SIAM J. APPL. MATH.
Vol. 78, No. 6, pp. 3060–3086

LATTICE DYNAMICS ON LARGE TIME SCALES AND DISPERSIVE EFFECTIVE EQUATIONS*

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Abstract. We investigate the long time behavior of waves in crystals. Starting from a linear wave equation on a discrete lattice with periodicity $\varepsilon > 0$, we derive the continuum limit equation for time scales of order ε^{-2}. The effective equation is a weakly dispersive wave equation of fourth order. Initial values with bounded support result in ring-like solutions, and we characterize the dispersive long time behavior of the radial profiles with a linearized KdV equation of third order.

Key words. lattice dynamics, continuum limit, dispersive effective equation

AMS subject classifications. 37K60, 35Q

DOI. 10.1137/17M1162184

1. Introduction.

Our aim is to describe the oscillations of atoms in a crystal lattice. In the simplest setting of the problem, the continuum limit provides a linear wave equation of second order. Due to its simple structure, this limit model fails to describe certain phenomena that can be observed in crystals: nonlinear behavior and dispersion. We concentrate here on the latter and justify a limit model that captures dispersive effects.

The discrete model is constructed from a regular rectangular lattice as follows. The lattice points are $\gamma \in \varepsilon\mathbb{Z}^d$, where $\varepsilon > 0$ is the periodicity and $d \geq 1$ the space dimension. Considering only displacements in one fixed direction, the unknown in the point γ at time $t \in [0, \infty)$ is $u^\varepsilon(\gamma, t) \in \mathbb{R}$. The evolution equation relates the acceleration $\partial_t^2 u^\varepsilon(\gamma, t)$ to the displacements $u^\varepsilon(\gamma', t)$ in neighboring points γ'. Restricting ourselves to a linear model, we consider

$$\partial_t^2 u^\varepsilon(\gamma, t) = \frac{1}{\varepsilon^2} \sum_{j \in \mathbb{Z}^d} a_j u^\varepsilon(\gamma + \varepsilon j, t) \tag{1.1}$$

for certain prescribed interaction coefficients $(a_j)_{j\in\mathbb{Z}^d}$, $a_j \in \mathbb{R}$ with $\sum_j a_j = 0$. If we prescribe low-frequency initial data $u_0^\varepsilon, u_1^\varepsilon : \mathbb{R}^d \to \mathbb{R}$ (e.g., by choosing the initial data independent of ε), then the continuum limit of (1.1) is the linear second order wave equation

$$\partial_t^2 u = AD_x^2 u\,. \tag{1.2}$$

Here, the elliptic operator $AD_x^2 = \sum_{i,j=1}^d A_{i,j}\partial_{x_i}\partial_{x_j}$ is given by an x-independent effective tensor $A \in \mathbb{R}^{d\times d}$ which can be calculated from the interaction coefficients $(a_j)_{j\in\mathbb{Z}^d}$. Equation (1.2) is a valid approximation for time intervals $[0, T]$; since the time interval is fixed, the equation fails to capture dispersive effects. Our interest is to derive a dispersive continuum limit equation that is valid on time intervals $[0, T\varepsilon^{-2}]$.

*Received by the editors December 20, 2017; accepted for publication (in revised form) September 17, 2018; published electronically November 15, 2018.
http://www.siam.org/journals/siap/78-6/M116218.html

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One-dimensional example. We have chosen here a natural scaling of the equation. The simplest example is obtained by considering the one-dimensional case $d = 1$ and the nearest-neighbor interaction with $a_1 = a_{-1} = 1$, $a_0 = -2$, and $a_j = 0$ for every j with $|j| > 1$. In this case, the right-hand side of (1.1) is the discrete Laplacian of $u^\varepsilon(., t)$ in the point γ. The homogenized equation is (1.2) with $A = 1$.

We consider two alternative approaches to characterize the evolution over time scales of order ε^{-2}:

(A) Derivation of continuum equations with ε-dependent coefficients. This is done is section 3. The advantage is that the effective equations have constant coefficients. The disadvantage is that the equations have to be solved on large time intervals.

(B) Effective equations with ε-independent coefficients (profile equations). Approximations to the solutions of the original lattice equation are obtained via ε-dependent transforms. This program is carried out in sections 4 and 5.

(A) Dispersive continuum limit. Our result is that the following weakly dispersive equation describes the long time behavior of the discrete system:

$$\partial_t^2 w^\varepsilon = AD_x^2 w^\varepsilon + \varepsilon^2 E D_x^2 \partial_t^2 w^\varepsilon - \varepsilon^2 F D_x^4 w^\varepsilon \,. \tag{1.3}$$

The limit system is continuous with x-independent coefficients; we may therefore call it a continuum limit of the discrete system. On the other hand, we note that the lattice constant ε is not set to zero, but it appears in the equations. The result of Theorem 3.1 is the following: Let u^ε be a solution of the discrete system (1.1) on a time interval $[0, T/\varepsilon^2]$, and let $w^\varepsilon : \mathbb{R}^d \times [0, T/\varepsilon^2] \to \mathbb{R}$ be a solution to (1.3) on the same time interval. Then the two solutions differ in the energy norm only in the order of ε. The estimate is relevant since the deviation of both functions from the solution of (1.2) is of order 1.

Our results can be compared to known results for the continuous problem. In the continuous case one starts from the wave equation with the coefficient $a(x/\varepsilon)$ in the elliptic operator. As for discrete systems, dispersive effects are observed [7, 8]. These can be understood with the help of the dispersion relation that is obtained with the help of Bloch expansions [16]. Also in the continuous setting, one can derive the dispersive limit equation (1.3) for large time scales [4, 5, 12].

In (1.3), the effective tensors $E \in \mathbb{R}^{d\times d}$ and $F \in \mathbb{R}^{d\times d\times d\times d}$ are symmetric and positive semidefinite, and we use the notation $FD_x^4 = \sum_{i,j,k,l=1}^d F_{i,j,k,l} \partial_{x_i} \partial_{x_j} \partial_{x_k} \partial_{x_l}$. The x-independent coefficient tensors are obtained from a Taylor expansion of the dispersion relation. In the one-dimensional case of [12], A, E, and F are positive real numbers. The quality of the effective equation (1.3) is also studied numerically; see [1, 2, 4, 5] for numerical studies and generalizations.

In the present work, we obtain a similar approximation result for the discrete wave equation (1.1). The approach is similar in the sense that we start from a representation formula, which is analyzed in the limit $\varepsilon \to 0$. In the discrete setting, the representation with Fourier transforms is possible, and we hence do not have to use Bloch transforms. This simplifies the proof of the approximation result considerably.

In classical approaches, the dynamics of (1.1) are analyzed in terms of the dispersion relation; see, e.g., [6, 14, 15]. In fact, our proofs are also based on an approximation of the dispersion relations of (1.1) and (1.3) to all relevant orders.

(B) Profile equations for ring-like solutions. In the second part of this work, starting with section 4, we characterize the long time behavior of solutions

with a family of one-dimensional profile equations. If initial data are given by a fixed $L^2(\mathbb{R}^d)$ function, then the solution at time $t = \tau/\varepsilon^2$ is large only in a neighborhood of a sphere with radius $r = ct = c\tau/\varepsilon^2$, where $c > 0$ is the effective wave velocity. We describe the profile of the wave in this neighborhood in dimension $d = 1$ and in dimension $d = 2$. Our results imply, in particular, that the profiles satisfy the linearized KdV equations

$$\partial_\tau V^\varepsilon(z,\tau;q) = b(q)\,\partial_z^3 V^\varepsilon(z,\tau;q)\,. \tag{1.4}$$

In this equation, $z \in \mathbb{R}$ is the radial parameter (the radial distance to the sphere $|x| = ct$), $\tau = \varepsilon^2 t$ is the rescaled time, and $q \in S^{d-1}$ is the direction of propagation. The effective coefficient $b(q) \in \mathbb{R}$ is obtained from the Taylor expansion of the dispersion relation of (1.1). The initial data for (1.4) are extracted from the initial data u_0 and u_1; the construction is dimension dependent.

Our results are quite strong as they provide approximate solutions v^ε. The functions v^ε are easy to calculate, and they approximate the lattice solution u^ε to (1.1). The construction of v^ε starts from the initial data u_0^ε and defines initial data V_0^ε for (1.4) from u_0^ε; the solutions of (1.4) are used as profiles to construct v^ε as a ring-type solution. Our results in Theorems 4.1 and 5.3 establish that v^ε is an approximation u^ε on time intervals $(0, T/\varepsilon^2)$.

The fact that a linearized KdV equation should describe the profile of a ring-like solution is well known. In the one-dimensional case, it is possible to study a solution to (1.3) in a frame of coordinates that moves with the effective speed $\sqrt{A}$. Formal calculations show that the shifted solution solves, approximately, the linearized KdV equation (1.4). The observation was made rigorous in [13, Theorem 2.7], where the error was shown to vanish in the limit $\varepsilon \to 0$. In space dimension $d = 2$, a result was established in Proposition 3.1 of [4]: If the wave profile of a solution w^ε to (1.3) converges in the sense of distributions, then the limiting profile satisfies a linearized KdV equation. We remark that, in [4], the height of the wave profile was not scaled with the factor $|ct|^{-1/2}$ that we use in (4.3); without this factor, one obtains the "cylindrical" KdV equation. Our result has the advantage that the initial data can be extracted explicitly from the initial data of the lattice equation.

Let us compare our result also with [9], where the dynamics of the nonlinear FPU-lattice are studied. For the nonlinear dynamics, a nonlinear KdV equation is obtained in [9], but we note that only solutions to profile equations can be compared in this case. Since we deal only with the linear case, our result can be more general: For fixed initial data in space dimension $d = 1$ or $d = 2$, the solution is a ring wave with a profile described by (1.4).

The works of [17] and [11] treat nonlinear continuous equations in the one-dimensional case. In [17] it is shown that a pair of uncoupled KdV equations is a rigorous approximation of long wavelength motions. In [11], the approximation of a general solution by two solitary waves is justified for large times.

2. Preliminaries.

We study (1.1) with initial conditions

$$u^\varepsilon(\gamma,0) = u_0^\varepsilon(\gamma)\,,\ \partial_t u^\varepsilon(\gamma,0) = u_1^\varepsilon(\gamma) \quad \forall\,\gamma \in \varepsilon\mathbb{Z}^d\,, \tag{2.1}$$

where $u_0^\varepsilon, u_1^\varepsilon : \mathbb{R}^d \to \mathbb{R}$ are given functions on all of $\mathbb{R}^d$.

Fourier transformation. We define the Fourier transform for functions on the lattice by setting, for $k \in \mathbb{R}^d$,

$$\hat{u}^\varepsilon(k,t) := \begin{cases} \varepsilon^d \sum_{\gamma \in \varepsilon\mathbb{Z}^d} e^{-ik\cdot\gamma} u^\varepsilon(\gamma,t) & \text{for } k \in \varepsilon^{-1}(-\pi,\pi)^d\,, \\ 0 & \text{else.} \end{cases} \tag{2.2}$$

The Fourier transform $\hat{u}^\varepsilon$ of the solution u^ε satisfies the following for every $k \in \mathbb{R}^d$:

$$\begin{aligned} \partial_t^2 \hat{u}^\varepsilon(k,t) &= \varepsilon^d \sum_{\gamma\in\varepsilon\mathbb{Z}^d} e^{-ik\cdot\gamma} \frac{1}{\varepsilon^2} \sum_{j\in\mathbb{Z}^d} a_j u^\varepsilon(\gamma+\varepsilon j,t) \\ &= \varepsilon^d \sum_{j\in\mathbb{Z}^d} \frac{1}{\varepsilon^2} a_j\, e^{ik\cdot\varepsilon j} \sum_{\gamma\in\varepsilon\mathbb{Z}^d} e^{-ik\cdot\varepsilon j - ik\cdot\gamma} u^\varepsilon(\gamma+\varepsilon j,t) \\ &= \frac{1}{\varepsilon^2}\left[\sum_{j\in\mathbb{Z}^d} a_j e^{ik\cdot\varepsilon j}\right] \varepsilon^d \sum_{\gamma'\in\varepsilon\mathbb{Z}^d} e^{-ik\cdot\gamma'} u^\varepsilon(\gamma',t)\,. \end{aligned}$$

We obtain, introducing the lattice dispersion relation $\omega_l : \mathbb{R}^d \to \mathbb{C}$,

$$\omega_l(\tilde{k})^2 := -\sum_{j\in\mathbb{Z}^d} a_j e^{i\tilde{k}\cdot j}\,, \tag{2.3}$$

the evolution equation for the Fourier transform in the simple form

$$\partial_t^2 \hat{u}^\varepsilon(k,t) = -\frac{\omega_l(\varepsilon k)^2}{\varepsilon^2}\, \hat{u}^\varepsilon(k,t)\,. \tag{2.4}$$

We illustrate the scaling with a calculation for the above one-dimensional example. We find $\omega_l(\varepsilon k)^2 = -\sum_{j\in\mathbb{Z}} a_j e^{ik\,\varepsilon j} = -(e^{ik\,\varepsilon} - 2 + e^{-ik\,\varepsilon}) = \varepsilon^2 k^2 - \frac{1}{12}\varepsilon^4 k^4 \pm \cdots$. To leading order, the dispersion relation is that of an isotropic wave equation, $\omega_0(\varepsilon k)^2 = \varepsilon^2 k^2$.

Let us collect the assumptions on the data of the discrete equation.

Assumption 2.1 (assumptions on coefficients and initial values). The coefficients $a : \mathbb{Z}^d \to \mathbb{R}$ have the following properties:

(a1) Neutrality: $\sum_j a_j = 0$.
(a2) Finite range: $a_j \neq 0$ only for finitely many $j \in \mathbb{Z}^d$.
(a3) Symmetry: $a_j = a_{-j}$.
(a4) Ellipticity: $\sum_{j\in\mathbb{Z}^d} (\xi\cdot j)^2 a_j > 0$ for $0 \neq \xi \in \mathbb{R}^d$.

In sections 4 and 5 we will additionally assume

(a5) Invariance: a is invariant under rotations and reflections of $\mathbb{Z}^d$.

The initial data $u_0^\varepsilon, u_1^\varepsilon : \mathbb{R}^d \to \mathbb{R}$ satisfy the following:

(u1) The Fourier transforms of initial values are supported in a compact set, independent of $\varepsilon > 0$: There exists a compact set $S_\psi \subset \mathbb{R}^d$ such that $k \in \mathbb{R}^d \setminus S_\psi$ implies $\hat{u}_0^\varepsilon(k) = 0$ and $\hat{u}_1^\varepsilon(k) = 0$.
(u2) The sequences u_0^ε and u_1^ε are bounded in $L^2(\mathbb{R}^d)$.

Remark 2.2. The assumptions on $(a_j)_j$ imply the following properties of the lat-

tice dispersion relation $\mathbb{R}^d \ni \tilde{k} \mapsto \omega_l^2(\tilde{k})$ from (2.3):

$\omega_l^2(0) = 0$ by (a1),

$\omega_l^2(.)$ is of class $C^6(\mathbb{R}^d)$ by (a2),

$\omega_l^2(.)$ is real and even, $D\omega_l^2(0) = 0$, all odd derivatives vanish in $\tilde{k} = 0$ by (a3),

$D^2\omega_l^2(0) : (\xi, \zeta) \mapsto \langle \xi, D^2\omega_l^2(0)\zeta \rangle = \sum_{j \in \mathbb{Z}^d} (\xi \cdot j)(\zeta \cdot j)\, a_j$

is symmetric and positive definite by (a4).

The invariance assumption (a5) implies that $A := \frac{1}{2} D^2\omega_l^2(0)$ is invariant under reflections and coordinate permutations, which implies that A is a scalar (a positive multiple of the identity matrix).

The assumptions on $(a_j)_j$ can be weakened: It is sufficient to require summability of $(a_j)_j$ and the above properties of the function $\tilde{k} \mapsto \omega_l^2(\tilde{k})$.

The assumption (u1) can be removed; see Corollary 3.2.

Diagonalization and explicit solution. The evolution law (2.4) can be diagonalized by introducing the functions

$$\hat{\psi}_\pm^{u,\varepsilon}(k,t) := \frac{1}{\sqrt{2}} \left(\frac{\omega_l(\varepsilon k)}{\varepsilon} \hat{u}^\varepsilon(k,t) \pm i \partial_t \hat{u}^\varepsilon(k,t) \right). \tag{2.5}$$

For the time derivative of these functions we find that by (2.4)

$$\begin{aligned} \partial_t \hat{\psi}_\pm^{u,\varepsilon}(k,t) &= \frac{1}{\sqrt{2}} \left(\frac{\omega_l(\varepsilon k)}{\varepsilon} \partial_t \hat{u}^\varepsilon(k,t) \mp i \frac{\omega_l(\varepsilon k)^2}{\varepsilon^2} \hat{u}^\varepsilon(k,t) \right) \\ &= -\frac{i\omega_l(\varepsilon k)}{\varepsilon} \frac{1}{\sqrt{2}} \left(i \partial_t \hat{u}^\varepsilon(k,t) \pm \frac{\omega_l(\varepsilon k)}{\varepsilon} \hat{u}^\varepsilon(k,t) \right) \\ &= \mp \frac{i\omega_l(\varepsilon k)}{\varepsilon} \hat{\psi}_\pm^{u,\varepsilon}(k,t)\,. \end{aligned}$$

This provides the explicit solution formula

$$\hat{\psi}_\pm^{u,\varepsilon}(k,t) = e^{\mp i[\omega_l(\varepsilon k)/\varepsilon]t}\, \hat{\psi}_\pm^{u,\varepsilon}(k,0)\,. \tag{2.6}$$

The initial data are given by $\hat{\psi}_\pm^{u,\varepsilon}(k,0) = \hat{\psi}_{\pm,0}^{u,\varepsilon}(k)$. These two functions are related to the initial data $\hat{u}_0^\varepsilon$ and $\hat{u}_1^\varepsilon$ via (2.5). Formula (2.5) expresses $\hat{\psi}_\pm^{u,\varepsilon}$ in terms of $\hat{u}^\varepsilon$ and $\partial_t \hat{u}^\varepsilon$, but we can also reconstruct with

$$\hat{u}^\varepsilon(k,t) = \frac{\varepsilon}{\omega_l(\varepsilon k)\sqrt{2}} \left(\hat{\psi}_+^{u,\varepsilon}(k,t) + \hat{\psi}_-^{u,\varepsilon}(k,t) \right), \tag{2.7}$$

and with a similar expression for $\partial_t \hat{u}^\varepsilon(k,t)$. The reconstruction formulas allow one also to write $\hat{u}_0^\varepsilon$ and $\hat{u}_1^\varepsilon$ in terms of $\hat{\psi}_{\pm,0}^{u,\varepsilon}(k)$.

What makes the transformation formulas slightly nontrivial is the fact that they involve explicit dependences on ε. For example, if we choose initial data $\hat{\psi}_{\pm,0}^{u,\varepsilon}$ without ε-dependence, then the initial data for u^ε are ε-dependent. Nevertheless, for $\hat{\psi}_{\pm,0}^{u,\varepsilon}$ with compact support, also the transformed quantities $\hat{u}_0^\varepsilon$ and $\hat{u}_1^\varepsilon$ are compactly supported.

The result of this work is a characterization of the dispersive long time behavior of u^ε with partial differential equations. We achieve this goal with the help of the Fourier representation (2.6) of solutions.

With the above Fourier transform and diagonalization procedure, we followed [10]; see in particular their evolution equation (2.11). The only difference is in the choice of the grid spacing: We have chosen a grid spacing ε in order that we do not have to rescale solutions. In particular, the initial data can be prescribed by ε-independent functions, and the dispersive limit equation turns out to be as in [4], where the oscillations of the medium are also on the spatial scale ε.

Expansion of the dispersion relation. We have to define the effective tensors. We start from a Taylor expansion of the (lattice) dispersion relation (2.3) in $k = 0$:

$$\omega_l^2(k) = A\,k \otimes k + C\,k \otimes k \otimes k \otimes k + O(|k|^6). \tag{2.8}$$

We emphasize that the second order tensor $A \in \mathbb{R}^{d\times d}$ and the fourth order tensor $C \in \mathbb{R}^{d\times d\times d\times d}$ are *defined* by the Taylor expansion (2.8). We observed in Remark 2.2 that our assumptions on the coefficients $(a_j)_j$ imply that the Taylor expansion can be performed to the required order, odd derivatives of ω_l^2 vanish in $k = 0$, and A is symmetric and positive definite.

With the help of A we define a (homogenized) dispersion relation through

$$\omega_0^2(k) := A\,k \otimes k\,. \tag{2.9}$$

We also need an approximation for $\omega_l(k) = \sqrt{\omega_l^2(k)}$. We write the function as $\omega_l(k) = (Ak \otimes k)^{1/2}\left(1 + (Ak \otimes k)^{-1} Ck \otimes k \otimes k \otimes k + O(|k|^4)\right)^{1/2}$ and use the approximation $(1+\eta)^{1/2} = 1 + \eta/2 + O(\eta^2)$ for $\eta \to 0$ to find

$$\omega_l(k) = \omega_0(k) + b(k) + O(|k|^5) \qquad \text{for} \quad |k| \ll 1\,, \tag{2.10}$$

where $b(k) = \frac{1}{2\,\omega_0(k)} Ck \otimes k \otimes k \otimes k$ is a 3-homogeneous function: $b(\mu k) = |\mu|^3 b(k)$ for every $\mu \in \mathbb{R}$ and $k \in \mathbb{R}^d$. Additionally, b is even: $b(k) = b(-k)$.

Remark 2.3. In general, the function b is not a polynomial in k. For example, in the simple case of a two-dimensional square lattice with nearest-neighbor interactions, we find $b(k) = -(k_1^4 + k_2^4)/(24\sqrt{k_1^2 + k_2^2})$. The example also shows that the function b is, in general, not isotropic.

3. Comparison with a weakly dispersive equation. In this section, we compare the solution u^ε of the lattice wave equation (1.1) with the solution w^ε of the weakly dispersive wave equation (1.3). On a fixed time interval $[0, T]$ the two solutions both coincide to leading order in ε with the solution of the homogenized wave equation (1.2). Here, we study time intervals $t \in [0, T/\varepsilon^2]$ on which (1.2) is not a valid approximation to leading order. Our result is that, on the other hand, the two solutions u^ε and w^ε coincide to leading order.

Considering expansion (2.8), it is tempting to compare the solution u^ε with the solution u to the equation $\partial_t^2 u = AD_x^2 u - \varepsilon^2 CD_x^4 u$. This latter equation has been suggested in another context already by Boussinesq and is named after him. In the one-dimensional case discussed above there holds $\omega_l(\tilde{k})^2 = \tilde{k}^2 - \frac{1}{12}\tilde{k}^4$, and hence $A = 1$ and $C = -\frac{1}{12}$; the Boussinesq equation reads $\partial_t^2 u = \partial_x^2 u + \frac{1}{12}\varepsilon^2 \partial_x^4 u$. This equation is called a "bad Boussinesq equation" since it cannot been solved easily: ∂_x^2 is a negative operator and ∂_x^4 is a positive operator. One way to proceed is to replace in the equation $\partial_x^4 u$ by $\partial_t^2 \partial_x^2 u$, which is correct to highest order [3].

We follow this idea of replacing the highest order term (at least in part) by mixed derivatives. We rely on Lemma 2.5 of [5], which provides the following: Given a

symmetric and positive definite $A \in \mathbb{R}^{d\times d}$ and a fourth order tensor $C \in \mathbb{R}^{d\times d\times d\times d}$, there exist symmetric and positive semidefinite tensors $E \in \mathbb{R}^{d\times d}$ and $F \in \mathbb{R}^{d\times d\times d\times d}$ such that

$$C\tilde{k}\otimes\tilde{k}\otimes\tilde{k}\otimes\tilde{k} = -E\tilde{k}\otimes\tilde{k}\; A\tilde{k}\otimes\tilde{k} + F\tilde{k}\otimes\tilde{k}\otimes\tilde{k}\otimes\tilde{k} \qquad \forall\, \tilde{k}\in\mathbb{R}^d\,. \tag{3.1}$$

We choose E and F as described. We conclude from (2.8) and (3.1) the approximation property

$$\omega_l^2(\tilde{k}) = A\tilde{k}\otimes\tilde{k} - E\tilde{k}\otimes\tilde{k}\;\omega_l^2(\tilde{k}) + F\tilde{k}\otimes\tilde{k}\otimes\tilde{k}\otimes\tilde{k} + O(|\tilde{k}|^6) \tag{3.2}$$

as $\tilde{k}\to 0$. Replacing once more Fourier symbols by derivatives, we arrive formally at the weakly dispersive wave equation (1.3).

Comparison of solutions. Let us now change the perspective: We consider the evolution equation (1.3) as given and want to analyze its solutions w^ε. Since solutions can be expanded in Fourier space, they can be characterized by the dispersion relation ω_d of (1.3) (the subscript d recalls that the equation is *dispersive*). We define ω_d implicitly through

$$\omega_d^2(\tilde{k}) = A\tilde{k}\otimes\tilde{k} - E\tilde{k}\otimes\tilde{k}\;\omega_d^2(\tilde{k}) + F\tilde{k}\otimes\tilde{k}\otimes\tilde{k}\otimes\tilde{k}\,, \tag{3.3}$$

which is solved by

$$\omega_d^2(\tilde{k}) = \frac{A\tilde{k}\otimes\tilde{k} + F\tilde{k}\otimes\tilde{k}\otimes\tilde{k}\otimes\tilde{k}}{1 + E\tilde{k}\otimes\tilde{k}}\,. \tag{3.4}$$

By property (3.1) of E and F, the function ω_d^2 has the same Taylor expansion to fourth order in 0 as ω_l^2, i.e., ω_d^2 satisfies (2.8). In particular, the difference satisfies $\omega_d^2(\tilde{k}) - \omega_l^2(\tilde{k}) = O(|\tilde{k}|^6)$ and the difference of the square roots satisfies

$$|\omega_d(\tilde{k}) - \omega_l(\tilde{k})| = O(|\tilde{k}|^4) \qquad \text{as } \tilde{k}\to 0\,. \tag{3.5}$$

We used here that A is positive definite.

We can now derive a representation formula for the solution w^ε. After a continuous Fourier transform in x, denoting the dual variable by k, the evolution equation (1.3) reads (D_x is replaced by ik)

$$\partial_t^2\hat{w}^\varepsilon(k,t) = -Ak\otimes k\,\hat{w}^\varepsilon(k,t) - \varepsilon^2 Ek\otimes k\,\partial_t^2\hat{w}^\varepsilon(k,t) - \varepsilon^2 Fk\otimes k\otimes k\otimes k\,\hat{w}^\varepsilon(k,t)\,. \tag{3.6}$$

We can solve (3.6) for $\partial_t^2\hat{w}^\varepsilon(k,t)$ and recover the expression of (3.4):

$$\partial_t^2\hat{w}^\varepsilon(k,t) = -\frac{\omega_d^2(\varepsilon k)}{\varepsilon^2}\hat{w}^\varepsilon(k,t)\,, \tag{3.7}$$

since

$$\frac{\omega_d^2(\varepsilon k)}{\varepsilon^2} = \frac{Ak\otimes k + \varepsilon^2 Fk\otimes k\otimes k\otimes k}{1+\varepsilon^2 Ek\otimes k}\,. \tag{3.8}$$

The evolution equation (3.7) is solved for every k on the time interval $t\in[0,T/\varepsilon^2]$. After a transformation as for u^ε, we define the transformed quantity $\hat{\psi}_\pm^{w,\varepsilon}$ as in (2.5):

$$\hat{\psi}_\pm^{w,\varepsilon}(k,t) := \frac{1}{\sqrt{2}}\left(\frac{1}{\varepsilon}\omega_d(\varepsilon k)\hat{w}^\varepsilon(k,t) \pm i\partial_t\hat{w}^\varepsilon(k,t)\right)\,. \tag{3.9}$$

The new unknowns have the simple explicit solution formula

$$\hat{\psi}_\pm^{w,\varepsilon}(k,t) = e^{\mp i[\omega_d(\varepsilon k)/\varepsilon]t}\hat{\psi}_{\pm,0}^{w,\varepsilon}(k)\,. \tag{3.10}$$

Initial values. The solution to the ε-problem (1.1) is determined by the initial values u_0^ε and u_1^ε. After a Fourier transform, (2.5) provides the two initial data $\hat\psi_{\pm,0}^{u,\varepsilon}(k)$ that are used in the solution formula (2.6).

The initial data w_0^ε and w_1^ε of the weakly dispersive equation (1.3) are treated accordingly: The continuous Fourier transform and (3.9) define the initial data $\psi_{\pm,0}^{w,\varepsilon}(k)$. We note that we can also reconstruct w from $\psi_\pm^{w,\varepsilon}$ with (2.7), except that we have to use the appropriate dispersion relation: $\hat w_0^\varepsilon(k) = \frac{\varepsilon}{\omega_d(\varepsilon k)\sqrt{2}}(\hat\psi_{+,0}^{u,\varepsilon}(k) + \hat\psi_{-,0}^{u,\varepsilon}(k))$. Similarly, we can reconstruct $\hat w_1^\varepsilon$.

THEOREM 3.1 (comparison of solutions). *Along a sequence $\varepsilon \searrow 0$ we consider the solutions $u^\varepsilon(.,t)$ to the lattice equation* (1.1). *We assume that the coefficients $(a_j)_j$ and the initial values $(u_0^\varepsilon, u_1^\varepsilon)$ satisfy Assumption* 2.1, (a1)–(a4) *and* (u1)–(u2). *Furthermore, let $(w_0^\varepsilon, w_1^\varepsilon)$ be a sequence of initial data such that, for $C_0 > 0$,*

$$\|\hat w_0^\varepsilon - \hat u_0^\varepsilon\|_{L^2(\mathbb{R}^d)} + \|\hat w_1^\varepsilon - \hat u_1^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C_0\varepsilon\,. \tag{3.11}$$

Let ω_l be as in (2.3), *the tensors A and C as in* (2.8), *and E and F as in* (3.1). *We recall that ω_l and ω_d from* (3.3) *satisfy* (3.5). *Let $w^\varepsilon(.,t)$ be the solution to the weakly dispersive equation* (1.3).

In this setting, the two functions u^ε and w^ε are comparable: The transformed solutions $\hat\psi_\pm^{w,\varepsilon}(.,t)$ of (3.9) *and $\hat\psi_\pm^{u,\varepsilon}(.,t)$ of* (2.5) *satisfy on large time intervals, for both signs, "+" and "−":*

$$\sup_{t\le T/\varepsilon^2} \left\| \hat\psi_\pm^{w,\varepsilon}(.,t) - \hat\psi_\pm^{u,\varepsilon}(.,t) \right\|_{L^2(\mathbb{R}^d)} \le C\varepsilon\,. \tag{3.12}$$

The original solutions are nearby in energy norm: With ω_0 from (2.9), *it holds that*

$$\sup_{t\le T/\varepsilon^2} \left\{ \left\| \frac{\omega_0(\varepsilon\,.)}{\varepsilon}[\hat w^\varepsilon(.,t) - \hat u^\varepsilon(.,t)] \right\|_{L^2(\mathbb{R}^d)} + \|\partial_t \hat w^\varepsilon(.,t) - \partial_t \hat u^\varepsilon(.,t)\|_{L^2(\mathbb{R}^d)} \right\} \le C\varepsilon\,. \tag{3.13}$$

The constant C depends only on ω_l, C_0, S_ψ, and T.

Proof. For the proof of (3.12) it suffices to insert the explicit solution formulas (2.6) and (3.10). The calculation for "+" is

$$\begin{aligned}
&\sup_{t\le T/\varepsilon^2} \left\| \hat\psi_+^{w,\varepsilon}(.,t) - \hat\psi_+^{u,\varepsilon}(.,t) \right\|_{L^2(\mathbb{R}^d)}^2 \\
&\quad = \sup_{t\le T/\varepsilon^2} \int_{\mathbb{R}^d} \left| e^{-i[\omega_d(\varepsilon k)/\varepsilon]t} \hat\psi_{+,0}^{w,\varepsilon}(k) - e^{-i[\omega_l(\varepsilon k)/\varepsilon]t} \hat\psi_{+,0}^{u,\varepsilon}(k) \right|^2 dk \\
&\quad = \sup_{t\le T/\varepsilon^2} \int_{\mathbb{R}^d} \left| e^{-i[\omega_d(\varepsilon k) - \omega_l(\varepsilon k)]t/\varepsilon} - 1 \right|^2 |\hat\psi_{+,0}^{w,\varepsilon}(k)|^2\, dk \\
&\qquad + \sup_{t\le T/\varepsilon^2} \int_{\mathbb{R}^d} \left| e^{-i\omega_l(\varepsilon k)t/\varepsilon} \right|^2 |\hat\psi_{+,0}^{w,\varepsilon}(k) - \hat\psi_{+,0}^{u,\varepsilon}(k)|^2\, dk \le C\varepsilon^2\,.
\end{aligned}$$

In the last step we used $\sup_{t\le T/\varepsilon^2}[\omega_d(\varepsilon k) - \omega_l(\varepsilon k)]t/\varepsilon \le C\varepsilon$ for every k in the compact set S_ψ, which is a consequence of (3.5). For the second term we used that initial values are nearby due to (3.11). The property for w and u translates into a corresponding property for ψ^w and ψ^u. The calculation for "−" is done accordingly.

In order to obtain (3.13), we use the reconstruction formulas; see (2.7). We calculate for the first term,

$$\sup_{t\le T/\varepsilon^2}\left\|\frac{\omega_0(\varepsilon\,.)}{\varepsilon}[\hat w^\varepsilon(.,t)-\hat u^\varepsilon(.,t)]\right\|^2_{L^2(\mathbb{R}^d)}$$
$$=\sup_{t\le T/\varepsilon^2}\frac12\left\|\frac{\omega_0(\varepsilon\,.)}{\varepsilon}\left[\frac{\varepsilon}{\omega_d(\varepsilon\,.)}(\hat\psi_+^{w,\varepsilon}+\hat\psi_-^{w,\varepsilon})-\frac{\varepsilon}{\omega_l(\varepsilon\,.)}(\hat\psi_+^{u,\varepsilon}+\hat\psi_-^{u,\varepsilon})\right]\right\|^2_{L^2(\mathbb{R}^d)}.$$

The expression coming from the contributions $\hat\psi_+^{w,\varepsilon}$ and $\hat\psi_+^{u,\varepsilon}$ is, up to the factor $1/2$,

$$\sup_{t\le T/\varepsilon^2}\left\|\omega_0(\varepsilon\,.)\left[\frac{\hat\psi_+^{w,\varepsilon}}{\omega_d(\varepsilon\,.)}-\frac{\hat\psi_+^{u,\varepsilon}}{\omega_l(\varepsilon\,.)}\right]\right\|^2_{L^2(\mathbb{R}^d)}$$
$$=\sup_{t\le T/\varepsilon^2}\left\|\frac{\omega_0(\varepsilon\,.)}{\omega_l(\varepsilon\,.)\omega_d(\varepsilon\,.)}(\omega_l(\varepsilon\,.)-\omega_d(\varepsilon\,.))\hat\psi_+^{w,\varepsilon}+\frac{\omega_0(\varepsilon\,.)}{\omega_l(\varepsilon\,.)}(\hat\psi_+^{w,\varepsilon}-\hat\psi_+^{u,\varepsilon})\right\|^2_{L^2(\mathbb{R}^d)}\le C\varepsilon^2\,.$$

We exploited in the last step (3.5), which provides $|\omega_l(\varepsilon\,.)-\omega_d(\varepsilon\,.)|\le C\varepsilon^4$; this implies that the first part is of order ε^2. The second part is of order ε by (3.12). We make use of the fact that only k-values in the compact set S_ψ must be considered. The argument for the contributions of $\hat\psi_-^{w,\varepsilon}$ and $\hat\psi_-^{u,\varepsilon}$ is analogous.

For the estimate of time derivatives in (3.13), we use the reconstruction formulas $\partial_t\hat u^\varepsilon(k,t)=-i(\hat\psi_+^{u,\varepsilon}-\hat\psi_-^{u,\varepsilon})/\sqrt2$ and $\partial_t\hat w^\varepsilon(k,t)=-i(\hat\psi_+^{w,\varepsilon}-\hat\psi_-^{w,\varepsilon})/\sqrt2$. Relation (3.12) provides immediately the second part of (3.13). ∎

The following corollary shows that the assumption on the bounded support in Fourier space can be removed if no approximation of the initial data is required. We recall that the function u is defined on the lattice, while w is defined on $\mathbb{R}^d$; in particular, it is easier to compare the Fourier transforms than to compare the original functions.

Corollary 3.2 (approximation result for nonsmooth initial data). *Let the setting be as in Theorem* 3.1*, with initial data* $(u_0^\varepsilon,u_1^\varepsilon)$ *and* $(w_0^\varepsilon,w_1^\varepsilon)$*. The assumptions are as in Theorem* 3.1*, but we do not demand* (u1) *of Assumption* 2.1*. Instead, we only demand* $\hat u_0^\varepsilon\to\hat u_0$*,* $\hat u_1^\varepsilon\to\hat u_1$*, and* $|k|\hat u_0^\varepsilon(k)\to|k|\hat u_0(k)$*, all convergences in* $L^2(\mathbb{R}^d)$*. We assume on the initial data for* w *the convergences* $\hat w_0^\varepsilon-\hat u_0^\varepsilon\to0$*,* $\hat w_1^\varepsilon-\hat u_1^\varepsilon\to0$*,* $|k|(\hat u_0^\varepsilon(k)-\hat u_0(k))\to0$ *in* $L^2(\mathbb{R}^d)$*, and that the supports of the Fourier transform are contained in a ball* $B_{C/\varepsilon}(0)$*. Then the solution* u^ε *of the lattice equation and the solution* w^ε *of the weakly dispersive equation satisfy, as* $\varepsilon\to0$*,*

(3.14)
$$\sup_{t\le T/\varepsilon^2}\left\{\left\|\frac{\omega_0(\varepsilon\,.)}{\varepsilon}[\hat w^\varepsilon(.,t)-\hat u^\varepsilon(.,t)]\right\|_{L^2(\mathbb{R}^d)}+\|\partial_t\hat w^\varepsilon(.,t)-\partial_t\hat u^\varepsilon(.,t)\|_{L^2(\mathbb{R}^d)}\right\}\to0\,.$$

Proof. All transformations are linear. Therefore, for the convergence result, it will be sufficient to analyze two arbitrary convergent sequences of functions $(u_0^\varepsilon,u_1^\varepsilon)$ and $(w_0^\varepsilon,w_1^\varepsilon)$ and the properties of their transformed quantities. We emphasize that, at this stage of the proof, these quantities are not the functions of the claim.

Step 1. *Transformations of* w*.* Let $(w_0^\varepsilon,w_1^\varepsilon)\to(w_0,w_1)$ in $H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ be arbitrary. The transformed quantities $\hat\psi_\pm^{w,\varepsilon}(.,t)$ of (3.9) satisfy, by the evolution

equation (3.10),

$$\sup_{t\le T/\varepsilon^2}\left\|\hat\psi_+^{w,\varepsilon}(.,t)\right\|^2_{L^2(\mathbb{R}^d)}\le\left\|\hat\psi_{+,0}^{w,\varepsilon}(.)\right\|^2_{L^2(\mathbb{R}^d)}\le C\left(\||k|\hat w_0^\varepsilon(k)\|^2_{L^2(\mathbb{R}^d)}+\|\hat w_1^\varepsilon(k)\|^2_{L^2(\mathbb{R}^d)}\right).$$

We exploited in the second inequality that $\hat w_0^\varepsilon$ is nonvanishing only for values of k satisfying $|k|\le C\varepsilon^{-1}$. This implies $\frac{1}{\varepsilon}\omega_d(\varepsilon k)|\le C|k|$.

The analogous calculation can be performed for $\hat\psi_-^{w,\varepsilon}(.,t)$.

We next consider the reconstruction to $\hat w^\varepsilon(.,t)$ and its norm:

$$\begin{aligned}\sup_{t\le T/\varepsilon^2}\left\|\frac{\omega_0(\varepsilon\,.)}{\varepsilon}\hat w^\varepsilon(.,t)\right\|^2_{L^2(\mathbb{R}^d)}&\le C\sup_{t\le T/\varepsilon^2}\left\|\frac{\omega_0(\varepsilon\,.)}{\omega_d(\varepsilon\,.)}(\hat\psi_+^{w,\varepsilon}(.,t)+\hat\psi_-^{w,\varepsilon}(.,t))\right\|^2_{L^2(\mathbb{R}^d)}\\ &\le C\sup_{t\le T/\varepsilon^2}\left\|\hat\psi_+^{w,\varepsilon}(.,t)+\hat\psi_-^{w,\varepsilon}(.,t)\right\|^2_{L^2(\mathbb{R}^d)}\le C\left(\||k|\hat w_0^\varepsilon(k)\|^2_{L^2(\mathbb{R}^d)}+\|\hat w_1^\varepsilon(k)\|^2_{L^2(\mathbb{R}^d)}\right),\end{aligned}$$

where we used $|k|\le C\varepsilon^{-1}$ in the second inequality and inserted the previous calculation in the third.

The analogous calculation can be performed to obtain the analogous bound for $\sup_{t\le T/\varepsilon^2}\|\partial_t\hat w^\varepsilon(.,t)\|^2_{L^2(\mathbb{R}^d)}$.

Step 2. *Transformations of u.* Let $(u_0^\varepsilon,u_1^\varepsilon)\to(u_0,u_1)$ in $H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ be arbitrary. Then there holds

$$\begin{aligned}&\sup_{t\le T/\varepsilon^2}\left\|\frac{\omega_0(\varepsilon\,.)}{\varepsilon}\hat u^\varepsilon(.,t)\right\|^2_{L^2(\mathbb{R}^d)}+\sup_{t\le T/\varepsilon^2}\|\partial_t\hat u^\varepsilon(.,t)\|^2_{L^2(\mathbb{R}^d)}\\ &\qquad\le C\left(\||k|\hat u_0^\varepsilon(k)\|^2_{L^2(\mathbb{R}^d)}+\|\hat u_1^\varepsilon(k)\|^2_{L^2(\mathbb{R}^d)}\right).\end{aligned}$$

The calculation is identical to that of Step 1. We exploit that the discrete Fourier transform was defined in (2.2) in such a way that only k with $|k|\le C\varepsilon^{-1}$ are in the support of the Fourier transforms $\hat u_0^\varepsilon(k)$ and $\hat u_1^\varepsilon(k)$.

Step 3. *Conclusion.* Now let $(u_0^\varepsilon,u_1^\varepsilon)$ and $(w_0^\varepsilon,w_1^\varepsilon)$ be as in the formulation of the corollary. Let an error quantity $\delta>0$ be fixed.

We select a small constant $c_0>0$ dependent on the constants of Steps 1 and 2. We find a bounded subset $S_\psi\subset\mathbb{R}^d$ and approximations $(\tilde u_0^\varepsilon,\tilde u_1^\varepsilon)$ such that

$$\||k|\,(\hat u_0^\varepsilon(k)-\tilde u_0^\varepsilon(k))\|_{L^2(\mathbb{R}^d)}+\|\hat u_1^\varepsilon(k)-\tilde u_1^\varepsilon(k)\|_{L^2(\mathbb{R}^d)}\le c_0\delta\,,$$

and $\tilde u_0^\varepsilon$ and $\tilde u_1^\varepsilon$ are supported in S_ψ. The strong convergence of the Fourier transforms imply that the approximations can be chosen with a uniformly bounded support. Analogously, possibly enlarging S_ψ, approximations $(\tilde w_0^\varepsilon,\tilde w_1^\varepsilon)$ can be found.

The rest of the proof is an application of the triangle inequality. We first choose approximations $(\tilde u_0^\varepsilon,\tilde u_1^\varepsilon)$ and $(\tilde w_0^\varepsilon,\tilde w_1^\varepsilon)$ of the Fourier transforms of the initial data. By Steps 1 and 2, this can be done in such a way that only small errors are introduced in the norms of (3.14).

Once the approximations and, most importantly, $S_\psi\subset\mathbb{R}^d$ are fixed, we apply the result of Theorem 3.1 to conclude that the two approximate solutions have distance $\delta/3$ in the same norm. The triangle inequality provides the result for the original initial values. ∎

4. Ring solutions and profile equations. The aim of this section is to derive a profile equation for the dispersive time scale. We will see that the evolution of a wave profile is given by a linearized KdV equation,

$$\partial_\tau V^\varepsilon(z,\tau;q) = b(q)\,\partial_z^3 V^\varepsilon(z,\tau;q)\,; \tag{4.1}$$

see (1.4). The profile is a function $V^\varepsilon : \mathbb{R}\times[0,T]\times S^{d-1}\to\mathbb{R}$, and we write $V^\varepsilon = V^\varepsilon(z,\tau;q)$. The independent variable $z\in\mathbb{R}$ denotes the distance to the sphere with radius ct. The variable $\tau\in[0,T]$ denotes a new time variable, obtained by rescaling through $t=\tau/\varepsilon^2$. The vector $q\in S^{d-1}$ denotes a direction of propagation. We emphasize that the parameter $\varepsilon>0$ does not appear in the evolution equation (4.1); it will enter through the initial values.

From now on, we exploit invariance assumption (a5) of Assumption 2.1. This implies that the matrix A is scalar, and hence $\omega_0(\tilde k)=c|\tilde k|$ holds for some number $c>0$; see Remark 2.2. We use the higher order expansion of ω_l from (2.10), which now reads

$$\omega_l(\tilde k) - \Big[c|\tilde k| + b(\tilde k)\Big] = O(|\tilde k|^5) \qquad \text{as } \tilde k\to 0 \tag{4.2}$$

with the 3-homogeneous function $b:\mathbb{R}^d\to\mathbb{R}$ given by $b(k)=\frac{1}{2\,\omega_0(k)}Ck\otimes k\otimes k\otimes k$. In most calculations below we only use directions $q\in S^{d-1}$ as arguments of b; we may therefore also consider the restriction $b:S^{d-1}\to\mathbb{R}$. In particular, this is how b should be understood in (4.1).

Interpretation of the profile variable V^ε. Loosely speaking, the distance variable z and the solution variable V^ε can be described as follows: We expect that the lattice solution u^ε has a ring-like shape; see Figure 2. For every $t>0$, an arbitrary point $x\in\mathbb{R}^2$ can be written uniquely as $x=q(ct+z)$ with a direction $q\in S^{d-1}$ and a signed distance $z\in\mathbb{R}$. We expect $u^\varepsilon(x,t)\sim V^\varepsilon(z,\varepsilon^2 t;q)$, up to an appropriate factor.

To formulate a mathematical statement, we now reverse the approach. We solve the one-dimensional linearized KdV equation (4.1) with appropriate initial data to obtain the function $V^\varepsilon:\mathbb{R}\times[0,T]\times S^{d-1}\to\mathbb{R}$. Using the shape function $V^\varepsilon = V^\varepsilon(z,\tau,q)$, we construct a function $v^\varepsilon(x,t)$ in such a way that the above interpretation of V^ε is respected. To be precise, the function v^ε is constructed from V^ε with the formula

$$v^\varepsilon(x,t) := \frac{1}{|x|^{(d-1)/2}} V^\varepsilon\left(|x|-ct,\varepsilon^2 t;\frac{x}{|x|}\right). \tag{4.3}$$

The prefactor $r^{-(d-1)/2}$ is introduced in order to keep the L^2-norm of the function v^ε always of order 1. Our aim is to compare the reconstructed approximate solution v^ε with the original solution u^ε. Our result will be that the two functions u^ε and v^ε are comparable to leading order in ε.

Initial values. We solve the evolution equation (4.1) with an initial condition of the form

$$V^\varepsilon(z,0;q) = V_0^\varepsilon(z;q)\,. \tag{4.4}$$

The initial values V_0^ε are extracted from the initial values u_0 of the lattice equation.

The construction of initial values is dimension dependent. In space dimension $d=1$, the vector $q\in S^{d-1}$ can have two values, $q\in S^0=\{+1,-1\}$. The function $V^\varepsilon(.,\tau;+1)$ describes the profile of a wave that travels in the positive x-direction. Similarly, $V^\varepsilon(.,\tau;-1)$ describes the profile of a wave in the negative x-direction.

Reducing to $\hat\psi^{u,\varepsilon}_{-,0} = 0$, introducing Q^u_ε and Q^v_ε. In the following subsections, we will only consider the case of initial values with $\psi^{u,\varepsilon}_{-,0} = 0$. Such initial values occur if $i\partial_t \hat u^\varepsilon(k) = (\omega_l(\varepsilon k)/\varepsilon)\, \hat u^\varepsilon(k)$ in $t = 0$ holds for every k. The opposite case (i.e., $\psi^{u,\varepsilon}_{+,0} = 0$) can be treated in an analogous way. Since the equations are linear, one can combine the two results; hence the approximation results are also valid for general initial data.

Once we have decided to study $\psi^{u,\varepsilon}_{-,0} = 0$, we know that every solution in Fourier space has a fast time dependence given by the factor $e^{-ic|k|\tau/\varepsilon^2}$, where c is the wave speed in $\omega_0(k) = c|k|$. In order to compensate for this factor, we will compare below the two expressions

$$Q^u_\varepsilon(k,\tau) := e^{ic|k|\tau/\varepsilon^2} \hat u^\varepsilon(k, \tau/\varepsilon^2) \quad \text{and} \quad Q^v_\varepsilon(k,\tau) := e^{ic|k|\tau/\varepsilon^2} \hat v^\varepsilon(k, \tau/\varepsilon^2)\,. \tag{4.5}$$

Remark. The factor $e^{ic|k|\tau/\varepsilon^2}$ can be interpreted as follows: In order to investigate the two solutions $\hat v^\varepsilon$ and $\hat u^\varepsilon$ at time t, we solve the linear wave equation backwards and compare the corresponding values in $t = 0$.

From the explicit solution formula (2.6) and the reconstruction rule (2.7) we find

$$\begin{aligned} Q^u_\varepsilon(k,\tau) &= e^{ic|k|\tau/\varepsilon^2} \hat u^\varepsilon(k, \tau/\varepsilon^2) \\ &= e^{ic|k|\tau/\varepsilon^2} \frac{\varepsilon}{\omega_l(\varepsilon\ .)\sqrt{2}} \left(\hat\psi^{u,\varepsilon}_+(k, \tau/\varepsilon^2) + \hat\psi^{u,\varepsilon}_-(k, \tau/\varepsilon^2) \right) \\ &= e^{ic|k|\tau/\varepsilon^2} \frac{\varepsilon}{\omega_l(\varepsilon\ .)\sqrt{2}} e^{-i[\omega_l(\varepsilon k)/\varepsilon]\tau/\varepsilon^2} \hat\psi_{+,0}(k) \\ &= e^{-i[\omega_l(\varepsilon k)/\varepsilon - c|k|]\tau/\varepsilon^2}\ \hat u^\varepsilon_0(k)\,. \end{aligned}$$

By the choice of b in (4.2) we can compare $\omega_l(\varepsilon k)/\varepsilon - c|k|$ with $\varepsilon^2 b(k)$. We find

$$Q^u_\varepsilon(k,\tau) = e^{-ib(k)\tau}\ \hat u^\varepsilon_0(k) + F_\varepsilon(k,\tau) \tag{4.6}$$

with an error function that satisfies

$$|F_\varepsilon(k,\tau)| = \left| e^{-i[\omega_l(\varepsilon k) - c|\varepsilon k| - ib(\varepsilon k)]\tau/\varepsilon^3} - 1 \right|\ |\hat u^\varepsilon_0(k)|\,. \tag{4.7}$$

Due to (4.2) there holds $[\omega_l(\varepsilon k) - c|\varepsilon k| - ib(\varepsilon k)]\tau/\varepsilon^3 = o(1)$, uniformly in $\tau \in [0,T]$ and uniformly in k for k ranging in a compact subset of $\mathbb{R}^d$. This implies smallness of the error function $F_\varepsilon(.,\tau) \in L^2(\mathbb{R}^d)$ for small $\varepsilon > 0$, uniformly in $\tau \in [0,T]$.

4.1. The KdV profile in one dimension. In space dimension $d = 1$, the initial data V^ε_0 can be defined easily in terms of the initial data $u^\varepsilon_0 : (\varepsilon\mathbb{Z})^d \to \mathbb{R}$ of class $u^\varepsilon_0 \in l^2((\varepsilon\mathbb{Z})^d; \mathbb{R})$. More precisely, we define the Fourier transform $\hat V^\varepsilon_0(.;q) \in L^2(\mathbb{R})$ of $V^\varepsilon_0 \in L^2(\mathbb{R})$ through the Fourier transform $\hat u^\varepsilon_0 \in L^2(\mathbb{R})$ of u^ε_0. We write $z \in \mathbb{R}$ for the spatial variable, $V^\varepsilon_0 = V^\varepsilon_0(z,q)$, and $\xi \in \mathbb{R}$ for the frequency variable, $\hat V^\varepsilon_0 = \hat V^\varepsilon_0(\xi;q)$.

In one space dimension, the initial data for the linearized KdV equation are

$$\begin{aligned} \hat V^\varepsilon_0(\xi;+1) &:= \begin{cases} \hat u^\varepsilon_0(\xi) & \text{for } \xi > 0\,, \\ 0 & \text{else,} \end{cases} \\ \hat V^\varepsilon_0(\xi;-1) &:= \begin{cases} \hat u^\varepsilon_0(-\xi) & \text{for } \xi > 0\,, \\ 0 & \text{else.} \end{cases} \end{aligned} \tag{4.8}$$

The solution of (4.1) can be expressed explicitly in Fourier space. Since the harmonic waves $e^{i\xi z}e^{-ib(|\xi|\,q)\tau}$ are solutions to the linearized KdV equation for $\xi \geq 0$, and since $\hat{V}_0^\varepsilon$ is supported on the positive half-line $[0,\infty)$, we find that $\hat{V}^\varepsilon(.,\tau;q) \in L^2(\mathbb{R})$ is given by

$$\hat{V}^\varepsilon(\xi,\tau;q) = e^{-ib(|\xi|\,q)\tau}\hat{V}_0^\varepsilon(\xi;q)\,. \tag{4.9}$$

The solution $V^\varepsilon(.,\tau;q) \in L^2(\mathbb{R})$ is obtained by an inverse Fourier transform.

In space dimension $d = 1$ the reconstruction formula (4.3) reads

$$v^\varepsilon(x,t) := \begin{cases} V^\varepsilon\left(|x| - ct, \varepsilon^2 t; +1\right) & \text{for } x > 0\,, \\ V^\varepsilon\left(|x| - ct, \varepsilon^2 t; -1\right) & \text{for } x < 0\,. \end{cases}$$

We can now calculate the complex number $Q_\varepsilon^v(k,\tau)$ of (4.5), using $t = \tau/\varepsilon^2$ to shorten the formulas. We use (4.5) in the first step, the Fourier transform in the second, and the reconstruction formula in the third step.

$$\begin{aligned} Q_\varepsilon^v(k,\tau) &= e^{ic|k|t}\hat{v}^\varepsilon(k,t) = e^{ic|k|t}\int_{-\infty}^{\infty} e^{-ik\cdot x}v^\varepsilon(x,t)\,dx \\ &= e^{ic|k|t}\sum_{q=\pm 1}\int_0^\infty e^{-ik\cdot qr}V^\varepsilon\left(r - ct, \varepsilon^2 t; q\right)\,dr \\ &= e^{ic|k|t}\sum_{q=\pm 1}\int_{\mathbb{R}} e^{-ik\cdot qct}e^{-ik\cdot qz}V^\varepsilon\left(z, \varepsilon^2 t; q\right)\,dz + G_\varepsilon(k,\tau) \\ &= e^{ic|k|t}\sum_{q=\pm 1} e^{-ik\cdot qct}\hat{V}^\varepsilon(k\cdot q, \varepsilon^2 t; q) + G_\varepsilon(k,\tau)\,, \end{aligned}$$

where we introduced the error function

$$G_\varepsilon(k,\tau) := -e^{ic|k|\tau/\varepsilon^2}\sum_{q=\pm 1}\int_{-\infty}^0 e^{-ik\cdot qr}V^\varepsilon\left(r - c\tau/\varepsilon^2, \tau; q\right)\,dr\,. \tag{4.10}$$

We exploit the representation (4.9) of $\hat{V}^\varepsilon$ to find

$$Q_\varepsilon^v(k,\tau) = e^{ic|k|t}\sum_{q=\pm 1} e^{-ik\cdot qct}e^{-ib(|k|\,q)\varepsilon^2 t}\hat{V}_0^\varepsilon(k\cdot q; q) + G_\varepsilon(k,\tau)\,. \tag{4.11}$$

We next expand the sum, using $q = +k/|k|$ and $q = -k/|k|$. For $q = k/|k|$, two exponential factors cancel. The other term vanishes since $\hat{V}_0^\varepsilon$ vanishes for negative first arguments.

$$\begin{aligned} Q_\varepsilon^v(k,\tau) &= e^{-ib(k)\tau}\ \hat{V}_0^\varepsilon(|k|; k/|k|) + G_\varepsilon(k,\tau) \\ &= e^{-ib(k)\tau}\ \hat{u}_0^\varepsilon(k) + G_\varepsilon(k,\tau)\,, \end{aligned} \tag{4.12}$$

where in the last equality we used the choice of initial data $\hat{V}_0^\varepsilon$ in (4.8).

Let us summarize the results. We have calculated $Q_\varepsilon^u(k,\tau)$ in (4.6) and $Q_\varepsilon^v(k,\tau)$ in (4.12). The leading order term is $e^{-ib(k)\tau}\ \hat{u}_0^\varepsilon(k)$ for both quantities. The difference satisfies

$$|Q_\varepsilon^u(k,\tau) - Q_\varepsilon^v(k,\tau)| \leq |F_\varepsilon(k,\tau)| + |G_\varepsilon(k,\tau)|\,. \tag{4.13}$$

This estimate allows us to conclude an approximation result.

THEOREM 4.1 (KdV profile equation in one space dimension). *We consider $d = 1$, coefficients $(a_j)_j$, and initial values $(u_0^\varepsilon, u_1^\varepsilon)$ that satisfy Assumption 2.1. We assume $\hat{u}_1^\varepsilon(.) = -i\varepsilon^{-1}\omega_l(\varepsilon.)\hat{u}_0^\varepsilon(.)$ on $\mathbb{R}$ and the convergence $\hat{u}_0^\varepsilon \to \hat{u}_0$ strongly in $L^2(\mathbb{R})$ as $\varepsilon \to 0$ for some limit $\hat{u}_0$.*

Let $u^\varepsilon(.,t)$ be the solution to the lattice equation (1.1), and let V^ε be the solution to the KdV-equation (4.1) with initial values (4.8) and coefficients c and b from (4.2). Let $v^\varepsilon(.,t)$ be given by the reconstruction formula (4.3). Then, for every $\tau \in (0,T]$,

$$\left\|\hat{u}^\varepsilon(.,\tau/\varepsilon^2) - \hat{v}^\varepsilon(.,\tau/\varepsilon^2)\right\|_{L^2(\mathbb{R})} \to 0 \tag{4.14}$$

as $\varepsilon \to 0$.

Proof. The error term G_ε of (4.10) can be written with a change of variables and with the characteristic function $\mathbf{1}_{\{s|s\le -ct\}}$ in the form (we use $t = \tau/\varepsilon^2$)

$$\begin{aligned} G_\varepsilon(k,\tau) &= -\sum_{q=\pm 1} e^{i|k|ct - ik\cdot qct} \int_{-\infty}^{\infty} e^{-ik\cdot qs} \mathbf{1}_{\{s\le -ct\}}(s) V^\varepsilon(s,\tau;q)\, ds \\ &= -\sum_{q=\pm 1} e^{i|k|ct - ik\cdot qct} \mathcal{F}\left(\mathbf{1}_{\{s\le -ct\}}(.) V^\varepsilon(.,\tau;q)\right)(k\cdot q)\,. \end{aligned}$$

We obtain

$$\|G_\varepsilon(.,\tau)\|^2_{L^2(dk)} \le 2\sum_{q=\pm 1} 2\pi \|\mathbf{1}_{\{s\le -ct\}}(.) V^\varepsilon(s,\tau;q)\|^2_{L^2(ds)}\,. \tag{4.15}$$

The Fourier transform of V^ε is given by (4.9). In the following we want to exploit this formula.

The initial values for the linearized KdV equation are given by (4.8): $\hat{V}_0^\varepsilon(\xi;+1) = \hat{u}_0^\varepsilon(\xi)$ for $\xi > 0$ and analogously for $\xi < 0$. The strong convergence of the (Fourier transform of the) initial values allows us to take $L^2(\mathbb{R}, d\xi)$-limits: $\hat{V}_0^\varepsilon(\xi;+1) \to \hat{V}_0^0(\xi;+1) := \hat{u}_0(\xi)$ for $\xi > 0$ and analogously for $\xi < 0$. The solution of the linearized KdV equation is given by (4.9). Since $\hat{u}_0$ has bounded support by assumption (u1) of Assumption 2.1, the formula allows us to form the limit $\varepsilon \to 0$ in $L^2(\mathbb{R}, d\xi)$: We find $\hat{V}^\varepsilon(\xi,\tau;q) \to \hat{V}^0(\xi,\tau;q) := e^{-ib(|\xi|\, q)\tau}\hat{V}_0^0(\xi;q)$. The convergence of the Fourier transforms implies the convergence of the difference:

$$\begin{aligned} 2\pi\|V^\varepsilon(s,\tau;q) - V^0(s,\tau;q)\|^2_{L^2(ds)} &= \|\hat{V}^\varepsilon(\xi,\tau;q) - \hat{V}^0(\xi,\tau;q)\|^2_{L^2(d\xi)} \\ &= \left\| e^{-ib(|\xi|\, q)\tau}\left(\hat{V}_0^\varepsilon(\xi;q) - \hat{V}_0^0(\xi;q)\right)\right\|^2_{L^2(d\xi)} \to 0\,, \end{aligned}$$

uniformly in $\tau \in [0,T]$. The triangle inequality allows us to continue from (4.15):

$$\begin{aligned} \|G_\varepsilon(.,\tau)\|^2_{L^2(dk)} &\le 4\pi \sum_{q=\pm 1} \|\mathbf{1}_{\{s\le -c\tau/\varepsilon^2\}}(s) V^\varepsilon(s,\tau;q)\|^2_{L^2(ds)} \\ &\le 8\pi \sum_{q=\pm 1} \|\mathbf{1}_{\{s\le -c\tau/\varepsilon^2\}}(s)\left(V^\varepsilon(s,\tau;q) - V^0(s,\tau;q)\right)\|^2_{L^2(ds)} \\ &\quad + 8\pi \sum_{q=\pm 1} \|\mathbf{1}_{\{s\le -c\tau/\varepsilon^2\}}(s) V^0(s,\tau;q)\|^2_{L^2(ds)} \to 0 \end{aligned}$$

for every $\tau \in (0,T]$, since the integral of the (ε-independent) L^1-function $|V^0(.,\tau;q)|^2$ over $(-\infty, -c\tau/\varepsilon^2)$ vanishes in the limit $\varepsilon \to 0$.

We have obtained that both error terms F_ε and G_ε vanish in the limit $\varepsilon \to 0$. We therefore conclude from (4.13), for every $\tau \in (0,T]$,

$$\begin{aligned}
&\|\hat u^\varepsilon(.,\tau/\varepsilon^2) - \hat v^\varepsilon(.,\tau/\varepsilon^2)\|^2_{L^2(dk)} \\
&\quad = \|e^{ic|k|t}\ \hat u^\varepsilon(.,\tau/\varepsilon^2) - e^{ic|k|t}\ \hat v^\varepsilon(.,\tau/\varepsilon^2)\|^2_{L^2(dk)} \\
&\quad = \|Q^u_\varepsilon(.,\tau) - Q^v_\varepsilon(.,\tau)\|^2_{L^2(dk)} \\
&\quad \le 2\|F_\varepsilon(.,\tau)\|^2_{L^2(dk)} + 2\|G_\varepsilon(.,\tau)\|^2_{L^2(dk)} \to 0
\end{aligned}$$

as $\varepsilon \to 0$. This was the claim in (4.14). ∎

Remark. The two error terms satisfy additionally bounds that are uniform in τ: $\sup_{\tau\in[0,T]} \|F_\varepsilon(.,\tau)\|_{L^2(dk)} \le C$ and $\sup_{\tau\in[0,T]} \|G_\varepsilon(.,\tau)\|_{L^2(dk)} \le C$. We therefore also obtain the convergence in (4.14) in an integral sense, e.g.,

$$\int_0^T \left\|\hat u^\varepsilon(.,\tau/\varepsilon^2) - \hat v^\varepsilon(.,\tau/\varepsilon^2)\right\|_{L^2(\mathbb{R})} d\tau = o(1)\,. \tag{4.16}$$

4.2. Ring solutions in arbitrary dimension. Also in dimension $d > 1$, the Fourier transform of the reconstructed solution v^ε can be calculated. We include here the formula which can be derived in arbitrary dimension $d \ge 1$. Starting with the next section, we have to restrict all further investigations to the two-dimensional case $d = 2$.

We assume in the following that $\hat V^\varepsilon_0(.;q)$ is supported on the positive half-line, $\xi \in [0,\infty)$. We then have $\xi^3 = |\xi|^3$ for all ξ in the support and can write the solution of the linearized KdV equation in Fourier space (exactly as in the one-dimensional case) as

$$\hat V^\varepsilon(\xi,\tau;q) = e^{-ib(|\xi|\,q)\tau} \hat V^\varepsilon_0(\xi;q)\,. \tag{4.17}$$

We now calculate the Fourier transform of the reconstruction $\hat v^\varepsilon$. In the following calculation we use the following: 1. The definition of the Fourier transform. 2. Polar coordinates $x = rq$, $r > 0$, $q \in S^{d-1}$. 3. The reconstruction formula (4.3). 4. The integral has its main contributions for the radial component $r \approx ct$; extending the integral and replacing r by ct introduces an error G_ε. We write $dS(q) = d\mathcal{H}^{d-1}(q)$ for the surface measure.

$$\begin{aligned}
\hat v^\varepsilon(k,t) &= \int_{\mathbb{R}^d} e^{-ik\cdot x} v^\varepsilon(x,t)\,dx \\
&= \int_{S^{d-1}} \int_0^\infty e^{-ik\cdot qr}\, v^\varepsilon\,(rq,t)\, r^{d-1}\,dr\,dS(q) \\
&= \int_{S^{d-1}} \int_0^\infty e^{-ik\cdot qr}\, V^\varepsilon\,(r-ct,\tau;q)\, r^{(d-1)/2}\,dr\,dS(q) \\
&= \int_{S^{d-1}} \int_{-\infty}^\infty e^{-ik\cdot qr}\, V^\varepsilon\,(r-ct,\tau;q)\, |ct|^{(d-1)/2}\,dr\,dS(q) + G_\varepsilon(k,\tau),
\end{aligned} \tag{4.18}$$

with the error function

$$G_\varepsilon(k,\tau) := \int_{S^{d-1}} \int_{-\infty}^\infty e^{-ik\cdot qr}\, V^\varepsilon\,(r-ct,\tau;q) \left[r^{(d-1)/2}\mathbf{1}_{r\ge 0} - |ct|^{(d-1)/2}\right] dr\,dS(q)\,. \tag{4.19}$$

Let us describe loosely why the error function $G_\varepsilon(k,\tau)$ might be small in comparison to $\hat v^\varepsilon(k,t)$. Let us assume that $V^\varepsilon(.,\tau;q)$ is supported in some compact interval

$[-M, M]$. In this case, the integrand is nonvanishing only for $|r - ct| \le M$, but then $r^{(d-1)/2} - |ct|^{(d-1)/2} \ll |ct|^{(d-1)/2}$, whence $G_\varepsilon(k,\tau)$ should be small in comparison to the function $\hat{v}^\varepsilon(k,t)$.

The details of the argument are involved: The relevant values of r are large (order ε^{-2}), and we deal with a highly oscillatory integral in q. Nevertheless, the argument can be made rigorous in two space dimensions. We will pursue this program in the next section.

5. Approximation result in two dimensions. We now consider the case of space dimension $d = 2$ in detail. To simplify formulas, we use ε-independent initial data $\hat{u}_0^\varepsilon = \hat{u}_0$ with bounded support $S_\psi \subset \mathbb{R}^2$. We note that the (discrete) Fourier expansion of the function $\hat{u}_0$ (understood as a periodic function on the Brillouin zone) provides the initial data u_0^ε. Since we want to work with $\psi_{-,0}^{u,\varepsilon} = 0$, we consider $\hat{u}_1^\varepsilon(.) = -i(\omega_l(\varepsilon.)/\varepsilon)\,\hat{u}_0^\varepsilon(.)$ on $\mathbb{R}^2$.

We next choose initial values $V_0^\varepsilon(z;q) = V_0(z;q)$ for the linearized KdV equation (4.1). Loosely speaking, we want to choose initial values

$$\hat{V}_0(\xi;q) \approx z_0^{-1}\sqrt{\xi}\;\hat{u}_0(\xi q)\;\forall \xi > 0 \quad \text{with} \quad z_0 := \sqrt{\pi}(1+i)\,. \tag{5.1}$$

The essential part is the factor $\sqrt{\xi}$, which was not present in the one-dimensional case; compare (4.8).

In order to obtain convergence results, we have to replace the square-root function by a smooth function W_ρ, where $\rho > 0$ is a small parameter. We require that

$$W_\rho \in C^\infty(\mathbb{R};\mathbb{R})\,, \quad W_\rho(\xi) = 0\;\forall \xi \le 0\,, \quad W_\rho(\xi) = \sqrt{\xi}\;\forall \xi \ge \rho/2\,. \tag{5.2}$$

Using the function W_ρ, we define the initial data for the linearized KdV equation with the help of the initial data u_0^ε by setting

$$\hat{V}_0^\varepsilon(\xi;q) := \begin{cases} z_0^{-1} W_\rho(\xi)\;\hat{u}_0^\varepsilon(\xi q) & \text{for } \xi > 0\,, \\ 0 & \text{else.} \end{cases} \tag{5.3}$$

5.1. Pointwise convergence. Our aim is to compare the Fourier transform of the lattice solution u^ε with the Fourier transform of the function v^ε, which was obtained by solving the linearized KdV equation and the reconstruction formula. We can compare these two functions on the basis of the calculation (4.18). The first step is to show smallness of the error term G_ε from (4.19), which, for $d = 2$, reads

$$G_\varepsilon(k,\tau) = \int_{S^1}\int_{-\infty}^{\infty} e^{-ik\cdot qr}\,V^\varepsilon\,(r - ct,\tau;q)\left[r^{1/2}\mathbf{1}_{r\ge 0} - (ct)^{1/2}\right]\,dr\,dS(q)\,. \tag{5.4}$$

LEMMA 5.1 (smallness of the error G_ε). *We consider $d = 2$ and assume that the solution $V^\varepsilon\,(s,\tau;q)$ to the linearized KdV equation (4.1) with initial values (5.3) has the following decay property for constants $C, \alpha, \varepsilon_0 > 0$:*

$$|V^\varepsilon\,(s,\tau;q)\,| \le C(1+|s|)^{-2-\alpha} \tag{5.5}$$

for all $s \in \mathbb{R}$, $\tau \in [0,T]$, $q \in S^1$, $\varepsilon \in (0,\varepsilon_0)$. Then the error function G_ε of (5.4) satisfies, for every $\tau \in (0,T]$,

$$G_\varepsilon(k,\tau) \to 0 \tag{5.6}$$

as $\varepsilon \to 0$, uniformly in $k \in \mathbb{R}^2$.

Proof of Lemma 5.1. We choose $\tau \in (0,T]$. We observe that S^1 has the finite measure 2π and that $e^{-ik\cdot qr}$ has norm 1. Therefore, in order to verify (5.6), it suffices to show (substituting $r = c\tau/\varepsilon^2 + s$)

$$\int_{-\infty}^{\infty} |V^\varepsilon(s,\tau;q)| \left| (c\tau/\varepsilon^2+s)^{1/2} \mathbf{1}_{s\geq -c\tau/\varepsilon^2} - |c\tau/\varepsilon^2|^{1/2} \right| ds \to 0\,, \tag{5.7}$$

uniformly in $q \in S^1$. We decompose the integral into two parts, distinguishing $|s| \leq \delta/\varepsilon$ and $|s| > \delta/\varepsilon$; the number $\delta > 0$ is chosen below. We only consider ε-values with $c\tau/\varepsilon > \delta$.

The integral over $\{|s| \leq \delta/\varepsilon\}$ is estimated using $\left|(c\tau/\varepsilon^2+s)^{1/2} - |c\tau/\varepsilon^2|^{1/2}\right| = \varepsilon^{-1}\left|(c\tau+s\varepsilon^2)^{1/2} - |c\tau|^{1/2}\right| \leq Cs\varepsilon \leq C\delta$:

$$\begin{aligned}\int_{\{|s|\leq\delta/\varepsilon\}} |V^\varepsilon(s,\tau;q)| \left| (c\tau/\varepsilon^2+s)^{1/2} \mathbf{1}_{s\geq -c\tau/\varepsilon^2} - |c\tau/\varepsilon^2|^{1/2} \right| ds \\ \leq C\delta \int_{-\infty}^{\infty} |V^\varepsilon(s,\tau;q)|\, ds \leq C\delta\,.\end{aligned}$$

The other integral concerns large values of $|s|$, and we consider $\{|s| > \delta/\varepsilon\}$. The integral is treated with the crude estimate $\left|(c\tau/\varepsilon^2+s)^{1/2}\mathbf{1}_{s\geq -c\tau/\varepsilon^2} - |c\tau/\varepsilon^2|^{1/2}\right| \leq C(\varepsilon^{-1} + |s|^{1/2})$. Assumption (5.5) allows us to calculate

$$\begin{aligned}&\int_{\{|s|>\delta/\varepsilon\}} |V^\varepsilon(s,\tau;q)| \left| (c\tau/\varepsilon^2+s)^{1/2} \mathbf{1}_{s\geq -c\tau/\varepsilon^2} - |c\tau/\varepsilon^2|^{1/2} \right| ds \\ &\quad\leq C \int_{\{|s|>\delta/\varepsilon\}} (\varepsilon^{-1} + |s|^{1/2})\, |V^\varepsilon(s,\tau;q)|\, ds \\ &\quad\leq C \int_{\{|s|>\delta/\varepsilon\}} (\varepsilon^{-1} + |s|^{1/2}) |s|^{-2-\alpha}\, ds \leq C\varepsilon^{-1}(\varepsilon/\delta)^{1+\alpha} + C(\varepsilon/\delta)^{1/2+\alpha}\,.\end{aligned}$$

Choosing first $\delta > 0$ to have smallness of the first integral, and then choosing $\varepsilon_0 > 0$ sufficiently small, we obtain smallness of both integrals. This yields (5.6). ∎

We now continue the calculation (4.18) of $\hat v^\varepsilon$ for $d = 2$. Substituting once more $r = ct + s$, we recognize the one-dimensional Fourier transform of V^ε. This allows us to insert the solution formula (4.17) in the last equality:

$$\begin{aligned}\hat v^\varepsilon(k,t) &= \int_{S^1} \int_{-\infty}^{\infty} e^{-ik\cdot q(ct+s)}\, V^\varepsilon(s,\tau;q)\, |ct|^{1/2}\, ds\, dS(q) + G_\varepsilon(k,\tau) \\ &= \int_{S^1} e^{-ik\cdot qct}\, \hat V^\varepsilon(k\cdot q,\tau;q)\, |ct|^{1/2}\, dS(q) + G_\varepsilon(k,\tau) \\ &= \int_{S^1} e^{-ik\cdot qct}\, e^{-ib((k\cdot q)\, q)\tau}\, \hat V_0^\varepsilon(k\cdot q;q) |ct|^{1/2}\, dS(q) + G_\varepsilon(k,\tau)\,.\end{aligned}$$

With this expression for $\hat v^\varepsilon(k,t)$ we can, as in the one-dimensional case, evaluate the function $Q_\varepsilon^v(k,\tau)$ of (4.5). We insert the initial conditions $\hat V_0^\varepsilon$ from (5.3) and find

$$\begin{aligned}Q_\varepsilon^v(k,\tau) &= e^{ic|k|\tau/\varepsilon^2} \hat v^\varepsilon(k,\tau/\varepsilon^2) \\ &= \int_{S^1\cap\{k\cdot q>0\}} e^{i(|k|-k\cdot q)c\tau/\varepsilon^2}\; e^{-ib((k\cdot q)\, q)\tau} z_0^{-1} W_\rho(k\cdot q)\, \hat u_0^\varepsilon((k\cdot q)q)\, \frac{\sqrt{c\tau}}{\varepsilon}\, dS(q) \\ &\qquad + e^{ic|k|\tau/\varepsilon^2} G_\varepsilon(k,\tau)\,.\end{aligned}$$

It remains to compare this expression with $Q^u_\varepsilon(k,\tau) \approx e^{-ib(k)\tau}\,\hat u^\varepsilon_0(k)$ from (4.6). With this aim we write the above integral in the form

$$Q^v_\varepsilon(k,\tau) = z_0^{-1}\int_{S^1} \Phi_\varepsilon(q;k,\tau)\,\varphi^\varepsilon(q;k,\tau)\,dS(q) + e^{ic|k|\tau/\varepsilon^2} G_\varepsilon(k,\tau) \tag{5.8}$$

with the two functions

$$\Phi_\varepsilon(q;k,\tau) := e^{i(|k|-k\cdot q)c\tau/\varepsilon^2}\, W_\rho(k\cdot q)\frac{\sqrt{c\tau}}{\varepsilon}\mathbf{1}_{\{k\cdot q>0\}}\,, \tag{5.9}$$

$$\varphi^\varepsilon(q;k,\tau) := e^{-ib((q\cdot k)\,q)\tau}\,\hat u^\varepsilon_0((k\cdot q)q)\,. \tag{5.10}$$

Loosely speaking, we will verify the following: If we consider $\Phi_\varepsilon(.;k,\tau)$ as a function in $q\in S^1$, this sequence is a Dirac sequence for the point $q_0 = k/|k|$. As a result, $Q^v_\varepsilon(k,\tau)$ from (5.8) can be compared with $\varphi^\varepsilon(q_0;k,\tau)$, which is essentially $Q^u_\varepsilon(k,\tau)$ from (4.6).

Since we have to evaluate the above expression for fixed k, it is no restriction to consider only a wave vector in the first coordinate direction, $k = k_1 e_1$. We abbreviate the subsequent calculation by setting $\eta = \varepsilon^2/(k_1 c\tau)$.

LEMMA 5.2 (Dirac sequence for $d=2$). *We consider $k_1 \ge \rho$ such that $W_\rho(k_1) = \sqrt{k_1}$. On the 1-sphere $S^1\subset\mathbb{R}^2\equiv\mathbb{C}$, points are denoted by $q=(q_1,q_2)\in S^1$, and the measure of integration is $dS(q) = d\mathcal{H}^1(q)$. We consider the following sequence of functions $\Phi^\eta(.): S^1\to\mathbb{C}$:*

$$\Phi^\eta(q) = e^{i(1-q_1)/\eta}\,\frac{1}{\sqrt{\eta}}\frac{W_\rho(k_1q_1)}{\sqrt{k_1}}\,\mathbf{1}_{\{q_1>0\}}\,. \tag{5.11}$$

Then there holds

$$\Phi^\eta(.) \to z_0\,\delta_{e_1}(.) \quad \textit{with} \quad z_0 = \sqrt{\pi}(1+i) \tag{5.12}$$

in the sense of distributions on S^1 as $\eta\to 0$. Convergence even holds for less regular test functions: For every test function $\varphi: S^1\to\mathbb{R}$ of class C^2 it holds that

$$\int_{S^1}\Phi^\eta(q)\,\varphi(q)\,dS(q) \to z_0\varphi(e_1)\,. \tag{5.13}$$

Lemma 5.2 is inspired by the classical stationary phase approximation. For example, let $\varepsilon>0$ be positive and let $f\in C^\infty((\vartheta_0-\varepsilon,\vartheta_0+\varepsilon))$ be a function with the properties $f(\vartheta_0)=0$, $f'(\vartheta)=0$ iff $\vartheta=\vartheta_0$, and $f''(\vartheta_0)\neq 0$. In this situation, for any $g\in C^\infty((\vartheta_0-\varepsilon,\vartheta_0+\varepsilon))$, it holds that

$$\lim_{\eta\to 0}\int_{\vartheta_0-\varepsilon}^{\vartheta_0+\varepsilon} e^{if(\vartheta)/\eta}\,\frac{1}{\sqrt{\eta}}\,g(\vartheta)\,d\vartheta = e^{i\pi/4}\sqrt{\frac{2\pi}{f''(\vartheta_0)}}\,g(\vartheta_0)\,; \tag{5.14}$$

see, e.g., section II.3 of [18]. Equation (5.14) can be applied to the formula at the end of Step 1 of the subsequent proof, and it delivers the claim of the lemma for C^∞-functions. In order to keep the presentation self-contained and reduce the differentiability assumptions, we present an independent proof.

Proof. Step 1: *Simplification.* Let $\varphi: S^1\to\mathbb{R}$ be twice continuously differentiable. We have to show (5.13). We parametrize the sphere with the map

$(-\pi,\pi) \ni \theta \mapsto q = e^{i\theta} \in S^1$, making use of the natural identification $\mathbb{C} \equiv \mathbb{R}^2$. In particular, there holds $q_1 = \cos(\theta)$.

To abbreviate calculations, we modify the test function and set

$$\tilde\varphi(\theta) := \frac{W_\rho(k_1 q_1)}{\sqrt{k_1}}\varphi(q) = \frac{W_\rho(k_1\cos(\theta))}{\sqrt{k_1}}\varphi(e^{i\theta}) \quad \text{and} \quad \bar\varphi(\theta) := \tilde\varphi(\theta) + \tilde\varphi(-\theta)\,.$$

The symmetrized variant $\bar\varphi$ allows us to consider only integrals over $\theta \in (0,\pi/2)$. The expression of interest now reads

$$\begin{aligned}\int_{S^1} \Phi^\eta(q)\,\varphi(q)\,dS(q) &= \int_{-\pi/2}^{\pi/2} e^{i(1-\cos(\theta))/\eta}\,\frac{1}{\sqrt{\eta}}\,\tilde\varphi(\theta)\,d\theta \\ &= \int_0^{\pi/2} e^{i(1-\cos(\theta))/\eta}\,\frac{1}{\sqrt{\eta}}\,\bar\varphi(\theta)\,d\theta\,.\end{aligned}$$

Step 2: *Decomposition.* We first calculate the contribution from the integration away from $\theta = 0$ (away from the point $q = e_1 \in S^1$). For arbitrary $\delta > 0$, we use a trivial extension of the integrand in order to recognize one term as a derivative:

$$\begin{aligned}\int_\delta^{\pi/2} e^{i(1-\cos(\theta))/\eta}\,\frac{1}{\sqrt{\eta}}\,\bar\varphi(\theta)\,d\theta &= \int_\delta^{\pi/2} e^{i(1-\cos(\theta))/\eta}\,\frac{i\sin(\theta)}{\eta}\,\sqrt{\eta}\,\frac{\bar\varphi(\theta)}{i\sin(\theta)}\,d\theta \\ &= \int_\delta^{\pi/2} \partial_\theta\left[e^{i(1-\cos(\theta))/\eta}\right]\,\sqrt{\eta}\,\frac{\bar\varphi(\theta)}{i\sin(\theta)}\,d\theta = \sqrt{\eta}\,O(1/\delta^2)\,. \qquad (5.15)\end{aligned}$$

In the last step we performed an integration by parts and the estimate $|\partial_\theta(1/\sin(\theta))| \le 1/\sin^2(\theta) \le C/\delta^2$. Our choice of $\delta > 0$ will ensure smallness of the error term in (5.15).

It remains to investigate the integral over small values of θ. We calculate with a Taylor expansion of $\bar\varphi$ in $\theta = 0$

$$\int_0^\delta e^{i(1-\cos(\theta))/\eta}\,\frac{1}{\sqrt{\eta}}\,\bar\varphi(\theta)\,d\theta = \bar\varphi(0)\int_0^\delta e^{i(1-\cos(\theta))/\eta}\,\frac{1}{\sqrt{\eta}}\,d\theta + \frac{O(\delta^3)}{\sqrt{\eta}}\,. \qquad (5.16)$$

In the estimate for the error we used that $\bar\varphi$ is symmetric, which provides $\bar\varphi'(0) = 0$, and that the interval of integration has length δ.

In view of (5.15) and (5.16) we choose $\delta := \eta^{1/5}$. With this choice, there hold $\sqrt{\eta}/\delta^2 = \eta^{1/2}\eta^{-2/5} = \eta^{1/10}$ and $\delta^3/\sqrt{\eta} = \eta^{3/5}\eta^{-1/2} = \eta^{1/10}$. This shows that the error terms in (5.15) and (5.16) are both of order $\eta^{1/10}$.

Step 3: *Limiting integral.* We note that the factor in front of the integral in (5.16) is $\bar\varphi(0) = 2\tilde\varphi(0) = 2W_\rho(k_1)/\sqrt{k_1}\varphi(e_1) = 2\varphi(e_1)$. It therefore only remains to evaluate the limit of the integral

$$I_\eta := \int_0^\delta e^{i(1-\cos(\theta))/\eta}\,\frac{1}{\sqrt{\eta}}\,d\theta\,. \qquad (5.17)$$

The integral can be written with the substitution $z = (1-\cos(\theta))/\eta$, leading to $d\theta = \eta/\sin(\theta)\,dz$ with the inverse function $\theta = \theta(z) := (1-\cos)^{-1}(\eta z) = \sqrt{2z\eta} + O((z\eta)^{3/2})$. We find

$$I_\eta = \int_0^{(1-\cos(\delta))/\eta} e^{iz}\,\frac{\sqrt{\eta}}{\sin(\theta)}\,dz\,. \qquad (5.18)$$

Regarding the domain of integration we find $R(\eta) := (1-\cos(\delta))/\eta \approx \frac{1}{2}\delta^2/\eta = \frac{1}{2}\eta^{-3/5} \to \infty$ as $\eta \to 0$. On the other hand, all values of z in the domain of integration satisfy $z\eta \le C\eta^{2/5}$, and hence $\theta = \sqrt{2z\eta} + O((z\eta)^{3/2}) \le C\eta^{1/5}$. We may therefore develop $\sin(\theta)$,

$$\sin(\theta) = \theta + O(\theta^3) = \sqrt{2z\eta} + O((z\eta)^{3/2})\,.$$

We find, as $\eta \to 0$,

$$\begin{aligned} I_\eta &= \int_0^{R(\eta)} e^{iz} \frac{1}{\sin(\theta)/\sqrt{\eta}}\,dz = \frac{1}{\sqrt{2}}\int_0^{R(\eta)} e^{iz}\frac{1}{\sqrt{z}+O(z^{3/2}\eta)}\,dz \\ &\to \frac{1}{\sqrt{2}}\int_0^\infty e^{iz}\frac{1}{\sqrt{z}}\,dz = \sqrt{2}\int_0^\infty e^{ip^2}\,dp = \frac{1}{2}\sqrt{\pi}(1+i) = \frac{1}{2}z_0\,. \end{aligned} \tag{5.19}$$

In the last line we used the substitution $z = p^2$ and Fresnel integrals: For real and imaginary parts there holds $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \sqrt{\pi}/(2\sqrt{2})$. Regarding the convergence of the integrals, for fixed $z \le R(\eta) \le C\eta^{-3/5}$, there holds $\frac{1}{\sqrt{z}+O(z^{3/2}\eta)} \to \frac{1}{\sqrt{z}}$ as $\eta \to 0$, and we therefore have pointwise convergence of the integrands. For large values of z we expand the fraction and find $1/(\sqrt{z}+O(z^{3/2}\eta)) = 1/\sqrt{z} + O(z^{1/2}\eta)$. Because of $\int_0^{R(\eta)} z^{1/2}\eta = O(R(\eta)^{3/2}\eta) = O(\eta^{-9/10}\eta) = O(\eta^{1/10})$, the integral of the error term is small. This justifies the limit in (5.19) and provides the claim of (5.12). ∎

We collect our results in the following theorem on pointwise convergence.

THEOREM 5.3 (pointwise convergence of the Fourier transforms). *We assume that the coefficients $(a_j)_j$ and the initial values $(u_0^\varepsilon, u_1^\varepsilon)$ satisfy Assumption 2.1, that $\hat{u}_0^\varepsilon = \hat{u}_0$ of class $C^2(\mathbb{R}^d)$ and ε-independent, and that $\hat{u}_1^\varepsilon(.) = -i(\omega_l(\varepsilon.)/\varepsilon)\,\hat{u}_0^\varepsilon(.)$ on $\mathbb{R}^2$. Let $u^\varepsilon(.,t)$ be the solution to the lattice equation (1.1).*

Let V^ε be the solution to the KdV equation (4.1) with initial values (5.3) for some parameter $\rho > 0$. We assume that V^ε satisfies the decay estimate (5.5). Let $v^\varepsilon(.,t)$ be given by the reconstruction formula (4.3).

Then there holds

$$|\hat{u}^\varepsilon(k,\tau/\varepsilon^2) - \hat{v}^\varepsilon(k,\tau/\varepsilon^2)| \to 0 \tag{5.20}$$

as $\varepsilon \to 0$ for every $\tau \in (0,T]$ and for every $k \in \mathbb{R}^2$ with $|k| \ge \rho$.

The decay estimate (5.5) is satisfied if we demand the regularity $\hat{u}_0 \in C^3(\mathbb{R}^d)$; see Remark 5.4 below.

Proof. We first compare the two expressions $Q_\varepsilon^u(k,\tau)$ and $Q_\varepsilon^v(k,\tau)$ of (4.5). We have obtained a simplified expression for $Q_\varepsilon^u(k,\tau)$ in (4.6), which implies that, for $\varepsilon \to 0$,

$$Q_\varepsilon^u(k,\tau) \to e^{-ib(k)\tau}\,\hat{u}_0(k)\,. \tag{5.21}$$

For $Q_\varepsilon^v(k,\tau)$, we calculated a simplification in (5.8). The error term G_ε was estimated in Lemma 5.1 using (5.5). Lemma 5.2 can be used to calculate the limit of the integral in (5.8), since $\varphi^\varepsilon = \varphi$ is C^2 and ε-independent. We emphasize that continuity of $\hat{u}_0^\varepsilon = \hat{u}_0$ is exploited here. We find, with $q_0 = k/|k|$,

$$Q_\varepsilon^v(k,\tau) \to \varphi(q_0;k,\tau) = e^{-ib((k\cdot q_0)\,q_0)\tau}\,\hat{u}_0((k\cdot q_0)q_0) = e^{-ib(k)\tau}\,\hat{u}_0(k)\,. \tag{5.22}$$

The limits in (5.21) and (5.22) are identical.

This implies also the pointwise convergence of the Fourier transforms of the two solutions:

$$|\hat{u}^\varepsilon(k,\tau/\varepsilon^2) - \hat{v}^\varepsilon(k,\tau/\varepsilon^2)| = |e^{ic|k|\tau/\varepsilon^2}\ \hat{u}^\varepsilon(k,\tau/\varepsilon^2) - e^{ic|k|\tau/\varepsilon^2}\ \hat{v}^\varepsilon(k,\tau/\varepsilon^2)|$$
$$= |Q^u_\varepsilon(k,\tau) - Q^v_\varepsilon(k,\tau)| \to 0\,.$$

This was the claim in (5.20). ∎

Remark 5.4. The assumptions on the initial data and on the decay of V^ε are related. We assume $\hat{u}_0 \in C^2_c(\mathbb{R}^d)$ in Theorem 5.3. This implies $\hat{V}^\varepsilon_0 \in C^2_c(\mathbb{R}^d \times S^1)$ for the initial values (5.3) of the linearized KdV equation. The solution formula (4.17) implies $\hat{V}^\varepsilon_0 \in C^2_c(\mathbb{R}^d \times [0,T] \times S^1)$. Since the inverse Fourier transform relates smoothness to decay properties, we find

$$|\xi|^2 \hat{V}^\varepsilon_0(\xi,\tau,q) \in H^1(\mathbb{R}, d\xi)\,. \tag{5.23}$$

We see that if we demand $\hat{u}_0 \in C^3_c(\mathbb{R}^d)$ in Theorem 5.3, the spatial decay property (5.5) of V^ε is guaranteed.

5.2. On strong convergence. The ultimate goal in the analysis of ring solutions in two dimensions is to show the strong convergence $u^\varepsilon - v^\varepsilon \to 0$ in $L^2(\mathbb{R}^2)$. This is a very challenging task.

Of course, since the Fourier transform is an isometry in L^2, the strong convergence in physical space can follow from strong convergence in Fourier space, $\hat{u}^\varepsilon - \hat{v}^\varepsilon \to 0$ in $L^2(\mathbb{R}^2)$. We recall that we have the pointwise convergence of (5.20) at our disposal. The main difficulty in proving the strong convergence is the behavior of the Fourier transform $\hat{v}^\varepsilon$ in a neighborhood of $k = 0$. We recall that the pointwise convergence of (5.20) is only valid for $|k| > \rho$. Indeed, since we have chosen a regularized form for V^ε_0, modifying the function for $|k| \le \rho$, we cannot expect any better pointwise convergence. The best result that we could obtain is therefore

$$\limsup_{\varepsilon\to 0} \|\hat{u}^\varepsilon(.,\tau/\varepsilon^2) - \hat{v}^\varepsilon(.,\tau/\varepsilon^2)\|_{L^2(\mathbb{R}^2)} \le h(\rho)\,, \tag{5.24}$$

where $h : [0,1) \to [0,\infty)$ is some function with $h(\rho) \to 0$ for $\rho \to 0$.

Comments on the convergence (5.24). We assume once more that $\hat{u}^\varepsilon_0 = \hat{u}_0$ is a smooth function with bounded support. Regarding large values of $|k|$ we note that the smoothness of V_0 implies differentiability properties of V^ε and hence of v^ε. These, in turn, imply decay properties of $\hat{v}^\varepsilon$. Since, on the other hand, $\hat{u}^\varepsilon$ has bounded support, we conclude that $|\hat{u}^\varepsilon(k,\tau/\varepsilon^2) - \hat{v}^\varepsilon(k,\tau/\varepsilon^2)|$ also has decay properties for $|k| \to \infty$.

An inspection of the proofs reveals that the convergence is also uniform on compact subsets of $k \in \mathbb{R}^2 \setminus B_\rho(0)$. All these considerations suggest that (5.24) holds at least in $L^2(\mathbb{R}^2 \setminus B_\rho(0))$.

The behavior of $\hat{v}^\varepsilon$ for small values of $|k|$ is much more intricate; a first hint of this fact is that for ring solutions we necessarily have $\hat{v}^\varepsilon(0) = \int_{\mathbb{R}^2} v^\varepsilon(x)\,dx \to \infty$ as $\varepsilon \to 0$. Nevertheless, a positive result can be expected, as the subsequent lemma suggests: The function w^ε is constructed as a ring solution, essentially as v^ε was constructed in (4.3).

LEMMA 5.5 (Fourier transform of a ring function near $k = 0$). *From a function $V \in C^\infty_c(\mathbb{R};\mathbb{R})$ we define $w^\varepsilon(x)$, $x \in \mathbb{R}^2$, by*

$$w^\varepsilon(x) := \frac{1}{|x|^{1/2}} V\left(|x| - \varepsilon^{-2}\right)\,. \tag{5.25}$$

Then there exist constants $\rho_0, \varepsilon_0, C > 0$ such that

$$\int_{B_\rho(0)} |\hat{w}^\varepsilon(k)|^2 \, dk \le C\rho + O(\varepsilon^2) \tag{5.26}$$

for every $\rho \in (0, \rho_0)$ and every $\varepsilon \in (0, \varepsilon_0)$.

Proof. We evaluate the Fourier transformation in polar coordinates and perform a calculation as in (4.18).

$$\begin{aligned}
\hat{w}^\varepsilon(k) &= \int_{\mathbb{R}^2} e^{-ik\cdot x} w^\varepsilon(x)\, dx \\
&= \int_{S^1} \int_0^\infty e^{-ik\cdot qr}\, w^\varepsilon(rq)\, r\, dr\, dS(q) \\
&= \int_{S^1} \int_0^\infty e^{-ik\cdot qr}\, V\left(r - \varepsilon^{-2}\right) r^{1/2}\, dr\, dS(q) \\
&= \int_{S^1} \int_{-\infty}^\infty e^{-ik\cdot q/\varepsilon^2} e^{-ik\cdot qz}\, V(z)\, \mathbf{1}_{\{z \ge -\varepsilon^{-2}\}} (\varepsilon^{-2} + z)^{1/2}\, dz\, dS(q) \\
&= \int_{S^1} e^{-ik\cdot q/\varepsilon^2} \int_{-\infty}^\infty e^{-ik\cdot qz}\, V(z)\, \frac{1}{\varepsilon}\, dz\, dS(q) + G_\varepsilon(k) \\
&= e^{-i|k|/\varepsilon^2} \int_{S^1} e^{i|k|/\varepsilon^2} e^{-ik\cdot q/\varepsilon^2} \hat{V}(k\cdot q)\, \frac{1}{\varepsilon}\, dS(q) + G_\varepsilon(k)\,.
\end{aligned}$$

Since the support of V is bounded, we have $|(\varepsilon^{-2}+z)^{1/2} - \varepsilon^{-1}| = O(\varepsilon)$ for all relevant values of z; this implies $G_\varepsilon = O(\varepsilon)$. The remaining integral of the last line is treated as in Lemma 5.2: Without loss of generality we consider $k = k_1 e_1$. With $\eta := \varepsilon^2/|k_1|$ we write

$$\begin{aligned}
\int_{S^1} e^{i|k|/\varepsilon^2} e^{-ik\cdot q/\varepsilon^2} \hat{V}(k\cdot q)\, \frac{1}{\varepsilon}\, dS(q) &= \int_{S^1} e^{i(1-q_1)k_1/\varepsilon^2} \hat{V}(k_1 q_1)\, \frac{1}{\varepsilon}\, dS(q) \\
&= \frac{1}{\sqrt{|k_1|}} \int_{S^1} e^{i(1-q_1)/\eta} \hat{V}(k_1 q_1)\, \frac{1}{\sqrt{\eta}}\, dS(q) \le \frac{C}{\sqrt{|k|}}\,,
\end{aligned}$$

where the last step follows with the calculations of Lemma 5.2 by uniform continuity of $\hat{V}$. We can therefore integrate the squared Fourier transform of w^ε,

$$\begin{aligned}
\int_{B_\rho(0)} |\hat{w}^\varepsilon(k)|^2\, dk &\le \int_{B_\rho(0)} \left(\frac{C}{\sqrt{|k|}}\right)^2 dk + O(\varepsilon^2) \\
&= \int_{B_\rho(0)} \frac{C^2}{|k|}\, dk + O(\varepsilon^2) = 2\pi\rho C^2 + O(\varepsilon^2)\,.
\end{aligned}$$

This implies (5.26). ∎

The estimate (5.26) for $\hat{v}^\varepsilon$ could be the key step in the derivation of (5.24): The k-values with small norm do not contribute much. The difficulty in deriving (5.26) for $\hat{v}^\varepsilon$ is that the cut-off parameter $\rho > 0$ enters in the construction of $\hat{V}^\varepsilon$ and hence in the decay properties of V^ε. For this reason our analysis ends with the result of pointwise convergence in Fourier space.

6. Numerical tests.

One space dimension. We test the validity of the linearized KdV equation for initial data $u_0(x) := u_0^\varepsilon(x) := e^{-|x|^2}$, $x \in \mathbb{R}$. The lattice model uses the three point discrete Laplacian, $a_1 = a_{-1} = 1$, $a_0 = -2$, and $a_j = 0$ for every j with $|j| > 1$. All calculations are performed with MATLAB.

Lattice model. The initial data u_1^ε are determined by the fact that we require $\psi_{-,0}^{u,\varepsilon} = 0$, or, equivalently, $\hat{u}_1^\varepsilon(.) = -i\omega_l(\varepsilon.)\varepsilon^{-1}\,\hat{u}_0^\varepsilon(.)$. The calculation of u_1 simplifies considerably in our case: u_0^ε is real and symmetric (invariant under $x \mapsto -x$), which implies that also $\hat{u}_0^\varepsilon$ is real and symmetric (invariant under $\xi \mapsto -\xi$). By its definition, $\hat{u}_1^\varepsilon$ is imaginary and symmetric, and hence u_1^ε is imaginary and symmetric. Since we are interested in the real part of the lattice solution, we can perform all calculations with the real parts of the initial data, i.e., with $u_0 = u_0^\varepsilon$ and $u_1 = 0$.

The lattice equations are solved for $\varepsilon = 1/6$ on the truncated domain $x \in (-100, 100)$ with the time interval $t \in [0, t_0]$, $t_0 = 80$. The homogenized wave speed is $c = 1$ such that the main pulses of $u(., t_0)$ are located near $x = \pm 80$. The part of the solution with $x \in (72, 88)$ is shown in the left panel of Figure 1.

Linearized KdV equation. On the other hand, we have solved numerically the linearized KdV equation. The initial values are determined by the definition of $\hat{V}_0^\varepsilon$ in (4.8). We use this definition, the formula for the inverse Fourier transform, and symmetry of $\hat{u}_0^\varepsilon$ to obtain

$$u_0^\varepsilon(x) = \frac{1}{2\pi}\int_{\{\xi>0\}} e^{i\xi x}\hat{V}_0^\varepsilon(\xi,1)\,d\xi + \frac{1}{2\pi}\int_{\{\xi<0\}} e^{i\xi x}\hat{V}_0^\varepsilon(-\xi,1)\,d\xi = 2\mathrm{Re}(V_0^\varepsilon(x,1))\,.$$

The result is that we must use the real initial values $V_0 = \frac{1}{2}u_0$ for the linearized KdV equation. The factor $b(q)$ of the linearized KdV equation is determined from the Taylor expansion of the dispersion relation as $b(q) = -\frac{1}{24}$ for $q \in \{\pm 1\}$. The numerical results for $\varepsilon = 1/6$ on the truncated domain $x \in (-8, 8)$ are shown in the right panel of Figure 1.

Two space dimensions. The two-dimensional tests are performed for $u_0(x) := u_0^\varepsilon(x) := e^{-|x|^2}$, $x \in \mathbb{R}^2$, and $\varepsilon = 1/6$. The lattice model uses the five-point discrete Laplacian, $a_{(1,0)} = a_{(-1,0)} = a_{(0,1)} = a_{(0,-1)} = 1$, $a_0 = -4$, and $a_j = 0$ for every $j \in \mathbb{Z}^2$ with $|j| > 1$. The dispersion relation for this lattice model is

$$\begin{aligned}\omega_l^2(k) &= -\sum_j a_j e^{ik\cdot j} = 4 - e^{ik_1} - e^{-ik_1} - e^{ik_2} - e^{-ik_2} = 4 - 2\cos(k_1) - 2\cos(k_2)\\ &\approx |k_1|^2 + |k_2|^2 - \frac{1}{12}(k_1^4 + k_2^4) = |k|^2 - \frac{1}{12}(k_1^4 + k_2^4)\,,\end{aligned}$$

where we approximated with a Taylor expansion. For the square root we find

$$\omega_l(k) \approx |k| - \frac{1}{24|k|}(k_1^4 + k_2^4)\,.$$

This determines c and b through (4.2). We find $c = 1$ and, for the two directions $q = e_1$ and $q = e_2$, the values $b(e_1) = b(e_2) = -1/24$.

We next calculate an approximation of $\hat{u}_0^\varepsilon$ by replacing the discrete Fourier transform with the continuous transform. Our choice of constants in the definition of the Fourier transform suggests we use

$$\hat{u}_0(\xi) = \int_{\mathbb{R}^2} e^{-|x|^2} e^{-ix\cdot\xi}\,dx = \pi e^{-|\xi|^2/4}\,. \tag{6.1}$$

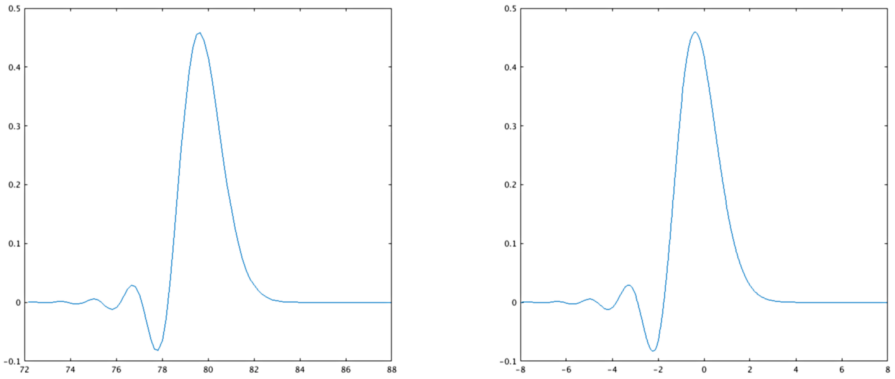

FIG. 1. One-dimensional model. *Left: The solution to the lattice equations with* $\varepsilon = 1/6$ *at time* $t_0 = 80$. *We show a zoom-in of the right-going wave-pulse. The solution is calculated with an explicit time discretization with time step size* $1 \cdot 10^{-5}$. *Right: Solution to the corresponding linearized KdV equation with* $b = -1/24$ *and initial data* $V_0(x) = \frac{1}{2}u_0(x) = \frac{1}{2}e^{-|x|^2}$ *at time* $t_0\varepsilon^2$. *We used an explicit Euler scheme with time step size* $4 \cdot 10^{-7}$.

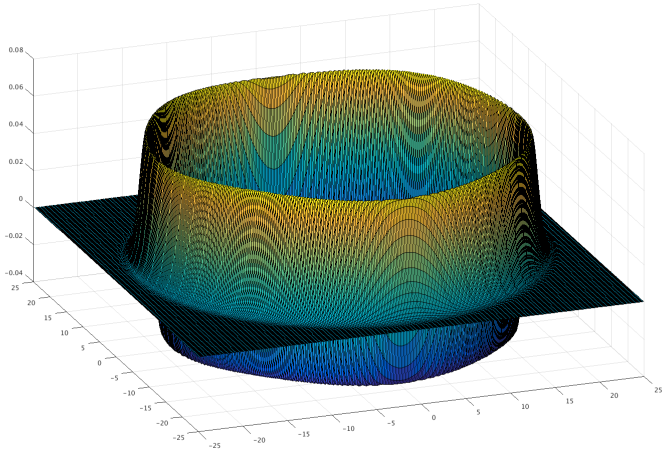

FIG. 2. *The two-dimensional solution* $x \mapsto u(x, t_0)$ *of the lattice equations for* $\varepsilon = 1/6$, $x \in (-25, 25)^2$, $t_0 = 20$, $u_0(x) = e^{-|x|^2}$. *It is calculated with an explicit Euler scheme for the original equations* (1.1) *with time step size* $5 \cdot 10^{-5}$. *We see that the solution has the shape of a ring.*

Lattice model. The initial data u_1^ε are determined by the fact that we require $\psi_{-,0}^{u,\varepsilon} = 0$, or, equivalently, $\hat{u}_1^\varepsilon(.) = -i\omega_l(\varepsilon.)\varepsilon^{-1}\,\hat{u}_0^\varepsilon(.)$. As in the one-dimensional case, the Fourier transform of u_1^ε is symmetric and purely imaginary, hence the same holds for u_1^ε. Since we calculate the solution to the lattice equations for real functions u_0 and u_1, we can use the homogeneous initial values $u_1 = 0$.

We present here two different numerical solutions for the two-dimensional lattice equations. Figure 2 shows the result for $\varepsilon = 1/6$, $t \in [0, t_0]$ with $t_0 = 20$, $x \in (-25, 25)^2$. Figures 3 and 4 show details from a calculation on a larger domain, $\varepsilon = 1/6$, $t_0 = 80$, $x \in (-120, 120)^2$. The homogenized wave speed is $c = 1$ such that

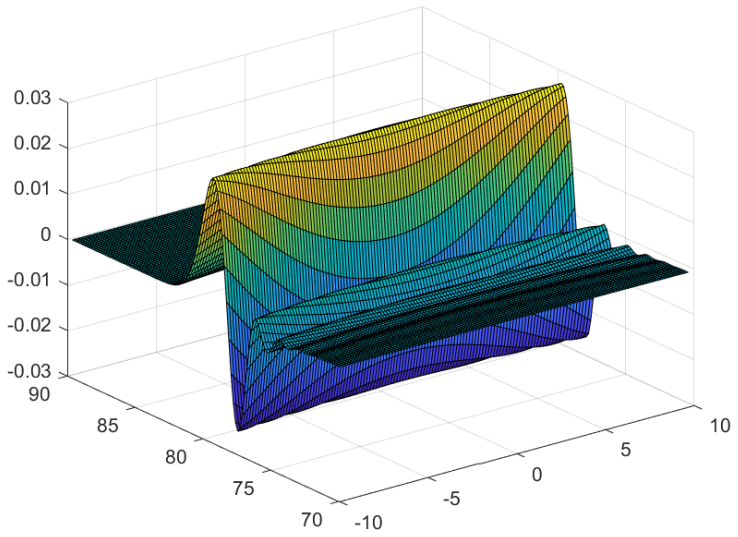

FIG. 3. *A detail of the two-dimensional solution $x \mapsto u(x, t_0)$ of the lattice equations for $\varepsilon = 1/6$, $t_0 = 80$, solved for $x \in (-120, 120)^2$ and shown for $x \in (70, 90) \times (-10, 10)$, $u_0(x) = e^{-|x|^2}$. We show a zoom into the front to make the profile visible. The calculation is based on the representation formula* (2.7) *and periodic solutions with periodicity cell of* 1441 *by* 1441.

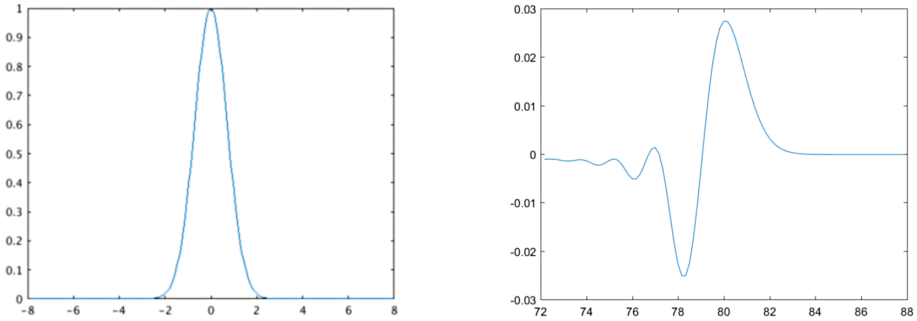

FIG. 4. Lattice solution. *Two-dimensional lattice model for $\varepsilon = 1/6$, solutions are shown along the line $\{(x_1, 0) | x_1 \in \mathbb{R}\}$. Left: Initial values u_0; plotted is the function $x_1 \mapsto u_0((x_1, 0))$. Right: The solution of the lattice model at time $t_0 = 80$; plotted is $x_1 \mapsto u((x_1, 0), t_0)$. The data are the same as for Figure* 3.

the main pulse of $u(., t_0)$ is near $|x| = t_0$. While Figure 2 is obtained from a calculation of the original lattice equation (1.1), we calculated Figure 3 from the representation formula (2.7) with periodic boundary conditions. The calculation in Fourier space allows us to calculate longer time intervals. In particular, we can observe at time $t_0 = 80$ the dispersive effects in two space dimensions; they are not yet visible for $t_0 = 20$ of Figure 2.

Linearized KdV equation. We use (6.1) to evaluate $\hat{V}_0$ from (5.1). For every $q \in S^1$ and $\xi > 0$ we find

$$\hat{V}_0(\xi; q) = (\sqrt{\pi}(1+i))^{-1}\sqrt{\xi}\,\hat{u}_0(\xi q) = \frac{1-i}{2\sqrt{\pi}}\sqrt{\xi}\,\hat{u}_0(\xi q) = \frac{1}{2}(1-i)\sqrt{\pi\xi}\,e^{-|\xi|^2/4}\,.$$

The real initial data for the linearized KdV equation are obtained by the one-dimensional inverse Fourier transform of the above function for arbitrary q. We obtain

$$V_0(x; q) = \frac{1}{4\sqrt{\pi}}\mathrm{Re}\int_{\{\xi>0\}}(1-i)\sqrt{\xi}\,e^{-|\xi|^2/4}\,e^{i\xi x}\,d\xi\,.$$

The left panel of Figure 5 shows the numerically determined initial values $x \mapsto V_0(x; q)$ for $q = e_1$. The right panel of Figure 5 shows the numerically obtained solution $V(., \tau_0; e_1)$ for $\tau_0 = t_0\varepsilon^2$. The evolution equation is solved for $q = e_1$, using $b(e_1) = -1/24$ as coefficient of the third order term and time step size $1 \cdot 10^{-7}$.

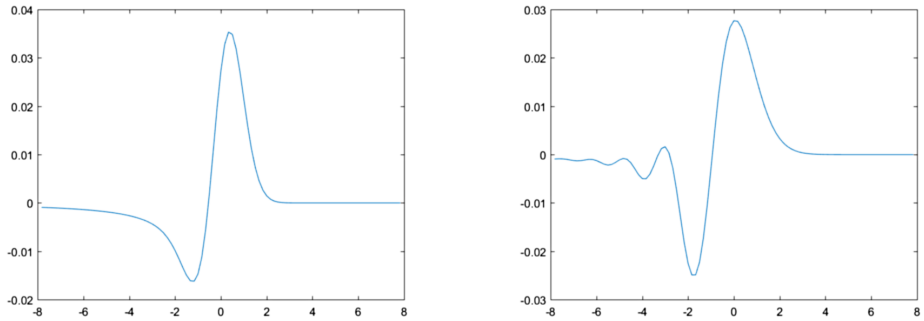

FIG. 5. KdV solution. *Two-dimensional model approximated with a one-dimensional KdV equation. Left: The initial values V_0, obtained from the Fourier transform of u_0; in the numerics we used a discretization width $2 \cdot 10^{-2}$. We plot $V_0/\sqrt{r_0}$ for the radius $r_0 = ct_0 = t_0 = 80$ in order to match the reconstruction formula* (4.3). *Right: The solution of the linearized KdV equation at time $t = t_0\varepsilon^2$, also divided by $\sqrt{r_0}$. We calculated with an explicit solver, using a time step size of $1 \cdot 10^{-7}$. The KdV solution matches almost perfectly the two-dimensional lattice solution; see the right panel of Figure* 4. *The above comparison makes clear that the initial wave trough results from the initial condition, not from dispersion.*

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