2!\,(\lambda_3-1)!}{\lambda_1!\,(\lambda_2-2)!\,(\lambda_3-1)!} = 2;$$

$$P_{h,\ell(A_1),\ell} = \frac{\lambda_1!\,(\lambda_2-1)!\,(\lambda_3-1)!}{\lambda_1!\,(\lambda_2-1)!\,(\lambda_3-1)!} = 1. \qquad \text{Hence,}$$

$$\xi_{\ell(A_1^2A_2),\ell(A_1)}\,\xi_{\ell(A_1),\ell} = 2\,\xi_{\ell(A_1^2A_2),\ell}, \text{ and } \xi_{h,\ell(A_1)}\,\xi_{\ell(A_1),\ell} = \xi_{h,\ell}.$$

Also, $P_{\ell(A_2)} = \dot{\bigcup}_{\mu=1,2} P_{\ell(A_1^2A_2),\ell(A_2)}\,\delta_\mu\, P_{\ell(A_2),\ell}$, where $\delta_1 = 1$ and $\delta_2 = (a_{22}\, a_{31})$.

But $\ell(A_1^2 A_2)\delta_2 = h$, and $P_{\ell(A_1^2A_2),\ell(A_2),\ell} = P_{h,\ell(A_2),\ell} = 1$. Thus

$$\xi_{\ell(A_1^2A_2),\ell(A_2)}\,\xi_{\ell(A_2),\ell} = \xi_{\ell(A_1^2A_2),\ell} + \xi_{h,\ell}.$$

Therefore, the first row of $F_2 F_1$ is

$$(a_1\,\xi_{\ell(A_1^2A_2),\ell(A_1)} + a_2\,\xi_{h,\ell(A_1)})\,\xi_{\ell(A_1),\ell} + a_3\,\xi_{\ell(A_1^2A_2),\ell(A_2)}\,\xi_{\ell(A_2),\ell} =$$

$$= (2a_1 + a_3)\,\xi_{\ell(A_1^2A_2),\ell} + (a_2 + a_3)\,\xi_{h,\ell}.$$

But, since $\xi_{\ell(A_1^2A_2),\ell}$ and $\xi_{h,\ell}$ are linearly independent elements of $S(B^+)$, this is zero iff

(13.7) $\qquad a_3 = -2a_1 \quad \text{and} \quad a_2 = 2a_1, \text{ any } a_1 \in k.$

Now we repeat this procedure for the second row of F_2F_1.

We have $P_{\ell(A_1)} = P_{\ell(A_1A_2^2),\,\ell(A_1)} P_{\ell(A_1),\ell}$, $\quad P_{\ell(A_2)} = P_{j,\ell(A_2)} P_{\ell(A_2),\ell}$, and

$P_{\ell(A_2)} = \dot{\bigcup}_{\mu=1,2} P_{\ell(A_1A_2^2),\,\ell(A_2)} \tau_\mu P_{\ell(A_2),\ell}$, where $\tau_1 = 1$, and $\tau_2 = (a_{21}\, a_{31})$. Also

$p_{\ell(A_1A_2^2),\,\ell(A_1),\,\ell} = p_{j,\,\ell(A_2),\ell} = 1$, and $p_{\ell(A_1A_2^2),\ell(A_2),\ell} = 2$.

Note that $\ell(A_1\, A_2^2)\tau_2\,(a_{31}\, a_{32}) = j$. Thus, $\xi_{\ell(A_1A_2^2)\tau_2,\,\ell} = \xi_{j,\ell}$, since $(a_{31}\, a_{32}) \in P_\ell$.

Therefore,

$$\xi_{\ell(A_1A_2^2),\,\ell(A_1)}\, \xi_{\ell(A_1),\ell} = \xi_{\ell(A_1A_2^2),\,\ell}; \quad \xi_{j,\ell(A_2)}\, \xi_{\ell(A_2),\ell} = \xi_{j,\ell};$$

$$\xi_{\ell(A_1A_2^2),\,\ell(A_2)}\, \xi_{\ell(A_2),\ell} = 2\xi_{\ell(A_1A_2^2),\,\ell} + \xi_{j,\ell}.$$

So, the second row of F_2F_1 is

$$a_4\, \xi_{\ell(A_1A_2^2),\,\ell(A_1)}\, \xi_{\ell(A_1),\ell} + (a_5\, \xi_{\ell(A_1A_2^2),\,\ell(A_2)} + a_6\, \xi_{j,\ell(A_2)})\, \xi_{\ell(A_2),\ell} =$$

$$= (a_4 + 2a_5)\, \xi_{\ell(A_1A_2^2),\,\ell} + (a_5 + a_6)\, \xi_{j,\ell}.$$

As, $\xi_{\ell(A_1A_2^2),\,\ell}$ and $\xi_{j,\ell}$ are linearly independent vectors, this is zero iff

(13.8) $\qquad a_4 = -2a_5$, and $a_6 = -a_5$, any $a_5 \in k$.

We make $a_1 = a_5 = -a_6 = 1$, and $-a_2 = a_3 = a_4 = -2$.

Then, F_2 is as defined in (13.3) and, since conditions (13.7) and (13.8) are satisfied, there holds

(13.9) $\qquad F_2F_1 = 0$, and $\operatorname{Im} \varphi_2 \subseteq \ker \varphi_1$.

Next we show that, in fact, we have dim $\operatorname{Im}\varphi_2 \geq \dim\ker\varphi_1$. Thus $\operatorname{Im}\varphi_2 = \ker\varphi_1$.

Let I_1, I_2, I_3 be the sets of all $i \in I(3,r)$ defined by the λ-tableaux (13.10), (13.11) and (13.12), respectively.

$$(13.10)\quad T_i^\lambda = \begin{array}{|l|}\hline 1\ 1\ \dots\ 1 \\ \hline \underbrace{1\ 1 \dots 1}_{b_{11}}\ 2\ \dots\ 2 \\ \hline 2\ \underbrace{1 \dots 1}_{b_{21}}\ \underbrace{2 \dots 2}_{b_{22}}\ \underbrace{3 \dots 3}_{b_{23}} \\ \hline \end{array}\ , \qquad \begin{array}{l} 2 \leq b_{11} \leq \lambda_2 \text{ and} \\ b_{21} + b_{22} + b_{23} = \lambda_3 - 1\ ; \end{array}$$

$$(13.11)\quad T_i^\lambda = \begin{array}{|l|}\hline 1\ 1\ \dots\ 1 \\ \hline 1\ 1\ \dots\ 1 \\ \hline \underbrace{1 .. 1}_{b_{21}}\ 2 \dots 2\ 3\ \dots\ 3 \\ \hline \end{array}\ , \qquad 1 \leq b_{21} \leq \lambda_3\ ;$$

$$(13.12)\quad T_i^\lambda = \begin{array}{|l|}\hline 1\ 1\ \dots\ 1 \\ \hline 1\ 2\ \dots\ 2 \\ \hline 2\ 2\ \underbrace{1 .. 1}_{b_{21}}\ \underbrace{2 .. 2}_{b_{22}}\ \underbrace{3 .. 3}_{b_{23}} \\ \hline \end{array}\ , \qquad b_{21}+b_{22}+b_{23} = \lambda_3-2.$$

(13.13) Remarks: (i) I_1, I_2 and I_3 are pairwise disjoint.

(ii) $I_1 \cup I_2 = I(A_1^2 A_2\lambda)$, and $I_3 \subseteq I(A_1 A_2^2\lambda)$. So, $\{\xi_{i,\ell(A_1^2A_2)} \mid i \in I_1 \cup I_2\}$ is a basis of $V_{A_1^2A_2\lambda}$, and $\{\xi_{i,\ell(A_1A_2^2)} \mid i \in I_3\}$ is contained in a basis of $V_{A_1A_2^2\lambda}$ (cf. (9.1) and (9.4)).

It is our aim to prove that

(13.14) The vectors $\varphi_2(\xi_{i,\ell(A_1^2A_2)})$, all $i \in I_1 \cup I_2$, and $\varphi_2(\xi_{i,\ell(A_1A_2^2)})$, all $i \in I_3$, are linearly independent.

From the definition of φ_2, we know that

(13.15) the components of $\varphi_2(\xi_{i,\ell(A_1^2A_2)})$ and $\varphi_2(\xi_{i,\ell(A_1A_2^2)})$ lying in $V_{A_2\lambda}$ are, respectively,

$$-2\, \xi_{i,\ell(A_1^2A_2)}\, \xi_{\ell(A_1^2A_2),\ell(A_2)} \quad , \text{if } \ i \in I_1 \cup I_2,$$

$$\xi_{i,\ell(A_1A_2^2)}\, \xi_{\ell(A_1A_2^2),\ell(A_2)} - \xi_{i,\ell(A_1A_2^2)}\, \xi_{j,\ell(A_2)}\, , \ \text{if } i \in I_3.$$

It is easy to see that

$$-2\, \xi_{i,\ell(A_1^2A_2)}\, \xi_{\ell(A_1^2A_2),\ell(A_2)} = \begin{cases} -b_{11}(b_{11}-1)\ \xi_{i,\ell(A_2)}\ ; & \text{if } i \in I_1 \\ -\lambda_2(\lambda_2+1)\ \ \xi_{i,\ell(A_2)}\ ; & \text{if } i \in I_2\ . \end{cases}$$

Also, if $i \in I_3$ has λ-tableau (13.12) then

$$\xi_{i,\ell(A_1A_2^2)}\, \xi_{\ell(A_1A_2^2),\ell(A_2)} = (b_{22}+1)\, \xi_{i,\ell(A_2)}.$$

To calculate $\xi_{i,\ell(A_1A_2^2)}\, \xi_{j,\ell(A_2)}$, we notice that $\ell(A_1 A_2^2)\ (a_{21}\, a_{32}) = j$. Thus, $\xi_{i,\ell(A_1A_2^2)} = \xi_{i'j}$, where $i' = i\ (a_{21}\, a_{32})$, i.e.,

(13.16) $$T_{i'}^{\lambda} = \begin{array}{|l|} \hline 1\ 1 \quad \ldots \quad 1 \\ \hline 2\ 2 \quad\quad \ldots \quad\quad 2 \\ \hline 2\ \underbrace{1 .. 1}_{b_{21}+1}\ \underbrace{2 .. 2}_{b_{22}}\ \underbrace{3 .. 3}_{b_{23}} \\ \hline \end{array}$$

Similarly to the previous cases, we have

$$\xi_{i,\ell(A_1 A_2^2)}\ \xi_{j,\ell(A_2)} = \xi_{i'j}\ \xi_{j,\ell(A_2)} = (b_{21}+1)\ \xi_{i',\ell(A_2)}.$$

Hence, by (13.15), we have

(3.17) (i) Let $i \in I_1 \cup I_2$, be defined by the λ-tableaux (13.10), or (13.11). Then, the component of $\varphi_2(\xi_{i,\ell(A_1^2 A_2)})$ lying in $V_{A_2\lambda}$ is

$$-b_{11}(b_{11}-1)\ \xi_{i,\ell(A_2)},\ \text{ if } i \in I_1\,;$$
$$-\lambda_2\,(\lambda_2+1)\ \xi_{i,\ell(A_2)},\ \text{ if } i \in I_2\,.$$

(ii) If $i \in I_3$ is defined by the λ-tableau (13.12) then the component of $\varphi_2(\xi_{i,\ell(A_1 A_2^2)})$ in $V_{A_2\lambda}$ is

$$(b_{22}+1)\ \xi_{i,\ell(A_2)} - (b_{21}+1)\ \xi_{i',\ell(A_2)}\,.$$

where i' is defined by the λ-tableau (13.16).

But $I_1 \cup I_2 = I(A_1^2\, A_2\lambda) \subseteq I(A_2\lambda)$,[9] and so the vectors $\xi_{i,\ell(A_2)}$ $(i \in I_1 \cup I_2)$ are linearly independent (since they are part of a basis of $V_{A_2\lambda}$).

Now, if we analyse $\xi_{i,\ell(A_2)}$ when T_i^λ is as in (13.12), we have

$$\xi_{i,\ell(A_2)} = \xi_{\bar{i},\ell(A_2)}.$$

where $\bar{i} \in A(3,r)$ is defined by the λ-tableau

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[9] This is a particular case of $I(A_1^m\,\alpha) \subseteq I(\alpha)$, for any $\alpha \in A(n,r)$, $0 \le m \le \alpha_2$.

$$T_1^{\lambda} = \begin{array}{|l|}\hline 1\ 1\ \ \ldots\ \ 1 \\ \hline 1\ 2\ \ \ \ \ldots\ \ \ \ \ 2 \\ \hline 2\ \underbrace{1\,..\,1}_{b_{21}}\ \underbrace{2\,..\,2}_{b_{22}+1}\ \underbrace{3..3}_{b_{23}} \\ \hline \end{array}\ , \qquad b_{21} + b_{22} + b_{23} = \lambda_3 - 1.$$

Clearly $\hat{\imath} \in I(A_2\lambda)$, but $\hat{\imath} \notin I(A_1^2A_2\lambda)$ (since $\hat{\imath}_{a_{22}} \neq 1$).

Hence, the vectors $\xi_{\iota,\ell(A_2)}$ $(\iota \in I_1 \cup I_2 \cup I_3)$ are linearly independent, and (13.14) follows from (13.17).

Now, as $I_1 \dot{\cup} I_2 = I(A_1^2 A_2\lambda)$, we have $\# I_1 + \# I_2 = \dim V_{A_1^2A_2\lambda} =$

$$= \frac{\lambda_2 \lambda_3 (\lambda_3 + 1)}{2} \qquad \text{(cf. (11.1))}.$$

Also, $\# I_3$ equals the number of distinct sequences of integers

$$\underbrace{1 \ldots 1}_{b_{21}} \quad \underbrace{2 \ldots 2}_{b_{22}} \quad \underbrace{3 \ldots 3}_{b_{23}} \quad ,$$

where $b_{2\mu} \geq 0$ $(\mu = 1,2,3)$ and $b_{21} + b_{22} + b_{23} = \lambda_3 - 2$.

Hence, $\# I_3 = \binom{\lambda_3}{2}$ and

$$\dim \operatorname{Im} \varphi_2 \geq \# I_1 + \# I_2 + \# I_3 = \tfrac{1}{2}[\lambda_2 \lambda_3(\lambda_3 + 1) + \lambda_3(\lambda_3 - 1)].$$

But,

$$\dim \ker \varphi_1 = \dim V_{A_1\lambda} + \dim V_{A_2\lambda} - \dim \operatorname{rad} V_\lambda =$$

$$= \tfrac{1}{2}[\lambda_2(\lambda_3 + 1)(\lambda_3 + 2) + (\lambda_2 + 2)(\lambda_3 + 1)\lambda_3 - (\lambda_2 + 1)(\lambda_3 + 1)(\lambda_3 + 2)$$

$$+ 2] = \tfrac{1}{2}[\lambda_2 \lambda_3(\lambda_3 + 1) + \lambda_3(\lambda_3 - 1)] \leq \dim \operatorname{Im} \varphi_2.$$

Therefore, $\operatorname{Im} \varphi_2 = \ker \varphi_1$ and we have the following result.

$$(13.18) \qquad V_{A_1^2A_2\lambda} \oplus V_{A_1A_2^2\lambda} \xrightarrow{\varphi_2} V_{A_1\lambda} \oplus V_{A_2\lambda} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0$$

is an exact sequence.

We now repeat this procedure to determine an $S(B^+)$-map $\varphi_3 : V_{A_1^2A_2^2\lambda} \to V_{A_1^2A_2\lambda} \oplus V_{A_1A_2^2\lambda}$, such that φ_3 is injective and $\operatorname{Im}\varphi_3 = \ker\varphi_2$.

This time we have

$$\dim (V_{A_1^2A_2^2\lambda},\, V_{A_1^2A_2\lambda})_{S(B^+)} = \dim (V_{A_1^2A_2^2\lambda},\, V_{A_1A_2^2\lambda})_{S(B^+)} = 1.$$

Hence, φ_3 is determined by a matrix of the type,

$$F = [\, b_1\, \xi_{\ell(A_1^2A_2^2),\, \ell(A_1^2A_2)} \qquad b_2\, \xi_{\ell(A_1^2A_2^2),\, \ell(A_1A_2^2)} \,], \qquad b_1, b_2 \in k.$$

Make $b_1 = b_2 = 1$. Then $F = F_3$ (as defined in (13.3)) and our next step is to show that $F_3F_2 = 0$.

The first column of F_3F_2 is

$$\xi_{\ell(A_1^2A_2^2),\, \ell(A_1^2A_2)} (\xi_{\ell(A_1^2A_2),\ell(A_1)} + 2\, \xi_{h,\ell(A_1)}) -$$
$$-2\xi_{\ell(A_1^2A_2^2),\, \ell(A_1A_2^2)}\, \xi_{\ell(A_1A_2^2),\, \ell(A_1)}\,.$$

Now, since $P_{\ell(A_1^2A_2)} = P_{\ell(A_1^2A_2^2),\ell(A_1^2A_2)}\, P_{\ell(A_1^2A_2),\ell(A_1)}$, and $p_{\ell(A_1^2A_2^2),\ell(A_1^2A_2),\ell(A_1)} = 2$, we have

$$\xi_{\ell(A_1^2A_2^2),\,\ell(A_1^2A_2)}\,\xi_{\ell(A_1^2A_2),\ell(A_1)} = 2\,\xi_{\ell(A_1^2A_2^2),\ell(A_1)}.$$

Also, $\ell(A_1^2A_2)\,(a_{22}\,a_{31}) = h$ and $\ell(A_1^2A_2^2)\,(a_{22}\,a_{31}) = c$, where

$$T_c^\lambda = \begin{array}{|l|}\hline 1\ 1\ \dots\ 1 \\ \hline 1\ 2\ \quad \dots \quad\ 2 \\ \hline 1\ 2\ 3\ \dots\ 3 \\ \hline \end{array}$$

Hence

$$\xi_{\ell(A_1^2A_2^2),\,\ell(A_1^2A_2)}\,\xi_{h,\ell(A_1)} = \xi_{c,h}\,\xi_{h,\ell(A_1)} = \xi_{c,\ell(A_1)}$$

(since $P_h = P_{c,h}\,P_{h,\ell(A_1)}$, and $p_{c,h,\ell(A_1)} = 1$). Finally, we have

$P_{\ell(A_1A_2^2)} = \dot{\bigcup}_{\mu=1,2} P_{\ell(A_1^2A_2^2),\ell(A_1A_2^2)}\ \delta_\mu\ P_{\ell(A_1A_2^2),\,\ell(A_1)}$, where $\delta_1 = 1$ and $\delta_2 = (a_{22}\,a_{31})$. Thus,

$$\xi_{\ell(A_1^2A_2^2),\ell(A_1A_2^2)}\,\xi_{\ell(A_1A_2^2),\,\ell(A_1)} = \xi_{\ell(A_1^2A_2^2),\,\ell(A_1)} + \xi_{c,\ell(A_1)}$$

(since $p_{\ell(A_1^2A_2^2),\ell(A_1A_2^2),\,\ell(A_1)} = p_{c,\ell(A_1A_2^2),\,\ell(A_1)} = 1$). Therefore, the first column of

$F_3\,F_2$ is

$$2\,\xi_{\ell(A_1^2A_2^2),\,\ell(A_1)} + 2\,\xi_{c,\ell(A_1)} - 2\,\xi_{\ell(A_1^2A_2^2),\,\ell(A_1)} - 2\,\xi_{c,\ell(A_1)} = 0.$$

Similar calculations show that

$$\xi_{\ell(A_1^2A_2^2),\,\ell(A_1^2A_2)}\,\xi_{\ell(A_1^2A_2),\,\ell(A_2)} = \xi_{\ell(A_1^2A_2^2),\,\ell(A_2)};$$

$$\xi_{\ell(A_1^2A_2^2),\,\ell(A_1A_2^2)}\,\xi_{\ell(A_1A_2^2),\ell(A_2)} = 2\,\xi_{\ell(A_1^2A_2^2),\,\ell(A_2)} + \xi_{d,\ell(A_2)};$$

$$\xi_{\ell(A_1^2A_2^2),\,\ell(A_1A_2^2)}\ \xi_{j,\ell(A_2)} = \xi_{d,\ell(A_2)};$$

where d is defined by the λ-tableau

$$T_d^\lambda = \begin{array}{|l|}\hline 1 \ \dots \ 1 \\ \hline 2\,1\,2 \ \dots \ 2 \\ \hline 2\,1\,3 \ \dots \ 3 \\ \hline \end{array}$$

Hence the second column of $F_3 F_2$ is

$$-2\,\xi_{\ell(A_1^2A_2^2),\,\ell(A_2)} + 2\,\xi_{\ell(A_1^2A_2^2),\,\ell(A_2)} + \xi_{d,\ell(A_2)} - \xi_{d,\ell(A_2)} = 0.$$

Therefore $F_3 F_2 = 0$.

Let φ_3 be defined by the matrix F_3. Then, $\varphi_2\,\varphi_3 = 0$ and next we show that

(13.19) $$\dim V_{A_1^2A_2^2\lambda} = \dim \operatorname{Im} \varphi_3 = \dim \ker \varphi_2 .$$

Thus, φ_3 is the map we were looking for.

$$V_{A_1^2A_2^2\lambda} \text{ has } k\text{-basis } \{\xi_{i,\ell(A_1^2A_2^2)} \mid i \in I(A_1^2A_2^2\lambda)\}.$$

By the structure of $T^\lambda_{\ell(A_1^2A_2^2)}$, we can see that $i \in I(A_1^2A_2^2\lambda)$ iff i is defined by one of the λ-tableaux (13.20), (13.21), or (13.22), below.

(13.20) $$T_i^\lambda = \begin{array}{|l|}\hline 1\,1 \ \dots \ 1 \\ \hline \underbrace{1\,1 \ \dots \ 1}_{b_{11}}\,2 \ \dots \ 2 \\ \hline 2\,2\,\underbrace{1 \dots 1}_{b_{21}}\,\underbrace{2 \dots 2}_{b_{22}}\,\underbrace{3 \dots 3}_{b_{23}} \\ \hline \end{array}, \qquad \begin{array}{l} 2 \le b_{11} \le \lambda_2 \text{, and} \\ b_{21} + b_{22} + b_{23} = \lambda_3 - 2; \end{array}$$

$$(13.21)\quad T_i^\lambda = \begin{array}{|l|}\hline 1\ 1\ \ \cdots\ \ 1 \\ \hline 1\ 1\ \ \ \cdots\ \ \ \ 1 \\ \hline 1\ 2\ \underbrace{1..1}_{b_{21}}\ \underbrace{2..2}_{b_{22}}\ \underbrace{3..3}_{b_{23}} \\ \hline \end{array} \qquad b_{21}+b_{22}+b_{23} = \lambda_3-2\,;$$

$$(13.22)\quad T_i^\lambda = \begin{array}{|l|}\hline 1\ 1\ \ \cdots\ \ 1 \\ \hline 1\ 1\ \ \ \cdots\ \ \ \ 1 \\ \hline 1\ 1\ \underbrace{1..1}_{b_{21}}\ \underbrace{2..2}_{b_{22}}\ \underbrace{3..3}_{b_{23}} \\ \hline \end{array} \qquad b_{21}+b_{22}+b_{23} = \lambda_3-2\,.$$

It follows from the definition of φ_3, that the component of $\varphi_3(\xi_{i,\ell(A_1^2A_2^2)})$ lying in $V_{A_1A_2^2\lambda}$ is $\xi_{i,\ell(A_1^2A_2^2)}\,\xi_{\ell(A_1^2A_2^2),\,\ell(A_1A_2^2)}$ $\quad (i \in I(A_1^2\,A_2^2\,\lambda))$.

Calculating this product we obtain

$$\xi_{i,\ell(A_1^2A_2^2)}\,\xi_{\ell(A_1^2A_2^2),\,\ell(A_1A_2^2)} = \begin{cases} (b_{11}-1)\ \xi_{i,\ell(A_1A_2^2)} & ;\ \text{if } T_i^\lambda \text{ is } (13.20) \\ \lambda_2\,\xi_{i,\ell(A_1A_2^2)} & ;\ \text{if } T_i^\lambda \text{ is } (13.21) \\ (\lambda_2+1)\,\xi_{i,\ell(A_1A_2^2)} & ;\ \text{if } T_i^\lambda \text{ if } (13.22) \end{cases}$$

But, since $I(A_1^2\,A_2^2\,\lambda) \subseteq I(A_1\,A_2^2\,\lambda)$, $\{\xi_{i,\ell(A_1A_2^2)} \mid i \in I(A_1^2\,A_2^2\,\lambda)\}$ is contained in a basis of $V_{A_1A_2^2\lambda}$. Hence, $\varphi_3(\xi_{i,\ell(A_1^2A_2^2)})$, for all $i \in I(A_1^2\,A_2^2\,\lambda)$, are linearly independent vectors and

(13.23) $\{\varphi_3(\xi_{i,\ell(A_1^2A_2^2)}) \mid i \in I(A_1^2A_2^2\lambda)\}$ is a basis of $\operatorname{Im}\varphi_3$.

Therefore, φ_3 is injective and $\dim \operatorname{Im}\varphi_3 = \dim V_{A_1^2A_2^2\lambda} = \dfrac{(\lambda_2+1)(\lambda_3-1)\lambda_3}{2}$.

Now, as $\dim \operatorname{Im}\varphi_2 = \frac{1}{2}[\lambda_2\lambda_3(\lambda_3+1) + \lambda_3(\lambda_3-1)]$ and

$\dim V_{A_1^2A_2\lambda} + \dim V_{A_1A_2^2\lambda} = \frac{1}{2}[\lambda_2\lambda_3(\lambda_3+1) + (\lambda_2+2)\lambda_3(\lambda_3-1)]$, we have

$$\dim \ker \varphi_2 = \frac{(\lambda_2+1)(\lambda_3-1)\lambda_3}{2} = \dim \operatorname{Im}\varphi_3.$$

Hence (13.19). This completes the proof of the following result.

(13.24) If $\lambda_2, \lambda_3 \geq 2$, the sequence below is a projective resolution of k_λ

$$0 \to V_{A_1^2A_2^2\lambda} \xrightarrow{\varphi_3} V_{A_1^2A_2\lambda} \oplus V_{A_1A_2^2\lambda} \xrightarrow{\varphi_2} V_{A_1\lambda} \oplus V_{A_2\lambda} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0.$$

Now we know, from (10.4), that $\ker\varphi_0 = \operatorname{rad} V_\lambda$, and $\ker\varphi_1 \subseteq \operatorname{rad}(V_{A_1\lambda} \oplus V_{A_2\lambda})$. So, to prove that the projective resolution in (13.24) is minimal it is enough to show that

(13.25) $$\ker\varphi_2 \subseteq \operatorname{rad} V_{A_1^2A_2\lambda} \oplus \operatorname{rad} V_{A_1A_2^2\lambda}.$$

By (13.23) and (13.24), $\ker\varphi_2$ has k-basis $\{\varphi_3(\xi_{i,\ell(A_1^2A_2^2)}) \mid i \in I(A_1^2A_2^2\lambda)\}$.

So, (13.25) is equivalent to

(13.26) $\varphi_3(\xi_{i,\ell(A_1^2A_2^2)}) \in \operatorname{rad} V_{A_1^2A_2\lambda} \oplus \operatorname{rad} V_{A_1A_2^2\lambda}$, all $i \in I(A_1^2A_2^2\lambda)$.

Let $i \in I(A_1^2A_2^2\lambda)$. Then, $i \le \ell(A_1^2A_2^2) < \ell(A_1^2A_2), \ell(A_1A_2^2)$. Thus,

(13.27) $i\delta \le \ell(A_1^2A_2^2)\delta = \ell(A_1^2A_2^2) < \ell(A_1^2A_2), \ell(A_1A_2^2)$, all $\delta \in P_{\ell(A_1^2A_2^2)}$.

But,

$$\varphi_3(\xi_{i,\ell(A_1^2A_2^2)}) = \xi_{i,\ell(A_1^2A_2^2)}\,\xi_{\ell(A_1^2A_2^2),\,\ell(A_1^2A_2)} +$$

$$+\ \xi_{i,\ell(A_1^2A_2^2)}\,\xi_{\ell(A_1^2A_2^2),\,\ell(A_1A_2^2)} = \sum_{\delta} a_\delta\,\xi_{i\delta,\ell(A_1^2A_2)} + \sum_{\delta'} a'_{\delta'}\,\xi_{i\delta',\,\ell(A_1A_2^2)},$$

where the sums are over subsets, $\{\delta\}$ and $\{\delta'\}$, of $P_{\ell(A_1^2A_2^2)}$, and $a_\delta, a'_{\delta'} \in \mathbf{Z}$.

And so, (13.26) follows from (13.27).[10]

With (13.24) and (13.25) we conclude the proof of the theorem (13.4) in the case $\lambda_2, \lambda_3 \ge 2$. The proof of the other cases is similar. □

§14. The case $n = 2$ and $\operatorname{char} k = p$

When k is a field of positive characteristic, the construction of minimal projective resolutions of k_λ becomes much more difficult than when characteristic of k is zero.

Now we shall give some results on this problem when $n = 2$.

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[10] We recall that, if $\alpha \in \Lambda(n,r)$ then $\xi_{i,\ell(\alpha)} \in \operatorname{rad} V_\alpha$, for all $i < \ell(\alpha)$ (cf. (9.4)).

Let $\lambda = (r - a, a)$ be an arbitrarily chosen element of $\Lambda(2,r)$, and write

$\lambda(1,m) = A_1^m \lambda$, $\ell(m) = \ell(A_1^m \lambda)$ $(0 \le m \le a)$.

Suppose char $k = p\ (\neq 0)$ and let

$$a = a_0 + a_1 p + \dots + a_d p^d, \text{ where } a_\mu \in \mathbb{Z}, 0 \le a_\mu < p \ (\mu = 0,\dots,d),\ a_d \neq 0.$$

Define an $S(B^+)$-map

$$\varphi_2 : \bigoplus_{m=1}^{d} (V_{\lambda(1,p^m)} \oplus V_{\lambda(1,1+p^m)} \oplus V_{\lambda(1,p+p^m)} \oplus \dots \oplus V_{\lambda(1,p^{m-1}+p^m)}) \to \bigoplus_{m=0}^{d} V_{\lambda(1,p^m)}$$

by

$$\varphi_2(\xi) = \begin{cases} \xi\, \xi_{\ell(p^m),\, \ell(p^{m-1})} & ; \text{ if } \xi \in V_{\lambda(1,p^m)} \\ -\xi\, \xi_{\ell(p^q+p^m),\, \ell(p^q)} + \xi\, \xi_{\ell(p^q+p^m),\ell(p^m)} & ; \text{ if } \xi \in V_{\lambda(1,p^q+p^m)} \quad (m \in \underline{d}, 0 \le q < m). \end{cases}$$

Then, if φ_0 and φ_1 are the maps defined in (10.1) and (10.2), respectively, we have the following result.

(14.1) Theorem: With the notation above,

$$\bigoplus_{m=1}^{d} (V_{\lambda(1,p^m)} \oplus V_{\lambda(1,1+p^m)} \oplus V_{\lambda(1,p+p^m)} \oplus \dots \oplus V_{\lambda(1,p^{m-1}+p^m)}) \to$$

$$\xrightarrow{\varphi_2} \bigoplus_{m=0}^{d} V_{\lambda(1,p^m)} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0,$$

are the first three terms of a minimal projective resolution of k_λ.

In the proof of (14.1) we will make use of the following two lemmas, which are easy consequences of (2.7) and (9.12), respectively.

(14.2) Lemma: Suppose b, c, d are non-negative integers satisfying $d \le c \le b \le a$, and consider the elements $\ell(b), \ell(c), \ell(d)$ of $I(2,r)$. Then, $\ell(b) \le \ell(c) \le \ell(d)$ and

$$\xi_{\ell(b),\ell(c)}\,\xi_{\ell(c),\ell(d)} = \binom{b-d}{b-c}\,\xi_{\ell(b),\ell(d)}\,.$$

(14.3) Lemma: Suppose $b = b_0 + b_1 p + \ldots + b_s p^s$, where $b_\mu \in \mathbf{Z}$, $0 \le b_\mu < p$ $(\mu = 0,\ldots,s)$ $b_s \neq 0$, and q, m are non-negative integers satisfying $q < m \le s$. Then

(i) $\quad p \nmid \binom{b - p^q}{b - p^m}$ iff $b_t = 0$, for all $q \le t < m$;

(ii) $\quad$ for $b \ge p^q + p^m$, $\quad p \nmid \binom{b - p^m}{b - p^q - p^m}$ iff $b_q \neq 0$.

Proof of (14.1): Assume the hypotheses of (14.1). Then, from (10.4), we know that

$$\bigoplus_{m=0}^{d} V_{\lambda(1,p^m)} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0$$

is exact and minimal. Thus, to prove the theorem we only need to show that

(14.4) (i) $\operatorname{Im}\varphi_2 = \ker\varphi_1$;

(ii) $\quad \ker\varphi_2 \subseteq \operatorname{rad} \bigoplus_{m \in \underline{d}} (V_{\lambda(1,p^m)} \oplus V_{\lambda(1,1+p^m)} \oplus \ldots \oplus V_{\lambda(1,p^{m-1}+p^m)})$.

We start by proving (14.4)(i).

From the definition of φ_2, we can see that $\varphi_1 \varphi_2 = 0$ iff

$$\varphi_1(\xi_{\ell(p^m),\ell(p^{m-1})}) = 0, \text{ and } \varphi_1(-\xi_{\ell(p^q+p^m),\ell(p^q)} + \xi_{\ell(p^q+p^m),\ell(p^m)}) = 0,$$

for all $m \in \underline{d}$, $0 \le q \le m-1$. But, by (14.2)

$$\varphi_1(\xi_{\ell(p^m),\ell(p^{m-1})}) = \xi_{\ell(p^m),\ell(p^{m-1})}\,\xi_{\ell(p^{m-1}),\ell} = \binom{p^m}{p^{m-1}}\xi_{\ell(p^m),\ell} = 0,$$

since $\binom{p^m}{p^{m-1}} \equiv 0 \pmod p$, and similarly,

$$\varphi_1(-\xi_{\ell(p^q+p^m),\ell(p^q)} + \xi_{\ell(p^q+p^m),\ell(p^m)}) = \left[-\binom{p^q+p^m}{p^m} + \binom{p^q+p^m}{p^q}\right]\xi_{\ell(p^q+p^m),\ell} = 0.$$

Hence we have $\varphi_1 \varphi_2 = 0$ and so $\operatorname{Im}\varphi_2 \subseteq \ker \varphi_1$.

Now let $m \in \underline{d}$ be fixed and consider any integer b such that $p^m \le b \le a$. Write

(14.5) $\qquad b = b_0 + b_1 p + \dots + b_s p^s \quad (b_\mu \in \mathbb{Z}, 0 \le b_\mu < p \ (\mu \in \underline{s}),\ b_s \ne 0).$

Suppose first that

(14.6) $\qquad b_0 = b_1 = \dots = b_{m-1} = 0.$

Then, as $\xi_{\ell(b),\ell(p^m)} \in V_{\lambda(1,p^m)}$, we have

(14.7) $$\varphi_2(\xi_{\ell(b),\ell(p^m)}) = \xi_{\ell(b),\ell(p^m)}\,\xi_{\ell(p^m),\ell(p^{m-1})}) =$$

$$= \binom{b-p^{m-1}}{b-p^m}\xi_{\ell(b),\ell(p^{m-1})}, \qquad \text{and } p \nmid \binom{b-p^{m-1}}{b-p^m} \quad \text{(cf. (14.3)(i)).}$$

Now suppose that

(14.8) $b_t \neq 0$, for some $0 \leq t \leq m-1$, and q is the smallest such t.

Then $b \geq p^q + p^m$ and $\xi_{\ell(b),\ell(p^q+p^m)} \in V_{\lambda(1,p^q+p^m)}$. So from the definition of φ_2 and (14.2), we have

$$\textbf{(14.9)} \quad \varphi_2\,(\xi_{\ell(b),\ell(p^q+p^m)}) = -\begin{pmatrix} b-q^q \\ b-q^q-p^m \end{pmatrix} \xi_{\ell(b),\ell(p^q)} + \begin{pmatrix} b-p^m \\ b-p^q-p^m \end{pmatrix} \xi_{\ell(b),\ell(p^m)},$$

and $p \nmid \begin{pmatrix} b-p^m \\ b-p^q-p^m \end{pmatrix}$ (since $b_q \neq 0$ (cf. (14.3)(ii)).

Write $f(m,b) = \begin{cases} p^m & \text{, if } b \text{ satisfies (14.6)} \\ p^q + p^m, & \text{if } b \text{ satisfies (14.8)} \end{cases}$.

Then, our next step is to prove that the set $\{\varphi_2(\xi_{\ell(b),\,\ell(f(m,b))}) \mid m \in \underline{d},\ p^m \leq b \leq a\}$ is linearly independent and so

$$\textbf{(14.10)} \qquad \dim \operatorname{Im} \varphi_2 \geq \sum_{m=1}^{d} (a - p^m + 1).$$

Suppose we have

$$\textbf{(14.11)} \qquad \sum_{m=1}^{d} \sum_{b=p^m}^{a} \gamma_{m,b}\, \varphi_2\,(\xi_{\ell(b),\ell(f(m,b))}) = 0, \text{ for some } \gamma_{m\,b} \in k.$$

We know from (14.7) and (14.9) that the component of (14.11) lying in $V_{\lambda(1,p^d)}$ is

$$\sum_b \gamma_{d,b} \binom{b-p^d}{b-p^q-p^d} \xi_{\ell(b),\ell(p^d)},$$

where the sum is over all $b \ge p^d$ satisfying (14.8) with $m = d$.

But, under these conditions, $p \nmid \binom{b-p^d}{b-p^q-p^d}$. Also, as the vectors $\xi_{\ell(b),\ell(p^d)}$ $(p^d \le b \le a)$ form a basis of $V_{\lambda(1,p^d)}$, they are linearly independent. So, we must have

(14.12) $\gamma_{d,b} = 0$, for all $b \ge p^d$ such that $b_t \ne 0$, for some $0 \le t < d$.

Hence, (14.11) and (14.12) imply (14.13), below

$$\textbf{(14.13)}\quad \sum_{m=1}^{d-1} \sum_{b=p^m}^{a} \gamma_{m,b}\, \varphi_2(\xi_{\ell(b),\ell(f(m,b))}) + \sum_{\substack{b=p^d\\ b_0=\cdots=b_{d-1}=0}}^{a} \gamma_{d,b}\, \varphi_2(\xi_{\ell(b),\ell(p^d)}) = 0.$$

Now, the component of (14.13) lying in $V_{\lambda(1,p^{d-1})}$ is

$$\textbf{(14.14)}\quad \sum_b \gamma_{d-1,b} \binom{b-p^{d-1}}{b-p^q-p^{d-1}} \xi_{\ell(b),\ell(p^{d-1})} + \sum_b \gamma_{d,b} \binom{b-p^{d-1}}{b-p^d} \xi_{\ell(b),\ell(p^{d-1})},$$

where the first sum is taken over all $b \ge p^{d-1}$ such that b satisfies (14.8) with $m = d-1$, and the second sum is over all $b \ge p^d$ satisfying $b_0 = \dots = b_{d-1} = 0$.

It is then clear that all vectors $\xi_{\ell(b),\ell(p^{d-1})}$, involved in (14.14), are linearly independent and, since $p \nmid \begin{pmatrix} b - p^{d-1} \\ b - p^q - p^{d-1} \end{pmatrix}$ in the first case, and $p \nmid \begin{pmatrix} b - p^{d-1} \\ b - p^d \end{pmatrix}$ in the second case, (14.13) and (14.14) imply

(14.15) $\gamma_{d,b} = 0$, for all $b \geq p^d$ such that $b_0 = \ldots = b_{d-1} = 0$. Also, $\gamma_{b,d-1} = 0$, for all $b \geq p^{d-1}$ satisfying $b_t \neq 0$, for some $0 \leq t < d-1$.

Proceeding like this, we can see that (14.11) implies $\gamma_{m,b} = 0$, for all $m \in \underline{d}$ and $p^m \leq b \leq a$, and so (14.10) holds. But

$$\dim \ker \varphi_1 = 1 - \dim V_\lambda + \sum_{m=0}^{d} \dim V_{\lambda(1,p^m)} = \sum_{m=1}^{d} (a - p^m + 1).$$

Thus, we must have $\operatorname{Im} \varphi_2 = \ker \varphi_1$.

Now we will turn our attention to (14.4)(ii).

Let

$$\xi = \sum_{m=1}^{d} \sum_{b=p^m}^{a} \gamma_{m,b}\, \xi_{\ell(b),\ell(p^m)} + \sum_{m=1}^{d} \sum_{q=0}^{m-1} \sum_{b=b^q+p^m}^{a} \gamma_{q,m,b}\, \xi_{\ell(b),\ell(p^q+p^m)},$$

where $\gamma_{m,b}, \gamma_{q,m,b} \in k$.

Suppose $\xi \notin \bigoplus_{m=1}^{d} (\operatorname{rad} V_{\lambda(1,p^m)} \oplus \operatorname{rad} V_{\lambda(1,1+p^m)} \oplus \ldots \oplus \operatorname{rad} V_{\lambda(1,p^{m-1}+p^m)})$.

Then, we know from (9.4), that $\gamma_{m,p^m} \neq 0$ or $\gamma_{q,m,p^q+p^m} \neq 0$, for some $m \in \underline{d}$ and some $0 \leq q < m$. Calculating $\varphi_2(\xi)$ we obtain

$$(14.16) \qquad \varphi_2(\xi) = \sum_{m=1}^{d} \sum_{b=p^m}^{a} \gamma_{m,b} \binom{b-p^{m-1}}{b-p^m} \xi_{\ell(b),\ell(p^{m-1})} +$$

$$+ \sum_{m=1}^{d} \sum_{q=0}^{m-1} \sum_{b=b^q+p^m}^{a} \gamma_{q,m,b} \left[- \binom{b-q^q}{b-p^q-p^m} \xi_{\ell(b),\ell(p^q)} + \binom{b-p^m}{b-p^q-p^m} \xi_{\ell(b),\ell(p^m)}\right]$$

and, for any $m \in \underline{d}$, the coefficient of $\xi_{\ell(p^m),\ell(p^{m-1})}$ in this expression is

$$\gamma_{m,p^m} + \sum_{q=0}^{m-2} \gamma_{q,m-1,p^m} \binom{p^m - p^{m-1}}{p^m - p^q - p^{m-1}}.$$

But, for any $0 \le q < m-1$, $\binom{p^m - p^{m-1}}{p^m - p^q - p^{m-1}} = 0 \pmod p$ (cf. (14.3)(ii)).

Hence, the coefficient of $\xi_{\ell(p^m),\ell(p^{m-1})}$ in (14.16) is γ_{m,p^m}. Thus if $\gamma_{m,p^m} \neq 0$, for some $m \in \underline{d}$, we must have $\varphi_2(\xi) \neq 0$.

In a similar way it can be seen that $\gamma_{q,m,p^q+p^m} \neq 0$, for some $m \in \underline{d}$ and some $0 \le q < m$, implies $\varphi_2(\xi) \neq 0$.

Hence (14.4)(ii) holds, and this ends the proof of the theorem. □

Unfortunately we are not able to construct the whole minimal projective resolution of k_λ when $n = 2$ and char $k = p$ ($\neq 0$). In our attempts to solve this problem we worked out some examples, which we shall now describe. We don't explain the calculations involved in the construction of these examples, since they are routine.

(14.17) Examples: Let φ_0, φ_1 and φ_2 be as in (14.1). Then the sequences below are minimal projective resolutions of k_λ.

(i) $\lambda = (r-6, 6)$ and char $k = 3$:

$$0 \to V_{\lambda(1,6)} \xrightarrow{\varphi_4} V_{\lambda(1,4)} \oplus V_{\lambda(1,6)} \xrightarrow{\varphi_3} V_{\lambda(1,3)} \oplus V_{\lambda(1,4)}$$

$$\xrightarrow{\varphi_2} V_{\lambda(1,1)} \oplus V_{\lambda(1,3)} \xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0,$$

where φ_3 and φ_4 are defined by the matrices

$$F_3 = \begin{bmatrix} \xi_{\ell(4),\ell(3)} & 0 \\ \xi_{\ell(6),\ell(3)} & \xi_{\ell(6),\ell(4)} \end{bmatrix}; \qquad F_4 = \begin{bmatrix} \xi_{\ell(6),\ell(4)} & 0 \end{bmatrix}$$

(ii) $\lambda = (r - 11, 11)$ and char $k = 3$:

$$0 \to V_{\lambda(1,10)} \xrightarrow{\varphi_7} V_{\lambda(1,9)} \oplus V_{\lambda(1,10)} \xrightarrow{\varphi_6} V_{\lambda(1,7)} \oplus V_{\lambda(1,9)} \xrightarrow{\varphi_5}$$

$$V_{\lambda(1,6)} \oplus V_{\lambda(1,7)} \xrightarrow{\varphi_4} V_{\lambda(1,4)} \oplus V_{\lambda(1,6)} \oplus V_{\lambda(1,10)} \to \xrightarrow{\varphi_3}$$

$$V_{\lambda(1,3)} \oplus V_{\lambda(1,4)} \oplus V_{\lambda(1,9)} \oplus V_{\lambda(1,10)} \xrightarrow{\varphi_2} V_{\lambda(1,1)} \oplus V_{\lambda(1,3)} \oplus V_{\lambda(1,9)} \xrightarrow{\varphi_1}$$

$$V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0,$$

where φ_μ is defined by matrix F_μ,

$$F_3 = \begin{bmatrix} \xi_{\ell(4),\ell(3)} & 0 & 0 & 0 \\ \xi_{\ell(6),\ell(3)} & \xi_{\ell(6),\ell(4)} & 0 & 0 \\ \xi_{\ell(10),\ell(3)} & \xi_{\ell(10),\ell(4)} & 2\xi_{\ell(10),\ell(9)} & 0 \end{bmatrix};$$

$$F_4 = \begin{bmatrix} \xi_{\ell(6),\ell(4)} & 0 & 0 \\ \xi_{\ell(7),\ell(4)} & 2\xi_{\ell(7),\ell(6)} & 0 \end{bmatrix}; \qquad F_5 = \begin{bmatrix} \xi_{\ell(7),\ell(6)} & 0 \\ \xi_{\ell(9),\ell(6)} & 2\xi_{\ell(9),\ell(7)} \end{bmatrix};$$

$$F_6 = \begin{bmatrix} \xi_{\ell(9),\ell(7)} & 0 \\ \xi_{\ell(10),\ell(7)} & 2\xi_{\ell(10),\ell(9)} \end{bmatrix}; \qquad F_7 = \begin{bmatrix} \xi_{\ell(10),\ell(9)} & 0 \end{bmatrix}.$$

(iii) $\lambda = (r-5, 5)$ and char $k = 2$:

$$0 \to V_{\lambda(1,5)} \xrightarrow{\varphi_5} V_{\lambda(1,4)} \oplus V_{\lambda(1,5)} \xrightarrow{\varphi_4} V_{\lambda(1,3)} \oplus V_{\lambda(1,4)} \oplus V_{\lambda(1,5)}$$

$$\xrightarrow{\varphi_3} V_{\lambda(1,2)} \oplus V_{\lambda(1,3)} \oplus V_{\lambda(1,4)} \oplus V_{\lambda(1,5)} \xrightarrow{\varphi_2} V_{\lambda(1,1)} \oplus V_{\lambda(1,2)} \oplus V_{\lambda(1,4)}$$

$$\xrightarrow{\varphi_1} V_\lambda \xrightarrow{\varphi_0} k_\lambda \to 0,$$

where φ_μ is defined by the matrix F_μ.

$$F_3 = \begin{bmatrix} \xi_{\ell(3),\ell(2)} & 0 & 0 & 0 \\ \xi_{\ell(4),\ell(2)} & \xi_{\ell(4),\ell(3)} & 0 & 0 \\ \xi_{\ell(5),\ell(2)} & \xi_{\ell(5),\ell(3)} & \xi_{\ell(5),\ell(4)} & 0 \end{bmatrix};$$

$$F_4 = \begin{bmatrix} \xi_{\ell(4),\ell(3)} & 0 & 0 \\ \xi_{\ell(5),\ell(3)} & \xi_{\ell(5),\ell(4)} & 0 \end{bmatrix}; \qquad F_5 = \begin{bmatrix} \xi_{\ell(5),\ell(4)} & 0 \end{bmatrix}.$$

§15 An application to S(G)

Consider the functors

$$F = S(G) \otimes_{S(B^+)} \cdot : \text{mod } S(B^+) \to \text{mod } S(G)$$

and

$$F' = \text{Hom}_{S(B^-)}(S(G), \cdot) : \text{mod } S(B^-) \to \text{mod } S(G).$$

In [W] it is proved the following

(15.1) Theorem: (D. Woodcock) Let $\alpha \in \Lambda^+(n,r)$. Then

$R^1F'(k_\alpha^-) = \mathrm{Ext}^1_{S(B^-)}(S(G), k_\alpha^-) = 0.$

We now apply this result to the sequences in theorems (13.1) and (14.1).

For the rest of this section we will fix $n = 2$ and use the notation of §14. However we will not demand $p = \mathrm{char}\, k$ to be different from zero.

Consider $\lambda = (\lambda_1, \lambda_2) \in \Lambda^+(2,r)$. If $p \neq 0$ write

$$\lambda_2 = a_0 + a_1 p + \ldots + a_d p^d, \text{ where } a_\mu \in \mathbb{Z},\ 0 \le a_\mu < p \quad (\mu = 0,\ldots,d),\ a_d \neq 0.$$

Let φ_0, φ_1 and φ_2 be the maps defined in §14 ($\varphi_2 = 0$ if $p = 0$), and let

$$f_{\alpha,1} : F(V_\alpha) \to S(G)\xi_{\alpha\alpha}, \quad \text{all } \alpha \in \Lambda(2,r)$$

be the $S(G)$-isomorphisms defined in (10.5). Define $S(G)$-maps ψ_0, ψ_1 and ψ_2 as follows

$$\psi_0 = F(\varphi_0)\, f^{-1}_{\lambda,1}; \quad \psi_1 = f_{\lambda,1}\, F(\varphi_1)\,\Big(\coprod_{m=0}^{d} f^{-1}_{\lambda(1,p^m),1}\Big); \text{ and}$$

$$\psi_2 = \Big(\coprod_{m=0}^{d} f_{\lambda(1,p^m),1}\Big)\, F(\varphi_2)\,\Big(\coprod_{m=1}^{d} (f^{-1}_{\lambda(1,p^m),1} \amalg \cdots \amalg f^{-1}_{\lambda(1,p^{m-1}+p^m),1}\Big)$$

(15.2) Theorem: Let $\lambda \in \Lambda^+(2,r)$. With the notation above, we have

(i) $$0 \to S(G)\xi_{\lambda(1,1)} \xrightarrow{\psi_1} S(G)\xi_\lambda \xrightarrow{\psi_0} K_\lambda \to 0$$

is a projective resolution of the Weyl module K_λ if char $k = 0$.

(ii) $$\coprod_{m=1}^{d} (S(G)\xi_{\lambda(1,p^m)} \amalg S(G)\xi_{\lambda(1,1+p^m)} \amalg S(G)\xi_{\lambda(1,p+p^m)} \amalg \cdots$$

$$\cdots \amalg S(G)\xi_{\lambda(1,p^{m-1}+p^m)}) \xrightarrow{\psi_2} \coprod_{m=0}^{d} S(G)\xi_{\lambda(1,p^m)} \xrightarrow{\psi_1} S(G)\xi_\lambda \xrightarrow{\psi_0} K_\lambda \to 0$$

are the first three terms of a projective resolution of K_λ if char $k = p > 0$.

Proof: Let char $k = p$ (≥ 0) and write $Y_0 = V_\lambda$,

$$Y_1 = \bigoplus_{m=0}^{d} V_{\lambda(1,p^m)}, \text{ and } Y_2 = \bigoplus_{m=1}^{d} (V_{\lambda(1,p^m)} \oplus V_{\lambda(1,1+p^m)} \oplus \cdots \oplus V_{\lambda(1,p^{m-1}+p^m)})$$

($Y_1 = V_{\lambda(1,1)}$, and $Y_2 = 0$ if $p = 0$).

By (13.1) and (14.1),

(15.3) $$Y_2 \xrightarrow{\varphi_2} Y_1 \xrightarrow{\varphi_1} Y_0 \xrightarrow{\varphi_0} k_\lambda \to 0$$

are the first terms of a minimal projective resolution of k_λ. Thus, taking duals (and since all the modules involved are finite dimensional over k) we have that

$$0 \to k_\lambda^\circ \xrightarrow{\varphi_0^*} Y_0^\circ \xrightarrow{\varphi_1^*} Y_1^\circ \xrightarrow{\varphi_2^*} Y_2^\circ$$ [11]

[11] If V, V' are k-vector spaces and $f \in \mathrm{Hom}_k(V, V')$, $f^* \in \mathrm{Hom}_k(V'^*, V^*)$ is the map defined by, $f^*(\theta) = \theta f$, for all $\theta \in V'^*$.

are the first three terms of an injective resolution of k_λ°. But , $k_\lambda^\circ \underset{S(B^-)}{\cong} k_\lambda^-$. Therefore, by (15.1), the sequence below is exact up to and including $F'(Y_1{}^\circ)$

$$0 \to F'(k_\lambda^\circ) \xrightarrow{F'(\varphi_0^*)} F'(Y_0{}^\circ) \xrightarrow{F'(\varphi_1^*)} F'(Y_1{}^\circ) \xrightarrow{F'(\varphi_2^*)} F'(Y_2{}^\circ).$$

Taking duals, once more, we obtain the exact sequence in mod $S(G)$

$$[F'(Y_2{}^\circ)]^\circ \xrightarrow{F'(\varphi_2^*)^*} [F'(Y_1{}^\circ)]^\circ \xrightarrow{F'(\varphi_1^*)^*} [F'(Y_0{}^\circ)]^\circ \xrightarrow{F'(\varphi_0^*)^*} [F'(k_\lambda^\circ)]^\circ \to 0$$

On the other hand, if we apply the functor F to the sequence (15.3), we obtain the following complex

$$F(Y_2) \xrightarrow{F(\varphi_2)} F(Y_1) \xrightarrow{F(\varphi_1)} F(Y_0) \xrightarrow{F(\varphi_0)} F(k_\lambda) \to 0.$$

But, from (5.6), we know that there is an $S(G)$-isomorphism

$$\theta_V : F(V^\circ) \to [F'(V)]^\circ$$

natural in $V \in \text{mod } S(B^-)$, i.e., $\{\theta_V \mid V \in \text{mod } S(B^-)\}$ is a class of $S(G)$-isomorphsims such that for any $V, V' \in \text{mod } S(B^-)$ and any $f \in \text{Hom}_{S(B^-)}(V, V')$ the diagram below commutes

$$\begin{tikzcd} F(V'^\circ) \arrow[r, "F(f^*)"] \arrow[d, "\theta_{V'}"'] & F(V^\circ) \arrow[d, "\theta_V"] \\ {[F'(V')]^\circ} \arrow[r, "F'(f)^*"'] & {[F'(V)]^\circ} \end{tikzcd}$$

.

It is also easy to see that the usual isomorphism $W \underset{S(B^+)}{\cong} (W^o)^o$ ($w \in W$ is taken to $e_w : W^o \to k$, defined by, $e_w(\delta) = \delta(w)$, for all $\delta \in W^o$) is natural in $W \in \operatorname{mod} S(B^+)$. Therefore, there are $S(G)$-isomorphisms $\eta, \eta_0, \eta_1, \eta_2$ such that the diagram below commutes.

(15.4)

$$\begin{tikzcd}
F(Y_2) \arrow[r, "F(\varphi_2)"] \arrow[d, "\eta_2"'] & F(Y_1) \arrow[r, "F(\varphi_1)"] \arrow[d, "\eta_1"'] & F(Y_0) \arrow[r, "F(\varphi_0)"] \arrow[d, "\eta_0"'] & F(k_\lambda) \arrow[r] \arrow[d, "\eta"] & 0 \\
{[F'(Y_2^o)]^o} \arrow[r, "F'(\varphi_2^o)^o"] & {[F'(Y_1^o)]^o} \arrow[r, "F'(\varphi_1^o)^o"] & {[F'(Y_0^o)]^o} \arrow[r, "F'(\varphi_0^o)^o"] & {[F'(k_\lambda^o)]^o} \arrow[r] & 0.
\end{tikzcd}$$

Hence, since the bottom row of (15.4) is exact, the top row is also exact.

Now, as $F(k_\lambda) = S(G) \otimes_{S(B^+)} k_\lambda$ is the Weyl module K_λ (cf. (7.2)), the theorem follows. □

(15.5) Remark: The sequence in (15.2)(i) is equivalent to the projective resolution of K_λ determined in [A] and [Z].

6. THE SCHUR ALGEGRA $S(U^+)$

In this chapter we consider the unipotent subgroup U^+ of B^+, and give some results on its Schur algebra $S(U^+) = S_k(n,r; U^+)$.

§16. A basis and the radical of $S(U^+)$

Let $\mu, \nu \in \underline{n}$, $\mu < \nu$. For each non-negative integer m, consider the elements $\Gamma^{(m)}_{\mu\nu}$ of $S(B^+)$, defined by

$$\Gamma^{(m)}_{\mu\nu} = \sum_{\alpha} \xi_{\ell(\mu,\nu,m,\alpha),\ell(\alpha)},$$

sum over all weights $\alpha \in \Lambda$ such that $m \le \alpha_\nu$.

Note that, since $0 \le \alpha_\nu \le r$ (all $\alpha \in \Lambda$), we have $\Gamma^{(0)}_{\mu\nu} = 1_{S(G)}$, and $\Gamma^{(m)}_{\mu\nu} = 0$ if $m > r$.

Let $u_{\mu\nu}(t)$ be the element of U^+ with 1's in the main diagonal, t in position (μ,ν), and zeros elsewhere $(t \in k)$. In (4.7) we proved that

(16.1) $$T_r(u_{\mu\nu}(t)) = \sum_{m=0}^{r} t^m \,\Gamma^{(m)}_{\mu\nu}\,.$$

As a consequence of this we have the following result.

(16.2) Lemma: (i) $\Gamma^{(m)}_{\mu\nu} \in S(U^+)$, all $\mu, \nu \in \underline{n}$, $\mu < \nu$; $m = 0,\ldots,r$.

(ii) $\Gamma^{(m)}_{\mu\nu}\,\Gamma^{(q)}_{\mu\nu} = \binom{m+q}{q}\,\Gamma^{(m+q)}_{\mu\nu}$, all $\mu, \nu \in \underline{n}$, $\mu < \nu$; $m, q = 0,\ldots,r$.

Proof: Let μ, ν be as above.

(i) As $u_{\mu\nu}(t) \in U^+$, $T_r(u_{\mu\nu}(t)) \in S(U^+)$, for all $t \in k$. Thus, since k is an infinite field, (16.1) implies $\Gamma^{(m)}_{\mu\nu} \in S(U^+)$, all $m = 0,\ldots,r$.

(ii) Let $t, t' \in k$. Then, $u_{\mu\nu}(t)\,u_{\mu\nu}(t') = u_{\mu\nu}(t+t')$. Hence

$T_r(u_{\mu\nu}(t))T_r(u_{\mu\nu}(t')) = T_r(u_{\mu\nu}(t+t'))$, i.e.,

$$\sum_{m=0}^{r}\sum_{q=0}^{r} t^m t'^q\,\Gamma^{(m)}_{\mu\nu}\,\Gamma^{(q)}_{\mu\nu} = \sum_{a=0}^{r} (t+t')^a\,\Gamma^{(a)}_{\mu\nu},$$

or equivalently,

$$\sum_{m=0}^{r}\sum_{q=0}^{r} t^m t'^q\,\Gamma^{(m)}_{\mu\nu}\,\Gamma^{(q)}_{\mu\nu} = \sum_{a=0}^{r}\sum_{b=0}^{a} \binom{a}{b}\, t^{a-b}\, t'^b\,\Gamma^{(a)}_{\mu\nu}.$$

As this holds for any $t, t' \in k$ (and k is infinite) we must have

$\Gamma^{(m)}_{\mu\nu}\,\Gamma^{(q)}_{\mu\nu} = \binom{m+q}{q}\,\Gamma^{(m+q)}_{\mu\nu}$, all $m, q = 0,\ldots,r$. □

It is well known that U^+ is generated by $\{u_{\nu\,\nu+1}(t) \mid \nu \in \underline{n-1},\ t \in k\}$. Thus, by (16.1) and (16.2),

(16.3) $S(U^+)$ is generated by $\{\Gamma^{(m)}_{\nu,\nu+1} \mid \nu \in \underline{n-1},\ m = 0,\ldots,r\}$.

We can refine this result as follows.

(16.4) Proposition: Suppose char $k = p(\geq 0)$. Then $S(U^+)$ is generated (as k-algebra) by $X = \{1_{S(G)}, \Gamma^{(p^d)}_{\nu,\nu+1} \mid \nu \in \underline{n-1},\ 1 \leq p^d \leq r\}$.

Proof: Let M be the subalgebra of $S(U^+)$ generated by X. Suppose we show that, for any $\nu \in \underline{n-1}$,

(16.5) $$\Gamma^{(m)}_{\nu,\nu+1} \in M, \quad m = 0,\dots,r.$$

Then the proposition follows from (16.3).

To prove (16.5) we use induction on m.

If $m = 0$, $\Gamma^{(m)}_{\nu,\nu+1} = 1_{S(G)} \in M$.

Now let $1 \leq m \leq r$, and suppose (16.5) holds, for any $q < m$.

If $p > 0$ there exists $b \in \mathbb{Z}$, $b \geq 0$, such that $p^b \leq m < p^{b+1}$, and so we may write $m = ap^b + s$, where $a, s \in \mathbb{Z}$, $1 \leq a < p$, $0 \leq s < p^b$ (if $p = 0$, we make $b = s = 0$, and $a = m$).

Suppose first that $s \neq 0$. Then by (16.2)(ii), $\Gamma^{(ap^b)}_{\nu,\nu+1}\ \Gamma^{(s)}_{\nu,\nu+1} = \binom{m}{s} \Gamma^{(m)}_{\nu,\nu+1}$. But, $p \nmid \binom{m}{s}$. Hence,

$$\Gamma^{(m)}_{\nu,\nu+1} = \frac{1}{\binom{m}{s}} \Gamma^{(ap^b)}_{\nu,\nu+1}\, \Gamma^{(s)}_{\nu,\nu+1}.$$

By the induction hypothesis both $\Gamma^{(ap^b)}_{\nu,\nu+1}$ and $\Gamma^{(s)}_{\nu,\nu+1}$ are in M. Thus $\Gamma^{(m)}_{\nu,\nu+1} \in M$.

Now suppose that $s = 0$. Then $\Gamma^{(m)}_{\nu,\nu+1} = \Gamma^{(ap^b)}_{\nu,\nu+1}$, and once more we have

$$\Gamma^{(m)}_{\nu,\nu+1} = \frac{1}{\binom{ap^b}{p^b}} \Gamma^{((a-1)p^b)}_{\nu,\nu+1}\Gamma^{(p^b)}_{\nu,\nu+1},$$

where $p \nmid \binom{ap^b}{p^b}$ (since $a < p$). So the result follows by the induction hypothesis. □

Our next step is to determine a basis for $S(U^+)$.

Let $u \in U^+$. Then $u_{\mu\nu} = 0$, unlesss $\mu \leq \nu$ $(\mu, \nu \in \underline{n})$.

Thus, $T_r(u) = \sum_{(i,j) \in \Omega} u_{i,j}\,\xi_{i,j} = \sum_{(i,j) \in \Omega'} u_{i,j}\,\xi_{i,j}$. [12]

(16.6) Definition: For any non-negative integer s, let Ω^*_s be a set of representatives of the $P(s)$-orbits of pairs (h, h') in $I(n,s) \times I(n,s)$ such that $h_1 < h'_1$, $h_2 < h'_2, \ldots, h_s < h'_s$.

Define $\Omega^* = \Omega^*_0 \cup \Omega^*_1 \cup \ldots \cup \Omega^*_r$.

Choose Ω so that if $(i,j) \in \Omega'$ then

$$i_1 < j_1,\ i_2 < j_2, \ldots, i_s < j_s,\ i_{s+1} = j_{s+1}, \ldots, i_r = j_r \text{ (some } s \geq 0).$$

Under these conditions, let c be the element of Ω^*_s satisfying $c \sim ((i_1,\ldots,i_s), (j_1,\ldots,j_s))$. Then, we say that c is the *core* of (i,j) (or of any element in the P-orbit of (i,j) in $I(n,r) \times I(n,r)$). For any $(i', j') \in I(n,r) \times I(n,r)$, $c(i', j')$ will denote the core of (i', j').

11 Recall that $\Omega' = \{(i,j) \in \Omega \mid i \leq j\}$.

Note that $c(i,j) \in \Omega^*_0$ iff $c(i,j)$ is "empty", i.e., iff $i = j$.

(16.7) Definition: If $c \in \Omega^*$ define the *core sum* ξ_c, by

$$\xi_c = \sum_{\substack{(i,j) \in \Omega' \\ c(i,j) = c}} \xi_{i,j}.$$

(16.8) Remarks: (i) Let $\mu, \nu \in \underline{n}$, $\mu < \nu$, and consider the element $c_m = ((\mu,..,\mu), (\nu,...,\nu))$ of Ω^*_m, $(m = 0,...,r)$. Then $\xi_{c_m} = \Gamma^{(m)}_{\mu\,\nu}$.

In particular, $c_0 \in \Omega^*_0$ and $\xi_{c_0} = 1_{S(G)}$.

(ii) Let $c = (h, h') \in \Omega^*_s$ $(s = 0,...,r)$. Then $c = c(i,j)$, for some $(i,j) \in \Omega'$.

In fact, let $i', j' \in I(n,r)$ be defined by

$$i'_\rho = \begin{cases} h_\rho, & \text{if } \rho \in \underline{s} \\ 1, & \text{if } \rho \in \{s+1,..,r\} \end{cases} \quad ; \quad j'_\rho = \begin{cases} h'_\rho, & \text{if } \rho \in \underline{s} \\ 1, & \text{if } \rho \in \{s+1,...,r\}. \end{cases}$$

Then $i' \le j'$ and $c(i', j') = c$. So if $(i,j) \in \Omega'$ and $(i', j') \sim (i,j)$ we have $c = c(i', j') = c(i,j)$.

(iii) By (ii) above, $\xi_c \neq 0$, all $c \in \Omega^*$.

It is clear that if $(i,j), (i', j') \in \Omega'$, and $c(i,j) = c(i'j')$, then for any $u \in U^+$ we have $u_{i,j} = u_{i',j'}$ (since $u_{\mu\mu} = 1$, $\mu \in \underline{n}$). Therefore

(16.9) $$T_r(u) = \sum_{(i,j) \in \Omega'} u_{i,j}\, \xi_{i,j} = \sum_{c \in \Omega^*} u_c\, \xi_c, \text{ for all } u \in U^+$$

(where $u_c = u_{i,j}$, for any $(i,j) \in \Omega'$ such that $c(i,j) = c$).

(16.10) Lemma: $\xi_c \in S(U^+)$, for all $c \in \Omega^*$.

Proof: Let $c = (h, h') \in \Omega^*$. If $c \in \Omega^*_0$ then $\xi_c = \sum_{(i,i) \in \Omega'} \xi_{i,i} = 1_{S(G)} \in S(U^+)$.

Now suppose that $c \in \Omega^*_s$ $(s \in \underline{r})$.

Let m be the number of distinct pairs (h_ρ, h'_ρ), $\rho \in \underline{s}$. Then there are $\mu_a, \nu_a \in \underline{n}$, $d_a \in \underline{s}$ $(a \in \underline{m})$ satisfying

(i) $\mu_a < \nu_a$, and $(\mu_a, \nu_a) \neq (\mu_b, \nu_b)$ if $a \neq b$ $(a,b \in \underline{m})$;

(ii) $\sum_{a \in \underline{m}} d_a = s$;

(iii) $c = (h, h') \sim ((\underbrace{\mu_1,\dots,\mu_1}_{d_1},\dots,\underbrace{\mu_m,\dots,\mu_m}_{d_m}), (\underbrace{\nu_1,\dots,\nu_1}_{d_1},\dots,\underbrace{\nu_m,\dots,\nu_m}_{d_m}))$

For each $t = (t_1,\dots,t_m) \in k^m$, define $u(t) \in U^+$, by

$$u(t)_{\mu,\nu} = \begin{cases} 1 & , \text{ if } \mu = \nu \\ t_a & , \text{ if } (\mu, \nu) = (\mu_a, \nu_a), \quad a \in \underline{m} \\ 0 & , \text{ otherwise} \end{cases} \quad ; \mu, \nu \in \underline{n}.$$

Then, for any $(i,j) \in \Omega'$, we have

$$u(t)_{i,j} = 0, \text{ unless } (i_\rho, j_\rho) \in \{(1,1),\dots,(n,n), (\mu_1, \nu_1),\dots,(\mu_m, \nu_m)\}, \text{ all } \rho \in \underline{r}.$$

If this last condition holds, and if $q_a = \#\{\rho \in \underline{r} \mid (i_\rho, j_\rho) = (\mu_a, \nu_a)\}$ $(a \in \underline{m})$, then $u(t)_{i,j} = t_1^{q_1} \dots t_m^{q_m}$.

Let $Q = \{q = (q_1,\dots,q_m) \in \mathbb{Z}^m \mid 0 \le q_a \le r \ (a \in \underline{m});\ \sum_{a \in \underline{m}} q_a \le r\}$.

For each $q \in Q$, let $c(q)$ be the element of Ω^* defined by,

$$c(q) \sim ((\underbrace{\mu_1,\dots,\mu_1}_{q_1},\dots,\underbrace{\mu_m,\dots,\mu_m}_{q_m}),\ (\underbrace{\nu_1,\dots,\nu_1}_{q_1},\dots,\underbrace{\nu_m,\dots,\nu_m}_{q_m})).$$

Then, we have just proved that, for any $(i,j) \in \Omega'$, there holds

$$u(t)_{i,j} = \begin{cases} t_1^{q_1} \cdots t_m^{q_m} & , \text{ if } c(i,j) = c(q), \text{ for some } q \in Q \\ 0 & , \text{ otherwise.} \end{cases}$$

Therefore,

$$\textbf{(16.11)} \quad T_r(u(t)) = \sum_{(i,j) \in \Omega'} u(t)_{i,j}\, \xi_{i,j} = \sum_{q \in Q} t_1^{q_1} \cdots t_m^{q_m}\, \xi_{c(q)}.$$

Since $T_r(u(t)) \in S(U^+)$, and (16.11) holds for any $t \in k^m$ (and k is infinite) we must have

$$\xi_{c(q)} \in S(U^+), \text{ for all } q \in Q.$$

But, in particular, $d = (d_1,\dots,d_m) \in Q$. Also $c = c(d)$. Hence $\xi_c = \xi_{c(d)} \in S(U^+)$. □

(16.12) Theorem: $S(U^+)$ has k-basis $Y = \{\xi_c \mid c \in \Omega^*\}$.

Proof: By (16.9) and the lemma (16.10), Y spans $S(U^+)$. Also from the definitions of Ω^* and of ξ_c it is clear that the elements of Y are linearly independent. □

Let $i,j \in I$ have weights α and β, respectively, and suppose that $i \leq j$. In §9 we defined the degree of $\xi_{i,j}$, $d(\xi_{i,j})$, by

$$d(\xi_{i,j}) = \alpha - \beta.$$

Also, if $\Psi = \{ \sum_{\mu \in \underline{n-1}} z_\mu \varepsilon_{\mu,\mu+1} \mid z_\mu \in \mathbb{Z}, z_\mu \geq 0 \ (\mu \in \underline{n-1}\} \quad$ (where

$\varepsilon_{\mu,\mu+1} = (0,\dots,\underset{(\mu)}{1}, \underset{(\mu+1)}{-1}, \dots, 0))$ and $S(B^+)_\zeta = \bigoplus_{\substack{(i,j) \in \Omega \\ d(\xi_{i,j}) = \zeta}} k\xi_{i,j} \ (\zeta \in \Psi)$ we proved that

$$S(B^+) = \bigoplus_{\zeta \in \Psi} S(B^+)_\zeta$$

is a grading of the algebra $S(B^+)$ (cf. (9.14)).

It is easy to see that if $(i,j), (i',j') \in \Omega'$, and $c(i,j) = c(i',j')$, then $d(\xi_{i,j}) = d(\xi_{i',j'})$. Thus, for any $c \in \Omega^*$, there holds

(6.13) (i) $\xi_c = \sum_{\substack{(i,j) \in \Omega' \\ c(i,j) = c}} \xi_{i,j}$ is homogeneous of degree $d(\xi_{i',j'})$, where $(i',j') \in \Omega'$ satisfies $c(i',j') = c$.

(ii) $d(\xi_c) = (0,\dots,0)$ iff $c \in \Omega^*_0$, i.e., iff $\xi_c = 1_{S(G)}$.

For each $\zeta \in \Psi$ let $S(U^+)_\zeta$ be the k-subspace of $S(U^+)$ spanned by all ξ_c $(c \in \Omega^*)$ of degree ζ.

By the remarks above,

$$S(U^+) = \bigoplus_{\zeta \in \Psi} S(U^+)_\zeta$$

is a grading of $S(U^+)$.

We now use this grading to determine the radical of $S(U^+)$.

(16.14) Theorem: The radical of $S(U^+)$ has k-basis $\{\xi_c \mid c \in \Omega^* \setminus \Omega^*_0\}$.
Thus, $S(U^+) = k\, 1_{S(G)} \oplus \operatorname{rad} S(U^+)$ is a local ring.

Proof: Let $N = \bigoplus_{c \in \Omega^* \setminus \Omega^*_0} k\xi_c$, and suppose we prove that

(1) N is a maximal left ideal of $S(U^+)$;

(2) N is a nil left ideal.

Then by (1), $\operatorname{rad} S(U^+) \subseteq N$ and by (2), $N \subseteq \operatorname{rad} S(U^+)$. Hence $N = \operatorname{rad} S(U^+)$, as desired.

Note that, by (16.13)(ii), $S(U^+)_{(0,\dots,0)} = k\, 1_{S(G)}$ and $\bigoplus_{\substack{\zeta \in \Psi \\ \zeta \neq (0,\dots,0)}} S(U^+)_\zeta = N$.

Hence (1) follows.

To prove (2) define, for each $\gamma = (\gamma_1,\dots,\gamma_n) \in \mathbb{Z}^n$,

$$\sigma(\gamma) = \sum_{\nu \in \underline{n}} \nu\gamma_\nu .$$

Clearly $\sigma(\gamma + \gamma') = \sigma(\gamma) + \sigma(\gamma')$, for all $\gamma, \gamma' \in \mathbb{Z}^n$. Thus, if $(i,j) \in \Omega'$ and $i \in \alpha$, $j \in \beta$ $(\alpha, \beta \in A)$ we have

(i) $\sigma(d(\xi_{i,j})) = \sigma(\alpha - \beta) = \sigma(\alpha) - \sigma(\beta) \geq -\sigma(\beta) \geq -nr;$

(ii) write $\alpha - \beta = \sum_{\mu \in \underline{n-1}} m_\mu \varepsilon_{\mu,\mu+1}$, where $m_1,\dots,m_{n-1}$ are non-negative integers.

Then, $\sigma(d(\xi_{i,j})) = \sigma(\alpha - \beta) = \sum_{\mu \in \underline{n-1}} \sigma(m_\mu \varepsilon_{\mu,\mu+1}) = -\sum_{\mu \in \underline{n-1}} m_\mu \leq 0$. Also, $\sigma(d(\xi_{i,j})) = 0$ iff $m_\mu = 0$ ($\mu \in \underline{n-1}$) iff $\alpha = \beta$, i.e., iff $i = j$ (since $i \leq j$).

Hence, if $c \in \Omega^*\backslash\Omega^*_0$, there holds

$$-r\,n \leq \sigma(d(\xi_c)) \leq -1. \tag{16.15}$$

Now let η be any element of N, and let $m \in \mathbb{Z}$ satisfy $m > r\,n$.

Then, if η^m is not zero, there are $c_1,\dots,c_m \in \Omega^*\backslash\Omega^*_0$ such that $\xi_{c_1} \dots \xi_{c_m} \neq 0$. But $\xi_{c_1} \dots \xi_{c_m}$ is homogeneous of degree $d(\xi_{c_1}) + \dots + d(\xi_{c_m})$. Also $\sigma(d(\xi_{c_1}) + \dots + d(\xi_{c_m})) = \sigma(d(\xi_{c_1})) + \dots + \sigma(d(\xi_{c_m})) \leq -m < -r\,n$. This contradicts (16.15). Hence $\eta^m = 0$, and (2) follows. □

§17. The natural epimorphism $S(T) \otimes S(U^+) \to S(B^+)$

Consider the subgroups T and U^+ of B^+.[13] As $B^+ = TU^+$ (semidirect product) we have $S(B^+) = S(T)\,S(U^+)$. Thus, there is a natural k-epimorphism

$$f : S(T) \otimes S(U^+) \longrightarrow S(B^+),$$

given by

$$f(\xi \otimes \eta) = \xi\,\eta, \quad \text{all } \xi \in S(T),\ \eta \in S(U^+).$$

We are interested in the kernel of f. From (3.8) and (16.12), we know that $S(T)$

[13] Recall that T is the group of all diagonal matrices in G.

and $S(U^+)$ have k-bases $\{\xi_\alpha \mid \alpha \in A(n,r)\}$ and $\{\xi_c \mid c \in \Omega^*\}$, respectively. So to calculate ker f we need to study the products $\xi_\alpha\, \xi_c$ $(\alpha \in A,\ c \in \Omega^*)$.

If $\alpha \in \Lambda(n,r)$ and $\beta \in \Lambda(n,s)$ $(s = 0,\ldots,r)$ we say that $\beta \subseteq \alpha$ if $\beta_\mu \le \alpha_\mu$, for all $\mu \in \underline{n}$.

(17.1) Definition: Let $c = (h,h') \in \Omega^*_s$ $(s = 0,\ldots,r)$. We define $\beta(c) \in \Lambda(n,s)$ to be the weight of h.

(17.2) Theorem: ker f has k-basis

$$\{\xi_\alpha \otimes \xi_c \mid \text{all } \alpha \in A,\ c \in \Omega^* \text{ such that } \beta(c) \not\subseteq \alpha\}.$$

Thus, there is a short exact sequence of k-spaces

$$0 \longrightarrow \bigoplus_{\substack{\alpha \in A,\ c \in \Omega^* \\ \beta(c) \not\subseteq \alpha}} k(\xi_\alpha \otimes \xi_c) \xrightarrow{\text{inc}} S(T) \otimes S(U^+) \xrightarrow{f} S(B^+) \longrightarrow 0.$$

Proof: Let $\alpha \in A$ and $c = (h,h') \in \Omega^*_s$ $(s = 0,\ldots,r)$. Define $A(\alpha,c) = \{(i,j) \in \Omega' \mid i \in \alpha \text{ and } c(i,j) = c\}$. Then

$$\xi_\alpha\, \xi_c = \sum_{\substack{(i,j) \in \Omega' \\ c(i,j) = c}} \xi_\alpha\, \xi_{i,j} = \sum_{(i,j) \in A(\alpha,c)} \xi_{i,j}\,. \tag{17.3}$$

Suppose that $(i,j) \in A(\alpha,c)$, and let $\gamma(i)_\nu = \#\{\rho \in \underline{r}\setminus\underline{s} \mid i_\rho = \nu\}$, for all $\nu \in \underline{n}$. Then, as $i \in \alpha$ and $(i_1,\ldots,i_s) \sim h$, we have

$$\alpha_\nu = \#\{\rho \in \underline{s} \mid i_\rho = \nu\} + \#\{\rho \in \underline{r} \setminus \underline{s} \mid i_\rho = \nu\} =$$

$$= \#\{\rho \in \underline{s} \mid h_\rho = \nu\} + \gamma(i)_\nu = \beta(c)_\nu + \gamma(i)_\nu, \text{ all } \nu \in \underline{n}.$$

Therefore

(17.4) $A(\alpha,c) \neq \emptyset$ implies $\beta(c) \subseteq \alpha$.

Now suppose that $A(\alpha,c) \neq \emptyset$, and let $(i,j), (i',j') \in A(\alpha,c)$. Since $c(i,j) = c(i',j') = c$, there is $\tau \in P(s)$ such that

$$i'_\rho = i_{\tau(\rho)} \text{ and } j'_\rho = j_{\tau(\rho)}, \text{ all } \rho \in \underline{s}.$$

As a consequence of this, and since $i, i' \in \alpha$, we must have $\gamma(i)_\nu = \gamma(i')_\nu$ ($\nu \in \underline{n}$). Hence, there is a bijection, $\sigma : \underline{r} \setminus \underline{s} \to \underline{r} \setminus \underline{s}$, such that

$$i'_\rho = i_{\sigma(\rho)}, \text{ all } \rho \in \underline{r} \setminus \underline{s}.$$

Define $\pi \in P(r)$ by, $\pi(\rho) = \tau(\rho)$ if $\rho \in \underline{s}$, while $\pi(\rho) = \sigma(\rho)$ if $\rho \in \underline{r} \setminus \underline{s}$. Clearly $i\pi = i'$. Also

$$j'_\rho = \begin{cases} j_{\tau(\rho)} = j_{\pi(\rho)} & , \text{ if } \rho \in \underline{s} \\ i'_\rho = i_{\sigma(\rho)} = j_{\sigma(\rho)} = j_{\pi(\rho)}, & \text{if } \rho \in \underline{r} \setminus \underline{s}. \end{cases}$$

Hence $(i,j) = (i',j')$. This proves that

(17.5) $A(\alpha,c)$ has at most one element.

Suppose now that $\beta(c) \subseteq \alpha$ and write $\gamma_\nu = \alpha_\nu - \beta(c)_\nu$, all $\nu \in \underline{n}$. As $\gamma_\nu \geq 0$ ($\nu \in \underline{n}$), we may define $i,j \in I(n,r)$ as follows

$$i = (h_1,\dots,h_s, \underbrace{1,\dots,1}_{\gamma_1}, \underbrace{2,\dots,2}_{\gamma_2},\dots, \underbrace{n,\dots,n}_{\gamma_n}) ; \; j = (h'_1,\dots,h'_s, \underbrace{1,\dots,1}_{\gamma_1}, \underbrace{2,\dots,2}_{\gamma_2},\dots, \underbrace{n,\dots,n}_{\gamma_n}).$$

It is clear that $i \in \alpha$, $i \leq j$, and $c(i,j) = (h, h') = c$. Thus, the element of Ω' which represents the P-orbit of (i,j) in $I \times I$ belongs to $A(\alpha,c)$. This together with (17.4) and (17.5) give the following

$$\# A(\alpha,c) = \begin{cases} 1, & \text{if } \beta(c) \subseteq \alpha \\ 0, & \text{if } \beta(c) \not\subseteq \alpha\text{; all } \alpha \in A, c \in \Omega^*. \end{cases}$$

If $\beta(c) \subseteq \alpha$ write $A(\alpha,c) = \{(i(\alpha,c), j(\beta,c))\}$. Then, by (17.3),

$$\text{(17.6)} \quad \xi_\alpha \xi_c = \begin{cases} \xi_{i(\alpha,c),j(\alpha,c)}, & \text{if } \beta(c) \subseteq \alpha \\ 0, & \text{if } \beta(c) \not\subseteq \alpha\text{; all } \alpha \in A, c \in \Omega^*. \end{cases}$$

Note that if $\alpha, \alpha' \in A$ and $c, c' \in \Omega^*$ satisfy $(\alpha,c) \neq (\alpha', c')$ then $\xi_{i(\alpha,c),j(\alpha,c)}$ and $\xi_{i(\alpha',c'),j(\alpha',c')}$ are linearly independent elements of $S(B^+)$. Hence, the theorem (17.2) follows from (17.6). □

INDEX OF NOTATION

Symbol	Meaning	Page of Definition
$K_{\lambda,J}$	$S(G_J^+) \otimes_{S(B^+)} k_\lambda$	2-13
$\ell(\lambda)$	The element of $I(n,r)$ defined by the λ-tableau (4.4)	1-14
$\ell(\nu,m)$	$\ell(A_\nu^m\lambda)$	3-3, 3-4
$\ell(\mu,\nu,m,\lambda)$	The element of $I(n,r)$ defined by the λ-tableau (4.5)	1-15
L_J	The standard Levi subgroup of G corresponding to the set J	1-10
mod S	The category of all S-modules which are finite dimensional over k	-
N_a	$\{m_{a-1}+1,\ldots,m_a\}$	1-2
$P(s)$	The symmetric group on $\{1,\ldots,s\}$	1-1
P	$P(r)$	1-1
P_i	The stabilizer of i in P	1-8
$P_{i,j}$	$P_i \cap P_j$	1-8
$S(H) = S_k(n,r,H)$	The Schur algebra for H, n, r and k	1-6
T	The group of all diagonal matrices in G	1-9
T^λ	The basic λ-tableau	1-4
T_i^λ	The λ-tableau iT^λ	1-5
T_r	The representation afforded by the kG-module $E^{\otimes r}$	1-6
$u_{\mu\nu}(t)$	The element of G with 1's in the main diagonal, t in position (μ,ν) and zeros elsewhere	1-9
U^+ (resp. U^-)	The group of all unipotent matrices in B^+ (resp. B^-)	1-9
V_λ	The projective indecomposable $S(B^+)$-module $S(B^+)\xi_\lambda$	2-1

Symbol	Meaning	Page of Definition
V^λ	The λ-weight space of V	1-14
V^o	The contravariant dual of V	1-18
V^*	The dual, $\mathrm{Hom}_k(V, k)$, of V	-
$V \otimes V'$	$V \otimes_k V'$	1-5
$\Gamma^{(m)}_{\mu\nu}$	$\sum_\lambda \xi_{\ell(\mu,\nu,m,\lambda),\, \ell(\lambda)}$ (sum over all $\lambda \in \Lambda$ such that $m \leq \lambda_\nu$)	1-16
$\varepsilon(\omega)$	The sign of the permutation ω	-
λ	Element of $\Lambda(n,r)$	-
$\lambda(\nu,m)$	$A^m_\nu \lambda$	3-4
$\Lambda = \Lambda(n,r)$	$\{\lambda = (\lambda_1,\ldots,\lambda_n) \mid \lambda_\nu \in \mathbb{Z}, \lambda_\nu \geq 0\ (\nu \in \underline{n}), \sum_{\nu \in \underline{n}} \lambda_\nu = r\}$	1-1
$\Lambda^+ = \Lambda^+(n,r)$	$\{\lambda \in \Lambda(n,r) \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}$	2-9
$\Lambda^+_J = \Lambda^+_J(n,r)$	$\{\lambda \in \Lambda(n,r) \mid \lambda_{m_{a-1}+1} \geq \ldots \geq \lambda_{m_a}$, all $a \in \underline{s}\}$	2-14
$\xi_{i,j}$	A basis element of $S(G)$	1-6
ξ_λ	$\xi_{i,i}$, where $i \in I(n,r)$ has weight λ	1-8
κ_λ	The representation afforded by the $S(B^+)$-module k_λ	2-3
$\omega(\lambda)$	$(\lambda_1 + \omega(1)-1,\ldots,\lambda_n + \omega(n)-n)$	4-11
ω_λ	$1_{S(G)} \otimes_{S(B^+)} 1_{k_\lambda}$	2-6
Ω	A transversal of the set of all P-orbits of $I \times I$	1-7
Ω'	$\{(i,j) \in \Omega \mid i \leq j\}$	2-1
$\underline{s}$	$\{1,\ldots,s\}$	1-1
$\mu \underset{J}{=} \nu$	μ and ν are in the same set N_a, for some $a \in \underline{s}$	1-2

Symbol	Meaning	Page of Definition
$\mu \underset{J}{\leq} \nu$	$\mu \leq \nu$ or $\mu \underset{J}{=} \nu$	1-2
$i \underset{J}{=} j$	$i_\rho \underset{J}{=} j_\rho$, all $\rho \in \underline{r}$	1-2
$i \underset{J}{\leq} j$	$i_\rho \underset{J}{\leq} j_\rho$, all $\rho \in \underline{r}$	1-2
$i \leq j$	$i_\rho \leq j_\rho$, all $\rho \in \underline{r}$	1-3
$i \sim j$	i and j are in the same P-orbit of I	1-1
$(i,j) \sim (i',j')$	(i,j) and (i',j') are in the same P-orbit of $I \times I$	1-1
$\trianglelefteq$	The dominance order on $\Lambda(n,r)$	1-3
$\oplus$	Internal direct sum	–
$\coprod$	External direct sum	–
$\#$	The cardinal	–
$\dot{\cup}$	Disjoint union	–

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