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Graphs without large bicliques and well-quasi-orderability by the induced subgraph relation*

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Abstract

Recently, Daligault, Rao and Thomassé asked in [3] if every hereditary class which is well-quasi-ordered by the induced subgraph relation is of bounded clique-width. While the question has been shown to have a negative answer in general [9], in the present paper we show that the statement is true for a family of hereditary classes of graphs that exclude large bicliques as subgraphs. In particular, this implies (through the use of Courcelle theorem [2]) that any problem definable in Monadic Second Order Logic can be solved in a polynomial time for all well-quasi-ordered hereditary classes of graphs that exclude large bicliques.

MSC codes: 05C75 Structural characterization of families of graphs; 05C85 Graph algorithms.

1 Introduction

Well-quasi-ordering is a highly desirable property and a frequently discovered concept in mathematics and theoretical computer science [6, 8]. One of the most remarkable recent results in this area is the proof of Wagner’s conjecture stating that the set of all finite graphs is well-quasi-ordered by the minor relation [12]. However, the subgraph or induced subgraph relation is not a well-quasi-order. On the other hand, each of these relations may become a well-quasi-order when restricted to graphs with some special properties.

A *graph property* (or a *class of graphs*) is a set of graphs closed under isomorphism. A property is *hereditary* if it is closed under taking induced subgraphs. It is well-known (and not difficult to see) that a graph property X is hereditary if and only if X can be described in terms of forbidden induced subgraphs. More formally, X is hereditary if and only if there is a set M of graphs such that no graph in X contains any graph from M as an induced subgraph. We call M the set of *forbidden induced subgraphs* for X and say that the graphs in X are M -free.

Of our particular interest in this paper are graphs *without large bicliques*. We say that the graphs in a hereditary class X are *without large bicliques* if there is a natural number t such that no graph in X contains $K_{t,t}$ as a (not necessarily induced) subgraph. Equivalently, there are q and r such $K_{q,q}$ and K_r appear in the set of forbidden induced subgraphs for X . According to [11], these are precisely the graphs with a subquadratic number of edges. This

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family of properties includes many important classes, such as graphs of bounded vertex degree, of bounded tree-width, all proper minor closed graph classes. In all these examples, the number of edges is bounded by a linear function in the number of vertices and all of the listed properties are rather small (see e.g. [10] for the number of graphs in proper minor closed graph classes). In the terminology of [1], they all are at most factorial. In fact, the family of classes without large bicliques is much richer and contains classes with a superfactorial speed of growth, such as projective plane graphs (or more generally C_4 -free bipartite graphs), in which case the number of edges is $\Theta(n^{\frac{3}{2}})$.

Recently, Daligault, Rao and Thomassé asked in [3] if every hereditary class which is well-quasi-ordered by the induced subgraph relation is of bounded clique-width. While the question has been shown to have a negative answer in general [9], the relationship holds true for some families of hereditary graph classes. Investigating such families is interesting because it connects two seemingly unrelated notions and leads to a strong algorithmic consequence. Indeed, it follows (through the use of Courcelle theorem [2]) that for such families any problem definable in Monadic Second Order Logic can be solved in a polynomial time on any class well-quasi-ordered by the induced subgraph relation.

In the present paper, we establish the relationship between well-quasi-ordering and boundedness of clique-width for graphs without large bicliques. More precisely, we prove that if a class X without large bicliques is well-quasi-ordered by the induced subgraph relation, then the graphs in X have bounded path-width, i.e. there is a constant c such that the path-width of any graph in X is at most c . Since bounded path-width implies bounded clique-width, the result affirmatively answers the question in [3] for graphs without large bicliques. Thus the above algorithmic consequence is confirmed e.g. for classes of graphs of bounded degree.

Section 2 contains all preliminary information related to the topic. In this section we define an infinite family of graphs pairwise incomparable by the induced subgraph relation, which we call *canonical graphs*. In Section 3 we prove our main combinatorial result, Theorem 1, stating that a graph without large bicliques and having a large path-width has a large induced canonical graph. A consequence of this result is that if a class X without large bicliques has unbounded path-width, then X contains an infinite subset of canonical graphs, i.e. an infinite antichain. This implies that classes of graphs without large bicliques that are well quasi-ordered by the induced subgraph relation must have bounded path-width.

2 Notation and definitions

In this work we will be using standard graph theory terminology and notation consistent with the book of Diestel [4]. In particular, K_n and P_n denote the complete graph and the chordless path with n vertices, respectively, and $K_{n,m}$ stands for a complete bipartite graph with parts of size n and m .

Throughout the text, whenever we say that G contains H , we mean that H is a subgraph of G , unless we explicitly say that H is an *induced* subgraph of G (or G contains H as an *induced* subgraph). If H is not an induced subgraph of G , we say that G is H -free. By $R = R(k, r, m)$, we denote the Ramsey number, i.e. the minimum R such that in every colouring of k -subsets of an R -set with r colours there is a monochromatic m -set, i.e. a set of m elements all of whose k -subsets have the same colour.

According to the celebrated Graph Minor Theorem of Robertson and Seymour, the set of all graphs is well-quasi-ordered by the graph minor relation [12]. This, however, is not the case

for the more restrictive relations such as subgraph or induced subgraph. Indeed, a sequence of graphs H_1, H_2, \dots , creates an infinite antichain with respect to both relations, where H_i is the graph represented in Figure 1.

By connecting two vertices of degree one having a common neighbour in H_i , we obtain a graph represented on the left of Figure 2. Let us denote this graph by H'_i . By further connecting the other pair of vertices of degree one we obtain the graph H''_i represented on the right of Figure 2.

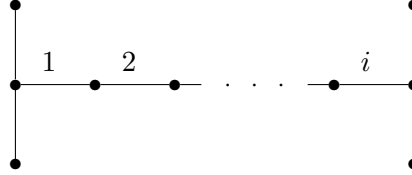


Figure 1: The graph H_i

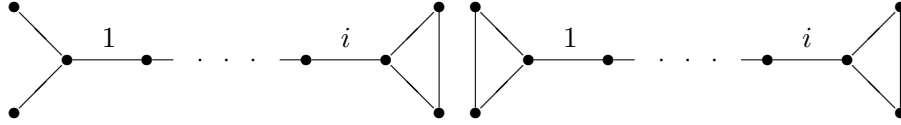


Figure 2: Graphs H'_i and H''_i

We call any graph of the form H_i , H'_i or H''_i an H -graph. Furthermore, we will refer to H''_i a *tight* H -graph and to H'_i a *semi-tight* H -graph. In an H -graph, the path connecting two vertices of degree 3 will be called the *body* of the graph, and the vertices which are not in the body the *wings*.

Following standard graph theory terminology, we call a chordless cycle of length at least four a *hole*. Let us denote by

\mathcal{C} the set of all holes and all H -graphs.

It is not difficult to see that any two distinct (i.e. non-isomorphic) graphs in \mathcal{C} are incomparable with respect to the induced subgraph relation. In other words,

Claim 1. \mathcal{C} is an antichain with respect to the induced subgraph relation.

Moreover, from the poof of Theorem 1 we will see that for classes of graphs without large bicliques which are of unbounded path-width this antichain is unavoidable, or *canonical*, in the terminology of [5]. Suggested by this observation, we introduce the following definition.

Definition 1. The graphs in the set \mathcal{C} will be called CANONICAL.

The *order* of a canonical graph G is either the number of its vertices, if G is a hole, or the number of vertices in its body, if G is an H -graph.

3 Main result

In this section we prove the following theorem which is the main result of the paper.

Theorem 1. *If X is a hereditary subclass of $(K_t, K_{q,q})$ -free graphs which is well-quasi-ordered by the induced subgraph relation, then graphs in X have a bounded path-width.*

To prove the theorem, we will show that a large path-width combined with the absence of large bicliques implies the existence of a large induced canonical graph, which is a much richer structural consequence than just the existence of a long induced path. An important part of showing the existence of a large canonical graph is verifying that its body (see Section 2 for the terminology) is induced. This will be done by application of the following theorem proved in [7].

Theorem 2. *For every s, t , and q , there is a number $Z = Z(s, t, q)$ such that every graph with a path of length at least Z contains either P_s or K_t or $K_{q,q}$ as an induced subgraph.*

A plan of the proof of Theorem 1 is outlined in Section 3.1. Sections 3.2, 3.3, 3.4, 3.5 contain various parts of the proof.

3.1 Plan of the proof

To prove Theorem 1 we will show that graphs of arbitrarily large path-width contain either arbitrarily large bicliques as subgraphs or arbitrarily large canonical graphs as induced subgraphs. The main notion in our proof is that of a *rake-graph*.

A *rake-graph* (or simply a *rake*) consists of a chordless path, the *base* of the rake, and a number of pendant vertices, called *teeth*, each having a private neighbour on the base. The only neighbour of a tooth on the base will be called the *root* of the tooth, and a rake with k teeth will be called a k -rake. We will say that a rake is ℓ -dense if any ℓ consecutive vertices of the base contain at least one root vertex. An example of a 1-dense 9-rake is given in Figure 3.

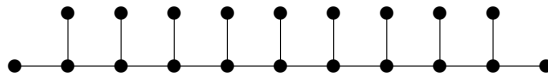


Figure 3: 1-dense 9-rake

We will prove Theorem 1 through a number of intermediate steps as follows.

1. In Section 3.2, we observe that any graph of large path-width contains a rake with many teeth as a subgraph.
2. In Section 3.3 we show that any graph containing a rake with many teeth as a subgraph contains either
 - a *dense* rake with many teeth as a subgraph or
 - a large canonical graph as an *induced* subgraph.
3. In Section 3.4 we prove that dense rake subgraphs necessarily imply either

- a large canonical graph as an *induced* subgraph or
- a large biclique as a subgraph.

4. In Section 3.5, we use the results of sections 3.2-3.4 to deduce Theorem 1.

3.2 Rake subgraphs in graphs of large path-width

Lemma 1. *For any natural k , there is a number $f(k)$ such that every graph of path-width at least $f(k)$ contains a k -rake as a subgraph.*

Proof. In [13], Robertson and Seymour has shown that for any tree T there is a constant c_T such that any graph of path-width is at least c_T contains T as a minor. Taking T to be some fixed k -rake, we obtain that there exist a constant $f(k)$ such that any graph of path-width at most $f(k)$ contains a k -rake as a minor. Finally, it is not hard to see that if a graph contains a k -rake as a minor, then it also contains a k -rake as a subgraph. This observation completes the proof. \square

3.3 From rake subgraphs to dense rake subgraphs

Lemma 2. *Let k and s be natural numbers. Every graph containing a $k + 2$ -rake as a subgraph contains either*

- *an $s + 5$ -dense k -rake as a subgraph or*
- *a canonical graph of order at least s as an induced subgraph.*

Proof. Consider a graph that contains a $k + 2$ -rake as a subgraph and choose such a $k + 2$ -rake with the minimal number of vertices. We denote the base of the rake by P . Let $\{u_1, u_2, \dots, u_{k+2}\}$ denote the roots of the rake that are indexed respecting the linear order of the path P , i.e. so that u_1 and u_{k+2} are the endpoints of P and the subpaths of P from u_i to u_{i+1} , which we denote by P_i , are all mutually disjoint apart from the endpoints. Note that by minimality of the rake it follows that each endpoint of the path P is indeed a root vertex of the rake and that each P_i is an induced path. If each P_i for $i = 2, 3, \dots, k$ has at most $s + 5$ vertices, then we have an $s + 5$ -dense k -rake as required. So assume now that P_i for some $i = 2, 3, \dots, k$ has size more than $s + 5$. To complete the proof we will show that this P_i gives rise to a canonical graph of order at least s as an induced subgraph. We proceed with some notation.

Let $P_i = w_1 w_2 \dots w_r$ with $w_1 = u_i$ and $w_r = u_{i+1}$. Extend P_i by adding the vertex w_0 of P_{i-1} that is adjacent to w_1 and the vertex w_{r+1} of P_{i+1} that is adjacent to w_r (unique choice as P_{i-1} and P_{i+1} are induced paths). Note that $w_0 w_1 w_2 \dots, w_r w_{r+1}$ is a subpath of P , the tooth v_i is adjacent to w_1 and the tooth v_{i+1} is adjacent to w_r . Let G be a graph induced by vertices $\{w_0, w_1, \dots, w_{r+1}\} \cup \{v_i, v_{i+1}\}$ and note that G contains an H -graph formed by edges $\{w_0 w_1, w_1 w_2, \dots, w_r w_{r+1}\} \cup \{v_i w_1, v_{i+1} w_r\}$ as a subgraph but not necessarily as an induced subgraph. Note that the body of the H -graph, spanned by vertices $\{w_1, w_2, \dots, w_r\}$, is a chordless path P_i . For the rest of the proof we will be arguing on the adjacencies of the wings of the H -graph in G , i.e. adjacencies of vertices w_0, w_r, v_i and v_{i+1} in G . It will follow G contains a canonical subgraph of order at least s as an induced subgraph.

We first claim that w_0 is not adjacent to w_l for any $l = 2, 3, \dots, r - 1$. Indeed, suppose for contradiction that w_0 is adjacent to some w_l for $l = 2, 3, \dots, r - 1$. Let a path P' be obtained from path P by replacing subpath $w_0 w_1 \dots w_r$ of P by path $w_0 w_l w_{l+1} \dots w_r$. The path P' has

smaller number of vertices than path P , and note that the missing root vertex w_1 can be replaced by w_l with the new tooth being w_{l-1} . This gives us a $k+2$ -rake that has smaller number of vertices than the original, which contradicts our minimality assumption.

Next, we show that v_i is not adjacent to w_4, w_5, \dots, w_r . Again, suppose for contradiction that v_i is adjacent to w_l for some $l = 4, 5, \dots, r$. Let the path P' be obtained from path P by replacing the subpath $w_1 w_2 \dots w_r$ of P by path $w_1 v_i w_l w_{l+1} \dots w_r$. Again, the path P' has fewer vertices than path P , all the root vertices of P remain in path P' , but as v_i is now in the path P' , we assign a new tooth w_2 to correspond to the root w_1 . Again, we obtain a $k+2$ -rake that has smaller number of vertices than the original, a contradiction.

By symmetry, we can show that w_{r+1} is not adjacent to w_l for any $l = 2, 3, \dots, r-1$ and v_{i+1} is not adjacent to any of w_1, w_2, \dots, w_{r-3} . We conclude that none of the wings of the H -graph are adjacent to any of w_4, w_5, \dots, w_{r-3} . In other words, vertices w_4, w_5, \dots, w_{r-3} are of degree 2 in G . If $w_4 w_5$ is a cut-edge of G , we have that no vertex of $\{w_0, w_1, w_2, w_3, v_i\}$ is adjacent to any of the vertex of $\{w_{r-2}, w_{r-1}, w_r, w_{r+1}, v_{i+1}\}$. Let $l \leq 3$ be the largest possible such that w_l has degree at least 3 in G , $p \geq r-2$ the smallest possible such that w_p has degree at least 3 in G . Taking the path $w_l w_{l+1} \dots w_p$ together with another two neighbours of w_l and w_p provides us with an induced H -graph whose base $w_l w_{l+1} \dots w_p$ has at least $s+1$ vertices. On the other hand, if $w_4 w_5$ is not a cut-edge in G , then there is a chordless cycle in G containing the edge $w_4 w_5$ and hence this cycle must contain $w_3 w_4 w_5 \dots w_{r-2}$ (because of vertices of degree 2). Therefore, we obtain an induced cycle of G with at least $r-4 \geq s+1$ vertices. Hence in both cases we obtain a canonical graph of order at least s as an induced subgraph. This finishes the proof. \square

3.4 Dense rake subgraphs

Lemma 3. *For every s, q and ℓ , there is a number $D = D(s, q, \ell)$ such that every graph containing an ℓ -dense D -rake as a subgraph contains either*

- *a canonical graph of order at least s as an induced subgraph or*
- *a biclique of order q as a subgraph.*

Proof. To define the number $D = D(s, q, \ell)$, we introduce intermediate notations as follows: $b := 2(q-1)s^q + 2sq + 4$ and $c := R(2, 2, \max(b, 2q))$, where R is the Ramsey number. With these notations the number D is defined as follows: $D = D(s, q, \ell) := Z(\ell c^2, 2q, q)$, where Z is the number defined in Theorem 2.

Consider a graph G containing an ℓ -dense D -rake R^0 as a subgraph. The base of this rake is a path P^0 of length at least D and hence, by Theorem 2, the subgraph of G induced by the base contains either a biclique of order at least q as a subgraph (in which case we are done) or an induced path P of length at least ℓc^2 . Let us call any (inclusionwise) maximal sequence of consecutive vertices of P^0 that belong to P a *block*. Assume the number of blocks is more than c . Let P' be the subpath of P induced by the first c blocks. Let w_1, \dots, w_c be the rightmost vertices of the blocks. Let v_1, \dots, v_c be the vertices such that each v_i is the vertex of P_0 immediately following w_i . Then P' together with v_1, \dots, v_c create a c -rake with P' being the induced base, v_1, \dots, v_c being the teeth and w_1, \dots, w_c being the respective roots. If the number of blocks is at most c , then P^0 must contain a block of size at least ℓc , in which case this block also forms an induced base of a c -rake (since R^0 is ℓ -dense). We see that in either case G has a c -rake with

an induced base. According to the definition of c , the c teeth of this rake induce a graph which has either a clique of size $2q$ (and hence a biclique of order q in which case we are done), or an independent set of size b . By ignoring the teeth outside this set we obtain a b -rake R with an induced base and with teeth forming an independent set.

Let us denote the base of R by U , its vertices by u_1, \dots, u_m (in the order of their appearances in the path), and the teeth of R by t_1, \dots, t_b (following the order of their root vertices).

Denote $r := (q - 1)s^q + 2$ and consider two sets of teeth $T_1 = \{t_2, t_3, \dots, t_r\}$ and $T_2 = \{t_{b-1}, t_{b-2}, \dots, t_{b-r+1}\}$. By definition of r and b , there are $2sq$ other teeth between t_r and t_{b-r+1} , and hence there is a set M of $2sq$ consecutive vertices of U between the root of t_r and the root of t_{b-r+1} . We partition M into $2q$ subsets (of consecutive vertices of U) of size s each and for $i = 1, \dots, 2q$ denote the i -th subset by M_i .

If each vertex of T_1 has a neighbour in each of the first q sets M_i , then by the Pigeonhole Principle there is a biclique of order q with q vertices in T_1 and q vertices in M . Similarly, a biclique of order q arises if each vertex of T_2 has a neighbour in each of the last q sets M_i . Therefore, we assume that there are two vertices $t_a \in T_1$ and $t_b \in T_2$ and two sets M_x and M_y with $x < y$ such that t_a has no neighbours in M_x , while t_b has no neighbours in M_y .

By definition, t_a has a neighbour in U (its root) on the left of M_x . If additionally t_a has a neighbour to the right of M_x , then a chordless cycle of length at least s arises (since $|M_x| = s$ and t_a has no neighbours in M_x), in which case the lemma is true. This restricts us to the case, when all neighbours of t_a in U are located to the left of M_x . By analogy, we assume that all neighbours of t_b in U are located to the right of M_y . Let u_i be the rightmost neighbour of t_a in U and u_j be the leftmost neighbour of t_b in U . According to the above discussion, $i < j$ and $j - i > 2s$. But then the vertices $t_a, t_b, u_{i-1}, u_i, \dots, u_j, u_{j+1}$ induce an H -graph (possibly tight or semi-tight) of order more than s (the existence of vertices u_{i-1} and u_{j+1} follows from the fact that T_1 does not include t_1 , while T_2 does not include t_b). \square

3.5 Proof of Theorem 1

Combining the results of Lemma 1, Lemma 2 and Lemma 3, we conclude that for every s, q , there is a number $X = X(s, q)$ such that every graph of path-width at least X contains either

- a canonical graph of order at least s as an induced subgraph or
- a biclique of order q as a subgraph.

From this it is not hard to conclude that a class of graphs with unbounded path-width that excludes a biclique of order q must contain an infinite family of distinct canonical graphs, hence the class must be not well-quasi-ordered. Therefore, well-quasi-ordered classes that exclude a biclique of order q for some q , must be of bounded path-width, as required.

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