<u>Non-Noetherian Unique Factorisation Rings.</u>

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CONTENTS

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SUMMARY

The main aim of this thesis is to produce and then study two generalizations of the unique factorisation domain of commutative algebra.When this has been done before, [2] and [5], it has always been assumed that the rings are Noetherian.It is our aim to show that this is not only unnatural but unnecessary.

Chapter 1 contains some well known results about rings and in particular about rings satisfying a polynomial identity.

In chapter 2 we define the unique factorisation ring (U.F.R.) and the unique factorisation domain (U.F.D.),show where these definitions come from and show what results can be obtained using only the definitions.

In chapter 3 we show that all the previously known results for Noetherian U.F.D.s can be proved for a U.F.D. which merely satisfies the Goldie condition.In particular we prove that a Goldie U.F.D. is a maximal order and that a bounded Goldie U.F.D. is either commutative or a Noetherian principal ideal ring.

In chapter 4 we look at U.F.R.s that satisfy a polynomial identity and show that these too are maximal orders.We also show that they are equal to the intersection of two rings ,one of which is a Noetherian principal ideal ring and the other of which is a simple Artinian ring.

In chapter 5 we look at the reflexive ideals of a U.F.R. which satisfies a polynomial identity and show that they are all principal.We also show that if T is a reflexive ideal of R then R/T has a quotient ring which is an Artinian principal ideal ring.

CONVENTIONS

All rings will be assumed to be associative with an identity element but will not necessarily be commutative.

When we refer to a two-sided ideal I as being principal we will mean that $I = aR = Ra$ for some $a \in R$.

The abbreviations A.C.C. and D.C.C. will stand for the ascending chain condition and descending chain condition respectively.

$\mathbb{N}$ will denote the natural numbers $\{1,2,3,...\}$.

$\mathbb{Z}$ will denote the integers.

$\mathbb{Q}$ will denote the field of fractions of $\mathbb{Z}$.

When we require a maximal ideal subject to certain properties these will be assumed to exist by Zorn's Lemma.

INTRODUCTION

The main aim of this thesis is to produce and then study a generalization of the unique factorisation domain (U.F.D.) of commutative algebra. In [2] and [5] Chatters and Jordan produce two generalizations of U.F.D. which they call the Noetherian U.F.D. and the Noetherian unique factorization ring (Noetherian U.F.R.). Since the Noetherian condition does not appear in the commutative definition it is better if we do not have to assume in the non-commutative definitions that the ring is Noetherian.It is our aim to show that we do not lose anything by omitting the Noetherian condition. This leaves us with the following definitions.

(1) A ring R is called a unique factorisation ring (U.F.R.) if it is prime and every non-zero prime ideal of R contains a non-zero principal prime ideal.

(2) A ring R is called a unique factorisation domain (U.F.D.) if it is a U.F.R. and also a domain in which each factor ring R/P is a domain where P is a principal prime ideal of R.

In Chapter 2 we shall show where these definitions come from and that they are true generalizations of the commutative U.F.D. We will also show in this chapter that a large amount of structure exists in a U.F.R. without assuming any further conditions and that U.F.R.s are closed under polynomial extensions and skew-polynomial extensions.

In Chapter 3 we show that all the previously known results for Noetherian U.F.D.s which appear in [2] and [8] can be proved for a U.F.D. which merely satisfies the Goldie condition. Therefore in the case of the U.F.D. incorporating Noetherian into the definition is not only too restricting from the point of view of generalization but also unnatural and all that is really needed to produce results is the Goldie condition. We prove in this chapter that a Goldie U.F.D. is a maximal order which is a generalization of the fact that a commutative U.F.D. is integrally closed. We also prove the surprising result that a bounded Goldie U.F.D. is either commutative or a Noetherian principal ideal ring. This was known for a Noetherian U.F.D. [8] but the surprising thing is that the Noetherian nature of the result is not dependant on the initial ring being Noetherian.

In Chapter 4 we take the previously known results for Noetherian U.F.R.s which appear in [5] and prove most of them are true for a U.F.R. which satisfies a polynomial identity. Therefore yet again it seems the results are not dependent on the Noetherian condition but depend to a great extent merely on the inner structure of a U.F.R. Thus it is again probably better to define a U.F.R. without assuming that it is Noetherian.

Having hopefully shown in Chapters 3 and 4 that the Noetherian condition is best left out of the definitions, we look deeper into the structure of U.F.R.s which satisfy a polynomial identity. Therefore in Chapter 5 we look at the reflexive ideals of a U.F.R. with a polynomial identity and show that they are all principal. We also show that if T is a reflexive ideal of R then R/T has a quotient ring which is an Artinian principal ideal ring. This result was previously only known for a prime Noetherian maximal order.

From results in Chapters 3, 4 and 5 it appears that not only is the Noetherian condition unnecessary in the definition of U.F.D.s and U.F.R.s but that many result which were previously known only for Noetherian rings can be proved if not for the whole class of U.F.R.s then at least for large families of them.

CHAPTER I

PRELIMINARIES

This chapter contains most of the known results which will be needed in later chapters. Firstly, we will deal with general ring theory terminology and localization. After this we will deal with the more specialized topic of rings which satisfy a polynomial identity.

In general proofs will be omitted only if they are readily accessible in the literature.

1.1 Terminology

A ring R is said to be *simple* if it contains no non-trivial two-sided ideals.

An ideal I of R is called *prime* if for all ideals A, B of R, $AB \subset I$ implies that either $A \subseteq I$ or $B \subseteq I$. This is equivalent to saying that an ideal I is prime if for all elements a, b of R, $aRb \subseteq I$ implies that either $a \in I$ or $b \in I$.

I is called *semi-prime* or *semi-simple* if for any ideal A of R and $n \in \mathbb{N}$, $A^n \subseteq I$ implies $A \subseteq I$.

R is called *prime* (respectively *semi-prime*) if 0 is a prime (respectively semi-prime) ideal of R.

We will need the following useful theorem concerning prime ideals.

1.2 Theorem

Let S be a multiplicatively closed set in a ring R and let I be an ideal in R maximal with respect to non-intersection with S, (which exists by Zorn's Lemma). Then I is a prime ideal.

Proof

Given $aRb \in I$ we must show that a or b lies in I.

Suppose this is not true.

Then the ideal I+RaR generated by I and a is strictly larger than I.

Therefore $(I+RaR) \cap S \neq \emptyset$.

Hence $s_1 = i_1 + \sum_{j=1}^{n} r_j a t_j$ for some $s_1 \in S, i_1 \in I, r_j, t_j \in R$.

Similarly we have

$$s_2 = i_2 + \sum_{k=1}^{m} x_k b y_k \text{ for some } s_2 \in S, i_2 \in I, x_k, y_k \in R.$$

Therefore

$$S \ni s_1 s_2 = i_1 i_2 + i_1 \sum_{k=1}^{m} x_k b y_k + (\sum_{j=1}^{n} r_j a t_j) i_2$$

$$+ (\sum_{j=1}^{n} r_j a t_j)(\sum_{k=1}^{m} x_k b y_k)$$

The first three terms on the right hand side are clearly in I. Also the fourth term is in I since $aRb \in I$.

Therefore $s_1 s_2 \in S \cap I$.

This is a contradiction therefore I is prime.

□

1.3 Definitions

An element c of R is said to be *left regular* if rc = 0 with $r \in R$ implies r = 0, *right regular* if cr = 0 implies r = 0, and *regular* if it is both left and right regular. R is called a *domain* if every non-zero element of R is regular.

If I is an ideal of R then $c \in R$ is (left or right) *regular modulo I* if c + I is (left or right) regular in the factor ring R/I.

The set of regular elements of a ring is denoted by $C_R(0)$, and the set of regular elements modulo I is denoted by $C_R(I)$. The subscript R will often be omitted where there is no ambiguity.

An element c of R is called a *left unit* of R if there exists $r \in R$ such that $rc = 1$. *Right unit* is defined analogously, and c is called a *unit* if it is both a left and right unit of R.

An element c of R is called *central* if $cr = rc$ for all $r \in R$, c is called *normal* if $cR = Rc$.

The set of all central elements of R is called the *centre* of R and is denoted by $Z(R)$.

A left ideal I of R is said to be *left principal* if $I = Ra$ for some element a of R. Right principal is defined analogously, and a two-sided ideal of R is called *principal* if it is both left and right principal. A ring is called a *principal ideal ring* if every two-sided ideal is principal.

A ring R is said to be *left Noetherian* if it has the ascending chain condition (A.C.C.) on left ideals. R is *left Artinian* if it has the descending chain condition (D.C.C.) on left ideals. Right Noetherian and Right Artinian are defined analogously. A ring is said to be *Noetherian* if it is both left and right Noetherian, and *Artinian* if it is left and right Artinian.

A prime ideal P of R is said to be *height-n* if there exists a chain $P \supset P_n \supset P_{n-1} \supset \dots \supset P_1$ of prime ideals of R and no longer such chain exists.

The *Jacobson radical* $J(R)$ is defined to be the intersection of all the maximal right ideals of R.

A ring R is said to be *local* if $R/J(R)$ is a simple Artinian ring. R is said to be *semi-local* if $R/J(R)$ is semi-simple Artinian.

An R-module M is said to have *finite Goldie dimension* if there does not exist an infinite direct sum of non-zero submodules of M. A submodule E of M is said to be *essential* if E has non-zero intersection with each non-zero submodule of M.

If $S \subseteq R$ then the set $L(S) = \{x \in R \mid xS = 0\}$ is a left ideal of R called the *left annihilator of S.* A left ideal is called a *left annihilator ideal* if it is the left annihilator of some set $S \subseteq R$. *Right annihilator ideals* are defined analogously.

A ring is called a *Goldie ring*, if it has finite Goldie dimension on both sides and has the ascending chain condition for both right and left annihilator ideals.

1.4 LOCALIZATION AND QUOTIENT RINGS

Let S be a subset of R. Then S is a *right Ore set* if and only if given $a \in R$ and $b \in S$ there exist $c \in R$, $d \in S$ such that $ad = bc$. *Left Ore set* is defined analogously, and S is called an *Ore set* if it is both a left and right Ore set.

Now let S be a multiplicatively closed subset of regular elements of R. Then a *left localization of R at S* is an overring $S^{-1}R$ of R such that

(i) each element of S is a unit in $S^{-1}R$;

(ii) each element of $S^{-1}R$ can be written in the form $s^{-1}r$ for some $s \in S$ and $r \in R$.

It is a well-known result that a left localization of R at S exists if and only if S is a left Ore subset of R. (see for example, [4]).

If S is the set of all regular elements of R and a left localization of R at S exists then $S^{-1}R$ is called the *left quotient ring of R*.

The *right quotient ring of R* is defined analogously.

If both the left and right quotient rings of R exist then they are equal (page 21,[4]) and we call them the *quotient ring of R*.

A prime ideal P of R is said to be *left localizable* if there exists a left localization of R at $C_R(P)$. *Right localizable* is defined analogously. An ideal is *localizable* if it is both left and right localizable. The localization at a prime ideal P if it exists will be denoted by $R_{C(P)}$. This is usually denoted by R_P but we will avoid this notation to prevent it being confused with the principal ideal generated by p.

1.5 Lemma

Let R be a ring and Q its quotient ring. If S is a subring of Q with $R \subseteq S \subset Q$ and every regular element of R is a unit of S, then S = Q.

Proof

Let $q \in Q$.

Since Q is the quotient ring of R we can write

$q = ac^{-1}$ for some $a \in R$, $c \in C_R(0)$.

Hence $qc = a$.

But $c \in C_R(0)$ and is therefore a unit in S.

Hence there exists $b \in S$ such that $cb = 1$.

Therefore $q = qcb = ab \in S$.

Thus $Q \subseteq S$.

Hence $Q = S$.

$\square$

Using the previous lemma it can be seen that if every regular element of R is a unit of R then R is its own quotient ring.

The next theorem is probably the most important result about quotient rings.

1.6 Theorem (Goldie)

Let R be any ring, then R has a right quotient ring which is (semi-)simple Artinian if and only if R is a (semi-) prime right Goldie ring.

Proof

(Theorem 1.27 and Theorem 1.28, [4]).

□

Any ring which is its own quotient ring is called a *quotient ring*.

A ring R is called a *(left) right order* if it has a (left) right quotient ring.

R is called an *order* if it is both a left and a right order.

Let Q be a quotient ring. Two orders R and S in Q are *equivalent* if there exist units of Q, a, b, c, d such that $aRb \subseteq S$ and $cSd \subseteq R$.

An order R in Q is called a *maximal order* if whenever S is an order of Q with $R \subseteq S$ and R equivalent to S, then $R = S$.

Before giving the most important theorem about maximal orders we need three definitions.

1.7 Definitions

Let R be an order in a simple Artinian quotient ring Q. A subset I of Q is called an *R-ideal* if

(i) I is an R-R bimodule

(ii) I contains a unit of Q.

(iii) There exist $u, v \in Q$ such that $uI \subseteq R$ and $Iv \subseteq R$.

For an R-ideal I define

$$O_l(I) = \{q \in Q \mid qI \subseteq I\}$$

and

$$O_r(I) = \{q \in Q \mid Iq \subseteq I\}.$$

Note

$O_l(I)$ and $O_r(I)$ are both subrings of Q which contain R.

1.8 Theorem

Let R be an order in an Artinian quoteint ring Q. Then the following properties are equivalent.

(a) R is a maximal order.

(b) For every R-ideal I, $O_l(I) = O_r(I) = R$.

(c) For every ideal I of R, $O_l(I) = O_r(I) = R$.

Proof

For the full proof see (Proposition 3.1, [14]) but in order to give a feel of the proof we will prove (c) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a) here.

(c) $\Rightarrow$ (b)

Let I be an R-ideal.

By definition there exists $\lambda \in Q$ such that $I\lambda \subseteq R$. It is easy to see that $I\lambda R$ is a two-sided ideal of R. Therefore $O_l(I\lambda R) = R$ by (c).

We have $R \subseteq O_l(I) \subseteq O_l(I\lambda R) = R$.

Hence $O_l(I) = R$.

Similarly for $O_r(I)$.

(b) $\Rightarrow$ (a)

First let M be a left R-module contained in Q and λ a unit of Q which is in R. Then if $\lambda M \subseteq R$ we have $M\lambda \subseteq R$.

Since $R\lambda RM\lambda = R\lambda M\lambda \subseteq R\lambda \subseteq R\lambda R$ and since $R\lambda R$ is an R-ideal then $O_r(R\lambda R) = R$ by (b).

Therefore $M\lambda \subseteq R$.

Now let R' be an order of Q containing and equivalent to R.

Therefore there exist $\lambda, \mu \in R$ with λ, μ units of Q such that $\lambda R'\mu \subseteq R$.

By the previous statement we have

$$R'\mu\lambda \subseteq R \text{ and } \mu\lambda R' \subseteq R.$$

Therefore R' is an R-ideal.

Hence $O_l(R') = R$.

But $R'R' \subseteq R'$.

Thus $R' \subseteq O_l(R') = R$.

Therefore $R' = R$.

□

Let R be a maximal order in an Artinian quotient ring Q and I a two-sided ideal of R.

Define $I^* = \{q \in Q \mid qI \subseteq R\}$.

We have $q \in I^* \iff qI \subseteq R$

$\iff IqI \subseteq R$

$\iff Iq \subseteq R$ by Theorem 1.8.

Therefore the definition of I^* is left-right symmetric.

Also note $I \subseteq I^{**} \subseteq R$.

<u>Note</u>

The concept of I^* and I^{**} will only be used in maximal orders.

<u>1.9 Definition</u>

I is a reflexive ideal if $I^{**} = I$.

I is an invertible ideal if $I^*I = II^* = R$.

Note

Clearly an invertible ideal is reflexive.

1.10 Theorem

Let R be a right Goldie ring and let $X_1,\ldots,X_s$ be prime ideals of R. Let K be a right ideal of R and assume for each i that K contains an element of $C(X_i)$, then there exists $c \in K$ such that $c \in C(X_i)$ for all i.

Proof

(Theorem 13.4, [4]).

□

1.11 RINGS SATISFYING A POLYNOMIAL IDENTITY

1.12 Definition

A ring R is said to *satisfy a polynomial identity* if there exists a polynomial in non-commuting variables $x_1,\ldots,x_d$ of the form $\sum_{\sigma\in Sd} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)}$, where the coefficients are ± 1, such that $\sum_{\sigma\in Sd} \alpha_\sigma r_{\sigma(1)} \cdots r_{\sigma(d)} = 0$ for all choices of $r_1,\ldots,r_d \in R$. (Sd denotes the d'th symmetric group). We say that R is of *degree* d if d is the least degree of a polynomial which R satisfies.

The following theorem is one of the most important structure theorems for rings satisfying a polynomial identity.

1.13 Theorem (Kaplansky).

Let R be a prime ring satisfying a polynomial identity of degree d. Then R is a central simple algebra of dimension $\left(\frac{d}{2}\right)^2$ over its centre.

Proof

(Theorem 6.3.1, [12]).

□

1.14 Corollary

Let R be a simple ring satisfying a polynomial identity. Then R is Artinian.

Proof

By Theorem 1.13 R is finitely generated over its centre. But the centre of R is a simple commutative ring and is therefore a field.

Therefore R is finitely generated over a field and so is Artinian.

□

1.15 Theorem (Amitsur)

Let R be a ring satisfying a polynomial identity and $a \in R$ with $r(a) = 0$. Then aR contains a non-zero ideal of R.

Proof

Among all a^kR, $k \in \mathbb{N}$ pick a^nR such that the degree of a^nR is minimal.

Replace a^n by a. Now the degree of aR equals the degree of a^2R.

Let g be a polynomial identity of minimal degree for aR.

Write $g = x_1 g_1(x_2,\ldots,x_d) + g_2(x_1,\ldots,x_d)$ where in g_2, x never appears on the left in any monomial and d is the degree of g.

Set $x_i = a^i r_i$ with arbitrary elements $r_i \in R$.

Then $0 = g(ar_1,\ldots,a^d r_d) = ar_1 g_1(a^2 r_2,\ldots,a^d r_d) + g_2(ar_1,\ldots,a^d r_d)$. Because g is of minimal degree for aR we can choose $r_2,\ldots,r_d$ such that $g_1(a^2 r_2,\ldots,a^d r_d) \neq 0$

Also ar_1 is not on the left of any monomial in g_2.

Therefore $g_2(ar_1,\ldots,a^d r_d) \in a^2 R$.

Hence $0 = ar_1 g_1(a^2 r_2,\ldots,a^d r_d) + a^2 s$ for some $s \in R$.

Thus $r_1 g_1(a^2 r_2,\ldots,a^d r_d) \in aR$ since $r(a) = 0$.

Therefore $Rg_1(a^2 r_2,\ldots,a^d r_d) \subseteq aR$ since r_1 was arbitrary.

Hence $Rg_1(a^2 r_2,\ldots,a^d r_d)R \subseteq aR$.

Since $r(a) = 0$, we have

$$Rg_1(a^2 r_2,\ldots,a^d r_d) \ni ag_1(a^2 r_2,\ldots,a^d r_d) \neq 0.$$

If $Rg_1(a^2 r_2,\ldots,a^d r_2)R = 0$ then $Rg_1(a^2 r_2,\ldots,a^d r_d)$ is already a non-zero ideal in aR, otherwise $Rg_1(a^2 r_2,\ldots,a^d r_d)R$ is a non-zero ideal in aR.

$\square$

Definitions

If R is a ring then a *central polynomial* for R is a polynomial all of whose evaluations on R are central, but which is not a polynomial identity on R.

A central polynomial is *m-central* if it is a central polynomial for $m \times m$ matrices over $\mathbb{Z}$.

1.16 Theorem (Formanek)

There exist m-central polynomials for each m.

Proof

This is the entirity of [6].

□

1.17 Theorem (Rowen, et al.)

Let R be a semi-prime ring which satisfies a polynomial identity. Then any non-zero ideal in R intersects the centre of R non-trivially.

Proof

This will be done in three stages:

(1) Semi-prime rings satisfying a polynomial identity have non-trivial centres.

(2) Any ideal in a semi-prime ring is semi-prime as a ring.

(3) The centre of an ideal of a semi-prime ring is contained in the centre of the ring.

Proof of (1)

Assume first that $0 = \bigcap_{M \lhd R} M_{max}$.

By Theorem 1.13 the R/M are finite dimensional central simple algebras of maximal pi-degree m = ,degree R.

Let g be an m-central polynomial.

There exits an M_o such that pi-degree $(R/M_o) = m$. Therefore there exist $r_i \in R$, $1 \le i \le m$ with

$$0 \neq g(r_1,\dots,r_m) + M_o \in Z(R/M_o).$$

Now if M is such that pi-degree $(R/M) < m$ then

$$g(r_1,\dots,r_m) + M = 0 \in R/M.$$

If M is such that pi-degree $(R/M) = m$ then $g(r_1,...,r_m)\ M \in Z(R/M)$.

Hence $O \neq g(r_1,...,r_m)$ is an element of the centre of R.

Now if R is an arbitrary semi-prime ring which satisfies a polynomial identity then R can be embedded in R[t] which has the property $O = \bigcap_{M \lhd R[t]} M_{max}$ and if Z is the centre of R then Z[t] is the centre of R[t] and so (1) is proved.

<u>Proof of (2)</u>

Let $I \lhd R$, $0 \neq N \lhd I$ with $N^2 = 0$.

Then if $NI = 0$

$N \subseteq L(I) \cap I = K$ say

and $K^2 = 0$ by definition.

Therefore $K = 0$ since R is semi-prime.

Hence $NI \neq 0$.

Put $(NI)^2 \subseteq N^2 = 0$.

Thus $NI = 0$ as R is semi-prime, but this is a contradiction.

<u>Proof of (3)</u>

Let $\alpha \in Z(I)$, $r \in R$ and $i \in I$.

Then $\alpha(ir) = (ir)\alpha$ since $\alpha \in Z(I)$.

Therefore $i\alpha r = ir\alpha$.

Hence $i(\alpha r - r\alpha) = 0$.

But this is true for any $i \in I$.

Therefore $(\alpha r - r\alpha) \in I \cap r\text{-ann } I = L$ say.

But $L^2 = 0$ therefore $L = 0$ since R is semi-prime.

Hence $\alpha r - r\alpha = 0$.

Thus $\alpha r = r\alpha$ and $\alpha \in Z(R)$.

$\square$

From the previous theorem we obtain.

1.18 Corollary

Let R be a semi-prime ring with a polynomial identity and $c \in R$ with $r(c) = 0$. Then cR contains a non-zero central element of R.

Proof

By theorem 1.15 cR contains a non-zero ideal of R.

Therefore by theorem 1.17 cR contains a non-zero central element of R.

□

We can now prove Posner's theorem which gives us several useful facts about prime rings satisfying a polynomial identity.

1.19 Theorem (Posner)

Let R be a prime ring which satisfies a polynomial identity. Then R has a ring of quotients Q obtained by inverting all the central regular elements. Also Q is simple Artinian and satisfies the same polynomial identity as R.

Proof

Let $S = Z(R)\setminus\{0\}$.

Define $Q = R_S = \{ac^{-1} \mid a \in R, c \in S\}$.

In order for Q to be a quotient ring of R it only needs to be shown that if c is regular in R then c is a unit in Q.

By Corollary 1.18 cR contains an element of S.

Therefore $cr = z$ for some $r \in R$, $z \in S$.

Hence $crz^{-1} = 1$ and so c is a right unit in Q. The fact that c is a left unit is proved similarly.

Q is simple by Theorem 1.17.

Q satisfies a polynomial identity since $Q = R \otimes_{Z(R)} Q(Z)$ where Q(Z) is the quotient field of Z(R).

Therefore Q is simple Artinian by Corollary 1.14.

$\square$

This produces several useful corollaries.

1.20 Corollary

If R satisfies a polynomial identity and P is a prime ideal of R then R/P is Goldie.

Proof

By 1.19 and Goldie's theorem 1.6.

$\square$

1.21 Corollary

Let R be a prime ring which satisfies a polynomial identity. Then R is bounded.

Proof

By 1.20 R is Goldie.

Therefore every essential one-sided ideal of R contains a regular element by

(Lemma 1.18 [4]).

Therefore by Theorem 1.15 every essential one-sided ideal contains a two sided ideal.

□

We will need only one more result about rings satisfying a polynomial identity.

1.22 Theorem (Cauchon)

Let R be a prime ring which satisfies a polynomial identity and has ACC on two-sided ideals. Then R is right and left Noetherian.

Proof (Goldie)

(Theorem 5,[9]).

□

CHAPTER 2

NON-NOETHERIAN UNIQUE FACTORISATION RINGS

In this chapter, we will define a unique factorisation ring (UFR) and a unique factorisation domain (UFD).We will also show where these definitions come from in terms of them being generalizations of the Unique Factorization Domain of commutative Algebra. We will then go on to show what preliminary results can be obtained using only the definitions.

In the last part of this chapter we will give some examples of UFR's and UFD's.

2.1 Definition

Let R be a commutative integral domain. An element $a \neq 0$ of R is said to be *irreducible* if it is a non-unit of R and if it is not a product of two non-units of R.

R is called a *commutative unique factorisation domain (UFD)* if every non-zero element is a product of a unit and of a finite number of irreducible elements and such a representation is unique up to order and units.

It is this definition of a commutative U.F.D. which we would like to generalize to the non-commutative case. The definition as it stands above is not in a very nice form for generalization to a non-commutative ring, but fortunately we have the following very useful theorem.

2.2 Theorem (Theorem 5, [13])

A commutative integral domain R is a U.F.D. if and only if every non-zero prime ideal of R contains a non-zero principal prime ideal.

Proof

$\Rightarrow$)

Let R be a commutative U.F.D. and P a non-zero prime ideal in R. Ignoring the trivial case when R is a field. P contains an element a which is not zero or a unit.Since R is a U.F.D. a can be written as a product of irreducible elements. Since P is prime and $a \in P$, P must contain one of these irreducible elements.

But in a U.F.D. an irreducible element generates a prime ideal. Therefore P contains a non-zero principal prime ideal.

$\Leftarrow$)

Let R be a commutative integral domain and every non-zero prime ideal in R contain a principal prime ideal.

Let S be the set of all products of prime elements and units of R.

It is enough to show that S contains every element of R which is not zero, as the uniqueness of expression is easy to show.

Suppose there exists c with $c \notin S$ and $cR \cap S \neq \emptyset$.

Therefore $cb = p_1 \dots p_n$ for some $b \in R$ and prime elements p_i

Choose c and b such that n is minimal.

Since p_1 is prime $p_1 | c$ or $p_1 | b$.

If $p_1 | b$ then $cb_1 = p_2 \dots p_n$ for some $b_1 \in R$ which contradicts the minimality of n.

Therefore $p_1 | c$.

Let $c = p_1 c_1$ for some $c_1 \in R$.

Hence $c_1 b = p_2 \cdots p_n$.

But $c_1 \notin S$ since $c \notin S$, and $c_1 R \cap S \neq \emptyset$ since $cR \cap S \neq \emptyset$.

Therefore we again have a contradiction of the minimality of n.

Hence $c \notin S$ implies $cR \cap S = \emptyset$.

Now suppose $c \notin S$.

Therefore $cR \cap S = \emptyset$.

Let I be the largest ideal containing cR with $I \cap S = \emptyset$.

By theorem 1.2 I is a prime ideal disjoint from S. But by assumption I contains a principal prime ideal and hence a prime element which is a contradiction. Therefore S contains every element in R that is not 0.

Since the elements of S are products of prime elements and units uniqueness of factorization is clear.

$\square$

Using this theorem as our definition it is much easier to produce a generalization of the commutative U.F.D.

2.3 Definition

A domain R is called a *unique factorisation domain* or *U.F.D.* if every non-zero prime ideal of R contains a non-zero principal prime ideal and every factor ring R/P is a domain where P is a principal prime ideal of R.

Clearly by Theorem 2.2 all commutative U.F.D.s are U.F.D.s under the generalized definition.

The requirement that the factor rings are also domains is necessary to produce the prime factorization of elements which we obtain in Corollary 2.10. Unfortunately this property is not as nice as the commutative case. For example it is not stable under the taking of polynomial extensions. However, the property that every prime ideal contains a principal prime ideal is a very stable one and this leads us to a further generalization.

<u>2.4 Definition</u>

A ring R is called a *unique factorisation ring* or *U.F.R.* if it is prime and every non-zero prime ideal of R contains a non-zero principal prime ideal.

Clearly all U.F.D.s are U.F.R.s and so the U.F.R. is a true generalization of the commutative U.F.D. by which we mean that every commutative U.F.R. which is a domain is a commutative U.F.D. as defined in commutative algebra.

We will now look at what can be deduced about the structure of U.F.D.s and U.F.R.s just from the definitions.

<u>2.5. Definition</u>

A *prime element* of a UFR is one that generates a principal prime ideal.

2.6 Lemma

In a UFR all the principal prime ideals are of height-1.

Proof

Let $pR = Rp$ be a principal prime ideal.

Suppose pR is not height -1

Therefore $pR \supsetneq Q \neq 0$ Q prime.

But R is a UFR,

hence $Q \supseteq qR$ q a prime element.

Therefore $pR \supsetneq qR$.

Thus $q = pr$ for some $r \in R$.

We now show p is regular modulo qR.

$RprR \subseteq qR$,

hence $pRrR \subseteq qR$.

Therefore $pR \subseteq qR$ or $rR \subseteq qR$ since qR is prime.

Thus $rR \subseteq qR = Rq$.

Therefore $r = xq$ for some $x \in R$.

Hence $q = pxq$.

Thus $px = 1$ since q regular.

Therefore $pR \supseteq R$.

Hence $pR = R$.

Therefore pR is of height -1.

□

From the above lemma it can be seen that we can replace the definition of a UFR by saying that every prime ideal must contain a height -1 prime ideal and that every height -1 prime ideal is principal.

2.7 Theorem

In a UFR any ideal contains a product of prime elements.

Proof

Let S be the semi-group generated by the prime elements.

Hence S is multiplicatively closed.

Let I be any ideal of R.

If $I \cap S = \emptyset$, maximise I with respect to not intersecting S.

By theorem 1.2 we obtain a prime ideal which does not intersect S.

But since we are in a UFR each prime ideal contains a prime element.

Therefore $P \cap S \neq \emptyset$.

This is a contradiction.

Hence $I \cap S \neq \emptyset$.

□

2.8 Corollary

In a UFR any non-zero element is contained in at most a finite number of height -1 primes.

Proof

Let x be an element of R contained in an infinite number of height - 1 primes.

RxR is an ideal of R.

Therefore $RxR \cap S \neq \emptyset$

Hence $p_1 \dots p_n = \sum_{k=1}^{m} r_k x s_k$ for some p_i prime elements of R,and $r_k, s_k \in R$.

Since x is in an infinite number of height - 1 primes

$x \in qR$ where qR is prime,and $qR \neq p_iR \quad \forall i$.

Hence $x = qt$ for some $r \in R$.

Therefore $p_1 \dots p_n = \sum_{k=1}^{m} r_k q t s_k$.

Thus $p_1 \dots p_n \in qR$ since q is normal.

Hence $p_1R\, p_2R \dots p_nR \subseteq qR$.

Therefore $p_iR \subset qR$ for some i since qR prime.

Hence $p_iR = qR$ since qR is height -1.

But this is a contradiction.

Therefore no element is contained in an infinite number of height-1 prime ideals.

$\square$

2.9 Lemma

In a UFR , $\cap P^n = 0$ for any height -1 prime P.

Proof

Let $I = \cap P^n$.

Take $A, B \not\subseteq I$.

Then $A \not\subseteq P^{n+1}$ for some n.

Let n be the integer such that $A \subseteq P^n$, $A \not\subseteq P^{n+1}$.

Let $X = \{r \in R \mid p^n r \in A\}$ where $pR = P$.

We have $p^n(r_1 + r_2) = p^n r_1 + p^n r_2 \in A$ if $r_1, r_2 \in X$.

Also $p^n(s_1 r_1) = s_2 p^n r_1$, $s_1 \in R$ since p^n normal.

Therefore $p^n(s_1 r_1) \in A$ if $r_1 \in X$.

Hence X is an ideal of R.

Also $X \not\subseteq P$ since $A \not\subseteq P^{n+1}$.

Therefore $A = P^n X$ for some $X \not\subseteq P$.

Similarly $B = YP^k$ for some $Y \not\subseteq P$.

Hence $AB = P^n XYP^k$

$= p^n XYp^k$

$\not\subseteq p^n pRp^k$ since pR is prime.

Therefore $AB \not\subseteq p^{n+k+1}R$.

Thus $AB \not\subseteq I$.

Hence I is prime.

Therefore $I = 0$ or contains some qR for some prime element q.

Assume $qR \subseteq I$.

Hence $qR \subseteq pR$.

Not possible unless $qR = pR$.

Therefore $pR \subseteq p^2R$.

Hence $p = p^2a$.

Thus $p(1-pa) = 0$.

But p is regular,

Therefore $I = 0$.

$\square$

2.10 Corollary

In a UFD any element can be expressed as a finite product of prime elements multiplied by some element in $C = \cap C(P)$ where P ranges over the height -1 primes.

Proof

Let x be any non-zero element in R.

If $x \in p_1R$ for some prime element p_1, then by Lemma 2.9 we can choose $a_1 \in \mathbb{N}$ such that $x = p_1^{a_1} r$ for some $r \in R/p_1R$.

Now if $r \in p_2R$ for some prime element p_2, then by Lemma 2.9 we can choose $a_2 \in \mathbb{N}$ such that $r = p_2^{a_2} s$ for some $s \in R/p_2R$.

Therefore $x = p_1^{a_1} p_2^{a_2} s$, with $s \in R/(p_1R+p_2R)$.

Now if $s \in p_3R$ for some prime element p_3 we continue as for r above.

By Corollary 2.8 x has only a finite number of different prime factors.

Therefore we must eventually obtain $x = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n} y$ where y is contained in no height-1 prime ideals.

Since R/p_iR is a domain for all p_i.

$y \in C(P_i)$ for all $P_i = p_iR$.

Therefore $y \in \cap C(P_i) = C$.

$\square$

2.11 Lemma

Let S be the partial quotient ring of R formed by inverting all the products of prime elements.Then S is simple.

Proof

Let $0 \neq I$ be an ideal of S.

Thus $(I \cap R) \triangleleft R$ and so contains a product of prime elements by Theorem 2.7.

Thus I contains a unit of S, and hence $I = S$.

$\square$

2.12 Lemma

Let R be a UFR and p,q prime elements of R. Then $pRqR = qRpR = qR \cap pR$.

Proof

The result is clearly true if $pR = qR$.

Therefore we can assume that pR and qR are distinct height -1 prime ideals.

Since pR and qR are ideals we have $pRqR \subseteq pR \cap qR$.

Let $a \in pR \cap qR$.

Therefore $a = pr$ for some $r \in R$, hence $pr \in qR$.

Thus $pRr \subseteq qR$ since p is normal.

Therefore $r \in qR$ since qR is prime and $p \notin qR$.

Hence $r = qr_1$ for some $r_1 \in R$.

Thus $a = pqr_1 \in pRqR$.

Therefore $pRqR = pR \cap qR$.

Similarly $pR \cap qR = qRpR$.

Hence $pRqR = qRpR = qR \cap pR$.

$\square$

2.13 Corollary

Let R be a UFR and p,q prime elements of R. Then $pq = uqp = qpw$ where u and w are units of R.

Proof

We have $pq \in pRqR = qRpR$ by previous lemma.

Hence $pq = uqp$ for some $u \in R$.

Also $uRqp = uqpR = pqR = RpRq = RqRp = Rqp$.

Therefore $uR = R$.

Hence u is a left unit of R.

Also $Ruqp = Rpq = RpRq = RqRp = Rqp$.

Thus $Ru = R$.

Hence u is a right unit and hence unit of R.

Similarly for $pq = qpw$.

$\square$

2.14 Lemma

If R is a UFR then every normal element of R is of the form $p_1 \ldots p_n u$ where u is a unit and the p_i's are prime elements of R.

Proof

Let x be a normal element of R.

Then xR is a two-sided ideal of R.

Hence xR contains a product of prime elements by Theorem 2.7.

Therefore $xr = p_1 \dots p_n$ for some prime elements p_i.

Choose r such that n is minimal.

If $r \in p_iR$ for some $1 \le i \le n$,

then $xr = xsp_i$ for some $s \in R$.

Also $p_1 \dots p_n = wp_1 \dots p_{i-1}\, p_{i+1} \dots p_n p_i$ by corollary 2.13.

Therefore $xs = wp_1 \dots p_{i-1}\, p_{i+1} \dots p_n$.

Thus $w^{-1}xs = p_1 \dots p_{i-1}\, p_{i+1} \dots p_n$ since w is a unit.

Therefore $xys = p_1 \dots p_{i-1}\, p_{i+1} \dots p_n$ for some $y \in R$ since x is normal.

This contradicts the minimality of n.

Therefore $r \notin p_iR$ for any $1 \le i \le n$.

Since $xr = p_1 \dots p_n \subseteq p_1R$ and x is normal we have

$xRr \subseteq p_1R$.

Therefore $x \in p_1R$ since p_1R is prime and $r \notin p_1R$.

Hence $x = p_1x_1$.

Thus $p_1x_1r = p_1 \dots p_n$.

Hence $x_1r = p_2 \dots p_n$.

Also $p_1x_1R = xR = Rx = Rp_1x_1 = p_1Rx_1$.

Hence $x_1R = Rx_1$ and so x_1 is normal.

Proceeding as above with $x_1r = p_2 \dots p_n$ we eventually obtain $x_{n-1}r = p_n$.

Therefore $x_{n-1} \in p_n R$ as above.

Hence $x_{n-1} = p_n x_n$ for some $x_n \in R$.

Thus $p_n x_n r = p_n$.

Therefore x_n is a unit u say.

Hence $x = p_1 \dots p_n u$.

$\square$

The proof of the next theorem is based on the proof of the Noetherian case by Chatters and Jordan (Theorem 3.1,[5]).

2.15 Theorem

If R is a UFR then R[x] is a UFR.

Proof

This proof relies on the use of S the simple partial quotient ring of R defined in Lemma 2.11.

Let $R^* = R[x]$ and $S^* = S[x]$.

Clearly since R is prime R^* is prime.

S^* is the partial quotient ring of R^* formed by inverting all the elements of R which are products of prime elements of R.

First we will show that every non-zero prime ideal of S^* is generated by a central element.

Let P be a non-zero prime ideal of S^*.

Let f be a non-zero element of P of least degree

$\deg(f) = n$.

The subset of S consisting of zero together with the leading coefficients of elements of P of degree n is a non-zero ideal of S, which therefore must equal S since S is simple by (Lemma 2.11).

Therefore we can suppose that f is monic.

Take $g \in P$

$g = fq + r$ for some $q,r \in S^*$ with $\deg(r) < \deg(f)$.

But $r \in P$.

Therefore by minimality of n, $r = 0$.

Hence $P = fS^*$.

Clearly $xf = fx$.

Also $\forall s \in S$, $sf - fs \in P$ and has degree less than $\deg(f)$.

Therefore by minimality of n, $sf = fs$.

Hence f is central in S^*.

Hence P is centrally generated.

We will now show that every prime ideal of R^* contains a principal prime ideal.

Let Q be a non-zero prime ideal of R^*.

Suppose $Q \cap R \neq 0$.

Then $Q \cap R$ is a non-zero prime ideal of R and so contains a prime element p.

Therefore Q contains the non-zero principal prime ideal

$pR^* = R^*p$ of R^*.

So next suppose $Q \cap R = 0$ so that $QS^* \neq S^*$ and is a non-zero prime ideal of S^*.

Then $QS^* = fS^*$ for some central element f of S^*.

Let D be the multiplicatively closed set of products of prime elements of R.

Since $fS^* = QS^*$,

$f = gd^{-1}$ for some $g \in Q$, $d \in D$.

$Rg = Rfd = fRd = fdR = gR$.

Also $gx = xg$.

Thus $gS^* = QS^*$ and $R^*g = gR^*$.

By corollary 2.8 g is contained in only a finite number of height-1 prime ideals.

Also by lemma 2.9 , $\cap P^n = 0$ for all height-1 prime ideals.

Therefore g has only a finite number of prime factors.

Choose g with a minimum number of prime factors such that $gS^* = QS^*$ and $R^*g = gR^*$.

Suppose that $gR^* \neq Q$.

Let $h \in Q/gR^*$.

Since $h \in QS^*$, $hd' \in gR^*$ for some $d' \in D$.

Because d' is a product of prime elements of R we can assume without loss of generality that $hp \in gR^*$ for some prime element $p \in R$.

Thus $hp = gb$ for some $b \in R^*$.

Therefore $gR^*b = R^*gb = R^*hp \subseteq R^*p$ where R^*p is prime in R^*.

Since $hp = gb$ and $h \notin gR^*$,

$b \notin R^*p$.

Therefore $g \in R^*p$.

Hence $g = g'p$ for some $g' \in R^*$.

But $g'R^*p = g'pR^* = gR^* \subseteq Q$

and $p \notin Q$ since $Q \cap R = 0$.

Therefore $g' \in Q$ since Q is prime.

Also $g'R^*p = gR^* = R^*g = R^*g'p$,

hence $g'R^* = R^*g'$ and

$g'S^* = gS^* = QS^*$ since p is invertible in S^*.

But g' has one less prime factor than g. This contradicts the choice of g.

Therefore $gR^* = Q$.

Thus in both cases Q contains a non-zero principal prime ideal.

$\square$

2.16 Corollary

If R is a UFR then $R[x_1,x_2,...]$ is a UFR.

Proof

Let Q be a non-zero prime ideal of $R[x_1,...]$.

Then Q contains a non-zero element in $R^+ = R[x_i]_{i \in I}$ for some finite set I.

$Q \cap R^+$ is a non-zero prime ideal in R^+.

R^+ is a UFR by repeated use of Theorem 2.15.

Therefore $Q \cap R^+$ contains a prime element p of R^+.

Hence Q contains the non-zero principal prime ideal

$pR[x_1,...] = R[x_1,...]\,p$ of $R[x_1,...]$.

$\square$

Having shown that a polynomial extension of a U.F.R. is a U.F.R. we will now look at what happens if the polynomial extension is skewed by an automorphism. We will show that if the automorphism is of finite order then a skew polynomial extension of a U.F.R. is itself a U.F.R.

2.17 Definitions

If R is a ring and α an automorphism then the *skew polynomial ring* $R[x,\alpha]$ consists of polynomials in x with coefficients from R written on the left and $xr = \alpha(r)x$ for all $r \in R$. Two polynomials are multiplied in the usual way ,term by term, using the above relation.

An α-*ideal* of R is an ideal I of R such that $\alpha(I) \subseteq I$.

An α-*prime ideal* of R is an α-ideal P such that if X and Y are α-ideals of R with $XY \subseteq P$ then $X \subseteq P$ or $Y \subseteq P$.

Note

We will always have the automorphism α of finite order in which case an ideal I is an α-ideal if and only if $\alpha(I) = I$.

Proof

Let n be the order of α.

Then we have $I = \alpha^n(I) \subseteq \alpha^{n-1}(I) \subseteq \ldots\ldots\ldots\ldots \subseteq \alpha(I) \subseteq I$

Therefore we must have equality throughout.

$\square$

We aim to show that if R is a UFR and α an automorphism of R of finite order then $R[x;\alpha]$ is a UFR.The method of proof is based on that of the polynomial extension earlier in this chapter.

2.18 Theorem

Let R be a UFR and α an automorphism of R of finite order. Then every non-zero α-prime ideal of R contains a non-zero principal α-prime ideal of R.

Proof

Let P be a non-zero α-prime ideal of R.

Since P is an ideal it contains a product of prime elements by Theorem 2.7.

Therefore $P \supseteq p_1 \ldots p_n R$ where the p_i are prime elements of R.

Let $p_i R = P_i$ for all $1 \leq i \leq n$.

Also let the order of α be m.

Hence we have

$$P \supseteq P_1 \ldots P_n \alpha(P_1) \ldots \alpha(P_n) \alpha^2(P_1) \ldots \alpha^2(P_n) \ldots \alpha^{m-1}(P_1) \ldots \alpha^{m-1}(P_n).$$

Since α maps ideals to ideals it must map height -1 prime ideals to height -1 prime ideals.

But if X and Y are height -1 prime ideals then

$$XY = X \cap Y = YX \text{ by lemma 2.12.}$$

Hence $P \supseteq P_1\alpha(P_1)\ldots\alpha^{m-1}(P_1).P_2\alpha(P_2)\ldots\alpha^{m-1}(P_2)\ldots P_n\alpha(P_n)\ldots\alpha^{m-1}(P_n)$.

Also $P_i\alpha(P_i)\ldots\alpha^{m-1}(P_i) = \alpha(P_i)\alpha^2(P_i)\ldots\alpha^{m-1}(P_i)P_i$.

Thus $P_i\alpha(P_i)\ldots\alpha^{m-1}(P_i)$ is an α-ideal for all $1 \leq i \leq n$.

Since P is an α-prime ideal of R

$$P \supseteq P_i\alpha(P_i)\ldots\alpha^{m-1}(P_i) \text{ for some } 1 \leq i \leq n.$$

Let t be the smallest integer such that $\alpha^t(P_i) = P_i$.

Then $P \supseteq P_i\alpha(P_i) \ldots \alpha^t(P_i)$ since P is an α-prime ideal.

Clearly $P_i\alpha(P_i) \ldots \alpha^t(P_i)$ is a principal α-ideal.

Also $P_i, \alpha(P_i), \ldots, \alpha^t(P_i)$ are all distinct height -1 prime ideals.

It just remains to show that $P_i\alpha(P_i) \ldots \alpha^t(P_i)$ is an α-prime ideal.

Let A, B be α-ideals of R with

$$AB \subseteq P_i\alpha(P_i) \ldots \alpha^t(P_i).$$

Therefore $AB \subseteq P_i$.

Hence $A \subseteq P_i$ or $B \subseteq P_i$ since P_i is a prime ideal.

We may assume $A \subseteq P_i$.

Also $A = \alpha^s(A) \subseteq \alpha^s(P_i)$ for all $1 \le s \le t$.

Therefore $A \subseteq P_i \cap \alpha(P_i) \cap \ldots \cap \alpha^t(P_i)$.

But since $P_i, \alpha(P_i), \ldots, \alpha^t(P_i)$ are distinct height -1 prime ideals

$$P_i \cap \alpha(P_i) \cap \ldots \cap \alpha^t(P_i) = P_i.\alpha(P_i) \ldots \alpha^t(P_i).$$

Hence $A \subseteq P_i\ \alpha(P_i) \ldots \alpha^t(P_i)$ and we have shown that $P_i\alpha(P_i) \ldots a^t(P_i)$ is a principal α-prime ideal contained in P.

□

It should be noted that the property that every α-prime ideal contains a principal α-prime ideal is analogous to the property used in the definition of a UFR that every prime ideal contains a principal prime ideal but ideals have been replaced by α-ideals. Several of the early results of this chapter can be proved for α-ideals instead of ideals by simply replacing ideals with α-ideals in the proofs.

2.19 Definition

An element $b \in R$ is called an α-*prime element* if $bR = Rb$ is a non-zero α-prime ideal of R.

2.20 Lemma

Let R be a UFR and α an automorphism of R of finite order. Then any α-ideal of R contains a product of α-prime elements of R.

Proof

By theorem 2.18 every non-zero α-prime ideal of R contains a non-zero principal α-prime ideal of R. Now use the proof of Theorem 2.7 with ideals replaced by α-ideals.

□

2.21 Corollary

Let R be a UFR and α an automorphism of R of finite order. Then any element of R is contained in at most a finite number of principal α-prime ideals.

Proof

By theorem 2.18 and the proof of Corollary 2.8 with ideals replaced by α-ideals.

□

2.22 Lemma

Let R be a UFR and α an automorphism of R of finite order. Then if P is a principal α-prime ideal of R , $\cap P^n = 0$.

Proof

By theorem 2.18 and the proof of Lemma 2.9 with ideals replaced by α-ideals.

$\square$

2.23 Lemma

Let R be a UFR and α an automorphism of R of finite order. Let S be the partial quotient ring of R formed by inverting all the products of α-prime elements of R. Then S is α-simple. That is S contains no non-trivial α-ideals, where α is extended from R to S, by $\alpha(rp^{-1}) = \alpha(r).\alpha(p)^{-1}$.

Proof

Let $0 \neq I$ be an α-ideal of S.

Therefore $I \cap R$ is an α-ideal of R.

Hence $I \cap R$ contains a product of α-prime elements of R by Lemma 2.19.

Therefore $I \cap R$ contains a unit of S.

Hence I = S.

$\square$

2.24 Theorem

Let R be a UFR and α an automorphism of R of finite order. Then the skew-polynomial ring $R[x;\alpha]$ is a UFR.

Proof

By theorem 2.18 we have that every non-zero α-prime ideal of R contains a non-zero principal α-prime ideal of R. We shall use the above fact to show that $R[x;\alpha]$ is a UFR by modifying the proof of Theorem 2.15.

Let S be the partial quotient ring of R formed by inverting all the products of α-prime elements of R. α can easily be extended to an automorphism of S which we shall also call α.

By Lemma 2.23 S is α-simple. That is 0 and S are the only α-ideals of S.

For ease of notation set $R^* = R[x;\alpha]$ and $S^* = S[x;\alpha]$.

Clearly R^* is a prime ring.

Also xS^* is a principal prime ideal of S^*.

Let P be any non-zero prime ideal of S^* such that $x \notin P$. We will show that P is principal.

Let f be a non-zero element of P of minimal degree. Let the degree of f be n.

Let L be the subset of S consisting of 0 together with the leading coefficients of elements of P of degree n.

Clearly L is an ideal of S.

Also if $a \in L$, $ax^n + bn^{n-1} + ... + cx + d \in P$.

Therefore , $x(ax^n + bx^{n-1} + ... + cx + d) \in P$.

Hence , $\alpha(a)x^{n+1} + \alpha(b)x^n + \ldots + \alpha(c)x^2 + \alpha(d)x \in P$.

Thus , $(\alpha(a)x^n + \alpha(b)x^{n-1} + \alpha(c)x + \alpha(d))x \in P$.

But x is regular modulo P.

Therefore $\alpha(a)x^n + \alpha(b)x^{n-1} + \ldots + \alpha(c)x + \alpha(d) \in P$.

Hence $\alpha(a) \in L$.

Therefore L is a non-zero α-ideal of S.

Hence $L = S$ since S is α-simple.

Thus we may suppose that f is monic.

Let $g \in P$.

Then $g = fh + j$ for some $h, j \in S^*$, with degree of $j <$ degree of f.

But $j = g - fh \in P$ and degree of $j <$ degree of f.

Therefore by minimality of the degree of f we have $j = 0$.

Hence $P = fS^*$.

Let $s \in S$ then $sf - f\alpha^n(s) \in P$ and has degree less than the degree of f.

Hence $sf = f\alpha^n(s)$.

Therefore $Sf = fS$.

Also $xf - fx = xh$ for some $h \in S$.

But $xh \in P$ and x is regular modulo P.

Hence $xf - fx = xh$ for some $h \in P$.

But h has degree less than the degree of f.

Thus $h = 0$.

Therefore $xf = fx$.

Hence $P = fS^* = S^*f$.

We will now show that every prime ideal of R^* contains a principal prime ideal and hence R^* is a UFR.

Let Q be a non-zero prime ideal of R^*.

If $x \in Q$ then Q contains xR^* which is principal and is prime because $\dfrac{R^*}{xR^*} \cong R$ which is prime.

Therefore suppose $x \notin Q$.

Suppose $Q \cap R \neq 0$.

We have $xQ = \alpha(Q)x \subseteq Q$.

Therefore $\alpha(Q) \subseteq Q$ since x is regular modulo Q.

Hence $\alpha(Q \cap R) \subseteq Q \cap R$.

Thus $Q \cap R$ is a non-zero α-prime ideal of R.

Therefore $Q \cap R$ contains an α-prime element q by Theorem 2.18.

Since qR is an α-prime ideal

$qR^* = R^*q$ is a prime ideal of R^*.

This only leaves the case when $Q \cap R = 0$.

In this case QS^* is a proper prime ideal of S^*.

If $x \in QS^*$, then $xs \in Q$ for some s a product of α-prime elements of R.

Therefore $xR^*s \subset Q$.

But since Q is prime and $x \notin Q$ we have that $0 \neq s \in Q \cap R$ which is a contradiction.

Hence $x \notin QS^*$.

Therefore as shown at the beginning of the proof $QS^* = fS^* = S^*f$ for some $f \in S^*$ with $xf = fx$ and $Rf = fR$.

Let $f = gd^{-1}$ for some $g \in Q$ and d a product of α-prime elements of R.

Since dR is an α-ideal of R, $\alpha(d) = du$ for some unit $u \in R$.

Hence $Rg = Rfd = fRd = fdR = gR$.

Also $xg = xfd = fxd = fdux = gux = gx\alpha^{-1}(u)$.

Therefore $R^*g = gR^*$.

To complete the proof we need to show that $Q = gR^*$.

The proof of this fact is as in the proof of Theorem 2.15 with prime elements replaced by α-prime elements.

Hence $R[x,\alpha]$ is a UFR.

□

2.25 Examples

Since the definition of a UFR is a generalization of both the definition of a commutative unique factorisation ring (2.1) and that of a Noetherian UFR [5] any example of either of these is a UFR. These include the universal enveloping algebras of finite dimensional solvable Lie algebras since they are Noetherian UFR's by (Corollary 5.6, [5]).

From the definition of a UFR it is also clear that any principal ideal ring is also a UFR.

Using Corollary 2.16 we have that $R[x_1,...]$ is a UFR whenever R is a UFR. This provides us with a method for producing non-Noetherian UFRs by simply taking a Noetherian UFR and adding on an infinite number of commuting indeterminates. Therefore $H[x_1,x_2,...]$ where H is the quaternions is a non-commutative non-Noetherian UFR.

Also it is easy to see that if R is a UFR then so is $M_n(R)$ the ring of $n \times n$ matrices over R. This along with the skew polynomial ring gives us two good methods for producing non-commutative examples of UFRs from commutative unique factorisation rings. Since there is a theorem of Gilmer's which states that there are commutative non-Noetherian unique factorization rings of every Krull dimension derived from group rings, we can take matrices of any size over one of these rings and produce an infinite number of strictly non-commutative non-Noetherian UFRs. These examples are important because since they are matrices over commutative rings they satisfy a polynomial identity (see Chapter 4). As a concrete example $M_2(\mathbb{Z}[x_1,x_2,...])$ is one of the easiest to see what is happening, and this will be used throughout later chapters to illustrate various results.

Closer examination of the proof of Theorem 2.15 shows that S[x] is a UFR whenever S is simple. This unfortunately will not produce examples of non-Noetherian UFRs with a polynomial identity since by corollary 1.14 any simple ring with a polynomial identity is automatically Artinian. However $S[x_1,...]$ is always a non-Noetherian UFR for every simple ring S.

CHAPTER 3

NON-NOETHERIAN UNIQUE FACTORISATION DOMAINS

In this chapter we will look at unique factorisation domains. In order to achieve any results we will need the Ore condition and so will only look at Goldie unique factorisation domains.

By looking at a particular partial quotient ring T of a Goldie UFD R (Theorem 3.2) we will show that R is a maximal order and that if R is bounded (i.e. every essential one sided ideal contains a two sided ideal) then R is either commutative or a Noetherian principal ideal ring. This theorem shows the difficulty in producing strictly non-commutative non-Noetherian examples of UFDs.

3.1 Lemma

If R is a UFD with the Goldie condition then R satisfies the Ore condition with respect to $C = \cap C(P)$ where P ranges over all the height -1 primes of R.

Proof

Let $a \in R$, $c \in C$.

By Goldie's theorem

$ax = cb$ for some $x, b \in R$.

But $x = dp_1 \dots p_n$ for some $d \in C$ where each p_i is a prime element of R by Corollary 2.10.

Thus $cb = adp_1 \dots p_n \in Rp_n$.

Since $c \in C(Rp_n)$, $b \in Rp_n$.

Therefore $b = b_n p_n$ for some $b_n \in R$.

Thus $adp_1 \dots p_n = cb_np_n$.

Hence $adp_1 \dots p_{n-1} = cb_n$.

Similarly since $c \in C(Rp_{n-1})$ we have $b_n = b_{n-1}p_{n-1}$.

Hence $adp_1 \dots p_{n-2} = cb_{n-1}$.

Repeating this process a finite number of times gives

$ad = cb_1$ for some $b_1 \in R$.

□

In the commutative case C consists of the units of R but even in the Noetherian case there are UFD's where this is not the case.Although if R is a prime Noetherian ring which satisfies a polynomial identity then the elements of C are units, [3].

The next theorem's proof is based on the proof for a Noetherian UFD (Theorem 2.7 [2]) but the order in which the various elements are proved has to be changed in order to avoid using the Noetherian condition.

3.2 Theorem

If R is a Goldie U.F.D. and T the partial quotient ring of R with respect to $C = \cap C(P)$ then

(1) T is a UFD.

(2) The elements of C(T) are units of T where $C(T) = \cap C(Q)$ when Q ranges over the height -1 prime ideals of T.

(3) Every one sided ideal of T is two-sided.

(4) $AB = BA$ for all ideals A and B of T.

Proof

First we will show that if p is a prime element of R then $pT = Tp$.

Take $t \in T$.

Then $pt = prc^{-1}$ for some $r \in R, c \in C$

$= r_1pc^{-1}$ for some $r_1 \in R$ since p is normal.

By the Ore condition $pc^{-1} = d^{-1}s$ for some $d \in C, s \in R$.

Hence $dp = sc$.

Therefore $sc \in pR = Rp$.

But $c \in C$ therefore $c \in C(pR)$.

Hence $s \in pR$.

We have $pt = r_1d^{-1}s$.

Therefore $pt = r_1d^{-1}r_2p$ for some $r_2 \in R$

Thus $pt \in Tp$.

Hence $pT \subseteq Tp$.

Similary $Tp \subseteq pT$.

Therefore $pT = Tp$.

We will next prove part (3).

Let x be a non-zero element of T.

Then $x = up_1 \ldots p_nc^{-1}$ where $u, c \in C$ and the p_i are prime elements of R.

Hence $xT = up_1 \ldots p_nc^{-1}T$

$= up_1 \ldots p_n Tc^{-1}$ since c^{-1} is a unit of T.

But as shown above $pT = Tp$ for any prime element p of R.

Hence $xT = uTp_1 \ldots p_nc^{-1}$.

Therefore $xT = Tup_1 \ldots p_nc^{-1}$ since $u \in C$ and hence is a unit of T.

Thus $xT = Tx$.

Hence every one-sided ideal of T is a two-sided ideal.

In order to show that the multiplication of ideals of T is commutative we first need to show that $pTqT = qTpT$.

But $pTqT = pqT = qpvT$ for some unit v of R by Corollary 2.13.

Therefore $pTqT = qpT = qTpT$ as required.

Now we will show that $AB = BA$ for any two ideals A and B of T.

Let $a \in A$ and $b \in B$.

By Corollary 2.10 $a = up_1...p_nc^{-1}$ and $b = wq_1...q_md^{-1}$, where the p_i and q_i are prime elements of R and $u,c,w,d \in C$.

$abT = up_1...p_nc^{-1}wq_1...q_md^{-1}T$.

Therefore $abT = Tp_1Tp_2T...p_nTq_1T...q_mT$ since u,c,w and d are units of T.

Hence $abT = Tq_1Tq_2...q_mTp_1T...p_nT$

$= wTq_1T...q_mTd^{-1}up_1T...p_nTc^{-1}$

$= wq_1...q_md^{-1}up_1...p_nc^{-1}T$

$= baT$.

Therefore $ab \in BA$.

Hence $AB \subseteq BA$.

Similarly $BA \subset AB$ and therefore $AB = BA$.

Thus we have proved part (4).

In order to prove (1) we need two results that are only known in general in the Noetherian case.

Firstly we need that if P is a prime ideal of T then $P \cap R$ is a prime ideal of R.

Let A and B be ideals of R with $AB \subseteq P \cap R$.

Therefore $TABT \subseteq P$.

But TA and BT are two-sided ideals by (3).

Hence we may assume $TA \subseteq P$ since P is prime.

Therefore $A \subseteq P \cap R$.

Hence $P \cap R$ is a prime ideal of R.

We also need that if Q is a height -1 prime ideal of R then QT is a prime ideal of T.

Let $aTb \subseteq QT = qT$ with $a,b \in T$.

We also have $a = c_1^{-1}r_1$, $b = r_2c_2^{-1}$ with $c_1, c_2 \in C$, $r_1, r_2 \in R$.

Hence $c_1^{-1}r_1\, Tr_2c_2^{-1} \subseteq QT$.

Therefore $r_1Tr_2 \subseteq QT$ since QT is an ideal of T by (3).

Hence $r_1Rr_2 \subseteq QT \cap R = Q$.

Therefore $r_1 \in Q$ or $r_2 \in Q$ since Q is prime.

Thus $c_1^{-1}r_1 \in QT$ or $r_2c_2^{-1} \in QT$.

That is $a \in QT$ or $b \in QT$.

Hence QT is a prime ideal of T.

Now let A be a height-1 prime ideal of T.

Therefore $A \cap R$ is a prime ideal of R as shown above.

Since R is a U.F.D. $A \cap R \supseteq B$ where B is a height-1 prime ideal of R.

Hence $A \supseteq BT$ which is a prime ideal of T as shown above.

Thus $A = BT$ since A is height-1.

Therefore every height-1 prime ideal of T is the extension of a height-1 prime ideal of R.

Now let Q be a height-1 prime ideal of R.

Therefore QT is a prime ideal of T.

Assume QT is not height-1.

Then $QT \supsetneq P$ a prime ideal of T.

Hence $QT \cap R \supseteq P \cap R \supseteq A$ a height-1 prime ideal of R since P is prime.

But $QT \cap R = Q$ since the elements inverted to form T are regular modulo Q.

Therefore $Q = A$ since Q is a height-1 prime ideal.

Hence $P \supseteq (P \cap R)T \supseteq AT = QT$.

This is a contradiction of $QT \supsetneq P$.

Therefore the extensions of height-1 prime ideals of R are height-1 prime ideals of T.

Combining these two results we obtain the result that the height -1 prime ideals of T are precisely the extensions to T of the height -1 prime ideals of R. We have already seen that $pT = Tp$ for any prime element p of R. Therefore the prime elements of R are the prime elements of T. Therefore every height -1 prime ideal of T is of the form $pT = Tp$ for some prime element p of R. In order to prove part (1) it only remains to show that T/pT is a domain for any prime element p of R.

Assume $ab \in pT$ where $a,b \in T$.

Let $a = c_1^{-1}r_1$ and $b = r_2c_2^{-1}$ for some $c_1, c_2 \in C$, $r_1, r_2 \in R$.

Therefore $c_1^{-1}r_1r_2c_2^{-1} \in pT$.

Hence $r_1r_2 \in pT$.

Thus $r_1r_2 \in pT \cap R = pR$.

But R/pR is a domain since R is a U.F.D.

Therefore $r_1 \in pR$ or $r_2 \in pR$.

Hence $a \in pT$ or $b \in pT$.

Thus T/pT is a domain and so we have proved that T is a U.F.D.

It only remains to prove part (2).

Let $t \in C(T)$ then $t = ac^{-1}$ for some $a \in R$, $c \in C$.

Thus $a = tc$ where c is a unit of T.

So $a \in C(T) \cap R$.

Therefore $a \in C(R)$ since the height -1 primes of T are extensions of height -1 primes of R.

Hence a is a unit of T and therefore so is t.

$\square$

The ring T will play a very important part in the proofs of the major theorems of this chapter. This is because as can be seen from the above theorem and also from the corollary to the following proposition T is very well behaved as a ring. This allows us to prove some important facts about R by first proving them for T and then showing that the properties of T intersect down to R.

3.3 Proposition

Let R be a strictly non-commutative Goldie U.F.D. Then every prime ideal of R with height greater than one contains an element of C.

Proof

Let P be a prime ideal of R with height greater than one.

Since R is a U.F.D. P contains a height -1 principal prime ideal.

First assume that P contains exactly one height -1 prime pR.

Choose $a \in P - pR$.

Then $a = cp_1 \dots p_n$ for some $c \in C$ and prime elements p_i.

Also $p_i \notin P$ for all $1 \leq i \leq n$.

We have $cp_1 \dots p_nR \subseteq P$.

Hence $cp_1 \dots p_{n-1}Rp_n \subseteq P$.

Therefore $cp_1 \dots p_{n-1} \in P$ since P is prime and $p_n \notin P$ by assumption.

Repeating this process $n-1$ times we obtain $c \in P$ as required.

Now suppose that P contains two distinct height -1 prime ideals pR and qR.

Choose an element $r \in R$.

For each positive integer n define

$$t_n = p + q(r + q^n) \in P.$$

Suppose the proposition is false.

Then $t_n \notin C$ for all n.

Therefore each $t_n \in T_n$ where T_n is a height -1 prime ideal.

By the argument used above we can assume that $T_n \subseteq P$ for every n.

If $q \in T_n$ then $p \in T_n$.

Therefore $qR = T_n = pR$ since qR, T_n and pR are all height -1 prime ideals.

Since $qR \neq pR$ we have $q \notin T_n$ for all n.

Suppose $T_m = T_n$ for some integers m and n with $m < n$. Then $t_m - t_n \in T_m$.

Hence $q^{m+1} - q^{n+1} \in T_m$.

Therefore $q^{m+1}(1-q^{n-m}) \in T_m$.

Therefore $(1-q^{n-m}) \in T_m$ since T_m is prime and q is normal with $q \notin T_m$.

Hence $1-q^{n-m} \in P$.

But $q \in P$ hence $1 \in P$ which is a contradiction. Therefore the set of ideals T_n is infinite. Since only finitely many height -1 primes lie over any non-zero element of

R by corollary 2.8 we have $\cap T_n = \{0\}$.

We will now obtain the desired contradiction by producing a non-zero element of $\cap T_n$.

This will be done in three cases.,

Case I

Suppose p and q are both central elements of R.

Choose r to be any non-central element of R.This exists since R is strictly non-commutative.

Also choose $s \in R$ with $sr \neq rs$.

Then $st_n - t_ns = sp+sq(r+q^n) - ps - q(r+q_n)s$

$= sqr - qrs$

$= q(sr-rs)$.

Hence $q(sr-rs) \in T_n$.

Therefore $qR(sr-rs) \subseteq T_n$.

Thus $sr-rs \in T_n$ since T_n is prime and $q \notin T_n$.

Hence $0 \neq sr-rs \in \cap T_n$.

Case II

Suppose $pq = qp$ but q is not a central element of R.

Choose $r \in R$ with $qr \neq rq$.

Let s be such that $rq = qs$.

Then $t_nq - qt_n = qrq-q^2r$

$= q^2(s-r)$.

Therefore $q^2(s-r) \in T_n$ for all n.

Hence $q^2R(s-r) \subseteq T_n$ for all n.

Thus $(s-r) \in T_n$ for all n since T_n is prime and $q \notin T_n$.

Therefore $0 \neq s-r \in \cap T_n$.

Case III

Suppose $pq \neq qp$.

Take $r = 0$.

Then $t_n q - q t_n = pq-qp$.

Hence $0 \neq pq-qp \in T_n$ for all n.

Thus $0 \neq pq-qp \in \cap T_n$.

□

3.4 Corollary

Let R be a strictly non-commutative Goldie U.F.D. Let T be the partial quotient ring of R with respect to C. Then every one-sided ideal of T is two sided and principal and T is Noetherian.

Proof

Let I be a one-sided ideal of T.

By Theorem 3.2 part (3) I is a two-sided ideal of T. If $I \neq T$ then I is contained in some maximal ideal M.

We have $M = (M \cap R)T$ and $M \cap R$ is a proper prime ideal of R. Therefore by Proposition 3.3 $M \cap R$ has height-1.

Hence as in the proof of Theorem 3.2 M also has height- 1. Thus I is contained in some height -1 prime ideal of T.

Let $p_1,\ldots,p_n$ be the prime elements of T with $I \subseteq p_i T$ for $1 \leq i \leq n$.

There are only a finite number by corollary 2.8.

Also let $a_1,\dots,a_n \in \mathbb{N}$ such that $I \subseteq p_i^{a_i}T$ but $I \not\subseteq p_i^{(a_i+1)}T$ for all $1 \le i \le n$.

This is possible since $\cap P^n = 0$ by lemma 2.9.

$I \subseteq p_1^{a_1}T \cap p_2^{a_2}T \cap \dots\dots\dots \cap p_n^{a_n}T = p_1^{a_1}p_2^{a_2}\dots\dots p_n^{a_n}T$.

Set $J = \{x \in T \mid p_1^{a_1} \dots p_n^{a_n} x \in I\}$.

Clearly $I = p_1^{a_1} \dots p_n^{a_n}J$

Then since the p_i are normal in T it is easy to see that $J \triangleleft T$.

But by the way we constructed J we have J not contained in any height -1 prime ideal of T,otherwise we would have a contradiction of the choice of elements p_i or a contradiction of the maximality of one of the a_i.

By the first part of the above proof this implies that $J = T$. Therefore $I = p_1^{a_1} \dots p_n^{a_n} T$.

All that remains is to show that T is Noetherian.

But every one-sided ideal of T is two-sided and principal.

Therefore T is Noetherian.

$\square$

3.5 Definition

A ring R is said to be *bounded* if every essential one-sided ideal of R contains a non-zero two-sided ideal of R.

Note

In the following lemma the intersection $T \cap S$ takes place inside the quotient ring of R which exists since R is a prime Goldie ring.

3.6 Lemma

Let R be a Goldie U.F.D., T the partial quotient ring of R with respect to C and S the partial quotient ring of R with respect to the multiplicatively closed set D generated by the prime elements of R. Then $R = T \cap S$.

Proof

Clearly $R \subseteq T \cap S$.

Let $u \in T \cap S$.

Since $u \in S$, $p_1 \dots p_n u \in R$ for some prime elements p_i of R.

Since $u \in T$, $p_2 \dots p_n u \in T$.

Therefore $p_2 \dots p_n uc \in R$ for some $c \in C$.

Hence $p_1 p_2 \dots p_n uc \in p_1 R$.

Therefore $p_1 p_2 \dots p_n u \in p_1 R$ since $c \in C(p_1 R)$ and $p_1 \dots p_n u \in R$.

Thus $p_2 \dots p_n u \in R$.

Repeating n−1 times gives $u \in R$ as required.

$\square$

3.7 Theorem

Let R be a strictly non-commutative bounded Goldie U.F.D. Then R is a Noetherian principal ideal domain.

Proof

Let S be as defined in Lemma 3.6.

Then $R = S \cap T$ by Lemma 3.6.

If we can now show that S is the full quotient ring of R, we will have $T \subseteq S$ and hence we will have $R = T$ and R will have the desired properties by Corollary 3.4.

Suppose $c \in R$ is a regular non-unit of R.

We aim to show that c is a unit of S.

We have $0 \neq cR \triangleleft_r R$.

Also cR is essential by (Lemma 1.11,[4]) since c is regular.

Hence since R is bounded we have $0 \neq I \triangleleft R$ with

$I \subseteq cR$.

Let D be the set of products of prime elements of R.

Since $I \cap D \neq \emptyset$ by Theorem 2.7 $\exists\, r \in R$, $d \in D$ with $d = cr$.

Since d is invertible in S, c is invertible in S.

Hence S is the full quotient ring of R as required.

$\square$

This result was previously only known in the case where R is a bounded Noetherian U.F.D. and the above proof is based on the Noetherian proof [8].

3.8 Corollary

Let R be a U.F.D. which satisfies a polynomial identity. Then R is either commutative or a Noetherian principal ideal domain.

Proof

This follows from the previous theorem and corollary 1.21 which shows that a prime polynomial identity ring is bounded.

$\square$

3.9 Corollary

Let R be a Goldie U.F.D. Then if the elements of $C = \cap C(P)$ are all units R is either commutative or a Noetherian principal ideal domain.

Proof

By Theorem 3.7 it is enough to show that R is bounded.

Let I be a non-zero right ideal of R.

Then there exists $0 \neq a \in I$.

$a = p_1 \ldots p_n c$ for some prime elements p_i and $c \in C$ by Corollary 2.10.

Hence $0 \neq p_1 \ldots p_n c \in I$.

Therefore $0 \neq p_1 \ldots p_n \in I$ since c is a unit.

Thus I contains the non-zero two-sided ideal $p_1 \ldots p_n R$.

The same proof works for a left ideal but you need $a = cp_1 \ldots p_n$.

$\square$

We will now prove another result which was only known in the Noetherian case, namely that if R is a Goldie U.F.D. then R is a maximal order. The proof is again based on the proof of the Noetherian case (Theorem 2.10 ,[2]) which seems to suggest that the condition of a U.F.D. being Noetherian is not really essential to produce strong theorems concerning U.F.D's. The Goldie condition however is essential as without it you cannot construct the partial quotient ring T and it is this which allows you to prove the various theorems.

In order to prove that a Goldie U.F.D. is a maximal order we first need the following.

3.10 Lemma

Let R be a Goldie U.F.D. and P a height -1 prime ideal of R. Then the classical localization $R_{C(P)}$ of R at P exists.

Proof

We have $P = pR$ for some prime element $p \in R$.

Also $C(P) = R-P$.

Let $a \in R$, $c \in C(P)$.

By Goldie's theorem $ax = cy$ for some non-zero $x, y \in R$.

Since $\cap P^n = 0$ by Lemma 2.9.

$x = dp^n$ for some $d \in C(P)$ and integer $n \geq 0$.

We need to show that $c \in C(P^n)$.

If $cs \in Rp^n$, then $cs \in Rp$.

Hence $s = s_1p$ for some $s_1 \in R$ since $c \in C(P)$.

Therefore $cs_1p \in Rp^n$.

Thus $cs_1 \in Rp^{n-1}$.

Therefore $s_1 = s_2p$ for some $s_2 \in R$ since $c \in C(P)$.

Continuing this way we get $s = s_np^n$ for some $s_n \in R$.

Hence we have $c \in C(P^n)$.

$cy = ax = adp^n$.

Therefore $y = bp^n$ for some $b \in R$, since $c \in C(P^n)$.

Thus $cbp^n = adp^n$.

Hence $cb = ad$.

Therefore R satisfies the Ore condition with respect to $C(P)$.

$\square$

3.11 Lemma

If R is a Goldie U.F.D.and P a height -1 prime ideal of R then $R_{C(P)}$ is a maximal order and a local domain.

Proof

By Lemma 3.10 $R_{C(P)}$ exists.

Let $I \lhd R_{C(P)}$.

If $I \not\subseteq PR_{C(P)}$ then there exists $a \in R-P$ and $c \in C(P)$ such that $ac^{-1} \in I$.

But c^{-1} is a unit in $R_{C(P)}$.

Hence $a \in I$.

But $a \in R-P = C(P)$,and is therefore a unit of $R_{C(P)}$.

Therefore $I = R_{C(P)}$

Therefore $PR_{C(P)}$ is the unique maximal ideal of $R_{C(P)}$.

Hence $PR_{C(P)}$ is the Jacobson radical of $R_{C(P)}$.

Also $\dfrac{R_{C(P)}}{PR_{C(P)}}$ is isomorphic to the full quotient ring of $\dfrac{R}{P}$ where the isomorphism is given by $(x+P)(c+P)^{-1} \longrightarrow xc^{-1} + PR_{C(P)}$. The fact that this is a well-defined isomorphism can easily be seen using the fact that R is Goldie.

Since $\dfrac{R}{P}$ is a prime Goldie ring then its quotient ring and hence $\dfrac{R_{C(P)}}{PR_{C(P)}}$ is a simple Artinian ring by Goldie's theorem (Theorem 1.6).

Therefore $R_{C(P)}$ is a local domain.

Let $0 \neq I \lhd R_{C(P)}$ then we have shown that if $I \neq R_{C(P)}$ then $I \subseteq PR_{C(P)} = pR_{C(P)}$ for some prime element p of R.

Assume $I \subseteq P^nR_{C(P)}$ for all $n \geq 0$.

Therefore $I \cap R \subseteq P^n R_{C(P)} \cap R$ for all $n \geq 0$.

But $P^n R_{C(P)} = p^n R_{C(P)}$.

Let $a \in p^n R_{C(P)} \cap R$,then $ba \in p^n R$ for some $b \in C(P)$.

Therefore $a \in p^n R$, since $b \in C(P)$.

Thus $P^n R_{C(P)} \cap R = p^n R = P^n$.

Therefore $I \cap R \subseteq P^n R_{C(P)} \cap R = P^n$ for all $n \geq 0$.

Hence $I \cap R \subseteq \cap P^n = 0$ by Lemma 2.9.

Therefore $I = (I \cap R)R_{C(P)} = 0$ which is a contradiction.

Hence we can choose n such that $I \subseteq P^n R_{C(P)}$ but $I \not\subseteq P^{n+1} R_{C(P)}$.

Let $J = \{x \in R_{C(P)} \mid p^n x \in I\}$.

Then it is easy to see that $J \lhd R_{C(P)}$.

Also$I = p^n J$.

Now $J \not\subseteq PR_{C(P)}$ by the choice of n.

Therefore $J = R_{C(P)}$ as shown at the start of the proof.

Thus $I = P^n R_{C(P)}$ for some positive integer n.

Hence $R_{C(P)}$ is a local domain in which every ideal is a power of the Jacobson radical.

Let I be any ideal and q be in the quotient ring of $R_{C(P)}$ with $qI \subseteq I$ then

$$qp^n R_{C(P)} \subseteq p^n R_{C(P)} = R_{C(P)} p^n \text{ for some integer n.}$$

Therefore $qp^n \in p^n R_{C(P)} = R_{C(P)} p^n$.

Hence $q \in R_{C(P)}$.

Thus $R_{C(P)}$ is a maximal order by Theorem 1.8.

$\square$

In order to show that a Goldie U.F.D. is a maximal order we first need to prove the following theorem.

3.12 Theorem

Let R be a Goldie U.F.D. and T the partial quotient ring of R with respect to C. Then T is a maximal order and $T = \bigcap_P R_{C(P)}$ where P ranges over the height -1 prime ideals of R.

Proof

Set $U = \bigcap_P R_{C(P)}$.

Then $T \subseteq U$ because $C \subseteq C(P)$ for every height -1 prime ideal P of R.

As was shown in the proof of Theorem 3.2 the height -1 prime ideals of T are of the form PT where P is a height -1 prime ideal of R.

Also $R_{C(P)} = T_{C(PT)}$.

Therefore $U = \bigcap_P T_{C(PT)}$ where P ranges over the height -1 prime ideals of R.

Let $u \in U$.

Then $xu \in T$ for some $x \in T$.

We have $x = cp_1 \dots p_n$ for some prime elements p_i of T and some $c \in C(T)$ by Theorem 3.2, part 1.

Therefore $cp_1 \dots p_n u \in T$.

But c is a unit of T by Theorem 3.2, part 2.

Hence $p_1 \dots p_n u \in T$.

Also $p_2 \dots p_n u \in U \subseteq T_{C(PT)}$.

Therefore $p_2 \dots p_n ud \in T$ for some element $d \in C(p_1 T)$.

Because $p_1 p_2 \dots p_n ud \in p_1 T$ and $d \in C(p_1 T)$,

we have $p_1 p_2 \dots p_n u \in p_1 T$.

Therefore $p_2 \dots p_n u \in T$.

Repeating this process for $p_2, p_3, \dots, p_n$ in turn gives $u \in T$.

Hence $U = T$ as required.

It just remains to show that T is a maximal order.

Let I be a non-zero ideal of T and q an element of the quotient ring of T such that $qI \subseteq I$.

Let P be any height -1 prime ideal of T.

Then $qIT_{C(P)} \subseteq IT_{C(P)}$.

As in the proof of Theorem 3.2, $IT_{C(P)}$ is a two-sided ideal of $T_{C(P)}$.

Also $T_{C(P)}$ is a maximal order by Lemma 3.12.

Therefore $q \in T_{C(P)}$ for all height -1 prime ideals P of T, by Theorem 1.8.

Hence $q \in \bigcap_P T_{C(P)} = T$.

Therefore T is a maximal order, by theorem 1.8.

$\square$

3.13 Theorem

Let R be a Goldie U.F.D. Then R is a maximal order.

Proof

Let D be the multiplicatively closed set generated by the prime elements of R.

Let S be the partial quotient of R with respect to D. Let I be a non-zero ideal of R.

$IS = S$ by Theorem 2.11.

Let q be an element of the quotient ring of R such that $qI \subseteq I$.

Therefore $qIS \subseteq IS$.

Hence $qS \subseteq S$.

Therefore $q \in S$.

Let T be the partial quotient ring of R with respect to C then

$qIT \subseteq IT$.

IT is a two-sided ideal of T by Theorem 3.2, part (3).

Therefore $q \in T$ since T is a maximal order by Theorem 3.12.

Hence $q \in T \cap S$.

Therefore $q \in R$ by Lemma 3.7.

Therefore R is a maximal order by Theorem 1.8.

$\square$

CHAPTER 4

UNIQUE FACTORISATION RINGS WITH A POLYNOMIAL IDENTITY

In this chapter we will look at unique factorisation rings that have the additional property of satisfying a polynomial identity as defined in Chapter 1.

We will show that if a U.F.R. with a polynomial identity has only a finite number of height-1 prime ideals then it is a semi-local Noetherian principal ideal ring.This result is not only of interest in itself but is used to prove the major result in chapter 5.

Two other major results will be proved in the course of this chapter. The first is that a UFR with a polynomial identity is a maximal order (Theorem 4.6). The second result is a modification of a result for Noetherian UFR's by M.P. Gilchrist in [7]. This gives that a UFR with a polynomial identity is equal to the intersection of two rings, one of which is a Noetherian ring in which every two-sided ideal is principal and the other is a simple Artinian ring.

Note that in the case of a UFD with a polynomial identity it was shown in Corollary 3.8 that this ring is either commutative or a Noetherian principal ideal domain.

In order to prove any of the results in this chapter it will frequently be necessary to localize at a height -1 prime ideal. It is not clear whether this localization exists as it is not clear what the elements regular modulo a height -1 prime are, let alone whether they satisfy the Ore condition. Even then it is not obvious that we will obtain a local ring by inverting these regular elements.

4.1 Lemma

Let R be a UFR with a polynomial identity and S the partial quotient ring of R formed by inverting all the products of prime elements in R. Then S is the full quotient ring of R.

Proof

S is contained in the quotient ring of R and hence satisfies a polynomial identity by theorem 1.19.

Also S is simple by Lemma 2.11.

Therefore S is Artinian by a theorem of Kaplansky (Corollary 1.14). Hence S is the full quotient ring of R.

$\square$

4.2 Theorem

Let R be a UFR with a polynomial identity. If R has only a finite number of height -1 prime ideals then R is a semi-local Noetherian principal ideal ring.

Proof

We will first prove that the height -1 prime ideals of R are precisely the maximal ideals.

Let $P_1,\ldots,P_n$ be the height -1 prime ideals of R. Let I be an ideal of R with $I \not\subseteq P_i$ for all $1 \leq i \leq n$.

Therefore $I \cap C(P_i) \neq \emptyset$ for all $1 \leq i \leq n$.

Hence by the Chinese Remainder Theorem (1.10)

$$I \cap \bigcap_{i=1}^{n} C(P_i) \neq \emptyset$$

Let $0 \neq c \in \bigcap_{i=1}^{n} C(P_i) \cap I$.

Since $c \in C(P_1) \subseteq C(0)$ we have $c \in C(0)$ therefore by corollary 1.18 cR contains a non-zero central element z.

Therefore $z = cr$ for some $r \in R$.

Since z is central it is normal.

Therefore by Lemma 2.14 $z = uq_1 \ldots q_m$ where u is a unit and the q_i are generators of height -1 prime ideals.

Therefore $cr = uq_1 \ldots q_m$.

But $c \in C(q_mR)$ hence $r = r_1q_m$ for some $r_1 \in R$.

Hence $cr_1q_m = uq_1 \cdots q_m$.

Thus $cr_1 = uq_1 \cdots q_{m-1}$.

Now $c \in C(q_{m-1}R)$ hence $r_1 = r_2q_{m-1}$ for some $r_2 \in R$.

Proceeding in this way we eventually obtain

$cr_m = u$ for some $r_m \in R$.

Therefore $u \in I$ and since u is a unit $I = R$.

Hence the height -1 prime ideals are precisely the maximal ideals of R.

Now let A be a non-trivial ideal of R.

Since $\cap P_i^s = 0$ for all $1 \leq i \leq n$ by Lemma 2.9 ,we can choose integers a_i ,$1 \leq i \leq n$ such that $A \subset P_i^{a_i}$ but $A \not\subseteq P_i^{a_i+1}$ where $P_i^0 = R$.

$$\text{Therefore } A \subseteq P_1^{a_1} \cap P_2^{a_2} \ldots \cap P_n^{a_n} = P_1^{a_1} P_2^{a_2} \ldots P_n^{a_n}$$
$$= p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}R.$$

Let $K = \{r \in R \mid p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n} r \in A\}$.

Since the p_i are normal it is easy to see that K is a two-sided ideal of R.

Also $A = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n} K$.

Therefore $K \not\subseteq P_i$ for any $1 \leq i \leq n$ otherwise this would contradict the maximality of one of the a_i.

But the P_i are the maximal ideals of R and therefore we must have $K = R$.

Therefore $A = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n} R$.

Hence R has A.C.C. on two-sided ideals, but R satisfies a polynomial identity hence by a theorem of Cauchon (Theorem 1.22) R is Noetherian.

Therefore we have shown that R is Noetherian and every two-sided ideal of R is principal.

We will now show that the Jacobson radical $J(R) = P_1P_2 \dots P_n$.

Since $J(R)$ is a two-sided ideal of R we have $J(R) = Q_1Q_2 \dots Q_m$ for some height-1 prime ideals Q_i.

These prime ideals must all be different since $J(R)$ is semi-prime and height-1 prime ideals commute by lemma 2.12.

We need that the Q_i are all the height-1 prime ideals.

Assume without loss of generality that P_1 is not one of the Q_i.

Let M be a maximal right ideal containing P_1.

We have $M \supseteq J(R) + P_1$.

But P_1 is a maximal ideal.

Therefore since $J(R) \not\subseteq P_1$ we have $J(R) + P_1 = R$.

Hence $M \supseteq R$ which is not possible.

Thus the Q_i consist of all the height-1 prime ideals without duplication.

Therefore reordering if necessary using lemma 2.12 we have $J(R) = P_1P_2 \dots P_n$.

We will now show that R is semi-local, that is we will show that $R/J(R)$ is Artinian.

Since P_i is a maximal ideal for $1 \le i \le n$ we have that R/P_i is simple for $1 \le i \le n$.

But R/P_i is a factor ring of a ring which satisfies a polynomial identity and therefore satisfies the same polynomial identity.

Hence by corollary 1.14 R/P_i is Artinian for $1 \le i \le n$.

$J(R) = P_1P_2...P_n$ therefore $R/J(R)$ embeds inside the ring

$R/P_1 \oplus R/P_2 \oplus .... \oplus R/P_n$ which since it is a direct sum of a finite number of Artinian rings is itself Artinian.

Therefore $R/J(R)$ is Artinian as required.

Also since $J(R)$ is invertible and R is Noetherian, all the one-sided ideals of R are principal by (Proposition 1.3 ,[10]).

$\square$

4.3 Theorem

If R is a UFR with a polynomial identity then R is localizable at any height -1 prime ideal.

Proof

Let P be a height -1 prime ideal of R.

To localize at P we invert all the prime elements which generate height -1 prime ideals other than P. We will show that this inverts all elements regular modulo P and that it produces a local ring.

Let R^+ be the partial quotient ring of R formed by inverting all the prime elements of R which generate height -1 prime ideals other than P.These prime elements form an Ore set since they are all normal.

We first show that every element of $C(P)$ is a unit in R^+.

Take $c \in C(P)$.

If $ca = 0$ for some $0 \neq a \in R$ then

$a \in P$ since $c \in C(P)$.

Since $\cap P^n = 0$ by Lemma 2.6, we have $a = bp^n$ for some integer n and $b \in R$ with $b \notin pR$.

Hence $0 = ca = cbp^n$.

But since p is regular we have $cb = 0$ and therefore $cb \in P$ with $b \notin pR$ which contradicts $c \in C(P)$.

Hence $c \in C(0)$.

Therefore by corollary 1.18 we have that cR contains a non-zero central element z.

Thus $z = cr$ for some $r \in R$.

Since $\cap P^n = 0$ by lemma 2.9 we have $z = sp^n$ for some $s \notin pR$.

Therefore s is normal since p is normal and z central.

We have $cr = sp^n$.

Hence $cr \in pR$.

Therefore $r \in pR$ since $c \in C(P)$.

Thus $r = r'p$ for some $r' \in R$,

so $cr'p = sp^n$.

Therefore $cr' = sp^{n-1}$.

But we still have $c \in C(P)$, therefore $r' \in pR$.

Continuing this way we obtain

$cx = s$ for some $x \in R$.

Since s is normal in R, sR is an ideal of R.

Hence sR contains a product of height -1 prime ideals by theorem 2.7.

Thus $sR \supseteq q_1 \dots q_n\, p^k R$ for some integer k and prime elements $q_i \neq p$.

Let k be the minimal integer such that this is true.

Hence $q_1 \dots q_n p^k = st$ for some $t \in R$.

Suppose that $k \neq 0$.

Therefore $st \in P$.

Hence $sRt \subseteq P$ since s is normal.

Thus $t \in P$ since P is prime and $s \notin P$.

Therefore $t = t'p$ for some $t' \in R$.

Hence $q_1 \dots q_n p^k = st'p$.

Thus $q_1 \dots q_n p^{k-1} = st' \in sR$.

This contradicts the minimality of k.

Therefore we have $k = 0$.

Hence $sR \ni q_1 \dots q_n$.

But $q_1 \dots q_n$ is a unit of R^+.

Therefore s is a unit of R^+.

Hence c is a unit of R^+.

Thus the elements of $C(P)$ are units of R^+.

It just remains to show that R^+ is a local ring.

Let I be an ideal of R^+ with $I \not\subseteq PR^+$.

Then $I \cap R$ is a non-zero ideal of R and $I \cap R \not\subseteq P$.

Therefore $((I \cap R) + P)/P$ is a non-zero ideal of R/P.

But R/P is prime Goldie since R satisfies a polynomial identity by corollary 1.20.

Hence $I \cap R$ contains an element of $C(P)$ (Lemma 1.18 ,[4]).

Thus I contains a unit of R^+.

Therefore $I = R^+$.

Hence PR^+ is the unique maximal ideal of R^+.

Also since R^+ is contained in the full quotient ring of R by Posner's theorem (Theorem 1.19) R^+ satisfies a polynomial identity.

Hence R^+/PR^+ is a simple p.i. ring.

Therefore R^+/PR^+ is Artinian by Kaplansky's theorem (Corollary 1.14).

Thus R^+ is a local ring with PR^+ the unique maximal ideal.

Hence R^+ is the localization of R at P.

$\square$

4.4 Lemma

If R is a UFR with a polynomial identity then R^+ the localization of R at a height -1 prime ideal P is a Noetherian ring in which every two-sided ideal is a power of the maximal ideal PR^+.

Proof

We first need to show that $pR^+ = R^+p$.

Let $rc^{-1} \in R^+$ with $c \in C(P)$.

Therefore by the Ore condition $prc^{-1} = d^{-1}x$ for some $x \in R$, $d \in C(P)$.

Hence $dpr = xc$.

Thus $xc \in pR$.

But $c \in C(P)$, therefore $x \in pR$.

Hence $x = sp$ for some $s \in R$.

Therefore $prc^{-1} = d^{-1}sp$.

Thus $pR^+ \subseteq R^+p$.

Similarly $R^+p \supseteq pR^+$.

Now let $I \neq R^+$ be a non-zero two-sided ideal of R^+, then $I \subseteq PR^+$ the unique maximal ideal of R^+.

Since $\cap P^n = 0$ by Lemma 2.9 ,we can choose an integer a such that $I \subseteq p^aR^+$ but $I \not\subseteq p^{a+1}R^+$.

Let $J = \{r \in R^+ \mid p^a r \in I\}$.

Since $PR^+ = pR^+ = R^+p$ it is easily seen that J is a non-zero two-sided ideal of R^+ and $I = p^aJ$.

If $J \neq R^+$ then $J \subset PR^+$ the unique maximal ideal of R^+.

Therefore $I = p^aJ \subseteq P^{a+1}R^+$, which contradicts the choice of a.

Hence $J = R^+$.

Thus $I = p^aR^+ = P^aR^+$.

Hence every ideal of R^+ is a power of PR^+ the maximal ideal.

Thus R^+ has A.C.C. on two-sided ideals and satisfies a polynomial identity, also R^+ is prime.

Therefore R^+ is Noetherian by a theorem of Cauchon (Theorem 1.22).

$\square$

We will now use these localizations to show that a UFR with a polynomial identity is a maximal order as defined in Chapter 1.

Note that all the localizations at height -1 prime ideals lie inside the quotient ring of R and this is where the intersection in the following theorem takes place.

4.5 Theorem

If R is a UFR with a polynomial identity then R is equal to the intersection of the localizations of R at its height -1 prime ideals.

Proof

Each localization exists by Theorem 4.3.

Let Q be the intersection of the localizations.

Clearly $R \subseteq Q$.

Let $q \in Q$.

Since q is in S the quotient ring of R, $q = r(p_1 \dots p_n)^{-1}$ for some $r \in R$, $p_1,\dots,p_n$ prime elements of R.

Choose r such that n is minimal.

Assume $n \neq 0$.

Since q is in the localization of R at p_iR for $1 \leq i \leq n$

$q = s_ic_i^{-1}$ for $s_i \in R$, $c_i \in C(P_i)$ for $1 \leq i \leq n$.

By the Ore condition there exists $t \in C(P_n)$ and $s \in R$ such that $c_ns = p_1 \dots p_nt$.

Since $c_n \in C(P_n)$ we have $s \in P_n$.

Hence $s = vp_n$ for some $v \in R$.

Therefore $rt = qp_1 \dots p_nt = qc_ns$

$$= qc_nvp_n$$

$$= s_nvp_n \in P_n.$$

Since $t \in C(P_n)$ we have $r \in P_n$.

Hence $r = r_1p_n$ for some $r_1 \in R$.

Therefore $q = r(p_1 \dots p_n)^{-1} = r_1(p_1 \dots p_{n-1})^{-1}$.

But this contradicts the minimality of n.

Therefore $n = 0$.

Therefore $q \in R$.

Hence $Q = R$ as required.

$\square$

4.6 Theorem

If R is a UFR with a polynomial identity then R is a maximal order.

Proof

Let I be a non-zero ideal of R and S be the quotient ring of R. By Theorem 1.8 we must show that if $q \in S$ such that $qI \subseteq I$ then $q \in R$.

Let P be a height -1 prime ideal of R.

Since R^+ the localization of R at P is Noetherian by Lemma 4.4, then IR^+ is a two-sided ideal of R^+ by (Theorem 1.31 ,[4]).

Hence $IR^+ = p^nR^+$ for some positive integer n by Lemma 4.4.

Since $qI \subseteq I$ we have $qIR^+ \subseteq IR^+$.

Therefore $qp^nR^+ \subseteq p^nR^+ = R^+p^n$.

Hence $q \in R^+$ since p is regular in S.

Therefore since P was chosen to be any height -1 prime ideal of R, q is an element of every localization of R at a height -1 prime ideal.

Hence $q \in R$ by Theorem 4.5.

Therefore R is a maximal order.

$\square$

Before going on to the next theorem we need to prove a result which is known in general for a Noetherian ring but seems to need all the power of a polynomial identity to prove it for a non-Noetherian ring.

4.7 Lemma

If R is a U.F.R. with a polynomial identity then any regular element of R is regular modulo all but finitely many height -1 prime ideals.

Proof

Let S be the quotient ring of R.

S is finitely generated over its centre by Posner's theorem (Theorem 1.19) and Kaplansky's theorem (Theorem 1.13).

Also the centre of S is the quotient ring of the centre of R and is therefore a field.

Let c be a regular element of R.

Then $c \in S$.

Since S is finitely generated over its centre c satisfies a monic equation with coefficients in the centre of S.

Hence c satisfies a non-monic equation with coefficients in the centre of R.

Thus c is algebraic over the centre of R.

If the constant terms of this equation is zero then we can cancel out a c since c is regular in R.

Therefore c satisfies a non-monic equation with coefficients in the centre of R with a non-zero constant term.

Since this constant term is central it is regular modulo all but a finite number of height -1 prime ideals by lemma 2.14.

Therefore c has the same property.

$\square$

The following theorem was proved for Noetherian UFR's by Gilchrist and the proof will follow the same lines as his proof [7], but a great deal of extra care has to be taken to either avoid the parts of the proof which use the Noetherian condition or to prove that they still work in the case of a UFR with a polynomial identity. This theorem and a version of this proof appears in a paper by Chatters, Gilchrist and myself which is currently in preparation.

In this proof the two rings T and S are both contained in the quotient ring of $R[x]$ and it is here that we will be taking our intersection.

4.8 Theorem

If R is a UFR with a polynomial identity then $R = T \cap S$ where S is simple and every two-sided ideal of T is principal.

Proof

First we shall construct the ring T which is a partial quotient ring of $R[x]$.

Consider the set $R[x] \supseteq G = \cap\, C(P[x])$ where the intersection ranges over the height -1 prime ideals P of R. T will be the partial quotient ring of $R[x]$ formed by inverting the elements of G. Therefore the first, and indeed most complicated thing we need to prove is that G is an Ore set.

Suppose $b(x) \in G$, $a(x) \in R[x]$.

Let $K = \{f(x) \mid a(x)f(x) \in b(x)R[x]\} \triangleleft_r R[x]$.

We need $b(x) \in C_{R[x]}(0)$.

Assume this is false.

Therefore $b(x).r(x) = 0$ for some $r(x) \in R[x]$.

By Lemma 2.9 $\cap P^n[x] = 0$.

Hence $r(x) = r_1(x)p^n$ for some $r_1(x) \in R[x]$, $n \geq 0$ with $r_1(x) \notin P[x]$.

Therefore $b(x).r_1(x)p^n = 0$.

Hence $b(x).r_1(x) = 0$ since p is regular.

But $r_1(x) \notin P[x]$ which contradicts the fact that $b(x) \in C(P[x])$.

Therefore $b(x) \in C_{R[x]}(0)$

Since $b(x) \in C_{R[x]}(0)$ and $C_{R[x]}(0)$ is an Ore set, $K \cap C_{R[x]}(0) \neq \emptyset$.

Also since R[x] is a UFR by Theorem 2.15 and P[x] is a height -1 prime ideal of R[x] by lemma 2.6, P[x] is localizable by Theorem 4.3.

Therefore C(P[x]) is an Ore set and since $b(x) \in C(P[x])$ we have

$K \cap C(P[x]) \neq \emptyset$ for each height -1 prime P of R.

We will now show that we can pick an element $c(x) \in K$ such that $c(x) = c_r x^r + \ldots + c_{n+r}x^{n+r}$ where c_r is regular in R.

Consider $K_0 = \{k_j \in R \mid f(x) = k_j x^j + \ldots + k_{j+m}x^{j+m} \in K\} \triangleleft_r R$.

Take $0 \neq \alpha \in R$ then

$a(x).\alpha \in R[x]$, $b(x) \in C_{R[x]}(0)$.

Therefore by right Ore condition

$a(x).\alpha.d(x) = b(x).e(x)$ for some $d(x) \subset C_{R[x]}(0)$, $e(x) \in R[x]$.

Therefore $\alpha.d(x) \in K$.

Since d(x) is regular in R[x], $\alpha.d(x) \neq 0$.

Hence the lowest coefficient of $\alpha.d(x)$ is in K_0.

That is $\alpha.d_i \in K_0$ for some $d_i \in R$.

Therefore K_0 is an essential right ideal of R.

Hence K_0 contains a regular element of R by (Theorem 1.10 ,[4]).

By Lemma 4.6; $c_r \in C(P)$ for all but possibly finitely many height -1 prime ideals, $P_1,\ldots,P_n$ of R.

Hence $c(x)$ is regular modulo $P[x]$ for all $P[x]$ except possibly $P_1[x],\ldots,P_n[x]$.

Now above we had $K \cap C(P_i[x]) \neq \emptyset$ for $i = 1,\ldots,n$.

Therefore by the Chinese Remainder Theorem (Theorem 1.10) we have

$$K \cap \left(\bigcap_{i=1}^{n} C(P_i[x])\right) \neq \emptyset$$

Hence choose $0 \neq d(x) \in K \cap \left(\bigcap_{i=1}^{n} C(P_i[x])\right)$.

Now let $f_j(x) = c(x) + d(x).x^j$ for all integers $j \geq r + 1$. Clearly $f_j(x) \in K$ for all $j \geq r+1$.

Also $f_j(x) \in C(P[x])$ for all height -1 primes $P \neq P_1,\ldots,P_n$ since the leading coefficient of f_j is c_r for all j.

Thus in order to show that $K \cap G \neq 0$ which is what we need for the right Ore condition on G, all we need to show is that for some j, $f_j(x) \in \bigcap_{i=1}^{n} C(P_i[x])$.

Suppose no $f_j(x) \in \bigcap_{i=1}^{n} C(P_i[x])$ then for each $j \geq r+1$ there is an $1 \leq i \leq n$ such that $f_j(x) \notin C(P_i[x])$.

Hence there is at least one height -1 prime P_i such that $J = \{j \mid f_j(x) \notin C(P_i[x])\}$ is an infinite subset of $\mathbb{N}$.

For each $a \in J$, $\exists\, r_a(x)$ such that $f_a(x).r_a(x) \in P_i[x]$ and $r_a(x) \notin P_i[x]$.

Since J is an infinite subset of $\mathbb{N}$ we can suppose $\{a_1,a_2,\ldots\} = J$.

Clearly $r_{a_1}(x)\dfrac{R[x]}{P_i[x]} \subseteq r_{a_1}(x)\dfrac{R[x]}{P_i[x]} + r_{a_2}(x)\dfrac{R[x]}{P_i[x]} \subseteq \ldots$.

Therefore since $\frac{R[x]}{P_i[x]}$ has finite Goldie dimension, by virtue of R[x] satisfying a polynomial identity, there exists $n \in \mathbb{N}$ such that $\sum_{j=1}^{n} r_{a_j}(x)\frac{R[x]}{P_i[x]}$ is essential in

$\sum_{j=1}^{n+1} r_{a_j}(x)\frac{R[x]}{P_i[x]}$

Thus by (Lemma 1.1 ,[4]) there exists an essential right ideal of $\frac{R[x]}{P_i[x]}$, E such that

$r_{a_{n+1}}(x)\, E \subseteq \sum_{j=1}^{n} r_{a_j}(x)\frac{R[x]}{P_i[x]}$.

Since E is essential and $\frac{R[x]}{P_i[x]}$ is a prime Goldie ring, E contains a regular element

$\gamma(x)$ of $\frac{R[x]}{P_i[x]}$ by (Lemma 1.18 ,[4]).

Therefore $r_{a_{n+1}}(x)\,\gamma(x) \subseteq \sum_{j=1}^{n} r_{a_j}(x)\frac{R[x]}{P_i[x]}$.

Hence $r_{a_{n+1}}(x)\,\gamma(x) = \sum_{j=1}^{n} r_{a_j}(x)\,\beta_j(x)$ in $\frac{R[x]}{P_i[x]}$.

Therefore $\sum_{j=1}^{n+1} r_{a_{n+1}}(x)\,\beta_j(x) \in P_i[x]$ where $\beta_{j+1}(x) = -\gamma(x)$ which is regular modulo $P_j[x]$.

Hence we may suppose that $\sum_{j=1}^{k} r_{a_j}(x)s_{a_j}(x) \in P_i[x]$ where at least one of the $s_{a_j}(x)$ is in $C(P_i[x])$ and k is the least integer for which this is true.

We shall now produce a contradiction of the minimality of k.

Since $\sum_{j=1}^{k} r_{a_j}(x)s_{a_j}(x) \in P_i[x]$ we have $c(x) \sum_{j=1}^{k} r_{a_j}(x)s_{a_j}(x) \in P_i[x]$ where $c(x)$ is as defined in the earlier part of this proof.

But since $c(x) = -d(x)x^{a_j} + f_{a_j}(x)$ for all $a_j \in J$ and $f_{a_j}(x)r_{a_j}(x) \in P_i[x]$ for all $a_j \in J$

we have $\sum_{j=1}^{k} d(x)\, x^{a_j} r_{a_j}(x)s_{a_j}(x) \in P_i[x]$.

By definition $d(x) \in C(P_i[x])$.

Therefore $\sum_{j=1}^{k} r_{a_j}(x)s_{a_j}(x)x^{a_j} \in P_i[x]$.

Suppose $s_{a_t}(x) \in C(P_i[x])$ then $s_{a_t}(x).(x^{a_t} - x^{a_j}) \in C(P_i[x])$ for any $j \neq t$.

But $\sum_{j=1}^{k} r_{a_j}(x)s_{a_j}(x) \in P_i[x]$ implies that

$$\sum_{j=1}^{k} r_{a_j}(x)s_{a_j}(x)x^{a_p} \in P_i[x] \text{ for some } p \neq t.$$

Hence $\sum_{j=1}^{k} (r_{a_j}(x)s_{a_j}(x)(x^{a_j} - x^{a_p})) \in P_i[x]$.

That is $\sum_{j=1}^{k}{}' (r_{a_j}(x)s_{a_j}(x)(x^{a_j} - x^{a_p})) \in P_i[x]$ where Σ' denotes that the p^{th} term is missing.

Renumbering if necessary we get

$$\sum_{j=1}^{k-1} r_{a_j}(x)s'_{a_j}(x) \in P_i[x] \text{ and } s'_{a_t}(x) \in C(P_i[x]) \text{ where } s'_{a_j}(x) = s_{a_j}(x)(x^{a_j} - x^{a_p}).$$

This is a contradiction of the minimality of k.

Hence there exists $j \in \mathbb{N}$ such that $f_j(x) \in \bigcap_{i=1}^{n} C(P_i[x])$ and hence $f_j(x) \in G$.

Therefore G satisfies the right Ore condition.

The proof of the left Ore condition is analogous to the one above.

Thus G is an Ore set.

Let $T = R[x]_G$ the partial quotient ring of R[x] formed by inverting the elements of G.

Also let S be the quotient ring of R formed by inverting all the products of prime elements in R.

Clearly $R \subseteq S \cap T$.

Now suppose

$$f(x).g(x)^{-1} = (p_1 \dots p_n)^{-1} r \in T \cap S \text{ where } f(x) \in R[x],\ g(x) \in G$$

and $p_1,\dots,p_n$ are prime elements of R.

Hence $p_1 \dots p_n f(x) = r.g(x)$.

But $g(x) \in C(P_i[x])$ for all height -1 prime ideals P_i and $P_i[x] = p_i R[x]$ for all i.

Hence $r = p_1 \dots p_n t$ for some $t \in R$.

Therefore $f(x) = t.g(x)$.

Thus $f(x)g(x)^{-1} = t \in R$.

Hence $R = T \cap S$.

It just remains to show that every two-sided ideal of T is principal.

Suppose $I \triangleleft R[x]_G = T$.

Using Corollary 2.8 and Lemma 2.9 we can choose prime elements $p_1,\dots,p_n$ and integers $a_1,\dots,a_n$ such that if we define $J = \{ r \in R[x]_G | p_1^{a_1} \dots p_n^{a_n} r \in I \}$ then $I = p_1^{a_1} \dots p_n^{a_n} J$ and $J \not\subseteq pR[x]_G$ for all prime elements p of R.

Also since the p_i are normal it is easy to see that $J \triangleleft R[x]_G$.

But then $J \cap C(pR[x]) \neq \emptyset$ for all prime elements p of R since J is essential in $\frac{R[x]}{pR[x]}$.

Then by the same argument we used on the ideal K at the very beginning of this proof $J \cap G \neq \emptyset$.

Therefore J contains a unit of $R[x]_G$.

Hence $J = R[x]_G = T$.

Thus if $I \lhd T$ then $I = p_1^{a_1}...p_n^{a_n}T = T\, p_1^{a_1}...p_n^{a_n}$ for some prime elements p_i and positive integers a_i.

$\square$

4.9 Corollary

If R is a UFR with a polynomial identity then $R = T \cap S$ where T is Noetherian with every two-sided ideal principal and S is simple Artinian.

Proof

In the proof of Theorem 4.8 T has A.C.C. on ideals and satisfies a polynomial identity and hence is Noetherian by theorem 1.22.

S is the simple Artinian quotient ring of R by lemma 4.1.

$\square$

In order to see what is going on we will give two examples and calculate G, T and S for these rings.

4.11 Examples

$R = \mathbb{Z}$

In order to produce T we first need to calculate G as in the proof.

$\mathbb{Z}[x] \supseteq G = \cap C(P[x])$ where P is a height - 1 prime ideal of $\mathbb{Z}$.

Thus $G = \cap C(p\mathbb{Z}[x])$ where p is a prime integer.

$$G = \mathbb{Z}[x] \setminus \Big(\bigcup_{p\,\text{prime}} p\mathbb{Z}[x] \Big).$$

Now $T = \mathbb{Z}[x]_G$.

Therefore a general element of T is of the form $\dfrac{(a_0 + a_1x + \ldots + a_nx^n)}{(b_0 + b_1x + \ldots + b_mx^m)}$ where

H.C.F. $(b_0, \ldots, b_m) = 1$.

Clearly $S = \mathbb{Q}$.

$$\text{If } \frac{a_0 + a_1x + \ldots + a_nx^n}{b_0 + b_1x + \ldots + b_mx^m} = \frac{a}{b} \in \mathbb{Q}.$$

Then $a_0 + a_1x + \ldots + a_nx^n = \frac{a}{b}(b_0 + b_1x + \ldots + b_mx^m)$.

Therefore b divides ab_i for all i, but H.C.F. $(b_0, \ldots, b_m) = 1$, hence b divides a.

Thus $\frac{a}{b} \in \mathbb{Z}$.

Therefore $\mathbb{Z} = T \cap S$.

It can easily be seen that T and S have the required properties.

$R = M_2(\mathbb{Z})$

$M_2(\mathbb{Z})[x] = M_2(\mathbb{Z}[x])$.

Therefore $G = \cap C(M_2(p\mathbb{Z}[x]))$ where p is a prime integer. It can be shown that $\quad G = \{q \in M_2(\mathbb{Z}[x]) \mid \text{coefficients of } \det(q) \text{ have H.C.F.} = 1\}$.

Hence a general element of T can be viewed as $\frac{a}{b}$ where $a \in M_2(\mathbb{Z}[x])$ and b is a polynomial in $\mathbb{Z}[x]$ whose coefficients have H.C.F. = 1.

Also S can be regarded $M_2(\mathbb{Q})$.

Clearly $M_2(\mathbb{Z}) \subseteq M_2(\mathbb{Q}) \cap T$ and it is not too difficult to see that $M_2(\mathbb{Z}) = M_2(\mathbb{Q}) \cap T$.

Again S and T obviously have the required properties.

CHAPTER 5

REFLEXIVE IDEALS

In this chapter we will look at the reflexive ideals of a U.F.R. with a polynomial identity.We will show that they are all of the form $p_1 \ldots p_n R$ where the p_i are prime elements of R.

In [11] Hajarnavis and Williams showed that if R is a prime Noetherian maximal order and $T \neq R$ a reflexive ideal of R, then the ring R/T has a quotient ring which is an Artinian principal ideal ring. In this chapter we will show that this is also true if R is a UFR with a polynomial identity.

Note

Using the notation of chapter 4 we have $R = T \cap S$.Therefore it is clear that $I \subseteq IT \cap IS$ for any ideal I of R, but for which ideals do we have $I = IT \cap IS$?

5.1 Lemma

Let R be a UFR with a polynomial identity and T be defined as in Theorem 4.7. Then if I is a two-sided ideal of R then IT is a two-sided ideal of T.

Proof

Since $T = R[x]_G$ as in Theorem 4.8 and $R[x]I = IR[x]$ it is enough to show that if $q \in G$ then $q^{-1}IT \subseteq IT$.

Since $q \in R[x]$, $qIT \subseteq R[x]IT = IR[x]T = IT$. Therefore $IT \subseteq q^{-1}IT$.

Hence $IT \subseteq q^{-1}IT \subseteq q^{-2}IT \subseteq q^{-3}IT \subseteq \ldots$.

But these are all right ideals of T and T is Noetherian by Corollary 4.9.

Therefore there exists a positive integer n such that

$$q^{-n}IT = q^{-(n+1)}IT.$$

Hence $IT = q^{-1}IT$.

□

5.2 Lemma

Let R be a UFR with a polynomial identity, I a two-sided ideal of R and T and S be as in Theorem 4.8. Then if $I = IT \cap IS$ we have $I = p_1 \ldots p_nR$ for some p_i prime elements of R.

Proof

Assume $I \neq 0$.

By Lemma 5.1 IT is a two-sided ideal of T.

Hence $IT = p_1 \ldots p_nT$ for some prime elements p_i of R by Theorem 4.8.

Since S is simple and IS is a two-sided ideal of S by (Theorem 1.31 ,[4])

$IS = S$.

Therefore $I = IT \cap S = p_1 \ldots p_nT \cap S$

Thus $I \supseteq p_1 \ldots p_nR$. Take $a \in I$.

Since $a \in I \subseteq IT$ we have

$a = p_1 \ldots p_n r(x)q(x)^{-1}$ for some $r(x) \in R[x]$, $q(x) \in G$

We have $q(x)$ is regular modulo $p_iR[x]$ for all i by definition of G.

Therefore $a \in p_iR$ for all i.

Hence $I \subseteq p_1 \ldots p_nR$.

Thus $I = p_1 \ldots p_nR$.

□

We will now use this lemma to determine precisely the reflexive ideals of R as defined in Chapter 1.

5.3 Theorem

If R is a UFR with a polynomial identity and I a reflexive ideal of R then I is principal and a product of height −1 prime ideals of R.

Proof

$I^*(S \cap IT) \subseteq I^*S \cap I^*IT = I^*S \cap T$.

I^*S is a two-sided ideal of S by (Theorem 1.31 ,[4]) and is therefore equal to S by the simplicity of S (Lemma 2.11).

Therefore $I^*(S \cap IT) \subset S \cap T = R$ by Theorem 4.7.

Hence $S \cap IT \subseteq I^{**} = I$.

Thus $I = S \cap IT$.

Therefore by Lemma 5.2 , $I = p_1 \dots p_n R$ for some prime elements p_i of R.

□

The next theorem is based on a theorem by Hajarnavis and Williams [11], which states that if R is a Noetherian maximal order and I a reflexive ideal of R then R/I has an Artinian quotient ring which is a principal ideal ring.Since we already know that a U.F.R. with a polynomial identity is a maximal order (Theorem 4.6) and we have shown in chapter 4 that most things that are true for a Noetherian U.F.R. are true in the polynomial identity case,it is not completely surprising that the following theorem is true.

5.4 Theorem

Let R be a UFR with a polynomial identity and let I be a reflexive ideal of R. Then R/I has an Artinian quotient ring which is a principal ideal ring.

Proof

By Theorem 5.2 $I = p_1 \dots p_n R$ where the p_i are prime element of R.

Let W be the ring obtained from R by inverting all the prime elements of R which do not generate prime ideals containing I.

The height -1 prime ideals which contain I are $\{p_i R \mid i = 1,\dots,n\}$.

First we show that $p_i W = W p_i$ for all $1 \le i \le n$.

Clearly $p_i R = R p_i$.

So therefore we need to show that $p_i q^{-1} \in W p_i$ where q generates a height -1 prime ideal of R not containing I.

We have $Rq = qR$.

Therefore $q^{-1}R = Rq^{-1}$.

Hence $p_i q^{-1} = q^{-1} r$ for some $r \in R$.

Thus $qp_i = rq$.

Therefore $rq \in p_i R$.

But $q \in C(p_i R)$ therefore $r = sp_i$ for some $s \in R$.

Hence $p_i q^{-1} = q^{-1} s p_i \in W p_i$.

Therefore $p_i W \subseteq W p_i$.

Similarly $W p_i \subset p_i W$.

Hence $W p_i = p_i W$.

We will now show that p_iW is a prime ideal of W for all $1 \leq i \leq n$.

Let $aWb \subseteq p_iW$ for some $a, b \in W$.

Let $a = rq^{-1}$ and $b = s^{-1}t$ for some $r,t \in R$ and q,s inverted elements in W.

Therefore $rq^{-1}Ws^{-1}t \subseteq p_iW$.

Hence $rWt \subseteq p_iW$.

Therefore $rRt \subseteq p_iR$.

Thus $r \in p_iR$ or $t \in p_iR$ since p_iR is prime.

Therefore $a \in p_iW$ or $b \in p_iW$ proving that p_iW is prime in W.

Now let Q be a non-zero prime ideal of W.

Therefore $Q \cap R$ is a non-zero ideal of R and so contains a product of prime elements by Theorem 2.7.

Rearranging these prime elements if necessary using corollary 2.13 we get $q_1 \ldots q_n s_1 \ldots s_t \in Q$ where the q_i and s_j are prime elements of R and the s_j are invertible in W.

Therefore $q_1 \ldots q_n \in Q$.

Hence $q_1 \ldots q_nW \in Q$ which gives us that $q_1Wq_2W \ldots q_mW \subseteq Q$ since $q_iW = Wq_i$ as shown previously.

Thus $q_iW \subseteq Q$ for some $1 \leq i \leq m$ since Q is prime.

But q_iW is prime, therefore every prime ideal of W contains a principal prime ideal.

Also W is a prime ring, thus W is a UFR.

The height -1 prime ideals of W are precisely the p_iW where p_i is a prime element generating a prime ideal which contains I.

Also W is contained in the quotient ring of R and so satisfies the same polynomial identity as R by theorem 1.19.

Therefore W is a UFR with a polynomial identity and only a finite number of height -1 prime ideals.

Hence W is a semi-local principal ideal ring by Theorem 5.2.

The Jacobson radical of W is $p_1 \dots p_n W$ as shown in theorem 5.2 therefore $W/p_1 \dots p_n W$ is an Artinian principal ideal ring.

It just remains to show that $W/p_1 \dots p_n W$ is isomorphic to the quotient ring of R/I.

We first need that $p_1 \dots p_n W \cap R = I$.

Let $p_1 \dots p_n r \; q_1^{-1} \dots q_m^{-1} = s$ with $r,s \in R$, $q_1, \dots, q_m$ prime.

Assume $s \notin p_1 \dots p_n R$.

Choose r such that m is minimal.

Therefore $p_1 \dots p_n r = s q_m \dots q_1$.

But the p_i are regular modulo Rq_1, hence $r = r_1 q_1$ for some $r_1 \in R$.

Thus $p_1 \dots p_n r_1 q_1 = s q_m \dots q_2 q_1$.

Hence $p_1 \dots p_n r_1 = s q_m \dots q_2$.

But this contradicts the minimality of m.

Hence $s \in p_1 \dots p_n R = I$.

Thus $p_1 \dots p_n \, W \cap R = I$.

Now let $\theta : R/I \to W/p_1 \dots p_n W$ be defined by

$$\theta(a + I) = a + p_1 \dots p_n W.$$

Since $p_1 \dots p_n W \cap R = I$, θ is well defined.

Let $\theta(a + I) = 0$.

Hence $a + p_1 \dots p_n W \subseteq p_1 \dots p_n W$.

Therefore $a \in p_1 \dots p_n W$.

Thus $a \in p_1 \dots p_n W \cap R$.

Hence $a \in I$.

Therefore θ is injective.

Hence $R/I \cong$ subring of $W/p_1 \dots p_n W$.

$$R/I \cong \{r + p_1 \dots p_n W \mid r \in R\}.$$

Let X be the ring obtained from R/I by inverting the prime elements of R which generate height -1 primes of R not containing I.

Then θ extends to an isomorphism between X and $W/p_1 \dots p_n W$.

For example if $rq^{-1} + I \in X$ and $\theta(rq^{-1} + I) = 0$.

We have $rq^{-1} \in p_1 \dots p_n W$.

Therefore

$$rq^{-1} = q_1^{-1} \dots q_m^{-1} s p_1 \dots p_n \text{ for some } s \in R \text{ and prime elements } q_i.$$

Thus $q_m \dots q_1 r = s p_1 \dots p_n q$.

Therefore since the q_i are regular modulo the p_j's we have $r = p_1 \dots p_n r_1$ for some $r_1 \in R$.

Therefore $r \in I$.

Hence $rq^{-1} + I = 0$ in X.

Finally we will show that X is the quotient ring of R/I.

Clearly $R/I \subseteq X$ and since $q^{-1}R = Rq^{-1}$ any element of X can be written as rq^{-1} with $r \in R/I$ and q regular in R/I.

It thus remains to be shown that if x is regular in R/I then x is a unit in X.

Let $x \in C(I)$.

Then $x \in \bigcap_{i=1}^{n} C(p_iR)$ otherwise if $xs \in p_iR$ then $xs\,p_1 \dots p_{i-1}p_{i+1}p_n \in I$ with $sp_1 \dots p_{i-1}p_{i+1} \dots p_n \notin I$.

Therefore $x \in \bigcap_{i=1}^{n} C(p_iW)$ since $C(p_iR) \subseteq C(p_iW)$.

We have x regular in W therefore by corollary 1.18 xW contains a non-zero central element z.

Therefore $z = xr$ for some $r \in W$.

Since z is central it is normal, therefore by Lemma 2.14 $z = uq_1 \dots q_m$ where u is a unit and the q_i are generators of height -1 prime ideals in W.

Therefore $xr = uq_1 \dots q_m$.

But $x \in C(q_jW)$ for all $1 \leq j \leq m$.

Therefore $r \in q_1W \cap \dots \cap q_mW = q_1 \dots q_mW$.

Hence $r = r_1q_1 \dots q_m$ for some $r_1 \in W$.

Thus $xr_1q_1 \dots q_m = uq_1 \dots q_m$.

Therefore $xr_1 = u$.

Hence x is a unit in W.

Therefore $x + p_1 \dots p_nW$ is a unit in $W/p_1 \dots p_nW$.

Thus if $x + I$ is regular in R/I

$$\theta(x + I) = x + p_1 \dots p_nW \text{ is a unit in } W/p_1 \dots p_nW.$$

Hence $\theta^{-1}(x + p_1 \dots p_nW)$ is a unit in X.

Therefore $x + I$ is a unit in X.

Hence we have shown that R/I has a quotient ring X which is isomorphic to $W/p_1 \dots p_nW$ and is therefore an Artinian principal ideal ring by Theorem 5.2.

□

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