Thermodynamic Formalism and Dimension gaps

by

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Declarations

The results given in chapter 6 of this thesis were obtained in collaboration with Jonathan Fraser and Thomas Jordan, and appear in the paper [FJJ] which is currently available as a preprint on the arXiv.

Except as noted above, I declare that the material in this thesis is my own except where otherwise indicated or cited in the text. This material has not been submitted for any other degree or qualification.

Abstract

Given an expanding Markov map $T : [0, 1] \to [0, 1]$ which admits an absolutely continuous invariant probability measure, we say that T gives rise to a dimension gap if there exists some $c > 0$ for which $\sup_{\mathbf{p}} \dim \mu_{\mathbf{p}} \leq 1 - c$, where $\mu_{\mathbf{p}}$ denotes the Bernoulli measure associated to the probability vector $\mathbf{p}$. We prove that under a 'non-linearity condition' on T, there is a dimension gap.

Our approach differs considerably to the approach of Kifer, Peres and Weiss in [KPW], who proved a similar result. The first part of our proof involves obtaining uniform lower estimates on the asymptotic variance of a class of potentials. Tools from the thermodynamic formalism of the countable shift play a key role in this part of the proof. The second part of our proof revolves around a 'mass redistribution' technique.

We also study a class of 'Käenmäki measures' which are supported on self-affine sets generated by a finite collection of diagonal and anti-diagonal matrices acting on the plane. We prove that such a measure is exact-dimensional and that its dimension satisfies a Ledrappier-Young formula. This is similar to the recent results of Bárány and Käenmäki [BK], who proved an analogous result for quasi-Bernoulli measures. While the measures we consider are not quasi-Bernoulli, which takes us out of the scope of [BK], we show that the measures can be written in terms of two quasi-Bernoulli measures on an associated subshift and use this to prove the result.

Chapter 1

Introduction

In this thesis we study the dimension of measures which appear in various dynamical settings. A common property of all the measures that are studied is that each one can be realised as the 'projection' of some measure which is defined on a symbolic space. Therefore, symbolic dynamics serves as an important model throughout the thesis. Chapters 3-5 are concerned with a 'dimension gap' problem for countable branch expanding maps of the interval. In chapter 6 we prove the exact-dimensionality of a measure which is supported on a 'self-affine set', and investigate its underlying structure. Paramount to this thesis is the use of tools from thermodynamic formalism to study the dimension of our measures.

In Chapter 2 we provide some preliminaries for the thesis. We begin by presenting some fundamental notions and results from ergodic theory, dynamical systems, measure theory and dimension theory. Next we introduce the notion of an iterated function system, which is an important tool for constructing fractal sets. In particular, we direct our attention to self-affine sets, which are one such class of fractal set which can be produced using this construction. The final part of our preliminaries is dedicated to the thermodynamic formalism of the shift map in three different settings: Hölder continuous potentials on the finite shift space, sub-additive potentials on the finite shift space and locally Hölder potentials on the countable shift space. The tools which are provided by thermodynamic formalism will be fundamental to the thesis, both to verify the existence of objects which we wish to study (such as Gibbs measures) and also to translate problems which appear in dynamics to the language of thermodynamic formalism, at which point we can then apply the extensive machinery which has been built up throughout the literature over the last fifty years. Varying amounts of detail are given within each account of the thermodynamic formalism, which reflects the depth of the theory that will be required from each setting.

In Chapters 3-5 we consider a 'dimension gap' problem, which forms the bulk of this thesis. Let $T : [0,1] \to [0,1]$ be an expanding Markov map. Under some regularity conditions, it is known that T has a unique absolutely continuous invariant probability measure μ_T. Under these conditions, we say that T gives rise to a *dimension gap* if for some $c > 0$

$$\sup_{\mathbf{p} \in \mathcal{P}} \dim \mu_{\mathbf{p}} \leqslant 1 - c \tag{1.1}$$

where $\mathcal{P}$ denotes the simplex of all probability vectors, $\mu_{\mathbf{p}}$ denotes the Bernoulli measure for the probability vector $\mathbf{p} \in \mathcal{P}$ (which has been projected to the real line in the usual way) and dim denotes Hausdorff dimension. Therefore 'dimension gap' is meant in the sense that there does not exist a Bernoulli measure of dimension greater than $1 - c$, where we notice that 1 is the dimension of μ_T.

We are interested in understanding the underlying geometric cause of the dimension gap. It is easy to show that in the case where T has finitely many branches, T has a dimension gap if and only if the absolutely continuous measure is not a Bernoulli measure. We are primarily interested in the countable branch analogue of this problem, where the facts that the dimension function is not necessarily upper semi-continuous and the set of Bernoulli measures is not compact pose difficulties which are not encountered in the finite branch case. It was shown by Kifer, Peres and Weiss [KPW] in 2001 that under some regularity conditions on T, (1.1) holds if and only if μ_T is not Bernoulli, where c is a constant which can be made explicit as long as the absolutely continuous measure μ_T is known. In particular, they showed that for the Gauss map $G(x) = \frac{1}{x} \mod 1$,

$$\sup_{\mathbf{p} \in \mathcal{P}} \dim \mu_{\mathbf{p}} \leqslant 1 - 10^{-7}.$$

Their proof relies on calculating the dimension of the set of points which see an exceptionally large deviation for the frequency of a certain word appearing in the symbolic coding of the point from the one prescribed by the absolutely continuous measure μ_T. The constant c then depends on how much weight μ_T assigns the cylinder corresponding to this word.

In Theorem 3.3.1, we prove that (1.1) holds if our map T satisfies a 'non-linearity condition', and in our case c depends on the the derivative of T at four periodic points, so the absolutely continuous measure need not be known for the calculation of explicit bounds on the 'gap' c. However, even though in principle c

can be estimated by using our approach, for the Gauss map this yields a particularly poor estimate of the gap (compared to the estimate of 10^{-7} found in [KPW]) and for this reason we do not include explicit estimates on c. The 'non-linearity condition' that we impose on our map turns out to be stronger than the analogous condition in [KPW], since we demand some non-linearity in the first two branches. This lack of generality is a byproduct of tools used in the proofs contained in Chapter 5. If one could get around these technical difficulties, we could obtain an equivalent result to [KPW].

While Theorem 3.3.1 and the analogous result in [KPW] are comparable, the proofs are completely different. We propose a new approach for proving this type of result which is largely based on ideas from thermodynamic formalism. Our approach offers some new techniques and an interesting link between dimension gaps and lower bounds on the (asymptotic) variance of certain potentials. We will now summarise the contents of chapters 3-5 in more detail.

Chapter 3 is dedicated to introducing the dimension gap problem for countable branch Markov maps and any tools and notation that will be required. We begin by discussing the finite branch analogue of this problem, which is markedly simpler to analyse but provides important intuition for the general setting. We then introduce the class of maps which we will be working with and provide some historical motivation, some important examples of such maps and their key properties. Next, we outline results due to Walters [W] and Kifer, Peres and Weiss [KPW] which are important milestones in the story of the problem. This is followed by a statement of our main result Theorem 3.3.1, along with a discussion of the conditions that are imposed and a comparison with the analogous result from [KPW]. The final part of this chapter lays out a description of the structure of the proof, which is separated into two distinct and independent parts, tackled in Chapters 4 and 5 respectively. Roughly speaking, the first part deals with the dimension of Bernoulli measures which assign some uniform amount of mass on the first two cylinders, while the second part considers measures where mass is concentrated in the tail.

In Chapter 4 we obtain a uniform upper bound on the dimension of Bernoulli measures which assign a uniform amount of mass on the first two cylinders. More precisely, we show that for each $\varepsilon > 0$ there exists some constant G_ε for which

$$\sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}} \leqslant 1 - G_\varepsilon$$

where $\mathcal{P}_\varepsilon$ can (for now) be thought of as the set $\{\mathbf{p} \in \mathcal{P} : p_1, p_2 \geqslant \varepsilon\}$. In order to obtain this bound, we relate the dimension of such a measure $\mu_{\mathbf{p}}$ to the derivative of the function $\beta_{\mathbf{p}} : [0,1] \to [0,1]$ at the point $t = 1$, where $\beta_{\mathbf{p}}$ is defined implicitly via the equation

$$P(-\beta_{\mathbf{p}}(t)\log|T'| + tf_{\mathbf{p}}) = 0. \tag{1.2}$$

In (1.2) $f_{\mathbf{p}}$ denotes the locally constant potential which when evaluated at the nth interval of monotonicity yields $\log p_n$ provided that $p_n > 0$, and 0 otherwise. P denotes the topological pressure. This reduces the problem to studying the convexity of $\beta_{\mathbf{p}}$. In particular, once we evaluate the derivatives of $\beta_{\mathbf{p}}$ we see that the second derivative of $\beta_{\mathbf{p}}$ is given in terms of the variance

$$\sigma^2_{\mu_{\mathbf{p},t}}(f_{\mathbf{p},t})$$

of a potential $f_{\mathbf{p},t}$ for an appropriate Gibbs measure $\mu_{\mathbf{p},t}$. Thus the heart of the problem is finding a lower bound on $\sigma^2_{\mu_{\mathbf{p},t}}(f_{\mathbf{p},t})$ which is uniform for $\mathbf{p} \in \mathcal{P}_\varepsilon$ and t in some compact interval. While it has been known for a long time that the variance plays an important role in several areas of dynamics, for instance appearing in many statistical properties of dynamical systems such as the central limit theorem, much less is known about estimates on the variance. By exploiting different characterisations of the variance it is generally easy to obtain *upper* bounds on the variance, but *lower* bounds have not previously been considered. The remainder of this chapter is dedicated to the description and execution of a method for obtaining a lower bound for the variance. One part of this method allows us to draw on tools from Hilbert-Birkhoff cone theory, which is discussed more thoroughly in Appendix A.

In Chapter 5 we conduct the second part of the proof, that is, we consider the case where $\mathbf{p} \notin \mathcal{P}_\varepsilon$. Here we exploit the formula $\dim \mu = \frac{h(\mu)}{\chi(\mu)}$ which (in our setting) holds for all finite-entropy ergodic measures, where $h(\cdot)$ denotes the measure-theoretic entropy and $\chi(\cdot)$ denotes the Lyapunov exponent. Rather than obtaining a direct upper bound for the dimension of such measures, we construct an algorithm which associates to each $\mathbf{p} \notin \mathcal{P}_\varepsilon$ some $\mathbf{p}_\varepsilon \in \mathcal{P}_\varepsilon$ for which we have uniform control over the difference in the entropy $h(\mu_{\mathbf{p}}) - h(\mu_{\mathbf{p}_\varepsilon})$. The focus then turns to controlling the difference in the Lyapunov exponents

$$\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}_\varepsilon}) = \int \log|T'|d\mu_{\mathbf{p}} - \int \log|T'|d\mu_{\mathbf{p}_\varepsilon}. \tag{1.3}$$

It is incredibly difficult to compare the integrals of $\log|T'|$ with respect to two distinct Bernoulli measures, due to the structure of the Bernoulli measure once it is projected to the real line. To this end, we present an approach which allows us to write (1.3) instead as the difference of two integrals of *distinct* functions - but importantly with respect to the same measure ν, which is a Bernoulli measure that is defined on a 'larger' symbolic space. In the case where T is orientation reversing, this allows us to write

$$\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}_\varepsilon}) \geqslant \int_E \log \left| \frac{(T^2)'(f_2(\mathbf{i}))}{(T^2)'(f_1(\mathbf{i}))} \right| d\nu(\mathbf{i}) \tag{1.4}$$

where E is an explicit set in the 'larger' symbolic space and f_i are projections from this space to $\mathbb{R}$. A similar expression holds whenever T preserves orientation. Importantly, (1.4) lends itself to explicit lower estimates. In fact, we will only need to use that the integral in (1.4) is non-negative, although we remark that in a recent joint work with Baker [BJ], better lower bounds on (1.4) were used in order to study a related problem (of determining the existence of a Bernoulli measure with maximal dimension amongst the Bernoulli measures). Consequently, combining (1.4) with the estimate on the difference in entropy, we deduce that there exists some constant E_ε such that for all pairs $(\mathbf{p}, \mathbf{p}_\varepsilon)$,

$$\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}_\varepsilon} + E_\varepsilon.$$

We conclude that for sufficiently small $\varepsilon > 0$

$$\sup_{\mathbf{p}} \dim \mu_{\mathbf{p}} \leqslant 1 - G_\varepsilon + E_\varepsilon \leqslant 1 - \frac{G_\varepsilon}{2}$$

completing the proof of Theorem 3.3.1.

Finally, Chapter 6 contains work on a different problem to the last three chapters. Rather than considering invariant measures for expanding maps, we consider measures which are supported on a self-affine set associated to a particular iterated function system. In particular, we restrict our attention to a class of self-affine carpets which was introduced by Fraser in [Fr1], intended to be a generalisation of the classical 'orientation preserving' self-affine carpets which have appeared in the literature beginning with the work of Bedford and McMullen [Be; Mc]. The goal of this chapter is to prove the exact-dimensionality of certain Gibbs measures μ which are supported on these self-affine carpets, meaning that the local dimension

$$\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{1.5}$$

exists and is constant μ-almost everywhere, and furthermore is given by a 'Ledrappier-

Young formula' which is an expression involving the entropy, Lyapunov exponents and dimension of projections of μ. This is the main result, Theorem 6.3.1. The Gibbs measure μ is sometimes known as the *Käenmäki measure*, referring to a paper of Käenmäki [K] where such measures were first considered in this context. μ is a Gibbs measure for a sub-multiplicative potential ϕ^s, known as the singular value function, which was introduced in a pioneering paper of Falconer in [F2] where it was used to evaluate the dimension of 'typical' self-affine sets. This interpretation of ϕ^s means that a direct consequence of Theorem 6.3.1 is the existence of an ergodic measure of maximal dimension for some self-affine sets belonging to our class.

The motivation for Theorem 6.3.1 was a recent paper of Bárány and Käenmäki [BK], who verified the exact dimensionality of quasi-Bernoulli measures for a very general class of self-affine sets. However, the multiplicative properties of ϕ^s combined with the nature of the self-affine carpets we consider (in particular, the fact that they do not fit into the classical 'orientation preserving' setting) means that in our case the conditions from [BK] are not satisfied and that furthermore our measure is *not* quasi-Bernoulli. This latter fact makes exact-dimensionality incredibly difficult to verify, since a proof of this property typically requires obtaining both upper and lower bounds for the measure $\mu(B(x,r))$ which appears in (1.5), and without a supermultiplicative property this makes it impossible to implement standard approaches. A solution to this issue emerges from an examination of the structure of the measure when considered on a 'larger' symbolic space. The interpretation of the measure on this 'larger' space captures the heart of this chapter and is both of independent interest as well as forming the backbone of the proofs. In particular, we show that μ can be written in terms of two quasi-Bernoulli measures on the new symbolic space. This allows us to derive an interesting characterisation for the Lyapunov exponents and aids us in obtaining the appropriate bounds for the measure $\mu(B(x,r))$ for a μ-typical point, allowing us to prove exact-dimensionality.

Chapter 2

Preliminaries

We begin by introducing some basic notions from dynamical systems and ergodic theory. Let the triple $(X, \mathcal{B}, \mu)$ denote the space X equipped with a σ-algebra $\mathcal{B}$ of measurable subsets of X and a probability measure μ. Let $T : X \to X$ be a transformation. Then we say that (T, X) is a dynamical system. Given a point $x \in X$ we say that $\{x, Tx, T^2x, \ldots\}$ is the orbit of x under T. For a subset $A \subset X$ denote $T^{-1}(A) = \{x \in X : T(x) \in A\}$. We say that T is measurable if for all $A \in \mathcal{B}$, $T^{-1}(A) \in \mathcal{B}$. We say that T is measure preserving if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$, and in this case we may also say that μ is T-invariant (or just invariant, whenever the choice of map is clear). We'll denote the set of all T-invariant measures by $\mathcal{M}_T(X)$.

We say that T is ergodic if for any $A \in \mathcal{B}$ which satisfies $T^{-1}(A) = A$ then either $\mu(A) = 0$ or $\mu(A) = 1$. Although T can have many ergodic measures, distinct ergodic measures μ_1 and μ_2 are mutually singular, meaning that there exists $A \in \mathcal{B}$ for which $\mu_1(A) = \mu_2(X \setminus A) = 1$. Given an ergodic transformation, we can deduce various statistical properties of T. The most well-known of these is the Birkhoff ergodic theorem, which connects the average of a potential f along the orbit of a μ-typical point with the space average of f.

Theorem 2.0.1 (Birkhoff Ergodic Theorem)**.** *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic measure preserving transformation such that $\mu(X) = 1$. Let $f \in L^1(\mu)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \int f d\mu$$

for μ almost every $x \in X$.

We denote $S_nf(x) = \sum_{k=0}^{n-1} f(T^k(x))$ and call this a Birkhoff sum and we call $\frac{1}{n}S_nf(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x))$ a Birkhoff average. Notice that if we define $f_n(x) =$

$S_n f(x)$ then the Birkhoff Ergodic Theorem tells one about the existence of the limit $\lim_{n\to\infty} \frac{f_n(x)}{n}$ for the additive sequence f_n. The following result, due to Kingsman, can be viewed as a generalisation of Birkhoff's ergodic theorem for sub-additive sequences. This will be useful for guaranteeing the existence of limits of sub-additive sequences of functions.

Theorem 2.0.2 (Sub-additive Ergodic Theorem)**.** *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic measure preserving transformation such that $\mu(X) = 1$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sub-additive sequence in $L^1(\mu)$, meaning that $f_{n+m}(x) \leqslant f_n(x) + f_m(T^n x)$ for all x. Then there exists some constant $\alpha \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} \frac{f_n(x)}{n} = \alpha$$

for μ almost every $x \in X$.

The measure-theoretic (or 'metric') entropy of a measure preserving transformation will play a central role throughout the thesis. Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure-preserving transformation of a probability space and α be a finite or countable partition of X into measurable sets.

The *entropy of the partition* α is defined as

$$H_\mu(\alpha) = -\sum_{A\in\alpha} \mu(A) \log \mu(A).$$

For two partitions α and β denote $\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}$ to be the join of α and β. We denote $T^{-1}\alpha = \{T^{-1}A : A \in \alpha\}$. Then we define the *entropy of T with respect to the countable or finite partition* α to be

$$h_\mu(T, \alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu \left(\bigvee_{j=0}^{n-1} T^{-j}\alpha \right).$$

Finally, the *measure-theoretic entropy of T with respect to μ* is defined as

$$h_\mu(T) = \sup\{h_\mu(T, \alpha)\}$$

where the supremum is taken over all finite or countable partitions α for which $H_\mu(\alpha) < \infty$.

By a well-known theorem of Kolmogorov and Sinai (e.g. [W2, Theorem 4.1.7]), $h_\mu(T) = h_\mu(T, \alpha)$ for any partition α such that $H_\mu(\alpha) < \infty$ and $\bigvee_{j=0}^{n-1} T^{-j}\alpha \to \mathcal{B}$ as $n \to \infty$. We also have the following important theorem which gives an alternative characterisation for the entropy when T is ergodic (see for instance the remark below Corollary 4.14.4 in [W2]).

Theorem 2.0.3 (Shannon-McMillan-Breiman Theorem)**.** *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic measure-preserving transformation of a probability space and let α be a finite partition of X. Let $B_n(x)$ denote the unique member of $\bigvee_{j=0}^{n-1} T^{-j}\alpha$ to which x belongs. Then*

$$-\lim_{n\to\infty} \frac{1}{n} \log \mu(B_n(x)) = h(T, \alpha)$$

for μ-almost every x.

2.1 Symbolic Dynamics

In this section we discuss symbolic dynamics, in particular topological Markov shifts which serve as an important model throughout this thesis. Let $\mathcal{A}$ be either a finite or countable alphabet, and we restrict to the cases where $\mathcal{A} = \{1, \ldots, m\}$ or $\mathcal{A} = \mathbb{N}$. Let A be a matrix with rows and columns indexed by the digits of $\mathcal{A}$, where for each pair i, j the entry $A_{i,j} = 0$ or 1. We say that A is *aperiodic* if there exists some $n \geqslant 1$ such that all the entries of A^n are positive. Define

$$\Sigma_A = \{\mathbf{i} = (i_n)_{n=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} : A_{i_n, i_{n+1}} = 1\}$$

and $\sigma : \Sigma_A \to \Sigma_A$ to be given by $\sigma((i_n)_{n=1}^{\infty}) = (i_{n+1})_{n=1}^{\infty}$. We call (Σ_A, σ) a one-sided subshift of finite type. When the matrix A has entries all equal to 1 we say this is the *full shift*. We say that $i_1 \ldots i_k$ is an *admissable* word if $A_{i_n, i_{n+1}} = 1$ for all $1 \leqslant n \leqslant k-1$. For an admissable word $i_1 \ldots i_k$ we define the *cylinder set* $[i_1 \ldots i_k]$ to be all $\mathbf{j} = (j_n)_{n=1}^{\infty}$ for which $j_n = i_n$ for $1 \leqslant n \leqslant k$. We sometimes call $[i_1 \ldots i_k]$ a *level k cylinder*. We also introduce the following notation: for $\mathbf{i} \in \Sigma_A$ we define $\mathbf{i}|_n$ to be the finite word obtained by truncating $\mathbf{i}$ after n symbols. Also we denote Σ^* to be the set of all finite admissable words over the alphabet $\mathcal{A}$. For any $\mathbf{i}, \mathbf{j} \in \Sigma^*$, we let $\mathbf{ij}$ denote the concatenation of $\mathbf{i}$ and $\mathbf{j}$. Note that $\mathbf{ij}$ is not automatically admissable, so it is not necessarily true that $\mathbf{ij} \in \Sigma^*$. Let $|\mathbf{i}|$ denote the 'length' of the finite word $\mathbf{i} \in \Sigma^*$, i.e. if $\mathbf{i} = i_1 \ldots i_k$ then $|\mathbf{i}| = k$. Given $\mathbf{i}, \mathbf{j} \in \Sigma$ we define $\mathbf{i} \wedge \mathbf{j} \in \Sigma \cup \Sigma^*$ to be the longest initial block common to both $\mathbf{i}$ and $\mathbf{j}$. We define a metric d on Σ by setting $d(\mathbf{i}, \mathbf{j}) = e^{-|\mathbf{i} \wedge \mathbf{j}|}$ if $|\mathbf{i} \wedge \mathbf{j}| < \infty$ and $d(\mathbf{i}, \mathbf{j}) = 0$ otherwise. The metric d defines a topology on Σ. The σ-algebra of open sets is generated by the set of cylinder sets. Σ is compact if $\mathcal{A}$ is finite and not compact if $\mathcal{A}$ is infinite.

A key object of our study will be invariant measures on the symbolic space. We will now introduce three important classes of shift-invariant measures.

2.1.1 Bernoulli measures

Let (Σ, σ) be the full shift on the alphabet $\mathcal{A}$, where $\mathcal{A} = \{1, \ldots, k\}$ or $\mathcal{A} = \mathbb{N}$. Let $\mathbf{p} = (p_1, p_2 \ldots)$ be a probability vector, that is, $0 \leqslant p_i \leqslant 1$ with $\sum_{i \in \mathcal{A}} p_i = 1$.

By the Kolmogorov extension theorem, to define a Borel measure on Σ it is sufficient to define a measure on the cylinder sets. We define the measure $m_{\mathbf{p}}$ on the cylinder sets of Σ by

$$m_{\mathbf{p}}([i_1 \ldots i_n]) = p_{i_1} \ldots p_{i_n}$$

and say that $m_{\mathbf{p}}$ is a Bernoulli measure for $\mathbf{p}$. Then $(\sigma, \Sigma, m_{\mathbf{p}})$ is an ergodic measure preserving system.

2.1.2 Gibbs measures

A probability measure m on Σ is called a Gibbs measure if there exists a continuous function $g : \Sigma \to \Sigma$ and constants $C > 0$, $P \in \mathbb{R}$ such that

$$C^{-1} \leqslant \frac{m([i_1, \ldots, i_n])}{e^{S_n g(\mathbf{i}) - nP}} \leqslant C \tag{2.1}$$

for all $\mathbf{i} \in \Sigma$ and $n \geqslant 1$. Then we say that μ is a Gibbs measure of g and we call g the *Gibbs potential*.

Note that in our definition, we do not require m to be invariant and indeed there are examples of Gibbs measures which are not invariant. We give some motivation for studying Gibbs measures in Section 2.5 and learn about sufficient conditions for their existence in various contexts.

2.1.3 Quasi-Bernoulli measures

We say that a measure m on Σ is *quasi-Bernoulli* if there exists a constant $0 < C < \infty$ such that for all $\mathbf{i}, \mathbf{j} \in \Sigma^*$,

$$C^{-1} m([\mathbf{i}]) m([\mathbf{j}]) \leqslant m([\mathbf{ij}]) \leqslant C m([\mathbf{i}]) m([\mathbf{j}]). \tag{2.2}$$

It is easy to see that any Gibbs measure for a Hölder continuous function f is a quasi-Bernoulli measure.

2.2 Expanding maps and Markov partitions

Let $T : [0,1] \to [0,1]$ be either a continuous map, or a continuous map when considered as a map of the circle $\mathbb{R}/\mathbb{Z}$. Then $T : [0,1] \to [0,1]$ consists of a finite or countable number of continuous branches, and we assume that each of them are C^1. We say that T is expanding if $|T'| > 1$. Since a symbolic dynamical system is easy to study, we are interested in what properties of T give rise to a 'symbolic coding', which we will shortly make precise. The answer lies with the idea of a Markov partition.

Markov partitions are a useful way of partitioning the space that a dynamical system acts on by providing a useful tool for developing a 'symbolic coding' of T. Let $T : [0,1] \to [0,1]$ satisfy the above conditions. We say that a (finite or countable) collection $\mathcal{M} = \{I_i : i \in \mathcal{A} \subset \mathbb{N}\}$ of open non-empty subintervals of $[0,1]$ is a *Markov partition* of T if

1. $T|_{\overline{I_i}} : \overline{I_i} \to T(\overline{I_i})$ is a homeomorphism
2. $\{I_i\}_{i\in\mathcal{A}}$ are pairwise disjoint
3. If $T(I_i) \cap I_j \neq \emptyset$ for some $i, j \in \mathcal{A}$, then $I_j \subset T(I_i)$, where $\emptyset$ denotes the empty set.

If T admits a Markov partition we say T is a *Markov map*. Then one can build a symbolic coding for T by a suitable subshift Σ_A, where the transition matrix A is constructed by considering pairs i, j which satisfy the third condition above.

Instead of describing this general construction in detail, we will just describe how to build a Markov partition and corresponding symbolic coding for the type of maps that are considered in this thesis. A more general version of the results below can be found for instance in [KMS, Theorem 1.2.26].

Let $\{\mathcal{I}_n\}_{n\in\mathbb{N}}$ be a countable collection of non-empty disjoint subintervals of $[0,1]$ such that $(0,1) \subset \bigcup_{n\in\mathbb{N}} \overline{\mathcal{I}}_n$ and let $T_n : \overline{\mathcal{I}}_n \to [0,1]$ be a sequence of expanding bijective C^1 maps. Define $T : \bigcup_{n\in\mathbb{N}} \overline{\mathcal{I}}_n \to [0,1]$ as

$$T(x) = T_n(x) \quad \text{if} \quad x \in \overline{\mathcal{I}}_n$$

where we put $T(x) = T_k(x)$ for $k = \min\{n : x \in \overline{\mathcal{I}}_n\}$ if x is a common endpoint of two intervals. In this case, the Markov partition is just the sequence $\{\mathcal{I}_i\}_{i=1}^{\infty}$.

Denoting Σ as the full shift on $\mathbb{N}$, we can define a canonical coding map $\Pi : \Sigma \to [0,1] \setminus \bigcup_{n=0}^{\infty} T^{-n}(\{0\})$ by

$$\Pi(\mathbf{i}) = \lim_{n\to\infty} T_{i_1}^{-1} \circ \cdots \circ T_{i_n}^{-1}([0,1]).$$

Then Π is a continuous surjection and $\Pi \circ \sigma = T \circ \Pi$. We say that T is *coded by the full shift (on the countable alphabet)*. Given $\mathbf{i} \in \Sigma^*$ we will call $\Pi([\mathbf{i}])$ a projected cylinder set or simply a cylinder set.

Finally, notice that $E = \bigcup_{n=0}^{\infty} T^{-n}(\{0\})$ is a countable set of points. We call

$$J = [0,1] \setminus \bigcup_{n=0}^{\infty} T^{-n}(\{0\})$$

the *repeller* of T. One can study the dynamics of (J,T) via the dynamics of (Σ, σ).

2.3 Dimension and measure theory

This thesis will primarily be concerned with studying the dimension theory of sets and measures. There are many different notions of the dimension of a set but, roughly speaking, they all provide some description of how much space a set fills and the amount of irregularity of a set when viewed at small scales. The dimension of a set can be used as a finer measure of the size of the set, for instance when the Lebesgue measure of the set is 0. In this thesis, we will only deal with the Hausdorff dimension of sets, which is the oldest notion based on a construction of Carathéodory.

For a non-empty subset $U \subset \mathbb{R}^d$, define the diameter of U as $|U| = \sup\{|x - y| : x, y \in U\}$, that is, the greatest possible distance between two points in U. Given a subset $F \subset \mathbb{R}^d$ and a finite or countable collection of subsets $\{U_n\}_{n\in\mathbb{N}}$ that cover F, i.e. $F \subset \bigcup_{n\in\mathbb{N}} U_n$, we say that $\{U_n\}_{n\in\mathbb{N}}$ is a δ-cover of F if $|U_n| \leqslant \delta$ for all $n \in \mathbb{N}$. For $s \geqslant 0$ define

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{n\in\mathbb{N}} |U_n|^s : \{U_n\}_{n\in\mathbb{N}} \text{ is a countable } \delta \text{ cover of } F\right\}$$

that is, we look at all covers of F by sets of diameter at most δ and seek to minimise the sth powers of the diameters of the sets in our cover. As δ decreases, the class of possible covers decreases, thus $\mathcal{H}^s_\delta(F)$ increases and then we define

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F) = \lim_{\delta\to 0}\inf\left\{\sum_{n\in\mathbb{N}} |U_n|^s : \{U_n\}_{n\in\mathbb{N}} \text{ is a countable } \delta \text{ cover of } F\right\}.$$

It is possible to show that for each s, $\mathcal{H}^s$ is a measure, and we call it the s-dimensional Hausdorff measure. $\mathcal{H}^s$ is clearly decreasing with s and is either infinite or 0 apart from one possible exception where it can be either 0, infinite, or positive and finite.

The Hausdorff dimension of A is then defined as this critical value and denoted by

$$\dim_{\mathrm{H}} A = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

In this thesis we shall be more focused on the dimension of measures, which again can be defined in several different ways. For a Borel probability measure μ the Hausdorff dimension is defined as

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} A : A \text{ Borel such that } \mu(A) = 1\}.$$

Roughly speaking, this corresponds to the dimension of the set that the measure 'sees'. Another perhaps more intuitive notion of the dimension of a measure is the local dimension of a measure, where it exists. The *upper* and *lower local dimensions* of a Borel probability measure μ at a point x in its support are defined by

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \;=\; \limsup_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r} \quad \text{and} \quad \underline{\dim}_{\mathrm{loc}}(\mu, x) \;=\; \liminf_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r}.$$

If the upper and lower local dimensions coincide, we call the common value the *local dimension* and denote it by $\dim_{\mathrm{loc}}(\mu, x)$. This describes the rate at which the measure of a small ball about a μ-typical point scales as the radius of the ball is decreased. This notion is particularly important because if there exists a constant α such that the local dimension exists and equals α at μ almost all points then we say the measure μ is *exact dimensional* and in particular, if μ is exact dimensional then all the definitions of the dimension of a measure coincide with the exact dimension α.

We now present some results from measure theory which will be used in the thesis.

If a measure μ is absolutely continuous with respect to a measure ν, we write $\mu \ll \nu$. The following proposition (e.g. [MMR, Lemma 2.4]) is useful to verify exact-dimensionality whenever we have a measure which is absolutely continuous with respect to an exact-dimensional measure.

Proposition 2.3.1. *Suppose ν is a non-null finite Borel measure on $\mathbb{R}^d$ with exact dimension α. Let μ be any non-null finite Borel measure μ on $\mathbb{R}^d$ with $\mu \ll \nu$. Then μ is exact dimensional with exact dimension α.*

The following results about derivatives of measures will also come in useful, see for instance [Ma, Theorem 2.12].

Proposition 2.3.2. *Let μ, λ be inner regular probability measures on $\mathbb{R}^d$ (by inner* *regular we mean that for each $m = \mu, \lambda$, $m(B)$ is the supremum of $m(K)$ over all compact subsets K of B, for any Borel set B). Then:*

1. *For λ-almost every $x \in \mathbb{R}^d$,*

$$\lim_{r \to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}$$

exists and is finite.

2. *$\mu \ll \lambda$ if and only if*

$$\liminf_{r \to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))} < \infty$$

for μ-almost every $x \in \mathbb{R}^d$.

We also recall a well-known result of Egorov about convergence of functions in a measure space.

Theorem 2.3.3 (Egorov's Theorem). *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued functions on a probability space $(X, \mathcal{B}, \mu)$ and suppose that $f_n \to f$ μ-almost everywhere. Then for all $\varepsilon > 0$ there exists a set $A \in \mathcal{B}$ with $\mu(A) > 1 - \varepsilon$ such that $f_n \to f$ uniformly on A.*

2.4 Iterated Function Systems

We say that $f : \mathbb{R}^d \to \mathbb{R}^d$ is a contraction if there exists a contraction ratio $0 < c < 1$ such that for all $x, y \in \mathbb{R}^d$

$$|f(x) - f(y)| \leqslant c|x - y|.$$

Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a finite collection of contractions $f_i : \mathbb{R}^d \to \mathbb{R}^d$. We call the family $\mathcal{F}$ an *iterated function system* (IFS). Iterated function systems are one of the main tools that are used to construct fractal sets, due to the following fundamental result which dates back to work of Hutchinson [H].

Theorem 2.4.1 (Hutchinson). *Let $\mathcal{F}$ be an iterated function system. Then there is a unique compact non-empty subset Λ such that*

$$\Lambda = \bigcup_{i=1}^{n} f_i(\Lambda).$$

In particular, we call the unique set whose existence is guaranteed by Theorem 2.4.1 the *attractor* of the IFS.

Often the attractor of an IFS has a more complex structure and is more difficult to analyse if the parts $\{f_i(\Lambda)\}$ overlap too much. Thus, separation conditions are often imposed, two of which we detail below.

Definition 2.4.2. *An IFS $\mathcal{F} = \{f_i : i \in \{1, \ldots n\}\}$ and its attractor Λ are said to satisfy the* Open Set Condition (OSC) *if there exists a non-empty open set U such that*

$$\bigcup_{i=1}^{n} f_i(U) \subset U$$

where the union is disjoint.

Definition 2.4.3. *An IFS $\mathcal{F} = \{f_i : i \in \{1, \ldots n\}\}$ and its attractor Λ are said to satisfy the* Strong Separation Condition (SSC) *if $f_i(\Lambda) \cap f_j(\Lambda) = \emptyset$ for $i \neq j$, where $\emptyset$ denotes the empty set.*

2.4.1 Symbolic coding of IFS and attractors

Typically, attractors of iterated function systems are studied by building a symbolic space from the index set, since the geometry of the symbolic space is more convenient to work with than the more complex geometry of the attractor. Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be an iterated function system and let Σ denote the full shift on the alphabet $\{1, \ldots, n\}$. For a finite word $\mathbf{i} = i_1 \ldots i_k$ define

$$f_{\mathbf{i}} = f_{i_1} \circ \cdots \circ f_{i_k}.$$

Then we define a natural projection $\Pi : \Sigma \to \Lambda$ from the symbolic space to the geometric space by

$$\Pi(\mathbf{i}) = \bigcap_{n \in \mathbb{N}} f_{\mathbf{i}|_n}(X)$$

which will allow us to move between the two spaces.

Importantly for us, the projection Π allows us to take a measure m on Σ and obtain an associated measure $\mu = m \circ \Pi^{-1}$ which is supported on Λ.

2.4.2 Self-similar sets

The simplest type of iterated function system is a self-similar iterated function system $\mathcal{F} = \{f_1, \ldots, f_n\}$ where all of the maps $f_i : \mathbb{R}^d \to \mathbb{R}^d$ are similarities, that is, for each i there exists a constant c_i such that for all $x, y \in \mathbb{R}^d$,

$$|f_i(x) - f_i(y)| = c_i|x - y|.$$

The resulting attractor is then called a self-similar set. A well-known example of such a set is the middle-third Cantor set, which is the attractor of the IFS $\mathcal{F}$ made up of the maps $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ given by $f_1(x) = \frac{x}{3}$, $f_2(x) = \frac{x}{3} + \frac{2}{3}$.

The dimension theory of self-similar sets is fairly well understood, at least in the non-overlapping setting. Given a self-similar IFS made up of k maps with similarity ratios c_i and associated attractor Λ, the similarity dimension $\dim_S \Lambda$ is defined to be the unique solution in s to the formula

$$\sum_{i=1}^{k} c_i^s = 1.$$

If $\mathcal{F}$ satisfies the OSC, $\dim_H \Lambda = \dim_S \Lambda$. In contrast, when the OSC is not satisfied $\dim_S \Lambda$ is still an upper bound for $\dim_H \Lambda$ but beyond that, the dimension theory is far from understood. For instance, it is known that a 'dimension drop' can occur if different iterates of maps in the IFS overlap exactly, but it is a difficult open problem to establish whether this is the only way in which the dimension can drop. Indeed, there is a folklore conjecture which says that a 'dimension drop' occurs only in the presence of exact overlaps. Hochman [Ho] recently made important progress towards proving this conjecture by verifying that exact overlaps are the only cause of a dimension drop in the case that the maps in the IFS are one-dimensional with algebraic parameters.

2.4.3 Self-affine sets

Self-similar sets which were discussed in the previous section are special examples of a much more general class of sets known as *self-affine sets*; the key difference being that in a self-affine iterated function system, the maps are now permitted to have different contraction ratios in different directions.

In particular, $\mathcal{F}$ is a self-affine IFS if $\mathcal{F}$ contains affine transformations, that is,

$$\mathcal{F} = \{A_i + t_i : i \in \{1, \ldots, n\}\}$$

where each A_i is a linear map and each t_i is a translation. In this case, we say that the attractor Λ is a self-affine set.

Allowing the maps in a self-affine IFS $\mathcal{F}$ to have different contraction ratios in different directions causes the self-affine attractor to be markedly more difficult to study. For this reason, research in the dimension theory of self-affine sets has been forced to follow one of two distinct lines of thought since the 1980s.

The first line of thought is the 'generic' approach, pioneered by the work of Falconer beginning with [F2]. With this outlook, one seeks to find the dimension of a generic self-affine set, in the sense of Lebesgue typical translations, while the linear parts of the maps are fixed from the outset. This allows one to get a value for the dimension of a 'typical' self-affine set.

In the other approach, one seeks to obtain a definite value for the dimension of a self-affine set (rather than an almost sure result as above), although this time it is at the cost of restricting to a specific type of self-affine set. This line of thought was pioneered by Bedford and McMullen [Be; Mc], and since then has revolved around the study of various constructions of '*self-affine carpets*'.

Although our focus in this thesis will be the dimension of *measures* that are supported on a self-affine set - rather than the self-affine set itself, there are elements from the history of both of the above perspectives which will be relevant to the problem studied in Chapter 6. Therefore we briefly summarise the important results, which will hopefully give the objects studied in Chapter 6 some context. For a more thorough survey of dimension results for self-affine sets, the reader is directed to [F4].

Dimension of a 'typical' self-affine set

Before we can discuss the results of Falconer [F2] and the work that followed, we must introduce the notion of the singular values of a linear map. Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map, so that A is a $d \times d$ matrix. The singular values of A are the positive square roots of the eigenvalues of $A^T A$ where A^T denotes the transpose of the matrix A. Geometrically, the singular values correspond to the lengths of the semi-axes of the image of the unit ball under A. Thus, the singular values represent how much the map A contracts or expands distances in different directions. Thus, one can see that it would make sense for the dimension of a self-affine set to be closely related to the singular values of the matrices which appear in the construction.

It is conventional to order the d singular values of a contracting linear map $A : \mathbb{R}^d \to \mathbb{R}^d$ like

$$1 > \alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_d > 0.$$

Let $\mathcal{F} = \{f_1, \ldots, f_k\}$ be a self-affine IFS, so each of the maps $f_i = A_i + b_i$ for some $d \times d$ matrix A_i and translation vector b_i. For a finite word $\mathbf{i} \in \Sigma^*$ denote $\alpha_r(\mathbf{i})$ to be the rth singular value of the linear part of $f_{\mathbf{i}}$, that is, the rth singular value of $A_{\mathbf{i}}$. Observe that with this notation, we can also take $\mathbf{i} \in \Sigma$ and write $\alpha_r(\mathbf{i}|_n)$ to be the rth singular value of $A_{\mathbf{i}|_n}$. (Note: we may use the notation $\alpha_r(A_{\mathbf{i}})$ and $\alpha_r(\mathbf{i})$ interchangeably for $\mathbf{i} \in \Sigma^*$).

Using this notation, we are ready to introduce the *singular value function* $\phi^s : \Sigma^* \to \mathbb{R}^+$ which was first introduced by Falconer in [F2]. For $s \in [0, d]$, and $\mathbf{i} \in \Sigma^*$, define $\phi^s(\mathbf{i})$ by

$$\phi^s(\mathbf{i}) = \alpha_1(\mathbf{i}) \cdots \alpha_{\lceil s \rceil - 1}(\mathbf{i}) \alpha_{\lceil s \rceil}^{s - \lceil s \rceil}.$$

(Note: as above, we may also sometimes use the notation $\phi^s(A_{\mathbf{i}})$ instead of $\phi^s(\mathbf{i})$).

Using this function, Falconer defined the *affinity dimension* (sometimes called the *singularity dimension*) $\dim_{\mathrm{A}} F$ of the self-affine attractor F to be

$$\begin{aligned} \dim_{\mathrm{A}} F &= \inf \left\{ s : \sum_{n=1}^{\infty} \sum_{\mathbf{i} \in \{1, \ldots, k\}^n} \phi^s(\mathbf{i}) < \infty \right\} \\ &= \inf \left\{ s : \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{\mathbf{i} \in \{1, \ldots, k\}^n} \phi^s(\mathbf{i}) \right) \leqslant 0 \right\} \end{aligned}$$

where the limit within the second displayed equation exists by the sub-additivity of ϕ^s. Using covering arguments, Falconer showed that $\dim_{\mathrm{A}} F$ was always an upper bound for the dimension of a self-affine set F. Moreover, by considering a natural cover of F he proved that for 'typical' translations it was equal to the Hausdorff dimension of the set.

Theorem 2.4.4. *Let $\{A_i\}_{i=1}^k$ be a collection of $d \times d$ matrices where each A_i satisfies the bound on its matrix norm $\|A_i\| = \alpha_1(A_i) < \frac{1}{2}$. Then for Lebesgue almost all translations $(t_1, \ldots, t_k) \in \mathbb{R}^{kd}$, the attractor F of the self-affine IFS $\mathcal{F} = \{A_i + t_i : i = 1, \ldots, k\}$ satisfies*

$$\dim_{\mathrm{H}} F = \min\{\dim_{\mathrm{A}} F, d\}.$$

In fact, initially the above result was proved in [F2] with the stronger assumption that all the norms $\|A_i\| < \frac{1}{3}$, but in [So], Solomyak weakened the condition to the current form. Moreover, an upper bound of $\frac{1}{2}$ was proved to be sharp by an example of Prztycyki and Urbański in [PU1].

The singular value function ϕ^s will be of particular importance to us since the measure whose dimension we study in Chapter 6 is the Gibbs measure for $\log \phi^s$.

Self-affine carpets

As discussed earlier, the other direction considered in the literature concerning the dimension of self-affine sets is the approach of restricting to a specific type of self-affine set and obtaining a value for the dimension. This strategy has centred around various constructions of 'self-affine carpets'. The pioneers of this line of research were Bedford and McMullen [Be; Mc], who independently considered the following construction.

Example 2.4.5 (Bedford-McMullen carpets)**.** *Take the unit square* $[0,1]^2$ *and divide it into a regular* $m \times n$ *grid, where the number of rows* m *is greater than 1 and is at most the number of columns* n*, i.e.* $1 < m \leqslant n$*.*

Select a family of rectangles $\{U_i\}_{i=1}^{l}$ *formed by this grid and define the affine map* S_i *to be the one that maps* $[0,1]^2$ *to* U_i*. We define the Bedford-McMullen carpet in this case to be the attractor of the IFS* $\mathcal{F} = \{S_i : i \in \{1, \ldots, l\}\}$*.*

Due to the simplicity of this model, this allowed the dimension of the attractor F to be calculated explicitly.

Theorem 2.4.6 (Bedford, McMullen)**.** *Let* F *be the attractor for the above construction. Then*

$$\dim_{\mathrm{H}} F = \frac{\log\left(\sum_{i=1}^{m} N_i^{\frac{\log m}{\log n}}\right)}{\log m} \tag{2.3}$$

where N_i *denotes the number of maps in the* i*th column.*

Observe that since the affinity dimension depends only on the number of maps and the contraction ratios (which depend on m and n), whereas the expression in (2.3) depends also on N_i (which is a quantity related to the translations of the maps in the IFS), the Hausdorff dimension of a Bedford-McMullen carpet can in fact be strictly less than the affinity dimension.

Pollicott and Weiss [PW2], Lalley and Gatzouras [GL], Barański [Ba1] and Feng and Wang [FW] studied variants on this construction, with one thing in common: the orientation of the maps in each of these models was always preserved. In [Fr1], Fraser introduced self-affine carpets where the maps were now allowed to have non-trivial rotational and reflectional components. By studying a modified version of the classical singular value function, Fraser was able to compute the box and packing dimensions of these carpets. This is in fact the family of self-affine sets that we will work with in Chapter 6, so we delay their precise definition till then.

2.4.4 Measures supported on attractors

In this thesis, we won't so much be concerned with the dimension of attractors themselves, but rather with the dimensions of measures which are supported on the attractors of iterated function systems (although of course sometimes this can give information about the dimension of the attractor itself).

Fix an IFS $\mathcal{F} = \{S_i : i = 1, \ldots, l\}$ with an attractor F, which is coded by the shift space Σ. Let m be a measure on Σ. Then we can use the coding map $\Pi : \Sigma \to \mathbb{R}$ to define a measure $\mu = m \circ \Pi^{-1}$ which is supported on the attractor F.

For example, given a Bernoulli measure m, we call $m \circ \Pi^{-1}$ a *self-similar measure* if F is a self-similar set. Similarly, we say that $m \circ \Pi^{-1}$ is a self-affine measure if F is a self-affine set. Therefore any self-affine (or self-similar) measure satisfies

$$\mu = \sum_{i=1}^{l} p_i \mu \circ S_i^{-1}$$

for some probability vector $(p_1, \ldots, p_l)$.

Another common type of measures which are studied are projected Gibbs measures. In some circumstances, the Gibbs measure may satisfy the quasi-Bernoulli property (2.2), for instance when the Gibbs potential is Hölder. In Chapter 6 we will consider projected Gibbs measures for the singular value function ϕ^s, which in our case will *not* be quasi-Bernoulli.

The appropriate notions of entropy and Lyapunov exponents for projected measures will play a key role in Chapter 6. Fix an ergodic measure m on Σ, and let $\mu = m \circ \Pi^{-1}$. The following important result due to Oseledets [O] guarantees the existence of *Lyapunov exponents.*

Theorem 2.4.7 (Oseledets)**.** *There exist positive constants, which we call* Lyapunov exponents $0 < \chi_1(\mu) \leqslant \chi_2(\mu) \leqslant \cdots \chi_d(\mu) < 1$ *such that for m almost all $\mathbf{i} \in \Sigma$ and $1 \leqslant j \leqslant d$ we have*

$$\chi_j(\mu) \;\; = \;\; -\lim_{n\to\infty} \frac{1}{n} \log \alpha_j(\mathbf{i}|_n).$$

Using the ergodicity of m, we can use the Shannon-McMillan-Breiman theorem 2.0.3 to define the *entropy* of μ.

Theorem 2.4.8. *Let m be an ergodic σ-invariant measure on Σ and $\mu = m \circ \Pi^{-1}$. We define the entropy of μ to be the constant $h(\mu)$ which satisfies*

$$h(\mu) = -\lim_{n\to\infty} \frac{1}{n} \log m([\mathbf{i}|_n])$$

for m almost every $\mathbf{i} \in \Sigma$.

The Lyapunov exponents and entropy of a measure are central objects in the dimension theory of measures supported on attractors of IFS and in many cases the dimension of a measure can be expressed purely in terms of these via a '*Ledrappier-Young formula*'. For more detail on this and an overview of the study of dimensions of measures which are supported on self-affine sets, refer to Chapter 6.

2.5 Thermodynamic formalism

In this section we introduce the tools from thermodynamic formalism which will be used extensively in the proofs of many results throughout this thesis.

Thermodynamic formalism is a branch of ergodic theory which originated in statistical mechanics. While the ergodic theorem provides us with a useful tool for studying the orbits of typical points, it does not provide us with a 'natural' invariant measure to equip the space with. Here 'natural' is used loosely and depends on the characteristic of the dynamical system that one is interested in, such as for instance an invariant measure which maximises *dimension* or *entropy*. Gibbs measures were translated from statistical mechanics to the setting of dynamical systems by Ruelle and Sinai beginning with [S], providing a class of invariant measures whose properties were closely connected with the properties of the Gibbs potential. The subsequent body of work that followed connecting Gibbs measures with other analogues of notions from statistical mechanics such as pressure, equilibrium states and entropy all in one beautiful and interwoven theory is now called *thermodynamic formalism*. The connections established by this theory have proved to be powerful tools in many areas of dynamical systems including its dimension theory, rates of mixing and statistical properties of dynamical systems. The monographs of Bowen and Ruelle [Bo; Ru] provide classical expositions of thermodynamic formalism in the original settings in which it was developed.

Of course this body of work has since grown and indeed, thermodynamic formalism will appear in a number of different settings in this thesis. Therefore this section will be split into three parts, each of which summarises the relevant results that will be used from each setting. Firstly, we briefly touch upon the thermodynamic formalism of Hölder continuous potentials for the subshift of finite type. Only very basic results from this setting will be used. Secondly, the most detailed account of results from the thermodynamic formalism will be given for the setting of the countable shift, which will be used throughout Chapters 3-5. Finally, we will give an overview that contains the analogue of these ideas in the sub-additive setting, that is, for sub-additive potentials on a subshift of finite type. Results from this setting will be used in Chapter 6.

2.5.1 Finite setting

The thermodynamic formalism for Hölder continuous potentials on a subshift of finite type is well understood and a detailed account can be found in [PP]. However, in this thesis the amount of results required from this setting is limited to knowing about the existence of Gibbs measures and their characterisation. Therefore, these are the only results we present here, all of which can be found in [PP].

Let A be an aperiodic matrix and (Σ, σ) be its associated subshift of finite type. For a continuous function $f : \Sigma \to \mathbb{C}$ and $n \geqslant 1$ we define

$$\mathrm{var}_n(f) = \sup\{|f(\mathbf{i}) - f(\mathbf{j})| : \mathbf{i}|_m = \mathbf{j}|_m \quad \text{for} \quad m < n\}$$

to be the nth variation of f. Let $0 < \delta < 1$. Define

$$F_\delta = \{f : f \text{continuous and for some } C > 0 \text{ and for all } n \in \mathbb{N}, \quad \mathrm{var}_n f \leqslant C\delta^n\}. \tag{2.4}$$

If $f \in F_\delta$ we say that f is δ-Hölder continuous. We say that a function $f : \Sigma \to \mathbb{R}$ is Hölder continuous if $f \in F_\delta$ for some $0 < \delta < 1$.

We begin by defining the topological pressure of a Hölder continuous function.

Definition 2.5.1 (Pressure)**.** *Let $f : \Sigma \to \mathbb{R}$ be a Hölder continuous potential. Then we define the* topological pressure *of f by*

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\mathbf{i}:\sigma^n \mathbf{i} = \mathbf{i}} \exp(S_n f(\mathbf{i})) \right).$$

In this setting, the pressure also satisfies a *variational principle*, providing us with another characterisation of the pressure.

Theorem 2.5.2 (Variational principle)**.** *For any Hölder continuous function f,*

$$P(f) = \sup \left\{ h(\mu) + \int f d\mu \right\} \tag{2.5}$$

where the supremum is taken over all σ-invariant probability measures.

We say that an invariant measure μ is an *equilibrium state* if it achieves the supremum in (2.5), that is, $P(f) = h(\mu) + \int f d\mu$.

Recall the definition of a Gibbs measure given in (2.1). An important result that will be used in this thesis is that, given a Hölder continuous potential f, there exists a unique invariant Gibbs measure that can be characterised using the pressure and variational principle.

Proposition 2.5.3. *Let $f : \Sigma \to \mathbb{R}$ be a Hölder continuous potential. Then there exists a unique invariant Gibbs measure μ_f for f and the constant P from (2.1) is given by $P = P(f)$. Moreoever, this is the unique equilibrium state for f.*

Given a continuous function $u : \Sigma \to \Sigma$ we say that $u - u \circ \sigma$ is a *coboundary.* We say that two functions $f, g : \Sigma \to \mathbb{R}$ are *cohomologous* (writing $f \sim g$) if there exists some continuous function $u : \Sigma \to \mathbb{R}$ for which

$$g = f + u - u \circ \sigma. \tag{2.6}$$

Note that two functions being cohomologous is an equivalence relation. Also observe that if two functions f and g are cohomologous then their Birkhoff sums coincide on periodic orbits, that is

$$S_n f(\mathbf{i}) = S_n g(\mathbf{i})$$

for any $\mathbf{i}$ such that $\sigma^n \mathbf{i} = \mathbf{i}$.

Coboundaries are useful since adding a coboundary to a function preserves thermodynamic quantities, as demonstrated by the following result, see for instance [PP, Proposition 3.6].

Proposition 2.5.4. *Two Hölder continuous functions f and g have the same equilibrium state if and only if $f \sim g + c$, where $c = P(f) - P(g)$.*

2.5.2 Countable setting

The thermodynamic formalism of the symbolic space was developed in the setting of the countable Markov shift by Mauldin and Urbański (e.g. [MU1; MU2; MU3]) and Sarig (e.g. [S1; S2]) in the turn of the 21st century. These references contain many different sufficient conditions on both the subshift and potentials which give results such as existence of Gibbs measures. However, in this thesis we will only be considering the full shift on the countable alphabet and this means that many of the results have a much simpler exposition. Therefore, in this section we will only summarise the necessary results for the full shift (Σ, σ) on a countable alphabet.

Since the ultimate purpose of this section is to provide applications to a countable branch expanding interval map, it is worth noting that equivalently if one is given such a map T which is coded by the full shift, then each result in this section could be rewritten by using the notation relating to the map T rather than the shift map σ. Indeed, throughout Chapters 3-5 we will use such a version of each notion and result that appears this section. Also, we remark that many years prior to the work of Mauldin, Urbański and Sarig (whose work was more focused on pushing the boundaries of how much conditions could be weakened while still preserving desirable thermodynamic formalism results) Walters [W] investigated the thermodynamic formalism of countable branch expanding interval maps *without* exploiting any symbolic coding, and a lot of the results that we will require could be gleaned from his 1978 paper instead.

A potential $f : \Sigma \to \mathbb{R}$ is said to be *locally Hölder continuous* if there exist constants $C > 0$ and $0 < \delta < 1$ such that for all $n \geqslant 1$ the variations $\mathrm{var}_n(f)$ decay exponentially, that is,

$$\mathrm{var}_n(f) = \sup_{i_1 \ldots i_n \in \mathbb{N}^n} \{|f(\mathbf{i}) - f(\mathbf{j})| : \mathbf{i}, \mathbf{j} \in [i_1, \ldots, i_n]\} \leqslant C\delta^n. \tag{2.7}$$

If f satisfies this hypothesis for $0 < \delta < 1$ we say that f is δ-locally Hölder continuous. We denote the space of all δ-locally Hölder continuous functions by F_δ. Notice that the key difference between this space and the space of Hölder functions as defined in (2.4) is that there is no assumption on the 0th variations of f, that is, f may not be bounded. Define the seminorm $[f]_\delta$ to be the smallest constant one can take in (2.7). Therefore $F_\delta = \{f : \Sigma \to \mathbb{R} : [f]_\delta < \infty\}$. Let $\mathcal{F}_\delta$ denote the space of all bounded and δ-locally Hölder functions. Define the norm $\|\cdot\|_\delta = [\cdot]_\delta + \|\cdot\|_\infty$ and observe that this makes $(\mathcal{F}_\delta, \|\cdot\|_\delta)$ a normed space.

We also say that a locally Hölder potential $f : \Sigma \to \mathbb{R}$ is *summable* if

$$\sum_{n \in \mathbb{N}} \exp(\sup f|_{[n]}) < \infty. \tag{2.8}$$

Both of the above conditions will be central for developing the thermodynamic formalism.

We can define the *topological pressure* of a potential f as follows.

Definition 2.5.5 (Pressure)**.** *Let $f : \Sigma \to \mathbb{R}$ be a locally Hölder potential. Then the pressure of f is given by*

$$P(f) = \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{\mathbf{i} : \sigma^n \mathbf{i} = \mathbf{i}} \exp(S_n f(\mathbf{i})) \right).$$

Notice that the pressure can either be finite or infinite. We have an analogue of the variational principle in our countable setting, see for instance [MU1, Theorem 2.1.8].

Theorem 2.5.6 (Variational Principle). *If f is locally Hölder then*

$$P(f) = \sup \left\{ h(\mu) + \int f d\mu \right\} \tag{2.9}$$

where the supremum is taken over all σ-invariant probability measures for which $-\int f d\mu < \infty$

Similarly, we say that a measure μ is an *equilibrium state* for f if $-\int f d\mu < \infty$ and $h(\mu) + \int f d\mu = P(f)$.

If f is a locally Hölder summable potential then we know about the existence of Gibbs measures, see for instance [MU2, Corollary 2.10] and [MU1, Theorem 2.2.9].

Proposition 2.5.7 (Existence of Gibbs measures). *Let $f : \Sigma \to \mathbb{R}$ be a locally Hölder summable potential. Then f has a unique σ-invariant Gibbs measure μ_f, and the constant $P = P(f)$. Moreoever, if $-\int f d\mu_f < \infty$ then μ_f is the unique equilibrium state for f.*

In the setting of the countable shift, we say that f and g are cohomologous if there exists some bounded locally Hölder continuous function $u : \Sigma \to \mathbb{R}$ for which

$$f = g + u - u \circ \sigma.$$

We have an analogue of Proposition 2.5.4 for our countable setting, see for instance [MU2, Theorem 3.3].

Proposition 2.5.8 (Equivalence of Gibbs measures). *Let $f, g : \Sigma \to \mathbb{R}$ be locally Hölder potentials that have σ-invariant Gibbs measures μ_f and μ_g. Then $\mu_f = \mu_g$ if and only if $f - g$ is cohomologous to a constant, that is, there exists some bounded locally Hölder continuous function $u : \Sigma \to \mathbb{R}$ for which*

$$f - g = c + u - u \circ \sigma.$$

It will be useful to understand the above notions from the point of view of operator theory.

Assume g is a summable locally Hölder potential $g \in F_\delta$. Then we can define the *transfer operator* (sometimes known as the Perron-Frobenius operator or Ruelle operator) as $\mathcal{L}_g : \mathcal{F}_\delta \to \mathcal{F}_\delta$ given by

$$(\mathcal{L}_g w)(\mathbf{i}) = \sum_{\sigma \mathbf{j} = \mathbf{i}} \exp(g(\mathbf{j})) w(\mathbf{j}). \tag{2.10}$$

Observe that the above sum is an infinite sum in our setting, hence the necessity that g be summable. Also, observe that the iterates of $\mathcal{L}_g$ are given by

$$(\mathcal{L}_g^n w)(\mathbf{i}) = \sum_{T^n \mathbf{j} = \mathbf{i}} \exp(S_n g(\mathbf{j})) w(\mathbf{j}).$$

We have a version of the well-known Ruelle-Perron-Frobenius theorem in our setting. See for instance [S1, Corollary 2] and [MU2, Corollary 2.10].

Theorem 2.5.9 (Ruelle-Perron-Frobenius theorem)**.** *Suppose g is a summable locally Hölder potential. Then the following statements hold.*

1. *There exists a unique eigenmeasure $\tilde{\mu}_g$ such that*

$$\int \mathcal{L}_g f d\tilde{\mu}_g = e^{P(g)} \int f d\tilde{\mu}_g$$

and there exists a unique positive continuous function h for which $\mathcal{L}_g h = e^{P(g)} h$ and $0 < \inf h < \sup h < \infty$.

2. *$\tilde{\mu}_g$ is a Gibbs measure for g.*

3. *There exists a unique invariant Gibbs measure μ_g for g. In particular, if we consider the* normalised operator *$\mathcal{M}_g : \mathcal{F}_\delta \to \mathcal{F}_\delta$ given by $\mathcal{M}_g w = e^{-P(g)} h^{-1} \mathcal{L}_g(hw)$ (so that $\mathcal{M}_g \mathbf{1} = \mathbf{1}$) then $d\mu_g = h d\tilde{\mu}_g$ and*

$$\int \mathcal{M}_g f d\mu_g = \int f d\mu_g.$$

The analytic properties of the pressure function P will come in useful. In order to discuss the analytic properties of pressure we must first introduce the (asymptotic) variance.

First, for $f, g, h : \Sigma \to \mathbb{R}$, where g is locally Hölder, define the *covariance* by

$$\sigma^2_{\mu_g}(f,h) = \lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{k=0}^{n-1} f(\sigma^k \mathbf{i}) - n \int f d\mu_g \right) \left(\sum_{k=0}^{n-1} h(\sigma^k \mathbf{i}) - n \int h d\mu_g \right) d\mu_g(\mathbf{i}) \tag{2.11}$$

whenever the limit exists. When $f = h$ in the above, we simply write $\sigma^2_{\mu_g}(f, f) = \sigma^2_{\mu_g}(f)$ as defined below.

Definition 2.5.10 (Variance)**.** *Let $f, g : \Sigma \to \mathbb{R}$ where g is locally Hölder. Then we define the* variance *by*

$$\sigma^2_{\mu_g}(f) = \lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{k=0}^{n-1} f(\sigma^k \mathbf{i}) - n \int f d\mu_g \right)^2 d\mu_g(\mathbf{i}) \tag{2.12}$$

whenever the limit exists.

The variance is invariant under adding a coboundary or a constant, that is, if $u : \Sigma \to \mathbb{R}$ is locally Hölder and $c \in \mathbb{R}$ then

$$\sigma^2_{\mu_g}(f) = \sigma^2_{\mu_g}(f + u - u \circ \sigma + c)$$

Also, it is a classical result that whenever the variance exists and $\int f d\mu = 0$, we can rewrite (2.12) as

$$\sigma^2_{\mu_g}(f) = \int f^2 d\mu_g + 2 \sum_{n=1}^{\infty} \left(\int f \cdot f \circ \sigma^n d\mu_g \right). \tag{2.13}$$

See for instance [PU2].

We are now ready to summarise the analytic properties of the pressure function in our setting. The following is given in a more general setting as [S1, Corollary 4].

Proposition 2.5.11 (Analyticity of pressure)**.** *Let $f : \Sigma \to \mathbb{R}$ be a locally Hölder continuous function and let*

$$\mathrm{Dir}(f) = \{g : g \text{ is locally Hölder and } P(f + tg) < \infty \text{ for } t \text{ in some neighbourhood } (-\varepsilon, \varepsilon)\}. \tag{2.14}$$

Let $g \in \mathrm{Dir}(f)$. Then $t \to P(f + tg)$ is real analytic in some neighbourhood $(-\varepsilon_0, \varepsilon_0)$ of t.

As a result of the above we can write down the derivatives of the pressure. The following two results can be found for instance in [MU1, Propositions 2.6.13, 2.6.14].

Proposition 2.5.12 (First derivative of pressure)**.** *Let f, g be locally Hölder and $t_0 \in \mathbb{R}$. Suppose that $t \to P(tf + g)$ is analytic in some neighbourhood of t_0. Then the first derivative of the pressure is given as*

$$\left.\frac{dP}{dt}\right|_{t=t_0} = \int f d\mu_{t_0 f+g}.$$

Proposition 2.5.13 (Second derivative of pressure)**.** *Let f, g be locally Hölder and $s_0, t_0 \in \mathbb{R}$. Suppose that $t \to P(sf + tg)$ is analytic for all pairs (s, t) in some neighbourhood of (s_0, t_0). Then*

$$\left.\frac{\partial^2 P}{\partial s \partial t}\right|_{(s,t)=(s_0,t_0)} = \sigma^2_{\mu_{s_0 f + t_0 g}}(f, g).$$

2.5.3 Sub-additive setting

In this section we present the thermodynamic formalism of the sub-additive potentials that will be considered in Chapter 6. Sub-additive thermodynamic formalism was developed as an extension of the standard thermodynamic formalism for additive potentials, in part due to its applications to the study of measures supported on self-affine sets, see for instance [F3], [K]. Essentially, this theory is concerned with generalising the classical results of Ruelle, Bowen and Walters which connects the topological pressure with the measure theoretic entropy and Lyapunov exponents via a 'variational principle', to the setting where additive potentials are replaced by sub-additive potentials on $\Sigma = \{1, \dots, l\}^\infty$ (or more generally, a compact metric space X, equipped with a continuous mapping T).

We say that a sequence $\mathcal{F} = \{\log f_n\}_{n=1}^\infty$ of functions on Σ is *sub-additive* if each f_n is a continuous non-negative function on Σ such that

$$0 \leqslant f_{n+m}(\mathbf{i}) \leqslant f_n(\mathbf{i}) f_m(\sigma^n \mathbf{i})$$

for each $\mathbf{i} \in \Sigma$ and $n, m \in \mathbb{N}$.

In this setting, the pressure of the sub-additive sequence $\mathcal{F}$ can be defined as

$$\begin{aligned} P(\mathcal{F}) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\mathbf{i}\in\{1,\ldots,l\}^n} f_n(\mathbf{i}) \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\mathbf{i}\in\{1,\ldots,l\}^n} \exp(\log f_n(\mathbf{i})) \right) \end{aligned}$$

where the limit exists by sub-additivity of the sequence $\mathcal{F}$.

Thus, it should be clear from the second displayed equation above, that in the sub-additive setting, the sequence $\log f_n$ plays the role of $S_n f$ in the classical setting. In particular, if we put $f_n = \exp S_n f$ for all $n \in \mathbb{N}$ then $P(\mathcal{F}) = P(f)$ and indeed we are in the additive case since

$$f_{n+m}(\mathbf{i}) = \exp(S_{n+m} f(\mathbf{i})) = \exp(S_n f(\mathbf{i})) \exp(S_m f(\sigma^n \mathbf{i})) = f_n(\mathbf{i}) f_m(\sigma^n \mathbf{i})$$

for each $\mathbf{i} \in \Sigma$ and $n, m \in \mathbb{N}$.

Once $P(\mathcal{F})$ is defined for a sub-additive potential, the goal is then to prove a variational principle, analogous to (2.5). We will be dealing with a very specific case of this theory which applies to the *singular value function* related to a family of matrices. Therefore, our presentation of the theory will differ from the general formulation provided above, with the hope of improving on the clarity of the exposition. For a more general treatment of sub-additive thermodynamic formalism, the reader is directed to [CFH].

Let $\{A_i\}_{i=1}^l$ be a family of invertible $d \times d$ matrices. We denote Σ to be the full shift on l symbols. Recall that the *singular value function* $\phi^s : \Sigma^* \to \mathbb{R}^+$ was defined for each $0 \leqslant s \leqslant d$ by

$$\phi^s(\mathbf{i}) = \alpha_1(\mathbf{i}) \cdots \alpha_{\lceil s \rceil - 1}(\mathbf{i}) \alpha_{\lceil s \rceil}^{s - \lceil s \rceil}$$

where $\alpha_k(\mathbf{i})$ denotes the kth singular value of the matrix $A_{\mathbf{i}}$.

It is known that ϕ^s is submultiplicative, that is,

$$\phi^s(\mathbf{ij}) \leqslant \phi^s(\mathbf{i}) \phi^s(\mathbf{j})$$

for each $\mathbf{i}, \mathbf{j} \in \Sigma^*$; see for instance [F2, Lemma 2.1].

This allows us to define the *sub-additive pressure*

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\mathbf{i}\in\{1,\ldots,l\}^n} \phi^s(\mathbf{i}) \right) \tag{2.15}$$

where the limit exists by the submultiplicativity of ϕ^s.

Remark 2.5.14. *To reformulate this in the 'standard' sub-additive language, notice that we can define $f_n(\mathbf{i}) = \phi^s(\mathbf{i}|_n)$, so that indeed each f_n is continuous and non-negative and*

$$f_{n+m}(\mathbf{i}) = \phi^s(\mathbf{i}|_{n+m}) \leqslant \phi^s(\mathbf{i}|_n)\phi^s(\sigma^n \mathbf{i}|_m) = f_n(\mathbf{i}) f_m(\sigma^n \mathbf{i})$$

so that $\mathcal{F} = \{\log f_n\}_{n=1}^\infty$ is a sub-additive sequence. Then immediately we see that

$$P(\mathcal{F}) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\mathbf{i}\in\{1,\ldots,l\}^n} f_n(\mathbf{i}) \right) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\mathbf{i}\in\{1,\ldots,l\}^n} \phi^s(\mathbf{i}) \right) = P(s).$$

For a σ-invariant probability measure $\mu \in \mathcal{M}_\sigma(\Sigma)$, we define

$$\phi_*^s(\mu) = \lim_{n\to\infty} \frac{1}{n} \sum_{\mathbf{i}\in\Sigma^n} \mu([\mathbf{i}]) \log \phi^s(\mathbf{i}) \tag{2.16}$$

where the limit converges by sub-additivity of $\log \phi^s$. This is sometimes referred to as the Lyapunov exponent of $\{A_i\}_{i=1}^l$ (e.g. [FK]) or the Lyapunov exponent of $\{\log \phi^s(\cdot|_n)\}_{n=1}^\infty$ (e.g. [CFH]), but since in our context the singular value function is closely related to the Lyapunov exponents of the matrices A_i we will avoid this terminology for the sake of clarity.

In [K], Käenmäki proved that we have the following *variational principle.*

Proposition 2.5.15 (Variational principle)**.** *For each $0 \leqslant s \leqslant d$ we have the variational principle*

$$P(s) = \sup\{\phi_*^s(\mu) + h(\mu) : \mu \in \mathcal{M}_\sigma(\Sigma)\}. \tag{2.17}$$

Moroever, the supremum is realised by an ergodic invariant measure.

We are interested in measures which realise the supremum in (2.17). In particular, we say that μ is an *s-equilibrium state* if $\mu \in \mathcal{M}_\sigma(\Sigma)$ and

$$P(s) = \phi_*^s(\mu) + h(\mu). \tag{2.18}$$

In [FK], Feng and Käenmäki proved that for each s, there in fact exists a *unique* s-equilibrium state, and proved that it has a Gibbs property.

Remark 2.5.16. *Note that in the sub-additive setting 'Gibbs' is meant in a different sense. We say that a measure μ has the Gibbs property for the sub-additive function* $\log \phi^s$ *if there exists a constant $C > 0$ such that*

$$C^{-1} \exp(-nP(s))\phi^s(\mathbf{i}) \leqslant \mu([\mathbf{i}]) \leqslant C \exp(-nP(s))\phi^s(\mathbf{i}) \tag{2.19}$$

for all $n \in \mathbb{N}$ and $\mathbf{i} \in \Sigma^n$.

Theorem 2.5.17. *Let $\{A_i\}_{i=1}^l$ be a collection of matrices in $\mathbb{R}^d$ which satisfy the following 'irreducibility' condition: there exists a constant $c > 0$ and $m \in \mathbb{N}$ such that for all $\mathbf{i}, \mathbf{j} \in \Sigma^*$ there exists some $\mathbf{k} \in \bigcup_{n=1}^m \Sigma^n$ for which*

$$\phi^s(\mathbf{ikj}) \geqslant c\phi^s(\mathbf{i})\phi^s(\mathbf{j}). \tag{2.20}$$

Then P has a unique s-equilibrium state m^s for each $s > 0$. Furthermore, m^s is ergodic and has the Gibbs property

$$C^{-1} \exp(-nP(s))\phi^s(\mathbf{i}) \leqslant m^s([\mathbf{i}]) \leqslant C \exp(-nP(s))\phi^s(\mathbf{i}) \tag{2.21}$$

for all $n \in \mathbb{N}$ and $\mathbf{i} \in \Sigma^n$. m^s is the unique invariant probability Gibbs measure for the potential $\log \phi^s$.

Therefore in order to use Theorem 2.5.17 one must determine whether the 'irreducibility' condition, which appears in Theorem 2.5.17 above, is satisfied. The following definition for an *irreducible* collection of matrices will be relevant to answering this question.

Definition 2.5.18. *A family of $d \times d$ matrices $\{A_i\}_{i=1}^l$ with entries in $\mathbb{R}$ is irreducible over $\mathbb{R}^d$ if there is no non-zero proper linear subspace V of $\mathbb{R}^d$ such that $S_i V \subset V$ for all $1 \leqslant i \leqslant l$.*

For $0 < s \leqslant 1$, the 'irreducibility' condition formulated in Theorem 2.5.17 is satisfied when our collection of matrices is irreducible, in the sense of Definition 2.5.18, see for instance [Fe]. Moreover, if we set $d = 2$, then by [FSl] this condition is satisfied if and only if $\{A_i\}_{i=1}^l$ is irreducible. This provides us with the following refinement of Theorem 2.5.17.

Corollary 2.5.19. *Let $\{A_i\}_{i=1}^l$ be an irreducible collection of matrices in $\mathbb{R}^2$. Then P has a unique s-equilibrium state m^s for each $s > 0$. Furthermore, m^s is ergodic* *and has the Gibbs property*

$$C^{-1}\exp(-nP(s))\phi^s(\mathbf{i}) \leqslant m^s([i]) \leqslant C\exp(-nP(s))\phi^s(\mathbf{i})$$

for all $n \in \mathbb{N}$ *and* $\mathbf{i} \in \Sigma^n$. m^s *is the unique invariant probability Gibbs measure for the potential* $\log \phi^s$.

Chapter 3

Dimension gap for Bernoulli measures

3.1 Introduction

In this chapter, we will be concerned with so-called 'dimension gaps' for T-invariant projected Bernoulli measures, where T is a countable (or finite) branch expanding map.

Let $\{\mathcal{I}_n\}_{n\in\Phi}$ be a sequence of disjoint open sub-intervals of $[0,1]$ where Φ is (for now) either a finite or countable index set. Let $T_n : \overline{\mathcal{I}_n} \to [0,1]$ be a sequence of bijective expanding C^2 maps which simply correspond to the branches of the map $T : [0,1] \to [0,1]$ which is defined as $T(x) = T_n(x)$ if $x \in \overline{\mathcal{I}_n}$ (taking care at the boundary points of intervals, which we'll make precise later) and $T(0) = 0$. By using Markov partitions, one can construct a symbolic coding of T by the full shift on either a finite or countable alphabet, depending on the cardinality of Φ. In particular, we can define a coding map $\Pi : \Sigma \to [0,1]$ given by

$$\Pi(\mathbf{i}) = \lim_{n\to\infty} T_{i_1}^{-1} \circ \cdots \circ T_{i_n}^{-1}([0,1]) \tag{3.1}$$

where $\Sigma = \{1,\ldots,k\}^{\mathbb{N}}$ if $\Phi = \{1,\ldots,k\}$ and $\Sigma = \mathbb{N}^{\mathbb{N}}$ if $\Phi = \mathbb{N}$, and Σ is equipped with the full shift σ.

The repeller of the map T is defined as the set

$$J = [0,1] \setminus \bigcup_{n=0}^{\infty} T^{-n}(\{0\}) \tag{3.2}$$

where the set of points that is removed is a countable set. For all $\mathbf{i} \in \Sigma$ we have

$T \circ \Pi = \Pi \circ \sigma$ and so the coding map Π allows us to move between the symbolic model and the geometric model and importantly allows us to project σ-invariant measures on Σ to T-invariant measures on $[0, 1]$. Our main focus will be (projected) Bernoulli measures. Throughout this chapter we will denote by $m_{\mathbf{p}}$ the Bernoulli measure associated to the probability vector $\mathbf{p} = (p_i)_{i\in\Phi} \in \mathcal{P}$, where $\mathcal{P}$ denotes the simplex of all probability vectors. We will denote the corresponding projected measure by $\mu_{\mathbf{p}} = m_{\mathbf{p}} \circ \Pi^{-1}$, and also refer to $\mu_{\mathbf{p}}$ as a Bernoulli measure.

We are interested in the Hausdorff dimension of Bernoulli measures, which from now we will just refer to as the dimension and denote by dim. In particular, we are interested in what properties of T give rise to a *dimension gap*, by which we mean that

$$\sup_{\mathbf{p}\in\mathcal{P}} \dim \mu_{\mathbf{p}} \leqslant 1 - c_0 \tag{3.3}$$

for some $c_0 > 0$. The reason we refer to this as a dimension gap is because, by the work of Walters [W] (which we'll expand on in more detail in Section 3.2), every map T that we'll consider will admit a unique absolutely continuous invariant probability measure μ_T, which corresponds to the unique invariant Gibbs measure for the potential $-\log|T'|$. Therefore since $\dim \mu_T = 1$, (3.3) refers to the fact that all Bernoulli measures have dimension which is uniformly bounded away from the maximal realised dimension of an invariant measure.

While we are primarily interested in the countable branch setting, in order to build up some intuition for the problem we begin by considering the simpler setting of the finite branch expanding system, where we have the index set $\Phi = \{1, ..., k\}$. Firstly, suppose that $-\log|T'| = \sum \log p_n \mathbf{1}_{\mathcal{I}_n} =: f_{\mathbf{p}}$ for some probability vector $\mathbf{p} = (p_1, \ldots, p_k)$, where the sum in the definition of $f_{\mathbf{p}}$ is taken over all $p_n \neq 0$. Notice that since $f_{\mathbf{p}}$ is locally constant, this means that T is necessarily a linear map. Since $\mu_{\mathbf{p}}$ coincides with the Gibbs measure $\mu_{f_{\mathbf{p}}}$, it follows that $\mu_T = \mu_{\mathbf{p}}$, that is, $\mu_{\mathbf{p}}$ is an absolutely continuous measure and $\dim \mu_{\mathbf{p}} = 1$.

It is easy to verify this by considering an example. For instance, if we consider the doubling map $T(x) = 2x \mod 1$ and fix $\mathbf{p} = (\frac{1}{2}, \frac{1}{2})$ which corresponds to the reciprocals of the slopes of the branches, then clearly $\mu_T = \mu_{\mathbf{p}} = \mathcal{L}|_{[0,1]}$, where $\mathcal{L}$ denotes Lebesgue measure.

Next, suppose that T is a map for which $-\log|T'|$ is cohomologous to the potential $f_{\mathbf{p}}$ for some $\mathbf{p} \in \mathcal{P}$, where $\mathcal{P}$ denotes the simplex of all probability vectors. By Proposition 2.5.4 we can verify that again $\mu_T = \mu_{\mathbf{p}}$, so the absolutely continuous measure is again a Bernoulli measure. Therefore, we see that a map does not necessarily have to be linear for the measure of maximal dimension to be Bernoulli. On the other hand, we can again use Proposition 2.5.4 and the variational principle to verify that this is the *only* instance in which a Bernoulli measure can have dimension 1. Indeed, if $-\log|T'|$ is *not* cohomologous to any potential $f_{\mathbf{p}}$ then this implies that $\mu_T \neq \mu_{\mathbf{p}}$ for any $\mathbf{p}$ by Proposition 2.5.4. Thus by the variational principle (2.5)

$$h(\mu_{\mathbf{p}}) - \int \log|T'|d\mu_{\mathbf{p}} < h(\mu_T) - \int \log|T'|d\mu_T = P(-\log|T'|).$$

Combining this with the fact that in the finite branch setting $\dim \mu_{\mathbf{p}} = \frac{h(\mu_{\mathbf{p}})}{\chi(\mu_{\mathbf{p}})}$ and $P(-\log|T'|) = 0$ (both facts will be stated more generally and attributed properly later in the chapter), it follows that $\dim \mu_{\mathbf{p}} < 1$ for any $\mathbf{p} \in \mathcal{P}$. In fact we can say more; since the map $m \to \dim m \circ \Pi^{-1}$ (restricted to $\mathcal{M}_\sigma(\Sigma)$) is upper semi-continuous and $\{m_{\mathbf{p}} : \mathbf{p} \in \mathcal{P}\}$ is a compact subset of $\mathcal{M}_\sigma(\Sigma)$, it follows that there exists $\mathbf{q} \in \mathcal{P}$ such that

$$\sup_{\mathbf{p}\in\mathcal{P}} \dim \mu_{\mathbf{p}} = \dim \mu_{\mathbf{q}} < 1$$

that is, we have a dimension gap. Note however, that this approach gives no quantitative information about the size of the gap and it is a difficult open problem to establish what $\mu_{\mathbf{q}}$ actually is.

We'd like to study the analogue of this problem in the countable branch setting. A key difficulty here is that the map $m \to \dim m \circ \Pi^{-1}$ (restricted to $\mathcal{M}_\sigma(\Sigma)$) is no longer necessarily upper semi-continuous and also $\{m_{\mathbf{p}} : \mathbf{p} \in \mathcal{P}\}$ is not compact. However, we will see that the problem still boils down to determining 'how different' the potentials $-\log|T'|$ and $f_{\mathbf{p}}$ are.

Therefore, for the rest of this chapter we fix $\Phi = \mathbb{N}$. The study of countable branch expanding interval maps originates with the Gauss map $G : [0, 1] \to [0, 1]$ given by $G(x) = \frac{1}{x} \mod 1$ if $x > 0$ and $G(0) = 0$. The Gauss map has attracted a lot of interest in the literature due to its close connections with the continued fraction expansions of real numbers. In the 1940s, Bissinger and Everett [Bis; E] were interested in generalisations of the Gauss map which could give rise to real representations of numbers, analogous to confinued fraction expansions, which they coined *f-expansions*. In 1957, Rényi [R] extended their work and showed that under some regularity conditions on f, one could determine the existence of an absolutely continuous invariant measure.

We will be working in the setting of *Expanding-Markov-Rényi maps* which we now introduce. From now let $\{\mathcal{I}_n\}_{n\in\mathbb{N}}$ be a countable collection of non-empty disjoint subintervals of $[0,1]$ and let $T_n : \overline{\mathcal{I}_n} \to [0,1]$ be a sequence of expanding bijective C^2 maps (so $|T_n'| > 1$). Define $T : [0,1] \to [0,1]$ as

$$\begin{aligned} T(x) &= T_n(x) \quad \text{if} \quad x \in \overline{\mathcal{I}}_n \\ T(0) &= 0 \end{aligned}$$

where we put $T(x) = T_k(x)$ for $k = \min\{n : x \in \overline{\mathcal{I}}_n\}$ if x is a common endpoint of two intervals. Similarly, we adopt the convention that $T'(x) = T_k'(x)$ where $k = \min\{n : x \in \overline{\mathcal{I}}_n\}$. For $\mathbf{n} = n_1 \ldots n_k \in \mathbb{N}^k$, we denote $T_{\mathbf{n}}^{-1} = T_{n_1}^{-1} \circ \ldots T_{n_k}^{-1}$.

Let $T : [0,1] \to [0,1]$ be a countable branch expanding map as described above for which $(0,1) \subset \bigcup_{n\in\mathbb{N}} \overline{\mathcal{I}_n}$. Then we say that T is an *Expanding-Markov-Rényi map* (or EMR map) if it satisfies the following conditions:

(1) **Some iterate of T is uniformly expanding.** There exists $l \in \mathbb{N}$ and $\Lambda > 1$ for which
$$|(T^l)'(x)| \geqslant \Lambda^l > 1$$
for all $x \in [0,1]$.

(2) **Markov.** T admits a Markov partition. (Note: under our assumptions this is already satisfied and moreoever T is coded by the full shift on $\mathbb{N}$).

(3) **Rényi condition.** There exists $\kappa < \infty$ such that
$$\sup_{n\in\mathbb{N}} \sup_{x,y,z\in\mathcal{I}_n} \left| \frac{T''(x)}{T'(y)T'(z)} \right| \leqslant \kappa < \infty.$$

Remark 3.1.1. *The thermodynamic formalism of maps that satisfy (1)-(3) were studied by Walters in [W]. In [PW] Pollicott and Weiss introduced the term 'Expanding-Markov-Rényi' to mean maps which satisfied the conditions given above except with a more general Markov structure. In [IJ1] and [IJ2], Iommi and Jordan adopted the term 'Expanding Markov Rényi' maps to mean ones which satisfied (1)-(3) and were coded by the full shift, but for which the union of the intervals $\overline{\mathcal{I}_n}$ no longer necessarily exhausted* $(0,1)$.

Before we proceed with attacking the dimension gap problem, we give some examples of EMR maps and their important properties.

3.1.1 Examples of EMR maps

We now give some examples of Expanding-Markov-Rényi maps. One of the most well-known examples of such a map is the *Gauss map*, sometimes known as the *continued fraction map* $G : [0,1] \to [0,1]$ given by

$$G(x) = \begin{cases} \frac{1}{x} \mod 1 & x > 0 \\ 0 & x = 0 \end{cases}$$

To verify that conditions (1)-(3) hold, notice that since $|G'(x)| \to 1$ as $x \to 1$, G itself is not uniformly expanding but if we consider the second iterate of the map we see that condition (1) holds for $l = 2$ and $\Lambda = \frac{3}{2}$. Since $|G'(x)| = \frac{1}{x^2}$ and $|G''(x)| = \frac{2}{x^3}$, the Rényi condition holds with $\kappa = 16$.

The Gauss map has been well studied due to its connections with the continued fraction expansion of a number. Given $x \in (0,1) \setminus \mathbb{Q}$, there exists a unique sequence $\{a_n\}_{n \in \mathbb{N}}$, known as the continued fraction expansion of x such that

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

where all of the digits $a_n \in \mathbb{N}$. Then the Gauss map generates the *continued fraction expansion* of a point $x \in (0,1) \setminus \mathbb{Q}$ in the sense that the orbit of a point under G encodes its continued fraction expansion via the intervals that it visits: $a_n = k \Leftrightarrow G^{n-1}(x) \in (\frac{1}{k+1}, \frac{1}{k})$.

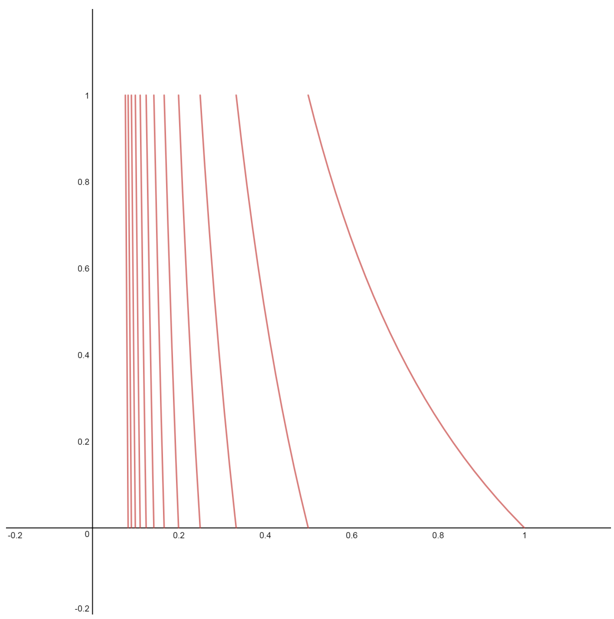

Figure 3.1: Gauss map.

α-Lüroth maps

The various Lüroth maps provide a class of linear examples of Expanding-Markov-Rényi maps.

Let $\alpha = \{A_n\}_{n\in\mathbb{N}}$ be a countable partition of $[0,1]$ consisting of intervals of the form $(x,y]$ ordered from right to left starting with A_1. Let $|A_n|$ denote the diameter of A_n and $t_n = \sum_{k=n+1}^{\infty} |A_k|$ denote the combined length of the sets in the partition that come after A_n. Then the α-Lüroth map is defined as

$$L_\alpha(x) = \begin{cases} \frac{t_n - x}{|A_n|} & \text{for } x \in A_n \\ 0 & x = 0 \end{cases}$$

These can be defined analogously to produce maps with monotone increasing branches.

Clearly L_α satisfies (1) for $l = 1$ and since it is a linear map, $T'' = 0$ so it follows that the Rényi condition is also satisfied.

If we take the partition $\alpha = \{(\frac{1}{n+1}, \frac{1}{n}]\}_{n\in\mathbb{N}}$ then we obtain the *alternating Lüroth map* $L : [0,1) \to [0,1)$

$$L(x) = \begin{cases} \left(\frac{1}{n} - x\right) n(n+1) & x \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \\ 0 & x = 0 \end{cases}$$

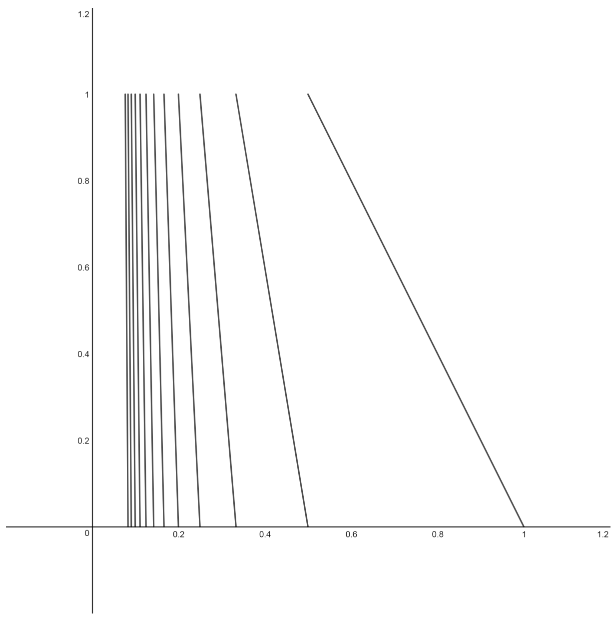

Figure 3.2: Alternating Lüroth map.

f-expansions

For examples of non-linear Expanding-Markov-Rényi maps, we can consider transformations $T : [0,1] \to [0,1]$ which arise from f-expansions. Let f either be a strictly decreasing continuous function $f : [1,\infty) \to (0,1]$ such that $f(1) = 1$, $\lim_{x\to\infty} f(x) = 0$ or let f be a strictly increasing continuous function $f : [0,\infty) \to [0,1)$ such that $f(0) = 0$, $\lim_{x\to\infty} f(x) = 1$ and define the map $T : [0,1] \to [0,1]$

$$T(x) \quad = \quad f^{-1}(x) \mod 1. \tag{3.4}$$

For $x \in (0,1)$ set $r_0(x) = x$ and $r_{i+1}(x) = f^{-1}(r_i(x)) \mod 1$ for $i \geqslant 0$. Let X be the set of all $x \in (0,1)$ such that $r_i(x) > 0$ for all $i \geqslant 0$ (which is the whole interval minus some countable set of points) and for $x \in X$ and $i \geqslant 1$ define

$$\alpha_i(x) = [f^{-1}(r_{i-1}(x))] \in \mathbb{N}$$

where $[\cdot]$ denotes the integer part of a number. Under some assumptions, one can deduce that

$$x \quad = \quad f(\alpha_1(x) + f(\alpha_2(x) + \cdots)) \tag{3.5}$$

for all $x \in (0,1)$, that is, the expansion on the right hand side of (3.5) converges to x. When the representation in (3.5) exists, we call it the *f-expansion* of x.

f-expansions were introduced in the decreasing case by Bissinger [Bis], and in the increasing case by Everett [E]. They were introduced as a generalisation of the continued fraction expansion, in order to investigate real representations of numbers. In [R], Rényi showed that under an assumption on the regularity of f, (3.5) holds for all $x \in (0,1)$. In particular, the condition in the decreasing case was that $|f(x) - f(y)| \leqslant |x - y|$ for all $x, y \geqslant 1$ and that there exists some $\lambda < 1$ such that for all $x, y > 1 + f(2)$, $|f(x) - f(y)| \leqslant \lambda |x - y|$. When f was increasing, the condition was that $|f(x) - f(y)| < |x - y|$ for all $x, y \geqslant 0$.

Note that in the decreasing case, when $f : [1,\infty) \to (0,1]$ is given by $f(x) = \frac{1}{x}$ then the map T is just the Gauss map and (3.5) just gives the usual continued fraction expansion of x.

If the map T arising from the f-expansion satisfies properties (1)-(3), then T is an EMR map.

3.1.2 Properties of EMR maps

In this section we summarise some useful properties of Expanding-Markov-Rényi maps which will be used extensively throughout the remainder of the chapter. Let $T : [0, 1] \to [0, 1]$ be an Expanding-Markov-Rényi map.

Firstly, the the uniformity of the *lower bound* for the rate of expansion of (the lth iterate of) the map T gives rise to a uniform *upper bound* for the diameters of projected cylinders. In this sense, the uniform rate of expansion of the map and the uniform rate of contraction of the cylinders are essentially two sides of the same coin.

Proposition 3.1.2. *For all $n \in \mathbb{N}$ and $(i_1, \ldots, i_n) \in \mathbb{N}^n$ the diameter $|\mathcal{I}_{i_1,\ldots,i_n}| \leqslant \frac{\Lambda^{l-1}}{\Lambda^n}$.*

Proof. It is enough to show that for all $n \in \mathbb{N}$, $(T^n(x))' \geqslant \frac{\Lambda^n}{\Lambda^{l-1}}$ for any $x \in \mathcal{I}_{i_1,\ldots,i_n}$. Let $n = kl + m$ where $k \in \mathbb{N}$ and $0 \leqslant m \leqslant l - 1$. Then by the chain rule,

$$\begin{aligned} |(T^n)'(x)| &= |(T^{kl})'(T^m x)| \cdot |(T^m)'(x)| \\ &\geqslant |(T^l)'(T^{(k-1)l+m}x)| \cdot |(T^l)'(T^{(k-2)l+m}x)| \cdots |(T^l)'(T^m x)| \\ &\geqslant \Lambda^{kl} \end{aligned}$$

where the second line follows because $|(T^m)'(x)| \geqslant |T'(x)| > 1$ and the third because $|(T^l)'(x)| \geqslant \Lambda^l$. Thus,

$$|(T^n)'(x)| \geqslant \Lambda^{kl} = \frac{\Lambda^n}{\Lambda^m} \geqslant \frac{\Lambda^n}{\Lambda^{l-1}}.$$

□

By a slight abuse of notation we will adopt the same notation that was used to denote the space of locally Hölder potentials on Σ in Section 2.5.2, in order to denote the analogous space of real potentials.

We say that $f : (0, 1) \to \mathbb{R}$ is δ*-locally Hölder* if there exists $C > 0$ such that for all $n \geqslant 1$ the variations $\mathrm{var}_n(f)$ decay exponentially, that is,

$$\mathrm{var}_n(f) = \sup_{i_1 \ldots i_n \in \mathbb{N}^n} \{|f(x) - f(y)| : x, y \in \mathcal{I}_{i_1,\ldots,i_n}\} \leqslant C\delta^n. \tag{3.6}$$

Denote the space of all δ-locally Hölder functions by F_δ. We say that f is locally Hölder continuous if f is δ-locally Hölder continuous for some $0 < \delta < 1$. Define the seminorm $[f]_\delta$ to be the smallest constant one can take in (2.7). Therefore $F_\delta = \{f : (0, 1) \to \mathbb{R} : [f]_\delta < \infty\}$. Let $\mathcal{F}_\delta$ denote the space of all bounded and

δ-locally Hölder functions. Define the norm $\|\cdot\|_\delta = [\cdot]_\delta + \|\cdot\|_\infty$ and observe that this makes $(\mathcal{F}_\delta, \|\cdot\|_\delta)$ a normed space.

By combining the uniform contraction rate of cylinders (equivalently the uniform expansion rate of the map) with the Rényi property we obtain the following important result which tells us that $\log|T'|$ is locally Hölder continuous.

Proposition 3.1.3. $\log|T'| \in F_{\Lambda^{-1}}$ *and moreover,* $[\log|T'|]_{\Lambda^{-1}} \leqslant \kappa\Lambda^l$.

Proof. Let $i_1, \dots, i_n \in \mathbb{N}^n$ for some $n \geqslant 1$ and $x, y \in \mathcal{I}_{i_1,\dots,i_n}$. Then

$$\begin{aligned}
|\log|T'(x)| - \log|T'(y)|| &\leqslant \sup_{w\in\mathcal{I}_{i_1,\dots,i_n}} \left|\frac{T''(w)}{T'(w)}\right| |x-y| \\
&\leqslant \sup_{w\in\mathcal{I}_{i_1,\dots,i_n}} \left|\frac{T''(w)}{T'(w)}\right| \sup_{u\in\mathcal{I}_{i_1,\dots,i_n}} \left|\frac{1}{(T^n)'(u)}\right| \\
&\leqslant \sup_{w\in\mathcal{I}_{i_1,\dots,i_n}} \left|\frac{T''(w)}{T'(w)}\right| \sup_{u\in\mathcal{I}_{i_1,\dots,i_n}} \left|\frac{1}{T'(u)}\right| \sup_{u\in\mathcal{I}_{i_1,\dots,i_n}} \left|\frac{1}{(T^{n-1})'(Tu)}\right| \\
&\leqslant \kappa\frac{\Lambda^{l+1-1}}{\Lambda^n} = \kappa\frac{\Lambda^l}{\Lambda^n}
\end{aligned}$$

where the third line follows by the chain rule. □

Finally, we obtain a property which is commonly known as a *bounded distortion property.* This allows us to replace the diameter of a cylinder $\mathcal{I}_{i_1,\dots,i_n}$ with the reciprocal of the derivative of the nth power of T at any point which has symbolic expansion beginning with the word $i_1, \dots, i_n$ (subject to some uniformly bounded error). The following derivation of the bounded distortion property from the Rényi condition is a classical result, a version of which can be found for instance in Lemma 2 of [CFS, Chapter 7.4].

Proposition 3.1.4 (Bounded distortion)**.** *For any* $n \geqslant 1$ *and cylinder* $\mathcal{I}_{i_1,\dots,i_n}$ *with* $x, y \in \mathcal{I}_{i_1,\dots,i_n}$ *we have*

$$\exp\left(\frac{-\kappa\Lambda^l}{\Lambda-1}\right) \leqslant \frac{(T^n)'(x)}{(T^n)'(y)} \leqslant \exp\left(\frac{\kappa\Lambda^l}{\Lambda-1}\right). \tag{3.7}$$

Also,

$$\exp\left(\frac{-\kappa\Lambda^l}{\Lambda-1}\right) \leqslant \frac{|\mathcal{I}_{i_1,\dots,i_n}|}{|(T^n)'(x)|^{-1}} \leqslant \exp\left(\frac{\kappa\Lambda^l}{\Lambda-1}\right). \tag{3.8}$$

Proof. We begin by proving (3.7). Fix some $n \geqslant 1$, some cylinder $\mathcal{I}_{i_1,\ldots,i_n}$ and $x, y \in \mathcal{I}_{i_1,\ldots,i_n}$. Then by the chain rule and Proposition 3.1.3,

$$\begin{aligned}
\frac{(T^n)'(x)}{(T^n)'(y)} &= \exp\left(\log|(T^n)'(x)| - \log|(T^n)'(y)|\right) \\
&= \exp\left(\sum_{k=0}^{n-1}\log|T'(T^k x)| - \sum_{k=0}^{n-1}\log|T'(T^k y)|\right) \\
&\leqslant \exp\left(\kappa\Lambda^l \sum_{k=1}^{n}\frac{1}{\Lambda^k}\right) \\
&\leqslant \exp\left(\frac{\kappa\Lambda^l}{\Lambda - 1}\right)
\end{aligned}$$

and since x and y were chosen arbitrarily the lower bound in (3.7) also follows from this. Then to see (3.8), fix some $y \in \mathcal{I}_{i_1,\ldots,i_n}$ and observe that

$$\exp\left(\frac{-\kappa}{1-\Lambda}\right)|\mathcal{I}_{i_1,\ldots,i_n}| \leqslant \int_{\mathcal{I}_{i_1,\ldots,i_n}} \frac{(T^n)'(x)}{(T^n)'(y)}dx \leqslant \exp\left(\frac{\kappa\Lambda^l}{\Lambda-1}\right)|\mathcal{I}_{i_1,\ldots,i_n}|.$$

It follows that

$$\begin{aligned}
\exp\left(\frac{-\kappa}{1-\Lambda}\right)|\mathcal{I}_{i_1,\ldots,i_n}||(T^n)'(y)| \leqslant 1 &= \int_{\mathcal{I}_{i_1,\ldots,i_n}} |(T^n)'(x)|dx \\
&\leqslant \exp\left(\frac{\kappa\Lambda^l}{\Lambda-1}\right)|\mathcal{I}_{i_1,\ldots,i_n}||(T^n)'(y)|
\end{aligned}$$

from which (3.8) follows. □

3.2 Previous work

We now outline some relevant prior results, including the work of Walters [W] and Kifer, Peres and Weiss [KPW], which are important to the story of the dimension gap problem.

We observe that by the Kolmogorov-Sinai theorem, the measure-theoretic entropy $h(\mu_{\mathbf{p}})$ has the simple form $h(\mu_{\mathbf{p}}) = -\sum_{n=1}^{\infty} p_n \log p_n$. We define the *Lyapunov exponent* of an ergodic measure μ (with respect to the map T) by

$$\chi(\mu) = \int \log|T'|d\mu$$

which measures the amount of expansion (or contraction) in the system from the point of view of the measure μ.

In 1966 Kinney and Pitcher [KP2] first proved that the dimension of any projected Bernoulli measure for the Gauss map was given by the formula

$$\dim \mu_{\mathbf{p}} = \frac{-\sum_{n=1}^{\infty} p_n \log p_n}{-\int 2 \log x d\mu_{\mathbf{p}}(x)} \tag{3.9}$$

provided that the entropy $h(\mu_{\mathbf{p}}) = -\sum_{n=1}^{\infty} p_n \log p_n < \infty$. Notice that is not clear from (3.9) whether or not $\dim \mu_{\mathbf{p}}$ is less than 1. (3.9) is now known to be a specific example of the more general result which says that for an ergodic invariant measure with finite entropy we have the following closed-form formula for the dimension, which links the dimension of the measure with the entropy and Lyapunov exponent of the measure (see for instance Theorem 4.4.2 in [MU1]).

Proposition 3.2.1 (Volume Lemma)**.** *If μ is an ergodic T-invariant probability measure on $[0,1]$ and $h(\mu) < \infty$ then the Hausdorff dimension of μ is given by*

$$\dim \mu = \frac{h(\mu)}{\chi(\mu)}.$$

It is also a classical result that the dimension of the repeller J is encoded as the zero of the pressure. This result in our setting was proved by Mauldin and Urbański, see for instance Theorem 3.15 in [MU3].

Proposition 3.2.2 (Bowen-Manning-McCluskey formula)**.** *Let J be the repeller of T. Then $\lambda = \dim J$ satisifes $P(-\lambda \log |T'|) = 0$.*

In 1978, Walters [W] developed the thermodynamic formalism of countable branch expanding maps, where he proved a generalised Ruelle-Perron-Frobenius theorem for countable branch maps and potentials with sufficient regularity. He used this to prove the existence of a unique Gibbs state μ_T for the potential $-\log |T'|$. Moreoever he proved that μ_T satisfied a variational principle and was the unique equilibrium state for $-\log |T'|$, that is, the unique absolutely continuous measure for the system. This means that for all invariant probability measures $\mu \neq \mu_T$ with $\int \log |T'| d\mu < \infty$,

$$h(\mu) - \int \log |T'| d\mu < h(\mu_T) - \int \log |T'| d\mu_T = P(-\log |T'|) = 0$$

where the last equality follows by Proposition 3.2.2. Notice that this implies that for any invariant measure μ for which $\int \log |T'| d\mu < \infty$, then

$$\dim \mu = \frac{h(\mu)}{\chi(\mu)} \leqslant 1$$

with equality if and only if $\mu_T = \mu$. Therefore if $\mu_T \neq \mu_{\mathbf{p}}$ for any $\mathbf{p} \in \mathcal{P}$, then provided $h(\mu_{\mathbf{p}}) < \infty$, Walters' work implies that $\dim \mu_{\mathbf{p}} < 1$. No further quantitative information about the size of $\dim \mu_{\mathbf{p}}$ can be obtained via this approach.

The next major breakthrough was in 2001, when Kifer, Peres and Weiss [KPW] showed that under some additional assumptions on the map T, $\sup_{\mathbf{p}\in\mathcal{P}} \dim \mu_{\mathbf{p}} < 1 - \psi$ for some constant ψ that could be made explicit. In particular, under the assumptions that

1. there exists some $s < 1$ for which
$$\sum_{n\in\mathbb{N}} |\mathcal{I}_n|^s < \infty,$$
2. the absolutely continuous measure μ_T is not a Bernoulli measure,

they proved that there existed a *dimension gap*, that is, $\sup_{\mathbf{p}\in\mathcal{P}} \dim \mu_{\mathbf{p}} < 1 - \psi$ for some constant $\psi > 0$. Importantly, their formula held even when $\mu_{\mathbf{p}}$ had infinite entropy.

In particular, they applied their results to the Gauss map, and obtained that
$$\sup_{\mathbf{p}\in\mathcal{P}} \dim \mu_{\mathbf{p}} < 1 - 10^{-7}.$$

They also gave a characterisation of when one is in the setting that μ_T is not Bernoulli: they showed that μ_T is Bernoulli if and only if
$$F \circ T \circ F^{-1} \qquad \text{is linear} \tag{3.10}$$
where F is the diffeomorphism $F(t) = \mu([0, t])$.

Their proof was separated into two cases, dependent on whether the entropy of the measure was finite or infinite. The infinite entropy case was tackled by looking at 'short expansion intervals' and the finite entropy case was tackled by looking at sets of large deviations for the frequency of certain digits from the one provided by μ_T. We will provide the proof for infinite entropy measures in section 3.4, and for now we describe their proof for the case where the entropy is finite.

The proof for the finite entropy case follows from looking at the dimension of sets of points whose symbolic coding sees a frequency of a certain digit appearing which differs from the one corresponding to the absolutely continuous measure μ_T. Their approach is as follows. Suppose that μ_T is not Bernoulli and let $\mu_{\mathbf{p}}$ be some

Bernoulli measure. For a finite word $\mathbf{w} \in \Sigma^*$ and $\delta > 0$, fix

$$\Gamma_{\mathbf{w}}^{\delta} = \left\{ x \in (0,1) : \limsup_{n \to \infty} \left| \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{\mathbf{w}}(T^i x) - \mu_T(\mathcal{I}_{\mathbf{w}}) \right| > \delta \right\}.$$

Observe that if $\mathbf{w} \in \Sigma^*$ is some word for which

$$|\mu_{\mathbf{p}}(\mathcal{I}_{\mathbf{w}}) - \mu_T(\mathcal{I}_{\mathbf{w}})| > \delta$$

then by the Birkhoff Ergodic Theorem, $\mu_{\mathbf{p}}(\Gamma_{\mathbf{w}}^{\delta}) = 1$ and so $\dim \mu_{\mathbf{p}} \leqslant \dim \Gamma_{\mathbf{w}}^{\delta}$.

Since μ_T is not Bernoulli, there exists some word $\mathbf{a} \in \Sigma^*$ for which $\mu_T(\mathcal{I}_{\mathbf{aa}}) \neq (\mu_T(\mathcal{I}_{\mathbf{a}}))^2$. Let

$$\delta_T = \frac{|\mu_T(\mathcal{I}_{\mathbf{aa}}) - (\mu_T(\mathcal{I}_{\mathbf{a}}))^2|}{3} > 0.$$

Fix $\mathbf{p}$ and put $\delta = |\mu_{\mathbf{p}}(\mathcal{I}_{\mathbf{a}}) - \mu_T(\mathcal{I}_{\mathbf{a}})|$. Then it follows that

$$\begin{aligned} |\mu_T(\mathcal{I}_{\mathbf{aa}}) - (\mu_{\mathbf{p}}(\mathcal{I}_{\mathbf{a}}))^2| &\geqslant |\mu_T(\mathcal{I}_{\mathbf{aa}}) - (\mu_T(\mathcal{I}_{\mathbf{a}}))^2| - |(\mu_T(\mathcal{I}_{\mathbf{a}}))^2 - (\mu_{\mathbf{p}}(\mathcal{I}_{\mathbf{a}}))^2| \\ &= |\mu_T(\mathcal{I}_{\mathbf{aa}}) - (\mu_T(\mathcal{I}_{\mathbf{a}}))^2| - |\mu_T(\mathcal{I}_{\mathbf{a}}) - \mu_{\mathbf{p}}(\mathcal{I}_{\mathbf{a}})||\mu_T(\mathcal{I}_{\mathbf{a}}) + \mu_{\mathbf{p}}(\mathcal{I}_{\mathbf{a}})| \\ &\geqslant 3\delta_T - 2\delta. \end{aligned}$$

Therefore, if $\delta < \delta_T$, $\dim \mu_{\mathbf{p}} \leqslant \dim \Gamma_{\mathbf{aa}}^{\delta_T}$ and if $\delta \geqslant \delta_T$, $\dim \mu_{\mathbf{p}} \leqslant \dim \Gamma_{\mathbf{a}}^{\delta} \leqslant \dim \Gamma_{\mathbf{a}}^{\delta_T}$. Therefore

$$\sup_{\mathbf{p} \in \mathcal{P}} \dim \mu_{\mathbf{p}} \leqslant \max\{\dim \Gamma_{\mathbf{aa}}^{\delta_T}, \dim \Gamma_{\mathbf{a}}^{\delta_T}\}. \tag{3.11}$$

Kifer, Peres and Weiss then proved that for any $\delta > 0$

$$\sup_{\mathbf{w}} \dim \Gamma_{\mathbf{w}}^{\delta} < 1$$

which in light of (3.11) implies the existence of a dimension gap.

In [KPW] the authors also present analogous results for k-step Markov measures. More recently, Rapaport [Rap] extended the work of [KPW] to the non-stationary case, to show that there is a uniform dimension gap for all measures with respect to which the digits of the f-expansion are independent but not necessarily i.i.d. Notice that these measures will no longer be invariant.

3.3 Main result

In this section we present and discuss the statement of our main result of this chapter; a theorem about the existence of a dimension gap under some different assumptions to the analogous theorem in [KPW].

Before we state the result, we introduce some additional notation: for a finite word $\mathbf{w} \in \Sigma^*$ we denote the periodic point in Σ obtained by repeating the finite word $\mathbf{w}$ by $(\mathbf{w})^\infty$ (note that since Σ is the full shift space this is well defined for any $\mathbf{w} \in \Sigma^*$). We denote the projection of this periodic point (which is periodic for T) by $z_{\mathbf{w}} = \Pi((\mathbf{w})^\infty)$.

For simplicity, in what follows we'll assume that if $T' > 0$ then $\mathcal{I}_1 = (0, a)$ for some $a < 1$ and if $T' < 0$ then $\mathcal{I}_1 = (b, 1)$ for some $b > 0$.

Theorem 3.3.1. *Let $T : \bigcup_{n\in\mathbb{N}} \overline{\mathcal{I}}_n \to [0,1]$ be an Expanding-Markov-Rényi map such that*

(1) ***Non-linearity condition.***

$$T'(z_1)T'(z_2) \neq T'(z_{12})T'(z_{21}). \tag{3.12}$$

(2) ***Polynomial decay of interval lengths.*** *For some $s < 1$,*

$$\sum_{n\in\mathbb{N}} |\mathcal{I}_n|^s < \infty.$$

(3) ***Technical assumptions on derivatives.*** *The derivative T' is monotone. Additionally: (a) if T is an increasing map ($T' > 0$) then $|T'(z_1)| > 1$ (b) if T is a decreasing map ($T' < 0$) then for all $n \in \mathbb{N}$, $(T^2)'|_{\mathcal{I}_n}$ must be monotone.*

Then

$$\sup_{\mathbf{p}\in\mathcal{P}} \dim \mu_{\mathbf{p}} \leqslant 1 - \psi$$

for some $\psi > 0$.

Remark 3.3.2. *In principle, ψ can be estimated in terms of Λ, κ, s and θ where*

$$\theta = \left|\log \frac{T'(z_1)T'(z_2)}{T'(z_{12})T'(z_{21})}\right| \neq 0.$$

However, unfortunately for the Gauss map this yields a very poor estimate for the gap compared to the one obtained by Kifer, Peres and Weiss. Essentially this is due to the fact that κ appears in several exponents in our estimate combined with the *fact that $\kappa = 16$ for the Gauss map. See Remark 4.6.4. For this reason, we choose not to include an explicit estimate of ψ.*

We now discuss the three conditions that we impose on our maps. First of all, the definitive ingredient which forces the dimension gap is the non-linearity condition. Clearly, if T was linear, the derivatives $T'(z_1) = T'(z_{12})$ and $T'(z_2) = T'(z_{21})$, and thus we'd have equality in (3.12). However, this condition should hold for a generic non-linear map and as such, θ can be thought of as a constant that measures the amount of non-linearity in the system.

The second condition is satisfied when the diameters of the intervals $|\mathcal{I}_n|$ are decaying polynomially. For instance, in the case of the Gauss map, it would suffice to take any constant $s > \frac{1}{2}$. The assumption that $\sum |\mathcal{I}_n|^s < \infty$ for some $0 < s < 1$ isn't usually equivalent to polynomial decay for the $|\mathcal{I}_n|$. (For instance, consider the example where $|\mathcal{I}_n| = \frac{c}{n^2}$ whenever n is not a power of 2, and $|\mathcal{I}_n| = \frac{c}{(\log n)^2}$ whenever n is a power of 2, where c is a normalising constant. Clearly $|\mathcal{I}_n|^s$ is summable but not polynomially decaying for any $s > \frac{1}{2}$). However, in our setting it turns out that these two assumptions are actually equivalent, since the monotonicity of T' forces $\{|\mathcal{I}_n|\}_{n\in\mathbb{N}}$ to be a decreasing sequence. In fact, if we define

$$s_0 := \inf\left\{s : \sum_{n=1}^{\infty} |\mathcal{I}_n|^s < \infty\right\} \tag{3.13}$$

and

$$t_0 := \inf\left\{t : |\mathcal{I}_n| \leqslant \frac{C}{n^{\frac{1}{t}}} \text{for some uniform constant } C\right\} \tag{3.14}$$

then $s_0 = t_0$. We provide a quick proof of this fact.

Lemma 3.3.3. *Let T be an EMR map that satisfies assumptions (1)-(3) and let s_0, t_0 be defined as in (3.13) and (3.14). Then $s_0 = t_0$.*

Proof. The first direction $s_0 \leqslant t_0$ is obvious. To see that $s_0 \geqslant t_0$, let $s > s_0$ so that $\sum_{n=1}^{\infty} |\mathcal{I}_n|^s < \infty$. So there exists a subsequence n_k for which $|\mathcal{I}_{n_k}|^s \leqslant \frac{1}{n_k}$ (if there were only finitely many such intervals it would contradict convergence of the sum). Let $\{n_k\}_{k\in\mathbb{N}}$ be exhaustive, in the sense that for any $n \notin \{n_k\}_{k\in\mathbb{N}}$, $|\mathcal{I}_n|^s > \frac{1}{n}$. We'll show that n_k is dense in the natural numbers, that is, we'll show that $\limsup_{k\to\infty} \frac{n_{k+1}}{n_k} = 1$. The desired result will then follow since for sufficiently large k we'll have $\frac{n_{k+1}}{n_k} \leqslant 2$ and so for any $n_k \leqslant n \leqslant n_{k+1}$

$$|\mathcal{I}_n|^s \leqslant \frac{1}{n_k} = \frac{1}{n_{k+1}} \cdot \frac{n_{k+1}}{n_k} \leqslant \frac{2}{n}$$

that is, $|\mathcal{I}_n| \leqslant \frac{C}{n^{\frac{1}{s}}}$ for some uniform constant C.

Suppose for a contradiction that instead $\limsup_{k\to\infty} \frac{n_{k+1}}{n_k} = c > 1$. Let $\varepsilon > 0$ such that $\frac{1}{c} + \varepsilon < 1$. Then there exists a subsequence n_{k_l} such that $\frac{1}{n_{k_l}} < \frac{\varepsilon}{2}$ for all l and $\frac{n_{k_l}}{n_{k_l+1}} \leqslant \frac{1}{c} + \frac{\varepsilon}{2}$. Moreover, for all $n_{k_l} \leqslant n < n_{k_l+1}$, $|\mathcal{I}_n|^s \geqslant \frac{1}{n_{k_l+1}}$ and so

$$\begin{aligned}
\sum_{n=1}^{\infty} |\mathcal{I}_n|^s \geqslant \sum_{l=1}^{\infty} \sum_{n=n_{k_l}}^{n_{k_l+1}-1} \frac{1}{n_{k_l+1}} &= \sum_{l=1}^{\infty} \frac{n_{k_l+1} - 1 - n_{k_l}}{n_{k_l+1}} \\
&\geqslant \sum_{l=1}^{\infty} 1 - \frac{1}{n_{k_l+1}} - \frac{n_{k_l}}{n_{k_l+1}} \\
&\geqslant \sum_{l=1}^{\infty} 1 - c\varepsilon = \infty
\end{aligned}$$

which contradicts the assumption that $s > s_0$. Thus the result follows. □

We also remark that (2) is equivalent to imposing the condition that

$$\sup_{x\in[0,1]} \sum_{n\in\mathbb{N}} \frac{1}{|T'(T_n^{-1}(x))|^s} < \infty \tag{3.15}$$

by the bounded distortion property (Proposition 3.1.4). Therefore (3.15) and condition (2) on the map will be used interchangeably.

Finally, we remark that (2) is a sharp condition. This boils down to the fact that if the interval lengths $|\mathcal{I}_n|$ were decaying too slowly, we could build a Bernoulli measure by distributing all the mass on the cylinders indexed by higher digits which are witnessing a slow decay, where the mass would be distributed proportional to the length of each cylinder. Since the branches of T which are indexed by higher digits appear increasingly linear, this results in a projected measure with high dimension. We make this precise in the following result. This result was pointed out to the author via personal communication with Thomas Jordan.

Lemma 3.3.4. *Suppose there does not exist $s < 1$ for which*

$$\sum_{n\in\mathbb{N}} |\mathcal{I}_n|^s < \infty.$$

Then for any $t < 1$ there exists a probability vector $\mathbf{p}$ for which $\dim \mu_{\mathbf{p}} > t$.

Proof. Let $t < 1$, then by assumption

$$\sum_{n=1}^{\infty} |\mathcal{I}_n|^t = \infty.$$

Thus, we can choose some large N for which

$$\sum_{n=N}^{k} |\mathcal{I}_n|^t \geqslant 1$$

for some $k > N$. Fix $\mathbf{p}_N = (p_1, p_2, \ldots)$ where

$$p_n = \begin{cases} 0 & n < N \quad \text{or} \quad n > k \\ c|\mathcal{I}_n|^t & N \leqslant p_n \leqslant k \end{cases}$$

where c is a normalising constant so that $\sum_{n=N}^{k} c|p_n|^t = 1$. Consider the Bernoulli measure $\mu_{\mathbf{p}_N}$. By Proposition 3.2.1, since $h(\mu_{\mathbf{p}_N}) < \infty$ it follows that the dimension $\dim \mu_{\mathbf{p}_N} = \frac{h(\mu_{\mathbf{p}_N})}{\chi(\mu_{\mathbf{p}_N})}$. By Proposition 3.1.4, $\log|T'(x)| \leqslant -\log|\mathcal{I}_n| + \frac{\kappa\Lambda^l}{1-\Lambda}$ for $x \in \mathcal{I}_n$. Therefore

$$\begin{aligned} \dim \mu_{\mathbf{p}_N} &\geqslant \frac{-\sum_{n=N}^{k} c|\mathcal{I}_n|^t \log c|\mathcal{I}_n|^t}{-\sum_{n=N}^{k} c|\mathcal{I}_n|^t (\log|\mathcal{I}_n| - \frac{\kappa\Lambda^l}{1-\Lambda})} \\ &= \frac{-t\sum_{n=N}^{k}(c|\mathcal{I}_n|^t \log|\mathcal{I}_n|) - \log c}{-\sum_{n=N}^{k}(c|\mathcal{I}_n|^t \log|\mathcal{I}_n|) + \frac{\kappa\Lambda^l}{1-\Lambda}}. \end{aligned}$$

Since N can be chosen arbitrarily large so that $-\sum_{n=N}^{k}(c|\mathcal{I}_n|^t \log|\mathcal{I}_n|)$ is arbitrarily large, we deduce that $\dim \mu_{\mathbf{p}_N} \to t$ as $N \to \infty$, and so we are done.

□

Now we move on to the third assumption which we make on the map T in Theorem 3.3.1. None of the conditions within this assumption are necessary for the main portion of our proof; indeed we obtain a uniform upper bound for the dimension of all Bernoulli measures for $\mathbf{p}$ for which p_1 and p_2 are uniformly bounded from below away from 0 *without* using any of the assumptions in (3). However, in order to extend the results to the whole simplex $\mathcal{P}$ we need to observe how entropy and Lyapunov exponents change when shifting small amounts of mass in a given Bernoulli measure. When T is an increasing map, we understand how the Lyapunov exponents behave under the technical assumption that T' is monotone, and when T is a decreasing map we understand how they behave under the assumption that both T' and $(T^2)'|_{\mathcal{I}_n}$ are monotone ($\forall n \in \mathbb{N}$). For this reason, it makes it necessary to impose these conditions on the map, since without them we are unable to use our current tools to understand how Lyapunov exponents change under redistribution of mass. The technical assumption that $|T'(z_1)| > 1$ whenever T is increasing is necessary in order to estimate $\dim \mu_{\mathbf{p}}$ for probability vectors $\mathbf{p}$ where p_1 is close to 1.

Essentially, this is down to the fact that this forces the Lyapunov exponent of such a measure to be bounded away from 0. An analogous condition is not required when T is decreasing since in this case it is immediate that $|T'(z_1)| > 1$ due to the geometry of the map. In light of the above, it seems as though the technical assumptions on derivatives in (3) of Theorem 3.3.1 should not be necessary in general, but are just a byproduct of the tools (which appear in Chapter 5) that are used.

We now compare our result to the analogous result of Kifer, Peres and Weiss in [KPW]. In their result, they require the absolutely continuous measure to not be a projected Bernoulli measure. By Proposition 2.5.8, μ_T is not Bernoulli if and only if $-\log|T'|$ is not cohomologous to a Bernoulli potential $f_{\mathbf{p}} = \sum_{n=1}^{\infty} \log p_n \mathbf{1}_{\mathcal{I}_n}$. Furthermore, by properties of cohomologous functions, this implies that if μ_T *were* Bernoulli, the Birkhoff sums of $-\log|T'|$ should coincide with the Birkhoff sums of $f_{\mathbf{p}}$ (for some probability vector $\mathbf{p}$) on all periodic orbits. Therefore, in order to verify that one is in the setting where results from [KPW] can be applied, it is enough to check that for any $\mathbf{p} \in \mathcal{P}$, there exists a periodic point x of period n such that

$$-S_n \log|T'(x)| \neq S_n \log f_{\mathbf{p}}(x) \tag{3.16}$$

which can now be verified purely by studying derivatives of the map at various points in the orbit of a periodic point.

Therefore, our assumptions are stronger than the assumptions in [KPW]. This is because our non-linearity condition (that $|T'(z_1)T'(z_2)| \neq |T'(z_{12})T'(z_{21})|$) implies that for all $\mathbf{p}$ there is a choice of $x \in \{z_1, z_2, z_{12}\}$ for which (3.16) holds, or in other words, we see some non-linearity in the first two branches of the map T. In contrast, a map where the first two branches were linear (but there was some non-linearity amongst the later branches of the map) would fail our condition but typically satisfy the condition from [KPW]. The reason why we have to impose this stronger condition is again a byproduct of the tools used in Chapter 5.

Another aspect in which Theorem 3.3.1 and the analogous result in [KPW] differ is that in order to use the result in [KPW] to obtain quantitative information about the size of the dimension gap, one needs to know what the absolutely continuous measure μ_T is, since the gap depends explicitly on how much mass μ_T attributes to certain cylinders. In light of this, one minor benefit of our result is that the estimate for the dimension gap can be quantified purely in terms of the derivative of the map at certain points, and thus no information about the absolutely continuous measure μ_T is required.

3.4 The infinite entropy case

In this section we obtain an upper bound for the dimension of Bernoulli measures with infinite entropy, where we cannot use the formula in Proposition 3.2.1 to calculate the dimension. We begin with a modification of Theorem 4.1 in [KPW] which provides an upper bound on the dimension of the set of points which belong to infinitely many cylinders whose diameters are contracting faster than some rate λ. In particular, define $J_n(x) = \mathcal{I}_{i_1 \ldots i_n}$ if $x \in \mathcal{I}_{i_1 \ldots i_n}$, that is, $J_n(x)$ is the 'level n' cylinder that x belongs to, and define

$$\mathcal{E}_\lambda = \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \{x \in (0,1) : |J_n(x)| \leqslant \exp(-\lambda n)\} \tag{3.17}$$

to be the set of all x whose cylinders shrink faster than $\exp(-\lambda)$ infinitely often. We begin by getting an upper bound on the dimension of $\mathcal{E}_\lambda$, which is closely related to the dimension of any measure with Lyapunov exponent greater than λ. This will then allow us to bound the dimension of any measure which has infinite Lyapunov exponent. As a consequence we will also be able to bound the dimension of any measure that has infinite entropy.

Denote

$$q = \log \left(\sup_{x \in (0,1)} \sum_{n \in \mathbb{N}} \frac{1}{|T'(T_n^{-1} x)|^s} \right) < \infty \tag{3.18}$$

where s is given by (2) of Theorem 3.3.1. In Section 4.4 we'll use the upper bound that this induces:

$$\sum_{n \in \mathbb{N}} \frac{1}{|T'(T_n^{-1} x)|^s} \leqslant e^q \tag{3.19}$$

for all $x \in [0,1]$.

The next result provides a bound which is sufficient for our purposes, although it is a rougher bound than the one given in Theorem 4.1 in [KPW]. However, this modification allows us to significantly simplify the proof.

Lemma 3.4.1. *Let $\lambda > 0$ and $\mathcal{E}_\lambda$ be defined as in (3.17). Then*

$$\dim \mathcal{E}_\lambda \leqslant s + \frac{q}{\lambda} \tag{3.20}$$

where q was defined in (3.18).

Proof. The idea behind the proof is straightforward: we choose the natural cover for $\mathcal{E}_\lambda$ and show that the sum of the uth powers of the diameters of sets in this cover is finite, for any $u > s + \frac{q}{\lambda}$.

In particular, define

$$\mathcal{S}_n = \{\mathcal{I}_{\mathbf{j}} : \mathbf{j} \in \mathbb{N}^n \quad \text{such that} \quad |\mathcal{I}_{\mathbf{j}}| \leqslant e^{-\lambda n}\}.$$

So $\mathcal{S}_n$ consists of all cylinders which are seeing the 'right decay rate' at time n. Notice that $\bigcup_{n=m}^{\infty} \mathcal{S}_n$ is a cover of $\mathcal{E}_\lambda$ for any $m \in \mathbb{N}$.

For any $s < u < 1$,

$$\sum_{I\in\mathcal{S}_n} |I|^u \leqslant e^{-\lambda n(u-s)} \sum_{I\in\mathcal{S}_n} |I|^s \leqslant e^{-\lambda n(u-s)} \int_0^1 |J_n(x)|^{s-1} dx \tag{3.21}$$

$$\leqslant C^{1-s} e^{-\lambda n(u-s)} \int_0^1 |(T^n)'(x)|^{1-s} dx \tag{3.22}$$

where the last inequality follows by Proposition 3.1.4 and $C = \exp\left(\frac{\kappa\Lambda^l}{\Lambda-1}\right)$ as in (3.1.4).

Next we estimate the integral $\int_0^1 |(T^n)'|^{1-s} dx$. By applying the chain rule and a change of variables,

$$\begin{aligned}\int_0^1 |(T^n)'(x)|^{1-s} dx &= \int_0^1 \left(\sum_{x\in T^{-1}y} |T'(x)|^{-s}\right) |(T^{n-1})'(y)|^{1-s} dy \\ &\leqslant e^q \int_0^1 |(T^{n-1})'(y)|^{1-s} dy\end{aligned}$$

where the second step follows by (3.19). Therefore by induction,

$$\int_0^1 |(T^n)'(x)|^{1-s} dx \leqslant e^{qn}. \tag{3.23}$$

By combining (3.22) and (3.23) we obtain

$$\sum_{I\in\mathcal{S}_n} |I|^u \leqslant C^{1-s} e^{n(q-\lambda(u-s))}. \tag{3.24}$$

Now, to see (3.20), first suppose that there exists some $s < u < 1$ for which $\lambda(u-s) > q$. Since this makes (3.24) summable in n and since $\bigcup_{n=m}^{\infty} \mathcal{S}_n$ is a cover for $\mathcal{E}_\lambda$, it follows that $\dim \mathcal{E}_\lambda \leqslant u$. In particular, setting u_0 to be the solution to $\lambda(u_0 - s) = q$, we have $\dim \mathcal{E}_\lambda \leqslant u_0 = s + \frac{q}{\lambda}$.

On the other hand, if $\lambda(u-s) \leqslant q$ for all $s < u < 1$, it follows that $\lambda(1-s) \leqslant q$

and therefore $s + \frac{q}{\lambda} \geqslant 1$ and so trivially $\dim \mathcal{E}_\lambda \leqslant s + \frac{q}{\lambda}$. □

Since any ergodic measure will see a specific rate of decay on a set of full measure, the above result allows us to get a good upper estimate on the dimension of any measure which has a large Lyapunov exponent. In particular, we can use this result to guarantee that any measure with infinite entropy (and therefore infinite Lyapunov exponent) will have dimension at most s.

Lemma 3.4.2. *Let $\mu_{\mathbf{p}}$ be a Bernoulli measure such that $h(\mu_{\mathbf{p}}) = \infty$. Then*

$$\dim \mu_{\mathbf{p}} \leqslant s.$$

Proof. We'll begin by showing that if $h(\mu_{\mathbf{p}}) = \infty$ then $\chi(\mu_{\mathbf{p}}) = \infty$. By Proposition 3.1.4,

$$\chi(\mu_{\mathbf{p}}) = \int \log |T'| \mathrm{d}\mu \geqslant \sum_{n=1}^{\infty} \mu_{\mathbf{p}}(\mathcal{I}_n) \log \frac{1}{|\mathcal{I}_n|} - C \tag{3.25}$$

for some constant $C > 0$. For any $N \in \mathbb{N}$ we have

$$\begin{aligned}
-\sum_{n=1}^{N} \mu_{\mathbf{p}}(\mathcal{I}_n) \log \mu_{\mathbf{p}}(\mathcal{I}_n) + \sum_{n=1}^{N} \mu_{\mathbf{p}}(\mathcal{I}_n) \log |\mathcal{I}_n| &= \sum_{n=1}^{N} \mu_{\mathbf{p}}(\mathcal{I}_n) \log \frac{|\mathcal{I}_n|}{\mu_{\mathbf{p}}(\mathcal{I}_n)} \\
&= \sum_{n=1}^{N} \mu_{\mathbf{p}}(\mathcal{I}_n) \cdot \sum_{n=1}^{N} \frac{\mu_{\mathbf{p}}(\mathcal{I}_n)}{\sum_{n=1}^{N} \mu_{\mathbf{p}}(\mathcal{I}_n)} \log \frac{|\mathcal{I}_n|}{\mu_{\mathbf{p}}(\mathcal{I}_n)} \\
&\leqslant \sum_{n=1}^{N} \mu_{\mathbf{p}}(\mathcal{I}_n) \cdot \log \Big(\sum_{n=1}^{N} \frac{|\mathcal{I}_n|}{\sum_{n=1}^{N} \mu_{\mathbf{p}}(\mathcal{I}_n)} \Big).
\end{aligned}$$

In the final step we used Jensen's inequality and the fact that log is a concave function. Since $\sum_{n=1}^{\infty} |\mathcal{I}_n| = 1$ the upper bound converges to 0 as $N \to \infty$. It follows that if $-\sum_{n=1}^{\infty} \mu_{\mathbf{p}}(\mathcal{I}_n) \log \mu_{\mathbf{p}}(\mathcal{I}_n) = \infty$ then $-\sum_{n=1}^{\infty} \mu_{\mathbf{p}}(\mathcal{I}_n) \log |\mathcal{I}_n| = \infty$. By (3.25) this implies that if $h(\mu_{\mathbf{p}}) = \infty$ then $\chi(\mu_{\mathbf{p}}) = \infty$. Thus for $\mu_{\mathbf{p}}$ almost every x,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |T'(T^k(x))| = \infty.$$

Fix arbitrary $\lambda > 0$. Then for $\mu_{\mathbf{p}}$ almost every x

$$\frac{1}{n} \sum_{k=0}^{n-1} \log |T'(T^k(x))| > 2\lambda \tag{3.26}$$

for all n sufficiently large. By rearranging (3.26) we obtain that for all x that satisfy (3.26), there exists a subsequence n_k such that

$$|(T^{n_k})'(x)|^{-1} < \exp(-2\lambda n_k)$$

for all $k \in \mathbb{N}$. By Lemma 3.1.4 this implies that

$$|J_{n_k}(x)| \leqslant \exp\left(\frac{\kappa\Lambda^l}{1-\Lambda}\right)|(T^{n_k})'(x)|^{-1} \leqslant \exp\left(\frac{\kappa\Lambda^l}{1-\Lambda}\right)\exp(-2\lambda n_k) \leqslant \exp(-\lambda n_k)$$

along the subsequence n_k. Therefore $x \in \mathcal{E}_\lambda$.

By Lemma 3.4.1 we know that $\dim \mathcal{E}_\lambda \leqslant s + \frac{q}{\lambda}$. Since $\mu_{\mathbf{p}}(\mathcal{E}_\lambda) = 1$ for all λ it follows that $\dim \mu_{\mathbf{p}} \leqslant s + \frac{q}{\lambda}$ where q is given by (3.18), and since λ was chosen to be arbitrarily large, the result follows. $\square$

3.5 Structure of proof of Theorem 3.3.1

Since we saw in Lemma 3.4.2 that the dimension of infinite entropy Bernoulli measures is bounded above by s, we can now restrict our attention to finite entropy Bernoulli measures. Our proof of Theorem 3.3.1 is split into two parts. Firstly, in Chapter 4, we use a thermodynamic formalism approach to study the dimension of Bernoulli measures $\mu_{\mathbf{p}}$ where $\mathbf{p}$ satisfies some conditions on its weights p_n. Given a probability vector which satisfies these conditions, we can obtain the dimension of the corresponding Bernoulli measure as the derivative of an associated function $\beta_{\mathbf{p}}$. The problem then reduces to obtaining a 'global bound' on the convexity of $\beta_{\mathbf{p}}$, that is, a uniform lower bound on the second derivative of $\beta_{\mathbf{p}}$. This approach was proposed by Kesseböhmer, Stratmann and Urbański and outlined in a talk given by Kesseböhmer in [Ke].

In Chapter 5 we consider Bernoulli measures $\mu_{\mathbf{p}}$ where $\mathbf{p}$ *does not* satisfy the conditions on the weights p_n which were required for the above approach. By 'redistributing mass' within such a Bernoulli measure $\mu_{\mathbf{p}}$, we can obtain an associated Bernoulli measure $\mu_{\mathbf{p}^*}$ which *does* satisfy the conditions. The problem then reduces to estimating the change in the entropy and Lyapunov exponent, in order to obtain an upper bound on $\dim \mu_{\mathbf{p}}$ in terms of $\dim \mu_{\mathbf{p}^*}$.

In order to make this precise, we need to introduce some notation pertaining to various subsets of the simplex $\mathcal{P}$.

Definition 3.5.1. *Define $\mathcal{P}_0$ to be the set of all probability vectors $\mathbf{p} = (p_1, p_2, \ldots)$ for which* *(a)* $\dim \mu_{\mathbf{p}} \geqslant \frac{2s+2}{s+3}$.

(b) $\mathbf{p}$ *has all strictly positive entries, possibly apart from a tail of zeroes. That is,* $p_n \neq 0$ *unless* $p_k = 0$ *for all* $k \geqslant n$.

(c) $\frac{p_n}{|\mathcal{I}_n|}$ *is bounded in* n.

Chapter 4 will be dedicated to studying the dimension of $\mu_{\mathbf{p}}$ for $\mathbf{p} \in \mathcal{P}_0$. Apart from (b), which is just a condition which makes the exposition of our arguments neater, the other conditions in Definition 3.5.1 are necessary in order to use the aforementioned thermodynamic formalism approach that we will employ in Chapter 4. In particular, conditions (a) and (c) will guarantee analyticity of $\beta_{\mathbf{p}}$ which is associated to the probability vector $\mathbf{p}$. Notice that (c) implies that the entries p_n decay polynomially at a rate that matches the polynomial decay rate of the intervals $|\mathcal{I}_n|$. Also, notice that by Lemma 3.4.2, for any $p \in \mathcal{P}_0$, the entropy $h(\mu_{\mathbf{p}}) < \infty$, since for $0 < s < 1$ we have $s < \frac{2s+2}{s+3}$. Therefore, by Proposition 3.2.1, for any $\mathbf{p} \in \mathcal{P}_0$, the dimension $\dim \mu_{\mathbf{p}} = \frac{h(\mu_{\mathbf{p}})}{\chi(\mu_{\mathbf{p}})}$. The importance of condition (a) is explained in Remark 4.1.4.

It will turn out that restricting to $\mathbf{p} \in \mathcal{P}_0$ will still not be enough to get a uniform upper bound for $\sup_{\mathbf{p} \in \mathcal{P}_0} \dim \mu_{\mathbf{p}}$ using only the thermodynamic formalism approach. In particular, we will only be able to get uniform upper bounds for $\dim \mu_{\mathbf{p}}$ if we restrict to a subset of $\mathcal{P}_0$ where the entries p_1 and p_2 are uniformly bounded away from 0, that is, $\mu_{\mathbf{p}}$ gives some uniform amount of mass to the first two cylinders. The relevant notation is introduced in the following definition.

Definition 3.5.2. *(i) Define* $\mathcal{P}^* \subset \mathcal{P}_0$ *to be the set of probability vectors with non-zero entries, that is, all* $\mathbf{p} = (p_1, p_2, \ldots)$ *for which* $p_n > 0$ *for all* n.

(ii) Define $\mathcal{P}^n \subset \mathcal{P}$ *to be all* $\mathbf{p} = (p_1, p_2, \ldots)$ *for which* $p_k > 0$ *for all* $k \leqslant n$ *and* $p_k = 0$ *for all* $k > n$*, which correspond to Bernoulli measures which are fully supported on the first* n *cylinders. Let* $\mathcal{P}^\infty = \bigcup_{n \in \mathbb{N}} \mathcal{P}^n$.

(iii) Define $\mathcal{P}_\varepsilon$ *to be all* $\mathbf{p} \in \mathcal{P}_0$ *for which* $p_1, p_2 \geqslant \varepsilon$ *and* $\mathcal{P}^*_\varepsilon = \mathcal{P}^* \cap \mathcal{P}_\varepsilon$.

(iv) For $n \geqslant 2$ *define* $\mathcal{P}^n_\varepsilon = \mathcal{P}_\varepsilon \cap \mathcal{P}^n$*. Finally, define*

$$\mathcal{P}^\infty_\varepsilon = \bigcup_{n=2}^{\infty} \mathcal{P}^n_\varepsilon$$

and notice that $\mathcal{P}_\varepsilon = \mathcal{P}^*_\varepsilon \cup \mathcal{P}^\infty_\varepsilon$.

Therefore the key piece of notation above is $\mathcal{P}_\varepsilon$ which denotes all $\mathbf{p} \in \mathcal{P}_0$ for which $p_1, p_2 \geqslant \varepsilon$. The other notation just describes how $\mathcal{P}_\varepsilon$ is divided into the set $\mathcal{P}_\varepsilon^*$ of probability vectors which have all non-zero entries and the set $\mathcal{P}_\varepsilon^\infty$ of probability vectors which end in a tail of zeroes. Notice that

$$\mathcal{P}_0 = \bigcup_{\varepsilon > 0} \mathcal{P}_\varepsilon. \tag{3.27}$$

As we alluded to earlier, we can only get a uniform upper bound for $\dim \mu_{\mathbf{p}}$ for $\mathbf{p} \in \mathcal{P}_\varepsilon$, whenever ε is fixed. This estimate will be dependent on ε, and approaches 1 as $\varepsilon \to 0$. Therefore, we are forced to calculate $\sup_{\mathbf{p} \in \mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}}$ for different values of $\varepsilon > 0$ and postpone a uniform result for $\sup_{\mathbf{p} \in \mathcal{P}_0} \dim \mu_{\mathbf{p}}$ till a later chapter.

In fact, our estimates for $\sup_{\mathbf{p} \in \mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}}$ will also depend on another parameter $\delta > 0$. This parameter emerges via the flexibility in the thermodynamic formalism approach; for the precise meaning of this parameter see Remark 4.1.5. For each $\delta > 0$ sufficiently small, we will be able to obtain a distinct upper bound for $\sup_{\mathbf{p} \in \mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}}$. The main result of Chapter 4 is the following theorem.

Theorem 3.5.3. *For all $\delta < \frac{1-s}{4}$ (where δ is defined in Remark 4.1.5) and ε sufficiently small there exists a 'gap constant' $G_{\varepsilon,\delta} > 0$ for which*

$$\sup_{\mathbf{p} \in \mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}} < 1 - G_{\varepsilon,\delta}.$$

By using the same tools as Chapter 4, we can also get an independent bound for the dimension of any measure $\mu_{\mathbf{p}}$ where $\mathbf{p} \in \mathcal{P}_0$ and its first entry p_1 is 'close to 1'. In particular, p_1 will assumed to be $p_1 > \xi$, where $\xi \in (0,1)$ is some fixed constant which is given explicitly in (B.1). Crucially, this estimate gives an upper bound on the dimension of a measure $\mu_{\mathbf{p}}$ for $\mathbf{p} \in \mathcal{P}_0$ with no further restrictions on the size of p_2. We obtain the following result.

Lemma 3.5.4. *There exist constants $0 < \phi < 1$ and $0 < \xi < 1$ such that for all $\mathbf{p} \in \mathcal{P}_0$ with $p_1 > \xi$, we have the bound*

$$\dim \mu_{\mathbf{p}} < 1 - \phi.$$

The proof of this lemma is a condensed and considerably more straightforward version of the arguments presented in Chapter 4, owing to the fact that in Chapter 4, at each step in the proof of Theorem 3.5.3 uniform estimates have to be obtained which hold for *all* $\mathbf{p}$ in the more general class $\mathcal{P}_\varepsilon$. For this reason, we prove Lemma 3.5.4 separately in Appendix B, where we also give an explicit value for ξ. This appendix should only be read after finishing Chapter 4.

Theorem 3.5.3 and Lemma 3.5.4 and their thermodynamic formalism proofs give us some partial estimates on $\dim \mu_{\mathbf{p}}$ for $\mathbf{p} \in \mathcal{P}_0$, but in Chapter 5 we turn to using a 'mass resdistribution' approach to tackle estimates on $\sup_{\mathbf{p}\in\mathcal{P}\setminus\mathcal{P}_0} \dim \mu_{\mathbf{p}}$ and to obtain a uniform upper bound for $\sup_{\mathbf{p}\in\mathcal{P}_0 : p_1<\xi} \dim \mu_{\mathbf{p}}$. First, we will prove the following result which means that in order to prove a dimension gap which is uniform over the whole simplex $\mathcal{P}$, it suffices to restrict our attention to $\mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$, for some $\varepsilon > 0$.

Theorem 3.5.5.

$$\sup_{\mathbf{p}\in\mathcal{P}\setminus\mathcal{P}_0} \dim \mu_{\mathbf{p}} \leqslant \sup_{\mathbf{p}\in\mathcal{P}_0} \dim \mu_{\mathbf{p}}.$$

Therefore, it remains to figure out how we can convert a uniform upper bound for $\sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}}$ into a uniform upper bound for $\sup_{\mathbf{p}\in\mathcal{P}_0 : p_1<\xi} \dim \mu_{\mathbf{p}}$. This is precisely where it comes in useful that we have upper bounds $\sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}} < 1 - G_{\varepsilon,\delta}$ for various values of the parameter $\delta > 0$ in Theorem 3.5.3. In particular, by using the 'mass redistribution technique', we will show that there exists some $\delta > 0$ and $\varepsilon > 0$, such that for any $\mathbf{p} \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ where $p_1 < \xi$, there exists $\mathbf{p}^* \in \mathcal{P}_\varepsilon$ with $\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*} + E(\varepsilon)$ and moreover, $E(\varepsilon) < \frac{1}{2} G_{\varepsilon,\delta}$. This is proved in the following theorem in Chapter 5.

Theorem 3.5.6. *There exists* $\delta < \frac{1-s}{4}$ *and some* $\varepsilon > 0$ *such that*

$$\sup_{\mathbf{p}\in\mathcal{P}_0\setminus\mathcal{P}_\varepsilon : p_1<\xi} \dim \mu_{\mathbf{p}} < 1 - \frac{1}{2} G_{\varepsilon,\delta}.$$

Clearly, Theorem 3.3.1 follows from Lemmas 3.4.2, 3.5.4 and the Theorems 3.5.3, 3.5.5 and 3.5.6.

Chapter 4

Estimating the variance

The main goal of this chapter is to prove Theorem 3.5.3, that is, to prove that for all ε sufficiently small and all $\delta < \frac{1-s}{4}$, there exists some constant $G_{\varepsilon,\delta} > 0$ for which

$$\sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}} < 1 - G_{\varepsilon,\delta}.$$

As the notation suggests, for each $\varepsilon > 0$ we get a range of different upper bounds for $\sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}}$, each of which depend on the initial choice of δ. One might wonder why we do not just present the best value of $G_{\varepsilon,\delta}$, that is, the one that yields the least upper bound for $\sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}}$. The reasoning behind this is that in Chapter 5 we will work on approximating $\sup_{\mathbf{p}\in\mathcal{P}_0\setminus\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}}$ in terms of $1 - G_{\varepsilon,\delta}$ and some term E_ε, which crucially will *only* depend on ε. In order to demonstrate the existence of a dimension gap, we need to ensure that $1 - G_{\varepsilon,\delta} + E_\varepsilon < 1$, and thus we need to have the flexibility to choose a suitable candidate for δ and ε which will guarantee this.

Recall that $\bigcup_{\varepsilon>0} \mathcal{P}_\varepsilon = \mathcal{P}_0$ and therefore throughout this chapter we will only consider $\mathbf{p} \in \mathcal{P}_0$. Initially, many results will be stated which hold uniformly for all $\mathbf{p} \in \mathcal{P}_0$, although later on in the chapter several results will be dependent on ε.

Consider a measure $\mu_{\mathbf{p}}$ for $\mathbf{p} \in \mathcal{P}_0$. In order to study the dimension of $\mu_{\mathbf{p}}$, in Section 4.1 we will reformulate the problem as a question concerning the analytic properties of a particular function $\beta_{\mathbf{p}}$. We will then see that the problem reduces to finding a lower bound for the variance of some potentials. In Section 4.2 we will rewrite the variance as an integral, which will provide us with a geometric framework within which we can develop a strategy to make estimates on the variance. At that point we will be ready to outline the structure of the remainder of the chapter, which will be split into proving various properties of the function and measure which are involved in the integral. These combined together will produce the desired lower bound.

4.1 Relating the problem to the study of the analytic properties of $\beta_{\mathbf{p}}$

In this section we reformulate the problem of finding an *upper bound* for the dimension of $\mu_{\mathbf{p}}$ for $\mathbf{p} \in \mathcal{P}_0$ into the problem of finding a *lower bound* for the second derivative of some function $\beta_{\mathbf{p}}$. Before we define $\beta_{\mathbf{p}}$, we introduce some notation.

For $\mathbf{p} \in \mathcal{P}_0$, let $\mathbb{N}_{\mathbf{p}} = \{n \in \mathbb{N} : p_n \neq 0\}$. By definition of $\mathcal{P}_0$, either $\mathbb{N}_{\mathbf{p}} = \mathbb{N}$ or $\mathbb{N}_{\mathbf{p}} = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$. Let $T_{\mathbf{p}}$ denote the map consisting only of branches T_n for which $n \in \mathbb{N}_{\mathbf{p}}$. This means that the map $T_{\mathbf{p}}$ will either be the map T or an approximation $T_{(N)}$ of T for some $N \in \mathbb{N}$, which defines a dynamical system on its corresponding repeller $J_{\mathbf{p}} = J_N$, where these objects are defined as follows. We define the Nth approximation $T_{(N)}$ of T to be the map made up of the first N branches $T_1, \ldots, T_N$. Similarly to (3.1) and (3.2), $T_{(N)}$ can be coded by the full shift on N symbols (Σ_N, σ) and its repeller is defined as

$$J_N = [0,1] \setminus \bigcup_{n=0}^{\infty} T^{-n} \left(\left\{ 0 \cup \mathcal{I}_{N+1} \bigcup_{n=N+2}^{\infty} \overline{\mathcal{I}_n} \right\} \right)$$

where $\Pi : \Sigma_N \to [0,1]$ given by

$$\Pi(\mathbf{i}) = \lim_{n \to \infty} T_{i_1}^{-1} \circ \cdots \circ T_{i_n}^{-1}([0,1])$$

sets up a correspondence between Σ_N and J_N.

For $\mathbf{p} \in \mathcal{P}_0$ define the Bernoulli potential $f_{\mathbf{p}} : J_{\mathbf{p}} \to (-\infty, 0]$ by

$$f_{\mathbf{p}} = \sum_{n \in \mathbb{N}_{\mathbf{p}}} \log p_n \mathbf{1}_{\mathcal{I}_n}.$$

Notice that $f_{\mathbf{p}}$ is the Gibbs potential for the Bernoulli measure $\mu_{\mathbf{p}}$. We are now ready to introduce the function $\beta_{\mathbf{p}}$.

Definition 4.1.1. *Fix a probability vector* $\mathbf{p} = (p_1, p_2, \ldots) \in \mathcal{P}_0$. *We can define the function* $\beta_{\mathbf{p}} : [0,1] \to [0,1]$ *where* $\beta_{\mathbf{p}}(t)$ *is defined implicitly as the solution to*

$$P(-\beta_{\mathbf{p}}(t) \log |T'| + t f_{\mathbf{p}}, T_{\mathbf{p}}) = 0 \tag{4.1}$$

where $P(\cdot, T_{\mathbf{p}})$ *denotes the usual pressure function for the map* $T_{\mathbf{p}}$.

Note that it is not immediately obvious that $\beta_{\mathbf{p}}$ should be well-defined; this fact will follow from Proposition 4.1.2.

We denote the function that appears inside the pressure in the definition of $\beta_{\mathbf{p}}$ as $g_{\mathbf{p},t} : J_{\mathbf{p}} \to \mathbb{R}$

$$g_{\mathbf{p},t} = -\beta_{\mathbf{p}}(t) \log |T'| + t f_{\mathbf{p}}. \tag{4.2}$$

Since by definition $P(g_{\mathbf{p},t}) = 0 < \infty$, by Proposition 2.5.7 we know that there exists a unique Gibbs measure for $g_{\mathbf{p},t}$ which we will denote by $\mu_{\mathbf{p},t}$. Clearly $\mu_{\mathbf{p},t}$ is supported on $J_{\mathbf{p}}$.

The function $\beta_{\mathbf{p}}$ will be the object of our focus throughout this section. In the following proposition we summarise its important properties. Recall that s was fixed in Theorem 3.3.1 to be some constant $0 < s < 1$ for which $\sum_{n\in\mathbb{N}} |\mathcal{I}_n|^s < \infty$.

Proposition 4.1.2. *Let* $\mathbf{p} \in \mathcal{P}_0$. *The function* $\beta_{\mathbf{p}} : [0,1] \to [0,1]$ *satisfies the following properties:*

1. $\beta_{\mathbf{p}}(t)$ *is convex and decreasing on* $[0,1]$.
2. $\beta_{\mathbf{p}}(t)$ *is analytic for* t *in (a) a neighbourhood of 1 and (b) for* $t \in (0, t_{\mathbf{p}})$ *where* $t_{\mathbf{p}} = \inf\{t : \beta_{\mathbf{p}}(t) \geqslant s\}$. *Moreover for these values of* t *the first derivative of* $\beta_{\mathbf{p}}$ *(with respect to* t*) is given by*

$$\beta'_{\mathbf{p}}(t) = \frac{-\int f_{\mathbf{p}} d\mu_{\mathbf{p},t}}{\int \log |T'| d\mu_{\mathbf{p},t}} \tag{4.3}$$

   and the second derivative is given by

$$\beta''_{\mathbf{p}}(t) = \frac{\sigma^2_{\mu_{\mathbf{p},t}}(-\beta'_{\mathbf{p}}(t) \log |T'| + f_{\mathbf{p}})}{\int \log |T'| d\mu_{\mathbf{p},t}} \tag{4.4}$$

   where the variance is associated to $T_{\mathbf{p}}$.
3. $0 < \beta_{\mathbf{p}}(0) \leqslant 1$ *and* $\beta_{\mathbf{p}}(1) = 0$.

Moreover, these properties determine the graph of $\beta_{\mathbf{p}}(t)$*; see Figure 4.1.*

Proof. For the first part, observe that

$$P(-\beta_{\mathbf{p}}(t) \log |T'| + t f_{\mathbf{p}}) = \lim_{n\to\infty} \left(\sum_{i_1,\dots i_n \in \mathbb{N}^n_{\mathbf{p}}} \frac{(p_{i_1} \dots p_{i_n})^t}{|(T^n)'(\Pi((i_1 \dots i_n)^\infty))|^{\beta_{\mathbf{p}}(t)}} \right).$$

For each n, as t increases the numerator of each term in the corresponding sum increases. Therefore by passing to the limit we see that as t increases, $\beta_{\mathbf{p}}(t)$ must decrease accordingly to ensure that $P(-\beta_{\mathbf{p}}(t)\log|T'|+tf_{\mathbf{p}})=0$ for all t. To see that $\beta_{\mathbf{p}}$ is convex, notice that for any $n\in\mathbb{N}$, and $a,u,t\in(0,1)$

$$\sum_{i_1,\ldots,i_n\in\mathbb{N}_{\mathbf{p}}^n}\frac{p_{i_1}^{at}\cdots p_{i_n}^{at}}{|(T^n)'(\Pi((i_1\ldots i_n)^\infty))|^{a\beta_{\mathbf{p}}(t)}}\frac{p_{i_1}^{(1-a)u}\cdots p_{i_n}^{(1-a)u}}{|(T^n)'(\Pi((i_1\ldots i_n)^\infty))|^{(1-a)\beta_{\mathbf{p}}(u)}}\leqslant$$

$$\left(\sum_{i_1,\ldots,i_n\in\mathbb{N}_{\mathbf{p}}^n}\frac{p_{i_1}^{t}\cdots p_{i_n}^{t}}{|(T^n)'(\Pi((i_1\ldots i_n)^\infty))|^{\beta_{\mathbf{p}}(t)}}\right)^a\left(\sum_{i_1,\ldots,i_n\in\mathbb{N}_{\mathbf{p}}^n}\frac{p_{i_1}^{u}\cdots p_{i_n}^{u}}{|(T^n)'(\Pi((i_1\ldots i_n)^\infty))|^{\beta_{\mathbf{p}}(u)}}\right)^{1-a}$$

by Hölder's inequality. Therefore

$$\begin{aligned}&P(-(a\beta_{\mathbf{p}}(t)+(1-a)\beta_{\mathbf{p}}(u))\log|T'|+(at+(1-a)u)f_{\mathbf{p}})\\&\qquad\leqslant aP(-\beta_{\mathbf{p}}(t)\log|T'|+tf_{\mathbf{p}})+(1-a)P(-\beta_{\mathbf{p}}(u)\log|T'|+uf_{\mathbf{p}})=0.\end{aligned}$$

Therefore it follows that $\beta_{\mathbf{p}}(at+(1-a)u)\leqslant a\beta_{\mathbf{p}}(t)+(1-a)\beta_{\mathbf{p}}(u)$ since when t is fixed, $P(-b\log|T'|+tf_{\mathbf{p}})$ is decreasing in b.

To prove the second part, we will use the Implicit Function theorem and Proposition 2.5.11. First, observe that for all $(t,\beta)\in[0,1]\times[s,1]$,

$$\begin{aligned}P(-\beta\log|T'|+tf_{\mathbf{p}})&\leqslant P(-\beta\log|T'|)\\&=\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{x:T^nx=x}\frac{1}{|(T^n)'(x)|^\beta}\right)\\&\leqslant\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{i_1,\ldots,i_n}\left(\exp\left(\frac{\kappa\Lambda^l}{1-\Lambda}\right)\right)^{n\beta}|\mathcal{I}_{i_1}|^\beta\cdots|\mathcal{I}_{i_n}|^\beta\right)\\&=\lim_{n\to\infty}\frac{1}{n}\log\left(\exp\left(\frac{\kappa\Lambda^l}{1-\Lambda}\right)^{n\beta}\left(\sum_{k=1}^\infty|\mathcal{I}_k|^\beta\right)^n\right)\\&=\log\left(\sum_{k=1}^\infty|\mathcal{I}_k|^\beta\right)+\left(\frac{\kappa\Lambda^l}{1-\Lambda}\right)^\beta<\infty.\end{aligned}$$

By Proposition 2.5.11, it follows that $P(-\beta\log|T'|+tf_{\mathbf{p}})$ is analytic for all $(t,\beta)\in[0,1]\times[s,1]$.

Similarly, we can also show that there exists $\varepsilon>0$ such that $P(-\beta\log|T'|+tf_{\mathbf{p}})$ is analytic for all $(t,\beta)\in[1-\varepsilon,1+\varepsilon]\times[-\varepsilon,\varepsilon]$. By our assumptions on T, there exists $r>0$ such that $\sum_{n\in\mathbb{N}}|\mathcal{I}_n|^{1-r}<\infty$. Also, since $\mathbf{p}\in\mathcal{P}_0$ there exists $C>0$

such that $\frac{p_n}{|\mathcal{I}_n|} \leqslant C$ for all $n \in \mathbb{N}$. Therefore,

$$\sum_{n\in\mathbb{N}} p_n^{1-r} \leqslant C^{1-r} \sum_{n\in\mathbb{N}} |\mathcal{I}_n|^{1-r} < \infty.$$

Let $(t,\beta) \in [1-\frac{r}{2}, 1+\frac{r}{2}] \times [-\frac{r}{2}, \frac{r}{2}]$. Then

$$\begin{aligned}
&P(-\beta \log|T'| + tf_{\mathbf{p}}) \\
&\qquad \leqslant \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_1\dots i_n \in \mathbb{N}^n} (p_{i_1}\dots p_{i_n})^{1-\frac{r}{2}} |(T^n)'(\Pi(i_1\dots i_n)^\infty)|^{\frac{r}{2}} \right) \\
&\qquad = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_1\dots i_n \in \mathbb{N}^n} (p_{i_1}\dots p_{i_n})^{1-r} (p_{i_1}\dots p_{i_n})^{\frac{r}{2}} |(T^n)'(\Pi(i_1\dots i_n)^\infty)|^{\frac{r}{2}} \right) \\
&\qquad \leqslant \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_1\dots i_n \in \mathbb{N}^n} (p_{i_1}\dots p_{i_n})^{1-r} \frac{(p_{i_1}\dots p_{i_n})^{\frac{r}{2}}}{(|\mathcal{I}_{i_1}|\dots|\mathcal{I}_{i_n}|)^{\frac{r}{2}}} \left(\exp\left(\frac{\kappa\Lambda^l}{1-\Lambda} \right) \right)^{n\frac{r}{2}} \right) \\
&\qquad \leqslant \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_1\dots i_n \in \mathbb{N}^n} (p_{i_1}\dots p_{i_n})^{1-r} C^{n\frac{r}{2}} \left(\exp\left(\frac{\kappa\Lambda^l}{1-\Lambda} \right) \right)^{n\frac{r}{2}} \right) \\
&\qquad = \frac{r}{2} \left(\frac{\kappa\Lambda^l}{1-\Lambda} \right) + \log C^{\frac{r}{2}} + \log \left(\sum_{k\in\mathbb{N}} p_k^{1-r} \right) < \infty.
\end{aligned}$$

Therefore, by Proposition 2.5.11, $P(-\beta \log|T'| + tf_{\mathbf{p}})$ is analytic for all $(t,\beta) \in [1-\frac{r}{2}, 1+\frac{r}{2}] \times [-\frac{r}{2}, \frac{r}{2}]$.

By the Implicit Function theorem, $\beta_{\mathbf{p}}(t)$ is analytic for $t \in (0, t_{\mathbf{p}})$ and in a neighbourhood of 1.

To verify (4.3) and (4.4) we follow the arguments of Ruelle [Ru]. To verify (4.3), we differentiate (4.1) and apply the Implicit Function theorem to deduce that

$$-\beta'_{\mathbf{p}}(t) \int \log|T'| d\mu_{\mathbf{p},t} + \int f_{\mathbf{p}} d\mu_{\mathbf{p},t} = 0. \tag{4.5}$$

To verify (4.4) we differentiate (4.5) to obtain

$$\beta''_{\mathbf{p}}(t) \int \log|T'| d\mu_{\mathbf{p},t} + \beta'_{\mathbf{p}}(t) \frac{d\left(\int \log|T'| d\mu_{\mathbf{p},t}\right)}{dt} - \frac{d\left(\int f_{\mathbf{p}} d\mu_{\mathbf{p},t}\right)}{dt} = 0.$$

By Proposition 2.5.13

$$\frac{d\left(\int \log|T'| d\mu_{\mathbf{p},t}\right)}{dt} = \sigma^2_{\mu_{\mathbf{p},t}}(-\beta'_{\mathbf{p}}(t)\log|T'| + f_{\mathbf{p}}, \log|T'|)$$

and

$$\frac{d\left(\int f_{\mathbf{p}} d\mu_{\mathbf{p},t}\right)}{dt} = \sigma^2_{\mu_{\mathbf{p},t}}(-\beta'_{\mathbf{p}}(t)\log|T'| + f_{\mathbf{p}}, f_{\mathbf{p}})$$

and therefore

$$\beta''_{\mathbf{p}}(t) = \frac{\sigma^2_{\mu_{\mathbf{p},t}}(-\beta'_{\mathbf{p}}(t)\log|T'| + f_{\mathbf{p}})}{\int \log|T'| d\mu_{\mathbf{p},t}} \geqslant 0. \tag{4.6}$$

For the third part, observe that by Proposition 3.2.2, $P(-s_{\mathbf{p}}\log|T'|, T_{\mathbf{p}}) = 0$ where $s_{\mathbf{p}} = \dim J_{\mathbf{p}} \leqslant 1$ (so in particular $\beta_{\mathbf{p}}(0) = 1$ when $\mathbf{p} \in \mathcal{P}^*$ since $T_{\mathbf{p}} = T$) and since $P(f_{\mathbf{p}}, T_{\mathbf{p}}) = 0$ it follows that $\beta_{\mathbf{p}}(1) = 0$.

□

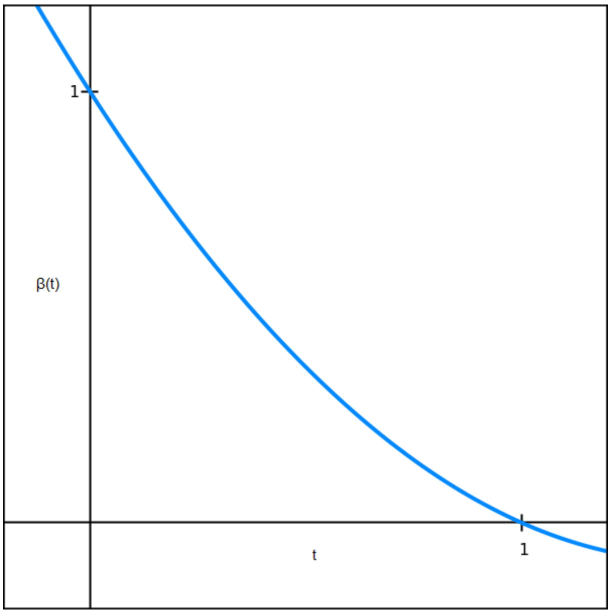

Figure 4.1: An example of the graph of $\beta_{\mathbf{p}}(t)$ when $\mathbf{p} \in \mathcal{P}^*$.

Notice that since $\beta_{\mathbf{p}}(1) = 0$, it follows that $\mu_{\mathbf{p},1} = \mu_{\mathbf{p}}$. By examining the formula (4.5) for the derivative $\beta'_{\mathbf{p}}(t)$ at $t = 1$ more closely, we observe that it is possible to write $\dim \mu_{\mathbf{p}}$ as $|\beta'_{\mathbf{p}}(1)|$.

Proposition 4.1.3. *For* $\mathbf{p} \in \mathcal{P}_0$

$$\dim \mu_{\mathbf{p}} = \frac{h(\mu_{\mathbf{p}})}{\chi(\mu_{\mathbf{p}})} = -\frac{\int f_{\mathbf{p}} d\mu_{\mathbf{p},1}}{\int \log|T'| d\mu_{\mathbf{p},1}} = -\beta'_{\mathbf{p}}(1) = |\beta'_{\mathbf{p}}(1)|.$$

Proof. This is a direct consequence of Proposition 3.2.1, Proposition 4.1.2 and the fact that $\mu_{\mathbf{p},1} = \mu_{\mathbf{p}}$. □

By rewriting $\dim \mu_{\mathbf{p}}$ as the absolute value of the derivative of a function, we are now able to exploit the tools of calculus to find an upper bound on $|\beta'_{\mathbf{p}}(1)| = \dim \mu_{\mathbf{p}}$. For a fixed $\varepsilon > 0$, we would like to show that $\beta'_{\mathbf{p}}(1)$ is bounded away from -1 uniformly for all $\mathbf{p} \in \mathcal{P}_\varepsilon$. Thus, we are interested in showing that $\beta_{\mathbf{p}}$ is convex in some compact interval of t, where the convexity is uniform over all $\mathbf{p} \in \mathcal{P}_\varepsilon$ and t in the chosen interval. By Proposition 4.1.2, our choice of compact interval for t is restricted to analytic domains of $\beta_{\mathbf{p}}(t)$, that is, the interval must lie either in a neighbourhood of 0 or a neighbourhood of 1. Since the size of the neighbourhood of 1 where $\beta_{\mathbf{p}}$ is analytic is dependent on the decay properties of p_n, whereas the size of the neighbourhood at 0 is dependent on the decay properties of the intervals $|\mathcal{I}_n|$, it will be beneficial for us to choose the interval to lie in a neighbourhood of 0.

Remark 4.1.4. *Notice that for all $t \in [0, \frac{1-s}{4}]$, $\beta_{\mathbf{p}}(t) \geqslant \frac{1+s}{2} > s$. To see this, suppose that for some $\mathbf{p} \in \mathcal{P}_0$, $\beta_{\mathbf{p}}\left(\frac{1-s}{4}\right) < \frac{1+s}{2}$. Then since $\beta''_{\mathbf{p}}(t) \geqslant 0$ for all t and since $\beta_{\mathbf{p}}(1) = 0$ it follows that*

$$\dim \mu_{\mathbf{p}} = |\beta'_{\mathbf{p}}(1)| \leqslant \frac{1+s}{2(1-\frac{1-s}{4})} = \frac{2s+2}{s+3}$$

which contradicts the fact that $\mathbf{p} \in \mathcal{P}_0$.

In many arguments throughout this chapter it will be important that $\beta_{\mathbf{p}}(t) \geqslant s$ for all $t \in [0, \frac{1-s}{4}]$, in order to ensure summability of $\sum_{n\in\mathbb{N}} \frac{1}{|T'(T_n^{-1}x)|^{\beta_{\mathbf{p}}(t)}}$ for $x \in [0,1]$. Thus in light of the above, we see that the assumption that $\dim \mu_{\mathbf{p}} < \frac{2s+2}{s+3}$ in Definition 3.5.1 guarantees that for all $\mathbf{p} \in \mathcal{P}_0$, $\beta_{\mathbf{p}}(t) \geqslant s$ for all $t \in [0, \frac{1-s}{4}]$, which allows us to employ this thermodynamic formalism approach.

Remark 4.1.5. *For technical reasons that will become clear later on we will be unable to obtain a uniform lower bound on $\beta''_{\mathbf{p}}(t)$ for t belonging to a neighbourhood of 0. Therefore, we will consider intervals of t of the form $[\frac{\delta}{2}, \delta]$ for $\delta < \frac{1-s}{4}$, and on each such interval obtain a uniform lower bound for $\beta''_{\mathbf{p}}(t)$. Each lower bound for $\beta''_{\mathbf{p}}(t)$ in $[\frac{\delta}{2}, \delta]$ will yield an upper bound of $1 - G_{\varepsilon,\delta}$ for $|\beta'_{\mathbf{p}}(1)|$.*

By (4.6), $\beta''_{\mathbf{p}}(t) = \frac{\sigma^2_{\mu_{\mathbf{p},t}}(-\beta'_{\mathbf{p}}(t)\log|T'|+f_{\mathbf{p}})}{\int \log|T'|d\mu_{\mathbf{p},t}}$. Therefore we are interested in finding an upper bound for the Lyapunov exponent $\chi(\mu_{\mathbf{p},t})$ and a lower bound for the variance $\sigma^2_{\mu_{\mathbf{p},t}}(-\beta'_{\mathbf{p}}(t)\log|T'| + f_{\mathbf{p}})$. The Lyapunov exponent is not difficult to estimate from above, but we will delay this until Lemma 4.6.2. Instead, our primary focus in this chapter will be obtaining a lower bound for the variance.

4.2 Rewriting the variance

In order to obtain the desired lower bound on $\beta''_{\mathbf{p}}(t)$, we need to obtain a lower bound for the variance $\sigma^2_{\mu_{\mathbf{p},t}}(-\beta'_{\mathbf{p}}(t)\log|T'| + f_{\mathbf{p}})$. From now on we shall denote $f_{\mathbf{p},t} : [0,1] \to \mathbb{R} \cup \{\infty\}$ by

$$f_{\mathbf{p},t} = -\beta'_{\mathbf{p}}(t)\log|T'| + f_{\mathbf{p}}. \tag{4.7}$$

This section is dedicated to rewriting $\sigma^2_{\mathbf{p},t}(f_{\mathbf{p},t})$ in a way that will allow us to estimate it from below. Note that $\int f_{\mathbf{p},t} d\mu_{\mathbf{p},t} = 0$ by (4.5). Recall that by (2.13) and the fact that the variance is invariant under adding a coboundary, it follows that

$$\sigma^2_{\mathbf{p},t}(f_{\mathbf{p},t}) = \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t} + 2\sum_{n=1}^{\infty}\left(\int \tilde{f}_{\mathbf{p},t} \cdot \tilde{f}_{\mathbf{p},t} \circ T^n_{\mathbf{p}} d\mu_{\mathbf{p},t}\right) \tag{4.8}$$

for any function $\tilde{f}_{\mathbf{p},t}$ which is cohomologous to $f_{\mathbf{p},t}$. The second term on the right hand side of (4.8) is what makes it difficult to study lower bounds on the variance. Therefore, our aim is to find a function $\tilde{f}_{\mathbf{p},t}$ which is cohomologous to $f_{\mathbf{p},t}$, that is, a coboundary $U_{\mathbf{p},t} - U_{\mathbf{p},t} \circ T_{\mathbf{p}}$ such that $\tilde{f}_{\mathbf{p},t} = f_{\mathbf{p},t} + U_{\mathbf{p},t} - U_{\mathbf{p},t} \circ T_{\mathbf{p}}$, for which the right hand term in (4.8) vanishes. Therefore, in the first part of this section we introduce a family of transfer operators which will aid us towards obtaining the appropriate function $U_{\mathbf{p},t}$ which achieves the above, that is, $\sigma^2_{\mathbf{p},t}(f_{\mathbf{p},t}) = \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$. Once we have rewritten the variance as the appropriate integral, we'll describe the strategy for obtaining a lower bound on this integral and state the main results from the remainder of the chapter which tie together to yield this lower bound.

In some sense, the rest of this chapter can be considered independently as an approach for getting *lower* estimates for the variance of some potentials. The variance is an important thermodynamic quantity that appears in many statistical properties of dynamical systems such as the central limit theorem. However, relatively little is known about explicit estimates for the variance and to our knowledge, lower bounds for the variance have not yet been studied.

When estimating the variance, often it is useful to introduce a transfer operator, particularly when using the characterisation (4.8). Generally, one has some flexibility over the choice of space on which to define a transfer operator. We choose to define it on the space of bounded locally Hölder continuous functions $\mathcal{F}_{\Lambda^{-1}}$ since, as it will become clear later on in the section, we will be interested in estimating the Hölder properties of certain potentials. In particular, we consider Λ^{-1}-locally Hölder continuous potentials since Λ^{-1} is the contraction rate of our cylinders, see

Proposition 3.1.2.

In our case we will need a family of transfer operators which are parametrised by $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$.

Definition 4.2.1. *For each $\mathbf{p} \in \mathcal{P}_0$ and $0 \leqslant t \leqslant \frac{1-s}{4}$ we define the bounded linear operator $\mathcal{L}_{\mathbf{p},t} : \mathcal{F}_{\Lambda^{-1}} \to \mathcal{F}_{\Lambda^{-1}}$ given by*

$$\mathcal{L}_{\mathbf{p},t}w(x) = \sum_{T_{\mathbf{p}}y=x} e^{g_{\mathbf{p},t}(y)}w(y).$$

Note that this can be written alternatively as

$$\mathcal{L}_{\mathbf{p},t}w(x) = \sum_{n\in\mathbb{N}_{\mathbf{p}}} e^{g_{\mathbf{p},t}(T_n^{-1}x)}w(T_n^{-1}x).$$

Notice that each operator in the family above is well-defined since for all $t \in [0, \frac{1-s}{4}]$, $\beta_{\mathbf{p}}(t) \geqslant s$ and so $\sum_{n\in\mathbb{N}_{\mathbf{p}}} e^{g_{\mathbf{p},t}(T_n^{-1}x)} < \infty$.

It will be more convenient for us to work with the normalised transfer operator, whose existence we know about as a result of Theorem 2.5.9.

Lemma 4.2.2. *For each $0 \leqslant t \leqslant \frac{1-s}{4}$ and $\mathbf{p} \in \mathcal{P}_0$ there exists a normalised operator*

$$\mathcal{M}_{\mathbf{p},t}w = h_{\mathbf{p},t}^{-1}\mathcal{L}_{\mathbf{p},t}(h_{\mathbf{p},t}w)$$

such that $\mathcal{M}_{\mathbf{p},t}\mathbf{1} = \mathbf{1}$, where $h_{\mathbf{p},t}$ is the unique fixed point of $\mathcal{L}_{\mathbf{p},t}$. Moreover, $\mathcal{M}_{\mathbf{p},t}^*\mu_{\mathbf{p},t} = \mu_{\mathbf{p},t}$ and $\mathrm{d}\mu_{\mathbf{p},t} = h_{\mathbf{p},t}\mathrm{d}\tilde{\mu}_{\mathbf{p},t}$ where $\mathcal{L}_{\mathbf{p},t}^*\tilde{\mu}_{\mathbf{p},t} = \tilde{\mu}_{\mathbf{p},t}$.

Proof. By Theorem 2.5.9 and the fact that $P(g_{\mathbf{p},t}) = 0$, for each $0 \leqslant t \leqslant \frac{1-s}{4}$ and $\mathbf{p} \in \mathcal{P}_0$ there exists a strictly positive function $h_{\mathbf{p},t} \in \mathcal{F}_{\Lambda^{-1}}$, $h_{\mathbf{p},t} : [0,1] \to \mathbb{R}$ such that $\mathcal{L}_{\mathbf{p},t}h_{\mathbf{p},t} = h_{\mathbf{p},t}$. Therefore we can define

$$\mathcal{M}_{\mathbf{p},t}w = h_{\mathbf{p},t}^{-1}\mathcal{L}_{\mathbf{p},t}(h_{\mathbf{p},t}w)$$

and it follows that $\mathcal{M}_{\mathbf{p},t}\mathbf{1} = \mathbf{1}$. Since $\mu_{\mathbf{p},t}$ is the unique invariant Gibbs measure for $g_{\mathbf{p},t}$, by Theorem 2.5.9 it follows that $\mathcal{M}_{\mathbf{p},t}^*\mu_{\mathbf{p},t} = \mu_{\mathbf{p},t}$. Moreover, by Theorem 2.5.9 $\mathrm{d}\mu_{\mathbf{p},t} = h_{\mathbf{p},t}\mathrm{d}\tilde{\mu}_{\mathbf{p},t}$ where $\mathcal{L}_{\mathbf{p},t}^*\tilde{\mu}_{\mathbf{p},t} = \tilde{\mu}_{\mathbf{p},t}$. □

The characterisation of the variance provided by (4.8) is more profitable for producing estimates than the various other characterisations. When seeking *upper* estimates, the second term on the right hand side in (4.8) can easily be dealt with, for instance one can bound it above by knowing an explicit rate for the decay of the correlation functions. However, when one is interested in *lower* estimates, this term makes the variance difficult to bound from below. Since (4.8) holds for any $\tilde{f}_{\mathbf{p},t}$ which is cohomologous to $f_{\mathbf{p},t}$, it would be useful if we could find some $\tilde{f}_{\mathbf{p},t} \sim f_{\mathbf{p},t}$ for which

$$\int (\tilde{f}_{\mathbf{p},t}) \cdot \tilde{f}_{\mathbf{p},t} \circ T_{\mathbf{p}}^n d\mu_{\mathbf{p},t} = 0$$

for all $n \in \mathbb{N}$. Using the properties of the transfer operator, we rewrite the above as

$$\begin{aligned} \int (\tilde{f}_{\mathbf{p},t}) \cdot \tilde{f}_{\mathbf{p},t} \circ T_{\mathbf{p}}^n d\mu_{\mathbf{p},t} &= \int \mathcal{M}_{\mathbf{p},t}^n (\tilde{f}_{\mathbf{p},t} \cdot \tilde{f}_{\mathbf{p},t} \circ T_{\mathbf{p}}^n) d\mu_{\mathbf{p},t} \\ &= \int \tilde{f}_{\mathbf{p},t} \cdot \mathcal{M}_{\mathbf{p},t}^n (\tilde{f}_{\mathbf{p},t}) d\mu_{\mathbf{p},t} = 0. \end{aligned}$$

Writing $\tilde{f}_{\mathbf{p},t} = f_{\mathbf{p},t} + U_{\mathbf{p},t} - U_{\mathbf{p},t} \circ T_{\mathbf{p}}$ for some coboundary $U_{\mathbf{p},t} - U_{\mathbf{p},t} \circ T_{\mathbf{p}}$, it transpires that the property we want is $\mathcal{M}_{\mathbf{p},t}(f_{\mathbf{p},t} + U_{\mathbf{p},t} - U_{\mathbf{p},t} \circ T_{\mathbf{p}}) = 0$. This leads us to the following definition for $U_{\mathbf{p},t}$, which we now fix.

Definition 4.2.3. *For each $0 \leqslant t \leqslant \frac{1-s}{4}$ and $\mathbf{p} \in \mathcal{P}_0$ define*

$$U_{\mathbf{p},t} = \sum_{n=1}^{\infty} \mathcal{M}_{\mathbf{p},t}^n (f_{\mathbf{p},t})$$

and

$$\tilde{f}_{\mathbf{p},t} = f_{\mathbf{p},t} + U_{\mathbf{p},t} - U_{\mathbf{p},t} \circ T_{\mathbf{p}}.$$

We will have to delay the proof that $U_{\mathbf{p},t}$ is well defined till Section 4.4 where we will show that $\|U_{\mathbf{p},t}\|_\infty < \infty$ for all $\mathbf{p} \in \mathcal{P}_0$ and $t \in (0, \delta]$.

As suggested above, it turns out that this definition for $U_{\mathbf{p},t}$ fits our purposes.

Lemma 4.2.4. *For all $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$,*

$$\mathcal{M}_{\mathbf{p},t}(\tilde{f}_{\mathbf{p},t}) = \mathcal{M}_{\mathbf{p},t}(f_{\mathbf{p},t} + U_{\mathbf{p},t} - U_{\mathbf{p},t} \circ T_{\mathbf{p}}) = 0.$$

Proof. It follows from definition that

$$\begin{aligned}
\mathcal{M}_{\mathbf{p},t}(\tilde{f}_{\mathbf{p},t}) &= \mathcal{M}_{\mathbf{p},t}(f_{\mathbf{p},t}) + \mathcal{M}_{\mathbf{p},t}(U_{\mathbf{p},t}) - \mathcal{M}_{\mathbf{p},t}(U_{\mathbf{p},t} \circ T_{\mathbf{p}}) \\
&= \mathcal{M}_{\mathbf{p},t}(f_{\mathbf{p},t}) + \sum_{n=2}^{\infty} \mathcal{M}^n_{\mathbf{p},t}(f_{\mathbf{p},t}) - \sum_{n=2}^{\infty} \mathcal{M}^n_{\mathbf{p},t}(f_{\mathbf{p},t} \circ T_{\mathbf{p}}) \\
&= \sum_{n=1}^{\infty} \mathcal{M}^n_{\mathbf{p},t}(f_{\mathbf{p},t}) - \sum_{n=2}^{\infty} \mathcal{M}^n_{\mathbf{p},t}(f_{\mathbf{p},t} \circ T_{\mathbf{p}}) \\
&= \sum_{n=1}^{\infty} \mathcal{M}^n_{\mathbf{p},t}(f_{\mathbf{p},t}) - \sum_{n=2}^{\infty} \mathcal{M}^{n-1}_{\mathbf{p},t}(\mathcal{M}_{\mathbf{p},t}(f_{\mathbf{p},t} \circ T_{\mathbf{p}})) \\
&= \sum_{n=1}^{\infty} \mathcal{M}^n_{\mathbf{p},t}(f_{\mathbf{p},t}) - \sum_{n=2}^{\infty} \mathcal{M}^{n-1}_{\mathbf{p},t}(f_{\mathbf{p},t} \cdot \mathcal{M}_{\mathbf{p},t}(\mathbf{1})) \\
&= 0.
\end{aligned}$$

□

As an immediate corollary to the above, we can write the variance as a single integral as we intended.

Corollary 4.2.5. *We can write*

$$\sigma^2_{\mathbf{p},t}(\tilde{f}_{\mathbf{p},t}) = \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}.$$

Proof. By (4.8)

$$\sigma^2_{\mathbf{p},t}(\tilde{f}_{\mathbf{p},t}) = \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t} + 2\sum_{n=1}^{\infty} \int \tilde{f}_{\mathbf{p},t} \cdot \tilde{f}_{\mathbf{p},t} \circ T^n_{\mathbf{p}} d\mu_{\mathbf{p},t}.$$

Therefore,

$$\begin{aligned}
\sigma^2_{\mathbf{p},t}(\tilde{f}_{\mathbf{p},t}) &= \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t} + 2\sum_{n=1}^{\infty} \int \tilde{f}_{\mathbf{p},t} \cdot \tilde{f}_{\mathbf{p},t} \circ T^n_{\mathbf{p}} d\mu_{\mathbf{p},t} \\
&= \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t} + 2\sum_{n=1}^{\infty} \int \mathcal{M}^n_{\mathbf{p},t}(\tilde{f}_{\mathbf{p},t} \cdot \tilde{f}_{\mathbf{p},t} \circ T^n_{\mathbf{p}}) d\mu_{\mathbf{p},t} \\
&= \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t} + 2\sum_{n=1}^{\infty} \int \tilde{f}_{\mathbf{p},t} \cdot \mathcal{M}^n_{\mathbf{p},t}(\tilde{f}_{\mathbf{p},t}) d\mu_{\mathbf{p},t} \\
&= \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}
\end{aligned}$$

since $\mathcal{M}^n_{\mathbf{p},t}(\tilde{f}_{\mathbf{p},t}) = 0$ for all $n \in \mathbb{N}$. $\square$

Now that we have managed to find a cohomologous function $\tilde{f}_{\mathbf{p},t} \sim f_{\mathbf{p},t}$ such that we can write the variance in the simple form $\sigma^2_{\mathbf{p},t}(f_{\mathbf{p},t}) = \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$, we can shift our focus to how we plan to estimate this integral in a way that will give us the 'gap constant' $G_{\varepsilon,\delta}$ that we are searching for.

We'll begin by fixing $\mathbf{p}$ and t and focus on a particular method of approximating $\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$ for this specific choice of $\mathbf{p}$ and t. Then, using this method as a blueprint, we'll identify the obstacles we face before we can transform this 'pointwise' estimate for $\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$ into a global estimate for more general $\mathbf{p}$ and t. We begin with the 'pointwise' estimate for $\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$ for a fixed $\mathbf{p}$ and t.

Lemma 4.2.6. *Fix $\mathbf{p} \in \mathcal{P}$ and $t \in (0, \frac{1-s}{4}]$. Suppose that*

(i) $[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}} < \infty$.

(ii) There exists a periodic point $\Pi(\mathbf{i}) = z \in J_{\mathbf{p}}$ of period n and $c \neq 0$ such that $\frac{1}{n} S_n f_{\mathbf{p},t}(z) = c$.

Then there exists $m \in \mathbb{N}$ and $1 \leqslant k \leqslant n$ such that

$$\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t} \geqslant \frac{c^2}{4} \mu_{\mathbf{p},t}(\mathcal{I}_{i_k,\ldots,i_{k+m-1}}). \tag{4.9}$$

Proof. Let z be a point of period n such that $\frac{1}{n} S_n f_{\mathbf{p},t}(z) = c \neq 0$. Since $\frac{1}{n} S_n f_{\mathbf{p},t}(z) = \frac{1}{n} S_n \tilde{f}_{\mathbf{p},t}(z)$ it follows that there exists $0 \leqslant k \leqslant n-1$ such that $|\tilde{f}_{\mathbf{p},t}(T^k(z))| \geqslant c$. Without loss of generality we can assume $k = 0$, that is $|\tilde{f}_{\mathbf{p},t}(z)| \geqslant c$. Since $[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}} < \infty$ we can choose m sufficiently large such that $\frac{[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}}}{\Lambda^m} \leqslant \frac{c}{2}$. Then for all $y \in \mathcal{I}_{i_1,\ldots,i_m}$ we have

$$\begin{aligned} |\tilde{f}_{\mathbf{p},t}(z) - \tilde{f}_{\mathbf{p},t}(y)| &\leqslant \frac{[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}}}{\Lambda^m} \\ &\leqslant \frac{c}{2}. \end{aligned}$$

Since $|\tilde{f}_{\mathbf{p},t}(z)| \geqslant c$ it follows that $|\tilde{f}_{\mathbf{p},t}(y)| \geqslant \frac{c}{2}$. Thus it follows that

$$\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t} \geqslant \frac{c^2}{4} \mu_{\mathbf{p},t}(\mathcal{I}_{i_1,\ldots,i_m}).$$

$\square$

Therefore, by relating the variance $\sigma^2_{\mathbf{p},t}(f_{\mathbf{p},t})$ to the integral $\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$, we can bound it from below by a 'strip' of the integral which is determined by an interval centred at an appropriate point in a periodic orbit which sees a large ergodic sum. One simply needs to find a periodic orbit along which the ergodic sum of $\tilde{f}_{\mathbf{p},t}$ is large, then make the interval width sufficiently small so that $\tilde{f}_{\mathbf{p},t}$ remains fairly large within the appropriate interval.

We would like to use the strategy outlined above to estimate $\int \tilde{f}_{\mathbf{p},t}^2 d\mu_{\mathbf{p},t}$ uniformly for more general $\mathbf{p}$ and t. We are using the word uniform in a loose way. In order to be more precise about this, we need to examine the statement and proof of Lemma 4.2.6 more carefully, in order to identify which estimates we'll need to make. By considering each of these in turn, we'll see how each estimate depends on $\mathbf{p}$ and t, so that at each stage we can choose the set of parameters for which the corresponding estimate can be made uniform.

Firstly, we need to make sure that there exists a uniform constant $c > 0$ such that for any choice of $\mathbf{p}$ and t we can find a periodic point $T^n z = z = \Pi(\mathbf{i})$ for which

$$\frac{1}{n}|S_n f_{\mathbf{p},t}(z)| \geqslant c. \tag{4.10}$$

If each pair of parameters $\mathbf{p}$ and t were to give rise to a distinct periodic point z, this would cause difficulties with finding uniform bounds for the Hölder norm and measure later on, thus we need to find a finite set $\mathcal{Z}$ of periodic points, such that for any $\mathbf{p}$ and t we can choose some $z \in \mathcal{Z}$ that satisfies (4.10).

As an easy consequence of the non-linearity condition in Theorem 3.3.1, we'll see that we can choose the set of periodic points $\mathcal{Z} = \{z_1, z_2, z_{12}\}$ and the constant $c = \frac{\theta}{8}$ such that for *any* $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$ we can choose some $z \in \mathcal{Z}$ such that (4.10) will hold. We can now provide the statement of the relevant result.

Lemma 4.2.7. *Let z_1, z_2, z_{12}, z_{21} be the periodic points fixed by (1) in Theorem 3.3.1. Recall that*

$$\theta = \left|\log \frac{T'(z_1)T'(z_2)}{T'(z_{12})T'(z_{21})}\right| > 0.$$

Then for any $t \in [0, \frac{1-s}{4}]$, $\mathbf{p} \in \mathcal{P}_0$, there exists $z \in \{z_1, z_2, z_{12}\}$ for which

$$\left|\frac{1}{2}S_2 f_{\mathbf{p},t}(z)\right| \geqslant \frac{\theta}{8}.$$

Secondly, we need to find an upper bound for $[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}}$ which is uniform over our set of parameters for $\mathbf{p}$ and t, so that we can also choose the cylinder length m uniformly.

Since $f_{\mathbf{p},t} = -\beta'_{\mathbf{p}}(t)\log|T'| + f_{\mathbf{p}}$ and $f_{\mathbf{p}}$ is locally constant, the regularity of $f_{\mathbf{p},t}$ essentially boils down to the regularity of $-\beta'_{\mathbf{p}}(t)\log|T'|$. For $\mathbf{p} \in \mathcal{P}_0$, $-\beta'_{\mathbf{p}}(t) \leqslant$

$\frac{1}{t}$. Although we can slightly improve on this bound by remembering that for $\mathbf{p} \in \mathcal{P}_0$, $\beta'_{\mathbf{p}}(\frac{1-s}{4}) \geqslant \frac{s+1}{2}$ (so that actually $-\beta'_{\mathbf{p}}(t) \leqslant \frac{1-s}{t}$), there is no way to make this estimate uniform for small t. This means that if we let t approach 0, we lose any uniformity on the bound. This is essentially the same problem that we face when we pass to approximating $[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}}$ instead. Therefore, for the estimate on $[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}}$, the best uniformity we can hope for is the existence of a uniform constant C_0 such that for all $\delta \in (0, \frac{1-s}{4}]$, $\mathbf{p} \in \mathcal{P}_0$ and $t \in [\frac{\delta}{2}, \delta]$

$$[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant \frac{C_0}{\delta}.$$

Once we have obtained the constant C_0, for each δ and $t \in [\frac{\delta}{2}, \delta]$ this will correspond to a cylinder length m. This quantifies how small a cylinder we should take in the symbolic space to ensure that $\tilde{f}_{\mathbf{p},t}(y) \geqslant \frac{c}{2}$ for all $y \in \mathcal{I}_{i_1 \dots i_m}$.

The following result is the main result that we prove which relates to finding an upper bound on the Hölder norm of $\tilde{f}_{\mathbf{p},t}$.

Lemma 4.2.8. *The function $U_{\mathbf{p},t} \in \mathcal{F}_{\Lambda^{-1}}$ for all $t \in (0, \frac{1-s}{4}]$ and $\mathbf{p} \in \mathcal{P}_0$. Moreover, given any $\delta < \frac{1-s}{4}$, the seminorm $[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant \frac{C_0}{\delta}$ for all $t \in [\frac{\delta}{2}, \delta]$, where C_0 is a uniform constant which is independent of $\mathbf{p}$, δ and t.*

Finally, we need a uniform lower bound for the measure of the cylinder $\mu_{\mathbf{p},t}(\mathcal{I}_{i_1 \dots i_m})$, where this time 'uniform' means over all $z = \Pi(\mathbf{i}) \in \mathcal{Z}$ as well as for $\mathbf{p}$ and t in our choice of parameter set. Since the cylinder length m will have been fixed uniformly by the previous lemma, we can focus on getting a uniform bound on the cylinder $\mu_{\mathbf{p},t}(\mathcal{I}_{i_1 \dots i_n})$ for any fixed n. Since $\mu_{\mathbf{p},t}$ is Gibbs,

$$\mu_{\mathbf{p},t}(\mathcal{I}_{i_1 \dots i_n}) \geqslant C_{\mathbf{p},t}^{-1} \frac{(p_{i_1} \cdots p_{i_n})^t}{(|T'(z)| \cdots |T'(T^{n-1}z)|)^{\beta_{\mathbf{p}}(t)}}$$

where $C_{\mathbf{p},t}$ are the constants coming from the Gibbs property for $\mu_{\mathbf{p},t}$. Clearly the measure of this cylinder cannot be bounded uniformly from below for all $\mathbf{p} \in \mathcal{P}_0$ since $p_{i_1}, \dots, p_{i_n}$ could be arbitrarily close to 0. As a consequence of Lemma 4.2.7, $p_{i_1}, \dots, p_{i_n} \in \{p_1, p_2\}$, thus we need to have some control over how small p_1 and p_2 can be. At this point it becomes necessary to fix some $\varepsilon > 0$ and we can only hope to get uniform estimates for $\mu_{\mathbf{p},t}(\mathcal{I}_{i_1 \dots i_n})$ over $\mathbf{p} \in \mathcal{P}_\varepsilon$ and $\Pi(\mathbf{i}) = z \in \mathcal{Z}$.

The main result relating to this estimate is stated below.

Lemma 4.2.9. *Define*

$$\tau = \inf_{x \in \mathcal{I}_1 \cup \mathcal{I}_2} \frac{1}{|T'(x)|}. \tag{4.11}$$

There exists some constant $C > 0$ such that for all $\varepsilon > 0$, $t \in [0, \frac{1-s}{4}]$ and $\mathbf{p} \in \mathcal{P}_\varepsilon$,

$$\min\{\mu_{\mathbf{p},t}(\mathcal{I}_{i_1,\ldots,i_n})\} \geqslant C^{-1}\varepsilon^{tn}\tau^n$$

for every $n \in \mathbb{N}$, where the minimum is taken over all $\Pi(\mathbf{i}) = z \in \{z_1, z_2, z_{12}\}$.

The remainder of the chapter is organised as follows. In Section 4.3 we prove Lemma 4.2.7. In Section 4.4 we introduce some Hilbert-Birkhoff cone theory and use this to prove Lemma 4.2.8, which comprises most of the work. In Section 4.5 we prove Lemma 4.2.9. Finally, in Section 4.6 we tie the proofs of the last three lemmas together to obtain a lower bound on the variance and use this to prove the main result of that chapter, that is, an upper bound for $\sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}}$.

4.3 Existence of good periodic orbit

In this short section we prove Lemma 4.2.7, that is, we show that there exists a finite set $\mathcal{Z}$ of periodic points and a constant $c > 0$ such that for any choice of $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$ we can find a periodic point $T^n z = z \in \mathcal{Z}$ such that the ergodic average $\frac{1}{n} S_n \tilde{f}_{\mathbf{p},t}(z) > c$. This choice of periodic orbit which sees a large ergodic sum will determine where we centre the 'strips' of the integral that will provide us with a lower bound for $\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$.

Importantly, since $f_{\mathbf{p},t}$ and $\tilde{f}_{\mathbf{p},t}$ are cohomologous, $S_n\tilde{f}_{\mathbf{p},t}(z) = S_n f_{\mathbf{p},t}(z)$ for any periodic point $T^n z = z$ which means that we only actually need to study how large $f_{\mathbf{p},t}$ is along periodic orbits.

The key ingredient for the proof of Lemma 4.3 is the non-linearity assumption in Theorem 3.3.1, and in fact, the proof of this lemma is the *only* place that this assumption will be used for the proof of Theorem 3.3.1. It is not difficult to see that existence of lower bounds for $f_{\mathbf{p},t}$ are related to the non-linearity of the map. For instance, if we consider a *linear* EMR map T and define the Bernoulli measure $\mu_{\mathbf{p}}$ where $\mathbf{p} = (|\mathcal{I}_1|, |\mathcal{I}_2|, \ldots)$, then clearly the Bernoulli potential for $\mathbf{p}$ and the geometric potential coincide, i.e. $f_{\mathbf{p}} = -\log|T'|$ (and so $\mu_{\mathbf{p}}$ is absolutely continuous). Therefore, $f_{\mathbf{p},t} = -\beta'_{\mathbf{p}}(t)\log|T'| + f_{\mathbf{p}} = \beta'_{\mathbf{p}}(t) f_{\mathbf{p}} + f_{\mathbf{p}} = -f_{\mathbf{p}} + f_{\mathbf{p}} = 0$, since $\beta'_{\mathbf{p}}(t) = -1$ for all t. Although $\beta'_{\mathbf{p}}(t)$ appears in the expression for $f_{\mathbf{p},t}$, this isn't the ingredient which plays a part in securing a lower bound for $f_{\mathbf{p},t}$ (and besides, this is essentially what we're trying to estimate). So the lower bound for $f_{\mathbf{p},t}$ is essentially a measure of how different $f_{\mathbf{p}}$ and $-\log|T'|$ are, which is made precise by finding a periodic orbit along which the difference is bounded from below.

The non-linearity assumption provided in Theorem 3.3.1 gives hints as to how we should construct our set of periodic orbits. By examining this assumption, we see that it forces some non-linearity in one of the first two branches, since otherwise we'd have $\theta = \left|\log \frac{T'(z_1)T'(z_2)}{T'(z_{12})T'(z_{21})}\right| = 0$. This suggests that we should build our set of periodic points so that their orbits visit the first and second cylinders which are known to see some non-linearity to guarantee that we don't end up in the same situation as we did above. By choosing $\mathcal{Z} = \{z_1, z_2, z_{12}\}$, which are precisely the periodic points whose iterates appear in the expression for θ, we will see that even if $f_{\mathbf{p}}$ and $-\log|T'|$ are close when evaluated at z_1 and z_2, $S_2 f_{\mathbf{p}}$ and $-S_2 \log|T'|$ *cannot* be close when evaluated at z_{12}, since $f_{\mathbf{p}}$ will be additive whereas $-\log|T'|$ will not be additive (since T is not linear on $\mathcal{I}_1 \cup \mathcal{I}_2$). Therefore they will differ by some multiple of $\theta = \left|\log \frac{T'(z_1)T'(z_2)}{T'(z_{12})T'(z_{21})}\right| > 0$.

Proof of Lemma 4.2.7. Fix $t \in [0, \frac{1-s}{4}]$ and $\mathbf{p} = (p_1, p_2, \ldots) \in \mathcal{P}_0$. Since $\mathbf{p} \in \mathcal{P}_0$ we have a lower bound on the dimension of $\mu_{\mathbf{p}}$, in particular, $\dim \mu_{\mathbf{p}} \geqslant \frac{2s+2}{s+3} > \frac{1}{2}$. Therefore $|\beta'_{\mathbf{p}}(t)| \geqslant |\beta'_{\mathbf{p}}(1)| = \dim \mu_{\mathbf{p}} \geqslant \frac{1}{2}$ by convexity of $\beta_{\mathbf{p}}$, which was proved in Proposition 4.1.2. Put

$$c = \frac{\theta}{8}.$$

Without loss of generality we can assume that both

$$|f_{\mathbf{p},t}(z_1)| = |-\beta'_{\mathbf{p}}(t) \log|T'(z_1)| + \log p_1| < c \tag{4.12}$$

and

$$|f_{\mathbf{p},t}(z_2)| = |-\beta'_{\mathbf{p}}(t) \log|T'(z_2)| + \log p_2| < c \tag{4.13}$$

since otherwise we are done. We will show that this forces $|\frac{1}{2} S_2 f_{\mathbf{p},t}(z_{12})| > c$, which will complete the proof.

By (4.12) and (4.13) it follows that

$$\frac{1}{2}|-\beta'_{\mathbf{p}}(t) \log|T'(z_1)T'(z_2)| + \log p_1 p_2| \leqslant c.$$

Moreover

$$\begin{aligned}4|\beta'_{\mathbf{p}}(t)|c &= \frac{|\beta'_{\mathbf{p}}(t)|}{2}\left|\log\frac{T'(z_1)T'(z_2)}{T'(z_{12})T'(z_{21})}\right| \\ &\leqslant \frac{1}{2}\left|-\beta'_{\mathbf{p}}(t)\log|T'(z_1)T'(z_2)|+\log p_1p_2\right| \\ &\quad+\frac{1}{2}\left|-\beta'_{\mathbf{p}}(t)\log|T'(z_{12})T'(z_{21})|+\log p_1p_2\right| \\ &\leqslant \frac{1}{2}\left|-\beta'_{\mathbf{p}}(t)\log|T'(z_{12})T'(z_{21})|+\log p_1p_2\right|+c.\end{aligned}$$

Therefore

$$\frac{1}{2}\left|-\beta'_{\mathbf{p}}(t)\log|T'(z_{12})T'(z_{21})|+\log p_1p_2\right| \geqslant 4|\beta'_{\mathbf{p}}(t)|c-c \geqslant c$$

where the final inequality is because $|\beta'_{\mathbf{p}}(t)| \geqslant \frac{1}{2}$. □

4.4 Decay of operator norms

In this section we prove Lemma 4.2.8, in particular we show that there exists a uniform constant $C_0 < \infty$ such that for all $\delta < \frac{1-s}{4}$, the seminorm $[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant \frac{C_0}{\delta}$ for all $t \in [\frac{\delta}{2}, \delta]$ and $\mathbf{p} \in \mathcal{P}_0$. Recall that we would like to estimate the integral $\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$ from below by a strip which is centred at an appropriate point in a chosen periodic orbit, which will be fixed for each $\mathbf{p}$ and t by Lemma 4.2.7. To this end, the constant C_0 will allow us to estimate the required *strip width*. In particular, we need the strip width to be sufficiently small so that $\tilde{f}_{\mathbf{p},t}$ does not drop 'too much' within each strip, when compared to the Birkhoff average along the periodic orbit that has been chosen.

Recall that for all $t \in [\frac{\delta}{2}, \delta]$ and $\mathbf{p} \in \mathcal{P}_0$, the seminorm $[f_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant \frac{\kappa\Lambda^l}{\delta}$, since $[\log|T'|]_{\Lambda^{-1}} \leqslant \kappa\Lambda^l$. Therefore, the difficulty is to prove that there is a uniform upper bound for $[U_{\mathbf{p},t}]_{\Lambda^{-1}}$. In fact we'll prove something stronger. Let $[f]_1$ denote the Lipschitz constant (or seminorm) of a function $f : [0,1] \to [0,1]$ which is defined by

$$[f]_1 = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|}.$$

We say that f is Lipschitz (continuous) if $[f]_1 < \infty$. Let $C^{0,1}([0,1])$ denote the space of all bounded Lipschitz continuous functions. Then $C^{0,1}([0,1])$ is a Banach space when equipped with the norm $\|\cdot\|_{0,1} = [\cdot]_1 + \|\cdot\|_\infty$.

Clearly, $f_{\mathbf{p},t} \notin C^{0,1}([0,1])$. However, we'll show that for all $n \in \mathbb{N}$, the iterates $\mathcal{M}^n_{\mathbf{p},t} f_{\mathbf{p},t} \in C^{0,1}([0,1])$, and as such we can calculate an upper bound for

$[U_{\mathbf{p},t}]_1$. Since $U_{\mathbf{p},t} = \sum_{n=1}^{\infty} \mathcal{M}_{\mathbf{p},t}^n f_{\mathbf{p},t}$, this suggests that it would be useful to find an explicit exponential decay rate for $[\mathcal{M}_{\mathbf{p},t}^n f_{\mathbf{p},t}]_1$ in n, which is precisely where the techniques of Hilbert-Birkhoff cone theory come in useful.

The remainder of this section is structured as follows. Firstly, in 4.4.1 we will introduce the key ideas from Hilbert-Birkhoff cone theory which will be used. 4.4.2 and 4.4.3 will be dedicated to using these tools to obtain the relevant exponential decay rate. In 4.4.4 we directly show that $\mathcal{M}_{\mathbf{p},t}^k f_{\mathbf{p},t} \in C^{0,1}([0,1])$ for each $1 \leqslant k \leqslant l$, which is a necessary step before we can apply the exponential decay rate to prove our main result. Finally in 4.4.5 we combine everything to obtain an upper bound for $[U_{\mathbf{p},t}]_1$, which consequently provides us with an upper bound for $[U_{\mathbf{p},t}]_{\Lambda^{-1}}$.

4.4.1 Hilbert-Birkhoff cone theory

In this section we'll introduce the key tools that will be used from Hilbert-Birkhoff cone theory. A more comprehensive overview of this area, including proofs of results, can be found in the appendix.

Hilbert-Birkhoff cone theory is concerned with the study of the action of an operator on a cone. Let $C \subset V \setminus \{0\}$ be a convex cone in a vector space V; this means that $\lambda w \in C$ and $w_1 + w_2 \in C$ for all $\lambda > 0$ and all $w, w_1, w_2 \in C$. We can define a partial ordering on V by

$$v \preceq w \quad \Leftrightarrow \quad w - v \in C \cup \{0\}.$$

Using this partial ordering, we can define a metric Θ on C, whose precise definition is given in the appendix. This metric is known as a *Hilbert* metric or *projective* metric. We say that the *diameter* D of a set A with respect to the metric Θ is

$$D = \sup\{\Theta(v, w) : v, w \in A\}.$$

The following proposition lays the foundation for the use of cone methods in the theory of transfer operators. This result tells us that if we have a linear operator $L : C \to C$ such that the image $L(C)$ has a finite diameter with respect to Θ, then the operator is a *strict* contraction with respect to the metric Θ and, moreover, the contraction ratio can be given in terms of the diameter of $L(C)$. The following result is a specific example of Proposition A.0.9 from the appendix, where a more general version of the result below is stated and proved.

Proposition 4.4.1. *Let $L : V \to V$ be a linear operator on a vector space and $C \subset V$ be a cone. Let $D = \sup\{\Theta(L(v), L(w)) : v, w \in C\}$ be the diameter of $L(C)$.*

Then, if $D < \infty$,

$$\Theta(L(v), L(w)) \leqslant (1 - e^{-D})\Theta(v, w).$$

Of course, one is typically not interested in how our operator behaves with respect to the abstract metric Θ. Therefore, for this result to be useful, we need a way of linking a contraction in Θ to similar behaviour in other norms - norms which we *are* interested in. The following lemma provides us with this tool. This is restated and proved as Proposition A.0.12 in the appendix.

Proposition 4.4.2. *Let $\|\cdot\|_1$, $\|\cdot\|_2$ be two norms on V and let $C \subset V$ be a convex cone which induces the partial ordering $\preceq$. Suppose there exists $C \geqslant 1$ such that for all $f, g \in V$*

$$-f \preceq g \preceq f \quad \Rightarrow \quad \begin{aligned} \|g\|_1 &\leqslant \|f\|_1 \\ \|g\|_2 &\leqslant C\|f\|_2. \end{aligned}$$

Then given any $f, g \in C$ for which $\|f\|_1 = \|g\|_1$,

$$\|f - g\|_2 \leqslant C^2(e^{\Theta(f,g)} - 1)\|f\|_2.$$

We will now hone in on the specific cones and operators which we will be working with when it comes to implementing the ideas discussed above. Let $C([0,1])$ denote the space of all continuous functions on $[0,1]$. We will be interested in cones of the type $\mathcal{C}_a$ for some $a > 0$ where

$$\mathcal{C}_a = \left\{ w \in C([0,1]) : w \geqslant 0 \text{ and } w(x) \leqslant e^{a|x-y|}w(y) \right\}.$$

It is easy to check that for each $a > 0$, $\mathcal{C}_a$ does indeed define a cone.

Observe that for each $a > 0$, $\mathcal{C}_a \subset C^{0,1}([0,1])$. To see this, let $f \in \mathcal{C}_a$. f is continuous on a compact set and therefore is bounded. Moreover,

$$-(e^{a|x-y|} - 1)f(x) \leqslant f(x) - f(y) \leqslant (e^{a|x-y|} - 1)f(y)$$

for all $x, y \in [0,1]$ which implies that

$$|f(x) - f(y)| \leqslant ae^a\|f\|_\infty|x - y|$$

that is, f is Lipschitz with Lipschitz constant $[f]_1 \leqslant ae^a\|f\|_\infty$. We'll use this fact several times throughout the rest of this section, therefore we state this as a proposition.

Proposition 4.4.3. *Suppose $f \in \mathcal{C}_a$ for some $a > 0$. Then $[f]_1 \leqslant ae^a \|f\|_\infty$.*

For these types of cone, we have a direct way of checking whether the image of a linear operator $L : \mathcal{C}_a \to \mathcal{C}_a$ has a finite diameter. The following lemma tells us that it is enough to check that $L(\mathcal{C}_a) \subset \mathcal{C}_{\lambda a}$ for some $\lambda < 1$, which combined with Proposition 4.4.1 allows us to deduce that L is a strict contraction. The next result is restated and proved as Proposition A.0.10 in the appendix.

Proposition 4.4.4. *Let $a > 0$ and $0 < \lambda < 1$. Let Θ be the metric associated to the cone $\mathcal{C}_a$. Then*

$$D_{\lambda,a} \quad := \quad \sup\{\Theta(v, w) : v, w \in \mathcal{C}_{\lambda a}\} < \infty. \tag{4.14}$$

Observe that if one can show that $L(\mathcal{C}_a) \subset \mathcal{C}_{\lambda a}$ for some $\lambda < 1$, then

$$\begin{aligned} \sup\{\Theta(L(v), L(w)) : v, w \in \mathcal{C}_a\} &\leqslant \sup\{\Theta(v, w) : v, w \in \mathcal{C}_{\lambda a}\} \\ &= D_{\lambda,a} < \infty \end{aligned}$$

and therefore by Proposition 4.4.1,

$$\Theta(L(v), L(w)) \leqslant (1 - e^{-D_{\lambda,a}})\Theta(v, w).$$

The following result allows us to use the partial ordering $\preceq$ induced by the cone $\mathcal{C}_a$ to control the norms of continuous functions. This is restated and proved as Proposition A.0.13 in the appendix.

Proposition 4.4.5. *Let $\preceq$ be the partial ordering induced by the cone $\mathcal{C}_a$ for some $a > 0$. Let m be a measure on $[0, 1]$ and let $L^1 = L^1(m)$. Then*

$$-f \preceq g \preceq f \quad \Rightarrow \quad \begin{aligned} \|g\|_\infty &\leqslant \|f\|_\infty \\ \|g\|_{L^1} &\leqslant \|f\|_{L^1} \\ \|g\|_{0,1} &\leqslant (a + 1)\|f\|_{0,1}. \end{aligned}$$

Observe that the result above allows us to employ Proposition 4.4.2 with the norms $\|\cdot\|_\infty, \|\cdot\|_{L^1}, \|\cdot\|_{0,1}$. In practise, we'll use the result for $\|\cdot\|_{L^1}$ with $m = \mu_{\mathbf{p},t}$ and $m = \tilde{\mu}_{\mathbf{p},t}$, where $\tilde{\mu}_{\mathbf{p},t} = \mathcal{L}^*_{\mathbf{p},t}\tilde{\mu}_{\mathbf{p},t}$ as defined in Lemma 4.2.2.

Since $U_{\mathbf{p},t} = \sum_{n=1}^\infty \mathcal{M}^n_{\mathbf{p},t} f_{\mathbf{p},t}$, this suggests that the correct operator to study would be (some iterate of) $\mathcal{M}_{\mathbf{p},t}$. In fact, we will choose our operator to be the

lth iterate $\mathcal{M}^l_{\mathbf{p},t} : \mathcal{C}_a \to \mathcal{C}_a$, where l is the integer that satisfies $|(T^l)'(x)| \geqslant \Lambda^l$ for all $x \in [0,1]$ as part of the expanding property of the map T. We choose the lth iterate since Hilbert-Birkhoff cone theory is implemented for uniformly expanding maps by showing that the associated transfer operator maps sufficiently large cones strictly inside themselves. Since we are considering T which may not necessarily be uniformly expanding (but T^l *is* uniformly expanding), we study the action of the lth iterate of the transfer operator on cones. In fact, before we can prove anything for the normalised operator, it will be necessary to study the lth iterate of the *non-normalised operator* $\mathcal{L}^l_{\mathbf{p},t} : \mathcal{C}_a \to \mathcal{C}_a$ first.

The remainder of this section is structured as follows. Firstly, we will need to prove that our operator $\mathcal{M}^l_{\mathbf{p},t}$ is a contraction with respect to the metric Θ. Therefore, in light of Propositions 4.4.1 and 4.4.4, we will need to show that $\mathcal{M}^l_{\mathbf{p},t}(\mathcal{C}_a) \subset \mathcal{C}_{\lambda a}$ for some $\lambda < 1$. This will be treated in 4.4.2.

Next, in 4.4.3, we'll see how we can use the contraction in Θ to study the behaviour of the operator $\mathcal{M}^l_{\mathbf{p},t}$ with respect to the norm $\|\cdot\|_{0,1}$. By using Proposition 4.4.2 with $\|\cdot\|_1 = \|\cdot\|_{L^1}$ and $\|\cdot\|_2 = \|\cdot\|_{0,1}$ along with Proposition 4.4.5, we'll show that $\|\mathcal{M}^{ln}_{\mathbf{p},t} f\|_{0,1}$ decays exponentially in n whenever $\int f d\mu_{\mathbf{p},t} = 0$ and $f \in C^{0,1}([0,1])$.

In 4.4.4, we deal with the issue that $f_{\mathbf{p},t} \notin C^{0,1}([0,1])$, which is preventing us from applying the exponential decay rate in our setting. In particular, we show that for each $1 \leqslant k \leqslant l$, the kth iterate $\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t} \in C^{0,1}([0,1])$ and we obtain upper bounds on $\|\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}$ for $1 \leqslant k \leqslant l$.

Finally in 4.4.5, we combine the upper bounds on $\|\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}$ and the exponential decay of $\|\mathcal{M}^{ln}_{\mathbf{p},t} f\|_{0,1}$ to prove Lemma 4.2.8.

4.4.2 Proving a contraction in Θ

The goal of this section is to find $a \in \mathbb{R}$ sufficiently large that $\mathcal{M}^l_{\mathbf{p},t}$ is a *strict contraction* on $\mathcal{C}_a$. Recall that by Proposition 4.4.1, it is enough to show that for some $a > 0$, $\sup\{\Theta(v,w) : v, w \in \mathcal{M}^l_{\mathbf{p},t}(\mathcal{C}_a)\} < \infty$. Moreover, by Proposition 4.4.4, it is sufficient to show that $\mathcal{M}^l_{\mathbf{p},t}(\mathcal{C}_a) \subset \mathcal{C}_{\lambda a}$ for some $\lambda < 1$.

First, we will need to prove that the non-normalised operator $\mathcal{L}^l_{\mathbf{p},t}$ is a contraction on some cone $\mathcal{C}_a$. Using this, we will then be able to deduce that for each $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$, the fixed point $h_{\mathbf{p},t}$ belongs to the cone $\mathcal{C}_a$. This will give us some regularity properties of the fixed point $h_{\mathbf{p},t}$ that hold for all $t \in [0, \frac{1-s}{4}]$ and $\mathbf{p} \in \mathcal{P}_0$. These uniform regularity properties are both necessary to prove the contraction in Θ for the normalised operator $\mathcal{M}^l_{\mathbf{p},t}$, but are also important properties in their own right which will be used at various points in the remainder of the chapter.

We begin with what is essentially a restatement of the locally Hölder properties of $g_{\mathbf{p},t}$.

Lemma 4.4.6. *Let* $\mathbf{p} \in \mathcal{P}_0$ *and* $t \in [0, \frac{1-s}{4}]$. *Then for all* $x, y \in [0,1]$ *and* $n_1, \ldots, n_l \in \mathbb{N}^l$,

$$|g^l_{\mathbf{p},t}(T^{-1}_{n_1,\ldots,n_l}x) - g^l_{\mathbf{p},t}(T^{-1}_{n_1,\ldots,n_l}y)| \leqslant l\kappa|x-y|. \tag{4.15}$$

Proof. Clearly it is enough to show that for any $1 \leqslant k \leqslant l$,

$$|g_{\mathbf{p},t}(T^{-1}_{n_1,\ldots,n_k}x) - g_{\mathbf{p},t}(T^{-1}_{n_1,\ldots,n_k}y)| \leqslant \kappa|x-y|. \tag{4.16}$$

Since $f_{\mathbf{p}}$ is locally constant, the left hand side of (4.16) equals

$$\begin{aligned}
\beta_{\mathbf{p}}(t)|\log|T'(T^{-1}_{n_1,\ldots,n_k}x)| - \log|T'(T^{-1}_{n_1,\ldots,n_k}y)|| &\leqslant \sup_{z\in\mathcal{I}_{n_1}}\left|\frac{T''_{n_1}(z)}{T'_{n_1}(z)}\right||T^{-1}_{n_1,\ldots,n_k}x - T^{-1}_{n_1,\ldots,n_k}y| \\
&\leqslant \sup_{z\in\mathcal{I}_{n_1}}\frac{|T''_{n_1}(z)|}{|T'_{n_1}(z)|}\sup_{z\in\mathcal{I}_{n_1}}\frac{1}{|T'_{n_1}(z)|}|x-y| \\
&\leqslant \kappa|x-y|.
\end{aligned}$$

Then (4.15) directly follows. $\square$

Using the regularity of $g_{\mathbf{p},t}$, we can deduce that for sufficiently large a_0 and some $\lambda_0 < 1$ (which is related to the contraction rate of the map T), then $\mathcal{L}^l_{\mathbf{p},t}\mathcal{C}_{a_0} \subset \mathcal{C}_{\lambda_0 a_0}$.

Lemma 4.4.7. *There exists* $\lambda_0 < 1$ *and* $a_0 > 0$ *such that for all* $\mathbf{p} \in \mathcal{P}_0$ *and* $t \in [0, \frac{1-s}{4}]$, $\mathcal{L}^l_{\mathbf{p},t}\mathcal{C}_{a_0} \subseteq \mathcal{C}_{\lambda_0 a_0}$.

Proof. Fix $t \in [0, \frac{1-s}{4}]$ and $\mathbf{p} \in \mathcal{P}_0$. Let $a_0 > 0$. Clearly, if $w \geqslant 0$ then $\mathcal{L}^l_{\mathbf{p},t}w \geqslant 0$. Let $w \in \mathcal{C}_{a_0}$ and $x, y \in [0,1]$. Recall that $|(T^l)'| \geqslant \Lambda^l$ on $[0,1]$. In particular, this means that any local inverse branch of T^l must be contracting by $\frac{1}{\Lambda^l}$. Using this fact and (4.15) we obtain

$$\begin{aligned}
(\mathcal{L}_{\mathbf{p},t}^{l}w)(x) &= \sum_{\mathbf{n}\in\mathbb{N}_{\mathbf{p}}^{l}} e^{g_{\mathbf{p},t}^{l}(T_{\mathbf{n}}^{-1}x)} w\left(T_{\mathbf{n}}^{-1}x\right) \\
&\leqslant \sum_{\mathbf{n}\in\mathbb{N}_{\mathbf{p}}^{l}} e^{g_{\mathbf{p},t}^{l}(T_{\mathbf{n}}^{-1}y)} w\left(T_{\mathbf{n}}^{-1}y\right) e^{(l\kappa|x-y|+a_0|T_{\mathbf{n}}^{-1}x-T_{\mathbf{n}}^{-1}y|)} \\
&\leqslant \sum_{\mathbf{n}\in\mathbb{N}_{\mathbf{p}}^{l}} e^{g_{\mathbf{p},t}^{l}(T_{\mathbf{n}}^{-1}y)} w\left(T_{\mathbf{n}}^{-1}y\right) e^{(l\kappa+\frac{a_0}{\Lambda^l})|x-y|} \\
&\leqslant \sum_{\mathbf{n}\in\mathbb{N}_{\mathbf{p}}^{l}} e^{g_{\mathbf{p},t}^{l}(T_{\mathbf{n}}^{-1}y)} w\left(T_{\mathbf{n}}^{-1}y\right) e^{(l\kappa+\frac{a_0}{\Lambda^l})|x-y|}
\end{aligned}$$

Choose $\frac{1}{\Lambda^l} < \lambda_0 < 1$ and $a_0 \geqslant \frac{l\kappa}{\lambda_0 - \frac{1}{\Lambda^l}}$. Then it follows that

$$(\mathcal{L}_{\mathbf{p},t}^{l}w)(x) \leqslant (\mathcal{L}_{\mathbf{p},t}^{l}w)(y)e^{a_0\lambda_0|x-y|}. \tag{4.17}$$

Finally, we verify that $\mathcal{L}_{\mathbf{p},t}^{l}w \in C([0,1])$. Recall that $\sup_x |(\mathcal{L}_{\mathbf{p},t}^{l}\mathbf{1})(x)| < \infty$ by our choice of $\mathbf{p}$ and t, so $\|\mathcal{L}_{\mathbf{p},t}^{l}w\|_\infty \leqslant \|w\|_\infty \sup_x |(\mathcal{L}_{\mathbf{p},t}^{l}\mathbf{1})(x)| < \infty$. By (4.17),

$$-(e^{a_0\lambda_0|x-y|}-1)\mathcal{L}_{\mathbf{p},t}^{l}(x) \leqslant \mathcal{L}_{\mathbf{p},t}^{l}(x) - \mathcal{L}_{\mathbf{p},t}^{l}(y) \leqslant (e^{a_0\lambda_0|x-y|}-1)\mathcal{L}_{\mathbf{p},t}^{l}(y)$$

and so

$$\begin{aligned}
|\mathcal{L}_{\mathbf{p},t}^{l}w(x) - \mathcal{L}_{\mathbf{p},t}^{l}w(y)| &\leqslant (e^{a_0\lambda_0|x-y|}-1)\mathcal{L}_{\mathbf{p},t}^{l}w(y) \\
&\leqslant a_0 e^{a_0}|x-y|\|\mathcal{L}_{\mathbf{p},t}^{l}w\|_\infty
\end{aligned}$$

which completes the proof. □

By Propositions 4.4.1 and 4.4.4, $\mathcal{L}_{\mathbf{p},t}^{l}$ is a contraction in Θ with contraction ratio $1 - e^{-D_{\lambda_0,a_0}}$, where

$$D_{\lambda_0,a_0} = \sup\{\Theta(v,w) : v,w \in \mathcal{C}_{\lambda_0 a_0}\} < \infty$$

is provided by Lemma 4.4.4.

Next, we would like to prove the analogous result for the normalised operator $\mathcal{M}_{\mathbf{p},t}^{l}$. By Lemma 2.5.9, we know that for each operator $\mathcal{L}_{\mathbf{p},t}$ there exists a unique fixed point $h_{\mathbf{p},t}$ and, moreover, $\mathcal{M}_{\mathbf{p},t} = h_{\mathbf{p},t}^{-1}\mathcal{L}_{\mathbf{p},t}(h_{\mathbf{p},t}\cdot)$. Therefore, before we can prove the analogous result for $\mathcal{M}_{\mathbf{p},t}^{l}$, we first require some regularity properties of the fixed point $h_{\mathbf{p},t}$, which is what the following lemma will provide. This is an important result in its own right, for instance, by using the uniform regularity properties of $h_{\mathbf{p},t}$ we will be able to obtain uniform Gibbs constants for the measures $\mu_{\mathbf{p},t}$ in Section 4.5.

Lemma 4.4.8. *Each $h_{\mathbf{p},t} \in \mathcal{C}_{a_0}$ where a_0 is fixed uniformly via Lemma 4.4.7. In particular,*

$$e^{-a} \leqslant \inf_{x,y} \frac{h_{\mathbf{p},t}(x)}{h_{\mathbf{p},t}(y)} \leqslant \sup_{x,y} \frac{h_{\mathbf{p},t}(x)}{h_{\mathbf{p},t}(y)} \leqslant e^{a}$$

and

$$\sup_x \frac{[h_{\mathbf{p},t}]_{0,1}}{h_{\mathbf{p},t}(x)} \leqslant ae^{2a}$$

for all $a \geqslant a_0$ and all $\mathbf{p} \in \mathcal{P}_0$, $t \in [0, \frac{1-s}{4}]$.

Proof. We will show that for each $\mathbf{p}$ and t we can find a fixed point of $\mathcal{L}_{\mathbf{p},t}$ inside $\mathcal{C}_{a_0}$. Since we know that the fixed point is unique, it will follow that this is $h_{\mathbf{p},t}$.

Let $\mathbf{p}$ and t be arbitrary and denote $\mathcal{L} = \mathcal{L}_{\mathbf{p},t}$. Let $N \in \mathbb{N}$ and consider integers $m, n \geqslant N$. Fix $D = D_{\lambda_0,a_0}$. For each $0 \leqslant k \leqslant l-1$, we can apply Propositions 4.4.1 and 4.4.4 to $\mathcal{L}^k\mathbf{1}$ and obtain

$$\Theta(\mathcal{L}^{ln+k}\mathbf{1}, \mathcal{L}^{lm+k}\mathbf{1}) \leqslant (1-e^{-D})^N \Theta(\mathcal{L}^{l(n-N)+k}\mathbf{1}, \mathcal{L}^{l(m-N)+k}\mathbf{1}) \leqslant D(1-e^{-D})^N.$$

Let $L^1 = L^1(\tilde{\mu}_{\mathbf{p},t})$. Since $\|\mathcal{L}^k\mathbf{1}\|_{L^1} = \|\mathbf{1}\|_{L^1}$ for all $k \in \mathbb{N}$, we can apply Proposition 4.4.2 to the norms $\|\cdot\|_1 = \|\cdot\|_{L^1}$ and $\|\cdot\|_2 = \|\cdot\|_\infty$ to deduce that for all $n, m \geqslant N$,

$$\begin{aligned} \|\mathcal{L}^{ln+k}\mathbf{1} - \mathcal{L}^{lm+k}\mathbf{1}\|_\infty &\leqslant (e^{\Theta(\mathcal{L}^{ln+k}\mathbf{1}, \mathcal{L}^{lm+k}\mathbf{1})} - 1)\|\mathbf{1}\|_\infty \\ &\leqslant e^{D(1-e^{-D})^N} - 1 \\ &\leqslant e^D D(1-e^{-D})^N. \end{aligned}$$

This implies that $\mathcal{L}^n\mathbf{1}$ is Cauchy for $\|\cdot\|_\infty$ since $1 - e^{-D} < 1$. Thus the limit $\lim_{n\to\infty} \mathcal{L}^n\mathbf{1} \in \mathcal{C}_{a_0}$ and is a fixed point of $\mathcal{L}$. By recalling that $\mathcal{L} = \mathcal{L}_{\mathbf{p},t}$ and since $\mathbf{p}$ and t were arbitrary, it follows that $h_{\mathbf{p},t} \in \mathcal{C}_{a_0}$ for all $\mathbf{p}$ and $t \in [0, \frac{1-s}{4}]$. Moreover, since $\mathcal{C}_{a_0} \subseteq \mathcal{C}_a$ for $a \geqslant a_0$ it follows that $h_{\mathbf{p},t} \in \mathcal{C}_a$ for all $\mathbf{p}$ and $t \in [0, \frac{1-s}{4}]$. By Proposition 4.4.3, $[h_{\mathbf{p},t}]_1 \leqslant ae^a\|h_{\mathbf{p},t}\|_\infty$. Therefore, $\sup_x \frac{[h_{\mathbf{p},t}]_1}{h_{\mathbf{p},t}(x)} \leqslant \sup_x ae^a \frac{\|h_{\mathbf{p},t}\|_\infty}{h_{\mathbf{p},t}(x)} \leqslant ae^{2a}$. □

Now, using the regularity properties of the fixed point, we can prove the analogue of Lemma 4.4.7 for the normalised operator $\mathcal{M}^l_{\mathbf{p},t}$.

Lemma 4.4.9. *There exists $\lambda_1 < 1$ and $a_1 > 0$ such that for all $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$, $\mathcal{M}^l_{\mathbf{p},t}\mathcal{C}_{a_1} \subseteq \mathcal{C}_{\lambda a_1}$. Moreoever,*

$$D_{\lambda_1,a_1} = \sup\{\Theta(v,w) : v, w \in \mathcal{C}_{\lambda_1 a_1}\} < \infty$$

and

$$\Theta(\mathcal{M}^l_{\mathbf{p},t}(v), \mathcal{M}^l_{\mathbf{p},t}(w)) \leqslant (1 - e^{-D_{\lambda_1, a_1}})\Theta(v, w).$$

Proof. Fix $t \in [0, \frac{1-s}{4}]$ and $\mathbf{p} \in \mathcal{P}_0$. Let a_0 be fixed by Lemma 4.4.7 and let $a_1 > 0$. Let $w \in \mathcal{C}_{a_1}$ and $x, y \in [0, 1]$. Similarly to Lemma 4.4.7 we see that

$$\begin{aligned}
(\mathcal{M}^l_{\mathbf{p},t} w)(x) &= h^{-1}_{\mathbf{p},t}(x) \sum_{\mathbf{n} \in \mathbb{N}^l_{\mathbf{p}}} e^{g^l_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} x)} w\left(T^{-1}_{\mathbf{n}} x\right) h_{\mathbf{p},t}\left(T^{-1}_{\mathbf{n}} x\right) \\
&\leqslant h^{-1}_{\mathbf{p},t}(x) \sum_{\mathbf{n} \in \mathbb{N}^l_{\mathbf{p}}} e^{g^l_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} y)} w\left(T^{-1}_{\mathbf{n}} y\right) h_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} y) e^{(l\kappa|x-y| + (a_0+a_1)|T^{-1}_{\mathbf{n}} x - T^{-1}_{\mathbf{n}} y|)} \\
&\leqslant h^{-1}_{\mathbf{p},t}(y) e^{a_0|x-y|} \sum_{\mathbf{n} \in \mathbb{N}^l_{\mathbf{p}}} e^{g^l_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} y)} w\left(T^{-1}_{\mathbf{n}} y\right) h_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} y) e^{(l\kappa + \frac{a_0+a_1}{\Lambda^l})|x-y|} \\
&\leqslant h^{-1}_{\mathbf{p},t}(y) \sum_{\mathbf{n} \in \mathbb{N}^l_{\mathbf{p}}} e^{g^l_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} y)} w\left(T^{-1}_{\mathbf{n}} y\right) h_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} y) e^{(l\kappa + \frac{(\Lambda^l+1)a_0 + a_1}{\Lambda^l})|x-y|}.
\end{aligned}$$

Choose $\frac{1}{\Lambda^l} < \lambda_1 < 1$ and $a_1 \geqslant \frac{l\kappa + (\frac{\Lambda^l+1}{\Lambda^l})a_0}{\lambda_1 - \frac{1}{\Lambda^l}}$. Then it follows that

$$(\mathcal{M}^l_{\mathbf{p},t} w)(x) \leqslant (\mathcal{M}^l_{\mathbf{p},t} w)(y) e^{a_1 \lambda_1 |x-y|}. \tag{4.18}$$

Clearly, since $h_{\mathbf{p},t} > 0$, if $w \geqslant 0$ then $\mathcal{M}^l_{\mathbf{p},t} w \geqslant 0$. We can verify that $\mathcal{M}^l_{\mathbf{p},t} w \in C([0, 1])$ in the same way that we did in Lemma 4.4.7, namely, we observe that $\sup_x (\mathcal{M}^l_{\mathbf{p},t} w)(x) \leqslant \|w\|_\infty$ and therefore by (4.18)

$$|\mathcal{M}^l_{\mathbf{p},t} w(x) - \mathcal{M}^l_{\mathbf{p},t} w(y)| \leqslant a_1 e^{a_1} \|w\|_\infty |x - y|.$$

Therefore, we have proved that $\mathcal{M}^l_{\mathbf{p},t} \mathcal{C}_{a_1} \subset \mathcal{C}_{\lambda_1 a_1}$.

By Proposition 4.4.4, $D_{\lambda_1, a_1} < \infty$ and by Proposition 4.4.1, the final part follows. □

For the remainder of this chapter, we fix $a = \max\{1, a_1\}$ and $D = D_{\lambda_1, a_1}$.

4.4.3 Exponential decay of $\|\mathcal{M}^{ln}_{\mathbf{p},t} f\|_{0,1}$

In the previous section, we showed that $\mathcal{M}^l_{\mathbf{p},t} : \mathcal{C}_a \to \mathcal{C}_a$ was a contraction in Θ. In this section, we will apply Propositions 4.4.2 and 4.4.5 and the contraction in Θ to study $\|\mathcal{M}^{ln}_{\mathbf{p},t} f\|_{0,1}$ instead. In particular, we will find an explicit rate of exponential decay for $\|\mathcal{M}^{ln}_{\mathbf{p},t} f\|_{0,1}$ whenever $f \in C^{0,1}([0, 1])$ and $\int f d\mu_{\mathbf{p},t} = 0$.

Before we can obtain an exponential decay rate for $\|\mathcal{M}^n_{\mathbf{p},t} f\|_{0,1}$, we need a uniform bound on the operator norms of $\mathcal{M}_{\mathbf{p},t}$, that is, some uniform constant A for which $\|\mathcal{M}_{\mathbf{p},t}f\|_{0,1} \leqslant A\|f\|_{0,1}$ for all $f \in C^{0,1}([0,1])$. In fact, although such A exists, we will only actually require a bound $\|\mathcal{M}_{\mathbf{p},t}f\|_{0,1} \leqslant A\|f\|_{0,1}$ which holds for any $f \in \mathcal{C}_a$. Therefore we will find this instead, since we can obtain this almost directly from the definition of $\mathcal{C}_a$. Note that alternatively, we could prove this by proving a 'Lasota-Yorke inequality'.

Lemma 4.4.10 (Uniformity of operator norms)**.** *There exists a uniform constant $A = 1 + ae^a$ such that for all $\mathbf{p} \in \mathcal{P}_0$, $t \in [0, \frac{1-s}{4}]$ and $n \in \mathbb{N}$,*

$$\|\mathcal{M}_{\mathbf{p},t}^{ln}f\|_{0,1} \leqslant A\|f\|_{0,1}$$

for all $f \in \mathcal{C}_a$.

Proof. Firstly, we can immediately see that $\|\mathcal{M}_{\mathbf{p},t}^{k}f\|_\infty \leqslant \|f\|_\infty$ for all $k \in \mathbb{N}$. Next, since $f \in \mathcal{C}_a$, by Lemma 4.4.9 it follows that $\mathcal{M}_{\mathbf{p},t}^{ln}f \in \mathcal{C}_a$ as well, and therefore by Proposition 4.4.3, $[\mathcal{M}_{\mathbf{p},t}^{ln}f]_1 \leqslant ae^a\|\mathcal{M}_{\mathbf{p},t}^{ln}f\|_\infty \leqslant ae^a\|f\|_\infty$.

Putting these together, we see that $\|\mathcal{M}_{\mathbf{p},t}f\|_{0,1} \leqslant (ae^a+1)\|f\|_\infty \leqslant (ae^a + 1)\|f\|_{0,1}$. □

Now we are ready to prove that $\|\mathcal{M}_{\mathbf{p},t}^{ln}f\|_{0,1}$ decays exponentially whenever $f \in C^{0,1}([0,1])$ and $\int f d\mu_{\mathbf{p},t} = 0$. This result will essentially be the backbone of the proof of Lemma 4.2.8.

Lemma 4.4.11. *Fix $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$. There exist uniform constants $0 < C < \infty$ and $0 < \rho < 1$ such that*

$$\|\mathcal{M}_{\mathbf{p},t}^{ln}f\|_{0,1} \leqslant C\rho^n\|f\|_{0,1}$$

for all $f \in C^{0,1}([0,1])$ such that $\int f d\mu_{\mathbf{p},t} = 0$.

Proof. Let $f \in C^{0,1}([0,1])$ for which $\int f d\mu_{\mathbf{p},t} = 0$. If f is constant, $f = 0$ since its integral is 0 and thus the result follows trivially. If f is not constant, $\|f\|_{0,1} > 0$. Let f_1 and f_2 be the positive and negative parts of f respectively, so that $f = f_1 - f_2$ with $f_1, f_2 \geqslant 0$. We can guarantee that they belong to a cone by adding a constant.

In particular, $f_i + \|f\|_{0,1} \in \mathcal{C}_1$ for each i since

$$\begin{aligned}
\frac{f_i(x)+\|f\|_{0,1}}{f_i(y)+\|f\|_{0,1}} &= \exp\left(\log\left(\frac{f_i(x)+\|f\|_{0,1}}{f_i(y)+\|f\|_{0,1}}\right)\right) \\
&= \exp\left(\log\left(\frac{f_i(x)-f_i(y)}{f_i(y)+\|f\|_{0,1}}+1\right)\right) \\
&\leqslant \exp\left(\log\left(\frac{\|f\|_{0,1}|x-y|}{f_i(y)+\|f\|_{0,1}}+1\right)\right) \\
&\leqslant \exp\left(\frac{\|f\|_{0,1}|x-y|}{f_i(y)+\|f\|_{0,1}}\right) \\
&\leqslant \exp\left(\frac{\|f\|_{0,1}|x-y|}{\|f\|_{0,1}}\right) \\
&= \exp(|x-y|)
\end{aligned}$$

where the fourth line follows because $\log(1+z) \leqslant z$ for any $z > -1$. Denote $\eta = \|f\|_{0,1}$. Then $f_i + \eta \in \mathcal{C}_a$, where a was fixed at the end of Section 4.4.2. Denote $\mathcal{M} = \mathcal{M}_{\mathbf{p},t}$. Then since $\int f_1 d\mu_{\mathbf{p},t} = \int f_2 d\mu_{\mathbf{p},t}$ we have

$$\begin{aligned}
\|\mathcal{M}^{ln} f\|_{0,1} &= \|\mathcal{M}^{ln} f_1 - \mathcal{M}^{ln} f_2\|_{0,1} \\
&= \|\mathcal{M}^{ln}(f_1+\eta) - \mathcal{M}^{ln}(f_2+\eta)\|_{0,1} \\
&\leqslant \left\|\mathcal{M}^{ln}(f_1+\eta) - \int (f_1+\eta) d\mu_{\mathbf{p},t}\mathbf{1}\right\|_{0,1} \\
&\quad + \left\|\mathcal{M}^{ln}(f_2+\eta) - \int (f_2+\eta) d\mu_{\mathbf{p},t}\mathbf{1}\right\|_{0,1}.
\end{aligned}$$

Now, since $\mathcal{M}^{ln}(\mathcal{C}_a) \subset \mathcal{C}_a$, we have $\mathcal{M}^{ln}(f_i+\eta) \in \mathcal{C}_a$ (and clearly $\int (f_i + \eta) d\mu_{\mathbf{p},t}\mathbf{1} \in \mathcal{C}_a$ as well). Moreover, denoting $L^1 = L^1(\mu_{\mathbf{p},t})$ it follows that $\|\mathcal{M}^{ln}(f_i + \eta)\|_{L^1} = \|\int (f_i+\eta) d\mu_{\mathbf{p},t}\mathbf{1}\|_{L_1}$ thus we can apply Propositions 4.4.2 and 4.4.5 for $\|\cdot\|_1 = \|\cdot\|_{L^1}$ and $\|\cdot\|_2 = \|\cdot\|_{0,1}$ to obtain

$$\begin{aligned}
\|\mathcal{M}^{ln} f\|_{0,1} \leqslant\ & (1+a)^2 (e^{\Theta(\mathcal{M}^{ln}(f_1+\eta),(\int f_1+\eta)\mathbf{1})} - 1)\|\mathcal{M}^{ln}(f_1+\eta)\|_{0,1} \\
& + (1+a)^2 (e^{\Theta(\mathcal{M}^{ln}(f_2+\eta),(\int f_2+\eta)\mathbf{1})} - 1)\|\mathcal{M}^{ln}(f_2+\eta)\|_{0,1}.
\end{aligned}$$

Next, since $\mathcal{M}^{ln}((\int f_i + \eta)\mathbf{1}) = (\int f_i + \eta)\mathbf{1}$ we can apply Theorem 4.4.1 to get

$$
\begin{aligned}
\|\mathcal{M}^{ln} f\|_{0,1} &\leqslant (1+a)^2(e^{(1-e^{-D})^n \Theta(f_1+\eta,(\int f_1+\eta)\mathbf{1})}-1)\|\mathcal{M}^{ln}(f_1+\eta)\|_{0,1} \\
&\quad +(1+a)^2(e^{(1-e^{-D})^n \Theta(f_2+\eta,(\int f_2+\eta)\mathbf{1})}-1)\|\mathcal{M}^{ln}(f_2+\eta)\|_{0,1} \\
&\leqslant (1+a)^2(e^{(1-e^{-D})^n D}-1)(\|\mathcal{M}^{ln}(f_1+\eta)\|_{0,1}+\|\mathcal{M}^{ln}(f_2+\eta)\|_{0,1}) \\
&\leqslant (1+a)^2 D(1-e^{-D})^n e^{D(1-e^{-D})^n}(\|\mathcal{M}^{ln}(f_1+\eta)\|_{0,1}+\|\mathcal{M}^{ln}(f_2+\eta)\|_{0,1}) \\
&\leqslant (1+a)^2 De^D(1-e^{-D})^n(\|\mathcal{M}^{ln}(f_1+\eta)\|_{0,1}+\|\mathcal{M}^{ln}(f_2+\eta)\|_{0,1}) \\
&\leqslant (1+a)^2 ADe^D(1-e^{-D})^n(\|f_1+\eta\|_{0,1}+\|f_2+\eta\|_{0,1}) \\
&\leqslant (1+a)^2 ADe^D(1-e^{-D})^n(\|f_1\|_{0,1}+\|f_2\|_{0,1}+2\eta) \\
&\leqslant 4(1+a)^2 ADe^D(1-e^{-D})^n\|f\|_{0,1}
\end{aligned}
$$

where A is the uniform constant from Lemma 4.4.10. □

4.4.4 Upper bounds on $\|\mathcal{M}^k_{\mathbf{p},t}(f_{\mathbf{p},t})\|_{0,1}$

Notice that we cannot immediately apply Lemma 4.4.11 to prove Lemma 4.2.8 since $f_{\mathbf{p},t} \notin C^{0,1}([0,1])$.

In this section we show that even though $f_{\mathbf{p},t} \notin C^{0,1}([0,1])$, for every $1 \leqslant k \leqslant l$ the functions $\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t} \in C^{0,1}([0,1])$. This will ultimately allow us find an upper bound for $[U_{\mathbf{p},t}]_1$ by applying Lemma 4.4.11 to $\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}$. Therefore, we need to get upper bounds on $\sup_{1\leqslant k\leqslant l}\|\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}$.

We'll begin with the easy upper bound for $\|\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}\|_\infty$. After this we'll calculate $[\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}]_1$. Finally, we'll bound $[\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}]_1$ above by a constant multiple of $\|\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}$ and use this to estimate $\sup_{1\leqslant k\leqslant l}\|\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}$.

Define

$$\vartheta = \frac{2}{e(1-s)}. \tag{4.19}$$

Let $\alpha > 0$ be fixed. Since $t = e^{\frac{1}{\alpha}}$ is a maximum on $[1,\infty)$ for $f(t) = \frac{\log t}{t^\alpha}$ and $f(e^{\frac{1}{\alpha}}) = \frac{1}{\alpha e}$, by setting $\alpha = \frac{1-s}{2}$ and $t = |T'(x)|$ it follows that

$$\frac{\log|T'(x)|}{|T'(x)|^{\frac{1-s}{2}}} \leqslant \vartheta \tag{4.20}$$

for all $x \in [0,1]$. Also, recall that

$$q = \log\left(\sup_{x\in(0,1)}\sum_{n=1}^{\infty}\frac{1}{|T'(T_n^{-1}x)|^s}\right). \tag{4.21}$$

Since for all $t \leqslant \frac{1-s}{4}$ and $\mathbf{p} \in \mathcal{P}_0$, $\beta_{\mathbf{p}}(t) \geqslant \frac{1+s}{2}$, the combination of (4.20) and (4.21) gives that

$$\sum_{n\in\mathbb{N}_{\mathbf{p}}} \sup_{x\in\mathcal{I}_n} \frac{\log|T'(x)|}{|T'(x)|^{\beta_{\mathbf{p}}(t)}} \leqslant \sum_{n\in\mathbb{N}_{\mathbf{p}}} \sup_{x\in\mathcal{I}_n} \frac{\log|T'(x)|}{|T'(x)|^{\frac{s+1}{2}}} = \sum_{n\in\mathbb{N}_{\mathbf{p}}} \sup_{x\in\mathcal{I}_n} \frac{\log|T'(x)|}{|T'(x)|^{\frac{1-s}{2}}} \frac{1}{|T'(x)|^s} \leqslant \vartheta e^q.$$

Finally, let $\delta < \frac{1-s}{4}$. Since $\beta_{\mathbf{p}}(t) \geqslant \frac{1+s}{2}$ for all $\mathbf{p} \in \mathcal{P}_0$ and $t \in [\frac{\delta}{2}, \delta]$,

$$-\beta'_{\mathbf{p}}(t) \leqslant \frac{1-\frac{s+1}{2}}{\frac{\delta}{2}} = \frac{1-s}{\delta} \leqslant \frac{1}{\delta} \tag{4.22}$$

for all $t \in [\frac{\delta}{2}, \delta]$. We will also need the following easy technical result, in order to bound $p_n^t \log p_n$.

Lemma 4.4.12. $|x^t \log x| \leqslant \frac{1}{\varepsilon}$ *for all* $x \in [0,1]$ *and* $t \geqslant \varepsilon$.

Proof. For a fixed $t \in (0,1)$ define $\alpha_t(x) = x^t \log x$ for $x \in [0,1]$. Differentiating with respect to x we obtain

$$\frac{d}{dx}(\alpha_t(x)) = tx^{t-1}\log x + x^{t-1} = x^{t-1}(t\log x + 1).$$

Clearly the only turning point in $[0,1]$ is $x = e^{-\frac{1}{t}}$ and since $\alpha_t(0) = \alpha_t(1) = 0$ this is a local minimum for α_t, that is, a local maximum for $|x^t \log x|$. Moreover, for $t \geqslant \varepsilon > 0$,

$$\alpha_t(e^{-\frac{1}{t}}) = e^{-1}\log e^{-\frac{1}{t}} = -\frac{1}{t}e^{-1} \geqslant -\frac{1}{\varepsilon}e^{-1} = \alpha_\varepsilon(e^{-\frac{1}{\varepsilon}}).$$

Therefore,

$$|x^t \log x| \leqslant |\alpha_t(e^{-\frac{1}{t}})| \leqslant |\alpha_\varepsilon(e^{-\frac{1}{\varepsilon}})| = \frac{1}{\varepsilon}e^{-1} \leqslant \frac{1}{\varepsilon}.$$

□

We are now ready to obtain an upper bound on $\|\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}\|_\infty$ which will hold for all $k \in \mathbb{N}$.

Lemma 4.4.13. *For all* $\mathbf{p} \in \mathcal{P}_0$, $t \in [\frac{\delta}{2}, \delta]$ *and* $k \geqslant 1$,

$$\|\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}\|_\infty \leqslant \frac{2e^{a+q}\vartheta}{\delta}.$$

Proof. First we fix $k = 1$. Then,

$$\begin{aligned}
\|\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}\|_\infty &= \sup_x \left| \frac{1}{h_{\mathbf{p},t}(x)} \sum_{n\in\mathbb{N}_\mathbf{p}} \frac{-p_n^t \beta'_\mathbf{p}(t) \log|T'(T_n^{-1}x)| + p_n^t \log p_n}{|T'(T_n^{-1}x)|^{\beta_\mathbf{p}(t)}} h_{\mathbf{p},t}(T_n^{-1}x) \right| \\
&\leqslant \sup_{x\neq y} \frac{h_{\mathbf{p},t}(x)}{h_{\mathbf{p},t}(y)} \frac{2}{\delta} \sup_x \left| \sum_{n\in\mathbb{N}_\mathbf{p}} \frac{\log|T'(T_n^{-1}x)| - 1}{|T'(T_n^{-1}x)|^{\beta_\mathbf{p}(t)}} \right| \\
&\leqslant \frac{2e^a}{\delta} \sum_{n\in\mathbb{N}_\mathbf{p}} \sup_{x\in\mathcal{I}_n} \frac{\log|T'(T_n^{-1}x)|}{|T'(T_n^{-1}x)|^{\beta_\mathbf{p}(t)}} \leqslant \frac{2e^{a+q}\vartheta}{\delta}.
\end{aligned}$$

For $k \geqslant 2$, we just need to observe that

$$\|\mathcal{M}_{\mathbf{p},t}^k f_{\mathbf{p},t}\|_\infty \leqslant \|\mathcal{M}_{\mathbf{p},t}^{k-1}(f_{\mathbf{p},t})\|_\infty \leqslant \|\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}\|_\infty \leqslant \frac{2e^{a+q}\vartheta}{\delta}.$$

□

We now begin the work towards bounding $[\mathcal{M}_{\mathbf{p},t}^k f_{\mathbf{p},t}]_1$. We will do this in two steps. First we bound $[\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}]_1$ directly from the definition of $\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}$. Then, using ideas from the proof of Lemma 4.4.11 we'll find a way of bounding $[\mathcal{M}_{\mathbf{p},t}^k f_{\mathbf{p},t}]_1$ from above by $\|\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}$.

The next result is a preparatory lemma for calculating $[\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}]_1$.

Lemma 4.4.14. *Let*

$$w_n := \frac{\log|T' \circ T_n^{-1}| - 1}{|T' \circ T_n^{-1}|^{\beta_\mathbf{p}(t)}}.$$

Then

$$[w_n]_1 \leqslant \kappa \sup_{z\in\mathcal{I}_n} \left| \frac{(1 - \beta_\mathbf{p}(t) + \beta_\mathbf{p}(t) \log|T'_n(z)|)}{|T'_n(z)|^{\beta_\mathbf{p}(t)}} \right|.$$

Proof. Putting $v_n = \frac{\log|T'_n| - 1}{|T'_n|^{\beta_\mathbf{p}(t)}}$ and differentiating, we get

$$|v'_n| = \left| \frac{T''_n(1 - \beta_\mathbf{p}(t) + \beta_\mathbf{p}(t) \log|T'_n|)}{|T'_n|^{\beta_\mathbf{p}+1}} \right|.$$

Therefore, for all x, y,

$$
\begin{aligned}
|w_n(x) - w_n(y)| &\leqslant \sup_{z \in \mathcal{I}_n} \left| \frac{|T_n''(z)|(1 - \beta_{\mathbf{p}}(t) + \beta_{\mathbf{p}}(t) \log |T_n'(z)|)}{|T_n'(z)|^{\beta_{\mathbf{p}}(t)+1}} \right| |T_n^{-1}x - T_n^{-1}y| \\
&\leqslant \sup_{z \in \mathcal{I}_n} \left| \frac{T_n''(z)(1 - \beta_{\mathbf{p}}(t) + \beta_{\mathbf{p}}(t) \log |T_n'(z)|)}{|T_n'(z)|^{\beta_{\mathbf{p}}(t)+1}} \right| \sup_{z \in \mathcal{I}_n} \frac{1}{T_n'(z)} |x - y| \\
&\leqslant \sup_{z, z' \in \mathcal{I}_n} \left| \frac{T_n''(z)(1 - \beta_{\mathbf{p}}(t) + \beta_{\mathbf{p}}(t) \log |T_n'(z)|)}{|T_n'(z)|^{\beta_{\mathbf{p}}(t)+1} |T_n'(z')|} \right| |x - y| \\
&\leqslant \kappa \sup_{z \in \mathcal{I}_n} \left| \frac{1 - \beta_{\mathbf{p}}(t) + \beta_{\mathbf{p}}(t) \log |T_n'(z)|}{|T_n'(z)|^{\beta_{\mathbf{p}}(t)}} \right| |x - y|.
\end{aligned}
$$

Therefore,

$$
[w_n]_1 \leqslant \kappa \sup_{z \in \mathcal{I}_n} \left| \frac{1 - \beta_{\mathbf{p}}(t) + \beta_{\mathbf{p}}(t) \log |T_n'(z)|}{|T_n'(z)|^{\beta_{\mathbf{p}}(t)}} \right|.
$$

□

Using this bound we can estimate $[\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}]_1$.

Lemma 4.4.15. *For all* $\mathbf{p} \in \mathcal{P}_0$ *and* $t \in [\frac{\delta}{2}, \delta]$,

$$
[\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}]_1 \leqslant \frac{6(1+\vartheta) a \kappa e^{2a+q}}{\delta}.
$$

Proof. Recall that

$$
\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}(x) = \frac{1}{h_{\mathbf{p},t}(x)} \sum_{n \in \mathbb{N}_{\mathbf{p}}} \frac{-\beta_{\mathbf{p}}'(t) p_n^t \log |T'(T_n^{-1}x)| + p_n^t \log p_n}{|T'(T_n^{-1}x)|^{\beta_{\mathbf{p}}(t)}} h_{\mathbf{p},t}(T_n^{-1}x). \quad (4.23)
$$

To obtain the desired bound we will make use of the straightforward inequality

$$
[uv]_1 \quad \leqslant \quad \|u\|_\infty [v]_1 + [u]_1 \|v\|_\infty. \quad (4.24)
$$

First, applying (4.24) to (4.23) with

$$
u = \tfrac{1}{h_{\mathbf{p},t}} \quad \text{and} \quad v = \textstyle\sum_{n \in \mathbb{N}_{\mathbf{p}}} \frac{-\beta_{\mathbf{p}}'(t) p_n^t \log |T' \circ T_n^{-1}| + p_n^t \log p_n}{|T' \circ T_n^{-1}|^{\beta_{\mathbf{p}}(t)}} h_{\mathbf{p},t} \circ T_n^{-1}
$$

we obtain

$$
[\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}]_1 \quad \leqslant \quad \|h_{\mathbf{p},t}^{-1}\|_\infty [v]_1 + [h_{\mathbf{p},t}^{-1}]_1 \|v\|_\infty. \quad (4.25)
$$

By an easy modification of the arguments in Lemma 4.4.13 we have $\|v\|_\infty \leqslant$

$\frac{2e^q\vartheta}{\delta}\|h_{\mathbf{p},t}\|_\infty$. Thus it remains to calculate

$$[v]_1 \leqslant \sum_{n\in\mathbb{N}_\mathbf{p}} \left[\frac{-\beta'_\mathbf{p}(t)p_n^t \log|T'\circ T_n^{-1}| + p_n^t \log p_n}{|T'\circ T_n^{-1}|^{\beta_\mathbf{p}(t)}} h_{\mathbf{p},t}\circ T_n^{-1} \right]_1. \tag{4.26}$$

Next, applying (4.24) again to each of the terms on the right hand side of (4.26) with

$$u_n = h_{\mathbf{p},t}\circ T_n^{-1} \quad \text{and} \quad v_n = \tfrac{-\beta'_\mathbf{p}(t)p_n^t \log|T'\circ T_n^{-1}| + p_n^t \log p_n}{|T'\circ T_n^{-1}|^{\beta_\mathbf{p}(t)}}$$

we obtain that

$$[u_n v_n]_1 \leqslant \|h_{\mathbf{p},t}\|_\infty [v_n]_1 + [h_{\mathbf{p},t}]_1 \|v_n\|_\infty.$$

By Lemma 4.4.14,

$$[v_n]_1 \leqslant \frac{2}{\delta}\left[\frac{\log|T'\circ T_n^{-1}| - 1}{|T'\circ T_n^{-1}|^{\beta_\mathbf{p}(t)}}\right]_1 \leqslant \frac{2\kappa}{\delta} \sup_{z\in\mathcal{I}_n} \left| \frac{(1-\beta_\mathbf{p}(t) + \beta_\mathbf{p}(t)\log|T'_n(z)|)}{|T'_n(z)|^{\beta_\mathbf{p}(t)}} \right|.$$

Therefore

$$\begin{aligned}
[v]_1 &\leqslant \|h_{\mathbf{p},t}\|_\infty \frac{2\kappa}{\delta} \sum_{n\in\mathbb{N}_\mathbf{p}} \sup_{z\in\mathcal{I}_n} \left| \frac{(1-\beta_\mathbf{p}(t) + \beta_\mathbf{p}(t)\log|T'_n(z)|)}{|T'_n(z)|^{\beta_\mathbf{p}(t)}} \right| \\
&\quad + [h_{\mathbf{p},t}]_1 \frac{2}{\delta} \sum_{n\in\mathbb{N}_\mathbf{p}} \sup_{x\in\mathcal{I}_n} \left| \frac{\log|T'(x)| - 1}{|T'(x)|^{\beta_\mathbf{p}(t)}} \right| \\
&\leqslant \frac{2(1+\vartheta)e^q\kappa}{\delta}\|h_{\mathbf{p},t}\|_\infty + \frac{2(1+\vartheta)e^q}{\delta}[h_{\mathbf{p},t}]_1.
\end{aligned}$$

Recall that by Lemma 4.4.8, $\|h_{\mathbf{p},t}\|_\infty \|h^{-1}_{\mathbf{p},t}\|_\infty \leqslant e^a$ and $[h_{\mathbf{p},t}]_1 \|h^{-1}_{\mathbf{p},t}\|_\infty \leqslant ae^a$. Also, observe that $[h^{-1}_{\mathbf{p},t}]_1 \leqslant \|h^{-1}_{\mathbf{p},t}\|^2_\infty [h_{\mathbf{p},t}]_1$. Plugging all of this into (4.25) we get

$$\begin{aligned}
[\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}]_1 &\leqslant \|h^{-1}_{\mathbf{p},t}\|_\infty \left(\frac{2(1+\vartheta)e^q\kappa}{\delta}\|h_{\mathbf{p},t}\|_\infty + \frac{2(1+\vartheta)e^q}{\delta}[h_{\mathbf{p},t}]_1 \right) \\
&\quad + \frac{2e^q\vartheta}{\delta}\|h_{\mathbf{p},t}\|_\infty [h^{-1}_{\mathbf{p},t}]_1 \\
&\leqslant \frac{2(1+\vartheta)e^{a+q}\kappa}{\delta} + \frac{2(1+\vartheta)ae^{a+q}}{\delta} + \frac{2ae^{2a+q}\vartheta}{\delta} \\
&\leqslant \frac{6(1+\vartheta)a\kappa e^{2a+q}}{\delta}.
\end{aligned}$$

□

Now, all that remains is to find an upper bound for $[\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}]_1$ for *all* $1 \leqslant k \leqslant l$. To save us having to bound it directly as we did above, we use some ideas from the proof of Lemma 4.4.11 to estimate $[\mathcal{M}_{\mathbf{p},t}^k f_{\mathbf{p},t}]_1$ in terms of $\|\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}$. In particular, we show that the positive and negative parts of $\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}$ (plus a constant) belong to a cone, and by applying the operator we show that each iterate of these also belongs to a (larger) cone. At this point, we will be able to apply Proposition 4.4.3 to bound their $[\cdot]_1$ seminorms. Note that alternatively, at this point we could instead deduce the desired result from Lemmas 4.4.13 and 4.4.15 by proving a 'Lasota-Yorke inequality'.

Lemma 4.4.16. *For all $\mathbf{p} \in \mathcal{P}_0$ and $t \in [\frac{\delta}{2}, \delta]$ and $1 \leqslant k \leqslant l$,*

$$[\mathcal{M}_{\mathbf{p},t}^k f_{\mathbf{p},t}]_1 \leqslant 2a' e^{a'} \|\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}$$

where $a' = (l-1)\kappa + 2a + 1$.

Proof. Let w_1, w_2 denote the positive and negative parts of $w = \mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}$ respectively, so that $w = w_1 - w_2$. In particular,

$$\begin{aligned} w_1(x) &= \frac{1}{h_{\mathbf{p},t}(x)} \sum_{n \in \mathbb{N}_{\mathbf{p}}} \frac{-\beta_{\mathbf{p}}'(t) p_n^t \log |T'(T_n^{-1}x)|}{|T'(T_n^{-1}x)|^{\beta_{\mathbf{p}}(t)}} h_{\mathbf{p},t}(T_n^{-1}x) > 0 \\ w_2(x) &= \frac{1}{h_{\mathbf{p},t}(x)} \sum_{n \in \mathbb{N}_{\mathbf{p}}} \frac{-p_n^t \log p_n}{|T'(T_n^{-1}x)|^{\beta_{\mathbf{p}}(t)}} h_{\mathbf{p},t}(T_n^{-1}x) > 0. \end{aligned}$$

Define $v_i = w_i + [w]_1$. Then by an argument similar to that which we used in the proof of Lemma 4.4.11, we can show that $v_i \in \mathcal{C}_1$. In particular

$$\begin{aligned} \frac{w_i(x) + [w]_1}{w_i(y) + [w]_1} &= \exp\left(\log\left(\frac{w_i(x) + [w]_1}{w_i(y) + [w]_1}\right)\right) \\ &= \exp\left(\log\left(\frac{w_i(x) - w_i(y)}{w_i(y) + [w]_1} + 1\right)\right) \\ &\leqslant \exp\left(\log\left(\frac{[w]_1 |x-y|}{w_i(y) + [w]_1} + 1\right)\right) \\ &\leqslant \exp\left(\frac{[w]_1 |x-y|}{w_i(y) + [w]_1}\right) \\ &\leqslant \exp\left(\frac{[w]_1 |x-y|}{[w]_1}\right) \\ &= \exp(|x-y|) \end{aligned}$$

where the fourth line follows because $\log(1+z) \leqslant z$ for any $z > -1$. Therefore, by adapting the argument in Lemma 4.4.9 we can see that $\mathcal{M}_{\mathbf{p},t}^k v_i \in \mathcal{C}_{(l-1)\kappa+2a+1}$ for each $1 \leqslant k \leqslant l-1$. In particular

$$\begin{aligned}
(\mathcal{M}^k_{\mathbf{p},t} v_i)(x) &= h^{-1}_{\mathbf{p},t}(x) \sum_{\mathbf{n} \in \mathbb{N}^k_{\mathbf{p}}} e^{g^k_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} x)} v_i(T^{-1}_{\mathbf{n}} x) h_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} x) \\
&\leqslant h^{-1}_{\mathbf{p},t}(y) \sum_{\mathbf{n} \in \mathbb{N}^k_{\mathbf{p}}} e^{g^k_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} y)} v_i(T^{-1}_{\mathbf{n}} y) h_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} y) e^{(k\kappa+a)|x-y| + (a+1)|T^{-1}_{\mathbf{n}} x - T^{-1}_{\mathbf{n}} y|} \\
&\leqslant h^{-1}_{\mathbf{p},t}(y) \sum_{\mathbf{n} \in \mathbb{N}^k_{\mathbf{p}}} e^{g^k_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} y)} v_i(T^{-1}_{\mathbf{n}} y) h_{\mathbf{p},t}(T^{-1}_{\mathbf{n}} y) e^{(k\kappa+2a+1)|x-y|} \\
&\leqslant (\mathcal{M}^k_{\mathbf{p},t} v_i)(y) e^{((l-1)\kappa+2a+1)|x-y|}.
\end{aligned}$$

Put $a' = (l-1)\kappa + 2a + 1$. It follows that for each $2 \leqslant k \leqslant l$,

$$\begin{aligned}
[\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}]_1 &= [\mathcal{M}^{k-1}_{\mathbf{p},t} v_1 - \mathcal{M}^{k-1}_{\mathbf{p},t} v_2]_1 \\
&\leqslant [\mathcal{M}^{k-1}_{\mathbf{p},t} v_1]_1 + [\mathcal{M}^{k-1}_{\mathbf{p},t} v_2]_1 \\
&\leqslant a' e^{a'} (\|v_1\|_\infty + \|v_2\|_\infty)
\end{aligned}$$

where the last line follows by Proposition 4.4.3. Since $\|v_i\|_\infty = \|w_i + [w]_1\|_\infty = \|w_i\|_\infty + [w]_1 = \|w\|_{0,1}$ we obtain

$$[\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}]_1 \leqslant 2a' e^{a'} \|\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}.$$

□

Finally, by tying the last few results together, we can find an upper bound for $\sup_{1 \leqslant k \leqslant l} \|\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}$.

Lemma 4.4.17. *There exists some uniform constant*

$$E = 2e^{2a+q}(1+\vartheta)(1 + 2a' e^{a'} + 6a\kappa)$$

such that for all $\mathbf{p} \in \mathcal{P}_0$, $\delta < \frac{1-s}{4}$ *and* $t \in [\frac{\delta}{2}, \delta]$,

$$\sup_{1 \leqslant k \leqslant l} \|\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1} \leqslant \frac{E}{\delta}.$$

Proof. By Lemmas 4.4.13 and 4.4.16,

$$\|\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1} \leqslant \frac{2e^{a+q}\vartheta}{\delta} + 2a' e^{a'} \|\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}.$$

By Lemma 4.4.15,

$$\|\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1} \leqslant \frac{2e^{a+q}\vartheta}{\delta} + 2a'e^{a'}\left(\frac{2e^{a+q}\vartheta}{\delta} + \frac{6(1+\vartheta)a\kappa e^{2a+q}}{\delta}\right).$$

□

4.4.5 Proof of Lemma 4.2.8

Lemma 4.4.11 provided us with an exponential decay rate for $\|\mathcal{M}^k_{\mathbf{p},t} f\|_{0,1}$ whenever the function f satisfied $f \in C^{0,1}([0,1])$ and $\int f d\mu_{\mathbf{p},t} = 0$. In Section 4.4.4, we showed that $\mathcal{M}^k_{\mathbf{p},t} f_{\mathbf{p},t} \in C^{0,1}([0,1])$ for each $1 \leqslant k \leqslant l$. Combining this with the fact that $\int f_{\mathbf{p},t} d\mu_{\mathbf{p},t} = 0$ by definition (see (4.5)), we are in the position to prove Lemma 4.2.8.

Proof of Lemma 4.2.8. By Theorem 4.4.11 there exist uniform constants C and ρ which are independent of $\mathbf{p}$ and t such that for all $\mathbf{p}$ and all $t \in [\frac{\delta}{2}, \delta]$,

$$\begin{aligned}
\|U_{\mathbf{p},t}\|_{0,1} &= \left\|\sum_{n=1}^{\infty} \mathcal{M}^n_{\mathbf{p},t}(f_{\mathbf{p},t})\right\|_{0,1} \\
&\leqslant \sum_{n=1}^{\infty} \|\mathcal{M}^n_{\mathbf{p},t}(f_{\mathbf{p},t})\|_{0,1} \\
&\leqslant \|\mathcal{M}_{\mathbf{p},t}(f_{\mathbf{p},t})\|_{0,1}(1 + \sum_{n=1}^{\infty} C\rho^n) \\
&\quad + \|\mathcal{M}^2_{\mathbf{p},t}(f_{\mathbf{p},t})\|_{0,1}(1 + \sum_{n=1}^{\infty} C\rho^n) \\
&\quad \vdots \\
&\quad + \|\mathcal{M}^l_{\mathbf{p},t}(f_{\mathbf{p},t})\|_{0,1}(1 + \sum_{n=1}^{\infty} C\rho^n) \\
&\leqslant \frac{C}{1-\rho}\left(\|\mathcal{M}_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1} + \cdots + \|\mathcal{M}^l_{\mathbf{p},t} f_{\mathbf{p},t}\|_{0,1}\right) \\
&\leqslant \frac{lCE}{1-\rho}\frac{1}{\delta}
\end{aligned}$$

where the third line follows by linearity of $\mathcal{M}_{\mathbf{p},t}$ and the final line follows by Lemma 4.4.17.

Since $[\log|T'|]_{\Lambda^{-1}} \leqslant \kappa\Lambda^l$ and $f_{\mathbf{p}}$ is locally constant

$$[f_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant |\beta'_{\mathbf{p}}(t)|\kappa\Lambda^l \leqslant |\beta'_{\mathbf{p}}(\frac{\delta}{2})|\kappa\Lambda^l.$$

By (4.22), $|\beta'_{\mathbf{p}}(\frac{\delta}{2})| \leqslant \frac{1}{\delta}$ and therefore

$$[f_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant \frac{\kappa\Lambda^l}{\delta}.$$

Put $C_1 = \frac{lCE}{1-\rho}$. Since $[U_{\mathbf{p},t}]_1 \leqslant \frac{C_1}{\delta}$, it follows that if $x, y \in \mathcal{I}_{i_1,\dots,i_n}$,

$$|U_{\mathbf{p},t}(x) - U_{\mathbf{p},t}(y)| \leqslant \frac{C_1}{\delta}|x-y| \leqslant \frac{C_1}{\delta}|\mathcal{I}_{i_1,\dots,i_n}| \leqslant \frac{C_1\Lambda^{l-1}}{\delta}\frac{1}{\Lambda^n}.$$

Therefore,

$$[U_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant \frac{C_1\Lambda^{l-1}}{\delta}.$$

Similarly, if $x, y \in \mathcal{I}_{i_1\dots i_n}$ for some $n \geqslant 2$,

$$|U_{\mathbf{p},t}(Tx) - U_{\mathbf{p},t}(Ty)| \leqslant \frac{C_1}{\delta}|Tx - Ty| \leqslant \frac{C_1}{\delta}|\mathcal{I}_{i_2,\dots,i_n}| \leqslant \frac{C_1\Lambda^l}{\delta}\frac{1}{\Lambda^n}$$

whereas if $x, y \in \mathcal{I}_k$ for some $k \in \mathbb{N}$,

$$|U_{\mathbf{p},t}(Tx) - U_{\mathbf{p},t}(Ty)| \leqslant \frac{C_1}{\delta}|Tx - Ty| \leqslant \frac{C_1}{\delta} = \frac{C_1}{\delta}\frac{\Lambda}{\Lambda}.$$

Therefore,

$$[U_{\mathbf{p},t} \circ T]_{\Lambda^{-1}} \leqslant \frac{C_1\Lambda^l}{\delta}.$$

Putting this all together we get

$$\begin{aligned}
[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}} &\leqslant [f_{\mathbf{p},t}]_{\Lambda^{-1}} + [U_{\mathbf{p},t} \circ T]_{\Lambda^{-1}} + [U_{\mathbf{p},t}]_{\Lambda^{-1}} \\
&\leqslant \frac{\kappa\Lambda^l}{\delta}(1 + 2C_1) =: \frac{C_0}{\delta}.
\end{aligned}$$

□

We return back to our goal of estimating the integral $\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$ from below. Fix some $\mathbf{p} \in \mathcal{P}_0$, $\delta < \frac{1-s}{4}$ and $t \in [\frac{\delta}{2}, \delta]$. Suppose that we have a point $z = \Pi(\mathbf{i})$ for which $|\tilde{f}_{\mathbf{p},t}(z)| \geqslant c$, and we are interested in finding a cylinder $\mathcal{I}_{i_1\dots i_n}$ such that for all $y \in \mathcal{I}_{i_1\dots i_n}$, $|\tilde{f}_{\mathbf{p},t}(y)| \geqslant \frac{c}{2}$. Then the above result tells us that it is enough to choose n large enough so that

$$\Lambda^n \geqslant \frac{2C_0}{\delta c}.$$

Then for all $x, y \in \mathcal{I}_{i_1\dots i_n}$,

$$|\tilde{f}_{\mathbf{p},t}(x) - \tilde{f}_{\mathbf{p},t}(y)| \leqslant \frac{C_0}{\delta}\frac{1}{\Lambda^n} \leqslant \frac{c}{2}.$$

Since by construction $z \in \mathcal{I}_{i_1 \dots i_n}$, it follows that $|\tilde{f}_{\mathbf{p},t}(y)| \geqslant \frac{c}{2}$ for any $y \in \mathcal{I}_{i_1 \dots i_n}$. Therefore, we will be interested in values of n where

$$n \geqslant \frac{\log\left(\frac{2C_0}{\delta c}\right)}{\log \Lambda}.$$

4.5 Estimates on the measure

In this short section we prove Lemma 4.2.9, that is, we find a lower bound for the measure $\mu_{\mathbf{p},t}(\mathcal{I}_{i_1 \dots i_n})$ of any cylinder $\mathcal{I}_{i_1 \dots i_n}$ that some $z \in \mathcal{Z}$ may belong to.

By definition, $\mu_{\mathbf{p},t}$ is a Gibbs measure for the potential $g_{\mathbf{p},t}$ and thus we know that for each $\mathbf{p}$ and t there exists a constant $0 < C_{\mathbf{p},t} < \infty$ for which

$$C_{\mathbf{p},t}^{-1} \frac{(p_{i_1} \cdots p_{i_n})^t}{|T'(z) \cdots T'(T^{n-1}z)|^{\beta_{\mathbf{p}}(t)}} \leqslant \mu_{\mathbf{p},t}(\mathcal{I}_{i_1 \dots i_n}) \leqslant C_{\mathbf{p},t} \frac{(p_{i_1} \cdots p_{i_n})^t}{|T'(z) \cdots T'(T^{n-1}z)|^{\beta_{\mathbf{p}}(t)}}. \tag{4.27}$$

It is easy to see that if $p_{i_1}, \dots, p_{i_n}$ are small, this produces a lower bound which is arbitrarily close to 0. In our setting, since the digits $i_j \in \{1, 2\}$, this means that we cannot get a uniform lower bound for the measure of such a cylinder unless we impose a lower bound on p_1 or p_2. Therefore, we fix $\varepsilon > 0$ and obtain uniform lower bounds across $\mathbf{p} \in \mathcal{P}_\varepsilon$.

Although (4.27) holds for each $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$ for some constant $C_{\mathbf{p},t}$, it is not obvious whether the constants can be chosen uniformly. Therefore, the main obstacle is to ensure that we can find a uniform upper bound for the constants $C_{\mathbf{p},t}$ for $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$. In order to calculate the Gibbs constants, we need bounds on $[g_{\mathbf{p},t}]_{\Lambda^{-1}}$ and $[\tilde{g}_{\mathbf{p},t}]_{\Lambda^{-1}}$ which we obtain in the next lemma. For $[\tilde{g}_{\mathbf{p},t}]_{\Lambda^{-1}}$ we'll utilise properties of the fixed point $h_{\mathbf{p},t}$ of $\mathcal{L}_{\mathbf{p},t}$, which we gathered in Lemma 4.4.8.

Lemma 4.5.1. *Let $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$. Then $[g_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant \kappa \Lambda^l$ and $[\tilde{g}_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant 3\Lambda^l a\kappa$.*

Proof. Since $f_{\mathbf{p}}$ is locally constant,

$$[g_{\mathbf{p},t}]_{\Lambda^{-1}} = [-\beta_{\mathbf{p}}(t) \log |T'|]_{\Lambda^{-1}} \leqslant [\log |T'|]_{\Lambda^{-1}} \leqslant \kappa \Lambda^l$$

by Lemma 3.1.3.

For the second part, notice that if $x, y \in \mathcal{I}_{i_1 \dots i_n}$ then

$$|\log h_{\mathbf{p},t}(x) - \log h_{\mathbf{p},t}(y)| = \left| \log \frac{h_{\mathbf{p},t}(x)}{h_{\mathbf{p},t}(y)} \right| \leqslant a|x - y| \leqslant \frac{a\Lambda^{l-1}}{\Lambda^n}$$

by Lemma 3.1.2. Therefore, $[\log h_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant a\Lambda^{l-1}$. Similarly, if $x, y \in \mathcal{I}_{i_1 \dots i_n}$ for $n \geqslant 2$ then

$$|\log h_{\mathbf{p},t}(Tx) - \log h_{\mathbf{p},t}(Ty)| \leqslant \frac{a\Lambda^{l-1}}{\Lambda^{n-1}} = \frac{a\Lambda^l}{\Lambda^n}.$$

If $x, y \in \mathcal{I}_{i_1}$ then

$$|\log h_{\mathbf{p},t}(Tx) - \log h_{\mathbf{p},t}(Ty)| \leqslant a \leqslant \frac{a\Lambda^l}{\Lambda}.$$

Therefore $[\log h_{\mathbf{p},t} \circ T]_{\Lambda^{-1}} \leqslant a\Lambda^l$.

Combining these together, we obtain

$$\begin{aligned} [\tilde{g}_{\mathbf{p},t}]_{\Lambda^{-1}} &\leqslant [g_{\mathbf{p},t}]_{\Lambda^{-1}} + [\log h_{\mathbf{p},t}]_{\Lambda^{-1}} + [\log h_{\mathbf{p},t} \circ T]_{\Lambda^{-1}} \\ &\leqslant \kappa\Lambda^l + a\Lambda^{l-1} + a\Lambda^l \leqslant 3\Lambda^l a\kappa. \end{aligned}$$

□

Now, by using the above properties of $[\tilde{g}_{\mathbf{p},t}]_{\Lambda^{-1}}$, we can obtain a uniform upper bound on the Gibbs constants $C_{\mathbf{p},t}$ which will hold for all $\mathbf{p} \in \mathcal{P}_0$ and $t \in [\frac{\delta}{2}, \delta]$. We do this by using properties of the (normalised) transfer operator, in particular, the property that $\mathcal{M}^*_{\mathbf{p},t}(\mu_{\mathbf{p},t}) = \mu_{\mathbf{p},t}$.

Lemma 4.5.2. *Let $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$. Then for any $n \in \mathbb{N}$, $i_1 \dots i_n \in \mathbb{N}^n_{\mathbf{p}}$ and $z \in \mathcal{I}_{i_1 \dots i_n}$,*

$$C^{-1}\frac{(p_{i_1} \cdots p_{i_n})^t}{|T'(z) \cdots T'(T^{n-1}z)|^{\beta_{\mathbf{p}}(t)}} \leqslant \mu_{\mathbf{p},t}(\mathcal{I}_{i_1 \dots i_n}) \leqslant C\frac{(p_{i_1} \cdots p_{i_n})^t}{|T'(z) \cdots T'(T^{n-1}z)|^{\beta_{\mathbf{p}}(t)}}$$

where

$$C = \exp\left(\frac{3\Lambda^l a\kappa}{\Lambda - 1} + a\right). \tag{4.28}$$

Proof. The proof of this lemma is based on arguments in [Bo]. Let $n \in \mathbb{N}$ and any

$i_1 \dots i_n \in \mathbb{N}^n_{\mathbf{p}}$. Then

$$\begin{aligned}
\mu_{\mathbf{p},t}(\mathcal{I}_{i_2,\dots,i_n}) &= \int \mathbf{1}_{\mathcal{I}_{i_2,\dots,i_n}}(x) d\mu_{\mathbf{p},t}(x) \\
&= \int \sum_{T_{\mathbf{p}}y=x} \mathbf{1}_{\mathcal{I}_{i_1,i_2,\dots,i_n}}(y) d\mu_{\mathbf{p},t}(x) \\
&= \int \sum_{T_{\mathbf{p}}y=x} e^{\tilde{g}_{\mathbf{p},t}(y)} \mathbf{1}_{\mathcal{I}_{i_1,\dots,i_n}}(y) e^{-\tilde{g}_{\mathbf{p},t}(y)} d\mu_{\mathbf{p},t}(x) \\
&= \int \mathcal{M}_{\mathbf{p},t}(\mathbf{1}_{\mathcal{I}_{i_1,\dots,i_n}}(x) e^{-\tilde{g}_{\mathbf{p},t}(x)}) d\mu_{\mathbf{p},t}(x) \\
&= \int_{\mathcal{I}_{i_1,\dots,i_n}} e^{-\tilde{g}_{\mathbf{p},t}(x)} d\mu_{\mathbf{p},t}(x)
\end{aligned}$$

where the final line follows because $\mathcal{M}^*_{\mathbf{p},t}\mu_{\mathbf{p},t} = \mu_{\mathbf{p},t}$.

Let $z \in \mathcal{I}_{i_1 \dots i_n}$. Then

$$\mu_{\mathbf{p},t}(\mathcal{I}_{i_2,\dots,i_n}) e^{\tilde{g}_{\mathbf{p},t}(z)} \leqslant e^{\frac{[\tilde{g}_{\mathbf{p},t}]_{\Lambda-1}}{\Lambda^l}} \mu_{\mathbf{p},t}(\mathcal{I}_{i_1,\dots,i_n})$$

so that

$$\frac{\mu_{\mathbf{p},t}(\mathcal{I}_{i_1,\dots,i_n})}{\mu_{\mathbf{p},t}(\mathcal{I}_{i_2,\dots,i_n})} e^{-\tilde{g}_{\mathbf{p},t}(z)} \geqslant e^{\frac{-[\tilde{g}_{\mathbf{p},t}]_{\Lambda-1}}{\Lambda^n}}.$$

Moreover, we can proceed to obtain the following sequence of inequalities

$$\begin{aligned}
\frac{\mu_{\mathbf{p},t}(\mathcal{I}_{i_2,\dots,i_n})}{\mu_{\mathbf{p},t}(\mathcal{I}_{i_3,\dots,i_n})} e^{-\tilde{g}_{\mathbf{p},t}(Tz)} &\geqslant e^{\frac{-[\tilde{g}_{\mathbf{p},t}]_{\Lambda-1}}{\Lambda^{n-1}}} \\
&\vdots \\
\mu_{\mathbf{p},t}(\mathcal{I}_{i_n}) e^{-\tilde{g}_{\mathbf{p},t}(T^{n-1}z)} &\geqslant e^{\frac{-[\tilde{g}_{\mathbf{p},t}]_{\Lambda-1}}{\Lambda}}.
\end{aligned}$$

Multiplying these all together we obtain

$$\frac{\mu_{\mathbf{p},t}(\mathcal{I}_{i_1,\dots,i_n})}{e^{S_n \tilde{g}_{\mathbf{p},t}(z)}} \geqslant e^{-\frac{[\tilde{g}_{\mathbf{p},t}]_{\Lambda-1}}{\Lambda-1}}. \tag{4.29}$$

Now,

$$\begin{aligned} S_n(\log h_{\mathbf{p},t} - \log h_{\mathbf{p},t} \circ T)(z) &= \log h_{\mathbf{p},t}(z) - \log h_{\mathbf{p},t}(Tz) \\ &\quad + \log h_{\mathbf{p},t}(Tz) - \log h_{\mathbf{p},t}(T^2 z) \\ &\quad \vdots \\ &\quad + \log h_{\mathbf{p},t}(T^{n-1}z) - \log h_{\mathbf{p},t}(T^n z) \\ &= \log \frac{h_{\mathbf{p},t}(z)}{h_{\mathbf{p},t}(T^n z)} \geqslant -a. \end{aligned}$$

Plugging this into (4.29) we obtain

$$\frac{\mu_{\mathbf{p},t}(\mathcal{I}_{i_1,\ldots,i_n})}{e^{S_n g_{\mathbf{p},t}(z)}} \geqslant e^{-\frac{[\tilde{g}_{\mathbf{p},t}]_{\Lambda-1}}{\Lambda-1} - a} \geqslant \exp\left(-\frac{3\Lambda^l a\kappa}{\Lambda - 1} - a\right). \tag{4.30}$$

By rearranging this inequality and expanding the ergodic sum we obtain the desired lower bound. The upper bound follows by an analogous argument. □

Next, we move onto the proof of Lemma 4.2.9. In particular, given $\mathbf{i}$ for which $\Pi(\mathbf{i}) \in \mathcal{Z}$, we would like to calculate a lower bound on the measure of a cylinder $\mu_{\mathbf{p},t}(\mathcal{I}_{i_1 \ldots i_n})$ which depends only on ε and n.

Recall that τ was defined in (4.11) to be

$$\tau = \inf_{x \in \mathcal{I}_1 \cup \mathcal{I}_2} \frac{1}{|T'(x)|}.$$

The proof of Lemma 4.2.9 is now a straightforward consequence of Lemma 4.5.2.

Proof of Lemma 4.2.9. Let $t \in [0, \frac{1-s}{4}]$, $\varepsilon > 0$ and $\mathbf{p} \in \mathcal{P}_\varepsilon$. Let $z = \Pi(\mathbf{i}) \in \{z_1, z_2, z_{12}\}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} \mu_{\mathbf{p},t}(\mathcal{I}_{i_1,\ldots,i_n}) &\geqslant C^{-1} e^{-S_n g_{\mathbf{p},t}(z)} \\ &= C^{-1} \frac{(p_{i_1} \cdots p_{i_n})^t}{|T'(z) \cdots T'(T^{n-1}z)|^{\beta_{\mathbf{p}}(t)}} \\ &\geqslant C^{-1} \varepsilon^{tn} \tau^n. \end{aligned}$$

□

4.6 Proof of Theorem 3.5.3

In this section we tie together Lemmas 4.2.7, 4.2.8 and 4.2.9 which were proved in the last three sections to obtain a lower bound on the variance $\sigma^2_{\mathbf{p},t}(f_{\mathbf{p},t})$, and consequently obtain upper bounds for $\sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}}$ for every $\varepsilon > 0$.

Recall that in Section 4.1 we showed that $|\beta'_{\mathbf{p}}(1)| = \dim \mu_{\mathbf{p}}$ for each $\mathbf{p} \in \mathcal{P}_0$. The objective was to obtain a lower bound on $\beta''_{\mathbf{p}}(t)$ on a compact subset $[\frac{\delta}{2}, \delta]$ of t in order to induce an upper bound on $|\beta'_{\mathbf{p}}(1)|$. In Sections 4.1 and 4.2 we saw that for $t \in [0, \frac{1-s}{4}]$,

$$\beta''_{\mathbf{p}}(t) = \frac{\sigma^2_{\mathbf{p},t}(f_{\mathbf{p},t})}{\int \log|T'|d\mu_{\mathbf{p},t}} = \frac{\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}}{\int \log|T'|d\mu_{\mathbf{p},t}}.$$

Suppose that $\varepsilon > 0$ and $\delta < \frac{1-s}{4}$ are fixed. We'll begin by finding a lower bound for $\sigma^2_{\mathbf{p},t}(f_{\mathbf{p},t}) = \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$ by following the blueprint provided by Lemma 4.2.6 along with Lemmas 4.2.7, 4.2.8 and 4.2.9. This will hold for all $\mathbf{p} \in \mathcal{P}_\varepsilon$ and $t \in [\frac{\delta}{2}, \delta]$. Next, as promised we'll bound $\int \log|T'|d\mu_{\mathbf{p},t}$ from above (uniformly for all $\mathbf{p} \in \mathcal{P}_0$ and $t \in [0, \frac{1-s}{4}]$. These two bounds combined together will give us a lower bound for $\beta''_{\mathbf{p}}(t)$ that holds for all $\mathbf{p} \in \mathcal{P}_\varepsilon$ and $t \in [\frac{\delta}{2}, \delta]$. Finally, this will allow us to prove the main result of this chapter, Theorem 3.5.3: for every $\varepsilon > 0$ and $\delta < \frac{1-s}{4}$ there exists some constant $G_{\varepsilon,\delta}$ for which $\sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}} < 1 - G_{\varepsilon,\delta}$. This will precisely be the upper bound on $|\beta'_{\mathbf{p}}(1)|$ which is yielded by estimating $\beta''_{\mathbf{p}}(t)$ from below on the interval $[\frac{\delta}{2}, \delta]$.

We begin by getting a lower bound on the variance $\sigma^2_{\mathbf{p},t}(f_{\mathbf{p},t}) = \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$. Here we essentially follow the proof of Lemma 4.2.6, but instead of treating one specific pair $\mathbf{p}, t$, we use the uniform bounds on $\inf_{z\in\mathcal{Z}} \frac{1}{2} S_2 f_{\mathbf{p},t}(z)$, $[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}}$ and $\inf_{\Pi(\mathbf{i})=z\in\mathcal{Z}} \mu_{\mathbf{p},t}(\mathcal{I}_{i_1\ldots i_n})$ which are provided by Lemmas 4.2.7, 4.2.8 and 4.2.9 to get uniform lower bounds on the variance for larger classes of $\mathbf{p}$ and t.

Lemma 4.6.1. *Let $\varepsilon > 0$ and let $\mathbf{p} \in \mathcal{P}_\varepsilon$, $t \in [0, \frac{1-s}{4}]$. Let*

$$m \geqslant \frac{1}{\log\Lambda} \log\left(\frac{2C_0}{\delta c}\right) \tag{4.31}$$

and $c = \frac{\theta}{4}\left|\log \frac{T'(z_1)T'(z_2)}{T'(z_{12})T'(z_{21})}\right| > 0$ as in Lemma 4.2.7. Then

$$\sigma^2_{\mathbf{p},t}(f_{\mathbf{p},t}) = \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t} \geqslant \frac{c^2}{4} C^{-1} \varepsilon^{mt} \tau^m$$

where C is given by Lemma 4.5.2.

Proof. Let $\mathbf{p} \in \mathcal{P}_\varepsilon$. Let z be the periodic point given by Lemma 4.2.7 that satisfies

$$|\frac{1}{2}S_2(\tilde{f}_{\mathbf{p},t}(z))| \geqslant c.$$

It follows that either $|\tilde{f}_{\mathbf{p},t}(z)| \geqslant c$ or $|\tilde{f}_{\mathbf{p},t}(T(z))| \geqslant c$. Let $\mathbf{i}$ be the symbolic coding for the point for which this holds. By Lemma 4.2.8, $[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant \frac{C_0}{\delta}$ and therefore for $m \geqslant \frac{1}{\log \Lambda} \log(\frac{2C_0}{\delta c})$ it follows that $[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant \frac{C_0}{\delta} \leqslant \frac{c}{2}\Lambda^m$, that is, for all $x \in \mathcal{I}_{i_1,\ldots,i_m}$, $\tilde{f}_{\mathbf{p},t}(x) \geqslant \frac{c}{2}$. Therefore by Lemma 4.2.9,

$$\begin{aligned} \int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t} &\geqslant \frac{c^2}{4}\mu_{\mathbf{p},t}(\mathcal{I}_{i_1,\ldots,i_m}) \\ &\geqslant \frac{c^2}{4}C^{-1}\varepsilon^{tm}\tau^m. \end{aligned}$$

□

Next, we obtain an upper bound for the Lyapunov exponent $\int \log|T'|d\mu_{\mathbf{p},t}$ which appears in the denominator of the expression for $\beta''_{\mathbf{p}}(t)$. Now that we have a uniform upper bound on the Gibbs constants for $\mu_{\mathbf{p},t}$, this result is straightforward.

Lemma 4.6.2. *Let $\mathbf{p} \in \mathcal{P}_0$, $t \in [0, \frac{1-s}{4}]$. Then*

$$\int \log|T'|d\mu_{\mathbf{p},t} \leqslant L$$

where $L = Ce^q\vartheta$ is some uniform constant independent of $\mathbf{p}$ and t.

Proof. By Lemma 4.5.2 we have

$$\begin{aligned} \int \log|T'|d\mu_{\mathbf{p},t} &\leqslant C\sum_{n\in\mathbb{N}_{\mathbf{p}}} \sup_{x\in\mathcal{I}_n} \frac{\log|T'(x)|}{|T'(x)|^{\beta_{\mathbf{p}}(t)}} \leqslant C\sum_{n\in\mathbb{N}_{\mathbf{p}}} \frac{\log|T'(x)|}{|T'(x)|^{\frac{s+1}{2}}} \\ &\leqslant Ce^q\vartheta = L. \end{aligned} \tag{4.32}$$

□

We are now ready to prove our main result of this chapter. The following theorem is essentially a restatement of Theorem 3.5.3, this time including the details about the exact form of $G_{\varepsilon,\delta}$.

Theorem 4.6.3. *Let $0 < \delta < \frac{1-s}{4}$ and $\varepsilon > 0$. Then*

$$\sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim\mu_{\mathbf{p}} \leqslant 1 - G_{\varepsilon,\delta} \tag{4.33}$$

where

$$G_{\varepsilon,\delta} = \rho_1 \delta^2 \varepsilon^{\rho_2 \delta(1-\log\delta)} \tau^{\rho_2 - \log\delta} \tag{4.34}$$

for some positive constants ρ_1, ρ_2 independent of ε and δ.

Proof. By Lemmas 4.6.1 and 4.6.2, for $t \in [\frac{\delta}{2}, \delta]$ and $\mathbf{p} \in \mathcal{P}_\varepsilon$,

$$\beta''_{\mathbf{p}}(t) \geqslant \gamma_{\varepsilon,\delta} := \frac{\frac{c^2}{4} C^{-1} \varepsilon^{m\delta} \tau^m}{L}.$$

Then it follows that

$$\dim \mu_{\mathbf{p}} = -\beta'_{\mathbf{p}}(1) \leqslant \frac{\beta_{\mathbf{p}}(\frac{1-s}{4})}{1 - \frac{1-s}{4}} = \frac{\int_0^{\frac{1-s}{4}} \beta'_{\mathbf{p}}(t)dt + 1}{1 - \frac{1-s}{4}} \leqslant \frac{1 - \frac{3}{8}\delta^2 \gamma_{\varepsilon,\delta} - \frac{1-s}{4} \dim \mu_{\mathbf{p}}}{1 - \frac{1-s}{4}}$$

where the first inequality follows by convexity of $\beta_{\mathbf{p}}$. Therefore,

$$\dim \mu_{\mathbf{p}} \leqslant 1 - \frac{3}{8}\delta^2 \gamma_{\varepsilon,\delta} \tag{4.35}$$

for all $\mathbf{p} \in \mathcal{P}_\varepsilon$. Therefore,

$$\begin{aligned}
\dim \mu_{\mathbf{p}} &\leqslant 1 - \frac{3c^2 C^{-1} \varepsilon^{m\delta} \tau^m}{32L} \delta^2 \\
&\leqslant 1 - \frac{3c^2 C^{-1}}{32L} \delta^2 \varepsilon^{(\frac{\delta}{\log\Lambda} \log(\frac{2C_0}{c\delta}))} \tau^{(\frac{1}{\log\Lambda} \log(\frac{2C_0}{c\delta}))} \\
&\leqslant 1 - \frac{3c^2 C^{-1}}{32L} \delta^2 \varepsilon^{(\log(\delta \frac{2C_0}{c\delta}))} \tau^{(\log(\frac{2C_0}{c\delta}))} \\
&\leqslant 1 - \frac{3c^2 C^{-1}}{32L} \delta^2 \varepsilon^{(\log \frac{2C_0}{c})\delta(1-\log\delta)} \tau^{\log(\frac{2C_0}{c}) - \log\delta}.
\end{aligned}$$

This yields (4.34) with $\rho_1 = \frac{3c^2 C^{-1}}{32L}$ and $\rho_2 = \log \frac{2C_0}{c}$. □

Remark 4.6.4. *Unfortunately, we quickly see that when T is the Gauss map, the upper bound that Theorem 4.6.3 yields for $\dim \mu_{\mathbf{p}}$ will be worse than $1 - 10^{-7}$ which was the corresponding result in [KPW]. The reason for this is that the Rényi constant for the Gauss map is $\kappa = 16$, which appears in several exponents, such as*

$$C = \exp\left(\frac{3\Lambda^l a\kappa}{\Lambda - 1} + a\right).$$

Recall that in Lemma 3.5.4 we obtain a uniform upper bound for $\dim \mu_{\mathbf{p}}$ for any $\mathbf{p} \in \mathcal{P}_0$ where p_1 is ‘close to 1’. The proof of this lemma appears in Appendix B and contains similar arguments to the ones presented in this chapter.

Chapter 5

Redistributing mass

In this chapter we introduce a 'mass redistribution technique' to compare Lyapunov exponents of projected Bernoulli measures for EMR maps. Recall that in Chapter 3 we assumed for simplicity that $T'' > 0$, meaning that in the orientation preserving case, $\mathcal{I}_1 = (0, a)$ for some $a \in (0, 1)$ and in the orientation reversing case $\mathcal{I}_1 = (b, 1)$ for some $b \in (0, 1)$.

We introduce a partial ordering on the simplex $\mathcal{P}$.

Definition 5.0.5. *Let $\mathbf{p}, \mathbf{q} \in \mathcal{P}$ where $\mathbf{p} = (p_1, p_2, \ldots)$ and $\mathbf{q} = (q_1, q_2, \ldots)$. We say that $\mathbf{p}$ can be obtained by $*$-transforming $\mathbf{q}$ if there exists some $n > m$ and $\varepsilon < \min\{q_n, 1 - q_m\}$ such that $\mathbf{p} = (q_1, \ldots, q_m + \varepsilon, \ldots, q_n - \varepsilon, \ldots)$.*

Then, we define a partial ordering $\prec$ on $\mathcal{P}$ by writing $\mathbf{p} \prec \mathbf{q}$ whenever $\mathbf{p}$ can be obtained by $*$-transforming $\mathbf{q}$ finitely or countably many times.

We'd like to be able to compare the Lyapunov exponents $\chi(\mu_{\mathbf{p}})$ and $\chi(\mu_{\mathbf{p}^*})$ whenever $\mathbf{p} \succ \mathbf{p}^*$. Comparing two integrals with respect to *distinct measures* is generally quite difficult, especially when the measures are Bernoulli, since they have a complicated structure once projected to $[0, 1]$ under Π. Therefore we would like to find a way to rewrite $\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*})$ as the integral of the difference of two *distinct functions*, over a common measure. We will do this by constructing a new measure ν which lives on a 'larger' space, and projects to $\mu_{\mathbf{p}}$ and $\mu_{\mathbf{p}^*}$ under some projections Π_1 and Π_2. In particular, this will allow us to verify that $\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*}) \geqslant 0$ whenever $\mathbf{p} \succ \mathbf{p}^*$ (although if it were required, we could also use this technique to provide more quantitative information about the size of $\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*})$). We will present these arguments in Section 5.1.

We'll use the arguments from Section 5.1 to help us complete the proof of Theorem 3.3.1. In particular, we'll use the fact that $\chi(\mu_{\mathbf{p}}) \geqslant \chi(\mu_{\mathbf{p}^*})$ whenever $\mathbf{p} \succ \mathbf{p}^*$ in order to prove Theorems 3.5.5 and 3.5.6. Roughly speaking, to prove

Theorem 3.5.5 we'll show that there exists $\mathbf{p}^* \in \mathcal{P}_0$ for which $\mathbf{p}^* \prec \mathbf{p}$, so that $\chi(\mu_{\mathbf{p}^*}) < \chi(\mu_{\mathbf{p}})$, and moreover, $\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*}$. Recall that ξ was introduced in Section 3.5 to be some explicit constant $0 < \xi < 1$ and is made precise in (B.1). To prove Theorem 3.5.6, we'll show that there exists some term E_ε which depends only on ε, such that for any $\mathbf{p} \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ with $p_1 < \xi$, we can find $\mathbf{p}^* \in \mathcal{P}_\varepsilon$ with $\mathbf{p}^* \prec \mathbf{p}$ and

$$\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*} + E_\varepsilon.$$

Finally, we'll show that for some $\delta > 0$ and $\varepsilon > 0$, $E_\varepsilon < \frac{1}{2} G_{\varepsilon,\delta}$, thus completing the proof of Theorem 3.5.6.

5.1 Estimating $\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*})$

In this section we study the difference $\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*})$ whenever $\mathbf{p} \succ \mathbf{p}^*$ by using a 'mass redistribution technique'.

Let $\mathbf{p} = (p_1, p_2, \ldots)$. For $n < m$ and $\varepsilon < \min\{p_m, 1 - p_n\}$ let

$$\mathbf{p}^* = (p_1, p_2, \ldots, p_{n-1}, p_n + \varepsilon, p_{n+1}, \ldots, p_{m-1}, p_m - \varepsilon, p_{m+1}, \ldots).$$

Let $\mu_{\mathbf{p}}$, $\mu_{\mathbf{p}^*}$ be the usual pushforward Bernoulli measures on $[0,1]$. Let $\Sigma_0 = (\{0\} \cup \mathbb{N})^{\mathbb{N}}$, equipped with the full shift map $\sigma_0 : \Sigma_0 \to \Sigma_0$.

We define two projections $\Pi_1 : \Sigma_0 \to \Sigma$ and $\Pi_2 : \Sigma_0 \to \Sigma$ given by

$$\Pi_1(x_1 x_2 \ldots) = y_1 y_2 \ldots \begin{cases} y_k = x_k & \text{if} \quad x_k \neq 0 \\ y_k = n & \text{if} \quad x_k = 0 \end{cases}$$

$$\Pi_2(x_1 x_2 \ldots) = y_1 y_2 \ldots \begin{cases} y_k = x_k & \text{if} \quad x_k \neq 0 \\ y_k = m & \text{if} \quad x_k = 0 \end{cases}$$

Let ν be the Bernoulli measure on Σ_0 associated to the probability vector $(q_0, q_1, \ldots) = (\varepsilon, p_1, \ldots, p_n, \ldots, p_m - \varepsilon, \ldots)$. Clearly $\Pi(\Pi_1(\nu)) = \mu_{\mathbf{p}^*}$ and $\Pi(\Pi_2(\nu)) = \mu_{\mathbf{p}}$.

The above technique of writing distinct measures as projections of some common measure is based on an idea of Anthony Quas which was communicated to me by Mark Pollicott.

We are interested in $\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*})$. When the branches of T preserve orientation, $(T' > 0)$ this is easy to understand.

Lemma 5.1.1. *Let T be an EMR map such that $T' > 0$. Let $\mathbf{p}$, $\mathbf{p}^*$, ν, Π_1, Π_2 be* *as above. Then*

$$\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*}) = \int \log \left| \frac{T'(\Pi(\Pi_2(\mathbf{i})))}{T'(\Pi(\Pi_1(\mathbf{i})))} \right| d\nu(\mathbf{i}) \geqslant 0.$$

Proof. Since the branches of T are orientation preserving,

$$\Pi \circ \Pi_2(\mathbf{i}) \geqslant \Pi \circ \Pi_1(\mathbf{i})$$

for all $\mathbf{i} \in \Sigma_0$. Since T' is increasing it follows that

$$\int \log |T' \circ \Pi \circ \Pi_2| d\nu \geqslant \int \log |T' \circ \Pi \circ \Pi_1| d\nu.$$

But $\nu \circ \Pi_1^{-1} \circ \Pi^{-1} = \mu_{\mathbf{p}^*}$ and $\nu \circ \Pi_2^{-1} \circ \Pi^{-1} = \mu_{\mathbf{p}}$ thus the result follows. $\square$

In the case that $T' < 0$, this is the precise point at which we utilise the fact that the derivative of the second iterate of T is monotone in 'level 1' cylinders. This allows us to obtain the following analogue of Lemma 5.1.1.

Lemma 5.1.2. *Let T be an EMR map that is orientation reversing ($T' < 0$). Let $\mathbf{p}$, $\mathbf{p}^*$, ν be as above. Then*

$$\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*}) = \sum_{\mathbf{w} \in \Sigma^*_{\text{even}}} \left(\sum_{n \in \mathbb{N}} \int_{[nw0]} \log \left| \frac{(T^2)'(\Pi(\Pi_2(\mathbf{i})))}{(T^2)'(\Pi(\Pi_1(\mathbf{i})))} \right| d\nu(\mathbf{i}) + \int_{[0w0]} \log \left| \frac{T'(T(\Pi(\Pi_2(\mathbf{i}))))}{T'(T(\Pi(\Pi_1(\mathbf{i}))))} \right| d\nu(\mathbf{i}) \right) \geqslant 0 \tag{5.1}$$

where Σ^*_{even} denotes all finite words over the alphabet $\mathbb{N}$ of even length.

Proof. We define $A, B \subset \Sigma_0$ to be the sets

$$A := \left\{ \mathbf{i} \in \Sigma_0 : \min_k \{i_k = 0\} \text{ is even} \right\}$$

and

$$B := \left\{ (\mathbf{i} \in \Sigma_0 : \min_k \{i_k = 0\} \text{ is odd} \right\}.$$

Therefore

$$A = \bigcup_{w \in \Sigma^*_{\text{odd}}} [w0]$$

and

$$B = \bigcup_{w\in\Sigma^*_{\text{even}}} [w0]$$

where Σ^*_{odd} denotes all finite words over the alphabet $\mathbb{N}$ of odd length, Σ^*_{even} denotes all finite words over the alphabet $\mathbb{N}$ of even length. Note that by the Birkhoff ergodic theorem, $\nu(\Sigma_0 \setminus A \cup B) = 0$.

Observe that for $\mathbf{i} \in A$, $\Pi(\Pi_1(\mathbf{i})) \leqslant \Pi(\Pi_2(\mathbf{i}))$ and for $\mathbf{i} \in B$, $\Pi(\Pi_2(\mathbf{i})) \leqslant \Pi(\Pi_1(\mathbf{i}))$. Now,

$$\begin{aligned}
\chi(\mu_{\mathbf{p}}) &= \int_{[0,1]} \log|T'|d\mu_{\mathbf{p}} = \int_{\Sigma} \log|T'\circ\Pi|d\Pi_2(\nu) = \int_{\Sigma_0} \log|T'\circ\Pi\circ\Pi_2|d\nu \\
&= \int_A \log|T'\circ\Pi\circ\Pi_2|d\nu + \int_B \log|T'\circ\Pi\circ\Pi_2|d\nu
\end{aligned}$$

and similarly

$$\begin{aligned}
\chi(\mu_{\mathbf{p}^*}) = \int_{[0,1]} \log|T'|d\mu_{\mathbf{p}} = \int_{\Sigma} \log|T'\circ\Pi|d\Pi_1(\nu) &= \int_{\Sigma_0} \log|T'\circ\Pi\circ\Pi_1|d\nu \\
&= \int_A \log|T'\circ\Pi\circ\Pi_1|d\nu + \int_B \log|T'\circ\Pi\circ\Pi_1|d\nu.
\end{aligned}$$

Thus, we need to find

$$\begin{aligned}
\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*}) &= \int_B \log\left|\frac{T'\circ\Pi\circ\Pi_2}{T'\circ\Pi\circ\Pi_1}\right|d\nu - \int_A \log\left|\frac{T'\circ\Pi\circ\Pi_1}{T'\circ\Pi\circ\Pi_2}\right|d\nu \\
&= \sum_{w\in\Sigma^*_{\text{even}}}\left(\int_{[w0]} \log\left|\frac{T'\circ\Pi\circ\Pi_2}{T'\circ\Pi\circ\Pi_1}\right|d\nu - \int_{\bigcup_{n\in\mathbb{N}}[nw0]} \log\left|\frac{T'\circ\Pi\circ\Pi_1}{T'\circ\Pi\circ\Pi_2}\right|d\nu\right).
\end{aligned}$$

Observe that both of the above integrals are positive since for $x < y$, $|T'(x)| > |T'(y)|$. Fix $w \in \Sigma^*_{\text{even}}$ (where w can be the 'null' word). Then since ν is σ_0 invariant,

$$\begin{aligned}
\int_{[w0]} \log\left|\frac{T'\circ\Pi\circ\Pi_2}{T'\circ\Pi\circ\Pi_1}\right|d\nu &= \int_{[w0]} \log\left|\frac{T'\circ\Pi\circ\Pi_2}{T'\circ\Pi\circ\Pi_1}\right|d\nu\circ\sigma_0^{-1} \\
&= \int_{\bigcup_{n\in\mathbb{N}_0}[nw0]} \log\left|\frac{T'\circ\Pi\circ\Pi_2\circ\sigma_0}{T'\circ\Pi\circ\Pi_1\circ\sigma_0}\right|d\nu \\
&= \int_{\bigcup_{n\in\mathbb{N}_0}[nw0]} \log\left|\frac{T'\circ T\circ\Pi\circ\Pi_2}{T'\circ T\circ\Pi\circ\Pi_1}\right|d\nu
\end{aligned}$$

where $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and the final line follows because $\Pi_1\circ\sigma_0 = \sigma\circ\Pi_1$, $\Pi_2\circ\sigma_0 = \sigma\circ\Pi_2$

and $\Pi \circ \sigma = T \circ \Pi$. It follows that

$$\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*}) =$$
$$\sum_{w \in \Sigma^*_{\text{even}}} \left(\sum_{n \in \mathbb{N}} \int_{[nw0]} \log \left| \frac{(T^2)'(\Pi(\Pi_2(\mathbf{i})))}{(T^2)'(\Pi(\Pi_1(\mathbf{i})))} \right| d\nu(\mathbf{i}) + \int_{[0w0]} \log \left| \frac{T'(T(\Pi(\Pi_2(\mathbf{i}))))}{T'(T(\Pi(\Pi_1(\mathbf{i}))))} \right| d\nu(\mathbf{i}) \right). \tag{5.2}$$

The first term is positive by the assumption that $(T^2)'(x) \geqslant (T^2)'(y)$ for any $x, y \in \mathcal{I}_n$ with $x > y$. The second term is positive since for all $\mathbf{i} \in [0w0]$, $T(\Pi \circ \Pi_2(\mathbf{i})) \leqslant T(\Pi \circ \Pi_1(\mathbf{i}))$ and so

$$\int_{[0w0]_0} \log |T' \circ T \circ \Pi \circ \Pi_2| - \log |T' \circ T \circ \Pi \circ \Pi_1| d\nu \geqslant 0.$$

□

This gives us the following important result which allows us to compare Lyapunov exponents via the partial ordering on probability vectors.

Corollary 5.1.3. *Suppose T is an EMR map where T' is monotone. If $\mathbf{p} \succ \mathbf{p}^*$ then $\chi(\mu_{\mathbf{p}}) \geqslant \chi(\mu_{\mathbf{p}^*})$.*

Remark 5.1.4. *Since $\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*})$ can always be expressed as a sum of non-negative terms, it is clear that better lower bounds should be available for the difference $\chi(\mu_{\mathbf{p}}) - \chi(\mu_{\mathbf{p}^*})$. Although 0 is a sufficient lower bound for our purposes, in a recent joint work with Baker [BJ], positive lower bounds were used to study the related problem of determining the existence of a Bernoulli measure with maximal dimension amongst Bernoulli measures.*

Also, we immediately obtain the following lower bound for the Lyapunov exponent of any Bernoulli measure in the setting where $T' < 0$.

Corollary 5.1.5. *Let $\mathbf{p} \in \mathcal{P}$ and T be an EMR map from the setting of Theorem 3.3.1 such that $T' < 0$. Then $\chi(\mu_{\mathbf{p}}) \geqslant \log |T'(\Pi((1)^\infty))| > 0$.*

For example, this means that for any Bernoulli measure $\mu_{\mathbf{p}}$ for the Gauss map, $\chi(\mu_{\mathbf{p}}) \geqslant -2\log(\phi - 1)$ where ϕ denotes the golden ratio.

5.2 Proofs of Theorems 3.5.5 and 3.5.6

In this section, we complete the proof of Theorem 3.3.1 by using the ideas from the previous section. Recall that $\mathcal{P}_0$ was defined to be all $\mathbf{p} \in \mathcal{P}$ such that

(a) $\dim \mu_{\mathbf{p}} \geqslant \frac{2s+2}{s+3}$.

(b) $\mathbf{p}$ has all strictly positive entries, possibly apart from a tail of zeroes. In other words, $p_n \neq 0$ unless $p_k = 0$ for all $k \geqslant n$.

(c) $\frac{p_n}{|\mathcal{I}_n|}$ is bounded in n.

Recall also that $\mathcal{P}_\varepsilon = \{\mathbf{p} \in \mathcal{P}_0 : p_1, p_2 \geqslant \varepsilon\}$, $\mathcal{P}^* = \{\mathbf{p} \in \mathcal{P}_0 : p_n > 0\ \forall n\}$ and $\mathcal{P}^\infty = \bigcup_{n\in\mathbb{N}} \mathcal{P}^n$ where $\mathcal{P}^n$ denoted the set of probability vectors which were fully supported on the first n digits.

We will begin by proving Theorem 3.5.5, that

$$\sup_{\mathbf{p}\in\mathcal{P}\setminus\mathcal{P}_0} \dim \mu_{\mathbf{p}} \leqslant \sup_{\mathbf{p}\in\mathcal{P}_0} \dim \mu_{\mathbf{p}}.$$

The proof of this will involve various arguments whereby we move mass in such a way that we obtain a measure $\mu_{\mathbf{p}^*}$ with $\mathbf{p}^* \in \mathcal{P}_0$, that is, whose dimension we *do* know about, and such that $\mathbf{p} \succ \mathbf{p}^*$ so that $\chi(\mu_{\mathbf{p}}) \geqslant \chi(\mu_{\mathbf{p}^*})$. By combining this with estimates on the change in entropy, we will be able to bound $\dim \mu_{\mathbf{p}}$ by $\dim \mu_{\mathbf{p}^*}$ and obtain the result. We will tackle this in Section 5.2.1.

After proving Theorem 3.5.5, this will allow us to restrict our attention to obtaining a uniform estimate for $\sup_{\mathbf{p}\in\mathcal{P}_0 : p_1<\xi} \dim \mu_{\mathbf{p}}$, that is, to the proof of Theorem 3.5.6. Recall that by Theorem 3.5.3, we know that for sufficiently small ε and δ,

$$\sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}} < 1 - G_{\varepsilon,\delta}.$$

In Section 5.2.2, we fix ε and construct an algorithm which will allow us to take $\mathbf{p} \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ with $p_1 < \xi$, and output $\mathbf{p}^* \in \mathcal{P}_\varepsilon$ where $\mathbf{p} \succ \mathbf{p}^*$ (so that $\chi(\mu_{\mathbf{p}}) \geqslant \chi(\mu_{\mathbf{p}^*})$) and the entropy change is bounded. This will let us find the term E_ε for which

$$\sup_{\mathbf{p}\in\mathcal{P}_0\setminus\mathcal{P}_\varepsilon : p_1<\xi} \dim \mu_{\mathbf{p}} \leqslant \sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}} + E_\varepsilon$$

and then finally choose $\varepsilon, \delta > 0$ for which $E_\varepsilon < \frac{1}{2} G_{\varepsilon,\delta}$, thus completing the proof.

5.2.1 Proof of Theorem 3.5.5

In this section we prove Theorem 3.5.5, that is, we show that

$$\sup_{\mathbf{p}\in\mathcal{P}\setminus\mathcal{P}_0} \dim \mu_{\mathbf{p}} \leqslant \sup_{\mathbf{p}\in\mathcal{P}_0} \dim \mu_{\mathbf{p}}.$$

If $\mathbf{p} \notin \mathcal{P}_0$ then either (a), (b) or (c) of Definition 3.5.1 is not satisfied. If (a) is not satisfied, then trivially $\dim \mu_{\mathbf{p}} \leqslant \frac{2s+2}{s+3} < \sup_{\mathbf{p}\in\mathcal{P}_0} \dim \mu_{\mathbf{p}}$ by definition of $\mathcal{P}_0$. Therefore it suffices to prove that if either $\mathbf{p} \in \mathcal{P} \setminus (\mathcal{P}^* \cup \mathcal{P}^\infty)$ or if $\frac{p_n}{|\mathcal{I}_n|}$ is unbounded in n, then

$$\dim \mu_{\mathbf{p}} \leqslant \sup_{\mathbf{p}\in\mathcal{P}_0} \dim \mu_{\mathbf{p}}.$$

We'll treat each of these cases separately, beginning with the assumption that $\mathbf{p} \in \mathcal{P} \setminus (\mathcal{P}^* \cup \mathcal{P}^\infty)$.

Suppose $\mathbf{p} \in \mathcal{P} \setminus (\mathcal{P}^* \cup \mathcal{P}^\infty)$. We define $\mathbf{p}^*$ to be the unique $\mathbf{p}^* \in \mathcal{P}^* \cup \mathcal{P}^\infty$ which can be obtained from $\mathbf{p}$ by replacing any finite string of zeroes in $\mathbf{p}$ by redistributing the mass from the next non-zero entry uniformly over the preceding zero entries and itself. For example, given

$$\mathbf{p} = (p_1, p_2, 0, 0, p_5, 0, p_7, p_8, \ldots)$$

where all of the terms are non-zero unless explicitly stated, then

$$\mathbf{p}^* = (p_1, p_2, \frac{p_5}{3}, \frac{p_5}{3}, \frac{p_5}{3}, \frac{p_7}{2}, \frac{p_7}{2}, p_8, \ldots).$$

The following lemma describes the change in entropy.

Lemma 5.2.1. *Let $\mathbf{p} \in \mathcal{P} \setminus (\mathcal{P}^* \cup \mathcal{P}^\infty)$. Define $\mathbf{p}^* \in \mathcal{P}^* \cup \mathcal{P}^\infty$ as described above. Then $h(\mu_{\mathbf{p}}) \leqslant h(\mu_{\mathbf{p}^*})$.*

Proof. Without loss of generality, we can assume that all entries are non-zero apart from one string of zeros (and possibly a tail of zeroes). Let's say

$$\mathbf{p} = (p_1, p_2, \ldots, p_n, 0, \ldots, 0, p_{n+m}, p_{n+m+1}, \ldots)$$

where entries (except possibly the tail) are all non-zero unless stated explicitly otherwise. Then we obtain $\mathbf{p}^*$ to be

$$\mathbf{p}^* = (p_1, p_2, \ldots, p_n, \frac{p_{n+m}}{m}, \frac{p_{n+m}}{m}, \ldots, \frac{p_{n+m}}{m}, p_{n+m+1}, \ldots).$$

Then

$$\begin{aligned} h(\mu_{\mathbf{p}}) - h(\mu_{\mathbf{p}^*}) &= -p_{n+m} \log p_{n+m} + m \cdot \frac{p_{n+m}}{m} \log \frac{p_{n+m}}{m} \\ &= -p_{n+m} \log p_{n+m} + p_{n+m} \log \frac{p_{n+m}}{m} \\ &= -p_{n+m} \log m \leqslant 0. \end{aligned}$$

The result follows. $\square$

Using Corollary 5.1.3 and Lemma 5.2.1 we immediately see that we can deal with the case where $\mathbf{p} \notin \mathcal{P}^* \cup \mathcal{P}^\infty$.

Lemma 5.2.2. *Suppose* $\mathbf{p} \notin \mathcal{P}^* \cup \mathcal{P}^\infty$. *Then there exists* $\mathbf{p}^* \in \mathcal{P}^* \cup \mathcal{P}^\infty$ *for which* $\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*}$.

Proof. Let $\mathbf{p} \in \mathcal{P} \setminus (\mathcal{P}^* \cup \mathcal{P}^\infty)$. Then using Lemma 5.2.1 we can find $\mathbf{p}^* \in \mathcal{P}^* \cup \mathcal{P}^\infty$ such that $\mathbf{p}^* \prec \mathbf{p}$ and $h(\mu_{\mathbf{p}}) \leqslant h(\mu_{\mathbf{p}^*})$. Therefore by Corollary 5.1.3

$$\dim \mu_{\mathbf{p}} = \frac{h(\mu_{\mathbf{p}})}{\chi(\mu_{\mathbf{p}})} \leqslant \frac{h(\mu_{\mathbf{p}^*})}{\chi(\mu_{\mathbf{p}^*})} = \dim \mu_{\mathbf{p}^*}.$$

$\square$

Next, we need to consider $\mathbf{p}$ where the weights p_n are decaying at a slower rate than $|\mathcal{I}_n|$. The following lemma provides the appropriate upper bound for $\dim \mu_{\mathbf{p}}$.

Lemma 5.2.3. *Suppose* $\frac{p_n}{|\mathcal{I}_n|}$ *is unbounded and* $\mathbf{p} \in \mathcal{P}^* \cup \mathcal{P}^\infty$. *Then*

$$\dim \mu_{\mathbf{p}} \leqslant \sup_{\mathbf{q} \in \mathcal{P}_0} \dim \mu_{\mathbf{q}}.$$

Proof. Suppose $\mathbf{p} = (p_1, p_2, \ldots)$ is such that $\mathbf{p} \in \mathcal{P}^* \cup \mathcal{P}^\infty$. The main point of this lemma is to show that we can approximate the dimension of $\mu_{\mathbf{p}}$ arbitrarily well by the dimension of a finitely supported measure $\mu_{\mathbf{p}^*}$ (so that $\mathbf{p}^*$ trivially satisfies the hypothesis that $\frac{p_n^*}{|\mathcal{I}_n|}$ is bounded and therefore $\mathbf{p}^* \in \mathcal{P}_0$). A consequence of this is that even if $\frac{p_n}{|\mathcal{I}_n|}$ is not bounded then $\dim \mu_{\mathbf{p}} \leqslant \sup_{\mathbf{q} \in \mathcal{P}_0} \dim \mu_{\mathbf{q}}$ as required. Denote

$$\varsigma = \sup_{\mathbf{q} \in \mathcal{P}_0} \dim \mu_{\mathbf{q}}.$$

Let $\alpha > 0$ be arbitrary and fix $\delta < \varsigma \alpha \phi_{\mathbf{p}}$ where $\phi_{\mathbf{p}} = (1 - p_1) \inf\{\log |T'(x)| : x \in \mathcal{I}_2\}$. Fix $\varepsilon > 0$ sufficiently small so that $-\varepsilon \log \varepsilon < \delta$ and N sufficiently large that $-\sum_{n=N}^\infty p_n \log p_n < \delta$ and $\sum_{n=N}^\infty p_n < \varepsilon$. Now define

$$\mathbf{p}^* = \left(p_1, p_2, \ldots, p_{N-1}, \sum_{n=N}^\infty p_n, 0, 0, \ldots \right).$$

Then since (b) and (c) of Definition 3.5.1 are satisfied, either $\dim \mu_{\mathbf{p}^*} \leqslant \frac{2s+2}{s+3}$ (that is, (a) fails) or $\mathbf{p}^* \in \mathcal{P}_0$. Notice that $\frac{\phi_{\mathbf{p}}}{\chi(\mu_{\mathbf{p}^*})} < 1$. Clearly $|h(\mu_{\mathbf{p}}) - h(\mu_{\mathbf{p}^*})| < \delta$ and by Corollary 5.1.3, $\chi(\mu_{\mathbf{p}}) \geqslant \chi(\mu_{\mathbf{p}^*})$. It follows that

$$\dim(\mu_{\mathbf{p}}) \leqslant \frac{h(\mu_{\mathbf{p}})}{\chi(\mu_{\mathbf{p}})} \leqslant \frac{h(\mu_{\mathbf{p}^*}) + \delta}{\chi(\mu_{\mathbf{p}^*})} \leqslant (1+\alpha)\varsigma.$$

Since $\alpha > 0$ was arbitrary, the result follows. □

Finally, by combining Lemmas 5.2.2 and 5.2.3 we can prove Theorem 3.5.5.

Proof of Theorem 3.5.5. Suppose $\mathbf{p} \notin \mathcal{P}_0$. We want to show that

$$\sup_{\mathbf{p}\in\mathcal{P}\setminus\mathcal{P}_0} \dim \mu_{\mathbf{p}} \leqslant \sup_{\mathbf{p}\in\mathcal{P}_0} \dim \mu_{\mathbf{p}}.$$

Recall that for $\mathbf{p} \in \mathcal{P}_0$ then (a), (b), (c) of Definition 3.5.1 must be satisfied. Suppose $\mathbf{p}$ does not satisfy (a). Then $\dim \mu_{\mathbf{p}} \leqslant \frac{2s+2}{s+3} < \sup_{\mathbf{p}\in\mathcal{P}_0} \dim \mu_{\mathbf{p}}$ and we are done.

Suppose $\mathbf{p}$ does not satisfy (b). Then by Lemma 5.2.2 there exists $\mathbf{p}^* \in \mathcal{P}^* \cup \mathcal{P}^\infty$ such that $\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*}$. If $\mathbf{p}^*$ does not satisfy (a), we are done. If $\mathbf{p}^*$ does not satisfy (c), by Lemma 5.2.3 we are done. Otherwise, $\mathbf{p}^*$ satisfies (a), (b), and (c) and so $\mathbf{p}^* \in \mathcal{P}_0$ and so we are done.

Suppose $\mathbf{p}$ does not satisfy (c). If $\mathbf{p}$ does not satisfy (a) or (b) then by the arguments above, we are done. Otherwise, by Lemma 5.2.3 we are also done. □

5.2.2 Proof of Theorem 3.5.6

In this section, we prove Theorem 3.5.6 and thus conclude the proof of Theorem 3.3.1. Recall that ξ was introduced in Section 3.5 to be some explicit constant $\xi \in (0,1)$ and is made precise in (B.1). In this section, we will prove that there exists $\delta < \frac{1-s}{4}$ and some $\varepsilon > 0$ such that

$$\sup_{\mathbf{p}\in\mathcal{P}_0\setminus\mathcal{P}_\varepsilon : p_1<\xi} \dim \mu_{\mathbf{p}} < 1 - \frac{1}{2}G_{\varepsilon,\delta}.$$

Therefore, throughout this section we will fix arbitrary $\varepsilon > 0$ and consider $\mathbf{p} \in \mathcal{P}_0\setminus\mathcal{P}_\varepsilon$ with $p_1 < \xi$, that is, some $\mathbf{p}$ for which either $p_1 < \varepsilon$ or $p_2 < \varepsilon$ (or both), with the restriction that p_1 must be smaller than ξ. The goal will be to find some term E_ε, which depends only on ε, such that for any $\mathbf{p} \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ with $p_1 < \xi$, there exists a $\mathbf{p}^* \in \mathcal{P}_\varepsilon$ such that

$$\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*} + E_\varepsilon.$$

The idea would then be to choose some $0 < \delta < \frac{1-s}{4}$ and $\varepsilon > 0$ such that $E_\varepsilon < \frac{1}{2}G_{\varepsilon,\delta}$. From here it would follow that $\dim \mu_{\mathbf{p}} \leqslant 1 - \frac{1}{2}G_{\varepsilon,\delta}$, thus completing the proof of Theorem 3.5.6.

We will begin by describing a general algorithm which allows us to input some probability vector $\mathbf{p}$ which satisfies some hypothesis, and produce some $\mathbf{p}^* \prec \mathbf{p}$ where a chosen co-ordinate in the new probability vector will now be uniformly bounded below by some chosen ε. Since $\mathbf{p}^* \prec \mathbf{p}$, by Corollary 5.1.3 we know that $\chi(\mu_{\mathbf{p}}) \geqslant \chi(\mu_{\mathbf{p}^*})$. Moreover, we will be able to get a uniform bound on the entropy change $h(\mu_{\mathbf{p}}) - h(\mu_{\mathbf{p}^*})$ that occurs after applying the algorithm to $\mathbf{p}$. The entropy change will be dependent on ε and on how sparsely the mass in $\mu_{\mathbf{p}}$ is distributed.

Next, we'll investigate the implications of the general results described above to the particular setting where $\mathbf{p} \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$. In particular, we'll find some term E_ε such that given $\mathbf{p} \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ with $p_1 < \xi$, we can find $\mathbf{p}^*$ (by an application of the algorithm) for which

$$\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*} + E_\varepsilon.$$

The choice of candidate $\mathbf{p}^*$ for $\mathbf{p}$ will depend on the specific details about the weights p_n, for instance, whether only one or both of p_1 and p_2 is less than ε. To this end, we will separate the general scenario that $\mathbf{p} \notin \mathcal{P}_\varepsilon$ into five cases, and in each we will apply the algorithm slightly differently. Consequently, we'll show that there exists a uniform $\lambda > 0$ such that for each $\mathbf{p}$ we can produce $\mathbf{p}^* \in \mathcal{P}_\varepsilon$ with $\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*} + \lambda(\varepsilon - \varepsilon \log \varepsilon)$ (so $E_\varepsilon = \lambda(\varepsilon - \varepsilon \log \varepsilon)$). Finally, by combining this with the exact form of $G_{\varepsilon,\delta}$ provided by Theorem 4.6.3, we can prove that there exists $\delta, \varepsilon > 0$ with $E_\varepsilon < \frac{1}{2}G_{\varepsilon,\delta}$, concluding the proof of Theorem 3.5.6.

An algorithm

We begin by describing a general algorithm which, given some chosen $\varepsilon > 0$ and some probability vector $\mathbf{p}$ that satisfies a hypothesis about the distribution of mass amongst its weights p_n, allows one to produce $\mathbf{p}^* \prec \mathbf{p}$ where a chosen co-ordinate p_k^* of the new probability vector satisfies $p_k^* \geqslant \varepsilon$. In order for the algorithm to be well defined, we need to make some assumptions on the weights of $\mathbf{p}$. For instance, the chosen weight p_k of the old probability vector should satisfy $p_k < \varepsilon$, and there should be enough mass amongst the weights 'further down' the probability vector $\mathbf{p}$ to make it possible to $*$-transform it and obtain a suitable $\mathbf{p}^*$. The hypothesis below makes this precise.

Hypothesis 5.2.4. *Fix $\varepsilon > 0$ and $1 \leqslant k \leqslant m$. We say that $\mu_{\mathbf{p}}$ satisfies the (ε, k, m) hypothesis if $p_k < \varepsilon$ and $\sum_{n=m+1}^{\infty} p_n \geqslant 3\varepsilon$.*

So the (ε, k, m) hypothesis basically ensures that some weight $p_k < \varepsilon$ and that we can 'correct this' by redistributing mass from the digits $(p_n)_{n \geqslant m+1}$.

Suppose $\mathbf{p}$ satisfies the (ε, k, m) hypothesis. Then the (ε, k, m) algorithm is as follows. We start with $\mathbf{p}$ and $n = m + 1$.

1. Move $\min\{\frac{p_n}{2}, \varepsilon\}$ mass from the nth co-ordinate to the kth co-ordinate.
2. Replace $\mathbf{p}$ with the output of 1.
3. If $p_k \geqslant \varepsilon$ then stop.
4. If $p_k < \varepsilon$ then replace n with $n + 1$ and return to 1.

We denote the final probability vector that we are left with once the algorithm stops by $\mathbf{p}^*$.

We define $N \in \mathbb{N}$ to be the first integer for which

$$p_k + \sum_{n=m+1}^{m+N} \min\left\{\frac{p_n}{2}, \varepsilon\right\} \quad \geqslant \quad \varepsilon. \tag{5.3}$$

Then N is just the number of weights that we move mass from before we stop the algorithm, or alternatively the number of times we repeat the algorithm. We call N the stopping time. By construction

$$\varepsilon \leqslant p_k + \sum_{n=m+1}^{m+N} \min\left\{\frac{p_n}{2}, \varepsilon\right\} = p_k + \sum_{n=m+1}^{m+N-1} \frac{p_n}{2} + \min\left\{\frac{p_{m+N}}{2}, \varepsilon\right\} \leqslant 2\varepsilon.$$

Then we can write $\mathbf{p}^*$ explicitly as

$$\begin{aligned}\mathbf{p}^* = \Bigg(p_1, \ldots, p_{k-1}, p_k + \sum_{n=m+1}^{m+N-1} \frac{p_n}{2} + \min\left\{\frac{p_{m+N}}{2}, \varepsilon\right\}, p_{k+1}, \ldots, p_m, \frac{p_{m+1}}{2}, \\ \ldots, \frac{p_{m+N-1}}{2}, p_{m+N} - \min\left\{\frac{p_{m+N}}{2}, \varepsilon\right\}, p_{m+N+1}, \ldots\Bigg).\end{aligned} \tag{5.4}$$

Observe that $\mathbf{p}^*$ is well-defined, i.e. the algorithm will stop, because by assumption $\sum_{n=m+1}^{\infty} p_n \geqslant 3\varepsilon$. Thus the reason we required that $\sum_{n=m+1}^{\infty} p_n \geqslant 3\varepsilon$ rather than just $\sum_{n=m+1}^{\infty} p_n \geqslant \varepsilon$ is because in order avoid some of the entries of $\mathbf{p}^*$ being 0, we only move at most half of each weights mass at a time. Also observe that the algorithm uniquely chooses $\mathbf{p}^*$. Finally observe that $\mathbf{p}^* \prec \mathbf{p}$.

In practise, we will consider $\mathbf{p}$ that satisfy Hypothesis 5.2.4 for the cases where $k = 1, 2$, that is, $p_1 < \varepsilon$ or $p_2 < \varepsilon$ and $m = k, k + 1$, that is, we start taking mass either from the next co-ordinate or the one after (depending on the setting).

The following lemma gives a bound on the change in entropy that might occur when applying the algorithm to some $\mathbf{p}$ which satisfies the (ε, k, m) hypothesis.

Lemma 5.2.5. *Let* $\mathbf{p}$ *satisfy the* (ε, k, m) *hypothesis. Apply the algorithm to obtain* $\mathbf{p}^*$ *and let* N *be the stopping time for the algorithm, i.e. such that (5.3) is first satisfied. Then* $h(\mu_{\mathbf{p}}) - h(\mu_{\mathbf{p}^*}) \leqslant 6\varepsilon \log(N+1) - 8\varepsilon \log \varepsilon$.

Proof. Suppose $\frac{p_{m+N}}{2} \leqslant \varepsilon$. Then

$$\begin{aligned}
|h(\mu_{\mathbf{p}}) - h(\mu_{\mathbf{p}^*})| = & \Big| p_k \log p_k + p_{m+1} \log p_{m+1} + \ldots p_{m+N} \log p_{m+N} \\
& - \frac{1}{2}(2p_k + p_{m+1} + \ldots p_{m+N}) \log \frac{1}{2}(2p_k + p_{m+1} + \ldots p_{m+N}) \\
& - \frac{1}{2}p_{m+1} \log \frac{1}{2}p_{m+1} - \ldots - \frac{1}{2}p_{m+N} \log \frac{1}{2}p_{m+N} \Big| \\
= & \Big| \log p_k^{p_k} p_{m+1}^{p_{m+1}} \ldots p_{m+N}^{p_{m+N}} \\
& - \log \left(\frac{1}{2}(2p_k + p_{m+1} + \ldots p_{m+N}) \right)^{\frac{1}{2}(2p_k + p_{m+1} + \ldots p_{m+N})} \\
& - \log \left(\frac{1}{2}p_{m+1} \right)^{\frac{1}{2}p_{m+1}} \cdots \left(\frac{1}{2}p_{m+N} \right)^{\frac{1}{2}p_{m+N}} \Big| .
\end{aligned}$$

First we consider the first term. Let S denote the sum

$$S = p_k + p_{m+1} + \ldots + p_{m+N} \leqslant 4\varepsilon.$$

Then

$$\begin{aligned}
-\log p_k^{p_k} p_{m+1}^{p_{m+1}} \ldots p_{m+N}^{p_{m+N}} & \leqslant -\log \left(\frac{S}{N+1} \right)^{\frac{S}{N+1} \cdot (N+1)} \\
& = -\log \left(\frac{S}{N+1} \right)^{S} \\
& \leqslant -\log \left(\frac{4\varepsilon}{N+1} \right)^{4\varepsilon} \\
& \leqslant 4\varepsilon \log(N+1) - 4\varepsilon \log \varepsilon
\end{aligned}$$

where the first line follows because entropy is maximised when weight is distributed uniformly.

Similarly, for the third term, observe that since $\frac{1}{2}(2p_k + p_{m+1} + \ldots p_{m+N}) \leqslant 2\varepsilon$ then,

$$-\log\left(\frac{1}{2}p_{m+1}\right)^{\frac{1}{2}p_{m+1}}\dots\left(\frac{1}{2}p_{m+N}\right)^{\frac{1}{2}p_{m+N}} \leqslant 2\varepsilon\log(N)-2\varepsilon\log\varepsilon.$$

Finally we consider the second term. Observe that since the derivative of x^x is $(1+\log x)x^x$ it follows that for $x<e^{-1}$, x^x is monotonically increasing to 1 as $x\to 0$. That is, $\log x^x$ monotonically increases to 0 as $x\to 0$.

Thus, it follows that since $\frac{1}{2}(2p_k+p_{m+1}+\dots p_{m+N})<2\varepsilon$,

$$-\log\left(\frac{1}{2}(2p_k+p_{m+1}+\dots p_{m+N})\right)^{\frac{1}{2}(2p_k+p_{m+1}+\dots p_{m+N})} \leqslant -\log(2\varepsilon)^{2\varepsilon}\leqslant -2\varepsilon\log\varepsilon.$$

Thus we get

$$|h(\mu_{\mathbf{p}})-h(\mu_{\mathbf{p}^*})|\leqslant 6\varepsilon\log(N+1)-8\varepsilon\log\varepsilon.$$

Now for the other case suppose that $p_{m+N}>\varepsilon$. Then

$$\begin{aligned}
|h(\mu_{\mathbf{p}})-h(\mu_{\mathbf{p}^*})| &= \\
&\Big|p_k\log p_k+p_{m+1}\log p_{m+1}+\dots p_{m+N}\log p_{m+N}\\
&-(\frac{1}{2}(2p_k+p_{m+1}+\dots p_{m+N-1})+\varepsilon)\log(\frac{1}{2}(2p_k+p_{m+1}+\dots p_{m+N-1})+\varepsilon)\\
&-\frac{1}{2}p_{m+1}\log\frac{1}{2}p_{m+1}-\dots-\frac{1}{2}p_{m+N-1}\log\frac{1}{2}p_{m+N-1}-(p_{m+N}-\varepsilon)\log(p_{m+N}-\varepsilon)\Big|\\
=&\Big|\log p_k^{p_k}p_{m+1}^{p_{m+1}}\dots p_{m+N-1}^{p_{m+N-1}}\\
&-\log(\frac{1}{2}(2p_k+p_{m+1}+\dots p_{m+N-1})+\varepsilon)^{\frac{1}{2}(2p_k+p_{m+1}+\dots p_{m+N-1})+\varepsilon}\\
&-\log(\frac{1}{2}p_{m+1})^{\frac{1}{2}p_{m+1}}\dots(\frac{1}{2}p_{m+N-1})^{\frac{1}{2}p_{m+N-1}}+\log\frac{p_{m+N}^{p_{m+N}}}{(p_{m+N}-\varepsilon)^{p_{m+N}-\varepsilon}}\Big|.
\end{aligned}$$

Similarly to before, since $p_k+p_{m+1}+\dots+p_{m+N-1}<2\varepsilon$ it follows that the first term satisfies

$$-\log p_k^{p_k}p_{m+1}^{p_{m+1}}\dots p_{m+N-1}^{p_{m+N-1}}\leqslant 2\varepsilon\log(N)-2\varepsilon\log\varepsilon.$$

The second term satisfies

$$-\log\left(\frac{1}{2}(2p_k+p_{m+1}+\dots p_{m+N-1})+\varepsilon\right)^{\frac{1}{2}(2p_k+p_{m+1}+\dots p_{m+N-1})+\varepsilon}\leqslant -2\varepsilon\log\varepsilon.$$

The third term satisfies

$$-\log(\frac{1}{2}p_{m+1})^{\frac{1}{2}p_{m+1}}\ldots(\frac{1}{2}p_{m+N-1})^{\frac{1}{2}p_{m+N-1}}\leqslant\varepsilon\log(N-1)-\varepsilon\log\varepsilon.$$

For the final term observe that $\frac{x^x}{(x-\varepsilon)^{x-\varepsilon}}$ is increasing on $(\varepsilon,1]$ since it has derivative

$$\frac{x^x(x-\varepsilon)^{x-\varepsilon}\log\frac{x}{x-\varepsilon}}{(x-\varepsilon)^{2(x-\varepsilon)}}>0$$

therefore

$$0>\log\frac{p_{m+N}^{p_{m+N}}}{(p_{m+N}-\varepsilon)^{p_{m+N}-\varepsilon}}\geqslant\varepsilon\log\varepsilon.$$

Therefore,

$$h(\mu_{\mathbf{p}})\leqslant h(\mu_{\mathbf{p}^*})+3\varepsilon\log(N)-6\varepsilon\log\varepsilon.$$

□

We are now ready to apply the more general ideas above to our specific case of interest: the case when $\mathbf{p}\notin\mathcal{P}_\varepsilon$. It is easy to see that we will not be able to apply the algorithm in a unanimous way, since the distribution of mass may vary e.g. for some $\mathbf{p}$ we may need to increase both the first and second weights, for others only the first or the second. Therefore, it will be necessary to split up the next part of this section into four separate cases: firstly where both $p_1,p_2<\varepsilon$, secondly where $p_1<\varepsilon$, $p_2\geqslant 2\varepsilon$, thirdly where $p_1<\varepsilon$, $\varepsilon<p_2\leqslant 2\varepsilon$, and finally where $\varepsilon\leqslant p_1<\xi$, $p_2<\varepsilon<\frac{1-\xi}{4}$. In each case, $\mathbf{p}$ will satisfy a slightly different variation of the hypothesis, and thus we shall apply the algorithm slightly differently.

Case 1: $p_1,p_2<\varepsilon$

Let $\mathbf{p}=(p_1,p_2,\ldots)\in\mathcal{P}_0\setminus\mathcal{P}_\varepsilon$ be such that $p_1,p_2<\varepsilon$. We can see that $\mathbf{p}$ satisfies the $(\varepsilon,1,2)$ hypothesis since $p_1<\varepsilon$ and $\sum_{n=3}^\infty p_n>1-2\varepsilon>3\varepsilon$. Thus, we can apply the algorithm (with $k=1$, $m=2$) and obtain $\mathbf{p}'$. Let N_1 denote the stopping time for this algorithm, that is, the first time N_1 that

$$p_1+\sum_{n=3}^{2+N_1}\min\left\{\frac{p_n}{2},\varepsilon\right\}\geqslant\varepsilon.$$

Observe that $p_1'\geqslant\varepsilon$ and $p_2'<\varepsilon$. Therefore $\mathbf{p}'$ satisfies the $(\varepsilon,2,N_1+1)$ hypothesis, and we can apply the algorithm again to obtain $\mathbf{p}^*$. Let N_2 denote the stopping time for this application of the algorithm, that is, the first time that

$$p_2 + \sum_{n=N_1+2}^{N_1+N_2+1} \min\left\{\frac{p_n}{2}, \varepsilon\right\} \geqslant \varepsilon.$$

Observe that $p_1^*, p_2^* \geqslant \varepsilon$ so either $\mathbf{p}^* \in \mathcal{P}_\varepsilon$ or $\dim \mu_{\mathbf{p}} \leqslant \frac{2s+2}{s+3}$. Also observe that $\mathbf{p}^* \succ \mathbf{p'} \succ \mathbf{p}$, so by Corollary 5.1.3, $\dim \mu_{\mathbf{p}} \geqslant \dim \mu_{\mathbf{p}^*}$.

By Lemma 5.2.5

$$h(\mu_{\mathbf{p}}) - h(\mu_{\mathbf{p'}}) \leqslant 6\varepsilon \log(N_1 + 1) - 8\varepsilon \log \varepsilon$$

and also

$$h(\mu_{\mathbf{p'}}) - h(\mu_{\mathbf{p}^*}) \leqslant 6\varepsilon \log(N_2 + 1) - 8\varepsilon \log \varepsilon.$$

Combining these two, we deduce that

$$h(\mu_{\mathbf{p}}) - h(\mu_{\mathbf{p}^*}) \leqslant 12\varepsilon \log(N_1 + N_2 + 1) - 16\varepsilon \log \varepsilon.$$

Also, since $p_1 + \frac{1}{2}p_3 + \cdots \frac{1}{2}p_{N_1+1} < \varepsilon$ this implies that

$$p_1 + p_3 + \cdots p_{N_1+1} < 2\varepsilon$$

and similarly, since $p_2 + \frac{1}{4}p_{N_1+2} + \frac{1}{2}p_{N_1+3} + \cdots \frac{1}{2}p_{N_1+N_2} < \varepsilon$, then

$$p_2 + p_{N_1+2} + p_{N_1+3} + \cdots p_{N_1+N_2} < 4\varepsilon.$$

Combining the two inequalities above, we deduce that $\sum_{n=N_1+N_2+1} p_n \geqslant 1 - 6\varepsilon$ and therefore $\sum_{n=N_1+N_2+1} p_n^* \geqslant 1 - 7\varepsilon$, that is, $\mu_{\mathbf{p}^*}$ supports at least $1 - 7\varepsilon$ mass on $\bigcup_{n=N_1+N_2+1} \mathcal{I}_n$.

The information about the change in entropy and Lyapunov exponent and distribution of mass in $\mu_{\mathbf{p}^*}$ allows us to get an upper bound on the dimension of $\mu_{\mathbf{p}}$ in terms of $\dim \mu_{\mathbf{p}^*}$ and ε.

Lemma 5.2.6. *Let $\mathbf{p} \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ be such that $p_1, p_2 < \varepsilon$, and use the notation given above. Then there exists a uniform constant $\lambda_1 > 0$ such that*

$$\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*} + \lambda_1(\varepsilon - \varepsilon \log \varepsilon).$$

Proof. Using the notation given above, by applying Corollary 5.1.3 and the above estimate for the entropy change,

$$\begin{aligned}\dim \mu_{\mathbf{p}} &\leqslant \frac{h(\mu_{\mathbf{p}^*}) + 12\varepsilon \log(N_1 + N_2 + 1) - 16\varepsilon \log \varepsilon}{\chi(\mu_{\mathbf{p}^*})} \\ &\leqslant \dim \mu_{\mathbf{p}^*} + \frac{12\varepsilon \log(N_1 + N_2 + 1) - 16\varepsilon \log \varepsilon}{(1 - 7\varepsilon) \inf\{|T'(x)| : x \in \mathcal{I}_{N_1+N_2+1}\}}.\end{aligned}$$

The result follows because $\frac{\log N}{\inf_{x \in \mathcal{I}_N} \log |T'(x)|}$ is uniformly bounded (which itself is true because $|\mathcal{I}_n|$ are polynomially decaying). □

Case 2: $p_1 < \varepsilon$**,** $p_2 \geqslant 2\varepsilon$

Let $\mathbf{p} = (p_1, p_2, \ldots) \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ be such that $p_1 < \varepsilon$ and $p_2 \geqslant 2\varepsilon$. We can see that $\mathbf{p}$ satisfies the $(\varepsilon, 1, 1)$ hypothesis since $p_1 < \varepsilon$ and $\sum_{n=2}^{\infty} p_n > 1 - \varepsilon > 3\varepsilon$. Thus, we can apply the algorithm (with $k = m = 1$) and obtain $\mathbf{p}^*$. Clearly the stopping time $N = 1$. Observe that $p_1^*, p_2^* \geqslant \varepsilon$ so either $\mathbf{p}^* \in \mathcal{P}_\varepsilon$ or $\dim \mu_{\mathbf{p}} \leqslant \frac{2s+2}{s+3}$. Also observe that $\mathbf{p}^* \succ \mathbf{p}$, so by Corollary 5.1.3, $\dim \mu_{\mathbf{p}} \geqslant \dim \mu_{\mathbf{p}^*}$.

By Lemma 5.2.5

$$h(\mu_{\mathbf{p}}) - h(\mu_{\mathbf{p}^*}) \leqslant 6\varepsilon \log(N + 1) - 8\varepsilon \log \varepsilon.$$

Also, since $p_1^* < 2\varepsilon$, $\sum_{n=2} p_n^* \geqslant 1 - 2\varepsilon$, that is, $\mu_{\mathbf{p}^*}$ supports at least $1 - 2\varepsilon$ mass on $\bigcup_{n=2} \mathcal{I}_n$.

The information about the change in entropy and Lyapunov exponent and distribution of mass in $\mu_{\mathbf{p}^*}$ allows us to get an upper bound on the dimension of $\mu_{\mathbf{p}}$ in terms of $\dim \mu_{\mathbf{p}^*}$ and ε.

Lemma 5.2.7. *Let $\mathbf{p} \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ be such that $p_1 < \varepsilon$ and $p_2 \geqslant 2\varepsilon$, and use the notation given above. Then there exists a uniform constant $\lambda_2 > 0$ such that*

$$\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*} + \lambda_2(\varepsilon - \varepsilon \log \varepsilon).$$

Proof. Using the notation given above, by applying Corollary 5.1.3 and the above estimate for the entropy change,

$$\begin{aligned}\dim \mu_{\mathbf{p}} &\leqslant \frac{h(\mu_{\mathbf{p}^*}) + 6\varepsilon \log(2) - 8\varepsilon \log \varepsilon}{\chi(\mu_{\mathbf{p}^*})} \\ &\leqslant \dim \mu_{\mathbf{p}^*} + \frac{6\varepsilon \log(2) - 8\varepsilon \log \varepsilon}{(1 - 2\varepsilon) \inf\{|T'(x)| : x \in \mathcal{I}_2\}}.\end{aligned}$$

The result follows because $\frac{\log N}{\inf_{x \in \mathcal{I}_N} \log |T'(x)|}$ is uniformly bounded (which itself is true because $|\mathcal{I}_n|$ are polynomially decaying). □

Case 3: $p_1 < \varepsilon$**,** $\varepsilon \leqslant p_2 < 2\varepsilon$

Let $\mathbf{p} = (p_1, p_2, \ldots) \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ be such that $p_1 < \varepsilon$ and $\varepsilon \leqslant p_2 < 2\varepsilon$. We can see that $\mathbf{p}$ satisfies the $(\varepsilon, 1, 2)$ hypothesis since $p_1 < \varepsilon$ and $\sum_{n=3}^{\infty} p_n > 1 - 3\varepsilon > 3\varepsilon$. Thus, we can apply the algorithm (with $k = 1$, $m = 2$) and obtain $\mathbf{p}^*$. Let N denote the stopping time for this algorithm, that is, the first time N that

$$p_1 + \sum_{n=3}^{2+N} \min\left\{\frac{p_n}{2}, \varepsilon\right\} \geqslant \varepsilon.$$

Observe that $p_1^*, p_2^* \geqslant \varepsilon$ so either $\mathbf{p}^* \in \mathcal{P}_\varepsilon$ or $\dim \mu_{\mathbf{p}} \leqslant \frac{2s+2}{s+3}$. Also observe that $\mathbf{p}^* \succ \mathbf{p}$, so by Corollary 5.1.3, $\dim \mu_{\mathbf{p}} \geqslant \dim \mu_{\mathbf{p}^*}$.

By Lemma 5.2.5

$$h(\mu_{\mathbf{p}}) - h(\mu_{\mathbf{p}^*}) \leqslant 6\varepsilon \log(N+1) - 8\varepsilon \log \varepsilon.$$

Also, since $p_1 + \frac{1}{2}p_3 + \cdots \frac{1}{2}p_{N+1} < \varepsilon$ this implies that

$$p_1 + p_3 + \cdots p_{N+1} < 2\varepsilon.$$

We deduce that $\sum_{n=N+2} p_n \geqslant 1 - 4\varepsilon$ and therefore $\sum_{n=N+2} p_n^* \geqslant 1 - 5\varepsilon$, that is, $\mu_{\mathbf{p}^*}$ supports at least $1 - 5\varepsilon$ mass on $\bigcup_{n=N+2} \mathcal{I}_n$.

The information about the change in entropy and Lyapunov exponent and distribution of mass in $\mu_{\mathbf{p}^*}$ allows us to get an upper bound on the dimension of $\mu_{\mathbf{p}}$ in terms of $\dim \mu_{\mathbf{p}^*}$ and ε.

Lemma 5.2.8. *Let $\mathbf{p} \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ be such that $p_1 < \varepsilon$ and $\varepsilon \leqslant p_2 < 2\varepsilon$, and use the notation given above. Then there exists a uniform constant $\lambda_3 > 0$ such that*

$$\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*} + \lambda_3(\varepsilon - \varepsilon \log \varepsilon).$$

Proof. Using the notation given above, by applying Corollary 5.1.3 and the above estimate for the entropy change,

$$\begin{aligned}
\dim \mu_{\mathbf{p}} &\leqslant \frac{h(\mu_{\mathbf{p}^*}) + 6\varepsilon \log(N+1) - 8\varepsilon \log \varepsilon}{\chi(\mu_{\mathbf{p}^*})} \\
&\leqslant \dim \mu_{\mathbf{p}^*} + \frac{6\varepsilon \log(N+1) - 8\varepsilon \log \varepsilon}{(1-5\varepsilon) \inf\{|T'(x)| : x \in \mathcal{I}_{N+2}\}}.
\end{aligned}$$

The result follows because $\frac{\log N}{\inf_{x\in\mathcal{I}_N}\log|T'(x)|}$ is uniformly bounded (which itself is true because $|\mathcal{I}_n|$ are polynomially decaying). $\square$

Case 4: $\varepsilon \leqslant p_1 < \xi$**,** $p_2 < \varepsilon < \frac{1-\xi}{4}$

Let $\mathbf{p} = (p_1, p_2, \ldots) \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ be such that $\varepsilon \leqslant p_1 < \xi$, $p_2 < \varepsilon < \frac{1-\xi}{4}$. We can see that $\mathbf{p}$ satisfies the $(\varepsilon, 2, 2)$ hypothesis since $p_2 < \varepsilon$ and $\sum_{n=3}^{\infty} p_n > 1 - \xi - \varepsilon > 3\varepsilon$. Thus, we can apply the algorithm (with $k = m = 2$) and obtain $\mathbf{p}^*$. Let N denote the stopping time for this algorithm, that is, the first time N that

$$p_2 + \sum_{n=3}^{2+N} \min\left\{\frac{p_n}{2}, \varepsilon\right\} \geqslant \varepsilon.$$

Observe that $p_1^*, p_2^* \geqslant \varepsilon$ so either $\mathbf{p}^* \in \mathcal{P}_\varepsilon$ or $\dim \mu_{\mathbf{p}} \leqslant \frac{2s+2}{s+3}$. Also observe that $\mathbf{p}^* \succ \mathbf{p}$, so by Corollary 5.1.3, $\chi(\mu_{\mathbf{p}}) \geqslant \chi(\mu_{\mathbf{p}^*})$.

By Lemma 5.2.5

$$h(\mu_{\mathbf{p}}) - h(\mu_{\mathbf{p}^*}) \leqslant 6\varepsilon\log(N+1) - 8\varepsilon\log\varepsilon.$$

Also, since $p_2 + \frac{1}{2}p_3 + \cdots \frac{1}{2}p_{N+1} < \varepsilon$ this implies that

$$p_2 + p_3 + \cdots p_{N+1} < 2\varepsilon.$$

We deduce that $\sum_{n=N+2} p_n \geqslant 1 - \xi - 2\varepsilon$ and therefore $\sum_{n=N+2} p_n^* \geqslant 1 - \xi - 3\varepsilon$, that is, $\mu_{\mathbf{p}^*}$ supports at least $1 - \xi - 3\varepsilon$ mass on $\bigcup_{n=N+2}^{\infty} \mathcal{I}_n$.

The information about the change in entropy and Lyapunov exponent and distribution of mass in $\mu_{\mathbf{p}^*}$ allows us to get an upper bound on the dimension of $\mu_{\mathbf{p}}$ in terms of $\dim \mu_{\mathbf{p}^*}$ and ε.

Lemma 5.2.9. *Let $\mathbf{p} \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$ be such that $\varepsilon \leqslant p_1 < 1 - \xi$, $p_2 < \varepsilon$, and use the notation given above. Then there exists a uniform constant $\lambda_4 > 0$ such that*

$$\dim \mu_{\mathbf{p}} \leqslant \dim \mu_{\mathbf{p}^*} + \lambda_4(\varepsilon - \varepsilon\log\varepsilon).$$

Proof. Using the notation given above, by applying Corollary 5.1.3 and the above estimate for the entropy change,

$$\begin{aligned}
\dim \mu_{\mathbf{p}} &\leqslant \frac{h(\mu_{\mathbf{p}^*}) + 6\varepsilon\log(N+1) - 8\varepsilon\log\varepsilon}{\chi(\mu_{\mathbf{p}^*})} \\
&\leqslant \dim \mu_{\mathbf{p}^*} + \frac{6\varepsilon\log(N+1) - 8\varepsilon\log\varepsilon}{(1-\xi-3\varepsilon)\inf\{|T'(x)| : x \in \mathcal{I}_{N+2}\}}.
\end{aligned}$$

The result follows because $\frac{\log N}{\inf_{x\in\mathcal{I}_N}\log|T'(x)|}$ is uniformly bounded (which itself is true because $|\mathcal{I}_n|$ are polynomially decaying). $\square$

By combining Lemmas 5.2.6, 5.2.7, 5.2.8 and 5.2.9 we immediately obtain the following corollary which provides us with the exact form of E_ε.

Corollary 5.2.10. *There exists a uniform constant $\lambda > 0$ independent of ε such that,*

$$\sup_{\mathbf{p}\in\mathcal{P}_0\setminus\mathcal{P}_\varepsilon : p_1<\xi} \dim \mu_{\mathbf{p}} \leqslant \sup_{\mathbf{p}\in\mathcal{P}_\varepsilon} \dim \mu_{\mathbf{p}} + \lambda(\varepsilon - \varepsilon\log\varepsilon).$$

Proof. This follows directly from the above. $\square$

Using Corollary 5.2.10 we can complete the proof of Theorem 3.5.6, that is, find $\varepsilon, \delta > 0$ for which $E_\varepsilon < \frac{1}{2}G_{\varepsilon,\delta}$ and so $\sup_{\mathbf{p}\in\mathcal{P}_0\setminus\mathcal{P}_\varepsilon : p_1<\xi} \dim \mu_{\mathbf{p}} \leqslant 1 - \frac{1}{2}G_{\varepsilon,\delta}$.

Proof of Theorem 3.5.6. Fix δ sufficiently small so that $\delta(1-\log\delta) < \frac{1}{\rho_2}$. We need to show that for some $0 < \varepsilon < \frac{1-\xi}{4}$ we have

$$1 - G_{\varepsilon,\delta} + \lambda(\varepsilon - \varepsilon\log\varepsilon) < 1.$$

It is enough to show that for our choice of $\delta > 0$,

$$\lim_{\varepsilon\to 0} \frac{\lambda(\varepsilon - \varepsilon\log\varepsilon)}{G_{\varepsilon,\delta}} = 0.$$

Substituting in for $G_{\varepsilon,\delta}$ we see that we need to prove that

$$\lim_{\varepsilon\to 0} \frac{\lambda(\varepsilon - \varepsilon\log\varepsilon)}{\rho_1\delta^2\varepsilon^{\rho_2\delta(1-\log\delta)}T^{\rho_2-\log\delta}} = 0.$$

It is enough to show that

$$\lim_{\varepsilon\to 0} \frac{1-\log\varepsilon}{\varepsilon^{\rho_2\delta(1-\log\delta)-1}} = 0 \tag{5.5}$$

which is the case whenever $\rho_2\delta(1-\log\delta) - 1 < 0$, that is, $\delta(1-\log\delta) < \frac{1}{\rho_2}$.

Now, fix $0 < \varepsilon_0 < \frac{1-\xi}{4}$ sufficiently small so that

$$\lambda(\varepsilon_0 - \varepsilon_0\log\varepsilon_0) < \frac{1}{2}G_{\varepsilon_0,\delta}.$$

Then we have that

$$\sup_{\mathbf{p}\in\mathcal{P}_0\setminus\mathcal{P}_\varepsilon : p_1<\xi} \dim \mu_{\mathbf{p}} \leqslant 1 - \frac{1}{2}G_{\varepsilon_0,\delta}$$

and so the result follows. $\square$

Chapter 6

Dimension of equilibrium measures

6.1 Introduction

Let $\{S_i\}_{i=1}^n$ be a self-affine IFS on $\mathbb{R}^d$ with attractor Λ. In particular, recall that this means each of the maps $S_i = A_i + t_i$ where A_i are $d \times d$ contracting non-singular matrices and $t_i \in \mathbb{R}^d$ are translations. See Section 2.4.3 for an introduction to self-affine sets. Let Σ be the full shift on n symbols and $\Pi : \Sigma \to \mathbb{R}^d$ be the canonical coding map. Recall that, given a measure m on Σ, we can define a measure $\mu = m \circ \Pi^{-1}$ on the self-affine set Λ.

In recent years, there has been considerable interest in studying the dimension of measures which are supported on self-affine sets. The analogue of this problem is well understood in the self-similar case. In particular, Feng and Hu [FH] proved the following very general result.

Theorem 6.1.1. *Let $\{S_i\}_{i=1}^n$ be a C^1 IFS on $X \subset \mathbb{R}^d$ (that is, each map S_i extends to a contracting C^1 diffeomorphism $S_i : U \to S_i(U) \subset U$ for some open set $U \supset X$). Let m be an ergodic measure on Σ and $\mu = m \circ \Pi^{-1}$. If $\chi_1(\mu) = \chi_d(\mu)$ then μ is exact dimensional and* $\dim \mu = \frac{h(\mu)}{\chi_1(\mu)}$.

Since $\alpha_1(\mathbf{i}|_n) = \alpha_d(\mathbf{i}|_n)$ for *all* $\mathbf{i} \in \Sigma$ whenever we are working with a self-similar IFS, the above result tells us that the projection $\mu = m \circ \Pi^{-1}$ of any ergodic measure m to a self-similar set is exact dimensional. More generally, it also tells us that if we have a self-affine system and ergodic (projected) measure where all of the Lyapunov exponents coincide, then that measure is also exact dimensional.

Once we move to the more general self-affine setting, the problem of determining the dimension of a measure becomes significantly more challenging. Essentially, this is down to the fact that the different contraction ratios in different directions makes the geometry of a self-affine attractor considerably more difficult to analyse.

The 'Ledrappier-Young formula' refers to a dimension formula originating with [LY1; LY2], which gives the exact dimension of a measure in a non-conformal setting in terms of entropy, Lyapunov exponents and dimensions of projected measures. The measures originally considered by Ledrappier and Young were invariant measures for C^2 diffeomorphisms, but the formula has shown up in various contexts since their original work. One of the contexts where this formula has cropped up is in the study of measures supported on self-affine sets.

Recall that we say that a measure μ is a self-affine measure if for some self-affine IFS $\{S_i\}_{i=1}^n$ with coding map $\Pi : \Sigma \to \mathbb{R}^d$, there exists some Bernoulli measure m on Σ such that $\mu = m \circ \Pi^{-1}$. In some sense, a self-affine measure is the simplest and most natural measure which a self-affine set Λ can support. However, it was not until [BK] that the problem of determining the exact-dimensionality of self-affine measures was resolved. Note that in the following results, to avoid introducing unnecessary notation we will not give the exact form for the Ledrappier-Young formula, since the expressions would be more complicated than the expression that we will be dealing with. In [BK], Bárány and Käenmäki proved the following result.

Theorem 6.1.2. *Let $\mu = m \circ \Pi^{-1}$ be a self-affine measure corresponding to the self-affine IFS $\{S_i\}_{i=1}^n$ on $\mathbb{R}^d$. If the Lyapunov exponents of μ are all distinct, that is,*

$$0 < \chi_1(\mu) < \cdots < \chi_d(\mu) < 1$$

then μ is exact-dimensional and satisfies (the appropriate version of) the Ledrappier-Young formula.

In the general $\mathbb{R}^d$ setting, the result of Bárány and Käenmäki leaves questions unanswered, such as how things stand when two Lyapunov exponents are *not* distinct. However, if we take $d = 2$ and combine their theorem with Theorem 6.1.1, then at least in the planar case we can verify the exact dimensionality of *any* self-affine measure.

Corollary 6.1.3. *Let $\mu = m \circ \Pi^{-1}$ be a self-affine measure corresponding to the self-affine IFS $\{S_i\}_{i=1}^n$ on $\mathbb{R}^2$. Then μ is exact dimensional and satisfies (the appropriate version of) the Ledrappier-Young formula.*

Proof. By Theorem 6.1.1, if $\chi_1(\mu) = \chi_2(\mu)$ then μ is exact-dimensional. The case where $\chi_1(\mu) < \chi_2(\mu)$ follows from Theorem 6.1.2. □

Another direction is to consider more general measures supported on self-affine sets. In [BK], Bárány and Käenmäki also considered quasi-Bernoulli measures, under some assumptions on the IFS. Before we state their result, we introduce the *Totally Dominated Splitting* condition, which is used in their result.

Definition 6.1.4. *Let $\{S_i\}_{i=1}^n = \{A_i + t_i\}_{i=1}^n$ be a self-affine system in $\mathbb{R}^d$. We say that the $d \times d$ matrices $\{A_i\}_{i=1}^n$ satisfy the Totally Dominated Splitting condition if there exist constants $C \geqslant 1$ and $0 < \lambda < 1$ such that for each $r \in \{1, \ldots, d-1\}$ either*

$$\frac{\alpha_{r+1}(A_{\mathbf{i}|_n})}{\alpha_r(A_{\mathbf{i}|_n})} \leqslant C\lambda^n \tag{6.1}$$

for all $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$ or

$$C^{-1} \leqslant \frac{\alpha_{r+1}(A_{\mathbf{i}|_n})}{\alpha_r(A_{\mathbf{i}|_n})} \tag{6.2}$$

for all $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$.

Essentially, the totally dominated splitting condition ensures that neighbouring singular values are either 'exponentially separated' from one another or essentially equal. Observe that for any ergodic measure m on Σ, the totally dominated splitting condition implies that for each $r \in \{1, \ldots, d\}$ either $\chi_r(m) = \chi_{r+1}(m)$ or $\chi_r(m) < \chi_r(m) - \log\lambda \leqslant \chi_{r+1}(m)$.

Under this condition, Bárány and Käenmäki [BK] proved an analogue of Theorem 6.1.2.

Theorem 6.1.5. *Let μ be a quasi-Bernoulli measure for a self-affine IFS $\{S_i\}_{i=1}^n$ where the matrices $\{A_i\}$ satisfy the Totally Dominated Splitting condition. Then μ is exact dimensional and satisfies (the appropriate version of) the Ledrappier-Young formula.*

In this chapter, we consider a natural class of measures which are *not* quasi-Bernoulli, and prove that they are exact dimensional and satisfy the appropriate version of the Ledrappier-Young formula. However, as in Theorem 6.1.5, we also pay the price of having to restrict to a specific class of self-affine sets. Our setting can be viewed as an extension of the self-affine carpets discussed in Section 2.4.3; in particular we consider planar self-affine sets generated by IFS where the contractions are all either diagonal or anti-diagonal matrices. We will define these in Section 6.1.1.

One of our main tools, which may be of independent interest, is that even though our measures are not quasi-Bernoulli, we show they are equal to the sum of (the pushforwards) of two quasi-Bernoulli measures on an associated subshift of finite type.

The chapter is structured as follows. In 6.1.1, we introduce the class of self-affine sets that will be considered.

In Section 6.2 we introduce the measure whose dimension we will be studying and conduct an analysis of its structure. This will involve introducing a related subshift of finite type and showing that our measure can be written in terms of two quasi-Bernoulli measures which are supported on this shift space. This characterisation forms the backbone of all subsequent proofs.

In Section 6.3 we state and discuss the main dimension related results of this chapter. In Section 6.4, using the description of the measure provided in Section 6.2, we prove these results.

6.1.1 Our class of planar self-affine sets

In this section, we introduce the class of self-affine sets that will be studied throughout this chapter.

Let $\mathcal{I} = \{1, \dots, d\}$ be a finite alphabet and $\{S_i\}_{i \in \mathcal{I}}$ be a finite collection of affine maps acting on the plane such that the associated linear parts are contracting non-singular 2×2 real-valued matrices with non-negative entries which are all either diagonal or anti-diagonal and we assume the collection contains at least one of each. In particular, we order the maps in the following way: let $S_i = A_i + t_i$ where

$$A_i = \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix}$$

for $i < l$ and

$$A_i = \begin{bmatrix} 0 & a_i \\ b_i & 0 \end{bmatrix}$$

for $i \geqslant l$ where $l - 1$ is the number of diagonal matrices in the IFS, that is, $1 < l \leqslant |\mathcal{I}| = d$. We will also assume that for some $1 \leqslant i \leqslant l - 1$ we have that $a_i \neq b_i$ (so our system is not self-similar).

We may assume for convenience that each S_i maps the unit square into itself. Recall that by Theorem 2.4.1 there exists a unique non-empty compact set $F \subseteq [0, 1]^2$ satisfying:

$$F \;=\; \bigcup_{i \in \mathcal{I}} S_i(F)$$

which we call the *self-affine set* corresponding to the iterated function system

$\{S_i\}_{i\in\mathcal{I}}$.

We provide a simple example of an IFS which belongs to this class, which we will return to throughout the chapter to illustrate some of the concepts that are introduced.

Example 6.1.6. *Let* $\{T_i\}_{i=1}^2$ *be the IFS with maps* T_1 *and* T_2 *given by*

$$T_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \qquad T_2 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \end{bmatrix} .$$

We will denote the linear parts of T_1 *and* T_2 *by* M_1 *and* M_2 *respectively.*

These self-affine sets were introduced in [Fr1] and were designed as a natural extension of the self-affine carpets introduced by Bedford-McMullen in [Be; Mc] (see Example 2.4.5) and developed by several others, such as [Ba1; GL; FW].

In general, *carpets* refer to planar self-affine sets generated by diagonal matrices. The key difference in the sets we consider here is the presence of anti-diagonal matrices. This causes the system to be irreducible - in the sense of Definition 2.5.18. To see this, observe that since some of the matrices A_i are antidiagonal, there cannot be a one-dimensional linear subspace of $\mathbb{R}^2$ that is preserved by all of the matrices A_i.

This is important, since it means that we can use Corollary 2.5.19 to define equilibrium measures in the sense of (2.18). We will make this precise in Section 6.2.1.

Also, observe that the presence of anti-diagonal matrices causes the system to fail the totally dominated splitting condition. To see this, first we'll show that (6.1) cannot be satisfied for $r = 1$ for any constants C, λ. Consider i where

$$A_i = \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix}$$

and $a_i \neq b_i$ and j where

$$A_j = \begin{bmatrix} 0 & a_j \\ b_j & 0 \end{bmatrix} .$$

In this case we have that for any $n \in \mathbb{N}$

$$A_i^n (A_j) A_i^n = \begin{bmatrix} 0 & a_i^n a_j b_i^n \\ b_i^n b_j a_i^n & 0 \end{bmatrix} .$$

Without loss of generality, we can assume that $a_j > b_j$ so that

$$\frac{\alpha_2(A_i^n(A_j)A_i^n)}{\alpha_1(A_i^n(A_j)A_i^n)} = \frac{b_j}{a_j}.$$

Therefore, clearly no constants C, λ exist for (6.1) to hold.

To see that (6.2) is not possible, let

$$A_i = \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix}$$

where $a_i \neq b_i$ as before, and without loss of generality we can assume that $a_i > b_i$. Then

$$\frac{\alpha_2(A_i^n)}{\alpha_1(A_i^n)} = \left(\frac{b_i}{a_i}\right)^n \to 0$$

and thus no constant C can exist for (6.2) to hold.

6.2 Description of equilibrium states

This section is dedicated to introducing the measures that will be studied throughout the chapter and investigating their measure theoretic and ergodic properties. In 6.2.1 and 6.2.2 we define the equilibrium measures m that we are interested in studying, and show that they are not quasi-Bernoulli. In 6.2.3 we introduce the subshift of finite type Σ_A and a mapping $\tau : \Sigma \to \Sigma_A$ which will play a key role in describing the structure of our measure. In 6.2.4 we introduce Gibbs measures m_1 and m_2 on Σ_A and outline some of their important properties. In 6.2.5, we prove that our equilibrium measure can be written in terms of the Gibbs measures m_1 and m_2, in particular that $m = m_1 \circ \tau + m_2 \circ \tau$. In 6.2.6 we obtain an interesting expression for the Lyapunov exponents of $\mu = m \circ \Pi^{-1}$ as a result of the structure of our measure m, and we prove that the Lyapunov exponents $\chi_1(\mu)$, $\chi_2(\mu)$ are distinct. In 6.2.7 we consider the projections of $m_1 \circ \tau$ and $m_2 \circ \tau$ to the x and y axes, and deduce their dimensional properties.

6.2.1 Definition of m via a variational principle

Recall that for a self-affine system, we can define the *singular value function* ϕ^s : $\Sigma^* \to \mathbb{R}^+$, which was initially introduced by Falconer to study the dimension of the attractor of a self-affine IFS. In this chapter, we are interested in the equilibrium states which emerge from the variational principle for ϕ^s.

For $s \in (0, 2]$, the singular value function has the simple form

$$\phi^s(\mathbf{i}) \;=\; \begin{cases} \alpha_1(\mathbf{i})^s & s \in (0,1) \\ \alpha_1(\mathbf{i})\,\alpha_2(\mathbf{i})^{s-1} & s \in [1,2] \end{cases}$$

Recall that $\phi^s(\mathbf{i})$ is submultiplicative. This allows the subadditive pressure to be defined as in (2.15) by

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\mathbf{i} \in \{1,\ldots,d\}^n} \phi^s(\mathbf{i}) \right).$$

Therefore, by Corollary 2.5.19 it follows that there exists a unique σ-invariant probability measure m^s that satisfies

$$P(s) = h(m^s) + \lim_{n\to\infty} \frac{1}{n} \sum_{\mathbf{i}\in\Sigma^n} m^s([\mathbf{i}]) \log \phi^s(\mathbf{i}).$$

Moreover, by this same result we know that this measure is ergodic and satisfies the Gibbs property, i.e. there exists a universal constant $C_0 \geqslant 1$ such that for all $\mathbf{i} \in \Sigma^*$ we have

$$C_0^{-1} e^{-P(s)|\mathbf{i}|}\, \phi^s([\mathbf{i}]) \;\leqslant\; m^s([\mathbf{i}]) \;\leqslant\; C_0 e^{-P(s)|\mathbf{i}|}\, \phi^s([\mathbf{i}]) \tag{6.3}$$

where $|\mathbf{i}|$ denotes the length of the string. We also know that this is the unique σ-invariant probability measure that is Gibbs for the potential ϕ^s.

Remark 6.2.1. *Observe that we use* 'Gibbs' *in the sense of Section 2.5.3.*

Remark 6.2.2. *Observe that we could use Theorem 2.5.17 instead of Corollary 2.5.19 to determine the existence of m, since the corresponding 'irreducibility' condition is easy to verify. In particular, for each $\mathbf{i}, \mathbf{j} \in \Sigma^*$, we can cleverly choose $k \in \{1, \ldots, d\}$ depending on whether $A_\mathbf{i}$, $A_\mathbf{j}$ are diagonal/ antidiagonal and guarantee that $\phi^s(\mathbf{i}k\mathbf{j}) \geqslant c\phi^s(\mathbf{i})\phi^s(\mathbf{j})$, where c can be taken as the smallest contraction ratio that appears amongst the matrices $\{A_i\}$.*

We fix $s < 2$ and let $\mu = \mu^s$ be the measure on F corresponding to $m = m^s$. We call m (and μ) a Käenmäki measure, following [K] where such measures were first considered in this context.

6.2.2 Proof that m is not quasi-Bernoulli

Recall that in Theorem 6.1.5, Bárány and Käenmäki already considered the dimension of quasi-Bernoulli measures. It is not immediately obvious as to whether or not m^s is quasi-Bernoulli. In this section we show that for each $s \in (0, 2)$, m^s is *not* quasi-Bernoulli (although it *is* submultiplicative). This confirms that we are working with a measure beyond the quasi-Bernoulli setting, which has already been considered in [BK].

Recall that a measure λ is a *quasi-Bernoulli measure* if there exists a universal constant $C \geqslant 1$ such that for all $\mathbf{i}, \mathbf{j} \in \Sigma^*$ we have

$$\frac{1}{C}\,\lambda([\mathbf{i}])\,\lambda([\mathbf{j}]) \;\leqslant\; \lambda([\mathbf{ij}]) \;\leqslant\; C\,\lambda([\mathbf{i}])\,\lambda([\mathbf{j}]). \tag{6.4}$$

In this case, we may also say that $\lambda \circ \Pi^{-1}$ is a quasi-Bernoulli measure.

Fix $s \in (0, 2)$ and set $m = m^s$. Since ϕ^s is submultiplicative, it follows that m is submultiplicative, that is, the right hand side of (6.4) always holds. Therefore, in order to show that m is *not* quasi-Bernoulli, we need to show that it is not *supermultiplicative*. This essentially boils down to the presence of both diagonal and anti-diagonal matrices amongst the $\{A_i\}_{i=1}^d$.

We will show that ϕ^s is not a supermultiplicative potential, and the desired result follows from this fact.

The proof is similar to the proof of irreducibility in Section 6.1.1. Consider i where

$$A_i = \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix}$$

and $a_i \neq b_i$ and j where

$$A_j = \begin{bmatrix} 0 & a_j \\ b_j & 0 \end{bmatrix}.$$

In this case we have that for $n \in \mathbb{N}$

$$A_i^n(A_j)A_i^n = \begin{bmatrix} 0 & a_i^n a_j b_i^n \\ b_i^n b_j a_i^n & 0 \end{bmatrix}.$$

Now, let's assume that $a_i > b_i$ and $a_j > b_j$ and that $0 < s \leqslant 1$. Then

$$\begin{aligned} \phi^s(A_i^n(A_j)A_i^n) &= (a_i^n b_i^n a_j)^s \\ \phi^s(A_i^n A_j) &= (a_i^n a_j)^s \\ \phi^s(A_i^n) &= (a_i^n)^s. \end{aligned}$$

Therefore,

$$\frac{\phi^s(A_i^n(A_j)A_i^n)}{\phi^s(A_i^n A_j)\phi^s(A_i^n)} = \left(\frac{a_i^n b_i^n a_j}{a_i^{2n} a_j}\right)^s = \left(\frac{b_i}{a_i}\right)^{sn} \to 0$$

as $n \to \infty$, and so in this case it is clear that no constant $c > 0$ exists for which $\phi^s(A_i^n(A_j)A_i^n) > c\phi^s(A_i^n A_j)\phi^s(A_i^n)$. The cases corresponding to alternative assumptions on a_i, b_i, a_j, b_j along with the case that $s \in (1,2)$ can be treated similarly, to deduce that for all $0 < s < 2$ we have

$$\lim_{n\to\infty} \frac{\phi^s(A_i^n(A_j)A_i^n)}{\phi^s(A_i^n A_j)\phi^s(A_i^n)} = 0.$$

Therefore, there does not exist $c > 0$ such that

$$\phi^s(A_i^n(A_j)A_i^n) \geqslant c\phi^s(A_i^n A_j)\phi^s(A_i^n)$$

for all $n \in \mathbb{N}$. Thus, ϕ^s is not supermultiplicative, and so if $s < 2$ the Käenmäki measure cannot be supermultiplicative and, in particular, cannot be quasi-Bernoulli.

6.2.3 The subshift Σ_A

We now introduce the subshift of finite type, Σ_A, and two maps $\tau, \omega : \Sigma \to \Sigma_A$ which play a key role in describing the structure of the measure $m = m^s$.

Let Σ_A the sub-shift of finite type on $\{1, \ldots, 2d\}^{\mathbb{N}}$ corresponding to the transition matrix A given by

$$A(i,j) = \begin{cases} 1 & i \in \{1, \ldots, l-1\} \cup \{d+l, \ldots, 2d\} \text{ and } j \leqslant d \\ 1 & i \in \{l, \ldots, d+l-1\} \text{ and } j > d \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the matrix A_* given by

$$A_* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

is a 'collapsed' version of the matrix A, in sense that the first row (respectively column) of A_* corresponds to the rows (respectively columns) of A indexed by $1 \leqslant i \leqslant l-1$, the second to $l \leqslant i \leqslant d$, the third to $d+1 \leqslant i \leqslant d+l-1$ and the last to $d+l \leqslant i \leqslant 2d$. Notice that A_*^2 has all positive entries, and therefore A^2 also has all positive entries.

Next, define $\tau : \Sigma \to \Sigma_A$ by $\tau(\mathbf{i}) = \tau(i_1 i_2 \ldots) = (\tau_1(\mathbf{i})\tau_2(\mathbf{i}) \ldots)$ where

$\tau_1(\mathbf{i}) = i_1$ and

$$\tau_m(\mathbf{i}) = \begin{cases} i_m & \text{if } card\{1 \leqslant j \leqslant m-1 : i_j \geqslant l\} \text{ even} \\ i_m + d & \text{if } card\{1 \leqslant j \leqslant m-1 : i_j \geqslant l\} \text{ odd.} \end{cases}$$

The purpose of this associated subshift of finite type is to precisely record at which times the orientation is preserved. More precisely, $\tau_m(\mathbf{i})$ is in the 'first half' of the double system if and only if the linear part of the map $S_{\mathbf{i}|_{m-1}}$ is a diagonal matrix. Note that τ is not a surjection (but it is an injection) and the image of τ is the subset of Σ_A consisting of sequences where the first digit is at most d.

It will be convenient to introduce $\omega : \Sigma \to \Sigma_A$ which is the projection to the complement of $\tau(\Sigma)$. Let $\omega : \Sigma \to \Sigma_A$ by $\omega(\mathbf{i}) = \omega(i_1 i_2 \dots) = (\omega_1(\mathbf{i})\omega_2(\mathbf{i}) \dots)$ where $\omega_1(\mathbf{i}) = i_1 + d$ and

$$\omega_m(\mathbf{i}) = \begin{cases} i_m + d & \text{if } card\{1 \leqslant j \leqslant m-1 : i_j \geqslant l\} \text{ even} \\ i_m & \text{if } card\{1 \leqslant j \leqslant m-1 : i_j \geqslant l\} \text{ odd.} \end{cases}$$

We then have that $\Sigma_A = \tau(\Sigma) \cup \omega(\Sigma)$ where the union is disjoint.

Example 6.2.3. *We illustrate these objects by returning to Example 6.1.6. In this case, observe that Σ_A is the subshift of finite type on $\{1, 2, 3, 4\}^{\mathbb{N}}$ corresponding to the transition matrix $A = A_*$, where the equality of A and A_* is down to the fact that we only have one diagonal matrix and one antidiagonal matrix in our system.*

Consider the periodic point $\mathbf{i} = (12)^\infty \in \Sigma$. Then $\tau(\mathbf{i}) = (1234)^\infty$ and $\omega(\mathbf{i}) = (3412)^\infty$.

6.2.4 Potentials and Gibbs measures

We are now ready to introduce the Gibbs measures m_1 and m_2 on Σ_A, which play a central part in relating the measure m to Σ_A. We will also introduce a measure ν on Σ, which is defined in terms of m_1 and m_2, and study some of its basic properties. It will turn out in a later section that in fact $\nu = m$.

We begin by defining locally constant potentials $f_{1,s}, f_{2,s} : \Sigma_A \to \mathbb{R}$ by

$$f_{1,s}(\mathbf{i}) = \begin{cases} s \log a_{i_1} & i_1 \leqslant d \\ s \log b_{i_1 - d} & i_1 \geqslant d+1 \end{cases}$$

and

$$f_{2,s}(\mathbf{i}) = \begin{cases} s \log b_{i_1} & i_1 \leqslant d \\ s \log a_{i_1 - d} & i_1 \geqslant d+1 \end{cases}$$

when $s \in (0,1)$ and similarly

$$f_{1,s}(\mathbf{i}) = \begin{cases} \log a_{i_1} + (s-1)\log b_{i_1} & i_1 \leqslant d \\ \log b_{i_1-d} + (s-1)\log a_{i_1-d} & i_1 \geqslant d+1 \end{cases}$$

and

$$f_{2,s}(\mathbf{i}) = \begin{cases} \log b_{i_1} + (s-1)\log a_{i_1} & i_1 \leqslant d \\ \log a_{i_1-d} + (s-1)\log b_{i_1-d} & i_1 \geqslant d+1 \end{cases}$$

when $s \in [1,2]$.

Observe that by symmetry, for all $\mathbf{i} \in \Sigma$

$$f_{1,s} \circ \tau = f_{2,s} \circ \omega \tag{6.5}$$

$$f_{1,s} \circ \omega = f_{2,s} \circ \tau. \tag{6.6}$$

In order to understand the underlying structure of the measure m, it is important to understand the role of the potentials $f_{1,s}$ and $f_{2,s}$. For the time being, we fix $s = 1$ and $\mathbf{i} \in \Sigma$. Observe that for all $n \in \mathbb{N}$, $S_n f_{1,1} \circ \tau(\mathbf{i})$ produces the (logarithm of) the length of the horizontal side of the rectangle $A_{\mathbf{i}|_n}([0,1]^2)$. analogously, $S_n f_{2,1} \circ \tau(\mathbf{i})$ produces the (logarithm of) the length of the vertical side of the rectangle $A_{\mathbf{i}|_n}([0,1]^2)$. By (6.6) we can also swap τ for ω and obtain the same statements, except with each instance of 'horizontal' switched with 'vertical'.

Example 6.2.4. *To give an example, we return to Example 6.1.6. Let* $\mathbf{i} = (12)^\infty \in \Sigma$. *Then*

$$M_{i_1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \quad M_{i_1 i_2} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{9} & 0 \end{bmatrix} \quad M_{i_1 i_2 i_3} = \begin{bmatrix} 0 & \frac{1}{12} \\ \frac{1}{18} & 0 \end{bmatrix}.$$

We saw earlier that $\tau((12)^\infty) = (1234)^\infty$, *and therefore it is easy to see that*

$$S_1 f_{1,1} \circ \tau(\mathbf{i}) = \log \frac{1}{2}$$

$$S_2 f_{1,1} \circ \tau(\mathbf{i}) = \log \frac{1}{2} + \log \frac{1}{2} = \log \frac{1}{4}$$

$$S_2 f_{1,1} \circ \tau(\mathbf{i}) = \log \frac{1}{2} + \log \frac{1}{2} + \log \frac{1}{3} = \log \frac{1}{12}$$

and

$$\begin{aligned} S_1 f_{2,1} \circ \tau(\mathbf{i}) &= \log \frac{1}{3} \\ S_2 f_{2,1} \circ \tau(\mathbf{i}) &= \log \frac{1}{3} + \log \frac{1}{3} = \log \frac{1}{9} \\ S_2 f_{2,1} \circ \tau(\mathbf{i}) &= \log \frac{1}{3} + \log \frac{1}{3} + \log \frac{1}{2} = \log \frac{1}{18} \end{aligned}$$

as expected.

For general s, $S_n f_{1,s} \circ \tau(\mathbf{i})$ and $S_n f_{2,s} \circ \tau(\mathbf{i})$ produce the analogue of this. In particular, $S_n f_{1,s} \circ \tau(\mathbf{i})$ produces the logarithm of the length of the horizontal side of the rectangle $A_{\mathbf{i}|_n}([0,1]^2)$ plus $s-1$ times the logarithm of the length of the vertical side of the rectangle $A_{\mathbf{i}|_n}([0,1]^2)$. $S_n f_{2,s} \circ \tau(\mathbf{i})$ produces the analogue of this, with all instances of 'horizontal' and 'vertical' switched.

It should now be easy to see that $\log \phi^s(\mathbf{i}|_n)$ will be the maximum of these.

Example 6.2.5. *Returning to Example 6.2.4 and* $\mathbf{i} = (12)^\infty \in \Sigma$*, we see that* $\phi^1(\mathbf{i}|_3) = \frac{1}{12} = \max\{\frac{1}{12}, \frac{1}{18}\} = \max\{\exp(S_3 f_{1,1} \tau(\mathbf{i})), \exp(S_3 f_{2,1} \tau(\mathbf{i}))\}$.

We state this important observation as a lemma.

Lemma 6.2.6. *For all* $\mathbf{i} \in \Sigma$ *and* $n \in \mathbb{N}$ *we have that*

$$\phi^s(\mathbf{i}|_n) = \max\{\exp S_n f_{1,s}(\tau(\mathbf{i})), \exp S_n f_{2,s}(\tau(\mathbf{i}))\}.$$

Proof. This follows immediately from the definitions. □

Next we move on to defining the associated Gibbs measures for these potentials. Since A is aperiodic (since all the entries in A^2 are positive), by Theorem 2.5.3 there exist unique invariant Gibbs probability measures m_1 and m_2 for the potentials $f_{1,s}$ and $f_{2,s}$. Therefore, denoting the topological pressure on Σ_A of $f_{1,s}$ and $f_{2,s}$ by $P_A(f_{1,s})$ and $P_A(f_{2,s})$ respectively,

$$m_1([\mathbf{i}|_n]) \asymp \exp(S_n f_{1,s}(\mathbf{i}) - nP_A(f_{1,s}))$$

and

$$m_2([\mathbf{i}|_n]) \asymp \exp(S_n f_{2,s}(\mathbf{i}) - nP_A(f_{2,s}))$$

where $a \asymp b$ means that $C^{-1}a \leqslant b \leqslant Ca$ for some universal constant C.

Observe that by symmetry, $P_A(f_{1,s}) = P_A(f_{2,s})$. To see this, consider the map $b : \Sigma_A \to \Sigma_A$ which sends

$$\begin{array}{lll} i_k \to i_k + d & \text{if} & i_k \in \{1, \dots, d\} \\ i_k \to i_k - d & \text{if} & i_k \in \{d+1, \dots, 2d\}. \end{array}$$

Using the transition matrix A, it is easy to check that this map is well defined and bijective. Therefore given a periodic point $\mathbf{i} = \sigma^n \mathbf{i} \in \Sigma_A$, $b(\mathbf{i})$ is also a periodic point of period n, and since both $S_n f_{1,s}(\mathbf{i}) = S_n f_{2,s}(b(\mathbf{i}))$ and $S_n f_{1,s}(b(\mathbf{i})) = S_n f_{2,s}(\mathbf{i})$, it follows that $P_A(f_{1,s}) = P_A(f_{2,s})$.

By symmetry, we also have

$$m_1 \circ \tau = m_2 \circ \omega \tag{6.7}$$

$$m_1 \circ \omega = m_2 \circ \tau. \tag{6.8}$$

To see this, consider again the bijection $b : \Sigma_A \to \Sigma_A$. Clearly b has the property that

$$b(\tau(\mathbf{i})) = \omega(\mathbf{i}) \quad \text{and} \quad b(\omega(\mathbf{i})) = \tau(\mathbf{i}). \tag{6.9}$$

It is easy to see that $m_1 \circ b$ is an invariant probability Gibbs measure for $f_{2,s}$ and so by uniqueness, $m_1 \circ b = m_2$. Thus (6.8) follows by (6.9).

Next, we move on to defining a measure ν on Σ in terms of the measures m_1 and m_2. Since τ is an injection, we can define a measure ν on Σ by $\nu(E) = m_1(\tau(E)) + m_2(\tau(E))$. ν is a probability measure since $\nu(\Sigma) = m_1(\tau(\Sigma)) + m_2(\tau(\Sigma)) = m_1(\tau(\Sigma)) + m_1(\omega(\Sigma)) = m_1(\Sigma_A) = 1$.

Also, although $m_1 \circ \tau$ and $m_2 \circ \tau$ are not invariant, ν *is* invariant, which we will prove next.

Lemma 6.2.7. *$\nu = m_1 \circ \tau + m_2 \circ \tau$ is shift invariant.*

Proof. By the Carathéodory extension theorem, it is enough to check that $\nu(\sigma^{-1}([\mathbf{i}|_n])) = \nu([\mathbf{i}|_n])$ for $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$. Let $\mathbf{i}|_n = i_1, \dots, i_n$ and fix $i_0 \in \mathcal{I}$. Then $\tau([i_0, \dots, i_n]) = [i_0, \tau_1(\mathbf{i}), \dots, \tau_n(\mathbf{i})]$ if $1 \leqslant i_0 \leqslant l - 1$ and $\omega([i_0, \dots, i_n]) = [i_0 + d, \tau_1(\mathbf{i}), \dots, \tau_n(\mathbf{i})]$ if $l \leqslant i_0 \leqslant d$.

Also, note that for $l \leqslant i_0 \leqslant d$,

$$m_1(\tau([i_0, \dots, i_n])) = m_2(\omega([i_0, \dots, i_n])) = m_2([i_0 + d, \tau_1(\mathbf{i}), \dots, \tau_n(\mathbf{i})])$$

and similarly

$$m_2(\tau([i_0,\dots,i_n])) = m_1(\omega([i_0,\dots,i_n])) = m_1([i_0+d,\tau_1(\mathbf{i}),\dots,\tau_n(\mathbf{i})]).$$

Therefore,

$$\begin{aligned} m_1(\tau(\sigma^{-1}([i_1\dots i_n]))) &= \sum_{i_0=1}^{l-1} m_1(\tau([i_0,\dots,i_n])) + \sum_{i_0=l}^{d} m_1(\tau([i_0,\dots,i_n])) \\ &= \sum_{i_0=1}^{l-1} m_1([i_0,\tau_1(\mathbf{i})\dots,\tau_n(\mathbf{i})]) + \sum_{i_0=l}^{d} m_2([i_0+d,\tau_1(\mathbf{i}),\dots,\tau_n(\mathbf{i})]) \end{aligned}$$

and similarly

$$\begin{aligned} m_2(\tau(\sigma^{-1}([i_1\dots i_n]))) &= \sum_{i_0=1}^{l-1} m_2(\tau([i_0,\dots,i_n])) + \sum_{i_0=l}^{d} m_2(\tau([i_0,\dots,i_n])) \\ &= \sum_{i_0=1}^{l-1} m_2([i_0,\tau_1(\mathbf{i})\dots,\tau_n(\mathbf{i})]) + \sum_{i_0=l}^{d} m_1([i_0+d,\tau_1(\mathbf{i}),\dots,\tau_n(\mathbf{i})]) \end{aligned}$$

so that

$$\begin{aligned} \nu(\sigma^{-1}([\mathbf{i}|_n])) &= m_1(\tau(\sigma^{-1}([i_1\dots i_n]))) + m_2(\tau(\sigma^{-1}([i_1\dots i_n]))) \\ &= \sum_{1\leqslant i_0\leqslant l-1} m_1([i_0,\tau_1(\mathbf{i})\dots,\tau_n(\mathbf{i})]) + \sum_{d+l\leqslant i_0\leqslant 2d} m_1([i_0,\tau_1(\mathbf{i})\dots,\tau_n(\mathbf{i})]) \\ &\quad + \sum_{1\leqslant i_0\leqslant l-1} m_2([i_0,\tau_1(\mathbf{i})\dots,\tau_n(\mathbf{i})]) + \sum_{d+l\leqslant i_0\leqslant 2d} m_2([i_0,\tau_1(\mathbf{i})\dots,\tau_n(\mathbf{i})]) \\ &= m_1(\sigma^{-1}\tau([\mathbf{i}|_n])) + m_2(\sigma^{-1}\tau([\mathbf{i}|_n])) = \nu([\mathbf{i}|_n]) \end{aligned}$$

where the last line follows by invariance of m_1 and m_2. □

6.2.5 Proof that $m = \nu$

In this section we prove that the measure ν that we have just constructed is in fact equal to the Käenmäki measure m. Since we have already shown that it is an invariant probability measure, it only remains to prove the Gibbs property for ν.

The Gibbs property essentially falls out as a direct consequence of the observation in Lemma 6.2.6, and thus we prove that $m = \nu$ in the next result. Additionally, we also show that $P(s) = P_A(f_{1,s}) = P_A(f_{2,s})$ all coincide.

Corollary 6.2.8. *There exists $C > 0$ such that for all $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$*

$$C^{-1} \leqslant \frac{\nu([\mathbf{i}|_n])}{\phi^s(\mathbf{i}|_n)\exp(-nP_A(f_{1,s}))} \leqslant C.$$

In particular, $\nu = m$ is the unique Käenmäki measure for ϕ^s and $P(s) = P_A(f_{1,s}) = P_A(f_{2,s})$.

Proof. It follows from the Gibbs properties for m_1 and m_2 and the fact that $P_A(f_{1,s}) = P_A(f_{2,s})$ that

$$\begin{aligned}
\nu([\mathbf{i}|_n]) &= m_1(\tau([\mathbf{i}|_n])) + m_2(\tau([\mathbf{i}|_n])) \\
&\asymp \exp(-nP_A(f_{1,s}))\exp(S_n f_{1,s}(\tau(\mathbf{i}))) + \exp(-nP_A(f_{2,s}))\exp(S_n f_{2,s}(\tau(\mathbf{i}))) \\
&\asymp \exp(-nP_A(f_{1,s}))\max\{\exp(S_n f_{1,s}(\tau(\mathbf{i}))), \exp(S_n f_{2,s}(\tau(\mathbf{i})))\}
\end{aligned}$$

and the first result now follows using Lemma 6.2.6. Finally, by combining this and the Gibbs property for the Käenmäki measure m we get

$$\frac{\nu([\mathbf{i}|_n])}{m([\mathbf{i}|_n])} \asymp \exp(n(P(s) - P_A(f_{1,s}))).$$

To see that $P_A(f_{1,s}) = P(s)$, assume that $P(s) > P_A(f_{1,s})$ instead. Then

$$f_n(\mathbf{i}) := \frac{\nu([\mathbf{i}|_n])}{m([\mathbf{i}|_n])} \to \infty$$

for all $\mathbf{i} \in \Sigma$. Thus $f_n \to \infty$ in m-measure, and therefore there exists a set A, with $m(A) \geqslant \frac{2}{3}$ with the property that $f_n(\mathbf{i}) \geqslant 2$ for all $\mathbf{i} \in A$. By construction, A must be a collection of cylinders of length n, thus

$$\nu(A) \geqslant 2m(A) \geqslant \frac{4}{3}$$

which contradicts the fact that ν is a probability measure. So $P(s) \leqslant P_A(f_{1,s})$. A symmetric argument implies that $P_A(f_{1,s}) \leqslant P(s)$, so the result follows.

Finally since ν and m are both invariant, it follows by the uniqueness of property (6.3) that $\nu = m$. □

6.2.6 Lyapunov exponents of μ

Recall that whenever the Lyapunov exponents $\chi_1(\mu) = \chi_2(\mu)$, we are in the setting of Theorem 6.1.1 where it is already known that μ is exact dimensional. However, in this section we show that our Lyapunov exponents are distinct. Furthermore, by using the fact that $m = \nu = m_1 \circ \tau + m_2 \circ \tau$, we show that in our setting we can express each Lyapunov exponent as the integral of one of the potentials $f_{1,1}, f_{2,1}$ with respect to m_1 (and similarly for m_2).

Remark 6.2.9. *Note that in the following two results we emphasise the dependence of $f_{1,s}$ and $f_{2,s}$ on s, so we can use $f_{1,1}$ and $f_{2,1}$ to express the Lyapunov exponents, but for simplicity of exposition we deliberately suppress this dependence when writing the measures m_1, m_2, ν and of course m and μ.*

In the following preparatory lemma we study the integrals of $f_{1,1}$ and $f_{1,2}$ with respect to the measures m_1 and m_2.

Lemma 6.2.10. *We have that*

$$\int f_{1,1} \mathrm{d}m_1 = \int f_{2,1} \ \mathrm{d}m_2 \geqslant \int f_{2,1} \mathrm{d}m_1 = \int f_{1,1} \mathrm{d}m_2.$$

Proof. The fact that $\int f_{1,1} \mathrm{d}m_1 = \int f_{2,1} \mathrm{d}m_2$ and $\int f_{2,1} \mathrm{d}m_1 = \int f_{1,1} \mathrm{d}m_2$ follows by the properties (6.6), (6.8). In particular,

$$\begin{aligned}
\int f_{1,1} dm_1 &= \int f_{1,1} \circ \tau dm_1 \circ \tau + \int f_{1,1} \circ \omega dm_1 \circ \omega \\
&= \int f_{2,1} \circ \omega dm_2 \circ \omega + \int f_{2,1} \circ \tau dm_2 \circ \tau = \int f_{2,1} dm_2
\end{aligned}$$

and similarly we also see that $\int f_{2,1} dm_1 = \int f_{1,1} dm_2$.

For m_1 almost all $\mathbf{i} \in \Sigma_A$ we have that

$$\lim_{n\to\infty} \frac{S_n f_{1,1}(\mathbf{i})}{n} = \int f_{1,1} \mathrm{d}m_1 = \int f_{2,1} \mathrm{d}m_2$$

and

$$\lim_{n\to\infty} \frac{S_n f_{2,1}(\mathbf{i})}{n} = \int f_{2,1} \mathrm{d}m_1 = \int f_{1,1} \mathrm{d}m_2.$$

To show that $\int f_{1,1} dm_1 \geqslant \int f_{2,1} dm_1$, suppose for a contradiction that $\int f_{2,1} dm_1 > \int f_{1,1} dm_1$ instead. Then we also have that $\int f_{2,s} dm_1 > \int f_{1,s} dm_1$. To see this, observe that if $0 < s \leqslant 1$ then it just follows from the fact that $f_{1,s} = s f_{1,1}$ and $f_{2,s} = s f_{2,1}$. If $1 < s < 2$ then $f_{1,s} = f_{1,1} + (s-1) f_{2,1}$ and $f_{2,s} = f_{2,1} + (s-1) f_{1,1}$, and so

$$\int f_{2,s} dm_1 - \int f_{1,s} dm_1 = (2-s) \int f_{2,1} dm_1 - (2-s) \int f_{1,1} > 0$$

since $s < 2$.

This means that by using the Gibbs property, the fact that $P_A(f_{1,s}) = P_A(f_{2,s})$ and the fact that

$$\lim_{n\to\infty} \frac{S_n f_{1,s}(\mathbf{i})}{n} = \int f_{1,s} \mathrm{d}m_1$$

and

$$\lim_{n\to\infty} \frac{S_n f_{2,s}(\mathbf{i})}{n} = \int f_{2,s} \mathrm{d}m_1$$

for m_1 almost all $\mathbf{i}$, it follows that

$$\lim_{n\to\infty} \frac{m_2([i_1, \ldots, i_n])}{m_1([i_1, \ldots, i_n])} = \lim_{n\to\infty} \exp\left(n\left(\int f_{2,s} dm_1 - \int f_{1,s} dm_1\right)\right) = \infty$$

for m_1 almost all $\mathbf{i}$. Since m_1 and m_2 are probability measures, we can use a similar argument to the one in the proof of Corollary 6.2.8 to obtain a contradiction. $\square$

Using the above result, we relate the Lyapunov exponents of μ to integrals of $f_{1,1}$ and $f_{2,1}$ with respect to the measures m_1 and m_2. Furthermore, we prove that $\chi_1(\mu) \neq \chi_2(\mu)$ which takes us away from the setting of Theorem 6.1.1.

Corollary 6.2.11. *We have that*

$$\chi_1(\mu) = -\int f_{1,1} \mathrm{d}m_1 < \chi_2(\mu) = -\int f_{2,1} \mathrm{d}m_1$$

and for ν-almost all $\mathbf{i} \in \Sigma$ we have that

$$\lim_{n\to\infty} \left(\frac{\alpha_2(\mathbf{i}|_n)}{\alpha_1(\mathbf{i}|_n)}\right) = 0.$$

Proof. We know that all $\mathbf{i} \in \Sigma$ satisfy

$$\alpha_1(\mathbf{i}|_n) = \max\{\exp(S_n f_{1,1}(\tau(\mathbf{i}))), \exp(S_n f_{2,1}(\tau(\mathbf{i})))\}$$

and

$$\alpha_2(\mathbf{i}|_n) = \min\{\exp(S_n f_{1,1}(\tau(\mathbf{i}))), \exp(S_n f_{2,1}(\tau(\mathbf{i})))\}.$$

Since we know by Lemma 6.2.10 that $\int f_{1,1} \mathrm{d}m_1 \geqslant \int f_{2,1} \mathrm{d}m_1$, and since $m_1 \circ \tau \ll m$, it follows that for $m_1 \circ \tau$ almost all $\mathbf{i} \in \Sigma$,

$$-\lim_{n\to\infty} \frac{1}{n} \log \alpha_1(\mathbf{i}|_n) \;=\; -\int f_{1,1} \mathrm{d}m_1$$

and

$$-\lim_{n\to\infty}\frac{1}{n}\log\alpha_2(\mathbf{i}|_n) \;=\; -\int f_{2,1}\mathrm{d}m_1.$$

Similarly it can be shown that for $m_2\circ\tau$ almost all $\mathbf{i}\in\Sigma$,

$$-\lim_{n\to\infty}\frac{1}{n}\log\alpha_1(\mathbf{i}|_n) \;=\; -\int f_{2,1}\mathrm{d}m_2$$

and

$$-\lim_{n\to\infty}\frac{1}{n}\log\alpha_2(\mathbf{i}|_n) \;=\; -\int f_{1,1}\mathrm{d}m_2.$$

Moreover, since $m=m_1\circ\tau+m_2\circ\tau$ and $-\int f_{1,1}\mathrm{d}m_1=-\int f_{2,1}\mathrm{d}m_2$ and $-\int f_{2,1}\mathrm{d}m_1=-\int f_{1,1}\mathrm{d}m_2$, then it follows that

$$\chi_1(\mu)=-\int f_{1,1}\mathrm{d}m_1=-\int f_{2,1}\mathrm{d}m_2 \quad\text{and}\quad \chi_2(\mu)=-\int f_{2,1}\mathrm{d}m_1=-\int f_{1,1}\mathrm{d}m_2.$$

To see why $\chi_1(\mu)<\chi_2(\mu)$, suppose that we have equality instead. Then it follows immediately that $\lim_{n\to\infty}\left(\frac{\alpha_1(\mathbf{i}|_n)}{\alpha_2(\mathbf{i}|_n)}\right)^{\frac{1}{n}}=1$. Therefore, there exist constants C', P' such that

$$\frac{1}{C'}e^{-nP'}\leqslant\frac{\alpha_1(\mathbf{i}|_n)}{\alpha_2(\mathbf{i}|_n)}\leqslant C'e^{nP'}.$$

Therefore, for all $\mathbf{i}\in\Sigma$ and $n\in\mathbb{N}$,

$$\begin{aligned} m(\mathbf{i}|_n) &\asymp \phi^s(\mathbf{i}|_n)e^{-nP(s)}\\ &\asymp \alpha_1(\mathbf{i}|_n)^{\frac{s}{2}}\alpha_2(\mathbf{i}|_n)^{\frac{s}{2}}e^{nP''} \end{aligned}$$

for some uniform constant P''. Thus, m is also an equilibrium state for the additive potential $\mathbf{i}\mapsto\frac{s}{2}\log\alpha_1(\mathbf{i})\alpha_2(\mathbf{i})$. This means that m is quasi-Bernoulli and we have already observed that in our setting this is not the case, so we obtain the desired contradiction.

Finally, $\chi_1(\mu)<\chi_2(\mu)$ implies that for m-almost every $\mathbf{i}\in\Sigma$,

$$\lim_{n\to\infty}\log\left(\frac{\alpha_2(\mathbf{i}|_n)}{\alpha_1(\mathbf{i}|_n)}\right)^{\frac{1}{n}}<0.$$

Thus, there exists $0<a<1$, $N\in\mathbb{N}$ such that for $n\geqslant N$,

$$\left(\frac{\alpha_2(\mathbf{i}|_n)}{\alpha_1(\mathbf{i}|_n)}\right)^{\frac{1}{n}}<a.$$

In particular, for all $n\geqslant N$, $\frac{\alpha_2(\mathbf{i}|_n)}{\alpha_1(\mathbf{i}|_n)}\leqslant a^n$, and the conclusion follows. $\square$

6.2.7 Projections of $m_1 \circ \tau$ and $m_2 \circ \tau$

The Ledrappier-Young formula gives the dimension of a measure in terms of the dimension of a projected measure (as well as the entropy and Lyapunov exponents). Therefore, since the identity $m = m_1 \circ \tau + m_2 \circ \tau$ will be central to our verification of exact-dimensionality, we need to consider the projections of $m_1 \circ \tau$ and $m_2 \circ \tau$ to the x and y axes.

The final piece of notation we introduce is $\mu_t = m_t \circ \tau \circ \Pi^{-1}$, the pushforward measure of $m_t \circ \tau$ onto F. Let π_1 denote the projection to the x co-ordinate and π_2 denote the projection to the y co-ordinate. Then we know about the exact dimensionality of the projections of μ_1 and μ_2 under m_1 and m_2.

Proposition 6.2.12. *All the measures $\pi_1(\mu_1)$, $\pi_1(\mu_2)$, $\pi_2(\mu_1)$ and $\pi_2(\mu_2)$ are exact dimensional and we have that $\dim \pi_1(\mu_1) = \dim \pi_2(\mu_2)$ and $\dim \pi_2(\mu_1) = \dim \pi_1(\mu_2)$.*

A proof of this is given in Appendix C.

6.3 Results

We now obtain our results by using the structure of the Käenmäki measure described in the previous section. For convenience, we assume that the underlying IFS satisfies the *strong separation property*, which means that for distinct $i, j \in \mathcal{I}$, we have $S_i(F) \cap S_j(F) = \emptyset$. Observe that this implies that there exists some $\delta > 0$ such that for any $i, j \in \{1, \ldots, d\}$ with $i \neq j$,

$$d(x, y) \geqslant \delta \tag{6.10}$$

for all $x \in S_i(F)$ and $y \in S_j(F)$, where d is the usual Euclidean metric.

Recall the definition of the entropy $h(\mu)$ of μ defined in Theorem 2.4.8. The following is our main result.

Theorem 6.3.1. *Assume the self-affine set F satisfies the strong separation property and let μ be any Käenmäki measure for F. Then μ is exact dimensional, with the exact dimension given by*

$$\dim \mu \;=\; \frac{h(\mu)}{\chi_2(\mu)} + \frac{\chi_2(\mu) - \chi_1(\mu)}{\chi_2(\mu)} \dim \pi_1(\mu_1).$$

Thus μ satisfies the appropriate version of the Ledrappier-Young formula.

We get the following corollary, which gives simpler formulae in the case where $\dim \pi_1(\mu_1)$ is what it is 'expected to be'.

Corollary 6.3.2. *If* $\dim \pi_1(\mu_1) = \min\left\{\frac{h(\mu)}{\chi_1(\mu)}, 1\right\}$ *then*

$$\dim \mu = \begin{cases} \frac{h(\mu)}{\chi_1(\mu)} & \text{if} \quad h(\mu) \leqslant \chi_1(\mu) \\ 1 + \frac{h(\mu)-\chi_1(\mu)}{\chi_2(\mu)} & \text{if} \quad h(\mu) > \chi_1(\mu). \end{cases}$$

Proof. We first suppose that $\dim \pi_1(\mu_1) = \frac{h(\mu)}{\chi_1(\mu)} \leqslant 1$. In this case we have

$$\begin{aligned} \dim \mu &= \frac{h(\mu)}{\chi_2(\mu)} + \frac{\chi_2(\mu) - \chi_1(\mu)}{\chi_2(\mu)} \dim \pi_1(\mu_1) \\ &= \frac{h(\mu)}{\chi_2(\mu)} + \frac{\chi_2(\mu) - \chi_1(\mu)}{\chi_2(\mu)} \frac{h(\mu)}{\chi_1(\mu)} \\ &= \frac{h(\mu)}{\chi_1(\mu)}. \end{aligned}$$

On the other hand if $\dim \pi_1(\mu_1) = 1 < \frac{h(\mu)}{\chi_1(\mu)}$ then

$$\begin{aligned} \dim \mu &= \frac{h(\mu)}{\chi_2(\mu)} + \frac{\chi_2(\mu) - \chi_1(\mu)}{\chi_2(\mu)} \dim \pi_1(\mu_1) \\ &= \frac{h(\mu)}{\chi_2(\mu)} + \frac{\chi_2(\mu) - \chi_1(\mu)}{\chi_2(\mu)} \\ &= 1 + \frac{h(\mu) - \chi_1(\mu)}{\chi_2(\mu)} \end{aligned}$$

completing the proof. □

Recall that the affinity dimension $\dim_{\mathrm{A}} F$ of F is defined as the value s_0 for which

$$P(s_0) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{\mathbf{i}\in\{1,\ldots,d\}^n} \phi^{s_0}(\mathbf{i})\right) = 0.$$

In [MS], Morris and Shmerkin showed that for a large class of attractors of the same self-affine systems that we have been considering in this chapter, the Hausdorff dimension of the attractor is given by the affinity dimension. In particular, they proved the following result.

Theorem 6.3.3. *Assume the set-up of Theorem 6.3.1. Additionally assume that:*

1. *All coefficients of* A_i *and* t_i *are algebraic.*

2. *For all* $n \in \mathbb{N}$ *and any* $\mathbf{i}, \mathbf{j} \in \{1, \ldots, d\}^n$ *for which either* $A_{\mathbf{i}}$ *and* $A_{\mathbf{j}}$ *are both diagonal or both antidiagonal, then* $\pi_1(S_{\mathbf{i}}(0)) \neq \pi_1(S_{\mathbf{j}}(0))$ *and* $\pi_2(S_{\mathbf{i}}(0)) \neq \pi_2(S_{\mathbf{j}}(0))$.

Then $\dim_{\mathrm{H}} F = s_0 = \dim_{\mathrm{A}} F$.

Moreover, our class of self-affine sets was recently considered by Morris in [Mo1], who derived an explicit formula for the pressure allowing very straightforward calculation of the affinity dimension. In particular, he showed that the affinity dimension is the unique real number $s > 0$ for which either:

1. $0 < s \leqslant 1$ and
$$\begin{bmatrix} \sum_{i=1}^{l-1} a_i^s & \sum_{i=l}^{d} a_i^s \\ \sum_{i=l}^{d} b_i^s & \sum_{i=1}^{l-1} b_i^s \end{bmatrix}$$
has spectral radius 1, or

2. $1 \leqslant s \leqslant 2$ and
$$\begin{bmatrix} \sum_{i=1}^{l-1} a_i b_i^{s-1} & \sum_{i=l}^{d} a_i b_i^{s-1} \\ \sum_{i=l}^{d} a_i^{s-1} b_i & \sum_{i=1}^{l-1} a_i^{s-1} b_i \end{bmatrix}$$
has spectral radius 1, or

3. $s \geqslant 2$ and
$$\sum_{i=1}^{d} (a_i b_i)^{\frac{s}{2}} = 1.$$

In light of this and Theorem 6.3.3, we see that a consequence of our main theorem is that under a condition on the dimension of the projected measure, such systems have an ergodic measure of maximal dimension.

To see this, suppose that $s_0 \leqslant 1$ and observe that by (2.17) and the fact that $\mu = \mu^{s_0}$ is ergodic,
$$P(s_0) = h(\mu) - s_0 \chi_1(\mu).$$
Rearranging this gives
$$s_0 = \frac{h(\mu)}{\chi_1(\mu)}$$
because $P(s_0) = 0$. Since $h(\mu) = s_0 \chi_1(\mu) \leqslant \chi_1(\mu)$ it follows that
$$\dim \mu = \frac{h(\mu)}{\chi_1(\mu)} = s_0.$$
The case where $s_0 > 1$ follows similarly.

6.4 Proof of Theorem 6.3.1

In this section we prove Theorem 6.3.1 by adapting an approach of Przytycki and Urbański from [PU1]. In [PU1], Przytycki and Urbański relate the dimension of a self-affine measure in two dimensions to the dimension of a self-similar measure in one dimension (in their case a Bernoulli convolution). In our case, we'll need to consider *two* measures in one dimension, in particular these will be $\pi_1(\mu_1)$ and $\pi_2(\mu_2)$. Rather than being strictly self-similar, these are Gibbs measures on a graph directed self-similar iterated function system. This approach is also similar to one used by Falconer and Kempton in [FK2].

At a couple of points in this section we will use Proposition 2.3.2 (2) to deduce that one measure 'dominates' another. In particular, notice that a straightforward consequence of this proposition is that whenever a measure μ is not absolutely continuous with respect to λ (where μ, λ are Radon measures on $\mathbb{R}^d$) then

$$\limsup_{r\to 0} \frac{\lambda(B(x,r))}{\mu(B(x,r))} = 0$$

for μ-almost every $x \in \mathbb{R}^d$, so that $\mu(B(x,r))$ 'dominates' $\lambda(B(x,r))$ at all small scales r.

The following lemma will allow us to also deduce a symbolic analogue of the above result.

Lemma 6.4.1. *For m-almost all $\mathbf{i} \in \Sigma$, there exists a constant $0 < C < \infty$ (independent of r) such that for all $r > 0$ there exists $n \in \mathbb{N}$ for which*

$$\frac{\mu_1(B(\Pi(\mathbf{i}),r))}{\mu_2(B(\Pi(\mathbf{i}),r))} \geqslant C\frac{m_1 \circ \tau([\mathbf{i}|_n])}{m_2 \circ \tau([\mathbf{i}|_n])}$$

and

$$\frac{\mu_2(B(\Pi(\mathbf{i}),r))}{\mu_1(B(\Pi(\mathbf{i}),r))} \geqslant C\frac{m_2 \circ \tau([\mathbf{i}|_n])}{m_1 \circ \tau([\mathbf{i}|_n])}.$$

Proof. Throughout this proof we denote $\delta > 0$ to be the constant given by (6.10). Let $\mathbf{i}$ belong to the full m-measure set for which

$$\lim_{n\to\infty} \frac{\alpha_2(\mathbf{i}|_n)}{\alpha_1(\mathbf{i}|_n)} = 0.$$

Let $x = \Pi(\mathbf{i})$ and fix some $r > 0$. Choose integers $m < n$ for which

$$\alpha_1(\mathbf{i}|_n) < \frac{r}{2} < \alpha_1(\mathbf{i}|_{n-1})$$

and

$$\delta\alpha_2(\mathbf{i}|_m) < r < \delta\alpha_2(\mathbf{i}|_{m-1}).$$

Then by construction, $\Pi([\mathbf{i}|_n]) \subset B(x,r) \subset \Pi([\mathbf{i}|_m])$. By the Gibbs property for m_1 and m_2,

$$\begin{aligned}
&m_1 \circ \tau([\mathbf{i}|_n]) \asymp \exp(-nP_A(f_{1,s}))\exp(S_n f_{1,s}(\tau(\mathbf{i})))\\
\asymp &\exp(-mP_A(f_{1,s}))\exp(S_m f_{1,s}(\tau(\mathbf{i}))) \cdot \exp(-(n-m)P_A(f_{1,s}))\exp(S_{n-m} f_{1,s}(\sigma^m \tau(\mathbf{i})))\\
\asymp &m_1 \circ \tau([\mathbf{i}|_m])\exp((n-m)c)
\end{aligned} \tag{6.11}$$

for some uniform constant c (which comes from the value of $P_A(f_{1,s})$ and the supremum norm of $f_{1,s}$).

We will show that the difference $n-m$ is bounded above by a constant that depends only on $\mathbf{i}$ (so in particular it is independent of r).

Let λ denote the maximum contraction ratio amongst the matrices $\{A_i\}$ that appear in the IFS. Then

$$\frac{r}{2} < \alpha_1(\mathbf{i}|_{n-1}) \leqslant \lambda^{n-m-1}\alpha_1(\mathbf{i}|_m) = \lambda^{n-m-1}\alpha_2(\mathbf{i}|_m)\frac{\alpha_1(\mathbf{i}|_m)}{\alpha_2(\mathbf{i}|_m)} \leqslant \lambda^{n-m-1}\frac{r}{\delta}\frac{\alpha_1(\mathbf{i}|_m)}{\alpha_2(\mathbf{i}|_m)}.$$

Therefore,

$$n-m \leqslant 1 + \frac{1}{\log\lambda}\log\left(\frac{\delta}{2}\cdot\frac{\alpha_2(\mathbf{i}|_m)}{\alpha_1(\mathbf{i}|_m)}\right).$$

Since

$$\lim_{k\to\infty}\frac{\alpha_2(\mathbf{i}|_k)}{\alpha_1(\mathbf{i}|_k)} = 0$$

it follows that $\sup_{k\in\mathbb{N}}\frac{\alpha_2(\mathbf{i}|_k)}{\alpha_1(\mathbf{i}|_k)} \leqslant c'$ for some constant c' that depends only on $\mathbf{i}$, and therefore $n-m$ is also bounded above by a constant that depends only on $\mathbf{i}$ (and so is independent of r). It follows that

$$\frac{\mu_1(B(x,r))}{\mu_2(B(x,r))} \geqslant \frac{m_1\circ\tau([\mathbf{i}|_n])}{m_2\circ\tau([\mathbf{i}|_m])} \geqslant C\frac{m_1\circ\tau([\mathbf{i}|_n])}{m_2\circ\tau([\mathbf{i}|_n])}$$

for some constant C that depends only on $\mathbf{i}$, where the final inequality follows by (6.11). The second result follows by a symmetric argument. □

This final section is structured as follows. In 6.4.1 we begin by collecting some 'ergodic properties' of the measures $m_1 \circ \tau$ and $m_2 \circ \tau$. Although these measures are not ergodic (since they are not even invariant), we can still establish equivalent statements about the behaviour of limits that are usually provided by the Shannon-McMillan-Breiman Theorem 2.4.8 and Oseledets Theorem 2.4.7 when one is in the ergodic setting.

Next, in 6.4.2 we make upper and lower estimates on the measure of a carefully defined 'strip' of F, in terms of the measure of an appropriate cylinder and the projected measure of the blow up of the 'strip'. This step will allow us to later connect the local dimension of μ with the entropy, Lyapunov exponents and dimension of the projected measure. The strategies used for the upper and lower estimates will be different, owing to the fact that m is submultiplicative but not supermultiplicative.

Next, in 6.4.3 we estimate the projected measure of the blow up of the strip at 'typical' times.

Finally, in 6.4.4 and 6.4.5 we combine the above estimates to compute the local dimension of μ.

6.4.1 'Ergodic properties' of $m_t \circ \tau$

The measures $m_t \circ \tau$ are not invariant and therefore cannot be ergodic. Therefore, the Lyapunov exponents and entropy are a priori not defined. This is problematic, since in order to evaluate the local dimension of μ in terms of the entropy and Lyapunov exponents, we need to know about the $m_t \circ \tau$-typical logarithmic growth rate of the measure and side lengths of the cylinders that appear as we go deeper into the construction of F.

Fortunately, we can still recover statements similar to Theorems 2.4.7 and 2.4.8, which will suffice for our purposes. Firstly, it turns out that we can still control the "Lyapunov exponents" of $m_t \circ \tau$. The following is essentially restating Corollary 6.2.11.

Lemma 6.4.2. *For $t = 1, 2$, $m_t \circ \tau$ almost all $\mathbf{i} \in \Sigma$ satisfy*

$$-\lim_{n\to\infty} \frac{1}{n} \log \alpha_1(\mathbf{i}|_n) \;=\; \chi_1(\mu) < -\lim_{n\to\infty} \frac{1}{n} \log \alpha_2(\mathbf{i}|_n) \;=\; \chi_2(\mu).$$

Thus for ν-almost every $\mathbf{i} \in \Sigma$,

$$\frac{\alpha_2(\mathbf{i}|_n)}{\alpha_1(\mathbf{i}|_n)} \to 0.$$

Proof. This follows because $m_t \circ \tau \ll m = \nu$. $\square$

Next, we move onto the entropy-type limit we are interested in. It also turns out that the 'entropy' of the measures $m_1 \circ \tau$ and $m_2 \circ \tau$ is equal to $h(\mu)$.

Lemma 6.4.3. *For $m_t \circ \tau$ almost every $\mathbf{i} \in \Sigma$,*

$$-\lim_{n\to\infty} \frac{1}{n} \log m_t \circ \tau([\mathbf{i}|_n]) = h(\mu).$$

Proof. Since m_1 and m_2 are distinct ergodic measures, they are mutually singular. Therefore μ_1 is not absolutely continuous with respect to μ_2. Thus by Proposition 2.3.2,

$$\limsup_{r\to 0} \frac{\mu_2(B(x,r))}{\mu_1(B(x,r))} = 0$$

for μ_1 almost every x. By Lemma 6.4.1,

$$\limsup_{n\to\infty} \frac{m_2 \circ \tau([\mathbf{i}|_n])}{m_1 \circ \tau([\mathbf{i}|_n])} = 0$$

for $m_1 \circ \tau$ almost every $\mathbf{i} \in \Sigma$. In other words, for $m_1 \circ \tau$ almost every $\mathbf{i}$, $m_1 \circ \tau([\mathbf{i}|_n])$ 'dominates' $m_2 \circ \tau([\mathbf{i}|_n])$ for all large n and

$$h(\mu) = -\lim_{n\to\infty} \frac{1}{n} \log m([\mathbf{i}|_n]) = -\lim_{n\to\infty} \frac{1}{n} \log m_1 \circ \tau([\mathbf{i}|_n])$$

for $m_1 \circ \tau$ almost all $\mathbf{i}$. By an analogous argument we obtain the same result for $m_2 \circ \tau$. $\square$

6.4.2 Estimates on the measure

In this section, we make estimates on the measure of a typical 'strip', which is the first step towards estimating the local dimension of μ. To this end, we need to introduce some notation and terminology.

It will be convenient to calculate the local dimension by measuring squares rather than balls in $\mathbb{R}^2$. Let $Q_1(x,r)$ denote the *one dimensional square* of side r centred at x, given by $Q_1(x,r) = [x - \frac{r}{2}, x + \frac{r}{2}]$. For $\mathbf{x} = (x,y)$ let $Q_2(\mathbf{x},r)$ denote the *2-dimensional square* of side r which is centred at $\mathbf{x}$, given by

$$Q_2(\mathbf{x},r) \;=\; \left\{(x',y') : |x-x'| \leqslant \frac{r}{2} \text{ and } |y-y'| \leqslant \frac{r}{2}\right\}.$$

Let $\mathbf{x} = (x,y) \in F$ with symbolic expansion $\mathbf{i} \in \Sigma$ and let $n \in \mathbb{N}$. Consider the cylinder $S_{\mathbf{i}|_n}([0,1]^2)$. Suppose the side lengths are distinct, so $\alpha_2(\mathbf{i}|_n) < \alpha_1(\mathbf{i}|_n)$. We shall call the longer side of $S_{\mathbf{i}|_n}([0,1]^2)$ the *primary side*. Additionally we shall call the axis parallel to this side the *primary axis* and denote the projection onto the primary axis by $\pi_p^{\mathbf{i},n}$. We may also call the direction of the primary axis the *primary direction*. So that this is all well-defined even when $\alpha_1(\mathbf{i}|_n) = \alpha_2(\mathbf{i}|_n)$ we agree that in this scenario the primary axis is the y axis. We denote the strip of all points inside $S_{\mathbf{i}|_n}([0,1]^2)$ that lie $\frac{r}{2}$-close to $\pi_p^{\mathbf{i},n}(\mathbf{x})$ in the primary direction by $B(\mathbf{x},n,r)$, and refer to this as the *primary strip*. We define the *secondary projection* to be the primary projection if the linear part of $S_{\mathbf{i}|_n}$ preserves each co-ordinate axis (i.e., an even number of the linear parts $\{A_{i_1}, \ldots, A_{i_n}\}$ are anti-diagonal matrices), and the other co-ordinate projection otherwise. We denote it by $\pi_s^{\mathbf{i},n}(\mathbf{x})$.

Remark 6.4.4. *To simplify the analysis, at several points in the proofs we will need to consider 'times' n when $A_{\mathbf{i}|_n}$ is diagonal. Suppose $\tau_{n+1}(\mathbf{i}) \in \{1, \ldots, d\}$. Then firstly, τ has the multiplicative property $\tau(\mathbf{i}) = \tau(\mathbf{i}|_n)\tau(\sigma^n\mathbf{i})$. Also, since $A_{\mathbf{i}|_n}$ is diagonal, we have $\pi_s^{\mathbf{i},n} = \pi_p^{\mathbf{i},n}$.*

The idea is to bound the measure of a primary strip $B(\mathbf{x}, n, r)$ by the measure of an appropriate cylinder (which contains $B(\mathbf{x}, n, r)$) and the projected measure of the 'blow up' of the strip.

In order to prove the desired lower bound for the local dimension of the measure (which corresponds to finding an appropriate upper bound for the measure of any given primary strip), we use sub-multiplicativity of the Käenmäki measure m. We can get an upper bound on the measure of a primary strip in terms of the product of the measure of an appropriate cylinder and an appropriate projected measure of the blow up of the strip.

Lemma 6.4.5. *Let $\mathbf{x} \in F$ with symbolic expansion $\mathbf{i} \in \Sigma$. For any $n \in \mathbb{N}$ and $r > 0$ we have*

$$\mu(B(\mathbf{x}, n, r)) \leqslant Cm([\mathbf{i}|_n])\pi_s^{\mathbf{i},n}(\mu)\left(Q_1\left(\pi_s^{\mathbf{i},n}(\Pi(\sigma^n(\mathbf{i})), \frac{r}{\alpha_1(\mathbf{i}|_n)}\right)\right)$$

where $C \geqslant 1$ is the uniform constant giving submultiplicativity of m.

Proof. Let

$$\mathcal{J} = \mathcal{J}(\mathbf{i}, n, r) = \left\{\mathbf{j} \in \Sigma^* : S_{\mathbf{i}|_n\mathbf{j}}([0,1]^2) \subseteq B(\mathbf{x}, n, r) \text{ and } S_{\mathbf{i}|_n\mathbf{j}^\dagger}([0,1]^2) \nsubseteq B(\mathbf{x}, n, r)\right\}$$

where $\mathbf{j}^\dagger$ is $\mathbf{j}$ with the last symbol removed. Note that by our separation assumption the family of rectangles $\{S_{\mathbf{i}|_n\mathbf{j}}([0,1]^2)\}_{\mathbf{j}\in\mathcal{J}}$ are pairwise disjoint and exhaust $B(\mathbf{x}, n, r)$ in measure. Thus

$$\begin{aligned}
\mu(B(\mathbf{x}, n, r)) &= \sum_{\mathbf{j}\in\mathcal{J}} m([\mathbf{i}|_n\mathbf{j}]) \\
&\leqslant Cm([\mathbf{i}|_n]) \sum_{\mathbf{j}\in\mathcal{J}} m([\mathbf{j}]) \\
&= Cm([\mathbf{i}|_n])\pi_s^{\mathbf{i},n}(\mu)\left(Q_1\left(\pi_s^{\mathbf{i},n}(\Pi(\sigma^n(\mathbf{i})), \frac{r}{\alpha_1(\mathbf{i}|_n)}\right)\right)
\end{aligned}$$

where the last equality follows since the family of rectangles $\{S_{\mathbf{j}}([0,1]^2)\}_{\mathbf{j}\in\mathcal{J}}$ are pairwise disjoint and exhaust $S_{\mathbf{i}|_n}^{-1}B(\mathbf{x},n,r)$ in measure. Then, noting that $S_{\mathbf{i}|_n}^{-1}B(\mathbf{x},n,r)$ is a strip with one side of length 1 and the other of length $r/\alpha_1(\mathbf{i}|_n)$ we have

$$\mu\left(S_{\mathbf{i}|_n}^{-1}(B(\mathbf{x},n,r))\right) = \pi_s^{\mathbf{i},n}(\mu)\left(Q_1\left(\pi_s^{\mathbf{i},n}(\Pi(\sigma^n(\mathbf{i})), \frac{r}{\alpha_1(\mathbf{i}|_n)}\right)\right)$$

as required. □

Next we prove an analogue of Lemma 6.4.5 giving an upper bound for the local dimension (so a lower bound for the measure). Since the measure μ is not supermultiplicative, we cannot use similar arguments to the ones we used above. Instead we employ the supermultiplicativity of m_t. Since we saw that τ only had the multiplicative property $\tau(\mathbf{i}) = \tau(\mathbf{i}|_n)\tau(\sigma^n\mathbf{i})$ when $\tau_{n+1}(\mathbf{i}) \in \{1,\ldots,d\}$, we will only be able to get our analogue of Lemma 6.4.5 along a suitable subsequence (for typical points).

Lemma 6.4.6. *Let $\mathbf{x} \in F$ with symbolic expansion $\mathbf{i} \in \Sigma$, such that $\mathbf{i}$ satisfies that $l \leqslant i_n \leqslant d$ for infinitely many n, i.e. infinitely many of the maps A_{i_n} are 'anti-diagonal'. (Observe that it is a direct consequence of the ergodic theorem that the set of such $\mathbf{i}$ is a set of full measure.) Let n_k be any subsequence for which $1 \leqslant \tau_{n_k+1}(\mathbf{i}) \leqslant d$ for all $k \in \mathbb{N}$. Then for any $r > 0$ and $k \in \mathbb{N}$,*

$$\begin{aligned}\nu\circ\Pi^{-1}(B(\mathbf{x},n_k,r)) &\geqslant \mu_t(B(\mathbf{x},n_k,r))\\ &\geqslant C m_t\circ\tau([\mathbf{i}|_{n_k}])\pi_s^{\mathbf{i},n_k}(\mu_t)\left(Q_1\left(\pi_s^{\mathbf{i},n_k}(\Pi(\sigma^{n_k}(\mathbf{i})), \frac{r}{\alpha_1(\mathbf{i}|_{n_k})}\right)\right)\end{aligned} \tag{6.12}$$

for $t = 1,2$ and where the constant C is independent of $\mathbf{x}$, k, r, t.

Proof. Let $\mathcal{J}$ be as in the proof of Lemma 6.4.5. Then, using the quasi-Bernoulli properties of μ_t we have

$$\begin{aligned}\mu_t(B(\mathbf{x},n_k,r)) &= \sum_{\mathbf{j}\in\mathcal{J}} m_t\circ\tau([\mathbf{i}|_{n_k}\mathbf{j}])\\ &= \sum_{\mathbf{j}\in\mathcal{J}} m_t(\tau([\mathbf{i}|_{n_k}])\tau([\mathbf{j}]))\\ &\geqslant c m_t\circ\tau([\mathbf{i}|_{n_k}])\sum_{\mathbf{j}\in\mathcal{J}} m_t\circ\tau([\mathbf{j}])\\ &= c m_t\circ\tau([\mathbf{i}|_{n_k}])\pi_s^{\mathbf{i},n_k}(\mu_t)\left(Q_1\left(\pi_s^{\mathbf{i},n_k}(\Pi(\sigma^{n_k}(\mathbf{i})), \frac{r}{\alpha_1(\mathbf{i}|_{n_k})}\right)\right)\end{aligned}$$

where the second equality holds because we are looking at times n_k when $\mathbf{i}|_{n_k}$ has seen an even number of rotations, c is a constant that comes from the quasi-Bernoulli properties of m_1 and m_2 and everything else follows by the same observations as in the proof of Lemma 6.4.5.

Finally, the result follows because $\nu \circ \Pi^{-1}(B(x, n_k, r)) \geqslant \mu_t(B(x, n_k, r))$. $\square$

6.4.3 Estimates on the projected measure

In this section, we obtain estimates for the projected measure of the blow up of a typical primary strip

$$\pi_s^{\mathbf{i},n_k}(\lambda)\left(Q_1\left(\pi_s^{\mathbf{i},n_k}(\Pi(\sigma^{n_k}(\mathbf{i})), \frac{r}{\alpha_1(\mathbf{i}|_{n_k})}\right)\right)$$

for $\lambda = \mu, \mu_1, \mu_2$, which appear in the upper and lower estimates in Lemmas 6.4.5 and 6.4.6.

Let $s_1 = \dim \pi_1(\mu_1)$. The key point of the proof of the next lemma is that an m_1-typical point $\tau(\mathbf{i})$ will regularly hit times n when the μ measure of $B(\overline{\Pi}(\sigma^n(\tau(\mathbf{i})), r)$ is sufficiently close to r^{s_1} and the matrix $A_{i_1} \cdots A_{i_n}$ is diagonal (and the same for the measure m_2).

Lemma 6.4.7. *For m-almost every $\mathbf{i} \in \Sigma$ there exists a choice of $t \in \{1, 2\}$ and a strictly increasing sequence of positive integers n_k for which simultaneously $\mu_t(B(\mathbf{x}, n_k, r))$ satisfies (6.12) for all $k \in \mathbb{N}$ and such that for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$, such that for all $k \geqslant N_\varepsilon$,*

$$\begin{aligned}(s_1 + \varepsilon) &\log \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})} \\ &\leqslant \log \pi_s^{\mathbf{i},n_k}(\mu_t)\left(Q_1\left(\pi_s^{\mathbf{i},n_k}(\Pi(\sigma^{n_k}\mathbf{i}), \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})}\right)\right) \leqslant (s_1 - \varepsilon) \log \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})}. \quad (6.13)\end{aligned}$$

Moreover, we can choose the sequence n_k such that

$$\lim_{k \to \infty} \frac{n_{k+1}}{n_k} \ \to \ 1. \quad (6.14)$$

Proof. Recall that $\tau : \Sigma \to \tau(\Sigma)$ is one-to-one and thus has an inverse. With slight abuse of notation, for $\mathbf{i} \in \tau(\Sigma)$ denote $\Pi(\mathbf{i}) = \Pi(\tau^{-1}(\mathbf{i}))$ and $\alpha_r(\mathbf{i}|_n) = \alpha_r(\tau^{-1}(\mathbf{i})|_n)$ for $r = 1, 2$. We will show that for each $t = 1, 2$, for m_t-almost every $\mathbf{i} \in \tau(\Sigma)$, there exists a sequence n_k such that (6.13) holds for μ_t. Then the result will follow because the union of the pre-images under τ of these full measure sets for m_1 and m_2 have full m-measure.

Fix $t \in \{1, 2\}$ and define the function $h_t^{(l)} : \tau(\Sigma) \to \mathbb{R}$ by

$$h_t^{(l)}(\mathbf{i}) = \frac{\log \pi_t(\mu_t) \left(Q_1 \left(\pi_t(\Pi(\mathbf{i})), \frac{1}{l}\right)\right)}{\log \frac{1}{l}}.$$

By Proposition 6.2.12 each of $\pi_t(\mu_t)$ are exact dimensional with dimension s_1 so that

$$\lim_{l \to \infty} h_t^{(l)}(\mathbf{i}) = s_1$$

for m_t-almost every $\mathbf{i} \in \tau(\Sigma)$. By Egorov's theorem, there exists a set $G_t \subset \tau(\Sigma)$ with measure $m_t(G_t) \geqslant m_t(\tau(\Sigma))/2 > 0$, for which $h_t^{(n)}$ converges uniformly to s_1 for all $\mathbf{i} \in G_t$. In particular this means that for all $\varepsilon > 0$, there exists $L_\varepsilon \in \mathbb{N}$ such that for $l \geqslant L_\varepsilon$,

$$s_1 - \varepsilon \leqslant \frac{\log \pi_t(\mu_t) \left(Q_1 \left(\pi_t(\Pi(\mathbf{i})), \frac{1}{l}\right)\right)}{\log \frac{1}{l}} \leqslant s_1 + \varepsilon$$

for all $\mathbf{i} \in G_t$. Moreover, by the Birkhoff Ergodic Theorem,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{G_t}(\sigma^k \mathbf{i}) \;=\; \int \mathbf{1}_{G_t} \, dm_t > 0$$

for m_t-almost every $\mathbf{i} \in \tau(\Sigma)$. In other words, for m_t-almost every $\mathbf{i} \in \tau(\Sigma)$ we have that $\sigma^n \mathbf{i} \in G_t$ with frequency greater than 0. Therefore, for such a fixed $\mathbf{i} \in \tau(\Sigma)$, we can choose n_k to be the subsequence of positive integers such that $\sigma^{n_k} \mathbf{i} \in G_t$ for all $k \in \mathbb{N}$. By Lemma 6.4.2 we can choose $N_\varepsilon \in \mathbb{N}$ such that for all $n \geqslant N_\varepsilon$,

$$\frac{\alpha_2(\mathbf{i}|_n)}{\alpha_1(\mathbf{i}|_n)} < \frac{1}{L_\varepsilon}.$$

Therefore, for $k \geqslant N_\varepsilon$, we have

$$(s_1{+}\varepsilon) \log \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})} \leqslant \log \pi_t(\mu_t) \left(Q_1 \left(\pi_t(\Pi(\sigma^{n_k} \mathbf{i}), \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})}\right)\right) \leqslant (s_1{-}\varepsilon) \log \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})}.$$

We now need to show that for m_t almost all such $\mathbf{i}$ and large enough k, we will have $\pi_s^{\mathbf{i}, n_k} = \pi_p^{\mathbf{i}, n_k} = \pi_t$. Since $\sigma^{n_k} \mathbf{i} \in \tau(\Sigma)$, $\mathbf{i}|_{n_k}$ is in the diagonal case which implies that $\pi_s^{\mathbf{i}, n_k} = \pi_p^{\mathbf{i}, n_k}$. Moreover it follows by Lemmas 6.2.11 and 6.2.10 that, for k sufficiently large, for m_t almost all $\mathbf{i}$

$$S_{n_k} f_{t,1}(\mathbf{i}) > S_{n_k} f_{t',1}(\mathbf{i}) \qquad (t' \neq t)$$

which in turn implies that the longer side of the rectangle $S_{\mathbf{i}|_{n_k}}([0,1]^2)$ is the horizontal side if $t = 1$ and vertical side if $t = 2$. Therefore $\pi_p^{\mathbf{i},n_k} = \pi_t$ as claimed and thus we obtain (6.13).

The fact that $\mu_t(B(\mathbf{x}, n_k, r))$ satisfies (6.12) follows because since $\sigma^{n_k}\mathbf{i} \in \tau(\Sigma)$, this implies that $1 \leqslant \tau_{n_k+1}(\mathbf{i}) \leqslant d$.

It only remains to prove that $\frac{n_{k+1}}{n_k} \to 1$ as $k \to \infty$. To see this let $S_{n_k} = \sum_{r=0}^{n_k} \chi_{G_t}(\sigma^r(\mathbf{i}))$. Then the ergodic theorem tells us that $\lim_{k\to\infty} \frac{S_{n_k}}{n_k} = m_t(G_t) > 0$. Moreover, clearly $\frac{S_{n_{k+1}}}{n_{k+1}} = \frac{S_{n_k}+1}{n_{k+1}}$. Now,

$$\left| \frac{S_{n_k}}{n_k} \left(\frac{n_k}{n_{k+1}} - 1 \right) + \frac{1}{n_{k+1}} \right| \quad = \quad \left| \frac{S_{n_k}+1}{n_{k+1}} - \frac{S_{n_k}}{n_k} \right| \to 0$$

as $k \to \infty$. Since $\frac{1}{n_{k+1}} \to 0$ as $k \to \infty$ and $\frac{S_{n_k}}{n_k} \to m_t(G_t) > 0$ as $k \to \infty$ it follows that $\frac{n_k}{n_{k+1}} - 1 \to 0$ as $k \to \infty$, in other words $\frac{n_k}{n_{k+1}}$ and $\frac{n_{k+1}}{n_k} \to 1$ as $k \to \infty$. $\square$

The next lemma is an analogue of Lemma 6.4.7 for μ instead of μ_t, and the proof is almost identical. However, for the sake of clarity we state it separately.

Lemma 6.4.8. *For m-almost every $\mathbf{i} \in \Sigma$ there exists a choice $t \in \{1, 2\}$ and a strictly increasing sequence of positive integers n_k for which simultaneously $\mu_t(B(\mathbf{x}, n_k, r))$ satisfies (6.12) for all $k \in \mathbb{N}$ and such that for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$, such that for all $k \geqslant N_\varepsilon$,*

$$\begin{aligned}(s_1 + \varepsilon) \log \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})} \\ \leqslant \log \pi_s^{\mathbf{i},n_k}(\mu) \left(Q_1 \left(\pi_s^{\mathbf{i},n_k}(\Pi(\sigma^{n_k}\mathbf{i}), \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})} \right) \right) \leqslant (s_1 - \varepsilon) \log \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})}.\end{aligned} \tag{6.15}$$

Moreover, we can choose the sequence n_k such that

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} \;\to\; 1. \tag{6.16}$$

Proof. Fix $t \in \{1, 2\}$ and define the function $h_t^{(l)} : \tau(\Sigma) \to \mathbb{R}$ by

$$h_t^{(l)}(\mathbf{i}) = \frac{\log \pi_t(\mu)\left(Q_1\left(\pi_t(\Pi(\mathbf{i})), \frac{1}{l}\right)\right)}{\log \frac{1}{l}}.$$

By Proposition 6.2.12 each of $\pi_t(\mu_t)$ are exact dimensional with dimension s_1. Since m_1 and m_2 are mutually singular, $\pi_1(\mu_1)$ is not absolutely continuous with respect to $\pi_1(\mu_2)$. Thus by Proposition 2.3.2, for m_1 almost every $\mathbf{i} \in \tau(\Sigma)$, $\pi_1(\mu_1)\left(Q_1\left(\pi_1(\Pi(\mathbf{i})), \frac{1}{l}\right)\right)$ 'dominates' $\pi_1(\mu_2)\left(Q_1\left(\pi_1(\Pi(\mathbf{i})), \frac{1}{l}\right)\right)$, that is,

$$\lim_{l \to \infty} \frac{\pi_1(\mu_2)\left(Q_1\left(\pi_1(\Pi(\mathbf{i})), \frac{1}{l}\right)\right)}{\pi_1(\mu_1)\left(Q_1\left(\pi_1(\Pi(\mathbf{i})), \frac{1}{l}\right)\right)} = 0$$

for m_1 almost every $\mathbf{i} \in \tau(\Sigma)$. Therefore,

$$\lim_{l \to \infty} h_1^{(l)}(\mathbf{i}) = \lim_{l \to \infty} \frac{\log \pi_1(\mu_1)\left(Q_1\left(\pi_t(\Pi(\mathbf{i})), \frac{1}{l}\right)\right)}{\log \frac{1}{l}} = s_1$$

for m_1 almost every $\mathbf{i} \in \tau(\Sigma)$. A symmetric argument proves the analogous statement for m_2. Therefore,

$$\lim_{l \to \infty} h_t^{(l)}(\mathbf{i}) = s_1$$

for m_t almost every $\mathbf{i} \in \tau(\Sigma)$. The rest of the argument follows identically to that for Lemma 6.4.7. □

To prove Theorem 6.3.1 it suffices to show that the local dimension of μ is what it should be at $x = \Pi(\mathbf{i})$ for $\mathbf{i}$ in a set of full m-measure. The proof will be split into two parts, concerning the lower and upper bound respectively.

6.4.4 The lower bound

Let $\mathbf{i} \in \Sigma$ belong to the set of full measure for which the conclusions of the Oseledets and Shannon-McMillan-Breiman theorems 2.4.7 and 2.4.8 and Lemma 6.4.8 hold simultaneously. In particular, let $t \in \{1, 2\}$ be such that Lemma 6.4.8 is satisfied for m_t. Write $\mathbf{x} = \Pi(\mathbf{i})$.

Since F satisfies the *strong separation property*, there exists $\delta > 0$ which satisfies (6.10).

Consider the square $Q_2(\mathbf{x}, \delta\alpha_2(\mathbf{i}|_n))$. Observe that since any cylinder on the nth level which is distinct from $S_{\mathbf{i}|_n}([0, 1]^2)$ must be at least $\delta\alpha_2(\mathbf{i}|_{n-1})$-separated from $S_{\mathbf{i}|_n}([0, 1]^2)$ and therefore $Q_2(\mathbf{x}, \delta\alpha_2(\mathbf{i}|_n))$ only intersects the cylinder $S_{\mathbf{i}|_n}([0, 1]^2)$.

Therefore, it is easy to see that

$$Q_2(\mathbf{x}, \delta\alpha_2(\mathbf{i}|_n)) \cap F \subseteq B(\mathbf{x}, n, \alpha_2(\mathbf{i}|_n)).$$

By Lemma 6.4.5, it follows that

$$\mu(Q_2(\mathbf{x}, \delta\alpha_2(\mathbf{i}|_n))) \leqslant m([\mathbf{i}|_n])\pi_s^{\mathbf{i},n}(\mu)\left(Q_1\left(\pi_s^{\mathbf{i},n}(\Pi(\sigma^n(\mathbf{i})), \frac{\alpha_2(\mathbf{i}|_n)}{\alpha_1(\mathbf{i}|_n)}\right)\right).$$

Fix $\varepsilon > 0$ and let n_k be the subsequence from Lemma 6.4.8. Observing that the sequence $\delta\alpha_2(\mathbf{i}|_{n_k})$ strictly decreases to zero, for any sufficiently small $r > 0$ we can choose $k \in \mathbb{N}$ large enough such that

$$\delta\alpha_2(\mathbf{i}|_{n_{k+1}}) \leqslant r \leqslant \delta\alpha_2(\mathbf{i}|_{n_k}).$$

Assume $r > 0$ is small enough to ensure $k \geqslant N_\varepsilon$ and then we have

$$\begin{aligned}
\frac{\log \mu\left(Q_2\left(\Pi(\mathbf{i}), r\right)\right)}{\log r} &\geqslant \frac{\log \mu\left(Q_2\left(\Pi(\mathbf{i}), \delta\alpha_2(\mathbf{i}|_{n_k})\right)\right)}{\log \delta\alpha_2(\mathbf{i}|_{n_{k+1}})} \\
&\geqslant \frac{\log m([\mathbf{i}|_{n_k}]) + \log \pi_s^{\mathbf{i},n_k}(\mu)\left(Q_1\left(\pi_s^{\mathbf{i},n_k}(\Pi(\sigma^{n_k}\mathbf{i})), \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})}\right)\right)}{\log \delta\alpha_2(\mathbf{i}|_{n_{k+1}})} \\
&\geqslant \frac{\log m([\mathbf{i}|_{n_k}]) + (s_1 - \varepsilon)\log \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})}}{\log \delta\alpha_2(\mathbf{i}|_{n_{k+1}})} \\
&= \frac{-\frac{1}{n_k}\log m([\mathbf{i}|_{n_k}]) - \frac{1}{n_k}(s_1 - \varepsilon)\log \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})}}{-\frac{1}{n_k}\log \delta - \frac{1}{n_{k+1}}\frac{n_{k+1}}{n_k}\log \alpha_2(\mathbf{i}|_{n_{k+1}})} \\
&\to \frac{h(\mu) + (s_1 - \varepsilon)(\chi_2(\mu) - \chi_1(\mu))}{\chi_2(\mu)}
\end{aligned}$$

as $r \to 0$ ($k \to \infty$). Finally, letting $\varepsilon \to 0$ yields the desired lower bound.

6.4.5 The upper bound

Let $\mathbf{i} \in \Sigma$ belong to the set of full measure for which the conclusions of the Oseledets and Shannon-McMillan-Breiman theorems 2.4.7 and 2.4.8 and Lemma 6.4.7 hold simultaneously. In particular, let $t \in \{1, 2\}$ be such that Lemma 6.4.7 is satisfied for m_t. Let n_k be the subsequence for $\mathbf{i}$ from Lemma 6.4.7. Write $\mathbf{x} = \Pi(\mathbf{i})$.

Consider the square $Q_2(\mathbf{x}, 2\alpha_2(\mathbf{i}|_{n_k}))$. Clearly

$$B(\mathbf{x}, n_k, \alpha_2(\mathbf{i}|_{n_k})) \subseteq Q_2(\mathbf{x}, 2\alpha_2(\mathbf{i}|_{n_k})) \cap F$$

and therefore by Lemma 6.4.6 it follows that

$$\nu \circ \Pi^{-1}(Q_2(\mathbf{x}, 2\alpha_2(\mathbf{i}|_{n_k}))) \geqslant C m_t \circ \tau([\mathbf{i}|_{n_k}]) \pi_s^{\mathbf{i}, n_k}(\mu_t) \left(Q_1 \left(\pi_s^{\mathbf{i}, n_k}(\Pi(\sigma^{n_k}(\mathbf{i})), \frac{\alpha_2(\mathbf{i}|_{n_k})}{\alpha_1(\mathbf{i}|_{n_k})} \right) \right).$$

Let $\varepsilon > 0$. Consider small $r > 0$, and since the sequence $2\alpha_2(\mathbf{i}|_{n_k})$ strictly decreases to zero we can choose k such that

$$2\alpha_2(\mathbf{i}|_{n_{k+1}}) \leqslant r \leqslant 2\alpha_2(\mathbf{i}|_{n_k}).$$

Assume r is small enough to guarantee $k \geqslant N_\varepsilon$. Then by Lemmas 6.4.6 and 6.4.7 we have

$$\begin{aligned}
&\frac{\log \nu \circ \Pi^{-1}\left(Q_2\left(\Pi(\mathbf{i}), r\right)\right)}{\log r} \\
\leqslant\ & \frac{\log \nu \circ \Pi^{-1}\left(Q_2\left(\Pi(\mathbf{i}), 2\alpha_2(\mathbf{i}|_{n_{k+1}})\right)\right)}{\log 2\alpha_2(\mathbf{i}|_{n_k})} \\
\leqslant\ & \frac{\log C m_t \circ \tau([\mathbf{i}|_{n_{k+1}}]) + \log \pi_s^{\mathbf{i}, n_{k+1}}(\mu_t)\left(Q_1\left(\pi_s^{\mathbf{i}, n_{k+1}}(\Pi(\sigma^{n_{k+1}}\mathbf{i})), \frac{\alpha_2(\mathbf{i}|_{n_{k+1}})}{\alpha_1(\mathbf{i}|_{n_{k+1}})}\right)\right)}{\log 2\alpha_2(\mathbf{i}|_{n_k})} \\
\leqslant\ & \frac{\log C + \log m_t \circ \tau([\mathbf{i}|_{n_{k+1}}]) + (s_1 + \varepsilon) \log \frac{\alpha_2(\mathbf{i}|_{n_{k+1}})}{\alpha_1(\mathbf{i}|_{n_{k+1}})}}{\log 2\alpha_2(\mathbf{i}|_{n_k})} \\
=\ & \frac{-\frac{1}{n_{k+1}} \log C - \frac{1}{n_{k+1}} \log\left(m_t \circ \tau([\mathbf{i}|_{n_{k+1}}])\right) - \frac{1}{n_{k+1}}(s_1 + \varepsilon) \log \frac{\alpha_2(\mathbf{i}|_{n_{k+1}})}{\alpha_1(\mathbf{i}|_{n_{k+1}})}}{-\frac{1}{n_{k+1}} \log 2 - \frac{1}{n_k} \frac{n_k}{n_{k+1}} \log \alpha_2(\mathbf{i}|_{n_k})} \\
\to\ & \frac{h(\mu) + (s_1 + \varepsilon)(\chi_2(\mu) - \chi_1(\mu))}{\chi_2(\mu)}
\end{aligned}$$

by Lemmas 6.4.2 and 6.4.3 as $r \to 0$ ($k \to \infty$). Finally, letting $\varepsilon \to 0$ yields the desired upper bound, and the result follows. $\square$

Appendix A

Hilbert-Birkhoff cone theory

In this appendix we give an overview of the use of Hilbert-Birkhoff cone theory in dynamics and prove the results of the statements in Section 4.4.1. The use of projective metrics associated to cones to express spectral properties of the transfer operator was introduced by Ferrero and Schmitt in [FS] in their proof of the Ruelle-Perron-Frobenius theorem. More recently, they were used to prove exponential rates of mixing for some expanding and hyperbolic systems, beginning with the paper of Liverani [L]. This approach was generalised by Viana whose book [V] contains an accessible exposition of the techniques of Hilbert-Birkhoff cone theory and their applications to dynamics.

Essentially, one can construct (Hilbert) metrics with respect to which the transfer operator is a contraction. This then allows one to obtain the invariant measure via a fixed point theorem rather than by a compactness argument from which one can, for instance, directly deduce exponential decay of correlations. For more on Hilbert-Birkhoff cone theory see [B], [L], [V].

We will begin by introducing the notions of a cone, the corresponding partial ordering and Hilbert metric. We also remind ourselves of the cone which was used in Chapter 4. From then on we will alternate between presenting the results for general cones that satisfy some conditions, and proving that our particular choice of cone satisfies the necessary conditions. As a result we will obtain all of the propositions 4.4.1, 4.4.2, 4.4.4 and 4.4.5 that were employed in Chapter 4. In particular, firstly we show that under a condition on the 'diameter' of the image of a cone under a linear operator, the linear operator is a contraction with respect to the Hilbert metric. This is followed by proving that our choice of cone and linear operator (some iterate of the transfer operator) satisfies the necessary condition. Next we show that if we equip the ambient vector space with norms that satisfy some conditions, then we can relate the norm of the difference of two appropriate functions to the Hilbert metric distance between the two functions. (Moreoever by using the previous result this will allow us to use the fact that the linear operator is contracting with respect to the Hilbert metric to deduce how our norms behave under the linear operator). This is followed by a proof that all the specific norms that were considered in Chapter 4 ($\|\cdot\|_\infty, \|\cdot\|_{L^1}, \|\cdot\|_{0,1}$) satisfy the necessary condition.

Let V be a vector space. We say that a subset $C \subset V \setminus \{0\}$ is a *cone* if for all $\lambda > 0$ and $f \in C$, then $\lambda f \in C$.

We say that a cone C is *convex* if for all $f, g \in C$ and $\lambda_1, \lambda_2 > 0$, then the sum $\lambda_1 f + \lambda_2 g \in C$.

We say that a cone C is *closed* if the set $C \cup \{0\}$ is closed. We will assume throughout that $C \cap -C = \emptyset$.

Given a closed convex cone $C \subset V$, we can define an *order relation* $\preceq$ on V by

$$f \preceq g \quad \Leftrightarrow \quad g - f \in C \cup \{0\}.$$

Observe that the partial ordering $\preceq$ will always depend implicitly on the cone we are working with. Moreover, the partial ordering is compatible with the vector space structure, that is, multiplication by positive scalars and addition.

Then we can define a (*Hilbert* or *projective*) *metric* Θ on C by setting

$$\begin{aligned}
\alpha(f,g) &= \sup\{t > 0 : (g - tf)(x) \in C\} = \sup\{t > 0 : tf \preceq g\} \\
\beta(f,g) &= \inf\{s > 0 : sf - g \in C\} = \inf\{s > 0 : g \preceq sf\} \\
\Theta(f,g) &= \log \frac{\beta(f,g)}{\alpha(f,g)}
\end{aligned}$$

where we take $\alpha = 0$ and $\beta = \infty$ if the corresponding sets are empty.

We will now give a couple of examples of cones of functions, and their corresponding Hilbert metrics.

The cone that we worked with in Section 4.4 was the set of non-negative continuous functions on the interval, whose logarithms are Lipschitz with Lipschitz constant less than a, that is,

$$\mathcal{C}_a = \left\{ f \in C([0,1]) : f \geqslant 0 \text{ and } f(x) \leqslant e^{a|x-y|} f(y) \right\}$$

where the parameter $a > 0$ is fixed. It is easy to check that $\mathcal{C}_a$ is a convex cone.

By definition, $\alpha(f,g)$ is the supremum of all $t > 0$ for which

(a) $(g-tf)(x) \geqslant 0$ for all $x \in [0,1]$, that is, $t \leqslant \frac{g(x)}{f(x)}$.

(b) $\frac{(g-tf)(x)}{(g-tf)(y)} \leqslant e^{a|x-y|}$ for all $x, y \in [0,1]$. Equivalently,

$$t \leqslant \frac{e^{a|x-y|}g(y) - g(x)}{e^{a|x-y|}f(y) - f(x)}$$

for all $x, y \in [0,1]$.

In particular, this means that

$$\alpha(f,g) \quad = \quad \inf\left\{\frac{g(x)}{f(x)}, \frac{e^{a|x-y|}g(x) - g(y)}{e^{a|x-y|}f(x) - f(y)} : x, y \in [0,1], x \neq y\right\}.$$

Similarly, we can obtain the expression

$$\beta(f,g) \quad = \quad \sup\left\{\frac{g(x)}{f(x)}, \frac{e^{a|x-y|}g(x) - g(y)}{e^{a|x-y|}f(x) - f(y)} : x, y \in [0,1], x \neq y\right\}$$

and then $\Theta(f,g) = \log \frac{\beta(f,g)}{\alpha(f,g)}$.

Another cone of functions is the set of all non-negative continuous functions on the interval, that is,

$$C_+ = \{f \in C([0,1]) : f(x) \geqslant 0\}.$$

Again, it is easy to check that this is a convex cone. We will denote its projective metric by $\Theta_+ = \log\left(\frac{\beta_+}{\alpha_+}\right)$. In light of the above, it is easy to see that

$$\begin{aligned}
\alpha_+(f,g) &= \inf\left\{\frac{g(x)}{f(x)}\right\} \\
\beta_+(f,g) &= \sup\left\{\frac{g(x)}{f(x)}\right\} \\
\Theta_+(f,g) &= \log\frac{\beta_+(f,g)}{\alpha_+(f,g)} = \log\sup\left\{\frac{g(x)f(y)}{f(x)g(y)} : x, y \in [0,1]\right\}.
\end{aligned}$$

We return back to the setting of a general cone C. Whenever one would like to use cone theory to study the behaviour of a linear operator under some chosen norm, one must first study the behaviour of the operator under the metric Θ. It is not difficult to show that under mild conditions on the operator it will be a contraction with respect to the metric Θ. In particular, we adopt the following general set up: let V_1, V_2 be vector spaces and $C_i \subset V_i$ be convex cones, where α_i, β_i, Θ_i correspond to the relevant projective metric for C_i. Let $L : V_1 \to V_2$ be a linear operator such that $L(C_1) \subset C_2$. Then

$$\begin{aligned}
\alpha_1(f,g) &= \sup\{t>0 : g - tf \in C_1\} \\
&\leqslant \sup\{t>0 : L(g-tf) \in C_2\} \\
&= \sup\{t>0 : L(g) - tL(f) \in C_1\} = \alpha_2(L(f), L(g))
\end{aligned}$$

where the second line follows because $L(C_1) \subset C_2$. analogously, we can also show that $\beta_1(f,g) \geqslant \beta_2(L(f), L(g))$ for all $f, g \in C_1$. Therefore

$$\Theta_2(L(f), L(g)) \leqslant \Theta_1(f,g).$$

However, in order for the contraction in Θ to yield useful results for norms that we may wish to study, the contraction in Θ has to be *strict*. The following classical result gives a sufficient condition for one to check whether an operator is a strict contraction with respect to the metric Θ. In particular, it tells us that if $L(C_1)$ has finite diameter with respect to Θ_2, then L is a strict contraction. The following result can be found in [V, Proposition 2.3].

Proposition A.0.9. *Let C_1, C_2 be as above and $D = \sup\{\Theta_2(Lf, Lg) : f, g \in C_1\}$. If $D < \infty$ then*

$$\Theta_2(Lf, Lg) \leqslant (1 - e^{-D})\Theta_2(f,g)$$

for all $f, g \in C_1$.

Proof. Without loss of generality, we can assume that $\alpha_1(f,g) > 0$ and $\beta_1(f,g) < \infty$ since otherwise we are done. Then there exist sequences $t_n \to \alpha_1(f,g)$ and $s_n \to \beta_1(f,g)$ such that $g - t_n f \in C_1$ and $s_n f - g \in C_1$ for all $n \in \mathbb{N}$. Thus, for all $n \geqslant 1$

$$\Theta_2(L(g - t_n f), L(s_n f - g)) \leqslant D.$$

Therefore, there exist sequences (T_n) and (S_n) such that $\lim_{n\to\infty} \log \frac{S_n}{T_n} \leqslant D$ and

$$L(s_n f - g) - T_n L(g - t_n f) \in C_2 \tag{A.1}$$

$$S_n L(g - t_n f) - L(s_n f - g) \in C_2. \tag{A.2}$$

By linearity of L, (A.1) implies that $(s_n + t_n T_n)L(f) - (1 + T_n)L(g) \in C_2$ so that

$$\beta_2(Lf, Lg) \leqslant \frac{s_n + t_n T_n}{1 + T_n} \tag{A.3}$$

and similarly (A.2) implies that

$$\alpha_2(Lf, Lg) \geqslant \frac{s_n + t_n S_n}{1 + S_n}. \tag{A.4}$$

Therefore,

$$\begin{aligned}
\Theta_2(Lf, Lg) &\leqslant \log\left(\frac{s_n + t_n T_n}{1 + T_n} \cdot \frac{1 + S_n}{s_n + t_n S_n}\right) \\
&= \log\left(\frac{s_n}{t_n} + T_n\right) - \log(1 + T_n) - \log\left(\frac{s_n}{t_n} + S_n\right) + \log(1 + S_n) \\
&= \int_0^{\log(\frac{s_n}{t_n})} \left(\frac{e^x dx}{e^x + T_n} - \frac{e^x dx}{e^x + S_n}\right) \\
&\leqslant \log\left(\frac{s_n}{t_n}\right) \cdot \left(1 - \frac{T_n}{S_n}\right).
\end{aligned}$$

Letting $n \to \infty$, we see that

$$\Theta_2(Lf, Lg) \leqslant \Theta_1(f, g)(1 - e^{-D}).$$

□

In order to save ourselves having to meddle with the definitions of the Hilbert metric Θ when checking whether a linear operator $L : \mathcal{C}_a \to \mathcal{C}_a$ is a strict contraction, the next result tells us that it is enough to just check that $L(\mathcal{C}_a) \subset \mathcal{C}_{\lambda a}$ for some $0 < \lambda < 1$. The following result can be found in [V, Proposition 2.5].

Proposition A.0.10. *Let $a > 0$ and $0 < \lambda < 1$ be arbitrary. Define*

$$D_{\lambda,a} = \sup\{\Theta(f, g) : f, g \in \mathcal{C}_{\lambda a}\}.$$

Then

$$D = D_{\lambda,a} < \infty.$$

Proof. Recall that

$$
\begin{aligned}
\alpha(f,g) &= \inf\left\{\frac{g(x)}{f(x)}, \frac{e^{a|x-y|}g(x)-g(y)}{e^{a|x-y|}f(x)-f(y)} : x,y\in[0,1], x\neq y\right\}\\
\beta(f,g) &= \sup\left\{\frac{g(x)}{f(x)}, \frac{e^{a|x-y|}g(x)-g(y)}{e^{a|x-y|}f(x)-f(y)} : x,y\in[0,1], x\neq y\right\}\\
\alpha_+(f,g) &= \inf\left\{\frac{g(x)}{f(x)}\right\}\\
\beta_+(f,g) &= \sup\left\{\frac{g(x)}{f(x)}\right\}\\
\Theta_+(f,g) &= \log\sup\left\{\frac{g(x)f(y)}{f(x)g(y)} : x,y\in[0,1]\right\}.
\end{aligned}
$$

We begin by showing that $D_{\lambda,a} \leqslant \sup\{\Theta_+(f,g) : f,g\in\mathcal{C}_{\lambda a}\} + C$, where C is some constant that depends only on λ.

Let $f,g\in\mathcal{C}_{\lambda a}$.

$$
\begin{aligned}
\frac{e^{a|x-y|}g(x)-g(y)}{e^{a|x-y|}f(x)-f(y)} &\geqslant \frac{g(x)}{f(x)}\frac{e^{a|x-y|}-e^{\lambda a|x-y|}}{e^{a|x-y|}-e^{-\lambda a|x-y|}}\\
&\geqslant K\frac{g(x)}{f(x)}
\end{aligned}
$$

where $K = \inf\{\frac{z-z^{\lambda}}{z-z^{-\lambda}} : z > 1\} \in (0,1)$. So $\alpha(f,g) \geqslant K\alpha_+(f,g)$. Similarly, $\beta(f,g) \leqslant L\beta_+(f,g)$, where $L = \sup\{\frac{z-z^{-\lambda}}{z-z^{\lambda}} : z>1\} \in (1,\infty)$. Thus,

$$
\begin{aligned}
\Theta(f,g) = \log\frac{\beta(f,g)}{\alpha(f,g)} &\leqslant \log\frac{L\beta_+(f,g)}{K\alpha_+(f,g)}\\
&= \Theta_+(f,g) + \log L - \log K.
\end{aligned}
$$

Now,

$$
\Theta_+(f,g) = \log\frac{\beta_+(f,g)}{\alpha_+(f,g)} = \log\sup\left\{\frac{g(x)}{f(x)}\frac{f(y)}{g(y)} : x,y\in[0,1]\right\}.
$$

But since $f,g\in\mathcal{C}_{\lambda a}$,

$$
\begin{aligned}
\frac{g(x)}{g(y)} &\leqslant e^{\lambda a|x-y|} \leqslant e^{\lambda a}\\
\frac{f(y)}{f(x)} &\leqslant e^{\lambda a|x-y|} \leqslant e^{\lambda a}
\end{aligned}
$$

for all $x,y\in[0,1]$. Thus $\Theta_+(f,g) \leqslant \log e^{2\lambda a} = 2\lambda a$, which implies $D_{\lambda,a} \leqslant 2\lambda a + \log L - \log K$. $\square$

Let $C_1 = C_2 = \mathcal{C}_a$ and $\Theta_1 = \Theta_2 = \Theta$ be the corresponding Hilbert metric. The above result tells us that if $L(\mathcal{C}_a) \subset \mathcal{C}_{\lambda a}$ for some $0 < \lambda < 1$, then

$$\sup\{\Theta(L(f), L(g)) : f, g \in \mathcal{C}_a\} \leqslant D_{\lambda,a} < \infty$$

and therefore, in light of Proposition A.0.9, L is a strict contraction with respect to Θ. Therefore, checking whether an operator is a Θ-contraction now reduces to proving results in the style of Lemmas 4.4.7 and 4.4.9.

Again, we return to the setting of a general cone C. Suppose $C \subset V$, where V is a vector space equipped with some norm $\|\cdot\|$. Given that one has established that a linear operator is a strict contraction with respect to some Hilbert metric Θ, we need a tool that will let us use the contraction in Θ to control the norm $\|\cdot\|$ of functions that belong to the cone. The following classical result in [L] tells us that under some hypothesis on $\|\cdot\|$, we can control $\|f - g\|$ by $\Theta(f, g)$ for $f, g \in C$.

Proposition A.0.11. *Let $\|\cdot\|$ be a norm on V, $C \subset V$ be a convex cone which induces the partial ordering $\preceq$ and suppose that for all $f, g \in V$,*

$$-f \preceq g \preceq f \quad \Rightarrow \quad \|g\| \leqslant \|f\|.$$

Then given any $f, g \in C$ for which $\|f\| = \|g\|$,

$$\|f - g\| \leqslant (e^{\Theta(f,g)} - 1)\|f\|.$$

We won't prove this result directly, but instead we'll state and prove a slight generalisation of this result, which we used several times in Section 4.4. Clearly, if we fix $C = 1$ and $\|\cdot\|_1 = \|\cdot\|_2 = \|\cdot\|$ in what follows, we recover Proposition A.0.11. Note also that the condition on $\|\cdot\|_1$ could also be relaxed, and a similar conclusion would follow. The proof of the following result is an adaptation of [L, Lemma 1.3].

Proposition A.0.12. *Let $\|\cdot\|_1$, $\|\cdot\|_2$ be two norms on V and let $C \subset V$ be a convex cone which induces the partial ordering $\preceq$. Suppose there exists $C \geqslant 1$ such that for all $f, g \in V$*

$$-f \preceq g \preceq f \quad \Rightarrow \quad \begin{aligned} &\|g\|_1 \leqslant \|f\|_1 \\ &\|g\|_2 \leqslant C\|f\|_2. \end{aligned}$$

Then given any $f, g \in C$ for which $\|f\|_1 = \|g\|_1$,

$$\|f - g\|_2 \leqslant C^2(e^{\Theta(f,g)} - 1)\|f\|_2$$

Proof. Denote $\alpha = \alpha(f, g)$ and $\beta = \beta(f, g)$ where α and β are the usual objects related to the projective metric Θ. In particular, we know that $\Theta(f, g) = \log\left(\frac{\beta}{\alpha}\right)$ where $\alpha f \preceq g$ and $g \preceq \beta f$. So in particular

$$-g \preceq 0 \preceq \alpha f \preceq g$$

so that $\|g\|_1 \geqslant \alpha\|f\|_1$. Since $\|f\|_1 = \|g\|_1$ this implies that $\alpha \leqslant 1$. analogously,

$$-\beta f \preceq 0 \preceq g \preceq \beta f$$

so that $\beta\|f\|_1 \geqslant \|g\|_1$, that is, $\beta \geqslant 1$.

Moreover, by the assumption on $\|\cdot\|_2$, this means that $\|f\|_2 \leqslant \frac{C}{\alpha}\|g\|_2$.

Using the fact that $\alpha \leqslant 1$ and $\beta \geqslant 1$, we obtain that

$$-(\beta - \alpha)f \preceq (\alpha - 1)f \preceq g - f \preceq (\beta - 1)f \preceq (\beta - \alpha)f$$

which implies that

$$\begin{aligned} \|g - f\|_2 &\leqslant (\beta - \alpha)C\|f\|_2 \leqslant \frac{\beta - \alpha}{\alpha}C^2\|g\|_2 \\ &\leqslant C^2(e^{\Theta(f,g)} - 1)\|g\|_2. \end{aligned}$$

□

In Propositions A.0.11 and A.0.12, it was important that the norms that we were working with had the property that if f and g were functions for which $-f \preceq g \preceq f$, then we had some control over $\|g\|$ in terms of $\|f\|$. In particular, we wanted there to exist some $C \geqslant 1$ for which

$$-f \preceq g \preceq f \quad \Rightarrow \quad \|g\| \leqslant C\|f\|.$$

Thus, the natural question arises: what norms satisfy some version of this hypothesis?

In Section 4.4 we used Proposition A.0.12 in various places, in the setting $C = \mathcal{C}_a$ and the norms $\|\cdot\|_\infty$, $\|\cdot\|_{L^1}$ and $\|\cdot\|_{0,1}$. Therefore, we now prove that these norms have the property (A.5) (with $C = 1$ for $\|\cdot\|_\infty$ and $\|\cdot\|_{L^1}$ and $C = 1 + a$ for $\|\cdot\|_{0,1}$) in the specific setting that $C = \mathcal{C}_a$. The following result is similar to [B, Lemma 2.2].

Proposition A.0.13. *Let $\preceq$ be the partial ordering induced by the cone $\mathcal{C}_a$ for some $a > 0$. Let m be a probability measure on $[0,1]$ and $L^1 = L^1(m)$. Then*

$$-f \preceq g \preceq f \quad \Rightarrow \quad \begin{array}{r} \|g\|_\infty \leqslant \|f\|_\infty \\ \|g\|_{L^1} \leqslant \|f\|_{L^1} \\ \|g\|_{0,1} \leqslant (a+1)\|f\|_{0,1}. \end{array}$$

Proof. Let $-f \preceq g \preceq f$. By assumption, $f - g, f + g \in \mathcal{C}_a$. By the positivity assumption in the definition of $\mathcal{C}_a$, it follows that $f \geqslant g$ and $f \geqslant -g$. Therefore, it immediately follows that $\|f\|_\infty \geqslant \|g\|_\infty$ and $\|f\|_{L^1} \geqslant \|g\|_{L^1}$.

Therefore, it remains to show that $\|g\|_{0,1} \leqslant (a+1)\|f\|_{0,1}$. We will do this by showing that $[g]_1 \leqslant a\|f\|_\infty$. To this end, let $x, y \in [0,1]$ such that $g(x) > g(y)$. Using the fact that $f - g, f + g \in \mathcal{C}_a$ we know that

$$\begin{aligned} f(y) - g(y) &\leqslant (f(x) - g(x))e^{a|x-y|} \\ f(x) + g(x) &\leqslant (f(y) + g(y))e^{a|x-y|}. \end{aligned}$$

Adding these inequalities together and rearranging, we obtain

$$(e^{a|x-y|} + 1)(g(x) - g(y)) \leqslant (f(x) + f(y))(e^{a|x-y|} - 1)$$

that is,

$$g(x) - g(y) \leqslant (f(x) + f(y))\frac{e^{a|x-y|} - 1}{e^{a|x-y|} + 1}.$$

Since $\frac{e^\delta - 1}{e^\delta + 1} \leqslant \frac{\delta}{2}$ for all $\delta > 0$,

$$g(x) - g(y) \leqslant (f(x) + f(y))\frac{a}{2}|x - y|$$

so that

$$\frac{g(x) - g(y)}{|x - y|} \leqslant a\left(\frac{f(x) + f(y)}{2}\right) \leqslant a\|f\|_\infty.$$

Since we chose x and y arbitrarily with the only restriction being that $g(x) > g(y)$, this implies $[g]_1 \leqslant a\|f\|_\infty$.

Therefore,

$$\|g\|_{0,1} = [g]_1 + \|g\|_\infty \leqslant (a+1)\|f\|_\infty \leqslant (a+1)\|f\|_{0,1}.$$

$\square$

Appendix B

Proof of Lemma 3.5.4

In this appendix we restate Lemma 3.5.4 with some additional detail and prove this result. Recall that Lemma 3.5.4 was concerned with bounding the dimension of a measure $\mu_{\mathbf{p}}$ where $\mathbf{p} \in \mathcal{P}_0$ and p_1 was 'close to 1'. In particular, we considered $p_1 > \xi$, where ξ was an undisclosed constant belonging to the unit interval.

We begin by giving an explicit value for ξ. Recall that $z_1 = \Pi((1)^\infty)$. Let ζ denote

$$\zeta = \log |T'(z_1)|$$

and observe that $\zeta > 0$ (this is always true when $T' < 0$, and in the case where $T' > 0$ this is a consequence of our assumptions on the map in Theorem 3.3.1). Then we define

$$\xi = \exp\left(-\frac{s+1}{s+3}\zeta\right) \in (0,1). \tag{B.1}$$

As we discussed in Chapter 3.1, the proof of Lemma 3.5.4 is a simplified version of the arguments presented in Chapter 2.12, owing to the fact that at each stage of the proof of Theorem 3.5.3 we had to get uniform bounds for the quantities being studied which held for all $\mathbf{p}$ belonging to the general class $\mathcal{P}_\varepsilon$. On the other hand, if we consider $\mathbf{p} \in \mathcal{P}_0$ where p_1 has a 'good' lower bound, these arguments can be simplified.

In particular, to prove this lemma we will make estimates on the integral $\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$ for $t \in [\frac{1-s}{8}, \frac{1-s}{4}]$. Then by following the proof of Theorem 4.6.3 we will be able to complete the proof. As in Chapter 2.12, in order to estimate the integral $\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t}$ from below, we need to make estimates on the ergodic sum of a chosen periodic point and the measure of an appropriate cylinder about that periodic point. Since we know that p_1 is large, it makes sense to choose z_1 to be our periodic point.

For the measure of the cylinder we utilise Lemma 4.5.2 and 4.2.8 (to calculate the size of the cylinder required).

We are now ready to state and prove a refinement of Lemma 3.5.4.

Lemma B.0.14. *Let $\mathbf{p} \in \mathcal{P}_0 \setminus \mathcal{P}_\varepsilon$, with $p_1 > \xi$. Then*

$$\dim \mu_{\mathbf{p}} \leqslant 1 - \frac{3}{8}\left(\frac{1-s}{4}\right)^2 \frac{\omega}{L}$$

for some constant uniform constant $\omega > 0$, where L is given in Lemma 4.6.2.

Proof. Let $\mathbf{i} \in \Sigma$ be the symbolic coding of z_1, i.e. $z_1 = \Pi(\mathbf{i})$. Fix some $\mathbf{p}$ that satisfies the assumptions. We follow the approach outlined in Lemma 4.2.6. Let $t \in [\frac{1-s}{8}, \frac{1-s}{4}]$. We begin by obtaining an estimate of the form

$$\int \tilde{f}^2_{\mathbf{p},t} d\mu_{\mathbf{p},t} \geqslant \frac{c^2}{4}\mu_{\mathbf{p},t}([i_1...i_m]) \tag{B.2}$$

for each t, where c will be a lower bound on $f_{\mathbf{p},t}(z_1)$ and m will be sufficiently large so that $\tilde{f}_{\mathbf{p},t}(x) \geqslant \frac{c}{2}$ for all $x \in \mathcal{I}_{i_1...i_m}$.

Recall that

$$f_{\mathbf{p},t}(z_1) = -\beta'_{\mathbf{p}}(t)\log|T'(z_1)| + \log p_1 = -\beta'_{\mathbf{p}}(t)\zeta + \log p_1.$$

Since $\mathbf{p} \in \mathcal{P}_0$, $\mu_{\mathbf{p}}$ has dimension $\dim \mu_{\mathbf{p}} \geqslant \frac{2s+2}{s+3}$, so that

$$-\beta'_{\mathbf{p}}(t) \geqslant -\beta'_{\mathbf{p}}(1) \geqslant \frac{2s+2}{s+3}.$$

Therefore, since $p_1 > \xi$ it follows that

$$f_{\mathbf{p},t}(z_1) \geqslant \left(\frac{2s+2}{s+3} - \frac{s+1}{s+3}\right)\zeta = \frac{s+1}{s+3}\zeta$$

for all $t \in [0, \frac{1-s}{4}]$. This gives us the value of $c = \frac{s+1}{s+3}\zeta$ in (B.2).

Next we estimate the measure of an arbitrary cylinder $[i_1...i_n]$ which contains $z_1 = \Pi(\mathbf{i})$. By Lemma 4.5.2,

$$\begin{aligned}\mu_{\mathbf{p},t}(\mathcal{I}_{i_1...i_n}) &\geqslant C^{-1}\frac{\xi^{tn}}{(|T'(z_1)\cdots T'(T^{n-1}(z_1))|)^{\beta_{\mathbf{p}}(t)}} \\ &= C^{-1}\exp\left(-\zeta n\left(\frac{s+1}{s+3}t + \beta_{\mathbf{p}}(t)\right)\right) \geqslant C^{-1}\exp(-\zeta n)\end{aligned}$$

where the last inequality is because

$$\frac{s+1}{s+3}t + \beta_{\mathbf{p}}(t) \leqslant t + \beta_{\mathbf{p}}(t) \leqslant 1$$

by convexity of $\beta_{\mathbf{p}}$.

Next, by Lemma 4.2.8, $[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}} \leqslant \frac{4C_0}{1-s}$ for all $t \in [\frac{1-s}{8}, \frac{1-s}{4}]$. By following the proof of Lemma 4.2.6, we see that

$$\int \tilde{f}_{\mathbf{p},t}^2 d\mu_{\mathbf{p},t} \geqslant \frac{1}{4}\left(\frac{s+1}{s+3}\zeta\right)^2 C^{-1}\exp(-\zeta m) =: \omega$$

where m is large enough so that $\frac{[\tilde{f}_{\mathbf{p},t}]_{\Lambda^{-1}}}{\Lambda^m} \leqslant \frac{1}{2}\frac{s+1}{s+3}\zeta$.

Finally, by following the proof of Lemma 4.6.3, we see that for all $t \in [\frac{1-s}{8}, \frac{1-s}{4}]$, $\beta''_{\mathbf{p}}(t) \geqslant \frac{\omega}{L}$ and

$$\dim \mu_{\mathbf{p}} \leqslant 1 - \frac{3}{8}\left(\frac{1-s}{4}\right)^2 \frac{\omega}{L}.$$

$\square$

Appendix C

Proof of Proposition 6.2.12

In this appendix we prove Proposition 6.2.12, which verifies the exact dimensionality of $\pi_1(\mu_1), \pi_2(\mu_2), \pi_1(\mu_2), \pi_2(\mu_1)$ and the links between their exact dimensions. Although these ideas fit into the framework of *graph-directed self-similar systems* (see for instance [F5]), we do not need to introduce this concept in order to prove the required results.

Define the maps $g_i : [0,1] \to [0,1]$ by $g_i(x) = a_i x + \tau_i^{(1)}$ for $1 \leqslant i \leqslant d$ and $g_i(x) = b_{i-d} x + \tau_{i-d}^{(2)}$ if $d+1 \leqslant i \leqslant 2d$. Define the projection $\overline{\Pi} : \Sigma_A \to \mathbb{R}$ by

$$\overline{\Pi}(\mathbf{i}) = \bigcap_{n=1}^{\infty} g_{\mathbf{i}|_n}([0,1])$$

which is well-defined since we assumed that the matrices A_i were contracting and each map S_i in our iterated function system mapped the unit square to itself. Then upon inspection, we see that for $\mathbf{i} \in \Sigma$

$$\pi_1(\Pi(\mathbf{i})) = \overline{\Pi}(\tau(\mathbf{i})) \tag{C.1}$$

and

$$\pi_2(\Pi(\mathbf{i})) = \overline{\Pi}(\omega(\mathbf{i})) \tag{C.2}$$

where π_1 denotes the projection to the x co-ordinate and π_2 denotes the projection to the y co-ordinate. This is essentially equivalent to the observation that was discussed before and after Example 6.2.4.

We claim that the measures $\overline{\Pi}(m_1)$ and $\overline{\Pi}(m_2)$ are exact dimensional. To see this, observe that these measures are supported on the set

$$\bigcup_{\mathbf{i}\in\Sigma_A}\bigcap_{n=1}^{\infty} g_{\mathbf{i}|_n}([0,1]) \subset \bigcup_{\mathbf{i}\in\Sigma_{2d}}\bigcap_{n=1}^{\infty} g_{\mathbf{i}|_n}([0,1]) \tag{C.3}$$

where Σ_{2d} denotes the full shift on $2d$ symbols and so the set on the right hand side of (C.3) is a self-similar set.

Consider $m_t \circ \overline{\Pi}^{-1}$ as a measure on Σ_{2d} and denote this by m_t'. Also let $\overline{\Pi}' : \Sigma_{2d} \to \mathbb{R}$ be the projection onto the self-similar set

$$\overline{\Pi}'(\mathbf{i}) = \bigcap_{n=1}^{\infty} g_{\mathbf{i}|_n}([0,1]).$$

Since m_t is an invariant ergodic measure on Σ, it is straightforward to show that m_t' is also invariant and ergodic for the full shift. Then by Theorem 6.1.1, $\overline{\Pi}'(m_t')$ is exact dimensional and therefore by the relationship between the pairs m_t, m_t' and $\overline{\Pi}, \overline{\Pi}'$ it immediately follows that $\overline{\Pi}(m_t)$ is also exact dimensional, completing the claim. We are now ready to prove Proposition 6.2.12.

Proof of Proposition 6.2.12. By the above discussion, we know that the measures $\overline{\Pi}(m_1)$ and $\overline{\Pi}(m_2)$ are exact dimensional.

The measures $\pi_1(\mu_1)$ and $\pi_2(\mu_2)$ are both absolutely continuous with respect to $\overline{\Pi}(m_1)$ and $\pi_1(\mu_2)$ and $\pi_2(\mu_1)$ are both absolutely continuous with respect to $\overline{\Pi}(m_2)$. To see this, notice that $\pi_1(\mu_1) = m_1 \circ \tau \circ \Pi^{-1} \circ \pi_1^{-1} = m_1 \circ \tau \circ (\pi_1 \circ \Pi)^{-1} = m_1 \circ \tau \circ (\overline{\Pi} \circ \tau)^{-1} = m_1 \circ \tau \circ \tau^{-1} \circ \overline{\Pi}^{-1} \ll m_1 \circ (\overline{\Pi})^{-1} = \overline{\Pi}(m_1)$, where the third equality follows by (C.1) and the absolute continuity is because τ is injective but not surjective. Similarly, $\pi_2(\mu_2) = m_2 \circ \tau \circ \Pi^{-1} \circ \pi_2^{-1} = m_2 \circ \tau \circ (\pi_2 \circ \Pi)^{-1} = m_2 \circ \tau \circ (\overline{\Pi} \circ \omega)^{-1} = m_2 \circ \tau \circ \omega^{-1} \circ \overline{\Pi}^{-1} = m_1 \circ \omega \circ \omega^{-1} \circ \overline{\Pi}^{-1} \ll m_1 \circ (\overline{\Pi})^{-1} = \overline{\Pi}(m_1)$ where the third equality follows by (C.2), the fifth by (6.8) and the absolute continuity is because ω is injective but not surjective. The other two cases follow similarly.

Finally the result follows by Proposition 2.3.1. $\square$

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