The Fermion Algebra in Quantum Statistical Mechanics:

Monodromy Fields on $\mathbb{Z}^2$ and Boson-Fermion Correspondence.

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Summary

Monodromy fields on $\mathbb{Z}^2$ are a family of lattice fields in two dimensions which are a natural generalization of the two dimensional Ising field occurring in the C^*-algebra approach to Statistical Mechanics. A criterion for the critical limit one point correlation of the monodromy field $\sigma_a(M)$ at $a \in \mathbb{Z}^2$,

$$\lim_{t \uparrow 1} \langle \sigma_a(M) \rangle,$$

is deduced for matrices $M \in GL(p, \mathbb{C})$ having non-negative eigenvalues.

Using this criterion a non-identity 2×2 matrix is found with a finite critical limit one point correlation. The general set of $p \times p$ matrices with finite critical limit one point correlations is also considered and a conjecture for the critical limit n point correlations postulated.

The boson-fermion correspondence for the representation of the CAR algebra over $L^2(S^1, \mathbb{C})$ defined by the (τ, β) KMS state with chemical potential μ is considered and the non-bijectivity shown. Using an alternative formulation the correlations are recalculated leading to a determinant identity reminiscent of Szego's Theorem.

Section 0
Introduction

0.1 Monodromy Fields on $\mathbf{Z}^2$.

The C^*-algebra approach to the Ising model via the transfer matrix is now well-known, see [A3], [C4], [E1]–[E3] and [L3] for example. Monodromy fields on $\mathbf{Z}^2$, introduced in [P2] are a family of lattice fields in two dimensions which are a natural generalisation of the two dimensional Ising field. They were inspired by [S1] and in a sense are lattice analogues of the continuum fields used in [S1,IV] and also in the Federbush and massless Thirring models, see [R3] and [C12] respectively. These lattice fields are interesting for several reasons. Firstly, by controlling the scaling limit, mathematically precise information on the continuum can be found and this approach was successfully used for the Ising field in [P8] and [P9], secondly there are numerous analogues of continuum structures suggesting a discrete theory on the lattice itself. For $M \in GL(p,\mathbf{C})$ and $a \in \mathbf{Z}^2$ it is possible to define the monodromy field $\sigma_a(M)$ at a. This is a generalisation of the Ising field in the sense that when M is the scalar -1 the vacuum expectation of a product $\sigma_{a_1}(-1)\dots\sigma_{a_n}(-1)$ gives the square of an Ising correlation. The motivation for the name 'monodromy field' is the fact that it is possible to 'create' monodromy M located at $a \in \mathbf{Z}^2$ in the solution to a certain linear difference equation on the lattice through a formula involving $\sigma_a(M)$.

In [P2] the one point correlations when M is a scalar were calculated using an elliptic substitution. Also the asymptotics of the correlations were examined in the scaling limit, that is the limit that sends the lattice spacing to zero and the 'temperature' to the critical point such that the correlation length remains fixed (massive scaling regime). In [P3] the critical scaling limit was studied, that is the large scale asymptotics of the correlations at the critical point (massless regime). However a limitation of the analysis carried out in [P2] was the fact that the monodromy fields had to appear in pairs, $\sigma_a(M)\sigma_b(M)^{-1}$, which was referred to as the twin problem. That is only correlations of the form

$$\langle \sigma_{a_1}(M_1)\sigma_{b_1}(M_1)^{-1}\dots\sigma_{a_n}(M_n)\sigma_{b_n}(M_n)^{-1}\rangle,$$

could be studied. Moreover the M_i had to have non-negative eigenvalues.

In order to find the large scale asymptotics at the critical point the following limit needs to be investigated:

$$\lim_{s\uparrow 1}\langle \sigma_{a_1}(M_1)\dots\sigma_{a_n}(M_n)\rangle.$$

This is non-trivial since the monodromy fields, $\sigma_a(M)$, are not defined for $s = 1$. A conjecture from [P3] was that the limit exists and is finite if $M_1\dots M_n = I$ and if $M_1\dots M_n \neq I$ then the limit is 0 or ∞. The second half of this conjecture is now shown to be false by an analysis of the limiting one point correlation:

$$\lim_{s\uparrow 1}\langle \sigma_a(M)\rangle.$$

A criterion for this limit is found enabling the existence of a non-identity M with finite critical limit correlation to be shown. However as is the case for the results in [P3] this is only true for M having non-negative eigenvalues. As for the general n point correlations a product formula, see [P3] or [P4], enables these to be written as the product of the individual one point correlations and a $\det_2$ term, see [S5] for a definition. This suggests that the one point correlations are sufficient though a proof is not available as yet.

The restriction on M to have non-negative eigenvalues is somewhat inconvenient since the Ising field case is given by the scalar -1 so non of the results are applicable to this case and the critical asymptotics for the two dimensional Ising model remain unknown.

0.2 Boson-Fermion Correspondence.

For many years physicists have written fermion fields as formal functions of certain boson fields in $1+1$ dimensions, see [C13] and [D2] for example. However these formulae are difficult to make sense of mathematically. In [F2] representations of certain infinite dimensional Lie algebras were constructed and in [F3] these were related to boson-fermion correspondence. Also in [C11], in $1+1$ dimensional field theory, representations of current algebras were obtained using automorphisms of the fermion or CAR algebra for the $1+1$ dimensional Dirac field. These were connected in [C7] via the work in [S2] using projective representations of infinite dimensional Lie groups (see also [C1], [C2]). Adopting a simplified version of fermions, namely the CAR algebra over $L^2(S^1, \mathbb{C})$ this enabled an explicit operator version of this correspondence for free bose fields. In [C5] this was taken further and the representations of loop groups which arise from representations of the CAR suggested by statistical mechanics were investigated and in [C6] projective representations of the gauge groups of $1+1$ dimensional quantum field theory.

So the basic idea of the boson-fermion correspondence in 1 space dimension is the following. Given a representation of the CAR in which the local gauge group G is implementable, by restricting to those maps in G which take their values in the maximal torus a representation of the CCR in Weyl form is obtained. For the other way, consider particular gauge group elements called 'blips' γ_ϵ. These depend on the real parameter ϵ such that they are singular at $\epsilon = 1$ and there exists a constant c_ϵ such that $c_\epsilon\Gamma(\gamma_\epsilon)$, where Γ is a representation of the gauge group, converges in a certain sense to a fermion field. This convergence is rather delicate, strong convergence on a dense domain of the approximate fermion fields has been shown in [C9].

Here the relation to statistical mechanics of [C5] is extended to include the chemical potential μ leading to some interesting technicalities concerning this correspondence. The prime reason for trying this extension was to investigate Bose-Einstein condensation, see [B2], [B3], [L1] and [L2] for example. However this did not prove very fruitful.

It ought to be mentioned that this is not the only construction referred to as boson-fermion correspondence. Hudson and Parthasarathy have developed a boson-fermion correspondence using quantum stochastic analysis, see [H2], [H3] and [P10], and a simple stochastic integral. Also Garbaczewski, [G1] and references therein, has yet another form however the connection of either with the above is unclear.

0.3 Infinite Complex Spin Groups.

Both of these objects are really examples of a general theory inspired by [S1]. From a mathematical point of view their work is not rigorous in the infinite dimensional case since it freely uses results only shown in the finite dimensional case. The generalization of their results is of interest for this reason and also in its own right to develop an infinite dimensional theory which may indeed go beyond that of the finite case. In [P1] certain results were extended rigorously to the infinite dimensional case and this has been developed in [C8] and [P4]–[P7] with a summary in [C3]. It concerns the existence of implementers $\Gamma_Q(G)$ on $A(W_+)$ such that

$$\Gamma_Q(G)F_Q(w)\Gamma_Q(G)^{-1} = F_Q(Gw),$$

where $F_Q(.)$ is a representation of $C(W, P)$ and G is an element of $O_{res}(W)$ where

$$O_{res}(W) = \{G : G \text{ orthogonal}, GQ - QG \text{ is Hilbert Schmidt}\}$$

is a subgroup of the complex orthogonal group

$$O(W) = \{G : PG^*P = P^{-1}\}.$$

The subgroup of $O_{res}(W)$ where G is a unitary is well known and the Hilbert Schmidt condition is the necessary and sufficient condition for the existence of $\Gamma_Q(G)$, [S4]. Some of the reasons for studying this are:

(1) As shown in 0.1 exactly solvable models in two dimensional quantum field theory (see [B1]) are connected with representations of $O_{res}(W)$ and its subgroups. For example the Federbush model [R3], [R4], the Luttinger model [C5], the massless Thirring model [C12] and the Ising and monodromy fields of 0.1. Note that these last two require the infinite dimensional analogues of [S1] and hence are not covered by the standard results on 'Bogoliubov transformations', see [R1], [R2], [C10] and [F1] for example.

(2) The representation of loop groups, vertex operators and string theory, see [P12], and hence by the comments in 0.2 boson-fermion correspondence.

(3) Segal and Wilson, [S3], used a subgroup of $O_{res}(W)$ in their study of the KdV equation. The work in [D1] suggests that this method can be extended to the Landau-Lifshitz equation using $O_{res}(W)$.

The basic reason for introducing $O_{res}(W)$ is simply to enlarge the group of Bogoliubov transformations studied to provide more 'room' in which to have approximations to the operators in [S1].

Here the results of [P1] are used to recalculate the correlations occurring in the boson-fermion correspondence constructed. This leads to a determinant identity similar in form to Szego's Theorem (see [M1,Chapter X] and [H1]).

From [P2] the one point correlation

$$\langle\sigma(\lambda)\rangle = \frac{\theta_3(-i/2\log\lambda, q)}{\theta_3(0,q)},$$

and from [C5] the same ratio of theta functions occurs in the state defining the representation used for the boson-fermion correspondence. This appears to be more than coincidence and the elliptic curve derivation of the formula above links this with the comments made in Section 2.3 of [C3] concerning a weak form of boson-fermion correspondence.

0.4 Outline of Thesis.

The format of the thesis is as follows:

(1) Section 1 introduces the basic definitions and notation.

(2) Section 2 introduces the definition of the monodromy field $\sigma_a(M)$.

(3) Section 3 deduces a criterion for the critical limit one point correlations based on the result of [P2] concerning the scalar case.

(4) Section 4 uses this criterion to find a non-trivial example of a matrix M with finite critical limit correlation. The structure of the set of such matrices is also studied.

(5) Section 5 poses a conjecture for the general n point correlations using a product formula. The variance/invariance of the correlations under the obvious action of S_n is also considered.

(6) Section 6 introduces the idea of boson-fermion correspondence as used in [C5].

(7) Section 7 extends this notion by the addition of another variable μ, the chemical potential, leading to some interesting results concerning the correspondence.

(8) Section 8 uses some results of [P1] to reformulate the first half of Section 7 and recalculate the correlations concerned leading to a determinant identity reminiscent of Szego's Theorem.

(9) The Appendix gives some general results and some proofs of facts used in Section 4.

Section 1
Preliminaries

1.1 Introduction.

This section will provide a brief description of the objects used throughout this thesis. They are fairly standard but do have slight variation. Hence the versions used are given here to set definitions and notation.

1.2 The Fermion Algebra.

1.2.1 Definition. Let H be a Hilbert space. The Fermion or CAR algebra over H, $A(H)$, is the C^*-algebra generated by the elements $\{a(f) : f \in H\}$ where a is a conjugate linear map from H into $A(H)$ satisfying the Canonical Anticommutation Relations:

$$a(f)a(g) + a(g)a(f) = 0,$$
$$a^*(f)a(g) + a(g)a^*(f) = (g, f)1,$$

for all f, g in H, where $a^*(f) = a(f)^*$.

1.2.2 Remark. There is an important representation of the CAR algebra known as the Fock representation which is as follows. Let $\Lambda(H)$ denote the Fock space (alternating tensor algebra) over H. Define the operators $a(f)$, $a^*(f)$ as follows:

$$a(f)(g_1 \wedge \cdots \wedge g_n) = n^{1/2}(f, g_1)(g_2 \wedge \cdots \wedge g_n),$$
$$a^*(f)(g_1 \wedge \cdots \wedge g_n) = (n+1)^{1/2}(f \wedge g_1 \wedge \cdots \wedge g_n).$$

Then if $\Omega = 1 \oplus 0 \oplus 0 \oplus \ldots$

$$a(f)\Omega = 0,$$
$$a^*(f)\Omega = f,$$

for all $f \in H$. These operators do indeed give a representation of the CAR algebra. Moreover this representation is unique in that it is the only irreducible representation for which a non-zero vector Ω exists such that $a(f)\Omega = 0$ for all $f \in H$. The vector Ω is called the vacuum vector and $a(.)$, $a^*(.)$ annihilation and creation operators respectively.

There are two other algebras which are essentially equivalent to this which will be of use in later sections thus their definitions are now given.

1.2.3 Definition. Let K be a Hilbert space and Γ an antiunitary involution on K. The self dual CAR algebra, $A_{SDC}(K,\Gamma)$, over (K,Γ) is the C^*-algebra generated by the elements $\{B(k) : k \in K\}$ where B is a conjugate linear map from K into $A_{SDC}(K,\Gamma)$ satisfying the following:

$$B(k)B(l)^* + B(l)^*B(k) = (k,l)1,$$
$$B(k)^* = B(\Gamma k).$$

1.2.4 Definition. Let W be a Hilbert space and P a conjugation on W. The Clifford algebra, $C(W,P)$, over W is the C^*-algebra generated by $\{c(w) : w \in W\}$ where c is a linear map from W into $C(W,P)$ satisfying the following:

$$c(u)c(v) + c(v)c(u) = (Pu, v)1.$$

Note that $c(w)$ will probably be identified with w.

1.2.5 Remark. With the definitions given above it is not difficult to show the following equivalences:

(1)

$$A_{SDC}(K,\Gamma) \cong C(K,\Gamma),$$

using the identification $B(h)^* = c(h)$.

(2)

$$A_{SDC}(K,\Gamma) \cong A(EK),$$

where E is a basis projection, that is a projection E with $\Gamma E \Gamma = 1 - E$, using the identification

$$B(h) = a(Eh) + a^*(E\Gamma h).$$

Acting on this algebra are a particular set of states (linear functionals) called the gauge-invariant quasi-free states which are analogues of Gaussian distributions in classical probability with the state completely determined by its two point functions.

1.2.6 Definition. A state ω on the CAR algebra is gauge-invariant if it is invariant under the group of gauge transformations

$$\tau_\theta(a(f)) = a(e^{i\theta} f), \quad \theta \in [0, 2\pi).$$

1.2.7 Definition. If R is a positive contraction then there is a unique gauge-invariant quasi-free state, denoted by ω_R, satisfying

$$\omega_R(a^*(f_m)\dots a^*(f_1)a(g_1)\dots a(g_n)) = \det[(g_i, Rf_j)]\,\delta_{nm}.$$

In particular

$$\omega_R(a^*(f)a(g)) = (g, Rf)$$

Moreover ω_R is pure if and only if R is a projection.

1.2.8 Notation. Let $(\mathcal{H}_R, \pi_R, \Omega_R)$ denote the GNS representation of the state ω_R.

Another set of states which will be of use later are the Q-Fock states on $C(W,P)$ which are defined as follows.

1.2.9 Definition. Suppose Q is a self adjoint operator on W with $Q^2 = 1$ and $QP + PQ = 0$. Then there exists a representation of $C(W,P)$ on the alternating tensor algebra, $\mathrm{A}(W_+)$, generated by

$$F_Q(w) = a^*(Q_+w) + a(PQ_-w),$$

where $a^*(.)$, $a(.)$ are creation and annihilation operators on $\mathrm{A}(W_+)$ with $Q_\pm = 1/2(1 \pm Q)$ and $W_+ = Q_+W$. The Q-Fock state ω_Q is then given by

$$\omega_Q(x) = (\Omega, F_Q(x)\Omega), \qquad x \in C(W,P).$$

1.2.10 Lemma. *The Q-Fock state on $C(W,P)$ is equivalent to the quasi-free state on $A_{SDC}(W,P)$ given by the basis projection $E = Q_-$. Alternatively the quasi-free state on $A_{SDC}(W,P)$ given by the basis projection E is equivalent to the Q'-Fock state on $C(W,P)$ where $Q' = 1 - 2E$.*

Proof: If E is a basis projection on W, π_E can be identified with

$$\pi_E(B(w)) = a_0((1-E)w) + a_0^*((1-E)Pw),$$

where $a_0(\cdot)$, $a_0^*(\cdot)$ are annihilation and creation operators on $\mathcal{A}((1-E)W)$. Using the equivalence of $A_{SDC}(W,P)$ and $C(W,P)$ given by $B(w)^* = c(w)$,

$$\begin{aligned}\pi_B(c(w)) &= \pi_B(B(w)^*) = \pi_B(B(w))^* \\ &= a_0^*((1-E)w) + a_0((1-E)Pw).\end{aligned}$$

Comparing this with the form given in Definition 1.2.9 for the representation associated with a Q-Fock state it is easy to see that

$$1 - E = Q_+.$$

That is $E = Q_-$ or alternatively $Q = 1 - 2E$.

Now there is a correspondence between basis projections, E, and Q's defining Q-Fock states given by the above. Namely if E is a basis projection then $Q = 1 - 2E$ defines a Q-Fock state and if Q defines a Q-Fock state then $E = Q_-$ is a basis projection.

With these two facts both versions of the Lemma are shown.

1.2.11 Remark. The above Lemma shows that Q-Fock states are in fact quasi-free states and moreover they are pure.

1.3 The CCR Algebra.

1.3.1 Definition. Let H be a pre-Hilbert space. The CCR algebra over H is the $*$-algebra generated by $\{a(f) : f \in H\}$ where a is a conjugate linear map satisfying the Canonical Commutation Relations:

$$\begin{aligned}a(f)a(g) - a(g)a(f) &= 0, \\ a^*(f)a(g) - a(g)a^*(f) &= (g,f)1,\end{aligned}$$

for all g, h in H, where $a^*(f) = a(f)^*$.

1.3.2 Definition. If the self adjoint operator $\Phi(f)$ is defined as

$$\Phi(f) = \frac{\overline{a(f) + a^*(f)}}{\sqrt{2}},$$

and $W(f)$ the unitary operator as

$$W(f) = \exp\{i\Phi(f)\},$$

then

$$W(f)W(g) = \exp\{-i/2\,\mathrm{Im}(f,g)\}\,W(f+g).$$

The operators $W(f)$ are called Weyl operators and the commutation relations

$$W(f)W(g) = e^{-i/2\,\mathrm{Im}(f,g)}W(f+g) = e^{-i\,\mathrm{Im}(f,g)}W(g)W(f),$$

the Weyl form of the Canonical Commutation Relations with the C^*-algebra generated by the Weyl operators called the Weyl form of the CCR algebra.

1.3.3 Remark. This form can in fact be slightly generalised to the following. Consider Weyl operators, $W(f)$, where f is an element of a real linear space H with a nondegenerate symplectic bilinear form σ,
[That is $\sigma : H \times H \to \mathbb{R}$ with $\sigma(f,g) = -\sigma(g,f)$ for all $f, g \in H$ and if $\sigma(f,g) = 0$ for all $f \in H$ then $g = 0$.]
and commutation relation

$$W(f)W(g) = \exp\{-i/2\sigma(f,g)\}\,W(f+g).$$

For example if H is a complex pre-Hilbert space and

$$\sigma(f,g) = \mathrm{Im}(f,g),$$

then the CCR algebra is obtained.

1.4 KMS states.

1.4.1 DEFINITION. Let h be a self adjoint operator on the Hilbert space H and assume that $\exp\{-\beta h\}$ is trace class. Let

$$\omega(A) = \frac{\mathrm{Trace}(e^{-\beta K_\mu} A)}{\mathrm{Trace}(e^{-\beta K_\mu})}$$

denote the Gibbs grand canonical equilibrium state over the CAR algebra $A(H)$ and

$$\tau_t(A) = e^{itK_\mu} A e^{-itK_\mu}$$

the evolution corresponding to the generalised Hamiltonian K_μ. So in particular

$$\tau_t(a(f)) = e^{-it\mu} a(e^{ith} f).$$

It follows that ω is the unique (τ, β) KMS state, and that this state is the gauge-invariant quasi-free state with two point function

$$\omega(a^*(f)a(g)) = \langle g, ze^{-\beta h}(1 + ze^{-\beta h})^{-1} f \rangle,$$

where $z = e^{\beta\mu}$.

For any further details concerning this and the rest of Section 1 see [A1], [A2] and [B4] for example.

Section 2
Monodromy Fields on $\mathbb{Z}^2$

2.1 Introduction.

This section gives the basic definitions for the study of monodromy fields on $\mathbb{Z}^2$.

2.2 Notation.

Let $H = L^2(S^1, \mathbb{C}^2)$ and p a positive integer with $H^p = H \oplus \ldots \oplus H = H \otimes \mathbb{C}^p$.

Let T be the multiplication operator on H defined by the 2×2 matrix:

$$Tf(\theta) = \begin{bmatrix} c^2/s - \cos\theta & s\sin\theta - i(c/s - c\cos\theta) \\ s\sin\theta + i(c/s - c\cos\theta) & c^2/s - \cos\theta \end{bmatrix} f(\theta),$$

where $c, s > 0$ and $c^2 - s^2 = 1$.

Let Q be the multiplication operator on H defined by:

$$Qf(\theta) = \begin{bmatrix} 0 & ie^{i\alpha(\theta)} \\ -ie^{-i\alpha(\theta)} & 0 \end{bmatrix} f(\theta),$$

where $\alpha(\theta)$ is determined by the following, with $\gamma(\theta) > 0$:

(1) $\alpha(0) = 0$,
(2) $\cosh\gamma(\theta) = c^2/s - \cos\theta$,
(3) $\sinh\gamma(\theta) e^{i\alpha(\theta)} = (c/s - c\cos\theta) + is\sin\theta$.

Note that

$$T(\theta) = \exp[-\gamma(\theta) Q(\theta)] = e^{-\gamma(\theta)} Q_+(\theta) + e^{\gamma(\theta)} Q_-(\theta),$$

where $Q_\pm = 1/2(1 \pm Q)$ with $Q^2 = 1$, Q self adjoint.

Also let z be the multiplication operator on H defined by:

$$zf(\theta) = e^{i\theta} f(\theta).$$

Now extend T, Q and z to operators on H^p in the obvious manner, namely tensoring by I^p i.e. the operator acts on each copy of H. With a slight abuse of notation call these operators T, Q and z. Let $W^p = H^p \oplus \overline{H}^p$, where $\overline{H}$ denotes the Hilbert space conjugate to H, and define a conjugation P on W^p by $P(x \oplus y) = y \oplus x$. If Q_W is the operator on W^p defined as $Q \oplus (-Q)$ then Q_W anticommutes with P and Q_W is self adjoint with $Q_W^2 = 1$.

Hence Q_W defines a Q_W-Fock state of the Clifford algebra $C(W^p, P)$ whose associated representation lives in the alternating tensor algebra $\Lambda(W_+^p)$ where $W_\pm^p = Q_W^\pm W^p$ and $Q_W^\pm = 1/2(1 \pm Q_W)$. The generators of this representation are given by:

$$F(w) = a^*(Q_W^+ w) + a(P Q_W^- w),$$

where $a^*(.)$, $a(.)$ are creation and annihilation operators on $\Lambda(W_+^p)$. Alternatively this representation can be thought of as the GNS representation of the self dual CAR algebra over W^p with antiunitary P defined by the basis projection Q_W^-, see Lemma 1.2.10 .

Now define the restricted general linear group $GL_Q(H^p)$ as the group of bounded, invertible linear maps on H^p with bounded inverses whose matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the $H_+^p \oplus H_-^p$ decomposition of H^p derived from Q have b, c Hilbert Schmidt and a, d Fredholm of index 0. Also define $GL_Q^0(H^p)$ as the subgroup with d 1+trace class and $GL_Q^1(H^p)$ as the subgroup with $b = c = 0$.

[P4], [P2] and [C8]— also a brief summary in [C3]— demonstrate the existence of a dense linear domain, $\mathcal{D} \subseteq \Lambda(W_+^p)$, together with two homomorphisms, $\Gamma_Q : GL_Q^0(H^p) \to L(\mathcal{D})$ and $\Gamma : GL_Q^1(H^p) \to L(\mathcal{D})$ where $L(\mathcal{D})$ denotes the linear maps from $\mathcal{D}$ to $\mathcal{D}$, such that:

$$\begin{aligned} \Gamma_Q(g)F(w) &= F(g \oplus g^{*^{-1}} w)\Gamma_Q(g), \qquad g \in GL_Q^0(H^p), \\ \Gamma(g)F(w) &= F(g \oplus g^{*^{-1}} w)\Gamma(g), \qquad g \in GL_Q^1(H^p), \\ \text{and } \Gamma_Q(hgh^{-1}) &= \Gamma(h)\Gamma_Q(g)\Gamma(h)^{-1}. \end{aligned}$$

Also if $GL_Q^0(H^p) \times GL_Q^1(H^p)$ is the semi-direct product with composition rule

$$g_1 \times h_1 . g_2 \times h_2 = g_1 h_1 g_2 h_1^{-1} \times h_1 h_2,$$

then the above gives $g \times h \to \Gamma_Q(g)\Gamma(h)$ is a homomorphism with kernel

$$K = \{g \times h : gh = 1 \text{ and } \det d(g) = 1\}.$$

Define $\widehat{GL_Q}(H^p) = GL_Q^0(H^p) \times GL_Q^1(H^p)/K$ then $(g \times h)K \to gh$ is a well defined homomorphism $T : \widehat{GL_Q}(H^p) \to GL_Q(H^p)$ with kernel $\mathbf{C}^*$. Identifying $\widehat{GL_Q}(H^p)$ with its image in $L(\mathcal{D})$:

$$gF(w) = F\left(T(g) \oplus T(g)^{*^{-1}} w\right) g, \qquad g \in \widehat{GL_Q}(H^p),$$

i.e. g is the implementer of $T(g)$. If Ω_Q is the vacuum vector of $\Lambda(W_+^p)$ define

$$(g)_Q = (\Omega_Q, g\Omega_Q), \qquad g \in \widehat{GL_Q}(H^p),$$

that is if $g = (g' \times h')K$

$$\begin{aligned} (g)_Q &= (\Omega_Q, \Gamma_Q(g')\Gamma(h')\Omega_Q) \\ &= (\Omega_Q, \Gamma_Q(g')\Omega_Q) \\ &= \det(d(g')). \end{aligned}$$

For more details of this, together with proofs, see [P2] and [C8].

In other words, if $g \in GL_Q(H^p)$ and has a decomposition as $g_0 g_1$ where $g_i \in GL_Q^i(H^p)$ for $i = 0, 1$ then $T((g_0 \times g_1)K) = g$, thus the implementer of the automorphism induced by g on the Clifford algebra is given by $\Gamma_Q(g_0)\Gamma(g_1)$ at least on the dense domain $\mathcal{D}$ and upto scalar multiple— a choice of factorisation at the $GL_Q(H^p)$ level being equivalent to a choice of normalisation at the $\widehat{GL_Q}(H^p)$ level.

With the structures defined above it is now possible to define the monodromy field $\sigma(M)$, where $M \in GL(p, \mathbf{C})$. Let M act on H^p as $I \otimes M$ and define ε as the convolution operator on H whose Fourier transform acts on $\ell^2(\mathbf{Z}_{1/2}, \mathbf{C}^2)$ as $\varepsilon f(k) = \operatorname{sgn}(k) f(k)$, for $k \in \mathbf{Z}_{1/2}$. Let $\varepsilon_\pm = (1 \pm \varepsilon)/2$ and define

$$s(M) = \varepsilon_- \otimes I_p + \varepsilon_+ \otimes M.$$

In [P2] it is shown that $s(M) \in GL_Q(H^p)$, then $\sigma(M)$ is essentially defined such that $T(\sigma(M)) = s(M)$. So from comments made above a factorisation of $s(M)$ into $g_0 g_1$ where $g_i \in GL_Q^i(H^p)$ is sufficient since $\sigma(M)$ may be defined as $\Gamma_Q(g_0)\Gamma(g_1)$. This factorisation is constructed in [P2] and using that notation $s(M) = \underline{g}(M)D(M)$ so that:

$$\sigma(M) = \Gamma_Q(\underline{g}(M))\Gamma(D(M)).$$

This definition can be extended to the points on a $\mathbf{Z}^2$ lattice as follows. Let $\Gamma(T)$ and $\Gamma(z)$ be the implementers of T and z then define

$$\begin{aligned} s_a(M) &= T^{a_2} z^{a_1} s(M) z^{-a_1} T^{-a_2}, \\ \text{and } \sigma_a(M) &= \Gamma(T)^{a_2}\Gamma(z)^{a_1}\sigma(M)\Gamma(z)^{-a_1}\Gamma(T)^{-a_2}, \end{aligned}$$

where $a = (a_1, a_2) \in \mathbf{Z}^2$. Call $\sigma_a(M)$ the monodromy field at a— so $\sigma(M)$ is the monodromy field at 0.

2.3 Remark. The mutiplication operator T introduced at the beginning of 2.2 Notation is the same as that for the study of the Ising model in [P8] and [P9] where $s = \cosh 2K^*$. Hence, using this connection, it is possible to consider the cases $s < 1$, $s = 1$ and $s > 1$ to correspond to below the critical temperature, at the critical temperature and above the critical temperature respectively.

2.4 Remark. One problem— which has been glossed over so far— is the fact that $s(M) \notin GL_Q(H^p)$ when $s = 1$ (Q depends on T and thus s) and consequently a limiting argument is required, which is the cause of the problems in this area.

2.5 Remark. The objects of study are the n point correlations:

$$\langle \sigma_{a_1}(M_1) \cdots \sigma_{a_n}(M_n) \rangle_Q.$$

and in particular their limit as $s \uparrow 1$, that is, their behaviour as they approach the critical temperature.

2.6 Remark. From now on it will be assumed that $s < 1$ and as the notation— $s \uparrow 1$— suggests the limit from below will be considered. Also the Q subscript in the correlation will be dropped.

Section 2
A Condition for the Existence of Limiting One Point Correlations

2.1 Introduction.

This section aims to classify the critical limit one point correlations, $\lim_{s\uparrow 1}\langle\sigma_s(M)\rangle$, for all matrices $M \in GL(p, \mathbb{C})$, $p \in \mathbb{N}$ with non-negative eigenvalues. The starting point for this classification is a result of **[P2]** concerning the one dimensional scalar case, that is $(p = 1)$.

2.1.1 Proposition. *Let $k = s^2$, $k'^2 = 1 - k^2$ and*

$$K^{(\prime)} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 + k^{(\prime)^2}\sin^2\theta}}.$$

Suppose $\lambda \in \mathbb{C}\setminus(-\infty, 0]$. If $s(\lambda) = \underline{g}(\lambda)\, D(\lambda)$ and $\underline{d}(\lambda) = d(\underline{g}(\lambda)) = Q_-\underline{g}(\lambda)Q_-$ then $\underline{d}(\lambda)$ is invertible and

$$\langle\sigma(\lambda)\rangle = \det \underline{d}(\lambda) = \prod_{l>0}\left[\frac{1 + \lambda^{-1}q^{2l}}{1 + q^{2l}}\,\frac{1 + \lambda q^{2l}}{1 + q^{2l}}\right],$$

where $l \in \mathbb{Z}_{1/2}$ and $q = \exp(-\pi K'/K)$.

Note. As $\underline{d}(\lambda)$ is invertible the correlation is non-zero.

Now assume $M \in GL(p, \mathbb{C})$ and has no negative eigenvalues. Hence there exists a matrix $S_M \in GL(p, \mathbb{C})$ such that $S_M M S_M^{-1} = J_M$ where J_M denotes the Jordan form of M. Thus

$$\begin{aligned} s(M) &= (1 \otimes S_M^{-1})\,(\epsilon_- \otimes I + \epsilon_+ \otimes J_M)(1 \otimes S_M) \\ &= (1 \otimes S_M^{-1})\, s(J_M)(1 \otimes S_M). \end{aligned}$$

Therefore if $s(J_M)$ is factorised as $\underline{g}(J_M)\,D(J_M)$, $s(M)$ may be factorised as

$$((1 \otimes S_M^{-1})\,\underline{g}(J_M)(1 \otimes S_M))\,((1 \otimes S_M^{-1})\,D(J_M)(1 \otimes S_M)).$$

Hence

$$\begin{aligned} \langle\sigma(M)\rangle &= \det d\left((1 \otimes S_M^{-1})\,\underline{g}(J_M)(1 \otimes S_M)\right) \\ &= \det\,(1 \otimes S_M^{-1})\,d(\underline{g}(J_M))(1 \otimes S_M) \\ &\qquad \text{as } Q_- = Q_- \otimes I \text{ commutes with } 1 \otimes S_M^{(-1)} \\ &= \det d(\underline{g}(J_M)) \\ &= \langle\sigma(J_M)\rangle. \end{aligned}$$

Appealing to [P2], in general the factorising terms are given by the following:

Suppose P_+, P_- are the orthogonal projections onto the subspaces of $L^2([-K, K], \mathbb{C})$ whose elements have fourier expansions in $\exp(\pi i l x/K)$ with no l negative, positive terms respectively. Then:

$$D(M) = I_+ \oplus (P_+ \otimes I_p + P_- \otimes M)_-\,,$$

where I_+ is the identity on H_+^p, $(P_+ \otimes I_p + P_- \otimes M)_-$ acts on $H_-^p \cong L^2([-K, K], \mathbb{C}) \otimes \mathbb{C}^p$ and

$$\underline{g}(M) = s(M)D(M)^{-1}, \qquad \text{with } d(\underline{g}(M)) \text{ 1+ trace class.}$$

So suppose the eigenvalues of M are $\lambda_1 \ldots \lambda_p$ then the Jordan form

$$J_M = \begin{bmatrix} \lambda_1 & \delta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \delta_{p-1} \\ & & & \lambda_p \end{bmatrix},$$

where $\lambda_i \in \mathbf{C} \setminus (-\infty, 0]$ for $i = 1, \ldots, p$ and $\delta_j = 0$ or 1 for $j = 1, \ldots, p-1$. Therefore

$$D(J_M) = \begin{bmatrix} 1 & \\ & \begin{bmatrix} D(\lambda_1) & x_1 & & \\ & \ddots & \ddots & \\ & & \ddots & x_{p-1} \\ & & & D(\lambda_p) \end{bmatrix} \end{bmatrix},$$

where $D(\lambda_i) = P_+ + \lambda_i P_-$ for $i = 1, \ldots, p$ and $x_j = 0$ or P_- for $j = 1, \ldots, p-1$. Denote the (Q_-, Q_-) term of $D(J_M)$ by D and let $s(J_M) = \begin{bmatrix} a(J_M) & b(J_M) \\ c(J_M) & d(J_M) \end{bmatrix}$ then

$$\underline{g}(J_M) = \begin{bmatrix} a(J_M) & b(J_M) \\ c(J_M) & d(J_M) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D^{-1} \end{bmatrix},$$

so

$$\begin{aligned} d(\underline{g}(J_M)) &= d(J_M) D^{-1} \\ &= \begin{bmatrix} d(\lambda_1) & y_1 & & \\ & \ddots & \ddots & \\ & & \ddots & y_{p-1} \\ & & & d(\lambda_p) \end{bmatrix} \begin{bmatrix} D(\lambda_1)^{-1} & \cdots & \cdots & \cdots \\ & D(\lambda_2)^{-1} & \cdots & \cdots \\ & & \ddots & \vdots \\ & & & D(\lambda_p)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \underline{d}(\lambda_1) & \cdots & \cdots & \cdots \\ & \underline{d}(\lambda_2) & \cdots & \cdots \\ & & \ddots & \vdots \\ & & & \underline{d}(\lambda_p) \end{bmatrix}. \end{aligned}$$

Therefore $\det d(\underline{g}(J_M)) = \prod_{i=1}^{p} \det \underline{d}(\lambda_i)$, that is:

$$\begin{aligned} \langle \sigma(M) \rangle = \langle \sigma(J_M) \rangle = \det d(\underline{g}(J_M)) &= \prod_{i=1}^{p} \det \underline{d}(\lambda_i) \\ &= \prod_{i=1}^{p} \prod_{l>0} \left[\frac{1 + \lambda_i^{-1} q^{2l}}{1 + q^{2l}} \frac{1 + \lambda_i q^{2l}}{1 + q^{2l}} \right]. \end{aligned}$$

Also note that for $a = (a_1, a_2) \in \mathbf{Z}^2$

$$\begin{aligned} \langle \sigma_a(M) \rangle &= \langle \Gamma(T)^{a_2} \Gamma(z)^{a_1} \sigma(M) \Gamma(z)^{-a_1} \Gamma(T)^{-a_2} \rangle \\ &= \langle \Gamma(T)^{a_2} \Gamma(z)^{a_1} \Gamma_Q(\underline{g}(M)) \, \Gamma(D(M)) \, \Gamma(z)^{-a_1} \Gamma(T)^{-a_2} \rangle \\ &= \langle \Gamma_Q\left(T^{a_2} z^{a_1} \underline{g}(M) z^{-a_1} T^{-a_2}\right) \Gamma\left(T^{a_2} z^{a_1} D(M) z^{-a_1} T^{-a_2}\right) \rangle \\ &= \langle \Gamma_Q\left(T^{a_2} z^{a_1} \underline{g}(M) z^{-a_1} T^{-a_2}\right) \rangle \\ &= \det d\left(T^{a_2} z^{a_1} \underline{g}(M) z^{-a_1} T^{-a_2}\right) \\ &= \det T^{a_2} z^{a_1} d(\underline{g}(M)) \, z^{-a_1} T^{-a_2} \qquad \text{by Lemma 3.1.3 below.} \\ &= \det d(\underline{g}(M)) \\ &= \langle \sigma(M) \rangle. \end{aligned}$$

So have shown the following lemma:

3.1.2 Lemma. *If $M \in GL(p,\mathbb{C})$ has no negative eigenvalues then*

$$\langle \sigma_a(M)\rangle = \prod_{i=1}^{p}\prod_{l>0}\left[\frac{1+\lambda_i^{-1}q^{2l}}{1+q^{2l}}\,\frac{1+\lambda_i q^{2l}}{1+q^{2l}}\right], \qquad \forall a \in \mathbb{Z}^2.$$

3.1.3 Lemma. *The operators Q, T and z commute.*

Proof: By definition

$$\begin{aligned} T &= e^{-\gamma(\theta)}Q_+ + e^{\gamma(\theta)}Q_- \\ &= \cosh\gamma(\theta)I - \sinh\gamma(\theta)Q \end{aligned}$$

giving T and Q commute.

Considering the fourier transforms of T and z on $\ell(\mathbb{Z}_{1/2},\mathbb{C}^2)$:

$$Tf(k) = T_- z^{-1}f(k) + T_0 f(k) + T_+ z f(k),$$

where

$$\begin{aligned} T_- &= -\frac{1}{2}\begin{bmatrix} 1 & -i(c+s) \\ i(c-s) & 1 \end{bmatrix}, \\ T_0 &= \frac{c}{s}\begin{bmatrix} c & -i \\ i & c \end{bmatrix}, \\ T_+ &= -\frac{1}{2}\begin{bmatrix} 1 & -i(c-s) \\ i(c+s) & 1 \end{bmatrix}, \end{aligned}$$

and $z^{\pm 1}f(k) = f(k \mp 1)$.

An easy calculation shows T and z commute. Thus, using the relation between T and Q above, it is easy to see z and Q commute.

Note. The lemma actually proves that T and z are in the domain of Γ which has been implied by the notation so far.

3.2 Convergence Argument.

The previous subsection showed that if $M \in GL(p,\mathbb{C})$ has no negative eigenvalues then

$$\langle \sigma_a(M)\rangle = \prod_{i=1}^{p}\prod_{l>0}\left[\frac{1+\lambda_i^{-1}q^{2l}}{1+q^{2l}}\,\frac{1+\lambda_i q^{2l}}{1+q^{2l}}\right], \qquad \forall a \in \mathbb{Z}^2$$

where $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of M. This may be rewritten as

$$\prod_{l>0}\prod_{i=1}^{p}\left[1+\frac{c(\lambda_i)q^{2l}}{(1+q^{2l})^2}\right] = \prod_{l>0}\left[1+\sum_{i=1}^{p}c_i\left(\frac{q^{2l}}{(1+q^{2l})^2}\right)^i\right],$$

where $c(\lambda_i) = \lambda_i + \lambda_i^{-1} - 2$ and $c_i = \sum_{1\le j_1<\cdots<j_i\le p} c(\lambda_{j_1})\ldots c(\lambda_{j_i})$.

3.2.1 Lemma. *Suppose $\lambda \in \mathbb{C}\backslash(-\infty,0]$. Define $f:[0,1]\to\mathbb{C}$ as $f(x) = 1+c(\lambda)\frac{x}{(1+x)^2}$ where $c(\lambda)$ is defined as above. Then $|f(x)| > \varepsilon > 0 \quad \forall x \in [0,1]$.*

Proof: If $c(\lambda) = a+ib$ then $|f(x)| = \sqrt{\left(1+a\frac{x}{(1+x)^2}\right)^2 + \left(b\frac{x}{(1+x)^2}\right)^2}$.

Now the term inside the square root is a product of two squares and thus ≥ 0, also it is zero if and only if both entries are zero which is not possible. This is due to the fact that if $x = 0$ then $\left(1+a\frac{x}{(1+x)^2}\right) = 1$ and if $b = 0$ then $\left(1+a\frac{x}{(1+x)^2}\right) > 0$ as $a > -4$ by the restriction on λ. Thus $|f(x)| > 0 \quad \forall x \in [0,1]$.

Therefore $|f(x)| > \varepsilon > 0 \quad \forall x \in [0,1]$.

3.2.2 Remark. By the lemma above $|1+\sum_{i=1}^{p} c_i \left(\frac{y}{(1+y)^2}\right)^i| > \epsilon > 0 \quad \forall y \in [0,1]$ so in particular it is true when $y = q^{2l} \quad \forall q \in (0,1), \quad \forall l \in \mathbf{Z}_{\frac{1}{2}}$. This lower bound will play an important part in the analysis that follows. Taking logarithms above leads to

$$\begin{aligned}\log(\sigma_s(M)) &= \sum_{l>0} \log\left\{1+\sum_{i=1}^{p} c_i \left(\frac{q^{2l}}{(1+q^{2l})^2}\right)^i\right\}, \qquad l \in \mathbf{Z}_{\frac{1}{2}} \\ &= \sum_{n=1}^{\infty} \log\left\{1+\sum_{i=1}^{p} c_i \left(\frac{q^{2n-1}}{(1+q^{2n-1})^2}\right)^i\right\}.\end{aligned}$$

Using the definition of q and the Taylor series expansion for $(1-s)^{-1/2}$ it is a simple calculation to show that $q \in (0,1)$ and as $s \uparrow 1$, $q \uparrow 1$.

3.2.3 Proposition. *Suppose $Z_q : [1,\infty) \to \mathbf{C}$ is defined as*

$$Z_q(x) = 1+\sum_{i=1}^{p} c_i \left(\frac{q^{2x-1}}{(1+q^{2x-1})^2}\right)^i,$$

where $q \in (0,1)$ and c_i are defined as above, and $Y_q : [1,\infty) \to \mathbf{C}$ as $Y_q(x) = \log Z_q(x)$. Then

$$\lim_{q \uparrow 1} \left(\frac{1}{2}\sum_{n=1}^{\infty} (Y_q(n) + Y_q(n+1)) - \int_1^{\infty} Y_q(x)\,dx\right) = 0$$

The proof of this proposition will follow after a series of lemmas which are of fundamental use in the proof.

3.2.4 Lemma. *Define the following functions from $\mathbf{N} \times (0,1) \times [0,1] \to \mathbf{C}$.*

$$\begin{aligned}X_1(n,q,u) &= 1+\sum_{i=1}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^i. \\ X_2(n,q,u) &= \sum_{i=1}^{p} c_i i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1}. \\ X_3(n,q,u) &= \sum_{i=1}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1} \sum_{j=0}^{i-1} \left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^j.\end{aligned}$$

Then $\exists C_i > 0$ for $i = 1,2,3$ and $Q \in (0,1)$ such that

(i)	$\lvert X_1(n,q,u)\rvert > C_1 > 0$	$\forall n \in \mathbf{N}, \forall q \in (0,1), \forall u \in [0,1]$.
(ii)	$\lvert X_2(n,q,u)\rvert < C_2$	$\forall n \in \mathbf{N}, \forall q \in (0,1), \forall u \in [0,1]$.
(iii)	$\lvert X_3(n,q,u)\rvert < C_3$	$\forall n \in \mathbf{N}, \forall q \in (Q,1), \forall u \in [0,1]$.

Proof: i): follows from remark 3.2.2 .

ii): follows from the fact that the function $f(y) = \dfrac{y}{(1+y)^2}$ on $[0,1]$ has maximum value $1/4$ so take

$$C_2 = \sum_{i=1}^{p} |c_i| \frac{i}{4^{i-1}}.$$

iii):

$$\frac{1}{q^{2u}}\frac{(1+q^{2n-1-2u})^2}{(1+q^{2n-1})^2} = 1 + \frac{(1-q^{2u})(1-q^{4n-2+2u})}{q^{2u}(1+q^{2n-1})^2}$$
$$< 1 + 1/q^2 \qquad \forall n \in \mathbb{N}, \forall u \in [0,1]$$
$$< 3 \qquad \forall q \in (Q,1) \quad \text{where } Q = \frac{1}{\sqrt{2}}.$$

So

$$\sum_{j=0}^{i-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^j < \sum_{j=0}^{i-1} 3^j = \frac{1}{2}(3^i - 1)$$

and

$$|X_3(n,q,u)| < \sum_{i=1}^{p} |c_i| \left(\frac{q^{2n-1-2u}}{(1+q^{2n-1-2u})^2}\right)^{i-1} \sum_{j=0}^{i-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^j$$
$$< \sum_{i=1}^{p} |c_i| \frac{1}{2^{i-1}} \frac{1}{2}(3^i - 1).$$

Let C_3 be as above, then $|X_3(n,q,u)| < C_3 \quad \forall n \in \mathbb{N}, \forall u \in [0,1], \forall q \in \left(\frac{1}{\sqrt{2}}, 1\right)$.

3.2.5 Lemma. *Define the following functions from* $\mathbb{N} \times (0,1) \times [0,1] \to \mathbb{C}$.

$$M_0(n,q,u) = \frac{q^{2n-1}(1-q^{2u})(1-q^{4n-2+2u})}{(1+q^{2n-1})^2(1+q^{2n-1+2u})^2} \cdot \frac{\displaystyle\sum_{i=1}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1} \sum_{j=0}^{i-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^j}{\displaystyle 1 + \sum_{i=1}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^i},$$

$$M_1(n,q,u) = \frac{1}{(1+q^{2n-1-2u})^2(1+q^{2n-1})^2} \frac{\displaystyle\sum_{i=1}^{p} c_i i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1}}{\displaystyle 1 + \sum_{i=1}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^i},$$

$$M_2(n,q,u) = \frac{(1-q^{4n-2+2u})^2}{q^{2u}(1+q^{2n-1})^4(1+q^{2n-1+2u})^2} \cdot \frac{\displaystyle\sum_{i=2}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1} \sum_{j=1}^{i-1}\sum_{k=0}^{j-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^k}{\displaystyle 1 + \sum_{i=1}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^i},$$

$$M_3(n,q,u) = \left(\frac{\displaystyle\sum_{i=1}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1} \sum_{j=0}^{i-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^j}{\displaystyle 1 + \sum_{i=1}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^i}\right)^2 \cdot \frac{q^{2n-1}(1-q^{4n-2+2u})^2}{(1+q^{2n-1})^4(1+q^{2n-1+2u})^4} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} M_0(n,q,u)^m}{m+2}.$$

Then $\exists \mathcal{M}_i$ and Q_i for $i = 1,2,3$ such that

$$|M_i(n,q,u)| < \mathcal{M}_i \qquad \forall n \in \mathbb{N}, \forall u \in [0,1], \forall q \in (Q_i, 1)$$

and $\exists Q_a$ such that

$$|M_a(n,q,u)| < 1 \qquad \forall n \in \mathbb{N}, \forall u \in [0,1], \forall q \in (Q_a, 1).$$

PROOF: 1): in the notation of lemma 3.2.4

$$M_0(n,q,u) = \frac{q^{2n-1}(1-q^{2u})(1-q^{4n-2+2u})}{(1+q^{2n-1})^2(1+q^{2n-1+2u})^2}\frac{X_2(n,q,u)}{X_1(n,q,u)}.$$

Therefore by lemma 3.2.4

$$\begin{aligned}|M_0(n,q,u)| &< (1-q^2)\frac{C_2}{C_1} \qquad \forall n \in \mathbb{N}, \forall u \in [0,1], \forall q \in \left(\frac{1}{\sqrt{2}}, 1\right) \\ &< 1 \qquad \forall n \in \mathbb{N}, \forall u \in [0,1], \forall q \in \left(\max\left\{\frac{1}{\sqrt{2}}, \sqrt{1-\frac{C_1}{C_2}}\right\}, 1\right).\end{aligned}$$

2): in the notation of lemma 3.2.4

$$M_1(n,q,u) = \frac{1}{(1+q^{2n-1-2u})^2(1+q^{2n-1})^2}\frac{X_2(n,q,u)}{X_1(n,q,u)}.$$

Therefore by lemma 3.2.4

$$|M_1(n,q,u)| < \frac{C_2}{C_1} \qquad \forall n \in \mathbb{N}, \forall u \in [0,1], \forall q \in (0,1).$$

Hence take $\mathcal{M}_1 = \frac{C_2}{C_1}$ and $Q_1 = 0$.

3): from the proof of lemma 3.2.4 part iii)

$$\sum_{k=0}^{j-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^k < \frac{(3^j-1)}{2} < \frac{3^j}{2} \qquad \forall n \in \mathbb{N}, \forall u \in [0,1] \forall q \in \left(\frac{1}{\sqrt{2}}, 1\right).$$

Therefore $\forall n \in \mathbb{N}$, $\forall u \in [0,1]$ and $\forall q \in \left(\frac{1}{\sqrt{2}}, 1\right)$

$$\sum_{j=1}^{i-1}\sum_{k=0}^{j-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^k < \sum_{j=1}^{i-1}\frac{3^j}{2} = \frac{3}{4}(3^{i-1}-1) < \frac{3^i}{4},$$

thus $\forall n \in \mathbb{N}$, $\forall u \in [0,1]$ and $\forall q \in \left(\frac{1}{\sqrt{2}}, 1\right)$

$$\left|\sum_{i=2}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1}\sum_{j=1}^{i-1}\sum_{k=0}^{j-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^k\right| < \sum_{i=2}^{p}|c_i|\left(\frac{3}{4}\right)^i.$$

Also $\forall n \in \mathbb{N}$, $\forall u \in [0,1]$ and $\forall q \in \left(\frac{1}{\sqrt{2}}, 1\right)$

$$\left|\frac{(1-q^{4n-2+2u})^2}{q^{2u}(1+q^{2n-1})^4(1+q^{2n-1+2u})^2}\right| < \frac{1}{q^2} < 2$$

so by the above and lemma 3.2.4

$$|M_2(n,q,u)| < 2\sum_{i=2}^{p} |c_i|(\frac{3}{4})^i.\frac{1}{C_1} \qquad \forall n \in \mathbf{N}, \forall u \in [0,1] \forall q \in \left(\frac{1}{\sqrt{2}}, 1\right).$$

Hence take $\mathcal{M}_2$ as above and $Q_2 = \frac{1}{\sqrt{2}}$.

4): in the notation of lemma 3.2.4

$$M_3(n,q,u) = \frac{q^{2n-1}(1-q^{4n-2+2u})^2}{(1+q^{2n-1})^4(1+q^{2n-1+2u})^4}\left(\frac{X_3(n,q,u)}{X_1(n,q,u)}\right)^2 \sum_{m=0}^{\infty} \frac{(-1)^{m+1}M_0(n,q,u)^m}{m+2}$$

so by lemma 3.2.4 and 1) above

$$\begin{aligned}
|M_3(n,q,u)| &< \left(\frac{C_3}{C_1}\right)^2 \sum_{m=0}^{\infty} \frac{\left((1-q^2)\frac{C_3}{C_1}\right)^m}{m+2} \qquad \forall n \in \mathbf{N}, \forall u \in [0,1] \forall q \in \left(\frac{1}{\sqrt{2}}, 1\right) \\
&< \left(\frac{C_3}{C_1}\right)^2 \sum_{m=0}^{\infty} \frac{1}{(m+2)2^m} \\
&\qquad \forall n \in \mathbf{N}, \forall u \in [0,1] \forall q \in \left(\max\left\{\frac{1}{\sqrt{2}}, \sqrt{1-\frac{C_1}{2C_3}}\right\}, 1\right) \\
&\stackrel{\text{def}}{=} \mathcal{M}_3.
\end{aligned}$$

3.2.6 **Lemma.**

$$\frac{Z_q(n)}{Z_q(n+u)} = 1 + \frac{q^{2n-1}(1-q^{2u})(1-q^{4n-2+2u})}{(1+q^{2n-1})^2(1+q^{2n-1+2u})^2}$$
$$\cdot\frac{\sum_{i=1}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1} \sum_{j=0}^{i-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^j}{1+\sum_{i=1}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^i}.$$

Proof: Straightforward calculation using definition of $Z_q(x)$ and the factorization

$$a^i - b^i = (a-b)(a^{i-1} + a^{i-2}b + \cdots + ab^{i-2} + b^{i-1}).$$

3.2.7 **Lemma.** *Define the function* $A : [1,\infty) \times (0,1) \times [0,1] \to \mathbf{R}$ *as*

$$A(n,q,u) = (1-q^{2u})(1-q^{4n-2+2u}) + u\log q^2.q^{2u}.\frac{(1-q^{2n-1+2u})(1+q^{2n-1})^2}{(1+q^{2n-1+2u})}.$$

Then $\exists Q \in (0,1)$ *such that* $\forall n \in [1,\infty)$, $\forall u \in [0,1]$, *and* $\forall q \in (Q,1)$

$$\begin{aligned}
|A(n,q,u)| \le \max\Big\{&(1-q^2) + q^2\log q^2, \\
&-\left[(1-q^2)(1-q^4) + \log q^2.q^2\frac{(1-q^3)(1+q)^2}{(1+q^3)}\right]\Big\}.
\end{aligned}$$

Proof: If $u = 0$, $A(n,q,u) = 0\ \forall n, q$ so forget this case.

Now

$$
\begin{aligned}
&\frac{\partial A(n,q,u)}{\partial n} \\
&= (1-q^{2u})\left(-4\log q . q^{4n-2+2u}\right) + u\log q^2 . q^{2u} . \frac{1}{(1+q^{2n-1+2u})^3}\Big\{(1+q^{2n-1+2u}) \\
&\quad . \left[-\log q^2 . q^{2n-1+2u}(1+q^{2n-1})^2 + (1-q^{2n-1+2u}) . 2(1+q^{2n-1})\log q^2 . q^{2n-1}\right] \\
&\quad - (1-q^{2n-1+2u})(1+q^{2n-1})^2\log q^2 . q^{2n-1+2u}\Big\} \\
&= -4\log q(1-q^{2u})q^{4n-2+2u} + \frac{u\log q^2 . q^{2u}}{(1+q^{2n-1+2u})^2} . 2\log q^2 . q^{2n-1}(1+q^{2n-1}) \\
&\quad . \left[(1+q^{2n-1+2u})(1-q^{2n-1+2u}) - q^{2u}(1+q^{2n-1})\right] \\
&= \frac{2\log q^2 . q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\Big\{u\log q^2(1+q^{2n-1})\left[(1-q^{2u}) - q^{2n-1+2u}(1+q^{2n-1+2u})\right] \\
&\qquad\qquad - (1-q^{2u})q^{2n-1}(1+q^{2n-1+2u})^2\Big\}
\end{aligned}
$$

If the functions $a, b, c : (0,1) \times (0,1] \to \mathbb{R}$ are defined as follows:

$$
a(q,u) = u\log q^2, \qquad b(q,u) = 1-q^{2u}, \qquad \text{and } c(q,u) = q^{2u},
$$

and the substitution $x = q^{2n-1}$ is made, the term inside the bracket becomes (dropping the q,u dependence in the notation for convenience):

$$
a(1+x)\left(b - cx(1+cx)\right) - bx(1+cx)^2 .
$$

In lemmas 3.2.8 and 3.2.9 it is shown that such a function is negative $\forall x \in [0,1]$, $\forall q \in (0,1)$ and $\forall u \in (0,1]$. Hence in particular it is true when $x = q^{2n-1}$ as $q^{2n-1} \in [0,1]$, $\forall q \in (0,1)$ and $\forall n \in [1,\infty)$, thus the term inside the bracket is negative $\forall n \in [1,\infty)$, $\forall q \in (0,1)$ and $\forall u \in (0,1]$. It is easy to see that the multiplying factor is also always negative. Hence $\dfrac{\partial A(n,q,u)}{\partial n}$ is positive $\forall n \in [1,\infty)$, $\forall q \in (0,1)$ and $\forall u \in (0,1]$.

So if q and u are fixed A increases as n increases. Thus the inequality

$$
A(1,q,u) \le A(n,q,u) \le A(\infty,q,u) \qquad \forall n \in [1,\infty), \forall q \in (0,1), \forall u \in (0,1]
$$

holds where $A(\infty,q,u)$ is the function defined from $(0,1) \times [0,1] \to \mathbb{R}$ by

$$
A(\infty,q,u) = (1-q^{2u}) + u\log q^2 . q^{2u}.
$$

So $|A(n,q,u)| \le \max\left(|A(1,q,u)|, |A(\infty,q,u)|\right)$.

Lemma 3.2.10 shows that

$$
|A(\infty,q,u)| \le (1-q^2) + q^2\log q^2 \qquad \forall q \in (0,1), \forall u \in [0,1].
$$

Lemmas 3.2.11 and 3.2.12 show that $\forall q \in (Q,1)$, $\forall u \in [0,1]$

$$
|A(1,q,u)| \le -\left[(1-q^2)(1-q^4) + \log q^2 . q^2\frac{(1-q^2)(1+q)^2}{(1+q^2)}\right]
$$

and the relevant Q is found.

Thus, as required, $\forall n \in [1,\infty)$, $\forall u \in [0,1]$, and $\forall q \in (Q,1)$

$$
\begin{aligned}
|A(n,q,u)| \le \max\Big\{&(1-q^2) + q^2\log q^2, \\
&-\left[(1-q^2)(1-q^4) + \log q^2 . q^2\frac{(1-q^2)(1+q)^2}{(1+q^2)}\right]\Big\}.
\end{aligned}
$$

3.2.8 Lemma. *Define the functions $a,b,c : (0,1) \times (0,1] \to \mathbb{R}$ by*

$$a(q,u) = u \log q^2, \qquad b(q,u) = 1 - q^{2u}, \qquad \text{and } c(q,u) = q^{2u},$$

then

$$\frac{a(c-b)+b}{-3(a+b)c} > 1 \qquad \forall q \in (0,1), \forall u \in (0,1].$$

Proof. First note that $a + b < 0 \quad \forall u \in (0,1], \forall q \in (0,1)$. This follows from the fact that $\dfrac{\partial(a+b)}{\partial u} = \log q^2(1-q^{2u}) < 0 \quad \forall u \in (0,1], \forall q \in (0,1)$ and $(a+b)(q,0) = 0$ gives a continuous extension.

So as $c > 0$ it is equivalent to prove $a(c-b) + b > -3(a+b)c$ which after some rearrangement and the observation that $b = 1 - c$ is equivalent to

$$4(a+b) - b(5a + 3b) > 0.$$

Therefore letting $f(q,u)$ denote this function it is necessary to prove that $f(q,u) > 0$ $\forall q \in (0,1)$, $\forall u \in (0,1]$ where

$$f(q,u) = 4\left(u \log q^2 + (1-q^{2u})\right) - (1-q^{2u})\left(5u \log q^2 + 3(1-q^{2u})\right).$$

Now

$$\begin{aligned}\frac{\partial f(q,u)}{\partial u} &= 4(\log q^2 - \log q^2.q^{2u}) + \log q^2.q^{2u}\left(5u \log q^2 + 3(1-q^{2u})\right)\\ &\quad - (1-q^{2u})(5\log q^2 - 3\log q^2.q^{2u})\\ &= \log q^2\left[(1-q^{2u})(6q^{2u}-1) + 5u \log q^2.q^{2u}\right]\end{aligned}$$

Making the substitution $x = q^{2u}$ and noting $\log x = u \log q^2$ the term inside the bracket becomes

$$(1-x)(6x-1) + 5x \log x.$$

Now consider the function $g : [0,1] \to \mathbb{R}$ defined by $g(x) = (1-x)(6x-1) + 5x \log x$. $g'(x) = 12(1-x) + 5\log x$ and $g'' = -12 + 5/x$ so $g(x)$ has a maximum point at 1 with value 0. Using simple properties of the functions $12(x-1)$ and $5\log x$ it is easy to see that $g'(x)$ has another unique zero in $(0,1)$. So $g(x)$ has only one more turning point in $[0,1]$ other than $x = 1$. Hence as $g(0) = -1$, $g(x) < 0\ \forall x \in [0,1)$.

Consequently

$$(1-q^{2u})(6q^{2u}-1) + 5u \log q^2.q^{2u} < 0 \qquad \forall q \in (0,1), \forall u \in (0,1]$$

Thus $\dfrac{\partial f(q,u)}{\partial u} > 0\ \forall q \in (0,1)$, $\forall u \in (0,1]$. Hence $f(q,u) > 0\ \forall q \in (0,1)$, $\forall u \in (0,1]$ as $f(q,0) = 0$ gives a continuous extension.

3.2.9 Lemma. *Define the functions a,b,c as in lemma 3.2.8 and define the function $H : [0,1] \times (0,1) \times (0,1] \to \mathbb{R}$ by*

$$H(x,q,u) = ab + (b(a-1) - ac)x - (a(1+c) + 2b)cx^2 - (a+b)c^2x^3,$$

then

$$H(x,q,u) < 0 \qquad \forall x \in [0,1], \forall q \in (0,1), \forall u \in (0,1].$$

PROOF: From the proof of lemma 3.2.8 $a + b < 0$, $\forall q \in (0,1)$, $\forall u \in (0,1]$. Therefore $H(x,q,u)$ is a cubic in x with a positive leading coefficient. Now

$$\frac{\partial H(x,q,u)}{\partial x} = b(a-1) - ac - 2\left(a(1+c)+2b\right)cx - 3(a+b)c^2x^2.$$

This is a quadratic in x and thus has at most 2 solutions for $\dfrac{\partial H(x,q,u)}{\partial x} = 0$. The discriminant '$B^2 - 4AC$' in this case is:

$$\begin{aligned}
&4c^2\left(a(1+c)+2b\right)^2 + 12(a+b)c^2\left(b(a-1)-ac\right)\\
&= 4c^2\left[a^2(1+c)^2 + 4ab(1+c) + 4b^2 + 3b(a+b)(a-1) - 3ac(a+b)\right]\\
&= 4c^2[a^2 - a^2c + a^2c^2 + ab + abc + b^2 + 3a^2b + 3ab^2]\\
&= 4c^2[a^2 + a^2b^2 + b^2 + 2ab + 2a^2b + 2ab^2] \qquad \text{using } c = 1-b\\
&= 4c^2\left[(a+b)^2 + a^2b^2 + 2ab(a+b)\right]\\
&= 4c^2\left[(a+b)+ab\right]^2
\end{aligned}$$

So if H is extended to a function on $\mathbb{R} \times (0,1) \times (0,1]$ in the obvious way $\dfrac{\partial H}{\partial x} = 0$ at

$$x_0^{\pm} = \frac{2\left(a(1+c)+2b\right)c \pm 2c.-\left[(a+b)+ab\right]}{-2.3(a+b)c^2} = \frac{a(1+c)+2b \mp (a+b+ab)}{-3(a+b)c}.$$

The negative sign in front of the $(a+b)+ab$ term follows from the fact that it is strictly negative. This is obvious as each part is negative, $(a+b)$ as mentioned above, ab trivially since $a < 0$ and $b > 0$. So

$$\begin{aligned}
x_0^+ &= \frac{a(c-b)+b}{-3(a+b)c},\\
x_0^- &= \frac{2a+3b+a(c+b)}{-3(a+b)c} = -\frac{1}{c}, \qquad \text{using } c+b=1.
\end{aligned}$$

From lemma 3.2.8 $x_0^+ > 1$ $\forall q \in (0,1)$, $\forall u \in (0,1]$. Also $x_0^- < -1$ $\forall q \in (0,1)$, $\forall u \in (0,1]$ as $0 < c < 1$.

Now $H(0,q,u) = ab < 0$ $\forall q \in (0,1)$, $\forall u \in (0,1]$ therefore

$$H(x,q,u) < 0 \qquad \forall x \in [0,1], \forall q \in (0,1), \forall u \in (0,1]$$

since H is negative at zero and both turning points are outside the interval $[0,1]$.

3.2.10 LEMMA. *Define the function* $K : (0,1) \times [0,1] \to \mathbb{R}$ *by*

$$K(q,u) = (1 - q^{2u}) + uq^{2u}\log q^2,$$

then

$$|K(q,u)| \leq (1-q^2) + q^2\log q^2 \qquad \forall q \in (0,1), \forall u \in [0,1].$$

PROOF: Suppose the function f is defined on $[0,1]$ by $f(q) = (1-q^2) + q^2\log q^2$. Then f has negative gradient $\forall q \in (0,1)$, a maximum point at 0 with value 1 and a minimum point at 1 with value 0. Hence $f(q) > 0$ for all $q \in (0,1)$.

$K(q,0) = 0$ $\forall q \in (0,1)$ hence the inequality is true in this case. Now

$$\frac{\partial K(q,u)}{\partial u} = (\log q^2)^2.uq^{2u} > 0 \qquad \forall q \in (0,1), \forall u \in (0,1].$$

So $K(q,u) > 0$ $\forall q \in (0,1)$, $\forall u \in (0,1]$ and

$$K(q,u) \leq K(q,1) = (1-q^2) + q^2\log q^2 \qquad \forall q \in (0,1), \forall u \in [0,1].$$

3.2.11 Lemma. *Define the function* $L : \mathbb{R} \setminus \{0\} \times (0,1) \to \mathbb{R}$ *by*

$$L(x,q) = 2q(1-x)(1+qx)(q^2x-1) - \log|x|(1+q)^2(1-2qx-q^2x^2).$$

Then if $q > \sqrt[3]{\sqrt{2}-1}$,

$$L(x,q) \neq 0 \qquad \forall x \in \left[q^2, 1\right).$$

Proof: Split L as l_1 and l_2 defined as:

$$\begin{aligned} l_1(x,q) &= 2q(1-x)(1+qx)(q^2x-1), \\ l_2(x,q) &= \log|x|(1+q)^2(1-2qx-q^2x^2). \end{aligned}$$

Then $L(x,q) = 0$ if and only if $l_1(x,q) = l_2(x,q)$.

Now $l_1(x,q)$ is a cubic in x with a negative leading coefficient and $l_1(x,q) = 0$ when $x = 1$, $x = -1/q < -1$ or $x = 1/q^2 > 1$ and $l_1(0,q) = -2q < 0$. Therefore $l_1(x,q) < 0$ for $x \in (0,1)$.

$l_2(x,q) = 0$ when $\log|x| = 0$ or when $(1-2qx-q^2x^2) = 0$, that is, when $x = \pm 1$ or $x = \frac{-(1+\sqrt{2})}{q} < -1$ or $x = \frac{(\sqrt{2}-1)}{q}$.

Using the fact that $\log x \begin{cases} < 0, & \text{for } |x| < 1 \\ > 0, & \text{for } |x| > 1 \end{cases}$ and sign properties of a negative leading coefficient quadratic it is easy to see that

$$l_2(x,q) < 0 \qquad \text{for} \qquad \begin{cases} x > \max\left\{1, \frac{(\sqrt{2}-1)}{q}\right\} \\ x < \frac{-(1+\sqrt{2})}{q} \\ -1 < x < \min\left\{1, \frac{(\sqrt{2}-1)}{q}\right\} \end{cases}$$

$$l_2(x,q) > 0 \qquad \text{for} \qquad \begin{cases} \frac{-(1+\sqrt{2})}{q} < x < -1 \\ \min\left\{1, \frac{(\sqrt{2}-1)}{q}\right\} > x > \max\left\{1, \frac{(\sqrt{2}-1)}{q}\right\} \end{cases}$$

Also as $x \to 0^{\pm}$ then $l_2(x,q) \to -\infty$, and as $x \to \pm\infty$ then $l_2(x,q) \to -\infty$.

So if $q > (\sqrt{2}-1)$ then $\frac{(\sqrt{2}-1)}{q} < 1$ and by the above $l_2(x,q) > 0$ for $\frac{(\sqrt{2}-1)}{q} < x < 1$. Therefore, as $l_2(x,q) \to -\infty$ as $x \to 0^+$, $l_1(x,q)$ and $l_2(x,q)$ must intersect somewhere in the interval $\left(0, \frac{(\sqrt{2}-1)}{q}\right)$, but not in the interval $\left(\frac{(\sqrt{2}-1)}{q}, 1\right)$.

Therefore, if the requirement that $q^2 > \frac{(\sqrt{2}-1)}{q}$ is added, which after simplification is $q > \sqrt[3]{\sqrt{2}-1}$, the functions $l_1(x,q)$ and $l_2(x,q)$ do not intersect in the interval $\left[q^2, 1\right)$, that is, $L(x,q) \neq 0 \ \forall x \in \left[q^2, 1\right)$ where $q > \sqrt[3]{\sqrt{2}-1}$.

3.2.12 Lemma. *Define the function* $J : (0,1) \times [0,1] \to \mathbf{R}$ *by*

$$J(q,u) = (1-q^{2u})(1-q^{2+2u}) + u \log q^2 \cdot \frac{q^{2u}(1-q^{1+2u})(1+q)^2}{(1+q^{1+2u})}.$$

Then $\exists Q \in (0,1)$ *such that* $\forall q \in (Q,1)$, $\forall u \in [0,1]$

$$|J(q,u)| \leq -\left((1-q^2)(1-q^4) + \log q^2 \cdot \frac{q^2(1-q^3)(1+q)^2}{(1+q^3)}\right).$$

Proof: Suppose $u \neq 0$ then

$$
\begin{aligned}
&\frac{\partial J(q,u)}{\partial u} \\
&= -\log q^2 . q^{2u}(1 - q^{2+2u}) - \log q^2\, q^{2+2u}(1 - q^{2u}) \\
&\quad + \log q^2 . \frac{q^{2u}(1 - q^{1+2u})(1+q)^2}{(1+q^{1+2u})} + u \log q^2 (1+q)^2 \log q^2 . q^{2u} \frac{(1 - q^{1+2u})}{(1+q^{1+2u})} \\
&\quad + \frac{u \log q^2 (1+q)^2 q^{2u}}{(1+q^{1+2u})^2} \left[-\log q^2 . q^{1+2u}(1+q^{1+2u}) - (1 - q^{1+2u}) \log q^2 . q^{1+2u} \right] \\
&= -\frac{\log q^2 . q^{2u}}{(1+q^{1+2u})^3} \Big\{ \left[(1 - q^{2+2u}) + q^2(1 - q^{2u}) \right] (1 + q^{1+2u})^2 \\
&\quad - (1+q)^2 (1 - q^{1+2u})(1 + q^{1+2u}) \\
&\quad - u \log q^2 (1+q)^2 \left[(1 - q^{1+2u})(1 + q^{1+2u}) - 2q^{1+2u} \right] \Big\} \\
&= -\frac{\log q^2 . q^{2u}}{(1+q^{1+2u})^3} \Big\{ (1 + q^{1+2u})[-2q + 2q^{1+2u} + 2q^{2+2u} - 2q^{3+4u}] \\
&\quad - u \log q^2 (1+q)^2 [1 - 2q^{1+2u} - q^{2+4u}] \Big\} \\
&= -\frac{\log q^2 . q^{2u}}{(1+q^{1+2u})^3} \Big\{ 2q(1 - q^{2u})(1 + q^{1+2u})(q^{2+2u} - 1) \\
&\quad - u \log q^2 (1+q)^2 [1 - 2q^{1+2u} - q^{2+4u}] \Big\}.
\end{aligned}
$$

If the substitution $x = q^{2u}$ is made inside the bracket, noting $\log x = u \log q^2$, it may be written as

$$2q(1-x)(1+qx)(q^2x - 1) - \log x (1+q)^2 (1 - 2qx - q^2x^2).$$

Now by lemma 3.2.11 if $q > \sqrt[3]{\sqrt{2} - 1}$ this is non-zero for $x \in [q^2, 1)$. In fact it is an easy consequence from the proof that $L(x) < 0$ for $x \in [q^2, 1)$. But the substitution $x = q^{2u}$ gives $x \in [q^2, 1)$ since $u \in (0, 1]$ as required. Also, as can be easily seen, the multiplier in front of the bracket is positive for all $q \in (0,1)$, for all $u \in (0,1]$. Thus

$$\frac{\partial J(q,u)}{\partial u} < 0 \qquad \forall q \in \left(\sqrt[3]{\sqrt{2} - 1}, 1 \right), \forall u \in (0,1].$$

So as $J(q, 0) = 0$ for all $q \in (0, 1)$

$$
\begin{aligned}
|J(q,u)| \leq |J(q,1)| = \left| (1 - q^2)(1 - q^4) + \log q^2 . \frac{q^2(1 - q^2)(1+q)^2}{(1+q^3)} \right| \\
\forall q \in \left(\sqrt[3]{\sqrt{2} - 1}, 1 \right), \forall u \in [0,1].
\end{aligned}
$$

Now the Taylor series expansion for $\log q^2 = \log(1 - (1 - q^2))$ gives $\log q^2 < -(1 - q^2)$. So

$$
\begin{aligned}
J(q,1) &< \frac{(1 - q^2)^2}{(1 + q^3)} \left[(1 + q^2)(1 + q^3) - q^2(1+q)(1 + q + q^2) \right] \\
&= \frac{(1 - q^2)^2}{(1 + q^3)} [1 - q^3 - 2q^4].
\end{aligned}
$$

A simple analysis of the function $1-q^2-2q^4$ gives that $J(q,1)<0$ if $q \in (Q_0,1)$ where Q_0 is the unique solution in $(0,1)$ to $1-q^2-2q^4=0$. If $q=\sqrt[4]{\sqrt{2}-1}$ then

$$1-q^2-2q^4=\sqrt{2}(\sqrt{2}-1)(1-\sqrt{2}\sqrt[4]{\sqrt{2}-1})<0,$$

as the last term is negative. So $Q_0<\sqrt[4]{\sqrt{2}-1}$ and $J(q,1)<0$ for all $q \in \left(\sqrt[4]{\sqrt{2}-1},1\right)$. Therefore

$$|J(q,1)|=-\left((1-q^2)(1-q^4)+\log q^2\frac{q^2(1-q^2)(1+q)^2}{(1+q^2)}\right) \qquad \forall q \in \left(\sqrt[4]{\sqrt{2}-1},1\right).$$

Hence have result, where $Q=\sqrt[4]{\sqrt{2}-1}$.

With the information accumulated in the previous lemmas 3.2.4-3.2.12 it is now possible to prove proposition 3.2.3 .

PROOF:

$$\begin{aligned}
&\frac{1}{2}\sum_{n=1}^{\infty}(Y_q(n)+Y_q(n+1))-\int_1^{\infty}Y_q(x)\,dx\\
&=\sum_{n=1}^{\infty}\left\{\frac{1}{2}(Y_q(n)+Y_q(n+1))-\int_n^{n+1}Y_q(x)\,dx\right\}\\
&=\sum_{n=1}^{\infty}\left\{\int_n^{n+1}(Y_q(n)-Y_q(x))\,dx+\frac{1}{2}(Y_q(n+1)-Y_q(n))\right\}\\
&=\sum_{n=1}^{\infty}\int_0^1\left\{Y_q(n)-Y_q(n+u)+u(Y_q(n+1)-Y_q(n))\right\}du\\
&=\sum_{n=1}^{\infty}\int_0^1\log\left\{\frac{Z_q(n)}{Z_q(n+u)}\cdot\left[\frac{Z_q(n+1)}{Z_q(n)}\right]^u\right\}du.
\end{aligned}$$

Now

$$\left|\int_0^1\log\left\{\frac{Z_q(n)}{Z_q(n+u)}\cdot\left[\frac{Z_q(n+1)}{Z_q(n)}\right]^u\right\}du\right|\le\max_{u\in[0,1]}\left\{\left|\log\left\{\frac{Z_q(n)}{Z_q(n+u)}\cdot\left[\frac{Z_q(n+1)}{Z_q(n)}\right]^u\right\}\right|\right\}$$

and when $u=0$ or 1

$$\log\left\{\frac{Z_q(n)}{Z_q(n+u)}\cdot\left[\frac{Z_q(n+1)}{Z_q(n)}\right]^u\right\}=0 \qquad \forall n\in\mathbb{N},\forall q\in(0,1).$$

So the maximum value occurs at a turning point of $\log\left\{\frac{Z_q(n)}{Z_q(n+u)}\cdot\left[\frac{Z_q(n+1)}{Z_q(n)}\right]^u\right\}$ with respect to u, that is

$$\frac{\partial}{\partial u}\left(\log\left\{\frac{Z_q(n)}{Z_q(n+u)}\cdot\left[\frac{Z_q(n+1)}{Z_q(n)}\right]^u\right\}\right)=\frac{-\frac{\partial}{\partial u}(Z_q(n+u))}{Z_q(n+u)}+\log\left\{\frac{Z_q(n+1)}{Z_q(n)}\right\}=0.$$

Therefore consider the function $X:\mathbb{N}\times(0,1)\times[0,1]\to\mathbb{C}$ defined by

$$X(n,q,u)=\log\left\{\frac{Z_q(n)}{Z_q(n+u)}\right\}+u\,\frac{\frac{\partial}{\partial u}(Z_q(n+u))}{Z_q(n+u)}.$$

Now

$$Z_q(n+u) = 1 + \sum_{i=1}^{p} c_i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2} \right)^i = X_1(n,q,u),$$

in the notation of lemma 3.2.4, so

$$\begin{aligned}
\frac{\partial Z_q(n+u)}{\partial u} &= \sum_{i=1}^{p} c_i i \left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2} \right)^{i-1} . \log q^2 . \frac{q^{2n-1+2u}(1-q^{2n-1+2u})}{(1+q^{2n-1+2u})^2} \\
&= \log q^2 . \frac{q^{2n-1+2u}(1-q^{2n-1+2u})}{(1+q^{2n-1+2u})^2} . X_2(n,q,u),
\end{aligned}$$

in the notation of lemma 3.2.4. Also from lemmas 3.2.6 and 3.2.5

$$\frac{Z_q(n)}{Z_q(n+u)} = 1 + M_0(n,q,u),$$

with

$$|M_0(n,q,u)| < 1 \qquad \forall n \in \mathbb{N}, \forall u \in [0,1], \forall q \in \left(\max\left\{ \frac{1}{\sqrt{2}}, \sqrt{1-\frac{C_1}{C_3}} \right\}, 1 \right).$$

Hence $\log\{1 + M_0(n,q,u)\}$ may be written as an infinite sum

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(M_0(n,q,u) \right)^m ,$$

provided q is large enough. Therefore X may be rewritten as

$$X(n,q,u) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(M_0(n,q,u) \right)^m + u \log q^2 . \frac{q^{2n-1+2u}(1-q^{2n-1+2u})}{(1+q^{2n-1+2u})^2} . \frac{X_2(n,q,u)}{X_1(n,q,u)}$$

giving

$$\begin{aligned}
|X(n,q,u)| \le &\left| M_0(n,q,u) + u \log q^2 . \frac{q^{2n-1+2u}(1-q^{2n-1+2u})}{(1+q^{2n-1+2u})^2} . \frac{X_2(n,q,u)}{X_1(n,q,u)} \right| \\
&+ \left| \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} \left(M_0(n,q,u) \right)^m \right| .
\end{aligned}$$

Now

$$\begin{aligned}
&M_0(n,q,u) + u \log q^2 . \frac{q^{2n-1+2u}(1-q^{2n-1+2u})}{(1+q^{2n-1+2u})^2} . \frac{X_2(n,q,u)}{X_1(n,q,u)} \\
&= \frac{q^{2n-1}(1-q^{2u})(1-q^{4n-2+2u})}{(1+q^{2n-1})^2(1+q^{2n-1+2u})^2} . \frac{X_3(n,q,u)}{X_1(n,q,u)} \\
&\quad + u \log q^2 . \frac{q^{2n-1+2u}(1-q^{2n-1+2u})}{(1+q^{2n-1+2u})^2} . \frac{X_2(n,q,u)}{X_1(n,q,u)}
\end{aligned}$$

see the proof of lemma 3.2.5

$$\begin{aligned}
&= \frac{q^{2n-1}}{(1+q^{2n-1})^2(1+q^{2n-1+2u})^2} . \frac{X_2(n,q,u)}{X_1(n,q,u)} \Bigg[(1-q^{2u})(1-q^{4n-2+2u}) \\
&\quad + u \log q^2 . \frac{q^{2u}(1-q^{2n-1+2u})(1+q^{2n-1})^2}{(1+q^{2n-1+2u})} \Bigg] \\
&\quad + \frac{q^{2n-1}(1-q^{2u})(1-q^{4n-2+2u})}{(1+q^{2n-1})^2(1+q^{2n-1+2u})^2} \frac{[X_3(n,q,u) - X_2(n,q,u)]}{X_1(n,q,u)} \\
&= q^{2n-1} M_1(n,q,u) A(n,q,u) \\
&\quad + \frac{q^{2n-1}(1-q^{2u})(1-q^{4n-2+2u})}{(1+q^{2n-1})^2(1+q^{2n-1+2u})^2} \frac{[X_3(n,q,u) - X_2(n,q,u)]}{X_1(n,q,u)} .
\end{aligned}$$

From lemma 3.2.4

$$
\begin{aligned}
&X_3(n,q,u)-X_2(n,q,u)\\
&=\sum_{i=1}^{p}c_i\left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1}\sum_{j=0}^{i-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^j\\
&\quad-\sum_{i=1}^{p}c_i i\left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1}\\
&=\sum_{i=1}^{p}c_i\left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1}\sum_{j=0}^{i-1}\left[\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^j-1\right]\\
&=\sum_{i=2}^{p}c_i\left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1}\sum_{j=1}^{i-1}\left[\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)-1\right]\sum_{k=0}^{j-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^k\\
&=\frac{(1-q^{2u})(1-q^{4n-2+2u})}{q^{2u}(1+q^{2n-1})^2}\sum_{i=2}^{p}c_i\left(\frac{q^{2n-1+2u}}{(1+q^{2n-1+2u})^2}\right)^{i-1}\sum_{j=1}^{i-1}\sum_{k=0}^{j-1}\left(\frac{(1+q^{2n-1+2u})^2}{q^{2u}(1+q^{2n-1})^2}\right)^k.
\end{aligned}
$$

Hence have

$$
q^{2n-1}M_1(n,q,u)A(n,q,u)+q^{2n-1}(1-q^{2u})^2M_2(n,q,u).
$$

Also

$$
\begin{aligned}
\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}(M_0(n,q,u))^m&=M_0(n,q,u)^2\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+2)}(M_0(n,q,u))^m\\
&=q^{2n-1}(1-q^{2u})^2M_3(n,q,u).
\end{aligned}
$$

So

$$
\begin{aligned}
&|X(n,q,u)|\\
&\le q^{2n-1}|M_1(n,q,u)A(n,q,u)|+q^{2n-1}|M_2(n,q,u)(1-q^{2u})^2|\\
&\quad+q^{2n-1}|M_3(n,q,u)(1-q^{2u})^2|\\
&<q^{2n-1}\mathcal{M}_1\max\left\{(1-q^2)+q^2\log q^2,-(1-q^2)(1-q^4)-\log q^2\cdot\frac{q^2(1-q^2)(1+q)^2}{(1+q^2)}\right\}\\
&\quad+q^{2n-1}\mathcal{M}_2(1-q^2)^2+q^{2n-1}\mathcal{M}_3(1-q^2)^2\\
&\quad\forall n\in\mathbb{N},\forall u\in[0,1],\forall q\in\left(\max\left\{\sqrt[4]{\sqrt{2}-1},\sqrt{1-\frac{C_1}{2C_2}}\right\},1\right)\\
&\quad\text{using the results of the previous lemmas}\\
&=q^{2n-1}\left[\mathcal{M}_1\max\left\{(1-q^2)+q^2\log q^2,-(1-q^2)(1-q^4)-\log q^2\cdot\frac{q^2(1-q^2)(1+q)^2}{(1+q^2)}\right\}\right.\\
&\quad\left.+(\mathcal{M}_2+\mathcal{M}_3)(1-q^2)^2\right].
\end{aligned}
$$

Therefore in particular

$$
\begin{aligned}
&\left|\int_0^1\log\left\{\frac{Z_q(n)}{Z_q(n+u)}\cdot\left[\frac{Z_q(n+1)}{Z_q(n)}\right]^u\right\}du\right|\\
&<q^{2n-1}\left[\mathcal{M}_1\max\left\{(1-q^2)+q^2\log q^2,-(1-q^2)(1-q^4)-\log q^2\cdot\frac{q^2(1-q^2)(1+q)^2}{(1+q^2)}\right\}\right.\\
&\quad\left.+(\mathcal{M}_2+\mathcal{M}_3)(1-q^2)^2\right]\\
&\quad\forall n\in\mathbb{N},\forall q\in\left(\max\left\{\sqrt[4]{\sqrt{2}-1},\sqrt{1-\frac{C_1}{2C_2}}\right\},1\right).
\end{aligned}
$$

So

$$\sum_{n=1}^{\infty}\left|\int_0^1 \log\left\{\frac{Z_q(n)}{Z_q(n+u)}.\left[\frac{Z_q(n+1)}{Z_q(n)}\right]^u\right\} du\right|$$
$$< q\Big[\mathcal{M}_1 \max\left\{1+\frac{q^2\log q^2}{(1-q^2)}, -\left[(1-q^4)+\frac{\log q^2}{(1-q^2)}.\frac{q^2(1-q^3)(1+q)^2}{(1+q^3)}\right]\right\}$$
$$+(\mathcal{M}_2+\mathcal{M}_3)(1-q^2)\Big]$$
$$\forall n\in \mathbb{N}, \forall q \in \left(\max\left\{\sqrt[3]{\sqrt{2}-1}, \sqrt{1-\frac{C_1}{2C_3}}\right\}, 1\right).$$

But $\lim_{q\uparrow 1}$RHS$=0$ as $\lim_{q\uparrow 1}\log q^2/(1-q^2)=-1$. Therefore

$$\lim_{q\uparrow 1}\sum_{n=1}^{\infty}\left|\int_0^1 \log\left\{\frac{Z_q(n)}{Z_q(n+u)}.\left[\frac{Z_q(n+1)}{Z_q(n)}\right]^u\right\} du\right| = 0.$$

So

$$\lim_{q\uparrow 1}\left[\frac{1}{2}\sum_{n=1}^{\infty}(Y_q(n)+Y_q(n+1)) - \int_1^{\infty} Y_q(x)\,dx\right] = 0.$$

3.2.13 Remark. The number $\sqrt{1-\frac{C_1}{2C_3}}$ may be replaced by $\sqrt{1-\frac{1}{3p}}$ in order to see that there do actually exist q in the range given.

3.3 Convergence Theorem.

The results of subsection 3.2 can now be used to examine the behaviour of one point correlations as the critical temperature is approached.

3.3.1 Theorem. *Suppose $M \in GL(p,\mathbb{C})$ with its eigenvalues denoted by $\lambda_1,\dots,\lambda_p$ and $\lambda_i \in \mathbb{C}\setminus(-\infty,0]$ for $i=1,\dots,p$. Define the complex numbers c_i for $i=1,\dots,p$ as*

$$c_i = \sum_{1\le j_1<\cdots<j_i\le p} c(\lambda_{j_1})\dots c(\lambda_{j_i}) \qquad \text{where } c(\lambda)=\lambda+\lambda^{-1}-2,$$

and the function $G:[0,1]\to\mathbb{C}$ by

$$G(y)=\log\left\{1+\sum_{i=1}^{p} c_i\left(\frac{y}{(1+y)^2}\right)^i\right\}.$$

Now let $I=\int_0^1\frac{G(y)}{y}\,dy$ then the following holds:

(1) *If* $\mathrm{Re}\, I>0$ *then* $\lim_{q\uparrow 1}(\sigma_a(M)) = +\infty.$

(2) *If* $\mathrm{Re}\, I<0$ *then* $\lim_{q\uparrow 1}(\sigma_a(M)) = 0.$

(3) *If* $I=0$ *then* $\lim_{q\uparrow 1}(\sigma_a(M)) = 1.$

3.3.2 Remark. The imaginary part of I determines the direction of the outward or inward spiral occuring in cases (1) and (2) which will be explained in the proof.

PROOF: From Remark 3.2.2

$$\begin{aligned}\log(\sigma_a(M)) &= \sum_{n=1}^{\infty} \log\left\{1+\sum_{i=1}^{p} c_i \left(\frac{q^{2n-1}}{(1+q^{2n-1})^2}\right)^i\right\} \\ &= \sum_{n=1}^{\infty} Y_q(n) \qquad \text{in the notation of Proposition 3.2.3} \\ &= \frac{Y_q(1)}{2} + \left(\sum_{n=1}^{\infty}\left\{\frac{Y_q(n)+Y_q(n+1)}{2}\right\} - \int_1^{\infty} Y_q(x)\,dx\right) \\ &\quad + \int_1^{\infty} Y_q(x)\,dx\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{q\uparrow 1}\{\log(\sigma_a(M))\} &= \lim_{q\uparrow 1}\frac{Y_q(1)}{2} + \lim_{q\uparrow 1}\int_1^{\infty} Y_q(x)\,dx \\ &\quad + \lim_{q\uparrow 1}\left(\sum_{n=1}^{\infty}\left\{\frac{Y_q(n)+Y_q(n+1)}{2}\right\} - \int_1^{\infty} Y_q(x)\,dx\right) \\ &= \frac{1}{2}\log\left\{1+\sum_{i=1}^{p}\frac{c_i}{4^i}\right\} + \lim_{q\uparrow 1}\int_1^{\infty} Y_q(x)\,dx + 0 \qquad (\dagger)\end{aligned}$$

the first term being straightforward, the last given by Proposition 3.2.3. Now

$$\int_1^{\infty} Y_q(x)\,dx = \int_1^{\infty} \log\left\{1+\sum_{i=1}^{p} c_i\left(\frac{q^{2x-1}}{(1+q^{2x-1})^2}\right)^i\right\} dx$$

so make the substitution $y = q^{2x-1}$ to get

$$\int_q^0 \frac{\log\left\{1+\sum_{i=1}^{p} c_i\left(\frac{y}{(1+y)^2}\right)^i\right\}}{y\log q^2}\,dy = \left(\frac{-1}{\log q^2}\right)\cdot\int_0^q \frac{G(y)}{y}\,dy.$$

Therefore

$$\lim_{q\uparrow 1}\int_1^{\infty} Y_q(x)\,dx = \lim_{q\uparrow 1}\left\{\left(\frac{-1}{\log q^2}\right)\cdot\int_0^q \frac{G(y)}{y}\,dy\right\}.$$

But

$$\lim_{q\uparrow 1}\int_0^q \frac{G(y)}{y}\,dy = \int_0^1 \frac{G(y)}{y}\,dy = I$$

which is some complex number $\operatorname{Re} I + i\operatorname{Im} I$, since it is a definite integral of a function continuous on $[0,1]$. So if $I \neq 0$ the $\left(\frac{-1}{\log q^2}\right)$ term will dominate and the behaviour is as follows:

(1) $\operatorname{Re} I > 0$; $\operatorname{Im} I > 0$: "$\lim_{q\uparrow 1}\int_1^{\infty} Y_q(x)\,dx = \infty + \infty i$"
That is $(\sigma_a(M))$ spirals outwards in an anticlockwise direction as $q \uparrow 1$.
(2) $\operatorname{Re} I > 0$; $\operatorname{Im} I < 0$: "$\lim_{q\uparrow 1}\int_1^{\infty} Y_q(x)\,dx = \infty - \infty i$"
That is $(\sigma_a(M))$ spirals outwards in a clockwise direction as $q \uparrow 1$.
(3) $\operatorname{Re} I < 0$; $\operatorname{Im} I > 0$: "$\lim_{q\uparrow 1}\int_1^{\infty} Y_q(x)\,dx = -\infty + \infty i$"
That is $(\sigma_a(M))$ spirals inwards to zero in an anticlockwise direction as $q \uparrow 1$.
(4) $\operatorname{Re} I < 0$; $\operatorname{Im} I < 0$: "$\lim_{q\uparrow 1}\int_1^{\infty} Y_q(x)\,dx = -\infty - \infty i$"
That is $(\sigma_a(M))$ spirals inwards to zero in a clockwise direction as $q \uparrow 1$.

(5) $\operatorname{Im} I = 0$ so $I = \operatorname{Re} I$ and "$\lim_{q\uparrow 1} \int_{\cdot}^{\infty} Y_q(s)\,ds = \operatorname{sgn}(I)\infty$"
That is if $I > 0$ then $\lim_{q\uparrow 1}\langle\sigma_a(M)\rangle = \infty$
and if $I < 0$ then $\lim_{q\uparrow 1}\langle\sigma_a(M)\rangle = 0$.

(6) $\operatorname{Re} I = 0$ so $I = \operatorname{Im} I i$ and "$\lim_{q\uparrow 1} \int_{\cdot}^{\infty} Y_q(s)\,ds = \operatorname{sgn}(\operatorname{Im} I)\infty i$"
That is if $\operatorname{Im} I > 0$ then $\langle\sigma_a(M)\rangle$ rotates anticlockwise around the circle of radius R
and if $\operatorname{Im} I < 0$ then $\langle\sigma_a(M)\rangle$ rotates clockwise around the circle of radius R
where $R = \exp\left\{\operatorname{Re}\left[\frac{1}{2}\log\left(1 + \sum_{i=1}^{p} c_i/4^i\right)\right]\right\}$.

Hence the interesting case occurs when $I = 0$.
Applying L'Hopital's Rule have

$$\lim_{q\uparrow 1}\frac{\int_0^q \frac{G(y)}{y}\,dy}{-\log q^2} = \lim_{q\uparrow 1}\frac{G(q)/q}{-2/q}$$
$$= -\frac{1}{2}G(1) = -\frac{1}{2}\log\left\{1 + \sum_{i=1}^{p}\frac{c_i}{4^i}\right\}.$$

So placing this in equation (†) have

$$\lim_{q\uparrow 1}\log\langle\sigma_a(M)\rangle = \frac{1}{2}\log\left\{1 + \sum_{i=1}^{p}\frac{c_i}{4^i}\right\} - \frac{1}{2}\log\left\{1 + \sum_{i=1}^{p}\frac{c_i}{4^i}\right\} + 0$$
$$= 0$$

Therefore $\lim_{q\uparrow 1}\langle\sigma_a(M)\rangle = 1$.

Theorem 3.3.1 gives a classification of the critical limit of one point correlations, which will be investigated more thoroughly in the next section. But first some remarks on the negative eigenvalue situation.

3.3.2 Remark. Firstly the special case when the eigenvalue is -1. From Proposition 3.1.1

$$\langle\sigma_a(-1)\rangle = \prod_{l>0}\left[\frac{1-q^{2l}}{1+q^{2l}}\right]^2.$$

This is the square of the spontaneous magnetization for the Ising model, thus its critical temperature behaviour is already known, see [O1], [O2], [Y1] and [M1,Chapter X] for example, namely

$$\lim_{q\uparrow 1}\langle\sigma_a(-1)\rangle = 0.$$

Consequently, in some cases, the value -1 could be added as a permissible value for an eigenvalue with its treatment being separate from the others. That is, suppose the eigenvalues of M are $\lambda_1, \ldots, \lambda_{p-1}$ and -1 with the reduced matrix M' having eigenvalues $\lambda_1, \ldots, \lambda_{p-1}$. If M' satisfies Theorem 3.3.1 cases (2) or (3) then

$$\lim_{q\uparrow 1}\langle\sigma_a(M)\rangle = 0.$$

If however M' satisfies case (1) then the limit is not clear. This is due to the fact that Lemma 3.2.1 fails if $\lambda = -1$. Consequently Remark 3.2.2 fails meaning Lemma 3.2.4 (i) fails and this bound plays a crucial role in the convergence argument. Having said this I would suggest that if the limit exists then it is zero since the individual entries of the infinite product tend to zero as q tends to 1.

For λ negative with $\lambda \neq -1$, defining the sequence

$$q_l = \exp\left[\frac{\operatorname{sgn}(1+\lambda)}{2l} . \log(-\lambda)\right], \quad \text{for } l \in \mathbb{Z}^+_o,$$

or equivalently

$$q_n = \exp\left[\frac{\operatorname{sgn}(1+\lambda)}{2n-1} . \log(-\lambda)\right], \quad \text{for } n \in \mathbb{N},$$

$q_n \in (0,1)$ for all n, $\lim_{n\to\infty} q_n = 1$ but for each q_n $(\sigma_s(\lambda))_{Q(q_n)} = 0$. This suggests that for matrices with such eigenvalues the limit as $s \uparrow 1$ is zero if any such limit actually exists.

SECTION 4
AN EXAMPLE OF A NON-TRIVIAL LIMITING ONE POINT CORRELATION

4.1 Introduction.

The previous section gave a condition for a matrix with non-negative eigenvalues to have a monodromy field which has non-degenerate, not zero or infinity, critical limit correlation. However, as yet, the existence of any non-trivial matrix which actually satisfies this condition has not been shown. It is this matter which is considered in this section.

4.1.1 NOTATION. Let $\mathbf{C}^p_R$ denote the permissible values of $c_1, \ldots, c_p$, that is,

$$\mathbf{C}^p_R = \Big\{(c_1, \ldots, c_p) \in \mathbf{C}^p : c_i = \sum_{1 \le j_1 < \cdots < j_i \le p} e(\lambda_{j_1}) \ldots e(\lambda_{j_i}) \text{ where}$$
$$e(\lambda_i) = \lambda_i + \lambda_i^{-1} - 2 \text{ and } \lambda_i \in \mathbf{C} \setminus (-\infty, 0]\, \forall i = 1, \ldots, p\Big\}$$

and define the map $I : \mathbf{C}^p_R \to \mathbf{C}$ by

$$I(c_1, \ldots, c_p) = \int_0^1 \frac{\log\left\{1 + \sum_{i=1}^p c_i \left(\frac{y}{(1+y)^2}\right)^i\right\}}{y}\, dy.$$

Then the object of interest is the set of points in $\mathbf{C}^p_R$ with $I(c_1, \ldots, c_p) = 0$ which will be denoted by $\mathcal{C}^p$.

4.2 Investigation of $\mathcal{C}^p$ for $p = 1, 2$.

4.2.1 PROPOSITION.

(1) $0 \in \mathcal{C}^p \quad \forall p \ge 1$.
(2) $\mathcal{C}^1 = \{0\}$.

PROOF: (1): $I(0, \ldots, 0) = 0$ is obvious. This is equivalent to M being the identity matrix and is the 'trivial' situation referred to above.

(2): $I(c) = \int_0^1 \frac{\log\left\{1 + c\frac{y}{(1+y)^2}\right\}}{y}\, dy$, where

$$c \in \mathbf{C}^1_R = \{c \in \mathbf{C} : c = \lambda + \lambda^{-1} - 2, \lambda \in \mathbf{C} \setminus (-\infty, 0]\}$$
$$\subsetneq \mathbf{C} \setminus (-\infty, -4].$$

Suppose $c = a + ib$ then

$$\arg\left(1 + c\frac{y}{(1+y)^2}\right) = \arctan\left(\frac{by}{(1+y)^2 + ay}\right),$$

so the argument is either in the interval $[0, \pi)$ or $(-\pi, 0]$ depending on whether $b > 0$ or $b < 0$ for all $y \in [0, 1]$. That is

$$\int_0^1 \frac{\arg\left(1 + c\frac{y}{(1+y)^2}\right)}{y}\, dy = 0 \Leftrightarrow b = 0.$$

If $b = 0$ then $c = a$ so c is real.
Since for all $y \in (0, 1]$

$$\log\left(1 + a\frac{y}{(1+y)^2}\right) \begin{cases} > 0 & \text{for } a > 0 \\ < 0 & \text{for } a < 0 \end{cases}$$

it is simple to see that

$$\int_0^1 \frac{\log\left(1 + a\frac{y}{(1+y)^2}\right)}{y}\, dy = 0 \Leftrightarrow a = 0.$$

That is $I(c) = 0$ if and only if $c = 0$. So $\mathcal{C}^1 = \{0\}$.

Note. In the above the definition $\log(z) = \log|z| + \imath \arg z$ was used where the branch of the logarithm was taken along the negative real axis.

4.2.2 Remark. This proposition may appear a bit discouraging as it says, for the scalar case (p=1), there exist no non-identity complex numbers $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ which have a critical limit correlation $\lim_{s\uparrow 1}(\sigma(\lambda))$.

However the reason for this is the 'lack of freedom' in the scalar case which will now be explained. From the proof of Proposition 4.2.1 to get $I(c)$ zero the imaginary part of c must be zero, thus $I(c)$ is now only determined by the real part of c and this one variable dependence is not sufficient to get a non-trivial solution. That is one variable does not provide sufficient 'freedom' for a non-trivial solution to exist. However for the larger dimensional cases ($p \geq 2$) there are more variables present and hence more 'freedom' so a non-trivial solution is possible. It is this that will now be shown by considering the simplest case $p = 2$.

4.2.3 Proposition.

$$C^2 \supsetneq \{0\}.$$

Proof: By Proposition 4.2.1 (1) it is sufficient to find non-zero $c_1, c_2 \in \mathbb{C}^2_{\mathbb{R}}$ such that $I(c_1, c_2) = 0$. To simplify this problem somewhat, consider the restriction where $c(\lambda_1)$ and $c(\lambda_2)$ are real thus forcing c_1 and c_2 to be real by definition.

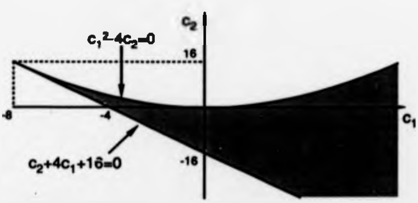
Now denote $c(\lambda_1)$ and $c(\lambda_2)$ by c and d respectively then

$$c, d \in (-4, \infty), \qquad c_1 = c + d \qquad \text{and } c_2 = cd.$$

Thus

$$\begin{aligned} -8 < \quad c+d \quad &= c_1 \\ 0 < (c+4)(d+4) &= c_2 + 4c_1 + 16, \\ \text{and } 0 \leq c_1^2 - 4c_2 \qquad & \text{as } c, d \text{ real.} \end{aligned}$$

Hence the permissible values of c_1 and c_2 are those in the shaded area of $\mathbb{R}^2$ shown below.



Define the function $F_{c_1,c_2} : [0,1] \to \mathbb{R}$ by

$$F_{c_1,c_2}(y) = \frac{y[c_1 + (2c_1 + c_2)y + c_1 y^2]}{(1+y)^4},$$

so that

$$I(c_1, c_2) = \int_0^1 \frac{\log\{1 + F_{c_1,c_2}(y)\}}{y}\, dy.$$

Now using $\log(1+x) < x$ for $x \neq 0$

$$\begin{aligned} I(c_1, c_2) &< \int_0^1 \frac{F_{c_1,c_2}(y)}{y}\, dy \\ &= \frac{(6c_1 + c_2)}{12} \qquad \text{by a simple calculation.} \end{aligned}$$

Thus if $c_2 = -6c_1$ then $I(c_1, c_2) < 0$.
By a simple analysis of the function $F_{c_1,c_2}(y)$, see Remark following this proof, if $c_2 = -4c_1$ then $F_{c_1,c_2}(y) > 0$ for all $y \in (0,1)$ so $I(c_1, c_2) > 0$.
Hence, by continuity, for each c_1 such that $0 < c_1 < 8$ there exists a $c_2(c_1)$ such that

$$-4c_1 < c_2(c_1) < -6c_1 \qquad \text{and } I(c_1, c_2(c_1)) = 0.$$

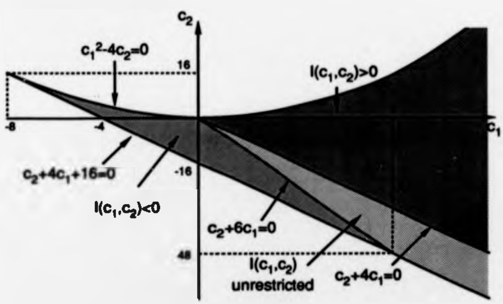
That is $(c_1, c_2(c_1)) \in C^2$ for $0 < c_1 < 8$ so $C^2 \neq \{0\}$.

4.2.4 REMARK. The proof of Proposition 4.2.3 demonstrated the existence of a particular set of non-zero points in C^2 without specifying if these were the only such points in C^2. This problem is now considered:

(1). Suppose c_1 and c_2 are real then either $c(\lambda_1), c(\lambda_2)$ are real or $c(\lambda_1) = \overline{c(\lambda_2)}$.
1): If $c(\lambda_1), c(\lambda_2)$ are real then the situation is as in Proposition 4.2.3 and it is fairly simple to see the following:

(a) If $c_1, c_2 > 0$ then $I(c_1, c_2) > 0$.
(b) If $c_1 = 0, c_2 \neq 0$ then $I(c_1, c_2)$ has the same sign as c_2.
(c) If $c_1 \neq 0, c_2 = 0$ then $I(c_1, c_2)$ has the same sign as c_1.
(d) If $c_1 < 0$ then $I(c_1, c_2) < 0$.
(e) If $c_1 > 0, 4c_1 + c_2 \geq 0$ then $I(c_1, c_2) > 0$.
(f) If $c_1 > 0, 6c_1 + c_2 \leq 0$ then $I(c_1, c_2) < 0$.

This gives the following picture.



This explains the restriction, in the proof of Proposition 4.2.3, of c_1 to the range $0 < c_1 < 8$ as the existence of a c_2 such that $I(c_1, c_2) < 0$ for $c_1 \geq 8$ is not known.
With $c_2 + 4c_1 < 0$, which is true in the area of interest, it is possible to deduce the following about the function $F_{c_1,c_2}(y)$ defined in the proof of Proposition 4.2.3 (See Appendix for the proof.):

(1) $F_{c_1,c_2}(0) = 0 : F'_{c_1,c_2}(0) > 0$.
(2) $0 > F_{c_1,c_2}(1) > -1 : F'_{c_1,c_2}(1) = 0 : F''_{c_1,c_2}(1) > 0$.
(3) $\forall c_1$ such that $0 < c_1 < 16$, $\exists y_0 \in (0,1)$ such that

$$0 < F_{c_1,c_2}(y_0) < 1 : F'_{c_1,c_2}(y_0) = 0 : F''_{c_1,c_2}(y_0) < 0.$$

Hence for $0 < c_1 < 16$ the logarithm in $I(c_1, c_2)$ can be rewritten as an infinite sum. Taking the integral through the sum this may be calculated leading to the following infinite sum evaluation of $I(c_1, c_2)$.

$$I(c_1, c_2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} c_1^n \left\{ \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} R(i,j,n) \right\},$$

where

$$R(i,j,n) = e^{i-j} \frac{(3n-i-j-1)!(n+i+j-1)!}{2^{4n-1}(4n-1)!} \sum_{k=n+i+j}^{4n-1} \binom{4n-1}{k},$$

with $e = 2 + c_2/c_1$.

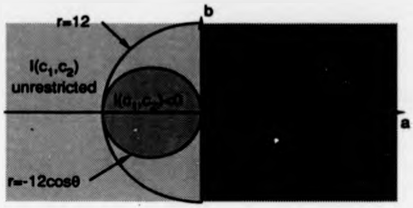
This infinite sum can then be used to approximate the $c_2(c_1)$ given in Proposition 4.2.3. For $c_1 = 1$ this gives an approximate value for c_2 of -5.93.
II): $c(\lambda_1) = \overline{c(\lambda_2)} = a + ib$ say, where $b \neq 0$.
Then $c_1 = 2a$ and $c_2 = a^2 + b^2$ so changing to polar coordinates
$c_1 = 2r\cos\theta$ and $c_2 = r^2$ and F_{c_1,c_2} becomes

$$G_{r,\theta}(y) = \frac{2r\cos\theta y}{(1+y)^2} + \frac{r^2y^2}{(1+y)^4},$$

where $r > 0$ and $\theta \in (-\pi, 0) \cup (0, \pi)$.
Hence if $\cos\theta \geq 0$ then $G_{r,\theta}(y) > 0$ for all $y \in (0,1)$ and $I(2r\cos\theta, r^2) > 0$.
Using $\log(1+x) < x$ for $x \neq 0$

$$\begin{aligned} I(2r\cos\theta, r^2) &< \int_0^1 \frac{G_{r,\theta}(y)}{y}\,dy \\ &= 2r\cos\theta \int_0^1 \frac{dy}{(1+y)^2} + r^2 \int_0^1 \frac{y}{(1+y)^4}\,dy \\ &= r\cos\theta + r^2/12 = r(\cos\theta + r/12). \end{aligned}$$

Therefore if $0 < r < 12$ and $\cos\theta \leq -r/12$ then $I(2r\cos\theta, r^2) < 0$.
This gives the following picture.



So, by continuity, for $0 < r < 12$ there exists a $\theta(r)$ such that

$$0 > \cos\theta(r) > -r/12 \text{ and } I(2r\cos\theta, r^2) = 0.$$

Solving the equation

$$\lambda + \lambda^{-1} = (x + iy), \qquad y \neq 0,$$

or alternatively

$$\lambda^2 - (x + iy)\lambda + 1 = 0,$$

for λ demonstrates that a λ exists that is strictly complex for all $x + iy$ with $y \neq 0$. Hence all the values $a + ib \in \mathbf{C} \setminus \mathbf{R}$ used in the previous argument are possible and the values of c_1 and c_2 given by $2r\cos\theta$ and r^2 respectively are permissible. Therefore the points given above are further points in C^2.

(2). One of c_1,c_2 is real the other is strictly complex.
This set has no points in C^2. This can be easily seen as follows:
The imaginary part of $I(c_1, c_2)$ equals

$$\int_0^1 \arg\left(\frac{\mathrm{Im}(c_1)y/(1+y)^2}{\cdots}\right) dy,$$

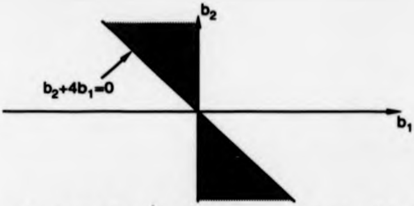
or

$$\int_0^1 \arg\left(\frac{\mathrm{Im}(c_2)y^2/(1+y)^4}{\cdots}\right) dy,$$

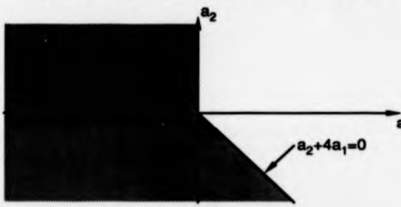
depending on which c_i is complex. An argument similar to that in Proposition 4.2.1 (2) gives this is non-zero and hence $(c_1, c_2) \notin C^2$.

(3). c_1 and c_2 are both strictly complex.
Cannot really say much about this particular case. One problem is the description of values which are permissible. If $\mathbf{C}^2$ is considered, ignoring the points where the logarithm would not be defined as these cannot be permissible, the following can be seen:
Suppose $c_j = a_j + ib_j$ for $j = 1, 2$

(i) b_1,b_2 have to be in the shaded area for the possibility that $(c_1, c_2) \in C^2$.



(ii) a_1, a_2 have to be in the shaded area for the possibility that $(c_1, c_2) \in C^2$.



4.2.5 REMARK. For the case (1) I): above, as c_2 approaches $-(4c_1 + 16)$ the logarithm tends to $-\infty$ at $y = 1$. This suggests that $I(c_1, c_2) < 0$ for some $c_2(c_1)$ in the interval

$$(-(4c_1 + 16) + \epsilon(c_1), -(4c_1 + 16)).$$

This in turn suggests that there exists a $c_2'(c_1)$ such that

$$I(c_1, c_2') = 0 \text{ for all } c_1 \geq 0.$$

Similarly for the case (1) II): as $\cos\theta$ approaches -1, i.e. θ tends to π, the minimum value of $G_{r,\theta}(y)$ tends to -1 causing the logarithm to tend to $-\infty$ at this point. This suggests that

$$I(2r\cos\theta, r^2) < 0 \text{ for some } \theta(r) \text{ with } \cos\theta(r) \in (-1, -1 + \epsilon(r)).$$

Hence this suggests that there exists a $\theta'(r)$ such that

$$I(2r\cos\theta', r^2) = 0 \text{ for all } r > 0.$$

These remarks show that C^2 contains points other than those given in the proof of Proposition 4.2.3. However its complete structure is still unclear.

4.3 Investigation of $\mathcal{C}^p$.

The structure of this set is not clear for $p \geq 2$, as shown by the previous analysis of $\mathcal{C}^2$, however the following properties can be seen fairly easily:

(1)
$$\{0\} \equiv \mathcal{C}^1 \subset \mathcal{C}^2 \subset \cdots \subset \mathcal{C}^p \subset \ldots$$

where the inclusion $\mathcal{C}^k \longrightarrow \mathcal{C}^{k+1}$ is the map which takes

$$(c_1, \ldots, c_k) \in \mathcal{C}^k \longmapsto (c_1, \ldots, c_k, 0) \in \mathcal{C}^{k+1}.$$

This is equivalent to embedding the $k \times k$ matrix corresponding to the element of $\mathcal{C}^k$ in a $(k+1) \times (k+1)$ matrix by adding a 1 on the diagonal.
Hence $\mathcal{C}^p \supsetneq \{0\}$ for all $p \geq 2$, that is, for $p \geq 2$ there exist non-trivial $p \times p$ matrices M_p whose monodromy one point correlation $\langle \sigma_a(M_p) \rangle$ has a critical limit.

(2)
$$\mathcal{C}^n \times \mathcal{C}^m \subset \mathcal{C}^{n+m}.$$

The map at this level is equivalent to placing the $n \times n$ matrix and the $m \times m$ matrix down the diagonal to form an $(n+m) \times (n+m)$ matrix. The formula at the $\mathcal{C}^n$ level is not given as it is not very illuminating. Note that the map given in (1) is a special case of this map when $n = k$ and $m = 1$.

(3) If

$$I_+ = \left\{ (c_1, \ldots, c_p) \in \mathbf{C}^p : \left| 1 + \sum_{i=1}^{p} c_i \left(\frac{y}{(1+y)^2} \right)^i \right| \geq 1 \quad \forall y \in [0,1] \right\},$$

and

$$I_- = \left\{ (c_1, \ldots, c_p) \in \mathbf{C}^p : \left| 1 + \sum_{i=1}^{p} c_i \left(\frac{y}{(1+y)^2} \right)^i \right| \leq 1 \quad \forall y \in [0,1] \right\},$$

then $\mathcal{C}^p \cap (I_+ \cup I_-) = \{0\}$.

(4) If for $j = 1, \ldots, p$ with $p \geq 2$

$$I_j = \{ (c_1, \ldots, c_p) \in \mathbf{C}^p : c_i \in \mathbf{C} \cap \mathbf{R}, i \neq j; \operatorname{Im}(c_j) \neq 0 \}$$

then $\mathcal{C}^p \cap \bigcup_{j=1}^{p} I_j = \emptyset$.
This follows from a similar argument to that in Proposition 4.2.1 for the imaginary part of $I(c_1, \ldots, c_p)$ since only one of the c_i's, namely c_j in I_j, has a non-zero imaginary part.

(5) A slight generalization of (4) is given by the following:
Suppose $c_j = a_j + ib_j$ for $j = 1, \ldots, p$. Then if the non-zero b_j's are all of the same sign the argument remains in $[0, \pi)$ or $(-\pi, 0]$ and thus $(c_1, \ldots, c_p) \notin \mathcal{C}^p$.

(6) In fact if for all $y \in (0,1)$ either

$$\sum_{i=1}^{p} b_i \left(\frac{y}{(1+y)^2} \right)^i < 0,$$

or

$$\sum_{i=1}^{p} b_i \left(\frac{y}{(1+y)^2} \right)^i > 0,$$

then it is easy to see that $(c_1,\dots,c_p) \notin C^p$.

(7) Using $\log(1+x) < x$ for $x \neq 0$, if $(c_1,\dots,c_p) \in \mathbf{C}^p_R \cap \mathbf{R}^p$

$$\begin{aligned}
I(c_1,\dots,c_p) &= \int_0^1 \frac{\log\left[1+\sum_{i=1}^p c_i\left(\frac{y}{(1+y)^2}\right)^i\right]}{y}\,dy \\
&< \sum_{i=1}^p c_i \int_0^1 \frac{y^{i-1}}{(1+y)^{2i}}\,dy \\
&= \sum_{i=1}^p c_i \frac{(i-1)!(i-1)!}{(2i-1)!2^{2i-1}} \sum_{k=i}^{2i-1} \binom{2i-1}{k} \qquad \text{see Appendix} \\
&= \sum_{i=1}^p c_i \frac{[(i-1)!]^2}{2(2i-1)!} \qquad \text{using } \sum_{k=i}^{2i-1}\binom{2i-1}{k} = 2^{2i-2}.
\end{aligned}$$

Thus if $\sum_{i=1}^p c_i \frac{[(i-1)!]^2}{2(2i-1)!} < 0$ then

$$I(c_1,\dots,c_p) < 0 \text{ and } (c_1,\dots,c_p) \notin C^p.$$

(8) In fact if

$$\left|\sum_{i=1}^p c_i\left(\frac{y}{(1+y)^2}\right)^i\right| < 1, \qquad \forall y \in [0,1]$$

then

$$I(c_1,\dots,c_p) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} S(n),$$

where

$$S(n) = \sum_{n_1+\cdots+n_p=n} \left\{\frac{n!}{n_1!n_2!\dots n_p!} c_1^{n_1}\dots c_p^{n_p} \frac{(N!)^2}{(M-1)!2^{M-1}} \sum_{k=N+1}^{M-1} \binom{M-1}{k}\right\},$$

$n_1,\dots,n_p$ are non-negative integers and

$$\begin{aligned}
N &= n_1 + 2n_2 + \cdots + pn_p - 1, \\
M &= 2n_1 + 4n_2 + \cdots + 2pn_p.
\end{aligned}$$

See Appendix for a proof of this.

NOTE. Each non-zero point of C^p, or indeed $\mathbf{C}^p_R$, actually corresponds to a family of matrices since it only depends on the eigenvalues of the matrix and is invariant under the transformation $\lambda \mapsto \lambda^{-1}$. The origin, however, only corresponds to the identity matrix.

Section 5
N Point Correlations

5.1 Introduction.

The two previous sections dealt with one point correlations and their critical limits. This section will consider higher order correlations. Unfortunately, as yet, there are no concrete results concerning critical limits only a conjecture. The starting point for this analysis is the 'product formula', see [P4] or [P3], which is stated below for reference.

5.1.1 Theorem. *Suppose that $g_k \in \widetilde{GL_Q}(H)$ for $k = 1, \ldots, N$ with*

$$T(g_k) = G_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \in GL_Q(H)$$

Suppose also that d_k is invertible for each $k = 1, \ldots, N$. Then

$$\langle g_1 \ldots g_N \rangle = \prod_{k=1}^{N} \langle g_k \rangle \det_2(1 + LR)$$

provided $\langle g_1 \ldots g_N \rangle \neq 0$, where:
L denotes the $N \times N$ block matrix with entries
for $i < k$,

$$\ell_{ik} = \begin{cases} -Q_+ & k = i+1 \\ -a_{i+1} Q_+ & k = i+2 \\ -a_{i+1} \ldots a_{k-1} Q_+ & k > i+2 \end{cases}$$

for $i > k$,

$$\ell_{ik} = \begin{cases} Q_- & i = k+1 \\ d_{k+1}^{-1} Q_- & i = k+2 \\ d_{i-1}^{-1} \ldots d_{k+1}^{-1} Q_- & i > k+2 \end{cases}$$

and for $i = k$

$$\ell_{ii} = 0$$

and $R = R_1 \oplus R_2 \oplus \cdots \oplus R_N$ where

$$R_k = \begin{bmatrix} -b_k d_k^{-1} c_k & b_k d_k^{-1} \\ d_k^{-1} c_k & 0 \end{bmatrix}.$$

5.2 Conjecture for limiting N point correlations.

The 'product formula' given in Theorem 5.1.1 gives rise to the following two Corollaries when applied to the specific case of monodromy fields.

5.2.1 Corollary. *Suppose $M_j \in GL(p, \mathbf{C})$ and has no negative eigenvalues for $j = 1, \ldots, n$. Then*

$$\langle \sigma_{a_1}(M_1) \ldots \sigma_{a_n}(M_n) \rangle = \prod_{k=1}^{n} \langle \sigma(M_k) \rangle \det_2(1 + LR)$$

where L and R have the structure defined above with

$$s_{a_k}(M_k) = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}.$$

Proof: By definition $T(\sigma_a(M)) = s_a(M)$. The condition on the eigenvalues of M_j implies that d_j is invertible for all $j = 1, \ldots, n$. Hence Theorem 5.1.1 gives the above as

$$\langle \sigma(M) \rangle = \langle \sigma_a(M) \rangle, \qquad \forall a \in \mathbf{Z}^2.$$

5.2.2 NOTATION. Let the expression $(1 + LR)$ present in Corollary 5.2.1 be denoted by

$$X(M_1, \ldots, M_n : a_1, \ldots, a_n)$$

and let

$$X(M_1, \ldots, M_n) \stackrel{\text{def}}{=} X(M_1, \ldots, M_n : a, \ldots, a).$$

5.2.3 COROLLARY. *Suppose $M_j \in GL(p, \mathbb{C})$ and has no negative eigenvalues for $j = 1, \ldots, n$. Then*

$$\langle \sigma_{a_1}(M_1) \ldots \sigma_{a_n}(M_n) \rangle = \langle \sigma(M_1 \ldots M_n) \rangle \frac{\det_2 X(M_1, \ldots, M_n : a_1, \ldots, a_n)}{\det_2 X(M_1, \ldots, M_n)}.$$

PROOF: Apply the product formula to

$$\langle \sigma(M_1 \ldots M_n) \rangle = \langle \sigma(M_1) \ldots \sigma(M_n) \rangle.$$

This together with Corollary 5.2.1 gives the result.

5.2.3 CONJECTURE. *If $\lim_{a \uparrow 1} \langle \sigma(M_1 \ldots M_n) \rangle$ exists then $\lim_{a \uparrow 1} \langle \sigma_{a_1}(M_1) \ldots \sigma_{a_n}(M_n) \rangle$ exists.*

5.2.4 REMARK. To prove this conjecture the existence of a limit for the determinant expression is required. This appears intractable at the present since $s_a(M) \notin GL_{Q_a}(H^p)$ where Q_a denotes the critical temperature Q, that is $Q_a = \lim_{a \uparrow 1} Q$. Note the conjecture is trivially true when $a_1 = \cdots = a_n$ as expressions are equal.

5.3 Order dependence of correlations.

Another problem to consider is the invariance/variance of the n point correlations under the obvious action of S_n, namely permuting the entries. In other words to what extent does the order of the monodromy fields in the correlation matter.

5.3.1 LEMMA. *Suppose $\pi_2(a - b) = 0$ and $\pi_1(a) \leq \pi_1(b)$. Then*

(1) $\sigma_a(M)\sigma_b(N) = \sigma_b(MNM^{-1})\sigma_a(M)$.
(2) $\sigma_b(N)\sigma_a(M) = \sigma_a(M)\sigma_b(M^{-1}NM)$.

PROOF: First note from [P2,(2.22)]

$$\begin{aligned} s(M) z s(M)^{-1} &= z + (I \otimes (M - I)) P_{1/2} z \\ &= z (I + (I \otimes (M - I)) P_{-1/2}), \end{aligned}$$

and

$$\begin{aligned} s(M) z^{-1} s(M)^{-1} &= z^{-1} + (I \otimes (M^{-1} - I)) P_{-1/2} z^{-1} \\ &= z^{-1} (I + (I \otimes (M^{-1} - I)) P_{1/2}). \end{aligned}$$

It is a simple consequence of this that for $p \geq 0$

$$\begin{aligned} s(M) z^p s(M)^{-1} &= z^p (I + (I \otimes (M - I)) P[-p, 0]), \\ s(M) z^{-p} s(M)^{-1} &= z^{-p} (I + (I \otimes (M^{-1} - I)) P[0, p]). \end{aligned}$$

From the defintion of $s(M)$ it is simple to deduce that for $l \leq 0$

$$\begin{aligned} (1) \qquad s(MN)(I + (M - I)P[l, 0]) &= (I + (M - I)P[l, 0]) s(MN) \\ &= \epsilon_- \otimes I_p + \epsilon_+ \otimes MN + P[l, 0] \otimes (M - I), \end{aligned}$$

and for $k \geq 0$

(2)
$$
\begin{aligned}
s(MN)\left(I + (M^{-1} - I)P[0,k]\right) &= \left(I + (MNM^{-1}N^{-1}M^{-1} - I)P[0,k]\right) s(MN) \\
&= \epsilon_- \otimes I_p + \epsilon_+ \otimes MN + P[0,k] \otimes MN(M^{-1} - I).
\end{aligned}
$$

So

$$
\begin{aligned}
s(MN)\left(I + (M - I)P[l,0]\right) D(MN)^{-1} &= \left(I + (M - I)P[l,0]\right) s(MN)D(MN)^{-1} \\
&= \left(I + (M - I)P[l,0]\right) \underline{s}(MN).
\end{aligned}
$$

But the right hand side is in the domain of Γ_Q so

$$\Gamma_Q\left(s(MN)\left(I + (M - I)P[l,0]\right) D(MN)^{-1}\right) = \Gamma_Q\left(\left(I + (M - I)P[l,0]\right) \underline{s}(MN)\right)$$

and it is simple to see that this can be rewritten as

$$
\begin{aligned}
&\Gamma_Q\left(\underline{s}(MN)\right)\Gamma\left(D(MN)\right)\Gamma_Q\left(I + (M - I)P[l,0]\right) \\
&\qquad = \Gamma_Q\left(I + (M - I)P[l,0]\right)\Gamma_Q\left(\underline{s}(MN)\right)\Gamma\left(D(MN)\right).
\end{aligned}
$$

That is, if l denotes $(l,0)$

$$\sigma(MN)\sigma_l(M)\sigma(M)^{-1} = \sigma_l(M)\sigma(M)^{-1}\sigma(MN).$$

But

$$
\begin{aligned}
\sigma(MN)\sigma_l(M)\sigma(M)^{-1} &= \sigma(MN)\sigma(M)^{-1}\sigma_l(M) \\
&= \sigma(MNM^{-1})\sigma_l(M),
\end{aligned}
$$

and

$$\sigma_l(M)\sigma(M)^{-1}\sigma(MN) = \sigma_l(M)\sigma(N),$$

so for $l \leq 0$

$$\sigma_l(M)\sigma(N) = \sigma(MNM^{-1})\sigma_l(M).$$

Here the following results of [P2] have been used

$$
\begin{aligned}
\sigma_m(M)\sigma_n(M)^{-1} &= (\det M)^{\pi_1(m-n)}\Gamma_Q\left(s_m(M)s_n(M)^{-1}\right), \\
\text{and } \sigma_m(M)\sigma_n(M)^{-1} &= \sigma_n(M)^{-1}\sigma_m(M) \text{ when } \pi_2(m-n) = 0.
\end{aligned}
$$

Now if x denotes $\Gamma(T)^{b_2} = \Gamma(T)^{a_2}$ then

$$
\begin{aligned}
\sigma_a(M)\sigma_b(N) &= x\Gamma(z)^{b_1}\sigma_{a-b}(M)\sigma(N)\Gamma(z)^{-b_1}x^{-1} \\
&= x\Gamma(z)^{b_1}\sigma(MNM^{-1})\sigma_{a-b}(M)\Gamma(z)^{-b_1}x^{-1} \qquad \text{by the above} \\
&= x\Gamma(z)^{b_1}\sigma(MNM^{-1})\Gamma(z)^{-b_1}\Gamma(z)^{a_1}\sigma(M)\Gamma(z)^{-a_1}x^{-1} \\
&= \sigma_b(MNM^{-1})\sigma_a(M).
\end{aligned}
$$

If N is replaced by $M^{-1}NM$ the other result follows.
By the same method the result

$$\sigma_a(M)\sigma_b(N) = \sigma_b(N)\sigma_a(N^{-1}MN),$$

where $\pi_1(a) \geq \pi_1(b)$ and $\pi_2(a-b) = 0$ can be obtained from the equality (2). However, since a,b,n and M can all vary this result can be obtained from the one given.

5.3.2 **Corollary**. *If $\pi_2(a-b)=0$ then*

$$\langle \sigma_a(M)\sigma_b(N)\rangle = \langle \sigma_b(N)\sigma_a(M)\rangle.$$

Proof: Assume $a \leq b$ (For $a > b$ switch roles of a and b.)
By definition

$$s(MNM^{-1}) = (1\otimes M)s(N)(1\otimes M^{-1}),$$

and $s(M)$ may be written as

$$s(M) = (1\otimes M)s(M)(1\otimes M^{-1}).$$

Thus the factorization of $\sigma(MNM^{-1})$ and $\sigma(M)$ can be written as

$$\begin{aligned}
\sigma(MNM^{-1}) &= \Gamma_Q\left((1\otimes M)\underline{g}(N)(1\otimes M^{-1})\right)\Gamma_Q\left((1\otimes M)D(N)(1\otimes M^{-1})\right),\\
\sigma(M) &= \Gamma_Q\left((1\otimes M)\underline{g}(M)(1\otimes M^{-1})\right)\Gamma_Q\left((1\otimes M)D(M)(1\otimes M^{-1})\right).
\end{aligned}$$

But $(1\otimes M)$ and $(1\otimes M^{-1})$ are trivially in the domain of Γ, hence the above can be rewritten as

$$\Gamma(1\otimes M).\Gamma_Q(\underline{g}(N))\,\Gamma(D(N))\,.\Gamma(1\otimes M^{-1}),$$

and

$$\Gamma(1\otimes M).\Gamma_Q(\underline{g}(M))\,\Gamma(D(N))\,.\Gamma(1\otimes M^{-1}),$$

respectively. Therefore as T and s commute with both $(1\otimes M)$ and $(1\otimes M^{-1})$

$$\sigma_b(MNM^{-1})\sigma_a(M) = \Gamma(1\otimes M)\sigma_b(N)\sigma_a(M)\Gamma(1\otimes M^{-1}),$$

so

$$\begin{aligned}
\langle \sigma_a(M)\sigma_b(N)\rangle &= \langle \sigma_b(MNM^{-1})\sigma_a(M)\rangle\\
&= \langle \Gamma(1\otimes M)\sigma_b(N)\sigma_a(M)\Gamma(1\otimes M^{-1})\rangle\\
&= \langle \sigma_b(N)\sigma_a(M)\rangle.
\end{aligned}$$

5.3.3 **Corollary**. *Suppose $\pi_2(a_i)=a$ and $M_i \in GL(p,\mathbf{C})$ is upper triangular for all $i=1,\ldots,n$. Let $\pi \in S_n$ and let*

$$\pi\left(\sigma_{a_1}(M_1)\ldots\sigma_{a_n}(M_n)\right)$$

denote the action of the permutation π on the monodromy fields. Then

$$\langle \sigma_{a_1}(M_1)\ldots\sigma_{a_n}(M_n)\rangle = \langle \pi\left(\sigma_{a_1}(M_1)\ldots\sigma_{a_n}(M_n)\right)\rangle.$$

Proof: Need only examine the case when π is a transposition. Hence consider

$$\sigma_{a_1}(M_1)\ldots\sigma_{a_{i-1}}(M_{i-1})\sigma_{a_j}(M_j)\sigma_{a_{i+1}}(M_{i+1})\ldots$$
$$\ldots\sigma_{a_{j-1}}(M_{j-1})\sigma_{a_i}(M_i)\sigma_{a_{j+1}}(M_{j+1})\ldots\sigma_{a_n}(M_n).$$

Using Lemma 5.3.1 repeatedly this may be rewritten as

$$\sigma_{a_1}(M_1)\ldots\sigma_{a_{i-1}}(M_{i-1})\sigma_{a_i}(\dot{M}_i)\sigma_{a_{i+1}}(\dot{M}_{i+1})\ldots$$
$$\ldots\sigma_{a_{j-1}}(\dot{M}_{j-1})\sigma_{a_j}(\dot{M}_j)\sigma_{a_{j+1}}(M_{j+1})\ldots\sigma_{a_n}(M_n).$$

where

$$\mathring{M}_i = X_j X_{i+1} \dots X_{j-1} M_i X_{j-1}^{-1} \dots X_{i+1}^{-1} X_j^{-1} \quad \text{with } X_k = \begin{cases} I & \text{if } a_i \leq a_k \\ M_k & \text{if } a_i > a_k \end{cases}$$

$$\mathring{M}_j = Y_{j-1}^{-1} \dots Y_{i+1}^{-1} \tilde{M}_j Y_{i+1} \dots Y_{j-1} \quad \text{with } Y_k = \begin{cases} I & \text{if } a_j \leq a_k \\ \tilde{M}_k & \text{if } a_j > a_k. \end{cases}$$

For $i+1 \leq k \leq j-1$,

$$\mathring{M}_k = \begin{cases} \tilde{M}_j \tilde{M}_k \tilde{M}_j^{-1} & \text{if } a_j \leq a_k \\ \tilde{M}_k & \text{if } a_j > a_k \end{cases}$$

with for $i+1 \leq k \leq j$,

$$\tilde{M}_k = \begin{cases} M_i M_k M_i^{-1} & \text{if } a_i \leq a_k \\ M_k & \text{if } a_i > a_k. \end{cases}$$

As the M_r for $r = 1, \dots, n$ are upper triangular $\mathring{M}_k$ are upper triangular and moreover for $k = i, \dots, j$

$$\operatorname{diag}(\mathring{M}_k) = \operatorname{diag}(M_k).$$

Therefore as the correlation can be expressed as the determinant of an upper triangular matrix in this case with the diagonal entries derived from the diagonal entries of the M_r

$$\begin{aligned}
(\sigma_{a_1}(M_1) \dots \sigma_{a_n}(M_n)) &= (\sigma_{a_1}(M_1) \dots \sigma_{a_{i-1}}(M_{i-1}) \sigma_{a_i}(\mathring{M}_i) \sigma_{a_{i+1}}(\mathring{M}_{i+1}) \dots \\
&\qquad \dots \sigma_{a_{j-1}}(\mathring{M}_{j-1}) \sigma_{a_j}(\mathring{M}_j) \sigma_{a_{j+1}}(M_{j+1}) \dots \sigma_{a_n}(M_n)) \\
&= (\sigma_{a_1}(M_1) \dots \sigma_{a_{i-1}}(M_{i-1}) \sigma_{a_i}(M_j) \sigma_{a_{i+1}}(M_{i+1}) \dots \\
&\qquad \dots \sigma_{a_{j-1}}(M_{j-1}) \sigma_{a_j}(M_i) \sigma_{a_{j+1}}(M_{j+1}) \dots \sigma_{a_n}(M_n)).
\end{aligned}$$

5.3.4 Remark. Corollary 5.3.3 suggests that for $n \geq 2$, $(\sigma_{a_1}(M_1) \dots \sigma_{a_n}(M_n))$ is not S_n invariant for general M_i, though it is if the M_i commute among themselves or there exists an $S \in GL(p, \mathbb{C})$ such that SM_iS^{-1} is upper triangular for all $i = 1, \dots, n$.

Section 6
Boson-Fermion Correspondence

6.1 Introduction.

Boson-Fermion correspondence is the term used by physicists to describe the linking of boson (CCR algebra) and fermi (CAR algebra) systems essentially through projective representations of loop groups. The particular situation of interest here is that of 'temperature states' on loop groups as described in [C5]. A brief summary of this now follows.

6.2 Summary.

Let $H = L^2(S^1, \mathbb{C})$ and $A(H)$ the CAR algebra over H. Define the one parameter group $\{\tau_t : t \in [0, 4\pi]\}$ by

$$\tau_t g(s) = e^{it/2} g(s+t), \qquad g \in H, s \in S^1 = [0, 2\pi)$$

and let h denote the generator so that

$$hg(s) = (-id/ds + 1/2)g(s).$$

Now if the operator A_β is defined as

$$e^{-\beta h}(1 + e^{-\beta h})^{-1}$$

the (τ, β) KMS (temperature) state ω_β, $\beta \in (0, \infty)$ is the quasi-free state determined by A_β, where τ refers to the evolution (automorphism group) of $A(H)$ induced by $\{\tau_t\}$.

Note that $A_\beta \to P_-$ uniformly as $\beta \to \infty$ where P_- is the projection onto the subspace of $L^2(S^1, \mathbb{C})$ whose elements have fourier expansions in $e^{ik\theta}$ with no k positive or zero terms. The projective representation of the loop group of $U(1)$ corresponding to P_- giving rise to a boson-fermion correspondence was studied in [C7].

Let π_β be the representation of $A(H)$ corresponding to the state ω_β. This is given in the usual "doubling up" manner, so that $\pi_\beta \simeq \pi_{P^\beta_-}|_{A(H\oplus 0)}$ where $A(H \oplus 0)$ is the subalgebra of $A(H \oplus H)$ and P_-^β is the projection on $H \oplus H$ given by the 2×2 matrix

$$\begin{pmatrix} A_\beta & A_\beta^{1/2}(1-A_\beta)^{1/2} \\ A_\beta^{1/2}(1-A_\beta)^{1/2} & 1 - A_\beta \end{pmatrix},$$

with $\pi_{P^\beta_-}$ denoting the Fock representation defined by P_-^β. Then $\pi_{P^\beta_-}$ and $\pi_{P^\infty_-}$ are equivalent where

$$P_-^\infty = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}, \qquad P_+ = 1 - P_-.$$

Also π_β and π_∞ are quasi-equivalent.

If ϕ_1, ϕ_2 are smooth maps from S^1 to $U(1)$ define the unitary operator $\tilde{\phi}$ on $K = H \oplus H$ to be multiplication by the function

$$\tilde{\phi}(s) = \begin{pmatrix} \phi_1(s) & 0 \\ 0 & \phi_2(s) \end{pmatrix}.$$

The multiplicative group of all such operators is denoted by $\mathrm{Map}\,(S^1, U(1) \times U(1))$. Each element of the group induces an automorphism of $A(K)$ which is implemented in the representation $\pi_{P^\beta_-}$. The subgroup of particular interest is $\mathrm{Map}\,(S^1, U(1))$ which consists of operators of the form $\begin{pmatrix} \phi(s) & 0 \\ 0 & 1 \end{pmatrix}$. For convenience denote this operator by $\tilde{\phi}$. Let $\Gamma_\beta(\phi)$ represent the implementer of this automorphism in π_{P_β} then the operators $\Gamma_\beta(\phi)$ for $\phi \in \mathrm{Map}\,(S^1, U(1))$ define a σ-representation of this group with the 2-cocycle, σ, being independent of β. Moreover if $L = \mathrm{Map}(S^1, \mathbb{R})$, $L \oplus L$ can be identified with the Lie algebra of $\mathrm{Map}\,(S^1, U(1) \times U(1))$ and

$$\tilde{\phi} \mapsto \Gamma_\beta(\tilde{\phi}), \qquad \tilde{\phi} \in \mathrm{Map}\,(S^1, U(1) \times U(1))$$

is a representation of the CCR algebra over $L \oplus L$, with

$$\phi \mapsto \Gamma_\beta(\phi), \qquad \phi \in \mathrm{Map}\,(S^1, U(1))$$

a representation of the CCR algebra over L. Also

$$\langle \Omega_\beta, \Gamma_\beta(\phi)\Omega_\beta \rangle = \delta_{n0}\theta_3(\alpha)\theta_3(0)^{-1} \exp\left(-\frac{1}{4\pi}\sum_{k\neq 0} k(1-e^{-\beta k})^{-1}|f_k|^2\right),$$

where

$$\phi(s) = \exp i(ns + \alpha + \sum f_k e^{iks}),$$

with the sum over non-zero k.

If the starting point is now reversed, that is, define a projective representation of $\mathrm{Map}\,(S^1, U(1))$ using the function given above. If a 'blip' $B_{\alpha,\lambda}$ is defined as

$$B_{\alpha,\lambda} = e^{i\alpha}(1-\lambda^2)^{-1/2}\Gamma_\beta(\gamma_{\alpha,0})\Gamma_\beta(\gamma_{\alpha,0}^{-1}\gamma_{\alpha,\lambda}),$$

where $\gamma_{\alpha,\lambda}$ denotes the 'kink' function defined by

$$\gamma_{\alpha,\lambda}(\theta) = \frac{(\lambda - e^{i(\theta-\alpha)})}{(\lambda e^{i(\theta-\alpha)} - 1)},$$

and $\Gamma_\beta(.)$ denotes the projective representation then as $\lambda \to 1$, $B_{\alpha,\lambda}$ converges to a fermion operator $B(g)$, $g \in L^2(S^1, \mathbb{C})$ in a suitable sense. Moreover the time dependence of the 'blips' is given by

$$[B_{\alpha,\lambda}]^t = e^{it/2}B_{\alpha-t,\lambda},$$

and this gives that the function defined above is a KMS state for the C^*-algebra generated by the $B(g)$, $g \in L^2(S^1, \mathbb{C})$ with evolution given by $\{r_t\}$. Hence by the uniqueness of KMS states on the CAR algebra this is the same as the state originally defined at the beginning through A_β. For more details of these constructions see [C5], [C6], [C7] and [C12] for example.

Section 7
Boson-Fermion Correspondence with Chemical Potential μ

7.1 Introduction.

This section will extend slightly the notion of Boson-Fermion correspondence given in section 6 to include an extra variable μ in the KMS state. This variable μ is referred to as the chemical potential and appears in the quantum statistical mechanics picture, particularly in the formulation of Bose-Einstein Condensation on the CCR side.

Consider the one parameter group $\{\tau_t^\mu : t \in [0, 4\pi]\}$ where

$$\tau_t^\mu g(s) = e^{-i\mu t} g(s+t), \qquad \mu \in \mathbb{R}, g \in L^2(S^1, \mathbb{C}).$$

Then h_μ defined by

$$h_\mu g(s) = (-id/ds - \mu)\, g(s)$$

is the generator of τ_t^μ and $\hat{h}_\mu g_n = (n-\mu) g_n$ where $\hat{\ }$ denotes the corresponding operator on the fourier transform space with

$$g(s) = (2\pi)^{-1/2} \sum_n g_n e^{ins}.$$

Each τ_t^μ induces an automorphism of $A(H)$, $H = L^2(S^1, \mathbb{C})$, via

$$a(g) \mapsto a(\tau_t^\mu g),$$

and hence there is a corresponding automorphism group τ_t^μ given by the above. So the (τ^μ, β) KMS state $\omega_{\beta,\mu}$ for $\beta \in (0, \infty)$, $\mu \in \mathbb{R}$ is the quasi-free state determined by $A_{\beta,\mu}$ where $A_{\beta,\mu}$ is the operator

$$e^{-\beta h_\mu}(1 + e^{-\beta h_\mu})^{-1}.$$

That is

$$\omega_{\beta,\mu}(a(g_1)^* a(g_2)) = \langle g_2, A_{\beta,\mu} g_1\rangle_H.$$

Taking fourier transforms gives

$$\hat{A}_{\beta,\mu} g_n = e^{-\beta(n-\mu)} \left(1 + e^{-\beta(n-\mu)}\right)^{-1} g_n.$$

7.1.1 Remark. As $\beta \to \infty$, $A_{\beta,\mu} \to P_{[\mu]}$ where

$$P_{[\mu]} g_n = \begin{cases} 0 & \text{for } n \geq [\mu] \\ 1 & \text{for } n < [\mu] \end{cases} \qquad \mu \notin \mathbb{Z},$$
$$= \begin{cases} 0 & \text{for } n > \mu \\ 1/2 & \text{for } n = \mu \\ 1 & \text{for } n < \mu \end{cases} \qquad \mu \in \mathbb{Z}.$$

with $[x]$ denoting the integer part of x. This is interpreted physically as the property that only particles with energy less than or equal to μ occur, which is described as the Fermi sea, see [B4,P55].

Let $\pi_{\beta,\mu}$ denote the representation of $A(H)$ determined by $\omega_{\beta,\mu}$. This may be realised by the usual 'doubling up' procedure [P11]. Set $K = H \oplus H$ and define the projection $P_-^{\beta,\mu} : K \to K$ by the 2×2 matrix

$$P_-^{\beta,\mu} = \begin{bmatrix} A_{\beta,\mu} & A_{\beta,\mu}^{1/2}(1 - A_{\beta,\mu})^{1/2} \\ A_{\beta,\mu}^{1/2}(1 - A_{\beta,\mu})^{1/2} & 1 - A_{\beta,\mu} \end{bmatrix},$$

then $\omega_{\beta,\mu}$ is the restriction to the subalgebra $A(H \oplus 0)$, isomorphic to $A(H)$, of the CAR algebra $A(K)$ over K of the Fock state $\omega_{P^{\beta,\mu}_-}$ on $A(K)$ defined by $P^{\beta,\mu}_-$. Moreover the cyclic vector $\Omega_{\beta,\mu}$ for the representation $\pi_{P^{\beta,\mu}_-}$ corresponding to $\omega_{P^{\beta,\mu}_-}$ is also cyclic and separating for $A(H \oplus 0)$ and

$$\pi_{\beta,\mu} \simeq \pi_{P^{\beta,\mu}_-}|_{A(H\oplus 0)}.$$

7.1.2 **Lemma**. *The representation $\pi_{P^{\beta,\mu}_-}$ of $A(K)$ is equivalent to the representation $\pi_{P^{\infty}_-}$ where*

$$P^{\infty}_- = \begin{bmatrix} P_- & 0 \\ 0 & P_+ \end{bmatrix},$$

with P_- the operator whose fourier transform acts as

$$\hat{P}_- g_n = \begin{cases} 0 & \text{for } n \geq 0 \\ g_n & \text{for } n < 0 \end{cases},$$

and $P_+ = 1 - P_-$.

Proof: $\pi_{P^{\beta,\mu}_-}$ and $\pi_{P^{\infty}_-}$ are equivalent if and only if $(P^{\beta,\mu}_- - P^{\infty}_-)$ is a Hilbert Schmidt operator [P11]. This is true if $P_- - A_{\beta,\mu}$ and $A^{1/2}_{\beta,\mu}(1 - A_{\beta,\mu})^{1/2}$ are Hilbert Schmidt operators. Now examining the fourier transforms of these operators it can be easily seen that

$$2\pi\,\mathrm{Trace}(P_- - A_{\beta,\mu}) = -\sum_{n=0}^{\infty} e^{-\beta(n-\mu)}\left(1 + e^{-\beta(n-\mu)}\right)^{-1} + \sum_{n=1}^{\infty}\left(1 + e^{-\beta(-n-\mu)}\right)^{-1},$$

and

$$2\pi\,\mathrm{Trace}(A_{\beta,\mu}(1 - A_{\beta,\mu})) = \sum_n e^{-\beta(n-\mu)}\left(1 + e^{-\beta(n-\mu)}\right)^{-2}.$$

So $(P_- - A_{\beta,\mu})$ and $(A_{\beta,\mu}(1 - A_{\beta,\mu}))$ are trace class as

$$\sum_{n=1}^{\infty}\left(1 + e^{\beta(n\pm\mu)}\right)^{-1} < \infty \qquad \forall\mu \in \mathbf{R}, \forall\beta \in (0,\infty),$$

by comparison with $\sum K/n^2$ where K is fixed by β and μ. Hence $(P_- - A_{\beta,\mu})$ and $A^{1/2}_{\beta,\mu}(1 - A_{\beta,\mu})^{1/2}$ are Hilbert Schmidt operators as required.

7.1.3 **Remark**. The representations $\pi_{\beta,\mu}$ and π_{P_-} are quasi-equivalent. This follows if and only if the operators $(1 - A_{\beta,\mu})^{1/2} - P_+$ and $A^{1/2}_{\beta,\mu} - P_-$ are Hilbert Schmidt. Now

$$2\pi\,\mathrm{Trace}(1 - A^{1/2}_{\beta,\mu})P_- = \sum_{n<0}\left(1 - (1 + e^{\beta(n-\mu)})^{-1/2}\right) < \infty,$$

$$2\pi\,\mathrm{Trace}(1 - (1 - A_{\beta,\mu})^{1/2})P_+ = \sum_{n\geq 0}\left(1 - (1 + e^{-\beta(n-\mu)})^{-1/2}\right) < \infty,$$

and

$$(1 - A_{\beta,\mu})P_- - A_{\beta,\mu}P_+ = P_- - A_{\beta,\mu}.$$

So the operators $(1 - A^{1/2}_{\beta,\mu})P_-$, $(1 - (1 - A_{\beta,\mu})^{1/2})P_+$, $(1 - A_{\beta,\mu})P_-$ and $A_{\beta,\mu}P_+$ are trace class. Using

$$(A^{1/2}_{\beta,\mu} - P_-)^*(A^{1/2}_{\beta,\mu} - P_-) = A_{\beta,\mu}P_+ + (1 - A^{1/2}_{\beta,\mu})^2 P_-,$$

and

$$((1 - A_{\beta,\mu})^{1/2} - P_+)^*((1 - A_{\beta,\mu})^{1/2} - P_+) = (1 - A_{\beta,\mu})P_- + (1 - (1 - A_{\beta,\mu})^{1/2})^2 P_+,$$

the result is obtained.

7.1.4 REMARK. The above lemma and remark were written in some detail as the fact that some operators are in fact trace class will be of great importance in a later section where the Hilbert Schmidt condition is not sufficient.

7.1.5 REMARK. The operator $W_{\beta,\mu}$ defined as

$$W_{\beta,\mu} = \begin{bmatrix} A_{\beta,\mu}^{1/2}P_- + (1-A_{\beta,\mu})^{1/2}P_+ & A_{\beta,\mu}^{1/2}P_+ - (1-A_{\beta,\mu})^{1/2}P_- \\ -A_{\beta,\mu}^{1/2}P_+ + (1-A_{\beta,\mu})^{1/2}P_- & A_{\beta,\mu}^{1/2}P_- + (1-A_{\beta,\mu})^{1/2}P_+ \end{bmatrix}$$

satisfies

$$W_{\beta,\mu}P_-^{\infty}W_{\beta,\mu}^* = P_-^{\beta,\mu}.$$

Moreover from the results of the previous remark 7.1.3 $(1-W_{\beta,\mu})$ is a trace class operator and $\det W_{\beta,\mu} = 1$.

7.2 The action of Map $(S^1, U(1))$.

Let ϕ_1, ϕ_2 be smooth maps from S^1 to $U(1)$, that is elements of Map $(S^1, U(1))$. Define the unitary operator $\tilde{\phi}$ on K to be multiplication by the function

$$\tilde{\phi}(s) = \begin{bmatrix} \phi_1(s) & 0 \\ 0 & \phi_2(s) \end{bmatrix},$$

and let Map $(S^1, U(1) \times U(1))$ denote the multiplicative group of such operators. These operators induce a corresponding Bogoliubov automorphism, $\tau(\tilde{\phi})$ on $A(K)$ given by

$$a(k) \mapsto a(\tilde{\phi}.k), k \in K, \qquad \text{where } \tilde{\phi}.k(s) = \tilde{\phi}(s)k(s).$$

These automorphisms are implemented in the representation $\pi_{P_-^{\infty}}$, see remark below, so there exists a unitary operator $\Gamma_{\infty}\left(\tilde{\phi}\right)$ on the representation space $\mathcal{H}$ of $\pi_{P_-^{\infty}}$ such that

$$\Gamma_{\infty}\left(\tilde{\phi}\right)\pi_{P_-^{\infty}}(a(k))\,\Gamma_{\infty}\left(\tilde{\phi}\right)^{-1} = \pi_{P_-^{\infty}}\left(a(\tilde{\phi}.k)\right), \qquad \forall k \in K.$$

7.2.1 REMARK. For implementability require that $(\tilde{\phi}P_-^{\infty} - P_-^{\infty}\tilde{\phi})$ is a Hilbert Schmidt operator [S4]. The property that

$$2\pi\,\mathrm{Trace}(P_-\phi^*P_+\phi P_-) = \sum_{k=1}^{\infty} k\phi_k^*\phi_k < \infty$$

for ϕ smooth, $\phi(s) = 1/\sqrt{2\pi}\sum_k \phi_k e^{iks}$, enables this to be deduced. This property will be of use in a later section which is the reason for its inclusion here.

Now by the irreducibility of the representation $\pi_{P_-^{\infty}}$ the map $\tilde{\phi} \mapsto \Gamma_{\infty}\left(\tilde{\phi}\right)$ defines a projective representation of Map $(S^1, U(1) \times U(1))$. That is, by fixing the phase of the implementing unitaries, a $U(1)$-valued 2-cocycle, σ, is defined on Map $(S^1, U(1) \times U(1))$ such that

$$\Gamma_{\infty}\left(\tilde{\phi}_1\right)\Gamma_{\infty}\left(\tilde{\phi}_2\right) = \sigma(\tilde{\phi}_1, \tilde{\phi}_2)\Gamma_{\infty}\left(\tilde{\phi}_1\tilde{\phi}_2\right).$$

But the representations $\pi_{P_-^{\infty}}$ and $\pi_{P_-^{\beta,\mu}}$ are equivalent, hence there exists a unitary $U_{\beta,\mu} \in \mathcal{H}$ such that

$$U_{\beta,\mu}\pi_{P_-^{\infty}}(a(k))U_{\beta,\mu}^{-1} = \pi_{P_-^{\beta,\mu}}(a(k)), \quad \forall k \in K.$$

Thus if the unitary operator $\Gamma_{\beta,\mu}\left(\tilde{\phi}\right)$ is defined as

$$\Gamma_{\beta,\mu}\left(\tilde{\phi}\right) = U_{\beta,\mu}\Gamma_{\infty}\left(\tilde{\phi}\right)U_{\beta,\mu}^{-1},$$

this satisfies

$$\Gamma_{\beta,\mu}\left(\tilde{\phi}\right)\pi_{\underline{\rho}\beta\mu}\left(a(k)\right)\Gamma_{\beta,\mu}\left(\tilde{\phi}\right)^{-1} = \pi_{\underline{\rho}\beta\mu}\left(a(\tilde{\phi}.k)\right).$$

That is $\Gamma_{\beta,\mu}\left(\tilde{\phi}\right)$ implements the automorphism $\tau\left(\tilde{\phi}\right)$ in the representation $\pi_{\underline{\rho}\beta\mu}$. A simple calculation also gives

$$\Gamma_{\beta,\mu}\left(\tilde{\phi}_1\right)\Gamma_{\beta,\mu}\left(\tilde{\phi}_2\right) = \sigma(\tilde{\phi}_1,\tilde{\phi}_2)\Gamma_{\beta,\mu}\left(\tilde{\phi}_1\tilde{\phi}_2\right).$$

So the map $\tilde{\phi} \mapsto \Gamma_{\beta,\mu}\left(\tilde{\phi}\right)$ defines a projective representation of $\mathrm{Map}\left(S^1, U(1)\times U(1)\right)$ with the same 2-cocycle, σ, as $\Gamma_{\infty}\left(\tilde{\phi}\right)$.

Map $\left(S^1, U(1)\right)$ denotes the subgroup of Map $\left(S^1, U(1)\times U(1)\right)$ consisting of multiplication operators of the form $\begin{pmatrix}\phi(s) & 0\\ 0 & 1\end{pmatrix}$ which will be denoted by ϕ. Then the above shows the following.

7.2.2 Remark. The operator $\Gamma_{\beta,\mu}(\phi)$, $\phi\in\mathrm{Map}\left(S^1,U(1)\right)$ defines a σ-representation of the group with the 2-cocycle σ being independent of both β and μ.

Let $\mathcal{M}$ denote the von-Neumann algebra generated by $\Gamma_{\beta,\mu}(\phi)$, $\phi\in\mathrm{Map}\left(S^1,U(1)\right)$. As $\omega_{\beta,\mu}$ is a (τ^{μ},β) KMS state, $\mathcal{M}$ is contained in $\left\{\tilde{A}(H\oplus 0)\right\}''$ where $\tilde{A}(H\oplus 0)$ is the C^*-algebra generated by the set

$$\left\{\Gamma_{\beta,\mu}(-1)\pi_{\underline{\rho}\beta\mu}(a(g)) : g\in H\oplus 0\right\},$$

with $\Gamma_{\beta,\mu}(-1)$ the implementer of the multiplication operator $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$, see [C5] for more details and proofs.

The evolution τ^{μ} of the system, that is $A(H)$, can be extended to an evolution $\tilde{\tau}^{\mu}$ of $A(K)$ by defining

$$\tilde{\tau}_t^{\mu}\left(a(k)\right) = a(\tilde{\tau}_t^{\mu}k), \qquad \forall k\in K$$

where

$$\tilde{\tau}_t^{\mu} = \tau_t^{\mu}\oplus\tau_{-t}^{\mu},$$

is the extension of τ_t^{μ} to K. These automorphisms are also implemented in the representation $\pi_{\underline{\rho}\beta\mu}$, by T_t say, thus giving a map on the σ-representation defined by

$$\Gamma_{\beta,\mu}(\phi)\mapsto T_t\Gamma_{\beta,\mu}(\phi)T_t^{-1}.$$

But $T_t\Gamma_{\beta,\mu}(\phi)T_t^{-1}$ and $\Gamma_{\beta,\mu}(\phi_t)$, where

$$\phi_t(s) = \tau_t^{\mu}\phi\tau_t^{\mu^{-1}}(s) = \phi(s+t),$$

both implement the same automorphism thus

$$T_t\Gamma_{\beta,\mu}(\phi)T_t^{-1} = c\,\Gamma_{\beta,\mu}(\phi_t),$$

where c is a complex number of unit modulus, dependent on ϕ and t. Denote this by $\tilde{\sigma}(\phi,t)$ then this satisfies the cocycle condition

$$(\dagger)\qquad \sigma(\phi,\psi)\tilde{\sigma}(\phi\psi,t) = \tilde{\sigma}(\phi,t)\tilde{\sigma}(\psi,t)\sigma(\phi_t,\psi_t)$$

and the following holds.

7.2.3 Remark. The modular automorphism group τ^{μ} corresponding to the state $\omega_{\beta,\mu}$ on $A(H)$ restricts on $\mathcal{M}$ to the one parameter group of automorphisms defined above and moreover the state $\omega_{\beta,\mu}|_{\mathcal{M}}$ is a (τ^{μ},β) KMS state.

7.3 Investigation of $\omega_{\beta,\mu}|_{\mathcal{M}}$.

First note that Map $(S^1, U(1))$ is the direct product of the subgroups

(1) $M_0 = \left\{ e^{if} : f \in \mathrm{Map}(S^1, \mathbb{R}), f(2\pi) = f(0), \int_0^{2\pi} f(s)\,ds = 0 \right\}$.

(2) M_c = subgroup generated by the constant functions and the functions given by $s \mapsto \exp(ins)$, $n \in \mathbb{Z}$.

Clearly M_c can be identified with $S^1 \times \mathbb{Z}$ hence let

$\mathcal{M}_0$ = von Neumann algebra generated by $\{\Gamma_{\beta,\mu}(\phi) : \phi \in M_0\}$,

$\mathcal{M}_c$ = von Neumann algebra generated by $\{\Gamma_{\beta,\mu}(\phi) : \phi \in S^1 \times \mathbb{Z}\}$.

The previous subsection 7.2 demonstrated that the 2-cocycle for the $\Gamma_{\beta,\mu}(.)$'s is the same as that for $\Gamma_{\infty}(.)$. Defining L as $\mathrm{Map}(S^1, \mathbb{R})$ then L is the Lie algebra of $\mathrm{Map}(S^1, U(1))$ and $L \oplus L$ the Lie algebra of Map $(S^1, U(1) \times U(1))$. Now the results in [C9] and [L4] imply the existence of a projective representation of $L \oplus L$, denoted by $\bar{f} \mapsto J_{\infty}(\bar{f})$ say, where $J_{\infty}(\bar{f})$ is a self adjoint operator with

$$(\Omega_{\infty}, J_{\infty}(\bar{f})\Omega_{\infty}) = 0,$$

and

$$\Gamma_{\infty}\left(\exp(i\bar{f})\right) = \exp iJ_{\infty}(\bar{f}).$$

Therefore defining $J_{\beta,\mu}(\bar{f})$ as

$$J_{\beta,\mu}(\bar{f}) = U_{\beta,\mu} J_{\infty}(\bar{f}) U_{\beta,\mu}^{-1}$$

gives a projective representation of $L \oplus L$ with $J_{\beta,\mu}(\bar{f})$ a self adjoint operator with

$$\Gamma_{\beta,\mu}\left(\exp(i\bar{f})\right) = \exp iJ_{\beta,\mu}(\bar{f}).$$

The results in [C9] also imply the existence of a self adjoint operator $\tilde{J}_{\beta,\mu}(\bar{f})$ with $\exp i\tilde{J}_{\beta,\mu}(\bar{f})$ differing by a phase from $\Gamma_{\beta,\mu}\left(\exp(i\bar{f})\right)$. The phase of $\Gamma_{\beta,\mu}\left(\exp(i\bar{f})\right)$ may be changed though, without changing the cocycle σ, to obtain this modified generator $\tilde{J}_{\beta,\mu}(\bar{f})$ with the properties

$$(\Omega_{\beta,\mu}, \tilde{J}_{\beta,\mu}(\bar{f})\Omega_{\beta,\mu}) = 0, \text{ and } \Gamma_{\beta,\mu}\left(\exp(i\bar{f})\right) = \exp i\tilde{J}_{\beta,\mu}(\bar{f}).$$

This choice of phase will now be assumed and the tilde dropped from the notation.

On the restriction to Map $(S^1, U(1))$ of the above the phase of $\Gamma_{\infty}(\phi)$ may be chosen for ϕ having arbitrary winding number $w(\phi)$ consistently with the zero winding number elements to give

$$\sigma(\phi_1, \phi_2) = \exp\left\{-\frac{i}{4\pi}\int_0^{2\pi} f_2(s)\,df_1(s)\right\} \phi_2(0)^{-w(\phi_1)/2},$$

where $\phi_j = \exp if_j$ for $j = 1, 2$, see [C7] for details. Hence

$$\Gamma_{\beta,\mu}(\phi_1)\Gamma_{\beta,\mu}(\phi_2) = \bar{\sigma}(\phi_1, \phi_2)\Gamma_{\beta,\mu}(\phi_2)\Gamma_{\beta,\mu}(\phi_1),$$

where

$$\partial(\phi_1,\phi_2) = \exp\left\{-\frac{i}{4\pi}\int_0^{2\pi}[f_2(s)df_1(s) - f_1(s)df_2(s)]\right\}$$
$$\cdot\exp\left\{-\frac{i}{4\pi}[f_1(2\pi)f_2(0) - f_2(2\pi)f_1(0)]\right\}.$$

For the special case when $\phi_j = \exp if_j$ and $f_j \in L$ for $j = 1,2$ this may be simplified to

$$\partial(\phi_1,\phi_2) = \exp\left\{-\frac{i}{2\pi}\int_0^{2\pi} f_2(s)df_1(s)\right\}.$$

But the factor in the exponential determines a non-degenerate symplectic form on L, hence the canonical commutation relations over L. Thus the map

$$\phi \mapsto \Gamma_{\beta,\mu}(\phi), \qquad \phi \in \mathrm{Map}\,(S^1, U(1))$$

gives a representation of the CCR algebra over L in Weyl form. Therefore the algebra $\mathcal{M}_0$ is generated by a representation of the CCR algebra.

From the definition of the cocycle above it is not difficult to deduce that

$$\Gamma_{\beta,\mu}(\phi_1)\,\Gamma_{\beta,\mu}(\phi_2) = \Gamma_{\beta,\mu}(\phi_2)\,\Gamma_{\beta,\mu}(\phi_1),$$

whenever $\phi_1 \in M_0$ and $\phi_2 \in M_s$ or vice versa. Therefore the algebras $\mathcal{M}_0$ and $\mathcal{M}_s$ are contained in the commutants of one another. Also the evolution τ^μ leaves these two algebras invariant. The expression given for the cocycle together with the condition (†) enables the determination of the expression $\partial(\phi,t)$, see [CS] Lemma 2.7. This is given by:

$$\partial(\phi,t) = \left(\phi(0)\phi(t)^{-1}e^{-ipt}\right)^{w(\phi)/2}, \qquad \text{for any } p \in \mathbb{R}$$

so that

$$\tau_t^\mu\left(\Gamma_{\beta,\mu}(\phi)\right) = \left(\phi(0)\phi(t)^{-1}e^{-ipt}\right)^{w(\phi)/2}\,\Gamma_{\beta,\mu}(\phi_t).$$

7.3.1 Lemma. *For each $\mu \in \mathbb{R}$, $\beta > 0$ the algebra $\mathcal{M}_s$ has a unique (τ^μ,β) KMS state $\omega^s_{\beta,\mu}$ whose generating functional is*

$$\omega^s_{\beta,\mu}\left(\Gamma_{\beta,\mu}(\psi)\right) = \delta_{n0}\theta(\alpha)\theta(0)^{-1},$$

where $\psi(s) = \exp i(ns+\alpha)$ and $\theta(\alpha) = \sum_k q^{k(k+p)}e^{ik\alpha}$ where the nome of the theta function is $q = e^{-\beta/2}$.

Proof: The proof in [CS] Lemma 2.8 for the special case $\mu = -1/2$ is sufficient as it is dependent only on the cocycle and the time evolution both of which are independent of β and μ.

The following proposition concerns the factorisation of a KMS state. It is taken from [CS]. See Proposition 2.9 in that paper for a proof.

7.3.2 Proposition. *Suppose B and C are von Neumann algebras of operators on the same space, each of which is in the commutant of the other, and each of which is invariant under the action of a one parameter group, $t \mapsto \tau_t$, of automorphisms of the algebra A they generate. Then any (τ,β) KMS state on A restricts to (τ,β) KMS states* *on B and C. Moreover if C has a unique (τ,β) KMS state ω_β^c then the original state factorises into a product of ω_β^c and a (τ,β) KMS state on B.*

7.3.3 THEOREM. *The (τ^μ,β) KMS state $\omega_{\beta,\mu}$ on the von Neumann algebra $\mathcal{M}$ has the form*

$$(\Omega_{\beta,\mu},\Gamma_{\beta,\mu}(\phi)\,\Omega_{\beta,\mu}) = \delta_{n0}\theta_3(\alpha)\theta_3(0)^{-1}\exp\left\{-\frac{1}{4\pi}\sum_{k\neq 0} k(1-e^{-\beta k})^{-1}|f_k|^2\right\},$$

where $\phi(s) = \exp\left(i(ns+\alpha+\sum_{k\neq 0} f_k e^{iks})\right)$.

PROOF: From the two preceding comments the form of $\omega_{\beta,\mu}$ on $\mathcal{M}_0$ is the only thing required as $\omega_{\beta,\mu}$ is the product of the unique KMS state on $\mathcal{M}_s$ given by Lemma 7.3.1 and a KMS state on $\mathcal{M}_0$, that is

$$\omega_{\beta,\mu}(m_0 m_s) = \omega^0_{\beta,\mu}(m_0)\omega^s_{\beta,\mu}(m_s).$$

By the methods of [C9] and [L4], for the representation $\tilde{f} \mapsto J_{\beta,\mu}(f)$ of $L \oplus L$

$$(\Omega_{\beta,\mu}, J_{\beta,\mu}(\tilde{f})J_{\beta,\mu}(\tilde{f}')\Omega_{\beta,\mu}) = \mathrm{Trace}(P_-^{\beta,\mu}\tilde{f}P_+^{\beta,\mu}\tilde{f}'P_-^{\beta,\mu}).$$

The case of interest involves the simplification

$$\tilde{f} = \tilde{f}' = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

in which case the two Lemmas following this proof give

$$(\Omega_{\beta,\mu}, J_{\beta,\mu}(\tilde{f})^2\Omega_{\beta,\mu}) = \frac{1}{2\pi}\sum_{k\neq 0} k(1-e^{-\beta k})^{-1}|f_k|^2.$$

Hence as $\mathcal{M}_0$ is generated by a representation of the CCR algebra, standard properties of this algebra lead to

$$(\Omega_{\beta,\mu},\Gamma_{\beta,\mu}(\phi)\,\Omega_{\beta,\mu}) = \exp\left\{-\frac{1}{4\pi}\sum_{k\neq 0} k(1-e^{-\beta k})^{-1}|f_k|^2\right\},$$

where $\phi = \exp if \in \mathcal{M}_0$.

Now from [R1]

$$(\Omega_{\beta,\mu},\Gamma_{\beta,\mu}\left(\tilde{\psi}\right)\Omega_{\beta,\mu}) = \det(1+B)^{-1/2},$$

where

$$B = (P_-^{\beta,\mu}\tilde{\psi}P_-^{\beta,\mu})^{-1}P_-^{\beta,\mu}\tilde{\psi}P_+^{\beta,\mu}\tilde{\psi}^*P_-^{\beta,\mu}(P_-^{\beta,\mu}\tilde{\psi}^*P_-^{\beta,\mu})^{-1}.$$

Therefore if $\tilde{\psi} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}$,

$$\begin{aligned}
&(\Omega_{\beta,\mu},\Gamma_{\beta,\mu}\left(\tilde{\psi}\right)\Omega_{\beta,\mu}) \\
&= \det(1+B)^{-1/2} \\
&= \det\left(\left(P_-^{\beta,\mu}\tilde{\psi}P_-^{\beta,\mu}\right)^{-1}\left(P_-^{\beta,\mu}\tilde{\psi}^*P_-^{\beta,\mu}\right)^{-1}\right)^{-1/2} \\
&= \left[\prod_n\left(1+2\cos\alpha e^{-\beta(n-\mu)}+e^{-2\beta(n-\mu)}\right)\left(1+e^{-\beta(n-\mu)}\right)^{-2}\right]^{1/2}.
\end{aligned}$$

Hence comparing this with the form given in Lemma 7.3.1 it can be seen that $\mu = 0$ and thus $\theta \equiv \theta_3$.

Combining this and the previous formula for $\omega_{\beta,\mu}$ on $\mathcal{M}_0$ gives the result.

7.3.4 Lemma. *Suppose $\mu \in \mathbb{R}$, $n \in \mathbb{Z} \setminus \{0\}$ and $\beta \in (0, \infty)$. Then*

$$\sum_{r \in \mathbb{Z}} \left(1 + e^{\beta(r-\mu)}\right)^{-1} \left(1 + e^{-\beta(r-n-\mu)}\right)^{-1} = -n(1 - e^{n\beta})^{-1}.$$

Proof:

$$\begin{aligned} &\left(1 + e^{\beta(r-\mu)}\right)^{-1} \left(1 + e^{-\beta(r-n-\mu)}\right)^{-1} \\ &= (1 - e^{n\beta})^{-1} \left[\left(1 + e^{\beta(r-\mu)}\right)^{-1} - \left(1 + e^{-\beta(r-n-\mu)}\right)^{-1}\right]. \end{aligned}$$

Hence the sum can be rewritten as

$$(1 - e^{n\beta})^{-1} \sum_{r \in \mathbb{Z}} \left[\left(1 + e^{\beta(r-\mu)}\right)^{-1} - \left(1 + e^{-\beta(r-n-\mu)}\right)^{-1}\right].$$

Now, suppose $N > n$ then

$$\begin{aligned} &\sum_{r=-N}^{N} \left[\left(1 + e^{\beta(r-\mu)}\right)^{-1} - \left(1 + e^{-\beta(r-n-\mu)}\right)^{-1}\right] \\ &= \underbrace{\frac{1}{1 + e^{\beta(N-n+1-\mu)}} + \cdots + \frac{1}{1 + e^{\beta(N-\mu)}}}_{n \text{ terms}} \\ &\quad - \underbrace{\frac{1}{1 + e^{\beta(-N-n-\mu)}} - \cdots - \frac{1}{1 + e^{\beta(-N-1-\mu)}}}_{n \text{ terms}} \end{aligned}$$

as other terms cancel. But each positive term tends to 0 as $N \to \infty$ and each negative term to -1. Thus

$$\lim_{N \to \infty} \sum_{r=-N}^{N} \left[\left(1 + e^{\beta(r-\mu)}\right)^{-1} - \left(1 + e^{-\beta(r-n-\mu)}\right)^{-1}\right] = -n$$

and the result follows.

7.3.5 Lemma. *Suppose $\phi(s) = \exp i(ns + \alpha + \sum_{k \neq 0} f_k e^{iks})$ and $f(s) = \sum_{k \neq 0} f_k e^{iks}$. Then if $\tilde{f}$ denotes the 2×2 matrix $\begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$,*

$$\mathrm{Trace}\left(P_-^{\beta,\mu} \tilde{f} P_+^{\beta,\mu} \tilde{f} P_-^{\beta,\mu}\right) = \frac{1}{2\pi} \sum_{k \neq 0} k(1 - e^{-\beta k})^{-1} \|f_k\|^2.$$

Proof: With $\tilde{f}$ as above, after simplification

$$\mathrm{Trace}\left(P_-^{\beta,\mu} \tilde{f} P_+^{\beta,\mu} \tilde{f} P_-^{\beta,\mu}\right) = \mathrm{Trace}\left(A_{\beta,\mu} f (1 - A_{\beta,\mu}) f\right).$$

Taking fourier transforms this can be shown to equal

$$\frac{1}{2\pi} \sum_{n \neq 0} \left\{ f_n f_{-n} \sum_{r \in \mathbb{Z}} \left(1 + e^{\beta(r-\mu)}\right)^{-1} \left(1 + e^{-\beta(r-n-\mu)}\right)^{-1} \right\}.$$

Now ϕ is a unitary operator, that is $\phi^* = \phi^{-1}$. But

$$\phi^{-1} = \exp -i(ns + \alpha + \sum_{k \neq 0} f_k e^{iks}),$$

and

$$\phi^* = \exp -i(ns + \alpha^* + \sum_{k \neq 0} f_k^* e^{-iks}).$$

Therefore $\alpha = \alpha^*$, that is α is real, and $f_k = f_{-k}^*$. Hence $f_n f_{-n} = f_{-n}^* f_{-n} = \|f_{-n}\|^2$.

These facts together with Lemma 7.3.4 give the result.

7.4 The Fermion algebra from Map $(S^1, U(1))$.

The other direction will now be considered, that is, with a projective representation of Map $(S^1, U(1))$ defined through a particular function, then the CAR algebra acts on the Hilbert space of this representation and the CAR elements are limits, in a certain sense, of the loop group elements.

Let $\omega_{\beta,\mu}$ denote the following function on the central extension of Map $(S^1, U(1))$ determined by the 2-cocycle, σ, given in the previous subsection 7.3

$$\omega_{\beta,\mu}(\phi) = \delta(w_\phi) \exp i f_0(1/2 + \mu)\theta_3(f_0)\theta_3(0)^{-1} \exp\left\{ -\frac{1}{4\pi} \sum_{k\neq 0} k(1 - e^{-\beta k})^{-1} |f_k|^2 \right\},$$

where $\phi(s) = \exp\left(i(ns + \sum_{k\in\mathbb{Z}} f_k e^{iks})\right)$.

This determines a σ-representation of Map $(S^1, U(1))$ as follows:
if

$$\omega_\beta(\phi) = \delta(w_\phi)\theta_3(f_0)\theta_3(0)^{-1} \exp\left\{ -\frac{1}{4\pi} \sum_{k\neq 0} k(1 - e^{-\beta k})^{-1} |f_k|^2 \right\}$$

subsection 7.3 gives that $\omega_\beta(\phi)$ determines a σ-representation of Map $(S^1, U(1))$ which will be denoted by $\Gamma_{\beta,\mu}(\phi)$. Note the function really has no dependence on μ so $\Gamma_{\beta,\mu}(\phi)$ could be written $\Gamma_\beta(\phi)$ see [C5]. This fact is the main point of subsection 7.3 . Hence

$$\omega_{\beta,\mu}(\phi) = \exp i f_0(1/2 + \mu)\omega_\beta(\phi)$$

determines a σ-representation via

$$\Gamma^{-}_{\beta,\mu}(\phi) = \exp i f_0(1/2 + \mu)\Gamma_{\beta,\mu}(\phi).$$

The elementary functions which will enable the construction of the CAR elements are given by the following:
let the special loops or 'kinks' be the functions defined by

$$\gamma_{\alpha,\lambda}(\theta) = \frac{(\lambda - e^{i(\theta-\alpha)})}{(\lambda e^{i(\theta-\alpha)} - 1)},$$

where $\lambda \in (0,1)$. These enable the approximate fermion operators or 'blips' to be defined as

$$\begin{aligned} B_{\alpha,\lambda} &= e^{i(1/2-\mu)\alpha}(1-\lambda^2)^{-1/2}\sigma(\gamma_{\alpha,0}, \gamma_{\alpha,0}^{-1}\gamma_{\alpha,\lambda})\Gamma_{\beta,\mu}(\gamma_{\alpha,\lambda}) \\ &= e^{i(1/2-\mu)\alpha}(1-\lambda^2)^{-1/2}\left(\frac{1-\lambda e^{-i\alpha}}{1-\lambda e^{i\alpha}}\right)^{1/2}\Gamma_{\beta,\mu}(\gamma_{\alpha,\lambda}) \end{aligned}$$

using the definition of the cocycle σ and the fact that $\gamma_{\alpha,\lambda}(\theta)$ can be written as

$$\gamma_{\alpha,\lambda}(\theta) = \exp\left\{ i\left[(\theta-\alpha) - i\sum_{n\neq 0}\lambda^{|n|}e^{in(\theta-\alpha)}/n\right]\right\}.$$

There now follows two propositions concerning the limit process used to obtain the fermion operators. One describes the limiting procedure and its domain while the other demonstrates the fermion operators. They are general results and may be found in [C12] for the computationally harder case of $\mathbb{R}$ instead of S^1 and also in [P12] in a slightly different form so no proofs are given here.

7.4.1 Proposition. *Set $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N)$ and*

$$\phi_{\underline{\lambda}}(G) = \int d\alpha_1 \ldots d\alpha_N G(\alpha_1, \ldots, \alpha_N). \prod_{i=1}^{N} B^{(*)}_{\alpha_i, \lambda_i} \Omega_{\beta,\mu}$$

where $(*)$ indicates that the adjoint may be substituted at any point and where G is a smooth function on S^1. Then $\phi_{\underline{\lambda}}(G)$ is a well defined vector in $\mathcal{H}$ and the strong limit as $\lambda_j \to 1$ $j = 1, \ldots, N$ exists independently of the order in which the λ_j are taken to 1. For g a smooth function of S^1 the operator $B(g)^{(*)}$ may therefore be defined on the domain consisting of polynomials in the blips and also inductively on the larger domain obtained by taking the span of all vectors of the form $\phi_{\underline{\lambda}}(G)$ via

$$B(g)^{(*)}\phi_{\underline{\lambda}}(G) = \operatorname*{s-lim}_{\lambda,\lambda_j \to 1} \int_0^{2\pi} d\alpha g(\alpha) B^{(*)}_{\alpha,\lambda} \phi_{\underline{\lambda}}(G).$$

7.4.2 Proposition. *Suppose $\phi = \exp if \in \mathrm{Map}\left(S^1, U(1)\right)$ and $\alpha, \zeta \in [0, 2\pi)$. Then*

(1) $\Gamma_{\beta,\mu}(\phi) B_\alpha \Gamma_{\beta,\mu}(\phi)^* = \phi(\alpha) B_\alpha,$
(2) $[B^*_\alpha, B_\zeta]_+ = 2\pi\delta(\alpha - \zeta)1,$
(3) $[B_\alpha, B_\zeta]_+ = 0 = \left[B^*_\alpha, B^*_\zeta\right]_+,$

where $B_\alpha = \lim_{\lambda \to 1} B_{\alpha,\lambda}$.

7.4.3 Remark. Proposition 7.4.1 defines an 'operator-valued distribution' and Proposition 7.4.2 is suppose to be understood in the sense of distributions. For example (2) means

$$B(f)B(g)^* + B(g)^* B(f) = \frac{1}{2\pi} \int_0^{2\pi} f(s)\overline{g(s)}\, ds.$$

Hence $B(g)^{(*)}$ can be defined for all $g \in L^2(S^1, \mathbf{C})$. Proposition 7.4.2 shows that the limiting distributions satisfy the anticommutation relations.

7.4.4 Remark. From subsection 7.2 the time evolution of the loop group elements, regarded as multiplication operators on $L^2(S^1, \mathbf{C})$, is given by $\phi \to \phi_t$ where $\phi_t(s) = \phi(s + t)$. Hence it is a simple calculation to show that the 'kinks' evolve according to

$$\gamma_{\alpha,\lambda} \to [\gamma_{\alpha,\lambda}]_t = \gamma_{\alpha-t,\lambda}.$$

This leads to the following Lemma.

7.4.5 Lemma. *The 'blips' evolve according to*

$$B_{\alpha,\lambda} \to [B_{\alpha,\lambda}]_t = e^{-i\mu t} B_{\alpha-t,\lambda}.$$

Proof: From subsection 7.3

$$[\Gamma_{\beta,\mu}(\gamma_{\alpha,\lambda})]_t = \left(\gamma_{\alpha,\lambda}(0)\gamma_{\alpha,\lambda}(t)^{-1}\right)^{1/2} \Gamma_{\beta,\mu}\left([\gamma_{\alpha,\lambda}]_t\right).$$

Hence

$$\begin{aligned}
&[B_{\alpha,\lambda}]_t \\
&= e^{i(1/2-\mu)\alpha}(1-\lambda^2)^{-1/2}\left(\frac{1-\lambda e^{-i\alpha}}{1-\lambda e^{i\alpha}}\right)^{1/2} \left(\gamma_{\alpha,\lambda}(0)\gamma_{\alpha,\lambda}(t)^{-1}\right)^{1/2} \Gamma_{\beta,\mu}(\gamma_{\alpha-t,\lambda}) \\
&= e^{i(1/2-\mu)\alpha}(1-\lambda^2)^{-1/2}\left(\frac{1-\lambda e^{-i\alpha}}{1-\lambda e^{i\alpha}} \cdot \frac{\lambda - e^{-i\alpha}}{\lambda e^{-i\alpha} - 1} \cdot \frac{\lambda e^{i(t-\alpha)} - 1}{\lambda - e^{i(t-\alpha)}}\right)^{1/2} \Gamma_{\beta,\mu}(\gamma_{\alpha-t,\lambda}) \\
&= e^{-it/2} e^{i(1/2-\mu)\alpha}(1-\lambda^2)^{-1/2}\left(\frac{1-\lambda e^{-i(\alpha-t)}}{1-\lambda e^{i(\alpha-t)}}\right)^{1/2} \Gamma_{\beta,\mu}(\gamma_{\alpha-t,\lambda}) \\
&= e^{-i\mu t} e^{i(1/2-\mu)(\alpha-t)}(1-\lambda^2)^{-1/2}\left(\frac{1-\lambda e^{-i(\alpha-t)}}{1-\lambda e^{i(\alpha-t)}}\right)^{1/2} \Gamma_{\beta,\mu}(\gamma_{\alpha-t,\lambda}) \\
&= e^{-i\mu t} B_{\alpha-t,\lambda}.
\end{aligned}$$

7.4.6 REMARK. This essentially gives the operator valued distribution B_α evolves according to

$$B_\alpha \to [B_\alpha]_t = e^{-i\mu t} B_{\alpha - t}.$$

7.4.7 THEOREM. *The state $\omega_{\beta,\mu}$ is a KMS state for the C^*-algebra generated by*

$$\{B(g) : g \in L^2(S^1, \mathbf{C})\}.$$

PROOF: The proof is essentially the same as that in [C5] Proposition 3.12, the only difference is that the 'blips' evolve somewhat differently. Using the notation of that Proposition, suppose X and Y are products of 'blips' and their adjoints and let

$$B^*_{\alpha_1}, \ldots, B^*_{\alpha_N}, B_{\zeta_1}, \ldots, B_{\zeta_M}$$

be the terms appearing in X and Y in the limit with $B^*_{\alpha_j}, B_{\zeta_k}$ elements of X if $j \in J$, $k \in K$ where $a = |J|$ and $b = |K|$.

So by the above $\omega_{\beta,\mu}(Y[X]_t)$ is $\exp(-i\mu(b-a)t)$ times the appropriate correlation function for the limits of the 'blips'. That is, α_j becomes $\alpha_j - t$ if $j \in J$ and ζ_k becomes $\zeta_k - t$ if $k \in K$. With the correlation function written explicitly in the form given by [C5] Proposition 3.8 together with the additional $\exp if_\alpha(1/2+\mu)$ term, which equals

$$\exp i(1/2+\mu)\sum_j(\alpha_j - \zeta_j)$$

in this case, the t dependent factors are

$$\exp i/2\left\{\sum_j(\zeta_j - \alpha_j) + (a-b)t\right\},$$

$$\exp i(1/2+\mu)\left\{\sum_j(\alpha_j - \zeta_j) + (b-a)t\right\},$$

$$\theta_3\left[\sum_j(\alpha_j - \zeta_j) + (b-a)t\right],$$

and θ_1 terms.

Now the $\exp(-i\mu(b-a)t)$, $\exp i/2(a-b)t$ and $\exp i(1/2+\mu)(b-a)t$ terms combine to leave 1, which is t independent. Thus the θ_3 and θ_1 terms are the only t dependent. Similarly these are the only t dependent terms in $\omega_{\beta,\mu}([X]_tY)$ and the proof of [C5] Proposition 3.12 covers the t dependence of these terms to give

$$\omega_{\beta,\mu}([X]_tY) = \omega_{\beta,\mu}(Y[X]_{t+i\beta}).$$

That is $\omega_{\beta,\mu}$ is a KMS state.

7.4.8 REMARK. Since $\omega_{\beta,\mu}$ is a KMS state on the CAR algebra with the time evolution given in the previous remark, by the uniqueness of KMS states for the CAR algebra this must coincide with the quasi-free state given at the beginning of this analysis. That is, the quasi-free state defined by $A_{\beta,\mu}$.

7.4.9 Remark. This analysis demonstrates that the Boson-Fermion correspondence as described is not as good as it could be. The only situation a bijective correspondence occurs in when $\mu + 1/2 = 0$ and it is this situation which is described in [C5]. Otherwise the process from loop group to CAR algebra is injective but from CAR algebra to CCR algebra it certainly is not. This may be due to the implicit choice of phase taken in the argument so that

$$(\Omega_{\beta,\mu}, J_{\beta,\mu}^{-}\Omega_{\beta,\mu}) = 0.$$

With another choice of phase the cocycle might be adjusted so that the 'rotation' $\exp i f_0(1/2 + \mu)$ occurs in the state.

7.4.10 Remark. This extra term $\exp i f_0(1/2 + \mu)$ appears to tie in with [C6] Section 4.3, certainly if $\mu \in \mathbb{Z}_{1/2}$ so that $(1/2 + \mu) \in \mathbb{Z}$, giving a connection with Bose-Einstein condensation.

Section 8
A Determinant Identity

8.1 Introduction.

This section uses results of [P1] to calculate the expression

$$(\Omega_{\beta,\mu}, \Gamma_{\beta,\mu}(\phi)\Omega_{\beta,\mu})$$

defined in section 7. Identifying this formula with that produced in the previous section leads to a determinant identity reminiscent of Szegö's theorem. A brief description of the structure and results of use from [P1] now follows.

Suppose W is an infinite dimensional complex Hilbert space with complex structure i and distinguished conjugation P. Let Q be a self adjoint operator such that $Q^2 = 1$ and $QP + PQ = 0$. The Q-Fock representation of the Clifford algebra $C(W, P)$ is given by

$$F_Q(w) = a^*(Q_+w) + a(PQ_-w),$$

where $a(.)$, $a^*(.)$ are annihilation and creation operators on the alternating tensor algebra $\Lambda(W_+)$. ($W_\pm = Q_\pm W$, $Q_\pm = 1/2(1 \pm Q)$, see section 2.2 for an example.)

8.1.1 Definition. Let $G(W, Q)$ denote the set of bounded operators g on $\Lambda(W_+)$ such that

$$gF(w) = F(T(g)w)g,$$

for some bounded, invertible, P-orthogonal $T(g)$ on W.

Note. An operator T on W is P-orthogonal if $PT^*P = T^{-1}$.

8.1.2 Theorem. *If T is unitary on W commuting with P and $TQ - QT$ is a Hilbert Schmidt operator on W then there exists a unitary $g \in G(W, Q)$ such that $T = T(g)$. Conversely if g is a unitary element of $G(W, Q)$ then $T(g)Q - QT(g)$ is a Hilbert Schmidt operator on W.*

8.1.3 Definition. The Q-representation of W is the representation where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad i = \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix},$$

where Λ is a complex structure.

8.1.4 Remark. If $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ is the Q-representation of W then:

$$T \text{ is } P\text{-orthogonal} \Leftrightarrow \begin{cases} T_{21}^*T_{11} + T_{11}^*T_{21} = 0 \\ T_{22}^*T_{12} + T_{12}^*T_{22} = 0 \\ T_{21}^*T_{12} + T_{11}^*T_{22} = I \\ T_{kk}\Lambda = \Lambda T_{kk} \qquad k = 1, 2 \\ T_{kj}\Lambda = -\Lambda T_{kj} \qquad j \neq k \end{cases}$$

This is a simple consequence of the facts $PT^*PT = 1$ and $iT = Ti$.

8.1.5 Definition. An element $g \in G(W, Q)$ is factorable if $Q_-T(g) + Q_+$ is invertible.

8.1.6 REMARK. If
$$T(g) = \begin{bmatrix} T_{11}(g) & T_{12}(g) \\ T_{21}(g) & T_{22}(g) \end{bmatrix}$$
in the Q-representation of W then
$$Q_-T(g) + Q_+ = \begin{bmatrix} I & 0 \\ T_{21}(g) & T_{22}(g) \end{bmatrix}.$$
Consequently $T(g)$ is factorable if and only if $T_{22}(g)$ is invertible in which case
$$(Q_-T(g) + Q_+)^{-1} = \begin{bmatrix} I & 0 \\ -T_{22}^{-1}(g)T_{21}(g) & T_{22}^{-1}(g) \end{bmatrix}.$$
8.1.7 REMARK. If T_{22} is invertible then $T = L(1)L(2)$ where
$$L(1) = \begin{bmatrix} T_{11}T_{22}^{*} & T_{12}T_{22}^{-1} \\ T_{21}T_{22}^{*} & I \end{bmatrix},$$
and
$$L(2) = \begin{bmatrix} T_{22}^{*-1} & 0 \\ 0 & T_{22} \end{bmatrix}.$$
If T is P-orthogonal then so are $L(1)$ and $L(2)$.

8.1.8 DEFINITION. Let
$$\begin{aligned} R = (T-1)(Q_-T+Q_+)^{-1} &= \begin{bmatrix} T_{22}^{*-1} - 1 & T_{12}T_{22}^{-1} \\ T_{22}^{-1}T_{21} & 1 - T_{22}^{-1} \end{bmatrix}, \\ R(1) = (L(1)-1)(Q_-L(1)+Q_+)^{-1} &= \begin{bmatrix} 0 & T_{12}T_{22}^{-1} \\ T_{21}T_{22}^{*} & 0 \end{bmatrix}, \\ R(2) = (L(2)-1)(Q_-L(2)+Q_+)^{-1} &= \begin{bmatrix} T_{22}^{*-1} - 1 & 0 \\ 0 & 1 - T_{22}^{-1} \end{bmatrix}. \end{aligned}$$
8.1.9 LEMMA. *Suppose g is a factorable element of $G(W,Q)$. Then $T(g)Q - QT(g)$ is a Hilbert Schmidt operator on W. Furthermore*
$$|\langle g\rangle|^2 = \|g\Omega_Q\|^2 \det\left(I + |T_{12}T_{22}^{-1}|^2\right)^{-1/2},$$
where $\langle g\rangle = \langle \Omega_Q, g\Omega_Q\rangle$ with $\Omega_Q = 1 \oplus 0 \oplus 0 \oplus \dots$ denoting the vacuum vector in $\Lambda(W_+)$.

8.1.10 THEOREM. *Suppose g_k is a factorable element of $G(W,Q)$ for each $k = 1,\dots,n$. Then if $\langle g_1 \dots g_n\rangle \neq 0$*
$$\langle g_1 \dots g_n\rangle^2 = \prod_{k=1}^{n} \langle g_k\rangle^2 \det{}_2(1 + L\Delta R),$$
where L is the $n \times n$ block matrix with entries
$$L_{lm} = \begin{cases} -Q_+L_{l+1}(2)\dots L_{m-1}(2) & \text{for } m \geq l+2 \\ -Q_+ & \text{for } m = l+1 \\ 0 & \text{for } m = l \\ -\overline{L_{ml}^{*}} & \text{for } m < l \end{cases}$$
and ΔR the $n \times n$ block diagonal matrix with entries
$$\Delta R_{lm} = \delta_{lm}\Delta R_m$$
where $\Delta R_m = R_m - R_m(2)$ and $\overline{X} = PXP$.

8.2 Basic Structure.

The necessary structure for the application of the results outlined in subsection 8.1 is now developed.

8.2.1 Lemma. *Suppose K is a Hilbert space. Let $\check{K} = K \oplus \overline{K}$ where $\overline{K}$ denotes the Hilbert space conjugate to K and where the inner product of $\check{K}$ is given by*

$$(f_1 \oplus \bar{f}_2, g_1 \oplus \bar{g}_2)_{\check{K}} = (f_1, g_1)_K + (\bar{f}_2, \bar{g}_2)_{\overline{K}}.$$

Let Γ denote the operator on $\check{K}$ defined by

$$\Gamma(x \oplus \bar{y}) = y \oplus \bar{x}.$$

*Then the CAR algebra over K, $A(K)$, is *-isomorphic to the self-dual CAR algebra over $\check{K}$ with antiunitary involution Γ, $A_{SDC}(\check{K}, \Gamma)$.*

Proof: $\Gamma^2 = 1$ is obvious.
If $f = f_1 \oplus \bar{f}_2$, $g = g_1 \oplus \bar{g}_2$ then

$$\begin{aligned}(\Gamma f, \Gamma g)_{\check{K}} &= (f_2 \oplus \bar{f}_1, g_2 \oplus \bar{g}_1)_{\check{K}} \\ &= (g, f)_{\check{K}}\end{aligned}$$

using the defintion of the inner product given above and

$$(x, y)_K = (\bar{y}, \bar{x})_{\overline{K}}$$

So Γ is an antiunitary involution, thus $A_{SDC}(\check{K}, \Gamma)$ can be defined.

Suppose $A_{SDC}(\check{K}, \Gamma)$ is generated by $B(f)^{(*)}$, $f \in \check{K}$ and $A(K)$ by $a(k)^{(*)}$, $k \in K$ then the *-isomorphism is given by the identification

$$B(x \oplus \bar{y}) = a(x) + a^*(y).$$

8.2.2 Lemma. *Suppose P is a projection on K and ω_P is the quasi-free state on $A(K)$ determined by P. Then ω_S, the quasi-free state on $A_{SDC}(\check{K}, \Gamma)$ determined by $S = P \oplus (1 - P)$, is equivalent to ω_P. That is, ω_P acting on a combination of $a(f_i)$'s and $a(g_j)^*$'s, $f_i, g_j \in K$ is equal to ω_S acting on the corresponding $B(x_k \oplus \bar{y}_k)$'s given by the *-isomorphism and vice versa.*

Proof: $\Gamma S \Gamma = 1 - S$ and $0 \leq S = S^* \leq 1$ so S does indeed determine a quasi-free state ω_S. Now

$$\begin{aligned}\omega_P\,(a^*(f)a(g)) &= (g, Pf)_K \\ &= (g, Pf)_K + (0, (1 - P)0)_{\overline{K}} \\ &= ((g \oplus 0), (P \oplus (1 - P))(f \oplus 0))_{\check{K}} \\ &= ((g \oplus 0), S(f \oplus 0))_{\check{K}} \\ &= \omega_S\,(B^*(f \oplus 0)B(g \oplus 0)) \qquad \text{definition of } \omega_S \\ &= \omega_S\,(B(0 \oplus \bar{f})B(g \oplus 0))\end{aligned}$$

and

$$\begin{aligned}\omega_S\,(B^*(f_1 \oplus \bar{f}_2)B(g_1 \oplus \bar{g}_2)) &= ((g_1 \oplus \bar{g}_2), S(f_1 \oplus \bar{f}_2))_{\check{K}} \\ &= (g_1, Pf_1)_K + (\bar{g}_2, (1 - P)\bar{f}_2)_{\overline{K}} \\ &= (g_1, Pf_1)_K + (f_2, g_2)_K - (f_2, Pg_2)_K \\ &\qquad \text{as } P \text{ is self adjoint} \\ &= \omega_P\,(a^*(f_1)a(g_1) + (f_2, g_2)_K - a^*(g_2)a(f_2)) \\ &\qquad \text{by the definition of } \omega_P \\ &= \omega_P\,(a^*(f_1)a(g_1) + a(f_2)a^*(g_2)) \\ &\qquad \text{using the canonical anticommutation relations} \\ &= \omega_P\big((a^*(f_1) + a(f_2))\,(a(g_1) + a^*(g_2))\big) \\ &\qquad \text{using the properties of a quasi-free state.}\end{aligned}$$

So the two point correlations agree. Hence as the two states are quasi-free they agree on any correlation as quasi-free states are determined by their two point correlations.

8.2.3 Remark. If R is an operator on K with $0 \le R \le 1$, R self adjoint then if T is defined as $R \oplus (1-R)$ the previous Lemma is true for ω_R and ω_T in place of ω_P and ω_S.

8.2.4 Lemma. *Suppose U is a unitary operator on K. Then the Bogoliubov automorphism of $A(K)$ given by*

$$\tau(U)a(k) = a(Uk), \qquad k \in K$$

is equivalent to the Bogoliubov automorphism of $A_{SDC}(\bar{K}, \Gamma)$ given by

$$\tau(\mathcal{U})B(f) = B(\mathcal{U}f), \qquad f \in \bar{K}$$

where $\mathcal{U} = U \oplus U$.

Proof: $\Gamma\mathcal{U}\Gamma = \mathcal{U}$ so $\tau(\mathcal{U})$ does indeed define an automorphism of $A_{SDC}(\bar{K}, \Gamma)$. The equivalence is obvious.

8.2.5 Remark. P is a projection, hence $S = P \oplus (1-P)$ is a projection and since $\Gamma S \Gamma = 1 - S$ it is a basis projection. Thus using the results in Section 1 the state ω_S is a Q-Fock state where $Q_+ = 1 - S, (S = Q_-)$ with its corresponding representation on the alternating tensor algebra $\mathrm{A}(Q_+\bar{K})$. Hence the results of subsection 8.1 are applicable.

8.3 Application of Basic Structure.

In this particular case $K = H \oplus H$ where $H = L^2(S^1, \mathbb{C})$ and $P = P_-^{\beta,\mu}$ is the projection on K given by the 2×2 matrix

$$P_-^{\beta,\mu} = \begin{bmatrix} A_{\beta,\mu} & A_{\beta,\mu}^{1/2}(1-A_{\beta,\mu})^{1/2} \\ A_{\beta,\mu}^{1/2}(1-A_{\beta,\mu})^{1/2} & 1 - A_{\beta,\mu} \end{bmatrix},$$

where

$$A_{\beta,\mu} = e^{-\beta h_\mu}(1 + e^{-\beta h_\mu})^{-1},$$

with

$$h_\mu g(s) = (-id/ds - \mu)g(s), \qquad g \in H.$$

So to examine the state $\omega_{P_-^{\beta,\mu}}$ on $A(K)$ and its associated representation $\pi_{P_-^{\beta,\mu}}$ the state ω_S on $A_{SDC}(\bar{K}, \Gamma)$ and its associated Q-Fock representation will be studied. To do this the 'Q-representation of W' in this case needs to be determined.

8.3.1 Notation. Let Q denote the operator on $\bar{K}$ given by the 2×2 matrix form

$$\begin{bmatrix} 1 - 2P_-^{\beta,\mu} & 0 \\ 0 & 2P_-^{\beta,\mu} - 1 \end{bmatrix}.$$

Then define $Q_\pm = 1/2(1 \pm Q)$ and $Q_+ = 1 - S$ so $Q_- = S$.
Also let

$$Q' = \begin{bmatrix} 1 - 2P_-^{\infty} & 0 \\ 0 & 2P_-^{\infty} - 1 \end{bmatrix}$$

with corresponding definitions for $Q'_\pm$ and S' as above where P_-^{∞} is given in Lemma 7.1.2 .

8.3.2 Lemma. *Let q denote the operator on $\tilde{K}$ given by the 4×4 matrix*

$$\begin{bmatrix} 0 & a & P_- & 0 \\ 0 & a^* & P_+ & 0 \\ P_- & 0 & 0 & a \\ P_+ & 0 & 0 & a^* \end{bmatrix}$$

where a is a partial isometry such that $aa^* = P_+$, $a^*a = P_-$.
For example

$$a : g_n \mapsto \begin{cases} g_{-(n+1)} & \text{for } n < 0 \\ 0 & \text{for } n \geq 0 \end{cases} \qquad a^* : g_n \mapsto \begin{cases} g_{-(n+1)} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Then q is a unitary and

(1) $q^* Q'_+ q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
(2) $q^* \Gamma q = \Gamma$.
(3) $q^* \Lambda q = \Lambda'$ *where*

$$\Lambda = \begin{pmatrix} i & & & \\ & i & & \\ & & -i & \\ & & & -i \end{pmatrix}, \qquad \Lambda' = \begin{pmatrix} -i & & & \\ & i & & \\ & & i & \\ & & & -i \end{pmatrix}.$$

Proof: Direct computation using the following properties of a and a^*:

$$P_- a = aP_+ = 0 \text{ and } P_+ a = aP_- = a,$$

$$a^* P_- = P_+ a^* = 0 \text{ and } a^* P_+ = P_- a^* = a^*.$$

8.3.3 Corollary. *Let $W_{\beta,\mu}$ be the operator on K defined in Remark 7.1.5 and $\tilde{W}_{\beta,\mu}$ the operator on $\tilde{K}$ given by*

$$\begin{pmatrix} W_{\beta,\mu} & 0 \\ 0 & W_{\beta,\mu} \end{pmatrix}.$$

Then

(1) $q^* \tilde{W}^*_{\beta,\mu} Q_+ \tilde{W}_{\beta,\mu} q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
(2) $q^* \tilde{W}^*_{\beta,\mu} \Gamma \tilde{W}_{\beta,\mu} = \Gamma$.
(3) $q^* \tilde{W}^*_{\beta,\mu} \Lambda \tilde{W}_{\beta,\mu} = \Lambda'$.

Proof: From Remark 7.1.5

$$W_{\beta,\mu} P_-^{\infty} W^*_{\beta,\mu} = P_-^{\beta,\mu},$$

hence

$$\tilde{W}^*_{\beta,\mu} Q_+ \tilde{W}_{\beta,\mu} = Q'_+,$$

and (1) follows from Lemma 8.3.2.

Since $\tilde{W}_{\beta,\mu}$ is a unitary and commutes with both Γ and Λ (2) and (3) also follow from Lemma 8.3.2.

8.3.4 Remark. From Corollary 8.3.3 above it can be deduced that

$$q^* W_{\beta,\mu}^* Q W_{\beta,\mu} q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

That is, the unitary $q^* W_{\beta,\mu}^*$ implements the Q-representation of $\bar{K}$. Note also that the unitary q^* implements the Q'-representation of $\bar{K}$ since

$$q^* Q' q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

8.3.5 Lemma. *Suppose U is a unitary on K. Then the following hold:*

(1) *$\tau(U)$ is implementable in $\pi_{P_-^\infty} \Leftrightarrow \tau(U)$ is implementable in $\pi_{P_-^{\beta,\mu}}$.*
(2) *$\tau(U)$ is implementable in $\pi_{P_-^\infty} \Leftrightarrow \tau(\mathcal{U})$ is implementable in $\pi_{S'}$.*
(3) *$\tau(U)$ is implementable in $\pi_{P_-^{\beta,\mu}} \Leftrightarrow \tau(\mathcal{U})$ is implementable in π_S.*
(4) *$\tau(\mathcal{U})$ is implementable in $\pi_{S'} \Leftrightarrow \tau(\mathcal{U})$ is implementable in π_S.*

Proof: (1) :

$\tau(U)$ is implementable in $\pi_{P_-^{\beta,\mu}}$
$\Leftrightarrow U P_-^{\beta,\mu} - P_-^{\beta,\mu} U$ is Hilbert Schmidt
$\Leftrightarrow U(W_{\beta,\mu} P_-^\infty W_{\beta,\mu}^*) - (W_{\beta,\mu} P_-^\infty W_{\beta,\mu}^*) U$ is Hilbert Schmidt

Now $(1 - W_{\beta,\mu})$ is a trace class operator (See Remark 7.1.5), T say, so $W_{\beta,\mu} = 1 - T$ hence

$\Leftrightarrow U(1-T)P_-^\infty(1-T^*) - (1-T)P_-^\infty(1-T^*)U$ is Hilbert Schmidt
$\Leftrightarrow U P_-^\infty - P_-^\infty U - \{(UTP_-^\infty - TP_-^\infty U) + (UP_-^\infty T^* - T^* P_-^\infty U)$
$\quad - (UTP_-^\infty T^* - TP_-^\infty T^* U)\}$ is Hilbert Schmidt
$\Leftrightarrow U P_-^\infty - P_-^\infty U$ is Hilbert Schmidt
$\Leftrightarrow \tau(U)$ is implementable in $\pi_{P_-^\infty}$.

(2) :

$\tau(U)$ is implementable in $\pi_{P_-^\infty}$
$\Leftrightarrow U P_-^\infty - P_-^\infty U$ is Hilbert Schmidt
$\Leftrightarrow \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} P_-^\infty & 0 \\ 0 & 1 - P_-^\infty \end{pmatrix} - \begin{pmatrix} P_-^\infty & 0 \\ 0 & 1 - P_-^\infty \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$ is Hilbert Schmidt
$\Leftrightarrow \mathcal{U}S' - S'\mathcal{U}$ is Hilbert Schmidt
$\Leftrightarrow \tau(\mathcal{U})$ is implementable in $\pi_{S'}$.

(3) : Same as (2) with P_-^∞ and S' replaced by $P_-^{\beta,\mu}$ and S respectively.
(4) : Follows from (2)$\Leftrightarrow$(1)$\Leftrightarrow$(3).

8.3.6 Remark. From the previous section the multiplication operator $\tilde{\phi} =$

where $\phi \in \mathrm{Map}\,(S^1, U(1))$ induces an automorphism of $A(K)$ which is implemented in both the representations $\pi_{P_-^\infty}$ and $\pi_{P_-^{\beta,\mu}}$ by $\Gamma_\infty(\phi)$ and $\Gamma_{\beta,\mu}(\phi)$ respectively. Hence the multiplication operator $\Phi = \begin{pmatrix} \bar{\phi} & 0 \\ 0 & \bar{\phi} \end{pmatrix}$ induces an automorphism of $A_{SDC}(\bar{K}, \Gamma)$ which is implemented in both the representations $\pi_{S'}$ and π_S. Let $\Gamma_{S'}(\Phi)$ and $\Gamma_S(\Phi)$ denote the respective implementers.

Now relationships between $\Gamma_S(\Phi)$, $\Gamma_{S'}(\Phi)$, $\Gamma_{\beta,\mu}(\phi)$ and $\Gamma_{\infty}(\phi)$ are required in order that the results obtained for $\Gamma_S(\Phi)$ are applicable to $\Gamma_{\beta,\mu}(\phi)$. This problem is considered below.

8.3.7 LEMMA. *If U is a unitary on K with $\tau(U)$ implementable in $\pi_{P_-^{\beta,\mu}}$ then*

$$
\begin{aligned}
(\Gamma_S(U)) &\stackrel{\text{def}}{=} (\Omega_S, \Gamma_S(U)\Omega_S) = (\Omega_{\beta,\mu}, \Gamma_{\beta,\mu}(U)\Omega_{\beta,\mu}), \\
(\Gamma_{S'}(U)) &\stackrel{\text{def}}{=} (\Omega_{S'}, \Gamma_{S'}(U)\Omega_{S'}) = (\Omega_{\infty}, \Gamma_{\infty}(U)\Omega_{\infty}).
\end{aligned}
$$

PROOF: The GNS representation, π_S, of the state ω_S on $A_{SDC}(\bar{K}, \Gamma)$ can be identified with the representation π_1 given by

$$
\pi_1\left(B(\tilde{f})\right) = A_0\left((1-S)\tilde{f}\right) + A_0^*\left((1-S)\Gamma\tilde{f}\right),
$$

where A_0, A_0^* are the annihilation and creation operators on the Fock space of $\bar{K}$, $F(\bar{K}) = F(K \oplus \overline{K})$, respectively. But

$$
F(K \oplus \overline{K}) \cong F(K) \otimes F(\overline{K}),
$$

allowing the identification

$$
A_0(f \oplus \bar{g}) = a_0(f) \otimes \gamma + 1 \otimes a_0(\bar{g}),
$$

where a_0, a_0^* are the annihilation and creation operators on $F(K)$, and γ is the self adjoint unitary such that

$$
\gamma a_0(f) = -a_0(f)\gamma,
$$

that is γ is the implementer of the Bogoliubov automorphism $\tau(-1)$, and $\gamma\Omega = \Omega$ where Ω is the vacuum vector.

Therefore if $\tilde{f} = f \oplus \bar{g} \in \bar{K}$

$$
\begin{aligned}
\pi_1\left(B(\tilde{f})\right) &= A_0\left((1-P_-^{\beta,\mu})f \oplus P_-^{\beta,\mu}\bar{g}\right) + A_0^*\left((1-P_-^{\beta,\mu})g \oplus P_-^{\beta,\mu}\bar{f}\right) \\
&= a_0\left((1-P_-^{\beta,\mu})f\right) \otimes \gamma + 1 \otimes a_0(\overline{P_-^{\beta,\mu}g}) \\
&\quad + a_0^*\left((1-P_-^{\beta,\mu})g\right) \otimes \gamma + 1 \otimes a_0^*(\overline{P_-^{\beta,\mu}f}) \\
&= \pi_2(a(f)) + \pi_2(a^*(g))
\end{aligned}
$$

where

$$
\pi_2(a(f)) = a_0\left((1-P_-^{\beta,\mu})f\right) \otimes \gamma + 1 \otimes a_0^*\left(\overline{P_-^{\beta,\mu}f}\right)
$$

is a representation of $A(K)$ on $F(K) \otimes F(\overline{K})$.

Hence if $\Gamma_2(U)$ is the implementer of $\tau(U)$ in the representation π_2 of $A(K)$ then $\Gamma_2(U)$ is the implementer of $\tau(U)$ in the representation π_1 of $A_{SDC}(\bar{K}, \Gamma)$. That is $\Gamma_1(U) = \Gamma_2(U)$ and

$$
(\Omega_1, \Gamma_1(U)\Omega_1) = (\Omega_2, \Gamma_2(U)\Omega_2).
$$

But the GNS representation, $\pi_{P_-^{\beta,\mu}}$, of the state $\omega_{P_-^{\beta,\mu}}$ on $A(K)$ can be identified with the representation π_2. Hence by the weak continuity of the inner product

$$
\begin{aligned}
(\Omega_S, \Gamma_S(U)\Omega_S) &= (\Omega_1, \Gamma_1(U)\Omega_1) \\
&= (\Omega_2, \Gamma_2(U)\Omega_2) \\
&= (\Omega_{\beta,\mu}, \Gamma_{\beta,\mu}(U)\Omega_{\beta,\mu}).
\end{aligned}
$$

The equality for S' and P_-^{∞} holds by exactly the same argument replacing S with S' and $P_-^{\beta,\mu}$ with P_-^{∞}.

A trivial application of this Lemma gives the following.

8.3.8 Corollary. *With the notation of Remark 8.3.6*

$$
\begin{aligned}
\langle \Gamma_S(\Phi)\rangle &\stackrel{\text{def}}{=} (\Omega_S, \Gamma_S(\Phi)\Omega_S) = (\Omega_{\beta,\mu}, \Gamma_{\beta,\mu}(\phi)\Omega_{\beta,\mu}), \\
\langle \Gamma_{S'}(\Phi)\rangle &\stackrel{\text{def}}{=} (\Omega_{S'}, \Gamma_{S'}(\Phi)\Omega_{S'}) = (\Omega_{\infty}, \Gamma_{\infty}(\phi)\Omega_{\infty}).
\end{aligned}
$$

8.3.9 Lemma. *If U is a unitary on K with $\tau(U)$ implementable in $\pi_{P_-^{\infty}}$ then*

$$
\begin{aligned}
\Gamma_{\beta,\mu}(U) &= \Gamma_{\infty}(W_{\beta,\mu})^* \,\Gamma_{\infty}(U)\,\Gamma_{\infty}(W_{\beta,\mu}), \\
\Gamma_S(U) &= \Gamma_{S'}(W_{\beta,\mu})^* \,\Gamma_{S'}(U)\,\Gamma_{S'}(W_{\beta,\mu}).
\end{aligned}
$$

Proof: From the previous section (7.2)

$$\Gamma_{\beta,\mu}(U) = U_{\beta,\mu}\Gamma_{\infty}(U)U_{\beta,\mu}^*,$$

where $U_{\beta,\mu}$ is a unitary such that

$$U_{\beta,\mu}\pi_{P_-^{\infty}}(a(k))U_{\beta,\mu}^* = \pi_{P_-^{\beta,\mu}}(a(k)), \qquad k \in K.$$

Claim: $U_{\beta,\mu} = \Gamma_{\infty}(W_{\beta,\mu})^*$.

Now $W_{\beta,\mu}$ is a unitary operator with $1 - W_{\beta,\mu}$ a trace class operator and $\det W_{\beta,\mu} = 1$, see Remark 7.1.5. Therefore the Bogoliubov automorphism $\tau(W_{\beta,\mu})$ is inner, so certainly implementable, in the representation $\pi_{P_-^{\infty}}$. Denote this implementer by $\Gamma_{\infty}(W_{\beta,\mu})$ then

$$\Gamma_{\infty}(W_{\beta,\mu})\pi_{P_-^{\infty}}(a(k))\Gamma_{\infty}(W_{\beta,\mu})^* = \pi_{P_-^{\infty}}(a(W_{\beta,\mu}k)), \qquad k \in K.$$

Replacing k by $W_{\beta,\mu}^*k$ and rearranging gives

$$
\begin{aligned}
\Gamma_{\infty}(W_{\beta,\mu})^*\pi_{P_-^{\infty}}(a(k))\Gamma_{\infty}(W_{\beta,\mu}) &= \pi_{P_-^{\infty}}(a(W_{\beta,\mu}^*k)) \\
&= \pi_{W_{\beta,\mu}P_-^{\infty}W_{\beta,\mu}^*}(a(k)) \\
&= \pi_{P_-^{\beta,\mu}}(a(k)).
\end{aligned}
$$

Therefore $U_{\beta,\mu} = \Gamma_{\infty}(W_{\beta,\mu})^*$ as required and the first equality is shown.

The same proof gives the second equality. π_S and $\pi_{S'}$ are unitarily equivalent by the equivalence of $\pi_{P_-^{\beta,\mu}}$ and $\pi_{P_-^{\infty}}$ and the equivalence unitary can be shown to be $\Gamma_{S'}(W_{\beta,\mu})^*$ using the same method and

$$W_{\beta,\mu}S'W_{\beta,\mu}^* = S.$$

Applying this Lemma to the particular case of interest.

8.3.10 COROLLARY. *With the notation of Remark 8.3.6*

$$\Gamma_{\beta,\mu}(\phi) = \Gamma_\infty(W_{\beta,\mu})^* \, \Gamma_\infty(\phi) \, \Gamma_\infty(W_{\beta,\mu}),$$
$$\Gamma_{\mathcal{B}}(\Phi) = \Gamma_{\mathcal{B}'}(W_{\beta,\mu})^* \, \Gamma_{\mathcal{B}'}(\Phi) \, \Gamma_{\mathcal{B}'}(W_{\beta,\mu}).$$

8.4 Factorability of Elements.

From Corollary 8.3.10

$$\langle \Gamma_{\mathcal{B}}(\Phi) \rangle = \langle \Gamma_{\mathcal{B}'}(W_{\beta,\mu})^* \, \Gamma_{\mathcal{B}'}(\Phi) \, \Gamma_{\mathcal{B}'}(W_{\beta,\mu}) \rangle.$$

In order to use the results of Palmer outlined in subsection 8.1, in particular Lemma 8.1.9 and Theorem 8.1.10, on the RHS correlation $\Gamma_{\mathcal{B}'}(W_{\beta,\mu})$, $\Gamma_{\mathcal{B}'}(W_{\beta,\mu})^*$ and $\Gamma_{\mathcal{B}'}(\Phi)$ need to be factorable. This is now considered.

8.4.1 LEMMA. *$\Gamma_{\mathcal{B}'}(W_{\beta,\mu})$ and $\Gamma_{\mathcal{B}'}(W_{\beta,\mu})^*$ are both factorable.*

PROOF: From Definition 8.1.5 and Remark 8.1.6 $\Gamma_{\mathcal{B}'}(W_{\beta,\mu})$ is factorable if $T(W_{\beta,\mu})_{22}$ is invertible where $T(W_{\beta,\mu})$ is the Q'-representation of $W_{\beta,\mu}$.

But

$$\begin{aligned} T(W_{\beta,\mu}) &= q^* W_{\beta,\mu} q \\ &= \begin{bmatrix} A^* & P^* \\ P^* & A^* \end{bmatrix} \begin{bmatrix} W_{\beta,\mu} & 0 \\ 0 & W_{\beta,\mu} \end{bmatrix} \begin{bmatrix} A & P \\ P & A \end{bmatrix} \\ &= \begin{bmatrix} X & Y \\ Y & X \end{bmatrix} \end{aligned}$$

where $P = \begin{bmatrix} P_- & 0 \\ P_+ & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & a \\ 0 & a^* \end{bmatrix}$ and

$$\begin{aligned} X &= P^* W_{\beta,\mu} P + A^* W_{\beta,\mu} A, \\ Y &= P^* W_{\beta,\mu} A + A^* W_{\beta,\mu} P. \end{aligned}$$

So

$$\begin{aligned} &T(W_{\beta,\mu})_{22} = X \\ &= P^* W_{\beta,\mu} P + A^* W_{\beta,\mu} A \\ &= \begin{bmatrix} P_- & P_+ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{\beta,\mu}^{1/2} P_- + (1 - A_{\beta,\mu})^{1/2} P_+ & A_{\beta,\mu}^{1/2} P_+ - (1 - A_{\beta,\mu})^{1/2} P_- \\ -A_{\beta,\mu}^{1/2} P_+ + (1 - A_{\beta,\mu})^{1/2} P_- & A_{\beta,\mu}^{1/2} P_- + (1 - A_{\beta,\mu})^{1/2} P_+ \end{bmatrix} \begin{bmatrix} P_- & 0 \\ P_+ & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 \\ a^* & a \end{bmatrix} \begin{bmatrix} A_{\beta,\mu}^{1/2} P_- + (1 - A_{\beta,\mu})^{1/2} P_+ & A_{\beta,\mu}^{1/2} P_+ - (1 - A_{\beta,\mu})^{1/2} P_- \\ -A_{\beta,\mu}^{1/2} P_+ + (1 - A_{\beta,\mu})^{1/2} P_- & A_{\beta,\mu}^{1/2} P_- + (1 - A_{\beta,\mu})^{1/2} P_+ \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & a^* \end{bmatrix} \\ &= \begin{bmatrix} A_{\beta,\mu}^{1/2} P_- + (1 - A_{\beta,\mu})^{1/2} P_+ & 0 \\ 0 & a^* (1 - A_{\beta,\mu})^{1/2} a + a A_{\beta,\mu}^{1/2} a^* \end{bmatrix} \end{aligned}$$

which is clearly invertible with

$$T(W_{\beta,\mu})_{22}^{-1} = \begin{bmatrix} A_{\beta,\mu}^{-1/2} P_- + (1 - A_{\beta,\mu})^{-1/2} P_+ & 0 \\ 0 & a^* (1 - A_{\beta,\mu})^{-1/2} a + a A_{\beta,\mu}^{-1/2} a^* \end{bmatrix}.$$

Hence $\Gamma_{\mathcal{B}'}(W_{\beta,\mu})$ is factorable.

Now $\Gamma_{\mathcal{B}'}(W_{\beta,\mu})^* = \Gamma_{\mathcal{B}'}\left(W_{\beta,\mu}^*\right)$ and

$$T(W_{\beta,\mu}^*) = q^* W_{\beta,\mu}^* q = T(W_{\beta,\mu})^* = \begin{bmatrix} X^* & Y^* \\ Y^* & X^* \end{bmatrix}.$$

So $T(W_{\beta,\mu}^*)_{22} = X^* = X$ by examination, which is invertible by the above. Hence $\Gamma_{\mathcal{B}'}(W_{\beta,\mu})^*$ is factorable.

Szego's Theorem will be of use when considering $\Gamma_{\mathcal{B}'}(\Phi)$ hence the relevant version will be stated here for this and future reference.

8.4.2 Szego's Theorem. *Suppose D_N is the $N \times N$ Toeplitz determinant*

$$\begin{vmatrix} c_0 & c_{-1} & \cdots & c_{-N+1} \\ c_1 & c_0 & \cdots & c_{-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N-1} & c_{N-2} & \cdots & c_0 \end{vmatrix}$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} C(e^{i\theta})\, d\theta.$$

Under suitable conditions (†)

$$\lim_{N\to\infty} \frac{D_N}{\rho^N} = \exp\left(\sum_{n=1}^{\infty} n g_{-n} g_n\right)$$

where

$$\rho = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \ln C(e^{i\theta})\, d\theta\right]$$

and

$$g_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} \ln C(e^{i\theta})\, d\theta.$$

The conditions (†) used in this particular case are the following:

(1) $\sum_{n=-\infty}^{\infty} |c_n| < \infty$.
(2) $\sum_{n=-\infty}^{\infty} n|c_n|^2 < \infty$.
(3) $C(\xi) \neq 0$ *for* $|\xi| = 1$.
(4) $\operatorname{Ind} C(\xi) = 0$.

For a proof of this version of Szego's Theorem see [H1].

8.4.3 Lemma. *If $\phi \in \mathrm{Map}\,(S^1, U(1))$ has a winding number of zero then $\Gamma_{g'}(\Phi)$ is factorable.*

Proof: As in Lemma 8.4.1 $\Gamma_{g'}(\Phi)$ is factorable if $T(\Phi)_{22}$ is invertible. Now

$$T(\Phi) = q^*\Phi q = \begin{bmatrix} C & D \\ D & C \end{bmatrix}$$

with

$$D = P^*\tilde{\phi}A + A^*\tilde{\phi}P,$$
$$C = P^*\tilde{\phi}P + A^*\tilde{\phi}A,$$

where P and A are defined as in the proof of Lemma 8.4.1 and $\tilde{\phi}$ is given in Remark 8.3.6 . Therefore

$$\begin{aligned} T(\Phi)_{22} &= P^*\tilde{\phi}P + A^*\tilde{\phi}A \\ &= \begin{bmatrix} P_- & P_+ \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} P_- & 0 \\ P_+ & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a^* & a \end{bmatrix}\begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & a \\ 0 & a^* \end{bmatrix} \\ &= \begin{bmatrix} P_-\phi P_- + P_+ & 0 \\ 0 & a^*\phi a + P_+ \end{bmatrix}. \end{aligned}$$

So $T(\Phi)_{22}$ is invertible provided there exist operators X, Y acting on P_-H such that

$$X = (P_-\phi P_-)^{-1},$$
$$Y = (a_-^*\phi a)^{-1}.$$

That is

$$P_- = P_-\phi P_- X P_- = P_- X P_- \phi P_-,$$
$$P_- = a^*\phi a a^* Y a \quad = a^* Y a a^* \phi a,$$

in which case

$$T(\Phi)_{22}^{-1} = \begin{bmatrix} P_- X P_- + P_+ & 0 \\ 0 & a^* Y a + P_+ \end{bmatrix}.$$

Now ϕ is a multiplication operator so $P_-\phi P_-$ has matrix form

$$\begin{bmatrix} \ddots & \ddots & \ddots & \vdots \\ \ddots & \phi_0 & \phi_{-1} & \phi_{-2} \\ \ddots & \phi_1 & \phi_0 & \phi_{-1} \\ \cdots & \phi_2 & \phi_1 & \phi_0 \end{bmatrix}$$

with respect to the basis

$$[e^{-i\theta} \quad e^{-2i\theta} \quad e^{-3i\theta} \quad \ldots]$$

of P_-H. But any cutoff of this gives a Toeplitz matrix so Szego's Theorem enables the determinant to be deduced.

If ϕ is written as

$$\exp i\left\{n\theta + f_0 + \sum_{r\neq 0} f_r e^{ir\theta}\right\},$$

then the winding number of ϕ is n. So by assumption $n = 0$. Also the term $\exp if_0$ is a constant and can be factored out as follows:
rewrite ϕ as $e^{if_0}\phi'$ then it is easy to see that

$$\phi \text{ is invertible on } P_-H \Leftrightarrow \phi' \text{ is invertible on } P_-H.$$

Therefore the situation to consider is when

$$\phi = \exp i \sum_{r\neq 0} f_r e^{ir\theta}.$$

This form for ϕ enables the fourier coefficients for $\ln\phi$ to be seen trivially, and in particular $\rho = 1$ in this case. The fact that ϕ satisfies the conditions (†) required for Szego's Theorem follows from the fact that $\phi \in \mathrm{Map}\left(S^1, U(1)\right)$ and the winding number of ϕ is zero by assumption. See Remark 7.2.1 for example for (2).

Applying Szego's Theorem shows that the determinant of $P_-\phi P_-$ is positive, as $g_{-n}g_n = -||f_n||^2$ in this case, hence in particular it is invertible in P_-H having non-zero determinant.

Similarly $a^*\phi a$ has matrix form

$$\begin{bmatrix} \ddots & \ddots & \ddots & \vdots \\ \ddots & \phi_0 & \phi_1 & \phi_2 \\ \ddots & \phi_{-1} & \phi_0 & \phi_1 \\ \cdots & \phi_{-2} & \phi_{-1} & \phi_0 \end{bmatrix}$$

with respect to the basis

$$[e^{-i\theta} \quad e^{-2i\theta} \quad e^{-3i\theta} \quad \ldots]$$

of P_-H. Consequently it is also invertible in P_-H as it is just the transpose of $P_-\phi P_-$ given above.

8.4.4 Remark. If $n > 0$ then $P_- e^{in\theta} P_- + P_+$ is not invertible as it has a non-trivial kernel. Similarly if $n < 0$ then $a^* e^{in\theta} a + P_+$ is not invertible as it has a non-trivial kernel. Both contain the function $e^{-i\theta}$.

8.4.5 Remark. It is easy to see that

$$a^* Y a = (P_- X P_-)^T = P_- X^T P_- = X^T.$$

Thus

$$Y = P_+ Y P_+ = a(a^* Y a) a^* = a X^T a^*$$

and

$$T(\Phi)_{22}^{-1} = \begin{bmatrix} P_- X P_- + P_+ & 0 \\ 0 & P_- X^T P_- + P_+ \end{bmatrix}.$$

This connection between X and Y will be of use later.

8.5 Calculation of One Point Correlations.

8.5.1 Remark. Using the results of subsection 8.1 together with the information in subsections 8.3 and 8.4 the following can be deduced.

$$\begin{aligned}
&\langle \Omega_{\beta,\mu}, \Gamma_{\beta,\mu}(\phi)\, \Omega_{\beta,\mu} \rangle^2 \\
&= \langle \Gamma_S(\Phi) \rangle^2 \\
&= \langle \Gamma_{S'}(\mathcal{W}_{\beta,\mu})^* \, \Gamma_{S'}(\Phi)\, \Gamma_{S'}(\mathcal{W}_{\beta,\mu}) \rangle^2 \\
&= \langle \Gamma_{S'}(\mathcal{W}_{\beta,\mu})^* \rangle^2 \langle \Gamma_{S'}(\Phi) \rangle^2 \langle \Gamma_{S'}(\mathcal{W}_{\beta,\mu}) \rangle^2 \mathrm{det}_2(1 + L\Delta R) \\
&= \langle \Omega_\infty, \Gamma_\infty(\phi)\, \Omega_\infty \rangle^2 \langle \Gamma_{S'}(\mathcal{W}_{\beta,\mu})^* \rangle^2 \langle \Gamma_{S'}(\mathcal{W}_{\beta,\mu}) \rangle^2 \mathrm{det}_2(1 + L\Delta R) \\
&= \langle \Omega_\infty, \Gamma_\infty(\phi)\, \Omega_\infty \rangle^2 \overline{\langle \Gamma_{S'}(\mathcal{W}_{\beta,\mu}) \rangle^2} \langle \Gamma_{S'}(\mathcal{W}_{\beta,\mu}) \rangle^2 \mathrm{det}_2(1 + L\Delta R) \\
&= \langle \Omega_\infty, \Gamma_\infty(\phi)\, \Omega_\infty \rangle^2 |\langle \Gamma_{S'}(\mathcal{W}_{\beta,\mu}) \rangle|^4 \mathrm{det}_2(1 + L\Delta R).
\end{aligned}$$

This section will be concerned with the one point correlations in the above equation, that is all but the det_2 term.

8.5.2 Proposition.

$$|\langle \Gamma_{S'}(\mathcal{W}_{\beta,\mu}) \rangle|^2 = \prod_{n \geq 0} \left(1 + e^{-\beta(n-\mu)}\right)^{-1} \left(1 + e^{-\beta(n+1+\mu)}\right)^{-1}.$$

Proof: By Lemma 8.1.9

$$|\langle \Gamma_{S'}(\mathcal{W}_{\beta,\mu}) \rangle|^2 = \|\Gamma_{S'}(\mathcal{W}_{\beta,\mu})\, \Omega_{S'}\|^2 \det\left(I + |T(\mathcal{W}_{\beta,\mu})_{12} T(\mathcal{W}_{\beta,\mu})_{22}^{-1}|^2\right)^{-1/2}.$$

But $\|\Gamma_{S'}(\mathcal{W}_{\beta,\mu})\, \Omega_{S'}\|^2 = 1$ as $\Gamma_{S'}(\mathcal{W}_{\beta,\mu})$ is a unitary implementer, hence

$$|\langle \Gamma_{S'}(\mathcal{W}_{\beta,\mu}) \rangle|^2 = \det\left(I + |T(\mathcal{W}_{\beta,\mu})_{12} T(\mathcal{W}_{\beta,\mu})_{22}^{-1}|^2\right)^{-1/2}.$$

Now from Lemma 8.4.1

$$\begin{aligned}
&T(\mathcal{W}_{\beta,\mu})_{12} \\
&= P^* \mathcal{W}_{\beta,\mu} A + A^* \mathcal{W}_{\beta,\mu} P \\
&= \begin{bmatrix} P_- & P_+ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{\beta,\mu}^{1/2} P_- + (1 - A_{\beta,\mu})^{1/2} P_+ & A_{\beta,\mu}^{1/2} P_+ - (1 - A_{\beta,\mu})^{1/2} P_- \\ -A_{\beta,\mu}^{1/2} P_+ + (1 - A_{\beta,\mu})^{1/2} P_- & A_{\beta,\mu}^{1/2} P_- + (1 - A_{\beta,\mu})^{1/2} P_+ \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & a^* \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & 0 \\ a^* & a \end{bmatrix} \begin{bmatrix} A_{\beta,\mu}^{1/2} P_- + (1 - A_{\beta,\mu})^{1/2} P_+ & A_{\beta,\mu}^{1/2} P_+ - (1 - A_{\beta,\mu})^{1/2} P_- \\ -A_{\beta,\mu}^{1/2} P_+ + (1 - A_{\beta,\mu})^{1/2} P_- & A_{\beta,\mu}^{1/2} P_- + (1 - A_{\beta,\mu})^{1/2} P_+ \end{bmatrix} \begin{bmatrix} P_- & 0 \\ P_+ & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -A_{\beta,\mu}^{1/2} a - (1 - A_{\beta,\mu})^{1/2} a^* \\ a^* A_{\beta,\mu}^{1/2} + a(1 - A_{\beta,\mu})^{1/2} & 0 \end{bmatrix}
\end{aligned}$$

and

$$T(W_{\beta,\mu})_{22}^{-1} = \begin{bmatrix} A_{\beta,\mu}^{-1/2}P_- + (1-A_{\beta,\mu})^{-1/2}P_+ & 0 \\ 0 & a^*(1-A_{\beta,\mu})^{-1/2}a + aA_{\beta,\mu}^{-1/2}a^* \end{bmatrix}.$$

Hence

$$T(W_{\beta,\mu})_{12}T(W_{\beta,\mu})_{22}^{-1} = \begin{bmatrix} 0 & F_1 \\ F_2 & 0 \end{bmatrix}$$

where

$$\begin{aligned} F_1 &= -A_{\beta,\mu}^{1/2}(1-A_{\beta,\mu})^{-1/2}a - (1-A_{\beta,\mu})^{1/2}A_{\beta,\mu}^{-1/2}a^*, \\ F_2 &= a(1-A_{\beta,\mu})^{1/2}A_{\beta,\mu}^{-1/2} + a^*A_{\beta,\mu}^{1/2}(1-A_{\beta,\mu})^{-1/2}. \end{aligned}$$

So

$$\begin{aligned} \left(T(W_{\beta,\mu})_{12}T(W_{\beta,\mu})_{22}^{-1}\right)^* &= \begin{bmatrix} 0 & F_2^* \\ F_1^* & 0 \end{bmatrix} \\ &= -\left(T(W_{\beta,\mu})_{12}T(W_{\beta,\mu})_{22}^{-1}\right) \end{aligned}$$

and

$$\begin{aligned} &|T(W_{\beta,\mu})_{12}T(W_{\beta,\mu})_{22}^{-1}|^2 \\ &= \left(T(W_{\beta,\mu})_{12}T(W_{\beta,\mu})_{22}^{-1}\right)^*\left(T(W_{\beta,\mu})_{12}T(W_{\beta,\mu})_{22}^{-1}\right) \\ &= \begin{bmatrix} F_3 & 0 \\ 0 & F_4 \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} F_3 &= A_{\beta,\mu}^{-1}(1-A_{\beta,\mu})P_- + A_{\beta,\mu}(1-A_{\beta,\mu})^{-1}P_+, \\ F_4 &= a^*A_{\beta,\mu}(1-A_{\beta,\mu})^{-1}a + aA_{\beta,\mu}^{-1}(1-A_{\beta,\mu})a^*. \end{aligned}$$

Therefore

$$\begin{aligned} &I + |T(W_{\beta,\mu})_{12}T(W_{\beta,\mu})_{22}^{-1}|^2 \\ &= 1 + \begin{bmatrix} F_3 & 0 \\ 0 & F_4 \end{bmatrix} \\ &= \begin{bmatrix} A_{\beta,\mu}^{-1}P_- + (1-A_{\beta,\mu})^{-1}P_+ & 0 \\ 0 & a^*(1-A_{\beta,\mu})^{-1}a + aA_{\beta,\mu}^{-1}a^* \end{bmatrix} \\ &= \begin{bmatrix} A_{\beta,\mu}P_- + (1-A_{\beta,\mu})P_+ & 0 \\ 0 & a^*(1-A_{\beta,\mu})a + aA_{\beta,\mu}a^* \end{bmatrix}^{-1}. \end{aligned}$$

So

$$\begin{aligned} &\det\left(I + |T(W_{\beta,\mu})_{12}T(W_{\beta,\mu})_{22}^{-1}|^2\right) \\ &= \det\left(\begin{bmatrix} A_{\beta,\mu}P_- + (1-A_{\beta,\mu})P_+ & 0 \\ 0 & a^*(1-A_{\beta,\mu})a + aA_{\beta,\mu}a^* \end{bmatrix}^{-1}\right) \\ &= \left[\det\left(A_{\beta,\mu}P_- + (1-A_{\beta,\mu})P_+\right)\left(a^*(1-A_{\beta,\mu})a + aA_{\beta,\mu}a^*\right)\right]^{-1} \\ &= \left[\det\left(A_{\beta,\mu}a^*(1-A_{\beta,\mu})a + (1-A_{\beta,\mu})aA_{\beta,\mu}a^*\right)\right]^{-1}. \end{aligned}$$

Hence

$$|(\Gamma_{\beta'}(W_{\beta,\mu}))|^2 = \left[\det(A_{\beta,\mu}a^*(1 - A_{\beta,\mu})a + (1 - A_{\beta,\mu})aA_{\beta,\mu}a^*)\right]^{1/2}.$$

Now at the fourier series level the operators $A_{\beta,\mu}$, a and a^* act as follows :

$$A_{\beta,\mu}g_n = e^{-\beta(n-\mu)}\left(1 + e^{-\beta(n-\mu)}\right)^{-1} g_n,$$

$$ag_n = \begin{cases} 0 & n < 0 \\ g_{-n-1} & n \geq 0 \end{cases}, \quad a^*g_n = \begin{cases} 0 & n \geq 0 \\ g_{-n-1} & n < 0 \end{cases}.$$

So it is not difficult to show that

$$A_{\beta,\mu}a^*(1 - A_{\beta,\mu})ag_n = \begin{cases} 0 & n \geq 0 \\ (1 + e^{\beta(n-\mu)})^{-1}(1 + e^{\beta(n+1+\mu)})^{-1} g_n & n < 0 \end{cases}$$

and

$$(1 - A_{\beta,\mu})aA_{\beta,\mu}a^*g_n = \begin{cases} 0 & n < 0 \\ (1 + e^{-\beta(n-\mu)})^{-1}(1 + e^{-\beta(n+1+\mu)})^{-1} g_n & n \geq 0 \end{cases}$$

giving

$$\begin{aligned} &(A_{\beta,\mu}a^*(1 - A_{\beta,\mu})a + (1 - A_{\beta,\mu})aA_{\beta,\mu}a^*)g_n \\ &= \begin{cases} (1 + e^{\beta(n-\mu)})^{-1}(1 + e^{\beta(n+1+\mu)})^{-1} g_n & n < 0 \\ (1 + e^{-\beta(n-\mu)})^{-1}(1 + e^{-\beta(n+1+\mu)})^{-1} g_n & n \geq 0. \end{cases} \end{aligned}$$

Therefore

$$\begin{aligned} &\det(A_{\beta,\mu}a^*(1 - A_{\beta,\mu})a + (1 - A_{\beta,\mu})aA_{\beta,\mu}a^*) \\ &= \prod_{n<0}\left(1 + e^{\beta(n-\mu)}\right)^{-1}\left(1 + e^{\beta(n+1+\mu)}\right)^{-1} \prod_{n\geq 0}\left(1 + e^{-\beta(n-\mu)}\right)^{-1}\left(1 + e^{-\beta(n+1+\mu)}\right)^{-1} \\ &= \left[\prod_{n\geq 0}\left(1 + e^{-\beta(n-\mu)}\right)^{-1}\left(1 + e^{-\beta(n+1+\mu)}\right)^{-1}\right]^2. \end{aligned}$$

So

$$\begin{aligned} &\left[\det(A_{\beta,\mu}a^*(1 - A_{\beta,\mu})a + (1 - A_{\beta,\mu})aA_{\beta,\mu}a^*)\right]^{1/2} \\ &= \prod_{n\geq 0}\left(1 + e^{-\beta(n-\mu)}\right)^{-1}\left(1 + e^{-\beta(n+1+\mu)}\right)^{-1}. \end{aligned}$$

8.5.3 Remark. Suppose $\phi = \exp i\sum_{k\in\mathbb{Z}} f_k e^{ik\theta}$ then from the previous Section (7.3.3)

$$(\Omega_{\beta,\mu}, \Gamma_{\beta,\mu}(\phi)\,\Omega_{\beta,\mu}) = \theta_3(f_0)\theta_3(0)^{-1}\exp\left\{-\frac{1}{4\pi}\sum_{k\neq 0} k(1 - e^{-\beta k})^{-1}|f_k|^2\right\},$$

and from [C7]

$$(\Omega_\infty, \Gamma_\infty(\phi)\,\Omega_\infty) = \exp\left\{-\frac{1}{4\pi}\sum_{k>0} k|f_k|^2\right\}.$$

8.5.4 Remark. Using the formulæ given in the previous remark it is not difficult to show that

$$\langle \Omega_{\beta,\mu}, \Gamma_{\beta,\mu}(\phi)\,\Omega_{\beta,\mu}\rangle = \langle \Omega_\infty, \Gamma_\infty(\phi)\,\Omega_\infty\rangle$$
$$.\theta_3(f_0)\theta_3(0)^{-1}\exp\left\{-\frac{1}{2\pi}\sum_{k=1}^{\infty} k(e^{k\beta}-1)^{-1}|f_k|^2\right\}.$$

8.6 The $\det_2(1+L\Delta R)$ term.

From Theorem 8.1.10 $(1+L\Delta R)$ is a 3×3 block matrix in this case with

$$L = \begin{bmatrix} 0 & -Q_+ & -Q_+L(2)_\bullet \\ \overline{-(-Q_+)^*} & 0 & -Q_+ \\ \overline{-(-Q_+L(2)_\bullet)^*} & \overline{-(-Q_+)^*} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -Q_+ & -Q_+L(2)_\bullet \\ \overline{Q_-} & 0 & -Q_+ \\ \overline{L(2)^*_\bullet}Q_- & Q_- & 0 \end{bmatrix} \qquad \text{where } \overline{C} = \Gamma C \Gamma,$$

and

$$\Delta R = \begin{bmatrix} \Delta R_{W^*_{\beta,\mu}} & & \\ & \Delta R_\bullet & \\ & & \Delta R_{W_{\beta,\mu}} \end{bmatrix},$$

so that

$$1 + L\Delta R = \begin{bmatrix} 1 & -Q_+\Delta R_\bullet & -Q_+L(2)_\bullet\Delta R_{W_{\beta,\mu}} \\ \overline{Q_-\Delta R_{W^*_{\beta,\mu}}} & 1 & -Q_+\Delta R_{W_{\beta,\mu}} \\ \overline{L(2)^*_\bullet}Q_-\Delta R_{W^*_{\beta,\mu}} & Q_-\Delta R_\bullet & 1 \end{bmatrix}.$$

The individual entries of this will now be calculated leading to a simplification of this term.

The ΔR terms.

From Definition 8.1.8 and Theorem 8.1.10, in the Q'-representation of $\tilde{K}$

$$\Delta R_G = \begin{bmatrix} 0 & T(G)_{12}T(G)_{22}^{-1} \\ T(G)_{22}^{-1}T(G)_{21} & 0 \end{bmatrix}.$$

Hence the ΔR terms can be calculated as follows:

1) $\Delta R_{W_{\beta,\mu}}$:

From Proposition 8.5.2

$$T(\mathcal{W}_{\beta,\mu})_{12}T(\mathcal{W}_{\beta,\mu})_{22}^{-1} = \begin{bmatrix} 0 & F_1 \\ F_2 & 0 \end{bmatrix}$$

where

$$F_1 = -A_{\beta,\mu}^{1/2}(1-A_{\beta,\mu})^{-1/2}a - (1-A_{\beta,\mu})^{1/2}A_{\beta,\mu}^{-1/2}a^*,$$
$$F_2 = a(1-A_{\beta,\mu})^{1/2}A_{\beta,\mu}^{-1/2} + a^*A_{\beta,\mu}^{1/2}(1-A_{\beta,\mu})^{-1/2}.$$

From Lemma 8.4.1 and Proposition 8.5.2

$$\begin{aligned}
&T(W_{\beta,\mu})_{22}^{-1}T(W_{\beta,\mu})_{21}\\
&= (P^*W_{\beta,\mu}P + A^*W_{\beta,\mu}A)^{-1}(P^*W_{\beta,\mu}A + A^*W_{\beta,\mu}P)\\
&= \begin{bmatrix} A_{\beta,\mu}^{-1/2}P_- + (1-A_{\beta,\mu})^{-1/2}P_+ & 0 \\ 0 & a^*(1-A_{\beta,\mu})^{-1/2}a + aA_{\beta,\mu}^{-1/2}a^* \end{bmatrix}\\
&\quad\cdot\begin{bmatrix} 0 & -A_{\beta,\mu}^{1/2}a - (1-A_{\beta,\mu})^{1/2}a^* \\ a^*A_{\beta,\mu}^{1/2} + a(1-A_{\beta,\mu})^{1/2} & 0 \end{bmatrix}\\
&= \begin{bmatrix} 0 & F_1 \\ F_2 & 0 \end{bmatrix}\\
&= T(W_{\beta,\mu})_{12}T(W_{\beta,\mu})_{22}^{-1}.
\end{aligned}$$

2) $\Delta R_{W^*_{\beta,\mu}}$:
From Lemma 8.4.1

$$T(W^*_{\beta,\mu}) = \begin{bmatrix} X^* & Y^* \\ Y^* & X^* \end{bmatrix}.$$

By examination $X^* = X$ and $Y^* = -Y$ hence

$$\begin{aligned}
T(W^*_{\beta,\mu})_{12}T(W^*_{\beta,\mu})_{22}^{-1} &= -YX^{-1}\\
&= -T(W_{\beta,\mu})_{12}T(W_{\beta,\mu})_{22}^{-1}
\end{aligned}$$

and it is a simple consequence that

$$\Delta R_{W^*_{\beta,\mu}} = -\Delta R_{W_{\beta,\mu}}.$$

8.6.1 Remark. The operators $\Delta R_{W_{\beta,\mu}}$ and $\Delta R_{W^*_{\beta,\mu}}$ are trace class. This follows from Remark 7.1.3 .

3) ΔR_Φ:
From Lemma 8.4.3

$$\begin{aligned}
T(\Phi)_{12} = T(\Phi)_{21} &= P^*\bar{\phi}A + a^*\bar{\phi}P\\
&= \begin{bmatrix} 0 & P_-\phi a \\ a^*\phi P_- & 0 \end{bmatrix}
\end{aligned}$$

and from Remark 8.4.5

$$T(\Phi)_{22}^{-1} = \begin{bmatrix} P_-XP_- + P_+ & 0 \\ 0 & P_-X^TP_- + P_+ \end{bmatrix}.$$

So

$$T(\Phi)_{12}T(\Phi)_{22}^{-1} = \begin{bmatrix} 0 & P_-\phi aX^TP_- \\ a^*\phi P_-XP_- & 0 \end{bmatrix},$$

and

$$T(\Phi)_{22}^{-1}T(\Phi)_{21} = \begin{bmatrix} 0 & P_-XP_-\phi a \\ P_-X^Ta^*\phi P_- & 0 \end{bmatrix}.$$

8.6.2 Remark. $P_-\phi a$ and $a^*\phi P_-$ have matrix forms

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & \phi_{-5} & \phi_{-4} & \phi_{-3} \\ \cdots & \phi_{-4} & \phi_{-3} & \phi_{-2} \\ \cdots & \phi_{-3} & \phi_{-2} & \phi_{-1} \end{bmatrix} \qquad \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & \phi_5 & \phi_4 & \phi_3 \\ \cdots & \phi_4 & \phi_3 & \phi_2 \\ \cdots & \phi_3 & \phi_2 & \phi_1 \end{bmatrix}$$

respectively. Hence both are trace class operators showing that ΔR_Φ is also a trace class operator.

The $L(2)$ terms.

Similarly in the Q'-representation of K

$$L(2)_G = \begin{bmatrix} T(G)^{*-1}_{22} & 0 \\ 0 & T(G)_{22} \end{bmatrix}$$

thus the $L(2)$ terms are as follows:

1) $L(2)_\Phi$:

$T(\Phi)_{22}$ is already known and it can be easily seen that

$$\begin{aligned} T(\Phi)^{*-1}_{22} &= \begin{bmatrix} P_- X^* P_- + P_+ & 0 \\ 0 & P_-(X^T)^* P_- + P_+ \end{bmatrix} \\ &= \begin{bmatrix} P_- \bar{X}^T P_- + P_+ & 0 \\ 0 & P_- \bar{X} P_- + P_+ \end{bmatrix} \end{aligned}$$

where $\bar{X}$ denotes the operator formed by complex conjugating the matrix elements of X.

2) $\overline{L(2)^*_\Phi}$:

$$\overline{L(2)^*_\Phi} = \Gamma L(2)^*_\Phi \Gamma = \begin{bmatrix} T(\Phi)^*_{22} & 0 \\ 0 & T(\Phi)^{-1}_{22} \end{bmatrix}$$

and these terms are already known.

The Individual Entries.

By the calculations above these are:

(1)

$$Q_- \Delta R_{W^*_{\beta,\mu}} = -\begin{bmatrix} 0 & 0 \\ M_1 & 0 \end{bmatrix}.$$

(2)

$$\overline{L(2)^*_\Phi} Q_- \Delta R_{W^*_{\beta,\mu}} = -\begin{bmatrix} 0 & 0 \\ L_1 M_1 & 0 \end{bmatrix}.$$

(3)

$$-Q_+ \Delta R_\Phi = -\begin{bmatrix} 0 & R_1 \\ 0 & 0 \end{bmatrix}.$$

(4)

$$Q_- \Delta R_\Phi = \begin{bmatrix} 0 & 0 \\ R_2 & 0 \end{bmatrix}.$$

(5)

$$-Q_+ L(2)_\Phi \Delta R_{W_{\beta,\mu}} = -\begin{bmatrix} 0 & L_2 M_1 \\ 0 & 0 \end{bmatrix}.$$

(6)

$$-Q_+ \Delta R_{W_{\beta,\mu}} = -\begin{bmatrix} 0 & M_1 \\ 0 & 0 \end{bmatrix}.$$

where

$$\begin{aligned} M_1 &= T(W_{\beta,\mu})_{12} T(W_{\beta,\mu})^{-1}_{22}. \\ L_1 &= T(\Phi)^{-1}_{22}. \\ L_2 &= T(\Phi)^{*-1}_{22}. \\ R_1 &= T(\Phi)_{12} T(\Phi)^{-1}_{22}. \\ R_2 &= T(\Phi)^{-1}_{22} T(\Phi)_{21}. \end{aligned}$$

thus $(1 + L\Delta R)$ has the form

$$\begin{bmatrix} 1 & 0 & 0 & -R_1 & 0 & -L_2M_1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -M_1 \\ -M_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -L_1M_1 & 0 & R_2 & 0 & 0 & 1 \end{bmatrix}.$$

Simplification of $\det_2$ term.

Given the form of $L\Delta R$ above and the Remarks 8.6.1 and 8.6.2 it is possible to deduce that $L\Delta R$ is trace class. Hence

$$\begin{aligned}
\det_2(1 + L\Delta R) &= \det(1 + L\Delta R), \text{ as } \mathrm{Trace}(L\Delta R) = 0 \\
&= \det\left\{\begin{bmatrix} 1 & 0 & 0 & -R_1 & 0 & -L_2M_1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -M_1 \\ -M_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -L_1M_1 & 0 & R_2 & 0 & 0 & 1 \end{bmatrix}\right\} \\
&= \det\left\{\begin{bmatrix} 1 & 0 & -R_1 & -L_2M_1 \\ 0 & 1 & 0 & -M_1 \\ -M_1 & 0 & 1 & 0 \\ -L_1M_1 & R_2 & 0 & 1 \end{bmatrix}\right\} \\
&= \det\left\{1 - \begin{bmatrix} M_1R_1 & M_1L_2M_1 \\ M_1R_1 & L_1M_1L_2M_1 - R_2M_1 \end{bmatrix}\right\}, \text{ as } L_1M_1R_1 = M_1R_1 \\
&= \det\{(1 - M_1R_1)(1 - L_1M_1L_2M_1 + R_2M_1) - M_1R_1M_1L_2M_1\} \\
&= \det\{1 - M_1R_1 - L_1M_1L_2M_1 + M_1R_1L_1M_1L_2M_1 + R_2M_1 \\
&\qquad - M_1R_1R_2M_1 - M_1R_1M_1L_2M_1\}.
\end{aligned}$$

Straightforward calculations using the formulae for M_1, $L_{1,2}$ and $R_{1,2}$ enable this to be rewritten as

$$\begin{aligned}
\det\Big\{&1 + A_{\beta,\mu}^{1/2}(1 - A_{\beta,\mu})^{-1/2}P_+\phi P_-XP_- + P_-XP_-\phi P_+(1 - A_{\beta,\mu})^{-1/2}A_{\beta,\mu}^{1/2} \\
&+ P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu}) + A_{\beta,\mu}^{1/2}(1 - A_{\beta,\mu})^{-1/2}P_+\phi P_-XP_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu}) \\
&+ A_{\beta,\mu}^{1/2}(1 - A_{\beta,\mu})^{-1/2}P_+Y^*P_+(1 - A_{\beta,\mu})^{-1/2}A_{\beta,\mu}^{1/2} \\
&+ A_{\beta,\mu}^{1/2}(1 - A_{\beta,\mu})^{-1/2}P_+\phi P_-XP_-XP_-\phi P_+(1 - A_{\beta,\mu})^{-1/2}A_{\beta,\mu}^{1/2} \\
&- A_{\beta,\mu}^{1/2}(1 - A_{\beta,\mu})^{-1/2}P_+\phi P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu})\Big\}
\end{aligned}$$

$$\begin{aligned}
\cdot\det\Big\{&1 - aA_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}\phi P_+Ya + aA_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}X^*A_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}a^* \\
&+ a^*YA_{\beta,\mu}(1 - A_{\beta,\mu})^{-1}a - aA_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}\phi P_+YP_+YA_{\beta,\mu}(1 - A_{\beta,\mu})^{-1}a \\
&- a^*YP_+\phi A_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}a^* \\
&+ aA_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}\phi P_+YP_+YP_+\phi A_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}a^* \\
&+ aA_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}\phi P_+YA_{\beta,\mu}(1 - A_{\beta,\mu})^{-1}a\Big\}.
\end{aligned}$$

Using the identity

$$\det(1 + A + B + AB) = \det(1 + A)\det(1 + B),$$

this may be simplified to

$$\det\{1 + A_{\beta,\mu}^{1/2}(1 - A_{\beta,\mu})^{-1/2}P_+Y^*P_+(1 - A_{\beta,\mu})^{-1/2}A_{\beta,\mu}^{1/2} + P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu})$$
$$+ P_-XP_-\phi P_+(1 - A_{\beta,\mu})^{-1/2}A_{\beta,\mu}^{1/2} - A_{\beta,\mu}^{1/2}(1 - A_{\beta,\mu})^{-1/2}P_+\phi P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu})\}$$

$$.\det\{1 + aA_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}X^*A_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}a^* + a^*YA_{\beta,\mu}(1 - A_{\beta,\mu})^{-1}a$$
$$- a^*YP_+\phi A_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}a^* + aA_{\beta,\mu}^{-1/2}(1 - A_{\beta,\mu})^{1/2}\phi P_+YA_{\beta,\mu}(1 - A_{\beta,\mu})^{-1}a\}.$$

With the connection between X and Y given in Remark 8.4.5 and the identity

$$aA_{\beta,\mu} = (1 - A_{\beta,-(1+\mu)})a,$$

this can be rewritten as

$$\det\{1 + A_{\beta,\mu}^{1/2}(1 - A_{\beta,\mu})^{-1/2}a\tilde{X}a^*(1 - A_{\beta,\mu})^{-1/2}A_{\beta,\mu}^{1/2} + P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu})$$
$$+ P_-XP_-\phi P_+(1 - A_{\beta,\mu})^{-1/2}A_{\beta,\mu}^{1/2} - A_{\beta,\mu}^{1/2}(1 - A_{\beta,\mu})^{-1/2}P_+\phi P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu})\}$$

$$.\det\{1 + (1 - A_{\beta,\lambda})^{-1/2}A_{\beta,\lambda}^{1/2}a\tilde{X}^Ta^*(1 - A_{\beta,\lambda})^{-1/2}A_{\beta,\lambda}^{1/2} + P_-X^TP_-(1 - A_{\beta,\lambda})A_{\beta,\lambda}^{-1} -$$
$$P_-X^TP_-\phi^TP_+(1-A_{\beta,\lambda})^{-1/2}A_{\beta,\lambda}^{1/2} + (1-A_{\beta,\lambda})^{-1/2}A_{\beta,\lambda}^{1/2}P_+\phi^TP_-X^TP_-(1-A_{\beta,\lambda})A_{\beta,\lambda}^{-1}\}$$

where $\lambda = -(1 + \mu)$. But

$$\det(1 + C^{-1}DC) = \det(1 + D),$$

hence this may be simplified to

$$\det\{1 + a\tilde{X}a^*(1 - A_{\beta,\mu})^{-1}A_{\beta,\mu} + P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu})$$
$$+ P_-XP_-\phi P_+(1 - A_{\beta,\mu})^{-1}A_{\beta,\mu} - P_+\phi P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu})\}$$

$$.\det\{1 + (1 - A_{\beta,\lambda})^{-1}A_{\beta,\lambda}a\tilde{X}^Ta^* + (1 - A_{\beta,\lambda})A_{\beta,\lambda}^{-1}P_-X^TP_-$$
$$- (1 - A_{\beta,\lambda})A_{\beta,\lambda}^{-1}P_-X^TP_-\phi^TP_+ + (1 - A_{\beta,\lambda})^{-1}A_{\beta,\lambda}P_+\phi^TP_-X^TP_-\}$$

where in the first determinant

$$C = P_- + (1 - A_{\beta,\mu})^{-1/2}A_{\beta,\mu}^{1/2}P_+,$$

and in the second

$$C = (1 - A_{\beta,\lambda})^{-1}A_{\beta,\lambda}P_- + A_{\beta,\lambda}^{-1/2}(1 - A_{\beta,\lambda})^{1/2}P_+.$$

This in turn can be rewritten as

$$\det\{1 + a\tilde{X}a^*(1 - A_{\beta,\mu})^{-1}A_{\beta,\mu} + P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu})$$
$$+ P_-XP_-\phi P_+(1 - A_{\beta,\mu})^{-1}A_{\beta,\mu} - P_+\phi P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu})\}$$

$$.\det\{[1 + a\tilde{X}a^*(1 - A_{\beta,\lambda})^{-1}A_{\beta,\lambda} + P_-XP_-A_{\beta,\lambda}^{-1}(1 - A_{\beta,\lambda})$$
$$+ P_-XP_-\phi P_+(1 - A_{\beta,\lambda})^{-1}A_{\beta,\lambda} - P_+\phi P_-XP_-A_{\beta,\lambda}^{-1}(1 - A_{\beta,\lambda})]^T\}.$$

Now $\det D = \det D^T$ hence this may be written as

$$\det F(X,\mu).\det F(X, -(1 + \mu)),$$

where

$$F(X,\mu) = 1 + a\tilde{X}a^*(1 - A_{\beta,\mu})^{-1}A_{\beta,\mu} + P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu})$$
$$+ P_-XP_-\phi P_+(1 - A_{\beta,\mu})^{-1}A_{\beta,\mu} - P_+\phi P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu}).$$

Hence the following Lemma has been shown.

8.6.3 Lemma.

$$\det_2(1 + L\Delta R) = \det F(X,\mu).\det F(X,-(1+\mu)),$$

where

$$F(X,\mu) = 1 + a\bar{X}a^*(1 - A_{\beta,\mu})^{-1}A_{\beta,\mu} + P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu})$$
$$+ P_-XP_-\phi P_+(1 - A_{\beta,\mu})^{-1}A_{\beta,\mu} - P_+\phi P_-XP_-A_{\beta,\mu}^{-1}(1 - A_{\beta,\mu}).$$

Note. The identity

$$aA_{\beta,\mu} = (1 - A_{\beta,-(1+\mu)})a$$

is a simple calculation similar to those at the end of the proof of Proposition 8.5.2 .

8.7 The Determinant Identity.

8.7.1 Remark. Using the Remarks 8.5.1, 8.5.3, Proposition 8.5.2 and Lemma 8.6.3

$$\left[\theta_3(f_0)\theta_3(0)^{-1}\exp\left\{-\frac{1}{2\pi}\sum_{n=1}^{\infty} n(e^{n\beta}-1)^{-1}|f_n|^2\right\}\right]^2 = M(X,\mu)M(X,-(1+\mu)),$$

where

$$M(X,\mu) = \prod_{n\geq 0}\left(1 + e^{-\beta(n-\mu)}\right)^{-2}\det F(X,\mu).$$

Note that the expression on the left hand side is independent of μ, similar to Lemma 7.3.4 .

8.7.2 Remark. From the previous Section, in the proof of Theorem 7.3.3

$$\theta_3(f_0)\theta_3(0)^{-1}$$
$$= \left[\prod_{n\in\mathbb{Z}}\left(1 + 2e^{-\beta(n-\mu)}\cos f_0 + (e^{-\beta(n-\mu)})^2\right)\left(1 + e^{-\beta(n-\mu)}\right)^{-2}\right]^{1/2}$$
$$= \left[\prod_{n\geq 0}\left(1 + 2\cos f_0 e^{-\beta(n-\mu)} + (e^{-\beta(n-\mu)})^2\right)\left(1 + e^{-\beta(n-\mu)}\right)^{-2}\right.$$
$$\left.\cdot\prod_{n\geq 0}\left(1 + 2\cos f_0 e^{-\beta(n+1+\mu)} + (e^{-\beta(n+1+\mu)})^2\right)\left(1 + e^{-\beta(n+1+\mu)}\right)^{-2}\right]^{1/2}.$$

Hence the identity can be rewritten as

$$\det F(X,\mu)\det F(X,-(1+\mu)) = \left[\exp\left\{-\frac{1}{2\pi}\sum_{n=1}^{\infty} n(e^{n\beta}-1)^{-1}|f_n|^2\right\}\right]^2$$
$$\cdot\prod_{n\geq 0}\left(1 + 2\cos f_0 e^{-\beta(n-\mu)} + (e^{-\beta(n-\mu)})^2\right)$$
$$\cdot\prod_{n\geq 0}\left(1 + 2\cos f_0 e^{-\beta(n+1+\mu)} + (e^{-\beta(n+1+\mu)})^2\right).$$

Thus when $\mu = -1/2$ this simplifies to

$$\det F(X,-1/2)$$
$$= \prod_{n\geq 0}\left(1 + 2\cos f_0 e^{-\beta(n+1/2)} + (e^{-\beta(n+1/2)})^2\right)\exp\left\{-\frac{1}{2\pi}\sum_{n=1}^{\infty} n(e^{n\beta}-1)^{-1}|f_n|^2\right\}$$

which is reminiscent of Szego's Theorem because of the following:

(1) Suppose $\phi = e^{if_0}$, i.e. the simplest case, then

$$F(X,\mu) = 1 + e^{if_0} P_+ (1 - A_{\beta,\mu})^{-1} A_{\beta,\mu} + e^{-if_0} P_- A_{\beta,\mu}^{-1} (1 - A_{\beta,\mu}),$$

and it is not difficult to show that

$$\det F(X,\mu) \det F(X,-(1+\mu)) = \prod_{n\geq 0} \left(1 + 2\cos f_0 e^{-\beta(n-\mu)} + (e^{-\beta(n-\mu)})^2\right) \cdot \prod_{n\geq 0} \left(1 + 2\cos f_0 e^{-\beta(n+1+\mu)} + (e^{-\beta(n+1+\mu)})^2\right).$$

Thus the two infinite product terms are similar to the ρ term in Szego's Theorem.

(2) As for the exponential. This is already similar to the exponential term in Szego's Theorem. Note that

$$|f_n|^2 = f_n f_n^* = f_n f_{-n}, \qquad \text{proof of Lemma 7.3.5},$$

and that $g_n = if_n$ in the application of Szego's Theorem, see Lemma 8.4.3, which explains the minus sign in the exponential. The $A_{\beta,\mu}$ terms presumably produce the extra term $(e^{n\beta} - 1)$.

Appendix

A1 Proposition. *For $0 \leq n \leq m-2$ let*

$$I_{n,m} = \int_0^q \frac{y^n}{(1+y)^m}\,dy,$$

then

$$I_{n,m} = \frac{((m-1)-(n+1))!\,n!}{(m-1)!(1+q)^{m-1}} \sum_{i=n+1}^{m-1} \binom{m-1}{i} q^i.$$

Proof: First note that

$$\begin{aligned}
I_{n,m} &= \int_0^q \frac{y^n}{(1+y)^m}\,dy \\
&= \left[\frac{y^n(1+y)^{-m+1}}{-m+1}\right]_0^q - \int_0^q \frac{ny^{n-1}(1+y)^{-m+1}}{-m+1}\,dy \\
&= \frac{-q^n}{(m-1)(1+q)^{m-1}} + \frac{n}{(m-1)} I_{n-1,m-1}.
\end{aligned}$$

So assuming the formula is true for $I_{n-1,m-1}$

$$\begin{aligned}
I_{n,m} &= \frac{-q^n}{(m-1)(1+q)^{m-1}} + \frac{n}{(m-1)} \frac{(m-n-2)!(n-1)!}{(m-2)!(1+q)^{m-2}} \sum_{i=n}^{m-2} \binom{m-2}{i} q^i \\
&= \frac{(m-n-2)!\,n!}{(m-1)!(1+q)^{m-1}} \left[(1+q) \sum_{i=n}^{m-2} \binom{m-2}{i} q^i - \frac{(m-2)!}{(m-n-2)!\,n!} q^n\right] \\
&= \frac{(m-n-2)!\,n!}{(m-1)!(1+q)^{m-1}} \left[\sum_{i=n+1}^{m-2} \left\{\binom{m-2}{i} + \binom{m-2}{i-1}\right\} q^i + q^{m-1}\right] \\
&= \frac{(m-n-2)!\,n!}{(m-1)!(1+q)^{m-1}} \sum_{i=n+1}^{m-1} \binom{m-1}{i} q^i
\end{aligned}$$

as required. Now for $p \geq 2$

$$\begin{aligned}
I_{0,p} &= \int_0^q \frac{dy}{(1+y)^p} = \left[\frac{(1+y)^{-(p-1)}}{-(p-1)}\right]_0^q \\
&= \frac{-1}{(p-1)(1+q)^{p-1}} + \frac{1}{(p-1)} \\
&= \frac{1}{(p-1)(1+q)^{p-1}} \{(1+q)^{p-1} - 1\} \\
&= \frac{(p-2)!}{(p-1)!(1+q)^{p-1}} \sum_{i=1}^{p-1} \binom{p-1}{i} q^i.
\end{aligned}$$

That is, the formula is true for $I_{0,p}$ for all $p \geq 2$. Therefore it is true by induction for $I_{n,m}$ where $0 \leq n \leq m-2$. (Start at $I_{0,m-n}$ and work upwards.)

Corollary.

$$\int_0^1 \frac{y^n}{(1+y)^m}\,dy = \frac{((m-1)-(n+1))!\,n!}{(m-1)!\,2^{m-1}} \sum_{i=n+1}^{m-1} \binom{m-1}{i}.$$

A2 Proposition. *Define the function $F_{c_1,c_2} : [0,1] \to \mathbb{R}$ by*

$$F_{c_1,c_2}(y) = \frac{y[c_1 + (2c_1 + c_2)y + c_1 y^2]}{(1+y)^4},$$

where $c_1 > 0$ and $c_2 < 0$. Suppose also that $c_2 + 4c_1 < 0$ and $c_2 + 4c_1 + 16 > 0$ then the following are true:

(1) $F_{c_1,c_2}(0) = 0 : F'_{c_1,c_2}(0) > 0$.
(2) $0 > F_{c_1,c_2}(1) > -1 : F'_{c_1,c_2}(1) = 0 : F''_{c_1,c_2}(1) > 0$.
(3) $\forall c_1$, $\exists$ a unique $y_0 \in (0,1)$ such that

$$F'_{c_1,c_2}(y_0) = 0 \text{ and } F''_{c_1,c_2}(y_0) < 0.$$

Moreover if $0 < c_1 < 16$ then $0 < F_{c_1,c_2}(y_0) < 1$.

Proof: (1) and (2) are obvious.

$$F'_{c_1,c_2}(y) = \frac{(1-y)}{(1+y)^5}\left(c_1 + 2(c_1+c_2)y + c_1y^2\right).$$

$$\begin{aligned} F''_{c_1,c_2}(y) &= \frac{2}{(1+y)^6}\left[c_2(1-y)^2 + (y-2)\left(c_1 + 2(c_1+c_2)y + c_1y^2\right)\right] \\ &= \frac{2}{(1+y)^6}\left[(c_2 - 2c_1) - 3(c_1 + 2c_2)y + 3c_2y^2 + c_1y^3\right]. \end{aligned}$$

(3) The roots of the equation $c_1 + 2(c_1+c_2)y + c_1y^2 = 0$ are

$$\frac{-(c_1+c_2) \pm \sqrt{c_2(c_2+2c_1)}}{c_1}$$

Using the conditions on c_1 and c_2 it is easy to see that

$$y_0 = \frac{-(c_1+c_2) - \sqrt{c_2(c_2+2c_1)}}{c_1}$$

and it is the unique root in $(0,1)$ since

$$\frac{-(c_1+c_2) + \sqrt{c_2(c_2+2c_1)}}{c_1} > 1.$$

So $F'_{c_1,c_2}(y_0) = 0$ by construction and

$$F''_{c_1,c_2}(y_0) = \frac{2c_2(1-y_0)^2}{(1+y_0)^6} < 0 \text{ as } c_2 < 0.$$

Now

$$\begin{aligned} F_{c_1,c_2}(y_0) &= \frac{c_1y_0}{(1+y_0)^2} + \frac{c_2y_o^2}{(1+y_0)^4} \\ &= \frac{-c_1^2}{4c_2} \qquad \text{using } (1+y_0)^2 = -2\frac{c_2}{c_1}y_0. \end{aligned}$$

Hence $0 < F_{c_1,c_2}(y_0) < 1$ if $c_1^2 + 4c_2 < 0$.
But the line $c_2 + 4c_1 = 0$ and the curve $c_1^2 + 4c_2 = 0$ intersect at the origin and the point $c_1 = 16$, $c_2 = -64$. Therefore if $0 < c_1 < 16$ the condition $c_2 + 4c_1 < 0$ gives $0 < F_{c_1,c_2}(y_0) < 1$.

Corollary. *If $F_{c_1,c_2}(y)$ is the function defined in the previous Proposition and the following hold:*

(1) $c_2 + 4c_1 < 0$,
(2) $c_2 + 4c_1 + 16 > 0$,
(3) $0 < c_1 < 16$,

then if $e = 2 + c_2/c_1$

$$\begin{aligned}\int_0^1 & \frac{\log\left(1 + F_{c_1,c_2}(y)\right)}{y}\,dy \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} c_1^n \left\{ \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i}\binom{i}{j} R(i,j,n) \right\},\end{aligned}$$

where

$$R(i,j,n) = e^{i-j} \frac{(3n-i-j-1)!(n+i+j-1)!}{2^{4n-1}(4n-1)!} \sum_{k=n+i+j}^{4n-1} \binom{4n-1}{k}.$$

Proof: From Proposition A2

$$|F_{c_1,c_2}(y)| < 1 \qquad \forall y \in [0,1].$$

Therefore the logarithm can be replaced by an infinite sum and the integral taken inside to get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^1 \frac{[F_{c_1,c_2}(y)]^n}{y}\,dy.$$

Then, using

$$\begin{aligned}[F_{c_1,c_2}(y)]^n &= \frac{y^n}{(1+y)^{4n}} c_1^n [1 + ey + y^2]^n \\ &= \frac{c_1^n y^n}{(1+y)^{4n}} \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i}\binom{i}{j} e^{i-j} y^{i+j}\end{aligned}$$

and the Corollary to Proposition A1 the result may be obtained.

A3 Proposition. *Suppose*

$$\left| \sum_{i=1}^{p} c_i \left(\frac{y}{(1+y)^2} \right)^i \right| < 1, \qquad \forall y \in [0,1].$$

If $I(c_1,\ldots,c_p)$ is defined as in 4.1 then

$$I(c_1,\ldots,c_p) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} S(n),$$

where

$$S(n) = \sum_{n_1+\cdots+n_p=n} \left\{ \frac{n!}{n_1!n_2!\ldots n_p!} c_1^{n_1} \ldots c_p^{n_p} \frac{(N!)^2}{(M-1)!2^{M-1}} \sum_{k=N+1}^{M-1} \binom{M-1}{k} \right\}.$$

$n_1, \ldots, n_p$ are non-negative integers and

$$N = n_1 + 2n_2 + \cdots + pn_p - 1,$$
$$M = 2n_1 + 4n_2 + \cdots + 2pn_p.$$

PROOF: The condition on the c_i's enables the logarithm to be written as an infinite sum. Passing the integral through this and the sum from the multinomial formula for

$$\left(\sum_{i=1}^{p} c_i \left(\frac{y}{(1+y)^2}\right)^i\right)^n$$

leads to terms of the form

$$\int_0^1 \frac{y^N}{(1+y^M)}\,dy.$$

Now $M - N \geq 2$ so the Corollary to Proposition A1 gives the relevant formula leading to the expression above.

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