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Central limit theorems for biased randomly trapped random walks on $\mathbb{Z}$

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Abstract

We prove CLTs for biased randomly trapped random walks in one dimension. By considering a sequence of regeneration times, we will establish an annealed invariance principle under a second moment condition on the trapping times. In the quenched setting, an environment dependent centring is necessary to achieve a central limit theorem. We determine a suitable expression for this centring. As our main motivation, we apply these results to biased walks on subcritical Galton–Watson trees conditioned to survive for a range of bias values.

MSC: primary 60K37; 60F05; 60F17; secondary 60J80

Keywords: Random walk; Random environment; Randomly trapped; Galton–Watson tree; Annealed; Quenched; Functional central limit theorem; Invariance principle

1. Introduction

In this paper, we investigate biased randomly trapped random walks (RTRWs) on $\mathbb{Z}$ and apply the results to subcritical Galton–Watson trees conditioned to survive. Randomly trapped random walks were first introduced in [3] where it is shown that the possible scaling limits belong to a certain class of time changed Brownian motions. The purpose of the RTRW is to generalise models such as the Bouchaud trap model (see [7,17] and [33]) and provide a framework for studying random walks on other random graphs in which trapping naturally occurs such as biased random walks on percolation clusters (see [15,18] and [30]) and random walk in random

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environment (see [22] and [31]). Higher dimensional ($d \geq 2$) unbiased randomly trapped random walks have been studied further in [11] where a complete classification of the possible scaling limits is given. In recent years there has been much progress in models involving trapping phenomena; a review of recent developments in a range of models of directionally transient and reversible random walks on underlying graphs such as supercritical GW-trees and supercritical percolation clusters is given in [4].

A subcritical GW-tree conditioned to survive consists of a semi-infinite path (called the backbone) with GW-trees as leaves. Typically, the leaves are quite short therefore the walk on the tree does not deviate far from the backbone. For this reason we have that the walk on the tree behaves like a randomly trapped random walk on $\mathbb{Z}$ with holding times distributed as excursion times in GW-trees. Biased walks on subcritical GW-trees are, therefore, a natural example of the randomly trapped random walk. Furthermore, they exhibit interesting behaviour as the relationship between the bias and offspring law influences the trapping. In this paper we are only concerned with ballistic walks. We note that the sub-ballistic regimes for the biased walk on the subcritical tree have been studied in [10] where it is shown that either a strong bias or heavy tails of the offspring law can slow the walk into a sub-ballistic phase.

Critical and supercritical GW-trees have also received much attention. In [13], it is shown that the walk on the critical GW-tree conditioned to survive is always sub-ballistic; this is studied further and it is shown that the walk belongs to the universality class of one-dimensional trapping models with slowly-varying tails. For the biased walk on the supercritical GW-tree, it is shown in [24] that when the bias is small the walk is recurrent, when the bias is large enough the walk is sub-ballistic and there is some intermediate range for the bias such that the walk is ballistic. The ballistic phase for this walk is studied further in [1] where an expression of the speed is given and in [5] where appropriate scaling sequences for the sub-ballistic phase are shown. The traps formed in the supercritical tree resemble those in the subcritical tree and it has been shown in [10] that the walks observe similar scaling regimes.

We next introduce the models of interest and state the main results. We consider the randomly trapped random walk model in which the embedded walk $(Y_k)_{k\geq 0}$ is a simple, biased random walk on $\mathbb{Z}$. That is, we write $Y_k := \sum_{j=1}^{k} \chi_j$ for a sequence of i.i.d. random variables $(\chi_j)_{j\geq 1}$ satisfying $P(\chi_j = -1) = (\beta+1)^{-1} = 1 - P(\chi_j = 1)$ where $\beta \geq 1$. For $x \in \mathbb{Z}$ write

$$\mathcal{L}(x,n) := \sum_{k=0}^{n} \mathbf{1}_{\{Y_k = x\}}$$

for the local time of Y at site x by time n. The random environment ω is a sequence of $(0,\infty)$-valued probability measures $(\omega_x)_{x\in\mathbb{Z}}$ with environment law $\mathbf{P} := \pi^{\otimes\mathbb{Z}}$ for some fixed law π. For a fixed environment ω, let $(\eta_{x,i})_{x\in\mathbb{Z}, i\geq 0}$ be independent with $\eta_{x,i} \sim \omega_x$. Writing

$$S_n := \sum_{x\in\mathbb{Z}} \sum_{i=1}^{\mathcal{L}(x,n-1)} \eta_{x,i} = \sum_{k=0}^{n-1} \eta_{Y_k, \mathcal{L}(Y_k,k)} \qquad \text{and} \qquad S_t^{-1} := \sup\{k \geq 0 : S_k \leq t\},$$

we then define the randomly trapped random walk by $X_t := Y_{S_t^{-1}}$.

This process is then a continuous time random walk on $\mathbb{Z}$ with k^{th} holding time $\eta_k := \eta_{Y_k, \mathcal{L}(Y_k,k)}$ and we write $\eta := (\eta_k)_{k\geq 0}$ to be the sequence of holding times. For convenience we will define $S_t = S_{\lfloor t \rfloor}$ where $\lfloor t \rfloor := \max\{k \in \mathbb{Z} : k \leq t\}$ for non-integer $t \in \mathbb{R}$. Let P^ω denote the law over X for fixed environment ω and $\mathbb{P}(\cdot) = \int P^\omega(\cdot)\mathbf{P}(d\omega)$ the annealed law. Furthermore, we denote by $D([0,\infty),\mathbb{R})$ the space of càdlàg functions mapping $[0,\infty)$ to $\mathbb{R}$ which we always equip with the Skorohod J_1 topology.

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We begin, in Section 2, by proving Theorem 1 which determines the ballistic range for the walk.

Theorem 1. *Suppose $\beta > 1$ and that $\mathbb{E}[\eta_0] < \infty$, then X_{nt}/n converges $\mathbb{P}$-a.s. on $D([0,\infty),\mathbb{R})$ to the process $\nu_\beta t$ where*

$$\nu_\beta := \frac{(\beta-1)}{\mathbb{E}[\eta_0](\beta+1)}.$$

Following this, we use a renewal argument, similar to [29], to prove Theorem 2 which is an annealed, functional central limit theorem for the walk.

Theorem 2. *Suppose that $\beta > 1$ and $\mathbb{E}[\eta_0^2] < \infty$ then there exists $\varsigma^2 \in (0,\infty)$ such that*

$$B_t^n := \frac{X_{nt} - nt\nu_\beta}{\varsigma\sqrt{n}}$$

converges in $\mathbb{P}$-distribution on $D([0,\infty),\mathbb{R})$ to a standard Brownian motion.

In Section 3 we adapt the technique used in [19] (to prove a quenched CLT for a random walk in random environment) to derive a quenched central limit theorem with an environment dependent centring for the randomly trapped random walk. This is Theorem 3.

Theorem 3. *Suppose $\beta > 1$, $\mathbb{E}[\eta_0^2] < \infty$ and for some $\varepsilon > 0$ we have that $\mathbf{E}[E^\omega[\eta_0]^{2+\varepsilon}] < \infty$, then there exists $\vartheta^2 \in (0,\infty)$ such that for $\mathbf{P}$-a.e. ω we have that*

$$P^\omega\left(\frac{X_t - \mathcal{G}^\omega(t)}{\vartheta\sqrt{t}} \le u\right) \to \Phi(u) := \int_{-\infty}^{u} \frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} \mathrm{d}v$$

uniformly in u as $t \to \infty$ where

$$\mathcal{G}^\omega(t) := \nu_\beta t - \nu_\beta \sum_{y=0}^{\lfloor \nu_\beta t - 1\rfloor} \frac{\beta+1}{\beta-1}(E^\omega[\eta_{y,0}] - \mathbb{E}[\eta_{y,0}]).$$

The function $\mathcal{G}^\omega(t)$ is the annealed, deterministic centring with an environment dependent correction. This correction is a sum of centred i.i.d. random variables with (typically) non-zero variance under the environment law. This shows that the correction obeys a central limit theorem under $\mathbf{P}$, thus has $\sqrt{t}$ fluctuations and is, therefore, necessary.

In Section 4 we apply these results to the biased random walk on a subcritical GW-tree conditioned to survive. Let $f(s) = \sum_{k=0}^{\infty} p_k s^k$ denote the generating function of a GW-process with mean $\mu \in (0,1)$ and variance $\sigma^2 < \infty$. Denote by Z_n the n^{th} generation size of a GW-process with this law. Such a process gives rise to a random rooted tree $\mathcal{T}^f$ where individuals in the process are represented by vertices (with the unique progenitor as the root ρ) and undirected edges connect individuals with their offspring. To avoid the trivial case in which no traps form we also assume that $p_0 + p_1 < 1$. We denote by ξ a random variable with the offspring law. It has been shown in [21] that there is a well defined probability measure $\mathbf{P}$ over f-GW trees conditioned to survive $\mathcal{T}$ which we describe in greater detail in Section 4.

For a fixed tree $\mathscr{T}$ rooted at ρ, we write $\overleftarrow{x}$ to denote the parent of $x \in \mathscr{T}$ and $c(x)$ the set of children of x. For $\beta \ge 1$, we then define the β biased random walk X as the Markov chain

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started from a fixed vertex z with transition probabilities

$$P_z^{\mathcal{T}}(X_{n+1}=y|X_n=x)=\begin{cases}\dfrac{1}{1+\beta|c(x)|}, & \text{if } y=\overleftarrow{x},\\ \dfrac{\beta}{1+\beta|c(x)|}, & \text{if } y\in c(x),\ x\neq\rho,\\ \dfrac{1}{|c(\rho)|}, & \text{if } y\in c(x),\ x=\rho,\\ 0, & \text{otherwise.}\end{cases} \tag{1.1}$$

As in the randomly trapped random walk case, we use $\mathbb{P}(\cdot)=\int P_\rho^{\mathcal{T}}(\cdot)\mathbf{P}(\mathrm{d}\mathcal{T})$ for the annealed law. This is the model of the biased random walk on a subcritical GW-tree conditioned to survive which is the focus of Theorems 4–6. Let $|X_n|$ denote the graph distance between the walk at time n and the root of the tree. In Theorem 4 we determine an explicit expression for the speed of the walk.

Theorem 4. *Suppose $\beta\mu<1,\ \sigma^2<\infty$ and $\beta>1$, then $|X_n|/n$ converges $\mathbb{P}$-a.s. to*

$$\nu_\beta := \frac{\mu(\beta-1)(1-\beta\mu)}{\mu(\beta+1)(1-\beta\mu)+2\beta(\sigma^2-\mu(1-\mu))}.$$

A short calculation shows that this speed ν_β is unimodal in the bias; an example of which is illustrated in Fig. 1. Proving this remains an open problem in the related models of random walks on supercritical GW-trees and supercritical percolation clusters (see [4]) where explicit expressions for the speed are, in general, not known.

Following this, we use Theorem 2 to prove Theorem 5 which is an annealed functional CLT for the walk on the tree.

Theorem 5. *If $\beta^2\mu<1,\ \beta>1$ and $\mathbf{E}[\xi^3]<\infty$ then there exists $\varsigma^2<\infty$ such that*

$$B_t^n := \frac{|X_{nt}|-nt\nu_\beta}{\varsigma\sqrt{n}}$$

converges in $\mathbb{P}$-distribution on $D([0,\infty),\mathbb{R})$ to a standard Brownian motion.

Finally, we use Theorem 3 to prove Theorem 6 which is a quenched CLT with an environment dependent centring $\mathcal{G}^{\mathcal{T}}$ (defined in (4.10)) for the walk on the tree.

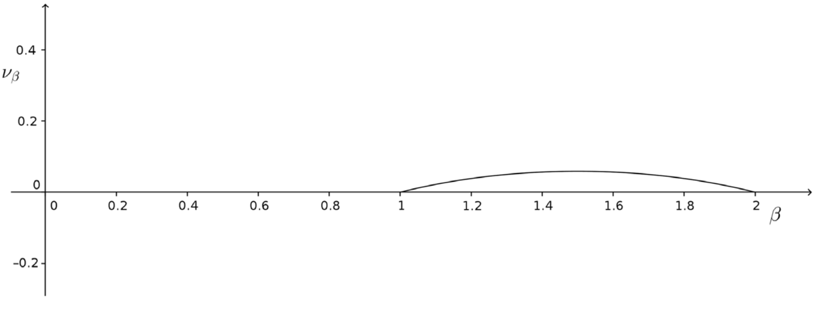

Fig. 1. An example of the speed ν_β relative to the bias for a fixed mean $\mu=1/2$ and variance $\sigma^2=1/2$.

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Theorem 6. *If $\beta^2\mu < 1$, $\beta > 1$ and $\mathbf{E}[\xi^{3+\delta}] < \infty$ for some $\delta > 0$ then there exists $\vartheta > 0$ such that for $\mathbf{P}$-a.e. $\mathcal{T}$ we have that*

$$P^{\mathcal{T}}\left(\frac{X_t - \mathcal{G}^{\mathcal{T}}(t)}{\vartheta\sqrt{t}} \leq x\right) \to \Phi(x)$$

uniformly in x as $n \to \infty$.

Further to these results, we also prove Einstein relations for both the randomly trapped random walk and the random walk on the subcritical GW-tree conditioned to survive. That is, we relate the diffusion of the unbiased walk with the derivative of the speed (with respect to the bias) as the bias tends to 1 (i.e. neutral bias).

A technique is developed in [6] that can be used to extend an annealed invariance principle to a quenched result. This is applied in [27] to prove a quenched functional central limit theorem for the walk on the supercritical tree when the offspring distribution has exponential moments and no deaths. The condition of exponential moments is purely technical. However, because the offspring law has no deaths, the supercritical tree does not have traps which represents a significant simplification of the problem. Due to the similarity of the traps in the supercritical and subcritical GW-trees with leaves, a key motivation of this paper is to be able to extend the result of [27] to allow for deaths in the offspring law.

The correction $\mathcal{G}^{\mathcal{T}}$ is the annealed centring with an additional sum of centred i.i.d. random variables with non-zero variance under the environment law. This term has fluctuations on the order of $\sqrt{t}$ which suggests that the annealed convergence in Theorem 5 cannot be extended to convergence under the quenched law as one may expect by comparison with the supercritical tree where it is possible to adapt the argument of [6]. This argument relies on multiple copies of the walk eventually having separate escape paths; this results in the randomness of the embedded walk mixing the environment encountered sufficiently to remove the dependency on the specific environment. In the subcritical GW-tree model, the walk is forced to escape along a single route and therefore visits every branch in the tree. This results in the walk accumulating environment dependent fluctuations.

We show in Lemma 4.5 that the duration of an excursion in a branch of a subcritical GW-tree has infinite variance when $\beta^2\mu \geq 1$. For this reason a central limit theorem should only hold when $\beta < \mu^{-1/2}$. This supports [4, Conjecture 3.1] which states that a quenched central limit will hold on the supercritical tree only when $\beta < \sqrt{\beta_c}$ where β_c is the critical upper bound on the bias for the walk to be ballistic. This will be shown in [8].

2. A law of large numbers and functional central limit theorem

The main aim of this section is to derive an annealed functional central limit theorem for the RTRW model with positive bias. That is, we prove Theorem 2 by considering a regeneration argument similar to that used in [29].

We begin, in Proposition 2.1, by using ergodicity of the sequence of holding times to show convergence of the scaled clock process. We then use this to deduce the speed result Theorem 1. Following this, by using a sequence of regeneration times, we approximate $X_{nt} - nt\nu_\beta$ by a sum of i.i.d. centred random variables. The result then follows straightforwardly from Donsker's theorem and continuity of the composition at continuous limits.

We now prove convergence of the clock process. Notice that we consider the unbiased case ($\beta = 1$) as well as the positive bias case. This will be used to prove an Einstein relation for the walk.

Proposition 2.1. *Suppose $\beta \geq 1$ and that $\mathbb{E}[\eta_0] < \infty$ then S_{nt}/n and S_{nt}^{-1}/n converge $\mathbb{P}$-a.s. on $D([0,\infty),\mathbb{R})$ to the deterministic processes $\mathcal{S}_t = \mathbb{E}[\eta_0]t$ and $\mathcal{S}_t^{-1} = \mathbb{E}[\eta_0]^{-1}t$ respectively.*

Proof. *By* [11, Lemma 2.1]*, the left shift on sequences ($\theta(\eta_0, \eta_1, \ldots) = (\eta_1, \eta_2, \ldots)$) acts ergodically on η under $\mathbb{P}$ for any non-degenerate random walk on a fixed environment with i.i.d. traps. This includes our embedded walk for any $\beta \geq 1$.*

We have that $f(\eta) := \eta_0$ is integrable because $\mathbb{E}[\eta_0] < \infty$. Therefore, since θ acts ergodically on η under $\mathbb{P}$, by the ergodic theorem

$$\lim_{n\to\infty} \frac{S_{nt}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{\lfloor nt\rfloor -1} \eta_k = \lim_{n\to\infty} t\frac{1}{nt} \sum_{k=0}^{\lfloor nt\rfloor -1} f(\theta^k \eta) = t\mathbb{E}[f(\eta)]$$

almost surely. The sequence of functions S_{nt}/n are increasing in t and the limit $\mathcal{S}_t = \mathbb{E}[\eta_0]t$ is continuous therefore the convergence holds uniformly over $t \in [0,T]$ for $T < \infty$.

Since $\mathcal{S}_t$ is strictly increasing we have the desired convergence of S_{nt}^{-1}/n by continuity of the inverse at strictly increasing functions ([32, Corollary 13.6.4]*).* □

Using Proposition 2.1, we are now able to complete the speed result for the walk.

Proof of Theorem 1. Notice that

$$\frac{X_{nt}}{n} = \frac{Y_{S^{-1}_{\lfloor nt\rfloor}}}{n} = \frac{Y_{S^{-1}_{\lfloor nt\rfloor}}}{S^{-1}_{\lfloor nt\rfloor}} \cdot \frac{S^{-1}_{\lfloor nt\rfloor}}{n}.$$

By the law of large numbers $n^{-1}Y_n$ converges a.s. to $(\beta-1)/(\beta+1)$ therefore, by Proposition 2.1 and using that $\mathcal{S}_t^{-1}$ is continuous, we indeed have the desired result. □

An additional result that can be deduced from Proposition 2.1 and Theorem 1 is that the following Einstein relation holds.

Corollary 2.2. *Suppose $\mathbb{E}[\eta_0] < \infty$. The unbiased ($\beta = 1$) walk $X_{\lfloor nt\rfloor}n^{-1/2}$ converges in $\mathbb{P}$-distribution on $D([0,\infty),\mathbb{R})$ to a scaled Brownian motion with variance $\Upsilon = \mathbb{E}[\eta_0]^{-1}$. Moreover,*

$$\lim_{\beta\to 1^+} \frac{\nu_\beta}{\beta - 1} = \frac{\Upsilon}{2}$$

where ν_β is the speed calculated in Theorem 1 for the β-biased walk.

Proof. *For $\beta = 1$ we have that Y_{nt} is the sum of i.i.d. copies of the random variable χ satisfying $P(\chi = 1) = 1/2 = P(\chi = -1)$ thus by Donsker's invariance principle $Y_{nt}n^{-1/2}$ converges in $\mathbb{P}$-distribution on $D([0,\infty),\mathbb{R})$ to a standard Brownian motion.*

By Proposition 2.1 we have that $S^{-1}_{\lfloor nt\rfloor}/n$ converges $\mathbb{P}$-a.s. to the deterministic process $t/\mathbb{E}[\eta_0]$ uniformly over $t \leq T$. By continuity of the limiting Brownian motion and continuity of composition at continuous limits ([32, Theorem 13.2.1]*), we have that*

$$\frac{X_{nt}}{\sqrt{n}} = \frac{Y_{S^{-1}_{\lfloor nt\rfloor}}}{\sqrt{n}} \quad \text{and} \quad \frac{Y_{nt/\mathbb{E}[\eta_0]}}{\sqrt{n}}$$

converge to the same limiting process. This is a scaled Brownian motion with variance $\Upsilon = \mathbb{E}[\eta_0]^{-1}$.

By Theorem 1 we have that, for $\beta > 1$,

$$\nu_\beta = \frac{(\beta-1)}{\mathbb{E}[\eta_0](\beta+1)},$$

and therefore we indeed have that

$$\lim_{\beta\to 1^+} \frac{\nu_\beta}{\beta-1} = \frac{\Upsilon}{2}. \quad \square$$

We now move on to proving Theorem 2. We want to approximate $X_{nt} - nt\nu_\beta$ by the sum of i.i.d. centred random variables with finite second moments. Let $\kappa_0 = 0$ and, for $j = 1, 2, \ldots$, define $\kappa_j := \inf\{m > \kappa_{j-1} : \{Y_l\}_{l=0}^{m-1} \cap \{Y_l\}_{l=m}^{\infty} = \emptyset\}$ to be the regeneration times of the walk Y. We then have that S_{κ_j} for $j \geq 1$ are regeneration times for X and we write $\varrho_j := Y_{\kappa_j} = X_{S_{\kappa_j}}$ to be the regeneration points. We then write

$$Z_j := \left(X_{S_{\kappa_j}} - X_{S_{\kappa_{j-1}}} - \left(S_{\kappa_j} - S_{\kappa_{j-1}}\right)\nu_\beta\right) = \left(\varrho_j - \varrho_{j-1} - \left(S_{\kappa_j} - S_{\kappa_{j-1}}\right)\nu_\beta\right).$$

By [14, Lemma 5.1] the time and distance between regenerations of Y have exponential moments when $\beta > 1$. That is, for any $j \geq 1$ and some constants C, c (depending on β),

$$\mathbb{P}(\varrho_{j+1} - \varrho_j > n),\ \mathbb{P}(\kappa_{j+1} - \kappa_j > n) \leq Ce^{-cn}. \tag{2.1}$$

Lemma 2.3. *Suppose that $\beta > 1$ and $\mathbb{E}[\eta_0] < \infty$ then $\{Z_j\}_{j\geq 2}$ are centred and i.i.d. under $\mathbb{P}$.*

Proof. *By [14] we have that the sections of the walk $(Y_{i+\kappa_j} - Y_{\kappa_j})_{i=1}^{\kappa_{j+1}-\kappa_j}$, for $j \geq 1$, are i.i.d. therefore, since the traps $(\omega_x)_{x\in\mathbb{Z}}$ are i.i.d. and independent of the walk Y, we have that the collections $(\eta_k)_{k=\kappa_j}^{\kappa_{j+1}-1}$ are i.i.d. It follows that $\{Z_j\}_{j\geq 2}$ are i.i.d. under $\mathbb{P}$.*

It remains to show that Z_j are centred. Since the distribution of a given holding time is independent of the regeneration times of Y and $\mathbb{E}[\eta_0] < \infty$ we have that

$$\nu_\beta \mathbb{E}[S_{\kappa_2} - S_{\kappa_1}] = \frac{\beta-1}{(\beta+1)\mathbb{E}[\eta_0]}\mathbb{E}\left[\sum_{k=\kappa_1}^{\kappa_2-1}\mathbb{E}[\eta_k|\kappa_2,\kappa_1]\right] = \frac{\beta-1}{\beta+1}\mathbb{E}[\kappa_2-\kappa_1]. \tag{2.2}$$

We want to show this is equal to $\mathbb{E}[\varrho_j - \varrho_{j-1}] = \mathbb{E}[Y_{\kappa_2} - Y_{\kappa_1}]$. By (2.1) the time between regenerations and distance between regeneration points have exponential moments hence, by the law of large numbers,

$$\frac{\sum_{j=2}^m Y_{\kappa_j} - Y_{\kappa_{j-1}}}{m} \to \mathbb{E}[Y_{\kappa_2} - Y_{\kappa_1}],$$

$$\frac{\sum_{j=2}^m \kappa_j - \kappa_{j-1}}{m} \to \mathbb{E}[\kappa_2 - \kappa_1]$$

and therefore

$$\frac{\sum_{j=2}^m Y_{\kappa_j} - Y_{\kappa_{j-1}}}{\sum_{j=2}^m \kappa_j - \kappa_{j-1}} \to \frac{\mathbb{E}[Y_{\kappa_2} - Y_{\kappa_1}]}{\mathbb{E}[\kappa_2 - \kappa_1]} \tag{2.3}$$

$\mathbb{P}$-a.s. as $m \to \infty$. However,

$$\frac{\sum_{j=2}^m Y_{\kappa_j} - Y_{\kappa_{j-1}}}{\sum_{j=2}^m \kappa_j - \kappa_{j-1}} = \frac{Y_{\kappa_m}}{\kappa_m}\left(1 + \frac{\kappa_1}{\kappa_m - \kappa_1}\right) - \frac{Y_{\kappa_1}}{\kappa_m - \kappa_1}$$

where $Y_{\kappa_1}/(\kappa_m-\kappa_1)$ and $\kappa_1/(\kappa_m-\kappa_1)$ converge $\mathbb{P}$-a.s. to 0. Furthermore, by the law of large numbers, Y_{κ_m}/κ_m converges $\mathbb{P}$-a.s. to $(\beta-1)/(\beta+1)$ therefore

$$\frac{\sum_{j=2}^{m} Y_{\kappa_j}-Y_{\kappa_{j-1}}}{\sum_{j=2}^{m}\kappa_j-\kappa_{j-1}} \to \frac{\beta-1}{\beta+1}. \tag{2.4}$$

By (2.2), (2.3) and (2.4) we then have that Z_j are centred since

$$\mathbb{E}[\varrho_2-\varrho_1]=\mathbb{E}[Y_{\kappa_2}-Y_{\kappa_1}]=\frac{\beta-1}{\beta+1}\mathbb{E}[\kappa_2-\kappa_1]=\nu_\beta\mathbb{E}[S_{\kappa_2}-S_{\kappa_1}]. \qquad \square$$

In Theorem 2 we show that B_t^n can be approximated by a sum of Z_j which, by Lemma 2.3, are i.i.d. centred random variables. With the aim of proving the central limit theorem, we now show that they also have finite second moments.

Lemma 2.4. *Suppose that $\beta>1$ and $\mathbb{E}[\eta_0^2]<\infty$ then $\mathbb{E}[Z_j^2]<\infty$ for $j\ge 2$.*

Proof. *Since $\{Z_j\}_{j\ge 2}$ are i.i.d. under $\mathbb{P}$ we have that $\mathbb{E}[Z_j^2]=\mathbb{E}[Z_2^2]$ for all $j\ge 2$. By properties of regenerations times $\varrho_2\ge\varrho_1$ and $S_{\kappa_2}\ge S_{\kappa_1}$ almost surely therefore we have that*

$$\mathbb{E}[Z_2^2]\le\mathbb{E}\left[(\varrho_2-\varrho_1)^2\right]+\nu_\beta^2\mathbb{E}\left[\left(S_{\kappa_2}-S_{\kappa_1}\right)^2\right]. \tag{2.5}$$

For the second term we have

$$\begin{aligned}\mathbb{E}\left[\left(S_{\kappa_2}-S_{\kappa_1}\right)^2\right]&=\mathbb{E}\left[\left(\sum_{x=\varrho_1}^{\varrho_2-1}\sum_{i=1}^{\mathcal{L}(x,\infty)}\eta_{x,i}\right)^2\right]\\&=\mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\left(\sum_{i=1}^{\mathcal{L}(x,\infty)}\eta_{x,i}\right)^2\right]\\&\quad+\mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\sum_{y=\varrho_1}^{\varrho_2-1}\mathbf{1}_{\{x\neq y\}}\left(\sum_{i=1}^{\mathcal{L}(x,\infty)}\eta_{x,i}\right)\left(\sum_{j=1}^{\mathcal{L}(y,\infty)}\eta_{y,j}\right)\right].\end{aligned} \tag{2.6}$$

By conditioning on Y we have that the holding times at separate vertices are independent therefore the second term in this expression can be written as

$$\begin{aligned}&\mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\sum_{y=\varrho_1}^{\varrho_2-1}\mathbf{1}_{\{x\neq y\}}\mathbb{E}\left[\sum_{i=1}^{\mathcal{L}(x,\infty)}\eta_{x,i}\middle|Y\right]\mathbb{E}\left[\sum_{j=1}^{\mathcal{L}(y,\infty)}\eta_{y,j}\middle|Y\right]\right]\\&\quad=\mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\sum_{y=\varrho_1}^{\varrho_2-1}\mathbf{1}_{\{x\neq y\}}\left(\sum_{i=1}^{\mathcal{L}(x,\infty)}\mathbb{E}[\eta_{x,i}]\right)\left(\sum_{j=1}^{\mathcal{L}(y,\infty)}\mathbb{E}[\eta_{y,j}]\right)\right]\\&\quad=\mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\sum_{y=\varrho_1}^{\varrho_2-1}\mathbf{1}_{\{x\neq y\}}\mathbb{E}[\eta_0]^2\mathcal{L}(x,\infty)\mathcal{L}(y,\infty)\right]\\&\quad\le\mathbb{E}[\eta_0]^2\mathbb{E}\left[\left(\sum_{x=\varrho_1}^{\varrho_2-1}\mathcal{L}(x,\infty)\right)^2\right]\\&\quad=\mathbb{E}[\eta_0]^2\mathbb{E}[(\kappa_2-\kappa_1)^2].\end{aligned}$$

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By (2.1), the time and distance between regenerations have exponential moments therefore

$$\mathbb{E}\left[(\varrho_2-\varrho_1)^2\right] \le \mathbb{E}\left[(\kappa_2-\kappa_1)^2\right] < \infty. \tag{2.7}$$

Combining (2.7) with (2.5) and (2.6), in order to show that $\mathbb{E}[Z_2^2]<\infty$ *it remains to show that*

$$\mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\left(\sum_{i=1}^{\mathcal{L}(x,\infty)}\eta_{x,i}\right)^2\right]<\infty.$$

Conditioning on Y *this expectation is equal to*

$$\mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\sum_{i,j=1}^{\mathcal{L}(x,\infty)}\mathbb{E}\left[\eta_{x,i}\eta_{x,j}\middle|Y\right]\right] \;\le\; \mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\mathcal{L}(x,\infty)^2\mathbb{E}\left[\eta_{x,1}^2\right]\right] \;\le\; \mathbb{E}[\eta_0^2]\mathbb{E}\left[(\kappa_2-\kappa_1)^2\right]$$

which is finite by assumption and (2.7). □

We now conclude the proof of the annealed functional central limit theorem by showing that B_t^n can be suitably approximated by a sum of Z_j.

Proof of Theorem 2. By Lemmas 2.3 and 2.4

$$\Sigma_m := \sum_{j=2}^m Z_j = \left(X_{S_{\kappa_m}}-S_{\kappa_m}\nu_\beta\right)-\left(X_{S_{\kappa_1}}-S_{\kappa_1}\nu_\beta\right) = \left(\varrho_m-S_{\kappa_m}\nu_\beta\right)-\left(\varrho_1-S_{\kappa_1}\nu_\beta\right)$$

for $m\ge 2$ is a sum of i.i.d. centred random variables with finite second moment.

Write $m_t := \sup\{j\ge 0 : S_{\kappa_j}\le t\}$ to be the number of regenerations by time $t>0$ then

$$\sup_{t\in[0,T]}\left|B_t^n-\frac{\Sigma_{m_{tn}}}{\varsigma\sqrt{n}}\right| \le \frac{\varrho_1+S_{\kappa_1}\nu_\beta+|\min_k Y_k|}{\varsigma\sqrt{n}} + \sup_{j=1,\dots,m_{Tn}}\frac{\varrho_{j+1}-\varrho_j+\left(S_{\kappa_{j+1}}-S_{\kappa_j}\right)\nu_\beta}{\varsigma\sqrt{n}}.$$

The random variables ϱ_1, S_{κ_1} and $|\min_k Y_k|$ are all almost surely finite therefore the first fraction converges to 0 $\mathbb{P}$-a.s. For $\varepsilon>0$, by a union bound and Markov's inequality

$$\begin{aligned}\mathbb{P}&\left(\sup_{j=1,\dots,m_{Tn}}\frac{S_{\kappa_{j+1}}-S_{\kappa_j}}{\sqrt{n}}>\varepsilon\right)\\ &\le \mathbb{P}\left(m_{Tn}>2Tn\mathbb{E}[\eta_0]^{-1}\right)+C_Tn\mathbb{P}\left(S_{\kappa_2}-S_{\kappa_1}>\varepsilon\sqrt{n}\right)\\ &\le \mathbb{P}\left(m_{Tn}>2Tn\mathbb{E}[\eta_0]^{-1}\right)+C_{T,\varepsilon}\mathbb{E}\left[(S_{\kappa_2}-S_{\kappa_1})^2\mathbf{1}_{\{S_{\kappa_2}-S_{\kappa_1}\ge\varepsilon\sqrt{n}\}}\right].\end{aligned}$$

By Proposition 2.1, since $S_t^{-1}\ge m_t$, we have that $\mathbb{P}\left(m_{Tn}>2Tn\mathbb{E}[\eta_0]^{-1}\right)\to 0$ as $n\to\infty$. By Lemma 2.4 we have that $\mathbb{E}\left[(S_{\kappa_2}-S_{\kappa_1})^2\right]<\infty$ therefore by dominated convergence

$$\mathbb{E}\left[(S_{\kappa_2}-S_{\kappa_1})^2\mathbf{1}_{\{S_{\kappa_2}-S_{\kappa_1}\ge\varepsilon\sqrt{n}\}}\right]\to 0$$

as $n\to\infty$. Similarly, by (2.7), $\mathbb{E}\left[(\varrho_2-\varrho_1)^2\right]<\infty$ hence we have that

$$\mathbb{P}\left(\sup_{j=1,\dots,m_{Tn}}\frac{\varrho_{j+1}-\varrho_j}{\sqrt{n}}>\varepsilon\right)\to 0$$

as $n \to \infty$, and the supremum distance between $(B_t^n)_{t\in[0,T]}$ and $(\Sigma_{m_{tn}}/\varsigma\sqrt{n})_{t\in[0,T]}$ converges to 0 in $\mathbb{P}$-probability. It therefore suffices to prove an invariance principle for $\Sigma_{m_{tn}}$.

For $s \in \mathbb{R}^+$ let Σ_s denote the linear interpolation of Σ_m then by Donsker's invariance principle we have that $(\Sigma_{tn}/\sqrt{n})_{t\in[0,T]}$ converges in distribution to a scaled Brownian motion.

By the law of large numbers we have that κ_n/n converges $\mathbb{P}$-a.s. to $\mathbb{E}[\kappa_2 - \kappa_1]$ as $n \to \infty$. Therefore, by continuity of the inverse at strictly increasing functions ([32, Corollary 13.6.4]), we have that m_{tn}/n converges $\mathbb{P}$-a.s. on $D([0,\infty),\mathbb{R})$ to the deterministic process $R_t := (\mathbb{E}[\eta_0]\mathbb{E}[\kappa_2-\kappa_1])^{-1}t$.

By continuity of composition at continuous limits ([32, Theorem 13.2.1]) it follows that the sequence $(\Sigma_{m_{tn}}/\sqrt{n})_{t\in[0,T]}$ converges to the same limit as $(\Sigma_{R_t n}/\sqrt{n})_{t\in[0,T]}$ which is a scaled Brownian motion. In particular, choosing

$$\varsigma^2 = \frac{\mathbb{E}[Z_2^2]}{\mathbb{E}[\eta_0]\mathbb{E}[\kappa_2-\kappa_1]} \tag{2.8}$$

we have that B_t^n converges to a standard Brownian motion. □

3. Quenched central limit theorem

In this section we prove Theorem 3 which is a quenched CLT for the RTRW. We do this by first proving a quenched CLT for the first hitting time of x. This involves an environment dependent centring $\mathcal{H}^\omega$ which we then approximate by a more suitable sum $\tilde{\mathcal{H}}^\omega$ of random variables which are i.i.d. under $\mathbf{P}$. We complete the result by controlling the variation of mean holding times at different vertices and using that the walk does not deviate too far from the furthest point reached.

Write $\tau_x := \inf\{t \geq 0 : X_t = x\}$ and, for ω fixed, $\mathcal{H}^\omega(x) := E^\omega[\tau_x]$. Let $\zeta_y := \tau_{y+1} - \tau_y$ be the time taken between hitting y and $y+1$ for the first time. The elements of $(\zeta_y)_{k\geq 1}$ are independent under P^ω and

$$\tau_x = \sum_{y=0}^{x-1} \zeta_y.$$

Lemma 3.1. *Suppose that $\beta > 1$ and $\mathbb{E}[\eta_0^2] < \infty$. For $\mathbf{P}$-a.e. ω we have that*

$$P^\omega\left(\frac{\tau_x - \mathcal{H}^\omega(x)}{\sigma\sqrt{x}} < t\right) \to \Phi(t)$$

uniformly in t as $x \to \infty$, where $\sigma^2 = \mathbf{E}[\mathrm{Var}_\omega(\tau_1)]$.

Proof. By definition of τ_x, $\mathcal{H}^\omega(x)$ and ζ_y

$$\frac{\tau_x - \mathcal{H}^\omega(x)}{\sigma\sqrt{x}} = \frac{\sum_{y=0}^{x-1}(\zeta_y - E^\omega[\zeta_y])}{\sigma\sqrt{x}}.$$

It therefore suffices to show that Lindeberg's conditions (see [16, Theorem 3.4.5]) hold:

1. for $\mathbf{P}$-a.e. ω, as $x \to \infty$

$$\sum_{y=0}^{x-1} E^\omega\left[\left(\frac{\zeta_y - E^\omega[\zeta_y]}{\sigma\sqrt{x}}\right)^2\right] \to 1;$$

2. for $\mathbf{P}$-a.e. ω, $\forall \varepsilon > 0$ as $x \to \infty$
$$\sum_{y=0}^{x-1} E^\omega\left[\left(\frac{\zeta_y - E^\omega[\zeta_y]}{\sigma\sqrt{x}}\right)^2 \mathbf{1}_{\{|\zeta_y - E^\omega[\zeta_y]| > \varepsilon\sqrt{x}\}}\right] \to 0.$$

Let $\tilde{\theta}$ denote the shift map on the environment ($(\tilde{\theta}\omega)_k = \omega_{k+1}$) which is ergodic because the environment is i.i.d. For the first condition we have that $\zeta_y - E^\omega[\zeta_y] = \tilde{\theta}^y(\zeta_0 - E^\omega[\zeta_0])$. These random variables are identically distributed under $\mathbf{P}$ and, similarly to Lemma 2.4, we have that $\mathbf{E}\left[E^\omega[(\zeta_0 - E^\omega[\zeta_0])^2]\right] \le \mathbb{E}[\tau_1^2] < \infty$ therefore
$$\sum_{y=0}^{x-1} \frac{\mathrm{Var}_\omega(\zeta_y)}{\sigma^2 x} = \sum_{y=0}^{x-1} \frac{\mathrm{Var}_{\tilde{\theta}^y\omega}(\zeta_0)}{\sigma^2 x}$$
which converges to $\mathbf{E}[\mathrm{Var}_\omega(\zeta_0)]\,\sigma^{-2} = 1$ for $\mathbf{P}$-a.e. ω by Birkhoff's ergodic theorem.

For the second condition write $U_K^\omega(\cdot) := E^\omega[(\cdot - E^\omega[\cdot])^2 \mathbf{1}_{\{|\cdot - E^\omega[\cdot]| > K\}}]$ then for all $\varepsilon > 0$ there exists $\exists N_{\varepsilon,K} \in \mathbb{N}$ such that $\varepsilon\sqrt{x} > K$ for all $x \ge N_{\varepsilon,K}$. Therefore, for x large
$$\sum_{y=0}^{x-1} E^\omega\left[\left(\frac{\zeta_y - E^\omega[\zeta_y]}{\sigma\sqrt{x}}\right)^2 \mathbf{1}_{\{|\zeta_y - E^\omega[\zeta_y]| > \varepsilon\sqrt{x}\}}\right] \le \sum_{y=0}^{x-1} \frac{U_K^\omega(\zeta_y)}{\sigma^2 x} = \sum_{y=0}^{x-1} \frac{U_K^{\tilde{\theta}^y\omega}(\zeta_0)}{\sigma^2 x}.$$
By Birkhoff's ergodic theorem, for $\mathbf{P}$-a.e. ω, this converges to
$$\frac{\mathbf{E}\left[U_K^\omega(\zeta_0)\right]}{\sigma^2} = \frac{\mathbf{E}\left[E^\omega\left[(\zeta_0 - E^\omega[\zeta_0])^2 \mathbf{1}_{\{|\zeta_0 - E^\omega[\zeta_0]| > K\}}\right]\right]}{\sigma^2}$$
which converges to 0 as $K \to \infty$ by dominated convergence. □

Write $\tau_x^Y := \inf\{m \ge 0 : \; Y_m = x\}$ to be the first hitting time of x by the embedded walk. The following lemma describes the probability that the embedded walk moves back to y before moving forward to x. The result is the classical Gambler's ruin therefore we omit the proof.

Lemma 3.2. *For integers $y < 0 < x$*
$$P_0(\tau_y^Y < \tau_x^Y) = \frac{\beta^x - 1}{\beta^{x-y} - 1}.$$

By Lemma 3.1 we have a central limit theorem for the first hitting time of vertex x. The environment dependent centring $\mathcal{H}^\omega(x)$ can be written as the sum of x identically distributed random variables $E^\omega[\zeta_y]$. These are not independent however; they are only locally dependent. Recall that $\eta_{y,i}$ is the i^{th} holding time at vertex y hence $\mathbb{E}[\zeta_0] = \mathbb{E}[\eta_{0,0}](\beta+1)/(\beta-1)$ then write
$$\tilde{\mathcal{H}}^\omega(x) := \sum_{y=0}^{x-1} \frac{\beta+1}{\beta-1} E^\omega[\eta_{y,0}].$$

We now show that $\mathcal{H}^\omega$ and $\tilde{\mathcal{H}}^\omega$ do not differ too much and therefore Lemma 3.1 also holds with $\mathcal{H}^\omega$ replaced by $\tilde{\mathcal{H}}^\omega$. Notice that this is the first point at which we introduce the extra $2+\varepsilon$ moment condition however we do require the condition later in Lemma 3.4 as well.

Lemma 3.3. *Suppose $\beta > 1$ and $\mathbf{E}\left[E^\omega[\eta_0]^{2+\varepsilon}\right] < \infty$ for some $\varepsilon > 0$. For $\mathbf{P}$-a.e. ω, as $x \to \infty$,*
$$\left|\frac{\tilde{\mathcal{H}}^\omega(x) - \mathcal{H}^\omega(x)}{\sqrt{x}}\right| \to 0.$$

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Proof. Recall that $\mathcal{L}(y,m)$ denotes the local time of Y at vertex y by time m and that the trapping times $\eta_{y,j}$ do not depend on the embedded walk, then

$$\mathcal{H}^\omega(x) = E^\omega\left[\sum_{y=-\infty}^{x-1}\sum_{j=1}^{\mathcal{L}(y,\tau_x^Y)}\eta_{y,j}\right] = \sum_{y=-\infty}^{x-1} E_0[\mathcal{L}(y,\tau_x^Y)]E^\omega[\eta_{y,0}].$$

We need to determine the expected local times at sites up to reaching x. By the strong Markov property $E_0[\mathcal{L}(y,\tau_x^Y)] = P_0(\tau_y^Y < \tau_x^Y)E_y[\mathcal{L}(y,\tau_x^Y)]$.

Let $(\tau_x^Y)^+ := \inf\{m > 0 : \ Y_m = x\}$ be the first return time to x by the embedded walk. By Lemma 3.2, $\mathcal{L}(y,\tau_x^Y)$ for a walk started at y is geometrically distributed with escape probability

$$P_y(\tau_x^Y < (\tau_y^Y)^+) = \frac{\beta}{1+\beta}\left(1 - P_0(\tau_{-1}^Y < \tau_{x-y-1}^Y)\right) = \frac{\beta^{x-y}(\beta-1)}{(\beta^{x-y}-1)(\beta+1)}.$$

Therefore,

$$E_y\left[\mathcal{L}(y,\tau_x^Y)\right] = \frac{(\beta^{x-y}-1)(\beta+1)}{\beta^{x-y}(\beta-1)} \qquad \text{and}$$

$$E_0\left[\mathcal{L}(y,\tau_x^Y)\right] = \begin{cases} \dfrac{(\beta^{x}-1)(\beta+1)}{\beta^{x-y}(\beta-1)} & \text{if } y < 0, \\ \dfrac{(\beta^{x-y}-1)(\beta+1)}{\beta^{x-y}(\beta-1)} & \text{if } y \geq 0. \end{cases}$$

For fixed $y < 0$, $E_0[\mathcal{L}(y,\tau_x^Y)]$ is increasing in x and converges to $\beta^y(\beta+1)/(\beta-1)$. In particular,

$$0 \leq \sum_{y=-\infty}^{-1} E_0[\mathcal{L}(y,\tau_x^Y)]E^\omega[\eta_{y,0}] \leq C\sum_{y=1}^{\infty}\beta^{-y}E^\omega[\eta_{-y,0}]$$

which is finite for $\mathbf{P}$-a.e. ω therefore $x^{-1/2}\sum_{y=-\infty}^{-1}E_0[\mathcal{L}(y,\tau_x^Y)]E^\omega[\eta_{y,0}]$ converges to 0 for $\mathbf{P}$-a.e. ω.

For $y \geq 0$ fixed, $E_0[\mathcal{L}(y,\tau_x^Y)]$ is increasing in x and converges to $(\beta+1)/(\beta-1)$. In particular,

$$\begin{aligned}
0 &\leq \sum_{y=0}^{\infty}\left(\frac{\beta+1}{\beta-1} - E_0[\mathcal{L}(y,\tau_x^Y)]\right)E^\omega[\eta_{y,0}] \\
&= \frac{\beta+1}{\beta-1}\sum_{y=0}^{x-1}\beta^{-(x-y)}E^\omega[\eta_{y,0}] \\
&= \frac{\beta+1}{\beta-1}\left(\sum_{y=0}^{x-\left\lfloor 2\frac{\log(x)}{\log(\beta)}\right\rfloor}\beta^{-(x-y)}E^\omega[\eta_{y,0}] + \sum_{y=x-\left\lfloor 2\frac{\log(x)}{\log(\beta)}\right\rfloor+1}^{x-1}\beta^{-(x-y)}E^\omega[\eta_{y,0}]\right) \\
&\leq C\left(\sum_{y=0}^{x-1}\beta^{-2\frac{\log(x)}{\log(\beta)}}E^\omega[\eta_{y,0}] + \sum_{y=x-\left\lfloor 2\frac{\log(x)}{\log(\beta)}\right\rfloor+1}^{x-1}E^\omega[\eta_{y,0}]\right)
\end{aligned}$$

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$$= C\left(\sum_{y=0}^{x-1}\frac{E^{\omega}[\eta_{y,0}]}{x^2} + \sum_{y=x-\left\lfloor 2\frac{\log(x)}{\log(\beta)}\right\rfloor+1}^{x-1} E^{\omega}[\eta_{y,0}]\right).$$

The first term converges to 0 for **P**-a.e. ω by the strong law of large numbers. For the second term we have that, for $\delta, \epsilon > 0$, by Markov's inequality

$$\begin{aligned}\mathbf{P}\left(\sum_{y=x-\left\lfloor 2\frac{\log(x)}{\log(\beta)}\right\rfloor+1}^{x-1} E^{\omega}[\eta_{y,0}] > \epsilon\sqrt{x}\right) &\leq 2\frac{\log(x)}{\log(\beta)}\mathbf{P}\left(E^{\omega}[\eta_0] > \frac{\epsilon\sqrt{x}\log(\beta)}{2\log(x)}\right)\\ &= 2\frac{\log(x)}{\log(\beta)}\mathbf{P}\left(E^{\omega}[\eta_0]^{2+\delta} > \frac{C_{\epsilon}x^{1+\delta/2}}{\log(x)^{2+\delta}}\right)\\ &\leq \frac{C\log(x)^{3+\delta}}{x^{1+\delta/2}}\end{aligned}$$

since we can choose $\delta > 0$ sufficiently small such that $\mathbf{E}\left[E^{\omega}[\eta_0]^{2+\delta}\right] < \infty$. By Borel–Cantelli we then have that

$$\sum_{y=x-\left\lfloor 2\frac{\log(x)}{\log(\beta)}\right\rfloor+1}^{x-1} \frac{E^{\omega}[\eta_{y,0}]}{\sqrt{x}}$$

converges to 0 for **P**-a.e. ω. □

We now prove a technical lemma that allows us to control the difference between $\tilde{\mathcal{H}}^{\omega}$ and its expected value under **P** which is important in proving the quenched CLT for the walk.

Lemma 3.4. *Let*

$$\mathcal{J}(x) := \sum_{y=0}^{x-1}\left(E^{\omega}[\eta_{y,0}] - \mathbb{E}[\eta_{y,0}]\right)$$

and

$$\mathcal{J}^*(x) := \max_{y\leq x}\mathcal{J}(y).$$

1. *Suppose* $\mathbf{E}\left[E^{\omega}[\eta_0]^2\right] < \infty$, *then for any* $c > 0$, $\mathcal{J}(x)n^{-\frac{1+c}{2}} \to 0$ *for* **P**-*a.e.* ω;
2. *Suppose* $\mathbf{E}\left[E^{\omega}[\eta_0]^{2+\varepsilon}\right] < \infty$ *for some* $\varepsilon > 0$, *then for* $\delta > 0$ *sufficiently small and some constant* C

$$\mathbf{E}\left[|\mathcal{J}^*(x)|^{2+2\delta}\right]^{\frac{1}{2+2\delta}} \leq Cx^{1/2}.$$

Proof. By [28, Theorem IX.3.17], if Z_x are i.i.d. centred random variables, a_x is an increasing, diverging sequence and

1. $\sum_{x=1}^{\infty}\mathbf{P}(|Z_1| \geq a_x) < \infty$;
2. $\sum_{y=x}^{\infty}a_y^{-2} = O\left(\frac{x}{a_x^2}\right)$;
3. $a_y/a_x \leq Cy/x$ for all $y \geq x$

then $\sum_{y=1}^{x}\frac{Z_y}{a_x}$ converges to 0, **P**-a.s.

Write $Z_x := E^\omega[\eta_{x,0}] - \mathbb{E}[\eta_{x,0}]$ then Z_x are i.i.d. and centred under $\mathbf{P}$; moreover, the sequence $a_x = x^{\frac{1+c}{2}}$ is increasing and diverges. By Chebyshev's inequality we have that

$$\sum_{x=1}^{\infty} \mathbf{P}(|Z_1| \geq a_x) \leq \sum_{x=1}^{\infty} \mathrm{Var}_{\mathbf{P}}(E^\omega[\eta_0]) x^{-(1+c)} < \infty$$

which gives condition 1. Since $x/a_x^2 = x^{-c}$, an integral test gives condition 2. For $y \geq x$ we have that $a_y/a_x = (y/x)^{\frac{1+c}{2}} \leq y/x$ so long as $c \leq 1$ which gives 3. We therefore have that for any $c > 0$, $\mathcal{J}(x) x^{-\frac{1+c}{2}} \to 0$ for $\mathbf{P}$-a.e. ω hence the first statement holds.

The process $\mathcal{J}(x)$ is a martingale therefore by the L^p-maximal inequality we have that

$$\mathbf{E}\left[\max_{y \leq x} |\mathcal{J}(y)|^{2+2\delta}\right] \leq \left(\frac{2+2\delta}{1+2\delta}\right)^{2+2\delta} \mathbf{E}\left[|\mathcal{J}(x)|^{2+2\delta}\right].$$

It therefore suffices to show that

$$\mathbf{E}\left[\left(\sum_{y=0}^{x-1} \frac{E^\omega[\eta_{y,0}] - \mathbb{E}[\eta_{y,0}]}{\sqrt{x}}\right)^{2+2\delta}\right]$$

is bounded above. By the Marcinkiewicz–Zygmund inequality [26, Theorem 5] we have that

$$\mathbf{E}\left[\left(\sum_{y=0}^{x-1} \frac{E^\omega[\eta_{y,0}] - \mathbb{E}[\eta_{y,0}]}{\sqrt{x}}\right)^{2+2\delta}\right] \leq C\mathbf{E}\left[\left(\sum_{y=0}^{x-1} \left(\frac{E^\omega[\eta_{y,0}] - \mathbb{E}[\eta_{y,0}]}{\sqrt{x}}\right)^2\right)^{1+\delta}\right]$$

which is bounded above by

$$C\mathbf{E}\left[\sum_{y=0}^{x-1} \frac{\left(E^\omega[\eta_{y,0}] - \mathbb{E}[\eta_{y,0}]\right)^{2+2\delta}}{n}\right] = C\mathbf{E}\left[(E^\omega[\eta_0] - \mathbb{E}[\eta_0])^{2+2\delta}\right]$$

using Jensen's inequality. Using that $\mathbf{E}\left[E^\omega[\eta_0]^{2+\varepsilon}\right] < \infty$ for some $\varepsilon > 0$, it then follows that for $\delta > 0$ sufficiently small and some constant C

$$\mathbf{E}\left[|\mathcal{J}^*(x)|^{2+2\delta}\right]^{\frac{1}{2+2\delta}} \leq C x^{1/2}. \quad \square$$

We now prove the main result of the section which is a quenched central limit theorem for the randomly trapped random walk. Recall that we use the centring

$$\mathcal{G}^\omega(t) := \nu_\beta t - \nu_\beta \sum_{y=0}^{\lfloor \nu_\beta t - 1 \rfloor} \frac{\beta+1}{\beta-1}(E^\omega[\eta_{y,0}] - \mathbb{E}[\eta_{y,0}]) \qquad \text{where} \qquad \nu_\beta = \frac{\beta-1}{(\beta+1)\mathbb{E}[\eta_0]}$$

is the $\mathbb{P}$-a.s. limit of X_n/n. Write $\vartheta := \sigma \nu_\beta^{3/2}$ where we recall that $\sigma^2 = \mathbf{E}[\mathrm{Var}_\omega(\tau_1)]$.

Proof of Theorem 3. Let $\overline{X}_t := \sup\{|X_s| : s \leq t\}$ be the furthest point reached by X up to time t; then $\tau_{\overline{X}_t} \leq t < \tau_{\overline{X}_t + 1}$ and $X_t \leq X_{\tau_{\overline{X}_t}} = \overline{X}_t$. We then have that $|X_t - \overline{X}_t| = |X_t - X_{\tau_{\overline{X}_t}}| \leq \sup_{s \geq \tau_{\overline{X}_t}} X_{\tau_{\overline{X}_t}} - X_s$. Write

$$A_x := \bigcap_{y=1}^{x} \left\{\inf\{Y_m : m \geq \tau_y^Y\} \geq y - C\log(x)\right\}$$

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to be the event that the walk never backtracks distance $C\log(x)$ up to reaching vertex x. By Lemma 3.2 we then have that

$$P(A_x^c) \;\le\; \overline{C}xP(\tau_{-\lfloor C\log(x)\rfloor}<\infty) \;\le\; \overline{C}x\beta^{-C\log(x)} \;=\; \overline{C}x^{1-C\log(\beta)}.$$

Therefore, choosing C such that $C\log(\beta)>2$, by Borel–Cantelli we have that there exists only finitely many x such that the walk backtracks distance $C\log(x)$ up to reaching level x. By Theorem 1 we then have that for t sufficiently large $|X_t-\overline{X}_t|t^{-1/2}\le C\log(t)t^{-1/2}$ which converges deterministically to 0. It therefore suffices to show that for $\mathbf{P}$-a.e. ω

$$\lim_{t\to\infty}P^\omega\left(\frac{\overline{X}_t-\mathcal{G}^\omega(t)}{\vartheta\sqrt{t}}\le u\right)=\Phi(u).$$

By monotonicity we have that $\{\overline{X}_t\le y\}=\{\tau_{y+1}>t\}$. Writing $\mathcal{I}^\omega(t):=\lfloor u\vartheta\sqrt{t}+\mathcal{G}^\omega(t)+1\rfloor$ it then follows that

$$\begin{aligned}P^\omega\left(\frac{\overline{X}_t-\mathcal{G}^\omega(t)}{\vartheta\sqrt{t}}<u\right) &= P^\omega\left(\tau_{\mathcal{I}^\omega(t)}>t\right)\\ &=P^\omega\left(\frac{\tau_{\mathcal{I}^\omega(t)}-\mathcal{H}^\omega(\mathcal{I}^\omega(t))}{\sigma\sqrt{\mathcal{I}^\omega(t)}}>\frac{t-\mathcal{H}^\omega(\mathcal{I}^\omega(t))}{\sigma\sqrt{t}}\cdot\sqrt{\frac{t}{\mathcal{I}^\omega(t)}}\right).\end{aligned}$$

The sequence $\mathcal{I}^\omega(t)$ is increasing in t and diverges; in particular, by the law of large numbers $t/\mathcal{I}^\omega(t)$ converges to ν_β^{-1} for $\mathbf{P}$-a.e. ω. The result then follows from Lemma 3.1 if $\mathcal{H}^\omega(\mathcal{I}^\omega(t))=t+\sigma\nu_\beta^{1/2}u\sqrt{t}+o_t$, where $o_t/\sqrt{t}$ converges to 0 for $\mathbf{P}$-a.e. ω.

Since $\mathcal{I}^\omega(t)$ diverges, by Lemma 3.3 it suffices to show that $\tilde{\mathcal{H}}^\omega(\mathcal{I}^\omega(t))=t+\sigma\nu_\beta^{1/2}x\sqrt{t}+o_t$. By definition of $\tilde{\mathcal{H}}^\omega$ and $\mathcal{I}^\omega(t)$ we have that there exists some $O_1:=O_1(\omega,t,u)$ such that $|O_1|\le\nu_\beta^{-1}$ and

$$\begin{aligned}\tilde{\mathcal{H}}^\omega(\mathcal{I}^\omega(t))&=\nu_\beta^{-1}\mathcal{I}^\omega(t)+\sum_{y=0}^{\mathcal{I}^\omega(t)-1}\frac{\beta+1}{\beta-1}(E^\omega[\eta_{y,0}]-\mathbb{E}[\eta_{y,0}])\\ &=t+\sigma\nu_\beta^{1/2}u\sqrt{t}-\sum_{y=0}^{\lfloor\nu_\beta t\rfloor-1}\frac{\beta+1}{\beta-1}(E^\omega[\eta_{y,0}]-\mathbb{E}[\eta_{y,0}])\\ &\quad+\sum_{y=0}^{\mathcal{I}^\omega(t)-1}\frac{\beta+1}{\beta-1}(E^\omega[\eta_{y,0}]-\mathbb{E}[\eta_{y,0}])+O_1.\end{aligned}$$

Moreover, for some $O_2:=O_2(\omega,t,u)$ satisfying $|O_2|\le 3$, we have that

$$\mathcal{I}^\omega(t)-\lfloor\nu_\beta t\rfloor=\vartheta u\sqrt{t}+\mathbb{E}[\eta_0]^{-1}\sum_{y=0}^{\lfloor\nu_\beta t-1\rfloor}\left(E^\omega[\eta_{y,0}]-\mathbb{E}[\eta_{y,0}]\right)+O_2.$$

By part 1 of Lemma 3.4 we have that $(\mathcal{I}^\omega(t)-\lfloor\nu_\beta t\rfloor)t^{-\frac{1+c}{2}}$ converges to 0 for $\mathbf{P}$-a.e. ω and any $c>0$. In order to show that $\tilde{\mathcal{H}}^\omega(\mathcal{I}^\omega(t))=t+\sigma\nu_\beta^{1/2}u\sqrt{t}+o_t$ it now suffices to show that for all $c>0$ suitably small

$$\mathcal{R}^\omega(x,c):=x^{-1/2}\max_{z\le x^{\frac{1+c}{2}}}\left|\sum_{y=x}^{x+z}\left(E^\omega[\eta_{y,0}]-\mathbb{E}[\eta_{y,0}]\right)\right|$$

converges to 0 as $x\to\infty$ for $\mathbf{P}$-a.e. ω.

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Suppose that $\mathcal{R}^\omega(x^2, 2c)$ converges to 0 for all $c > 0$ suitably small and **P**-a.e. ω then for $v = 1, \ldots, 2x$ (that is, $x^2 < x^2 + v < (x+1)^2$) we have that

$$\begin{aligned}\mathcal{R}^\omega(x^2+v, c) &= (x^2+v)^{-1/2} \max_{m \leq (x^2+v)^{\frac{1+c}{2}}} \left| \sum_{k=x^2+v}^{x^2+v+z} \left(E^\omega[\eta_{y,0}] - \mathbb{E}[\eta_{y,0}]\right) \right| \\ &\leq (x^2+v)^{-1/2} \max_{z \leq (x^2+v)^{\frac{1+c}{2}}} \left(\left| \sum_{y=x^2}^{x^2+v+z} \left(E^\omega[\eta_{y,0}] - \mathbb{E}[\eta_{y,0}]\right) \right| \right. \\ &\quad \left. + \left| \sum_{y=x^2}^{x^2+v-1} \left(E^\omega[\eta_{y,0}] - \mathbb{E}[\eta_{y,0}]\right) \right| \right).\end{aligned}$$

Since $v + z < x^{1+2c}$ for x suitably large we then have that $\mathcal{R}^\omega(x^2+v, c) \leq 2\mathcal{R}^\omega(x^2, 2c)$ for all $v = 1, \ldots, 2x$ thus it suffices to show that $\mathcal{R}^\omega(x^2, 2c)$ converges to 0 for all $c > 0$ suitably small and **P**-a.e. ω. For $\epsilon > 0$, by Markov's inequality

$$\begin{aligned}\mathbf{P}\left(\mathcal{R}^\omega(x^2, 2c) > \epsilon\right) &\leq \mathbf{E}\left[\mathcal{R}^\omega(x^2, 2c)^{2+2\delta}\right] \epsilon^{-(2+2\delta)} \\ &= C_\epsilon \mathbf{E}\left[\left(x^{-1} \max_{z \leq x^{1+2c}} \left| \sum_{y=x^2}^{x^2+z} \left(E^\omega[\eta_{y,0}] - \mathbb{E}[\eta_{y,0}]\right) \right| \right)^{2+2\delta} \right] \\ &\leq C_\epsilon x^{(1+2c)(1+\delta)-2(1+\delta)}\end{aligned}$$

by part 2 of Lemma 3.4. Choosing $c < \delta/(2+2\delta)$ gives us that

$$\sum_{x=1}^{\infty} \mathbf{P}\left(\mathcal{R}^\omega(x^2, 2c) > \epsilon\right) < \infty$$

therefore by Borel–Cantelli we have the desired result. □

4. Subcritical Galton–Watson trees

A subcritical Galton–Watson tree conditioned to survive consists of a semi-infinite path with GW-trees as leaves. The walk on the tree does not deviate too far from this path and therefore behaves like a randomly trapped random walk on $\mathbb{Z}$ with holding times distributed as excursion times in GW-trees. In this section we prove Theorems 4–6. Our strategy is to couple the walk on the tree with a randomly trapped random walk in such a way that these results can be deduced from Theorems 1–3 along with the appropriate moment bounds on the excursion times in random trees.

Recall that f denotes the generating function of a GW-process with mean $\mu \in (0, 1)$ and variance $\sigma^2 < \infty$, Z_n is the n^{th} generation size of a GW-process with this law and $\mathcal{T}^f$ the associated tree. Furthermore, recall that we denote by ξ a random variable with the offspring law and then define ξ^* to be a random variable with the size-biased law given by the probabilities $\mathbf{P}(\xi^* = k) = kp_k\mu^{-1}$.

It has been shown in [21] that there is a well defined probability measure **P** over f-GW trees conditioned to survive which arises as the limit as $n \to \infty$ of probability measures over GW-trees conditioned to survive up to generation n. It can be seen (e.g. [20]) that the tree can be constructed by attaching i.i.d. finite trees to a single semi-infinite path $\mathcal{Y} := (\rho_0 = \rho, \rho_1, \ldots)$.

More specifically, starting with a single special vertex ρ_0, at each generation let every normal vertex give birth to normal vertices according to independent copies of the original offspring distribution and every special vertex give birth to vertices according to independent copies of the size-biased distribution, one of which is chosen uniformly at random to be special (and denoted ρ_k in the k^{th} generation). We will use $\mathcal{T}$ to denote an f-GW-tree conditioned to survive and $\mathcal{T}_x^{*-}$ the branch rooted at $x \in \mathcal{Y}$; that is, the descendants of x which are not in the descendant tree of the unique child of x on $\mathcal{Y}$ (see Fig. 2).

Recall that a β-biased random walk on a fixed, rooted tree $\mathscr{T}$ is a random walk X on $\mathscr{T}$ which is β-times more likely to make a transition to a given child of the current vertex than the parent (i.e. X is the Markov chain given by the transition probabilities (1.1)) and that we use $\mathbb{P}(\cdot) = \int P_\rho^{\mathcal{T}}(\cdot)\mathbf{P}(\mathrm{d}\mathcal{T})$ for the annealed law obtained by averaging the quenched law $P_\rho^{\mathcal{T}}$ over the law $\mathbf{P}$ on f-GW-trees conditioned to survive (with a fixed root ρ). Unless stated otherwise we will assume that $\beta > 1$ so that the walk is $\mathbf{P}$-a.s. transient.

We now construct an almost equivalent model which allows us to consider the walk on the GW-tree in our randomly trapped random walk framework. We begin by constructing the holding times of the randomly trapped random walk via a sequence of i.i.d. trees. Start with an initial vertex ρ and a unique ancestor $\overline{\rho}$. Attach $\xi^* - 1$ offspring to ρ where ξ^* is size-biased as above then attach independent f-GW trees to the offspring of ρ. This creates a tree $\overline{\mathcal{T}}$ which has the distribution of a branch with an additional vertex connected to the root (see Fig. 3).

Consider a walk $(W_n)_{n\geq 0}$ on $\overline{\mathcal{T}}$ with transition probabilities

$$P^{\overline{\mathcal{T}}}(W_{n+1} = y | W_n = x) = \begin{cases} 1, & \text{if } x = y = \overline{\rho}, \\ \dfrac{\beta+1}{1+\beta(|c(x)|+1)}, & \text{if } x = \rho,\ y = \overline{\rho}, \\ \dfrac{\beta}{1+\beta(|c(x)|+1)}, & \text{if } x = \rho,\ y \in c(x), \\ \dfrac{\beta}{1+\beta|c(x)|}, & \text{if } x \notin \{\rho, \overline{\rho}\},\ y \in c(x), \\ \dfrac{1}{1+\beta|c(x)|}, & \text{if } x \notin \{\rho, \overline{\rho}\},\ y = \overleftarrow{x}, \\ 0, & \text{otherwise.} \end{cases}$$

An excursion in $\overline{\mathcal{T}}$ started from ρ until absorption in $\overline{\rho}$ has the same distribution as the time taken to move between backbone vertices of $\mathcal{T}$ (except at the root of $\mathcal{T}$). Let $\omega = (\overline{\mathcal{T}}_x)_{x\in\mathbb{Z}}$ denote a sequence of independent trees with this law. For ω fixed let $(\eta_{x,i})_{x\in\mathbb{Z}, i\geq 0}$ be independent with

$$P^\omega\left(\eta_{x,i} = k\right) = P_\rho^{\overline{\mathcal{T}}_x}\left(\min\{n > 0 : W_n = \overline{\rho}\} = k\right)$$

where $\rho, \overline{\rho}$ are the vertices in $\overline{\mathcal{T}}_x$ corresponding with the construction. For convenience we often write $\overline{\mathcal{T}}$ for $\overline{\mathcal{T}}_0$. We then consider the randomly trapped random walk with these holding times.

A two-sided tree can be constructed as the extension of a subcritical GW-tree by using the infinite backbone $\mathcal{Y} = (\ldots, \rho_{-1}, \rho_0, \rho_1, \ldots)$ and i.i.d. branches. The backbone is homoeomorphic to $\mathbb{Z}$ therefore the randomly trapped random walk we consider is equal in distribution to the projection of a random walk X_n on a two-sided tree onto the unique value $x \in \mathbb{Z}$ satisfying $d(X_n, \rho_x) = \min_{y\in\mathcal{Y}} d(X_n, y)$. By transience of the embedded walk on the backbone, the walk on the two sided tree almost surely spends a finite amount of time outside the sub-tree rooted at ρ_0. Moreover, for C large, up to level n there will be no branches of height greater than $C\log(n)$ with

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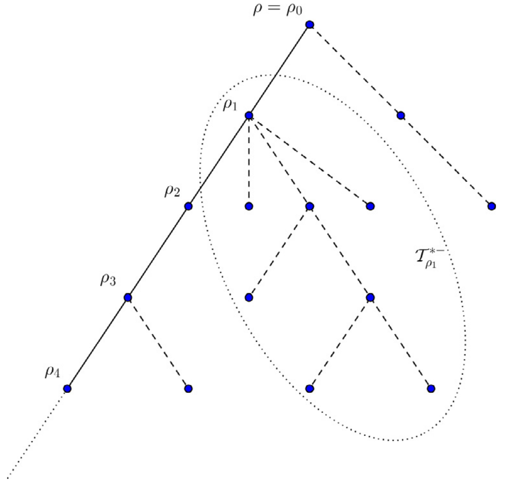

Fig. 2. A sample subcritical GW-tree conditioned to survive $\mathcal{T}$ with the backbone $\mathcal{Y}$ represented by solid lines and the buds and traps connected by dashed lines.

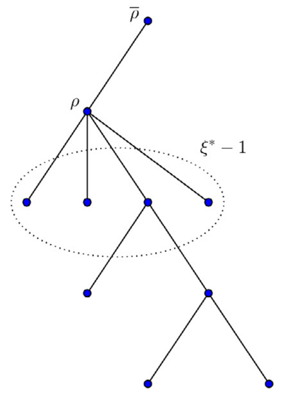

Fig. 3. A tree $\overline{\mathcal{T}}$ with fixed vertices $\rho, \overline{\rho}$ and $\xi^* - 1$ independent f-GW-trees attached to ρ.

high probability. It follows that the walk on the one-sided subcritical GW-tree can be coupled to a randomly trapped random walk so that the two walks deviate by at most $C \log(n)$ up to time n for n large (see [9] for more detail). Since we consider polynomial scaling, it suffices to consider this randomly trapped random walk. Without loss of generality, we continue to denote by X the randomly trapped random walk.

Before discussing the walk in detail we give several useful asymptotic properties of the generation sizes of GW-trees. To begin, by [23, Theorem B] the sequence $\mathbf{P}(Z_n > 0)\mu^{-n}$ is decreasing and converges since $\mu \in (0, 1)$ and $\sigma^2 < \infty$. In particular, we write

$$c_\mu := \lim_{n\to\infty} \frac{\mathbf{P}(Z_n > 0)}{\mu^n}. \tag{4.1}$$

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For a fixed ρ-rooted tree $\mathscr{T}$ associated to Z_n with $Z_1 > 0$ it follows from hitting time identities in electrical network theory (e.g. [25, Proposition 2.20]) that

$$E_\rho^{\mathscr{T}}[\tau_\rho^+] = 2\sum_{n\geq 1}\frac{Z_n\beta^{n-1}}{Z_1} \tag{4.2}$$

where $\tau_x^+ := \inf\{k > 0 : X_k = x\}$ is the first return time to a vertex x by a β biased random walk X. Lemma 4.1 shows bounds on the expected moments of the generation sizes. Parts 1 and 2 follow from [2] and part 3 is a simple extension (full details of which are given in [9]) therefore we omit the proof.

Lemma 4.1. *Let Z_n denote an f-GW-process with offspring distribution ξ and mean $\mu \in (0, 1)$. Then,*

1. $\mathbf{E}[Z_n] = \mu^n$;
2. *If* $\mathbf{E}[\xi^2] < \infty$ *and* $m \geq n$ *then* $\mathbf{E}[Z_n Z_m] \leq C\mu^m$ *for some constant* C;
3. *If* $\mathbf{E}[\xi^3] < \infty$ *and* $l \geq m \geq n$ *then* $\mathbf{E}[Z_n Z_m Z_l] \leq C\mu^l$ *for some constant* C.

4.1. *The speed of the walk*

We now prove Theorem 4 by proving a bound on the expected holding time for the randomly trapped random walk and applying Theorem 1.

Proof of Theorem 4. The quantity η_0 is distributed as the first hitting time of $\overline{\rho}$ in the random tree $\overline{\mathcal{T}}$ by the walk W started from ρ. Let

$$N = \sum_{k=1}^{\tau_{\overline{\rho}}}\mathbf{1}_{\{W_k=\rho\}} \tag{4.3}$$

be the number of return times to the root ρ before reaching its unique ancestor $\overline{\rho}$. That is, N is the number of excursions to the trees attached to ρ before the walk reaches $\overline{\rho}$. Let $\tau_x^{(0)} := 0$ then for $k = 1, \ldots, N$ write $\tau_x^{(k)} := \min\{n > \tau_x^{(k-1)} : W_n = x\}$ to be the hitting times of x and $\zeta_k := \tau_\rho^{(k)} - \tau_\rho^{(k-1)}$ the duration of the kth excursion. We want to determine the expected value of

$$\eta_0 = 1 + \sum_{k=1}^{N}\zeta_k. \tag{4.4}$$

Letting Z_n denote the generation sizes of an f-GW-tree $\mathcal{T}^f$ we have that

$$\mathbb{E}[\eta_0] = 1 + \mathbb{E}\left[\sum_{k=1}^{N}\mathbb{E}[\zeta_k|N]\right] = 1 + \mathbb{E}[N]\mathbf{E}\left[E_\rho^{\mathcal{T}^f}[\tau_\rho^+]|Z_1 = 1\right].$$

The number of excursions N is geometrically distributed with termination probability $1 - p_{ex}$ where

$$p_{ex} := P^{\overline{\mathcal{T}}}(W_1 \neq \overline{\rho}) = \frac{\beta(\xi^* - 1)}{\beta\xi^* + 1}.$$

It therefore follows that

$$\mathbb{E}[N] = \mathbb{E}\left[\frac{\beta(\xi^* - 1)}{\beta + 1}\right] = \frac{\beta}{\beta + 1}\left(\sum_{k=1}^{\infty}\frac{k^2 p_k}{\mu} - 1\right) = \frac{\beta(\sigma^2 - \mu(1 - \mu))}{(\beta + 1)\mu}.$$

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Using the formula (4.2) for the expected time spent in a fixed tree and statement 1 of Lemma 4.1 for the expected size of the k^{th} generation we have that

$$\mathbf{E}\left[E_\rho^{\mathcal{T}^f}[\tau_\rho^+]|Z_1=1\right] = \mathbf{E}\left[2\sum_{k\geq 1}\frac{Z_k\beta^{k-1}}{Z_1}|Z_1=1\right]$$
$$= 2\sum_{k\geq 1}\mathbf{E}[Z_k|Z_1=1]\beta^{k-1} = 2\sum_{k\geq 1}(\beta\mu)^{k-1}.$$

Since $\beta < \mu^{-1}$ then this is equal to $2/(1-\beta\mu)$; otherwise, the sum does not converge. It follows that

$$\mathbb{E}[\eta_0] = \frac{\mu(\beta+1)(1-\beta\mu)+2\beta(\sigma^2-\mu(1-\mu))}{\mu(\beta+1)(1-\beta\mu)} < \infty. \tag{4.5}$$

The result then follows from Theorem 1. $\square$

The following corollary extends the Einstein relation for the randomly trapped random walk to the walk on the GW-tree. This is a non-trivial extension because, in the tree model, the bias influences the trapping times and the unbiased walk is significantly influenced by the restriction to the half line. For this reason we observe convergence to a reflected Brownian motion and cannot simply apply Corollary 2.2. Despite this, we omit the proof which follows by a straightforward adaptation of standard techniques (similar to [12]) using [3, Theorem 2.9] and (4.5). Furthermore, the full details are given in [9].

Corollary 4.2. *Suppose $\mu < 1$ and $\sigma^2 < \infty$. The unbiased ($\beta = 1$) walk $X_{\lfloor nt\rfloor}n^{-1/2}$ converges in $\mathbb{P}$-distribution on $D([0,\infty),\mathbb{R})$ to $|B_t|$ where B_t is a scaled Brownian motion with variance $\Upsilon = \mathbb{E}[\eta_0]^{-1}$. Moreover,*

$$\lim_{\beta\to 1^+}\frac{\nu_\beta}{\beta-1} = \frac{\Upsilon}{2}.$$

4.2. An annealed functional central limit theorem

We now prove Theorem 5 by using the annealed invariance principle Theorem 2. That is, we show conditions on the tree and the bias which ensure that $\mathbb{E}[\eta_0^2] < \infty$.

In order to show this we will use a decomposition which counts the number of visits to each vertex. For $z \in \mathscr{T}$ and $A, B \subset \mathscr{T}$ write

$$q_z(A,B) := P_z^{\mathscr{T}}(\tau_A^+ < \tau_B^+)$$

to be the probability that the walk started from z hits A before B.

Let $\mathscr{T}_{x,y}$ denote a tree with root ρ in which every vertex has a single offspring except the vertices w, x, y where w has two offspring and x, y have none. Denote these offspring w_x, w_y then let x, y be descendants of w_x, w_y respectively (possibly w_x, w_y) (see Fig. 4).

Lemma 4.3 gives the probability that the walk started at w reaches ρ before x or y. Alternatively, this can be shown by comparing with an electrical network with conductances β^k between vertices in generations $k, k+1$ and then using network reduction (see, for example, [25, Chapter 2]).

Lemma 4.3. *For any $\mathscr{T}_{x,y}$,*

$$q_w(\rho,\{x,y\}) = \frac{(\beta^{|y|-|w|}-1)(\beta^{|x|-|w|}-1)}{2\beta^{|y|+|x|-|w|}-\beta^{|y|+|x|-2|w|}-\beta^{|x|}-\beta^{|y|}+1}.$$

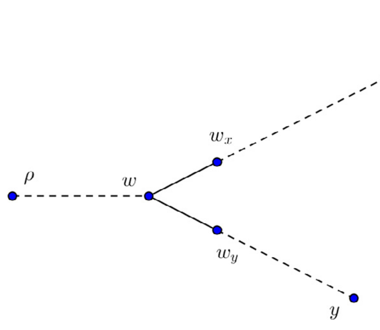

Fig. 4. The tree $\mathscr{T}_{x,y}$ with single branching point w and extremal points ρ, x, y.

Proof. Write w_ρ as the parent of w then

$$\begin{aligned}
q_w(\rho,\{x,y\}) &= \frac{1}{2\beta+1} q_{w_\rho}(\rho,\{x,y\}) + \frac{\beta}{2\beta+1} q_{w_x}(\rho,\{x,y\}) + \frac{\beta}{2\beta+1} q_{w_y}(\rho,\{x,y\}) \\
q_{w_\rho}(\rho,\{x,y\}) &= q_w(\rho,\{x,y\}) q_{w_\rho}(w,\rho) + q_{w_\rho}(\rho,w) \\
q_{w_x}(\rho,\{x,y\}) &= q_w(\rho,\{x,y\}) q_{w_x}(w,x) \\
q_{w_y}(\rho,\{x,y\}) &= q_w(\rho,\{x,y\}) q_{w_y}(w,y).
\end{aligned}$$

Combining these gives us that

$$\begin{aligned}
q_w(\rho,\{x,y\}) &= \frac{q_w(\rho,\{x,y\})}{2\beta+1}\left(q_{w_\rho}(w,\rho) + \beta q_{w_x}(w,x) + \beta q_{w_y}(w,y)\right) + \frac{q_{w_\rho}(\rho,w)}{2\beta+1} \\
&= \frac{q_{w_\rho}(\rho,w)}{2\beta + q_{w_\rho}(\rho,w) - q_{w_x}(w,x) - q_{w_y}(w,y)} \\
&= \frac{\frac{\beta-1}{\beta^{|w|}-1}}{2\beta + \frac{\beta-1}{\beta^{|w|}-1} - \frac{\beta^{|x|-|w|}-\beta}{\beta^{|x|-|w|}-1} - \frac{\beta^{|y|-|w|}-\beta}{\beta^{|y|-|w|}-1}}
\end{aligned}$$

by Lemma 3.2. Rearranging gives the result. □

Let $\mathscr{T}$ be a fixed tree and $(X_n)_{n\geq 1}$ a β-biased walk on $\mathscr{T}$. For $x \in \mathscr{T}$ let

$$v_x := \sum_{k=1}^{\tau_\rho^+} \mathbf{1}_{\{X_k = x\}}$$

denote the number of visits to x before returning to the root. Then $\tau_\rho^+ = \sum_{x\in\mathscr{T}} v_x$ and

$$E_\rho^{\mathscr{T}}\left[(\tau_\rho^+)^2\right] = \sum_{x,y\in\mathscr{T}} E_\rho^{\mathscr{T}}[v_x v_y]. \tag{4.6}$$

For any $x, y \in \mathscr{T}$ there exists a unique vertex $w_{x,y}$ which is the closest ancestor of both x and y. We will often write w instead of $w_{x,y}$ when it is clear to which vertices we are referring. Moreover

$$E_\rho^{\mathscr{T}}[v_x v_y] = P_\rho^{\mathscr{T}}(\tau_{w_{x,y}}^+ < \tau_\rho^+) E_{w_{x,y}}^{\mathscr{T}}[v_x v_y]$$

where, by comparison with a simple biased random walk on $\mathbb{Z}$, we have that $P^{\mathscr{T}}(\tau^+_{w_{x,y}} < \tau^+_\rho) \in [1-\beta^{-1}, 1]$. We now prove a bound on $E^{\mathscr{T}}_w[v_x v_y]$ following a similar method to that used in [21] for the unbiased case. Recall that $c(x)$ is the set of children of x in $\mathscr{T}$.

Lemma 4.4. *For $\beta > 1$, there exists a constant C_β such that for any finite tree $\mathscr{T}$,*

$$E^{\mathscr{T}}_\rho[v_x v_y] \le E^{\mathscr{T}}_w[v_x v_y] \le C_\beta(|c(x)|\beta + 1)(|c(y)|\beta + 1)\beta^{|x|+|y|}.$$

Proof. When $w = \rho$ at least one of x and y is never reached therefore $v_x v_y = 0$ and we may assume $|w| \ge 1$. There are now three cases to consider; these are:

1. $x = y = w_{x,y}$;
2. $x = w_{x,y} \ne y$;
3. $x \ne w_{x,y} \ne y$.

In case 1 we have that v_x is geometrically distributed with termination probability $q_x(\rho, x)$ therefore

$$E^{\mathscr{T}}_{w_{x,y}}[v_x v_y] = E^{\mathscr{T}}_x[v_x^2] = \frac{q_x(x,\rho) + 1}{q_x(\rho, x)^2}.$$

For $x \notin c(\rho)$ we have that $\beta/(1+\beta) \le q_x(x, \rho) \le 1$ and by Lemma 3.2

$$q_x(\rho, x) = \frac{1-\beta^{-1}}{(|c(x)|\beta + 1)(\beta^{|x|-1} - \beta^{-1})}.$$

We therefore have that

$$E^{\mathscr{T}}_x[v_x^2] \le C_\beta(|c(x)|\beta + 1)^2\beta^{2|x|}.$$

In case 2, the number of visits to x from x is geometrically distributed as in case 1. For each visit to x (except the last) the walk reaches y before returning to x with probability $q_x(y, x)/q_x(x, \rho)$ since, due to the tree structure, the walk cannot move from ρ to y without hitting x. From y, the walk returns to y a geometric number of times before returning to x. More specifically,

$$E^{\mathscr{T}}_{w_{x,y}}[v_x v_y] = E^{\mathscr{T}}_x[v_x v_y] = \sum_{j=1}^\infty j q_x(\rho, x) q_x(x, \rho)^{j-1} E^{\mathscr{T}}_x[v_y | v_x = j]$$

where, conditional on the event $\{v_x = j\}$, we have that v_y is equal in distribution to the sum of $B^j_{x,y} \sim Bin(j-1, q_x(y,x)/q_x(x,\rho))$ independent geometric random variables $G^i_{x,y} \sim Geo(q_y(x, y))$. Under $P^{\mathscr{T}}$ the number of excursions are independent therefore

$$E^{\mathscr{T}}_x[v_y | v_x = j] = (j-1)\frac{q_x(y,x)}{q_x(x,\rho)} \cdot \frac{1}{q_y(x,y)}.$$

We therefore have that

$$\begin{aligned}
E^{\mathscr{T}}_{w_{x,y}}[v_x v_y] &= \frac{q_x(y,x)q_x(\rho,x)}{q_x(x,\rho)q_y(x,y)} \sum_{j=1}^\infty j(j-1) q_x(x,\rho)^{j-1} \\
&= \frac{q_x(y,x)q_x(\rho,x)}{q_x(x,\rho)q_y(x,y)} \cdot \frac{2q_x(x,\rho)}{q_x(\rho,x)^3} \\
&= \frac{2q_x(y,x)}{q_y(x,y)q_x(\rho,x)^2}.
\end{aligned} \tag{4.7}$$

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Using Lemma 3.2 we then have that

$$q_x(y,x) = \frac{\beta}{|c(x)|\beta+1}\cdot\frac{1-\beta^{-1}}{1-\beta^{|x|-|y|}},$$
$$q_y(x,y) = \frac{1}{|c(y)|\beta+1}\cdot\frac{\beta-1}{\beta^{|y|-|x|}-1},$$
$$q_x(\rho,x) = \frac{1}{|c(x)|\beta+1}\cdot\frac{\beta-1}{\beta^{|x|}-1}.$$

Combining these with (4.7) we have that

$$E^{\mathscr{T}}_{w_{x,y}}[v_x v_y] \leq C_\beta(|c(x)|\beta+1)(|c(y)|\beta+1)\beta^{|x|+|y|}.$$

In case 3, started from $w_{x,y}$, the walk reaches either x or y before returning to ρ with probability $q_{w_{x,y}}(\{x,y\},\rho)$. From x the walk has a geometric number of returns to x before returning to $w_{x,y}$. Moreover, from x, the walk must return to $w_{x,y}$ before reaching either ρ or y by definition of $w_{x,y}$. The same also holds switching x and y. Letting

$$\overline{q}_w(x,y) = P^{\mathscr{T}}_w\left(\tau^+_x < \tau^+_y | \tau^+_{\{x,y\}} < \tau^+_\rho\right) \qquad \text{and}$$
$$\overline{q}_w(y,x) = P^{\mathscr{T}}_w\left(\tau^+_y < \tau^+_x | \tau^+_{\{x,y\}} < \tau^+_\rho\right)$$

we then have that

$$E^{\mathscr{T}}_w[v_x v_y] = \sum_{j=0}^{\infty} q_w(\{x,y\},\rho)^j q_w(\rho,\{x,y\}) \times \sum_{k=0}^{j}\overline{q}_w(x,y)^k\overline{q}_w(y,x)^{j-k}\binom{j}{k}\frac{k(j-k)}{q_x(w,x)q_y(w,y)} \tag{4.8}$$

since $q_x(w,x)^{-1}$ is the expected number of visits to x (started from x) before returning to w (and similarly for y) which are independent. Rearranging gives

$$\begin{aligned}
&\sum_{k=0}^{j}\overline{q}_w(x,y)^k\overline{q}_w(y,x)^{j-k}\binom{j}{k}\frac{k(j-k)}{q_x(w,x)q_y(w,y)} \\
&\quad = \frac{1}{q_x(w,x)q_y(w,y)}\sum_{k=1}^{j-1}\overline{q}_w(x,y)^k\overline{q}_w(y,x)^{j-k}\frac{j!}{(k-1)!(j-k-1)!} \\
&\quad = j(j-1)\frac{\overline{q}_w(x,y)\overline{q}_w(y,x)}{q_x(w,x)q_y(w,y)}\sum_{l=0}^{j-2}\overline{q}_w(x,y)^l\overline{q}_w(y,x)^{j-2-l}\frac{(j-2)!}{l!(j-2-l)!} \\
&\quad = j(j-1)\frac{\overline{q}_w(x,y)\overline{q}_w(y,x)}{q_x(w,x)q_y(w,y)}.
\end{aligned}$$

Substituting back into (4.8) it follows that

$$\begin{aligned}
E^{\mathscr{T}}_w[v_x v_y] &= \frac{\overline{q}_w(x,y)\overline{q}_w(y,x)q_w(\rho,\{x,y\})}{q_x(w,x)q_y(w,y)}\sum_{j=0}^{\infty} j(j-1)q_w(\{x,y\},\rho)^j \\
&= \frac{\overline{q}_w(x,y)\overline{q}_w(y,x)q_w(\rho,\{x,y\})}{q_x(w,x)q_y(w,y)}\cdot\frac{2q_w(\{x,y\},\rho)^2}{q_w(\rho,\{x,y\})^3} \\
&= \frac{2q_w(\{x,y\},\rho)^2\overline{q}_w(x,y)\overline{q}_w(y,x)}{q_w(\rho,\{x,y\})^2 q_x(w,x)q_y(w,y)}.
\end{aligned}$$

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The terms in the numerator can all be bounded below by half of the escape probability $1-\beta^{-1}$ therefore we gain nothing using their exact expressions and bound them above by 1. Using Lemmas 3.2 and 4.3 for the other terms we have that

$$\begin{aligned}
q_w(\rho,\{x,y\}) &= \frac{(\beta^{|y|-|w|}-1)(\beta^{|x|-|w|}-1)}{2\beta^{|y|+|x|-|w|}-\beta^{|y|+|x|-2|w|}-\beta^{|x|}-\beta^{|y|}+1},\\
q_x(w,x) &= \frac{1}{|c(x)|\beta+1}\cdot\frac{\beta-1}{\beta^{|x|-|w|}-1},\\
q_y(w,y) &= \frac{1}{|c(y)|\beta+1}\cdot\frac{\beta-1}{\beta^{|y|-|w|}-1}.
\end{aligned}$$

Since $|y|\geq 1$ we have that $\beta^{|y|}\geq 1$ therefore

$$q_w(\rho,\{x,y\}) \geq \frac{(\beta^{|y|-|w|}-1)(\beta^{|x|-|w|}-1)}{2\beta^{|y|+|x|-|w|}}$$

and

$$E_w^{\mathscr{T}}[v_x v_y] \leq \frac{2}{q_w(\rho,\{x,y\})^2 q_x(w,x) q_y(w,y)} \leq C_\beta(|c(x)|\beta+1)(|c(y)|\beta+1)\beta^{|x|+|y|}. \qquad \square$$

Proof of Theorem 5. By Theorem 2 it suffices to show that $\mathbf{E}\left[E^\omega\left[\eta_0^2\right]\right]<\infty$.

Recall from (4.3) that N is the number of return times to the root ρ before reaching its unique ancestor $\overline{\rho}$. This is geometrically distributed with termination probability $1-p_{ex}$; that is,

$$P^{\overline{\mathcal{T}}}(N=k) = p_{ex}^k(1-p_{ex}) \quad \text{where} \quad p_{ex} = \frac{\beta(\xi^*-1)}{\beta\xi^*+1} \tag{4.9}$$

and ξ^*+1 is the number of neighbours attached to ρ in $\overline{\mathcal{T}}$. By (4.4) and convexity we have that

$$E^\omega\left[\eta_0^2\right] = E^{\overline{\mathcal{T}}}\left[\left(1+\sum_{k=1}^N \zeta_k\right)^2\right] \leq E^{\overline{\mathcal{T}}}\left[(N+1)\left(1+\sum_{k=1}^N \zeta_k^2\right)\right]$$

where ζ_k is the duration of the k^{th} excursion.

Noting that $N\geq 1$ with positive probability and

$$\mathbf{E}\left[E^{\overline{\mathcal{T}}}[N+1]\right] = 1+\frac{\beta(\mathbf{E}[\xi^*]-1)}{\beta+1} < \infty$$

since $\mathbf{E}[\xi^3]<\infty$, it suffices to show that

$$\mathbf{E}\left[E^{\overline{\mathcal{T}}}\left[N\sum_{k=1}^N \zeta_k^2\right]\right] < \infty.$$

Recall that $\overline{\mathcal{T}}^\circ$ denotes the tree $\overline{\mathcal{T}}$ without the ancestor of the root $\overline{\rho}$. Since the separate excursions are independent under $P^{\overline{\mathcal{T}}}$ and N is geometrically distributed we have that

$$\mathbf{E}\left[E^{\overline{\mathcal{T}}}\left[N\sum_{k=1}^N \zeta_k^2\right]\right] \leq C_\beta \mathbf{E}\left[\left(\xi^*\right)^2 E_\rho^{\overline{\mathcal{T}}^\circ}\left[\zeta_1^2\right]\right].$$

Labelling $\rho_1,\dots,\rho_{\xi^*-1}$ as the neighbours of ρ in $\overline{\mathcal{T}}^\circ$, and $\overline{\mathcal{T}}^\circ_{\rho_k}$ as the tree consisting of ρ, ρ_k and the descendants of ρ_k we have that

$$E_\rho^{\overline{\mathcal{T}}^\circ}\left[\zeta_1^2\right] = \sum_{k=1}^{\xi^*-1} \frac{E_\rho^{\overline{\mathcal{T}}^\circ_{\rho_k}}\left[(\tau_\rho^+)^2\right]}{\xi^*-1}$$

when $\xi^* \neq 1$ and 0 otherwise. Moreover, it then follows that

$$\mathbf{E}\left[E^{\overline{\mathcal{T}}}\left[N\sum_{k=1}^{N}\zeta_k^2\right]\right] \leq C_\beta \mathbf{E}\left[\xi^*\sum_{k=1}^{\xi^*-1} E^{\overline{\mathcal{T}}^\circ_{\rho_k}}\left[(\tau_\rho^+)^2\right]\right] \leq C_\beta \mathbf{E}[(\xi^*)^2]\mathbf{E}\left[E^{\overline{\mathcal{T}}^\circ_{\rho_1}}\left[(\tau_\rho^+)^2\right]\right]$$

since the subtraps are independent. We have that $\mathbf{E}[(\xi^*)^2] \leq C\mathbf{E}[\xi^3] < \infty$ thus it suffices to show that

$$\mathbf{E}\left[E_\rho^{\widetilde{\mathcal{T}}}\left[(\tau_\rho^+)^2\right]\right] < \infty$$

where $\widetilde{\mathcal{T}}$ is a tree (equal in distribution to $\overline{\mathcal{T}}^\circ_{\rho_1}$) with root ρ, single first generation vertex $\overrightarrow{\rho}$ and, under $\mathbf{P}$, the subtree rooted at $\overrightarrow{\rho}$ is a subcritical GW-tree with the original offspring distribution.

By (4.6) and Lemma 4.4 we have that

$$\mathbf{E}\left[E_\rho^{\widetilde{\mathcal{T}}}\left[(\tau_\rho^+)^2\right]\right] = \mathbf{E}\left[\sum_{x,y\in\widetilde{\mathcal{T}}} E_\rho^{\widetilde{\mathcal{T}}}[v_x v_y]\right]$$
$$\leq C_\beta \mathbf{E}\left[\left(\sum_{x\in\widetilde{\mathcal{T}}}(|c(x)|\beta+1)\,\beta^{|x|}\right)\left(\sum_{y\in\widetilde{\mathcal{T}}}(|c(y)|\beta+1)\,\beta^{|y|}\right)\right].$$

By collecting terms in the k^{th} generation we have that

$$\sum_{x\in\widetilde{\mathcal{T}}}(|c(x)|\beta+1)\,\beta^{|x|} \;=\; 1+\sum_{k\geq 1} Z_k^{\widetilde{\mathcal{T}}}(\beta^k+\beta^{k-1}) \;\leq\; (1-\beta^{-1})\sum_{k\geq 0} Z_k^{\widetilde{\mathcal{T}}}\beta^k$$

where $Z_k^{\widetilde{\mathcal{T}}}$ is the size of the k^{th} generation of $\widetilde{\mathcal{T}}$. For $k \geq 0$ the tree $\widetilde{\mathcal{T}}$ satisfies $Z_{k+1}^{\widetilde{\mathcal{T}}} = Z_k$ for a GW-process Z_k with $Z_0 = 1$. Therefore, using that $\beta^2\mu < 1$ and Lemma 4.1, we have that $\mathbf{E}\left[Z_k^{\widetilde{\mathcal{T}}} Z_j^{\widetilde{\mathcal{T}}}\right] \leq C\mu^j$, for $j \geq k$. In particular,

$$\mathbf{E}\left[E_\rho^{\widetilde{\mathcal{T}}}\left[(\tau_\rho^+)^2\right]\right] \leq C_\beta \sum_{k\geq 0}\beta^k \sum_{j\geq k}\beta^j \mathbf{E}\left[Z_k^{\widetilde{\mathcal{T}}} Z_j^{\widetilde{\mathcal{T}}}\right] \leq C_\beta \sum_{k\geq 0}\beta^k\sum_{j\geq k}(\mu\beta)^j$$
$$\leq C_{\beta,\mu}\sum_{k\geq 0}(\mu\beta^2)^k < \infty. \qquad \square$$

Recall that the expression (2.8) for ς^2 was given in Theorem 2 in terms of the moments of the distance and time between regenerations. We can therefore use this to write the corresponding form in the GW-tree model as

$$\varsigma^2 = \frac{\mathbb{E}\left[\left(Y_{\kappa_2} - Y_{\kappa_1} - \nu_\beta \sum_{j=\kappa_1}^{\kappa_2-1}\eta_j\right)^2\right]}{\mathbb{E}[\eta_0]\mathbb{E}[\kappa_2-\kappa_1]}$$

where κ_j are the regeneration times of the walk Y.

We now show that both of the conditions $\beta^2\mu < 1$ and $\mathbf{E}[\xi^3] < \infty$ are necessary in order to apply Theorem 2. This suggests that these conditions are required to obtain an annealed functional central limit theorem for the walk on the subcritical GW-tree conditioned to survive.

Lemma 4.5. *If* $\beta^2\mu \geq 1$ *or* $\mathbf{E}[\xi^3] = \infty$ *then*

$$\mathbb{E}[\eta_0^2] \geq \mathbf{E}\left[E^{\overline{\mathcal{T}}}[\eta_0]^2\right] = \infty.$$

Proof. Recall that η_0 is the first hitting time of $\overline{\rho}$ by W_n started from the root ρ in $\overline{\mathcal{T}}$. With positive probability ρ has neighbours other than $\overline{\rho}$ and the walk moves to one on the first step. Until returning to ρ the walk is equal in distribution to a β-biased random walk on an f-GW-tree conditioned to have a single first generation vertex. In particular, it suffices to show that for a β-biased walk

$$\mathbf{E}\left[E_\rho^{\mathcal{T}^f}[\tau_\rho^+]^2 | Z_1 = 1\right] = \infty$$

where $\mathcal{T}^f$ is an f-GW-tree rooted at ρ. Using the formula for the expected time spent in a tree (4.2) we have that

$$\mathbf{E}\left[E_\rho^{\mathcal{T}^f}[\tau_\rho^+]^2 | Z_1 = 1\right] = \frac{4}{\beta^2}\mathbf{E}\left[\left(\sum_{k\geq 1}\beta^k Z_k\right)^2 | Z_1 = 1\right] \geq \frac{4}{\beta^2}\sum_{k\geq 1}\beta^{2k}\mathbf{E}[Z_k^2 | Z_1 = 1].$$

Since Z_k takes nonnegative values in $\mathbb{Z}$ we have that

$$\mathbf{E}\left[Z_k^2 | Z_1 = 1\right] \geq \mathbf{E}\left[Z_k | Z_1 = 1\right] = \mu^{k-1}$$

by statement 1 of Lemma 4.1. We therefore have that

$$\mathbf{E}\left[E_\rho^{\mathcal{T}^f}[\tau_\rho^+]^2 | Z_1 = 1\right] \geq c\sum_{k\geq 1}(\beta^2\mu)^k$$

which is infinite if $\beta^2\mu \geq 1$.

The first hitting time of $\overline{\rho}$ is at least the number of visits to the offspring of ρ. From ρ, the walk takes a geometric number of visits (with termination probability $1 - p_{ex}$, see (4.9)) to these vertices before reaching $\overline{\rho}$. Using properties of geometric random variables we then have that

$$\mathbf{E}\left[E_\rho^{\overline{\mathcal{T}}}[\eta_0]^2\right] \geq \mathbf{E}\left[\left(\frac{(\xi^* - 1)\beta}{\beta + 1}\right)^2\right] \geq c\left(\mathbf{E}[(\xi^*)^2] - 1\right) = c\left(\mu^{-1}\mathbf{E}[\xi^3] - 1\right). \quad \square$$

4.3. *A quenched central limit theorem*

We now prove a quenched central limit theorem for the biased walk on the subcritical GW-tree conditioned to survive. As in the annealed case, it will suffice to show the result holds for the corresponding randomly trapped random walk and we obtain the result by using Theorem 3. Define

$$\mathcal{G}^{\mathcal{T}}(t) = \nu_\beta t - \nu_\beta \sum_{k=1}^{\lfloor \nu_\beta t\rfloor} \frac{\beta + 1}{\beta - 1}\left(E^{\mathcal{T}}[\eta_{\rho_k,0}] - \mathbb{E}[\eta_0]\right). \tag{4.10}$$

Notice that, under the assumptions of Theorem 6, $\nu_\beta t - \mathcal{G}^{\mathcal{T}}(t)$ is a sum of i.i.d. centred random variables with positive, finite variance. It therefore follows that this expression (scaled by $\sqrt{t}$) converges in distribution with respect to $\mathbf{P}$ to a Gaussian random variable. In particular, this means that the environment dependent centring is necessary.

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Proof of Theorem 6. By Theorem 5 we have that $\mathbb{E}[\eta_0^2] < \infty$ therefore by Theorem 3 it suffices to show that for some $\varepsilon \in (0, 1)$

$$\mathbf{E}\left[E^{\overline{\mathcal{T}}}[\eta_0]^{2+\varepsilon}\right] < \infty.$$

Recall that

$$N = \sum_{n=1}^{\tau^+_{\overleftarrow{\rho}}} \mathbf{1}_{\{W_n=\rho\}}$$

is the number of hitting times of the root ρ before reaching $\overleftarrow{\rho}$ (for the walk started at ρ) and $\overline{\mathcal{T}}^\circ$ is the tree $\overline{\mathcal{T}}$ with $\overleftarrow{\rho}$ removed. Then

$$\begin{aligned} E^{\overline{\mathcal{T}}}_\rho[\eta_0] = 1 + E^{\overline{\mathcal{T}}}[N]E^{\overline{\mathcal{T}}^\circ}_\rho\left[\tau^+_\rho\right] &= 1 + 2E^{\overline{\mathcal{T}}}[N]\sum_{n\geq 1}\frac{Z_n}{Z_1}\beta^{n-1} \\ &\leq 2(E^{\overline{\mathcal{T}}}[N]+1)\sum_{n\geq 1}\frac{Z_n}{Z_1}\beta^{n-1} \end{aligned}$$

by (4.2) where Z_n is the n^{th} generation size of $\overline{\mathcal{T}}^\circ$ since the walk on $\overline{\mathcal{T}}^\circ$ is β-biased.

For a fixed tree, N is geometrically distributed with excursion probability p_{ex} (see (4.9)) therefore $E^{\overline{\mathcal{T}}}[N] \leq Z_1$. By conditioning on Z_1 we therefore have that

$$\begin{aligned} \mathbf{E}\left[E^{\overline{\mathcal{T}}}[\eta_0]^{2+\varepsilon}\right] &\leq C\mathbf{E}\left[Z_1^{2+\varepsilon}\mathbf{E}\left[\left(\sum_{n\geq 1}\frac{Z_n}{Z_1}\beta^{n-1}\right)^{2+\varepsilon}\middle| Z_1\right]\right] \\ &= C\mathbf{E}\left[\left(\sum_{n\geq 1}Z_n\beta^{n-1}\right)^{2+\varepsilon}\right]. \end{aligned}$$

We can write

$$Z_n = \sum_{j=1}^{Z_1} Z^{(j)}_{n-1}$$

where $Z^{(j)}$ are independent GW-processes. Therefore, by convexity,

$$\begin{aligned} \mathbf{E}\left[\left(\sum_{n\geq 1}Z_n\beta^{n-1}\right)^{2+\varepsilon}\right] &= \mathbf{E}\left[Z_1^{2+\varepsilon}\left(\sum_{j=1}^{Z_1}\sum_{n\geq 1}\frac{Z^{(j)}_{n-1}}{Z_1}\beta^{n-1}\right)^{2+\varepsilon}\right] \\ &\leq \mathbf{E}\left[Z_1^{1+\varepsilon}\sum_{j=1}^{Z_1}\left(\sum_{n\geq 1}Z^{(j)}_{n-1}\beta^{n-1}\right)^{2+\varepsilon}\right] \\ &= \mathbf{E}[(\xi^*-1)^{2+\varepsilon}]\mathbf{E}\left[\left(\sum_{n\geq 1}Z^{(1)}_{n-1}\beta^{n-1}\right)^{2+\varepsilon}\right]. \end{aligned}$$

By the assumptions of the theorem we have that $\mathbf{E}[(\xi^*-1)^{2+\varepsilon}] \leq \mu^{-1}\mathbf{E}[\xi^{3+\varepsilon}] < \infty$ whenever $\varepsilon < \delta$ hence it suffices to show that

$$\mathbf{E}\left[\left(\sum_{n\geq 0} Z_n\beta^n\right)^{2+\varepsilon}\right] < \infty$$

where Z_n now denotes the n^{th} generation size of an f-GW-process.

For $\varepsilon < \delta$, by conditioning on the height $\mathcal{H} := \max\{n \geq 0 : Z_n > 0\}$ of the tree we have that

$$\begin{aligned}
\mathbf{E}\left[\left(\sum_{n\geq 0} Z_n\beta^n\right)^{2+\varepsilon}\right] &\leq \mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}\mathbf{E}\left[\left(\sum_{n\geq 0} Z_n\right)^{2+\varepsilon}\Big|\mathcal{H}\right]\right] \\
&\leq \mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}(\mathcal{H}+1)^{2+\varepsilon}\mathbf{E}\left[\max_{n\leq\mathcal{H}} Z_n^{2+\varepsilon}\Big|\mathcal{H}\right]\right] \\
&\leq \mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}(\mathcal{H}+1)^{2+\varepsilon}\sum_{n=0}^{\mathcal{H}}\mathbf{E}\left[Z_n^{2+\varepsilon}\Big|\mathcal{H}\right]\right] \\
&= \sum_{n=0}^{\infty}\mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}(\mathcal{H}+1)^{2+\varepsilon}Z_n^{2+\varepsilon}\right] \\
&= \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} j^{2+\varepsilon}\mathbf{P}(Z_n=j)\mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}(\mathcal{H}+1)^{2+\varepsilon}|Z_n=j\right]. \qquad (4.11)
\end{aligned}$$

From (4.1) we have that $\mathbf{P}(\mathcal{H} \geq n) \sim c\mu^n$ therefore $\mathbf{P}(\mathcal{H} \geq n) \leq C\mu^n$ for some constant C hence

$$\begin{aligned}
\mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}(\mathcal{H}+1)^{2+\varepsilon}|Z_n=j\right] &= \mathbf{E}\left[\beta^{(2+\varepsilon)(\mathcal{H}+n+1)}(\mathcal{H}+n)^{2+\varepsilon}|Z_0=j\right] \\
&= \sum_{i=1}^{\infty}\beta^{(2+\varepsilon)(i+n)}(i+n+1)^{2+\varepsilon}\mathbf{P}(\mathcal{H}=i|Z_0=j) \\
&\leq \sum_{i=1}^{\infty}\beta^{(2+\varepsilon)(i+n)}(i+n+1)^{2+\varepsilon}\mathbf{P}(\mathcal{H}\geq i|Z_0=j) \\
&\leq C\sum_{i=1}^{\infty}\beta^{(2+\varepsilon)(i+n)}(i+n+1)^{2+\varepsilon}j\mu^i \\
&\leq Cj\beta^{(2+\varepsilon)n}(n+2)^{2+\varepsilon}\sum_{i=1}^{\infty}i^{2+\varepsilon}(\beta^{2+\varepsilon}\mu)^i.
\end{aligned}$$

Since $\beta^2\mu < 1$ we can choose $\varepsilon > 0$ sufficiently small so that $\beta^{2+\varepsilon}\mu < 1$ therefore

$$\sum_{i=1}^{\infty}i^{2+\varepsilon}(\beta^{2+\varepsilon}\mu)^i < \infty.$$

Substituting back into (4.11) we have that

$$\begin{aligned}
\mathbf{E}\left[\left(\sum_{n\geq 0} Z_n\beta^n\right)^{2+\varepsilon}\right] &\leq C\sum_{n=0}^{\infty}\beta^{n(2+\varepsilon)}(n+2)^{2+\varepsilon}\sum_{j=1}^{\infty}j^{3+\varepsilon}\mathbf{P}(Z_n=j) \\
&= C\sum_{n=0}^{\infty}\beta^{n(2+\varepsilon)}(n+2)^{2+\varepsilon}\mathbf{E}[Z_n^{3+\varepsilon}]. \qquad (4.12)
\end{aligned}$$

Using a telescoping sum we can write

$$Z_n = \mu^n + \sum_{k=0}^{n-1}(Z_{n-k} - \mu Z_{n-(k+1)})\mu^k.$$

Using convexity we then have that

$$\begin{aligned}\mathbf{E}[Z_n^{3+\varepsilon}] &= (n+1)^{3+\varepsilon}\mathbf{E}\left[\left(\frac{\mu^n}{n+1} + \sum_{k=0}^{n-1}\frac{(Z_{n-k} - \mu Z_{n-(k+1)})\mu^k}{n+1}\right)^{3+\varepsilon}\right] \\ &\leq (n+1)^{3+\varepsilon}\mathbf{E}\left[\frac{\mu^{n(3+\varepsilon)}}{n+1} + \sum_{k=0}^{n-1}\frac{\left((Z_{n-k} - \mu Z_{n-(k+1)})\mu^k\right)^{3+\varepsilon}}{n+1}\right] \\ &= (n+1)^{2+\varepsilon}\mu^{n(3+\varepsilon)} + (n+1)^{2+\varepsilon}\sum_{k=0}^{n-1}\mu^{k(3+\varepsilon)}\mathbf{E}\left[\left(Z_{n-k} - \mu Z_{n-(k+1)}\right)^{3+\varepsilon}\right].\end{aligned} \tag{4.13}$$

Let ξ_j be independent copies of ξ then using the Marcinkiewicz–Zygmund inequality and convexity we have that

$$\begin{aligned}\mathbf{E}\left[\left(Z_{n-k} - \mu Z_{n-(k+1)}\right)^{3+\varepsilon}\right] &= \mathbf{E}\left[\mathbf{E}\left[\left(\sum_{j=1}^{Z_{n-(k+1)}}(\xi_j - \mu)\right)^{3+\varepsilon}\middle| Z_{n-(k+1)}\right]\right] \\ &\leq C\mathbf{E}\left[\mathbf{E}\left[\left(\sum_{j=1}^{Z_{n-(k+1)}}(\xi_j - \mu)^2\right)^{\frac{3+\varepsilon}{2}}\middle| Z_{n-(k+1)}\right]\right] \\ &= C\mathbf{E}\left[\mathbf{E}\left[\left(\sum_{j=1}^{Z_{n-(k+1)}}\frac{(\xi_j - \mu)^2}{Z_{n-(k+1)}}\right)^{\frac{3+\varepsilon}{2}}\middle| Z_{n-(k+1)}\right]Z_{n-(k+1)}^{\frac{3+\varepsilon}{2}}\right] \\ &\leq C\mathbf{E}\left[\mathbf{E}\left[\sum_{j=1}^{Z_{n-(k+1)}}\frac{|\xi_j - \mu|^{3+\varepsilon}}{Z_{n-(k+1)}}\middle| Z_{n-(k+1)}\right]Z_{n-(k+1)}^{\frac{3+\varepsilon}{2}}\right] \\ &= C\mathbf{E}\left[|\xi - \mu|^{3+\varepsilon}\right]\mathbf{E}\left[Z_{n-(k+1)}^{\frac{3+\varepsilon}{2}}\right] \\ &\leq C\mathbf{E}\left[|\xi - \mu|^{3+\varepsilon}\right]\mathbf{E}\left[Z_{n-(k+1)}^2\right].\end{aligned}$$

By Lemma 4.1 we have that $\mathbf{E}\left[Z_{n-(k+1)}^2\right] \leq C\mu^{n-(k+1)}$ where C is independent of n, k therefore substituting into (4.13) we have that

$$\mathbf{E}[Z_n^{3+\varepsilon}] \leq (n+1)^{2+\varepsilon}\mu^{n(3+\varepsilon)} + C(n+1)^{2+\varepsilon}\mu^n\sum_{k=0}^{n-1}\mu^{k(2+\varepsilon)} \leq C(n+1)^{2+\varepsilon}\mu^n.$$

Combining with (4.12) we then have that

$$\mathbf{E}\left[\left(\sum_{n\geq 0}Z_n\beta^n\right)^{2+\varepsilon}\right] \leq C\sum_{n=1}^{\infty}(n+2)^{4+2\varepsilon}(\beta^{2+\varepsilon}\mu)^n$$

which is finite since we have chosen $\varepsilon > 0$ sufficiently small so that $\beta^{2+\varepsilon}\mu < 1$. □

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