IRREDUCIBLE CHARACTERS OF THE UNITRIANGULAR GROUP AND COADJOINT ORBITS

Carlos Alberto Martins André

A thesis submitted for the degree of Doctor of Philosophy of the University of Warwick

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DECLARATION

None of the original material in this thesis (see p. xi) has been used by me before, nor is any of it the result of joint collaboration.

Carlos Alberto Martins André

ABSTRACT

The method of coadjoint orbits was introduced by Kirillov to study the unitary irreducible representations of a nilpotent Lie groups. Afterwards Kazhdan adapted this method to determine the irreducible complex characters of a finite unipotent group. We use this method to study the irreducible complex characters of any finite unitriangular group.

In chapters 2 and 5 we established an orthogonal decomposition of the regular character of any finite unitriangular group.

Chapters 3 and 4 are concerned with coadjoint orbits of any unitriangular group defined over an algebraically closed field. Chapter 3 is essentially the orbit version of chapter 2. In fact we obtain a decomposition of the dual space of the niltriangular Lie algebra as a disjoint union of invariant subvarieties. In a certain sense this decomposition corresponds to the one obtained in chapter 2 and 5.

INTRODUCTION

The main purpose of this thesis is the study of the irreducible (complex) characters of the finite group $U_n(q)$ consisting of all upper unitriangular matrices of size n with coefficients in the finite field $\mathbb{F}_q$ (throughout the thesis q will denote a power of a prime number p). Our approach is based on Kirillov's method of coadjoint orbits (see [Ki1], [Ki2] or [CG]). This method was introduced in the context of nilpotent Lie groups and was adapted by Kazhdan to the context of finite unipotent groups (see [Ka]). It gives a very useful way of constructing the irreducible characters of $U_n(q)$ and it runs as follows.

Let K be the algebraic closure of $\mathbb{F}_q$. Then we may realize the finite group $U_n(q)$ as a subgroup of $U_n(K)$. In fact $U_n(q)$ is canonically isomorphic to the subgroup of $U_n(K)$ which consists of all fixed elements of the *Frobenius map* $F: U_n(K) \to U_n(K)$ - by definition $F(x)=(x_{ij}^q)$ for all $x=(x_{ij}) \in U_n(K)$. The linear algebraic group $U_n(K)$ acts on its Lie algebra $\mathfrak{u}_n(K)$ via the *adjoint representation* - we recall that the Lie algebra $\mathfrak{u}_n(K)$ consists of all upper niltriangular matrices of size n over K. Therefore $U_n(K)$ acts on the dual vector space $\mathfrak{u}_n(K)^*$ of $\mathfrak{u}_n(K)$ via the contragradient representation. This representation is called the *coadjoint representation* and its $U_n(K)$-orbits are the *coadjoint $U_n(K)$-orbits.*

A similar rule to the one above defines a Frobenius map on $\mathfrak{u}_n(K)$ which we denote also by F. Then (using a basis of $\mathfrak{u}_n(K)$ consisting of F-fixed elements) we may define in a natural way the Frobenius map on $\mathfrak{u}_n(K)^*$ and we may consider F-stable coadjoint $U_n(K)$-orbits. In order to define the exponential map $exp: \mathfrak{u}_n(K) \to U_n(K)$ we assume that $p \geq n$ (we note that with this assumption the Campbell-Hausdorf formula holds). Then any F-stable coadjoint $U_n(K)$-orbit $O \subseteq \mathfrak{u}_n(K)^*$ determines a character χ_O of $U_n(q)$ which is defined by

$$\chi_O(\exp a)=\frac{1}{\sqrt{|O^F|}}\sum_{g\in O^F}\psi_0(g(a))$$

where ψ_0 is an arbitrary (but fixed) non-trivial linear (complex) character of the additive group F_q^+ (we note that any coadjoint $U_n(K)$-orbit is an irreducible algebraic variety of even dimension). This character is irreducible and so we obtain a correspondence $O\to\chi_O$ from the set of all F-stable coadjoint $U_n(K)$-orbits to the set of all irreducible characters of $U_n(q)$. The irreducible character χ_O was defined by Kazhdan in his paper [Ka] (see also [Sr; chapter 7]). It was also proved by Kazhdan that any irreducible character of $U_n(q)$ has the form χ_O for some F-stable coadjoint $U_n(K)$-orbit $O\subseteq\mathfrak{u}_n(K)^*$ and that two F-stable coadjoint $U_n(K)$-orbits are distinct if and only if the corresponding irreducible characters of $U_n(q)$ are distinct. Therefore the above correspondence $O\to\chi_O$ is one-to-one and so we have a parametrization of all irreducible characters of $U_n(q)$ in terms of F-stable coadjoint $U_n(K)$-orbits.

Another approach (which is closer to the original construction of Kirillov) to the characters χ_O of $U_n(q)$ (here O is an F-stable coadjoint $U_n(K)$-orbit) is as follows. Let $f\in\mathfrak{u}_n(K)^*$ be an arbitrary F-fixed element. Then we may define a skew-symmetric K-bilinear form B_f on $\mathfrak{u}_n(K)$ by $B_f(a,b)=f([ab])$ for all $a,b\in\mathfrak{u}_n(K)$. Hence we may consider maximal isotropic subspaces of $\mathfrak{u}_n(K)$ (with respect to that skew-symmetric form). A general result asserts that there exists at least one maximal isotropic subspace $\mathfrak{h}_n(K)$ of $\mathfrak{u}_n(K)$ which is also a (Lie) subalgebra of $\mathfrak{u}_n(K)$ and which is F-stable. Therefore $\mathfrak{h}_n(K)$ defines (via the exponential map) a subgroup $H_n(K)$ of $U_n(K)$ which is also F-stable, hence the finite set of all its F-fixed elements $H_n(q)$ is a subgroup of $U_n(q)$. Then the correspondence $a\to\psi_0 f(a)$, $a\in\mathfrak{u}_n(K)$, defines a linear character λ_f of $H_n(q)$ and we may obtain the irreducible character χ_O which corresponds to the F-stable $U_n(K)$-orbit of f as the induced character $\lambda_f^{U_n(q)}$. Therefore we conclude that any irreducible character of $U_n(q)$ is induced from a linear character of some "admissible" subgroup of $U_n(q)$. This result was proved independently by Gutkin [Gu] for an arbitrary prime p. In particular we deduce that each irreducible character of $U_n(q)$ has degree q^m for some integer $m\geq 0$ (see

[Gu]). In fact (by definition) we see that $\chi_O(1)=\sqrt{q^{\dim O}}$ for all F-stable $U_n(K)$-orbit $O\subseteq u_n(K)^*$.

Although Kazhdan's results reduce the classification of the irreducible characters of $U_n(q)$ to the classification of coadjoint $U_n(K)$-orbits they do not give a constructive method to obtain those characters. In fact they do not allow a systematic method to construct those characters. In the paper [Le] Lehrer used a different method which is based on Clifford theory and which is valid for an arbitrary prime number p. This method was also used by Lambert and Dijk [LD] in the context of real Lie groups. It is completely constructive and it allows the construction of all irreducible characters of $U_n(q)$ once we know the irreducible characters of $U_{n-1}(q)$ and of some of its subgroups. In fact the group $U_{n-1}(q)$ is canonically isomorphic to a subgroup of $U_n(q)$ which has a normal complement $A_n(q)$. Therefore (by Clifford theory - see theorem 2.1.1) each irreducible character χ of $U_n(q)$ is determined by a linear character λ of $A_n(q)$ and by an irreducible character ϕ of centralizer $C_{U_{n-1}(q)}(\lambda)$ of λ in $U_{n-1}(q)$ - we note that $U_{n-1}(q)$ acts in a natural way on the set of all characters of its normal complement $A_n(q)$. This correspondence is not one-to-one because the irreducible character χ of $U_n(q)$ is also determined by any pair (λ^x,ϕ^x) for $x\in U_{n-1}(q)$. However the $U_{n-1}(K)$-orbit of λ contains a certain canonical character for which the subgroup $A_n(q)C_{U_{n-1}(q)}(\lambda)$ is the subgroup $U_\omega(q)=U_n(q)\cap\omega^{-1}U_n(q)\omega$ where $\omega\in S_n$ is a permutation of the form $\omega=(n\text{-}1\ \dots\ i\text{+}1\ i)$ for some $i\in\{1,\dots,n\text{-}1\}$ (if $i=n\text{-}1$ then $\omega=1$). Therefore we conclude that each irreducible character of $U_n(q)$ is induced by some irreducible character of $U_\omega(q)$. If we restrict our attention to the linear characters of $U_\omega(q)$ then we obtain a family of irreducible characters of $U_n(q)$ for which the Lie algebra $u_\omega(K)=u_n(K)\cap\omega^{-1}u_n(K)\omega$ is a maximal isotropic subspace of $u_n(K)$ with respect to a certain (F-fixed) element $f\in u_n(K)^*$. This family includes all the irreducible characters of $U_n(q)$ which correspond to the (F-stable) $U_n(K)$-orbit $O_{in}(\alpha)$ of the element $\alpha e_{in}{}^*\in u_n(K)^*$ for $\alpha\in\mathbb{F}_q$ - here $e_{in}{}^*\in u_n(K)^*$ is the linear map defined by $e_{in}{}^*(a)=a_{in}$ for all $a=(a_{rs})\in u_n(K)$. These orbits lie in a larger family of orbits which we will call the *elementary orbits*. In general if (i,j) is any pair in the set

$\Phi(n)=\{(a,b);\ 1\leq a,b\leq n\}$ we may define the *(i,j)-th elementary $U_n(K)$-orbit associated with an element $\alpha\in K$* to be the $U_n(K)$-orbit $O_{ij}(\alpha)$ of the element $\alpha e_{ij}^*\in u_n(K)^*$ - here $e_{ij}^*\in u_n(K)^*$ is the linear map defined by $e_{ij}^*(a)=a_{ij}$ for all $a=(a_{rs})\in u_n(K)$. If $\alpha\in \mathbf{F}_q$ then the orbit $O_{ij}(\alpha)$ is F-stable and so it determines an irreducible character $\xi_{ij}(\alpha)$ of $U_n(q)$. A character with this form will be called an *elementary character* of $U_n(q)$. They are the characters of the irreducible representations defined by Lehrer in [Le]. Their construction is described in section 2.1 where we also prove that each elementary character $\xi_{ij}(\alpha)$, $(i,j)\in\Phi(n)$, $\alpha\in\mathbf{F}_q$, is induced by the linear character $\lambda_{ij}(\alpha)$ of the subgroup $U_\omega(q)=U_n(q)\cap\omega^{-1}U_n(q)\omega$, $\omega=(j\text{-}1\ \dots\ i\text{+}1\ i)\in S_n$ - this linear character is defined by $\lambda_{ij}(\alpha)(x)=\alpha x_{ij}$ for all $x=(x_{rs})\in U_\omega(q)$. In particular the Lie algebra $u_\omega(K)=u_n(K)\cap\omega^{-1}u_n(K)\omega$ is an (F-stable) maximal isotropic subspace of $u_n(K)$ (with respect to the element $\alpha e_{ij}^*\in u_n(K)^*$).

Following Lehrer's work we consider products of elementary characters. In fact, in [Le], Lehrer obtained a decomposition of restriction of any irreducible discrete series complex representation of the general linear group $GL_n(q)$ to $U_n(q)$ as sum of certain tensor products of irreducible representations whose characters are elementary. This decomposition can be improved if we apply our results of section 2.2. In fact we will prove that any product of elementary characters can be decomposed as a sum of basic characters (proposition 2.2.13). By definition a *basic subset D* of $\Phi(n)$ is any subset of the form $\Phi(n)\cap\omega(\Delta)$ where $\Delta=\{(1,2),(2,3),\dots,(n\text{-}1,n)\}$ and ω is any element of S_n. Then we define the basic character $\xi_D(\varphi)$ associated with a basic subset D of $\Phi(n)$ and a map $\varphi:D\to\mathbf{F}_q^*=\mathbf{F}_q\setminus\{0\}$ to be product of all the elementary characters $\xi_{ij}(\varphi(i,j))$ for $(i,j)\in\Phi(n)$. The main result of this thesis asserts that any irreducible character χ of $U_n(q)$ is a component of a unique basic character (see theorem 2.2.1). Moreover the regular character of $U_n(q)$ decomposes as the sum of all basic characters of $U_n(q)$, and the multiplicity of a given basic character $\xi_D(\varphi)$ as a component of the regular character is a power of q which depends only on the basic subset D (see theorem 5.2.1).

On the other hand we consider the closely related problem of describing the coadjoint $U_n(K)$-orbits. Our results in this direction are established for an arbitrary algebraically closed field (with the restriction $p\geq n$ if K has prime characteristic p). In this more general context we consider sums of elementary $U_n(K)$-orbits. In fact a general result (corollary 1.3.11) asserts that (if K has characteristic $p\geq n$) the irreducible components of a product of irreducible characters $\chi_1,\dots,\chi_r$ are in one-to-one correspondence with the F-stable $U_n(K)$-orbits which are contained in the sum $O_1+\dots+O_r$ where O_i is the f-stable $U_n(K)$-orbits which corresponds to the irreducible character χ_i $(1\leq i\leq r)$. Therefore we consider basic sums of elementary $U_n(K)$-orbits. If D is a basic subset of $\Phi(n)$ and $\varphi:D\rightarrow K^*=K\backslash\{0\}$ is a map then we define the subset $O_D(\varphi)$ of $u_n(K)^*$ to be the sum of all the elementary orbits $O_{ij}(\varphi(i,j))$ for $(i,j)\in D$. The orbit version of the result mentioned above states that $u_n(K)^*$ is the disjoint union of all basic sums $O_D(\varphi)$ where D is any map (see theorem 3.1.7). Moreover each $O_D(\varphi)$ is a $U_n(K)$-invariant irreducible subvariety of $u_n(K)^*$, hence there should exist a finite set $P_1,\dots,P_m$ of $U_n(K)$-invariant polynomial functions defined on $u_n(K)^*$ such that $O_D(\varphi)=\{f\in u_n(K)^*;\ P_i(f)=k_i,\ 1\leq i\leq m\}$ where $k_1,\dots,k_m\in K$. These functions are defined in section 3.2 and they are indexed by a certain subset of $\Phi(n)$. This subset depends only on the set D and it will be denoted by $R(D)$. Its definition is purely combinatorial. In fact a pair $(i,j)\in\Phi(n)$ lies in the set $R(D)$ if and only if $(i,k)\notin D$ for all $k\in\{j+1,\dots,n\}$ and $(l,j)\notin D$ for all $l\in\{1,\dots,j-1\}$ - in particular we have $D\subseteq R(D)$. For each $(i,j)\in R(D)$ the $U_n(K)$-invariant polynomial function corresponding to (i,j) will be denoted by Δ_{ij}^D. Their definition was motivated by the work of Dixmier [Di] - in fact the functions $\Delta_1,\dots,\Delta_r$ (where $n=2r$ or $n=2r+1$) defined in [Di] are our functions Δ_{ij}^D where $D=\{(1,n),(2,n-1),(3,n-2),\dots\}$ (in this case $R(D)=D$).

In general the subvariety $O_D(\varphi)$ of $u_n(K)^*$ is not a single $U_n(K)$-orbit and its decomposition as a union of orbits seems to be very difficult to obtain. An attempt is made in chapter 4 where we give a decomposition of certain varieties $O_D(\varphi)$ which depend on the pairs $(a,i)\in\Phi(n)$ where $(i,n)\in D$ for some $i\in\{1,\dots,n-1\}$. This decomposition suggests also that it might be possible to find an algorithm to describe the coadjoint $U_n(K)$-orbits once we know the coadjoint $U_{n-1}(K)$-orbits. In fact using the permutation $\omega=(n-1\dots i+1\ i)$ one may define a certain basic subset D_ω of $\mathcal{U}_n(K)^*$ and a map $\varphi_\omega: D_\omega \to K^*$ which depend on the initial pair (D,φ) and which satisfy $(n-1,n)\in D_\omega$. Hence the variety $O_{D_\omega}(\varphi_\omega)$ of $\mathcal{U}_n(K)^*$ is canonically isomorphic to the subvariety $O_{D_{\omega,0}}(\varphi_{\omega,0})$ where $D_{\omega,0}=D_\omega\setminus\{(n-1,n)\}$ and $\varphi_{\omega,0}$ is the restriction of φ_ω to $D_{\omega,0}$. Unfortunately the pair $(D_\omega,\varphi_\omega)$ depends on a particular element $f\in O_D(\varphi)$ (which satisfy $f(e_{an})=f(e_{ia})=0$ for all $a\in\{i+1,\dots,n-1\}$) and in many cases different elements in $O_D(\varphi)$ (satisfying that condition) may determine different pairs $(D_\omega,\varphi_\omega)$. In fact $(D_\omega,\varphi_\omega)$ is the unique pair such that $f_\omega\in O_{D_\omega}(\varphi_\omega)$ where $f_\omega\in\mathcal{U}_n(K)^*$ is defined by $f_\omega(e_{ab})=f(e_{\omega^{-1}(a),\omega^{-1}(b)})$ for all $(a,b)\in\Phi(n)$ with $\omega^{-1}(a)<\omega^{-1}(b)$, and $f_\omega(e_{ab})=0$ if $(a,b)\in\Phi(n)$ with $\omega^{-1}(a)>\omega^{-1}(b)$. The pair $(D_\omega,\varphi_\omega)$ can be also defined step-by-step applying the simple reflections $(i\ i+1),\dots,(n-2\ n-1)$ (in this order) to the element f. At each stage we define a pair (D_a,φ_a) $(1\le a\le n-i-1)$ such that $(D_\omega,\varphi_\omega)=(D_{n-i-1},\varphi_{n-i-1})$. The pair (D_1,φ_1) (corresponding to the reflection $(i\ i+1)$) is determined in section 5.1 (see theorem 5.1.7), and in section 5.2 we give an application of this method to the character theory of $U_n(q)$ (see the proof of proposition 5.2.2).

More precisely our work in this thesis is organized in the following way.

Chapter 1 is an introductory chapter. It consists of three sections. The first contains the basic notation and definitions which will be used throughout this thesis. In the second we describe Kazhdan's parametrization of the irreducible complex characters of $U_n(q)$. Finally the third section is concerned with the operations of induction and restriction of irreducible characters.

In chapter 2 we define elementary characters and elementary coadjoint orbits. Then we study products of elementary characters and we prove the main result of this thesis. We also describe the irreducible characters of $U_n(q)$ for $n\le 5$.

Chapter 3 is dedicated to the study of basic sums of elementary orbits. It has three sections. In the first we define certain subvarieties $V_D(\varphi)$ of $\mathcal{U}_n(K)^*$ and we prove that they coincide with the sum of elementary orbits. In this section we also prove the orbit version of our main result. In the second we determine the dimension of each variety $V_D(\varphi)$ and we derive some number-theoretical consequences. Finally in the third section we determine the pairs (D,φ) for which $V_D(\varphi)$ is a single coadjoint orbit.

Chapter 4 is mainly concerned with a certain decomposition of $V_D(\varphi)$. Finally in chapter 5 we use this decomposition to describe a certain inductive process which is used to obtain an additive decomposition of the regular character of $U_n(q)$.

As far as we know all the results presented in this thesis are original with exception of the major part of the results of the chapter 1. Section 1.2 follows [Sr; chapter 7] and section 1.3 is an adaptation of Kirillov theory as it can be found in [Ki2] and [CG] (however some of our proofs are different). The applications to the character theory in section 1.3 (namely theorems 1.3.8, 1.3.9 and 1.3.10 and corollary 1.3.11) are also original.

CHAPTER 1

GENERAL THEORY

This chapter is concerned with the general background of our thesis.

In section 1.1 we introduce the basic notation and we discuss briefly a certain family of unipotent algebraic groups. These groups can be obtained as exponential images of nilpotent Lie algebras and so they will be called exponential (unipotent) groups.

In section 1.2 we construct the irreducible characters of the finite exponential groups. In particular we prove that these characters are induced by linear characters of some subgroups which depend on the existence of certain maximal isotropic subspaces of the Lie algebra associated with the given unipotent group. We also establish the one-to-one correspondence between the irreducible characters of this unipotent group and the orbits of its coadjoint representation (on the dual space of its Lie algebra).

Finally in section 1.3 we translate to the orbit language the well-known operations of restriction and induction of characters. In particular we will consider subgroups of "codimension" one. Since these subgroups are normal this section is closely related with the Clifford theory of finite groups.

Although the major part of the results of this chapter can be found in the literature (except as far as we know the applications to the finite groups in section 1.3) we will give a detailed proof of each result. This is done for the convenience of the reader and in order to make this thesis the most self-contained it can be. However a reference will be given for all non-original results and proofs.

1.1. Generalities

Let K be any field and let n be a positive integer. Throughout this work we will denote by $U_n(K)$ the (upper) *unitriangular group* of degree n over K. By definition this group consists of all square matrices $x=(x_{ij})$ of size n with coefficients in the field K satisfying $x_{ii}=1$ ($1\leq i\leq n$) and $x_{ij}=0$ ($1\leq j<i\leq n$). If K is the finite field $\mathbf{F}_q$ with q elements (where $q=p^e$, $e>0$, is the e-th power of the prime number p) we will write $U_n(q)$ instead of $U_n(\mathbf{F}_q)$. It is well-known that $U_n(q)$ is a finite group of order $q^{n(n-1)/2}$. For our purposes it is convenient to realize the finite group $U_n(q)$ as a subgroup of the infinite group $U_n(K)$ where K is the algebraic closure of $\mathbf{F}_q$. For we use the *Frobenius map* $F=F_q:U_n(K)\to U_n(K)$ which is defined by

(1.1.1) $$F(x)=(x_{ij}^{\,q})$$

for all $x=(x_{ij})\in U_n(K)$. The set

$$U_n(K)^F=\{x\in U_n(K);\, F(x)=x\}$$

consisting of all the F-fixed elements of $U_n(K)$ is a finite subgroup of $U_n(K)$ and we have a canonical isomorphism

$$U_n(K)^F\cong U_n(q).$$

For the basic properties of the Frobenius map we refer to Carter's book [Ca]. We note that, if the field K is algebraically closed, the group $U_n(K)$ has a structure of affine algebraic variety ([1]). In fact it is isomorphic to the affine space $K^{n(n-1)/2}$ of dimension $\frac{n(n-1)}{2}$ ([2]). Since the multiplication and the inversion maps are morphisms of algebraic varieties ([3]) we conclude that $U_n(K)$ is an algebraic group.

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[1] For the basic theory of affine algebraic varieties and of algebraic groups we refer to Humphreys' book [Hu1].

[2] The coordinate ring of $U_n(K)$ is the polynomial ring $K[T_{ij};\, 1\leq i<j\leq n]$ in $\frac{n(n-1)}{2}$ indeterminates T_{ij} ($1\leq i<j\leq n$) over K.

[3] The expression "algebraic variety" abbreviates "affine algebraic variety". This abbreviation will be kept throughout our thesis.

Our main goal is the study of the (complex) character theory of the finite group $U_n(q)$. The starting point of our work is Kazhdan's construction (see [Ka] or [Sr, pg. 114-118]) of the irreducible characters of the finite groups consisting of all the fixed elements of a Frobenius map defined on a unipotent algebraic group over an algebraically closed field of prime characteristic p. This construction uses the method of coadjoint orbits developed by Kirillov to study the unitary representations of nilpotent Lie groups (see [Ki1], [Ki2] or [CG]). Therefore a restriction has to be imposed on the prime p in order to realize our algebraic group as the exponential image of its Lie algebra (even though not all unipotent algebraic groups can be obtained by this process). In the following we discuss this general situation. Since any unipotent algebraic group is isomorphic to a closed subgroup of some unitriangular group (see [Hu1; corollary 17.5]) we may define a *unipotent group* to be a closed subgoup of some $U_n(K)$. Then the Lie algebra of a unipotent group is a subalgebra ([1]) of Lie algebra of $U_n(K)$. This Lie algebra consists of all square matrices $x=(x_{ij})$ of size n over K satisfying $x_{ij}=0$ $(0\leq j\leq i\leq n)$. It is called the (upper) *niltriangular Lie algebra* and it will be denoted by $\mathcal{U}_n(K)$. It can also be defined for an arbitrary field K. In particular if K is the finite field $\mathbf{F}_q$ we will write $\mathcal{U}_n(q)$ instead of $\mathcal{U}_n(\mathbf{F}_q)$. As in the case of the unitriangular group we may identify $\mathcal{U}_n(q)$ with the subalgebra

$$\mathcal{U}_n(K)^F=\{a\in \mathcal{U}_n(K);\, F(a)=a\}$$

of $\mathcal{U}_n(K)$ where K is the algebraic closure of $\mathbf{F}_q$ and $F=F_q: \mathcal{U}_n(K)\to \mathcal{U}_n(K)$ is the Frobenius map. As before F is defined by

(1.1.2) $$F(a_{ij})=(a_{ij}{}^q)$$

for all $a=(a_{ij})\in \mathcal{U}_n(K)$. We note that $\mathcal{U}_n(q)$ is a vector space over $\mathbf{F}_q$ and that we have a canonical isomorphism of vector spaces

(1.1.3) $$\mathcal{U}_n(K)\cong \mathcal{U}_n(q)\otimes_{\mathbf{F}_q} K.$$

Now we assume that $p\geq n$ whenever K has prime characteristic p. Then we may

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[1] By a subalgebra we understand a Lie subalgebra.

define the *exponential map* $exp: u_n(K) \to U_n(K)$ by

$$exp\, a = \sum_{i=0}^{n-1} \frac{1}{i!} a^i \tag{1.1.4}$$

for all $a \in u_n(K)$ - we note that $a^n=0$ for all $a \in u_n(K)$. exp is bijective and its inverse is the *logarithm map* $ln: U_n(K) \to u_n(K)$ which is defined by

$$ln\, x = \sum_{i=0}^{n-1} \frac{(-1)^{i+1}}{i} (x-1)^i \tag{1.1.5}$$

for all $x \in U_n(K)$ - since $x-1 \in u_n(K)$ we have $ln\, x \in u_n(K)$ for all $x \in U_n(K)$. Moreover let $\theta: u_n(K) \times u_n(K) \to u_n(K)$ be the map defined by

$$\theta(a,b) = ln(exp\, a\ exp\, b)$$

for all $a,b \in u_n(K)$. Then

$$exp\, a\ exp\, b = exp(\theta(a,b))$$

for all $a,b \in u_n(K)$ and the following equality holds for all $a,b \in u_n(K)$

$$\theta(a,b) = \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i} \sum_{\substack{p_a, q_a = 0 \\ p_a + q_a > 0 \\ 1 \le a \le i}}^{n-1} \frac{[a^{p_1} b^{q_1} \dots a^{p_i} b^{q_i}]}{\left(\sum_{a=1}^{i} (p_a + q_a)\right) p_1! q_1! \dots p_i! q_i!} \tag{1.1.6}$$

where $[a_1 a_2 \dots a_n] = [a_1 [a_2 \dots [a_{n-1} a_n]] \dots]$ for all $a_1, a_2, \dots, a_n \in u_n(K)$. This equality is known as the *Campbell-Hausdorf formula* ([1]).

As a consequence of the Campbell-Hausdorf formula we deduce that the exponential image $exp\, u$ of any subalgebra u of $u_n(K)$ is a subgroup of $U_n(K)$. A subgroup of this form will be called an *exponential group* ([2]).

The "smallest" examples of exponential groups are obtained when we consider subalgebras of $u_n(K)$ of dimension one. Let a be an arbitrary non-zero element of $u_n(K)$. Then the subspace Ka of $u_n(K)$ is a subalgebra of $u_n(K)$ and we may consider the exponential group

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[1] For the proof of the Campbell-Hausdorf formula (as well as for the proofs of the properties of the maps exp and ln) we refer to Jacobson [Ja; pg. 170-174] where this formula is established in the ring of formal power series $K\{X,Y\}$ in two non-commutative variables over a field of characteristic zero (we calculate the sum of all homogeneous components of degree less than n of the formal power series $ln(expX\ expY)$ and then we specialise X to a and Y to b). The result for a field K of prime characteristic $p \ge n$ can be obtained starting with the polynomial ring $\mathbb{Q}\{X,Y\}$ over the field $\mathbb{Q}$, then reducing to the polynomial ring $\mathbb{Z}\{X,Y\}$ over the ring $\mathbb{Z}$ and finally tensoring (over $\mathbb{Z}$) with K. This process establishes the Campbell-Hausdorf formula in the polynomial ring $K\{X,Y\}$ over K. (1.1.6) is deduced specializing X to a and Y to b.

[2] A more general definition of exponential group can be found in Kirillov's book [Ki1].

$$exp(Ka)=\{exp(\alpha a);\ \alpha\in K\}.$$

It is easy to verify that

(1.1.7) $$exp(\alpha a)exp(\beta a)=exp((\alpha+\beta)a)$$

for all $\alpha,\beta\in K$.

In particular we obtain the root subgroups of $U_n(K)$. These are defined as follows. Throughout this work a pair (i,j) with $1\leq i<j\leq n$ will be called a *root* ([1]) and the set of all roots will be denoted by $\Phi(n)$ ([2]). Let $(i,j)\in\Phi(n)$. Then the (i,j)*-th root vector* e_{ij} of $\mathfrak{u}_n(K)$ is the matrix

(1.1.8) $$e_{ij}=(\delta_{ai}\delta_{bj})_{1\leq a,b\leq n}$$

(here δ_{ab}, $1\leq a,b\leq n$, is the usual Kronecker symbol). The (i,j)*-th root subalgebra* of $\mathfrak{u}_n(K)$ is the subspace

(1.1.9) $$\mathfrak{X}_{ij}(K)=Ke_{ij}$$

of $\mathfrak{u}_n(K)$. Finally the (i,j)*-th root subgroup* of $U_n(K)$ is the exponential group

(1.1.10) $$X_{ij}(K)=exp\ \mathfrak{X}_{ij}(K).$$

As usual for each $\alpha\in K$ the element $exp(\alpha e_{ij})$ of $X_{ij}(K)$ will be denoted by $x_{ij}(\alpha)$ and we have

(1.1.11) $$x_{ij}(\alpha)=1+\alpha e_{ij}.$$

It is obvious that

$$x_{ij}(\alpha)x_{ij}(\beta)=x_{ij}(\alpha+\beta)$$

for all $\alpha,\beta\in K$ (we note that this is precisely the equality (1.1.7) with $a=e_{ij}$). If K is the finite field $\mathbf{F}_q$ we will write $\mathfrak{X}_{ij}(q)$ and $X_{ij}(q)$ instead of $X_{ij}(K)$ and $\mathfrak{X}_{ij}(K)$ respectively.

Other examples of exponential groups are the subgroups of $U_n(K)$ associated with elements of the symmetric group S_n of degree n. In fact for any $\omega\in S_n$ we define the subset $\Phi_\omega(n)$ of $\Phi(n)$ by

$$\Phi_\omega(n)=\{(i,j)\in\Phi(n);\ \omega(i)<\omega(j)\}.$$

We denote by $\mathfrak{u}_\omega(K)$ the vector subspace of $\mathfrak{u}_n(K)$ generated by the set $\{e_{ij};\ (i,j)\in\Phi_\omega(n)\}$. Then

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[1] In the standard terminology our expression "root" means "positive root".

[2] This notation simplifies the standard notation $\Phi^+(n)$ for the set of positive roots.

$$u_\omega(K)=\sum_{(i,j)\in\Phi_\omega(n)} \mathfrak{X}_{ij}(K). \tag{1.1.12}$$

We have also

$$u_\omega(K)=u_n(K)\cap P(\omega^{-1})u_n(K)P(\omega) \tag{1.1.13}$$

where $P(\omega)=(\delta_{i\omega(j)})_{1\le i,j\le n}\in GL_n(K)$ ([1]) is the permutation matrix associated with $\omega\in S_n$. Usually we write $\omega^{-1}u_n(K)\omega$ instead of $P(\omega^{-1})u_n(K)P(\omega)$.

We denote by $U_\omega(K)$ the exponential group $exp\ u_\omega(K)$. If K is the finite field $\mathbb{F}_q$ we will write $U_\omega(q)$ and $u_\omega(q)$ instead of $U_\omega(K)$ and $u_\omega(K)$ respectively. In the general situation we have

$$U_\omega(K)=\prod_{(i,j)\in\Phi_\omega(n)} X_{ij}(K) \tag{1.1.14}$$

and also

$$U_\omega(K)=U_n(K)\cap P(\omega^{-1})U_n(K)P(\omega). \tag{1.1.15}$$

Usually we write $\omega^{-1}U_n(K)\omega$ instead of $P(\omega^{-1})U_n(K)P(\omega)$.

In particular if $\omega=1\in S_n$ then

$$u_\omega(K)=u_n(K) \quad \text{and} \quad U_\omega(K)=U_n(K).$$

On the other extreme let $\omega_0\in S_n$ be defined by

$$\omega_0=\begin{cases}(1n)(2\,n\text{-}1)\ldots(r\,r{+}1) & \text{if } n=2r \text{ is an even number}\\ (1n)(2\,n\text{-}1)\ldots(r\,r{+}2) & \text{if } n=2r+1 \text{ is an odd number}\end{cases} \tag{1.1.16}$$

Then

$$u_{\omega_0}(K)=\{0\} \quad \text{and} \quad U_{\omega_0}(K)=\{1\}.$$

Moreover for each $\omega\in S_n$ we have

$$\Phi_{\omega_0\omega}(n)=\{(i,j)\in\Phi(n);\ \omega(i)>\omega(j)\}$$

and so $\Phi(n)$ is the disjoint union

$$\Phi(n)=\Phi_\omega(n)\cup\Phi_{\omega_0\omega}(n).$$

Thus

$$u_n(K)=u_\omega(K)\oplus u_{\omega_0\omega}(K)$$

and

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[1] We denote by $GL_n(K)$ the general linear group of degree n over K. This group consists of all non-singular $n\times n$ matrices with coefficients in K.

$$U_n(K)=U_{\omega}(K)U_{\omega_0\omega}(K)$$

(although $U_{\omega}(K)\cap U_{\omega_0\omega}(K)=\{1\}$ in general this product is not semidirect).

Now we assume that the field K is algebraically closed. Then both $U_n(K)$ and $\mathcal{U}_n(K)$ are algebraic varieties (both isomorphic to the affine space of dimension $\frac{n(n-1)}{2}$). Moreover the exponential map $exp: \mathcal{U}_n(K) \to U_n(K)$ is a morphism of algebraic varieties. Since its inverse $ln: U_n(K) \to \mathcal{U}_n(K)$ is also a morphism exp is in fact an isomorphism. It follows that the exponential image of any subalgebra $\mathcal{U}$ of $\mathcal{U}_n(K)$ is a closed subgroup of $U_n(K)$, hence a unipotent (algebraic) group (see [Hu1]). On the other hand $exp\ \mathcal{U}$ is a connected group because $\mathcal{U}$ is an irreducible variety (see [Hu1; proposition 1.3.A]). Finally (by the usual derivation rules for the exponential map) the Lie algebra of $exp\ \mathcal{U}$ is $\mathcal{U}$ itself. Therefore any nilpotent Lie algebra determines a connected unipotent group. However it is not true that all connected unipotent algebraic groups are exponential groups. For example let K have characteristic $p\neq 2$ and let U be the subgroup of $U_3(K)$ consisting of all matrices $x=(x_{ij})\in U_3(K)$ satisfying

$$Q(x_{12},x_{13},x_{23})=(x_{12})^p-x_{23}=0.$$

A generic element of U has the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & x^p \\ 0 & 0 & 1 \end{pmatrix}$$

where $x,y\in K$. U is connected closed subgroup of $U_3(K)$ and its Lie algebra $\mathcal{U}$ consists of all matrices $a=(a_{ij})\in \mathcal{U}_3(K)$ satisfying

$$p(a_{12})^p-a_{23}=-a_{23}=0.$$

Therefore a generic element of $\mathcal{U}$ has the form

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $x,y\in K$. However the exponential group $exp\,\mathcal{U}$ determined by the Lie algebra U is

$$exp\ \mathcal{U}=\left\{\begin{pmatrix}1 & x & y\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix};\ x,y\in K\right\}.$$

As a final remark we note that the exponential map $exp:\mathcal{U}\rightarrow U$ (where $\mathcal{U}$ is a subalgebra of $\mathcal{U}_n(K)$ and $U=exp\ \mathcal{U}$) may sometimes be defined if K has prime characteristic $p<n$. In fact it is enough to assume that $p\geq m$ where m is the smallest integer such that $a^m=0$ for all $a\in\mathcal{U}$. However for the purposes of this work (where we are mainly interested in the case $U=U_n(K)$) we will always assume that $p\geq n$.

1.2. Coadjoint orbits and irreducible characters

Let K be any field and let $\mathcal{U}$ be a nilpotent Lie algebra. We assume that $\mathcal{U}$ is a subalgebra of $\mathcal{U}_n(K)$ for some n. For the results of this section we refer either to [CG] or to [Sr]. Although the results in [CG] are stated for nilpotent Lie groups they can be adapted to our situation.

Let

$$\mathcal{U}^* = Hom_K(\mathcal{U}, K)$$

be the dual vector space of $\mathcal{U}$. For each $f \in \mathcal{U}^*$ we define a K-bilinear form $B_f: \mathcal{U} \times \mathcal{U} \to K$ on $\mathcal{U}$ by

$$B_f(a,b) = f([ab]) \tag{1.2.1}$$

for all $a, b \in \mathcal{U}$. By the axioms of the Lie product B_f is a skew-symmetric bilinear form. Hence $\mathcal{U}$ has the structure of a symplectic space ([1]). Thus we may consider isotropic subspaces of $\mathcal{U}$. By definition a subspace V of $\mathcal{U}$ is called *isotropic* (with respect to B_f) if

$$B_f(a,b) = f([ab]) = 0$$

for all $a, b \in V$. To simplify the notation we will say that a subspace V of $\mathcal{U}$ is *f-isotropic* if it is isotropic with respect to B_f. By Witt's theorem (see [Ar; theorems 3.10 and 3.11]) all maximal f-isotropic subspaces of $\mathcal{U}$ have the same dimension

$$\frac{1}{2} dim\, \mathcal{U}/_{\mathfrak{r}(f)} + dim\, \mathfrak{r}(f) = \frac{1}{2}(dim\, \mathcal{U} + dim\, \mathfrak{r}(f))$$

where

$$\mathfrak{r}(f) = \{ a \in \mathcal{U};\ B_f(a,b) = 0 \text{ for all } b \in \mathcal{U} \} \tag{1.2.2}$$

is the *radical* of B_f.

Let $\mathfrak{v}$ be a subalgebra of $\mathcal{U}$. We say that $\mathfrak{v}$ is *subordinate to f* if it is an f-isotropic subspace of $\mathcal{U}$. If $\mathfrak{v}$ is a maximal f-isotropic subspace of $\mathcal{U}$ then $\mathfrak{v}$ is called a *polarization for f*.

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[1] For the basic notions of symplectic spaces we refer to Artin's book [Ar].

We now establish the existence of a polarization for any element of $\mathcal{U}^*$. The construction of this polarization is due to M. Vergne (see [Ve]).

Proposition 1.2.1. ([CG; theorem 1.3.5]) *Let $f\in\mathcal{U}^*$. Let $m=\dim\mathcal{U}$ and let $\mathcal{U}=\mathcal{U}_m\supseteq\ldots\supseteq\mathcal{U}_1\supseteq\mathcal{U}_0=(0)$ be a chain of ideals of $\mathcal{U}$ such that $\dim\mathcal{U}_i=i$ $(0\leq i\leq m)$. For each $i\in\{1,\ldots,m\}$ let*

$$\mathfrak{h}_i=\{a\in\mathcal{U}_i;\, B_f(a,b)=0 \text{ for all } b\in\mathcal{U}_i\}.$$

Then

$$\mathfrak{h}=\mathfrak{h}_1+\mathfrak{h}_2+\ldots+\mathfrak{h}_m$$

is a polarization $\mathfrak{h}$ for f.

Proof. Firstly we note that the given chain of ideals of $\mathcal{U}$ exists by Engel's theorem on nilpotent Lie algebras (see [Hu2; theorem 3.2]).

Let $i,j\in\{1,\ldots,m\}$ and let $a\in\mathfrak{h}_i$, $b\in\mathfrak{h}_j$ be arbitrary. Without loss of generality we may assume that $i\leq j$. Then $a\in\mathcal{U}_j$ and so

$$f([ab])=0$$

(by definition of $\mathfrak{h}_j$). It follows that $\mathfrak{h}$ is f-isotropic. To prove that $\mathfrak{h}$ is a subalgebra of $\mathcal{U}$ we claim that $[ab]\in\mathfrak{h}_i$. Since $\mathcal{U}_i$ is an ideal of $\mathcal{U}$ we have $[ab]\in\mathcal{U}_i$. Let $c\in\mathcal{U}_i$ be arbitrary. Then (by Jacobi's identity)

$$f([[ab]c])=f([[ac]b])+f([a[bc]]).$$

Since $[ac]\in\mathcal{U}_i\subseteq\mathcal{U}_j$ and $b\in\mathfrak{h}_j$ we have

$$f([[ac]b])=0.$$

On the other hand $a\in\mathfrak{h}_i$ and $[bc]\in\mathcal{U}_i$(because $\mathcal{U}_i$ is an ideal). Thus

$$f([a[bc]])=0.$$

It follows that

$$f([[ab]c])=0$$

and so $[ab]\in\mathfrak{h}_j$ (because $c\in\mathcal{U}_i$ is arbitrary).

Finally we consider the dimension of $\mathfrak{h}$. We claim that

$$\dim \mathfrak{h}=\frac{1}{2}(\dim \mathfrak{u}+\dim \mathfrak{r}(f)).$$

For we proceed by induction on m. If $m=1$ then $\mathfrak{h}=\mathfrak{u}=\mathfrak{r}(f)$ and the claim is trivial. Now suppose that the result is proved for all nilpotent Lie algebras of dimension less than m. We consider the ideal $\mathfrak{u}_{m-1}$ and we define

$$\mathfrak{h}'=\mathfrak{h}_1+\mathfrak{h}_2+\ldots+\mathfrak{h}_{m-1}.$$

By induction we have

$$\dim \mathfrak{h}'=\frac{1}{2}(\dim \mathfrak{u}_{m-1}+\dim \mathfrak{r}(f'))$$

where f' is the restriction of f to $\mathfrak{u}_{m-1}$.

Now let $(e_1,\ldots,e_m)$ be a K-basis of $\mathfrak{u}$ such that $(e_1,\ldots,e_{m-1})$ is a basis of $\mathfrak{u}_{m-1}$. Let $M=(f([e_i,e_j]))_{1\le i,j\le m}$ be the $m\times m$ matrix which represents the bilinear form B_f with respect to the basis $(e_1,\ldots,e_m)$. Then M is skew-symmetric, i.e.

$$M=-M^{\mathrm{T}} \quad ([1]).$$

Therefore M has even rank (see [Co; theorem 8.6.1]) and in fact

$$\operatorname{rank} M=\dim \mathfrak{u}-\dim \mathfrak{r}(f).$$

Consider the $(m-1)\times(m-1)$ submatrix $M'=(f([e_i,e_j]))_{1\le i,j\le m-1}$ of M. Then M' represents the bilinear form $B_{f'}:\mathfrak{u}_{m-1}\times\mathfrak{u}_{m-1}\to K$. Then

$$\operatorname{rank} M'=\dim \mathfrak{u}_{m-1}-\dim \mathfrak{r}(f').$$

Since $\operatorname{rank} M$ and $\operatorname{rank} M'$ are even, either $\operatorname{rank} M=\operatorname{rank} M'$ or $\operatorname{rank} M=\operatorname{rank} M'-1$. It follows that either $\dim \mathfrak{r}(f)=\dim \mathfrak{r}(f')+1$ or $\dim \mathfrak{r}(f)=\dim \mathfrak{r}(f')-1$ (because $\dim \mathfrak{u}_{m-1}=\dim \mathfrak{u}-1$).

Now suppose that $\dim \mathfrak{r}(f)=\dim \mathfrak{r}(f')-1$. Then

$$\dim \mathfrak{h}'=\frac{1}{2}(\dim \mathfrak{u}-1+\dim \mathfrak{r}(f)+1)=\frac{1}{2}(\dim \mathfrak{u}+\dim \mathfrak{r}(f)).$$

Thus $\mathfrak{h}'$ is a maximal f-isotropic subspace of $\mathfrak{u}$. Since $\mathfrak{h}'\subseteq\mathfrak{h}$ we conclude that $\mathfrak{h}'=\mathfrak{h}$ and our claim follows.

On the other hand suppose that $\dim \mathfrak{r}(f)=\dim \mathfrak{r}(f')+1$. Then

$$\dim \mathfrak{h}'=\frac{1}{2}(\dim \mathfrak{u}-1+\dim \mathfrak{r}(f)-1)=\frac{1}{2}(\dim \mathfrak{u}+\dim \mathfrak{r}(f))-1.$$

Since $\mathfrak{h}'\subseteq\mathfrak{h}$ we have $\dim \mathfrak{h}'\le\dim \mathfrak{h}$. If $\dim \mathfrak{h}'=\dim \mathfrak{h}$ then $\mathfrak{h}'=\mathfrak{h}$ and so $\mathfrak{r}(f)\subseteq\mathfrak{h}'\subseteq\mathfrak{u}_{m-1}$. In

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[1] If A is any matrix we denote by A^{T} the transpose of A.

this case we must have $\mathfrak{r}(f)\subseteq\mathfrak{r}(f')$ and so $dim\,\mathfrak{r}(f)\leq dim\,\mathfrak{r}(f')$, a contradiction. It follows that

$$dim\,\mathfrak{h}'<dim\,\mathfrak{h}.$$

Since $\mathfrak{h}$ is an f-isotropic subspace of $\mathfrak{u}$ we deduce that

$$dim\,\mathfrak{h}'\leq dim\,\mathfrak{h}-1\leq\frac{1}{2}(dim\,\mathfrak{u}+dim\,\mathfrak{r}(f))-1=dim\,\mathfrak{h}'.$$

Our claim follows and the proof of the proposition is complete. ♦

Subsequently we assume that $p\geq n$ if the field K has prime characteristic p. Then we may consider the exponential group $U=exp\,\mathfrak{u}$ ([1]).

Let $ad:\mathfrak{u}\rightarrow\mathfrak{gl}(\mathfrak{u})$ be the *adjoint representation* of the Lie algebra $\mathfrak{u}$ (as usual $\mathfrak{gl}(\mathfrak{u})$ is the general linear algebra consisting of all endomorphisms of $\mathfrak{u}$ (as a vector space over K)). This representation is defined by

$$ad\,a(b)=[ab]$$

for all $a,b\in\mathfrak{u}$. Let $a\in\mathfrak{u}$ be arbitrary. Since $(ad\,a)^n=0$ the element

$$exp(ad\,a)=1+ad\,a+\frac{1}{2!}(ad\,a)^2+\ldots+\frac{1}{(n-1)!}(ad\,a)^{n-1} \tag{1.2.3}$$

is a well-defined element of $\mathfrak{gl}(\mathfrak{u})$ and in fact it is an element of $GL(\mathfrak{u})$ (here $GL(\mathfrak{u})$ is the general linear group consisting of all non-singular endomorphisms of $\mathfrak{u}$). Moreover we have

$$exp(ad\,a)(b)=(exp\,a)^{-1}b(exp\,a) \tag{1.2.4}$$

for all $b\in\mathfrak{u}$. Therefore the map $Ad:U\rightarrow GL(\mathfrak{u})$ defined by

$$Ad(exp\,a)=exp(ad\,a)$$

for all $a\in\mathfrak{u}$ is a representation of U over $\mathfrak{u}$. Ad is called the *adjoint representation* of U. Since U is a subgroup of $U_n(K)$ we have

$$Ad\,x(b)=x^{-1}bx \tag{1.2.5}$$

for all $x\in U$ and all $b\in\mathfrak{u}$.

Now we consider the contragradient representation of Ad. This is (by definition) the

[1] We note that U is a subgroup of $U_n(K)$ because $\mathfrak{u}$ is a subalgebra of $\mathfrak{u}_n(K)$.

homomorphism $Ad^*:U \to GL(\mathcal{U}^*)$ defined by

(1.2.6) $$(Ad^*x(f))(b)=f(Adx^{-1}(b))=f(xbx^{-1})$$

for all $x \in U$, all $f \in \mathcal{U}^*$ and all $b \in \mathcal{U}$. Ad^* is called the *coadjoint representation* of U. For simplicity we will write $x \cdot f$ instead of $Ad^*x(f)$ for all $x \in U_n(K)$ and all $f \in \mathcal{U}_n(K)^*$.

The *coadjoint U-orbits* (i.e. the orbits of the coadjoint representation) will be fundamental for our work. Let $f \in \mathcal{U}^*$. Then the U-orbit of f will be denoted by $O(f)$. By definition

$$O(f)=\{x \cdot f \in \mathcal{U}^*; x \in U\}.$$

If K is algebraically closed then $O(f)$ is a locally closed subset of the algebraic variety $\mathcal{U}^*$ (see [Hu1; proposition 8.3]). Moreover since U is a unipotent algebraic group we conclude that $O(f)$ is a closed subset of $\mathcal{U}^*$ (see [St; proposition 2.5]). Since U is connected (because it is the image of a morphism of irreducible algebraic varieties) and since it acts transitively on $O(f)$ we conclude that $O(f)$ is an irreducible algebraic variety (see [Hu1; proposition 8.2(d)]). Hence we may consider the dimension of $O(f)$. It is well-known that

$$dimO(f)=dimU-dimC_U(f)$$

where

$$C_U(f)=\{x \in U; x \cdot f=f\}$$

is the centralizer of f in U - the equality above follows easily from [Hu1; theorem 4.3] applied to the morphism $\vartheta:U \to O(f)$, $\vartheta(x)=x \cdot f$ $(x \in U)$, because all the fibres of this morphism have the same dimension. However $dimO(f)$ can be expressed in terms of the bilinear form B_f. In fact:-

Proposition 1.2.2. ([CG; lemma 1.3.1]) *Let $f \in \mathcal{U}^*$. Then*

$$C_U(f)=exp\,\mathfrak{r}(f).$$

Hence $C_U(f)$ is an exponential subgroup of U. In particular $C_U(f)$ is connected.

Proof. Let $a,b\in U$ be arbitrary. Since $Ad(exp\,a)=exp(ad\,a)$ we have

$$f(Ad(exp\,a)(b))=\sum_{i=0}^{n-1}\frac{f((ad\,a)^i(b))}{i!}$$

where $(ad\,a)^0(b)=b$ and $(ad\,a)^i(b)=(ad\,a)^{i-1}([ab])$ for all $i\in\{1,\dots,n-1\}$.

Now suppose that $a\in \tau(f)$. Then

$$f((ad\,a)^i(b))=f([a,(ad\,a)^{i-1}(b)])=0$$

for all $i\in\{1,\dots,n-1\}$. It follows that

$$f(Ad(exp\,a)(b))=f(b).$$

Since $b\in U$ is arbitrary we conclude that $exp\,a\in C_U(f)$. This implies that

$$exp\,\tau(f)\subseteq C_U(f).$$

Conversely let $x\in C_U(f)$ be arbitrary and let $a\in U$ be such that $x=exp\,a$. Let t be an indeterminate over K and let $b\in U$ be arbitrary. We consider the polynomial

$$P(t)=(exp(ta))\cdot f)(b)=\sum_{i=0}^{n-1}\frac{f((ad\,a)^i(b))}{i!}t^i\in K[t].$$

Since $C_U(f)$ is a subgroup of U we have

$$x^k=(exp\,a)^k\in C_U(f).$$

Since $(exp\,a)^k=exp(ka)$ we deduce that

$$(exp(ka))\cdot f)(b)=f(b)$$

for all $k\in\{0,1,2,3,\dots\}$. Therefore the elements $0,1,2,\dots$ of K are roots of the polynomial $P(t)-f(b)\in K[t]$. If K has characteristic zero this implies immediately that

$$P(t)=f(b).$$

On the other hand suppose that K has characteristic p. Then the integers $0,1,2,\dots,p-1$ define p distinct elements of K. Thus $P(t)-f(b)$ has at least p distinct roots. Since $P(t)-f(b)$ has degree n and $p\geq n$ (by assumption) we also conclude that

$$P(t)=f(b).$$

In particular we obtain

$$f([ab])=0$$

- we note that f($[ab]$) is the coefficient of t in the polynomial $P(t)-f(b)$. Since $b\in U_0$ is arbitrary it follows that $a\in\tau(f)$ and the proof is complete. ◆

Corollary 1.2.3. ([CG; lemma 1.3.2]) *Let $f \in \mathfrak{u}^*$. Then the U-orbit $O(f)$ of f has even dimension. Moreover if $\mathfrak{h} \subseteq \mathfrak{u}$ is a polarization for f then*

$$\dim \mathfrak{h} = \dim \mathfrak{u} - \frac{1}{2} \dim O(f).$$

Proof. The first assertion is clear because $\dim \mathfrak{u} - \dim \mathfrak{r}(f)$ is even (see [Ar; theorem 3.11]). Now let $\mathfrak{h}$ be any polarization for f. Then

$$\dim \mathfrak{h} = \frac{1}{2}(\dim \mathfrak{u} + \dim \mathfrak{r}(f)) = \frac{1}{2}(\dim \mathfrak{u} + \dim \mathfrak{u} - \dim O(f)) = \dim \mathfrak{u} - \frac{1}{2} \dim O(f)$$

as required. ♦

Henceforth we assume that K is algebraically closed of characteristic $p \geq n$. Let $q = p^e$ ($e > 0$) and let F be the Frobenius map on $U_n(K)$ as defined in (1.1.1). We assume that the exponential group $U = \exp \mathfrak{u}$ is F-stable, i.e. $F(U) = U$. As usual we denote by U^F the subgroup of U consisting of all F-fixed elements in U. Then U^F is a finite group of order $q^{\dim U}$ (see below).

Our aim is to construct the irreducible characters of the finite group U^F. These will depend on certain coadjoint U-orbits. As one should expect the relevant orbits must be F-stable for a suitable action of F. Moreover the set of F-fixed elements of a given orbit should be itself an orbit under the action of U^F.(1) In the following we will define the "Frobenius" map $F: \mathfrak{u}^* \to \mathfrak{u}^*$. We recall that a Frobenius map F can be defined on the Lie algebra $\mathfrak{u}_n(K)$ so that

$$\exp F(a) = F(\exp a)$$

for all $a \in \mathfrak{u}_n(K)$ (see (1.1.2)). It follows that $U = \exp \mathfrak{u}$ is F-stable if and only if $\mathfrak{u}$ is F-stable. Therefore we may consider the subalgebra $\mathfrak{u}^F$ of $\mathfrak{u}$ consisting of all F-fixed elements of $\mathfrak{u}$ - then U^F is the exponential subgroup $\exp \mathfrak{u}^F$ of $U_n(q) = U_n(K)^F$. The subalgebra $\mathfrak{u}^F$ is finite and it has cardinality $q^{\dim \mathfrak{u}}$ (hence the finite group U^F has cardinality $q^{\dim U}$). In fact $\mathfrak{u}^F$ is a vector space over the finite field $\mathbb{F}_q$ and we have a

(1) We note that an F-stable U-orbit contains at least an F-fixed element (see [Bo; corollary 16.5]).

canonical isomorphism

$$\mathcal{U}\cong\mathcal{U}^F\otimes_{\mathbb{F}_q}K.$$

In particular $\mathcal{U}$ has a K-basis consisting of F-stable elements. Let $(e_1,\dots,e_m)$ be such a basis (hence $m=\dim\mathcal{U}$). Then we define the *Frobenius map* $F^*=F_q^*:\mathcal{U}^*\to\mathcal{U}^*$ by

(1.2.7) $$F^*(f)(e_i)=f(e_i)^q$$

for all $f\in\mathcal{U}^*$ and all $i\in\{1,\dots,m\}$. If $a=\alpha_1e_1+\dots+\alpha_me_m\in\mathcal{U}$ ($\alpha_i\in K$, $1\le i\le m$) then

$$F(a)=\alpha_1^qe_1+\dots+\alpha_m^qe_m$$

and $F^*(f)(F(a))=\alpha_1^qf(e_1)^q+\dots+\alpha_m^qf(e_m)^q$, therefore

(1.2.8) $$F^*(f)(F(a))=f(a)^q$$

for all $a\in\mathcal{U}^*$. In particular

$$F^*(f)(a)=f(a)^q$$

for all $a\in\mathcal{U}^F$. It follows that the map F^* is independent of the choice of the F-fixed basis for $\mathcal{U}$. Moreover the set

$$(\mathcal{U}^*)^{F^*}=\{f\in\mathcal{U}^*;F^*(f)=f\}$$

is finite (of cardinality $q^{\dim\mathcal{U}^*}=q^{\dim\mathcal{U}}$) and we have

$$(\mathcal{U}^F)^*=(\mathcal{U}^*)^{F^*}$$

- in fact $\mathcal{U}^*$ can be regarded as the tensor product $(\mathcal{U}^F)^*\otimes_{\mathbb{F}_q}K$. Since there is no ambiguity we will write F instead of F^*.

We now prove the following:

Proposition 1.2.4. ([Ka; lemma 1]) *Let $f\in(\mathcal{U}^*)^F$. Then $\mathcal{U}$ contains an F-stable polarization $\mathfrak{h}$ for f.* -

Proof. Let $\mathcal{U}=\mathcal{U}_m\supseteq\dots\supseteq\mathcal{U}_1\supseteq\mathcal{U}_0=(0)$ be a chain of F-stable ideals of $\mathcal{U}$ such that $\dim\mathcal{U}_i=\dim\mathcal{U}_{i-1}+1$ ($1\le i\le m$) and let $\mathfrak{h}=\mathfrak{h}_1+\mathfrak{h}_2+\dots+\mathfrak{h}_m$ be the polarization for f defined as in proposition 1.2.1. We claim that $\mathfrak{h}$ is F-stable. For it is enough to show that $\mathfrak{h}_i$ is F-stable for all $i\in\{1,\dots,m\}$. Let $i\in\{1,\dots,m\}$ and let $a\in\mathfrak{h}_i$. Then

$$f([ab])=0$$

for all $b\in \mathfrak{u}_i$. Let $b\in \mathfrak{u}_i$ be F-fixed. Then

$$[F(a),b]=F([ab])$$

and so

$$f([F(a),b])=f(F([ab]))=f([ab])^q=0$$

(by (1.2.8) because f is F-fixed). Since $\mathfrak{u}_i$ contains a basis consisting of F-fixed elements we conclude that $F(a)\in \mathfrak{h}_i$ and this implies that $\mathfrak{h}_i$ is F-stable. ♦

Next we construct an irreducible character of U^F associated with a given element $f\in(\mathfrak{u}^*)^F$. We start with an F-stable polarization $\mathfrak{h}\subseteq\mathfrak{u}$ for f and we let $H=exp\ \mathfrak{h}$ be the exponential subgroup of U defined by $\mathfrak{h}$. Then H is an F-stable connected closed subgroup of U.

We denote by K^+ the additive group of the field K and we define the function $\phi_f:H\to K^+$ by

(1.2.9) $$\phi_f(exp\ a)=f(a)$$

for all $a\in\mathfrak{h}$. Then

$$\phi_f(exp\ a\ exp\ b)=\phi_f(exp\ \theta(a,b))=f(\theta(a,b))$$

for all $a,b\in\mathfrak{h}$ - here $\theta(a,b)\in\mathfrak{h}$ is the element defined by the Campbell-Hausdorf formula (1.1.6). Since $f([\mathfrak{h}\mathfrak{h}])=0$ this formula shows that

$$f(\theta(a,b))=f(a+b)$$

for all $a,b\in\mathfrak{h}$. Thus ϕ_f is an homomorphism from H into K^+ (of course $\phi_f(1)=0$).

Now we choose (and fix) an arbitrary non-trivial linear character $\psi_0:K^+\to\mathbb{C}^*$ of K^+ and we define the function $\psi_f:H\to\mathbb{C}^*$ by

$$\psi_f(exp\ a)=\psi_0(f(a))$$

for all $a\in\mathfrak{h}$. Since $\psi_f=\psi_0\phi_f$ the function ψ_f is a linear character of H. Therefore the restriction $(\psi_f)_{H^F}$ of ψ_f to the (finite) subgroup H^F of H is a linear character of H^F. We denote this character by λ_f. Then $\lambda_f:H^F\to\mathbb{C}^*$ is the linear character of H^F defined by

(1.2.10) $$\lambda_f(exp\ a)=\psi_0(f(a))$$

for all $a \in \mathfrak{h}^F$ [1].

Finally we define $\chi_f: U^F \to \mathbb{C}^*$ to be the induced character

(1.2.11) $$\chi_f = (\lambda_f)^{U^F}.$$

We will prove several properties of the characters χ_f ($f \in (\mathfrak{u}^*)^F$) namely that they are irreducible, independent of the choice of the U-conjugate of f and that they exhaust the set of irreducible characters of U^F (hence we will get a parametrization of the irreducible characters of U^F by means of F-stable coadjoint U-orbits on $\mathfrak{u}^*$). Firstly for any F-stable U-orbit $O \subseteq \mathfrak{u}^*$ we define the function $\chi_O: U^F \to \mathbb{C}^*$ by

(1.2.12) $$\chi_O(\exp a) = \frac{1}{\sqrt{|O^F|}} \sum_{g \in O^F} \psi_0(g(a))$$

for all $a \in \mathfrak{u}^F$. Since $1 = \exp 0$ we have

$$\chi_O(1) = \frac{1}{\sqrt{|O^F|}} \cdot |O^F| = \sqrt{|O^F|}$$

We have the following rule:

Proposition 1.2.5. ([Ka; propositions 1 and 2], [Sr; theorem 7.7]) *Let $f \in (\mathfrak{u}^*)^F$. Then the U-orbit $O(f)$ is F-stable and*

$$\chi_f(x) = \chi_{O(f)}(x)$$

for all $x \in U^F$. In particular we have

(i) $\chi_f(1) = \chi_{O(f)}(1) = \sqrt{|O^F|}$;

(ii) $\chi_g = \chi_f$ *for all* $g \in O(f)^F$;

(iii) χ_f *is independent of the polarization* $\mathfrak{h} \subseteq \mathfrak{u}$ *for f.*

Proof. Let $x \in U$ and let $(e_1, \ldots, e_m)$ be a K-basis of $\mathfrak{u}$ consisting of F-fixed elements. Then for each $i \in \{1, \ldots, m\}$

$$F(x \cdot f)(e_i) = (x \cdot f)(e_i)^q = f(xe_ix^{-1})^q = F(f)(F(xe_ix^{-1})) = f(F(x)e_iF(x)^{-1}) = (F(x) \cdot f)(e_i)$$

- we note that (by (1.2.8)) $F(f)(F(a)) = f(a)^q$ for all $a \in \mathfrak{u}$. It follows that

[1] We note that $f(a) \in \mathbb{F}_q$ for all $a \in \mathfrak{h}^F$ and that the restriction of ψ_0 to $\mathbb{F}_q^+$ is a linear character of $\mathbb{F}_q^+$.

(1.2.13) $$F(x\cdot f)=F(x)\cdot f$$

for all $x\in U$. This implies that $O(f)$ is F-stable. Thus it makes sense to consider the set $O(f)^F$ of all F-fixed elements of $O(f)$. This set is finite and its cardinality is $q^{\dim O(f)}$.

Now let $\mathfrak{h}\subseteq\mathfrak{u}$ be a polarization for f and and let $H=\exp\mathfrak{h}$. We define the function $\nu_f:\mathfrak{u}^F\to\mathbb{C}^*$ by

$$\nu_f(a)=\begin{cases}\psi_0(f(a))=\lambda_f(\exp a) & \text{if } a\in\mathfrak{h}^F\\ 0 & \text{otherwise}\end{cases}$$

Then (by definition of induced characters)

(1.2.14) $$\chi_f(\exp a)=\frac{1}{|H^F|}\sum_{x\in U^F}\nu_f(xax^{-1})$$

for all $a\in\mathfrak{u}^F$ - we note that $x(\exp a)x^{-1}=\exp(xax^{-1})$ for all $x\in U^F$ and all $a\in\mathfrak{u}^F$.

Let l be the affine subspace of $\mathfrak{u}^*$ consisting of all $g\in\mathfrak{u}^*$ such that $g(a)=f(a)$ for all $a\in\mathfrak{h}$. Then

$$l=f+\mathfrak{h}^{\perp}$$

where

$$\mathfrak{h}^{\perp}=\{h\in\mathfrak{u}^*;\ h(a)=0 \text{ for all } a\in\mathfrak{h}\}$$

is the annihilator of $\mathfrak{h}$ in $\mathfrak{u}^*$. We claim that

(1.2.15) $$\nu_f(a)=\frac{1}{|l^F|}\sum_{g\in l^F}\psi_0(g(a))$$

for all $a\in\mathfrak{u}^F$. If $a\in\mathfrak{h}^F$ then $g(a)=f(a)$ for all $g\in l^F$ and the equality is clear in this case. On the other hand suppose that $a\notin\mathfrak{h}^F$ and let $(e_1,\dots,e_s)$ be a K-basis of $\mathfrak{h}$ consisting of F-fixed elements. Then the system of vectors $(e_1,\dots,e_s,a)$ is linearly independent and we can extend it to a basis $(e_1,\dots,e_s,a,f_1,\dots f_t)$ of $\mathfrak{u}$ where $f_1,\dots f_t\in\mathfrak{u}$ are F-fixed elements (since $\dim\mathfrak{u}=m$ we have $t=m$-s-1). Let $(e_1^*,\dots,e_s^*,a^*,f_1^*,\dots,f_t^*)$ be the dual basis of $\mathfrak{u}^*$ ([1]). Then $(a^*,f_1^*,\dots,f_t^*)$ is a basis of $(\mathfrak{h}^{\perp})^F$ and any element $g\in l^F$ can be written as a sum

$$g=f+\alpha a^*+\beta_1 f_1^*+\dots+\beta_t f_t^*$$

where $\alpha,\beta_1,\dots,\beta_t\in\mathbb{F}_q$ are uniquely determined. Therefore l^F is the disjoint union

[1] If $(u_1,\dots,u_r)$ is a K-basis of $\mathfrak{u}$ then, for each $i\in\{1,\dots,r\}$, the dual vector $u_i^*\in\mathfrak{u}^*$ is defined by $u_i^*(u_j)=\delta_{ij}$ for all $j\in\{1,\dots,r\}$. The system $(u_1^*,\dots,u_r^*)$ is called the dual basis of $(u_1,\dots,u_r)$.

$$l^F = \bigcup_{\beta_1,\dots,\beta_t \in F_q} X(\beta_1,\dots,\beta_t)$$

where

$$X(\beta_1,\dots,\beta_t) = \{f+\alpha a^* + \beta_1 f_1^* + \dots + \beta_t f_t^*;\ \alpha \in F_q\}.$$

for all $\beta_1,\dots,\beta_t \in F_q$. It follows that

$$\sum_{g \in l^F} \psi_0(g(a)) = \sum_{\beta_1,\dots,\beta_t \in F_q} \sum_{g \in X(\beta_1,\dots,\beta_t)} \psi_0(g(a)).$$

Now let $\beta_1,\dots,\beta_t \in F_q$ be arbitrary. Then

$$(f+\alpha a^* + \beta_1 f_1^* + \dots + \beta_t f_t^*)(a) = f(a) + \alpha$$

for all $\alpha \in F_q$. Moreover the correspondence $\alpha \to f(a)+\alpha$ defines a bijective map from F_q onto itself. Therefore

$$\sum_{g \in X(\beta_1,\dots,\beta_t)} \psi_0(g(a)) = \sum_{\gamma \in F_q} \psi_0(\gamma).$$

Since ψ_0 is a non-trivial character of F_q we have

$$0 = (\psi_0, 1_{F_q})_{F_q} = \frac{1}{|F_q|} \sum_{\gamma \in F_q} \psi_0(\gamma)$$

where 1_{F_q} is the unit character of F_q and $(.,.)$ is the Frobenius scalar product of characters. It follows that

$$\sum_{g \in l^F} \psi_0(g(a)) = 0$$

and this completes the proof of the equality (1.2.15).

Now by (1.2.14) and (1.2.15) we obtain

$$\chi_f(\exp a) = \frac{1}{|H^F|} \frac{1}{|l^F|} \sum_{x \in U^F} \sum_{g \in l^F} \psi_0(g(xax^{-1})).$$

In other words

$$\chi_f(\exp a) = \frac{1}{|H^F|} \frac{1}{|l^F|} \sum_{x \in U^F} \sum_{g \in l^F} \psi_0((x \cdot g)(a)). \tag{1.2.16}$$

Next we will prove that H^F acts transitively on l^F. We start by proving that H acts transitively on l. For let $a,b \in \mathfrak{h}$ and let $g \in l$. Then

$$g((\exp a)^{-1} b (\exp a)) = g(\exp(ad\, a)(b)) = \sum_{i=0}^{n-1} \frac{g((ad\, a)^i(b))}{i!}.$$

Since $(ad\, a)^i(b) \in \mathfrak{h}$ we have

$$g((ad\,a)^i(b))=f((ad\,a)^i(b))$$

for all $i\in\{0,1,\ldots,n-1\}$. On the other hand

$$f((ad\,a)^i(b))=f([a,(ad\,a)^{i-1}(b)])=0$$

for any $i\in\{1,\ldots,n-1\}$ - we recall that $\mathfrak{h}$ is a maximal f-isotropic subspace of $\mathfrak{u}$. Therefore

$$g((exp\,a)^{-1}b(exp\,a))=f(b).$$

Since $H=exp\,\mathfrak{h}$ we conclude that $\mathfrak{l}$ is H-invariant. Now we define $\mu{:}H\to\mathfrak{l}$ by

$$\mu(x)=x\cdot f$$

for all $x\in H$. μ is a morphism of algebraic varieties and it induces an injective morphism $\mu'{:}H/H_0\to\mathfrak{l}$ where H_0 is the centralizer $C_U(f)$ of f - we note that H_0 is a subgroup of H because $H_0=exp\,\mathfrak{r}(f)$ (by proposition 1.2.2) and $\mathfrak{r}(f)\subseteq\mathfrak{h}$ (we recall that $\mathfrak{h}$ is a maximal f-isotropic subspace). Now we have

$$dim\,H/H_0=dimH-dimH_0=dim\,\mathfrak{u}-\frac{1}{2}dimO(f)-dim\,\mathfrak{u}+dimO(f)$$
$$=\frac{1}{2}dimO(f)=dim\,\mathfrak{u}-dim\,\mathfrak{h}=dim\,\mathfrak{h}^{\perp}=dim\,\mathfrak{l}$$

(using corollary 1.2.3). Therefore μ is surjective and this implies that H acts transitively on $\mathfrak{l}$.

Now we consider the H^F-orbits of elements of $\mathfrak{l}^F$. Let $x\in H$ be such that $x\cdot f\in\mathfrak{l}^F$. Then

$$x\cdot f=F(x\cdot f)=F(x)\cdot f$$

(by (1.2.13)) so $x^{-1}F(x)\in H_0$. Let O' be the H^F-orbit of $x\cdot f$ and let $y\in H$ be such that $y\cdot f\in\mathfrak{l}^F\cap O'$. Then there exists $z\in H^F$ such that

$$y\cdot f=z\cdot(x\cdot f)=(zx)\cdot f.$$

We have $y^{-1}zx\in H_0$ and

$$y^{-1}zx(x^{-1}F(x))F(x^{-1}z^{-1}y)=y^{-1}zF(z^{-1})F(y)=y^{-1}F(y)$$

(because $z^{-1}\in H^F$, hence $F(z^{-1})=z^{-1}$). This means that the elements $x^{-1}F(x)$ and $y^{-1}F(y)$ of H_0 are F-conjugate ([1]). Therefore we have a bijective map between H^F-orbits on $\mathfrak{l}^F$ and F-conjugacy classes of H_0. But F-conjugacy classes of H_0 are in one-to-one correspondence with F-conjugacy classes of the quotient group $H_0/(H_0)^\circ$ where $(H_0)^\circ$ is

[1] For the definition of F-conjugacy we refer to [Sr] or to [Ca].

the connected component of H_0 (see [Sr; lemma 2.5]). Since H_0 is connected we conclude that there exists a unique F-conjugacy class in H_0. Hence there exists a unique H^F-orbit on l^F, i.e. H^F acts transitively on l^F. It follows that

$$\sum_{g\in l^F}\psi_0((x\cdot g)(a))=\frac{1}{|H_0{}^F|}\sum_{y\in H^F}\psi_0((xy)\cdot f)(a))$$

for all $x\in U^F$ (we note that $H_0{}^F$ is a subgroup of H^F). Therefore (by (1.2.16))

$$\chi_f(\exp a)=\frac{1}{|H^F|}\,\frac{1}{|l^F|}\,\frac{|H^F|}{|H_0{}^F|}\sum_{x\in U^F}\psi_0((x\cdot f)(a)).$$

Since $H_0{}^F$ is the centralizer of f in U^F we conclude that

$$\chi_f(\exp a)=\frac{1}{|l^F|}\sum_{g\in O(f)^F}\psi_0(g(a)).$$

The result follows because $\dim l=\frac{1}{2}\dim O(f)$ and

$$|l^F|=q^{\dim l}=\sqrt{q^{\dim O(f)}}=\sqrt{|O(f)^F|}. \qquad \blacklozenge$$

Now we can prove the main result of this section:

Theorem 1.2.6. ([Ka; proposition 2]; [Sr; theorem 7.7]) *Let χ be an arbitrary (complex) character of U^F. Then χ is irreducible if and only if there exists an F-stable U-orbit $O\subseteq\mathfrak{u}^*$ such that $\chi=\chi_O$. Moreover if $O,O'\subseteq\mathfrak{u}^*$ are F-stable U-orbits then $\chi_O=\chi_{O'}$ if and only if $O=O'$.*

Proof. Let $O\subseteq\mathfrak{u}^*$ be any F-stable U-orbit. We claim that χ_O is irreducible. For it is enough to prove that $(\chi_O,\chi_O)=1$ where $(.,.)$ is the Frobenius inner product of characters. In fact (by proposition 1.2.5)

$$(\chi_O,\chi_O)=\frac{1}{|O^F|}\,\frac{1}{|\mathfrak{u}^F|}\sum_{a\in\mathfrak{u}^F}\Big|\sum_{g\in O^F}\psi_0(g(a))\Big|^2$$

$$\leq\frac{1}{|O^F|}\,\frac{1}{|\mathfrak{u}^F|}\sum_{a\in\mathfrak{u}^F}\sum_{g\in O^F}|\psi_0(g(a))|^2$$

$$=\frac{1}{|O^F|}\sum_{g\in O^F}\left(\frac{1}{|\mathfrak{u}^F|}\sum_{a\in\mathfrak{u}^F}|\psi_0(g(a))|^2\right)$$

$$=\frac{1}{|O^F|}\sum_{g\in O^F}(\psi_0 g,\psi_0 g)_{U^F}.$$

Since $\psi_0 g: U^F \to \mathbb{C}^*$ is a linear character of the finite abelian group U^F (under addition) we have

$$(\psi_0 g,\psi_0 g)_{U^F}=1.$$

The claim follows.

Now suppose that $\chi_O=\chi_{O'}$ where $O'\subseteq U^*$ is an F-stable U-orbit. Since $(exp\, a)^{-1}=exp(-a)$ for all $a\in U$, we have

$$(\chi_O,\chi_{O'})=\frac{1}{|U^F|}\sum_{a\in U^F}\chi_O(exp\, a)\,\chi_{O'}(exp(-a))$$

$$=\frac{1}{\sqrt{|O^F|}}\frac{1}{\sqrt{|O'^F|}}\sum_{f\in O^F}\sum_{f'\in O'^F}\left(\frac{1}{|U^F|}\sum_{a\in U^F}\psi_0(f(a))\psi_0(f'(-a))\right)$$

$$=\frac{1}{\sqrt{|O^F|}}\frac{1}{\sqrt{|O'^F|}}\sum_{g\in O^F}\sum_{g'\in O'^F}(\psi_0 g,\psi_0 g')_{U^F}.$$

If $g\in O$ and $g'\in O'$ are distinct the characters $\psi_0 g$ and $\psi_0 g'$ (of the additive group U^F) are orthogonal, i.e.

$$(\psi_0 g,\psi_0 g')_{U^F}=0$$

- in fact since ψ_0 is a non-trivial character of $\mathbb{F}_q^+$ there exists $a\in U^F$ such that $g(a)\neq g'(a)$ and $\psi_0(g(a))\neq\psi_0(g'(a))$. It follows that

$$(\psi_0 g,\psi_0 g')_{U^F}=\delta_{gg'}.$$

Since $\chi_O=\chi_{O'}$ we have $(\chi_O,\chi_{O'})=1$. Therefore the intersection $O\cap O'$ is non-empty, hence $O=O'$.

Finally by proposition 1.2.5 we have $\chi_O(1)=\sqrt{|O^F|}$ and this implies that

$$\sum_O \chi_O(1)^2=\sum_O |O^F|=|U^F|$$

where the sum is over all F-stable U-orbits $O\subseteq U^*$. The proof is complete. ♦

1.3. Subgroups of codimension one

In this section we keep the notation of section 1.2. K will denote an algebraically closed field and $\mathcal{U}$ a nilpotent Lie algebra (regarded as a subalgebra of $\mathcal{U}_n(K)$ for some n). Moreover if K has prime characteristic p we will assume that $p \geq n$. Then U will denote the exponential subgroup $exp\,\mathcal{U}$ (U is a subgroup of $U_n(K)$). In this section we refer to [CG] or to [Ki2]. The results in both references are proved for nilpotent Lie groups but they can be adapted to the context of our thesis.

Let $\mathcal{U}_0$ be a subalgebra of $\mathcal{U}$ with codimension 1, i.e.

$$dim\,\mathcal{U}_0 = dim\,\mathcal{U} - 1.$$

Then $\mathcal{U}_0$ is an ideal of $\mathcal{U}$ and $[\mathcal{U}\mathcal{U}] \subseteq \mathcal{U}_0$ (see [CG; lemma 1.1.8] - the proof of this lemma does not depend on the characteristic of the field).

Let $\pi: \mathcal{U}^* \rightarrow \mathcal{U}_0{}^*$ be the natural projection. By definition

$$\pi(f)(a) = f(a)$$

for all $f \in \mathcal{U}^*$ and all $a \in \mathcal{U}$. The kernel of π is the subspace

$$\mathcal{U}_0{}^{\perp} = \{g \in \mathcal{U}^*;\ g(a) = 0 \text{ for all } a \in \mathcal{U}_0\}$$

of $\mathcal{U}$. On the other hand for any $f \in \mathcal{U}^*$ the fibre $\pi^{-1}(\pi(f))$ of the $\pi(f) \in \mathcal{U}_0{}^*$ is the subset

$$l(f) = \{g \in \mathcal{U}^*;\ g(a) = f(a) \text{ for all } a \in \mathcal{U}_0\}$$

of $\mathcal{U}^*$. It is clear that

$$l(f) = f + \mathcal{U}_0{}^{\perp} = \{f + h;\ h \in \mathcal{U}_0{}^{\perp}\}$$

for all $f \in \mathcal{U}^*$. Moreover $l(f)$ ($f \in \mathcal{U}^*$) is an irreducible algebraic variety of dimension one (because $\mathcal{U}_0$ has codimension one, hence $dim\,\mathcal{U}_0{}^{\perp} = 1$).

In the following we will fix the element $f \in \mathcal{U}^*$ and we will denote by f_0 the image $\pi(f) \in \mathcal{U}_0{}^*$. We consider the intersection $l(f) \cap O(f)$ of the fibre $l(f)$ with the coadjoint U-orbit $O(f)$. Since $f \in l(f)$ it is obvious that

$$l(f) \cap O(f) \neq \emptyset.$$

Let

$$U'=\{x\in U; x\cdot f\in l(f)\}\cdot$$

Then

$$l(f)\cap O(f)=\{x\cdot f; x\in U'\}$$

and we have:

Lemma 1.3.1. ([Ki2; pg. 76-77]) *$U'=C_U(f_0)$ is a closed connected subgroup of U. In fact $U'=exp\ \mathcal{U}'$ where*

$$\mathcal{U}'=\{a\in \mathcal{U}; f([ab])=0 \text{ for all } b\in \mathcal{U}_0\}.$$

Proof. Since $\mathcal{U}_0$ is an ideal of $\mathcal{U}$ we have $xax^{-1}\in \mathcal{U}_0$ for all $x\in U$ and all $a\in \mathcal{U}_0$. Therefore U acts on $\mathcal{U}_0$ (via the adjoint representation), hence it acts on $\mathcal{U}_0$*. This action is defined by

$$(x\cdot g_0)(a)=g_0(xax^{-1})$$

for all $x\in U$, all $g_0\in \mathcal{U}_0^*$ and all $a\in \mathcal{U}_0$. Moreover we have

(1.3.1) $$\pi(x\cdot g)=x\cdot\pi(g)$$

for all $x\in U$ and all $g\in \mathcal{U}^*$. It follows that, for all $x\in U$,

$$x\cdot f\in l(f) \iff x\cdot f_0=f_0.$$

Thus

$$U'=\{x\in U; x\cdot f_0=f_0\}=C_U(f_0).$$

Therefore U' is a closed subgroup of U (see [Hu1; proposition 8.2(b)]).

Now we claim that $U'=exp\ \mathcal{U}'$. Let $a\in \mathcal{U}'$ and let $b\in \mathcal{U}_0$. Then

$$((exp\ a)\cdot f)(b)=f((exp\ a)b(exp\ a)^{-1})=f(exp(ad\ a)(b))=f(b)+\sum_{i=1}^{n-1}\frac{f((ad\ a)^i(b))}{i!}.$$

Since $\mathcal{U}_0$ is an ideal of $\mathcal{U}$ and $a\in \mathcal{U}'$ we have

$$f((ad\ a)^i(b))=0$$

for all $i\in\{1,\ldots,n-1\}$. Hence

$$((exp\ a)\cdot f)(b)=f(b).$$

Since $b\in \mathcal{U}_0$ is arbitrary we conclude that

$$(exp\ a)\cdot f_0 = f_0.$$

Therefore $exp\ a \in U'$. Conversely let $x \in U'$ and let $a \in \mathcal{U}$ be such that $x = exp\ a$ (we recall that $U = exp\ \mathcal{U}$). Let t be an indeterminate over K and let $b \in \mathcal{U}_0$. We consider the polynomial

$$P(t) = (exp(ta))\cdot f_0)(b) = \sum_{i=0}^{n-1} \frac{f_0((ad\ a)^i(b))}{i!} t^i \in K[t].$$

The argument used in the proof of proposition 1.2.2 may be repeated to show that

$$P(t) = f(b).$$

In particular we obtain

$$f([ab]) = 0.$$

Since $b \in \mathcal{U}_0$ is arbitrary it follows that $a \in \mathcal{U}'$ and the proof is complete. ◆

Now let h be a non-zero element of $\mathcal{U}_0^{\perp}$. Since $dim\,\mathcal{U}_0^{\perp} = 1$ we have

$$\mathcal{U}_0^{\perp} = \{\, \alpha h;\ \alpha \in K \,\}.$$

Since $x\cdot f \in l(f) = f + \mathcal{U}_0^{\perp}$ for all $x \in U'$, we may define a map $\varphi: U' \to K$ by

$$x\cdot f = f + \varphi(x)h$$

for all $x \in U'$.

Let $x \in U$. Since $\mathcal{U}_0$ is U-invariant we have $xax^{-1} \in \mathcal{U}_0$ for all $a \in \mathcal{U}_0$. Hence

$$(x\cdot h)(a) = h(xax^{-1}) = 0$$

for all $a \in \mathcal{U}_0$. Therefore $x\cdot h \in \mathcal{U}_0^{\perp}$ and so

$$x\cdot h = \alpha h$$

for some $\alpha \in K$, i.e. α is an eigenvalue of the linear map $Ad^*x: \mathcal{U}_0^{\perp} \to \mathcal{U}_0^{\perp}$. Since Ad^*x is unipotent all its eigenvalues are equal to $1 \in K$. Hence we must have $\alpha = 1$ and so

$$x\cdot h = h.$$

Since this equality holds for all $x \in U$ we deduce that

$$(xy)\cdot f = x\cdot(y\cdot f) = x\cdot(f + \varphi(y)h) = x\cdot f + \varphi(y)(x\cdot h) = f + \varphi(x)h + \varphi(y)h = f + (\varphi(x) + \varphi(y))h$$

for all $x,y \in U'$. It follows that

$$\varphi(xy) = \varphi(x) + \varphi(y)$$

for all $x,y \in U'$, i.e. φ is an homomorphism from U' into the additive group K^+ of the field K. Moreover φ is a morphism of algebraic varieties. Therefore φ is an homomorphism of algebraic groups. Since U' is connected the image $\varphi(U')$ is also connected (see [Hu1; proposition 1.3.A]). So either φ is identically zero or $\varphi(U')=K$. It follows that

$$l(f)\cap O(f)=\begin{cases}\{f\} & \text{if } \varphi \text{ is identically zero}\\ l(f) & \text{if } \varphi(U')=K\end{cases}.$$

This completes the proof of the first assertion of the following:

Lemma 1.3.2. ([Ki2; lemma 6.2]; [CG; lemma 3.1.3]) *Let $f\in \mathcal{U}^*$. Then either $l(f)\cap O(f)=\{f\}$ or $l(f)\subseteq O(f)$. Moreover let $g\in O(f)$. Then $l(f)\cap O(f)=\{f\}$ if and only if $l(g)\cap O(f)=\{g\}$. Hence $l(f)\subseteq O(f)$ if and only if $l(g)\subseteq O(f)$.*

Proof. Let $x\in U$ be such that $g=x\cdot f$. Then $l(g)=x\cdot l(f)$ and

$$l(g)\cap O(f)=x\cdot(l(f)\cap O(f)) \quad ([1]).$$

The result follows. ♦

This lemma allows us to say that a U-orbit $O\subseteq \mathcal{U}^*$ is of the *first kind with respect to* $\mathcal{U}_0$ if $l(f)\cap O=\{f\}$ for all $f\in O$. Otherwise we will say that O is of the *second kind with respect to* $\mathcal{U}_0$.

Let O be a U-orbit in $\mathcal{U}^*$ and consider the image $\pi(O)\subseteq \mathcal{U}_0{}^*$. Since π is U-invariant (see (1.3.1)) $\pi(O)$ is a U-invariant subset of $\mathcal{U}_0{}^*$. Moreover U acts transitively on $\pi(O)$. Since U is unipotent we conclude that $\pi(O)$ is a closed subset of $\mathcal{U}_0{}^*$ (see [St; proposition 2.5]). Since O is an irreducible closed subset of $\mathcal{U}^*$ and π is a morphism of algebraic varieties we deduce that $\pi(O)$ is an irreducible closed subset of $\mathcal{U}_0{}^*$ (see [Hu1; proposition 1.3.A]). Therefore $\pi: O\to\pi(O)$ is a surjective morphism between irreducible algebraic varieties. It follows that

$$dim\,\pi(O)\leq dim\,O.$$

Let $r=dim\,O-dim\,\pi(O)$. Then there exists a non-empty open subset A of $\pi(O)$ such that

[1] By definition $x\cdot A=\{x\cdot g;\ g\in A\}$ for all subsets $A\subseteq\mathcal{U}^*$.

$$dim\pi^{-1}(a)=r$$

for all $a\in A$ (see [Hu1; theorem 4.3]). Let $f\in O$. Then

$$\pi^{-1}(\pi(f))=l(f)\cap O.$$

It follows that

$$\pi^{-1}(\pi(f))=\begin{cases}\{f\} & \text{if } O \text{ is of the first kind}\\ l(f) & \text{if } O \text{ is of the second kind}\end{cases}.$$

Thus

$$dim\pi^{-1}(\pi(f))=\begin{cases}0 & \text{if } O \text{ is of the first kind}\\ 1 & \text{if } O \text{ is of the second kind}\end{cases}.$$

Since $f\in O$ is arbitrary we obtain:

Proposition 1.3.3. *Let $O\subseteq \mathfrak{u}^*$ be a U-orbit. Then:*

(i) *$dim\pi(O)=dimO$ if and only if O is of the first kind (with respect to $\mathfrak{u}_0$);*

(ii) *$dim\pi(O)=dimO-1$ if and only if O is of the second kind (with respect to $\mathfrak{u}_0$).*

Now let $U_0=exp\,\mathfrak{u}_0$. Then U_0 is a connected closed subgroup of U. Since $\pi(O)$ is U-invariant U_0 acts on $\pi(O)$. Therefore $\pi(O)$ is a disjoint union of U_0-orbits. Our purpose is to obtain this decomposition of $\pi(O)$. Firstly we relate the dimension of O with the dimension of any U_0-orbit in $\pi(O)$.

Let $f\in O$ and let O_0 be the U_0-orbit of $f_0=\pi(f)\in \mathfrak{u}_0^*$. Then O_0 is an irreducible closed subset of $\pi(O)$ and so

(1.3.2) $$dimO_0\leq dim\pi(O)\leq dimO.$$

Now let $(e_1,\dots,e_m)$ be a K-basis of $\mathfrak{u}$ and let $M(f)$ be the $m\times m$ matrix which represents the K-bilinear form B_f with respect to the basis $(e_1,\dots,e_m)$. For $i,j\in\{1,\dots,m\}$ the (i,j)-th entry of $M(f)$ is $f([e_i,e_j])$. Then $M(f)$ is skew-symmetric matrix and so it has even rank (see [Co; theorem 8.6.1]). In fact

$$rankM(f)=dim\,\mathfrak{u}-dim\,\mathfrak{r}(f).$$

By lemma 1.2.2 $dim\,\tau(f)=dimC_U(f)$, hence

(1.3.3) $$rankM(f)=dimO(f)$$

(we note that $O=O(f)$).

Since U_0 is a subspace of U with codimension one we may choose the basis $(e_1,\dots,e_m)$ such that $e_i\in U_0$ for all $i\in\{1,\dots,m-1\}$. Then (with respect to this basis) the matrix $M(f)$ has the form

$$M(f)=\begin{pmatrix} M(f_0) & -v \\ v & 0 \end{pmatrix}$$

where $M(f_0)$ is the $(m-1)\times(m-1)$ matrix which represents the K-bilinear form B_{f_0} with respect to the basis $(e_1,\dots,e_{m-1})$ of U_0 and v is the row vector

$$v=(f([e_m,e_1])\ \dots\ f([e_m,e_{m-1}])).$$

Since O_0 is the U_0-orbit of f_0 we have

$$dimO_0=rankM(f_0)$$

(by (1.3.3)). Since $M(f)$ and $M(f_0)$ are skew-symmetric they have even ranks and so

$$rankM(f)\in\{rankM(f_0),rankM(f_0)+2\}.$$

It follows that

(1.3.4) $$dimO\in\{dimO_0,dimO_0+2\}.$$

We now prove the following:

Proposition 1.3.4. *Let $O\subseteq U^*$ be a U-orbit and let $O_0\subseteq\pi(O)$ be a U_0-orbit. Then:*

(i) *$dimO=dimO_0$ if and only if O is of the first kind (with respect to U_0);*

(ii) *$dimO=dimO_0+2$ if and only if O is of the second kind (with respect to U_0).*

Proof. It is enough to prove (i).

Suppose that $dimO=dimO_0$. Then

$$dim\pi(O)=dimO$$

(by (1.3.2)) and so O is of the first kind (by proposition 1.3.3).

Conversely suppose that O is of the first kind. Then (by proposition 1.3.3)

$$dim\pi(O)=dimO.$$

Since $C_{U_0}(f_0)=U_0\cap C_U(f_0)$ we have

$$dimC_{U_0}(f_0)\leq dimC_U(f_0).$$

Since U acts transitively on $\pi(O)$ we deduce that

$$dim\pi(O)=dimU-dimC_U(f_0)\leq dimU-dimC_{U_0}(f_0)=dimU_0+1-dimC_{U_0}(f_0)=dimO_0+1.$$

Thus

$$dimO=dim\pi(O)\leq dimO_0+1.$$

(i) follows by (1.3.4). ◆

The following result gives morecharacterizations of the orbit O (with respect to the subalgebra $\mathcal{U}_0$):

Lemma 1.3.5. ([CG; proposition 1.3.4]) *Let $O\subseteq\mathcal{U}^*$ be a U-orbit and let $f\in O$. Let $f_0=\pi(f)\in\mathcal{U}_0^*$ and let $O_0\subseteq\mathcal{U}_0^*$ be the U_0-orbit of f_0. Then:*

(i) *The following conditions are equivalent:*

(a) *O is of the first kind (with respect to $\mathcal{U}_0$);*

(b) $\tau(f)\cap\mathcal{U}_0\neq\tau(f)$;

(c) $\tau(f_0)\subset\tau(f)$;

(d) $dim\,\tau(f)=dim\,\tau(f_0)+1$;

(ii) *The following conditons are equivalent:*

(a) *O is of the second kind (with respect to $\mathcal{U}_0$);*

(b) $\tau(f)\cap\mathcal{U}_0=\tau(f)$;

(c) $\tau(f)\subset\tau(f_0)$;

(d) $dim\,\tau(f)=dim\,\tau(f_0)-1$.

Proof. Let V be a subspace of $\mathcal{U}_0$ such that

$$\mathcal{U}_0=V\oplus\tau(f_0)$$

and suppose that $dimV=r$. Let $(e_1,\ldots,e_{m-1})$ be a K-basis of $\mathcal{U}_0$ such that $(e_{r+1},\ldots,e_{m-1})$ is a K-basis for $\tau(f_0)$. Then (with respect to this basis) the matrix $M(f_0)$ has the form

$$M(f_0)=\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

where A is an $r \times r$ non-singular skew-symmetric matrix.

Suppose that O is of the first kind. Then (by proposition 1.3.4) $dimO=dimO_0$ and so

$$rankM(f)=rankM(f_0)$$

(by (1.3.3)). Therefore the m-th row of $M(f)$ is a linear combination of the remaining rows. In particular we must have

$$f([e_m,e_i])=0$$

for all $i\in\{r+1,\ldots,m-1\}$. Since $(e_{r+1},\ldots,e_{m-1})$ is a basis of $\tau(f_0)$ we conclude that

$$\tau(f_0)\subseteq\tau(f).$$

On the other hand consider the subspace $V\oplus Ke_m$ of $\mathcal{U}$. The restriction of B_f to this subspace determines a bilinear form which is represented (with respect to the basis $(e_1,\ldots,e_r,e_m)$) by a skew-symmetric matrix of odd size $r+1$ (we note that r is an even number because $r=rankM(f)=rankM(f_0)$). Therefore there exists a vector

$$a=b+\alpha e_m\in V\oplus Ke_m \qquad (b\in V,\ \alpha\in K^*)$$

such that

$$B_f(a,e_i)=B_f(b,e_i)+\alpha B_f(e_m,e_i)=0$$

for all $i\in\{1,\ldots,r\}$. Since $B_f(e_m,e_i)=f([e_m,e_i])=0$ for all $i\in\{r+1,\ldots,m-1\}$ we conclude that $a\in\tau(f)$. Since $a\notin\mathcal{U}_0$ we deduce that $\tau(f)$ is not contained in $\mathcal{U}_0$, i.e.

$$\tau(f)\cap\mathcal{U}_0\neq\tau(f).$$

Moreover $(e_{r+1},\ldots,e_{m-1},a)$ is a basis of $\tau(f)$. Hence

$$dim\,\tau(f)=dim\,\tau(f_0)+1.$$

We have proved that (a) implies (b), (c) and (d) in (i). The equivalencies in (i) and in (ii) will follow once we have proved that (a) implies (b), (c) and (d) in (ii). For suppose that O is of the second kind. Then (by proposition 1.3.4) $dimO=dimO_0+2$ and so

$$rankM(f)=rankM(f_0)+2$$

(by (1.3.3)). In this case the m-th row of the matrix $M(f)$ is linearly independent of the remaining rows. If $f([e_m,e_i])=0$ for all $i\in\{r+1,\dots,m-1\}$ then the previous argument shows that

$$rankM(f)=rankM(f_0)=r.$$

This contradiction implies that there exists at least one $i\in\{r+1,\dots,m-1\}$ such that $f([e_m,e_i])\neq0$. Without loss of generality we may assume

$$f([e_m,e_{r+1}])\neq0.$$

Then for each $i\in\{r+2,\dots,m-1\}$ we define the vector

$$a_i=f([e_m,e_{r+1}])e_i-f([e_m,e_i])e_{r+1}.$$

Since $e_{r+1},e_i\in\mathfrak{r}(f_0)$ we have

$$B_f(e_j,a_i)=0$$

for all $j\in\{1,\dots,m-1\}$. Also

$$B_f(e_m,a_i)=f([e_m,e_{r+1}])f([e_m,e_i])-f([e_m,e_i])f([e_m,e_{r+1}])=0.$$

It follows that $a_i\in\mathfrak{r}(f)$ for all $i\in\{r+2,\dots,m-1\}$. Moreover $a_i\in\mathfrak{u}_0$ for all $i\in\{r+2,\dots,m-1\}$. Since $(a_{r+2},\dots,a_{m-1})$ is linearly independent and $rankM(f)=r+2$, $(a_{r+2},\dots,a_{m-1})$ is a basis of $\mathfrak{r}(f)$. This implies that

$$\mathfrak{r}(f)\subseteq\mathfrak{r}(f_0)\subseteq\mathfrak{u}_0.$$

Moreover

$$dim\,\mathfrak{r}(f)=dim\,\mathfrak{r}(f_0)-1.$$

The proof of the lemma is complete. ♦

We are now able to give the required decomposition of $\pi(O)$ into U_0-orbits.

Proposition 1.3.6. ([Ki2; lemma 6.2(a),(b)]; [CG; theorem 2.5.1]) *Let $O\subseteq\mathfrak{u}^*$ be a U-orbit, let $f\in O$ and let $O_0\subseteq\mathfrak{u}_0^*$ be the U_0-orbit of $f_0=\pi(f)\in\mathfrak{u}_0^*$. Then*

(i) *If O is of the first kind (with respect to $\mathfrak{u}_0$), U_0 acts transitively on $\pi(O)$ - hence $\pi(O)=O_0$.*

(ii) *If O is of the second kind (with respect to $\mathfrak{u}_0$), $\pi(O)$ is the disjoint union*

$$\pi(O)=\bigcup_{\alpha\in K}O_\alpha$$

where, for each $\alpha \in K$, $O_\alpha \subseteq \mathfrak{u}_0{}^*$ is the U_0-orbit of the element $(exp(\alpha a)) \cdot f_0 \in \mathfrak{u}_0{}^*$ and $a \in \mathfrak{u}$ is an arbitrary vector such that $a \notin \mathfrak{u}_0$. In fact

$$O_\alpha = (exp(\alpha a)) \cdot O_0 = \{ (exp(\alpha a)) \cdot g_0;\ g_0 \in O_0 \}$$

for all $\alpha \in K$. Moreover we have

$$dim O_\alpha = dim O - 2$$

for all $\alpha \in K$.

Proof. (i) follows from propositions 1.3.3 and 1.3.4 (we recall that O_0 is an irreducible subvariety of $\pi(O)$).

Suppose that O is of the second kind and let $a \in \mathfrak{u} \setminus \mathfrak{u}_0$ be arbitrary. We consider the one-dimensional subgroup

$$X = \{ exp(\alpha a);\ \alpha \in K \} \subseteq U.$$

Since $a \notin \mathfrak{u}_0$ we have

$$U_0 \cap X = \{1\}.$$

On the other hand U_0 is a normal subgroup of U - because $dim U_0 = dim U - 1$ (see [Hu1; proposition 17.4]). Therefore $U_0 X$ is a subgroup of U. Since U_0 and X are connected $U_0 X$ is also connected (see [Hu1; corollary 7.5]). Since $U_0 \neq U_0 X$ we have

$$dim U_0 < dim(U_0 X) \leq dim U.$$

Since U is connected and $dim U_0 = dim U - 1$ we conclude that

$$U = U_0 X \tag{1.3.5}$$

(moreover this product is semidirect).

Now let $g \in O$ and let $x \in U$ be such that $f = x \cdot g$. Since $U = U_0 X$ there exist $x_0 \in U_0$ and $\alpha \in K$ such that $x = x_0 exp(\alpha a)$. Then

$$f_0 = \pi(f) = x_0 \cdot \pi((exp(\alpha a)) \cdot g)$$

(because π is U-invariant). This implies that

$$(exp(\alpha a)) \cdot \pi(g) = \pi((exp(\alpha a)) \cdot g) \in O_0,$$

so

$$\pi(g) \in (exp(-\alpha a)) \cdot O_0 = O_{-\alpha}$$

(because $exp(\alpha a)^{-1}=exp(-\alpha a)$). Since $g\in O$ is arbitrary we conclude that

$$\pi(O)\subseteq \bigcup_{\alpha\in K} O_{\alpha}.$$

Conversely let $\alpha\in K$ and let $g\in \mathcal{U}^*$ be such that $(exp(\alpha a))\cdot\pi(g)\in O_0$ (i.e. $\pi(g)\in O_{-\alpha}$).

Then there exists $x_0\in U_0$ such that

$$f_0=x_0\cdot((exp(\alpha a))\cdot\pi(g))=\pi((x_0(exp(\alpha a))\cdot g).$$

Hence

$$(x_0(exp(\alpha a))\cdot g\in l(f).$$

Since O is of the second kind we have $l(f)\subseteq O$ so

$$(x_0(exp(\alpha a))\cdot g\in O.$$

It follows that $g\in O$ and this completes the proof of the inclusion $O_{-\alpha}\subseteq\pi(O)$. Since $\alpha\in K$ is arbitrary we conclude that

$$\pi(O)=\bigcup_{\alpha\in K} O_{\alpha}.$$

To conclude the proof we claim that for all $\alpha,\beta\in$ K

$$O_{\alpha}=O_{\beta} \iff \alpha=\beta.$$

In fact let $\alpha,\beta\in K$ be two distinct elements and suppose that $O_{\alpha}=O_{\beta}$. Then the elements $x=exp(\alpha a)$ and $y=exp(\beta a)$ of X are distinct. So

$$z=xy^{-1}\neq 1$$

and we have

$$z\cdot O_{\alpha}=z\cdot(x\cdot O_0)=z\cdot O_{\alpha}=z\cdot O_{\beta}=z\cdot(y\cdot O_0)=(zy)\cdot O_0=x\cdot O_0=O_{\alpha}.$$

Let $g_0\in O_{\alpha}$. Since O_{α} is a U_0-orbit (because U_0 is a normal subgroup of U) and $z\cdot g_0\in O_{\alpha}$, there exists $x_0\in U_0$ such that

$$z\cdot g_0=x_0\cdot g_0.$$

Therefore

$$x_0^{-1}z\in C_U(g_0).$$

Now let $b\in \mathcal{U}$ be such that $x_0^{-1}z=exp\,b$ and consider the subgroup

$$Y=\{exp(\alpha b);\ \alpha\in K\}\subseteq U.$$

Since $x_0\in U_0$ and $z\in X$ we have $x_0^{-1}z\neq 1$, hence $b\neq 0$. Let $g\in O$ be such that $g_0=\pi(g)$. Then (by lemma 1.3.1) $C_U(g_0)=exp\,\mathcal{U}'$ where

$$\mathcal{U}'=\{c\in\mathcal{U};\, g([cd])=0 \text{ for all } d\in\mathcal{U}_0\}.$$

Since $x_0^{-1}z=\exp b\in C_U(g_0)$ we have $b\in\mathcal{U}'$. Therefore $\{\alpha b;\ \alpha\in K\}\subseteq\mathcal{U}'$ and this implies that

$$Y\subseteq C_U(g_0).$$

On the other hand we have

$$U=U_0Y$$

(since $b\in\mathcal{U}_0$ (because $x_0^{-1}z\in U_0$) this equality has exactly the same proof as (1.3.5)). Since Y centralizes g_0 and O_α is the U_0-orbit of g_0 we conclude that O_α is U-invariant and this implies that

$$O_\alpha=\pi(O)$$

(we recall that U acts transitively on $\pi(O)$). Finally we have

$$\dim\pi(O)=\dim O-1$$

(by proposition 1.3.3 because O is of the second kind).Therefore

$$\dim O_\alpha=\dim O-1.$$

This is impossible because both O and O_α have even dimension (corollary 1.2.3). This contradiction completes the proof of our claim and also the proof of the lemma. ◆

Next we consider the inverse image $\pi^{-1}(O_0)$ of the U_0-orbit $O_0\subseteq\mathcal{U}_0^*$ of $f_0=\pi(f)$. Let $g\in\pi^{-1}(O_0)$. Then $\pi(g)\in O_0$ so there exists $x\in U_0$ such that

$$\pi(f)=f_0=x\cdot\pi(g)=\pi(x\cdot g).$$

It follows that

$$x\cdot g\in l(f)=f+\mathcal{U}_0^{\perp}.$$

Let h be a non-zero vector of $\mathcal{U}_0^{\perp}$. Then

$$x\cdot g=f+\alpha h$$

for some $\alpha\in K$, i.e. $g\in O(f+\alpha h)$. Therefore

(1.3.6) $$\pi^{-1}(O_0)\subseteq\bigcup_{\alpha\in K}O(f+\alpha h).$$

Since $f+\alpha h\in\pi^{-1}(O_0)$ for all $\alpha\in K$ we have

$$O(f+\alpha h)\cap\pi^{-1}(O_0)\neq\emptyset.$$

On the other hand let O' be a U-orbit such that $O'\cap\pi^{-1}(O_0)\neq\emptyset$. Then the above argument shows that there exists $\alpha\in K$ such that $O'=O(f+\alpha h)$. However in general the inclusion (1.3.6) is not an equality. In fact suppose that O is of the second kind (with respect to $\mathcal{U}_0$). Then $l(f)\subseteq O(f)$ and so $f+\alpha h\in O$ for all $\alpha\in K$. Hence

$$O(f+\alpha h)=O$$

for all $\alpha\in K$. On the other hand we have

$$\pi\pi^{-1}(O_0)=O_0$$

(because $\pi:\mathcal{U}^*\to\mathcal{U}_0{}^*$ is surjective) and the previous proposition implies that

$$\pi^{-1}(O_0)\neq O$$

(we note that in this case $\pi^{-1}(O_0)\subseteq O$). However the next result shows that $\pi^{-1}(O_0)$ is U-invariant whenever O is of the first kind (with respect to $\mathcal{U}_0$).

Proposition 1.3.7. ([Ki2; lemma 6.2(c),(d)]; [CG; theorem 2.5.1]) *Let $O\subseteq\mathcal{U}^*$ be a U-orbit, let $f\in O$ and let $O_0\subseteq\mathcal{U}_0{}^*$ be the U_0-orbit of $f_0=\pi(f)$. Then:*

(i) *If O is of the first kind (with respect to $\mathcal{U}_0$) the inverse image $\pi^{-1}(O_0)$ is the disjoint union*

$$\pi^{-1}(O_0)=\bigcup_{\alpha\in K}O(f+\alpha h)$$

where h is any non-zero vector of $\mathcal{U}_0{}^{\perp}$. Moreover for each $\alpha\in K$ the U-orbit $O(f+\alpha h)$ is of the first kind (with respect to $\mathcal{U}_0$) and we have

$$\dim O(f+\alpha h)=\dim O=\dim O_0.$$

Also

$$\pi(O(f+\alpha h)=O_0$$

for all $\alpha\in K$.

(ii) *If O is of the second kind (with respect to $\mathcal{U}_0$) O is the unique U-orbit which intersects $\pi^{-1}(O_0)$ (hence $\pi^{-1}(O_0)\subseteq O$ and this inclusion is proper).*

Proof. (ii) have been proved above.

Suppose that O is of the first kind and let $\alpha\in K$. Let $(e_1,\ldots,e_m)$ be a basis of $\mathcal{U}$

and consider the skew-symmetric matrix $M(f+\alpha h)$ which represents the bilinear form $B_{f+\alpha h}$ with respect to this basis. Since $[\mathcal{U}\mathcal{U}]\subseteq\mathcal{U}_0$, we have

$$(f+\alpha h)([ab])=f([ab])$$

for all $a,b\in\mathcal{U}$. Therefore

$$M(f+\alpha h)=M(f).$$

By (1.3.3) we get

$$dim O(f+\alpha h)=dim O.$$

On the other hand we have $dim O=dim O_0$ (by proposition 1.3.4). Therefore

$$dim O(f+\alpha h)=dim O_0$$

and this implies that $O(f+\alpha h)$ is of the first kind (again by proposition 1.3.4). Now we apply the previous proposition to conclude that

$$\pi(O(f+\alpha h))=O_0.$$

We have justified above that each element $g\in\pi^{-1}(O_0)$ lies in $O(f+\alpha h)$ for some $\alpha\in K$. Conversely the equality $\pi(O(f+\alpha h))=O_0$ implies that

$$O(f+\alpha h)\subseteq\pi^{-1}(O_0)$$

for all $\alpha\in K$. Therefore

$$\pi^{-1}(O_0)=\bigcup_{\alpha\in K}O(f+\alpha h).$$

To prove that this union is disjoint let $\alpha,\beta\in K$ and suppose that $O(f+\alpha h)=O(f+\beta h)$. Then

$$(f+\alpha h)+(\beta-\alpha)h=f+\beta h\in O(f+\beta h)=O(f+\alpha h).$$

Since $O(f+\alpha h)$ is of the first kind we have

$$l(f+\alpha h)\cap O(f+\alpha h)=\{f+\alpha h\}$$

so $\alpha-\beta=0$, i.e. $\alpha=\beta$.

The proof is complete. ♦

Subsequently we assume that K is the algebraic closure of $\mathbf{F}_q$ and we consider the Frobenius map $F=F_q:\mathcal{U}_n(K)\rightarrow\mathcal{U}_n(K)$. We also assume that $\mathcal{U}$ and $\mathcal{U}_0$ (hence U and U_0) are F-stable.

Let $f\in(\mathcal{U}^*)^F$ and let $O=O(f)$ be the U-orbit of f. Since f is F-fixed O is F-stable. As before we let $O_0\subseteq\mathcal{U}_0{}^*$ be the U_0-orbit of the element $f_0=\pi(f)$. If we choose a K-basis $(e_1,\ldots,e_{m-1},e_m)$ of $\mathcal{U}$ such that $(e_1,\ldots,e_{m-1})$ is a basis of $\mathcal{U}_0$ and such that $F(e_i)=e_i$ for all $i\in\{1,\ldots,m\}$ then we can easily verify that $f_0\in(\mathcal{U}_0{}^*)^F$. Hence O_0 is an F-stable U_0-orbit.

Theorem 1.3.8. *Let $O\subseteq\mathcal{U}^*$ be an F-stable U-orbit, let $f\in O^F$ and let $O_0\in\mathcal{U}_0{}^*$ be the (F-stable) U_0-orbit of $f_0=\pi(f)$. Let $\chi=\chi_O$ be the irreducible character of U^F which corresponds to the U-orbit O and let $\chi_0=\chi_{O_0}$ be the irreducible character of $U_0{}^F$ which corresponds to the U_0-orbit O_0.*

(i) *If O is of the first kind (with respect of $\mathcal{U}_0$) then $O_0{}^F=\pi(O^F)$. Hence*

$$\chi_{U_0{}^F}=\chi_0$$

is an irreducible character of $U_0{}^F$.

(ii) *If O is of the second kind (with respect to $\mathcal{U}_0$) then $\pi(O^F)$ is the disjoint union*

$$\pi(O^F)=\bigcup_{\alpha\in\mathbf{F}_q} O_\alpha{}^F$$

where, for each $\alpha\in\mathbf{F}_q$,

$$O_\alpha=(\exp\alpha a)\cdot O_0$$

and $a\in\mathcal{U}$ is an arbitrary F-fixed element such that $a\notin\mathcal{U}_0$. On the other hand $\chi_{U_0{}^F}$ is the direct sum

$$\chi_{U_0{}^F}=\sum_{\alpha\in\mathbf{F}_q}\chi_\alpha$$

where, for each $\alpha\in\mathbf{F}_q$, $\chi_\alpha=\chi_{O_\alpha}$ is the irreducible character which corresponds to the U_0-orbit O_α. Moreover

$$\chi_\alpha=(\chi_0)^{\exp(\alpha a)}$$

for all $\alpha\in\mathbf{F}_q$

Proof. (i) By proposition 1.3.6 we have $\pi(O)=O_0$. Thus the map $\pi:O\to O_0$ is surjective. On the other hand let $g,g'\in O$ be such that $\pi(g)=\pi(g')$. Then $g'\in l(g)$. Since O is of the first kind we have $l(g)\cap O(g)=\{g\}$ and so $g'=g$. It follows that $\pi:O\to O_0=\pi(O)$

is injective, hence it is bijective. Therefore

$$|\pi(O^F)|=|O^F|=q^{dimO}.$$

Since $dimO=dimO_0$ (by proposition 1.3.4) we conclude that

$$|\pi(O^F)|=q^{dimO_0}=|O_0{}^F|.$$

Finally we get

$$\pi(O^F)=O_0{}^F$$

because $\pi(O^F)\subseteq\pi(O)^F=O_0{}^F$.

Now we consider that character χ. By definition (see (1.2.12)) we have

$$\chi(exp\, b)=\frac{1}{\sqrt{|O^F|}}\sum_{g\in O^F}\psi_0(g(b))$$

for all $b\in \mathcal{U}$. Since $\pi:O\to O_0=\pi(O)$ is bijective we deduce that

$$\sum_{g\in O^F}\psi_0(g(b))=\sum_{g_0\in O_0{}^F}\psi_0(g_0(b))$$

for all $b\in \mathcal{U}_0$ (we note that $\pi(g)(b)=g(b)$ for all $b\in \mathcal{U}_0$). Since $|O^F|=|O_0{}^F|$ we obtain (by (1.2.12))

$$\chi(exp\, b)=\frac{1}{\sqrt{|O_0{}^F|}}\sum_{g_0\in O_0{}^F}\psi_0(g_0(b))=\chi_0(exp\, b)$$

for all $b\in \mathcal{U}_0$. (i) follows because $U_0=exp\, \mathcal{U}_0$.

(ii) Let a be an F-fixed element in $\mathcal{U}\setminus\mathcal{U}_0$. Then the element $(exp\, \alpha a)\cdot f_0\in \mathcal{U}_0{}^*$ is F-fixed for all $\alpha\in \boldsymbol{F}_q$. Hence the U_0-orbit O_α is F-stable for all $\alpha\in \boldsymbol{F}_q$. Now it is clear that

$$\pi(O^F)\subseteq\pi(O)^F.$$

Since O is of the second kind we have $dim\pi(O)=dimO$-1 (by proposition 1.3.3) and so

$$|\pi(O^F)|\leq|\pi(O)^F|=q^{dim\pi(O)}=q^{dimO-1}.$$

On the other hand the element $(exp\, \alpha a)\cdot f\in O$ is F-fixed for all $\alpha\in \boldsymbol{F}_q$ (we recall that a and f are F-fixed). Hence

$$(exp\, \alpha a)\cdot f_0=\pi((exp\, \alpha a)\cdot f)\in \pi(O^F).$$

Since $\pi(O^F)$ is U^F-invariant and $O_\alpha{}^F$ is the $U_0{}^F$-orbit of $(exp\, \alpha a)\cdot f_0$ we conclude that

$$O_\alpha{}^F\subseteq\pi(O^F)$$

for all $\alpha\in \boldsymbol{F}_q$. Therefore

$$\sum_{\alpha\in \boldsymbol{F}_q}|O_\alpha{}^F|\leq|\pi(O^F)|\leq q^{dimO-1}$$

(we recall that the union $\bigcup_{\alpha\in K} O_\alpha$ is disjoint). Now

$$dimO_\alpha = dimO_0 = dimO - 2$$

(by proposition 1.3.4 because O is of the second kind). Thus

$$\sum_{\alpha\in F_q} |O_\alpha{}^F| = \sum_{\alpha\in F_q} q^{dimO_0} = qq^{dimO-2} = q^{dimO-1}.$$

It follows that

$$\pi(O^F) = \bigcup_{\alpha\in F_q} O_\alpha{}^F$$

(and this union is disjoint).

Now we consider the irreducible character χ of U^F. As before we have

$$\chi(exp\, b) = \frac{1}{\sqrt{|O^F|}} \sum_{g\in O^F} \psi_0(g(b))$$

for all $b\in \mathcal{U}$. Suppose that $b\in \mathcal{U}_0$. Since O is of the second kind we have $l(g)\subseteq O$ for all $g\in O$. Therefore there exist $g_1,\ldots,g_r\in O^F$ such that O^F is the disjoint union

$$O^F = \bigcup_{i=1}^{r} l(g_i)^F$$

(we note that each $l(g_i)$ is F-stable because g_i is F-fixed). It follows that

$$\sum_{g\in O^F} \psi_0(g(b)) = \sum_{i=1}^{r} \sum_{g\in l(g_i)^F} \psi_0(g(b)).$$

Since $b\in \mathcal{U}_0$ we have

$$g(b) = g_i(b)$$

for all $g\in l(g_i)$ and all $i\in \{1,\ldots,r\}$. Therefore

$$\sum_{g\in O^F} \psi_0(g(b)) = \sum_{i=1}^{r} |l(g_i)^F|\, \psi_0(g_i(b)).$$

Since

$$|l(g_i)^F| = q^{dim\, l(g_i)} = q$$

for all $i\in \{1,\ldots,r\}$ (we recall that $dim\, l(g)=1$ for all $g\in \mathcal{U}^*$) we conclude that

$$\sum_{g\in O^F} \psi_0(g(b)) = q^{-1} \sum_{i=1}^{r} \psi_0(g_i(b)).$$

Now we clearly have

$$\pi(O^F) = \{\pi(g_i);\ 1\le i\le r\}.$$

Since the elements $\pi(g_i)$ $(1\le i\le r)$ are all distinct we deduce that

$$\sum_{g\in O^F}\psi_0(g(b))=q^{-1}\sum_{g_0\in\pi(O^F)}\psi_0(g_0(b)).$$

By the first assertion of (ii) we have

$$\sum_{g_0\in\pi(O^F)}\psi_0(g_0(b))=\sum_{\alpha\in F_q}\sum_{g_0\in O_\alpha^F}\psi_0(g_0(b)).$$

On the other hand (by (1.2.12))

$$\sum_{g_0\in O_\alpha^F}\psi_0(g_0(b))=\sqrt{|O_\alpha^F|}\,\chi_\alpha(exp\,b).$$

Therefore

$$\chi(exp\,b)=\frac{q^{-1}}{\sqrt{|O^F|}}\sum_{\alpha\in F_q}\sqrt{|O_\alpha^F|}\,\chi_\alpha(exp\,b).$$

Since $|O^F|=|O_\alpha^F|q^2$ we conclude that

$$\chi(exp\,b)=\sum_{\alpha\in F_q}\chi_\alpha(exp\,b).$$

Since $b\in u_0$ is arbitrary and $U_0=exp\,u_0$ it follows that

$$\chi_{U^F}=\sum_{\alpha\in F_q}\chi_\alpha$$

as required.

Finally let $\alpha\in F_q$, let $x=exp(\alpha a)$ and let $b\in u_0$. Then

$$\chi_\alpha(exp\,b)=\frac{1}{\sqrt{|O_\alpha^F|}}\sum_{g\in O_\alpha^F}\psi_0(g(b)).$$

Since $O_\alpha=x\cdot O_0$ we obtain

$$\begin{aligned}\chi_\alpha(exp\,b)&=\frac{1}{\sqrt{|O_0^F|}}\sum_{g\in O_0^F}\psi_0((x\cdot g)(b))\\&=\frac{1}{\sqrt{|O_0^F|}}\sum_{g\in O_0^F}\psi_0(g(xbx^{-1}))\\&=\chi_0(exp(xbx^{-1})).\end{aligned}$$

Since $exp(xbx^{-1})=x(expb)x^{-1}$ we conclude that

$$\chi_\alpha=(\chi_0)^x$$

and the proof of the theorem is complete. ♦

We now consider induction of characters from U_0^F to U^F. We let the notation be as in the previous theorem.

Firstly suppose that O is of the second kind. Then (by proposition 1.3.6) O is the unique U-orbit which intersects the inverse image $\pi^{-1}(O_0)$ (in fact $\pi^{-1}(O_0)\subseteq O$ and this inclusion is proper). Therefore O is the unique F-stable U-orbit such that

$$O^F\cap\pi^{-1}(O_0^F)\neq\emptyset.$$

Hence

$$\pi^{-1}(O_0^F)\subseteq O^F$$

and this inclusion is proper.

Now consider the character $\chi=\chi_O$. Let $\alpha\in\mathbf{F}_q$ be arbitrary. Then (by the previous theorem) the irreducible character χ_α of U_0^F is a component of $\chi_{U_0^F}$. Hence (by Frobenius reciprocity) χ is an irreducible component of the induced character $(\chi_\alpha)^{U^F}$. Since

$$\chi(1)=q\chi_\alpha(1)=(\chi_\alpha)^{U^F}(1),$$

we conclude that

$$\chi=(\chi_\alpha)^{U^F}$$

for all $\alpha\in\mathbf{F}_q$.

On the other hand suppose that O is of the first kind. For each $\alpha\in K$ we consider the element $f+\alpha h\in\mathcal{U}^*$ where $h\in\mathcal{U}_0^\perp$ is an arbitrary non-zero F-fixed element (this element exists because $\mathcal{U}_0$ is F-stable). Then (by proposition 1.3.7) the inverse image $\pi^{-1}(O_0)$ is the disjoint union

$$\pi^{-1}(O_0)=\bigcup_{\alpha\in K}O(f+\alpha h)$$

where $O(f+\alpha h)$ is the U-orbit of $f+\alpha h$. Now if $\alpha\in\mathbf{F}_q$ then the element $f+\alpha h$ is F-fixed, hence the U-orbit $O(f+\alpha h)$ is F-stable. We claim that

$$\pi^{-1}(O_0^F)=\bigcup_{\alpha\in\mathbf{F}_q}O(f+\alpha h)^F.$$

In fact since $O(f+\alpha h)\subseteq\pi^{-1}(O_0)$ we have

$$O(f+\alpha h)^F\subseteq\pi^{-1}(O_0^F)$$

for all $\alpha\in \mathbf{F}_q$. Thus

(1.3.7) $$\bigcup_{\alpha\in \mathbf{F}_q} O(f+\alpha h)^F \subseteq \pi^{-1}(O_0{}^F).$$

To prove that the equality holds we observe that

(1.3.8) $$\pi^{-1}(O_0)=O+u_0{}^{\perp}.$$

In fact let $g\in\pi^{-1}(O_0)$. Then $\pi(g)\in O_0$. Since O is of the first kind we have $\pi(O)=O_0$ (by proposition 1.3.6). Thus there exists $g'\in O$ such that $\pi(g)=\pi(g')$. It follows that

$$g\in l(g')=g'+u_0{}^{\perp}\subseteq O+u_0{}^{\perp}.$$

Conversely if $g\in O+u_0{}^{\perp}$ then $g\in l(g')$ for some $g'\in O$. Hence $\pi(g)=\pi(g')\in O_0$ because $O_0=\pi(O)$. Now the equality (1.3.8) implies that

$$\pi^{-1}(O_0{}^F)=O^F+(u_0{}^{\perp})^F=O^F+\mathbf{F}_q h$$

(we note that $u_0{}^{\perp}=Kh$). Since O is of the first kind its intersection with the fibre of any $g_0\in O_0$ consists of a unique element (we recall that the fibre of $g_0=\pi(g)$ ($g\in u^*$) is $l(g)$). It follows that

$$|\pi^{-1}(O_0{}^F)|=|O^F+\mathbf{F}_q h|=q|O^F|.$$

Finally we have

$$\dim O(f+\alpha h)=\dim O$$

for all $\alpha\in K$ (by proposition 1.3.6). So

$$\sum_{\alpha\in \mathbf{F}_q} |O(f+\alpha h)^F|=q|O^F|=|\pi^{-1}(O_0{}^F)|$$

and the inclusion (1.3.7) is in fact an equality as claimed.

Now we consider the induced character $(\chi_0)^{U^F}$. Since $\chi_{U_0{}^F}=\chi_0$ (by theorem 1.3.9) we have

$$(\chi,(\chi_0)^{U^F})_{U^F}=(\chi_{U_0{}^F},\chi_0)_{U_0{}^F}=(\chi_0,\chi_0)_{U_0{}^F}=1,$$

i.e. χ occurs with multiplicity 1 as a component of the induced character $(\chi_0)^{U^F}$. On the other hand let $\alpha\in\mathbf{F}_q$ and let $\chi(\alpha)$ denote the irreducible character $\chi_{O(f+\alpha h)}$ of U^F which corresponds to the (F-stable) U-orbit $O(f+\alpha h)$. By proposition 1.3.7 $O(f+\alpha h)$ is of the first kind (with respect to u_0) and $\pi(O(f+\alpha h))=O_0$. If we replace O by $O(f+\alpha h)$ in the above argument we conclude that

$$(\chi(\alpha),(\chi_0)^{U^F})_{U^F}=1.$$

Moreover

$$\chi(\alpha)(1)=\sqrt{|O(f+\alpha h)^F|}=\sqrt{|O^F|}=\chi(1)$$

because $dimO(f+\alpha h)=dimO$ (by proposition 1.3.7). It follows that

$$\sum_{\alpha\in F_q}\chi(\alpha)(1)=q\chi(1)=q\chi_0(1)=(\chi_0)^{U^F}(1).$$

Hence

$$(\chi_0)^{U^F}=\sum_{\alpha\in F_q}\chi(\alpha).$$

We have finished the proof of the following:

Theorem 1.3.9. *Let $O\subseteq U^*$ be an F-stable U-orbit, let $f\in O^F$ and let O_0 be the U_0-orbit of the element $f_0=\pi(f)\in U_0^*$. Then:*

(i) *If O is of the first kind (with respect to U_0), $\pi^{-1}(O_0^F)$ is the disjoint union*

$$\pi^{-1}(O_0^F)=\bigcup_{\alpha\in F_q}O(f+\alpha h)^F$$

where $h\in(U_0)^\perp$ is an arbitrary non-zero F-fixed element. On the other hand

$$(\chi_0)^{U^F}=\sum_{\alpha\in F_q}\chi(\alpha)$$

where for each $\alpha\in F_q$

$$\chi(\alpha)=\chi_{O(f+\alpha h)}$$

is the irreducible character of U^F which corresponds to the (F-stable) U-orbit $O(f+\alpha h)$.

(ii) *If O is of the second kind (with respect to U_0) then O is the unique U-orbit such that $O^F\cap\pi^{-1}(O_0^F)\neq\emptyset$ (hence $\pi^{-1}(O_0^F)\subseteq O^F$ and this inclusion is proper). Moreover*

$$(\chi_0)^{U^F}=\chi.$$

Now we use theorems 1.3.8 and 1.3.9 to prove the following result:

Theorem 1.3.10. *Let $\mathfrak{v}$ be an F-stable subalgebra of $\mathfrak{u}$ and let $V=\exp\mathfrak{v}$. Let χ be an irreducible character of U^F and let ϕ be an irreducible character of V^F. Let $O\subseteq\mathfrak{u}^*$ be the F-stable U-orbit such that $\chi=\chi_O$ and let $O'\subseteq\mathfrak{v}^*$ be the F-stable V-orbit such that $\phi=\chi_{O'}$. Finally let $\pi:\mathfrak{u}^*\to\mathfrak{v}^*$ be the natural projection (i.e. $\pi(f)(a)=f(a)$ for all $f\in\mathfrak{u}^*$ and all $a\in\mathfrak{v}$). Then:*

(i) *ϕ is a component of the restriction χ_{V^F} of χ to V^F if and only if $O'\subseteq\pi(O)$.*

(ii) *χ is a component of the induced character ϕ^{U^F} if and only if $O\cap\pi^{-1}(O')\neq\varnothing$.*

Proof. (i) We proceed by induction on $\dim U=\dim\mathfrak{u}$. If $\dim U=1$ then either $V=\{1\}$ or $V=U$ and the result is trivial in this case. Suppose that $\dim U>1$ and let $\mathfrak{u}_0$ be an F-stable subalgebra of $\mathfrak{u}$ such that $\mathfrak{v}\subseteq\mathfrak{u}_0$ and $\dim\mathfrak{u}=\dim\mathfrak{u}_0$-1 [1]. Let $\pi_0:\mathfrak{u}^*\to\mathfrak{u}_0{}^*$ be the natural projection and let $\{O_i;\ i\in I\}$ be a complete set of F-stable U_0-orbits satisfying $O_i\subseteq\pi_0(O)$ for all $i\in I$. Then (by theorem 1.3.9)

$$\chi_{U_0{}^F}=\sum_{i\in I}\chi_{O_i}.$$

Therefore $(\phi,\chi_{V^F})_{V^F}\neq0$ if and only if there exists $i\in I$ such that $(\phi,(\chi_{O_i})_{V^F})_{V^F}\neq0$. Now let $\pi':\mathfrak{u}_0{}^*\to\mathfrak{v}^*$ be the natural projection. Then (by induction) $(\phi,(\chi_{O_i})_{V^F})_{V^F}\neq0$ if and only if $O'\subseteq\pi'(O_i)$. The result follows because $\pi=\pi'\pi_0$.

(ii) By Frobenius reciprocity

$$(\chi,\phi^{U^F})_{U^F}=(\chi_{V^F},\phi)_{V^F}.$$

Therefore (by (i)) $(\chi,\phi^{U^F})_{U^F}\neq0$ if and only if $O'\subseteq\pi(O)$. The result follows because $O'\subseteq\pi(O)$ if and only if let $O\cap\pi^{-1}(O')\neq\varnothing$. In fact suppose that $O\cap\pi^{-1}(O')\neq\varnothing$ and let $f\in O\cap\pi^{-1}(O')$. Then $\pi(f)\in O'$. Since $\pi(f)\in\pi(O)$, we conclude that $O'\cap\pi(O)\neq\varnothing$. The inclusion $O'\subseteq\pi(O)$ follows because $\pi(O)$ is V-invariant. The implication $O'\subseteq\pi(O)\Rightarrow O'\cap\pi(O)\neq\varnothing$ is clear. ◆

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[1] The subalgebra $\mathfrak{u}_0$ exists. In fact let $(0)=l_0\subseteq l_1\subseteq l_2\subseteq\ldots\subseteq l_m=\mathfrak{u}$ be a chain of F-stable ideals of $\mathfrak{u}$ such that $\dim l_{a+1}=\dim l_a+1$ for all $a\in\{0,1,\ldots,m-1\}$. Since $\mathfrak{v}$ is F-stable, the subspace $\mathfrak{v}_a=\mathfrak{v}+l_a$ is an F-stable subalgebra of $\mathfrak{u}$ for all $a\in\{0,1,\ldots,m-1\}$. Let $a\in\{1,\ldots,m-1\}$ be such that $\mathfrak{u}=\mathfrak{v}_m=\mathfrak{v}_{m-1}=\ldots=\mathfrak{v}_{a+1}$ and $\mathfrak{v}_a\neq\mathfrak{u}$. Then $\dim\mathfrak{u}=\dim\mathfrak{v}_a+1$ so we may take $\mathfrak{u}_0=\mathfrak{v}_a$.

Finally we prove the following corollary which is of great importance for our work.

Corollary 1.3.11. *Let $O_1,\ldots,O_r \subseteq \mathcal{U}^*$ be F-stable U-orbits and define*

$$O_1+\ldots+O_r=\{f_1+\ldots+f_r \in \mathcal{U}^*; f_i \in O_i, 1\le i\le r\}.$$

For each $i \in \{1,\ldots,r\}$ let $\chi_i=\chi_{O_i}$ be the irreducible character of U^F which corresponds to O_i and let χ be an irreducible character of U^F. Let $O \subseteq \mathcal{U}^*$ be the F-stable U-orbit such that $\chi=\chi_O$. Then χ is a component of the character $\chi_1\ldots\chi_r$ ([1]) if and only if $O \subseteq O_1+\ldots+O_r$.

Proof. We consider the direct product

$$U^r=\{(x_1,\ldots,x_r); x_i \in U, 1\le i\le r\}.$$

Then

$$U^r=exp(\mathcal{U}^r)$$

where

$$\mathcal{U}^r=\{(a_1,\ldots,a_r); a_i \in \mathcal{U}, 1\le i\le r\}$$

(we note that $\mathcal{U}^r$ can be regarded as a subalgebra of $\mathcal{U}_{rn}(K)$ and it is clear that the exponential map $exp: \mathcal{U}^r \to U^r$ can be defined component-by-component). The dual space $(\mathcal{U}^r)^*$ of $\mathcal{U}^r$ is naturally isomorphic to

$$(\mathcal{U}^*)^r=\{(f_1,\ldots f_r); f_i \in \mathcal{U}^*, 1\le i\le r\}.$$

Moreover $O_1\times\ldots\times O_r$ is an F-stable U^r-orbit of $(\mathcal{U}^r)^*$. It corresponds the irreducible character $\chi_1\times\ldots\times\chi_r$ of $(U^r)^F$ ([2]).

Now we identify the group U with the diagonal subgroup

$$U'=\{(x,\ldots,x)\ ;\ x \in U\} \subseteq U^r.$$

This subgroup is exponential. In fact

$$U'=exp\,\mathcal{U}'$$

where

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[1] The character $\chi_1\ldots\chi_r$ is defined by $(\chi_1\ldots\chi_r)(x)=\chi_1(x)\ldots\chi_r(x)$ for all $x \in U^F$.

[2] The character $\chi_1\times\ldots\times\chi_r$ (of $(U^r)^F$) is defined by $(\chi_1\times\ldots\times\chi_r)(x_1,\ldots,x_r)=\chi_1(x_1)\ldots\chi_r(x_r)$ for all $x_i \in U^F$ (we note that $(U^r)^F=(U^F)^r$).

$$U^r = \{(a,\ldots,a);\ a \in U\} \subseteq U^r.$$

Then the character $\chi_1 \ldots \chi_r$ is identified with the restriction $(\chi_1 \times \ldots \times \chi_r)_{U^r}$ of $\chi_1 \times \ldots \times \chi_r$ to U^r and the sum $O_1 + \ldots + O_r$ is identified with the subset $\pi(O_1 \times \ldots \times O_r)$ of $(U^r)^*$ where $\pi:(U^*)^r \to (U^r)^*$ is the natural projection. The result follows from the previous theorem. ♦

CHAPTER 2

BASIC CHARACTERS

The main goal of this chapter is to establish the existence of a certain equivalence relation on the set $Irr(U_n(q))$ of all the irreducible complex characters of $U_n(q)$ (as before q a power of a prime number p). The equivalence classes of this relation are parametrized by pairs (D,φ) where D is a basic subset of $\Phi(n)$ and $\varphi:D \to \mathbf{F}_q{}^*=\mathbf{F}_q\setminus\{0\}$ is a map. By definition a subset $D\subseteq\Phi(n)$ is *basic* if it contains at most one root from each row and at most one root from each column ([1]).

Our work was motivated by Lehrer's decomposition of the restriction to $U_n(q)$ of any irreducible discrete series representation of the general linear group $GL_n(q)$ (see [Le]). This decomposition envolves tensor products of certain irreducible representations of $U_n(q)$ which are associated with the roots $(i,j)\in\Phi(n)$ and with the non-zero elements $\alpha\in\mathbf{F}_q$ (see [Le; theorem 4.6]). We will denote by $\xi_{ij}(\alpha)$ the character of the irreducible representation associated with $(i,j)\in\Phi(n)$ and with $\alpha\in\mathbf{F}_q$. In the notation of [Le] $\xi_{ij}(\alpha)$ is the character of the representation $\alpha_{ij}(\chi)$ where χ is the linear character of $\mathbf{F}_q{}^+$ defined by

$$\chi(\beta)=\psi_0(\alpha\beta)$$

for all $\beta\in\mathbf{F}_q$ and ψ_0 is as in section 1.2. A rigorous definition of the characters $\xi_{ij}(\alpha)$ is given in section 2.1. Then in section 2.2 we will define, for each basic subset D of $\Phi(n)$ and each map $\varphi:D\to\mathbf{F}_q{}^*$, the character $\xi_D(\varphi)$ to be the product of all the irreducible characters $\xi_{ij}(\varphi(i,j))$ with $(i,j)\in D$. We denote by $I_D(\varphi)$ the set consisting of all the irreducible components of $\xi_D(\varphi)$. Then we will prove that the family of all the sets $I_D(\varphi)$ is a partition of $Irr(U_n(q))$ (see theorem 2.1.1). Therefore each set $I_D(\varphi)$ is an equivalence class of a well-determined relation defined on $Irr(U_n(q))$.

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[1] For each $i\in\{1,\dots,n\}$ the i-th row of $\Phi(n)$ is the subset $\{(i,i+1),\dots,(i,n)\}$ and the i-th column of $\Phi(n)$ is the subset $\{(1,i),\dots,(i-1,i)\}$.

2.1. Elementary characters and elementary orbits

In this section we construct for each root $(i,j)\in\Phi(n)$ and each non-zero element $\alpha\in \boldsymbol{F}_q$ the irreducible character $\xi_{ij}(\alpha)$ of $U_n(q)$ mentioned in the introduction. First we analyse the roots in the n-th column $\{(i,n);\ 1\leq i\leq n\text{-}1\}$. Our construction is independent of the orbit language, hence it is valid for any prime p. We follow very closely the work of Lehrer [Le]. Independently Lambert and van Dijk [LD] have used a similar construction to describe the characters of the Lie group $U_n(\boldsymbol{R})$ over the field $\boldsymbol{R}$ of real numbers.

We denote by $A_n(q)$ the subgroup of $U_n(q)$ consisting of all matrices having zeros in all non-diagonal entries except in the last column. $A_n(q)$ is an abelian normal subgroup of $U_n(q)$ and it has a complement H consisting of all matrices in $U_n(q)$ whose non-diagonal entries in the last column are zero. This group H is clearly isomorphic to $U_{n\text{-}1}(q)$ so we will identify these two groups. Then we obtain a decomposition

$$U_n(q)=A_n(q)U_{n\text{-}1}(q)$$

of $U_n(q)$ as a semidirect product. We now apply Clifford's theory to conclude the following:

Theorem 2.1.1. (i) *Let ψ be a linear character of $A_n(q)$, let ϕ be an irreducible character of the centralizer $C_{U_{n-1}(q)}(\psi)$ of ψ in $U_{n\text{-}1}(q)$ and let $\psi\phi$ be the irreducible character of $A_n(q)C_{U_{n-1}(q)}(\psi)$ defined by*

$$(\psi\phi)(ax)=\psi(a)\phi(x)$$

for all $a\in A_n(q)$ and all $x\in C_{U_{n-1}(q)}(\psi)$. Then the induced character $(\psi\phi)^{U_n(q)}$ is an irreducible character of $U_n(q)$.

(ii) *Let χ be an irreducible character of $U_n(q)$. Then there exist a linear character ψ of $A_n(q)$ and an irreducible character ϕ of $C_{U_{n-1}(q)}(\psi)$ such that*

$$\chi=(\psi\phi)^{U_n(q)}.$$

(iii) *Let ψ and ϕ be as in (i). Let $x\in U_{n-1}(q)$ and let $\psi'=\psi^x$ be the linear character of $A_n(q)$ defined by*

$$\psi^x(a)=\psi(xax^{-1})$$

for all $a\in A_n(q)$. Then

$$\chi=(\psi'\phi')^{U_n(q)}$$

where $\phi'=\phi^x$ is the character of $C_{U_{n-1}(q)}(\psi')=x^{-1}C_{U_{n-1}(q)}(\psi)x$ defined by

$$\phi'(x^{-1}yx)=\phi(y)$$

for all $y\in C_{U_{n-1}(q)}(\psi)$.

Proof. (i) and (ii) are applications of [CR; theorem 11.5]) and of ([CR; proposition 11.8]) (respectively) and (iii) can be proved easily using the definition of induced characters. ◆

Now we consider the irreducible characters of $A_n(q)$. This group is isomorphic to a direct sum of n-1 copies of the additive group $\boldsymbol{F}_q^{\,+}$ of the field $\boldsymbol{F}_q$. Therefore the irreducible characters of $A_n(q)$ correspond to sequences $\psi=(\psi_1,\ldots,\psi_{n-1})$ of n-1 linear characters of $\boldsymbol{F}_q^{\,+}$ where

$$\psi(x)=\psi_1(x_{1n})\ldots\psi_{n-1}(x_{n-1n})$$

for all $x=(x_{rs})\in A_n(q)$. Let ψ_0 be a (fixed) non-trivial linear character of $\boldsymbol{F}_q^{\,+}$. Then any irreducible character of $\boldsymbol{F}_q^{\,+}$ is of the form $\alpha\psi_0$ for some $\alpha\in\boldsymbol{F}_q$ where $\alpha\psi_0$ is defined by

$$(\alpha\psi_0)(\beta)=\psi_0(\alpha\beta)$$

for all $\beta\in\boldsymbol{F}_q$. Thus the irreducible characters of $A_n(q)$ are in one-to-one correspondence with sequences $(\alpha_1,\ldots,\alpha_{n-1})\in\boldsymbol{F}_q^{\,n-1}$. If an irreducible character ψ of $A_n(q)$ corresponds to the sequence $(\alpha_1,\ldots,\alpha_{n-1})\in\boldsymbol{F}_q^{\,n-1}$ then we will identify ψ with the row vector $(\alpha_1\ \ldots\ \alpha_{n-1})$. As a trivial example the unit character $1_{A_n(q)}$ of $A_n(q)$ corresponds to the sequence $(0,\ldots,0)$, hence

$$1_{A_n(q)}=(0\ \ldots\ 0).$$

Lemma 2.1.2. *Let $\psi=(\alpha_1 \dots \alpha_{n-1})$ ($\alpha_k \in \mathbf{F}_q$, $1 \le k \le n-1$) be a non-trivial linear character of $A_n(q)$ and let $i \in \{1,\dots,n-1\}$ be the smallest integer such that $\alpha_i \ne 0$. Then the $U_{n-1}(q)$-orbit of ψ consists of all row vectors $(0 \dots 0\ \alpha_i\ \beta_{i+1} \dots \beta_{n-1})$ where $\beta_k \in \mathbf{F}_q$ for all $k \in \{i+1,\dots,n-1\}$. In particular this orbit contains a unique character with the form $(0 \dots 0\ \alpha_i\ 0 \dots 0)$.*

Proof. Let $x=(x_{rs}) \in U_{n-1}(q)$ and let $a=(a_{rs}) \in A_n(q)$. Then

$$\begin{aligned}
\psi^x(a)&=\psi(xax^{-1})\\
&=(\alpha_1\psi_0)((xax^{-1})_{1n})\dots(\alpha_{n-1}\psi_0)((xax^{-1})_{n-1n})\\
&=(\alpha_1\psi_0)\left(\sum_{k=1}^{n-1}x_{1k}a_{kn}\right)\cdot(\alpha_2\psi_0)\left(\sum_{k=2}^{n-1}x_{2k}a_{kn}\right)\cdot\dots\cdot(\alpha_{n-1}\psi_0)(a_{n-1n})\\
&=\psi_0\left(\left(\sum_{k=1}^{n-1}\alpha_1x_{1k}a_{kn}\right)+\left(\sum_{k=2}^{n-1}\alpha_2x_{2k}a_{kn}\right)+\dots+\alpha_{n-1}a_{n-1n}\right)\\
&=\psi_0\left(\alpha_1a_{1n}+(\alpha_1x_{12}+\alpha_2)a_{2n}+\dots+\left(\sum_{k=1}^{n-1}\alpha_kx_{kn-1}\right)a_{n-1n}\right)\\
&=(\alpha_1\psi_0)(a_{1n})\cdot((\alpha_1x_{12}+\alpha_2)\psi_0)(a_{2n})\cdot\dots\cdot\left(\left(\sum_{k=1}^{n-1}\alpha_kx_{kn-1}\right)\psi_0\right)(a_{n-1n})\\
&=\left(\alpha_1\psi_0,(\alpha_1x_{12}+\alpha_2)\psi_0,\dots,\left(\sum_{k=1}^{n-1}\alpha_kx_{kn-1}\right)\psi_0\right)(a).
\end{aligned}$$

Therefore

$$\psi^x=(\alpha_1 \dots \alpha_{n-1})\,y$$

where y is a unitriangular matrix of order $n-1$ such that

$$x=\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}.$$

Now let $\beta_{i+1},\dots,\beta_{n-1} \in \mathbf{F}_q$ and let $x=(x_{rs}) \in U_{n-1}(q)$ be the matrix defined by

$$x_{rs}=\begin{cases}\alpha_i^{-1}(\beta_s-\alpha_s) & \text{if } r=i \text{ and } s\ne n\\ 0 & \text{otherwise}\end{cases}$$

for all $(r,s) \in \Phi(n)$. Then

$$(\alpha_1 \dots \alpha_{n-1})\,y=(0 \dots 0\ \alpha_i\ \beta_{i+1} \dots \beta_{n-1})$$

where y is as above. The lemma follows. ◆

In the notation of the previous lemma the character $(0\dots0\,\alpha_i\,0\dots0)$ will be called the *canonical character* in the $U_{n-1}(q)$-orbit of ψ and the index i will be referred to as the *type of* ψ. It is clear that all $U_{n-1}(q)$-conjugates of ψ have the same type. The unit character of $A_n(q)$ will be called the *canonical character of type 0.*

Lemma 2.1.3. *Let* $\psi=(0\dots0\,\alpha\,0\dots0)$ *be a canonical character of* $A_n(q)$ *of type i. Then the centralizer* $C_{U_{n-1}(q)}(\psi)$ *of* ψ *in* $U_{n-1}(q)$ *consists of all matrices* $x=(x_{rs})\in U_{n-1}(q)$ *which satisfy* $x_{ii+1}=\dots=x_{in-1}=0$. *Therefore*

$$C_{U_{n-1}(q)}(\psi)=U_{n-1}(q)\cap\omega^{-1}U_{n-1}(q)\omega$$

where $\omega\in S_n$ *is the permutation* $(n\text{-}1\dots i\text{+}1\,i)$.

Proof. The first assertion follows easily because $\psi^x=\psi$ if and only if

$$(0\dots0\,\alpha\,0\dots0)\,y=(0\dots0\,\alpha\,0\dots0)$$

for all $x=\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}\in U_n(q)$. The second assertion is a consequence of the decomposition

$$U_{n-1}(q)\cap\omega^{-1}U_{n-1}(q)\omega=\prod_{\substack{1\le r<s\le n-1\\ \omega(r)<\omega(s)}}X_{rs}(q)$$

(cf (1.1.14) and (1.1.15)). ♦

Let χ be an arbitrary irreducible character of $U_n(q)$. Then (by theorem 2.1.1)

$$\chi=(\psi\phi)^{U_n(q)}$$

where

$$\psi=(0\dots0\,\alpha\,0\dots0)$$

is a canonical character of $A_n(q)$ of type i and ϕ is an irreducible character of $U_{n-1}(q)\cap\omega^{-1}U_{n-1}(q)\omega$, $\omega=(n\text{-}1\dots i\text{+}1\,i)\in S_n$ (if $i=0$ then $\psi=(0\dots0)$ and we put $\omega=1\in S_n$). Since

$$A_n(q)(U_{n-1}(q)\cap\omega^{-1}U_{n-1}(q)\omega)=U_n(q)\cap\omega^{-1}U_n(q)\omega=U_\omega(q)$$

(cf. (1.1.14) and (1.1.15)) we conclude that χ is induced by the irreducible character $\psi\phi$ of $U_\omega(q)$.

Now suppose that ϕ is the unit character $1_{U_{n-1}(q)\cap\omega^{-1}U_{n-1}(q)\omega}$ of $U_{n-1}(q)\cap\omega^{-1}U_{n-1}(q)\omega$. Then we obtain a linear character

$$\lambda_{in}(\alpha)=\psi 1_{U_{n-1}(q)\cap\omega^{-1}U_{n-1}(q)\omega}$$

of $U_\omega(q)$. By definition we have

(2.1.1) $$\lambda_{in}(\alpha)(x)=(\alpha\psi_0)(x_{in})=\psi_0(\alpha x_{in})$$

for all $x=(x_{rs})\in U_\omega(q)$. Then the induced character

(2.1.2) $$\xi_{in}(\alpha)=(\lambda_{in}(\alpha))^{U_n(q)}$$

is an irreducible character of $U_n(q)$ (by theorem 2.1.1). This character will be referred to as the *(i,n)-th elementary character of $U_n(q)$ associated with $\alpha\in \mathbf{F}_q^*$*. We have:

Proposition 2.1.4. *Let $i\in\{1,\dots,n-1\}$, let $\omega=(n-1 \dots i+1\, i)\in S_n$ and let $\alpha\in\mathbf{F}_q^*$. Then the function $\lambda_{in}(\alpha):U_\omega(q)\to\mathbb{C}^*$ defined by (2.1.1) is a linear character of $U_\omega(q)$ and the induced character $\xi_{in}(\alpha)=(\lambda_{in}(\alpha))^{U_n(q)}$ is an irreducible character of $U_n(q)$.*

In general we may associate with any root $(i,j)\in\Phi(n)$ and any element $\alpha\in\mathbf{F}_q$ an irreducible character $\xi_{ij}(\alpha)$ of $U_n(q)$ as follows:

Proposition 2.1.5. *Let $(i,j)\in\Phi(n)$, let $\omega=(j-1 \dots i+1\, i)\in S_n$ and let $\alpha\in\mathbf{F}_q^*$. Then the function $\lambda_{ij}(\alpha):U_\omega(q)\to\mathbb{C}^*$ defined by*

(2.1.3) $$\lambda_{ij}(\alpha)(x)=\psi_0(\alpha x_{ij})$$

for all $x=(x_{rs})\in U_\omega(q)$ is a linear character of $U_\omega(q)$ and the induced character

(2.1.4) $$\xi_{ij}(\alpha)=(\lambda_{ij}(\alpha))^{U_n(q)}$$

is an irreducible character of $U_n(q)$.

Proof. We apply proposition 2.1.4 to the group $U_j(q)$. This group may be identified with the subgroup of $U_n(q)$ consisting of all matrices whose non-diagonal entries of the last $n-j-1$ columns are zero. Since ω may be considered as an element of the symmetric group S_j we have

$$U_{\omega}(q)\cap U_j(q)=U_j(q)\cap\omega^{-1}U_j(q)\omega.$$

By propositon 2.1.4 the restriction of $\lambda_{ij}(\alpha)$ to $U_{\omega}(q)\cap U_j(q)$ is a linear character of $U_{\omega}(q)\cap U_j(q)$. We denote this character by $\mu_{ij}(\alpha)$. Now the subgroup $U_{\omega}(q)\cap U_j(q)$ of $U_{\omega}(q)$ has a normal complement, namely the subgroup

$$\bar{A}_{j+1}(q)=A_n(q)A_{n-1}(q)\ldots A_{j+1}(q)$$

where for each $k\in\{1,\ldots,n\}$ $A_k(q)$ denotes the subgroup of $U_n(q)$ consisting of all matrices whose non-diagonal entries are zero except in the k-th col umn. Therefore we may consider the lifting $\mu_{ij}(\alpha)^*$ of $\mu_{ij}(\alpha)$ from $U_{\omega}(q)\cap U_j(q)$ to $U_{\omega}(q)$. By definition we have

$$\mu_{ij}(\alpha)^*(xa)=\mu_{ij}(\alpha)(x)=\lambda_{ij}(\alpha)(x)$$

for all $x\in U_{\omega}(q)\cap U_j(q)$ and all $a\in\bar{A}_{j+1}(q)$. It follows that

$$\mu_{ij}(\alpha)^*=\lambda_{ij}(\alpha)$$

so $\lambda_{ij}(\alpha)$ is a linear character of $U_{\omega}(q)$.

On the other hand the induced character

$$\zeta_{ij}(\alpha)=(\mu_{ij}(\alpha))^{U_j(q)}$$

is an irreducible character of $U_j(q)$ (proposition 2.1.4). Since $\bar{A}_{j+1}(q)$ is a normal complement of $U_j(q)$ in $U_n(q)$ the lifting $\zeta_{ij}(\alpha)^*$ from $U_j(q)$ to $U_n(q)$ is an irreducible character of $U_n(q)$. Since lifting commutes with induction we deduce that

$$\zeta_{ij}(\alpha)^*=(\mu_{ij}(\alpha)^*)^{U_n(q)}=(\lambda_{ij}(\alpha))^{U_n(q)}$$

is an irreducible character of $U_n(q)$ as required. ♦

For each root $(i,j)\in\Phi(n)$ and each element $\alpha\in\boldsymbol{F}_q{}^*$ the irreducible character $\xi_{ij}(\alpha)$ of $U_n(q)$ defined in the previous proposition will be called the *(i,j)-th elementary character associated with α.*

Next we prove that the elementary characters are all distinct. Therefore $U_n(q)$ has $(q-1)^{\frac{n(n-1)}{2}}$ distinct elementary characters.

Lemma 2.1.6. *Let $(i,j),(k,l)\in\Phi(n)$ and let $\alpha,\beta\in \boldsymbol{F}_q{}^*$.* Then $\xi_{ij}(\alpha)=\xi_{kl}(\beta)$ *if and only if $(i,j)=(k,l)$ and $\alpha=\beta$.*

Proof. Suppose that $\xi_{ij}(\alpha)=\xi_{kl}(\beta)$. Without loss of generality we may assume that $l\leq j=n$. Then the irreducible character $\xi_{in}(\alpha)$ is uniquely determined by the canonical character

$$\psi=(0\ldots 0\,\alpha\,0\ldots 0)$$

of $A_n(q)$ of type i. Suppose that $l<n$. Then (in the notation of the previous proof)

$$\xi_{kl}(\beta)=\zeta_{kl}(\beta)^*.$$

It follows that the unit character $1_{A_n(q)}$ is the unique component of the restriction of $\xi_{kl}(\beta)$ to $A_n(q)$. On the other hand ψ is a component of the restriction of $\xi_{jn}(\alpha)$ to $A_n(q)$ (see [CR; proposition 11.8]). So $\psi=1_{A_n(q)}$. This contradiction implies that $l=j=n$. Hence $\xi_{kn}(\beta)$ is uniquely determined by the canonical character

$$\phi=(0\ldots 0\,\beta\,0\ldots 0)$$

of $A_n(q)$ of type k. Therefore ϕ is a component of the restriction of $\xi_{kn}(\beta)$ to $A_n(q)$ and so ϕ is $U_n(q)$-conjugate to ψ. By lemma 2.1.3 we conclude that $k=i$ and $\beta=\alpha$. ♦

Now we let K be the algebraic closure of the field $\boldsymbol{F}_q$ and we realize the group $U_n(q)$ as the group $U_n(K)^F$ consisting of all fixed elements of the Frobenius map $F=F_q:U_n(K)\rightarrow U_n(K)$.

Let $(i,j)\in\Phi(n)$ and let $\alpha\in \boldsymbol{F}_q{}^*$. Since the elementary character $\xi_{ij}(\alpha)$ is an irreducible character of $U_n(q)$ there exists an F-stable $U_n(K)$-orbit $O\subseteq\mathcal{U}_n(K)^*$ such that

$$\xi_{ij}(\alpha)=\chi_O$$

(see (1.2.12)). We will prove that O is the $U_n(K)$-orbit of the element $\alpha e_{ij}{}^*\in\mathcal{U}_n(K)^*$ where $e_{ij}{}^*:\mathcal{U}_n(K)\rightarrow K$ is the dual vector of the root vector $e_{ij}\in\mathcal{U}_n(K)$. In general for any $\alpha\in K$ the $U_n(K)$-orbit of the element $\alpha e_{ij}{}^*$ will be denoted by $O_{ij}(\alpha)$ and it will be referred to as the *(i,j)-th elementary $U_n(K)$-orbit associated with α*. We note that

$$O_{ij}(0)=\{0\}$$

so $O_{ij}(0)$ corresponds to the unit character $1_{U_n(q)}$ of $U_n(q)$. Also for any $\alpha\in \mathbf{F}_q$ the $U_n(K)$-orbit $O_{ij}(\alpha)$ is F-stable.

Next we follow the construction of section 1.2 to determine the irreducible character of $U_n(q)$ which corresponds to $O_{ij}(\alpha)$. Firstly we find a polarization for αe_{ij}^*.

Lemma 2.1.7. *Let $(i,j)\in\Phi(n)$, let $\omega=(j\ \ldots\ i+1\,i)\in S_n$ and let $\alpha\in K^*$. Then the subalgebra $\mathfrak{u}_\omega(K)=\mathfrak{u}_n(K)\cap\omega^{-1}\mathfrak{u}_n(K)\omega$ of $\mathfrak{u}_n(K)$ is a maximal (αe_{ij}^*)-isotropic subspace of $\mathfrak{u}_n(K)$.*

Proof. Let $a,b\in \mathfrak{u}_\omega(K)$ be arbitrary. Then

$$[ab]_{ij}=(ab)_{ij}-(ba)_{ij}=\sum_{k=i+1}^{j-1}(a_{ik}b_{kj}-b_{ik}a_{kj})=0$$

because $a_{ik}=b_{ik}=0$ for all $k\in\{i+1,\ldots,j-1\}$. Therefore

$$\alpha e_{ij}^*([ab])=\alpha[ab]_{ij}=0.$$

It follows that $\mathfrak{u}_\omega(K)$ is an (αe_{ij}^*)-isotropic subspace of $\mathfrak{u}_n(K)$.

Now

$$dim\,\mathfrak{u}_\omega(K)=dim\,\mathfrak{u}_n(K)-(j-i-1).$$

On the other hand, with respect to the basis

$$(e_{ik},e_{kj};\ i+1\le k\le j-1)\cup(e_{rs};\ 1\le r<s\le n,\ r\ne i,\ s\ne j)\cup(e_{rj};\ 1\le r\le i)\cup(e_{is};\ j+1\le s\le n)$$

the matrix $M(\alpha e_{ij}^*)$ has the form

$$M(\alpha e_{ij}^*)=\begin{pmatrix} X & 0\\ 0 & 0\end{pmatrix}$$

where X is the square matrix of size $2(j-i-1)$

$$X=\begin{pmatrix} J_\alpha & 0 & \ldots & 0\\ 0 & J_\alpha & \ldots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \ldots & J_\alpha\end{pmatrix}$$

and

$$J_\alpha = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}.$$

Since $\alpha \neq 0$ the matrix X is non-singular and so $M(\alpha e_{ij}^*)$ has rank $2(j-i-1)$. It follows that

$$\dim O_{ij}(\alpha) = 2(j-i-1)$$

(by (1.3.3)). By corollary 1.2.3 we conclude that $\mathfrak{u}_\omega(K)$ is an (αe_{ij}^*)-maximal isotropic of $\mathfrak{u}_n(K)$ as required. ◆

Proposition 2.1.8. *Let $(i,j) \in \Phi(n)$ and let $\alpha \in \mathbf{F}_q^*$. Then the $U_n(K)$-orbit $O_{ij}(\alpha)$ is F-stable and it corresponds to the irreducible character $\xi_{ij}(\alpha)$ of $U_n(q)$, i.e.*

$$\xi_{ij}(\alpha) = \chi_{O_{ij}(\alpha)}.$$

The correspondence $\xi_{ij}(\alpha) \to O_{ij}(\alpha)$ is one-to-one between elementary characters of $U_n(q)$ and non-trivial elementary F-stable $U_n(K)$-orbits on $\mathfrak{u}_n(K)^*$.

Proof. Since $\alpha \in \mathbf{F}_q^*$ the element $\alpha e_{ij}^* \in O_{ij}(\alpha)$ is F-fixed and the $U_n(K)$-orbit is F-stable. Now let $\omega = (j-1 \ldots i+1\, i) \in S_n$. Then $U_\omega(K) = \exp \mathfrak{u}_\omega(K)$ is F-stable and $U_\omega(q) = \exp \mathfrak{u}_\omega(q)$. Since $\mathfrak{u}_\omega(K)$ is a polarization for αe_{ij}^* and $\xi_{ij}(\alpha)$ is induced by the linear character $\lambda_{ij}(\alpha)$ of $U_\omega(q)$ it is enough to show that this linear character is defined by the element $\alpha e_{ij}^* \in \mathfrak{u}_n(K)^*$. In fact the homomorphism $\phi_f: U_\omega(K) \to K^+$ is defined by

$$\phi_f(\exp a) = f(a) = \alpha a_{ij}$$

for all $a = (a_{rs}) \in \mathfrak{u}_\omega(K)$ (cf. (1.2.9)). However

$$(\exp a)_{ij} = \left(1 + a + \frac{a^2}{2!} + \ldots + \frac{a^{n-1}}{(n-1)!}\right)_{ij} = a_{ij}$$

because

$$(a^k)_{ij} = \sum_{r=i+1}^{j-1} a_{ir}(a^{k-1})_{rj}$$

and $a_{ir} = 0$ for all $r \in \{i+1, \ldots, j-1\}$. Therefore

$$\phi_f(x) = \alpha x_{ij}$$

for all $x = (x_{rs}) \in U_\omega(K)$. It follows that the character $\lambda_f: U_\omega(q) \to \mathbf{C}^*$ is given by

$$\lambda_f(x) = \psi_0 \phi_f(x) = \psi_0(\alpha x_{ij})$$

for all $x=(x_{rs})\in U_{\alpha}(q)$ (cf. (1.2.10)). So

$$\lambda_f=\lambda_{ij}(\alpha)$$

and the lemma follows. ♦

Let $(i,j)\in\Phi(n)$. Then the previous proposition allows the definition of the *(i,j)-th elementary character $\xi_{ij}(0)$ associated with* $0\in\boldsymbol{F}_q$ to be the irreducible character which corresponds to the F-stable $U_n(K)$-orbit $O_{ij}(0)$. Since $O_{ij}(0)=\{0\}$ we have

(2.1.5) $$\xi_{ij}(0)=1_{U_n(q)}.$$

2.2. Products of elementary characters

We recall that a subset D of $\Phi(n)$ is *basic* if it has at most one root from each column and one root from each row. More precisely $D\subseteq\Phi(n)$ is *basic* if it has the form

$$D=\omega(\Delta)\cap\Phi(n)$$

where $\omega\in S_n$ and

$$\Delta=\{(1,2),\ldots,(n-1,n)\}$$

is the set of simple roots. In particular the empty set is a basic subset of $\Phi(n)$. In fact

$$\emptyset=\omega_0(\Delta)\cap\Phi(n)$$

where $\omega_0\in S_n$ is the permutation defined in (1.1.16). On the other hand if D is a non-empty basic subset of $\Phi(n)$ then

$$D=\{(i_1,j_1),(i_2,j_2),\ldots,(i_r,j_r)\}$$

where $(i_1,j_1),(i_2,j_2),\ldots,(i_r,j_r)\in\Phi(n)$ are such that $j_1<j_2<\ldots<j_r$ and $i_s\neq i_{s'}$ for all $s,s'\in\{1,\ldots,r\}$. Conversely any subset of $\Phi(n)$ with this form is basic.

Let D be a non-empty basic subset of $\Phi(n)$ and let $\varphi:D\to\boldsymbol{F}_q{}^*$ be a map. Then we denote by $\xi_D(\varphi)$ the character of $U_n(q)$ defined by

$$\xi_D(\varphi)=\prod_{(i,j)\in D}\xi_{ij}(\varphi(i,j)). \tag{2.2.1}$$

On the other hand if $D\subseteq\Phi(n)$ is the empty subset then $\varphi:D\to\boldsymbol{F}_q{}^*$ is the empty function ([1]) and we define $\xi_D(\varphi)$ to be the unit character $1_{U_n(q)}$ of $U_n(q)$.

The main goal of this section is the proof the following result:

Theorem 2.2.1. *Let χ be an irreducible character of $U_n(q)$. Then there exists a unique basic subset D of $\Phi(n)$ and a unique map $\varphi:D\to\boldsymbol{F}_q{}^*$ such that χ is a component of $\xi_D(\varphi)$.*

---

[1] We note that, if X and Y are two sets, a function $\varphi:X\to Y$ may be considered as a subset of the Cartesian product $X\times Y$.

This theorem will be proved later in the section. Firstly we will discuss some examples which illustrate the general situation and which will be useful in some steps of our proof. As one should expect our arguments are based on the general results of chapter 1. In particular corollary 1.3.11 is fundamental for the study of the characters $\xi_D(\varphi)$. In fact it implies that for any non-empty basic subset D of $\Phi(n)$ and any map $\varphi: D \to \mathbf{F}_q^*$ the irreducible components of $\xi_D(\varphi)$ are in one-to-one correspondence with the F-stable $U_n(K)$-orbits which are contained in the sum

(2.2.2) $$O_D(\varphi) = \sum_{(i,j)\in D} O_{ij}(\varphi(i,j)).$$

If $D=\varnothing$ then we define $O_D(\varphi)$ (φ is the empty function) to be the orbit $\{0\}$ of the element $0\in \mathcal{U}_n(K)^*$. The (irreducible) varieties $O_D(\varphi)$ will be described in section 3.1 where we will obtain a decomposition of $\mathcal{U}_n(K)^*$ which generalizes theorem 2.2.1 (cf. theorem 3.1.7).

Our first example is trivial. We describe the linear characters of $U_n(q)$ in terms of products of the form (2.2.1). It is well-known that the commutator subgroup $U_n(q)'$ of $U_n(q)$ consists of all matrices $x=(x_{ij})\in U_n(q)$ which satisfy $x_{12}=\ldots=x_{n-1n}=0$. Moreover the quotient group $U_n(q)/U_n(q)'$ is isomorphic to a direct sum of n-1 copies of the additive group $\mathbf{F}_q^+$. Therefore $U_n(q)$ has q^{n-1} distinct linear characters which are in one-to-one correspondence with sequences $(\alpha_1,\ldots,\alpha_{n-1})$ of elements $\alpha_i\in \mathbf{F}_q$ $(1\le i\le n-1)$. These characters can also described as follows:

Lemma 2.2.2. *Let $\alpha_1,\ldots,\alpha_{n-1}\in \mathbf{F}_q$. Then the linear character of $U_n(q)$ which corresponds to the sequence $(\alpha_1,\ldots,\alpha_{n-1})$ is $\xi_{12}(\alpha_1)\ldots\xi_{n-1n}(\alpha_{n-1})$. Therefore the linear characters of $U_n(q)$ are in one-to-one correspondence with pairs (D,φ) where D is a subset of $\Delta=\{(1,2),\ldots,(n-1,n)\}$ and $\varphi: D\to \mathbf{F}_q^*$ is a map.*

Proof. It is clear that the character $\xi_{12}(\alpha_1)\ldots\xi_{n-1n}(\alpha_{n-1})$ is linear. Therefore it is irreducible and it corresponds to an F-stable $U_n(K)$-orbit O of dimension zero. By corollary 1.3.11 we must have

$$O=O_{12}(\alpha_1)+\ldots+O_{n-1n}(\alpha_{n-1}).$$

Hence O contains the element

$$\alpha_1 e_{12}{}^*+\ldots+\alpha_{n-1}e_{n-1n}{}^*\in U_n(K)^*.$$

Since O is connected it contains only this element. Thus

$$\xi_{12}(\alpha_1)\ldots\xi_{n-1n}(\alpha_{n-1})\neq\xi_{12}(\beta_1)\ldots\xi_{n-1n}(\beta_{n-1})$$

whenever $(\alpha_1,\ldots,\alpha_{n-1})\neq(\beta_1,\ldots,\beta_{n-1})$ $(\alpha_i,\beta_i\in\mathbf{F}_q,\ 1\leq i\leq n-1)$. The lemma follows. ♦

In the next lemma we identify some irreducible characters which have the form (2.2.1). This result was proved by Lehrer (see [Le; theorem 5.2]) using Mackey's theorem on the decomposition of tensor products. We give a different proof which can be extended to establish that (with the same conditions on the set D) the variety $O_D(\varphi)$ is a single $U_n(K)$-orbit (cf. theorem 3.3.3).

Lemma 2.2.3. *Let $D=\{(i_1,j_1),\ldots,(i_r,j_r)\}$ be a basic subset of $\Phi(n)$ and suppose that $i_1<i_2<\ldots<i_r$ and that $j_1>j_2>\ldots>j_r$. Let $\varphi:D\to\mathbf{F}_q{}^*$ be any map. Then the character $\xi_D(\varphi)$ is irreducible.*

Proof. We consider the sum $O_D(\varphi)$. Let $\alpha_s=\varphi(i_s,j_s)$ $(1\leq s\leq r)$. Since $O_{i_sj_s}(\alpha_s)$ is the $U_n(K)$-orbit of $\alpha_s e_{i_sj_s}{}^*$ $(1\leq s\leq r)$, the element

$$f=\sum_{s=1}^{r}\alpha_s(e_{i_sj_s})^*$$

lies in $O_D(\varphi)$. Thus

$$O(f)\subseteq O_D(\varphi)$$

(we recall that $O(f)$ denotes the $U_n(K)$-orbit of f). Since f is F-fixed $O(f)$ is F-stable. Therefore (by corollary 1.3.11) the irreducible character $\chi=\chi_{O(f)}$ of $U_n(q)$ is a component of $\xi_D(\varphi)$. We claim that

$$\chi=\xi_D(\varphi).$$

For we compare the degrees of these two characters. On the one hand we have

$$\xi_D(\varphi)(1)=q^m,\quad m=\sum_{s=1}^{r}(j_s-i_s-1),$$

because $\xi_{i_s j_s}(\alpha_s)$ has degree $q^{j_s-i_s-1}$ $(1\leq s\leq r)$. On the other hand (by proposition 1.2.5)

$$\chi(1)=\sqrt{|O(f)^F|}=\sqrt{q^{dimO}}.$$

By (1.3.3) we have

$$dimO(f)=rankM(f)$$

where $M(f)$ is the matrix which represents the bilinear form B_f with respect to the basis $(e_{ij};(i,j)\in\Phi(n))$. Now for each $s\in\{1,\ldots,r\}$ and each $k\in\{i_s+1,\ldots,j_s-1\}$ the plane

$$H_{sk}=Ke_{i_s k}+Ke_{kj_s}$$

is non-singular because

$$f([e_{i_s k},e_{kj_s}])=f(e_{i_s j_s})=\alpha_s\neq 0.$$

Moreover the subspace

$$V=\sum_{s=1}^{r}\sum_{k=i_s+1}^{j_s-1}H_{sk}$$

is an orthogonal sum of non-singular planes. It follows that V is non-singular so

$$rankM(f)\geq dimV.$$

Since $dimV=2m$ we deduce that

$$\chi(1)=\sqrt{q^{dimO}}=\sqrt{q^{rankM(f)}}\geq\sqrt{q^{dimV}}=q^m.$$

Finally since χ is a component of $\xi_D(\varphi)$ we conclude that

$$\chi(1)=\xi_D(\varphi)(1)$$

so $\chi=\xi_D(\varphi)$ as claimed. The lemma is proved. ◆

Corollary 2.2.4. *Let $D=\{(1,n),(2,n-1),\ldots,(r,n-r+1)\}$ where either $n=2r$ or $n=2r+1$. Let $\varphi:D\rightarrow\mathbb{F}_q^*$ be any map. Then the character $\xi_D(\varphi)$ is irreducible. It has degree $q^{\mu(n)}$ where $\mu(n)=$ $(n-2)+(n-4)+\ldots$* [1]*. Moreover if $n=2r$ is even, the characters $\xi_D(\varphi)$ where $D=\{(1,n),(2,n-1),\ldots,(r-1,n-r+2)\}$ and $\varphi:D\rightarrow\mathbb{F}_q^*$ is any map have degree $q^{\mu(n)}$. Therefore $U_n(q)$ has at least $(q-1)^r$ irreducible characters of degree $q^{\mu(n)}$, if $n=2r+1$ is odd,*

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(1) We have $\mu(n)=r^2-r$ if $n=2r$ is even, and $\mu(n)=r^2$ if $n=2r+1$ is odd

and at least $q(q-1)^{r-1}$ irreducible characters of degree $q^{\mu(n)}$, if $n=2r$ is even.

Proof. It remains only to prove that the given characters are distinct. For we let $\alpha_1,\dots,\alpha_r \in \boldsymbol{F}_q$ be arbitrary and we consider the element

$$f=\alpha_1 e_{1n}{}^*+\dots+\alpha_r e_{rn-r+1}{}^* \in \mathcal{U}_n(K)^*.$$

Then

$$(x\cdot f)(a)=f(xax^{-1})=\alpha_1(xax^{-1})_{1n}+\dots+\alpha_r(xax^{-1})_{rn-r+1}.$$

Let $\beta_1,\dots,\beta_r \in \boldsymbol{F}_q$ and suppose that the element

$$g=\beta_1 e_{1n}{}^*+\dots+\beta_r e_{rn-r+1}{}^* \in \mathcal{U}_n(K)^*$$

lies in the $U_n(K)$-orbit of f. Then there exists $x\in U_n(K)$ such that

$$g(a)=f(xax^{-1})$$

for all $a\in \mathcal{U}_n(K)$. In particular let $a=e_{in-i+1}$ $(1\leq i\leq r)$. Then

$$(xe_{in-i+1}x^{-1})_{jn-j+1}=\delta_{ij}$$

for all $j\in\{1,\dots,r\}$. Therefore we get

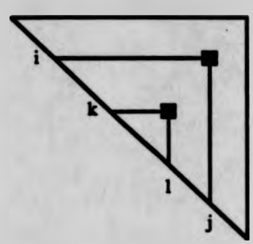
$$\beta_i=g(e_{in-i+1})=f(xe_{in-i+1}x^{-1})=\alpha_i$$

for all $i\in\{1,\dots,r\}$. The lemma follows. ♦

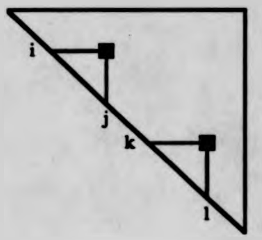
It was proved by Lehrer ([Le; corollary 5.2']) that $q^{\mu(n)}$ is the maximal degree of the irreducible characters of $U_n(q)$. This result will follow as a corollary of theorem 2.2.1 (see corollary 2.2.15). Moreover theorem 2.2.1 implies also that any irreducible character of $U_n(q)$ of maximal degree has the form $\xi_D(\varphi)$ where D and φ are as in previous corollary.

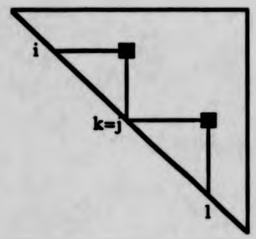
In the next lemmas we study the decomposition of the products $\xi_{ij}(\alpha)\xi_{kl}(\beta)$ where $(i,j),(k,l)\in\Phi(n)$ and $\alpha,\beta\in \boldsymbol{F}_q{}^*$. It will turn out that this decomposition depends on the relative position of the two roots (i,j) and (k,l).

By the previous lemma $\xi_{ij}(\alpha)\xi_{kl}(\beta)$ is irreducible whenever $i<k<l<j$. The adjacent diagram illustrates this situation - the symbol ■ represents a root in D (this symbology will be used throughout the thesis).



Other possibilities are illustrated by the following pictures.



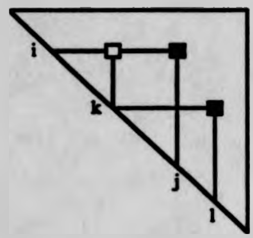


Our next result asserts that in both situations any character $\xi_{ij}(\alpha)\xi_{kl}(\beta)$ is irreducible (its proof is an immitation of the proof of lemma 2.2.3).

Lemma 2.2.5. *Let $(i,j),(k,l)\in \Phi(n)$ and suppose that $j\leq k$. Let $\alpha,\beta\in \mathbf{F}_q{}^*$. Then $\xi_{ij}(\alpha)\xi_{kl}(\beta)$ is an irreducible character of $U_n(q)$.*

Our next example shows that in general the character $\xi_D(\varphi)$ is not irreducible.

Lemma 2.2.6. *Let $(i,j),(k,l)\in \Phi(n)$ and suppose that $i<k<j<l$. Let $\alpha,\beta\in \mathbf{F}_q$. Then $\xi_{ij}(\alpha)\xi_{kl}(\beta)$ is a reducible character of $U_n(q)$. In fact let $\gamma\in \mathbf{F}_q$ and let $\chi(\gamma)$ be the irreducible character of $U_n(q)$ which corresponds to the F-stable $U_n(K)$-orbit of the element $\alpha e_{ij}{}^*+\beta e_{kl}{}^*+\gamma e_{ik}{}^*\in u_n(K)^*$. Then $\chi(\gamma)$ is a component of $\xi_{ij}(\alpha)\xi_{kl}(\beta)$ of degree $q^{(l+k)-(j+l)-3}$. Moreover the correspondence $\gamma\rightarrow\chi(\gamma)$ is one-to-one between the elements of $\mathbf{F}_q$ and the irreducible components of $\xi_{ij}(\alpha)\xi_{kl}(\beta)$.*



Proof. The adjacent diagram illustrates the situation of the lemma. Here the symbol □ means that the root (i,k) is associated with any element of the field (this root determines the irreducible components of the given character).

In order to use corollary 1.3.11 we consider the sum $O_{ij}(\alpha)+O_{kl}(\beta)$ of the elementary $U_n(K)$-orbits $O_{ij}(\alpha)$ and $O_{kl}(\beta)$. Let $\gamma\in K$ and let

$$x=x_{kj}(-\alpha^{-1}\gamma)\in X_{kj}(q).$$

Then

$$(x\cdot(\alpha e_{ij}^*))(a)=\alpha e_{ij}^*(xax^{-1})=\alpha(xax^{-1})_{ij}=\alpha(a_{ij}+\alpha^{-1}\gamma a_{ik})=(\alpha e_{ij}^*+\gamma e_{ik}^*)(a)$$

for all $a=(a_{rs})\in u_n(K)$. Since $O_{ij}(\alpha)$ is the $U_n(K)$-orbit of $\alpha e_{ij}^*\in u_n(K)^*$ we conclude that

$$\alpha e_{ij}^*+\gamma e_{ik}^*\in O_{ij}(\alpha).$$

On the other hand $O_{kl}(\beta)$ is the $U_n(K)$-orbit of $\beta e_{kl}^*\in u_n(K)^*$. Thus

$$\alpha e_{ij}^*+\gamma e_{ik}^*+\beta e_{kl}^*\in O_{ij}(\alpha)+O_{kl}(\beta).$$

Since $O_{ij}(\alpha)+O_{kl}(\beta)$ is $U_n(K)$-invariant we conclude that

$$O(\gamma)\subseteq O_{ij}(\alpha)+O_{kl}(\beta)$$

where $O(\gamma)$ is the $U_n(K)$-orbit of $\alpha e_{ij}^*+\gamma e_{ik}^*+\beta e_{kl}^*$.

Now let $\gamma,\gamma'\in \boldsymbol{F}_q$ and suppose that $O(\gamma)=O(\gamma')$. Then there exists $x=(x_{rs})\in U_n(K)$ such that

$$\alpha e_{ij}^*+\gamma' e_{ik}^*+\beta e_{kl}^*=x\cdot(\alpha e_{ij}^*+\gamma e_{ik}^*+\beta e_{kl}^*).$$

Applying this function to an arbitrary $a=(a_{rs})\in u_n(K)$ we obtain

$$\alpha a_{ij}+\gamma' a_{ik}+\beta a_{kl}=\alpha(xax^{-1})_{ij}+\gamma'(xax^{-1})_{ik}+\beta(xax^{-1})_{kl}.$$

In particular let $a=e_{ml}$ where $m\in\{k+1,\dots,l-1\}$. Then we get

$$0=\alpha x_{im}(x^{-1})_{lj}+\gamma' x_{im}(x^{-1})_{lk}+\beta x_{km}(x^{-1})_{ll}=\beta x_{km}$$

because $j<l$ and $k<l$. Since $\beta\neq 0$ it follows that

$$x_{km}=0$$

for all $m\in\{k+1,\dots,l-1\}$. On the other hand if $a=e_{ik}\in u_n(K)$ we obtain

$$\gamma'=\alpha(x^{-1})_{kj}+\gamma.$$

Since $(x^{-1})_{kj}=0$ we conclude that $\gamma'=\gamma$. Therefore the $U_n(K)$-orbits $O(\gamma)$ $(\gamma\in K)$ are all distinct.

Finally we consider the dimension of the $U_n(K)$-orbits $O(\gamma)$ for $\gamma\in \boldsymbol{F}_q$. For each $\gamma\in \boldsymbol{F}_q$ we denote by $M(\gamma)$ the matrix $M(\alpha e_{ij}{}^*+\gamma e_{ik}{}^*+\beta e_{kl}{}^*)$. The system of vectors $(e_{kr},e_{rl};\, k+1\leq r\leq l-1)\cup(e_{ir},e_{rj};\, i+1\leq r\leq j-1,\, r\neq k)$ span a non-singular subspace V of $u_n(K)$. Therefore

$$dimO(\gamma)=rankM(\gamma)\geq 2(l-k-1)+2(j-i-2)$$

and so

$$\chi(\gamma)(1)\geq q^{(l-k-1)+(j-i-2)}.$$

It follows that

$$\sum_{\gamma\in \boldsymbol{F}_q}\chi(\gamma)(1)\geq q^{(l-k-1)+(j-i-2)+1}.$$

Since

$$(\xi_{ij}(\alpha)\xi_{kl}(\beta))(1)=q^{l-k-1}q^{j-i-1}=q^{(l-k-1)+(j-i-1)}$$

we conclude that

$$\xi_{ij}(\alpha)\xi_{kl}(\beta)=\sum_{\gamma\in \boldsymbol{F}_q}\chi(\gamma)$$

and the lemma is proved. ♦

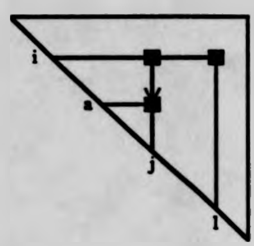
The next lemma will be fundamental for the proof of theorem 2.2.1. We note that in the situation of this lemma (as well as in the subsequent lemmas) the roots involved do not constitute a basic set.

Lemma 2.2.7. *Let $(i,j),(i,l)\in \Phi(n)$ and suppose that $j<l$. Let $\alpha,\beta\in \boldsymbol{F}_q{}^*$. Then*

$$\xi_{ij}(\alpha)\xi_{il}(\beta)=\xi_{il}(\beta)+\sum_{a=i+1}^{j-1}\sum_{\gamma\in \boldsymbol{F}_q{}^*}\xi_{aj}(\gamma)\xi_{il}(\beta)$$

is the decomposition $\xi_{ij}(\alpha)\xi_{il}(\beta)$ into irreducible components. Therefore $\xi_{ij}(\alpha)\xi_{il}(\beta)$ has one irreducible component of degree q^{l-i-1} and, for each $a\in\{i+1,\ldots,j-1\}$, q-1 irreducible components of degree $q^{(j-a-1)+(l-i-1)}$.

Proof. The situation of this lemma is illustrated by the adjacent diagram The symbol ■ represents the initial root (which does not occur in the decomposition) and the arrow means that there exists a certain "dislocation" of this root.



In the proof of the previous lemma we have justified that

$$\beta e_{il}^{*}-\alpha e_{ij}^{*}\in O_{il}(\beta).$$

Therefore

$$\beta e_{il}^{*}=\beta e_{il}^{*}-\alpha e_{ij}^{*}+\alpha e_{ij}^{*}\in O_{il}(\beta)+O_{ij}(\alpha).$$

This proves that the irreducible character $\xi_{il}(\beta)$ is a component of $\xi_{ij}(\alpha)\xi_{il}(\beta)$.

On the other hand let $a\in\{i+1,\dots,j-1\}$ and let $\gamma\in F_q^{*}$. By lemma 2.2.3 the character $\xi_{aj}(\gamma)\xi_{il}(\beta)$ is irreducible. This character corresponds to the F-stable $U_n(K)$-orbit $O_{aj}(\gamma)+O_{il}(\beta)$. Let us consider the element $\gamma e_{aj}^{*}+\beta e_{il}^{*}\in O_{aj}(\gamma)+O_{il}(\beta)$. Let $x=x_{ia}(\alpha^{-1}\gamma)\in X_{ia}(q)$. Then $\alpha e_{ij}^{*}+\gamma e_{aj}^{*}=x\cdot(\alpha e_{ij}^{*})$ (see the proof of the previous lemma) and so

$$\alpha e_{ij}^{*}+\gamma e_{aj}^{*}\in O_{ij}(\alpha).$$

It follows that

$$\gamma e_{aj}^{*}+\beta e_{ij}^{*}=\alpha e_{ij}^{*}+\gamma e_{aj}^{*}+\beta e_{il}^{*}-\alpha e_{ij}^{*}\in O_{ij}(\alpha)+O_{il}(\beta)$$

- we recall that $\beta e_{il}^{*}-\alpha e_{ij}^{*}\in O_{il}(\beta)$. Therefore

$$O_{aj}(\gamma)+O_{il}(\beta)\subseteq O_{ij}(\alpha)+O_{il}(\beta).$$

By corollary 1.3.11 the character $\xi_{aj}(\gamma)\xi_{il}(\beta)$ is an irreducible component of $\xi_{ij}(\alpha)\xi_{il}(\beta)$.

Finally we consider character degrees. We have

$$\xi_{il}(\beta)(1)=q^{l-i-1} \quad\text{and}\quad (\xi_{aj}(\gamma)\xi_{il}(\beta))(1)=q^{j-a-1}q^{l-i-1}$$

for all $a\in\{i+1,\dots,j-1\}$ and all $\gamma\in F_q^{*}$. On the other hand

$$q^{l-i-1}+\sum_{a=i+1}^{j-1}(q-1)q^{j-a-1}q^{l-i-1}=q^{l-i-1}\left(1+(q-1)\sum_{a=i+1}^{j-1}q^{j-a-1}\right)=q^{l-i-1}q^{j-i-1}.$$

The lemma follows because

$$(\xi_{ij}(\alpha)\xi_{il}(\beta))(1)=q^{l-i-1}q^{j-i-1}$$

and because the characters $\xi_{aj}(\gamma)\xi_{il}(\beta)$ ($i+1\leq a\leq j-1$, $\gamma\in \boldsymbol{F}_q^*$) are all distinct (this can be proved using a similar argument to the one used in the proof of lemma 2.1.6; we note that $\xi_{aj}(\gamma)\xi_{il}(\beta)\neq\xi_{bj}(\gamma)\xi_{il}(\beta)$ whenever $a\neq b$ because they have different degrees). ♦

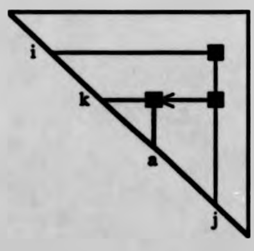
A similar proof can be used to obtain the following:

Lemma 2.2.8. *Let $(i,j),(k,j)\in \Phi(n)$ and suppose that $i<k$. Let $\alpha,\beta\in \boldsymbol{F}_q^*$. Then*

$$\xi_{ij}(\alpha)\xi_{kj}(\beta)=\xi_{ij}(\alpha)+\sum_{a=k+1}^{j-1}\sum_{\gamma\in \boldsymbol{F}_q^*}\xi_{ij}(\alpha)\xi_{ka}(\gamma)$$

is the decomposition $\xi_{ij}(\alpha)\xi_{kj}(\beta)$ into irreducible characters. Therefore $\xi_{ij}(\alpha)\xi_{kj}(\beta)$ has one irreducible component of degree q^{j-i-1} and, for each $a\in\{k+1,\ldots,j-1\}$, $q-1$ irreducible components of degree $q^{(j-i-1)+(j-a-1)}$.

As in the previous cases we illustrate the situation of the lemma by the adjacent diagram. The symbology is as in the previous case.



Finally we consider the products $\xi_{ij}(\alpha)\xi_{ij}(\beta)$ where $(i,j)\in\Phi(n)$ and $\alpha,\beta\in \boldsymbol{F}_q^*$. We start with the case $\beta=-\alpha$.

Lemma 2.2.9. *Let $(i,j)\in \Phi(n)$ and let $\alpha\in \boldsymbol{F}_q^*$. Then*

$$\xi_{ij}(\alpha)\xi_{ij}(-\alpha)=1_{U_n(q)}+\sum_{a=i+1}^{j-1}\sum_{\beta\in \boldsymbol{F}_q^*}\xi_{aj}(\beta)+\sum_{b=i+1}^{j-1}\sum_{\gamma\in \boldsymbol{F}_q^*}\xi_{ib}(\gamma)+\sum_{a,b=i+1}^{j-1}\sum_{\beta,\gamma\in \boldsymbol{F}_q^*}\xi_{aj}(\beta)\xi_{ib}(\gamma).$$

Therefore each irreducible component of $\xi_{ij}(\alpha)\xi_{ij}(-\alpha)$ occurs with multiplicity one ([1]).

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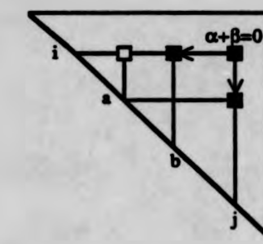
[1] We note that the characters appearing in the given sum are not (in general) the irreducible components of $\xi_{ij}(\alpha)\xi_{ij}(-\alpha)$.



Proof. Here the situation is as in the adjacent diagram. The arrows mean that the root (i,j) "dislocates" in the directions indicated. The root (i,a) determines the decomposition of any component $\xi_{aj}(\beta)\xi_{ib}(\gamma)$ (cf. lemma 2.2.6). The equality $\alpha+\beta=0$ means that the root (i,j) is associated with two elementary characters which correspond to $\alpha,\beta\in \boldsymbol{F}_q{}^*$ satisfying that equality.

Since

$$0=\alpha e_{ij}{}^*-\alpha e_{ij}{}^*\in O_{ij}(\alpha)+O_{ij}(-\alpha),$$

the unit character $1_{U_n(q)}$ of $U_n(q)$ is a component of $\xi_{ij}(\alpha)\xi_{ij}(-\alpha)$. On the other hand let $a\in\{i+1,\ldots,j-1\}$ and let $\beta\in \boldsymbol{F}_q{}^*$. Then

$$\alpha e_{ij}{}^*+\beta e_{aj}{}^*\in O_{ij}(\alpha)$$

(see the proof of lemma 2.2.7). Thus

$$\beta e_{aj}{}^*=\alpha e_{ij}{}^*+\beta e_{aj}{}^*-\alpha e_{ij}{}^*\in O_{ij}(\alpha)+O_{ij}(-\alpha)$$

and this implies that $\xi_{aj}(\beta)$ is a component of $\xi_{ij}(\alpha)\xi_{ij}(-\alpha)$. Similarly for each $b\in\{i+1,\ldots,j-1\}$ and each $\gamma\in \boldsymbol{F}_q{}^*$ the character $\xi_{ib}(\beta)$ is a component of $\xi_{ij}(\alpha)\xi_{ij}(-\alpha)$.

Now let $a,b\in\{i+1,\ldots,j-1\}$ and let $\beta,\gamma\in \boldsymbol{F}_q{}^*$. If $b<a$ the character $\xi_{aj}(\beta)\xi_{ib}(\gamma)$ is irreducible (by lemma 2.2.5) and it corresponds to the orbit $O_{aj}(\beta)+O_{ib}(\gamma)$ of the element

$$\beta e_{aj}{}^*+\gamma e_{ib}{}^*=\alpha e_{ij}{}^*+\beta e_{aj}{}^*-\alpha e_{ij}{}^*+\gamma e_{ib}{}^*\in O_{ij}(\alpha)+O_{ij}(-\alpha).$$

It follows that $\xi_{aj}(\beta)\xi_{ib}(\gamma)$ is a component of $\xi_{ij}(\alpha)\xi_{ij}(-\alpha)$. On the other hand suppose that $a<b$. Then (by lemma 2.2.6) the character $\xi_{aj}(\beta)\xi_{ib}(\gamma)$ is reducible and its components are parametrized by the elements of $\boldsymbol{F}_q$. For each $\delta\in \boldsymbol{F}_q$ the irreducible component associated with δ corresponds to the $U_n(K)$-orbit of the element

$$\beta e_{aj}{}^*+\gamma e_{ib}{}^*+\delta e_{ia}{}^*\in O_{aj}(\beta)+O_{ib}(\gamma).$$

Since

$$\alpha e_{ij}{}^*+\beta e_{aj}{}^*\in O_{ij}(\alpha) \quad\text{and}\quad -\alpha e_{ij}{}^*+\gamma e_{ib}{}^*+\delta e_{ia}{}^*\in O_{ij}(-\alpha)$$

(see proof of lemma 2.2.6) we deduce that

$$\beta e_{aj}{}^*+\gamma e_{ib}{}^*+\delta e_{ia}{}^*\in O_{ij}(\alpha)+O_{ij}(-\alpha).$$

Therefore each irreducible component of $\xi_{aj}(\beta)\xi_{ib}(\gamma)$ is also a component of $\xi_{ij}(\alpha)\xi_{ij}(-\alpha)$. Since $\xi_{aj}(\beta)\xi_{ib}(\gamma)$ is multiplicity free we conclude that $\xi_{aj}(\beta)\xi_{ib}(\gamma)$ is a component of $\xi_{ij}(\alpha)\xi_{ij}(-\alpha)$.

Finally we consider character degrees (we note that the characters involved have no components in common as we can prove using the argument of the previous cases). If we denote by ϕ the character of the rigth hand side of the required identity then we have

$$\begin{aligned}
\phi(1)&=1+(q-1)\sum_{a=i+1}^{j-1} q^{j-a-1}+(q-1)\sum_{b=i+1}^{j-1} q^{b-i-1}+(q-1)^2\sum_{a,b=i+1}^{j-1} q^{j-a-1}q^{b-i-1}\\
&=1+(q-1)\sum_{a=i+1}^{j-1} q^{j-a-1}+(q-1)\sum_{b=i+1}^{j-1} q^{b-i-1}\left(1+(q-1)\sum_{a=i+1}^{j-1} q^{j-a-1}\right)\\
&=q^{j-i-1}+(q-1)\sum_{b=i+1}^{j-1} q^{b-i-1}q^{j-i-1}\\
&=q^{j-i-1}\left(1+(q-1)\sum_{a=i+1}^{j-1} q^{j-a-1}\right)\\
&=q^{j-i-1}q^{j-i-1}\\
&=(\xi_{ij}(\alpha)\xi_{ij}(-\alpha))(1)
\end{aligned}$$

and the result follows. ♦

The next two lemmas will be used to decompose the character $\xi_{ij}(\alpha)\xi_{ij}(\beta)$ where $(i,j)\in\Phi(n)$ and $\alpha,\beta\in \boldsymbol{F}_q{}^*$, $\beta\neq-\alpha$. The first will also be used in the proof of theorem 2.2.1. Here we consider $U_{n-1}(q)$ as a subgroup of $U_n(q)$ (cf. section 2.1).

Lemma 2.2.10. $(1_{U_{n-1}(q)})^{U_n(q)}=1_{U_n(q)}+\sum_{i=1}^{n-1}\sum_{\alpha\in \boldsymbol{F}_q{}^*}\xi_{in}(\alpha)$.

Proof. Let $i\in\{1,\ldots,n-1\}$ and let $\alpha\in \boldsymbol{F}_q{}^*$. By Frobenius reciprocity $\xi_{in}(\alpha)$ is a component of the induced character $(1_{U_{n-1}(q)})^{U_n(q)}$ if and only if $1_{U_{n-1}(q)}$ is a component of the restriction $\xi_{in}(\alpha)_{U_{n-1}(q)}$. Let $\pi: u_n(K)^*\rightarrow u_{n-1}(K)^*$ be the map sending $f\in u_n(K)^*$ to its restriction $\pi(f)$ to $u_{n-1}(K)$ and consider the image $\pi(O_{in}(\alpha))$ of the (i,n)-th elementary orbit $O_{in}(\alpha)$. Since $O_{in}(\alpha)$ is the $U_n(K)$-orbit of the element $\alpha e_{in}{}^*\in u_n(K)$ and

$\pi(\alpha e_{in}^*)=0\in \mathcal{U}_{n-1}(K)^*$ we have

$$0\in \pi(O_{in}(\alpha)).$$

Since $\{0\}\subseteq \mathcal{U}_n(K)^*$ is the $U_{n-1}(q)$-orbit which corresponds to $1_{U_{n-1}(q)}$ and $O_{in}(\alpha)$ is the $U_n(K)$-orbit which corresponds to $\xi_{in}(\alpha)$ (by proposition 2.1.8) we conclude

$$(1_{U_{n-1}(q)},\xi_{in}(\alpha)_{U_{n-1}(q)})_{U_{n-1}(q)}\neq 0$$

(by theorem 1.3.10). It follows that

$$(\xi_{in}(\alpha),(1_{U_{n-1}(q)})^{U_n(q)})_{U_n(q)}\neq 0.$$

Finally we consider character degrees. It is clear that

$$(1_{U_{n-1}(q)})^{U_n(q)}(1)=q^{n-1}.$$

On the other hand

$$1_{U_n(q)}(1)+\sum_{i=1}^{n-1}\sum_{\alpha\in \mathbf{F}_q^*}\xi_{in}(\alpha)(1)=1+\sum_{i=1}^{n-1}(q-1)q^{n-i-1}=q^{n-1}$$

and the proof is complete (we note that the characters $\xi_{in}(\alpha)$, $\alpha\in \mathbf{F}_q$, are all distinct). ♦

On the other hand we have:

Lemma 2.2.11. *Let $i\in\{1,\ldots,n-1\}$ and let $\alpha\in \mathbf{F}_q^*$. Then*

$$\xi_{in}(\alpha)_{U_{n-1}(q)}=1_{U_{n-1}(q)}+\sum_{j=i+1}^{n-1}\sum_{\beta\in \mathbf{F}_q^*}\zeta_{ij}(\beta)$$

is the decomposition of the restriction $\xi_{in}(\alpha)_{U_{n-1}(q)}$ of $\xi_{in}(\alpha)$ to $U_{n-1}(q)$ into irreducible components. Here $\zeta_{ij}(\beta)$ denotes the (i,j)-th elementary character of $U_{n-1}(q)$ associated with β.

Proof. Let $\pi: \mathcal{U}_n(K)^*\to \mathcal{U}_{n-1}(K)^*$ be the natural projection. By theorem 1.3.10 the irreducible components of $\xi_{in}(\alpha)_{U_{n-1}(q)}$ are in one-to-one correspondence with the $U_{n-1}(K)$-orbits which are contained in the image $\pi(O_{in}(\alpha))$. Since $O_{in}(\alpha)$ is the $U_n(K)$-orbit of the element $\alpha e_{in}^*\in \mathcal{U}_n(K)^*$, we have

$$0=\pi(e_{in}^*)\in \pi(O_{in}(\alpha)).$$

Therefore

$$(1_{U_{n-1}(q)}, \xi_{in}(\alpha)_{U_{n-1}(q)})_{U_{n-1}(q)} \neq 0.$$

On the other hand let $j\in\{i+1,\dots,n-1\}$ and let $\beta\in F_q^*$. Then

$$\alpha e_{in}^* + \beta e_{ij}^* \in O_{in}(\alpha)$$

(see the proof of lemma 2.2.6). Hence

$$\beta e_{ij}^* = \pi(\alpha e_{in}^* + \beta e_{ij}^*) \in \pi(O_{in}(\alpha)).$$

Since $\zeta_{ij}(\beta)$ corresponds to the $U_{n-1}(K)$-orbit of $\beta e_{ij}^* \in u_{n-1}(K)^*$, we conclude that

$$(\zeta_{ij}(\beta), \xi_{in}(\alpha)_{U_{n-1}(q)})_{U_{n-1}(q)} \neq 0.$$

Since

$$1+(q-1)\sum_{j=i+1}^{n-1} q^{j-i-1} = q^{n-i-1},$$

the lemma follows because the characters $\zeta_{ij}(\beta)$, $\beta\in F_q$, are all distinct. ♦

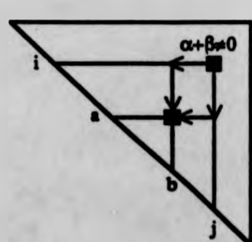
Finally we may prove the following:

Lemma 2.2.12. *Let $(i,j)\in\Phi(n)$ and let $\alpha,\beta\in F_q^*$ be such that $\beta\neq\alpha$. Then*

$$\xi_{ij}(\alpha)\xi_{ij}(\beta) = (1+(q-1)(j-i-1))\xi_{ij}(\alpha+\beta) + \sum_{a=i+1}^{j-2}\sum_{b=a+1}^{j-1}\sum_{\gamma\in F_q^*}(q-1)\xi_{ij}(\alpha+\beta)\xi_{ab}(\gamma)$$

is the decomposition of $\xi_{ij}(\alpha)\xi_{ij}(\beta)$ into irreducible components.

Proof. Here the situation is as in the adjacent picture. The arrows and the inequality $\alpha+\beta\neq 0$ have the same meaning as in lemma 2.2.9. We note that the root (i,j) occurs in the final situation.



First we prove that the irreducible characters $\xi_{ij}(\alpha+\beta)$ and $\xi_{ij}(\alpha+\beta)\xi_{ab}(\gamma)$ ($i+1\leq a<b\leq j-1$, $\gamma\in F_q^*$) occur as components of $\xi_{ij}(\alpha)\xi_{ij}(\beta)$. In fact (by corollary 1.3.11)

$$(\xi_{ij}(\alpha+\beta), \xi_{ij}(\alpha)\xi_{ij}(\beta)) \neq 0$$

because

$$\alpha e_{ij}^*+\beta e_{ij}^*\in O_{ij}(\alpha)+O_{ij}(\beta).$$

On the other hand let $a,b\in\{i+1,\ldots,j-1\}$, $a<b$, and let $\gamma\in \boldsymbol{F}_q^*$. As in the proof of lemma 2.2.6 we have

$$\alpha e_{ij}^*+e_{ib}^*\in O_{ij}(\alpha).$$

Let $x=x_{ia}(\gamma)\in X_{ia}(q)$. Then the element

$$x\cdot(\alpha e_{ij}^*+e_{ib}^*)=\alpha e_{ij}^*+e_{ib}^*+\alpha\gamma e_{aj}^*+\gamma e_{ab}^*$$

lies in the $U_n(K)$-orbit $O_{ij}(\alpha)$. Since $\beta e_{ij}^*-e_{ib}^*\in O_{ij}(\beta)$ we conclude that

$$(\alpha+\beta)e_{ij}^*+\alpha\gamma e_{aj}^*+\gamma e_{ab}^*\in O_{ij}(\alpha)+O_{ij}(\beta).$$

But $(\alpha+\beta)e_{ij}^*+\alpha\gamma e_{aj}^*\in O_{ij}(\alpha+\beta)$, so

$$(\alpha+\beta)e_{ij}^*+\alpha\gamma e_{aj}^*+\gamma e_{ab}^*\in O_{ij}(\alpha+\beta)+O_{ab}(\gamma).$$

Since $O_{ij}(\alpha)+O_{ij}(\beta)$ is $U_n(K)$-invariant we deduce that

$$O_{ij}(\alpha+\beta)+O_{ab}(\gamma)\subseteq O_{ij}(\alpha)+O_{ij}(\beta)$$

- we note that $O_{ij}(\alpha+\beta)+O_{ab}(\gamma)$ is the $U_n(K)$-orbit of $(\alpha+\beta)e_{ij}^*+\alpha\gamma e_{aj}^*+\gamma e_{ab}^*$. Finally corollary 1.3.11 implies that

$$(\xi_{ij}(\alpha+\beta)\xi_{ab}(\gamma),\xi_{ij}(\alpha)\xi_{ij}(\beta))\neq 0.$$

Next we consider the multiplicities of the irreducible characters $\xi_{ij}(\alpha+\beta)$ and $\xi_{ij}(\alpha+\beta)\xi_{ab}(\gamma)$ ($i+1\leq a<b\leq j-1$, $\gamma\in\boldsymbol{F}_q^*$) as components of $\xi_{ij}(\alpha)\xi_{ij}(\beta)$. If $j<n$ then all the elementary characters involved are the lifting from $U_j(q)$ ([1]) to $U_n(q)$ of the corresponding elementary characters of $U_j(q)$. Therefore those multiplicities may be calculated in the group $U_j(q)$. Hence without loss of generality we may assume that $j=n$.

Firstly we consider the character $\xi_{in}(\alpha+\beta)$. We calculate the multiplicity of this character as a component of the product $\xi_{in}(\alpha)(1_{U_{n-1}(q)})^{U_n(q)}$. By lemma 2.2.10 we have

$$\xi_{in}(\alpha)(1_{U_{n-1}(q)})^{U_n(q)}=\xi_{in}(\alpha)+\sum_{r=1}^{n-1}\sum_{\nu\in\boldsymbol{F}_q^*}\xi_{in}(\alpha)\xi_{rn}(\nu).$$

Since $\beta\neq 0$ the irreducible characters $\xi_{in}(\alpha+\beta)$ and $\xi_{in}(\alpha)$ are distinct. Therefore

$$(\xi_{in}(\alpha+\beta),\xi_{in}(\alpha))=0.$$

On the other hand we have (by lemmas 2.2.7 and 2.2.8)

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[1] $U_j(q)$ is identified with a subgroup of $U_n(q)$ as in section 2.1.

$$(\xi_{in}(\alpha+\beta),\xi_{in}(\alpha)\xi_{rn}(\nu))=0$$

for all $r\in\{1,\dots,n-1\}$, $r\neq i$, and all $\nu\in F_q{}^*$. Finally let $\nu\in F_q{}^*$, $\nu\neq\beta$, and let $f\in O_{in}(\alpha)+O_{in}(\nu)$. Then

$$f(e_{in})=\alpha+\nu.$$

Also

$$g(e_{in})=\alpha+\beta$$

for all $g\in O_{in}(\alpha+\beta)$. Since $\nu\neq\beta$ we conclude that

$$O_{in}(\alpha+\beta)\cap(O_{in}(\alpha)+O_{in}(\nu))=\varnothing.$$

Therefore

$$(\xi_{in}(\alpha+\beta),\xi_{in}(\alpha)\xi_{in}(\nu))=0$$

(by corollary 1.3.11). It follows that

$$(\xi_{in}(\alpha+\beta),\xi_{in}(\alpha)\xi_{in}(\beta))=(\xi_{in}(\alpha+\beta),\xi_{in}(\alpha)(1_{U_{n-1}(q)})^{U_n(q)}).$$

Now we have

$$(\xi_{in}(\alpha+\beta),\xi_{in}(\alpha)(1_{U_{n-1}(q)})^{U_n(q)})=(\overline{\xi_{in}(\alpha)}\xi_{in}(\alpha+\beta),(1_{U_{n-1}(q)})^{U_n(q)}).$$

By Frobenius reciprocity we conclude that

$$(\xi_{in}(\alpha+\beta),\xi_{in}(\alpha)\xi_{in}(\beta))_{U_n(q)}=(\overline{\xi_{in}(\alpha)}_{U_{n-1}(q)}\xi_{in}(\alpha+\beta)_{U_{n-1}(q)},1_{U_{n-1}(q)})_{U_{n-1}(q)}.$$

Hence

$$(\xi_{in}(\alpha+\beta),\xi_{in}(\alpha)\xi_{in}(\beta))_{U_n(q)}=(\xi_{in}(\alpha+\beta)_{U_{n-1}(q)},\xi_{in}(\alpha)_{U_{n-1}(q)})_{U_{n-1}(q)}.$$

By lemma 2.2.11 (applied to the characters $\xi_{in}(\alpha)$ and $\xi_{in}(\alpha+\beta)$) we have

$$\xi_{in}(\alpha)_{U_{n-1}(q)}=\xi_{in}(\alpha+\beta)_{U_{n-1}(q)}=1_{U_{n-1}(q)}+\sum_{r=i+1}^{n-1}\sum_{\nu\in F_q{}^*}\zeta_{ir}(\nu).$$

Since the characters in this sum are all irreducible and distinct we conclude that

$$(\xi_{in}(\alpha+\beta),\xi_{in}(\alpha)\xi_{in}(\beta))_{U_n(q)}=1+(n-i-1)(q-1).$$

Now let $a,b\in\{1,\dots,n-1\}$, $i<a<b$, and let $\gamma\in F_q{}^*$. We use a similar argument to calculate the multiplicity of the (irreducible) character $\xi_{in}(\alpha+\beta)\xi_{ab}(\gamma)$ as a component of $\xi_{in}(\alpha)\xi_{in}(\beta)$. As before we obtain

$$(\xi_{in}(\alpha+\beta)\xi_{ab}(\gamma),\xi_{in}(\alpha)\xi_{in}(\beta))_{U_n(q)}=(\xi_{in}(\alpha+\beta)_{U_{n-1}(q)}\zeta_{ab}(\gamma),\xi_{in}(\alpha)_{U_{n-1}(q)})_{U_{n-1}(q)}$$

- we note that $\xi_{ab}(\gamma)=\zeta_{ab}(\gamma)^*$ and so

$$\xi_{ab}(\gamma)_{U_{n-1}(q)}=\zeta_{ab}(\gamma).$$

By lemma 2.2.11, we have

$$\xi_{in}(\alpha)_{U_{n-1}(q)}=\xi_{in}(\alpha+\beta)_{U_{n-1}(q)}=1_{U_{n-1}(q)}+\sum_{r=i+1}^{n-1}\sum_{\nu\in F_q^*}\zeta_{ir}(\nu).$$

Therefore $\xi_{in}(\alpha+\beta)_{U_{n-1}(q)}\zeta_{ab}(\gamma)$ has the decomposition

$$\xi_{in}(\alpha+\beta)_{U_{n-1}(q)}\zeta_{ab}(\gamma)=\zeta_{ab}(\gamma)+\sum_{s=i+1}^{n-1}\sum_{\mu\in F_q^*}\zeta_{is}(\mu)\zeta_{ab}(\gamma)$$

where the characters $\zeta_{is}(\mu)\zeta_{ab}(\gamma)$ are not in general irreducible (however, since $a>i$, all of them are multiplicity free). By lemmas 2.2.3, 2.2.5 and 2.2.6 we have

$$(1_{U_{n-1}(q)},\xi_{in}(\alpha+\beta)_{U_{n-1}(q)}\zeta_{ab}(\gamma))=0.$$

On the other hand, let $r\in\{i+1,\dots,n-1\}$ and let $\nu\in F_q^*$. Then

$$(\zeta_{ir}(\nu),\xi_{in}(\alpha+\beta)_{U_{n-1}(q)}\zeta_{ab}(\gamma))\neq 0$$

if and only if

$$(\zeta_{ir}(\nu),\zeta_{is}(\mu)\zeta_{ab}(\gamma))\neq 0$$

for some $s\in\{i+1,\dots,n-1\}$ and some $\mu\in F_q^*$ (we note that $a>i$). By lemma 2.2.8 this happens if and only if $s=r=b$ and $\mu=\nu$ and, if this is the case, we have

$$(\zeta_{ib}(\nu),\zeta_{ib}(\nu)\zeta_{ab}(\gamma))_{U_{n-1}(q)}=1.$$

It follows that

$$(\xi_{in}(\alpha+\beta)\xi_{ab}(\gamma),\xi_{in}(\alpha)\xi_{in}(\beta))_{U_n(q)}=\sum_{\nu\in F_q^*}(\zeta_{ib}(\nu),\zeta_{ib}(\nu)\zeta_{ab}(\gamma))_{U_{n-1}(q)}=q-1.$$

To conclude the proof we calculate the degree of the character of the right hand side of the required equality. We denote this character by ϕ. Then

$$\phi(1)=(1+(q-1)(n-i-1))\xi_{in}(\alpha+\beta)(1)+\sum_{a=i+1}^{n-2}\sum_{b=a+1}^{n-1}\sum_{\gamma\in F_q^*}(q-1)(\xi_{in}(\alpha+\beta)\xi_{ab}(\gamma))(1)$$

$$=(1+(q-1)(n-i-1))q^{n-i-1}+\sum_{a=i+1}^{n-2}\sum_{b=a+1}^{n-1}\sum_{\gamma\in F_q^*}(q-1)q^{n-i-1}q^{b-a-1}$$

$$=q^{n-i-1}\left((1+(q-1)(n-i-1))+(q-1)^2\sum_{a=i+1}^{n-2}\sum_{b=a+1}^{n-1}q^{b-a-1}\right)$$

Since

$$(q-1)\sum_{b=a+1}^{n-1} q^{b-a-1}=q^{n-a-1}-1$$

for all $a\in\{i+1,\ldots,n-2\}$, we obtain

$$\begin{aligned}\phi(1)&=q^{n-i-1}\left((1+(q-1)(n-i-1))+(q-1)\sum_{a=i+1}^{n-2}(q^{n-a-1}-1)\right)\\&=q^{n-i-1}(1+(q-1)(n-i-1))+q^{n-i-1}-1-(q-1)(n-i-1))\\&=q^{n-i-1}\,q^{n-i-1}\\&=(\xi_{in}(\alpha)\xi_{in}(\beta))(1).\end{aligned}$$

The lemma is proved. ◆

Using the previous lemmas we now prove the following result:

Proposition 2.2.13. *Let ϕ be a character of $U_n(q)$ which can be expressed as a product of elementary characters. Then ϕ can be decomposed as a sum of characters with the form (2.2.1).*

Proof. We define the total order $<$ on the set of roots $\Phi(n)$ as follows. If $(i,j),(k,l)\in\Phi(n)$ then $(i,j)<(k,l)$ if and only if either $j>l$ or $j=l$ and $i<k$. Hence we may define a total order in the set $\mathcal{U}_n(\mathbb{N}_0)$ consisting of all matrices $a=(a_{ij})_{1\le i,j\le n}$ satisfying $a_{ij}=0$ if $(i,j)\notin\Phi(n)$, and $a_{ij}\in\mathbb{N}_0$ if $(i,j)\in\Phi(n)$ - here $\mathbb{N}_0$ denotes the set of all non-negative integers. This order is defined as follows. Let $a,b\in\mathcal{U}_n(\mathbb{N}_0)$, $a=(a_{ij})_{1\le i,j\le n}$, $b=(b_{ij})_{1\le i,j\le n}$. Then $a<b$ if and only if there exists $(i,j)\in\Phi(n)$ such that $a_{kl}=b_{kl}$ for all $(k,l)\in\Phi(n)$ with $(k,l)<(i,j)$, and $a_{ij}<b_{ij}$. This order allows us to prove the proposition by induction on the set $\mathcal{U}_n(\mathbb{N}_0)$. In fact the character ϕ determines a matrix $a_\phi=(a_{ij})\in\mathcal{U}_n(\mathbb{N}_0)$ as follows. By hypothesis there exists a non-empty subset A of $\Phi(n)$ such that

$$\phi=\prod_{(i,j)\in A}\prod_{k_{ij}=1}^{t_{ij}}\xi_{ij}(\alpha_{ijk_{ij}})$$

where, for each $(i,j)\in A$ and each $k\in\{1,\ldots,t_{ij}\}$, α_{ijk} is a non-zero element of the field $\mathbb{F}_q$. Then for any $(i,j)\in\Phi(n)$ the (i,j)-th coefficient of a_ϕ is

$$a_{ij}=\begin{cases}t_{ij} & \text{if } (i,j)\in A\\ 0 & \text{otherwise}\end{cases}$$

Suppose that ϕ corresponds to the smallest matrix in $\mathcal{U}_n(\mathbb{N}_0)$, i.e. all the entries of $a_\phi=(a_{ij})$ is such that $a_{ij}=0$ for all $(i,j)\in\Phi(n)\setminus\{(1,2)\}$, and $a_{12}=1$. Then $\phi=\xi_{12}(\alpha)$ for some $\alpha\in\mathbb{F}_q$ and the result is trivial. Now let ϕ be such that a_ϕ is not the smallest element of $\mathcal{U}_n(\mathbb{N}_0)$ and suppose that the result has been proved for all character ψ (as in the proposition) such that $a_\psi<a_\phi$. Let (i,j) be the smallest root of A. Without loss of generality we may assume that $j=n$ - otherwise $\phi=\psi^*$ for some character ψ of $U_j(q)$ and we may prove the proposition in this "smaller" group. Then

$$\phi=\xi_{in}(\alpha_1)\ldots\xi_{in}(\alpha_t)\prod_{(i,j)\in A_0}\prod_{k_{ij}=1}^{t_{ij}}\xi_{ij}(\alpha_{ijk_{ij}})$$

where $t=t_{in}$, $\alpha_k=\alpha_{ink}$ $(1\le a\le t_{in})$ and $A_0=A\setminus\{(i,n)\}$ (possibly A_0 is empty). Suppose that $t>1$. Then after a finite number of applications of either lemma 2.2.9 or lemma 2.2.12 we obtain a decomposition of ϕ as a sum

$$\phi=\phi_1+\ldots+\phi_r$$

where for each $s\in\{1,\ldots,r\}$ there exists a subset A_s of $\Phi(n)$ such that $(i,n)\in A_s$ and

$$\phi_s=\xi_{in}(\alpha_1+\ldots+\alpha_t)\prod_{(i,j)\in A_s}\prod_{k_{ij}=1}^{t_{ij}}\xi_{ij}(\beta_{ijk_{ij}})$$

for some $\beta_{ijk_{ij}}\in\mathbb{F}_q^*$ $(1\le k_{ij}\le t_{ij},\ (i,j)\in A_s)$ - we note that we may have $\alpha_1+\ldots+\alpha_t=0$. Since $t>1$ we have $a_{\phi_s}<a_\phi$ for all $s\in\{1,\ldots,r\}$. Thus (by induction) the result is true for each character ϕ_s $(1\le s\le r)$, hence it is also true for the character ϕ.

Now suppose that $t=1$. Then

$$\phi=\xi_{in}(\alpha)\prod_{(i,j)\in A_0}\prod_{k_{ij}=1}^{t_{ij}}\xi_{ij}(\alpha_{ijk_{ij}})$$

for some $\alpha\in\mathbb{F}_q^*$. If A_0 is empty then ϕ is an elementary character and the result is trivial. Suppose that A_0 is non-empty and let

$$\phi_0=\prod_{(i,j)\in A_0}\prod_{k_{ij}=1}^{t_{ij}}\xi_{ij}(\alpha_{ijk_{ij}}).$$

We have $a_{\phi_0}<a_\phi$, hence (by induction) the character ϕ_0 decomposes as a sum of characters with the form (2.2.1). If $\xi_D(\varphi)$ is one of these components (hence D is a basic subset of $\Phi(n)$ and $\varphi:D\to\mathbb{F}_q^*$ is a map) then the character $\xi_{in}(\alpha)\xi_D(\varphi)$ is a component of ϕ. Moreover ϕ decomposes as a sum of characters with this form. Let ϕ_D denote the component $\xi_{in}(\alpha)\xi_D(\varphi)$ of ϕ. If $D\cup\{(i,n)\}$ is basic then ϕ_D is one of the required

components of ϕ. On the other hand suppose that $D\cup\{(i,n)\}$ is not basic. Suppose also that there exists $j\in\{1,\ldots,n-1\}$ such that $(j,n)\in D$. Then $(j,n)\in A_0$ - we note that (by the argument above applied to the character ϕ_0) the smallest root of D is greater or equal than the smallest root of A_0. It follows that $(j,n)\in A$ and so $i<j$. Now (by lemma 2.2.8) we obtain

$$\phi_D=\xi_{in}(\alpha)\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{D_0}(\varphi_0)+\sum_{k=j+1}^{n-1}\sum_{\beta\in F_q^*}\xi_{in}(\alpha)\xi_{jk}(\beta)\xi_{D_0}(\varphi_0)$$

where $D_0=D\setminus\{(j,n)\}$ and φ_0 is the restriction of φ to D_0. Since $(j,n)\in A$ and $(j,n)\notin D_0$ the characters $\xi_{in}(\alpha)\xi_{D_0}(\varphi_0)$ and $\xi_{in}(\alpha)\xi_{jk}(\beta)\xi_{D_0}(\varphi_0)$ $(j+1\leq k\leq n-1,\ \beta\in F_q^*)$ determine matrices in $\mathcal{U}_n(N_0)$ which are smaller than a_ϕ. By induction we the proposition is true for these characters, hence it is true for the character ϕ_D.

Finally suppose that $(j,n)\notin D$ for all $j\in\{1,\ldots,n-1\}$. Then $(i,k)\in D$ for some $k\in\{i+1,\ldots,n-1\}$ and (by lemma 2.2.7)

$$\phi_D=\xi_{in}(\alpha)\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{D_0}(\varphi_0)+\sum_{l=i+1}^{k-1}\sum_{\beta\in F_q^*}\xi_{in}(\alpha)\xi_{lk}(\beta)\xi_{D_0}(\varphi_0)$$

where $D_0=D\setminus\{(i,k)\}$ and φ_0 is the restriction of φ to D_0. The result follows by induction because the characters $\xi_{in}(\alpha)\xi_{D_0}(\varphi_0)$ and $\xi_{in}(\alpha)\xi_{lk}(\beta)\xi_{D_0}(\varphi_0)$ $(i+1\leq l\leq n-1,\ \beta\in F_q^*)$ determine matrices in $\mathcal{U}_n(N_0)$ which are smaller than a_ϕ. In fact our construction shows that at each stage we are constructing a family of characters which determine matrices in $\mathcal{U}_n(N_0)$ which are smaller than the previous one. In particular the matrix determined by $\xi_D(\varphi)$ is smaller than the matrix determined by ϕ_0.

The proof is complete. ♦

As a corollary we obtain:

Corollary 2.2.14. *Let χ be an irreducible character of $U_n(q)$. Then there exists a basic subset D of $\Phi(n)$ and a map $\varphi: D \to \mathbf{F}_q{}^*$ such that $(\chi, \xi_D(\varphi)) \neq 0$.*

Proof. Let $O \subseteq \mathcal{U}_n(K)^*$ be the F-stable $U_n(K)$-orbit which corresponds to χ and let $f \in O^F$. Then

$$f = \sum_{(i,j)\in\Phi(n)} f(e_{ij})\, e_{ij}{}^*$$

(since f is F-fixed we have $f(e_{ij}) \in \mathbf{F}_q$ for all $(i,j) \in \Phi(n)$). Let

$$A = \{(i,j) \in \Phi(n); f(e_{ij}) \neq 0\}.$$

Then

$$f \in \sum_{(i,j)\in A} O_{ij}(f(e_{ij}))$$

and (by corollary 1.3.11) we conclude that

$$(\chi, \prod_{(i,j)\in A} \xi_{ij}(\alpha_{ij})) \neq 0.$$

Since χ is irreducible the result follows by the previous proposition. ♦

Corollary 2.2.15. *Let $\mu(n) = (n-2)+(n-4)+\ldots+1$ (cf. corollary 2.2.4). Then $q^{\mu(n)}$ is the maximal degree of the irreducible characters of $U_n(q)$. Moreover if $n=2r+1$ is odd, $U_n(q)$ has exactly $(q-1)^r$ irreducible characters of degree $q^{\mu(n)}$, whereas if $n=2r$ is even, $U_n(q)$ has exactly $q(q-1)^{r-1}$ irreducible characters of degree $q^{\mu(n)}$.*

Proof. Let χ be an arbitrary irreducible character of $U_n(q)$. By the previous corollary there exist a basic subset D of $\Phi(n)$ and a map $\varphi: D \to \mathbf{F}_q{}^*$ such that

$$(\chi, \xi_D(\varphi)) \neq 0.$$

Therefore

$$\chi(1) \leq \xi_D(\varphi)(1) = \prod_{(i,j)\in D} q^{j-i-1}.$$

Since D contains at most one root of each column and at most one root of each row we conclude that

$$\sum_{(i,j)\in D} (j-i-1) \leq \mu(n)$$

and the corollary follows. ♦

The proof of theorem 2.2.1 will be complete with the following:

Proposition 2.2.16. *Let D and D' be basic subsets of $\Phi(n)$ and let $\varphi: D \rightarrow \mathbf{F}_q^*$ and $\varphi': D' \rightarrow \mathbf{F}_q^*$ be maps. Then $(\xi_D(\varphi), \xi_{D'}(\varphi')) \neq 0$ if and only if $D=D'$ and $\varphi=\varphi'$.*

Proof. We proceed by induction on n. If $n=2$ then all irreducible characters of $U_2(q)$ are linear and the result is obvious. Hence let $n>2$ and assume that the result is proved for n-1. We consider $U_{n-1}(q)$ as a subgroup of $U_n(q)$. For each $(i,j) \in \Phi(n)$ such that $j<n$ we denote by $\zeta_{ij}(\alpha)$ the (i,j)-th elementary character of $U_{n-1}(q)$ associated with $\alpha \in \mathbf{F}_q$ and for each basic subset $D \subseteq \{(i,j) \in \Phi(n);\ 1 \leq i<j \leq n-1\}$ and any map $\varphi: D \rightarrow \mathbf{F}_q^*$ we denote by $\zeta_D(\varphi)$ the character of $U_{n-1}(q)$ which is the product of the elementary characters $\zeta_{ij}(\varphi(i,j))$ for $(i,j) \in D$. Then

$$\xi_D(\varphi)=\zeta_D(\varphi)^*$$

is the lifting of $\zeta_D(\varphi)$ from $U_{n-1}(q)$ to $U_n(K)$.

Now let D, D', φ and φ' be as given in the proposition and suppose that $(\xi_D(\varphi), \xi_{D'}(\varphi')) \neq 0$. Let $(i,j) \in D$ be smallest root of D. If $j<n$ then

$$D \subseteq \{(i,j);\ 1 \leq i<j \leq n-1\}.$$

We claim that $D' \subseteq \{(i,j);\ 1 \leq i<j \leq n-1\}$. For suppose that $(k,n) \in D'$ for some $k \in \{1,\ldots,n-1\}$ and let χ be an irreducible character of $U_n(q)$ such that

$$(\chi, \xi_D(\varphi)) \neq 0 \text{ and } (\chi, \xi_{D'}(\varphi')) \neq 0.$$

Let $O \subseteq \mathfrak{u}_n(K)^*$ be the F-stable $U_n(K)$-orbit which corresponds to χ. Then

$$O \subseteq O_D(\varphi) \cap O_{D'}(\varphi').$$

Let $f \in O$. Then

$$f \in O_{D'}(\varphi')=O_{kn}(\beta)+O_{D'_0}(\varphi'_0)$$

where $\beta=\varphi'(k,n)$, $D'_0=D' \setminus \{(k,n)\}$ and φ'_0 is the restriction of φ' to D'_0. Since $O_{kn}(\beta)$ is the $U_n(K)$-orbit of βe_{kn}^* and $O_{D'_0}(\varphi'_0)$ is $U_n(K)$-invariant, there exists $x \in U_n(K)$ such that

$$x \cdot f=\beta e_{kn}^*+f_0$$

for some $f_0 \in O_{D'_0}(\varphi'_0)$. Since $O_{D'_0}(\varphi'_0) \subseteq \mathcal{U}_{n-1}(K)^*$ we have

$$f_0 \in \mathcal{U}_{n-1}(K)^*.$$

On the other hand $O_D(\varphi) \subseteq \mathcal{U}_{n-1}(K)^*$. Since $O_D(\varphi)$ is $\mathcal{U}_n(K)$-invariant, we obtain

$$x \cdot f \in \mathcal{U}_{n-1}(K)^*.$$

It follows that $\beta=0$ and this contradiction implies that

$$D' \subseteq \{(i,j);\ 1 \le i < j \le n-1\}$$

as required. Therefore we have

$$\xi_D(\varphi)=\zeta_D(\varphi)^* \quad \text{and} \quad \xi_{D'}(\varphi')=\zeta_{D'}(\varphi')^*.$$

It follows that

$$(\zeta_D(\varphi),\zeta_{D'}(\varphi'))_{U_{n-1}(q)}=(\xi_D(\varphi),\xi_{D'}(\varphi'))_{U_n(q)} \neq 0.$$

By induction we conclude that

$$(D,\varphi)=(D',\varphi').$$

Now suppose that $j=n$ (hence $(i,n) \in D$). By the above argument we conclude that $(k,n) \in D'$ for some $k \in \{1,\ldots,n-1\}$. Moreover let

$$f \in O_D(\varphi) \cap O_{D'}(\varphi').$$

Then f is $\mathcal{U}_n(K)$-conjugate to $\alpha e_{in}^* + f_0$ and to $\beta e_{in}^* + g_0$,

where $\alpha=\varphi(i,n)$, $\beta=\varphi(k,n)$ and $f_0, g_0 \in \mathcal{U}_{n-1}(K)^*$. It follows that $i=k$ and that $\alpha=\beta$. Therefore

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{D_0}(\varphi_0) \quad \text{and} \quad \xi_{D'}(\varphi')=\xi_{in}(\alpha)\xi_{D'_0}(\varphi'_0)$$

where $D_0=D\setminus\{(i,n)\}$, $D'_0=D'\setminus\{(i,n)\}$, φ_0 is the restriction of φ to D_0 and φ'_0 is the restriction of φ' to D'_0. We have

$$(\xi_D(\varphi),\xi_{D'}(\varphi'))=(\xi_{D_0}(\varphi_0),\overline{\xi_{in}(\alpha)}\xi_{in}(\alpha)\xi_{D'_0}(\varphi'_0))$$

where $\overline{\xi_{in}(\alpha)}$ is the character of $U_n(q)$ defined by

$$\overline{\xi_{in}(\alpha)}(x)=\xi_{in}(\alpha)(x^{-1})$$

for all $x \in U_n(q)$. We claim that

$$\overline{\xi_{in}(\alpha)}=\xi_{in}(-\alpha).$$

We recall that (by definition)

$$\xi_{in}(\alpha)=\lambda_{in}(\alpha)^{U_n(q)}$$

where $\lambda_{in}(\alpha)$ is the character of $U_\omega(q)$ ($\omega=(n\text{-}1\ldots i\text{+}1\ i)\in S_n$) defined by

$$\lambda_{in}(\alpha)(x)=\psi_0(\alpha x_{in})$$

for all $x=(x_{rs})\in U_\omega(q)$. Since

$$\overline{\xi_{in}(\alpha)}=\overline{\lambda_{in}(\alpha)}^{U_n(q)}$$

we must prove that

$$\overline{\lambda_{in}(\alpha)}=\lambda_{in}(-\alpha).$$

In fact let $x=(x_{rs})\in U_\omega(q)$. Then

$$\overline{\lambda_{in}(\alpha)}(x)=\lambda_{in}(\alpha)(x^{-1})=\psi_0(\alpha(x^{-1})_{in}).$$

Since $x_{ii+1}=\ldots=x_{in-1}=0$ we have $(x^{-1})_{in}=-x_{in}$. Therefore

$$\overline{\lambda_{in}(\alpha)}(x)=\psi_0((-\alpha)x_{in})=\lambda_{in}(-\alpha)(x).$$

Our claim follows. Hence

$$(\xi_D(\varphi),\xi_{D'}(\varphi'))=(\xi_{D_0}(\varphi_0),\xi_{in}(-\alpha)\xi_{in}(\alpha)\xi_{D'_0}(\varphi'_0)).$$

By lemma 2.2.9, $\xi_{in}(\alpha)\xi_{in}(-\alpha)$ has the following components:

(a) $1_{U_n(q)}$;

(b) $\xi_{an}(\beta)$ for all $a\in\{i+1,\ldots,n-1\}$ and all $\beta\in \boldsymbol{F}_q^*$;

(c) $\xi_{ib}(\gamma)$ for all $b\in\{i+1,\ldots,n-1\}$ and all $\gamma\in \boldsymbol{F}_q^*$;

(d) $\xi_{an}(\beta)\xi_{ib}(\gamma)$ for all $a,b\in\{i+1,\ldots,n-1\}$ and all $\beta,\gamma\in \boldsymbol{F}_q^*$.

Since $D_0\subseteq\{(i,j);\ 1\le i<j\le n-1\}$ we must have

$$(\xi_{D_0}(\varphi_0),\xi_{an}(\beta)\xi_{D'_0}(\varphi'_0))=0$$

for all $a\in\{i+1,\ldots,n-1\}$ and all $\beta\in \boldsymbol{F}_q^*$. Also

$$(\xi_{D_0}(\varphi_0),\xi_{an}(\beta)\xi_{ib}(\gamma)\xi_{D'_0}(\varphi'_0))=0$$

for all $a,b\in\{i+1,\ldots,n-1\}$ and all $\beta,\gamma\in \boldsymbol{F}_q^*$.

Now let $b\in\{i+1,\ldots,n-1\}$ and let $\gamma\in \boldsymbol{F}_q^*$. Suppose that $(j,b)\notin D'_0$ for all $j\in\{1,\ldots,b-1\}$. Then the subset $D'_0\cup\{(i,b)\}$ of $\Phi(n)$ is basic (because $D'_0\cup\{(i,n)\}$ is basic). Moreover

$$D_0\neq D'_0\cup\{(i,b)\}$$

(otherwise $(i,b)\in D_0$ and this is impossible because $D_0\cup\{(i,n)\}$ is basic). By induction we conclude that

$$(\xi_{D_0}(\varphi_0),\xi_{ib}(\gamma)\xi_{D'_0}(\varphi'_0))=0.$$

On the other hand suppose that there exists $j\in\{1,\ldots,b-1\}$ with $(j,b)\in D'_0$. If $i<j$ then (by lemma 2.2.9)

$$\xi_{ib}(\gamma)\xi_{jb}(\nu)=\xi_{ib}(\gamma)+\sum_{k=j+1}^{b-1}\sum_{\mu\in F_q^*}\xi_{ib}(\gamma)\xi_{jk}(\mu)$$

where $\nu=\varphi'(j,b)$. Therefore

$$\xi_{ib}(\gamma)\xi_{D'_0}(\varphi'_0)=\xi_{ib}(\gamma)\xi_{D''_0}(\varphi''_0)+\sum_{k=j+1}^{b-1}\sum_{\mu\in F_q^*}\xi_{ib}(\gamma)\xi_{jk}(\mu)\xi_{D''_0}(\varphi''_0)$$

where $D''_0=D'_0\setminus\{(j,b)\}$ and φ''_0 is the restriction of φ'_0 to D''_0. Since $D''_0\cup\{(i,b)\}$ is basic we have

$$(\xi_{D_0}(\varphi_0),\xi_{ib}(\gamma)\xi_{D''_0}(\varphi''_0))=0$$

(by induction because $(i,b)\in D$). On the other hand let $k\in\{j+1,\ldots,b-1\}$ and let $\mu\in F_q^*$. By proposition 2.2.13 there exists a set $\{D_1,\ldots,D_t\}$ of basic subsets of $\Phi(n)$ such that

$$\xi_{ib}(\gamma)\xi_{jk}(\mu)\xi_{D''_0}(\varphi''_0)=\sum_{s=1}^{t}\xi_{D_s}(\varphi_s)$$

where for each $s\in\{1,\ldots,t\}$ $\varphi_s:D_s\rightarrow F_q^*$ is a map. For each $s\in\{1,\ldots,t\}$ the set D_s is obtained after a finite number of applications of lemma 2.2.8. At each stage we obtain a decomposition of the character $\xi_{ib}(\gamma)\xi_{jk}(\mu)\xi_{D''_0}(\varphi''_0)$ as a sum of products of the form

$$\xi_{ib}(\gamma)\prod_{(r,s)\in A}\xi_{rs}(\alpha_{rs})$$

where A is a subset of $\Phi(n)$ such that each root $(r,s)\in A$ is in the same row of some root of D'_0 and $\alpha_{rs}\in F_q^*$. Since D'_0 does not contain roots of the i-th row, we conclude that $(i,b)\in D_s$ for all $s\in\{1,\ldots,t\}$. It follows that

$$(\xi_{D_0}(\varphi_0),\xi_{D_s}(\varphi_s))=0$$

for all $s\in\{1,\ldots,t\}$. Therefore

$$(\xi_{D_0}(\varphi_0),\xi_{ib}(\gamma)\xi_{jk}(\mu)\xi_{D''_0}(\varphi''_0))=0.$$

Now suppose that $j<i$. Then (by lemma 2.2.8)

$$\xi_{ib}(\gamma)\xi_{jb}(\nu)=\xi_{jb}(\nu)+\sum_{k=i+1}^{b-1}\sum_{\mu\in F_q^*}\xi_{jb}(\gamma)\xi_{ik}(\mu)$$

where $v=\varphi'(j,b)$. Therefore in this case

$$\xi_{ib}(\gamma)\xi_{D'_0}(\varphi'_0)=\xi_{D'_0}(\varphi'_0)+\sum_{k=i+1}^{b-1}\sum_{\mu\in F_q^*}\xi_{ik}(\mu)\xi_{D'_0}(\varphi'_0).$$

Now we repeat either the argument of the previous paragraph or of this paragraph to study the scalar products

$$(\xi_{D_0}(\varphi_0),\xi_{ik}(\mu)\xi_{D'_0}(\varphi'_0))$$

for each $k\in\{i+1,\ldots,b-1\}$ and each $\mu\in F_q^*$. After a finite number of steps we eventually obtain

$$(\xi_{D_0}(\varphi_0),\xi_{ib}(\gamma)\xi_{D'_0}(\varphi'_0))=m_b(\xi_{D_0}(\varphi_0),\xi_{D'_0}(\varphi'_0))$$

where m_b is a positive integer which depends of b but not of γ.

Finally we conclude that

(2.2.3) $$(\xi_D(\varphi),\xi_{D'}(\varphi'))=m(\xi_{D_0}(\varphi_0),\xi_{D'_0}(\varphi'_0))$$

where m is a positive integer (in fact $m=1+(q-1)(m_{i+1}+\ldots+m_{n-1})$). Since

$$(\xi_D(\varphi),\xi_{D'}(\varphi'))\neq 0,$$

we deduce that

$$(\xi_{D_0}(\varphi_0),\xi_{D'_0}(\varphi'_0))\neq 0$$

and so (by induction)

$$(D_0,\varphi_0)=(D'_0,\varphi'_0).$$

The result follows. ♦

The proof of theorem 2.2.1 is now complete. The existence of the pair (D,φ) is given by corollary 2.2.14 and its unicity by the previous proposition.

The last result of this section is a corollary of the proof of the previous proposition. It is concerned with the scalar product $(\xi_D(\varphi),\xi_D(\varphi))$ where D is a basic subset of $\Phi(n)$ and $\varphi:D\to F_q^*$ is a map. This value can be expressed as a power of q where the exponent depends only on the set D. Firstly we give some definitions.

For each root $(i,j)\in\Phi$ we define the set

(2.2.4) $$S(i,j)=S^{(c)}(i,j)\cup S^{(r)}(i,j)$$

where

$$S^{(c)}(i,j)=\{(k,j);\, i<k<j\} \quad\text{and}\quad S^{(r)}(i,j)=\{(i,k);\, i<k<j\}.$$

The elements of $S(i,j)$ are called the *(i,j)-th singular roots*. It is clear that the (i,j)-th elementary $\xi_{ij}(\alpha)$ of $U_n(q)$ associated with $\alpha\in F_q{}^*$ has degree

$$\xi_{ij}(\alpha)(1)=q^{s(i,j)-l(i,j)}$$

where

(2.2.5) $$s(i,j)=|S(i,j)| \quad\text{and}\quad l(i,j)=|S^{(c)}(i,j)|=|S^{(r)}(i,j)|.$$

Now let D be a basic subset of $\Phi(n)$. Then we define the set

(2.2.6) $$S(D)=\bigcup_{(i,j)\in D} S(i,j)$$

where

$$S^{(c)}(D)=\bigcup_{(i,j)\in D} S^{(c)}(i,j) \quad\text{and}\quad S^{(r)}(D)=\bigcup_{(i,j)\in D} S^{(r)}(i,j).$$

A root $(a,b)\in\Phi(n)$ is called a *D-singular root* if $(a,b)\in S(D)$. We have

$$\xi_D(\varphi)(1)=q^{l(D)}=q^{s(D)}q^{l(D)-s(D)}$$

where

(2.2.7) $$s(D)=|S(D)| \quad\text{and}\quad l(D)=|S^{(c)}(D)|=|S^{(r)}(D)|.$$

Finally a root $(a,b)\in\Phi(n)$ is called a *D-regular root* if $(a,b)\notin S(D)$. We denote by $R(D)$ the set of all D-regular roots, i.e.

(2.2.8) $$R(D)=\Phi(n)\backslash S(D).$$

Therefore $\Phi(n)$ is the disjoint union

$$\Phi(n)=S(D)\cup R(D).$$

Moreover

$$D\subseteq R(D).$$

Corollary 2.2.17. *Let D be a basic subset of $\Phi(n)$ and let $\varphi: D\to F_q{}^*$. Then*

$$(\xi_D(\varphi),\xi_D(\varphi))=q^{2l(D)-s(D)}=q^{s(D)}q^{2(l(D)-s(D))}.$$

Proof. We proceed by induction on n. The result is clear if $n=2$. So we assume that $n>2$ and that the result is proved for n-1. If $D\subseteq\{(i,j);\ 1\leq i<j\leq n-1\}$ then

$$\xi_D(\varphi)=\zeta_D(\varphi)^*$$

(here the notation is as before). Therefore

$$(\xi_D(\varphi),\xi_D(\varphi))_{U_n(q)}=(\zeta_D(\varphi),\zeta_D(\varphi))_{U_{n-1}(q)}$$

and the result follows by induction. On the other hand, assume that $(i,n)\in D$ for some $i\in\{1,\ldots,n-1\}$ and let $D_0=D\setminus\{(i,n)\}$. Then (by (2.2.3))

$$(\xi_D(\varphi),\xi_D(\varphi))=m(\xi_{D_0}(\varphi_0),\xi_{D_0}(\varphi_0))$$

where

$$m=1+\sum_{b\in B}\sum_{\gamma\in F_q^*}m_b(\gamma),$$

$$B=\{b;\ i<b<n,\ (i,b)\in S^{(r)}(i,n)\cap S^{(c)}(D)\}$$

and for each $b\in B$ and each $\gamma\in F_q^*$

$$(\xi_{D_0}(\varphi_0),\xi_{ib}(\gamma)\xi_{D_0}(\varphi_0))=m_b(\gamma)(\xi_{D_0}(\varphi_0),\xi_{D_0}(\varphi_0)).$$

Since $m_b(\gamma)$ is independent of $\gamma\in F_q^*$ we have

$$m=1+\sum_{b\in B}(q-1)m_b$$

where $m_b=m_b(\gamma)$ for all $\gamma\in F_q^*$. Let

$$B=\{b_1,\ldots,b_t\}$$

where $b_1<\ldots<b_t$. We claim that

$$m_{b_s}=q^{s-1}$$

for all $s\in\{1,\ldots,t\}$. In fact we see from the previous proof that

$$(\xi_{D_0}(\varphi_0),\xi_{ib_1}(\gamma)\xi_{D_0}(\varphi_0))=(\xi_{D_0}(\varphi_0),\xi_{D_0}(\varphi_0))$$

so

$$m_{b_1}=1=q^0.$$

On the other hand if $s\in\{2,\ldots,t\}$ we get (using the argument of the previous proof)

$$(\xi_{D_0}(\varphi_0),\xi_{ib_s}(\gamma)\xi_{D_0}(\varphi_0))=(\xi_{D_0}(\varphi_0),\xi_{D_0}(\varphi_0))+\sum_{r=1}^{s-1}\sum_{\nu\in F_q^*}(\xi_{D_0}(\varphi_0),\xi_{ib_r}(\nu)\xi_{D_0}(\varphi_0)).$$

Hence

$$(\xi_{D_0}(\varphi_0),\xi_{ib_s}(\gamma)\xi_{D_0}(\varphi_0))=\left(1+\sum_{r=1}^{s-1}(q-1)m_{b_r}\right)(\xi_{D_0}(\varphi_0),\xi_{D_0}(\varphi_0))$$

so

$$m_{b_s}=1+\sum_{r=1}^{s-1}(q-1)m_{b_r}.$$

By induction we have $m_{b_r}=q^{r-1}$ for all $r\in\{1,\dots,s-1\}$. Therefore

$$m_{b_s}=1+\sum_{r=1}^{s-1}(q-1)q^{r-1}=q^{s-1}$$

as required.

It follows that

$$m=1+\sum_{s=1}^{t}(q-1)q^{s-1}=q^{t}.$$

Now

$$t=|S^{(r)}(i,n)\cap S^{(c)}(D)|$$

and (by induction)

$$(\xi_{D_0}(\varphi_0),\xi_{D_0}(\varphi_0))=q^{2l(D_0)-s(D_0)}.$$

Therefore

$$(\xi_D(\varphi),\xi_D(\varphi))=q^t q^{2l(D_0)-s(D_0)}=q^{2l(D)-s(D)}$$

because $l(D)=l(D_0)+l(i,n)$ and $s(D)=s(D_0)+s(i,n)-t$. The proof is complete. ♦

2.3. The irreducible characters of $U_n(q)$ for $n \leq 5$

Let D be a basic subset of $\phi(n)$ and let $\varphi: D \to F_q{}^*$ be a map. Then an irreducible character χ of $U_n(q)$ such that $(\chi, \xi_D(\varphi)) \neq 0$ will be referred to as an *irreducible character of type D*. The type D of the irreducible character χ may be represented by a "matrix" with entries (i,j) $(1 \leq i,j \leq n)$ which are occupied by the symbols $\cdot$, ■, • or o as follows: if $(i,j) \in \Phi(n)$ then the (i,j)-th entry is occupied by $\cdot$; if $(i,j) \in D$ then the (i,j)-th entry is occupied by ■; if $(i,j) \in S(D)$ then the (i,j)-th entry is occupied by •; finally if $(i,j) \in R(D)$ then the (i,j)-th entry is occupied by o. For example if $n=6$ then the irreducible characters of $U_6(q)$ of maximal degree have types

$$\begin{pmatrix} \cdot & \bullet & \bullet & \bullet & \bullet & \blacksquare \\ \cdot & \cdot & \bullet & \bullet & \blacksquare & \bullet \\ \cdot & \cdot & \cdot & \blacksquare & \bullet & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot & \bullet \end{pmatrix} \text{ or } \begin{pmatrix} \cdot & \bullet & \bullet & \bullet & \bullet & \blacksquare \\ \cdot & \cdot & \bullet & \bullet & \blacksquare & \bullet \\ \cdot & \cdot & \cdot & \circ & \bullet & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot & \bullet \end{pmatrix}.$$

In this section we describe the irreducible characters of $U_n(q)$ for $n \leq 5$.

Example 2.3.1. Let $n=2$. Then all irreducible characters of $U_2(q)$ are linear. In fact $U_2(q)$ is isomorphic to the additive group $F_q{}^+$, so the irreducible characters of $U_2(q)$ are in one-to-one correspondence with the elements of F_q. For each $\alpha \in F_q$ the irreducible character associated with α is the (1,2)-elementary character $\xi_{12}(\alpha)$ - we recall that $\xi_{12}(0)$ is the unit character of $U_2(q)$.

Example 2.3.2. We determine the irreducible characters of the group $U_3(q)$.

By lemma 2.2.2 $U_3(q)$ has q^2 linear characters which are in one-to-one correspondence with pairs (α,β) of elements of F_q. For $\alpha,\beta \in F_q$ the linear character which corresponds to (α,β) is $\xi_{12}(\alpha)\xi_{23}(\beta)$.

On the other hand for each non-zero element $\alpha \in F_q$ the (1,3)-th elementary character $\xi_{13}(\alpha)$ is irreducible and has degree q. Therefore $U_3(q)$ has q-1 irreduducible characters of degree q. These characters have type

$$\begin{pmatrix} \cdot & \bullet & \blacksquare \\ \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

Example 2.3.3. Here we consider the irreducible characters of the group $U_4(q)$.

By lemma 2.2.2 $U_4(q)$ has q^3 linear characters, namely the products $\xi_{12}(\alpha)\xi_{23}(\beta)\xi_{34}(\gamma)$ where α,β,γ are arbitrary elements of $\boldsymbol{F}_q$.

Now consider the irreducible characters of degree q. The following types

$$\begin{pmatrix} \cdot & \bullet & \blacksquare & \circ \\ \cdot & \cdot & \bullet & \circ \\ \cdot & \cdot & \cdot & \circ \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \circ & \circ & \circ \\ \cdot & \cdot & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \bullet & \blacksquare & \circ \\ \cdot & \cdot & \bullet & \circ \\ \cdot & \cdot & \cdot & \blacksquare \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \text{and} \begin{pmatrix} \cdot & \blacksquare & \circ & \circ \\ \cdot & \cdot & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

give $(q\text{-}1)+(q\text{-}1)+(q\text{-}1)^2+(q\text{-}1)^2=2q(q\text{-}1)$ irreducible characters of degree q. On the other hand consider the type

$$\begin{pmatrix} \cdot & \bullet & \blacksquare & \circ \\ \cdot & \cdot & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

By lemma 2.2.6 the character $\xi_{13}(\alpha)\xi_{24}(\beta)$ $(\alpha,\beta\in \boldsymbol{F}_q{}^*)$ has q distinct irreducible components of degree q. Therefore we obtain $q(q\text{-}1)^2$ irreducible characters of degree q of type $D=\{(1,3),(2,4)\}$.

Finally $U_4(q)$ has $q(q\text{-}1)$ irreducible characters of maximal degree q^2. These characters have one of the types

$$\begin{pmatrix} \cdot & \bullet & \bullet & \blacksquare \\ \cdot & \cdot & \blacksquare & \bullet \\ \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \text{or} \begin{pmatrix} \cdot & \bullet & \bullet & \blacksquare \\ \cdot & \cdot & \circ & \bullet \\ \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Example 2.3.4. We now consider the irreducible characters of $U_5(q)$.

By lemma 2.2.2 $U_5(q)$ has q^4 linear characters, namely the characters $\xi_D(\varphi)$ where D is a subset of $\Delta=\{(1,2),(2,3),(3,4),(4,5)\}$ and $\varphi:D\to \boldsymbol{F}_q{}^*$ is a map.

$U_5(q)$ has $3q^2(q-1)$ distinct irreducible characters of degree q corresponding to the following types

$$\begin{pmatrix} \cdot & \bullet & \blacksquare & \circ & \circ \\ \cdot & \cdot & \bullet & \circ & \circ \\ \cdot & \cdot & \cdot & \square & \circ \\ \cdot & \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \square & \circ & \circ & \circ \\ \cdot & \cdot & \bullet & \blacksquare & \circ \\ \cdot & \cdot & \cdot & \bullet & \circ \\ \cdot & \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \text{ and } \begin{pmatrix} \cdot & \square & \circ & \circ & \circ \\ \cdot & \cdot & \square & \circ & \circ \\ \cdot & \cdot & \cdot & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

where the symbol $\square$ in an entry (i,j) means that that entry may be occupied either by $\blacksquare$ or by $\circ$.

On the other hand consider the irreducible characters of type

$$\begin{pmatrix} \cdot & \bullet & \blacksquare & \circ & \circ \\ \cdot & \cdot & \bullet & \blacksquare & \circ \\ \cdot & \cdot & \cdot & \bullet & \circ \\ \cdot & \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Let $\alpha,\beta\in F_q^*$ be arbitrary. By lemma 2.2.6 the character $\xi_{13}(\alpha)\xi_{24}(\beta)$ has q distinct irreducible components of degree q. Thus for any $\gamma\in F_q$ the character $\xi_{13}(\alpha)\xi_{24}(\beta)\xi_{45}(\gamma)$ has q distinct irreducible components (because $\xi_{45}(\gamma)$ is linear). Therefore we obtain $q^2(q-1)^2$ distinct irreducible characters of the given type. All these characters have degree q.

Similarly $U_5(q)$ have $q^2(q-1)^2$ irreducible characters of degree q of type

$$\begin{pmatrix} \cdot & \square & \circ & \circ & \circ \\ \cdot & \cdot & \bullet & \blacksquare & \circ \\ \cdot & \cdot & \cdot & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

So far we have obtained $3q^2(q-1)+2q^2(q-1)^2=q^2(q-1)(2q+1)$ irreducible characters of degree q. We will see that these are all the irreducible characters of degree q.

Now we consider the irreducible characters of degree q^2. The types

$$\begin{pmatrix} \cdot & \bullet & \bullet & \blacksquare & \circ \\ \cdot & \cdot & \square & \bullet & \circ \\ \cdot & \cdot & \cdot & \bullet & \circ \\ \cdot & \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \text{ and } \begin{pmatrix} \cdot & \square & \circ & \circ & \circ \\ \cdot & \cdot & \bullet & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \square & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

give $2q^2(q-1)$ irreducible characters of degree q^2. Using lemma 2.2.6 we conclude that the types

$$\begin{pmatrix} \cdot & \bullet & \blacksquare & \circ & \circ \\ \cdot & \cdot & \bullet & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \square & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \text{ and } \begin{pmatrix} \cdot & \bullet & \bullet & \blacksquare & \circ \\ \cdot & \cdot & \square & \bullet & \circ \\ \cdot & \cdot & \cdot & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

determine $2q^2(q-1)^2$ irreducible characters of degree q^2. On the other hand we obtain another set consisting of $(q-1)^2$ irreducible characters of degree q^2 if we consider the type

$$\begin{pmatrix} \cdot & \bullet & \blacksquare & \circ & \circ \\ \cdot & \cdot & \bullet & \circ & \circ \\ \cdot & \cdot & \cdot & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Finally let $D=\{(1,3),(2,4),(3,5)\}$ - this type corresponds to the diagram

$$\begin{pmatrix} \cdot & \bullet & \blacksquare & \circ & \circ \\ \cdot & \cdot & \bullet & \blacksquare & \circ \\ \cdot & \cdot & \cdot & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

We claim that for each map $\varphi: D \to K^*$ the character $\xi_D(\varphi)$ has a unique irreducible component which occurs with multiplicity q. In fact let $\alpha,\beta,\gamma \in \mathbb{F}_q^*$ and consider the element

$$f=\alpha e_{13}^*+\beta e_{24}^*+\gamma e_{35}^* \in u_5(K)^*.$$

The matrix $M(f)$ which represents the bilinear form B_f with respect to the basis $(e_{12},e_{23},e_{34},e_{45})\cup(e_{ij};\ 1\le i<j\le 5, j-i\ge 2)$ has the form

$$M(f)=\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$A=\begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & \beta & 0 \\ 0 & -\beta & 0 & \gamma \\ 0 & 0 & -\gamma & 0 \end{pmatrix}.$$

It is clear that this matrix is non-singular. Thus

$$\operatorname{rank} M(f)=4$$

and (by (1.3.3))

$$dimO(f)=4$$

where $O(f)$ is the $U_5(K)$-orbit of f. It follows that

$$\chi_{O(f)}(1)=q^2$$

where $\chi_{O(f)}$ is the irreducible character of $U_5(q)$ which corresponds to the orbit $O(f)$. Now we consider the irreducible character χ_0 of $U_4(q)$ which corresponds to the $U_4(K)$-orbit of the element

$$\alpha e_{13}^*+\beta e_{24}^*\in u_4(K)^*.$$

By theorem 1.3.10 χ_0 is a component of the restriction $\chi_{U_4(q)}$ of χ to $U_4(q)$. We claim that

$$\chi=\chi_0^*\xi_{35}(\gamma).$$

Since the $U_5(q)$-orbit which corresponds to χ_0^* contains the element $\alpha e_{13}^*+\beta e_{24}^*\in u_5(K)^*$ we conclude that χ is a component of $\chi_0^*\xi_{35}(\gamma)$ (by corollary 1.3.11). On the other hand the character χ_0^* is an irreducible component of $\xi_{13}(\alpha)\xi_{24}(\beta)$. Thus it has degree q. Since $\xi_{35}(\gamma)(1)=q$ we get

$$\chi(1)=q^2=(\chi_0^*\xi_{35}(\gamma))(1)$$

and the claim follows. Finally we have

$$(\chi,\xi_{13}(\alpha)\xi_{24}(\beta)\xi_{35}(\gamma))=(\chi_0^*,\xi_{13}(\alpha)\xi_{24}(\beta)\xi_{35}(\gamma)\overline{\xi_{35}(\gamma)}).$$

Since $\overline{\xi_{35}(\gamma)}=\xi_{35}(-\gamma)$ (see the proof of proposition 2.2.16) we may use lemma 2.2.9 to obtain the decomposition

$$\xi_{35}(\gamma)\xi_{35}(-\gamma)=1_{U_5(q)}+\sum_{\nu\in F_q^*}\xi_{34}(\nu)+\sum_{\mu\in F_q^*}\xi_{45}(\mu)+\sum_{\nu\in F_q^*}\sum_{\mu\in F_q^*}\xi_{34}(\nu)\xi_{45}(\mu).$$

Since χ_0^* is a component of $\xi_{13}(\alpha)\xi_{24}(\beta)$ we have

$$(\chi_0^*,\xi_{13}(\alpha)\xi_{24}(\beta)\xi_{45}(\mu))=0$$

for all $\mu\in F_q^*$. On the other hand (by lemma 2.2.8)

$$\xi_{24}(\beta)\xi_{34}(\nu)=\xi_{24}(\beta)$$

for all $\nu\in F_q^*$. It follows that

$$(\chi,\xi_{13}(\alpha)\xi_{24}(\beta)\xi_{35}(\gamma))=q(\chi_0^*,\xi_{13}(\alpha)\xi_{24}(\beta))=q$$

and so

$$\xi_{13}(\alpha)\xi_{24}(\beta)\xi_{35}(\gamma)=q\chi$$

as required. Therefore $U_5(q)$ has $(q-1)^3$ irreducible characters of type

$$\begin{pmatrix} \cdot & \bullet & \blacksquare & \circ & \circ \\ \cdot & \cdot & \bullet & \blacksquare & \circ \\ \cdot & \cdot & \cdot & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

All these characters have degree q^2.

Now we consider the irreducible characters of $U_5(q)$ of degree q^3. On one hand the set of all irreducible characters of type

$$\begin{pmatrix} \cdot & \bullet & \bullet & \blacksquare & \circ \\ \cdot & \cdot & \bullet & \bullet & \blacksquare \\ \cdot & \cdot & \cdot & \bullet & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

contains $q(q-1)^2$ elements. All these characters have degree q^3. On the other hand there are $q^2(q-1)$ irreducible characters of type

$$\begin{pmatrix} \cdot & \bullet & \bullet & \bullet & \blacksquare \\ \cdot & \cdot & \square & \circ & \bullet \\ \cdot & \cdot & \cdot & \square & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

These characters have also degree q^3.

Finally (by corollary 2.2.15) $U_5(q)$ has $(q-1)^2$ irreducible characters of maximal degree q^4. They are of type

$$\begin{pmatrix} \cdot & \bullet & \bullet & \bullet & \blacksquare \\ \cdot & \cdot & \bullet & \blacksquare & \bullet \\ \cdot & \cdot & \cdot & \bullet & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Now it is easy to conclude (calculating the sum of the squares of the character degrees) that we have obtained all the irreducible characters of $U_5(q)$.

CHAPTER 3

BASIC SUMS OF ORBITS

In this chapter we generalize the results of chapter 2. In fact we will define for each basic subset D of $\Phi(n)$ and each map $\varphi:D\to K^*$ an irreducible subvariety $V_D(\varphi)$ of $u_n(K)^*$ which will turn out to be the sum $O_D(\varphi)$ of all the elementary $U_n(K)$-orbits $O_{ij}(\varphi(i,j))$ for $(i,j)\in D$. We will also prove that the family consisting of all these varieties defines a partition of $u_n(K)^*$. This is the purpose of section 3.1.

In section 3.2 we determine the dimension of the variety $V_D(\varphi)$. By definition the dimension of an irreducible algebraic variety is the transcendence degree over K of its field of rational functions, i.e. the field of fractions of the ring of polynomial functions defined on the given variety. Since $V_D(\varphi)$ is defined by certain polynomial functions which are parametrized by the D-regular roots the dimension of $V_D(\varphi)$ is related to the cardinality $s(D)$ of the set $S(D)$ consisting of all D-singular roots. In fact we will prove that the dimension of $V_D(\varphi)$ is exactly $s(D)$. Then (using this knowledge of dimensions) we will be able to decide about the transitivity of the $U_n(K)$-action on $V_D(\varphi)$. This will be done in section 3.3 where we will give a necessary and sufficient condition for $V_D(\varphi)$ to be a single $U_n(K)$-orbit. This condition is purely combinatorial and depends on the existence of some special chains of roots in D. If K has characteristic $p\geq n$, we will translate these results to decide about the irreducibility of the character $\xi_D(\varphi)$ of the finite group $U_n(q)$ (cf. chapter 2).

3.1. Sums of elementary orbits

Throughout this section K will denote an algebraically closed field either of characteristic zero or of prime characteristic $p\geq n$. We will consider the sum

(3.1.1) $$O_D(\varphi)=\sum_{(i,j)\in D} O_{ij}(\varphi(i,j))$$

where D is a basic subset of $\Phi(n)$ and $\varphi:D\to K^*=K\setminus\{0\}$ is a map (cf. (2.2.2)). Since $O_D(\varphi)$ is an algebraic subvariety of $\mathcal{U}_n(K)^*$, there should exist polynomial functions $P_1,\ldots,P_m:\mathcal{U}_n(K)^*\to K$ such that

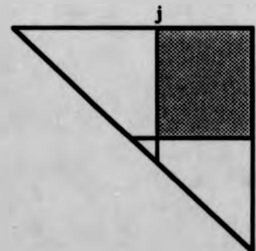
$$O_D(\varphi)=\{f\in \mathcal{U}_n(K)^*;\, P_i(f)=0,\ 1\leq i\leq m\}.$$

Our purpose is to identify these functions. We will see that m is the cardinality of the set $R(D)$ consisting of all D-regular roots [1]. In fact we will define a polynomial function

$$\Delta_{ij}^D:\mathcal{U}_n(K)^*\to K$$

for each D-regular root $(i,j)\in R(D)$ [2].

Let $(i,j)\in\Phi(n)$ be any root. We say that a root $(r,s)\in\Phi(n)$ is *dominated* by (i,j) if $r<i$ and $j<s$. Therefore the roots which are dominated by (i,j) lie in the dotted region of the adjacent picture.



Now let D be a basic subset of $\Phi(n)$ and let $(i,j)\in\Phi(n)$. Then we denote by $\bar{D}(i,j)$ the subset of $\Phi(n)$ consisting of all roots in D which are dominated by (i,j), i.e.

(3.1.2) $$D(i,j)=\{(r,s)\in D;\ 1\leq r<i,\ j<s\leq n\}.$$

It is clear that $D(i,j)$ is a basic subset of $\Phi(n)$. Let

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[1] We recall that a root $(i,j)\in\Phi(n)$ is called D-regular if $(i,k)\in D$ $(j+1\leq k\leq n)$ and $(k,j)\in D$ $(1\leq k\leq i-1)$.

[2] In section 3.2, we will prove that these functions are algebraically independent. Then it will follow that $O_D(\varphi)$ has dimension $\dim \mathcal{U}_n(K)^*-r(D)=s(D)$ where $r(D)=|R(D)|$ and $s(D)$ is the cardinality of the set of all D-singular roots. This is the natural conjecture.

$$D(i,j)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$$

and suppose that $j_1<\ldots<j_r$. Let $\sigma\in S_r$ be the permutation such that $i_{\sigma(1)}<i_{\sigma(2)}<\ldots<i_{\sigma(r)}$. Then we define the function $\Delta_{ij}^{D}: \mathcal{U}_n(K)^* \to K$ by

$$\Delta_{ij}^{D}(f)=\det\begin{pmatrix} f(e_{i_{\sigma(1)}j}) & f(e_{i_{\sigma(1)}j_1}) & \cdots & f(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & & \vdots \\ f(e_{i_{\sigma(r)}j}) & f(e_{i_{\sigma(r)}j_1}) & \cdots & f(e_{i_{\sigma(r)}j_r}) \\ f(e_{ij}) & f(e_{ij_1}) & \cdots & f(e_{ij_r}) \end{pmatrix} \tag{3.1.3}$$

for all $f\in \mathcal{U}_n(K)^*$.

Our first result describes the (i,j)-th elementary $U_n(K)$-orbit associated $O_{ij}(\alpha)$ with a non-zero element $\alpha\in K$.

Lemma 3.1.1. *Let $(i,j)\in\Phi(n)$ and let $\alpha\in K^*$. Then $O_{ij}(\alpha)$ consists of all elements $f\in \mathcal{U}_n(K)^*$ which satisfy the equations*

$$\Delta_{ab}^{D}(f)=\begin{cases}\alpha & \text{if } (a,b)=(i,j)\\ 0 & \text{if } (a,b)\neq(i,j)\end{cases} \tag{3.1.4}$$

for all $(a,b)\in R(i,j)$. In particular $f\in O_{ij}(\alpha)$ if and only if

$$f(e_{ab})=\begin{cases}\alpha & \text{if } (a,b)=(i,j)\\ \alpha^{-1}f(e_{aj})f(e_{ib}) & \text{if } (a,b) \text{ dominates } (i,j)\\ 0 & \text{otherwise}\end{cases} \tag{3.1.5}$$

for all $(a,b)\in R(D)$ ([1]).

Proof. Let V be the subset of $\mathcal{U}_n(K)^*$ consisting of all matrices which satisfy (3.1.5). We claim that $V\subseteq O_{ij}(\alpha)$. Let $f\in V$ and consider the element $\alpha e_{ij}^*\in O_{ij}(\alpha)$. Let $x=(x_{rs})\in U_n(K)$ be the element defined by

$$x_{rs}=\begin{cases}-\alpha^{-1}f(e_{ir}) & \text{if } i<r<j \text{ and } s=j\\ \alpha^{-1}f(e_{sj}) & \text{if } r=i \text{ and } i<s<j\\ 1 & \text{if } r=s\\ 0 & \text{otherwise}\end{cases} \tag{3.1.6}$$

[1] We note that if (a,b) dominates (i,j) then $i+1\leq a<b\leq j-1$, so the roots (a,j) and (i,b) are D-singular.

Then

$$(x\cdot(\alpha e_{ij}{}^*))(e_{ab})=(\alpha e_{ij}{}^*)(xe_{ab}x^{-1})=\alpha(xe_{ab}x^{-1})_{ij}=\alpha x_{ia}(x^{-1})_{bj}$$

for all $(a,b)\in\Phi(n)$. This value is non-zero only if $i\leq a<b\leq j$. If $(a,b)=(i,j)$, we have

$$(x\cdot(\alpha e_{ij}{}^*))(e_{ab})=\alpha=f(e_{ij}).$$

If $i<b<j$ then

$$(x\cdot(\alpha e_{ij}{}^*))(e_{ib})=\alpha(x^{-1})_{bj}=-\alpha x_{bj}=\alpha\alpha^{-1}f(e_{ib})=f(e_{ib}).$$

On the other hand

$$(x\cdot(\alpha e_{ij}{}^*))(e_{aj})=\alpha x_{ia}=\alpha\alpha^{-1}f(e_{aj})=f(e_{aj})$$

whenever $i<a<j$. Finally suppose that $(a,b)\in\Phi(n)$ dominates (i,j). Then

$$(x\cdot(\alpha e_{ij}{}^*))(e_{ab})=\alpha x_{ia}(x^{-1})_{bj}=\alpha\alpha^{-1}f(e_{aj})\alpha^{-1}f(e_{ib})=f(e_{ab})$$

It follows that

$$f=x\cdot(\alpha e_{ij}{}^*)\in O_{ij}(\alpha).$$

Since $f\in V$ is arbitrary we conclude

$$V\subseteq O_{ij}(\alpha).$$

Now the map $\phi:V\to K^{2(j-i-1)}$ defined by

$$\phi(f)=(f(e_{ii+1}),\dots f(e_{ij-1})f(e_{i+1j}),\dots f(e_{j-1j}))$$

for all $f\in V$, is an isomorphism of algebraic varieties. Therefore V is an irreducible variety of dimension $2(j-i-1)$. Since $dimO_{ij}(\alpha)=2(j-i-1)$, we conclude that

$$V=O_{ij}(\alpha).$$

The lemma follows. ♦

Proposition 3.1.2. *Let D be a basic subset of $\Phi(n)$ and let $\varphi:D\to K^*$ be a map. Let $f\in O_D(\varphi)$ and let $(i,j)\in R(D)$. Then*

$$\Delta_{ij}^D(f)=\begin{cases}0 & \text{if } (i,j)\in D\\ (-1)^r sgn(\sigma)\varphi(i,j)\prod_{s=1}^{r}\varphi(i_s,j_s) & \text{if } (i,j)\in D\end{cases}\tag{3.1.7}$$

where $D(i,j)=\{(i_1,j_1),\dots,(i_r,j_r)\}$, $j_1<\dots<j_r$, and $\sigma\in S_r$ is such that $i_{\sigma(1)}<\dots<i_{\sigma(r)}$.

Proof. Let $D'=D\setminus D(i,j)$ and let φ' be the restriction of φ to D'. Then

$$O_D(\varphi)=O_{D'}(\varphi')+\sum_{s=1}^{r} O_{i_s j_s}(\alpha_s)$$

where $\alpha_s=\varphi(i_s,j_s)$ $(1\le s\le r)$. Therefore the element $f\in O_D(\varphi)$ can be written in the form

$$f=f'+\sum_{s=1}^{r} f^{(s)}$$

where $f'\in O_{D'}(\varphi')$ and $f^{(s)}\in O_{i_s j_s}(\alpha_s)$ for all $s\in\{1,\dots,r\}$. By the previous lemma we have

$$f(e_{ab})=f'(e_{ab})+\sum_{s=1}^{r} f^{(s)}(e_{ab})=f'(e_{ab})+\sum_{s=1}^{r} \alpha_s^{-1}f^{(s)}(e_{aj_s})f^{(s)}(e_{i_s b})$$

for all $(a,b)\in\Phi(n)$. Now suppose that $(a,b)\in\Phi(n)$ is dominated by (i,j). Then

$$f'(e_{ab})=0 \quad\text{and}\quad f(e_{ab})=u_a v_b$$

where u_a is the row vector of length r

$$u_a=\left(\alpha_1^{-1}f^{(1)}(e_{aj_1}) \;\dots\; \alpha_r^{-1}f^{(r)}(e_{aj_r})\right)$$

and v_b is the column vector of length r

$$v_b=\begin{pmatrix} f^{(1)}(e_{i_1 b}) \\ \vdots \\ f^{(r)}(e_{i_r b}) \end{pmatrix}$$

On the other hand we have

$$f'(e_{ij})=\begin{cases} 0 & \text{if } (i,j)\in D \\ \varphi(i,j)=\alpha & \text{if } (i,j)\in D \end{cases}$$

Therefore

$$f(e_{ij})=\begin{cases} uv & \text{if } (i,j)\in D \\ \alpha+uv & \text{if } (i,j)\in D \end{cases}$$

where u is the row vector of length r

$$u=\left(\alpha_1^{-1}f^{(1)}(e_{ij_1}) \;\dots\; \alpha_r^{-1}f^{(r)}(e_{ij_r})\right)$$

and v is the column vector of length r

$$v=\begin{pmatrix} f^{(1)}(e_{i_1 j}) \\ \vdots \\ f^{(r)}(e_{i_r j}) \end{pmatrix}$$

Now suppose that $(i,j)\in D$. Let A be the square matrix of size r

$$A=\begin{pmatrix} u_{i_1} \\ \vdots \\ u_{i_r} \end{pmatrix}$$

and let B be the square matrix of size r

$$B=\begin{pmatrix} v_{j_1} & \dots & v_{j_r} \end{pmatrix}.$$

Then

$$\Delta_{ij}^{D}(f)=det\left(\begin{pmatrix} P(\sigma^{-1}) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} Av & AB \\ uv & uB \end{pmatrix}\right)=sgn(\sigma)det(A'B')$$

where $P(\sigma^{-1})$ is the permutation matrix (of order r) associated with $\sigma^{-1}\in S_r$ (hence the (a,b)-th entry of $P(\sigma^{-1})$ is $\delta_{a\sigma^{-1}(b)}$ $(1\leq a,b\leq r)$), A' is the matrix of type $(r+1)\times r$

$$A'=\begin{pmatrix} A \\ u \end{pmatrix}$$

and B' is the matrix of type $r\times(r+1)$

$$B'=\begin{pmatrix} v & B \end{pmatrix}.$$

Since A' and B' have rank less or equal than r, we deduce that

$$rank(A'B')\leq min\{rankA',rankB'\}\leq r.$$

It follows that

$$det(A'B')=0$$

and so

$$\Delta_{ij}^{D}(f)=0.$$

On the other hand suppose that $(i,j)\in D$. In this case

$$\Delta_{ij}^{D}(f)=det\left(\begin{pmatrix} P(\sigma^{-1}) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} Av & AB \\ \alpha+uv & uB \end{pmatrix}\right)=sgn(\sigma)\left(det\begin{pmatrix} 0 & AB \\ \alpha & uB \end{pmatrix}+det\begin{pmatrix} Av & AB \\ uv & uB \end{pmatrix}\right)$$

where $P(\sigma^{-1})$, A' and B' are as above. Since

$$det\begin{pmatrix} Av & AB \\ uv & uB \end{pmatrix}=0$$

(by the argument above) we conclude that

$$\Delta_{ij}^{D}(f)=sgn(\sigma)det\begin{pmatrix}0 & AB\\ \alpha & uB\end{pmatrix}=(-1)^{r}sgn(\sigma)\alpha\, det(AB)=(-1)^{r}sgn(\sigma)\alpha\, det A\, det B.$$

Consider the matrix B. For $s,t\in\{1,\ldots,r\}$ the (s,t)-th entry of B is

$$b_{st}=f^{(s)}(e_{i_s j_t}).$$

Since $f^{(s)}\in O_{i_s j_s}(\alpha_s)$, we have

$$f^{(s)}(e_{i_s b})=0$$

for all $b\in\{j_s+1,\ldots,n\}$ (by the previous lemma). Therefore

$$f^{(s)}(e_{i_s j_t})=0$$

whenever $s<t$ (we recall that $j_1<\ldots<j_r$). It follows that B is lower-triangular. Moreover the diagonal entries of B are

$$b_{ss}=f^{(s)}(e_{i_s j_s})=\alpha_s$$

for all $s\in\{1,\ldots,r\}$. Hence

$$det\, B=\prod_{s=1}^{r}\alpha_s.$$

Finally we consider the matrix A. For $s,t\in\{1,\ldots,r\}$ the (s,t)-th entry of A is

$$a_{st}=\alpha_t^{-1}f^{(t)}(e_{i_s j_t}).$$

Therefore the (s,t)-th entry of $P(\sigma^{-1})AP(\sigma)$ is

$$a_{\sigma(s)\sigma(t)}=\alpha_{\sigma(t)}^{-1}f^{(\sigma(t))}(e_{i_{\sigma(s)}j_{\sigma(t)}}).$$

Now (by the previous lemma)

$$f^{(\sigma(t))}(e_{a j_{\sigma(t)}})=0$$

for all $a\in\{1,\ldots,i_{\sigma(t)}-1\}$ and

$$f^{(\sigma(t))}(e_{i_{\sigma(t)}j_{\sigma(t)}})=\alpha_{\sigma(t)}$$

for all $t\in\{1,\ldots,r\}$. Since $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$, we conclude that the s-th row of $P(\sigma^{-1})AP(\sigma)$ is

$$\left(\alpha_{\sigma(1)}^{-1}f^{(\sigma(1))}(e_{i_{\sigma(s)}j_{\sigma(1)}})\ \ldots\ \alpha_{\sigma(s-1)}^{-1}f^{(\sigma(s-1))}(e_{i_{\sigma(s)}j_{\sigma(s-1)}})\ 1\ 0\ \ldots\ 0\right).$$

Therefore

$$det\, A=det(P(\sigma^{-1})AP(\sigma))=1$$

and the result follows. ♦

If D is a basic subset of $\Phi(n)$ and $\varphi:D\rightarrow K^*$ is a map we denote by $V_D(\varphi)$ the subvariety of $U_n(K)^*$ consisting of all $f\in U_n(K)^*$ which satisfy the equations (3.1.7) for all $(i,j)\in R(D)$. By the previous proposition we have

$$O_D(\varphi)\subseteq V_D(\varphi).$$

Our next result asserts that this inclusion is in fact an equality.

Proposition 3.1.3. *Let D be a basic subset of $\Phi(n)$, let $\varphi:D\rightarrow K^*$ be a map and let $f\in V_D(\varphi)$. Then $f\in O_D(\varphi)$, hence $V_D(\varphi)=O_D(\varphi)$.*

Proof. We proceed by induction on $|D|$. If $|D|=1$ the result folows by lemma 3.1.1. Thus we suppose that D contains at least two elements and that the result is proved for all basic subsets D_0 of $\Phi(n)$ such that $|D_0|<|D|$.

Let $f\in V_D(\varphi)$ be arbitrary and let $(i,j)\in D$ be such that $l<j$ for all $(k,l)\in D\setminus\{(i,j)\}$. We put $\alpha=\varphi(i,j)$ and we define $f'\in U_n(K)^*$ by

$$f'(e_{ab})=\begin{cases} f(e_{ib}) & \text{if } a=i \text{ and } i<b<j \\ f(e_{aj}) & \text{if } b=j \text{ and } i<a<j \\ \alpha^{-1}f(e_{aj})f(e_{ib}) & \text{if } i<a<b<j \\ 0 & \text{otherwise} \end{cases}$$

for all $(a,b)\in\Phi(n)$. Then

$$f'\in O_{ij}(\alpha)$$

and

$$f=f'+f_0$$

for some $f_0\in U_n(K)^*$. We claim that

(3.1.8) $$f_0\in V_{D_0}(\varphi_0)$$

where $D_0=D-\{(i,j)\}$ and φ_0 is the restriction of φ to D_0. For let $(a,b)\in R(D_0)$. If (a,b) does not dominate (i,j) then (by definition)

$$\Delta^{D_0}_{ab}(f_0)=\begin{cases}\Delta^{D}_{ab}(f) & \text{if } a\neq i \text{ or } b\neq j \\ 0 & \text{if } a=i \text{ or } b=j\end{cases}$$

- we note that

$$\Delta^{D_0}_{aj}(f_0)=f_0(e_{aj})=0 \quad \text{and} \quad \Delta^{D_0}_{ib}(f_0)=0$$

because $f_0(e_{ik})=0$ for all $k\in\{i+1,\ldots,n\}$. Since $f\in V_D(\varphi)$ we conclude that the equations (3.1.7) (for the pair (D_0,φ_0)) hold in this case.

Now suppose that (a,b) dominates (i,j) (i.e. $i<a<b<j$). Let

$$D(a,b)=\{(i_1,j_1),\ldots,(i_r,j_r)\},\ j_1<\ldots<j_r$$

and let $\sigma\in S_r$ be the permutation such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. Then $(i,j)=(i_r,j_r)$ (by choice of (i,j)) and

$$\Delta^D_{ab}(f)=sgn(\tau\sigma)det\begin{pmatrix} u & \alpha \\ A & v\end{pmatrix}$$

where $\tau\in S_r$ is the permutation $\tau=(1\,2\ldots r)$ (we note that $P(\sigma^{-1})P(\tau^{-1})=P((\tau\sigma)^{-1})$), u is the row vector (of length r)

$$u=\big(f(e_{ib})\, f(e_{ij_1})\ \ldots\ f(e_{ij_{r-1}})\big),$$

v is the column vector (of length r)

$$v=\begin{pmatrix} f(e_{i_1j}) \\ \vdots \\ f(e_{i_{r-1}j})\end{pmatrix}$$

and A is the $r\times r$ matrix

$$A=\begin{pmatrix} f(e_{i_1b}) & f(e_{i_1j_1}) & \ldots & f(e_{i_1j_{r-1}}) \\ \vdots & \vdots & & \vdots \\ f(e_{i_{r-1}b}) & f(e_{i_{r-1}j_{r-1}}) & \ldots & f(e_{i_{r-1}j_{r-1}}) \\ f(e_{ab}) & f(e_{aj_1}) & \ldots & f(e_{aj_{r-1}})\end{pmatrix}.$$

Since

$$\begin{pmatrix} u & \alpha \\ A & v\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ \alpha^{-1}v & I\end{pmatrix}\begin{pmatrix} u & \alpha \\ A-\alpha^{-1}vu & 0\end{pmatrix},$$

we conclude that

$$\Delta^D_{ab}(f)=sgn(\tau\sigma)det\begin{pmatrix} u & \alpha \\ A-\alpha^{-1}vu & 0\end{pmatrix}=(-1)^r sgn(\tau\sigma)\alpha\, det(A-\alpha^{-1}vu).$$

Now let $A-\alpha^{-1}vu=(a_{st})_{1\le s,t\le r}$. Then

$$a_{st}=\begin{cases} f(e_{i_sb})-\alpha^{-1}f(e_{i_sj})f(e_{ib})=f_0(e_{i_sb}) & \text{if } t=1 \text{ and } 1\le s\le r-1 \\ f(e_{aj_{t-1}})-\alpha^{-1}f(e_{aj})f(e_{ij_{t-1}})=f_0(e_{aj_{t-1}}) & \text{if } s=r \text{ and } 2\le t\le r \\ f(e_{i_sj_{t-1}})-\alpha^{-1}f(e_{i_sj})f(e_{ij_{t-1}})=f_0(e_{i_sj_{t-1}}) & \text{if } 1\le s\le r-1 \text{ and } 2\le t\le r \\ f(e_{ab})-\alpha^{-1}f(e_{aj})f(e_{ib})=f_0(e_{ab}) & \text{if } (s,t)=(r,1) \end{cases}$$

Since

$$D_0(a,b)=\{(i_1,j_1),\ldots,(i_{r-1},j_{r-1})\}$$

we conclude that

$$det(A-\alpha^{-1}vu)=sgn(\sigma_0)\Delta_{ab}^{D_0}(f_0)$$

where $\sigma_0\in S_{r-1}$ is such that $i_{\sigma_0(1)}<\ldots<i_{\sigma_0(r-1)}$. Hence

$$\Delta_{ab}^{D}(f)=(-1)^r\,\alpha\, sgn(\tau\sigma)\, sgn(\sigma_0)\,\Delta_{ab}^{D_0}(f_0).$$

Now suppose that $(a,b)\notin D_0$. Then $(a,b)\notin D$ and

$$\Delta_{ab}^{D}(f)=0$$

(by equation (3.1.7)). So

$$\Delta_{ab}^{D_0}(f_0)=0.$$

On the other hand suppose that $(a,b)\in D_0$. Then $(a,b)\in D$ and (by (3.1.7))

$$\Delta_{ab}^{D}(f)=(-1)^r sgn(\sigma)\varphi(a,b)\prod_{s=1}^{r}\alpha_s$$

where $\alpha_s=\varphi(i_s,j_s)$ $(1\le s\le r-1)$ and $\alpha_r=\alpha$. Since $sgn(\tau)=(-1)^{r-1}$ and $sgn(\tau\sigma)=sgn(\tau)sgn(\sigma)$ we conclude that

$$\Delta_{ab}^{D_0}(f_0)=(-1)^{r-1}sgn(\sigma_0)\varphi(a,b)\prod_{s=1}^{r-1}\alpha_s.$$

This completes the proof of (3.1.8).

Finally (by induction) we have

$$f_0\in O_{D_0}(\varphi_0).$$

Hence

$$f=f'+f_0\in O_{ij}(\alpha)+O_{D_0}(\varphi_0)=O_D(\varphi)$$

and the proof of the proposition is complete. ♦

Our next result generalizes the corollary 2.2.14. It asserts that $U_n(K)^*$ is the union of the subvarieties $V_D(\varphi)$ for all basic subsets D of $\Phi(n)$ and all maps $\varphi: D \rightarrow K^*$.

Proposition 3.1.4. *Let $f \in U_n(K)^*$. Then there exists a basic subset D of $\Phi(n)$ and a map $\varphi: D \rightarrow K^*$ such that $f \in V_D(\varphi)$.*

Proof. In order to construct the set D and the map φ we proceed by recursion on the set of all roots $\Phi(n)$ endowed with the total order $<$ defined in the proof of proposition 2.2.13.

If $f=0$ then we let D be the empty set. On the other hand suppose $f \neq 0$. Then the set

$$R_1=\{(i,j) \in \Phi(n); f(e_{ij}) \neq 0\}$$

is non-empty and we may choose its first element (i_1,j_1). We define

$$D_1=\{(i_1,j_1)\}$$

and $\varphi_1: D_1 \rightarrow K^*$ by

$$\varphi_1(i_1,j_1)=f(e_{i_1j_1}).$$

Now either $f \in V_{D_1}(\varphi_1)$ (and the proof is complete) or the set

$$R_2=\{(i,j) \in R(D_1); (i,j)>(i_1,j_1) \text{ and } \Delta_{ij}^{D_1}(f) \neq 0\}$$

is non-empty. If this is the case we choose the first element $(i_2,j_2) \in R_2$ and we define

$$D_2=\{(i_1,j_1),(i_2,j_2)\}$$

and $\varphi_2: D_2 \rightarrow K^*$ by

$$\varphi_2(i_1,j_1)=\varphi_1(i_1,j_1)$$

and

$$\varphi(i_2,j_2)=\begin{cases} \Delta_{i_2j_2}^{D_1}(f)=f(e_{i_2j_2}) & \text{if } (i_2,j_2) \text{ does not dominate } (i_1,j_1) \\ -\varphi_1(i_1,j_1)^{-1}\Delta_{i_2j_2}^{D_1}(f) & \text{if } (i_2,j_2) \text{ dominates } (i_1,j_1) \end{cases}$$

It is clear that D_2 is a basic subset of $\Phi(n)$ and that

$$\Delta_{ij}^{D_2}(f)=\Delta_{ij}^{D_1}(f)$$

for all $(i,j) \in R(D_2)$ such that $(i,j) \leq (i_2,j_2)$. Hence the proof is complete if $f \in V_{D_2}(\varphi_2)$. On the other hand if $f \in V_{D_2}(\varphi_2)$ we use the functions $\Delta_{ij}^{D_2}: U_n(K)^* \rightarrow K$, $(i,j) \in R(D_2)$,

$(i,j)>(i_2,j_2)$, to define a new basic subset D_3 of $\Phi(n)$ which includes D_2 and a new map $\varphi_3{:}D_3\to K^*$ whose restriction to D_2 is φ_2.

In general let $r>2$ and suppose that we have constructed a basic subset

$$D_{r-1}=\{(i_1,j_1),\ldots,(i_{r-1},j_{r-1})\}\subset\Phi(n),\ j_{r-1}<\ldots<j_1,$$

and a map $\varphi_{r-1}{:}D_{r-1}\to K^*$ such that f satisfies the equation (3.1.7) (for the pair (D_{r-1},φ_{r-1})) whenever $(i,j)\in R(D_{r-1})$ is such that $(i,j)\leq(i_{r-1},j_{r-1})$. Then either $f\in V_{D_{r-1}}(\varphi_{r-1})$ (and the proof is complete) or the set

$$R_r=\{(i,j)\in R(D_{r-1});\ (i,j)>(i_{r-1},j_{r-1})\text{ and }\Delta_{ij}^{D_{r-1}}(f)\neq0\}$$

is non-empty. We assume that this situation happens and we let (i_r,j_r) be the first element of R_r. Then $j_r<j_{r-1}$ so $D_r=D_{r-1}\cup\{(i_r,j_r)\}$ is a basic set of roots. We define $\varphi_r{:}D_r\to K^*$ by

$$\varphi_r(i_s,j_s)=\begin{cases}\varphi_{r-1}(i_s,j_s) & 1\leq s\leq r-1\\ (-1)^t sgn(\sigma)\prod\limits_{\alpha=1}^{t}\varphi(i_{s_\alpha},j_{s_\alpha})^{-1}\Delta_{i_rj_r}^{D_{r-1}}(f) & \text{if } s=r\end{cases}$$

where $D_r(i_r,j_r)=\{(i_{s_1},j_{s_1}),\ldots,(i_{s_t},j_{s_t})\}$, $j_{s_1}<\ldots<j_{s_t}$ and $\sigma\in S_t$ is such that $i_{s_{\sigma(1)}}<\ldots<i_{s_{\sigma(t)}}$.

Now we consider the variety $V_{D_r}(\varphi_r)$. Since $D_{r-1}(i_r,j_r)=\{(i_{s_1},j_{s_1}),\ldots,(i_{s_t},j_{s_t})\}$ we have

$$\Delta_{ij}^{D_r}(f)=\Delta_{ij}^{D_{r-1}}(f)$$

for all $(i,j)\in R(D_r)$ such that $(i,j)<(i_r,j_r)$. If $f\in V_{D_r}(\varphi_r)$ the proof is complete. On the other hand if $f\in V_{D_r}(\varphi_r)$ we continue the construction until we eventually get a basic subset $D=D_t$ of $\Phi(n)$ and a map $\varphi=\varphi_t{:}D\to K^*$ such that $f\in V_D(\varphi)$. ♦

Now we generalize proposition 2.2.16.

Proposition 3.1.5. *Let D and D' be basic subsets of $\Phi(n)$ and let $\varphi{:}D\to K^*$ and $\varphi'{:}D'\to K^*$ be maps. Then $V_D(\varphi)\cap V_{D'}(\varphi')\neq\varnothing$ if and only if $D=D'$ and $\varphi=\varphi'$.*

Proof. Suppose that $V_D(\varphi)\cap V_{D'}(\varphi')\neq\varnothing$ and let $f\in V_D(\varphi)\cap V_{D'}(\varphi')$. Let (i,j) be the smallest root in D (the order in $\Phi(n)$ is the same as in the previous proof). Then

$\Delta^D_{ij}(f)=f(e_{ij})=\varphi(i,j)\neq 0$ and $\Delta^D_{ab}(f)=f(e_{ab})=0$ for all $(a,b)\in\Phi(n)$ such that $(a,b)<(i,j)$. It follows that

$$\Delta^{D'}_{ab}(f)=\Delta^D_{ab}(f)$$

for all $(a,b)\in\Phi(n)$ such that $(a,b)\leq(i,j)$. Hence (i,j) is the smallest element in D' and $\varphi(i,j)=\varphi'(i,j)$.

Now we proceed by induction on $|D|$. Suppose that $|D|=1$ and that $|D'|\geq 2$. Let $(a,b)\in D'$ be the smallest root in D' such that $(a,b)>(i,j)$ (this root exists because (i,j) is the smallest root in D'). Since $D'\subseteq R(D')$ (a,b) is a D'-regular root. If (a,b) does not dominate (i,j) then

$$\Delta^{D'}_{ab}(f)=f(e_{ab})=\Delta^D_{ab}(f)=0$$

so $\varphi'(a,b)=0$, a contradiction. Therefore (a,b) dominates (i,j). In this case we have

$$\Delta^{D'}_{ab}(f)=det\begin{pmatrix} f(e_{ib}) & f(e_{ij}) \\ f(e_{ab}) & f(e_{aj}) \end{pmatrix}=\Delta^D_{ab}(f)=0.$$

By proposition 3.1.2 we conclude that $(a,b)\in D'$, another contradiction. It follows that $|D'|=1$ and so $(D,\varphi)=(D',\varphi')$.

Now suppose that $|D|>1$ and let $D_0=D-\{(i,j)\}$. Since $V_D(\varphi)=O_D(\varphi)$, there exist $f'\in O_{ij}(\alpha)$ and $f_0\in O_{D_0}(\varphi_0)$ such that

$$f=f'+f_0$$

(here $\alpha=\varphi(i,j)$ and φ_0 is the restriction of φ to D_0). On the other hand since

$$f\in V_{D'}(\varphi')=O_{D'}(\varphi')=O_{ij}(\alpha)+O_{D'_0}(\varphi'_0)$$

(where $D'_0=D'-\{(i,j)\}$ and φ'_0 is the restriction of φ' to D'_0) there exists $x\in U_n(K)$ such that

$$x\cdot f=\alpha e_{ij}{}^*+x\cdot f_0.$$

Moreover

$$x\cdot f_0\in O_{D_0}(\varphi_0)$$

because $O_{D_0}(\varphi_0)$ is $U_n(K)$-invariant. Since $(a,n)\in D_0$ for all $a\in\{1,\ldots,n-1\}$ we deduce that

$$x\cdot f_0\in u_{n-1}(K)^*.$$

On the other hand since $O_{D'}(\varphi')$ is $U_n(K)$-invariant we have

$$x\cdot f=\alpha e_{ij}^*+x\cdot f_0\in O_{D'}(\varphi')=O_{ij}(\alpha)+O_{D'_0}(\varphi'_0).$$

Since $x\cdot f_0\in \mathcal{U}_{n-1}(K)^*$ it follows that

$$x\cdot f_0\in O_{D'_0}(\varphi'_0).$$

Hence

$$f_0\in O_{D'_0}(\varphi'_0)$$

because $O_{D'_0}(\varphi'_0)$ is $U_n(K)$-invariant. This proves that

$$O_{D_0}(\varphi_0)\cap O_{D'_0}(\varphi'_0)\neq\emptyset.$$

By proposition 3.1.3 we deduce that

$$V_{D_0}(\varphi_0)\cap V_{D'_0}(\varphi'_0)\neq\emptyset$$

and so (by induction) $(D_0,\varphi_0)=(D'_0,\varphi'_0)$. Therefore $(D,\varphi)=(D',\varphi')$ as required. ♦

We have just finished the proof of the following:

Theorem 3.1.6. *Let D be a basic subset of $\Phi(n)$ and let $\varphi:D\to K^*$ be a map. Then $O_D(\varphi)=V_D(\varphi)$ where $V_D(\varphi)$ is the algebraic variety consisting of all $f\in \mathcal{U}_n(K)^*$ satisfying the equations (3.1.7) for all $(i,j)\in R(D)$. Moreover we have a decomposition of $\mathcal{U}_n(K)^*$ into disjoint subvarieties*

$$\mathcal{U}_n(K)^*=\bigcup_{D,\varphi} O_D(\varphi)=\bigcup_{D,\varphi} V_D(\varphi)$$

where the unions are over all basic subsets D of $\Phi(n)$ and all maps $\varphi:D\to K^*$.

Now suppose that K has prime characteristic $p\geq n$ and let $F=F_q:\mathcal{U}_n(K)^*\to \mathcal{U}_n(K)^*$ (q is a power of p) be the usual Frobenius map . Let D be a basic subset of $\Phi(n)$ and let $\varphi:D\to K^*$ be a map such that $\varphi(D)\subseteq \mathbf{F}_q^*$. Then the variety $V_D(\varphi)$ is F-stable. In fact (by (3.1.7))

$$\Delta_{ij}^D(F(f))=(\Delta_{ij}^D(f))^q=\Delta_{ij}^D(f)$$

for all $f\in V_D(\varphi)$ and all $(i,j)\in R(D)$ (we note that $\Delta_{ij}^D(f)\in \mathbf{F}_q$ because $\varphi(D)\subseteq\mathbf{F}_q^*$). Therefore we may consider the (finite) set

(3.1.9) $$V_D(\varphi)^F=O_D(\varphi)^F$$

consisting of all F-fixed elements of $V_D(\varphi)=O_D(\varphi)$. By the previous theorem we get

(3.1.10) $$U_n(q)^*=\bigcup_{D,\varphi} O_D(\varphi)^F=\bigcup_{D,\varphi} V_D(\varphi)^F$$

where the unions are over all basic subsets D of $\Phi(n)$ and all maps $\varphi:D\to\mathbf{F}_q^*$. On the other hand since $O_D(\varphi)$ is $U_n(K)$-invariant we have

$$O_D(\varphi)^F=\bigcup_{O} O^F$$

where the union is over all F-stable $U_n(K)$-orbits O such that $O\subseteq O_D(\varphi)$. Applying corollary 1.3.11 we obtain theorem 2.2.1 as a corollary of the previous theorem.

Corollary 3.1.7. *Let χ be an irreducible character of $U_n(q)$. Then there exist a unique basic set of roots D and a unique map $\varphi:D\to\mathbf{F}_q^*$ such that $(\chi,\xi_D(\varphi))\neq 0$.*

3.2. The dimension of $V_D(\varphi)$

In this section we determine the dimension of the irreducible subvarieties $V_D(\varphi)$ of $\mathcal{U}_n(K)^*$ [1]. Our results are independent of the characteristic of the field K so we will assume that K is an algebraically closed field of arbitrary characteristic. We also fix an arbitrary basic subset D of $\Phi(n)$ and an arbitrary map $\varphi: D \to K^*$. We start the section with some generalities about polynomial functions defined on an arbitrary finite-dimensional vector space over K.

Let V be a vector space over K of dimension m and let $(e_1,\ldots,e_m)$ be a basis of V. Let $P: V \to K$ be a polynomial function on V. Then there exists a polynomial $P(T_1,\ldots,T_m) \in K[T_1,\ldots,T_m]$ in m indeterminates $T_1,\ldots,T_m$ over K such that

$$P(v)=P(v_1,\ldots,v_m) \quad [2]$$

for all $v=v_1e_1+\ldots+v_me_m \in V$ $(v_1,\ldots,v_m \in K)$. For each $i \in \{1,\ldots,m\}$ we denote by $\partial_i P(T_1,\ldots,T_m)$ the *i-th partial derivative* of $P(T_1,\ldots,T_m)$, i.e.

$$\partial_i P(T_1,\ldots,T_m)=\frac{\partial}{\partial T_i}P(T_1,\ldots,T_m) \in K[T_1,\ldots,T_m]$$

and we consider the polynomial function $\partial_i P: V \to K$ associated with the polynomial $\partial_i P(T_1,\ldots,T_m)$. Then for each $v \in V$ we define the differential of the polynomial function $P: V \to K$ at the vector v as follows. Let $V^*=Hom_K(V,K)$ be the dual space of V and for each $i \in \{1,\ldots,m\}$, let $e_i^* \in V^*$ be defined by

$$e_i^*(e_j)=\delta_{ij}$$

for all $j \in \{1,\ldots,m\}$. Then $(e_1^*,\ldots,e_m^*)$ is a basis of V^* and we define the map $dP: V \to V^*$ by

$$(dP)(v)=(\partial_1 P)(v)e_1^*+\ldots+(\partial_m P)(v)e_m^*$$

for all $v \in V$. The vector $(dP)(v) \in V^*$ is called *the differential of P at $v \in V$*.

---

[1] We note that $V_D(\varphi)=O_D(\varphi)$ (by theorem 3.1.7) and that $O_D(\varphi)$ is an irreducible variety because it is the image of an irreducible variety under a morphism of algebraic varieties (cf. corollary 1.3.11)

[2] We abuse the notation and use the symbol P to denote both the polynomial function and the polynomial associated with it.

Now we prove the following general result:

Proposition 3.2.1. *Let the notation be as above and let $P_1,\ldots,P_r$ be $r\leq m$ polynomial functions defined on V. Suppose that there exists a non-empty open set $U\subseteq V$ such that for each $v\in U$ the vectors $(dP_1)(v),\ldots,(dP_r)(v)$ are linearly independent. Then the functions $P_1,\ldots,P_r$ are algebraically independent.*

Proof. Let $Y_1,\ldots,Y_r$ be r indeterminates over K and suppose that there exists a non-zero polynomial $F\in K[Y_1,\ldots,Y_r]$ such that

$$F(P_1,\ldots,P_r)=0.$$

Without loss of generality we may assume that F has minimal degree among all the polynomials with this property. Let $P:V\to K^r$ be the function defined by

$$P(v)=(P_1(v),\ldots,P_r(v))$$

for all $v\in V$. Then the composite function $F\circ P:V\to K$ is identically zero and so

$$\frac{\partial(F\circ P)}{\partial T_j}(v)=0$$

for all $j\in\{1,\ldots,m\}$ and all $v\in V$. By the chain rule

$$\frac{\partial(F\circ P)}{\partial T_j}(v)=\sum_{i=1}^{r}\frac{\partial F}{\partial Y_i}(P(v))\,\frac{\partial P_i}{\partial T_j}(v)$$

for all $v\in V$. Therefore

$$0=\sum_{j=1}^{m}\frac{\partial(F\circ P)}{\partial T_j}(v)\,e_j^*=\sum_{j=1}^{m}\sum_{i=1}^{r}\frac{\partial F}{\partial Y_i}(P(v))\,\frac{\partial P_i}{\partial T_j}(v)\,e_j^*=\sum_{i=1}^{r}\frac{\partial F}{\partial Y_i}(P(v))\,(dP_i)(v)$$

for all $v\in V$.

Now suppose that $v\in U$. Then (by hypothesis) the vectors $(dP_1)(v),\ldots,(dP_r)(v)$ are linearly independent. Thus

$$\frac{\partial F}{\partial Y_i}(P(v))=0$$

for all $i\in\{1,\ldots,r\}$. So U is contained in the closed subset

$$W=\left\{v\in V;\ \frac{\partial F}{\partial Y_i}(P(v))=0,\ 1\leq i\leq r\right\}$$

of V. Since U is dense we conclude that $W=V$, hence

$$\frac{\partial F}{\partial Y_i}(P(v))=0$$

for all $v \in V$ and all $i \in \{1,\ldots,r\}$. It follows that for each $i \in \{1,\ldots,r\}$ the polynomial $\frac{\partial F}{\partial Y_i} \in K[Y_1,\ldots,Y_r]$ is such that

$$\frac{\partial F}{\partial Y_i}(P_1,\ldots,P_r)=0.$$

Since these polynomials have degree smaller than F we conclude that

$$\frac{\partial F}{\partial Y_i}(Y_1,\ldots,Y_r)=0$$

for all $i \in \{1,\ldots,r\}$.

If K has characteristic zero this implies that

$$F(Y_1,\ldots,Y_r)=\alpha \in K$$

is a constant. Since $F(P_1,\ldots,P_r)=0$ we conclude that $\alpha=0$. This is in contradiction with the choice of F. Hence the functions $P_1,\ldots,P_r$ are algebraically independent.

Finally suppose that K has prime characteristic p. Then the polynomial F has the form

$$F=\sum_{i_1=0}^{n_1} \ldots \sum_{i_r=0}^{n_r} a_{i_1\ldots i_r} Y_1^{pi_1} \ldots Y_r^{pi_r} \qquad (a_{i_1\ldots i_r} \in K)$$

for some non-negative integers $n_1,\ldots,n_r$. Now for each sequence $(i_1,\ldots i_r)$ $(0 \le i_s \le n_s,$ $1 \le s \le r)$ there exists an element $b_{i_1\ldots i_r} \in K$ such that

$$a_{i_1\ldots i_r}=(b_{i_1\ldots i_r})^p$$

(we recall that K is algebraically closed). It follows that

$$F=G^p$$

for some polynomial $G \in K[Y_1,\ldots,Y_r]$. Since G has degree smaller than F we conclude that $G=0$, hence $F=0$. As before this contradiction implies that the functions $P_1,\ldots,P_r$ are algebraically independent and the proof is complete. ♦

Now we consider the functions $\Delta_{ij}^D: U_n(K)^* \to K$ for $(i,j) \in R(D)$. These functions are polynomial. In fact for each $(i,j) \in R(D)$ Δ_{ij}^D is associated with the polynomial

$$(3.2.1)\qquad \Delta^{D}_{ij}(T_{ab};(a,b)\in\Phi(n))=\det\begin{pmatrix} T_{i_{\sigma(1)}j} & T_{i_{\sigma(1)}j_1} & \cdots & T_{i_{\sigma(1)}j_r}\\ \vdots & \vdots & & \vdots\\ T_{i_{\sigma(r)}j} & T_{i_{\sigma(r)}j_1} & \cdots & T_{i_{\sigma(r)}j_r}\\ T_{ij} & T_{ij_1} & \cdots & T_{ij_r}\end{pmatrix}$$

in $\frac{n(n-1)}{2}$ indeterminates T_{ab}, $(a,b)\in\Phi(n)$, over K (here $D(i,j)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$, $j_1<\ldots<j_r$, and $\sigma\in S_n$ is such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$). We note that for each $(a,b)\in\Phi(n)$ the indeterminate T_{ab} determines the polynomial function $T_{ab}: U_n(K)^*\to K$ defined by

$$T_{ab}(f)=f(e_{ab})$$

for all $f\in U_n(K)^*$.

Let $(a,b)\in\Phi(n)$ and consider the (a,b)-th partial derivative

$$\partial_{ab}\Delta^{D}_{ij}=\frac{\partial\Delta^{D}_{ij}}{\partial T_{ab}}\in K[T_{ab};(a,b)\in\Phi(n)].$$

For simplicity we introduce the following notation.

Let $A=(a_{uv})_{1\le u,v\le m}$ be any square matrix (with coefficients in any ring) and let $i_1,\ldots,i_r,j_1,\ldots,j_r\in\{1,\ldots,m\}$. Then we denote by $\Delta^{i_1\ldots i_r}_{j_1\ldots j_r}(A)$ the determinant

$$(3.2.2)\qquad \Delta^{i_1\ldots i_r}_{j_1\ldots j_r}(A)=\det\begin{pmatrix} a_{i_1j_1} & \cdots & a_{i_1j_r}\\ \vdots & & \vdots\\ a_{i_rj_1} & \cdots & a_{i_rj_r}\end{pmatrix}$$

In particular let

$$(3.2.3)\qquad A=\begin{pmatrix} 0 & T_{12} & \cdots & T_{1n-1} & T_{1n}\\ 0 & 0 & \cdots & T_{2n-1} & T_{2n}\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & T_{n-1n}\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}.$$

Then for all $i_1,\ldots,i_r,j_1,\ldots,j_r\in\{1,\ldots,n\}$ the determinant $\Delta^{i_1\ldots i_r}_{j_1\ldots j_r}(A)$ is a polynomial in the indeterminates T_{ab}, $(a,b)\in\Phi(n)$, and we will write

$$(3.2.4)\qquad \Delta^{i_1\ldots i_r}_{j_1\ldots j_r}(T_{ab};(a,b)\in\Phi(n))=\Delta^{i_1\ldots i_r}_{j_1\ldots j_r}(A).$$

This polynomial determines a polynomial function $\Delta^{i_1\ldots i_r}_{j_1\ldots j_r}: \mathcal{U}_n(K)^* \to K$ which is defined by

$$(3.2.5) \qquad \Delta^{i_1\ldots i_r}_{j_1\ldots j_r}(f) = det \begin{pmatrix} f(e_{i_1j_1}) & \cdots & f(e_{i_1j_r}) \\ \vdots & & \vdots \\ f(e_{i_rj_1}) & \cdots & f(e_{i_rj_r}) \end{pmatrix}$$

for all $f \in \mathcal{U}_n(K)^*$ (we put $f(e_{ab})=0$ whenever $1 \le b \le a \le n$).

Now let $(a,b) \in \Phi(n)$. Then

$$(3.2.6) \qquad \partial_{ab}\Delta^D_{ij}(f) = \begin{cases} (-1)^r \Delta^{i_{\sigma(1)}\cdots i_{\sigma(r)}}_{j_1\cdots j_r}(f) & \text{if } (a,b)=(i,j) \\ (-1)^{s+1} \Delta^{i_{\sigma(1)}\cdots i_{\sigma(s-1)} i_{\sigma(s+1)}\cdots i_{\sigma(r)} i}_{j_1\cdots j_r}(f) & \text{if } (a,b)=(i_{\sigma(s)},j) \ (1 \le s \le r) \\ (-1)^{r+t+1} \Delta^{i_{\sigma(1)}\cdots i_{\sigma(r)}}_{j_1\cdots j_{t-1} j_{t+1}\cdots j_r j}(f) & \text{if } (a,b)=(i,j_t) \ (1 \le t \le r) \\ (-1)^{r+t+1} \Delta^{i_{\sigma(1)}\cdots i_{\sigma(s-1)} i_{\sigma(s+1)}\cdots i_{\sigma(r)} i}_{j_1\cdots j_{t-1} j_{t+1}\cdots j_r j}(f) & \text{if } (a,b)=(i_{\sigma(s)},j_t) \ (1 \le s,t \le r) \\ 0 & \text{otherwise} \end{cases}$$

for all $f \in \mathcal{U}_n(K)^*$.

Next we consider the differential of the polynomial function $\Delta^D_{ij}: \mathcal{U}_n(K)^* \to K$. Since we have a canonical identification

$$(\mathcal{U}_n(K)^*)^* = \mathcal{U}_n(K),$$

the differential of Δ^D_{ij} at an element $f \in \mathcal{U}_n(K)^*$ is the vector $(d\Delta^D_{ij})(f) \in \mathcal{U}_n(K)$ defined by

$$(3.2.7) \quad (d\Delta^D_{ij})(f) = c_{ij}(f)e_{ij} + \sum_{s=1}^{r} c_{i_{\sigma(s)}j}(f)e_{i_{\sigma(s)}j} + \sum_{t=1}^{r} c_{ij_t}(f)e_{ij_t} + \sum_{s=1}^{r}\sum_{t=1}^{r} c_{i_{\sigma(s)}j_t}(f)e_{i_{\sigma(s)}j_t}$$

where for any $(a,b) \in \Phi(n)$ $c_{ab}(f) = \partial_{ab}\Delta^D_{ij}(f)$.

Finally let $\Delta: \mathcal{U}_n(K)^* \to K$ be the polynomial function defined by

$$\Delta(f) = \prod_{(i,j) \in D} \Delta^D_{ij}(f)$$

for all $f \in \mathcal{U}_n(K)^*$. Then

$$U=\{f\in u_n(K)^*;\ \Delta(f)\neq 0\}$$

is a non-empty dense open subset of $u_n(K)^*$. Moreover (by (3.1.7))

(3.2.8) $$V_D(\varphi)\subseteq U.$$

We claim that for all $f\in U$ the vectors $(d\Delta_{ij}^D)(f)$, $(i,j)\in R(D)$, are linearly independent. For let $f\in U$ be arbitrary and consider the matrix

$$B=\left((\partial_{ab}\Delta_{ij}^D)(f)\right)_{(a,b),(i,j)\in R(D)}$$

(we consider the order in $\Phi(n)$ introduced in the proof of proposition 2.2.13). By (3.2.7) this matrix is upper-triangular. On the other hand let $(i,j)\in R(D)$, let $D(i,j)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$ and let $\sigma\in S_r$ be such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. Then

$$\Delta_{j_1\ldots j_r}^{i_{\sigma(1)}\ldots i_{\sigma(r)}}(f)=(-1)^{r-1}\,sgn(\sigma)\prod_{s=1}^{r}\varphi(i_s,j_s)\neq 0$$

(see the proof of proposition 3.1.2). It follows that the diagonal entries of B are non-zero. Thus

$$\det B\neq 0$$

and this implies that the vectors $(d\Delta_{ij}^D)(f)$, $(i,j)\in R(D)$, are linearly independent. By proposition 3.2.1 we conclude that:

Proposition 3.2.2. *The polynomial functions $\Delta_{ij}^D: u_n(K)^*\to K$, $(i,j)\in R(D)$, are algebraically independent over K.*

In the next result we determine the dimension of the varieties $V_D(\varphi)$.

Theorem 3.2.3. *Let D be a basic subset of $\Phi(n)$ and let $\varphi: D\to K^*$ be any map. Then $\dim V_D(\varphi)=s(D)$* [1].

Proof. We define the map $\vartheta: u_n(K)^*\to K^{r(D)}$ [2] by

---

[1] We recall that $s(D)=|S(D)|$.
[2] We recall that $r(D)=|R(D)|$.

$$\vartheta(f)=(\Delta^D_{ij}(f);\,(i,j)\in R(D))\quad[1]$$

for all $f\in \mathcal{U}_n(K)^*$. ϑ is a morphism of algebraic varieties. We claim that it is dominant, i.e. the image

$$Y=\vartheta(\mathcal{U}_n(K)^*)$$

is dense in $K^{r(D)}$. In fact if Y is not dense there exists a non-zero polynomial function $P:K^{r(D)}\to K$ such that

$$Y\subseteq\{a\in K^{r(D)};\,P(a)=0\}.$$

Then

$$P(\vartheta(f))=P(\Delta^D_{ij}(f);\,(i,j)\in R(D))=0$$

for all $f\in \mathcal{U}_n(K)^*$. Therefore

$$P(\Delta^D_{ij};\,(i,j)\in R(D))=0$$

where $P(T_{ij};\,(i,j)\in R(D))$ is a polynomial in the indeterminates T_{ij}, $(i,j)\in R(D)$. Since the functions Δ^D_{ij}, $(i,j)\in R(D)$, are algebraically independent (by the previous proposition), we conclude that

$$P(T_{ij};\,(i,j)\in R(D))=0.$$

Hence $P(a)=0$ for all $a\in K^{r(D)}$. This contradiction implies that the morphism ϑ is dominant. By [Hu1;Theorem 4.1] we conclude that for any $a\in Y$ and any irreducible component X of $\vartheta^{-1}(a)$

$$\dim X\geq \dim \mathcal{U}_n(K)^*-r(D)=s(D).$$

Since $V_D(\varphi)$ is irreducible and

$$V_D(\varphi)=\vartheta^{-1}(a)$$

for a well-determined $a\in X$ (by (3.1.7)) we obtain

(3.2.9) $$\dim V_D(\varphi)\geq s(D).$$

To prove that the equality holds we consider the ring $K[V_D(\varphi)]$ of all polynomial functions defined on $V_D(\varphi)$. For each $(i,j)\in \Phi(n)$ let $t_{ij}:V_D(\varphi)\to K$ be the polynomial function defined by

---

[1] We order the roots $(i,j)\in\Phi$ as in the proof of proposition 2.2.13.

$$t_{ij}(f)=f(e_{ij})$$

for all $f\in V_D(\varphi)$. Then

$$K[V_D(\varphi)]=K[t_{ij};\,(i,j)\in\Phi(n)]$$

is the K-algebra generated by the functions t_{ij}, $(i,j)\in\Phi(n)$. Since $V_D(\varphi)$ is an irreducible variety the ring $K[t_{ij};\,(i,j)\in\Phi(n)]$ is an integral domain (see [Hu1; proposition 1.3C]). Hence we may form its field of fractions $K(t_{ij};\,(i,j)\in\Phi(n))$. We claim that

$$tr.deg_K K(t_{ij};\,(i,j)\in\Phi(n))\leq s(D)$$

(here $tr.deg_K K(t_{ij};\,(i,j)\in\Phi(n))$ is the transcendence degree of $K(t_{ij};\,(i,j)\in\Phi(n))$ over K). For we fix an arbitrary element $f\in V_D(\varphi)$. Then

$$\Delta_{ij}^{D}(f)=(-1)^r f(e_{ij})\Delta_{j_1\ldots j_r}^{i_{\sigma(1)}\ldots i_{\sigma(r)}}(f)+\sum_{a=1}^{r}(-1)^{r+a+1}f(e_{ij_a})\Delta_{j_1\ldots j_{a-1}j_{a+1}\ldots j_r j}^{i_{\sigma(1)}\ldots i_{\sigma(r)}}(f)$$

where $D(i,j)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$, $j_1<\ldots<j_r$, and $\sigma\in S_r$ is such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. Let Y_{ij},Y_{ab}, $(a,b)\in S(D)$, be $s(D)+1$ indeterminates over K and consider the polynomial

$$R_{ij}^{(f)}=\Delta_{ij}^{D}(f)+c_{ij}(f)Y_{ij}+\sum_{a=1}^{r}c_{ij_a}(f)Y_{ij_a}\in K[Y_{ij},Y_{ab};\,(a,b)\in S(D)]$$

where

$$c_{ib}(f)=\begin{cases}(-1)^{r+1}\Delta_{j_1\ldots j_r}^{i_{\sigma(1)}\ldots i_{\sigma(r)}}(f) & \text{if } b=j\\[2ex] (-1)^{r+a}\Delta_{j_1\ldots j_{a-1}j_{a+1}\ldots j_r j}^{i_{\sigma(1)}\ldots i_{\sigma(r)}}(f) & \text{if } b=j_a\ (1\leq a\leq r)\end{cases}$$

Since $f\in V_D(\varphi)$ we have

$$c_{ij}(f)\neq 0$$

(see the proof of proposition 3.1.2). Thus the polynomial $R_{ij}^{(f)}$ is non-zero. We claim that the polynomial function $R_{ij}^{(f)}(t_{ij},t_{ab};\,(a,b)\in S(D)):V_D(\varphi)\rightarrow K$ determined by the polynomial $R_{ij}^{(f)}$ is identically zero. In fact let $g\in V_D(\varphi)$. Then

$$R_{ij}^{(f)}(g)=\Delta_{ij}^{D}(f)+c_{ij}(f)g(e_{ij})+\sum_{a=1}^{r}c_{ij_a}(f)g(e_{ij_a})=\Delta_{ij}^{D}(f)-det\begin{pmatrix}f(e_{i_{\sigma(1)}j}) & f(e_{i_{\sigma(1)}j_1}) & \ldots & f(e_{i_{\sigma(1)}j_r})\\ \vdots & \vdots & & \vdots\\ f(e_{i_{\sigma(r)}j}) & f(e_{i_{\sigma(r)}j_1}) & \ldots & f(e_{i_{\sigma(r)}j_r})\\ g(e_{ij}) & g(e_{ij_1}) & \ldots & g(e_{ij_r})\end{pmatrix}$$

Since $\Delta_{j_1\ldots j_r}^{i_{\sigma(1)}\ldots i_{\sigma(r)}}(f)\neq 0$ the row vectors $(f(e_{i_a j})\,f(e_{i_a j_1})\ldots f(e_{i_a j_r}))$ $(1\leq a\leq r)$ are linearly independent. Moreover the vector space (over K) generated by these vectors is also generated by the row vectors $(g(e_{i_a j})\ g(e_{i_a j_1})\ ...\ g(e_{i_a j_r}))$ $(1\le a\le r)$ (because $g\in V_D(\varphi)$, hence $\Delta^{i_{\sigma(1)}\cdots i_{\sigma(r)}}_{j_1\cdots j_r}(g)\neq 0$). Now suppose that $(i,j)\in D$. Then

$$\Delta^D_{ij}(g)=0$$

so the row vector $(g(e_{ij})\ g(e_{ij_1})\ ...\ g(e_{ij_r}))$ is a linear combination of the vectors $(g(e_{i_a j})\ g(e_{i_a j_1})\ ...\ g(e_{i_a j_r}))$ $(1\le a\le r)$, hence it is a linear combination of the vectors $(f(e_{i_a j})\ f(e_{i_a j_1})\ ...\ f(e_{i_a j_r}))$ $(1\le a\le r)$. It follows that

$$det\begin{pmatrix} f(e_{i_{\sigma(1)}j}) & f(e_{i_{\sigma(1)}j_1}) & \cdots & f(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & & \vdots \\ f(e_{i_{\sigma(r)}j}) & f(e_{i_{\sigma(r)}j_1}) & \cdots & f(e_{i_{\sigma(r)}j_r}) \\ g(e_{ij}) & g(e_{ij_1}) & \cdots & g(e_{ij_r}) \end{pmatrix}=0.$$

Since $\Delta^D_{ij}(f)=0$ we conclude that

$$R^{(f)}_{ij}(g)=0.$$

On the other hand suppose that $(i,j)\in D$. Then (as in the proof of proposition 3.1.2) the row vector $(g(e_{ij})-\varphi(i,j)\ g(e_{ij_1})\ ...\ g(e_{ij_r}))$ is a linear combination of the vectors $(g(e_{i_a j})\ g(e_{i_a j_1})\ ...\ g(e_{i_a j_r}))$ $(1\le a\le r)$ and the argument above shows that

$$det\begin{pmatrix} f(e_{i_{\sigma(1)}j}) & f(e_{i_{\sigma(1)}j_1}) & \cdots & f(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & & \vdots \\ f(e_{i_{\sigma(r)}j}) & f(e_{i_{\sigma(r)}j_1}) & \cdots & f(e_{i_{\sigma(r)}j_r}) \\ g(e_{ij}) & g(e_{ij_1}) & \cdots & g(e_{ij_r}) \end{pmatrix}=(-1)^r\varphi(i,j)\Delta^{i_{\sigma(1)}\cdots i_{\sigma(r)}}_{j_1\cdots j_r}(f).$$

Since

$$\Delta^D_{ij}(f)=(-1)^r\varphi(i,j)\Delta^{i_{\sigma(1)}\cdots i_{\sigma(r)}}_{j_1\cdots j_r}(f)$$

(see the proof ofproposition 3.1.2) we deduce that

$$R^{(f)}_{ij}(g)=0.$$

Our claim follows.

Finally since $R^{(f)}_{ij}\in K[Y_{ij},Y_{ab},(a,b)\in S(D)]$ is a non-zero polynomial we conclude that the polynomial function t_{ij} is algebraically dependent of the functions t_{ab}, $(a,b)\in S(D)$. Since $(i,j)\in R(D)$ is arbitrary this means that

$$tr.deg_K K(t_{ij},(i,j)\in\Phi(n))\leq s(D)$$

as required. It follows that

$$dimV_D(\varphi)=tr.deg_K K(t_{ij},(i,j)\in\Phi(n))\leq s(D)$$

By (3.2.9) we deduce that

$$dimV_D(\varphi)=s(D)$$

and the proof is complete. ♦

Now we assume that K is the algebraic closure of $\boldsymbol{F}_q$ where q is a power of a prime number $p\geq n$. We let $F=F_q:\mathcal{U}_n(K)^*\to\mathcal{U}_n(K)^*$ be the usual Frobenius map. Moreover we assume that $\varphi(D)\subseteq\boldsymbol{F}_q$. Then (by (3.1.7)) the variety $V_D(\varphi)$ is F-stable and the (finite) set $V_D(\varphi)^F$ is the disjoint union

$$V_D(\varphi)^F=\bigcup_O O^F$$

where O runs over all F-stable $U_n(K)$-orbits which are contained in $V_D(\varphi)$. Since

$$|V_D(\varphi)^F|=q^{dimV_D(\varphi)}=q^{s(D)}$$

(by the previous theorem) and

$$\chi_O(1)=\sqrt{|O^F|}=\sqrt{q^{dimO}}$$

(by proposition 1.2.5) for all F-stable $U_n(K)$-orbit $O\subseteq\mathcal{U}_n(K)^*$ [1] we conclude the following:

Corollary 3.2.4. *Let D be a basic subset of $\Phi(n)$ and let $\varphi:D\to\boldsymbol{F}_q^*$ be a map. Then*

$$\sum_{\chi\in I_D(\varphi)}\chi(1)^2=q^{s(D)}$$

where $I_D(\varphi)$ is the set consisting of all irreducible components of the character $\xi_D(\varphi)$ of $U_n(q)$.

Another corollary is the following number-theoretical equality:

---

[1] We recall that χ_O is the irreducible character of $U_n(q)$ which corresponds to O.

Corollary 3.2.5. *Let t be an indeterminate over the ring $\mathbb{Z}$ and let n be a positive integer. Then the identity*

$$\sum_{\substack{D\subseteq\Phi(n)\\ D \text{ basic}}} (t-1)^{|D|}\, t^{s(D)} = t^{n(n-1)/2}$$

holds in the polynomial ring $\mathbb{Z}[t]$.

Proof. Let $p\geq n$ be any prime number and consider the finite group $U_n(p)$. Then

$$p^{n(n-1)/2}=|U_n(p)|=\sum_{\chi\in Irr(U_n(p))} \chi(1)^2$$

where $Irr(U_n(p))$ is the set of all the irreducible characters of $U_n(p)$. By theorem 2.2.1 $Irr(U_n(p))$ is the disjoint union

$$Irr(U_n(p))=\bigcup_{D,\varphi} I_D(\varphi)$$

where the union is over all basic subsets D of $\Phi(n)$ and all maps $\varphi:D\rightarrow \mathbb{F}_p{}^*$. Therefore

$$p^{n(n-1)/2}=\sum_{D,\varphi}\ \sum_{\chi\in I_D(\varphi)} \chi(1)^2$$

where the sum is over all basic subsets D of $\Phi(n)$ and all maps $\varphi:D\rightarrow \mathbb{F}_p{}^*$. By the previous corollary we obtain

$$p^{n(n-1)/2}=\sum_{D,\varphi} p^{s(D)}$$

where the sum is as above. Finally, for each basic subset D of $\Phi(n)$, there are exactly $(p-1)^{|D|}$ distinct maps $\varphi:D\rightarrow \mathbb{F}_p{}^*$. Hence

$$p^{n(n-1)/2}=\sum_{\substack{D\subseteq\Phi(n)\\ D \text{ basic}}} (p-1)^{|D|}\, \mathrm{p}^{|S(D)|}$$

and the result follows because the set of all prime numbers is infinite. ♦

We will give a different proof of this result which is independent of the varieties $V_D(\varphi)$. In fact we will establish a more general identity. Firstly we introduce some notation and we recall some well-known facts. Let $\omega\in S_n$ and consider the set

$$\Phi_{\omega_0\omega}(n)=\{(i,j)\in\Phi(n);\ \omega(i)>\omega(j)\}$$

where $\omega_0 \in S_n$ is the permutation defined in (1.1.16). It is well-known that

$$|\Phi_{\omega_0\omega}(n)| = \ell(\omega)$$

where $\ell(\omega)$ is the length of ω [1]. Moreover ω_0 is the unique permutation of maximal length and we have

$$\ell(\omega_0) = \frac{n(n-1)}{2}.$$

Finally let D be a basic subset of $\Phi(n)$ and suppose that $D \subseteq \Phi_{\omega_0\omega}(n)$. Then we define

$$S_{\omega_0\omega}(D) = S(D) \cap \Phi_{\omega_0\omega}(n).$$

Proposition 3.2.6. *Let t be an indeterminate over $\mathbb{Z}$ and let $\omega \in S_n$. Then the identity*

$$\sum_{\substack{D \subseteq \Phi_{\omega_0\omega}(n) \\ D \text{ basic}}} (t-1)^{|D|}\, t^{|S_{\omega_0\omega}(D)|} = t^{\ell(\omega)}$$

holds in the polynomial ring $\mathbb{Z}[t]$. In particular if $\omega = \omega_0$ we obtain the identity of corollary 3.2.5.

Proof. We proceed by induction on the length $\ell(\omega)$ of ω. If $\ell(\omega)=0$ then $\omega=1$. In this case

$$\Phi_{\omega_0\omega}(n) = \varnothing$$

and (obviously) the empty set is the unique basic subset of $\Phi_{\omega_0\omega}(n)$. The required identity is trivial. If $\ell(\omega)=1$ then ω is a simple reflection, i.e. $\omega=(i\; i+1)$ for some $i \in \{1,\dots,n\}$. In this case

$$\Phi_{\omega_0\omega}(n) = \{(i, i+1)\}$$

and there exist two basic subsets of $\Phi_{\omega_0\omega}(n)$, namely the empty set and the set $\Phi_{\omega_0\omega}(n)$. Since

$$S_{\omega_0\omega}(\varnothing) = S_{\omega_0\omega}(\Phi_{\omega_0\omega}(n)) = \varnothing,$$

the required identity reads

$$1 + (t-1) = t.$$

Now suppose that $\ell(\omega) > 1$ and assume that the result is proved for all $\omega' \in S_n$ such

[1] By definition $\ell(\omega)$ is the minimal length of an expression of ω as product of simple reflections, i.e. transpositions of the form $(i\; i+1)$ $(1 \leq i \leq n)$.

that $\ell(\omega')<\ell(\omega)$. Let $\tau=(i\,i+1)\in S_n$ $(1\leq i\leq n-1)$ be a simple reflection such that

$$\ell(\omega\tau)=\ell(\omega)-1.$$

Then

$$\Phi_{\omega_0\omega}(n)=\{(i,i+1)\}\cup\tau(\Phi_{\omega_0\omega'}(n))$$

where $\omega'=\omega\tau$. Let D be a basic subset of $\Phi_{\omega_0\omega}(n)$. Then $(i,i+1)\in D$ if and only if $\tau(D)\subseteq\Phi_{\omega_0\omega'}(n)$. On the other hand if $(i,i+1)\in D$ then $D=\{(i,i+1)\}\cup D_0$ where D_0 is a basic subset of $\Phi_{\omega_0\omega}(n)$ such that $\tau(D_0)\subseteq\Phi_{\omega_0\omega'}(n)$ and $(i,i+1)\in S(D_0)$. Moreover for any basic subset of roots D' of $\Phi_{\omega_0\omega'}(n)$ such that $(i,i+1)\in S(\tau(D'))$ the subset $\{(i,i+1)\}\cup\tau(D')$ of $\Phi_{\omega_0\omega'}(n)$ is basic. It follows that

$$\sum_{\substack{D\subseteq\Phi_{\omega_0\omega}(n)\\ D\ basic}}(t-1)^{|D|}t^{|S_{\omega_0\omega}(D)|}=\sum_{\substack{D\subseteq\Phi_{\omega_0\omega'}(n)\\ D\ basic}}(t-1)^{|D|}t^{|S_{\omega_0\omega}(\tau(D))|}+\sum_{\substack{D\subseteq\Phi_{\omega_0\omega'}(n)\\ D\ basic\\ (i,i+1)\in S(\tau(D))}}(t-1)^{|D|+1}t^{|S_{\omega_0\omega}(\tau(D))|}.$$

Now let D be a basic subset of $\Phi_{\omega_0\omega'}(n)$. Suppose that $(i,i+1)\in S(\tau(D))$. Then

$$S(\tau(D))=\begin{cases}\tau(S(D)) & \text{if } (i,i+1)\in S(D)\\ \tau(S(D))\setminus\{(i+1,i)\} & \text{if } (i,i+1)\in S(D)\end{cases}.$$

In both cases we have

$$S_{\omega_0\omega}(\tau(D))=\tau(S_{\omega_0\omega'}(D)),$$

hence

$$|S_{\omega_0\omega}(\tau(D))|=|S_{\omega_0\omega'}(D)|.$$

On the other hand suppose that $(i,i+1)\in S(\tau(D))$. Then

$$S(\tau(D))=\begin{cases}\tau(S(D))\cup\{(i,i+1)\} & \text{if } (i,i+1)\in S(D)\\ (\tau(S(D))\setminus\{(i+1,i)\})\cup\{(i,i+1)\} & \text{if } (i,i+1)\in S(D)\end{cases}.$$

In this case we obtain

$$S_{\omega_0\omega}(\tau(D))=\{(i,i+1)\}\cup\tau(S_{\omega_0\omega'}(D)),$$

hence

$$|S_{\omega_0\omega}(\tau(D))|=|S_{\omega_0\omega'}(D)|+1.$$

It follows that

$$\sum_{\substack{D\subseteq\Phi_{\omega_0\omega'}(n)\\ D\ basic}}(t-1)^{|D|}t^{|S_{\omega_0\omega}(\tau(D))|}=\sum_{\substack{D\subseteq\Phi_{\omega_0\omega'}(n)\\ D\ basic\\ (i,i+1)\in S(\tau(D))}}(t-1)^{|D|}t^{|S_{\omega_0\omega'}(D)|}+\sum_{\substack{D\subseteq\Phi_{\omega_0\omega'}(n)\\ D\ basic\\ (i,i+1)\in S(\tau(D))}}(t-1)^{|D|}t^{|S_{\omega_0\omega'}(D)|+1}$$

whereas

$$\sum_{\substack{D\subseteq\Phi_{\omega_0\omega'}(n)\\ D\ basic\\ (i,i+1)\in S(\tau(D))}} (t-1)^{|D|+1}\, t^{|S_{\omega_0\omega'}(\tau(D))|} = \sum_{\substack{D\subseteq\Phi_{\omega_0\omega'}(n)\\ D\ basic\\ (i,i+1)\in S(\tau(D))}} (t-1)^{|D|+1}\, t^{|S_{\omega_0\omega'}(\tau(D))|}.$$

Therefore

$$\sum_{\substack{D\subseteq\Phi_{\omega_0\omega'}(n)\\ D\ basic}} (t-1)^{|D|}\, t^{|S_{\omega_0\omega'}(D)|} = t\left(\sum_{\substack{D\subseteq\Phi_{\omega_0\omega'}(n)\\ D\ basic}} (t-1)^{|D|}\, t^{|S_{\omega_0\omega'}(D)|}\right) = t\, t^{\ell(\omega')} = t^{\ell(\omega)}$$

because

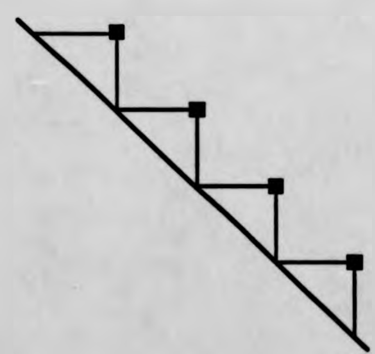
$$\sum_{\substack{D\subseteq\Phi_{\omega_0\omega'}(n)\\ D\ basic}} (t-1)^{|D|}\, t^{|S_{\omega_0\omega'}(D)|} = t^{\ell(\omega')}$$

(by induction because $\ell(\omega')=\ell(\omega)-1$). The proof is complete. ♦

3.3. Homogeneous $V_D(\varphi)$'s

This section is concerned with the transitivity of the action of $U_n(K)$ on the varieties $V_D(\varphi)$ (as usual D is a basic subset of $\Phi(n)$ and $\varphi{:}D\to K^*$ is a map). To be more specific we will determine all the pairs (D,φ) for which the variety $V_D(\varphi)$ is a single $U_n(K)$-orbit. The answer to this problem is purely combinatorial and it depends only on the "geometrical configuration" of the basic set D. As a consequence we will also obtain a necessary and sufficient conditon to decide about the pairs (D,φ) for which the character $\xi_D(\varphi)$ of $U_n(q)$ has a unique irreducible component (here the image of the map φ has to be a subset of $\mathbf{F}_q{}^*$). In general the homogeneity of an F-stable variety $V_D(\varphi)$ does not imply that the corresponding character $\xi_D(\varphi)$ is irreducible. In fact the unique irreducible component of $\xi_D(\varphi)$ may occur with multiplicity greater than one. However it is not very difficult to calculate this multiplicity (cf. corollary 2.2.17).

A subset C of $\Phi(n)$ is called a *chain* if $C=\{(i_1,i_2),(i_2,i_3),\dots,(i_{r-1},i_r)\}$. Since $(i_a,i_{a+1})\in\Phi(n)$ for all $a\in\{1,\dots,r-1\}$, we have $i_1<i_2<\dots<i_r$. The cardinality $|C|$ of C will be referred to as the *length of* C. In the adjacent picture we show a chain of length 4 (as usual the symbol ■ represents a root in the chain).



It is clear that a chain C is a basic subset of $\Phi(n)$. Hence the variety $V_C(\varphi)$ is defined for all maps $\varphi{:}C\to K^*$. We have:

Lemma 3.3.1. *Let $C\subseteq\Phi(n)$ be a chain and let $\varphi{:}C\to K^*$ be a map. Then $V_C(\varphi)$ is a single $U_n(K)$-orbit.*

Proof. Let $C=\{(i_1,i_2),(i_2,i_3),\dots,(i_{r-1},i_r)\}$. By theorem 3.2.3

$$dimV_C(\varphi)=s(C)=\sum_{a=1}^{r-1} 2(i_{a+1}-i_a-1)=2(i_r-i_1)-2(r-1).$$

On the other hand the element

$$f=\sum_{a=1}^{r-1} \varphi(i_a,i_{a+1})\, e_{i_a i_{a+1}}{}^{*}\in \mathcal{U}_n(K)^{*}$$

lies in $V_C(\varphi)$. By (1.3.3) we have

$$dimO(f)=rankM(f)$$

where $O(f)$ is the $U_n(K)$-orbit of f and $M(f)$ is the skew-symmetric matrix which represents the bilinear form B_f with respect to the canonical basis of $\mathcal{U}_n(K)$. A similar argument to the one used in the proof of lemma 2.2.3 shows that

$$rankM(f)\geq 2(i_r-i_1)-2(r-1).$$

Therefore

$$dimO(f)=dimV_C(\varphi).$$

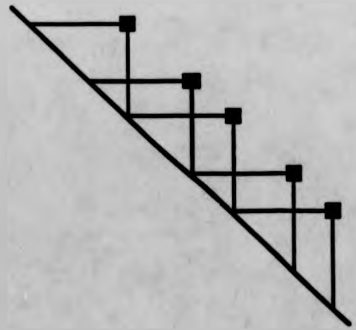
Since $O(f)$ and $V_C(\varphi)$ are irreducible varieties we conclude that $V_C(\varphi)=O(f)$ and the result follows. ♦

Let $C=\{(i_1,i_2),(i_2,i_3),\dots,(i_{r-1},i_r)\}$ and $C'=\{(j_1,j_2),(j_2,j_3),\dots,(j_{s-1},j_s)\}$ be two chains in $\Phi(n)$ and suppose that $i_1<j_1$. We say that the chains C and C' *intertwine* if one (and only one) of the following conditions is satisfied:

(a) $r=s$ and $i_1<j_1<i_2<j_2<\dots<i_r<j_r$;

(b) $r=s+1$ and $i_1<j_1<i_2<j_2<\dots<i_{r-1}<j_{r-1}<i_r$.

In the adjacent picture we show a pair of intertwining chains (with $r=3$ and $s=2$) (as before the roots in both chains are represented by the symbol ■).



It is clear that the union of two intertwinning chains is a basic set of roots. We have:

Proposition 3.3.2. *Let $C, C' \subseteq \Phi(n)$ be a pair of intertwining chains. We put $C = \{(i_1,i_2),(i_2,i_3),\ldots,(i_{r-1},i_r)\}$ and $C' = \{(j_1,j_2),(j_2,j_3),\ldots,(j_{s-1},j_s)\}$ and we suppose that $i_1 < j_1$. Let $D = C \cup C'$ and let $\varphi: D \to K^*$. Then:*

(i) *If $r=s+1$ $V_D(\varphi)$ is a single $U_n(K)$-orbit and we have*

$$\dim V_D(\varphi) = 2(i_r - i_1) + 2(j_{r-1} - j_1) - 6(r-1) + 4.$$

(ii) *If $r=s$*

$$V_D(\varphi) = \bigcup_{\alpha \in K} O(\alpha)$$

is a disjoint union of $U_n(K)$-orbits $O(\alpha)$ $(\alpha \in K)$. Moreover

$$\dim O(\alpha) = 2(i_r - i_1) + 2(j_r - j_1) - 6(r-1)$$

for all $\alpha \in K$.

Proof. In both cases let

$$f = \sum_{a=1}^{r-1} \varphi(i_a, i_{a+1}) e_{i_a i_{a+1}}{}^* + \sum_{b=1}^{s-1} \varphi(j_b, j_{b+1}) e_{j_b j_{b+1}}{}^* \in u_n(K)^*.$$

It is clear that $f \in V_D(\varphi)$. We consider the dimension of the $U_n(K)$-orbit $O(f)$.

On the one hand let $(i,j) \in R(D)$. Then

$$f([e_{ij}, e_{jk}]) = f(e_{ik}) = 0$$

for all $k \in \{j+1,\ldots,n\}$ (otherwise $(i,k) \in D$ and $(i,j) \in S(D)$). Similarly

$$f([e_{ki}, e_{ij}]) = f(e_{kj}) = 0$$

for all $k \in \{1,\ldots,i-1\}$ (otherwise $(k,j) \in D$ and $(i,j) \in S(D)$). Therefore the subspace

$$\mathfrak{r} = \sum_{(i,j) \in R(D)} K e_{ij}$$

of $u_n(K)$ is contained in the radical of the bilinear form B_f.

On the other hand let

$$S' = \begin{cases} \{(i_1,j_1),(j_1,i_2),\ldots,(i_{r-1},j_{r-1}),(j_{r-1},i_r)\} & \text{if } r=s+1 \\ \{(i_1,j_1),(j_1,i_2),\ldots,(i_{r-1},j_{r-1}),(j_{r-1},i_r),(i_r,j_r)\} & \text{if } r=s \end{cases}$$

and let

$$\mathfrak{v} = \sum_{(i,j) \in S(D) \setminus S'} K e_{ij}.$$

Let

$$v = \sum_{(i,j) \in S(D) \setminus S'} \alpha_{ij} e_{ij} \in \mathfrak{v}$$

be an arbitrary non-zero vector and let $(i,j)\in S(D)\setminus S'$ be such that $\alpha_{ij}\neq 0$. Then either $i\in\{i_a;\ 1\leq a\leq r-1\}\cup\{j_b;\ 1\leq b\leq s-1\}$ or $j\in\{i_a;\ 2\leq a\leq r\}\cup\{j_b;\ 2\leq b\leq s\}$. Suppose that $i=i_a$ for some $a\in\{1,\dots,r-1\}$. Then

$$f([v,e_{ji_{a+1}}])=\alpha_{i_aj}f(e_{i_ai_{a+1}})=\alpha_{i_aj}\varphi(i_a,i_{a+1})\neq 0.$$

Similarly we have $f([v,e_{ij_{b+1}}])\neq 0$ (if $i=j_b$, $1\leq b\leq s-1$), $f([v,e_{i_{a-1}i}])\neq 0$ (if $j=i_a$, $2\leq a\leq r$) and $f([v,e_{j_{b-1}i}])\neq 0$ (if $j=j_b$, $2\leq b\leq s$). Since v is arbitrary we conclude that $\mathcal{V}$ is a non-degenerate subspace of $\mathcal{U}_n(K)$. Its dimension is

$$\dim\mathcal{V}=2(i_r-i_1)+2(j_s-j_1)-4(r+s-2).$$

Moreover $\mathcal{V}$ is orthogonal to the subspace

$$\mathcal{V}'=\sum_{(i,j)\in S'}Ke_{ij}.$$

In fact consider a root $(i_a,j)\in S(D)\setminus S'$ $(1\leq a\leq r-1)$. Then for all $(r,s)\in\Phi(n)$ $f([e_{i_aj},e_{rs}])\neq 0$ if and only if either $j=r$ and $s=i_{a+1}$ or $s=i_a$ and $(r,j)\in D$. Suppose that $j=r$ and $s=i_{a+1}$. Then $(r,s)=(j,i_{a+1})\in S'$ if and only if $j=j_a$. This is impossible because $(i_a,j_a)\in S'$ (hence $(i_a,j)\in S(D)\setminus S'$). On the other hand suppose that $s=i_a$ and $(r,j)=(i_b,i_{b+1})$ $(1\leq b\leq r-1)$. Then $(r,s)=(i_b,i_a)\in S'$ (in fact $(i_b,i_a)\in S(D)$). Finally if $s=i_a$ and $(r,j)=(j_b,j_{b+1})$ $(1\leq b\leq s-1)$ then $(r,s)=(j_b,i_a)\in S'$ if and only if $b=a-1$ (so $a>1$). This is impossible because $(i_a,j)=(i_a,j_a)\in S'$ (hence $(i_a,j)\in S(D)\setminus S'$). The other possibilities for the root $(i,j)\in S(D)\setminus S'$ are discussed similarly. Therefore in order to determine the dimension of the orbit $O(f)$ it is enough to determine the rank of the skew-symmetric submatrix $M_0(f)$ of $M(f)$ whose entries correspond to the pairs $((i,j),(k,l))$ where $(i,j),(k,l)\in S'$ - the $((i,j),(k,l))$-th entry of $M_0(f)$ is $f([e_{ij},e_{kl}])$. Now for a suitable ordering of these roots, the matrix $M_0(f)$ has the form

$$M_0(f)=\begin{pmatrix}0 & A\\ -A^{T} & 0\end{pmatrix}$$

where

$$A=\begin{pmatrix}\alpha_1 & 0 & \cdots & 0 & 0\\ -\beta_1 & \alpha_2 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & \alpha_{r-2} & 0\\ 0 & 0 & \cdots & -\beta_{r-1} & \alpha_{r-1}\end{pmatrix}\text{(if } r=s+1\text{)}\quad\text{or}\quad A=\begin{pmatrix}\alpha_1 & 0 & \cdots & 0 & 0\\ -\beta_1 & \alpha_2 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & -\beta_{r-2} & \alpha_{r-1}\\ 0 & 0 & \cdots & 0 & -\beta_{r-1}\end{pmatrix}\text{(if } r=s\text{)}.$$

In the first case the matrix A is non-singular and in the second case it has rank r-1. Therefore in both cases

$$dimO(f)=2(i_r-i_1)+2(j_s-j_1)-4(r+s-2)+2(r-1)=2(i_r-i_1)+2(j_s-j_1)-2(r+2s-3)$$

as required in our statement.

Now if $r=s+1$ then

$$dimV_D(\varphi)=2(i_r-i_1)+2(j_{r-1}-j_1)-6(r-1)+4$$

so $V_D(\varphi)=O(f)$. On the other hand suppose that $r=s$. Then

$$dimV_D(\varphi)=2(i_r-i_1)+2(j_r-j_1)-6(r-1)+1$$

so $O(f)$ is a proper subvariety of $V_D(\varphi)$. However for each $\alpha\in K$ the element

$$f+\alpha e_{i_1j_1}{}^*\in u_n(K)^*$$

lies in $V_D(\varphi)$. We denote by $O(\alpha)$ the $U_n(K)$-orbit of $f+\alpha e_{i_1j_1}{}^*$ (then $O(0)=O(f)$). We claim that $O(\alpha)\neq O(\beta)$ whenever $\alpha,\beta\in K$ are distinct. Suppose that $O(\alpha)=O(\beta)$ and let $x\in U_n(K)$ be such that

$$x\cdot(f+\beta e_{i_1j_1}{}^*)=f+\alpha e_{i_1j_1}{}^*.$$

Then

$$\alpha=(f+\alpha e_{i_1j_1}{}^*)(e_{i_1j_1})=(x\cdot(f+\beta e_{i_1j_1}{}^*))(e_{i_1j_1})=(f+\beta e_{i_1j_1}{}^*(xe_{i_1j_1}x^{-1})$$
$$=f(xe_{i_1j_1}x^{-1})+\beta(xe_{i_1j_1}x^{-1})_{i_1j_1}=f(xe_{i_1j_1}x^{-1})+\beta.$$

On the other hand

$$f(xe_{i_1j_1}x^{-1})=\sum_{a=1}^{r-1}\varphi(i_a,i_{a+1})x_{i_ai_1}(x^{-1})_{j_1i_{a+1}}+\sum_{b=1}^{s-1}\varphi(j_b,j_{b+1})x_{j_bi_1}(x^{-1})_{j_1j_{b+1}}.$$

Now for $1\leq a\leq r-1$ $x_{i_ai_1}\neq 0$ only if $i_a\leq i_1$ and this happens if and only if $a=1$. Since $j_b\geq j_1>i_1$ for all $b\in\{1,\ldots,r-1\}$ we conclude that

$$f(xe_{i_1j_1}x^{-1})=\varphi(i_1,i_2)(x^{-1})_{j_1i_2}.$$

A similar argument shows that

$$0=(f+\alpha e_{i_1j_1}{}^*)(e_{i_1k})=f(xe_{i_1k}x^{-1})=\varphi(i_1,i_2)(x^{-1})_{ki_2}$$

for all $k\in\{j_1+1,\ldots,i_2-1\}$. Hence $(x^{-1})_{ki_2}=0$ for all $k\in\{j_1+1,\ldots,i_2-1\}$ and this implies that

$$(x^{-1})_{j_1i_2}=-x_{j_1i_2}.$$

Now let $s\in\{2,\ldots,r-1\}$. Then

$$0=(f+\alpha e_{i_1j_1}{}^*)(e_{i_1j_s})=f(xe_{i_1k}x^{-1})=\varphi(i_s,i_{s+1})(x^{-1})_{j_1j_{s+1}}+\varphi(j_{s-1},j_s)x_{j_{s-1}i_s}$$

- we note that for $1\leq a\leq r-1$ $i_a\leq i_s$ and $j_s\leq i_{a+1}$ if and only if $a=s$; also for $1\leq b\leq r-1$, $j_b\leq i_s$ and

$j_s \leq j_{b+1}$ if and only if $b=s-1$. As before we can prove that

$$(x^{-1})_{j_s j_{s+1}} = -x_{j_s j_{s+1}}.$$

Finally we have

$$0=(f+\alpha e_{i_1 j_1}{}^*)(e_{i_r j_r})=f(xe_{i_r j_r}x^{-1})=\varphi(j_{r-1},j_r)x_{j_{r-1} i_r}.$$

Hence we have obtained the equations

$$\begin{cases} \alpha=-\varphi(i_1,i_2)x_{j_1 i_2}+\beta \\ 0=-\varphi(i_s,i_{s+1})x_{j_s j_{s+1}}+\varphi(j_{s-1},j_s)x_{j_{s-1} i_s} & (2\leq s\leq r-1) \\ 0=\varphi(j_{r-1},j_r)x_{j_{r-1} i_r} \end{cases}.$$

This system has a solution if and only if $\alpha=\beta$. Therefore $O(\alpha)=O(\beta)$ if and only if $\alpha=\beta$.

Now consider the dimension of any $U_n(K)$-orbit $O(\alpha)$ ($\alpha\in K$). The argument used above shows that the space υ is non-degenerate of dimension

$$dim\,\upsilon=2(i_r-i_1)+2(j_r-j_1)-7(r-1).$$

Moreover $dim\,\upsilon'=r-1$. Therefore

$$dimO(\alpha)\geq 2(i_r-i_1)+2(j_r-j_1)-6(r-1).$$

On the other hand

$$dimV_D(\varphi)=2(i_r-i_1)+2(j_r-j_1)-6(r-1)+1.$$

Since $dimO(\alpha)\leq dimV_D(\varphi)$ and $O(\alpha)$ is even-dimensional (by corollary 1.2.3) we conclude that

$$dimO(\alpha)=2(i_r-i_1)+2(j_r-j_1)-6(r-1)$$

as required.

Finally we consider the disjoint union of all the $U_n(K)$-orbits $O(\alpha)$ ($\alpha\in K$). This union is a subvariety V of $V_D(\varphi)$. To prove that $V=V_D(\varphi)$ let V' be an irreducible component of V. Since the algebraic group $U_n(K)$ is connected V' is $U_n(K)$-invariant (see [Hu1 ; proposition 8.2]). Hence V' is a union of some of the orbits $O(\alpha)$ ($\alpha\in K$). Let $\alpha\in K$ be such that

$$O(\alpha)\subseteq V'.$$

Since $O(\alpha)$, V' and $V_D(\varphi)$ are irreducible varieties we have

$$dimO(\alpha)\leq dimV'\leq dimV_D(\varphi).$$

It follows that either $V'=O(\alpha)$ or $V'=V_D(\varphi)$. Since there are finitely many irreducible components of V we may allways assume that V' includes at least two orbits $O(\alpha)$ and

$O(\beta)$ where $\beta \in K$, $\beta \neq \alpha$. It follows that $V'=V_D(\varphi)$, hence $V=V_D(\varphi)$.

The proof is complete. ♦

Let $C=\{(i_1,i_2),\ldots,(i_{r-1},i_r)\}$ and $C'=\{(j_1,j_2),\ldots,(j_{r-1},j_r)\}$ be two intertwining chains of the same length and suppose that $i_1<j_1$. Then the root (i_1,j_1) is called the (C,C')-*derived root*. More generally we may define the D-derived roots for any basic subset D of $\Phi(n)$ as follows.

Let $C=\{(i_1,i_2),\ldots,(i_{r-1},i_r)\}$ and $C'=\{(j_1,j_2),\ldots,(j_{s-1},j_s)\}$ be two chains in D. Then the pair (C,C') is called *special* if the following four conditions are satisfied:

(i) C and C' have the same length, i.e. $r=s$.

(ii) C and C' intertwine, i.e. $i_1<j_1<i_2<j_2<\ldots<i_r<j_r$.

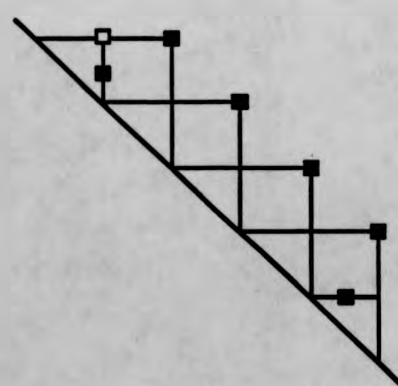
(iii) If there exists $j_0 \in \{1,\ldots,j_1-1\}$ such that $(j_0,j_1)\in D$ then $i_1<j_0$.

(iv) If there exists $i_{r+1} \in \{i_r+1,\ldots,n\}$ such that $(i_r,i_{r+1})\in D$ then $i_{r+1}<j_r$.

The root (i_1,j_1) is an example of a D-derived root (we note that this root is the (C,C')-derived root). In general a root $(i,j)\in\Phi(n)$ is called a D-*derived root* if there exists a special pair of chains (C,C') in D such that (i,j) is a (C,C')-derived root. The set of all D-derived roots is called the *derived set* of D and it will be denoted by D'. It is clear that

(3.3.1) $$D' \subseteq S(D).$$

In the adjacent picture we show a special pair of chains of length 2 (the roots represented by ■ and by ■ are in D and the chains C and C' correspond to the roots represented by ■; the symbol □ represents a derived root).



The main result of this section is the following:

Theorem 3.3.3. *Let $D\subseteq\Phi(n)$ be a basic subset of $\Phi(n)$ and let $\varphi:D\to K^*$ be a map. Then $V_D(\varphi)$ is a single $U_n(K)$-orbit if and only if the derived set D' of D is empty.*

Proof. We will use the argument suggested by the proof of proposition 3.3.2. We consider the element

$$f=\sum_{(i,j)\in D}\varphi(i,j)e_{ij}^*\in \mathcal{U}_n(K)^*.$$

It is clear that this element lies in $V_D(\varphi)$. Therefore $V_D(\varphi)$ is a single $U_n(K)$-orbit if and only if $V_D(\varphi)=O(f)$. Since $V_D(\varphi)$ and $O(f)$ are irreducible varieties we have

$$V_D(\varphi)=O(f) \iff dimO(f)=s(D)$$

(by theorem 3.2.3). Now (by (1.3.3))

$$dimO(f)=rankM(f)$$

where $M(f)$ is the skew-symmetric matrix which represents B_f with respect to the canonical basis of $\mathcal{U}_n(K)$. Therefore

$$V_D(\varphi)=O(f) \iff rankM(f)=s(D).$$

Since $e_{ij}\in r(f)$ for all $(i,j)\in R(D)$ we have

$$rankM(f)\leq dim\,\mathcal{U}_n(K)-r(D)=s(D).$$

To prove that the equality holds we consider the matrix

$$M'=(f([e_{ij},e_{kl}]))_{(i,j),(k,l)\in S(D)}.$$

Then there exists a permutation matrix P (of size $|\Phi(n)|$) such that

$$P^{-1}M(f)P=\begin{pmatrix} M' & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$rankM(f)=rankM'.$$

Hence

$$rankM(f)=s(D) \iff M' \textit{ is non-singular}$$

and so

$$V_D(\varphi)=O(f) \iff M' \textit{ is non-singular}.$$

Now we define an equivalence relation $\sim$ on the set $S(D)$ as follows. Let $(i,j),(k,l)\in S(D)$. Then $(i,j)\sim(k,l)$ if and only if either $(i,j)=(k,l)$ or there exists a sequence

$$(i_1,j_1)=(i,j),(i_2,j_2),\ldots,(i_{r-1},j_{r-1}),(i_r,j_r)=(k,l)$$

of D-singular roots such that

$$f([e_{i_sj_s},e_{i_{s+1}j_{s+1}}])\neq 0$$

for all $s\in\{1,\ldots,r-1\}$. Let $C_1,\ldots,C_t\subseteq S(D)$ be all the equivalence classes of this relation. Then $S(D)$ is the disjoint union

(3.3.2) $$S(D)=C_1\cup\ldots\cup C_t.$$

By definition of the relation $\sim$ we have

$$f([e_{ij},e_{kl}])=0$$

whenever $(i,j)\in C_s$, $(k,l)\in C_{s'}$ and $s\neq s'$ $(1\leq s,s'\leq t)$. Therefore the partition (3.3.2) of $S(D)$ implies that there exists a permutation matrix Q (of size $s(D)=|S(D)|$) such that

$$Q^{-1}M'Q=\begin{pmatrix} M_1 & 0 & \ldots & 0 \\ 0 & M_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_t \end{pmatrix}$$

where for each $s\in\{1,\ldots,t\}$ M_s is the matrix (of size $|C_s|$)

$$M_s=(f([e_{ij},e_{kl}]))_{(i,j),(k,l)\in C_s}.$$

It follows that

$$rankM'=rankM_1+\ldots+rankM_t,$$

hence

$$V_D(\varphi)=O(f) \iff M_1,\ldots,M_t \text{ are non-singular.}$$

Now let $s\in\{1,\ldots,t\}$ and consider the equivalence class C_s. We claim that C_s is a chain. For let (i_1,i_2) be the largest root in C_s (we consider the order $<$ introduced in the proof of proposition 2.2.13). Since $C_s\subseteq S(D)$ the root (i_1,i_2) is D-singular. Therefore at least one of the following cases occurs:

(i) there exists $a\in\{1,\ldots,i_1-1\}$ such that $(a,i_2)\in D$;

(ii) there exists $b\in\{i_2+1,\ldots,n\}$ such that $(i_1,b)\in D$.

If case (i) occurs then $(a,i_1)\in S(D)$ and

$$f([e_{ai_1},e_{i_1i_2}])=f(e_{ai_2})\neq 0$$

(because $(a,i_2)\in D$). So $(a,i_1)\sim(i_1,i_2)$ and $(a,i_1)\in C_s$. Since $i_1<i_2$ we have $(i_1,i_2)<(a,i_1)$ which is in contradiction with the choice of (i_1,i_2). Thus case (ii) occurs. In this case the root (i_2,b) is D-singular and

$$f([e_{i_1i_2},e_{i_2b}])=f(e_{i_1b})\neq 0.$$

Therefore $(i_1,b)\in C_s$, hence we have constructed a chain

$$C_s^{(2)}=\{(i_1,i_2),(i_2,i_3)\}$$

(where $i_3=b$) such that

$$C_s^{(2)}\subseteq C_s.$$

If $C_s=C_s^{(2)}$ then the claim is proved. On the other hand suppose that $C_s\setminus C_s^{(2)}\neq\emptyset$. By induction we assume that $r'\geq 2$ and that we have constructed a chain

$$C_s^{(r')}=\{(i_1,i_2),\ldots,(i_{r'},i_{r'+1})\}$$

such that

$$C_s^{(r')}\subseteq C_s.$$

If $C_s=C_s^{(r')}$ our claim is proved. On the other hand suppose that $C_s\setminus C_s^{(r')}\neq\emptyset$ and let $(i,j)\in C_s\setminus C_s^{(r')}$. Then (by definition of $\sim$) there exists a sequence

$$(i_{r'},i_{r'+1}),(a_1,b_1),\ldots,(a_u,b_u)=(i,j)$$

such that

$$f([e_{i_{r'}i_{r'+1}},e_{a_1b_1}])\neq 0 \quad\text{and}\quad f([e_{a_{v-1}b_{v-1}},e_{a_vb_v}])\neq 0\ (2\leq v\leq u).$$

Without loss of generality we may assume that the roots in this sequence are all distinct. For simplicity we write $(a,b)=(a_1,b_1)$. Since $f([e_{i_{r'}i_{r'+1}},e_{ab}])\neq 0$ either $a=i_{r'+1}$ and $(i_{r'},b)\in D$ or $b=i_{r'}$ and $(a,i_{r'+1})\in D$. If the second case occurs then $a=i_{r'-1}$ (because $f([e_{i_{r'-1}i_{r'}},e_{i_{r'}i_{r'+1}}])\neq 0$, so $(i_{r'-1},i_{r'})\in D$). Thus $(a,b)=(i_{r'-1},i_{r'})$ and an inductive argument shows that

$$(i,j)>(i_1,i_2)$$

(because $(i,j)\in C_s^{(r')}$). This is contrary to the choice of (i_1,i_2), hence we conclude that

$$a=i_{r'+1} \quad\text{and}\quad (i_{r'},b)\in D.$$

Now we put $b=i_{r'+2}$ and

$$C_s^{(r'+1)}=\{(i_1,i_2),\ldots,(i_{r'},i_{r'+1}),(i_{r'+1},i_{r'+2})\}.$$

It is clear that

$$C_s^{(r'+1)} \subseteq C_s.$$

Moreover $C_s^{(r')}$ is a proper subchain of $C_s^{(r'+1)}$. Therefore (repeating this process a finite number of steps) we eventually get

$$C_s = C_s^{(r-1)} = \{(i_1,i_2),\ldots,(i_{r-1},i_r)\}$$

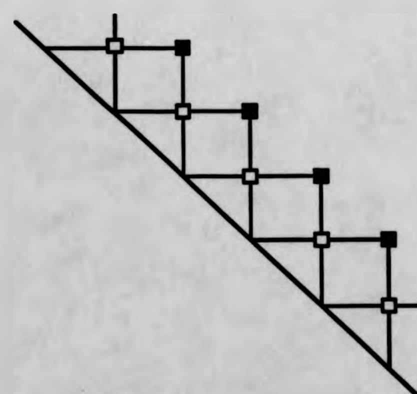
for some $r \geq 2$. This proves our claim. Moreover C_s determines a pair of intertwinning chains $C_{s1}, C_{s2} \subseteq D$ where

$$\begin{cases} C_{s1} = \{(i_1,i_3),(i_3,i_5),\ldots,(i_{r-3},i_{r-1})\} \\ C_{s2} = \{(i_2,i_4),(i_4,i_6),\ldots,(i_{r-2},i_r)\} \end{cases} \qquad \text{(if } r \text{ is even)}$$

or

$$\begin{cases} C_{s1} = \{(i_1,i_3),(i_3,i_5),\ldots,(i_{r-2},i_r)\} \\ C_{s2} = \{(i_2,i_4),(i_4,i_6),\ldots,(i_{r-3},i_{r-1})\} \end{cases} \qquad \text{(if } r \text{ is odd)}$$

The adjacent picture illustrates this situation (here the roots in C_s correspond to the symbol □, the roots in C_{s1} to the symbol ■ and the roots in C_{s2} to the symbol ■ ; we note that all the roots in C_{s1} and in C_{s2} lie in D). We note that either $|C_{s1}|=|C_{s2}|$ (if r is even) or $|C_{s1}|=|C_{s2}|+1$ (if r is odd). By the previous proposition we conclude that:



$$M_s \text{ is non-singular} \iff |C_{s1}|=|C_{s2}|+1 \iff C_s \text{ has even length.}$$

Finally if C_s $(1 \leq s \leq t)$ has odd length (hence r is even) the pair (C_{s1},C_{s2}) is special and the root (i_1,i_2) is D-derived, i.e. $(i_1,i_2) \in D'$. In fact we have

$$C_s \text{ has odd length} \iff (i_1,i_2) \in D'.$$

Conversely if $(i,j) \in D'$ then $(i,j) \in S(D)$ and so $(i,j) \in C_s$ for some $s \in \{1,\ldots,t\}$. Moreover (in the above notation) $(i,j)=(i_1,i_2)$ and the pair (C_{s1},C_{s2}) is the unique special pair in D which determines (i,j). It follows that

$$C_1,\ldots,C_t \text{ have even lengths } \Leftrightarrow D'=\varnothing.$$

Hence

$$V_D(\varphi)=O(f) \Leftrightarrow D'=\varnothing$$

and the proof is complete. ♦

Next we translate the previous theorem to the character theory of the finite group $U_n(q)$ (as usual q is a power of the prime number $p \geq n$). We let K be the algebraic closure of the finite field $\boldsymbol{F}_q$ and we realize $U_n(q)$ as the subgroup of $U_n(K)$ consisting of all fixed elements of the Frobenius map $F=F_q:U_n(K) \rightarrow U_n(K)$.

Theorem 3.3.4. *Let D be a basic subset of $\Phi(n)$ and let $\varphi:D \rightarrow \boldsymbol{F}_q^*$ be a map. Then the character $\xi_D(\varphi)$ of $U_n(q)$ has a unique irreducible component if and only if the derived set D' is empty. Moreover this component has multiplicity $\frac{q^{l(D)}}{\sqrt{q^{s(D)}}}$ where $l(D)$ and $s(D)$ are as in (2.2.7).*

Proof. The first part follows from theorem 3.3.3 and the second part follows from corollary 2.2.17. ♦

CHAPTER 4

A DECOMPOSITION OF $V_D(\varphi)$

In this chapter we discuss a certain decomposition of the variety $V_D(\varphi)$ into $U_n(K)$-invariant subvarieties (as usual D is a basic subset of $\Phi(n)$ and $\varphi: D \rightarrow K^*$ is any map). By theorem 3.3.3 $V_D(\varphi)$ is a single $U_n(K)$-orbit if and only if the derived set D' is empty. Therefore we may assume that D' contains at least one element. The construction of the required subvarieties is similar to the construction of the varieties $V_D(\varphi)$. Firstly we define certain $U_n(K)$-invariant polynomial functions on the variety $V_D(\varphi)$ (instead of $u_n(K)^*$). These functions are associated with some D-singular roots (in fact with D-derived roots) and their definition is recursive as in the case of the functions $\Delta_{ij}^D: u_n(K)^* \rightarrow K$ $((i,j) \in R(D))$. However a different method is used to prove that the new functions are $U_n(K)$-invariant.

The decomposition of $V_D(\varphi)$ will be obtained in section 4.2. Firstly in section 4.1 we discuss some examples which suggest the use of permutation matrices to study the variety $V_D(\varphi)$. They suggest also that a knowledge of the coadjoint orbits of the groups $U_\omega(K)$, $\omega \in S_n$, could be of fundamental importance for the understanding of the coadjoint orbits of $U_n(K)$. In fact the conjugation by a certain permutation matrix allows the "reduction" of our problem to the same problem in the smaller group $U_{n-1}(K)$. A similar method will be used in chapter 5 to establish the decomposition of the regular character of $U_n(q)$ as the sum of all the basic characters $\xi_D(\varphi)$ (see theorem 5.2.1).

4.1. Some examples

In this section we discuss some examples which will motivate our subsequent work. We denote by K an algebraically closed field of arbitrary characteristic. We let n be a positive integer and we suppose that $n=2r$ is even.

Let $f\in U_n(K)^*$ be such that

$$\Delta^{1\ldots r}_{r+1\ldots n}(f)\neq 0.$$

Let D be the (unique) basic subset of $\Phi(n)$ and let $\varphi: D\to K^*$ be the (unique) map such that $f\in V_D(\varphi)$. Then (by proposition 3.1.2) we must have

$$D=\{(1,\tau(1)),\ldots,(r,\tau(r))\}$$

for some permutation $\tau\in S_r$.

We consider the $U_n(K)$-orbit $O(f)$ of f. For pratical reasons we define for each $g\in U_n(K)^*$ the upper triangular matrix $A(g)=(a_{ij}(g))_{1\leq i,j\leq n}$ by

$$a_{ij}(g)=\begin{cases} g(e_{ij}) & \text{if } (i,j)\in\Phi(n)\\ 0 & \text{otherwise}\end{cases}$$

for all $i,j\in\{1,\ldots,n\}$. Let $x=(x_{ij})\in U_n(K)$ be arbitrary. Then

$$(x\cdot f)(e_{ij})=f(xe_{ij}x^{-1})=\sum_{a=1}^{i}\sum_{b=j}^{n}x_{ai}(x^{-1})_{jb}f(e_{ab})$$

for all $(i,j)\in\Phi(n)$. Therefore

(4.1.1) $$A(x\cdot f)=p_n(x^{\mathrm{T}}A(f)(x^{-1})^{\mathrm{T}})=p_n(x^{\mathrm{T}}A(f)(x^{\mathrm{T}})^{-1})$$

where for any matrix $X\in M_n(K)$ ([1]) $p_n(X)=(y_{ij})\in M_n(K)$ is the upper triangular matrix defined by

$$y_{ij}=\begin{cases} x_{ij} & \text{if } (i,j)\in\Phi(n)\\ 0 & \text{otherwise}\end{cases}$$

for all $i,j\in\{1,\ldots,n\}$. Since x^{T} is lower triangular we conclude the following:

---

[1] We denote by $M_n(K)$ the set of all square matrices of size n with coefficients in the field K.

Lemma 4.1.1. *Let $g\in U_n(K)^*$ and let $x\in U_n(K)$. Then $g=x\cdot f$ if and only if $p_n(x^{\mathrm{T}}A(f))=p_n(A(g)x^{\mathrm{T}})$.*

Using this lemma one can show that:

Lemma 4.1.2. *The $U_n(K)$-orbit of f contains an element $g\in U_n(K)^*$ such that*

$$A(g)=\begin{pmatrix} A & HP(\tau) \\ 0 & 0 \end{pmatrix}$$

where $A\in M_r(K)$ is upper triangular, $H=(h_{ij})\in M_r(K)$ is the diagonal matrix such that

$$h_{ii}=\varphi(i,\tau(i))$$

for all $i\in\{1,\ldots,r\}$, and $P(\tau)$ is the permutation matrix associated with $\tau\in S_r$.

Proof. Let

$$A(f)=\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where $A_{11},A_{12},A_{22}\in M_r(K)$ (obviously A_{11} and A_{22} are upper niltriangular). Let $x\in U_n(K)$ be arbitrary and let $x_{11},x_{22}\in U_r(K)$, $x_{12}\in M_r(K)$ be such that

$$x=\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix}.$$

Then

$$p_n(x^{\mathrm{T}}A(f))=\begin{pmatrix} p_r(x_{11}{}^{\mathrm{T}}A_{11}) & x_{11}{}^{\mathrm{T}}A_{12} \\ 0 & p_r(x_{12}{}^{\mathrm{T}}A_{12}+x_{22}{}^{\mathrm{T}}A_{22}) \end{pmatrix}$$

and

$$p_n(A(g)x^{\mathrm{T}})=\begin{pmatrix} p_r(Ax_{11}{}^{\mathrm{T}}+HP(\tau)x_{12}{}^{\mathrm{T}}) & HP(\tau)x_{22}{}^{\mathrm{T}} \\ 0 & 0 \end{pmatrix}.$$

Therefore (by the previous lemma) $g=x\cdot f$ if and only if the following equalities hold:

(i) $p_r(x_{11}{}^{\mathrm{T}}A_{11})=p_r(Ax_{11}{}^{\mathrm{T}}+HP(\tau)x_{12}{}^{\mathrm{T}})$

(ii) $x_{11}{}^{\mathrm{T}}A_{12}=HP(\tau)x_{22}{}^{\mathrm{T}}$

(iii) $p_r(x_{12}{}^{\mathrm{T}}A_{12}+x_{22}{}^{\mathrm{T}}A_{22})=0.$

Consider the equation (ii). Since

$$\det A_{12}=\Delta^{1\ldots r}_{r+1\ldots n}(f)\neq 0$$

we have $A_{12}\in GL_r(K)$. By Bruhat's decomposition of $GL_n(K)$ there exist $\sigma\in S_r$ such that

$$A_{12}\in U_r(K)^- H_r(K)P(\sigma)U_r(K)^-$$

where $H_r(K)$ is the subgroup of $GL_r(K)$ consisting of all non-singular diagonal matrices and $U_r(K)^-$ is the subgroup of $GL_r(K)$ consisting of all lower unitriangular matrices. Therefore there exist $H'\in H_r(K)$ and $y,z\in U_r(K)^-$ such that

$$A_{12}=yH'P(\sigma)z.$$

We now claim that $H'=H$ and $\sigma=\tau$. In fact the element

$$x'=\begin{pmatrix} y^{\mathrm{T}} & 0 \\ 0 & z^{\mathrm{T}} \end{pmatrix}\in U_n(K)$$

transforms the element $f\in \mathcal{U}_n(K)^*$ into an element $g'\in \mathcal{U}_n(K)$ such that

$$A(g')=\begin{pmatrix} A' & H'P(\sigma) \\ 0 & B' \end{pmatrix}$$

where $A',B'\in M_r(K)$ are upper triangular. It follows that $g'\in O(f)$. Since $f\in V_D(\varphi)$ we conclude that $g'\in V_D(\varphi)$. Finally proposition 3.1.2 (see also its proof) implies that $H'=H$ and that $\sigma=\tau$.

Now equation (ii) is satisfied if we take $x_{11}=(y^{-1})^{\mathrm{T}}$ and $x_{22}=z^{\mathrm{T}}$. Finally equation (iii) is clearly satisfied if we define $x_{12}\in GL_r(K)$ by

$$x_{12}^{\mathrm{T}}=-x_{22}^{\mathrm{T}}A_{22}A_{12}^{-1}.$$

The lemma follows. ♦

Now we may assume that $f\in \mathcal{U}_n(K)^*$ is such that

$$A(f)=\begin{pmatrix} A(f') & HP(\tau) \\ 0 & 0 \end{pmatrix}$$

where $f'\in \mathcal{U}_r(K)^*$. We have:

Lemma 4.1.3. *$V_D(\varphi)=O(f)$ if and only if $\tau\in S_r$ is the element defined by (1.1.16).*

Proof. By theorem 3.3.3 $V_D(\varphi)=O(f)$ if and only if the derived set D' is empty. The lemma is clear because $D'=\varnothing$ if and only if

$$P(\tau)=\begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}\in GL_r(K). \qquad \blacklozenge$$

Next we consider particular elements $\tau\in S_r$.

Lemma 4.1.4. *Let $\tau=1\in S_r$ and let $g\in \mathcal{U}_r(K)^*$ be such that*

$$A(g)=\begin{pmatrix} A(g') & H \\ 0 & 0 \end{pmatrix}$$

where $g'\in \mathcal{U}_r(K)^*$ (we note that $P(\tau)=I_r$). Then $g\in O(f)$ if and only if $g'\in O(f')$ - here $O(f')$ is the $U_r(K)$-orbit of $f'\in \mathcal{U}_r(K)^*$.

Proof. Let $x=\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix}\in U_n(K)$ $(x_{11},x_{22}\in U_r(K), x_{12}\in M_r(K))$. Then $g=x\cdot f$ if and only if

(i) $p_r(x_{11}{}^{\mathrm{T}}A(f'))=p_r(A(g')x_{11}{}^{\mathrm{T}}+Hx_{12}{}^{\mathrm{T}})$

(ii) $x_{11}{}^{\mathrm{T}}H=Hx_{22}{}^{\mathrm{T}}$

(iii) $p_r(x_{12}{}^{\mathrm{T}}H)=0$.

Suppose that these equations are satisfied (hence $g\in O(f)$). Since H is diagonal we have $p_r(x_{12}{}^{\mathrm{T}}H)=0$ if and only if $x_{12}{}^{\mathrm{T}}$ is lower triangular. Therefore $Hx_{12}{}^{\mathrm{T}}$ is lower triangular and

$$p_r(x_{11}{}^{\mathrm{T}}A(f'))=p_r(A(g')x_{11}{}^{\mathrm{T}}).$$

It follows that $g'=x_{11}\cdot f'$, so $g'\in O(f')$.

Conversely suppose that $g'=x_{11}\cdot f'$. Then equations (i), (ii) and (iii) are satisfied with $x_{22}{}^{\mathrm{T}}=H^{-1}x_{11}{}^{\mathrm{T}}H$ and $x_{12}{}^{\mathrm{T}}=0$. Thus $g\in O(f)$ and the lemma is proved. $\blacklozenge$

By this lemma (and by lemma 4.1.2) we deduce that:

Corollary 4.1.5. *Let $\tau=1\in S_r$. Then the $U_n(K)$-orbits in $V_D(\varphi)$ are in one-to-one correspondence with the $U_r(K)$-orbits in $u_r(K)^*$.*

This corollary might suggest that a decomposition of $V_D(\varphi)$ may be obtained by arguments analogous to the ones used in the previous chapter. However in general this is "impossible" as the next example shows.

Let $r=st$, $s,t\geq 2$, and consider the element $\tau\in S_r$ such that $P(\tau)$ has the form

$$P(\tau)=\begin{pmatrix} J_s & 0 & \dots & 0 \\ 0 & J_s & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & J_s \end{pmatrix}$$

where $J_s\in M_s(K)$ is the matrix

$$J_s=\begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}.$$

We have:

Lemma 4.1.6. *Let $r=st$ and let $\tau\in S_r$ be as above. Let $A=(A_{ij})_{1\leq i,j\leq t}$ where $A_{ij}\in M_s(K)$ $(1\leq i,j\leq t)$ and $A_{ij}=0$ for all $i,j\in\{1,\dots,t\}$ such that $i>j$. Let $g\in O(f)$ and suppose that*

$$A(g)=\begin{pmatrix} B & P(\tau) \\ 0 & 0 \end{pmatrix}$$

where $B\in M_r(K)$ is upper triangular. Suppose also that $B=(B_{ij})_{1\leq i,j\leq t}$ where $B_{ij}\in M_s(K)$ $(1\leq i,j\leq t)$ and $B_{ij}=0$ for all $i,j\in\{1,\dots,t\}$ such that $i>j$. Then $A_{1t}=B_{1t}$.

Proof. Let $x=\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix} \in U_n(K)$ $(x_{11}, x_{22} \in U_r(K), x_{12} \in M_r(K))$ be arbitrary. Then $g=x \cdot f$ if and only if the following equalities hold

(i) $p_r(x_{11}{}^{T}A)=p_r(Bx_{11}{}^{T}+P(\tau)x_{12}{}^{T})$

(ii) $x_{11}{}^{T}P(\tau)=P(\tau)x_{22}{}^{T}$

(iii) $p_r(x_{12}{}^{T}P(\tau))=0$.

Consider the equation (iii). Let $x_{12}{}^{T}=(y_{ij})_{1\le i,j\le t}$ where $y_{ij} \in M_s(K)$ $(1\le i,j\le t)$. Then

$$x_{12}{}^{T}P(\tau)=(y_{ij}J_s)_{1\le i,j\le t}\,.$$

We have $p_r(x_{12}{}^{T}P(\tau))=0$ if and only if $x_{12}{}^{T}P(\tau)$ is lower triangular. Thus $y_{ij}J_s=0$ and this implies that $y_{ij}=0$ for all $i,j \in \{1,\ldots,t\}$, $i>j$ (because J_s is non-singular) . On the other hand

$$p_s(y_{ii}J_s)=0$$

for all $i \in \{1,\ldots,t\}$. Therefore $y_{ii}J_s$ is lower triangular for all $i \in \{1,\ldots,t\}$.

Now we consider the equation (i). Let $x_{11}{}^{T}=(z_{ij})_{1\le i,j\le t}$ where $z_{ij} \in M_s(K)$ $(1\le i,j\le t)$ - since $x_{11}{}^{T}$ is lower triangular we have $z_{ij}=0$ $(1\le i<j\le t)$ and $z_{ii} \in U_s(K)^{-}$ $(1\le i\le t)$. Let $x_{11}{}^{T}A=(u_{ij})_{1\le i,j\le t}$ where $u_{ij} \in M_s(K)$ $(1\le i,j\le t)$. Then

$$u_{1t}=\sum_{k=1}^{t} z_{1k}A_{kt}=z_{11}A_{1t}.$$

On the other hand let $Bx_{11}{}^{T}=(v_{ij})_{1\le i,j\le t}$ where $v_{ij} \in M_s(K)$ $(1\le i,j\le t)$. Then

$$v_{1t}=\sum_{k=1}^{t} B_{1k}z_{kt}=B_{1t}z_{tt}.$$

Since $P(\tau)x_{12}{}^{T}=(J_sy_{ij})_{1\le i,j\le t}$ and $y_{1t}=0$, we conclude that

$$z_{11}A_{1t}=B_{1t}z_{tt}.$$

Finally consider the equation (ii). We have $x_{11}{}^{T}P(\tau)=(z_{ij}J_s)_{1\le i,j\le t}$ and $P(\tau)x_{22}{}^{T}=(J_sw_{ij})_{1\le i,j\le t}$ where $x_{22}{}^{T}=(w_{ij})_{1\le i,j\le t}$, $w_{ij} \in M_s(K)$ $(1\le i,j\le t)$. Since $x_{11}{}^{T}P(\tau)=P(\tau)x_{T1}{}^{T}$ we obtain

$$z_{ii}J_s=J_sw_{ii}$$

for all $i \in \{1,\ldots,t\}$. Since z_{ii} and w_{ii} are lower unitriangular we conclude that

$$z_{ii}=w_{ii}=I_s$$

for all $i \in \{1,\ldots,t\}$.

The lemma follows. ◆

The previous lemma shows that in general a decomposition of $V_D(\varphi)$ cannot be obtained using the methods of chapter 3. In fact the eventual definition of a new subset of roots (which corresponds to the notion of basic subset of $\Phi(n)$) has to include a general condition allowing the existence of more than one root from each row and more than one root from each column (in the example above the columns $r\text{-}s+1,\dots,r$ may contain more than one root (i,j) such that $f(e_{ij})\neq 0$). In the general case it seems to be very difficult to guess what condition has to be imposed. The natural conjecture is that it envolves the rank of certain matrices whose entries are elements $f(e_{ij})$ for some roots (i,j) in the required set. In the previous example the orbits which are contained in $V_D(\varphi)$ depend on the rank of the matrix A_{1t} (the notation is as in lemma 4.1.6) - we note that (by lemma 4.1.6) this rank is an invariant for the action of $U_n(K)$ on $V_D(\varphi)$. In fact:

Lemma 4.1.7. *Let the notation be as in lemma 4.1.6.*

(i) *For each $i\in\{1,\dots,s\}$ we denote by $c_i(A_{1t-1})$ (resp. $c_i(B_{1t-1})$) the i-th column of the matrix A_{1t-1} (resp. B_{1t-1}). Then the vector $c_i(A_{1t-1})-c_i(B_{1t-1})$ is a linear combination of the columns of the matrix A_{1t}.*

(ii) *For each $i\in\{1,\dots,s\}$ we denote by $r_i(A_{2t})$ (resp. $r_i(B_{2t})$) the i-th row of the matrix A_{2t} (resp. B_{2t}). Then the vector $r_i(A_{2t})-r_i(B_{2t})$ is a linear combination of the rows of the matrix A_{1t}.*

Proof. We keep the notation of the proof of the lemma 4.1.6. Let $x\in U_n(K)$ be such that $g=x\cdot f$ and consider the equation

$$p_r(x_{11}^{\mathrm{T}}A)=p_r(Bx_{11}^{\mathrm{T}}+P(\tau)x_{12}^{\mathrm{T}})=p_r(Bx_{11}^{\mathrm{T}}).$$

Since $u_{1t-1}=A_{1t-1}$ and $v_{1t-1}=B_{1t-1}+A_{1t}z_{tt-1}$ we have

$$A_{1t-1}-B_{1t-1}=A_{1t}z_{tt-1}$$

and (i) follows immediately.

Similarly we have

$$B_{2t}-A_{2t}=z_{21}A_{1t}$$

because $u_{2t}=z_{21}A_{1t}+A_{2t}$ and $v_{2t}=B_{2t}$. (ii) follows. ♦

Another attempt to find the condition mentioned above is to consider the coadjoint orbits of the subgroups $U_\omega(K)$ of $U_n(K)$ for an arbitrary $\omega\in S_n$. In fact in the case $\tau=1$ the orbits contained in $V_D(\varphi)$ are in one-to-one correspondence with the orbits of $U_\omega(K)$ for $\omega\in S_n$ such that

$$P(\omega)=\begin{pmatrix} 0 & J_r \\ I_r & 0 \end{pmatrix}$$

where

$$J_r=\begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}$$

- we note that $U_\omega(K)=U_n(K)\cap P(\omega^{-1})U_n(K)P(\omega)$ (see (1.1.15)).

On the other hand we have:

Lemma 4.1.8. *Let $\tau\in S_r$ be as in lemmas 4.1.6 and 4.1.7. Then the $U_n(K)$-orbits in $V_D(\varphi)$ are in one-to-one correspondence with the $U_\omega(K)$-orbits in $\mathfrak{u}_\omega(K)^*$ where $\omega\in S_n$ is such that*

$$P(\omega)=\begin{pmatrix} 0 & J_r \\ P(\tau) & 0 \end{pmatrix}.$$

Proof. We note that $U_\omega(K)$ consists of all matrices $\begin{pmatrix} x_{11} & 0 \\ 0 & I_r \end{pmatrix}\in U_n(K)$ such that $x_{11}\in U_r(K)\cap P(\tau)U_r(K)P(\tau^{-1})$ (hence x_{11} has the form

$$x_{11}=\begin{pmatrix} I_s & * & \dots & * \\ 0 & I_s & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & I_s \end{pmatrix}$$

where the symbol $*$ means that the corresponding block is any matrix in $M_s(K)$.

Now for this proof we keep the notation of the proof of the lemma 4.1.6. We claim that there exists $g \in O(f)$ such that

$$A(g)=\begin{pmatrix} B & P(\tau) \\ 0 & 0 \end{pmatrix}$$

where $B=(B_{ij})_{1\le i,j\le t}\in M_r(K)$ $(B_{ij}\in M_s(K),\ 1\le i,j\le t)$ satisfies $B_{ii}=0$ for all $i\in\{1,\dots,t\}$. In fact consider the equation

$$p_r(x_{11}{}^{\mathrm{T}}A)=p_r(Bx_{11}{}^{\mathrm{T}}+P(\tau)x_{12}{}^{\mathrm{T}}).$$

Let $i\in\{1,\dots,t\}$. Then

$$u_{ii}=\sum_{k=1}^{t} z_{ik}A_{ki}=z_{i1}A_{1i}+\dots+z_{ii-1}A_{i-1i}+A_{ii}$$

(we recall that $z_{ii}=I_s$) and

$$v_{ii}=\sum_{k=1}^{t} B_{ik}z_{ki}=B_{ii}+B_{ii+1}z_{i+1i}+\dots+B_{it}z_{ti}.$$

On the other hand $y_{ii}J_s$ is lower triangular so $J_sy_{ii}=J_sy_{ii}J_sJ_s$ is upper triangular. It follows that

$$p_s(z_{i1}A_{1i}+\dots+z_{ii-1}A_{i-1i}+A_{ii})=p_s(B_{ii}+B_{ii+1}z_{i+1i}+\dots+B_{it}z_{ti})+J_sy_{ii}.$$

Since J_s is non-singular we can allways choose $y_{ii}\in M_s(K)$ such that this equation is satisfied with $B_{ii}=0$ (for any choice of $z_{i1},\dots,z_{ii-1},z_{i+1i},\dots,z_{ti}$). Our claim follows.

Now let us assume that f is such that

$$A(f)=\begin{pmatrix} A & P(\tau) \\ 0 & 0 \end{pmatrix}$$

where $A=(A_{ij})_{1\le i,j\le t}$ where $A_{ij}\in M_s(K)$ $(1\le i,j\le t)$ and $A_{ii}=0$ for all $i\in\{1,\dots,t\}$. Let $g\in U_n(K)^*$ and suppose that $A(g)$ has the form

$$A(g)=\begin{pmatrix} B & P(\tau) \\ 0 & 0 \end{pmatrix}$$

where $B=(B_{ij})_{1\le i,j\le t}$, $B_{ij}\in M_s(K)$ $(1\le i,j\le t)$ is upper triangular and $B_{ii}=0$ for all $i\in\{1,\dots,t\}$. Then $g\in O(f)$ if and only if the equalities (i), (ii) and (iii) hold with

$x=\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix}\in U_n(K)$ such that $g=x\cdot f$. In particular we have $u_{ij}=v_{ij}$ for all $i,j\in\{1,\ldots,t\}$, $i>j$. This means that the element $x_{11}\in U_n(K)$ satisfies

$$g'=x_{11}\cdot f'$$

where $f',g'\in \mathcal{U}_\omega(K)^*$ are defined by

$$f'=\pi_\omega(f) \text{ and } g'=\pi_\omega(g)$$

where $\pi_\omega: \mathcal{U}_n(K)^* \to \mathcal{U}_\omega(K)^*$ is the natural projection (i.e., for any $f\in \mathcal{U}_\omega(K)^*$, $\pi_\omega(f)$ is the restriction of f to $\mathcal{U}_\omega(K)$). We conclude that g' is $U_\omega(K)$-conjugate to f'.

Conversely let $f',g'\in \mathcal{U}_\omega(K)^*$ and suppose that g' is $U_\omega(K)$-conjugate to f'. Then there exists $x_{11}\in U_\omega(K)$ such that $g'=x_{11}\cdot f'$. By the argument of the first paragraph of this proof we can allways choose $x_{12}\in M_r(K)$ such that the equality

$$p_r(x_{11}{}^{\mathrm{T}}A)=p_r(Bx_{11}{}^{\mathrm{T}}+P(\tau)x_{12}{}^{\mathrm{T}})$$

is satisfied. Moreover x_{12} can be choosed so that

$$p_r(x_{12}{}^{\mathrm{T}}P(\tau))=0.$$

Finally we define $x_{12}\in U_r(K)$ by

$$x_{22}{}^{\mathrm{T}}=P(\tau)^{-1}x_{11}{}^{\mathrm{T}}P(\tau).$$

The lemma follows because any $U_n(K)$-orbit on $V_D(\varphi)$ contains an element $f\in \mathcal{U}_n(K)^*$ such that

$$A(f)=\begin{pmatrix} A & P(\tau) \\ 0 & 0 \end{pmatrix}$$

for some upper triangular matrix $A\in M_r(K)$. ♦

In general case we have:

Proposition 4.1.9. *Let $\tau\in S_r$ and suppose that $\varphi(i,j)=1$ for all $(i,j)\in D$. Then the $U_n(K)$-orbits in $V_D(\varphi)$ are in one-to-one correspondence with the $U_\omega(K)$-orbits in $\mathcal{U}_\omega(K)^*$ where $\omega\in S_n$ is such that*

$$P(\omega)=\begin{pmatrix} 0 & J_r \\ P(\tau^{-1}) & 0 \end{pmatrix}$$

Proof. In this general case $U_\omega(K)$ consists of all matrices $\begin{pmatrix} x_{11} & 0 \\ 0 & I_r \end{pmatrix} \in U_n(K)$ such that

$x_{11} \in U_r(K) \cap P(\tau)U_r(K)P(\tau^{-1})$.

Let $f \in V_D(\varphi)$. Then (by lemma 4.1.2) we may assume that

$$A(f)=\begin{pmatrix} A & P(\tau) \\ 0 & 0 \end{pmatrix}$$

where $A \in M_r(K)$ is upper triangular. We claim that there exists $g \in O(f)$ such that

$$A(g)=\begin{pmatrix} B & P(\tau) \\ 0 & 0 \end{pmatrix}$$

where $B \in U_r(K) \cap P(\tau)U_r(K)P(\tau^{-1})$. In fact let $g \in U_n(K)^*$ be such that $A(g)$ has this form and let $x=\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix} \in U_n(K)$ $(x_{11}, x_{22} \in U_r(K),\ x_{12} \in M_r(K))$ be arbitrary. Then (as in the previous cases $g=x \cdot f$ if and only if the following equalities hold

(i) $p_r(x_{11}{}^{T}A)=p_r(Bx_{11}{}^{T}+P(\tau)x_{12}{}^{T})$

(ii) $x_{11}{}^{T}P(\tau)=P(\tau)x_{22}{}^{T}$

(iii) $p_r(x_{12}{}^{T}P(\tau))=0$.

The equation (iii) is satisfied if and only if the matrix $x_{12}{}^{T}P(\tau)$ is lower triangular. On the other hand the equation (ii) is satisfied if and only if

$$x_{11}{}^{T}=P(\tau)x_{22}{}^{T}P(\tau^{-1}) \in U_r(K) \cap P(\tau)U_r(K)^{-}P(\tau^{-1})$$

where $U_r(K)^{-}$ is the subgroup of $GL_r(K)$ consisting of all lower unitriangular matrices.

Now consider the equation (i). Since $A \in U_r(K)$ there exist $B \in U_r(K) \cap P(\tau)U_r(K)P(\tau^{-1})$ and $C \in U_r(K) \cap P(\tau)U_r(K)^{-}P(\tau^{-1})$ such that

$$A=B+C$$

(see pg. 6). Now the equation (i) is trivially satisfied if we define

$$x_{11}{}^{T}=I_r \quad \text{and} \quad x_{12}{}^{T}=P(\tau^{-1})C.$$

Since $C \in U_r(K) \cap P(\tau)U_r(K)^{-}P(\tau^{-1})$ we conclude that

$$x_{12}{}^{T}P(\tau)=P(\tau^{-1})P(\tau)x_{12}{}^{T}P(\tau)=P(\tau^{-1})CP(\tau) \in P(\tau^{-1})P(\tau)U_r(K)^{-}P(\tau^{-1})P(\tau)=U_r(K)^{-}..$$

Hence $x_{12}{}^{T}P(\tau)$ is lower triangular and (iii) is satisfied. Finally (ii) is trivially satisfied if we define $x_{22}^{\mathrm{T}}=I_r$. Our claim follows.

Now we assume that

$$A(f)=\begin{pmatrix} A & P(\tau) \\ 0 & 0 \end{pmatrix}$$

where $A\in U_r(K)\cap P(\tau)U_\tau(K)P(\tau^{-1})$. Let $x\in U_n(K)$ be as above and suppose that

$$A(x\cdot f)=\begin{pmatrix} B & P(\tau) \\ 0 & 0 \end{pmatrix}$$

where $B\in U_r(K)\cap P(\tau)U_\tau(K)P(\tau^{-1})$. Then the above equations (i), (ii) and (iii) are satisfied.

Let $\pi_\omega: U_n(K)^* \to U_\omega(K)^*$ be the natural projection (i.e. for any $f\in U_n(K)^*$, $\pi_\omega(f)$ is the restriction of f to $U_\omega(K)$). We denote by $p_\omega: U_n(K)\to U_\omega(K)$ the canonical projection (we recall that $U_\omega(K)$ is a direct summand of $U_n(K)$). Then, for any $f',g'\in U_\omega(K)^*$, we have $g'=y\cdot f'$ if and only if $p_\omega(y^{\mathrm{T}}A(g'))=p_\omega(A(f')y^{\mathrm{T}})$.

Now consider the equation (i), i.e.

$$p_r(x_{11}{}^{\mathrm{T}}A)=p_r(Bx_{11}{}^{\mathrm{T}}+P(\tau)x_{12}{}^{\mathrm{T}})=p_r(Bx_{11}{}^{\mathrm{T}})+p_r(P(\tau)x_{12}{}^{\mathrm{T}}).$$

Then

$$p_\tau(x_{11}{}^{\mathrm{T}}A)=p_\tau(Bx_{11}{}^{\mathrm{T}})+p_\tau(P(\tau)x_{12}{}^{\mathrm{T}})$$

where $p_\tau: U_r(K)\to U_\tau(K)$ is the canonical projection. Since $p_r(x_{12}{}^{\mathrm{T}}P(\tau))=0$ (by the equation (iii)) the matrix $x_{12}{}^{\mathrm{T}}P(\tau)$ is lower triangular. Hence

$$P(\tau)x_{12}{}^{\mathrm{T}}=P(\tau)x_{12}{}^{\mathrm{T}}P(\tau)P(\tau^{-1})\in P(\tau)U_r(K)^-P(\tau^{-1})$$

and

$$p_r(P(\tau)x_{12}{}^{\mathrm{T}})\in U_r(K)\cap P(\tau)U_r(K)^-P(\tau^{-1}).$$

It follows that

$$p_\tau(P(\tau)x_{12}{}^{\mathrm{T}})=0$$

so

$$p_\tau(x_{11}{}^{\mathrm{T}}A)=p_\tau(Bx_{11}{}^{\mathrm{T}}).$$

Since $x_{11}{}^{\mathrm{T}}\in U_r(K)\cap P(\tau)U_r(K)^-P(\tau^{-1})$ (see the first part of the proof) we have

$$x_{11}\in U_r(K)\cap P(\tau)U_r(K)P(\tau^{-1}).$$

It follows that

$$\pi_\omega(g)=y\cdot\pi_\omega(f)$$

where $y=\begin{pmatrix} x_{11} & 0 \\ 0 & I_s \end{pmatrix}\in U_\omega(K)$.

Conversely suppose that $\pi_\omega(g)=y\cdot\pi_\omega(f)$ for some $y\in U_\omega(K)$. Then

$$y=\begin{pmatrix} x_{11} & 0 \\ 0 & I_s \end{pmatrix}$$

where $x_{11}\in \mathcal{U}_r(K)\cap P(\tau)\mathcal{U}_r(K)P(\tau^{-1})$. Then

$$p_\tau(x_{11}{}^{T}A)=p_\tau(Bx_{11}{}^{T}).$$

Hence there exists $C\in \mathcal{U}_r(K)\cap P(\tau)\mathcal{U}_r(K)^{-}P(\tau^{-1})$ such that

$$p_r(x_{11}{}^{T}A)=p_r(Bx_{11}{}^{T})+C.$$

Then we may define $x_{12}{}^{T}=P(\tau^{-1})C$ and (i) is clearly satisfied. Moreover (as before) $x_{12}{}^{T}P(\tau)\in \mathcal{U}_r(K)^{-}$ and (iii) is also satisfied. Finally (ii) holds because $x_{11}{}^{T}\in \mathcal{U}_r(K)\cap P(\tau)\mathcal{U}_r(K)^{-}P(\tau^{-1})$. It follows that $g\in O(f)$.

The proposition is proved. ◆

Corollary 4.1.10. *Let $\tau\in S_r$ and let $\varphi:D\to K^*$ be arbitrary. Then the $U_n(K)$-orbits in $V_D(\varphi)$ are in one-to-one correspondence with the $U_\omega(K)$-orbits in $\mathcal{U}_\omega(K)^*$ where $\omega\in S_n$ is such that*

$$P(\omega)=\begin{pmatrix} 0 & J_r \\ P(\tau^{-1}) & 0 \end{pmatrix}.$$

Proof. Let $\psi:D\to K^*$ be the map defined by $\psi(i,j)=1$ for all $(i,j)\in D$. We define the map $\vartheta:V_D(\psi)\to V_D(\varphi)$ as follows. If $f\in V_D(\psi)$ then $\vartheta(f)\in V_D(\varphi)$ is the element such that

$$A(\vartheta(f))=xA(f)x^{-1}$$

where $x\in GL_n(K)$ is the diagonal matrix whose diagonal entries are

$$x_{ii}=\begin{cases}\varphi(i,\tau(i)) & \text{if } 1\le i\le r\\ 1 & \text{otherwise}\end{cases}.$$

Then ϑ is an isomorphism of algebraic varieties. The result follows by the previous proposition because

$$\vartheta(y\cdot f)=(xyx^{-1})\cdot\vartheta(f)$$

for all $y\in U_n(K)$ and all $f\in V_D(\psi)$. In fact

$$A(\vartheta(y\cdot f))=xA(y\cdot f)x^{-1}=xp_n(y^{\mathrm{T}}A(f)(y^{\mathrm{T}})^{-1})x^{-1}=p_n(xy^{\mathrm{T}}A(f)(y^{\mathrm{T}})^{-1}x^{-1})$$
$$=p_n(xy^{\mathrm{T}}x^{-1}xA(f)x^{-1}x(y^{\mathrm{T}})^{-1}x^{-1})=p_n(xy^{\mathrm{T}}x^{-1}A(\vartheta(f))x^{-1}x(y^{\mathrm{T}})^{-1}x^{-1})$$
$$=p_n((xyx^{-1})^{\mathrm{T}}xA(f)x^{-1}((xyx^{-1})^{\mathrm{T}})^{-1})=A((xyx^{-1})\cdot\vartheta(f))$$

for all $y\in U_n(K)$ and all $f\in V_D(\psi)$. ◆

4.2. A decomposition of $V_D(\varphi)$

In this section we fix a basic subset D of $\Phi(n)$ and a map $\varphi: D \to K^*$. As usual K is an algebraically closed field either of characteristic zero or of prime characteristic $p \geq n$. We assume that the smallest root of D lies in the n-th column, i.e. there exists $i \in \{1,\dots,n\}$ such that $(i,n) \in D$. Moreover we assume that $i<n-2$ and that the set

$$D'(i)=D' \cap \{(a,i);\ 1 \leq a < i-1\}$$

is non-empty (here D' is the derived set of D). Then (by theorem 3.3.3) the variety $V_D(\varphi)$ contains at least two $U_n(K)$-orbits.

Our aim is to obtain a decomposition of the variety $V_D(\varphi)$. This decomposition depends on the roots which lie in the set $D'(i)$. In fact we will associate to each root in $D'(i)$ a certain polynomial function (defined on $u_n(K)^*$). Moreover if the root is chosen conveniently we will show that the function is $U_n(K)$-invariant (hence it can be used to define a proper subvariety of $V_D(\varphi)$). In the following we motivate the introduction of these functions.

Let $f \in V_D(\varphi)$ be arbitrary and let $x=(x_{ab}) \in U_n(K)$ be the element defined by

$$(4.2.1) \qquad x_{ab}=\begin{cases} -\varphi(i,n)^{-1} f(e_{ia}) & \text{if } i<a<n \text{ and } b=n \\ \varphi(i,n)^{-1} f(e_{bn}) & \text{if } a=i \text{ and } i<b<n \\ 1 & \text{if } a=b \\ 0 & \text{otherwise} \end{cases}$$

(cf. (3.1.6)). Then the element $x \cdot f \in V_D(\varphi)$ satisfies

$$(x \cdot f)(e_{in})=\varphi(i,n) \quad \text{and} \quad (x \cdot f)(e_{an})=(x \cdot f)(e_{ib})=0 \ (i<a,b<n).$$

We let $g=x \cdot f$. Since $V_D(\varphi)$ is $U_n(K)$-invariant we have $g \in V_D(\varphi)$. Now let $\omega=(n-1 \dots i+1\ i) \in S_n$. Then the matrix $A(g)$ (as defined in section 4.1) lies in the subalgebra $u_\omega(K)=u_n(K) \cap \omega^{-1} u_n(K) \omega$ (cf. (1.1.13)). We define the matrix $A_\omega(g)$ by

$$A_\omega(g)=P(\omega)A(g)P(\omega^{-1})$$

where $P(\omega)$ is the permutation matrix associated with $\omega\in S_n$. Then

$$A_\omega(g)\in U_n(K)\cap\omega U_n(K)\omega^{-1}=U_{\omega^1}(K).$$

Let

$$A(g)=\begin{pmatrix} A & a & C & 0\\ 0 & 0 & 0 & \alpha\\ 0 & 0 & B & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

where $A\in U_{i-1}(K)$, $B\in U_{n-i-1}(K)$, C is a matrix of type $(i-1)\times(n-i-1)$, a is a row vector of lenght i-1 and $\alpha=\varphi(i,n)$. Since

$$P(\omega)=\begin{pmatrix} I_{i-1} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & I_{n-i-1} & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

we have

$$A_\omega(g)=\begin{pmatrix} A & C & a & 0\\ 0 & B & 0 & 0\\ 0 & 0 & 0 & \alpha\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Therefore $A_\omega(g)$ defines an element $g_\omega\in U_n(K)^*$, i.e.

$$A_\omega(g)=A(g_\omega).$$

By theorem 3.1.7 there exists a unique basic subset D_ω of $\Phi(n)$ and a unique map $\varphi_\omega:D_\omega\to K^*$ such that $g_\omega\in V_{D_\omega}(\varphi_\omega)$. It is clear that the smallest root in D_ω is $(n-1,n)$. Therefore the variety $V_{D_\omega}(\varphi_\omega)$ is canonically isomorphic to the variety $V_{D_{\omega,0}}(\varphi_{\omega,0})$ where $D_{\omega,0}=D_\omega\cap\Phi(n-1)$ and $\varphi_{\omega,0}$ is the restriction of φ_ω to $D_{\omega,0}$.

Now we consider the set $D_{\omega,0}$. For simplicity we write $D_1=D_{\omega,0}$ and $\varphi_1=\varphi_{\omega,0}$. Let $(r,s)\in\Phi(n)$ be the smallest root in D_1 and suppose that $s=n-1$. Then we must have $r<i-1$ and $g_\omega(e_{rn-1})=g(e_{ri})$. We know that the function $\Delta^{D_1}_{rn-1}:V_{D_1}(\varphi_1)\to K$ is $U_n(K)$-invariant (we note that this function is defined by $\Delta^{D_1}_{rn-1}(f)=f(e_{rn-1})=\varphi_1(r,n-1)$ for all $f\in V_{D_1}(\varphi_1)$. Moreover for any $a\in\{1,\dots,r-1\}$ the function $\Delta^{D_1}_{an-1}:V_{D_1}(\varphi_1)\to K$ is $U_n(K)$-invariant and we have $\Delta^{D_1}_{an-1}(f)=0$ for all $f\in V_{D_1}(\varphi_1)$. Therefore $g(e_{ai})=g_\omega(e_{an-1})=\Delta^{D_1}_{an-1}(g_\omega)=0$. Next we will prove that for any $a\in\{1,\dots,r\}$ there exists a $U_n(K)$-invariant polynomial function

$\Psi_{ai}:V_D(\varphi)\rightarrow K$ such that $\Psi_{ai}(h)=g(e_{ai})$ for all $h\in V_D(\varphi)$. In particular we must have $\Psi_{ri}(h)=\varphi_1(r,n\text{-}1)\neq 0$ and $\Psi_{ai}(h)=0$ for all $a\in\{1,\dots,r\text{-}1\}$ and all $h\in V_D(\varphi)$.

Let $f\in V_D(\varphi)$ be our initial element and let $x\in U_n(K)$ be as in (4.2.1). Then $x\cdot f=g$ and

$$g(e_{ai})=(x\cdot f)(e_{ai})=-\varphi(i,n)^{-1}\left(\sum_{b=i}^{n-1} f(e_{ab})f(e_{bn})\right)$$

for all $a\in\{1,\dots,i\text{-}1\}$. Therefore we may define $\Psi_{ai}:V_D(\varphi)\rightarrow K$ by

$$\Psi_{ai}(h)=-\varphi(i,n)^{-1}\left(\sum_{b=i}^{n-1} h(e_{ab})h(e_{bn})\right)$$

for all $h\in V_D(\varphi)$. However we will consider the polynomial function $\Theta_{ai}:\mathcal{U}_n(K)^*\rightarrow K$ which is determined by the polynomial

$$\Theta_{ai}(T_{rs};(r,s)\in\Phi(n))=\sum_{b=i}^{n-1} T_{ab}T_{bn}\in K[T_{rs};(r,s)\in\Phi(n)]$$

where T_{rs}, $(r,s)\in\Phi(n)$, are $\frac{n(n-1)}{2}$ indeterminates over K. Then we have

$$\Theta_{ai}(h)=-\varphi(i,n)\Psi_{ai}(h)$$

for all $h\in V_D(\varphi)$. It follows that $\Theta_{ai}:V_D(\varphi)\rightarrow K$ is $U_n(K)$-invariant if and only if $\Psi_{ai}:V_D(\varphi)\rightarrow K$ is $U_n(K)$-invariant.

In order to prove that the function $\Theta_{ai}:V_D(\varphi)\rightarrow K$ is $U_n(K)$-invariant we establish a general result which can be applied in later cases. Let V be $U_n(K)$-invariant subvariety of $\mathcal{U}_n(K)^*$ and let $P:V\rightarrow K$ be a polynomial function. Then there exists a polynomial $P(T_{ab};(a,b)\in\Phi(n))$ in $\frac{n(n-1)}{2}$ indeterminates T_{ab}, $(a,b)\in\Phi(n)$, such that

$$P(f)=P(f(e_{ab});(a,b)\in\Phi(n)).$$

Therefore we may define the differential map $dP:V\rightarrow\mathcal{U}_n(K)$ by the rule, i.e.

$$(dP)(f)=\sum_{(a,b)\in\Phi(n)}(\partial_{ab}P)(f)\,e_{ab}$$

for all $f\in V$ (we recall that $\partial_{ab}P$ denotes the (a,b)-th partial derivative $\frac{\partial P}{\partial T_{ab}}$ of the polynomial P).

Theorem 4.2.1. *Let V be any $U_n(K)$-invariant subvariety of $u_n(K)^*$ and let $P:V\to K$ be a polynomial function. For each $(i,j)\in\Phi(n)$ let n_{ij} be the degree of the polynomial*

$$P_{ij}(t)=P(f(e_{ab})+tf([e_{ij},e_{ab}]);\ (a,b)\in\Phi(n))\in K[t].$$

If K has prime characteristic p, we assume that $p\geq max\{n_{ij};\ (i,j)\in\Phi(n)\}$. Then P is $U_n(K)$-invariant if and only if $(dP)(f)\in\mathfrak{r}(f)$ for all $f\in V$ ([1]*).*

Since every element of $U_n(K)$ can be written as a product of elements

$$x_{ij}(\alpha)=1+\alpha e_{ij}$$

where $(i,j)\in\Phi(n)$ and $\alpha\in K$, the function $P:V\to K$ is $U_n(K)$-invariant if and only if

$$P(x_{ij}(\alpha)\cdot f)=P(f)$$

for all $(i,j)\in\Phi(n)$, all $\alpha\in K$ and all $f\in V$. In other words P is $U_n(K)$-invariant if and only if P is $X_{ij}(K)$-invariant for all $(i,j)\in\Phi(n)$ (we recall that $X_{ij}(K)$ denotes the (i,j)-th root subgroup of $U_n(K)$). Therefore the theorem is a consequence of the following:

Proposition 4.2.2. *Let $(i,j)\in\Phi(n)$ be arbitrary. Let V be a $X_{ij}(K)$-invariant subvariety of $u_n(K)$ and let $P:V\to K$ be a polynomial function. For each $(i,j)\in\Phi(n)$ we let n_{ij} be as in the theorem and we assume that $p\geq max\{n_{ij};\ (i,j)\in\Phi(n)\}$ whenever K has prime characteristic p. Then P is $X_{ij}(K)$-invariant if and only if $f([e_{ij},(dP)(f)])=0$ for all $f\in V$.*

Proof. Let t be an indeterminate over K. For each $f\in V$ and each $(a,b)\in\Phi(n)$ we define the polynomial $\phi_{ab}^{(f)}(t)\in K[t]$ by

$$\phi_{ab}^{(f)}(t)=f(e_{ab})+tf([e_{ij},e_{ab}]).$$

Then

$$\phi_{ab}^{(f)}(\alpha)=f(e_{ab})+\alpha f([e_{ij},e_{ab}])=(x_{ij}(\alpha)\cdot f)(e_{ab})$$

for all $\alpha\in K$ and all $(a,b)\in\Phi(n)$. Therefore

$$P(x_{ij}(\alpha)\cdot f)=P(\phi_{ab}^{(f)}(\alpha);\ (a,b)\in\Phi(n))$$

for all $\alpha\in K$. Let $\phi^{(f)}(t)\in K[t]$ be the polynomial

$$\phi^{(f)}(t)=P(\phi_{ab}^{(f)}(t);\ (a,b)\in\Phi(n)).$$

[1] We recall that for any $f\in u_n(K)^*$ $\mathfrak{r}(f)$ denotes the radical of the bilinear form B_f.

It follows that

$$P \text{ is } X_{ij}(K)\text{-invariant} \iff \phi^{(f)}(\alpha)=\phi^{(f)}(0)=P(f) \text{ for all } f\in V.$$

Since K is infinite we conclude that

$$P \text{ is } X_{ij}(K)\text{-invariant} \iff \phi^{(f)}(t)=P(f) \text{ for all } f\in V.$$

Now suppose that P is $X_{ij}(K)$-invariant (hence $\phi^{(f)}(t)$ is constant) and let $f\in V$ be arbitrary. For simplicity we will write $\phi(t)$ and $\phi_{ab}(t)$ instead of $\phi^{(f)}(t)$ and of $\phi^{(f)}_{ab}(t)$ respectively. We also denote by $\phi'(t)$ and by $(\phi_{ab})'(t)$ $((a,b)\in\Phi(n))$ the derivatives of $\phi(t)$ and of $\phi_{ab}(t)$ $((a,b)\in\Phi(n))$ respectively. Then (by the chain rule) we have

$$\begin{aligned}\phi'(t)&=\sum_{(a,b)\in\Phi(n)}(\partial_{ab}P)(\phi_{ab}(t);(a,b)\in\Phi(n))\,(\phi_{ab})'(t)\\ &=\sum_{(a,b)\in\Phi(n)}(\partial_{ab}P)(\phi_{ab}(t);(a,b)\in\Phi(n))f([e_{ij},e_{ab}]).\end{aligned}$$

Since $\phi(t)$ is constant we have $\phi'(t)=0$ and so

$$\begin{aligned}0=\phi'(0)&=\sum_{(a,b)\in\Phi(n)}(\partial_{ab}P)(\phi_{ab}(0);(a,b)\in\Phi(n))f([e_{ij},e_{ab}])\\ &=\sum_{(a,b)\in\Phi(n)}(\partial_{ab}P)(f(e_{ab});(a,b)\in\Phi(n))f([e_{ij},e_{ab}])\\ &=f\left(\left[e_{ij},\sum_{(a,b)\in\Phi(n)}\partial_{ab}P(f)e_{ab}\right]\right)\\ &=f([e_{ij},(dP)(f)]).\end{aligned}$$

Conversely suppose that

$$f([e_{ij},(dP)(f)])=0$$

for all $f\in V$. We let $f\in V$ be arbitrary and we define

$$\phi(t)=\phi^{(f)}(t)\in K[t]$$

as above. We claim that $\phi'(t)=0$. By the chain rule we have

$$\phi'(t)=\sum_{(a,b)\in\Phi(n)}(\partial_{ab}P)(\phi_{ab}(t);(a,b)\in\Phi(n))f([e_{ij},e_{ab}])$$

where for each $(a,b)\in\Phi(n)$ the polynomial

$$\phi_{ab}(t)=\phi^{(f)}_{ab}(t)\in K[t]$$

is defined above. Now let $\alpha\in K$ be arbitrary. Then

$$\begin{aligned}(dP)(x_{ij}(\alpha)\cdot f)&=\sum_{(a,b)\in\Phi(n)}(\partial_{ab}P)(x_{ij}(\alpha)\cdot f)\,e_{ab}\\ &=\sum_{(a,b)\in\Phi(n)}(\partial_{ab}P)(\phi_{ab}(t);(a,b)\in\Phi(n))\,e_{ab}\end{aligned}$$

and

$$(x_{ij}(\alpha)\cdot f)([e_{ij},v])=f([e_{ij},v])+\alpha f([e_{ij},[e_{ij},v]])=f([e_{ij},v])$$

for all $v\in \mathcal{U}_n(K)$. In particular consider the vector

$$(dP)(x_{ij}(\alpha)\cdot f)\in \mathcal{U}_n(K).$$

Then

$$(x_{ij}(\alpha)\cdot f)([e_{ij},(dP)(x_{ij}(\alpha)\cdot f)])=f([e_{ij},(dP)(x_{ij}(\alpha)\cdot f)]).$$

Since V is $X_{ij}(K)$-invariant we have

$$x_{ij}(\alpha)\cdot f\in V.$$

Therefore (by hypothesis)

$$(x_{ij}(\alpha)\cdot f)([e_{ij},(dP)(x_{ij}(\alpha)\cdot f)])=0.$$

It follows that

$$\phi'(\alpha)=\sum_{(a,b)\in\Phi(n)}(\partial_{ab}P)(\phi_{ab}(\alpha);(a,b)\in\Phi(n))f([e_{ij},e_{ab}])$$
$$=f([e_{ij},(dP)(x_{ij}(\alpha)\cdot f)])=0.$$

Since $\alpha\in K$ is arbitrary and K is an infinite field we conclude that $\phi'(t)=0$ as claimed. This implies that the polynomial $\phi(t)\in K[t]$ is constant - we note that (by our assumption) p is larger than the degree of $\phi(t)$ whenever K has prime characteristic p.

The result follows. ◆

Now we consider the differential

$$d\Theta_{ai}:\mathcal{U}_n(K)^*\to \mathcal{U}_n(K)$$

of the polynomial map $\Theta_{ai}:\mathcal{U}_n(K)^*\to K$ $(1\le a<i)$. For any $f\in \mathcal{U}_n(K)^*$ and any $a\in\{1,\dots,i\text{-}1\}$ we have

$$(d\Theta_{ai})(f)=\sum_{b=i}^{n-1}f(e_{ab})e_{bn}+\sum_{b=i}^{n-1}f(e_{bn})e_{ab}.$$

Lemma 4.2.3. *Let $f\in V_D(\varphi)$. Then*

$$f([e_{rs},(d\Theta_{ai})(f)])=\begin{cases}\Theta_{ri}(f) & \text{if } s=a\\ 0 & \text{otherwise}\end{cases}$$

for all $a\in\{1,\dots,i\text{-}1\}$ and all $(r,s)\in\Phi(n)$.

Proof. Let $a\in\{1,\ldots,i-1\}$ and $(r,s)\in\Phi(n)$ be arbitrary. Then

$$f([e_{rs},(d\Theta_{ai})(f)])=\sum_{b=i}^{n-1}f(e_{ab})f([e_{rs},e_{bn}])+\sum_{b=i}^{n-1}f(e_{bn})f([e_{rs},e_{ab}]).$$

Suppose that $s\notin\{a,i,\ldots,n-1\}$. If $r\notin\{i,\ldots,n\}$, it is clear that

$$f([e_{rs},(d\Theta_{ai})(f)])=0.$$

If $r\in\{i,\ldots,n\}$ then we must have $s=n$ (because $i\leq r<s$). Hence $r<n$ and

$$f([e_{rn},(d\Theta_{ai})(f)])=-f(e_{rn})f(e_{an})=0$$

(because $a<i$ and $f\in V_D(\varphi)$). On the other hand suppose that $s\in\{i,\ldots,n-1\}$. Then $r\notin\{i,\ldots,n\}$ (because $r<s<i$) so

$$f([e_{rs},(d\Theta_{ai})(f)])=f(e_{as})f(e_{rn})=0$$

(because $r<i$ and $f\in V_D(\varphi)$). Finally suppose that $s=a$. Then $r\notin\{i,\ldots,n\}$ (because $a<i$) and

$$f([e_{ra},(d\Theta_{ai})(f)])=\sum_{b=i}^{n-1}f(e_{bn})f(e_{rb})=\Theta_{ri}(f).$$

The lemma is proved. ◆

For each $a\in\{1,\ldots,i-1\}$ we define the subvariety V_a of $V_D(\varphi)$ by

(4.2.2) $$V_a=\{f\in V_D(\varphi);\ \Theta_{1i}(f)=\Theta_{2i}(f)=\ldots=\Theta_{ai}(f)=0\}.$$

We claim that V_a is $U_n(K)$-invariant for all $a\in\{1,\ldots,i-1\}$. Since

$$V_{i-1}\subseteq\ldots\subseteq V_1\subseteq V_0=V_D(\varphi),$$

this claim is a corollary of the following:

Lemma 4.2.4. *Let $a\in\{1,\ldots,i-1\}$ be arbitrary. Then the polynomial function $\Theta_{ai}:V_{a-1}\to K$ is $U_n(K)$-invariant.*

Proof. By the previous lemma we have

$$(d\Theta_{ai})(f)\in\tau(f)$$

for all $f\in V_{a-1}$. If K has characteristic zero this is enough to conclude the proof (by theorem 4.2.1). On the other hand suppose that K has prime characteristic p. Then in order to apply theorem 4.2.1 we must prove that for all $f\in V_{b-1}$ and all $(r,s)\in\Phi(n)$ the polynomial

$$\phi(t)=\Theta_{ai}(f(e_{uv})+tf([e_{rs},e_{uv}]);\,(u,v)\in\Phi(n))\in K[t]$$

has degree less than p. In fact for each $b\in\{i,\ldots,n-1\}$ the monomial

$$\phi_b(t)=(f(e_{ab})+tf([e_{rs},e_{ab}]))(f(e_{bn})+tf([e_{rs},e_{bn}]))\in K[t]$$

has degree at most one (because either $f([e_{rs},e_{ab}])=0$ or $f([e_{rs},e_{bn}])=0$). Since

$$\phi(t)=\sum_{b=i}^{n-1}\phi_b(t),$$

we conclude that $\phi(t)$ has degree at most one. The lemma follows. ◆

Corollary 4.2.5. *Let $a\in\{1,\ldots,i-1\}$ be arbitrary. Then the variety $V_a\subseteq V_D(\varphi)$ is $U_n(K)$-invariant. Moreover suppose that the function $\Theta_{ia}:V_{a-1}\to K$ is not identically zero (here $V_0=V_D(\varphi)$). Then for each $\alpha\in K$, the subvariety*

$$V_a(\alpha)=\{f\in V_{a-1};\,\Theta_{ai}(f)=\alpha\}$$

of V_{a-1} is $U_n(K)$-invariant (we note that $V_a(0)=V_a$).

Next we consider the minimum number of equations necessary to describe the variety V_a ($1\leq a<i$). In fact there could exist $a\in\{1,\ldots,i-1\}$ such that $\Theta_{ai}(f)=0$ for all $f\in V_{a-1}$. For example if $(1,i)\in R(D)$ we clearly have $\Theta_{1i}(f)=0$ for all $f\in V_D(\varphi)$. More generally we have:

Lemma 4.2.6. *Let $a\in\{1,\ldots,i-1\}$ and suppose that $(a,i)\in R(D)$. Then*

$$V_a=\begin{cases}\varnothing & \text{if } (a,i)\in D\\ V_{a-1} & \text{if } (a,i)\notin D\end{cases}.$$

Proof. Let $f\in V_{a-1}$ be arbitrary and let $x\in U_n(K)$ be such that the element $g=x\cdot f\in u_n(K)^*$ satisfies $g(e_{bn})=0$ for all $b\in\{i+1,\ldots,n-1\}$. Then

$$\Theta_{ci}(g)=g(e_{ci})g(e_{in})=g(e_{ci})\varphi(i,n)$$

for all $c\in\{1,\ldots,i-1\}$. Since V_{a-1} is $U_n(K)$-invariant we have $g\in V_{a-1}$ so $\Theta_{ci}(g)=0$ for all $c\in\{1,\ldots,a-1\}$. Since $\varphi(i,n)\neq0$ we deduce that

(4.2.3) $$g(e_{ci})=0$$

for all $c \in \{1,\ldots,a-1\}$.

Now we consider the polynomial function $\Delta^D_{ai}: U_n(K)^* \to K$. Let $D(a,i)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$, $j_1<\ldots<j_r$, and let $\sigma \in S_r$ be such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. Then (by (4.2.3))

(4.2.4) $$\Delta^D_{ai}(g)=(-1)^r g(e_{ai})\Delta^{i_{\sigma(1)}\ldots i_{\sigma(r)}}_{j_1\ldots j_r}(g).$$

Suppose that $(a,i)\notin D$. Then $\Delta^D_{ai}(g)=0$. Hence $g(e_{ai})=0$ and

$$\Theta_{ai}(g)=g(e_{ai})g(e_{in})=0.$$

Since $\Theta_{ai}:V_{a-1}\to K$ is $U_n(K)$-invariant we deduce that

$$\Theta_{ai}(f)=\Theta_{ai}(x\cdot f)=\Theta_{ai}(g)=0.$$

Since f is arbitrary we conclude that $V_a=V_{a-1}$ as required.

On the other hand suppose that $(a,i)\in D$. Then

$$\Delta^D_{ai}(g)=(-1)^r\varphi(a,i)\Delta^{i_{\sigma(1)}\ldots i_{\sigma(r)}}_{j_1\ldots j_r}(g).$$

Thus (by (4.2.4)) $g(e_{ai})=\varphi(a,i)$ and so $\Theta_{ai}(g)=\varphi(a,i)\varphi(i,n)\neq 0$. As before we deduce that

$$\Theta_{ai}(f)=\Theta_{ai}(x\cdot f)=\Theta_{ai}(g)=\varphi(a,i)\varphi(i,n)\neq 0$$

and this implies that $V_a=\emptyset$.

The proof is complete. ◆

Lemma 4.2.7. *Let $a\in\{1,\ldots,i-1\}$ such that $(a,i)\notin D'(i)$. Suppose that V_{a-1} is non-empty. Then for each $\alpha\in K$ the variety $V_a(\alpha)$ is non-empty and V_{a-1} is the disjoint union*

$$V_{a-1}=\bigcup_{\alpha\in K} V_a(\alpha).$$

Proof. Let $\alpha\in K$ and let $f\in V_{a-1}$ be such that $f(e_{bn})=0$ for all $b\in\{i+1,\ldots,n-1\}$. Then the element

$$g=f+(\varphi(i,n)^{-1}\alpha-f(e_{ai})).e_{ai}{}^*$$

satisfies

$$\Theta_{ai}(g)=g(e_{ai})g(e_{in})=(f(e_{ai})+\varphi(i,n)^{-1}\alpha-f(e_{ai}))\varphi(i,n)=\alpha.$$

Hence $g \in V_a(\alpha)$ and so the variety $V_a(\alpha)$ is non-empty. The remaining assertions are clear because $\Theta_{ai}: V_a \to K$ is $U_n(K)$-invariant. ♦

The following result is an obvious consequence of the previous results (its proof uses an easy argument of induction):

Theorem 4.2.8. *Let D be a basic subset of roots and let $\varphi: D \to K^*$. Suppose that $(i,n) \in D$ and that $D'(i) = D' \cap \{(k,i);\ 1 \leq k < i\} \neq \varnothing$. Let*

$$D'(i) = \{(k_1,i),\ldots,(k_t,i)\},\ k_1 < \ldots < k_t.$$

Let $s \in \{1,\ldots,t\}$. Then

$$V_{k_s} = \{f \in V_D(\varphi);\ \Theta_{k_1 i}(f) = \ldots = \Theta_{k_s i}(f) = 0\}.$$

On the other hand let $\alpha \in K$. Then the set

$$V_{k_s}(\alpha) = \{f \in V_{k_{s-1}};\ \Theta_{k_s i}(f) = \alpha\}$$

is a non-empty $U_n(K)$-invariant subvariety of $V_{k_{s-1}}$ (we put $V_{k_0} = V_D(\varphi)$). Moreover $V_{k_{s-1}}$ is the disjoint union

$$V_{k_{s-1}} = \bigcup_{\alpha \in K} V_{k_s}(\alpha).$$

Finally suppose that there exists $a \in \{1,\ldots,i-1\}$ such that $(a,i) \in D$. Then

$$V_a(\alpha) = \begin{cases} \varnothing & \text{if } \alpha \neq \varphi(a,i)\varphi(i,n) \\ V_{k_t} & \text{if } \alpha = \varphi(a,i)\varphi(i,n) \end{cases}$$

for all $\alpha \in K$. In particular $V_a = \varnothing$.

Since $V_{k_1} \subseteq V_{k_2} \subseteq \ldots \subseteq V_{k_t} \subseteq V_D(\varphi)$ (the notation is as in the theorem) we conclude the following:

Corollary 4.2.9. *Let the notation be as in the previous theorem. Then the variety $V_D(\varphi)$ is the disjoint union*

$$V_D(\varphi) = V_{k_t} \cup \left(\bigcup_{s=1}^{t} \bigcup_{\alpha_s \in K^*} V_{k_s}(\alpha_s) \right).$$

Now let $f \in V_D(\varphi)$ and suppose that there exists $a \in \{1,\dots,i\text{-}1\}$ such that $f \in V_a(\alpha)$ for some $\alpha \in K^*$. Then we have $\Theta_{bi}(f)=0$ $(1 \le b \le a)$ and $\Theta_{ai}(f)=\alpha \ne 0$.

Hence a is the smallest integer $b \in \{1,\dots,i\text{-}1\}$ such that $\Theta_{bi}(f) \ne 0$. Moreover (by theorem 4.2.8) either $(a,i) \in D$ or $(a,i) \in D'(i) \subseteq D'$. If $(a,i) \in D$ then

$$\alpha=\varphi(a,i)\varphi(i,n).$$

On the other hand suppose that $(a,i) \in D'(i)$. For convenience we put $a=a_1$ and $\alpha=\alpha_1$. Since $(a_1,i) \in D'$ there exists $b_1 \in \{i+1,\dots,n\text{-}1\}$ such that $(a_1,b_1) \in D$. Our aim is to decompose the variety $V_{a_1}(\alpha_1)$ as a disjoint union of $U_n(K)$-invariant subvarieties. For we define a new set of polynomial functions on $\mathcal{U}_n(K)^*$ and we imitate the arguments used before.

Let $g \in V_D(\varphi)$ be such that $g(e_{jn})=0$ $(1 \le j \le n\text{-}1)$ and $g(e_{in})=\varphi(i,n)=\alpha$. Let $\omega=(n\text{-}1 \dots i+1\ i) \in S_n$ and define $g_\omega \in \mathcal{U}_n(K)^*$, $D_1 \subseteq \Phi(n\text{-}1)$ and $\varphi_1{:}D_1 \to K^*$ as before. Let $a \in \{a_1+1,\dots,i\text{-}1\}$ and suppose that the root $(a,b_1\text{-}1)$ is D_1-regular (we note that the root $(a_1,b_1\text{-}1)$ is D_1-singular because $(a_1,n\text{-}1) \in D_1$; however there does not exist $c \in \{1,\dots,a_1\text{-}1\}$ such that $(c,b_1\text{-}1) \in D_1$). We consider the function $\Delta^{D_1}_{ab_1\text{-}1}{:}V_{D_1}(\varphi_1) \to K$. This function is $U_n(K)$-invariant and we have

$$\Delta^{D_1}_{ab_1\text{-}1}(g_\omega)=\pm\det\begin{pmatrix} g_{i_{\sigma(1)}i} & g(e_{i_{\sigma(1)}j_1}) & \cdots & g(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & & \vdots \\ g_{i_{\sigma(r)}i} & g(e_{i_{\sigma(r)}j_1}) & \cdots & g(e_{i_{\sigma(r)}j_r}) \\ g_{ai} & g(e_{aj_1}) & \cdots & g(e_{aj_r}) \end{pmatrix}$$

where $(i_1,j_1)=(a_1,b_1)$, $D_1(a,b_1\text{-}1)=\{(i_2,j_2),\dots,(i_r,j_r),(a_1,n\text{-}1)=(i_1,n\text{-}1)\}$, $j_2<\dots<j_r<n\text{-}1$, and $\sigma \in S_n$ is such that $i_{\sigma(1)}<i_{\sigma(2)}<\dots<i_{\sigma(r)}$. As in the previous case this suggests the definition of a polynomial function $\Lambda_{ai}{:}\mathcal{U}_n(K)^* \to K$ as follows.

Let $a \in \{a_1+1,\dots,i\text{-}1\}$. Let

$$D(a,b_1\text{-}1)=\{(i_1,j_1),\dots,(i_r,j_r)\},\ j_1<\dots<j_r$$

(we note that $(i_1,j_1)=(a_1,b_1)$) and let $\sigma \in S_r$ be such that $i_{\sigma(1)}<\dots<i_{\sigma(r)}$. We define the polynomial function $\Lambda_{ai}{:}\mathcal{U}_n(K)^* \to K$ by

(4.2.5)
$$\Lambda_{ai}(f)=det\begin{pmatrix} \Theta_{i_{\sigma(1)}i}(f) & f(e_{i_{\sigma(1)}j_1}) & \dots & f(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & & \vdots \\ \Theta_{i_{\sigma(r)}i}(f) & f(e_{i_{\sigma(r)}j_1}) & \dots & f(e_{i_{\sigma(r)}j_r}) \\ \Theta_{ai}(f) & f(e_{aj_1}) & \dots & f(e_{aj_r}) \end{pmatrix}$$

for all $f\in u_n(K)^*$.

As in the previous case the first step is to show that for each $a\in\{a_1+1,\dots,i-1\}$ the function $\Lambda_{ai}: u_n(K)^*\to K$ is $U_n(K)$-invariant when restricted to a certain subvariety of $V_{a_1}(\alpha_1)$. This subvariety is defined by

$$V_{a_1a}(\alpha_1)=\{f\in V_{a_1}(\alpha_1);\ \Lambda_{bi}(f)=0,\ a_1+1\le b\le a,\ (b,b_1)\in S^{(r)}(D)\}$$

(for the definition of the set $S^{(r)}(D)$ see pg. 85) ([1]). We note that

$$V_{a_1a}(\alpha_1)=V_{a_1a-1}(\alpha_1)$$

whenever $(a,b_1)\in S^{(r)}(D)$. We have:

Proposition 4.2.10. *Let $a\in\{a_1+2,\dots,a-1\}$ be such that $(a,b_1)\in S^{(r)}(D)$ and let $f\in V_{a_1a-1}(\alpha_1)$. Then the vector $(d\Lambda_{ai})(f)\in u_n(K)^*$ lies in the radical $\mathfrak{r}(f)$ of the bilinear form B_f, i.e.*

$$f([e_{rs},(d\Phi_{ai})(f)])=0$$

for all $(r,s)\in\Phi(n)$.

Proof. For each root $(r,s)\in\Phi(n)$, we define the polynomial function $\vartheta_{rs}: u_n(K)^*\to K$ by

$$\vartheta_{rs}(f)=\begin{cases}\Theta_{ri}(f) & \text{if } s=i\\ f(e_{rs}) & \text{otherwise}\end{cases}$$

for all $f\in u_n(K)^*$. Then we obtain a morphism of algebraic varieties $\vartheta: u_n(K)^*\to u_n(K)^*$ if we define

$$\vartheta(f)(e_{rs})=\vartheta_{rs}(f)$$

for all $f\in u_n(K)^*$ and all $(r,s)\in\Phi(n)$. Moreover if $D(a,b_1-1)=\{(i_1,j_1),\dots,(i_r,j_r)\}$, $j_1<\dots<j_r$,

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[1] We note that the condition $(b,b_1)\in S^{r}(D)$ corresponds to the condition $(b,b_1)\in R(D_1)$.

and if $\sigma\in S_r$ is such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$, we have

$$\Lambda_{ai}(f)=\Delta^{i_{\sigma(1)}\ldots i_{\sigma(r)}a}_{ij_1\ldots j_r}(\vartheta(f))$$

for all $f\in \mathcal{U}_n(K)^*$. For simplicity we will denote by Δ the polynomial function $\Delta^{i_{\sigma(1)}\ldots i_{\sigma(r)}a}_{ij_1\ldots j_r}:\mathcal{U}_n(K)^*\to K$.

Let $f\in \mathcal{U}_n(K)^*$. Then (by the chain rule)

$$(\partial_{rs}\Lambda_{ai})(f)=\sum_{(u,v)\in\Phi(n)}(\partial_{uv}\Delta)(\vartheta(f))\,(\partial_{rs}\vartheta_{uv})(f)$$

for all $(r,s)\in\Phi(n)$. Therefore

$$\sum_{(r,s)\in\Phi(n)}(\partial_{rs}\Lambda_{ai})(f)\,e_{rs}=\sum_{(r,s)\in\Phi(n)}\left(\sum_{(u,v)\in\Phi(n)}(\partial_{uv}\Delta)(\vartheta(f))\,(\partial_{rs}\vartheta_{uv})(f)\right)e_{rs}$$

$$=\sum_{(u,v)\in\Phi(n)}(\partial_{uv}\Delta)(\vartheta(f))\left(\sum_{(r,s)\in\Phi(n)}(\partial_{rs}\vartheta_{uv})(f)\,e_{rs}\right),$$

i.e.

$$(d\Lambda_{ai})(f)=\sum_{(u,v)\in\Phi(n)}(\partial_{uv}\Delta)(\vartheta(f))\,(d\vartheta_{uv})(f).$$

Since

$$(d\vartheta_{uv})(f)=\begin{cases}(d\Theta_{ui})(f) & \text{if } v=i\\ e_{uv} & \text{otherwise}\end{cases}$$

we conclude that

$$(d\Lambda_{ai})(f)=c_{ai}(f)(d\Theta_{ai})(f)+\sum_{u=1}^{r}c_{i_{\sigma(u)}i}(f)(d\Theta_{i_{\sigma(u)}i})(f)+\sum_{v=1}^{r}c_{aj_v}(f)e_{aj_v}+\sum_{u=1}^{r}\sum_{v=1}^{r}c_{i_{\sigma(u)}j_v}(f)e_{i_{\sigma(u)}j_v}$$

where for each $k\in\{a,i_1,\ldots,i_r\}$ and each $l\in\{i,j_1,\ldots,j_r\}$ $c_{kl}(f)\in K$ is the $r\times r$ minor of $\Lambda_{ai}(f)$ "complementary" to the position (k,l).

Now let $f\in V_{a_1a}(\alpha_1)$ and let $(r,s)\in\Phi(n)$. We compute the "scalar" product $f([e_{rs},(d\Phi_{ai})(f)])$. By lemma 4.2.3 we have

$$f([e_{rs},(d\Theta_{bi})(f)])=\delta_{sb}\Theta_{ri}(f)$$

for each $b\in\{1,\ldots,i\text{-}1\}$.

Suppose that $s\notin\{a,i_1,\ldots,i_r\}$. Then

$$f([e_{rs},(d\Theta_{bi})(f)])=0$$

for all $b\in\{i,j_1,\ldots,j_r\}$. If $r\in\{j_1,\ldots,j_r\}$, it is obvious that

$$f([e_{rs},(d\Lambda_{ai})(f)])=0.$$

Now let $r=j_v$ for some $v\in\{1,\ldots,r\}$. Then

$$f([e_{j_v s},(d\Lambda_{ai})(f)])=-c_{aj_v}(f)f(e_{as})-\sum_{v=1}^{r} c_{i_{\sigma(u)}j_v}(f)f(e_{i_{\sigma(u)}s})=-\Delta^{i_{\sigma(1)}\ldots i_{\sigma(r)}a}_{ij_1\ldots j_{v-1}sj_{v+1}\ldots j_r}(\vartheta(f)).$$

We claim that this determinant is zero. For we will show that the column-vector

$$w_s=(f(e_{i_{\sigma(1)}s})\ \ldots\ f(e_{i_{\sigma(r)}s})\ f(e_{as}))^{\mathrm{T}}$$

is a linear combination of the column-vectors

$$w_{j_u}=(f(e_{i_{\sigma(1)}j_u})\ \ldots\ f(e_{i_{\sigma(r)}j_u})\ f(e_{aj_u}))^{\mathrm{T}}$$

for $u\in\{v+1,\ldots,r\}$ (we note that $j_v=r<s$). This is obvious if $s=j_u$ for some $u\in\{v+1,\ldots,r\}$. On the other hand if $j_r<s$ we have

$$f(e_{i_{\sigma(1)}s})=\ldots=f(e_{i_{\sigma(r)}s})=f(e_{as})=0$$

(we note that $(a,b_1)\in S^{(r)}(D)$ so $(a,s)\in R(D)\setminus D$) and the claim is obvious. Finally let $u\in\{v+1,\ldots,r\}$ be such that $j_{u-1}<s<j_u$. Then

$$D(a,s)=\{(i_u,j_u),\ldots,(i_r,j_r)\}$$

and

$$(i_1,s),\ldots,(i_{u-1},s)\in R(D).$$

By theorem 3.1.7 we have

$$V_D(\varphi)=\sum_{t=u}^{r} O_{i_tj_t}(\varphi(i_t,j_t))+V_{D_0}(\varphi_0)$$

where $O_{i_tj_t}(\varphi(i_t,j_t))$ is the (i_t,j_t)-th elementary $U_n(K)$-orbit associated with $\varphi(i_t,j_t)\in K^*$ $(u\le t\le r)$, $D_0=D\setminus\{(i_u,j_u),\ldots,(i_r,j_r)\}$ and φ_0 is the restriction of φ to D_0. Therefore

$$f=f_u+\ldots+f_r+f'$$

where $f_t\in O_{i_tj_t}(\varphi(i_t,j_t))$ $(u\le t\le r)$ and $f'\in V_{D_0}(\varphi_0)$. Thus for each $b\in\{s,j_u,\ldots,j_r\}$ the vector w_b is a linear combination of the vectors

$$w_{tb}=(f_t(e_{i_{\sigma(1)}b})\ \ldots\ f_t(e_{i_{\sigma(r)}b})\ f_t(e_{ab}))^{\mathrm{T}}$$

for $t\in\{u,\ldots,r\}$ (we note that $(a,s)\in R(D)\setminus D$). On the other hand (by lemma 3.1.1) for each $t\in\{u,\ldots,r\}$ and each $b\in\{s,j_u,\ldots,j_r\}$ the vector w_{tb} is a scalar multiple of w_{tj_t}. It follows that

$$w_b\in\sum_{t=u}^{r}Kw_{tb}=\sum_{t=u}^{r}Kw_{tj_t}$$

for all $b\in\{s,j_u,\ldots,j_r\}$. Thus

$$dimV\leq r-u+1$$

where

$$V=Kw_s+\sum_{t=u}^{r}Kw_{j_t}.$$

Since $\Delta_{j_u\ldots j_r}^{i_u\ldots i_r}(f)\neq 0$ (see the proof of proposition 3.1.2), the vectors $w_{j_u},\ldots,w_{j_r}$ are linearly independent. So

$$dimV=r-u+1$$

and this implies that w_s is a linear combination of the vectors $w_{j_u},\ldots,w_{j_r}$ as required. It follows that

$$\Delta_{ij_1\ldots j_{v-1}sj_{v+1}\ldots j_r}^{i_{\sigma(1)}\ldots i_{\sigma(r)}a}(\vartheta(f))=0$$

so

$$f([e_{j_v s},(d\Lambda_{ai})(f)])=0.$$

Now suppose that $s=a$. Then $r<s=a<i<j_u$ $(1\leq u\leq r)$, so $r\notin\{j_1,\ldots,j_r\}$ and

$$f([e_{ra},(d\Lambda_{ai})(f)])=c_{ai}(f)\Theta_{ri}(f)+\sum_{v=1}^{r}c_{aj_v}(f)f(e_{rj_v})$$

(we recall that $f([e_{ra},(d\Theta_{ai})(f)])=\Theta_{ri}(f)$ (by lemma 4.2.3)). Therefore

$$f([e_{ra},(d\Lambda_{ai})(f)])=\Delta_{ij_1\ldots j_r}^{i_{\sigma(1)}\ldots i_{\sigma(r)}r}(\vartheta(f)).$$

As before we claim that this determinant is zero. This is obvious if $r\in\{i_1,\ldots,i_r\}$ so we assume that $r\notin\{i_1,\ldots,i_r\}$. If $i_{\sigma(r)}<r$ then

$$\Delta_{ij_1\ldots j_r}^{i_{\sigma(1)}\ldots i_{\sigma(r)}r}(\vartheta(f))=\Phi_{ri}(f)=0$$

because $r<s=a$ and $(r,b_1)\notin S^{(r)}(D)$ (otherwise $r\in\{i_1,\ldots,i_r\}$). On the other extreme if $r<i_{\sigma(1)}$ then $f(e_{rb})=0$ for all $b\in\{i,\ldots,n\}$ and

$$\Theta_{ri}(f)=f(e_{ri})f(e_{in})=0.$$

Therefore we clearly have

$$\Delta_{ij_1\ldots j_r}^{i_{\sigma(1)}\ldots i_{\sigma(r)}r}(\vartheta(f))=0.$$

Finally we suppose that $i_{\sigma(u)}<r<i_{\sigma(u+1)}$ for some $u\in\{1,\dots,r-1\}$ and we consider the matrix

$$A=\begin{pmatrix} \Theta_{i_{\sigma(1)}i}(f) & f(e_{i_{\sigma(1)}j_1}) & \dots & f(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & & \vdots \\ \Theta_{i_{\sigma(u)}i}(f) & f(e_{i_{\sigma(u)}j_1}) & \dots & f(e_{i_{\sigma(u)}j_r}) \\ \Theta_{ri}(f) & f(e_{rj_1}) & \dots & f(e_{rj_r}) \end{pmatrix}.$$

We claim that A has rank u. Since $\Delta^{i_{\sigma(1)}\dots i_{\sigma(r)}}_{j_{\sigma(1)}\dots j_{\sigma(r)}}(f)\neq0$ (see the proof of proposition 3.1.2) we have

$$rankA\in\{u,u+1\}.$$

Suppose that $rankA=u+1$. Then there exists a non-singular submatrix A' of A of size $u+1$. Since $(r,j_v)\in R(D)$ for all $v\in\{1,\dots,r\}\setminus\{\sigma(1),\dots,\sigma(u)\}$, the column-vector

$$w_{j_v}=(f(e_{i_{\sigma(1)}j_v}) \dots f(e_{i_{\sigma(u)}j_v})\, f(e_{rj_v}))^{\mathrm{T}}$$

is a linear combination of the vectors

$$w_{j_{\sigma(t)}}=(f(e_{i_{\sigma(1)}j_{\sigma(t)}}) \dots f(e_{i_{\sigma(u)}j_{\sigma(t)}})\, f(e_{rj_{\sigma(t)}}))^{\mathrm{T}}$$

for $t\in\{1,\dots,u\}$ (see the previous paragraph). Thus the submatrix A' (if it exists) has to be

$$A'=\begin{pmatrix} \Theta_{i_{\sigma(1)}i}(f) & f(e_{i_{\sigma(1)}j_{\sigma(1)}}) & \dots & f(e_{i_{\sigma(1)}j_{\sigma(u)}}) \\ \vdots & \vdots & & \vdots \\ \Theta_{i_{\sigma(u)}i}(f) & f(e_{i_{\sigma(u)}j_{\sigma(1)}}) & \dots & f(e_{i_{\sigma(u)}j_{\sigma(u)}}) \\ \Theta_{ri}(f) & f(e_{rj_{\sigma(1)}}) & \dots & f(e_{rj_{\sigma(u)}}) \end{pmatrix}.$$

Now if $r<a_1$ then

$$\Theta_{i_{\sigma(1)}i}(f)=\dots=\Theta_{i_{\sigma(u)}i}(f)=\Theta_{ri}(f)=0$$

(by definition of a_T - we note that $i_{\sigma(v)}<r$ for all $v\in\{1,\dots,u\}$). Therefore $detA'=0$ and we must have $a_1<r$. Then

$$\{(i_{\sigma(1)},j_{\sigma(1)}),\dots,(i_{\sigma(u)},j_{\sigma(u)})\}=\{(a_1,b_1)\}\cup D(r,b_1)$$

Thus

$$detA'=\pm\Lambda_{ri}(f)=0$$

because $r<a$ and $(r,b_1)\in S^{(r)}(D)$. We conclude that the non-singular submatrix A' of A

does not exist and so $rankA=u$ as required. This implies that

$$\Delta_{ij_1\ldots j_r}^{i_{\alpha(1)}\ldots i_{\alpha(r)}r}(\vartheta(f))=0$$

as it was also required. Therefore

$$f([e_{ra},(d\Phi_{ai})(f)])=0.$$

Finally suppose that $s=i_{\alpha(u)}$ for some $u\in\{1,\ldots,r\}$. Since $r<s=i_u<a<i<j_v$ $(1\le v\le r)$, we have $r\notin\{j_1,\ldots,j_r\}$. Therefore

$$f([e_{ri_{\alpha(u)}},(d\Phi_{ai})(f)])=c_{i_{\alpha(u)}i}(f)\Theta_{ri}(f)+\sum_{v=1}^{r}c_{i_{\alpha(u)}j_v}(f)f(e_{rj_v})$$
$$=\Delta_{ij_1\ldots j_r}^{i_{\alpha(1)}\ldots i_{\alpha(u-1)}ri_{\alpha(r+1)}\ldots i_{\alpha(r)}a}(\vartheta(f))$$

and we may repeat the above argument to conclude that this determinant is zero.

The proof is complete. ◆

As a corollary we obtain:

Lemma 4.2.11. *Let* $a\in\{a_1+2,\ldots,i-1\}$ *be such that* $(a,b_1)\in S^{(r)}(D)$. *Then the polynomial function* $\Lambda_{ai}:V_{a_1a-1}(\alpha_1)\to K$ *is* $U_n(K)$*-invariant.*

Proof. If K has characteristic zero the lemma follows immediately from the previous proposition and from theorem 4.2.1.

On the other hand suppose that K has prime characteristic and let us determine the maximal degree of the polynomials

$$\phi(t)=\Lambda_{ai}(f(e_{pq})+tf([e_{rs},e_{pq}]);\ (p,q)\in\Phi(n))\in K[t]$$

where $f\in V_{a_1a-1}(\alpha_1)$ and $(r,s)\in\Phi(n)$ are arbitrary. By definition we have

$$\phi(t)=\sum_{\tau\in S_{r+1}}\sum_{b=i}^{n-1}\mathrm{M}_{b,\tau}(f(e_{pq})+tf([e_{rs},e_{pq}]);\ (p,q)\in\Phi(n))$$

where for each $b\in\{i,\ldots,n-1\}$ and each $\tau\in S_{r+1}$

$$M_{b,\tau}(T_{pq};\ (p,q)\in\Phi(n))\in K[T_{pq};\ (p,q)\in\Phi(n)]$$

is the monomial (in the indeterminates T_{pq}, $(p,q)\in\Phi(n)$)

$$M_{b,\tau}(T_{pq};(p,q)\in\Phi(n))=sgn(\tau)T_{bn}T_{i_{\tau(1)}b}T_{i_{\tau(2)}j_1}\dots T_{i_{\tau(r+1)}j_r}$$

where $D(a,b_1-1)=\{(i_1,j_1),\dots,(i_r,j_r)\}$, $j_1<\dots<j_r$, and $i_{r+1}=a$. Let $b\in\{i,\dots,n-1\}$ and let $\tau\in S_{r+1}$. Then the polynomial $M_{b,\tau}(T_{pq};(p,q)\in\Phi(n))$ is not constant if and only if there exists $(p,q)\in\{(b,n),(i_{\tau(1)},b),(i_{\tau(2)},j_1),\dots,(i_{\tau(r+1)},j_r)\}$ such that $f([e_{rs},e_{pq}])\neq 0$ and this is possible only if $s\in\{b,i_{\tau(1)},\dots,i_{\tau(r+1)}\}$ or $r\in\{b,j_1,\dots,j_r\}$ (we note that $r<s\leq n$).

Suppose that $s\in\{b,i_{\tau(1)},\dots,i_{\tau(r+1)}\}$ and that $r\in\{b,j_1,\dots,j_r\}$. Then the polynomial

$$M_{b,\tau}(f(e_{pq})+tf([e_{rs},e_{pq}]);(p,q)\in\Phi(n))\in K[t]$$

is not constant and its degree is less or equal to two. Suppose that it has degree two. Then $r=b=j_u$ for some $u\in\{1,\dots,r\}$ and the coefficient of t^2 is

$$sgn(\tau)f(e_{j_un})f(e_{i_{\tau(1)}s})f(e_{i_{\tau(2)}j_1})\dots f(e_{i_{\tau(u)}j_{u-1}})f(e_{i_{\tau(u+1)}s})f(e_{i_{\tau(u+2)}j_{u+1}})\dots f(e_{i_{\tau(r+1)}j_r}).$$

On the other hand let ω be the transposition $(1\ u+1)\in S_{r+1}$ and consider the polynomial

$$M_{j_u,\tau\omega}(f(e_{pq})+tf([e_{j_us},e_{pq}]);(p,q)\in\Phi(n))\in K[t].$$

This polynomial has degree two and the coefficient of t^2 is

$$-sgn(\tau)f(e_{j_un})f(e_{i_{\tau(u+1)}s})f(e_{i_{\tau(2)}j_1})\dots f(e_{i_{\tau(u)}j_{u-1}})f(e_{i_{\tau(1)}s})f(e_{i_{\tau(u+2)}j_{u+1}})\dots f(e_{i_{\tau(r+1)}j_r}).$$

Now suppose that $s\in\{b,i_{\tau(1)},\dots,i_{\tau(r+1)}\}$. If $s=i_{\tau(u)}$ for some $u\in\{1,\dots,r+1\}$, then $r\notin\{b,j_1,\dots,j_r\}$ (because $r<s=i_{\tau(u)}<a<i$, $i\leq b\leq n-1$ and $i<j_1<\dots<j_r$) and the polynomial

$$M_{b,\tau}(f(e_{pq})+tf([e_{ri_{\tau(u)}},e_{pq}]);(p,q)\in\Phi(n))\in K[t]$$

has degree at most one. On the other hand suppose that $s=b$ and that

$$M_{b,\tau}(f(e_{pq})+tf([e_{rb},e_{pq}]);(p,q)\in\Phi(n))\in K[t]$$

has degree two. Then we must have $r=j_u$ for some $u\in\{1,\dots,r\}$, and the coefficient of t^2 in

$$M_{b,\tau}(f(e_{pq})+tf([e_{j_ub},e_{pq}]);(p,q)\in\Phi(n))\in K[t]$$

is

$$sgn(\tau)f(e_{j_un})f(e_{i_{\tau(1)}b})f(e_{i_{\tau(2)}j_1})\dots f(e_{i_{\tau(u)}j_{u-1}})f(e_{i_{\tau(u+1)}b})f(e_{i_{\tau(u+2)}j_{u+1}})\dots f(e_{i_{\tau(r+1)}j_r}).$$

As before we may consider the transposition $\omega=(1\ u+1)$ and the polynomial

$$M_{j_u,\tau\omega}(f(e_{pq})+tf([e_{j_ub},e_{pq}]);(p,q)\in\Phi(n))\in K[t].$$

This polynomial has degree two and the coefficient of t^2 is

$$-sgn(\tau)f(e_{j_un})f(e_{i_{\tau(1)}b})f(e_{i_{\tau(2)}j_1})\dots f(e_{i_{\tau(u)}j_{u-1}})f(e_{i_{\tau(u+1)}b})f(e_{i_{\tau(u+2)}j_{u+1}})\dots f(e_{i_{\tau(r+1)}j_r}).$$

It follows that the polynomial $\phi(t)$ has degree at most one. Finally since the root $(r,s)\in\Phi(n)$ is arbitrary we may apply the previous proposition and theorem 4.2.1 to conclude the proof. ◆

As in the initial situation the next step is to show that

$$V_{a_1a}(\alpha_1)=V_{a_1a-1}(\alpha_1)$$

whenever $(a,i)\in R(D)\setminus D$. In fact:

Lemma 4.2.12. *Let $a\in\{a_1+2,\ldots,i-1\}$ and suppose that (a,i) is D-regular (hence $(a,b_1)\in S^{(r)}(D)$). Let $f\in V_{a_1a-1}(\alpha_1)$. Then $\Lambda_{ai}(f)\neq 0$ if and only if $(a,i)\in D$.*

Proof. As before let $D(a,b_1-1)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$, $j_1<\ldots<j_r$, and let $\sigma\in S_r$ be such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. Then (by definition)

$$\Lambda_{ai}(f)=\Delta^{i_{\sigma(1)}\ldots i_{\sigma(r)}a}_{ij_1\ldots j_r}(\vartheta(f))$$

where $\vartheta:U_n(K)^*\to U_n(K)^*$ is defined as in the proof of proposition 4.2.10.

We claim that there exists an element $x=(x_{uv})\in U_n(K)$ such that $x\cdot f$ satisfies

$$(x\cdot f)(e_{aj_u})=0$$

for all $u\in\{1,\ldots,r\}$ and

$$(x\cdot f)(e_{ai})=\begin{cases}\varphi(a,i) & \text{if } (a,i)\in D\\ 0 & \text{if } (a,i)\notin D\end{cases}.$$

In fact let $D(a,i)=\{(r_1,s_1),\ldots,(r_p,s_p)\}$, $s_1<\ldots<s_p$, let $\tau\in S_p$ be such that $r_{\tau(1)}<\ldots<r_{\tau(p)}$ and consider the system of linear equations

$$(y_1\ldots y_p)A=(-f(e_{r_1a})\ldots -f(e_{r_pa}))$$

where

$$A=\begin{pmatrix} f(e_{r_1s_1}) & \ldots & f(e_{r_1s_p})\\ \vdots & & \vdots\\ f(e_{r_ps_1}) & \ldots & f(e_{r_ps_p})\end{pmatrix}$$

Since

$$\det A=sgn(\tau)\Delta^{r_{\tau(1)}\ldots r_{\tau(p)}}_{s_1\ldots s_p}(f)\neq 0$$

this system has a unique solution $(\beta_1 \dots \beta_p)$. We define $x=(x_{uv})\in U_n(K)$ by

$$x_{uv}=\begin{cases}\beta_q & \text{if } (u,v)=(r_q,a),\ 1\le q\le p\\ 1 & \text{if } u=v\\ 0 & \text{otherwise}\end{cases}.$$

Then

$$(x\cdot f)(e_{ab})=f(xe_{ab}x^{-1})=f(e_{ab})+\sum_{q=1}^{p} x_{r_q a}f(e_{r_q b})$$

for all $b\in\{a+1,\dots,n\}$. Therefore

$$(x\cdot f)(e_{ab})=0$$

for all $b\in\{s_1,\dots,s_p\}$. If $b\in\{i,\dots,n\}\setminus\{s_1,\dots,s_p\}$ then we consider the function $\Delta^D_{ab}:V_D(\varphi)\to K$ (we note that $(a,b)\in R(D)$). The last row of the determinant $\Delta^D_{ab}(x\cdot f)$ is

$$(\,(x\cdot f)(e_{ab})\ (x\cdot f)(e_{as_q})\ \dots\ (x\cdot f)(e_{as_p})\,)=(\,(x\cdot f)(e_{ab})\ 0\ \dots\ 0\,)$$

where $s_{q-1}<b<s_q$ and $s_0=i$. So

$$\Delta^D_{ab}(x\cdot f)=(-1)^{p-q+1}(x\cdot f)(e_{ab})\Delta^{r_{\tau(q)}\dots r_{\tau(p)}}_{s_q\dots s_p}(f)$$

where $\tau\in S_{p-q+1}$ is such that $r_{\tau(q)}<\dots<r_{\tau(p)}$. Now suppose that $i<b$. Then $(a,b)\in D$ and

$$\Delta^D_{ab}(x\cdot f)=0.$$

Since $\Delta^{r_{\tau(q)}\dots r_{\tau(p)}}_{s_q\dots s_p}(f)\neq 0$ we conclude that

$$(x\cdot f)(e_{ab})=0.$$

The same argument justifies that

$$(x\cdot f)(e_{ai})=0$$

whenever $(a,i)\in D$. Finally suppose that $(a,i)\in D$. Then

$$\Delta^D_{ab}(x\cdot f)=(-1)^p\varphi(a,i)\Delta^{r_{\tau(1)}\dots r_{\tau(p)}}_{s_1\dots s_p}(f)$$

(by proposition 3.1.2) and $(x\cdot f)(e_{ai})=\varphi(a,i)$.

Now (by the previous proposition) the function $\Lambda_{ai}:V_{a_1a-1}(\alpha_1)\to K$ is $U_n(K)$-invariant. Therefore

$$\Lambda_{ai}(f)=\Lambda_{ai}(x\cdot f)=(-1)^r\,\Theta_{ai}(x\cdot f)\Delta^{i_{\alpha(1)}\dots i_{\alpha(r)}}_{j_1\dots j_r}(x\cdot f).$$

By definition of the polynomial function $\Theta_{ai}: U_n(K)^* \to K$ we have

$$\Theta_{ai}(x\cdot f)=(x\cdot f)(e_{ai})(x\cdot f)(e_{in})=\begin{cases}\varphi(a,i)\varphi(i,n) & \text{if } (a,i)\in D\\ 0 & \text{if } (a,i)\notin D\end{cases}.$$

Since $\Delta^{i_{\sigma(1)},\ldots,i_{\sigma(r)}}_{j_1\ldots j_r}(x\cdot f)\neq 0$ the proof is complete. ◆

This lemma allows us to write a similar version of theorem 4.2.8:

Theorem 4.2.13. *Let D be a basic subset of $\Phi(n)$ and suppose that $(i,n),(a_1,b_1)\in D$ where $a_1<i<b_1$. Suppose that the set $X=\{(k,i)\in D';\, a_1<k,\, (b_1,k)\in S^{(r)}(D)\}$ is non-empty and let*

$$X=\{(k_1,i),\ldots,(k_t,i)\}$$

where $k_1<\ldots<k_t$. Let $\alpha_1\in K^*$. Then for all $s\in\{1,\ldots,t\}$ and all $\beta\in K$ we have

$$V_{a_1k_s}(\alpha_1)=\{f\in V_{a_1}(\alpha_1);\, \Lambda_{k_1i}(f)=\ldots=\Lambda_{k_si}(f)=0\}$$

for all $s\in\{1,\ldots,t\}$. Moreover for each $\beta\in K$ and each $s\in\{1,\ldots,t\}$ the subvariety

$$V_{a_1k_s}(\alpha_1,\beta)=\{f\in V_{a_1k_{s-1}}(\alpha_1);\, \Lambda_{k_{s-1}i}(f)=\beta\}$$

of $V_{a_1}(\alpha_1)$ is non-empty and $U_n(K)$-invariant, and $V_{a_1k_{s-1}}(\alpha_1)$ is the disjoint union

$$V_{a_1k_{s-1}}(\alpha_1)=\bigcup_{\beta\in K} V_{a_1k_s}(\alpha_1,\beta)$$

(here $V_{a_1k_0}(\alpha_1)=V_{a_1}(\alpha_1)$). Finally suppose that there exists $a\in\{k_t+1,\ldots,i-1\}$ such that $(a,i)\in D$. Let $D(a,b_1-1)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$, $j_1<\ldots<j_r$, and let $\sigma\in S_r$ be such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. Then

$$V_{a_1a}(\alpha_1,\beta)=\begin{cases}V_{a_1k_t}(\alpha_1) & \text{if } \beta=(-1)^r sgn(\sigma)\varphi(a,i)\varphi(i,n)\prod_{k=1}^{r}\varphi(i_k,j_k)\\ \varnothing & \text{otherwise}\end{cases}$$

for all $\alpha\in K$. In particular $V_{a_1a}(\alpha_1)=\varnothing$.

Proof. Let $s\in\{1,\ldots,t\}$ and let $\beta\in K$. We claim that the variety $V_{a_1k_s}(\alpha_1,\beta)$ is non-empty. By induction we suppose that the variety $V_{a_1k_{s-1}}(\alpha_1)$ is non-empty. Let $f\in V_{a_1k_{s-1}}(\alpha_1)$ be an element such that $f(e_{jn})=0$ for all $j\in\{1\ldots,n-1\}$. Then the element $f+\gamma e_{k_si}{}^*\in U_n(K)$ lies in the variety $V_{a_1k_{s-1}}(\alpha_1)$ for all $\gamma\in K$. Since we may choose γ such that $\Lambda_{k_{s-1}i}(f)=\beta$ we conclude that $V_{a_1k_s}(\alpha_1,\beta)$ is non-empty. The remaining assertions of the theorem are consequences of the previous lemmas except the statement about $V_{a_1a}(\alpha_1,\beta)$ when $(a,i)\in D$. However this follows from the proof of the lemma 4.2.12 and also from the proof of proposition 3.1.2. In fact

$$\Delta_{j_1\dots j_r}^{i_{\sigma(1)}\dots i_{\sigma(r)}}(x\cdot f)=sgn(\sigma)\prod_{k=1}^{r}\varphi(i_k,j_k)$$

for all $f\in V_D(\varphi)$. ♦

Next we will describe the situation on the whole of the i-th column. As before we assume that $(i,n)\in D$ for some $i\in\{1,\dots,n-1\}$. We suppose also that the set

$$D'(i)=D'\cap\{(a,i);\ 1\le a<i\}$$

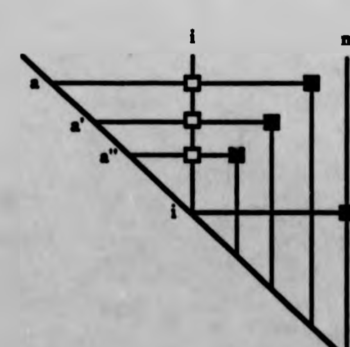
has at least $t\ge 2$ elements and that there exists a sequence $(a_1,a_2,\dots,a_t)$ with the following properties:

(i) $1\le a_1<a_2<\dots<a_t<i$.

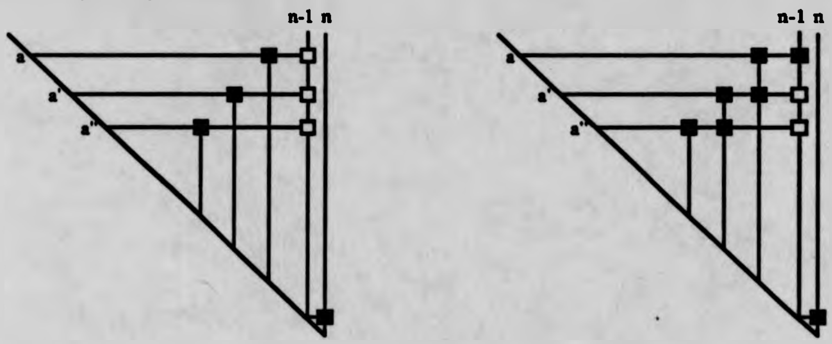
(ii) $(a_s,i)\in D'$ for all $s\in\{1,\dots,t\}$.

(iii) If $b_s\in\{1,\dots,n\}$ is such that $(a_s,b_s)\in D$ $(1\le s\le t)$ then $b_1>b_2>\dots>b_t$.

We note that (by (ii)) we have $b_1<n$, $i<b_t$ and $a_t<a$ whenever $(a,i)\in D$ for some $a\in\{1,\dots,i-1\}$. A sequence satisfying properties (i)-(iii) will be referred to as a *special sequence* (with respect to the basic subset D of $\Phi(n)$ and to the root $(i,n)\in D$).



In the adjacent diagram we show a special sequence with 3 elements. The roots in this sequence are represented by the symbol □ (as usual the symbol ■ represents a root in D). If we apply the element $\omega=(n\text{-}1\ \dots\ i\text{+}1\,i)\in S_n$ to an element corresponding to this diagram we obtain the diagram shown below on the left. The diagram on the right corresponds to the situation in D_ω. In this diagram the symbol ■ indicates that the corresponding root was "transformed" in the root of D_ω which lies in the same column (the roots in D_ω are represented by the symbol ■).



Now let $(a,i)\in D'$ $(1\leq a<i)$ and let $b\in\{i+1,\ldots,n-1\}$ be such that $(a,b)\in D$. Let $c\in\{a+1,\ldots,i-1\}$, let $D(c,b-1)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$, $j_1<\ldots<j_r$, and let $\sigma\in S_r$ be such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. Then we define the polynomial function $\Lambda_{ci}^{(a,i)}:U_n(K)^*\to K$ by

$$\Lambda_{ci}^{(a,i)}(f)=\det\begin{pmatrix}\Theta_{i_{\sigma(1)}i}(f) & f(e_{i_{\sigma(1)}j_1}) & \cdots & f(e_{i_{\sigma(1)}j_r})\\ \vdots & \vdots & & \vdots\\ \Theta_{i_{\sigma(r)}i}(f) & f(e_{i_{\sigma(r)}j_1}) & \cdots & f(e_{i_{\sigma(r)}j_r})\\ \Theta_{ci}(f) & f(e_{cj_1}) & \cdots & f(e_{cj_r})\end{pmatrix} \tag{4.2.6}$$

for all $f\in U_n(K)^*$.

Now we consider a special sequence $(a_1,a_2,\ldots,a_t)$ as above. For simplicity we write $\Lambda_{ci}^{(s)}$ instead of $\Lambda_{ci}^{(a_s,i)}$ for all $(c,i)\in D'$ such that $a_s<c$ $(1\leq s\leq t)$. Let $\alpha_1,\alpha_2,\ldots,\alpha_t$ be non-zero elements of K. Then we define the variety $V_{a_1a_2\ldots a_t}(\alpha_1,\ldots,\alpha_t)$ recursively as follows. If $t=1$ then $V_{a_1}(\alpha_1)$ is defined as before. If $s\in\{2,\ldots,t\}$ then $V_{a_1\ldots a_s}(\alpha_1,\alpha_2,\ldots,\alpha_s)$ is the variety consisting of all $f\in V_{a_1\ldots a_{s-1}}(\alpha_1,\ldots,\alpha_{s-1})$ which satisfy the equations

$$\Lambda_{ci}^{(s)}(f)=0$$

for all $c\in\{a_{s-1}+1,\dots,a_s\}$ such that $(c,i)\in D'$ and $(c,b_s)\in S^{(r)}(D)$, and

$$\Lambda^{(s)}_{a_s i}(f)=\alpha_s.$$

By induction we assume that the varieties $V_{a_1a_2\dots a_s}(\alpha_1,\dots,\alpha_s)$ $(1\le s\le t)$ are $U_n(K)$-invariant.

For each $a\in\{a_t+1,\dots,i-1\}$ we define the subvariety $V_{a_1\dots a_t a}(\alpha_1,\dots,\alpha_t)$ of $V_{a_1\dots a_t}(\alpha_1,\dots,\alpha_t)$ by

$$V_{a_1\dots a_t a}(\alpha_1,\dots,\alpha_t)=\{f\in V_{a_1\dots a_t}(\alpha_1,\dots,\alpha_t);\ \Lambda^{(t)}_{bi}(f)=0,\ a_t<b\le a,\ (c,b_t)\in S^{(r)}(D)\}$$

(as above $\Lambda^{(t)}_{bi}$ simplifies $\Lambda^{(a_t,i)}_{bi}$).

In order to prove the similar version of proposition 4.2.10 we define for each $b\in\{1,\dots,a_t-1\}$ a new polynomial function $\eta_{bi}:\mathcal{U}_n(K)^*\to K$ as follows. We let $D(b,b_t)=\{(i_1,j_1),\dots,(i_r,j_r)\}$, $j_1<\dots<j_r$, and let $\sigma\in S_r$ be such that $i_{\sigma(1)}<\dots<i_{\sigma(r)}$. Then

$$\eta_{bi}(f)=\begin{pmatrix} \Theta_{i_{\sigma(1)}i}(f) & f(e_{i_{\sigma(1)}j_1}) & \cdots & f(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & & \vdots \\ \Theta_{i_{\sigma(r)}i}(f) & f(e_{i_{\sigma(r)}j_1}) & \cdots & f(e_{i_{\sigma(r)}j_r}) \\ \Theta_{bi}(f) & f(e_{bj_1}) & \cdots & f(e_{bj_r}) \end{pmatrix} \tag{4.2.7}$$

for all $f\in\mathcal{U}_n(K)^*$. This function has the following property:

Lemma 4.2.14. *Let $b\in\{1,\dots,a_t-1\}$ and suppose that $(b,b_t)\in R(D)$ $(1\le b<a_t)$. Then $\eta_{bi}(f)=0$ for all $f\in V_{a_1a_2\dots a_t}(\alpha_1,\alpha_2,\dots,\alpha_t)$.*

Proof. We proceed by induction on b. Let $D(b,b_t)=\{(i_1,j_1),\dots,(i_r,j_r)\}$, $j_1<\dots<j_r$, and let $\sigma\in S_r$ be such that $i_{\sigma(1)}<\dots<i_{\sigma(r)}$.

Firstly we suppose that there is no $c\in\{1,\dots,b-1\}$ with $(c,b_t)\in R(D)$. We claim that

$$(d\eta_{bi})(f) \in \bar{r}(f)$$

for all $f \in V_{a_1a_2\ldots a_t}(\alpha_1,\alpha_2,\ldots,\alpha_t)$. For we have (using the argument of the proof of proposition 4.2.10)

$$(d\eta_{bi})(f)=c_{bi}(f)(d\Theta_{bi})(f)+\sum_{u=1}^{r} c_{i_{\sigma(u)}i}(f)(d\Theta_{i_{\sigma(u)}i})(f)+\sum_{v=1}^{r} c_{bj_v}(f)e_{bj_v}+\sum_{u=1}^{r}\sum_{v=1}^{r} c_{i_{\sigma(u)}j_v}(f)e_{i_{\sigma(u)}j_v}$$

where, for each $p \in \{b,i_1,\ldots,i_r\}$ and each $q \in \{i,j_1,\ldots,j_r\}$, $c_{pq}(f) \in K$ is the $r \times r$ minor of $\eta_{ci}(f)$ "complementary" to the position (p,q). Now let $(k,l) \in \Phi(n)$ be arbitrary. If $k \notin \{j_1,\ldots,j_r\}$ and $l \notin \{b,i_1,\ldots,i_r\}$ then

$$f([e_{kl},(d\eta_{bi})(f)])=0.$$

If $k=j_s$ for some $s \in \{1,\ldots,r\}$ then $i_p<b<i<j_s=k<l$ for all $p \in \{1,\ldots,r\}$, so

$$f([e_{j_sl},(d\eta_{bi})(f)])=-\Delta^{i_{\sigma(1)}\cdots i_{\sigma(r)}b}_{ij_1\cdots j_{s-1}lj_{s+1}\cdots j_r}(\vartheta(f))$$

where $\vartheta: \mathcal{U}_n(K)^* \to \mathcal{U}_n(K)^*$ is the morphism defined in the proof of proposition 4.2.10 Since the column

$$(f(e_{i_{\sigma(1)}l}) \;\ldots\; f(e_{i_{\sigma(r)}l})\; f(e_{bl})\;)^{\mathrm{T}}$$

is a linear combination of the columns

$$(f(e_{i_{\sigma(1)}j_u}) \;\ldots\; f(e_{i_{\sigma(r)}j_u})\; f(e_{bj_u})\;)^{\mathrm{T}}$$

for $u \in \{s,\ldots,r\}$ (see the proof of proposition 4.2.10), we conclude that

$$\Delta^{i_{\sigma(1)}\cdots i_{\sigma(r)}b}_{ij_1\cdots j_{s-1}lj_{s+1}\cdots j_r}(\vartheta(f))=0,$$

i.e.

$$f([e_{j_sl},(d\eta_{bi})(f)])=0.$$

Suppose that $k \notin \{j_1,\ldots,j_r\}$ and that $l=b$. Then

$$f([e_{kb},(d\eta_{bi})(f)])=\Delta^{i_{\sigma(1)}\cdots i_{\sigma(r)}k}_{ij_1\cdots j_r}(\vartheta(f)).$$

By choice of b the root (b,b_r) is the smallest D-regular root in the b_r-th column. So $(k,b_r) \in S(D)$ and this implies that $k \in \{i_1,\ldots,i_r\}$. It follows that

$$\Delta^{i_{\sigma(1)}\cdots i_{\sigma(r)}k}_{ij_1\cdots j_r}(\vartheta(f))=0,$$

i.e.

$$f([e_{kb},(d\eta_{bi})(f)])=0.$$

The same argument applies to the case $k=i_s$ $(1 \le s \le r)$. This concludes the proof of our claim.

Now the argument of the proof of lemma 4.2.11 shows that the function $\eta_{bi}:V_{a_1a_2\ldots a_t}(\alpha_1,\alpha_2,\ldots,\alpha_t)\to K$ is $U_n(K)$-invariant.

Finally let $f\in V_{a_1a_2\ldots a_t}(\alpha_1,\alpha_2,\ldots,\alpha_t)$ be arbitrary. Then (as in the proof of lemma 4.2.12) there exists $x\in U_n(K)$ such that the element $x{\cdot}f\in u_n(K)^*$ satisfies

$$(x{\cdot}f)(e_{bj_u})=0$$

for all $u\in\{1,\ldots,r\}$. Since the variety $V_{a_1a_2\ldots a_t}(\alpha_1,\alpha_2,\ldots,\alpha_t)$ is $U_n(K)$-invariant we have $x{\cdot}f\in V_{a_1a_2\ldots a_t}(\alpha_1,\alpha_2,\ldots,\alpha_t)$ and so

$$\eta_{bi}(f)=\eta_{bi}(x{\cdot}f).$$

Since $(x{\cdot}f)(e_{bj_u})=0$ for all $u\in\{1,\ldots,r\}$ we conclude that

$$\eta_{bi}(f)=(-1)^r\Theta_{bi}(f)\Delta_{j_1\ldots j_r}^{i_{\sigma(1)}\ldots i_{\sigma(r)}}(\vartheta(x{\cdot}f)). \tag{4.2.8}$$

We claim that

$$\Theta_{bi}(x{\cdot}f)=0.$$

If $b<a_1$ this follows because $V_{a_1}(\alpha_1)$ is $U_n(K)$-invariant and $f\in V_{a_1}(\alpha_1)$. So suppose that $a_1<b$ and let $u\in\{1,\ldots,t\text{-}1\}$ be such that $a_u<b<a_{u+1}$ (we cannot have $b\in\{a_1,\ldots,a_t\}$ because $(b_t,b)\in R(D)$ and $b_1>\ldots>b_t$). We consider the function $\Lambda_{bi}^{(u)}:u_n(K)^*\to K$. Since $x{\cdot}f\in V_{a_1a_2\ldots a_u}(\alpha_1,\alpha_2,\ldots,\alpha_u)$ [1] we have

$$\Lambda_{bi}^{(u)}(x{\cdot}f)=0.$$

But $(x{\cdot}f)(e_{bj_u})=0$ for all $u\in\{1,\ldots,r\}$ so

$$\Lambda_{bi}^{(u)}(x{\cdot}f)=\Theta_{bi}(f)\mathrm{c}(f)$$

where $\mathrm{c}(f)\in K$ is non-zero. It follows that $\Theta_{bi}(f)=0$ as required. By (4.2.8) we conclude that

$$\eta_{bi}(f)=\eta_{bi}(x{\cdot}f)=0$$

and the first step of the induction is complete.

Now let $(b,b_r)\in R(D)$ $(1\leq b<a_t)$ be arbitrary and assume that

$$\eta_{b'i}(f)=0$$

[1] We note that $V_{a_1a_2\ldots a_u}(\alpha_1,\alpha_2,\ldots,\alpha_u)$ is $U_n(K)$-invariant and $f\in V_{a_1a_2\ldots a_u}(\alpha_1,\alpha_2,\ldots,\alpha_u)$ (because $V_{a_1a_2\ldots a_t}(\alpha_1,\alpha_2,\ldots,\alpha_t)\subseteq V_{a_1a_2\ldots a_u}(\alpha_1,\alpha_2,\ldots,\alpha_u)$).

for all $f \in V_{a_1a_2\ldots a_t}(\alpha_1,\alpha_2,\ldots,\alpha_t)$ and all $b' \in \{1,\ldots,b-1\}$ such that $(b',b_r) \in R(D)$. As before we claim that

$$(d\eta_{bi})(f) \in \tau(f)$$

for all $f \in V_{a_1a_2\ldots a_t}(\alpha_1,\alpha_2,\ldots,\alpha_t)$. Let $(k,l) \in \Phi(n)$ be arbitrary and consider the scalar product $f([e_{kl},(d\eta_{bi})(f)])$. We keep the notation of the first step of the induction. If $k \notin \{j_1,\ldots,j_r\}$ we may repeat the previous argument to conclude that

$$f([e_{kl},(d\eta_{bi})(f)])=0.$$

Now suppose that $k \in \{j_1,\ldots,j_r\}$ and that $l=b$. Then

$$f([e_{kb},(d\eta_{bi})(f)])=\Delta^{i_{\sigma(1)}\ldots i_{\sigma(r)}k}_{ij_1\ldots j_r}(\vartheta(f)).$$

We claim that the row-vector $(\Theta_{ki}(f)\; f(e_{kj_1})\;\ldots\; f(e_{kj_r}))$ is a linear combination of the remaining rows. This is clearly true if either $k \in \{i_1,\ldots,i_r\}$ or $k<i_{\sigma(1)}$. If $k>i_{\sigma(r)}$ then

$$\Delta^{i_{\sigma(1)}\ldots i_{\sigma(r)}k}_{ij_1\ldots j_r}(\vartheta(f))=\eta_{ki}(f)=0$$

(by induction because $k<l=b$ and $(k,b_t) \in R(D)$ - otherwise $k \in \{i_1,\ldots,i_r\}$). On the other hand, suppose that $i_{\sigma(u)}<k<i_{\sigma(u+1)}$ for some $u \in \{1,\ldots,r-1\}$. We consider the matrix

$$A=\begin{pmatrix} \Theta_{i_{\sigma(1)}i}(f) & f(e_{i_{\sigma(1)}j_1}) & \cdots & f(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & & \vdots \\ \Theta_{i_{\sigma(u)}i}(f) & f(e_{i_{\sigma(u)}j_1}) & \cdots & f(e_{i_{\sigma(u)}j_r}) \\ \Theta_{ki}(f) & f(e_{kj_1}) & \cdots & f(e_{kj_r}) \end{pmatrix}.$$

The argument used in the proof of proposition 4.2.10 shows that this matrix has rank $u+1$ if and only if the matrix

$$A'=\begin{pmatrix} \Theta_{i_{\sigma(1)}i}(f) & f(e_{i_{\sigma(1)}j_{\sigma(1)}}) & \cdots & f(e_{i_{\sigma(1)}j_{\sigma(u)}}) \\ \vdots & \vdots & & \vdots \\ \Theta_{i_{\sigma(u)}i}(f) & f(e_{i_{\sigma(u)}j_{\sigma(1)}}) & \cdots & f(e_{i_{\sigma(u)}j_{\sigma(u)}}) \\ \Theta_{ki}(f) & f(e_{kj_{\sigma(1)}}) & \cdots & f(e_{kj_{\sigma(u)}}) \end{pmatrix}$$

is non-singular. Since $(k,b_t) \in R(D)$ (because $k \notin \{i_1,\ldots,i_r\}$) and

$$D(k,b_t)=\{(i_{\sigma(1)}j_{\sigma(1)}),\ldots,(i_{\sigma(u)}j_{\sigma(u)})\}$$

we have

$$det A' = \pm \eta_{ki}(f).$$

Since $\eta_{ki}(f)=0$ (by induction because $k<b$) we conclude that $det A'=0$ and this implies that $rank A=u$. It follows that the row-vector

$$(\Theta_{ki}(f)\; f(e_{kj_1})\; \dots\; f(e_{kj_r}))$$

is a linear combination of the rows

$$(\Theta_{i_{\sigma(v)}i}(f)\; f(e_{i_{\sigma(v)}j_1})\; \dots\; f(e_{i_{\sigma(v)}j_r}))$$

for $v \in \{1,\dots,u\}$ as required. The case $l=j_s$ for some $s \in \{1,\dots,r\}$ is discussed similarly.

Finally we repeat the proof of lemma 4.2.11 to show that the function $\eta_{bi}: V_{a_1a_2\dots a_t}(\alpha_1,\alpha_2,\dots,\alpha_t) \to K$ is $U_n(K)$-invariant. Then for each $f \in V_{a_1a_2\dots a_t}(\alpha_1,\alpha_2,\dots,\alpha_t)$ we choose the element $x \in U_n(K)$ as in the first step of the induction and we imitate that argument to complete the proof. ♦

We are now able to establish the similar version of proposition 4.2.10:

Proposition 4.2.15. *Let $a \in \{a_t+2,\dots,i-1\}$ be such that $(a,b_t) \in S^{(r)}(D)$ and let $f \in V_{a_1a_2\dots a_t a-1}(\alpha_1,\alpha_2,\dots,\alpha_t)$.* Then

$$f([e_{kl},(d\Lambda_{ai}^{(t)})(f)])=0$$

for all $(k,l) \in \Phi(n)$.

Proof. Let $D(a,b_t-1)=\{(i_1,j_1),\dots,(i_r,j_r)\}$, $j_1<\dots<j_r$, and let $\sigma \in S_r$ be such that $i_{\sigma(1)}<\dots<i_{\sigma(r)}$. In this case we have

$$(d\Lambda_{ai}^{(t)})(f)=c_{ai}(f)(d\Theta_{ai})(f)+\sum_{u=1}^{r} c_{i_{\sigma(u)}i}(f)(d\Theta_{i_{\sigma(u)}i})(f)+\sum_{v=1}^{r} c_{aj_v}(f)e_{aj_v}+\sum_{u=1}^{r}\sum_{v=1}^{r} c_{i_{\sigma(u)}j_v}(f)e_{i_{\sigma(u)}j_v}$$

where, for each $p \in \{b,i_1,\dots,i_r\}$ and each $q \in \{i,j_1,\dots,j_r\}$, $c_{pq}(f) \in K$ is the $r \times r$ minor of $\Phi_{ai}^{(t)}(f)$ "complementary" to the position (p,q).

Let $(k,l) \in \Phi(n)$ be arbitrary. We claim that

$$f([e_{kl},(d\Lambda_{ai}^{(t)})(f)])=0.$$

The argument used in the proof of proposition 4.2.10 applies here (with obvious minor changes) except in the case $l\in\{a,i_1,\dots,i_r\}$. Then $k\in\{j_1,\dots,j_r\}$ (because $k<l\le a<i<j_1<\dots<j_r$). Suppose that $l=a$. We have

$$f([e_{ka},(d\Lambda_{ai}^{(t)})(f)])=\Delta_{ij_1\dots j_r}^{i_{\sigma(1)}\dots i_{\sigma(r)}k}(\vartheta(f))$$

where the morphism $\vartheta: U_n(K)^*\to U_n(K)^*$ is as before. We claim that the row-vector $(\Theta_{ki}(f)\,f(e_{kj_1})\dots f(e_{kj_r}))$ is a linear combination of the remaining rows. This is clearly true if either $k\in\{i_1,\dots,i_r\}$ or $k<i_{\sigma(1)}$. If $k>i_{\sigma(r)}\ge a_t$ then

$$\Delta_{ij_1\dots j_r}^{i_{\sigma(1)}\dots i_{\sigma(r)}k}(\vartheta(f))=\Lambda_{ki}^{(t)}(f)=0$$

(because $f\in V_{a_1a_2\dots a_\rho\text{-}1}(\alpha_1,\alpha_2,\dots,\alpha_t)$, $k<l=a$ and $(k,b_t)\in R(D)$ - otherwise $k\in\{i_1,\dots,i_r\}$). Now suppose that $i_{\sigma(u)}<k<i_{\sigma(u+1)}$ for some $u\in\{1,\dots,r-1\}$. As before, we consider the matrix

$$A=\begin{pmatrix} \Theta_{i_{\sigma(1)}i}(f) & f(e_{i_{\sigma(1)}j_1}) & \dots & f(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & & \vdots \\ \Theta_{i_{\sigma(u)}i}(f) & f(e_{i_{\sigma(u)}j_1}) & \dots & f(e_{i_{\sigma(u)}j_r}) \\ \Theta_{ki}(f) & f(e_{kj_1}) & \dots & f(e_{kj_r}) \end{pmatrix}.$$

This matrix has rank $u+1$ if and only if the matrix

$$A'=\begin{pmatrix} \Theta_{i_{\sigma(1)}i}(f) & f(e_{i_{\sigma(1)}j_{\sigma(1)}}) & \dots & f(e_{i_{\sigma(1)}j_{\sigma(u)}}) \\ \vdots & \vdots & & \vdots \\ \Theta_{i_{\sigma(u)}i}(f) & f(e_{i_{\sigma(u)}j_{\sigma(1)}}) & \dots & f(e_{i_{\sigma(u)}j_{\sigma(u)}}) \\ \Theta_{ki}(f) & f(e_{kj_{\sigma(1)}}) & \dots & f(e_{kj_{\sigma(u)}}) \end{pmatrix}$$

is non-singular (the justification of this assertion is the same as in the proof of proposition 4.2.10). If $k>a_t$ then

$$D(k,b_t\text{-}1)=\{(i_{\sigma(1)},j_{\sigma(1)}),\dots,(i_{\sigma(u)},j_{\sigma(u)})\}$$

and we have that

$$\det A'=\pm\Lambda_{ki}^{(t)}(f)$$

(we note that $(k,b_t)\in R(D)$ because $k\in\{i_1,\dots,i_r\}$). Since $f\in V_{a_1a_2\dots a_\rho\text{-}1}(\alpha_1,\alpha_2,\dots,\alpha_t)$ and $k<l=a$ we have $\Lambda_{ki}^{(t)}(f)=0$. Finally suppose that $k<a_t$. Then

$$D(k,b_t)=\{(i_{\sigma(1)},j_{\sigma(1)}),\dots,(i_{\sigma(u)},j_{\sigma(u)})\}$$

and so

$$det\, A'=\eta_{ki}(f)=0$$

(by the previous lemma). It follows that

$$rank A=u$$

and (as before) this implies that

$$f([e_{ka'},(d\Lambda_{ai}^{(t)})(f)])=\Delta_{ij_1\dots j_r}^{i_{\sigma(1)}\dots i_{\sigma(r)}k}(\vartheta(f))=0.$$

The case $l=j_s$ for some $s\in\{1,\dots,r\}$ is discussed similarly. ◆

The next result has the "same" proof as lemma 4.2.11.

Lemma 4.2.16. *Let $a\in\{a_t+2,\dots,i-1\}$ be such that $(a,b_t)\in S^{(r)}(D)$. Then the function $\Lambda_{ai}^{(t)}:V_{a_1a_2\dots a_\rho-1}(\alpha_1,\alpha_2,\dots,\alpha_t)\rightarrow K$ is $U_n(K)$-invariant.*

This lemma implies that

$$V_{a_1a_2\dots a_\rho}(\alpha_1,\alpha_2,\dots,\alpha_t)=V_{a_1a_2\dots a_\rho-1}(\alpha_1,\alpha_2,\dots,\alpha_t)$$

whenever $(a,i)\in R(D)\backslash D$. In fact we have the similar version of lemma 4.2.12:

Lemma 4.2.17. *Let $a\in\{a_t+2,\dots,i-1\}$ be such that $(a,b_t)\in S^{(r)}(D)$ and suppose that $(a,i)\in R(D)$. Let $f\in V_{a_1a_2\dots a_\rho-1}(\alpha_1,\alpha_2,\dots,\alpha_t)$. Then $\Lambda_{ai}^{(t)}(f)\neq 0$ if and only if $(a,i)\in D$.*

Finally we state the generalization of theorem 4.2.13 (its proof is similar to the proof of that theorem).

Theorem 4.2.18. *Let D be a basic subset of $\Phi(n)$ and suppose that $(i,n),(a_1,b_1),\dots,(a_t,b_t)\in D$ are such that $(a_1,\dots,a_t)$ is a special sequence (with respect to D and to (i,n)). Suppose that the set $X=D'\cap\{(k,l);\, a_t<k<i,\, (k,b_t)\in S^{(r)}(D)\}$ is non-empty and let*

$$X=\{(k_1,l),(k_2,l),\dots,(k_r,l)\}$$

where $k_1<k_2<\ldots<k_r$. *Let* $\alpha_1,\ldots,\alpha_t\in K^*$. *Then*

$$V_{a_1\ldots a_tk_s}(\alpha_1,\ldots,\alpha_t)=\{f\in V_{a_1\ldots a_tk_{s-1}}(\alpha_1,\ldots,\alpha_t);\ \Lambda^{(i)}_{k_s}(f)=0\}$$

for all $s\in\{1,\ldots,r\}$ *(here* $V_{a_1\ldots a_tk_0}(\alpha_1,\ldots,\alpha_t)=V_{a_1\ldots a_t}(\alpha_1,\ldots,\alpha_t)$*). Moreover for each* $\beta\in K$ *and each* $s\in\{1,\ldots,r\}$ *the set*

$$V_{a_1\ldots a_tk_s}(\alpha_1,\ldots,\alpha_t,\beta)=\{f\in V_{a_1\ldots a_tk_{s-1}}(\alpha_1,\ldots,\alpha_t);\ \Lambda^{(i)}_{k_s}(f)=\beta\}$$

is a non-empty $U_n(K)$*-invariant subvariety of* $V_{a_1\ldots a_tk_{s-1}}(\alpha_1,\ldots,\alpha_t)$ *and* $V_{a_1\ldots a_tk_{s-1}}(\alpha_1,\ldots,\alpha_t)$ *is the disjoint union*

$$V_{a_1\ldots a_tk_{s-1}}(\alpha_1,\ldots,\alpha_t)=\bigcup_{\beta\in K}V_{a_1\ldots a_tk_s}(\alpha_1,\ldots,\alpha_t,\beta).$$

Finally suppose that there exists $a\in\{a_t+1,\ldots,i-1\}$ *such that* $(a,i)\in D$. *Let* $D(a,b_t-1)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$, $j_1<\ldots<j_r$, *and let* $\sigma\in S_r$ *be such that* $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. *Then*

$$V_{a_1\ldots a_ta}(\alpha_1,\ldots,\alpha_t,\beta)\neq\emptyset\ \Leftrightarrow\ \beta=(-1)^r sgn(\sigma)\varphi(a,i)\varphi(i,n)\prod_{k=1}^{r}\varphi(i_k,j_k).$$

Moreover

$$V_{a_1\ldots a_ta}(\alpha_1,\ldots,\alpha_t,\beta)=V_{a_1\ldots a_t}(\alpha_1,\ldots,\alpha_t)$$

where $\beta=(-1)^r sgn(\sigma)\varphi(a,i)\varphi(i,n)\prod_{k=1}^{r}\varphi(i_k,j_k)$.

Now let $a=(a_1,\ldots,a_t)$ be a special sequence. We define the subset $D^{(a)}$ of $\Phi(n)$ by

$$D^{(a)}=D\cup\{(a_s,i);\ 1\leq s\leq t\}$$

and we consider maps $\varphi^{(a)}:D^{(a)}\to K^*$ which satisfy

$$\varphi^{(a)}(u,v)=\varphi(u,v)$$

for all $(u,v)\in D$. Then we will denote by

$$V_{D^{(a)}}(\varphi^{(a)})$$

the $U_n(K)$-invariant subvariety $V_{a_1\ldots a_t}(\alpha_1,\ldots,\alpha_t)$ of $V_D(\varphi)$ where

$$\alpha_s=\Lambda^{(a_s,i)}_{a_j}(f)\quad(1\leq s\leq t)$$

and $f\in V_D(\varphi)$ is such that $f(e_{jn})=0$ for all $j\in\{i+1,\ldots,n-1\}$ and

$$f(e_{ia_s})=\varphi^{(a)}(i,a_s)\quad(1\leq s\leq t).$$

We have:

Theorem 4.2.19. *Let D be a basic subset of $\Phi(n)$ and let $\varphi: D \to K$ be a map. Suppose that $D'(i)$ is non-empty and let $a=(a_1,\ldots,a_t)$ be a special sequence. Then $V_{D^{(a)}}(\varphi^{(a)})$ is a non-empty $U_n(K)$-invariant proper subvariety of $V_D(\varphi)$.*

Finally theorem 4.2.19 implies the following:

Theorem 4.2.20. *Let D be a basic subset of $\Phi(n)$ and let $\varphi: D \to K$ be a map. Let $f \in V_D(\varphi)$. Then there exists a unique special sequence $a=(a_1,\ldots,a_t)$ and a unique map $\varphi^{(a)}: D^{(a)} \to K^*$ such that $f \in V_{D^{(a)}}(\varphi^{(a)})$.*

CHAPTER 5

A DECOMPOSITION OF THE REGULAR CHARACTER

In this chapter we establish the decomposition of the regular character ρ of $U_n(q)$ as the sum of all basic characters $\xi_D(\varphi)$ (as usual D is a basic subset of $\Phi(n)$ and $\varphi: D \to K^*$ is a map). We prove this result using an inductive argument suggested by the methods of chapter 4. In fact we may define a total order on the set of all basic subsets of $\Phi(n)$ (see next page) and the "action" of a convenient element of S_n allows the definition of a basic subset D_1 of $\Phi(n)$ which is smaller than a given subset D. In section 5.1 we construct this basic subset. Then in section 5.2 we prove the result mentioned above. For this proof we establish a series of lemmas which relate the multiplicity of any irreducible component of $\xi_D(\varphi)$ and the multiplicity of a corresponding irreducible component of $\xi_{D_1}(\varphi_1)$ where $\varphi_1 : D_1 \to \mathbb{F}_q{}^*$ is a map which depends on a given map $\varphi: D \to \mathbb{F}_q{}^*$. Each lemma is concerned with a different case depending on the relative position of the roots in D (and in D_1).

5.1. Weyl conjugation of orbits

In this section we fix a basic subset D of $\Phi(n)$ and a map $\varphi: D \to K^*$ (n and K are as before). Without loss of generality we assume that $(i,n)\in D$ for some $i\in\{1,\dots,n-1\}$. Moreover we assume that $i\le n-2$ [1].

Our aim is to describe an inductive process which allows us to get information about a given $U_n(K)$-orbit $O\subseteq V_D(\varphi)$ once we have some knowledge about the $U_n(K)$-orbits O_1 which are contained in a "smaller" variety $V_{D_1}(\varphi_1)$. We consider the total order $<$ on the set of roots $\Phi(n)$ as defined in the proof of proposition 2.2.13. Then the set of all subsets of $\Phi(n)$ is totally ordered as follows. Let $A=\{(a_1,b_1),\dots,(a_r,b_r)\}$, $(a_1,b_1)<\dots<(a_r,b_r)$, and $B=\{(c_1,d_1),\dots,(c_s,d_s)\}$, $(c_1,d_1)<\dots<(c_s,d_s)$, be arbitrary subsets of $\Phi(n)$. Then we define $A<B$ if one of the following conditions is satisfied:

(i) $r<s$ and $(a_1,b_1)=(c_1,d_1),\dots,(a_r,b_r)=(c_r,d_r)$;

(ii) there exists $t\in\{1,\dots,r\}$ such that $(a_1,b_1)=(c_1,d_1),\dots,(a_{t-1},b_{t-1})=(c_{t-1},d_{t-1})$ and $(a_t,b_t)>(c_t,d_t)$.

Now we consider the simple reflection $\omega=(i\,i{+}1)\in S_n$. Let $O\subseteq V_D(\varphi)$ be any $U_n(K)$-orbit and let $f\in O$ be an element satisfying

$$f(e_{i+1\,n})=0$$

(the existence of this element was justified in the proof of lemma 3.1.1). Then we define the element $f_\omega\in \mathfrak{u}_n(K)^*$ by

$$(5.1.1)\qquad f_\omega(e_{ab})=\begin{cases} f(e_{\omega(a)\omega(b)}) & \text{if } (a,b)\neq(i,i+1)\\ 0 & \text{if } (a,b)=(i,i+1)\end{cases}$$

(cf. section 4.2).By theorem 3.1.7 there exists a unique basic set of roots $D_\omega\subseteq\Phi(n)$ and a

[1] If $i=n-1$ then the variety $V_D(\varphi)$ is the sum of the zero-dimensional orbit $O_{n-1n}(\varphi(n-1,n))$ and the variety $V_{D_0}(\varphi_0)$ where $D_0=D\setminus\{(n-1,n)\}$ and φ_0 is the restriction to D_0 of the map φ. Therefore the $U_n(K)$-orbits on $V_D(\varphi)$ are in one-to-one correspondence with the $U_{n-1}(K)$-orbits on $V_{D_0}(\varphi_0)$.

unique map $\varphi_\omega : D_\omega \to K^*$ such that

$$f_\omega = V_{D_\omega}(\varphi_\omega).$$

In the next lemmas we will determine the set D_ω and the map φ_ω. Since the set D_ω depends on the choice of the element $f \in O$ we will fix this element throughout the section.

The set D_ω is closely related with the set

$$\omega(D) = \{(\omega(a), \omega(b)); (a,b) \in D\}$$

and the map φ_ω with the map $\varphi \circ \omega : \omega(D) \to K^*$ - in fact in many cases $D_\omega = \omega(D)$ and $\varphi_\omega = \varphi \circ \omega$.

Lemma 5.1.1. *Let $(a,b) \in \Phi(n)$ be such that $i+1<b$. Then*

$$(a,b) \in D_\omega \iff \omega(a,b) \in D.$$

Moreover

$$\varphi_\omega(a,b) = \varphi \circ \omega(a,b)$$

whenever $(a,b) \in D_\omega$.

Proof. We proceed by recursion on the set $\{(a,b) \in \Phi(n); i+1<b\}$. We have $f_\omega(e_{an}) = f(e_{an}) = 0$ for all $a \in \{1,\ldots,i-1\}$ and $f_\omega(e_{in}) = f(e_{i+1n}) = 0$ (by the choice of f). On the other hand $f_\omega(e_{i+1n}) = f(e_{in}) = \varphi(i,n) \neq 0$. Hence the lemma is true whenever $b=n$. Now let $(a,b) \in \Phi(n)$ be such that $i+1<b<n$ and assume that the result is proved for all $(a',b') \in \Phi(n)$ with $(a',b') < (a,b)$.

Firstly we claim that

$$(a,b) \in R(D_\omega) \iff (\omega(a),b) \in R(D).$$

For we will prove that

$$(a,b) \in S(D_\omega) \iff (\omega(a),b) \in S(D)$$

(this statement is equivalent to the previous one).

Suppose that $(\omega(a),b) \in S(D)$. If $(\omega(a),b') \in D$ for some $b' \in \{b+1,\ldots,n\}$ then $(\omega(a),b') < (a,b)$ (because $b'>b$). By induction we obtain $(a,b') \in D_\omega$. Thus $(a,b) \in S(D_\omega)$. On the other hand suppose that $(a',b) \in D$ for some $a' \in \{1,\ldots,\omega(a)-1\}$. If $\omega(a')>a=\omega\omega(a)$ then $(a',\omega(a))=(i,i+1)$. Since $(i,n) \in D$ we conclude that $b=n$. This contradiction implies that $\omega(a')<a$. Since $(a',b)\in D$ we obtain (by induction) $(\omega(a'),b)\in D_\omega$. So $(a,b)\in S(D_\omega)$.

Conversely suppose that $(a,b)\in S(D_\omega)$. If $(a,b')\in D$ for some $b'\in\{b+1,\ldots,n\}$ then $(\omega(a),b')\in D$ (by induction) so $(\omega(a),b)\in S(D)$. On the other hand suppose that $(a',b)\in D_\omega$ for some $a'\in\{1,\ldots,a-1\}$. Then (by induction) $(\omega(a'),b)\in D$. Now if $\omega(a')>a=\omega\omega(a)$ then $(a',\omega(a))=(i,i+1)$. Thus $a'=i=a$. This contradiction implies that $\omega(a')<a$ so $(a,b)\in S(D)$. Our claim is proved.

Now we consider the functions $\Delta^{D}_{\omega(a)b}:U_n(K)^*\to K$ and $\Delta^{D_\omega}_{ab}:U_n(K)^*\to K$. If $(a,b)\in D(i,i+1)$ then $\omega(a)=a$. Moreover $D_\omega(a,b)=D(a,b)$ and $f_\omega(e_{rs})=f(e_{rs})$ for all $(r,s)\in D_\omega(a,b)$. Therefore

$$\Delta^{D_\omega}_{ab}(f_\omega)=\Delta^{D}_{ab}(f).$$

It follows that

$$\Delta^{D_\omega}_{ab}(f_\omega)\neq 0 \;\Leftrightarrow\; \Delta^{D}_{ab}(f)\neq 0$$

and this implies that

$$(a,b)\in D_\omega \;\Leftrightarrow\; (a,b)\in D.$$

Moreover (by (3.1.7)) we have

$$\varphi_\omega(a,b)=\varphi(a,b)=\varphi\circ\omega(a,b)$$

for all $(a,b)\in D_\omega$.

Now suppose that $a=i$. By induction we have $D(i+1,b)=D_\omega(i,b)\cup\{(i,n)\}$ and so

$$\Delta^{D}_{i+1b}(f)=-f(e_{in})\Delta^{D_\omega}_{ib}(f_\omega).$$

Since $f(e_{in})\neq 0$ we conclude that

$$\Delta^{D_\omega}_{ib}(f_\omega)\neq 0 \;\Leftrightarrow\; \Delta^{D}_{i+1b}(f)\neq 0,$$

i.e.

$$(i,b)\in D_\omega \;\Leftrightarrow\; (i+1,b)\in D.$$

Suppose that $(i,b)\in D_\omega$ (hence $(i+1,b)\in D$). Then (by (3.1.7)) we have

$$\Delta^{D_\omega}_{ib}(f_\omega)=(-1)^r\varphi_\omega(i,b)c(f_\omega) \quad\text{and}\quad \Delta^{D}_{i+1b}(f)=(-1)^{r+1}\varphi(i+1,b)c(f)$$

where $r=|D_\omega(i,b)|=|D(i+1,b)|-1$ and $c:U_n(K)^*\to K$ is a well-determined polynomial function which satisfies $c(f_\omega)=c(f)$. It follows that

$$\varphi_\omega(i,b)=\varphi(i+1,b)=\varphi\circ\omega(i,b).$$

Finally suppose that $a>i$. Then we must have $a>i+1$ (otherwise $b=n$). In this case

$$D_{\omega}(a,b)=\omega(D(a,b)).$$

If there is no $b'\in\{b+1,\dots,n-1\}$ with $(i,b')\in D_{\omega}$ (hence there is no $b'\in\{b+1,\dots,n-1\}$ with $(i+1,b')\in D$) then

$$\Delta_{ab}^{D_{\omega}}(f_{\omega})=\Delta_{ab}^{D}(f)$$

and (as before) we conclude that

$$(a,b)\in D_{\omega} \iff (a,b)\in D.$$

Moreover

$$\varphi_{\omega}(a,b)=\varphi(a,b)=\varphi\circ\omega(a,b)$$

for all $(a,b)\in D_{\omega}$. On the other hand if $(i,b')\in D_{\omega}$ for some $b'\in\{b+1,\dots,n-1\}$ (hence $(i+1,b')\in D$ for some $b'\in\{b+1,\dots,n-1\}$) then the determinant $\Delta_{ab}^{D_{\omega}}(f_{\omega})$ is obtained from $\Delta_{ab}^{D}(f)$ by permuting the columns corresponding to the roots (i,n) and $(i+1,b)$ of D. Therefore

$$\Delta_{ab}^{D_{\omega}}(f_{\omega})=-\Delta_{ab}^{D}(f).$$

As before, we deduce that

$$(a,b)\in D_{\omega} \iff (a,b)\in D.$$

Suppose that $(a,b)\in D_{\omega}$. Then (by (3.1.7)) we have

$$\Delta_{ab}^{D_{\omega}}(f_{\omega})=(-1)^{r}\varphi_{\omega}(a,b)c(f_{\omega}) \quad\text{and}\quad \Delta_{ab}^{D}(f)=(-1)^{r}\varphi(a,b)c(f)$$

where $r=|D_{\omega}(a,b)|=|D(a,b)|$ and $c\colon U_n(K)^{*}\to K$ is a polynomial function which satisfy $c(f_{\omega})=-c(f)$. It follows

$$\varphi_{\omega}(i,b)=\varphi(i+1,b)=\varphi\circ\omega(i,b).$$

The proof is complete. ♦

Next we consider the roots in the i-th and in the $(i+1)$-th columns. By theorem 4.2.20 there exists a unique special sequence $\alpha=(a_1,\dots,a_t)$ (with respect to D and to (i,n)) and a unique map $\varphi^{(\alpha)}\colon D^{(\alpha)}\to K^{*}$ such that

$$f\in V_{D^{(\alpha)}}(\varphi^{(\alpha)}).$$

For each $i\in\{1,\dots,t\}$ we let $b_i\in\{i+1,\dots,n-1\}$ be such that $(a_i,b_i)\in D$.

Lemma 5.1.2. *Let the notation be as above.*

(i) *If $b_t=i+1$ then $(a_t,i+1)\in D_\omega$ and*

$$\varphi_\omega(a_r,i+1)=\varphi^{(a)}(a_r,i).$$

(ii) *If $b_t\neq i+1$ then, for all $a\in\{1,\ldots,i-1\}$,*

$$(a,i+1)\in D_\omega \iff (a,i)\in D.$$

Moreover if $(a,i+1)\in D_\omega$ we have $a_t<a$ and

$$\varphi_\omega(a,i+1)=\varphi(a,i).$$

Proof. Firstly we suppose that $(b,i+1)\in D$ for all $b\in\{1,\ldots,i-1\}$. Let $a\in\{1,\ldots,i-1\}$ be arbitrary. We claim

$$(a,i+1)\in R(D_\omega) \iff (a,i)\in R(D).$$

For we will prove that

$$(a,i+1)\in S(D_\omega) \iff (a,i)\in S(D).$$

We proceed by induction on a.

Suppose that $(1,i+1)\in S(D_\omega)$. Then $(1,c)\in D_\omega$ for some $c\in\{i+2,\ldots,n\}$. By the previous lemma we have $(1,b)\in D$. Since $b>i+1>i$, we conclude that $(1,i+1)\in S(D)$. Conversely suppose that $(1,i+1)\in S(D)$. Then $(1,c)\in D$ for some $c\in\{i+1,\ldots,n\}$. Since $(1,i+1)\in D$ (by our hypothesis), we have $c>i+1$. By the previous lemma we have $(1,c)\in D_\omega$ and so $(1,i+1)\in S(D_\omega)$.

Now we assume that $a>1$ and that the claim is proved for all $a'\in\{1,\ldots,a-1\}$. Suppose that $(a,i+1)\in S(D_\omega)$. If $(a,c)\in D_\omega$ for some $c\in\{i+2,\ldots,n\}$ then we repeat the argument above to conclude that $(a,i)\in S(D)$. On the other hand suppose that $(a',i+1)\in D_\omega$ for some $a'\in\{1,\ldots,a-1\}$. Then $(a',i)\in R(D)$ (by induction and by the previous lemma) we have

$$\Delta^D_{a'i}(f)=\Delta^{D_\omega}_{a'i+1}(f_\omega)$$

(because there is no $b\in\{1,\ldots,i-1\}$ with $(b,i+1)\in D$). Since $(a',i+1)\in D_\omega$ we have

$$\Delta^D_{a'i}(f)=\Delta^{D_\omega}_{a'i+1}(f_\omega)\neq 0$$

Therefore $(a',i)\in D$ and this implies that $(a,i)\in S(D)$.

Conversely suppose that $(a,i)\in S(D)$. If $(a,c)\in D$ for some $c\in\{i+1,\dots,n\}$ we must have $c>i+2$ (by our hypothesis). Hence (by the previous lemma) $(a,c)\in D_\omega$, so $(a,i+1)\in S(D_\omega)$. On the other hand suppose that $(a',i)\in D$ for some $a'\in\{1,\dots,a-1\}$. Then $(a',i+1)\in R(D_\omega)$ (by induction) and we have

$$\Delta^{D}_{a'i}(f)=\Delta^{D_\omega}_{a'i+1}(f_\omega).$$

Now an imitation of the argument used in the previous paragraph justifies that $(a',i+1)\in D_\omega$ and this implies that $(a,i+1)\in S(D_\omega)$. This completes the proof of the claim.

Now we may use the equality

$$\Delta^{D}_{ai}(f)=\Delta^{D_\omega}_{ai+1}(f_\omega)$$

to conclude that

$$(a,i+1)\in D_\omega \iff (a,i)\in D.$$

Moreover (by (3.1.7)) we obtain

$$\varphi_\omega(a,i+1)=\varphi(a,i)$$

whenever $(a,i+1)\in D_\omega$. The assertion $a_t<a$ follows from the definition of special sequence.

Now we suppose that there exists $b\in\{1,\dots,i-1\}$ such that $(b,i+1)\in D$. Then we have $b=a_t$ if and only if $b_t=i+1$. Let $a\in\{1,\dots,b-1\}$. Then the argument of the previous case can be repeated to prove that

$$(a,i+1)\in R(D_\omega) \iff (a,i)\in R(D).$$

Moreover we have

$$\Delta^{D}_{ai}(f)=\Delta^{D_\omega}_{ai+1}(f_\omega)$$

and so

$$(a,i+1)\in D_\omega \iff (a,i)\in D.$$

If this is the case then $a_t<a$ (by definition of special sequence) and

$$\varphi_\omega(a,i+1)=\varphi(a,i)$$

(by (3.1.7)).

On the other hand suppose that $(a,i+1)\in D_\omega$ (hence $(a,i)\in D$) for all $a\in\{1,\dots,b-1\}$. We claim that $(b,i+1)\in R(D_\omega)$. In fact if $(b,i+1)\in S(D_\omega)$ then $(b,c)\in D_\omega$ for some $c\in\{i+2,\dots,n-1\}$. By lemma 5.1.1 we have $(b,c)\in D$ and this is in contradiction with

$(b,i+1)\in D$. Now let $D_\omega(b,i+1)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$, $j_1<\ldots<j_r$ and let $\sigma\in S_r$ be such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. Then

$$\Delta^{D_\omega}_{bi+1}(f_\omega)=\Delta^{i_{\sigma(1)}\cdots i_{\sigma(r)}b}_{ij_1\cdots j_r}(f).$$

Since $V_D(\varphi)=O_D(\varphi)$ (theorem 3.1.7) we have

$$f=g+h$$

where $g\in O_{bi+1}(\alpha)$, $\alpha=\varphi(b,i+1)$, $h\in V_{D_0}(\varphi_0)$, $D_0=D\backslash\{(b,i+1)\}$ and φ_0 is the restriction of φ to D_0. Therefore we get

$$(5.1.2)\qquad \Delta^{D_\omega}_{bi+1}(f_\omega)=\Delta^{D_0}_{bi}(\mathrm{h})+det\begin{pmatrix} f(e_{i_{\sigma(1)}i}) & f(e_{i_{\sigma(1)}j_1}) & \cdots & f(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & & \vdots \\ f(e_{i_{\sigma(r)}i}) & f(e_{i_{\sigma(r)}j_1}) & \cdots & f(e_{i_{\sigma(r)}j_r}) \\ g(e_{bi}) & 0 & \cdots & 0 \end{pmatrix}.$$

Since $(b,i)\in R(D_0)\backslash D_0$ we have

$$\Delta^{D_0}_{bi}(\mathrm{h})=0.$$

On the other hand

$$\Delta^{i_{\sigma(1)}\cdots i_{\sigma(r)}}_{j_1\cdots j_r}(f)\neq 0$$

(see the proof of proposition 3.1.2). Thus

$$\Delta^{D_\omega}_{bi+1}(f_\omega)\neq 0 \iff g(e_{bi})\neq 0,$$

and (by (3.1.7)) this means that

$$(b,i+1)\in D_\omega \iff g(e_{bi})\neq 0.$$

Now we claim that

$$(b,i+1)\in D_\omega \iff b=a_t.$$

Firstly we assume that $b\neq a_t$. Then $b_t\neq i+1$. To prove that $(b,i+1)\notin D_\omega$ we must show that $g(e_{bi})=0$. For we consider the function $\Theta_{bi}:\mathcal{U}_n(K)^*\to K$. We have

$$\begin{aligned}\Theta_{bi}(f)&=\sum_{c=i}^{n-1}f(e_{bc})f(e_{cn})\\ &=g(e_{bi})f(e_{in})+h(e_{bi})f(e_{in})+\sum_{c=i+1}^{n-1}f(e_{bc})f(e_{cn})\\ &=g(e_{bi})f(e_{in})+\Theta_{bi}(h).\end{aligned}$$

If $b<a_1$ then

$$\Theta_{bi}(f)=0$$

(we recall that $a=(a_1,\ldots,a_t)$). On the other hand the sequence $a=(a_1,\ldots,a_t)$ is special with respect to D_0 (and to (i,n)). Moreover it is clear that

$$h\in V_{D_0^{(a)}}(\varphi_0^{(a)})$$

where $\varphi_0^{(a)}$ is the restriction of $\varphi^{(a)}$ to $D_0^{(a)}$ (we note that $D_0^{(a)}=D^{(a)}\setminus\{(b,i+1)\}$). Since $(b,i)\in R(D_0)$ we have

$$\Theta_{bi}(h)=0$$

(by lemma 4.2.6). Since $f(e_{in})\neq 0$ we conclude that

$$g(e_{bi})=0.$$

Now we suppose that $a_s<b<a_{s+1}$ for some $s\in\{1,\ldots,t\}$ (if $s=t$ we put $a_{t+1}=a$, if $(a,i)\in D$ for some $a\in\{a_t+1,\ldots,i-1\}$ and $a_{t+1}=i$, otherwise). We note that

$$b\notin\{a_1,\ldots,a_t\}$$

because $(b,i+1)\in D$ and $b\neq a_t$. We consider the function $\Lambda_{bi}^{(s-1)}:U_n(K)^*\rightarrow K$ (the notation is as in section 4.2, pg. 172). Let $D_\omega(b,b_s-1)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$, $j_1<\ldots<j_r$, and let $\sigma\in S_r$ be such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. Then

$$\Lambda_{bi}^{(s-1)}(f)=\Lambda_{bi}^{(s-1)}(h)+f(e_{in})\det\begin{pmatrix}\Theta_{i_{\sigma(1)}i}(f) & f(e_{i_{\sigma(1)}j_1}) & \ldots & f(e_{i_{\sigma(1)}j_r})\\ \vdots & \vdots & & \vdots\\ \Theta_{i_{\sigma(r)}i}(f) & f(e_{i_{\sigma(r)}j_1}) & \ldots & f(e_{i_{\sigma(1)}j_r})\\ g(e_{bi}) & 0 & \ldots & 0\end{pmatrix}. \tag{5.1.3}$$

As above

$$h\in V_{D_0^{(a)}}(\varphi_0^{(a)}).$$

Since $(b,i)\in R(D_0)$ we have

$$\Lambda_{bi}^{(s-1)}(h)=0$$

(by lemma 4.2.12). Since $\Lambda_{bi}^{(s-1)}(f)=0$ (because $b\neq a_t$), $f(e_{in})\neq 0$ and $\Delta^{i_{\sigma(1)}\ldots i_{\sigma(r)}}_{j_1\ldots j_r}(f)\neq 0$ we conclude that

$$g(e_{bi})=0.$$

We have proved that

$$(b,i+1)\in D_\omega \Rightarrow b=a_t.$$

Conversely suppose that $b=a_t$. Then we repeat the above argument to conclude that

$$\Phi_{a,i}^{(t-1)}(f)=(-1)^r f(e_{in})g(e_{a_ti})\Delta^{i_{\sigma(1)}\ldots i_{\sigma(r)}}_{j_1\ldots j_r}(f)$$

where $D_\omega(a_t,b_{t-1}-1)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$, $j_1<\ldots<j_r$, and $\sigma\in S_r$ is such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. Since $\Lambda_{a,i}^{(s-1)}(f)\neq 0$ we deduce that $g(e_{a_ti})\neq 0$, so $(a_t,i+1)\in D_\omega$. This completes the proof of our claim, i.e.

$$(b,i+1)\in D_\omega \iff b=a_t.$$

Now suppose that $b=a_t$. Then $(a_t,i+1)\in D_\omega$. By (5.1.2) we have $\varphi_\omega(a_t,i+1)=g(e_{a_ti})$ and by (5.1.3) $g(e_{a_ti})=\varphi^{(a)}(a_t,i)$. So

$$\varphi_\omega(a_t,i+1)=\varphi^{(a)}(a_t,i)$$

and part (i) of the lemma is proved.

Finally we suppose that $b\neq a_t$ (we note that by assumption $(b,i+1)\in D$). We consider the roots $(a,i+1)$ for an arbitrary $a\in\{b+1,\ldots,i-1\}$. An inductive argument (analogous to the one used in the first part of the proof) shows that

$$(a,i+1)\in R(D_\omega) \iff (a,i)\in R(D_0).$$

On the other hand we have

$$g(e_{ai})=\alpha^{-1}g(e_{bi})g(e_{ai+1})$$

(see lemma 3.1.1) Since $g(e_{bi})=0$ we deduce that $f(e_{ai})=h(e_{ai})+g(e_{ai})=h(e_{ai})$. Moreover (by (5.1.2))

$$\Delta^{D_\omega}_{ai+1}(f_\omega)=\Delta^{D_0}_{ai}(h).$$

This implies that

$$(a,i+1)\in D_\omega \iff (a,i)\in D_0.$$

Since $D_0=D\setminus\{(b,i+1)\}$ we conclude that

$$(a,i+1)\in D_\omega \iff (a,i)\in D.$$

Finally if $(a,i+1)\in D_\omega$ (hence $(a,i)\in D$) we must have $a>a_t$ and

$$\varphi_\omega(a,i+1)=\varphi_0(a,i)=\varphi(i,a).$$

The proof of the lemma is complete. ◆

Remark 5.1.3. Let the notation be as above and suppose that $(i,i+1)\in R(D_\omega)$. Then $(i,i+1)\in D_\omega$ if and only if $\Delta^{D_\omega}_{ii+1}(f_\omega)\neq 0$. If this is the case then there exists a unique $\alpha\in K^*$ such that

$$\Delta^{D_\omega}_{ii+1}(f_\omega+\alpha e_{ii+1}{}^*)=0$$

(in fact $\alpha=-\varphi(i,i+1)$). Moreover we have

$$f_\omega+\alpha e_{ii+1}{}^*\in V_{D_\omega\setminus\{(i,i+1)\}}((\varphi_\omega)_0)$$

where $(\varphi_\omega)_0$ is the restriction of φ_ω to $D_\omega\setminus\{(i,i+1)\}$.

In the next result we consider the roots in the i-th column.

Lemma 5.1.4. *Let the notation be as above.*

(i) *If $b_t=i+1$ then, for all $a\in\{1,\dots,i-1\}$,*

$$(a,i)\in D_\omega \iff (a,i)\in D.$$

If $(a,i)\in D_\omega$ then $a>a_t$ and

$$\varphi_\omega(a,i)=-\varphi^{(a)}(a_t,i)^{-1}\varphi(a,i)\varphi(a_t,i+1).$$

(ii) If $b_t\neq i+1$ then, for all $a\in\{1,\dots,i-1\}$,

$$(a,i)\in D_\omega \iff (a,i+1)\in D.$$

If $(a,i)\in D_\omega$ then

$$\varphi_\omega(a,i)=\varphi(a,i+1).$$

Proof. Suppose that $b_t=i+1$.

We claim that $(a,i)\in D_\omega$ for all $a\in\{1,\dots,a_t-1\}$. In fact let $a\in\{1,\dots,a_t-1\}$ and suppose that $(a,i)\in R(D_\omega)$. Since $a<a_t$ and $(a_t,i+1)\in D$ there is no $a'\in\{1,\dots,a-1\}$ with $(a',i+1)\in D$. On the other hand since $(a,i)\in R(D_\omega)$ there is no $b'\in\{i+1,\dots,n\}$ with $(a,b')\in D$ (otherwise $(a,b')\in D_\omega$ - by lemma 5.1.1). Therefore $(a,i+1)\in R(D)$. Since $(a_t,i+1)\in D_\omega$ (by the previous lemma) we have $(a',i+1)\in D_\omega$ for all $a'\in\{1,\dots,a-1\}$. Thus

$$\Delta^{D_\omega}_{ai}(f_\omega)=\Delta^{D}_{ai+1}(f).$$

Since $(a,i+1)\in D$ (because $a<a_t$) we obtain

$$\Delta^{D_\omega}_{ai}(f_\omega)=\Delta^{D}_{ai+1}(f)=0.$$

and this implies that $(a,i)\in D_\omega$ as required. On the other hand we have $(a_t,i)\in D_\omega$ because $(a_t,i+1)\in D_\omega$ (by the previous lemma).

Finally let $a\in\{a_t+1,\dots,i-1\}$ and suppose that $(a,i)\in D_\omega$. We claim that $(a,i)\in R(D)$. In fact suppose that $(a,i)\in S(D)$. If $(a,b)\in D$ for some $b\in\{i+1,\dots,n\}$ then $b>i+1$ (because $a>a_t$ and $(a_t,i+1)\in D$) and $(a,b)\in D_\omega$ (by lemma 5.1.1). This is in contradiction with $(a,i)\in D_\omega$. On the other hand suppose that there exists $a'\in\{1,\dots,a-1\}$ such that $(a',i)\in D$. Then $a_t<a'$ (by definition of special sequence). If $(a',i)\in S(D_\omega)$ then there exists $b'\in\{i+1,\dots,n\}$ such that $(a',b')\in D_\omega$ (we note that $(a,i)\in D_\omega$). Since $(a_t,i+1)\in D_\omega$ (by the previous lemma) we have $b'>i+1$, so $(a',b')\in D$ (by lemma 5.1.1). This contradicts the assumption $(a',i)\in D$. Therefore $(a',i)\in R(D_\omega)$. Now the determinant $\Delta^{D_\omega}_{a'i}(f_\omega)$ is obtained from the determinant $\Delta^{D}_{a'i}(f)$ by permuting the columns corresponding to the roots (a',i) and $(a_t,i+1)$. It follows that

$$\Delta^{D_\omega}_{a'i}(f_\omega)=-\Delta^{D}_{a'i}(f).$$

Since $(a',i)\in D$ we obtain

$$\Delta^{D_\omega}_{a'i}(f_\omega)=-\Delta^{D}_{a'i}(f)\neq 0.$$

This implies that $(a',i)\in D_\omega$ which is in contradiction with $(a,i)\in D_\omega$. It follows that $(a,i)\in R(D)$ as required.

Now we have (as above)

$$\Delta^{D_\omega}_{ai}(f_\omega)=-\Delta^{D}_{ai}(f).$$

Since $(a,i)\in D_\omega$ we conclude that

$$\Delta^{D}_{ai}(f)\neq 0$$

and so $(a,i)\in D$. Moreover (using (3.1.7)) we deduce that

$$\varphi_\omega(a,i)\varphi_\omega(a_r,i+1)=-\varphi(a,i)\varphi(a_r,i+1).$$

By the previous lemma we have

$$\varphi_\omega(a_r,i+1)=\varphi^{(\alpha)}(a_r,i).$$

Thus

$$\varphi_{\omega}(a,i)=-\varphi^{(\bullet)}(a_r,i)^{-1}\varphi(a,i)\varphi(a_r,i+1).$$

To conclude the proof of part (i) it remains to show that for any $a\in\{1,\dots,i-1\}$ $(a,i)\in D \Rightarrow (a,i)\in D_{\omega}$. For suppose that $(a,i)\in D$. As before we claim that $(a,i)\in R(D_{\omega})$. In fact if $(a,b)\in D_{\omega}$ for some $b\in\{i+1,\dots,n\}$ then $b>i+2$ (because $(a_r,i+1)\in D_{\omega}$) and (by lemma 5.1.1) $(a,b)\in D$ which contradicts the assumption $(a,i)\in D$. On the other hand, if $(a',i)\in D_{\omega}$ for some $a'\in\{1,\dots,a-1\}$ then $(a',i)\in D$ (by the first part of the proof) and this is also in contradiction with $(a,i)\in D$. It follows that $(a,i)\in R(D_{\omega})$ as claimed. Finally we use the equality

$$\Delta_{ai}^{D_{\omega}}(f_{\omega})=-\Delta_{ai}^{D}(f)$$

(see above) to conclude that $(a,i)\in D_{\omega}$.

Now we assume that $b_i\neq i+1$ and we let $a\in\{1,\dots,i-1\}$. We have two distinct cases: either $(b,i+1)\in D_{\omega}$ for all $b\in\{1,\dots,a-1\}$ or $(b,i+1)\in D_{\omega}$ for some $b\in\{1,\dots,a-1\}$.

Firstly we assume that $(b,i+1)\in D_{\omega}$ for all $b\in\{1,\dots,a-1\}$. Suppose that $(a,i)\in D_{\omega}$. As in the previous cases we claim that $(a,i+1)\in R(D)$. For suppose that $(a,i+1)\in S(D)$. If $(a,b')\in D$ for some $b'\in\{i+2,\dots,n\}$ then $(a,b')\in D_{\omega}$ (by lemma 5.1.1) which is in contradiction with $(a,i)\in D_{\omega}$. On the other hand suppose that $(a',i+1)\in D$ for some $a'\in\{1,\dots,a-1\}$. If $(a',i)\in S(D_{\omega})$ then there exists $b'\in\{i+1,\dots,n\}$ such that $(a',b')\in D_{\omega}$. By our hypothesis $b'>i+2$, so $(a',b')\in D$ (by lemma 5.1.1) and this is in contradiction with $(a',i+1)\in D$. It follows that $(a',i)\in R(D_{\omega})$. Now we have

$$\Delta_{a'i}^{D_{\omega}}(f_{\omega})=\Delta_{a'i+1}^{D}(f).$$

Since $(a',i+1)\in D$ we obtain

$$\Delta_{a'i}^{D_{\omega}}(f_{\omega})=\Delta_{a'i+1}^{D}(f)\neq 0$$

and this implies that $(a',i)\in D_{\omega}$ which is in contradiction with $(a,i)\in D_{\omega}$. It follows that $(a,i+1)\in R(D)$.

Finally we have

$$\Delta_{ai}^{D_{\omega}}(f_{\omega})=\Delta_{ai+1}^{D}(f).$$

Since $(a,i)\in D_{\omega}$ we obtain

$$\Delta^{D}_{ai+1}(f)=\Delta^{D_\omega}_{ai}(f_\omega)\neq 0$$

and so $(a,i+1)\in D$. Moreover (3.1.7) implies that

$$\varphi_\omega(a,i)=\varphi(a,i+1).$$

To complete the proof in this case suppose that $(a,i+1)\in D$ and suppose also that $(a,i)\in S(D_\omega)$. If $(a,b')\in D_\omega$ for some $b'\in\{i+1,\dots,n\}$ then $b'>i+1$ (by hypothesis) and $(a,b')\in D$ (by lemma 5.1.1) which contradicts the assumption $(a,i+1)\in D$. On the other hand if $(a',i)\in D_\omega$ for some $a'\in\{1,\dots,a-1\}$ then $(a',i+1)\in$ D (by the reverse implication) and this is also in contradiction with $(a,i+1)\in D$. Thus $(a,i)\in R(D_\omega)$. Now the equality

$$\Delta^{D_\omega}_{ai}(f_\omega)=\Delta^{D}_{ai+1}(f)$$

implies that $(a,i)\in D_\omega$ and the proof is complete in this case.

On the other hand suppose that there exists $b\in\{1,\dots,a-1\}$ such that $(b,i+1)\in D_\omega$. Suppose that $(a,i)\in D_\omega$. As before we have $(a,i+1)\in S(D)$ if and only if $(a',i+1)\in D$ for some $a'\in\{1,\dots,a-1\}$. Suppose that this is so. Then

$$(a',i)\in S(D_\omega)\iff(a',i+1)\in D_\omega.$$

In fact if $(a',b')\in D_\omega$ for some $b'\in\{i+2,\dots,n\}$ then $(a',b')\in D$ (by lemma 5.1.1) and this is impossible because $(a',i+1)\in D$. By lemma 5.1.2 we have

$$(a',i+1)\in D_\omega\iff(a',i)\in D.$$

Since $(a',i+1)\in D$ we conclude that $(a',i)\in R(D_\omega)$. If $a'<b$ then

$$\Delta^{D_\omega}_{a'i}(f_\omega)=\Delta^{D}_{a'i+1}(f)$$

and we deduce that $(a',i)\in D_\omega$ because $(a',i+1)\in D$. This is in contradiction with $(a,i)\in D_\omega$ Therefore $b<a'$. Since $(b,i+1)\in D_\omega$, we have $(b,i)\in D$ (by lemma 5.1.2) so

$$\Delta^{D_\omega}_{a'i}(f_\omega)=-\det\begin{pmatrix} f(e_{i_{\sigma(1)}i}) & f(e_{i_{\sigma(1)}i+1}) & f(e_{i_{\sigma(1)}j_2}) & \dots & f(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & \vdots & & \vdots \\ f(e_{i_{\sigma(r)}i}) & f(e_{i_{\sigma(r)}i+1}) & f(e_{i_{\sigma(r)}j_2}) & \dots & f(e_{i_{\sigma(r)}j_r}) \\ f(e_{a'i}) & f(e_{a'i+1}) & f(e_{a'j_2}) & \dots & f(e_{a'j_r}) \end{pmatrix}$$

where $i_1=b$, $D(a',i+1)=D_\omega(a',i+1)=\{(i_2,j_2),\dots,(i_r,j_r)\}$ and $\sigma\in S_r$ is such that $i_{\sigma(1)}<\dots<i_{\sigma(r)}$. Since $(b,i),(a',i+1)\in D$ we conclude that

$$\Delta^{D_\omega}_{a'i}(f_\omega)\neq 0$$

(see the proof of proposition 3.1.2). It follows that $(a',i)\in D_\omega$ which is in contradiction with $(a,i)\in D_\omega$. This contradiction implies that $(a,i+1)\in R(D)$.

Finally to conclude that $(a,i+1)\in D$ we use the equality

$$(5.1.4)\qquad \Delta^{D_\omega}_{ai}(f_\omega)=-\det\begin{pmatrix} f(e_{i_{\sigma(1)}i}) & f(e_{i_{\sigma(1)}i+1}) & f(e_{i_{\sigma(1)}j_2}) & \cdots & f(e_{i_{\sigma(1)}j_r}) \\ \vdots & \vdots & \vdots & & \vdots \\ f(e_{i_{\sigma(r)}i}) & f(e_{i_{\sigma(r)}i+1}) & f(e_{i_{\sigma(r)}j_2}) & \cdots & f(e_{i_{\sigma(r)}j_r}) \\ f(e_{ai}) & f(e_{ai+1}) & f(e_{aj_2}) & \cdots & f(e_{aj_r}) \end{pmatrix}$$

where $i_1=b$, $D(a,i+1)=D_\omega(a,i+1)=\{(i_2,j_2),\ldots,(i_r,j_r)\}$ and $\sigma\in S_r$ is such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. If $(a,i+1)\in D$ the second column of this determinant is a linear combination of the remaining columns (see the proof of proposition 4.2.10) so

$$\Delta^{D_\omega}_{ai}(f_\omega)=0$$

which is in contradiction with $(a,i)\in D_\omega$. Thus $(a,i+1)\in D$.

Now the determinant of the rigth hand side of the equality (5.1.4) is equal to

$$sgn(\sigma\omega)\varphi(b,i)\varphi(a,i+1)\varphi(i_2,j_2)\ldots\varphi(i_r,j_r)$$

(see the proof of proposition 3.1.2) whereas

$$\Delta^{D_\omega}_{ai}(f_\omega)=sgn(\sigma)\varphi_\omega(a,i)\varphi_\omega(b,i+1)\varphi(i_2,j_2)\ldots\varphi(i_r,j_r)$$

(by (3.1.7) and by lemma 5.1.1). Since $sgn(\sigma)=-sgn(\sigma\omega)$ we deduce that

$$\varphi_\omega(a,i)\varphi_\omega(b,i+1)=\varphi(b,i)\varphi(a,i+1).$$

Since $\varphi_\omega(b,i+1)=\varphi(b,i)$ (by lemma 5.1.2) we obtain

$$\varphi_\omega(a,i)=\varphi(a,i+1).$$

To finish the proof we must justify that

$$(a,i+1)\in D \;\Rightarrow\; (a,i)\in D_\omega$$

In fact suppose that $(a,i+1)\in D$ and that $(a,i)\in S(D_\omega)$. If $(a,b')\in D_\omega$ for some $b'\in\{i+2,\ldots,n\}$ then $(a,b')\in D$ (by lemma 5.1.1) which contradicts $(a,i+1)\in D$. If $(a,i+1)\in D_\omega$ then $(a,i)\in D$ (by lemma 5.1.2) which is also in contradiction with $(a,i+1)\in D$. Finally if $(a',i)\in D_\omega$ for some $a'\in\{1,\ldots,a-1\}$ then $(a',i)\in D$ (by the reverse implication) and this is again in contradiction with $(a,i+1)\in D$. It follows that $(a,i)\in R(D_\omega)$. Now we use the equality (5.1.4) to conclude that $(a,i)\in D_\omega$ as required.

The proof of the lemma is complete. ♦

Finally we consider roots $(a,b)\in \Phi(n)$ with $b<i$.

Lemma 5.1.5. *Let the notation be as before and let $(a,b)\in \Phi(n)$ be such that $b<i$. Then*

$$(a,b)\in D_\omega \iff (a,b)\in D.$$

If $(a,b)\in D_\omega$ then

$$\varphi_\omega(a,b)=\varphi(a,b).$$

Proof. We proceed by induction on the set $\{(a,b)\in \Phi(n);\, b<i\}$. The smallest root in this set is $(1,i-1)$. Suppose that $(1,i-1)\in S(D_\omega)$. Then there exists $b'\in \{i,\ldots,n\}$ such that $(1,b')\in D_\omega$. If $b'>i+1$ then $(1,b')\in D$ (by lemma 5.1.2) so $(1,i-1)\in S(D)$. If $b'=i+1$ then (by lemma 5.1.2) either $(1,i)\in D$ (if $b_t=i+1$) or $(1,i+1)\in D$ (if $b_t\neq i+1$). In both cases we conclude that $(1,i-1)\in S(D)$. Finally if $b'=i$ then (by lemma 5.1.3) either $(1,i+1)\in D$ (if $b_t\neq i+1$) or $(1,i)\in D$ (if $b_t=i+1$). In both cases we also conclude that $(1,i-1)\in S(D)$. The implication $(1,i-1)\in S(D) \Rightarrow (1,i-1)\in S(D_\omega)$ is proved similarly. It follows that

$$(1,i-1)\in R(D_\omega) \iff (1,i-1)\in R(D).$$

Now we have $(1,i-1)\in D_\omega$ if and only if $f_\omega(e_{1i})=\ldots=f(e_{1n})=0$ and $f_\omega(e_{1i-1})\neq 0$. $f_\omega(e_{1b})=f(e_{1\alpha(b)})$ for all $b\in\{2,\ldots,n\}$ we conclude that

$$(1,i-1)\in D_\omega \iff (1,i-1)\in D.$$

Moreover

$$\varphi_\omega(1,i-1)=f_\omega(e_{1i-1})=f(e_{1i-1})=\varphi(1,i-1)$$

and the lemma is proved for the root $(1,i-1)$.

Suppose that $(a,b)>(1,i-1)$ and that the lemma is proved for all $(a',b')\in \Phi(n)$ such that $(1,i-1)\leq(a',b')<(a,b)$. A similar argument to the one used above shows that

$$(a,b)\in S(D_\omega) \iff (a,b)\in S(D)$$

(we note that (by induction) for all $a'\in\{1,\ldots,a-1\}$ $(a',b)\in D_\omega$ if and only if $(a',b)\in D$). It follows that

$$(a,b)\in R(D_\omega) \Leftrightarrow (a,b)\in R(D).$$

Now we assume that $b_t\neq i+1$.

On the one hand suppose that there exist $a',b'\in\{1,\ldots,a-1\}$ such that (a',i) and $(b',i+1)$ lie in $D_\omega(a,b)$. Then $(a',i+1),(b',i)\in D$ (by lemmas 5.1.2 and 5.1.3) and we have

$$\Delta_{ab}^{D_\omega}(f_\omega)=-\Delta_{ab}^{D}(f)$$

because the determinant $\Delta_{ab}^{D_\omega}(f_\omega)$ is obtained from $\Delta_{ab}^{D}(f)$ by permuting the columns which correspond to the roots $(a',i+1),(b',i)\in D$. This implies that

$$(a,b)\in D_\omega \Leftrightarrow (a,b)\in D.$$

By (3.1.7) (and by the previous lemmas) we conclude that

$$\varphi_\omega(a,b)=\varphi(a,b)$$

whenever $(a,b)\in D_\omega$.

On the other hand suppose that at least one of the following cases occurs:

(i) $(a',i)\in D_\omega$ for all $a'\in\{1,\ldots,a-1\}$ - hence (by lemma 5.1.4) $(a',i+1)\in D$ for all $a'\in\{1,\ldots,a-1\}$;

(ii) $(b',i+1)\in D_\omega$ for all $b'\in\{1,\ldots,a-1\}$ - hence (by lemma 5.1.2) $(b',i)\in D$ for all $b'\in\{1,\ldots,a-1\}$.

Then we have (in both cases)

$$\Delta_{ab}^{D_\omega}(f_\omega)=\Delta_{ab}^{D}(f)$$

and the conclusion of the proof is as in the previous case.

Finally suppose that $b_t=i+1$. Then (by lemma 5.1.2) $(a_t,i+1)\in D_\omega$.

If $a<a_t$ then we have

$$\Delta_{ab}^{D_\omega}(f_\omega)=\Delta_{ab}^{D}(f)$$

and the lemma follows as above. On the other hand suppose that $a>a_t$. Then one (and only one) of the following cases occurs:

(i) $(a',i)\in D_\omega$ for some $a'\in\{a_t+1,\ldots,i-1\}$ - hence (by lemma 5.1.4) $(a',i)\in D$;

(ii) $(a',i)\in D_\omega$ for all $a'\in\{1,\ldots,i-1\}$ - hence (by lemma 5.1.4) $(a',i)\in D$ for all

$a' \in \{1,\ldots,i-1\}$.

If case (i) occurs and $a>a'$ then

$$\Delta_{ab}^{D_\omega}(f_\omega)=-\Delta_{ab}^{D}(f)$$

and the result follows as before. Finally suppose that either the first case occurs and $a<a'$ or the second case occurs. Let $D_\omega(a,b)=\{(i_1,j_1),\ldots,(i_r,j_r)\}$, $j_1<,\ldots<j_r$, and let $\sigma\in S_r$ be such that $i_{\sigma(1)}<\ldots<i_{\sigma(r)}$. Then $(a_t,i+1)=(i_s,j_s)$ for some $s\in\{1,\ldots,r\}$. Since $(a,i)\in R(D)\backslash D$ the column-vector

$$c_i=(f(e_{i_{\sigma(1)}i}) \ \ldots \ f(e_{i_{\sigma(r)}i}) \ f(e_{ai}) \ )^{\mathrm{T}}$$

is a linear combination of the vectors

$$c_{i+1}=(f(e_{i_{\sigma(1)}i+1}) \ \ldots \ f(e_{i_{\sigma(r)}i+1}) \ f(e_{ai+1}) \ )^{\mathrm{T}}$$

and

$$c_{j_{s'}}=(f(e_{i_{\sigma(1)}j_{s'}}) \ \ldots \ f(e_{i_{\sigma(r)}j_{s'}}) \ f(e_{aj_{s'}}) \ )^{\mathrm{T}}$$

for $s'\in\{s+1,\ldots,r\}$ (see the proof of proposition 4.2.10). Therefore the vector space spanned by the vectors $c_i,c_{j_{s+1}},\ldots,c_{j_r}$ is also spanned by the vectors $c_{i+1},c_{j_{s+1}},\ldots,c_{j_r}$. It follows that

$$\Delta_{ab}^{D_\omega}(f_\omega)\neq 0 \iff \Delta_{ab}^{D}(f)\neq 0$$

and this implies that

$$(a,b)\in D_\omega \iff (a,b)\in D.$$

To conclude the proof let $\alpha,\alpha_{s+1},\ldots,\alpha_r\in K$ be such that

$$c_i=\alpha c_{i+1}+\alpha_{s+1}c_{j_{s+1}}+\ldots+\alpha_r c_{j_r}.$$

We claim that

$$\alpha=\varphi(a_t,i+1)^{-1}\varphi^{(\omega)}(a_t,i).$$

For let $D_0=D\backslash\{(a_t,i+1)\}$ and and let φ_0 is the restriction of φ to D_0. Then there exist $f^{(t)}\in O_{a_t i+1}(\varphi(a_t,i+1))$ and $f'\in V_{D_0}(\varphi_0)$ such that

$$f=f^{(t)}+f'.$$

Therefore

$$c_{i+1}=c^{(t)}+c'$$

where

$$c^{(t)}=(0\ \dots\ 0\ \varphi(a_t,i+1)\ f^{(t)}(e_{i_{\sigma(s'+1)}i+1})\ \dots\ f^{(t)}(e_{i_{\sigma(r)}i+1})\ f^{(t)}(e_{ai+1})\ )^{\mathrm{T}},$$

$s'=\sigma^{-1}(s)$ and

$$c'=(f'(e_{i_{\sigma(1)}i+1})\ \dots\ f'(e_{i_{\sigma(r)}i+1})\ f'(e_{ai+1})\ )^{\mathrm{T}}.$$

Now we have

$$f(e_{a'i})=f^{(t)}(e_{a'i})+f'(e_{a'i})=\varphi(a_t,i+1)^{-1}f^{(t)}(e_{a_ti})f^{(t)}(e_{a'i})+f'(e_{a'i})$$

for all $a'\in\{1,\dots,i-1\}$. It follows that

$$c_i=\varphi(a_t,i+1)^{-1}f^{(t)}(e_{a_ti})c^{(t)}+c'=\varphi(a_t,i+1)^{-1}f^{(t)}(e_{a_ti})c_{i+1}+\beta c'$$

where

$$\beta=1-\varphi(a_t,i+1)^{-1}f^{(t)}(e_{a_ti}).$$

Since $(a',i+1)\in D_0$ for all $a'\in\{1,\dots,i\}$ the vector c' is a linear combination of the vectors $c_{j_{s+1}},\dots,c_{j_r}$ (see the proof of proposition 4.2.10). Since the vectors $c_{i+1},c_{j_{s+1}},\dots,c_{j_t}$ are linearly independent we conclude that

$$\alpha=\varphi(a_t,i+1)^{-1}f^{(t)}(e_{a_ti}).$$

Our claim follows because

$$f^{(t)}(e_{a_ti})=\varphi^{(a)}(a_t,i)$$

(see the proof of lemma 5.1.2).

Now we conclude that

$$\Delta_{ab}^{D_\omega}(f_\omega)=\varphi(a_t,i+1)^{-1}\varphi^{(a)}(a_t,i)\Delta_{ab}^{D}(f).$$

Finally suppose that $(a,b)\in D_\omega$ (hence $(a,b)\in D$). Then (by (3.1.7))

$$\Delta_{ab}^{D_\omega}(f_\omega)=(-1)^r sgn(\sigma)\varphi_\omega(a,b)\varphi_\omega(a_t,i+1)\prod_{s'=1,s'\neq s}^{r}\varphi_\omega(i_{s'},j_{s'})$$

and

$$\Delta_{ab}^{D}(f)=(-1)^r sgn(\sigma)\varphi(a,b)\varphi(a_t,i+1)\prod_{s'=1,s'\neq s}^{r}\varphi(i_{s'},j_{s'}).$$

By the previous lemmas (and also by induction) we have

$$\varphi_\omega(i_{s'},j_{s'})=\varphi(i_{s'},j_{s'})$$

for all $s'\in\{1,\dots,t\}$ with $s'\neq s$. Moreover (by lemma 5.1.2)

$$\varphi_\omega(a_t,i+1)=\varphi^{(a)}(a_t,i).$$

It follows that

$$\varphi_\omega(a,b)=\varphi(a,b)$$

as required. ◆

To finish this section we collect our lemmas in the following:

Theorem 5.1.7. *Let D be a basic subset of $\Phi(n)$ and suppose that $(i,n)\in D$ for some $j\in\{1,\ldots,n-2\}$. Let $\varphi:D\to K^*$ be any map and let $f\in V_D(\varphi)$ be such that $f(e_{i+1n})=f(e_{ii+1})=0$. Let $a=(a_1,\ldots,a_r)$ be the (unique) special sequence such that $f\in V_{D(a)}(\varphi^{(a)})$. Let $\omega=(i\,i+1)\in S_n$ and define $f_\omega\in U_n(K)^*$ by*

$$f_\omega(e_{ab})=\begin{cases} f(e_{\omega(a)\omega(b)}) & \text{if } (a,b)\neq(i,i+1)\\ 0 & \text{if } (a,b)=(i,i+1)\end{cases}$$

for all $(a,b)\in\Phi(n)$. Let D_ω be the basic subset of $\Phi(n)$ and let $\varphi_\omega:D_\omega\to K^*$ be the map such that $f_\omega\in V_{D_\omega}(\varphi_\omega)$. Then:

(i) *If either $a=\varnothing$ or $(a_t,i+1)\in D$,*

$$D_\omega=\begin{cases}\omega(D)\cup\{(i,i+1)\} & \text{if } (i,i+1)\in R(\omega(D)) \text{ and } \Delta_{ii+1}^{\omega(D)}(f_\omega)\neq0\\ \omega(D) & \text{otherwise}\end{cases}$$

and

$$\varphi_\omega(u,v)=\varphi(\omega(u),\omega(v))$$

for all $(u,v)\in\omega(D)\subseteq D_\omega$. If $(i,i+1)\in D_\omega$ then $\varphi_\omega(i,i+1)$ is well-determined by the value $\Delta_{ii+1}^{\omega(D)}(f_\omega)$ and there exists a unique $\alpha\in K$ such that

$$\Delta_{ii+1}^{\omega(D)}(f_\omega+\alpha\, e_{ii+1}{}^*)=0.$$

(ii) *If $(a_t,i+1)\in D$ and there is no $a\in\{1,\ldots,i-1\}$ such that $(a,i)\in D$,*

$$D_\omega=(\omega(D)\setminus\{(a_t,i)\})\cup\{(a_t,i+1)\}$$

and

$$\varphi_\omega(u,v)=\begin{cases}\varphi^{(a)}(a_t,i) & \text{if } (u,v)=(a_t,i+1)\\ \varphi(\omega(u),\omega(v)) & \text{otherwise}\end{cases}$$

for all $(u,v)\in D_\omega$.

(iii) *If* $(a_t,i+1)\in D$ *and there exists* $a\in\{1,\ldots,i-1\}$ *such that* $(a,i)\in D$,

$$D_\omega=(\omega(D)\setminus\{(a_t,i),(a,i+1)\})\cup\{(a_t,i+1),(a,i)\}$$

and

$$\varphi_\omega(u,v)=\begin{cases}\varphi^{(a)}(a_t,i) & \text{if } (u,v)=(a_t,i+1)\\ -\varphi^{(a)}(a_t,i)^{-1}\varphi(a,i)\varphi(a_t,i+1) & \text{if } (u,v)=(a,i)\\ \varphi(\omega(u),\omega(v)) & \text{otherwise}\end{cases}$$

for all if $(u,v)\in D_\omega$.

5.2. A decomposition of the regular character

The aim of this section is to prove the following result.

Theorem 5.2.1. *Let ρ be the regular character of $U_n(q)$. For each basic subset D of $\Phi(n)$ we define the character*

$$\xi_D=\sum_{\varphi}\xi_D(\varphi)$$

where the sum is over all maps $\varphi:D\to \boldsymbol{F}_q{}^*$. Then

$$\rho=\sum_{\substack{D\subseteq\Phi(n)\\ D\ \text{basic}}} q^{s(D)-l(D)}\,\xi_D$$

where for each basic subset D of $\Phi(n)$ $s(D)=|S(D)|$ and $l(D)=|S^{(c)}(D)|=|S^{(r)}(D)|$ (see (2.2.7)).

Here $q=p^e$, $e\geq 1$, is the e-th power of the prime number p. As usual we let K be the algebraic closure of the finite field $\boldsymbol{F}_q$ and we realize the finite group $U_n(q)$ as the subgroup of $U_n(K)$ consisting of all fixed elements of the Frobenius map $F=F_q:U_n(K)\to U_n(K)$ (see (1.1.1)).

Theorem 5.2.1 will be a consequence of the following result (which is in fact equivalent to it):

Proposition 5.2.2. *Let D be a basic subset of $\Phi(n)$ and let $\varphi:D\to \boldsymbol{F}_q{}^*$ be any map. Let χ be an irreducible component of $\xi_D(\varphi)$. Then*

$$\chi(1)=q^{s(D)-l(D)}(\chi,\xi_D(\varphi))$$

where $(.,.)$ denotes the Frobenius product between characters.

To prove this proposition we will use induction on the set of all basic subsets of $\Phi(n)$ (with the order introduced in section 5.1). If $D\subseteq\Phi(n)$ is the empty set then

$\varphi: D \to \boldsymbol{F}_q{}^*$ is the empty map and

$$\xi_D(\varphi)=1_{U_n(q)}$$

is the unit character of $U_n(q)$. The result is obvious because $s(\emptyset)=l(\emptyset)=0$.

If $D=\{(i,j)\}$ consists only of one root $(i,j)\in\Phi(n)$ then for any map $\varphi: D\to\boldsymbol{F}_q{}^*$

$$\xi_D(\varphi)=\xi_{ij}(\alpha)$$

is the (i,j)-th elementary character associated with $\alpha=\varphi(i,j)\in\boldsymbol{F}_q{}^*$. By definition this character is irreducible and

$$\xi_{ij}(\alpha)(1)=q^{j-i-1}.$$

The result is also clear because $s(D)-l(D)=2(j-i-1)-(j-i-1)=j-i-1$.

Now suppose that $|D|\geq 2$ and let (i,j) be the smallest root in D (hence $b<i$ whenever $(a,b)\in D$). If $j<n$ then the proposition follows by induction on n. On the other hand suppose that $(i,n)\in D$ for some $i\in\{1,\ldots,n-1\}$. If $i=n-1$ then there exists a unique irreducible character χ_1 of $U_n(q)$ such that

$$\chi=\xi_{n-1n}(\alpha)\chi_1$$

where $\alpha=\varphi(n-1,n)$. In fact since $\xi_{n-1n}(-\alpha)$ is linear the character

$$\chi_1=\xi_{n-1n}(-\alpha)\chi$$

is irreducible. By lemma 2.2.9 we have

$$\xi_{n-1n}(-\alpha)\xi_{n-1n}(\alpha)=1_{U_n(q)}.$$

On the other hand let $D_1=D\setminus\{(n-1,n)\}$ and let φ_1 be the restriction of φ to D_1. Since $\xi_{n-1n}(\alpha)$ is linear and

$$\overline{\xi_{n-1n}(\alpha)}=\xi_{n-1n}(-\alpha)$$

(see the proof of proposition 2.2.16) we have

$$(\chi_1,\xi_{D_1}(\varphi_1))=(\xi_{n-1n}(\alpha)\chi_1,\xi_{n-1n}(\alpha)\xi_{D_1}(\varphi_1))=(\chi,\xi_D(\varphi)).$$

The result follows by induction on n because $s(D_1)=s(D)$ and $l(D_1)=l(D)$.

Finally suppose that $i<n-1$. Let $f\in V_D(\varphi)$ be an F-fixed element such that the irreducible character χ corresponds to the (F-stable) $U_n(K)$-orbit $O(f)$, i.e. $\chi=\chi_{O(f)}$. Let $\omega\in S_n$ be the transposition $\omega=(i\ i+1)$ and let $f_\omega\in u_n(K)^*$ be the element defined by (5.1.1).

Let $(D_\omega,\varphi_\omega)$ be the pair defined as in theorem 5.1.7. Then $f_\omega\in V_{D_\omega}(\varphi_\omega)$. Moreover f_ω is an F-fixed element hence its $U_n(K)$-orbit $O(f_\omega)$ is F-stable (we note that $V_{D_\omega}(\varphi_\omega)$ is also F-stable because $\varphi_\omega(D_\omega)\subseteq F_q{}^*$ - in fact for $(a,b)\in D_\omega$, the value $\varphi_\omega(a,b)$ is a function of the entries of the element f_ω. Let

$$\chi_\omega=\chi_{O(f_\omega)}$$

be the irreducible character of $U_n(q)$ which corresponds to $O(f_\omega)$. Since $O(f_\omega)\subseteq V_{D_\omega}(\varphi_\omega)$ (because $V_{D_\omega}(\varphi_\omega)$ is $U_n(K)$-invariant, $f_\omega\in V_{D_\omega}(\varphi_\omega)$) and

$$V_{D_\omega}(\varphi_\omega)=O_{D_\omega}(\varphi_\omega)$$

(by theorem 3.1.7) we have

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))\neq 0$$

(see corollary 1.3.11). Since $D_\omega<D$, we may assume (by induction) that the result is proved for the basic subset D_ω of $\Phi(n)$. Therefore

(5.2.1) $$\chi_\omega(1)=q^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)).$$

Now we prove the following:

Lemma 5.2.3. *The $U_n(K)$-orbit $O(f)$ is of the second kind with respect to the normal subgroup $U_\omega(K)$.*

Proof. By definition $O(f)$ is of the second kind if $f+\alpha e_{ii+1}{}^*\in O(f)$ for all $\alpha\in K$. Let $\alpha\in K$ be arbitrary and consider the element

$$x_{i+1n}(-\alpha f(e_{in})^{-1})=1-\alpha f(e_{in})^{-1}e_{i+1n}\in U_n(K).$$

Then

$$(x_{in}(-\alpha f(e_{in})^{-1})\cdot f)(e_{rs})=\begin{cases} f(e_{rs}) & \text{if } (r,s)\neq(i,i+1)\\ f(e_{ii+1})+\alpha & \text{if } (r,s)=(i,i+1)\end{cases}$$

and the lemma follows. ♦

Since the subgroup $U_\omega(K)$ is F-stable we conclude that there exists a unique irreducible character θ of $U_\omega(q)$ such that

(5.2.2) $$\chi=\theta^{U_n(q)}$$

(see theorem 1.3.9). Therefore (by Frobenius reciprocity) we deduce that

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta,\zeta_D(\varphi))_{U_\omega(q)}$$

where

$$\zeta_D(\varphi)=\xi_D(\varphi)_{U_\omega(q)}$$

is the restriction of $\xi_D(\varphi)$ to $U_\omega(q)$. Since

$$\xi_D(\varphi)=\prod_{(a,b)\in D}\xi_{ab}(\varphi(a,b))$$

we have

$$\zeta_D(\varphi)=\prod_{(a,b)\in D}\zeta_{ab}(\varphi(a,b))$$

where for each $(r,s)\in\Phi(n)$ and each $\alpha\in\mathbf{F}_q$

$$\zeta_{rs}(\alpha)=\xi_{rs}(\alpha)_{U_\omega(q)}$$

is the restriction of $\xi_{rs}(\alpha)$to $U_\omega(q)$.

Next we consider the decomposition (into irreducible components) of any character $\zeta_{rs}(\alpha)$ where $(r,s)\in\Phi(n)$ and $\alpha\in\mathbf{F}_q$. Since $\xi_{rs}(0)=1_{U_n(q)}$ we have

$$\zeta_{rs}(0)=1_{U_\omega(q)}.$$

On the other hand we have the following:

Lemma 5.2.4. *Let $(r,s)\in\Phi(n)$ and let $\alpha\in\mathbf{F}_q^*$. Then:*

(i) *If $r\neq i$ and $s\neq i+1$, $\zeta_{rs}(\alpha)$ is an irreducible character of $U_\omega(q)$.*

(ii) *If $s\neq i+1$,*

$$\zeta_{is}(\alpha)=\sum_{\beta\in\mathbf{F}_q}\zeta_{is}(\alpha,\beta)$$

where for each $\beta\in\mathbf{F}_q$ $\zeta_{is}(\alpha,\beta)$ denotes the irreducible character of $U_\omega(q)$ which corresponds to the (F-stable) $U_\omega(K)$-orbit of the element $\alpha e_{is}^*+\beta e_{i+1s}^*\in U_\omega(K)^*$. Moreover

$$\zeta_{is}(\alpha,\beta)\neq\zeta_{is}(\alpha,\beta')$$

whenever $\beta,\beta'\in\mathbf{F}_q$ *are distinct.*

(iii) *If* $r\neq i$,

$$\zeta_{ri+1}(\alpha)=\sum_{\beta\in\mathbf{F}_q}\zeta_{ri+1}(\alpha,\beta)$$

where for each $\beta\in\mathbf{F}_q$ $\zeta_{ri+1}(\alpha,\beta)$ *denotes the irreducible character of* $U_\omega(q)$ *which corresponds to the (F-stable)* $U_\omega(K)$*-orbit of the element* $\alpha e_{ri+1}{}^*+\beta e_{ri}{}^*\in u_\omega(K)^*$. *Moreover*

$$\zeta_{ri+1}(\alpha,\beta)\neq\zeta_{ri+1}(\alpha,\beta')$$

whenever $\beta,\beta'\in\mathbf{F}_q$ *are distinct.*

(iv) $\zeta_{ii+1}(\alpha)=1_{U_\omega(q)}$ *is the unit character of* $U_\omega(q)$.

Proof. By proposition 2.1.8

$$\xi_{rs}(\alpha)=\chi_{O_{rs}(\alpha)}$$

is the irreducible character of $U_n(q)$ corresponds to the $U_n(K)$-orbit $O_{rs}(\alpha)$ of the element $\alpha e_{rs}{}^*\in u_n(K)^*$.

Let $\pi: u_n(K)^*\to u_\omega(K)^*$ be the natural projection. Then (by theorem 1.3.10) an irreducible character of $U_\omega(q)$ is an irreducible component of $\zeta_{rs}(\alpha)$ if and only if it corresponds to an (F-stable) $U_\omega(K)$-orbit which is contained in the image $\pi(O_{rs}(\alpha))$. Since $O_{ii+1}(\alpha)=\{\alpha e_{ii+1}{}^*\}$ we have

$$\pi(O_{ii+1}(\alpha))=\{0\}.$$

Therefore the unit character $1_{U_\omega(q)}$ of $U_\omega(q)$ is the unique (irreducible) component of $\zeta_{ii+1}(\alpha)$. (iv) follows because

$$\zeta_{ii+1}(\alpha)(1)=\xi_{ii+1}(\alpha)(1)=1.$$

Now suppose that $(r,s)\neq(i,i+1)$. Then

$$\pi(\alpha e_{rs}{}^*)=\alpha e_{rs}{}^*\in u_\omega(K)^*.$$

For an arbitrary $\beta\in\mathbf{F}_q$ we consider the element

$$\alpha e_{rs}{}^*+\beta e_{ii+1}{}^*\in u_n(K)^*.$$

By lemma 3.1.1 this element is $U_n(K)$-conjugate to $\alpha e_{rs}{}^*$ if and only if either $r=i$ or

$s=i+1$. By definition we conclude that the $U_n(K)$-orbit $O_{rs}(\alpha)$ is of the second kind (with respect to $U_\omega(K)$) if and only if either $r=i$ or $s=i+1$. By theorem 1.3.8 we deduce that the character $\zeta_{rs}(\alpha)$ is reducible if and only if either $r=i$ or $s=i+1$. The proof of (i) is complete.

On the other hand suppose that either $r=i$ or $s=i+1$. Then (by theorem 1.3.8)

$$\zeta_{rs}(\alpha)=\sum_{\beta\in F_q}\zeta_{rs}(\alpha,0)^{x_{ii+1}(\beta)}$$

where $x_{ii+1}(\beta)=1+\beta e_{ii+1}$ for all $\beta\in F_q$. Moreover for each $\beta\in F_q$ the irreducible character $\zeta_{rs}(\alpha,0)^{x_{ii+1}(\beta)}$ of $U_\omega(q)$ corresponds to the $U_\omega(K)$-orbit of the element

$$x_{ii+1}(\beta)\cdot(\alpha e_{rs}{}^*)\in \mathfrak{u}_\omega(K)^*.$$

The lemma follows because

$$x_{ii+1}(\beta)\cdot(\alpha e_{rs}{}^*)=\begin{cases}\alpha e_{is}{}^*+\beta e_{i+1s}{}^* & \text{if } r=i\\ \alpha e_{ri+1}{}^*+\beta e_{ri}{}^* & \text{if } s=i+1\end{cases}$$

for all $\beta\in F_q$. ♦

On the other hand we have:

Corollary 5.2.5. *Let* $(r,s)\in\Phi(n)$ *and let* $\alpha\in F_q{}^*$. *Then:*

(i) *If* $r\neq i$ *and* $s\neq i+1$,

$$\zeta_{rs}(\alpha)^{U_n(q)}=\sum_{\beta\in F_q}\xi_{rs}(\alpha)\xi_{ii+1}(\beta).$$

- We note that (by theorem 2.2.1) the characters $\xi_{rs}(\alpha)\xi_{ii+1}(\beta)$, $\beta\in F_q$, *are all distinct.*

(ii) *If* $(r,s)\neq(i,i+1)$ *and either* $r=i$ *or* $s=i+1$,

$$\zeta_{rs}(\alpha,\beta)^{U_n(q)}=\xi_{rs}(\alpha)$$

for all $\beta\in F_q$.

(iii) $(1_{U_\omega(q)})^{U_n(q)}=\sum_{\beta\in F_q}\xi_{ii+1}(\beta).$

Proof. Suppose that $r\neq i$ and $s\neq i+1$. Then (by the previous lemma) $\zeta_{rs}(\alpha)$ is irreducible and $\zeta_{ii+1}(\beta)=1_{U_\omega(q)}$. Therefore (by Frobenius reciprocity) we have

$$(\zeta_{rs}(\alpha)^{U_n(q)},\xi_{rs}(\alpha)\xi_{ii+1}(\beta))_{U_n(q)}=(\zeta_{rs}(\alpha),\zeta_{rs}(\alpha)\zeta_{ii+1}(\beta))_{U_\omega(q)}=(\zeta_{rs}(\alpha),\zeta_{rs}(\alpha))_{U_\omega(q)}=1$$

for all $\beta\in \boldsymbol{F}_q$. The proof of (i) is complete.

For (ii) by the previous lemma we have

$$(\zeta_{rs}(\alpha,\beta)^{U_n(q)},\xi_{rs}(\alpha))_{U_n(q)}=(\zeta_{rs}(\alpha,\beta),\zeta_{rs}(\alpha))_{U_\omega(q)}=1$$

for all $\beta\in \boldsymbol{F}_q$. Hence $\xi_{rs}(\alpha)$ is an irreducible component of $\zeta_{rs}(\alpha,\beta)^{U_n(q)}$ for all $\beta\in \boldsymbol{F}_q$. The result follows because

$$\xi_{rs}(\alpha)(1)=q^{s-r-1} \quad \text{and} \quad \zeta_{rs}(\alpha,\beta)=q^{s-r-2} \ (\beta\in \boldsymbol{F}_q).$$

The proof of (iii) is analogous to the proof of (i). ♦

In order to complete the proof of proposition 5.2.2 we establish a series of lemmas relating the multiplicity $(\chi,\xi_D(\varphi))$ with the multiplicity $(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))$. Each lemma depends on the "type" of the root $(i,i+1)$. Firstly we prove the following:

Lemma 5.2.6. *Let θ^ω be the irreducible character of $U_\omega(q)$ defined by*

$$\theta^\omega(x)=\theta(\omega x\omega^{-1})$$

for all $x\in U_\omega(q)$. Then

$$(\chi_\omega,(\theta^\omega)^{U_n(q)})=1$$

and χ_ω is the unique irreducible component of $(\theta^\omega)^{U_n(q)}$ with the property

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))\neq 0.$$

In fact we have:

(i) *If the $U_n(K)$-orbit $O(f_\omega)$ of f_ω is of the first kind with respect to $U_\omega(K)$ then*

$$(\chi_\omega)_{U_\omega(q)}=\theta^\omega$$

and

$$(\theta^\omega)^{U_n(q)}=\sum_{\alpha\in \boldsymbol{F}_q}\chi_\omega\,\xi_{ii+1}(\alpha).$$

(ii) *If the $U_n(K)$-orbit $O(f_\omega)$ of f_ω is of the second kind with respect to $U_\omega(K)$ then*

$$\chi_\omega=(\theta^\omega)^{U_n(q)}$$

and

$$(\chi_\omega)_{U_\omega(q)}=\sum_{\alpha\in \boldsymbol{F}_q}\theta^\omega(\alpha)$$

where

$$\theta^{\omega}(\alpha)=(\theta^{\omega})^{\chi_{ii+1}(\alpha)}.$$

Proof. We claim that θ^{ω} corresponds to the (F-stable) $U_{\omega}(K)$-orbit $O(\pi(f_{\omega}))$ of the element $\pi(f_{\omega})\in \mathcal{U}_{\omega}(K)^*$ (here $\pi:\mathcal{U}_n(K)^*\to \mathcal{U}_{\omega}(K)^*$ is the natural projection). For we shall use proposition 1.2.5. In fact let $g\in \mathcal{U}_{\omega}(K)^*$ and define $\omega^{-1}g\omega\in \mathcal{U}_{\omega}(K)^*$ by

$$(\omega^{-1}g\omega)(a)=g(\omega a\omega^{-1})$$

for all $a\in \mathcal{U}_{\omega}(K)$. We note that

$$\pi(f_{\omega})=\omega^{-1}\pi(f)\omega.$$

Then it is not difficult to show that

$$g\in O(\pi(f)) \iff \omega^{-1}g\omega\in O(\pi(f_{\omega}))$$

where $O(\pi(f))$ denotes the $U_{\omega}(K)$-orbit of the element $\pi(f)\in \mathcal{U}_{\omega}(K)^*$. It follows that

$$\sum_{g\in O(\pi(f))^F} \psi_0(g(\omega a\omega^{-1}))=\sum_{\omega^{-1}g\omega\in O(\pi(f_{\omega}))^F} \psi_0((\omega^{-1}g\omega)(a))$$

for all $a\in \mathcal{U}_{\omega}(K)=\mathcal{U}_{\omega}(K)\cap\omega^{-1}\mathcal{U}_{\omega}(K)\omega$ (here ψ_0 is a non-trivial irreducible character of $\boldsymbol{F}_q^{+}$). By proposition 1.2.5 we conclude that

$$\theta^{\omega}(\exp a)=\theta(\omega(\exp a)\omega^{-1})=\theta(\exp(\omega a\omega^{-1}))$$
$$=\frac{1}{\sqrt{|O(\pi(f))^F|}}\sum_{\omega^{-1}g\omega\in O(\pi(f_{\omega}))^F}\psi_0((\omega^{-1}g\omega)(a))$$

and our claim follows because

$$|O(\pi(f))|=|O(\pi(f_{\omega}))|.$$

Now suppose that the $U_n(K)$-orbit $O(f_{\omega})$ is of the first kind (with respect to $U_{\omega}(K)$). Then (by theorem 1.3.8) the character $(\chi_{\omega})_{U_{\omega}(q)}$ is irreducible. Since $\pi(f_{\omega})\in\pi(O(f_{\omega}))$ we conclude that

$$(\chi_{\omega})_{U_{\omega}(q)}=\theta^{\omega}.$$

On the other hand we have (by lemma 5.2.4)

$$\zeta_{ii+1}(\alpha)=1_{U_{\omega}(q)}$$

for all $\alpha\in\boldsymbol{F}_q$. Therefore (by Frobenius reciprocity)

$$(\chi_{\omega}\xi_{ii+1}(\alpha),(\theta^{\omega})^{U_n(q)})_{U_n(q)}=((\chi_{\omega})_{U_{\omega}(q)},\theta^{\omega})_{U_{\omega}(q)}=1$$

for all $\alpha\in \boldsymbol{F}_q$. (i) follows by degree considerations.

Now for each $\alpha\in \boldsymbol{F}_q$ we have

$$(\chi_\omega \xi_{ii+1}(\alpha),\xi_{D_\omega}(\varphi_\omega))=(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)\xi_{ii+1}(-\alpha))$$

because

$$\overline{\xi_{ii+1}(\alpha)}=\xi_{ii+1}(-\alpha)$$

(see the proof of the proposition 2.2.16).

Suppose that $(i,i+1)\notin D_\omega$ and that $\alpha\in \boldsymbol{F}_q$ is non-zero. Then

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)\xi_{ii+1}(-\alpha))=0$$

(by theorem 2.2.1 because $(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))\neq 0$). Hence

$$(\chi_\omega \xi_{ii+1}(\alpha),\xi_{D_\omega}(\varphi_\omega))=0.$$

On the other hand suppose that $(i,i+1)\in D_\omega$. First we consider $\alpha=\varphi_\omega(i,i+1)$. Then (by lemma 2.2.9)

$$\xi_{D_\omega}(\varphi_\omega)\xi_{ii+1}(-\alpha)=\xi_{(D_\omega)_0}((\varphi_\omega)_0)$$

where $(D_\omega)_0=D_\omega\backslash\{(i,i+1)\}$ and $(\varphi_\omega)_0$ is the restriction of φ_ω to $(D_\omega)_0$. It follows that

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)\xi_{ii+1}(-\alpha))=0$$

(by theorem 2.2.1 because $D_\omega\neq(D_\omega)_0$), hence

$$(\chi_\omega \xi_{ii+1}(\alpha),\xi_{D_\omega}(\varphi_\omega))=0.$$

Finally suppose that $\alpha\in \boldsymbol{F}_q\backslash\{0,\varphi_\omega(i,i+1)\}$. Then (by lemma 2.2.12)

$$\xi_{D_\omega}(\varphi_\omega)\xi_{ii+1}(-\alpha)=\xi_{(D_\omega)_0}((\varphi_\omega)_0)\xi_{ii+1}(\varphi_\omega(i,i+1)-\alpha)$$

where $(D_\omega)_0$ and $(\varphi_\omega)_0$ are as above. By theorem 2.2.1 we conclude that

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)\xi_{ii+1}(-\alpha))=0$$

(because $\varphi_\omega(i,i+1)\neq\varphi_\omega(i,i+1)-\alpha$), hence

$$(\chi_\omega \xi_{ii+1}(\alpha),\xi_{D_\omega}(\varphi_\omega))=0.$$

This completes the proof of the lemma in case (i).

Now suppose that $O(f_\omega)$ is of the second kind (with respect to $U_\omega(K)$). Then (by theorem 1.3.9) the character $(\theta^\omega)^{U_n(q)}$ is irreducible so

$$\chi_\omega=(\theta^\omega)^{U_n(q)}.$$

The statement about the decomposition of $(\chi_\omega)_{U_\omega(q)}$ follows from theorem 1.3.8. The remaining assertion of the lemma is obvious in this case.

The proof is complete. ♦

Corollary 5.2.7. *Let the notation be as before.*

(i) *If $O(f_\omega)$ is of the first kind with respect to $U_\omega(K)$ then*

$$\chi_\omega(1)=q^{-1}\chi(1).$$

(ii) *If $O(f_\omega)$ is of the second kind with respect to $U_\omega(K)$ then*

$$\chi_\omega(1)=\chi(1).$$

Proof. Suppose that $O(f_\omega)$ is of the first kind. Then (by the previous lemma) $\chi_\omega(1)=\theta^\omega(1)$. (i) follows because $\theta^\omega(1)=\theta(1)$ and $\chi(1)=q\theta(1)$ (by (5.2.2)).

On the other hand suppose that $O(f_\omega)$ is of the second kind. Then

$$\chi_\omega(1)=q\theta^\omega(1)=q\theta(1)=\chi(1)$$

(by the previous lemma and by (5.2.2)). ♦

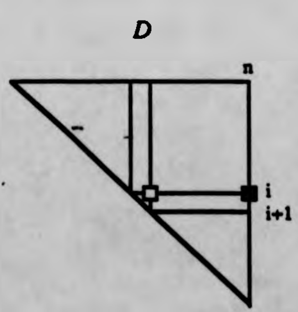
Lemma 5.2.8. *Suppose that $(i,i+1)\in R(D_\omega)$ and that $(i,i+1)\in S^{(c)}(D)$. Then*

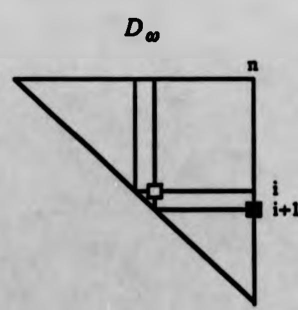
$$\chi(1)=q\chi_\omega(1)$$

and

$$(\chi,\xi_D(\varphi))=(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)).$$

Proof. In this lemma we are concerned with the following situation





The first assertion follows because the $U_n(K)$-orbit $O(f_\omega)$ is of the first kind with respect to $U_\omega(K)$. In fact let $\alpha\in K^*$ be arbitrary and consider the element $f_\omega+\alpha e_{ii+1}{}^*\in \mathcal{U}_n(K)^*$. Since

$$\Delta_{ii+1}^{\omega(D)}(f_\omega+\alpha e_{ii+1}{}^*)\neq\Delta_{ii+1}^{\omega(D)}(f_\omega)$$

(because $(i,i+1)\in R(D_\omega)$) we have

$$f_\omega+\alpha e_{ii+1}{}^*\in V_{D_\omega}(\varphi_\omega).$$

Since $V_{D_\omega}(\varphi_\omega)$ is $U_n(K)$-invariant we conclude that

$$f_\omega+\alpha e_{ii+1}{}^*\in O(f_\omega)$$

for all $\alpha\in K^*$.

Now (by theorem 5.1.7)

$$D_\omega=\begin{cases}\omega(D) & \text{if } \Delta_{ii+1}^{\omega(D)}(f_\omega)=0\\ \omega(D)\cup\{(i,i+1)\} & \text{if } \Delta_{ii+1}^{\omega(D)}(f_\omega)\neq0\end{cases}.$$

Moreover $(a,i),(a',i+1),(i+1,b)\in D$ for all $a\in\{1,\dots,i\text{-}1\}$, all $a'\in\{1,\dots,i\}$ and all $b\in\{i+2,\dots,n\}$. Therefore

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{D_0}(\varphi_0)$$

where $\alpha=\varphi(i,n)$, $D_0=D\setminus\{(i,n)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\begin{cases}\xi_{i+1n}(\alpha)\xi_{D_0}(\varphi_0) & \text{if } D_\omega=\omega(D)\\ \xi_{ii+1}(\varphi_\omega(i,i+1))\xi_{i+1n}(\alpha)\xi_{D_0}(\varphi_0) & \text{if } D_\omega=\omega(D)\cup\{(i,i+1)\}\end{cases}.$$

Next we consider the multiplicity $(\chi,\xi_D(\varphi))$. By Frobenius reciprocity

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta^{U_n(q)},\xi_D(\varphi))_{U_n(q)}=(\theta,\zeta_D(\varphi))_{U_\omega(q)}.$$

By lemma 5.2.4 we have

$$\zeta_{in}(\alpha)=\sum_{\beta\in\mathbf{F}_q}\zeta_{in}(\alpha,\beta)$$

where for each $\beta\in\mathbf{F}_q$, $\zeta_{in}(\alpha,\beta)$ is the irreducible character of $U_\omega(q)$ which corresponds to the (F-stable) $U_\omega(K)$-orbit of the element $\alpha e_{in}{}^*+\beta e_{i+1n}{}^*\in\mathcal{U}_\omega(K)^*$. Therefore

$$\zeta_D(\varphi)=\zeta_{in}(\alpha)\zeta_{D_0}(\varphi_0)=\sum_{\beta\in\mathbf{F}_q}\zeta_{in}(\alpha,\beta)\zeta_{D_0}(\varphi_0)$$

Let $\beta\in\mathbf{F}_q$ be arbitrary. Then

$$(\theta,\zeta_{in}(\alpha,\beta)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=(\theta^\omega,(\zeta_{in}(\alpha,\beta)\zeta_{D_0}(\varphi_0))^\omega)_{U_\omega(q)}$$

$$=(\theta^{\omega},\zeta_{in}(\alpha,\beta)^{\omega}\zeta_{D_0}(\varphi_0)^{\omega})_{U_\omega(q)}.$$

Since $\zeta_{in}(\alpha,\beta)$ corresponds to the $U_\omega(K)$-orbit of the element $\alpha e_{in}{}^*+\beta e_{i+1n}{}^*\in u_\omega(K)^*$ the character $\zeta_{in}(\alpha,\beta)^{\omega}$ corresponds to the $U_\omega(K)$-orbit of the element $\beta e_{in}{}^*+\alpha e_{i+1n}{}^*\in u_\omega(K)^*$ (see the first paragraph of the proof of lemma 5.2.6). It follows that

$$\zeta_{in}(\alpha,\beta)^{\omega}=\zeta_{in}(\beta,\alpha).$$

Since $\omega(D_0)=D_0$ we have

$$\zeta_{ab}(\varphi(a,b))^{\omega}=\zeta_{ab}(\varphi(a,b))$$

for all $(a,b)\in D_0$ - because $\zeta_{ab}(\varphi(a,b))$ and $\zeta_{ab}(\varphi(a,b))^{\omega}$ both correspond to the $U_\omega(K)$-orbit of the element $\varphi(a,b)e_{ab}{}^*\in u_\omega(K)^*$. Therefore

$$\zeta_{D_0}(\varphi_0)^{\omega}=\zeta_{D_0}(\varphi_0)$$

and

$$(\zeta_{in}(\beta,\alpha)\zeta_{D_0}(\varphi_0))^{U_n(q)}=\zeta_{in}(\beta,\alpha)^{U_n(q)}\xi_{D_0}(\varphi_0)$$

- because $\xi_{D_0}(\varphi_0)_{U_\omega(q)}=\zeta_{D_0}(\varphi_0)$. Since $(\chi_\omega)_{U_\omega(q)}=\theta^{\omega}$ we conclude that

$$\begin{aligned}(\theta^{\omega},\zeta_{in}(\alpha,\beta)^{\omega}\zeta_{D_0}(\varphi_0)^{\omega})_{U_\omega(q)}&=(\theta^{\omega},\zeta_{in}(\beta,\alpha)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}\\&=(\chi_\omega,(\zeta_{in}(\beta,\alpha)\zeta_{D_0}(\varphi_0))^{U_n(q)})_{U_n(q)}\\&=(\chi_\omega,\zeta_{in}(\beta,\alpha)^{U_n(q)}\xi_{D_0}(\varphi_0))_{U_n(q)}.\end{aligned}$$

If β is non-zero the character $\zeta_{in}(\beta,\alpha)^{U_n(q)}$ is irreducible (by corollary 5.2.5) and in fact

$$\zeta_{in}(\beta,\alpha)^{U_n(q)}=\xi_{in}(\beta).$$

Since $(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))\neq 0$ and $(i,n)\notin D_\omega$ theorem 2.2.1 implies that

$$(\chi_\omega,\zeta_{in}(\beta,\alpha)^{U_n(q)}\xi_{D_0}(\varphi_0))_{U_n(q)}=0.$$

On the other hand we have

$$\zeta_{in}(0,\alpha)=\zeta_{i+1n}(\alpha)$$

because $\zeta_{ni+1}(\alpha)=\bar{\xi}_{ni+1}(\alpha)_{U_\omega(q)}$ is irreducible (by lemma 5.2.4) and it corresponds to the $U_\omega(K)$-orbit of $\alpha e_{i+1n}{}^*\in u_\omega(K)^*$. By corollary 5.2.5 we conclude that

$$\zeta_{in}(0,\alpha)^{U_n(q)}=\sum_{\gamma\in F_q}\xi_{ii+1}(\gamma)\xi_{i+1n}(\alpha).$$

Therefore

$$(\chi_\omega, \zeta_{in}(0,\alpha)^{U_n(q)} \xi_{D_0}(\varphi_0))_{U_n(q)} = \sum_{\gamma\in F_q} (\chi_\omega, \xi_{ii+1}(\gamma)\xi_{i+1n}(\alpha)\xi_{D_0}(\varphi_0))_{U_n(q)}.$$

If $(i,i+1)\in D_\omega$ then (by theorem 2.2.1)

$$(\chi_\omega, \xi_{ii+1}(\gamma)\xi_{i+1n}(\alpha)\xi_{D_0}(\varphi_0))_{U_n(q)}=0$$

for all $\gamma\in F_q{}^*$. On the other hand suppose that $(i,i+1)\in D_\omega$. Then (by lemmas 2.2.9 and 2.2.12)

$$\xi_{ii+1}(\gamma)\xi_{ii+1}(\varphi_\omega(i,i+1))=\xi_{ii+1}(\gamma+\varphi_\omega(i,i+1))$$

for all $\gamma\in F_q$. As before theorem 2.2.1 implies that

$$(\chi_\omega, \xi_{ii+1}(\gamma)\xi_{i+1n}(\alpha)\xi_{D_0}(\varphi_0))_{U_n(q)}=0$$

for all $\gamma\in F_q{}^*$. Therefore (in both cases) we conclude that

$$(\chi_\omega, \zeta_{in}(0,\alpha)^{U_n(q)} \xi_{D_0}(\varphi_0))_{U_n(q)}=(\chi_\omega, \xi_{D_\omega}(\varphi_\omega))_{U_n(q)}$$

because $\xi_{ii+1}(0)=1_{U_n(q)}$.

The lemma is proved. ♦

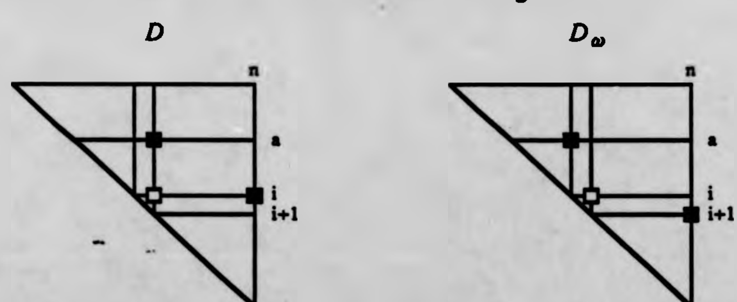
Lemma 5.2.9. *Suppose that* $(i,i+1)\in R(D_\omega)$ *and that* $(i,i+1)\in S^{(c)}(D)$. *Then*

$$\chi(1)=q\chi_\omega(1)$$

and

$$(\chi, \xi_D(\varphi))=(\chi_\omega, \xi_{D_\omega}(\varphi_\omega)).$$

Proof. In this lemma we are concerned with the following situation



As in the previous lemma the $U_n(K)$-orbit $O(f_\omega)$ is of the first kind with respect to $U_\omega(K)$. This implies that

$$\chi(1)=q\chi_\omega(1)$$

(by corollary 5.2.7).

Now (by theorem 5.1.7)

$$D_\omega=\begin{cases}\alpha(D) & \text{if } \Delta^{\alpha(D)}_{ii+1}(f_\omega)=0\\ \alpha(D)\cup\{(i,i+1)\} & \text{if } \Delta^{\alpha(D)}_{ii+1}(f_\omega)\neq 0\end{cases}.$$

Also $(a,i),(i+1,b)\in D$ for all $a\in\{1,\ldots,i-1\}$ and all $b\in\{i+2,\ldots,n\}$. However there exists $a\in\{1,\ldots,i-1\}$ such that $(a,i+1)\in D$ (hence $(a,i)\in D_\omega$). Therefore

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{ai+1}(\beta)\xi_{D_0}(\varphi_0)$$

where $\alpha=\varphi(i,n)$, $\beta=\varphi(a,i+1)$, $D_0=D\setminus\{(i,n),(a,i+1)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\begin{cases}\xi_{i+1n}(\alpha)\xi_{ai}(\beta)\xi_{D_0}(\varphi_0) & \text{if } D_\omega=\alpha(D)\\ \xi_{ii+1}(\varphi_\omega(i,i+1))\xi_{i+1n}(\alpha)\xi_{ai}(\beta)\xi_{D_0}(\varphi_0) & \text{if } D_\omega=\alpha(D)\cup\{(i,i+1)\}\end{cases}.$$

As before we have

$$\begin{aligned}(\chi,\xi_D(\varphi))_{U_n(q)}&=(\theta,\zeta_{in}(\alpha)\zeta_{ai+1}(\beta)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}\\ &=\sum_{\alpha'\in F_q}\sum_{\beta'\in F_q}(\theta,\zeta_{in}(\alpha,\alpha')\zeta_{ai+1}(\beta,\beta')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}\\ &=\sum_{\alpha'\in F_q}\sum_{\beta'\in F_q}(\theta^\omega,\zeta_{in}(\alpha,\alpha')^\omega\zeta_{ai+1}(\beta,\beta')^\omega\zeta_{D_0}(\varphi_0)^\omega)_{U_\omega(q)}\\ &=\sum_{\alpha'\in F_q}\sum_{\beta'\in F_q}(\theta^\omega,\zeta_{in}(\alpha',\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}\end{aligned}$$

Let $\alpha',\beta'\in F_q$ and suppose that α' is non-zero. Let ϕ be an irreducible component of $\zeta_{in}(\alpha',\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{D_0}(\varphi_0)$ and let O be the (F-stable) $U_\omega(K)$-orbit which corresponds to ϕ. Then (by corollary 1.3.11)

$$O\subseteq O_{in}(\alpha',\alpha)+O_{ai+1}(\beta',\beta)+\pi(V_{D_0}(\varphi_0))$$

where $O_{in}(\alpha',\alpha)$ is the $U_\omega(K)$-orbit which corresponds to $\zeta_{in}(\alpha',\alpha)$, $O_{ai+1}(\beta',\beta)$ is the $U_\omega(K)$-orbit which corresponds to $\zeta_{ai+1}(\beta',\beta)$ and $\pi:u_n(K)^*\to u_\omega(K)^*$ is the natural projection. Moreover $\pi(V_{D_0}(\varphi_0))$ is the sum of all $U_\omega(K)$-orbits of the elements $\varphi(r,s)e_{rs}{}^*\in u_\omega(K)^*$ with $(r,s)\in D_0$. Therefore any element $f\in O_0$ satisfies $f(e_{1n})=\ldots=f(e_{i-1n})=0$ and $f(e_{in})=\alpha'\neq 0$. Since θ^ω corresponds to the $U_\omega(K)$-orbit of the element $\pi(f_\omega)\in u_\omega(K)^*$ and $f_\omega(e_{in})=0$ we conclude that

$$(\theta^{\omega},\zeta_{in}(\alpha',\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0.$$

Since

$$(\chi_\omega)_{U_\omega(q)}=\theta^{\omega} \quad \text{and} \quad \zeta_{D_0}(\varphi_0)=\xi_{D_0}(\varphi_0)_{U_\omega(q)}$$

we obtain

$$(\chi,\xi_D(\varphi))_{U_n(q)}=\sum_{\beta'\in F_q}(\theta^{\omega},\zeta_{in}(0,\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}$$
$$=\sum_{\beta'\in F_q}(\chi_{\omega},(\zeta_{in}(0,\alpha)\zeta_{ai+1}(\beta',\beta))^{U_n(q)}\xi_{D_0}(\varphi_0))_{U_n(q)}.$$

Since

$$\zeta_{in}(0,\alpha)=\zeta_{i+1n}(\alpha)=\xi_{i+1n}(\alpha)_{U_\omega(q)}$$

we have

$$(\zeta_{in}(0,\alpha)\zeta_{ai+1}(\beta',\beta))^{U_n(q)}=\xi_{i+1n}(\alpha)\zeta_{ai+1}(\beta',\beta)^{U_n(q)}$$

for all $\beta'\in F_q$. Let $\beta'\in F_q$ be non-zero. Then (by corollary 5.2.5)

$$\zeta_{ai+1}(\beta',\beta)^{U_n(q)}=\xi_{ai+1}(\beta')$$

is an irreducible character of $U_n(q)$. Since $(a,i+1)\in D_\omega$ we conclude that

$$(\chi_{\omega},(\zeta_{in}(0,\alpha)\zeta_{ai+1}(\beta',\beta))^{U_n(q)}\xi_{D_0}(\varphi_0))_{U_n(q)}=0.$$

Thus

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\chi_{\omega},(\zeta_{in}(0,\alpha)\zeta_{ai+1}(0,\beta))^{U_n(q)}\xi_{D_0}(\varphi_0))_{U_n(q)}.$$

Since

$$\zeta_{in}(0,\alpha)=\zeta_{i+1n}(\alpha)=\xi_{i+1n}(\alpha)_{U_n(q)} \quad \text{and} \quad \zeta_{ai+1}(0,\beta)=\zeta_{ai}(\beta)$$

we have

$$(\zeta_{in}(0,\alpha)\zeta_{ai+1}(0,\beta))^{U_n(q)}=\xi_{i+1n}(\alpha)\zeta_{ai}(\beta)^{U_n(q)}.$$

Since

$$\zeta_{ai}(\beta)^{U_n(q)}=\sum_{\gamma\in F_q}\xi_{ai}(\beta)\xi_{ii+1}(\gamma)$$

(by corollary 5.2.5) we conclude that

$$(\chi,\xi_D(\varphi))_{U_n(q)}=\sum_{\gamma\in F_q}(\chi_{\omega},\xi_{i+1n}(\alpha)\xi_{ai}(\beta)\xi_{ii+1}(\gamma)\xi_{D_0}(\varphi_0))_{U_n(q)}.$$

Finally we may repeat the argument of the final paragraph of the previous proof to conclude that

$$(\chi_{\omega},\xi_{i+1n}(\alpha)\xi_{ai}(\beta)\xi_{ii+1}(\gamma)\xi_{D_0}(\varphi_0))_{U_n(q)}=0$$

for all $\gamma\in F_q$. Therefore

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\chi_\omega,\xi_{i+1n}(\alpha)\xi_{ai}(\beta)\xi_{D_0}(\varphi_0))_{U_n(q)}=(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}$$

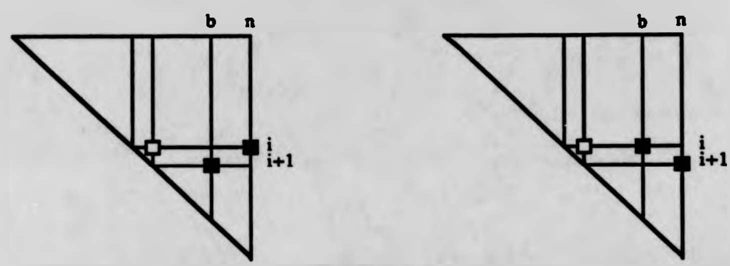
and the proof is complete. ◆

Lemma 5.2.10. *Suppose that* $(i,i+1)\in S^{(r)}(D_\omega)$, $(i,i+1)\in S^{(c)}(D_\omega)$ *and* $(i,i+1)\in S^{(c)}(D)$. *Then*

$$(\chi,\xi_D(\varphi))=\begin{cases}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)) & \text{if } \chi(1)=q\chi_\omega(1)\\ q^{-1}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)) & \text{if } \chi(1)=\chi_\omega(1)\end{cases}.$$

Proof. In this lemma we are concerned with the following situation

D $\qquad\qquad$ D_ω



By theorem 5.1.7 the condition $(i,i+1)\in S^{(c)}(D_\omega)$ implies that there is no $a\in\{1,\ldots,i-1\}$ such that $(a,i)\in D$. Moreover the condition $(i,i+1)\in S^{(r)}(D_\omega)$ implies that there exists $b\in\{i+2,\ldots,n-1\}$ such that $(i+1,b)\in D$. Therefore

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{i+1b}(\beta)\xi_{D_0}(\varphi_0)$$

where $\alpha=\varphi(i,n)$, $\beta=\varphi(i+1,b)$ and $D_0=D\setminus\{(i,n),(i+1,b)\}$. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ib}(\beta)\xi_{D_0}(\varphi_0).$$

Now the argument of the previous proof shows that

$$(\chi,\xi_D(\varphi))_{U_n(q)}=\sum_{\gamma\in F_q}(\theta^\omega,\zeta_{in}(\gamma,\alpha)\zeta_{ib}(\beta,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}$$

- we note that $\zeta_{i+1b}(\beta)$ is the irreducible character of $U_\omega(q)$ (by lemma 5.2.4) which corresponds to the $U_\omega(K)$-orbit of $\beta e_{i+1b}{}^*\in u_\omega(K)^*$, hence $\zeta_{i+1b}(\beta)^\omega$ corresponds to the $U_\omega(K)$-orbit of $\beta e_{ib}{}^*\in u_\omega(K)^*$, i.e.

$$\zeta_{i+1b}(\beta)^\omega=\zeta_{ib}(\beta,0).$$

Let $\gamma \in \boldsymbol{F}_q$ be non-zero, let ϕ be an irreducible component of $\zeta_{in}(\gamma,\alpha)\zeta_{ib}(\beta,0)\zeta_{D_0}(\varphi_0)$ and let O be the $U_\omega(K)$-orbit which corresponds to ϕ. Then we may use corollary 1.3.11 to conclude that any $f \in O$ satisfies $f(e_{1n})=\ldots=f(e_{i-1n})=0$ and $f(e_{in})=\gamma\neq 0$. Therefore

$$(\theta^\omega,\zeta_{in}(\gamma,\alpha)\zeta_{ib}(\beta,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

(because $f_\omega(e_{in})=0$). Since

$$\zeta_{in}(0,\alpha)=\zeta_{i+1n}(\alpha)$$

we obtain

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Now suppose that $\chi(1)=q\chi_\omega(1)$. Then (by corollary 5.2.7) the $U_n(K)$-orbit $O(f_\omega)$ is of the first kind with respect to $U_\omega(K)$. Thus

$$(\chi_\omega)_{U_\omega(q)}=\theta^\omega$$

and

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\chi_\omega,\xi_{i+1n}(\alpha)\zeta_{ib}(\beta,0)^{U_n(q)}\xi_{D_0}(\varphi_0))_{U_n(q)}$$

because

$$(\xi_{i+1n}(\alpha)\xi_{D_0}(\varphi_0))_{U_\omega(q)}=\zeta_{i+1n}(\alpha)\zeta_{D_0}(\varphi_0).$$

Since

$$\zeta_{ib}(\beta,0)^{U_n(q)}=\xi_{ib}(\beta)$$

(by corollary 5.2.5) we conclude that

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}$$

as required.

On the other hand suppose that $\chi(1)=\chi_\omega(1)$. Then (by corollary 5.2.7) the $U_n(K)$-orbit $O(f_\omega)$ is of the second kind with respect to $U_\omega(K)$ so

$$\chi_\omega=(\theta^\omega)^{U_n(q)}.$$

In this case we calculate the multiplicity $(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}$. By lemma 5.2.4 we have

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=\sum_{\gamma\in \boldsymbol{F}_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Now we claim that

$$\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,\gamma)=\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,0)$$

for all $\gamma\in \boldsymbol{F}_q$. Let $\gamma\in \boldsymbol{F}_q$ be arbitrary. Since

$$\xi_{i+1n}(\alpha)_{U_\omega(q)}=\zeta_{i+1n}(\alpha) \quad \text{and} \quad \zeta_{ib}(\beta,\gamma)^{U_n(q)}=\xi_{ib}(\beta)$$

(by lemma 5.2.4 and by corollary 5.2.5 respectively) we have

$$(\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,\gamma))^{U_n(q)}=\xi_{i+1n}(\alpha)\xi_{ib}(\beta).$$

By lemma 2.2.6 (we note that $i<i+1<b<n$) the character $\xi_{i+1n}(\alpha)\xi_{ib}(\beta)$ has q irreducible components which correspond to the $U_n(K)$-orbits of the q distinct elements

$$\alpha e_{i+1n}^*+\beta e_{bi}^*+\mu e_{ii+1}^*\in u_n(K)^*$$

where $\mu\in F_q$. All these components have the same restriction ϕ to $U_\omega(q)$. Moreover ϕ is an irreducible character of $U_\omega(q)$ (because the $U_n(K)$-orbit of $\alpha e_{i+1n}^*+\beta e_{bi}^*+\mu e_{ii+1}^*$ is of the first kind with respect to $U_\omega(K)$ - otherwise $\alpha e_{i+1n}^*+\beta e_{bi}^*+\mu e_{ii+1}^*$ is $U_n(K)$-conjugate to $\alpha e_{i+1n}^*+\beta e_{bi}^*$). It follows that

$$(\xi_{i+1n}(\alpha)\xi_{ib}(\beta))_{U_\omega(q)}=q\phi.$$

Since

$$(\xi_{i+1n}(\alpha)\xi_{ib}(\beta))_{U_\omega(q)}=\sum_{\gamma\in F_q}\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,\gamma)$$

(by lemma 5.2.4) we conclude that

$$\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,\gamma)=\phi$$

and our claim is proved.

Finally we deduce that

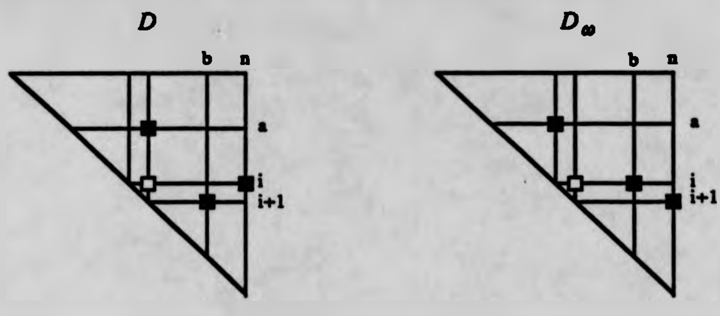
$$\begin{aligned}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}&=q(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}\\&=q(\chi,\xi_D(\varphi))_{U_n(q)}.\end{aligned}$$

The proof of the lemma is complete. ◆

Lemma 5.2.11. *Suppose that* $(i,i+1)\in S^{(r)}(D_\omega)$, $(i,i+1)\in S^{(c)}(D_\omega)$ *and* $(i,i+1)\in S^{(c)}(D)$. *Then*

$$(\chi,\xi_D(\varphi))=\begin{cases}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)) & \text{if } \chi(1)=q\chi_\omega(1)\\ q^{-1}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)) & \text{if } \chi(1)=\chi_\omega(1)\end{cases}$$

Proof. In this lemma we are concerned with the following situation



We follow the proof of the previous lemma. In this case there exists $a\in\{1,\dots,i-1\}$ such that $(a,i+1)\in D$ and

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{i+1b}(\beta)\xi_{ai+1}(\gamma)\xi_{D_0}(\varphi_0)$$

where $\alpha=\varphi(i,n)$, $\beta=\varphi(i+1,b)$, $\gamma=\varphi(a,i+1)$, $D_0=D\setminus\{(i,n),(i+1,b),(a,i+1)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ib}(\beta)\xi_{ai}(\gamma)\xi_{D_0}(\varphi_0).$$

We have

$$(\chi,\xi_D(\varphi))_{U_n(q)}=\sum_{\alpha'\in F_q}\sum_{\gamma'\in F_q}(\theta^\omega,\zeta_{in}(\alpha',\alpha)\zeta_{ib}(\beta,0)\zeta_{ai+1}(\gamma',\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

As before

$$(\theta^\omega,\zeta_{in}(\alpha',\alpha)\zeta_{ib}(\beta,0)\zeta_{ai+1}(\gamma',\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\alpha'\in F_q{}^*$. Thus

$$(\chi,\xi_D(\varphi))_{U_n(q)}=\sum_{\gamma'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,0)\zeta_{ai+1}(\gamma',\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Let $\gamma'\in F_q{}^*$ be arbitrary and let ϕ be any irreducible component of $\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,0)\zeta_{ai+1}(\gamma',\gamma)\zeta_{D_0}(\varphi_0)$. By corollary 1.3.11 the $U_\omega(K)$-orbit O which corresponds to ϕ is contained in the sum of the $U_\omega(K)$-orbits of the elements $\alpha e_{i+1n}{}^*$, $\beta e_{ib}{}^*$, $\gamma' e_{ai+1}{}^*+\gamma e_{ai}{}^*$ and $\varphi(u,v)e_{uv}{}^*$ for all $(u,v)\in D_0$. It follows that any element $g\in O$ satisfies the equation

$$\Delta^{D_\omega}_{ai+1}(g)\neq 0$$

(cf. proposition 3.1.2). Since $f_\omega\in V_{D_\omega}(\varphi_\omega)$ and $(a,i+1)\in R(D_\omega)\setminus D_\omega$ we have

$$\Delta^{D_\omega}_{ai+1}(f_\omega)=0$$

(by (3.1.7)). Therefore

$$(\theta^{\omega},\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,0)\zeta_{ai+1}(\gamma',\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0.$$

Since $\gamma'\in \boldsymbol{F}_q{}^*$ is arbitrary and

$$\zeta_{ai+1}(0,\gamma)=\zeta_{ai}(\gamma)$$

we conclude that

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta^{\omega},\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,0)\zeta_{ai}(\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Now suppose that $\chi(1)=q\chi_\omega(1)$. Then (by corollary 5.2.7) the $U_n(K)$-orbit $O(f_\omega)$ is of the first kind with respect to $U_\omega(K)$. So (by lemma 5.2.6)

$$(\chi_\omega)_{U_\omega(q)}=\theta^{\omega}$$

and

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\chi_\omega,\xi_{i+1n}(\alpha)\zeta_{ib}(\beta,0)^{U_n(q)}\xi_{ai}(\gamma)\xi_{D_0}(\varphi_0))_{U_n(q)}$$

because

$$(\xi_{i+1n}(\alpha)\xi_{ai}(\gamma)\xi_{D_0}(\varphi_0))_{U_\omega(q)}=\zeta_{i+1n}(\alpha)\zeta_{ai}(\gamma)\zeta_{D_0}(\varphi_0).$$

Since

$$\zeta_{ib}(\beta,0)^{U_n(q)}=\xi_{ib}(\beta)$$

we conclude that

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}.$$

On the other hand suppose that $\chi(1)=\chi_\omega(1)$. Then $O(f_\omega)$ is of the first kind with respect to $U_\omega(K)$ so

$$\chi_\omega=(\theta^{\omega})^{U_n(q)}.$$

In this case we repeat the proof of the previous lemma to conclude that

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=q(\theta^{\omega},\zeta_{i+1n}(\alpha)\zeta_{ib}(\beta,0)\zeta_{ai}(\gamma)_{D_0}(\varphi_0))_{U_\omega(q)}$$
$$=q(\chi,\xi_D(\varphi))_{U_n(q)}.$$

The proof of the lemma is complete. ♦

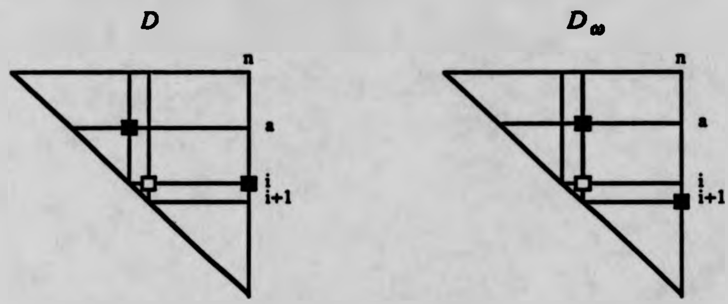
Lemma 5.2.12. *Suppose that* $(i,i+1)\in S^{(c)}(D_\omega)$, $(i,i+1)\in S^{(r)}(D_\omega)$ *and* $(i,i+1)\in S^{(c)}(D)$. *Then*

$$\chi(1)=\chi_\omega(1)$$

and

$$(\chi,\xi_D(\varphi))=(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)).$$

Proof. In this lemma we are concerned with the following situation



Since $(i,i+1)\in S^{(c)}(D)$ there is no $a'\in\{1,\dots,i-1\}$ such that $(a',i+1)\in D$. Therefore $(i,i+1)\in S^{(c)}(D_\omega)$ if and only if there exists $a\in\{1,\dots,i-1\}$ such that $(a,i+1)\in D_\omega$ and $(a,i)\in D$. Moreover the condition $(i,i+1)\in S^{(r)}(D_\omega)$ implies that there is no $b\in\{i+2,\dots,n-1\}$ such that $(i+1,b)\in D$. It follows that

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{ai}(\beta)\xi_{D_0}(\varphi_0)$$

where $\alpha=\varphi(i,n)$, $\beta=\varphi(a,i)$, $D_0=D\setminus\{(i,n),(a,i)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ai+1}(\beta)\xi_{D_0}(\varphi_0).$$

To prove the first assertion of the lemma we claim that the $U_n(K)$-orbit $O(f_\omega)$ is of the second kind with respect to $U_\omega(K)$. For we suppose that $O(f_\omega)$ is of the first kind. Then (by proposition 1.3.6) the image $\pi(O(f_\omega))$ is a single $U_\omega(K)$-orbit in $u_\omega(K)^*$ (here $\pi: u_n(K)^*\to u_\omega(K)^*$ is the natural projection). Let $\lambda\in K^*$ and consider the element $x_{ii+1}(\lambda)=1+\lambda e_{ii+1}\in U_n(K)$. Then

$$g_\omega(\lambda)=\pi(x_{ii+1}(\lambda)\cdot f_\omega)$$

is $U_\omega(K)$-conjugate to $g_\omega=\pi(f_\omega)$. Thus $\omega^{-1}g_\omega(\lambda)\omega$ is $U_\omega(K)$-conjugate to $\omega^{-1}g_\omega\omega=\pi(f)$ (for the definition of $\omega^{-1}h\omega$, $h\in u_\omega(K)^*$, see the proof of lemma 5.2.6). Let $f(\lambda)\in u_n(K)^*$ be the (unique) element satisfying

$$\pi(f(\lambda))=\omega^{-1}g_\omega(\lambda)\omega \quad\text{and}\quad f(\lambda)(e_{ii+1})=0.$$

Since the $U_n(K)$-orbit O(f) is of the second kind with respect to $U_\omega(K)$ we conclude that $f(\lambda)$ is $U_n(K)$-conjugate to f (by proposition 1.3.7). Since

$$x_{ii+1}(\lambda)e_{rs}x_{ii+1}(-\lambda)=e_{rs}+\lambda[e_{ii+1},e_{rs}]$$

we have

$$g_\omega(\lambda)(e_{rs})=\begin{cases} f(e_{rs}) & \text{if } r\neq i+1 \text{ and } s\neq i \\ f(e_{is})+\lambda f(e_{i+1s}) & \text{if } r=i+1 \\ f(e_{ri+1})-\lambda f(e_{ri}) & \text{if } s=i \end{cases}$$

for all $(r,s)\in\Phi(n)\setminus\{(i,i+1)\}$. It follows that

$$f(\lambda)(e_{rs})=(\omega^{-1}g_\omega(\lambda)\omega)(e_{rs})=g_\omega(\lambda)(e_{\omega(r)\omega(s)})=\begin{cases} f(e_{rs}) & \text{if } r\neq i+1 \text{ and } s\neq i \\ f(e_{is})+\lambda f(e_{i+1s}) & \text{if } r=i \\ f(e_{ri+1})-\lambda f(e_{ri}) & \text{if } s=i+1 \end{cases}$$

for all $(r,s)\in\Phi(n)\setminus\{(i,i+1)\}$. Now we consider the function $\Delta^D_{ai+1}:U_n(K)^*\rightarrow K$. Since $(a,i+1)\in R(D)\setminus D$ we have

$$\Delta^D_{ai+1}(f(\lambda))=\Delta^D_{ai+1}(f)=0$$

(we recall that $f\in V_D(\varphi)$). On the other hand

$$\Delta^D_{ai+1}(f(\lambda))=\Delta^D_{ai+1}(f)+\lambda\Delta^D_{ai}(f)$$

so

$$\lambda\Delta^D_{ai}(f)=0.$$

This is a contradiction because $\lambda\neq 0$ and $(a,i)\in D$ (hence $\Delta^D_{ai}(f)\neq 0$). It follows that $O(f_\omega)$ is of the second kind with respect to $U_\omega(K)$ and this implies that

$$\chi(1)=\chi_\omega(1)$$

(by corollary 5.2.7).

Now we have (repeating the usual argument)

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

On the other hand (by lemma 5.2.6)

$$\chi_\omega=(\theta^\omega)^{U_n(q)}.$$

So

$$\begin{aligned}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}&=(\theta^\omega,\zeta_{D_\omega}(\varphi_\omega))_{U_\omega(q)}\\ &=\sum_{\beta'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta,\beta')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.\end{aligned}$$

To conclude the proof of the lemma we claim that

$$(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta,\beta')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\beta' \in F_q^*$. Let $\beta' \in F_q^*$ be arbitrary. Then

$$(\theta^{\omega}, \zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta,\beta')\zeta_{D_0}(\varphi_0))_{U_\omega(q)} = (\theta, \zeta_{in}(\alpha,0)\zeta_{ai+1}(\beta',\beta)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Let ϕ be any irreducible component of $\zeta_{in}(\alpha,0)\zeta_{ai+1}(\beta',\beta)\zeta_{D_0}(\varphi_0)$ and let O be the $U_\omega(K)$-orbit which corresponds to ϕ. Let $g \in u_n(K)^*$ be such that $\pi(g) \in O$. Then $g(e_{1n}) = \ldots = g(e_{i-1n}) = 0$ and $g(e_{in}) = \alpha \neq 0$. Thus the $U_n(K)$-orbit $O(g)$ is of the second kind with respect to $U_\omega(K)$ (see lemma 5.2.3). Therefore $\phi^{U_n(q)}$ is an irreducible character of $U_n(q)$ (by theorem 1.3.9). Since

$$(\zeta_{in}(\alpha,0)\zeta_{ai+1}(\beta',\beta))^{U_n(q)} = \xi_{in}(\alpha)\xi_{ai+1}(\beta')$$

(see the proof of the previous lemma) and

$$\xi_{D_0}(\varphi_0)_{U_\omega(q)} = \zeta_{D_0}(\varphi_0)$$

we conclude that $\phi^{U_n(q)}$ is an irreducible component of

$$(\zeta_{in}(\alpha,0)\zeta_{ai+1}(\beta',\beta)\zeta_{D_0}(\varphi_0))^{U_n(q)} = \xi_{in}(\alpha)\xi_{ai+1}(\beta')\xi_{D_0}(\varphi_0).$$

It follows that

$$\Delta^D_{ai+1}(g) \neq 0$$

for all $g \in O$. Since

$$\Delta^D_{ai+1}(f) = 0$$

(because $(a,i+1) \in R(D) \setminus D$) we conclude that

$$(\theta, \zeta_{in}(\alpha,0)\zeta_{ai+1}(\beta',\beta)\zeta_{D_0}(\varphi_0))_{U_\omega(q)} = 0$$

(because θ corresponds to the $U_\omega(K)$-orbit of the element $\pi(f) \in u_\omega(K)^*$). It follows that

$$(\chi_\omega, \xi_{D_\omega}(\varphi_\omega))_{U_n(q)} = (\theta^{\omega}, \zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}$$

and the proof of the lemma is complete. ♦

Lemma 5.2.13. *Suppose that $(i,i+1) \in S^{(c)}(D_\omega)$, $(i,i+1) \in S^{(r)}(D_\omega)$ and $(i,i+1) \in S^{(c)}(D)$. Suppose also that there is no $b \in \{1,\ldots,i-1\}$ such that $(b,i) \in D$. Then*

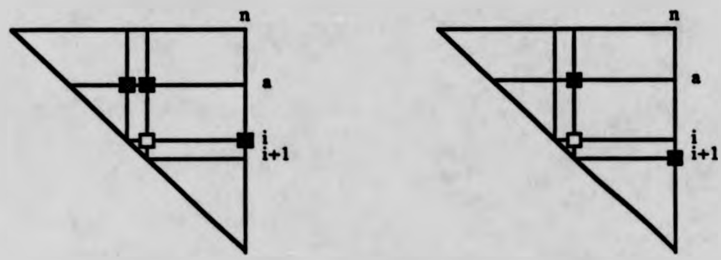
$$\chi(1) = \chi_\omega(1)$$

and

$$(\chi, \xi_D(\varphi)) = (\chi_\omega, \xi_{D_\omega}(\varphi_\omega)).$$

Proof. In this lemma we are concerned with the following situation (the symbol ■ represents a D-derived root)

D $\qquad\qquad\qquad\qquad$ D_ω



In this case there exists $a\in\{1,\ldots,i-1\}$ such that $(a,i+1)\in D$ and

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{ai+1}(\beta)\xi_{D_0}(\varphi_0)$$

where $\alpha=\varphi(i,n)$, $\beta=\varphi(a,i+1)$, $D_0=D\setminus\{(i,n),(a,i+1)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ai+1}(\gamma)\xi_{D_0}(\varphi_0)$$

where $\gamma=\varphi_\omega(a,i+1)$ (we note that $(a,i+1)\in D\cap D_\omega$ (see theorem 5.1.7)).

To prove the first assertion we claim that the $U_n(K)$-orbit $O(f_\omega)$ is of the second kind with respect to $U_\omega(K)$. For we suppose that $O(f_\omega)$ is of the first kind and we repeat the construction of the previous proof to obtain for each $\lambda\in K^*$ the element $f(\lambda)\in O(f)$. Then

$$\Delta^D_{ai+1}(f(\lambda))=\Delta^D_{ai+1}(f)\neq 0$$

(we note that $(a,i+1)\in D$). On the other hand

$$\Delta^D_{ai+1}(f(\lambda))=\Delta^D_{ai+1}(f)+\lambda\Delta^{D_\omega}_{ai+1}(f).$$

Therefore

$$\lambda\Delta^{D_\omega}_{ai+1}(f)=0.$$

This is a contradiction because $\lambda\in K^*$ and

$$\Delta^{D_\omega}_{ai+1}(f)=\Delta^{D_\omega}_{ai+1}(f_\omega)\neq 0$$

(we note that $(a,i+1)\in D_\omega$). This contradiction implies that $O(f_\omega)$ is of the second kind with respect to $U_\omega(K)$. By corollary 5.2.7 we conclude that

$$\chi(1)=\chi_\omega(1).$$

Now (by the usual argument)

$$(\chi,\xi_D(\varphi))_{U_n(q)}=\sum_{\beta'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Let $\beta'\in F_q$ be arbitrary. Let ϕ be any irreducible component of $\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{D_0}(\varphi_0)$ and let O be the $U_\omega(K)$-orbit which corresponds to ϕ. Then there exists a constant $c(\beta')\in F_q$ such that

$$\Delta_{ai+1}^{D_\omega}(g)=c(\beta')$$

for all $g\in O$ (see (3.1.7)). Moreover for all $\beta',\beta''\in F_q$

$$c(\beta')=c(\beta'') \iff \beta'=\beta''.$$

Since

$$\Delta_{ai+1}^{D_\omega}(f_\omega)=c(\gamma)$$

we conclude that

$$(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\beta'\in F_q\backslash\{\beta\}$. Therefore

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\gamma,\beta)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

On the other hand (by lemma 5.2.6)

$$\chi_\omega=(\theta^\omega)^{U_n(q)}.$$

Thus

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=\sum_{\gamma'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\gamma,\gamma')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

To conclude the proof of the lemma we claim that

$$(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\gamma,\gamma')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\gamma'\in F_q\backslash\{\gamma\}$. In fact

$$(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\gamma,\gamma')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=(\theta,\zeta_{in}(\alpha,0)\zeta_{ai+1}(\gamma',\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}$$

and an argument similar to the one used in the previous paragraph (using the function Δ_{ai+1}^{D}) shows that

$$(\theta,\zeta_{in}(\alpha,0)\zeta_{ai+1}(\gamma',\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\gamma'\in F_q\backslash\{\beta\}$.

The proof of the lemma is complete. ♦

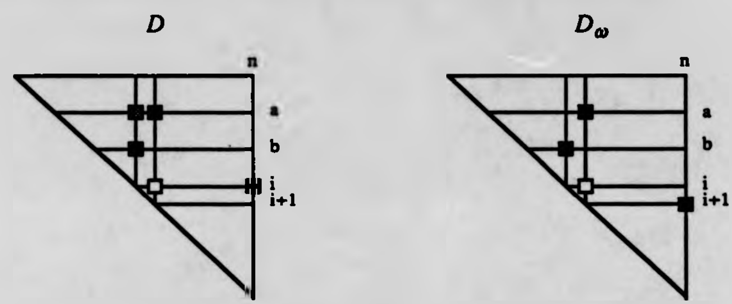
Lemma 5.2.14. *Suppose that* $(i,i+1)\in S^{(c)}(D_\omega)$, $(i,i+1)\in S^{(r)}(D_\omega)$ *and* $(i,i+1)\in S^{(c)}(D)$. *Suppose also that* $(b,i)\in D\cap D_\omega$ *for some* $b\in\{1,\ldots,i-1\}$. *Then*

$$\chi(1)=\chi_\omega(1)$$

and

$$(\chi,\xi_D(\varphi))=(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)).$$

Proof. In this lemma we are concerned with the following situation



In this case there exists $a\in\{1,\ldots,i-1\}$ such that $(a,i+1)\in D$ (by theorem 5.1.7 we must have $a<b$). Hence

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{ai+1}(\beta)\xi_{bi}(\gamma)\xi_{D_0}(\varphi_0)$$

where $\alpha=\varphi(i,n)$, $\beta=\varphi(a,i+1)$, $\gamma=\varphi(b,i)$, $D_0=D\setminus\{(i,n),(a,i+1),(b,i)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ai+1}(\mu)\xi_{bi}(\nu)\xi_{D_0}(\varphi_0)$$

where $\mu=\varphi_\omega(a,i+1)$ and $\nu=\varphi_\omega(b,i)$ (we note that $(a,i+1)\in D\cap D_\omega$).

An imitation of the previous proof shows that the $U_n(K)$-orbit which corresponds to the character χ_ω is of the second kind with respect to $U_\omega(K)$. Thus (by corollary 5.2.7)

$$\chi(1)=\chi_\omega(1).$$

Now (by the usual argument)

$$(\chi,\xi_D(\varphi))_{U_n(q)}=\sum_{\beta'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

As in the previous case we have

$$(\theta^{\omega},\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\beta'\in F_q\setminus\{\mu\}$. Hence

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta^{\omega},\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

On the other hand consider the character $\xi_{D_\omega}(\varphi_\omega)$. We claim that

$$\xi_{D_\omega}(\varphi_\omega)=(\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))^{U_n(q)}.$$

Since

$$\xi_{i+1n}(\alpha)_{U_\omega(q)}=\zeta_{i+1n}(\alpha) \quad \text{and} \quad \xi_{D_0}(\varphi_0)_{U_\omega(q)}=\zeta_{D_0}(\varphi_0)$$

the rigth hand side of that equality is equal to

$$\xi_{i+1n}(\alpha)(\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0))^{U_n(q)}\xi_{D_0}(\varphi_0).$$

Therefore our claim will follow once we prove the equality

$$(\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0))^{U_n(q)}=\xi_{ai+1}(\mu)\xi_{bi}(\nu).$$

Since $a<b$, the character $\xi_{ai+1}(\mu)\xi_{bi}(\nu)$ of $U_n(q)$ is irreducible (by lemma 2.2.3). Moreover (by corollary 1.3.11) it corresponds to the $U_n(K)$-orbit $O_{ai+1}(\mu)+O_{bi}(\nu)$ of the element $\mu e_{ai+1}^*+\nu e_{bi}^*\in u_n(K)^*$. By lemma 3.1.1 this orbit consists of all the elements $g\in u_n(K)^*$ which satisfy $g(e_{uv})=0$ for all $(u,v)\in\Phi(n)$ such that $u<a$ or $i+1<v$, $g(e_{ai+1})=\mu$ and

$$det\begin{pmatrix} g(e_{av}) & g(e_{ai+1}) \\ g(e_{uv}) & g(e_{ui+1}) \end{pmatrix}=-\delta_{ub}\delta_{vi}\mu\nu$$

for all $(u,v)\in D(a,i+1)$. Let ϕ be the irreducible component of $\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0)$ which corresponds to the $U_\omega(K)$-orbit O of the element

$$\pi(\mu e_{ai+1}^*+\beta e_{ai}^*+\gamma e_{bi+1}^*)\in u_\omega(K)^*.$$

Since $\beta\gamma=-\mu\nu$ (by theorem 5.1.7) the element $\mu e_{ai+1}^*+\beta e_{ai}^*+\gamma e_{bi+1}^*\in u_n(K)^*$ satisfies the above equations. Therefore

$$\mu e_{ai+1}^*+\beta e_{ai}^*+\gamma e_{bi+1}^*\in\pi^{-1}(O)\cap(O_{ai+1}(\mu)+O_{bi}(\nu)).$$

By theorem 1.3.9 we conclude that

$$(\xi_{ai+1}(\mu)\xi_{bi}(\nu),\phi^{U_n(q)})\neq0.$$

Since $\xi_{ai+1}(\mu)\xi_{bi}(\nu)$ is irreducible and ϕ is a component of $\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0)$ we deduce that

$$(\xi_{ai+1}(\mu)\xi_{bi}(\nu),(\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0))^{U_n(q)})\neq 0.$$

On the other hand

$$(\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0))^{U_n(q)}(1)=q\,\zeta_{ai+1}(\mu,\beta)(1)\,\zeta_{bi+1}(\gamma,0)(1)$$
$$=qq^{i-a}q^{i-b}=q^{i-a}q^{i-b-1}$$
$$=\xi_{ai+1}(\mu)(1)\,\xi_{bi}(\nu)(1)$$

which completes the proof of the required equality.

Finally

$$(\chi_\omega\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=(\chi_\omega(\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))^{U_n(q)})_{U_n(q)}$$
$$=((\chi_\omega)_{U_\omega(q)},\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Since $O(f_\omega)$ is of the second kind with respect to $U_\omega(K)$ we have (by lemma 5.2.6)

$$(\chi_\omega)_{U_\omega(q)}=\sum_{\lambda\in F_q}\theta^\omega(\lambda)$$

where

$$\theta^\omega(\lambda)=(\theta^\omega)^{x_{ii+1}(\lambda)}$$

for all $\lambda\in F_q$. Hence

$$(\chi_\omega\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=\sum_{\lambda\in F_q}(\theta^\omega(\lambda),\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

To conclude the proof of the lemma we claim that

$$(\theta^\omega(\lambda),\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\mu,\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\lambda\in F_q^*$. Since

$$\theta(\lambda)=(\theta^\omega(\lambda))^\omega$$

this equivalent to prove that

$$(\theta(\lambda),\zeta_{in}(\alpha,0)\zeta_{ai+1}(\beta,\mu)\zeta_{bi}(\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\lambda\in F_q^*$. Let ϕ be an irreducible component of $\zeta_{in}(\alpha,0)\zeta_{ai+1}(\beta,\mu)\zeta_{bi}(\gamma)\zeta_{D_0}(\varphi_0)$ and let O be the $U_\omega(K)$-orbit which corresponds to ϕ. Then (by (3.1.7)) there exists an element $c(\beta)\in F_q$ such that

$$\Delta^D_{ai+1}(g)=c(\beta)$$

for all $g\in O$. On the other hand let $\lambda\in F_q$ be arbitrary and let $O(\lambda)$ be the $U_\omega(K)$-orbit which corresponds to the irreducible character $\theta(\lambda)$. Then (by (3.1.7)) there exists an element $c'(\lambda)\in F_q$ (depending on λ) such that

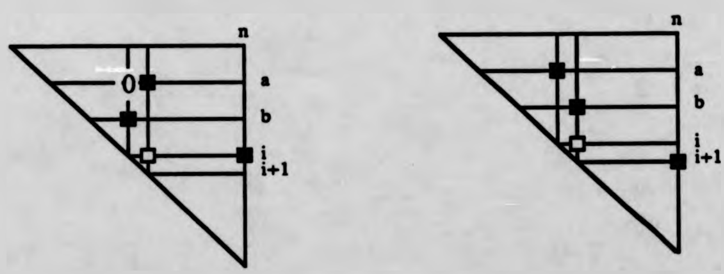
$$\Delta^{D}_{ai+1}(g)=c'(\lambda)$$

for all $g\in O(\lambda)$. Moreover $c'(\lambda)\neq c'(\lambda')$ for all $\lambda'\in \mathbf{F}_q\backslash\{\lambda'\}$. Our claim follows because $c(\beta)=c'(0)$ (as one can easily check). Since $\theta(0)=\theta$ the proof of the lemma is complete. ♦

Lemma 5.2.15. *Suppose that $(i,i+1)\in S^{(c)}(D_\omega)$, $(i,i+1)\in S^{(r)}(D_\omega)$ and $(i,i+1)\in S^{(c)}(D)$. Suppose also that there exists $b\in\{1,\ldots,i-1\}$ such that $(b,i)\in D$ and $(b,i)\in D_\omega$. Let $a\in\{1,\ldots,i-1\}$ be such that $(a,i+1)\in D$ and assume that $a<b$. Then*

$$(\chi,\xi_D(\varphi))=\begin{cases}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)) & \text{if } \chi(1)=q\chi_\omega(1)\\ q^{-1}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)) & \text{if } \chi(1)=\chi_\omega(1)\end{cases}$$

Proof. In this lemma we are concerned with the following situation



In this case we have

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{ai+1}(\beta)\xi_{bi}(\gamma)\xi_{D_0}(\varphi_0)$$

where $\alpha=\varphi(i,n)$, $\beta=\varphi(a,i+1)$, $\gamma=\varphi(b,i)$, $D_0=D\backslash\{(i,n),(a,i+1),(b,i)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ai}(\beta)\xi_{bi+1}(\gamma)\xi_{D_0}(\varphi_0)$$

because $D_\omega=\omega(D)$ and $\varphi_\omega=\varphi\omega$ (by theorem 5.1.7).

As usual we have

$$(\chi,\xi_D(\varphi))_{U_n(q)}=\sum_{\beta'\in\mathbf{F}_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_n(q)}.$$

Since $(a,i)\in D_\omega$ a similar argument to the one used in the previous proof shows that

$$(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_n(q)}=0$$

for all $\beta' \in F_q^*$. Therefore

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta^{\omega},\zeta_{i+1n}(\alpha)\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}$$

because

$$\zeta_{ai+1}(0,\beta)=\zeta_{ai}(\beta).$$

Now suppose that $\chi(1)=\chi_\omega(1)$. Then

$$\chi_\omega=(\theta^{\omega})^{U_n(q)}$$

and so

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=\sum_{\gamma'\in F_q}(\theta^{\omega},\zeta_{i+1n}(\alpha)\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,\gamma')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

An argument similar to the one used in the proof of lemma 5.2.10 shows that

$$\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,\gamma')=\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,0)$$

for all $\gamma' \in F_q$. Therefore

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=q(\theta^{\omega},\zeta_{i+1n}(\alpha)\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}$$
$$=q(\chi,\xi_D(\varphi))_{U_n(q)}$$

as required.

On the other hand suppose that $\chi(1)=q\chi_\omega(1)$. Then

$$(\chi_\omega)_{U_\omega(q)}=\theta^{\omega}$$

and so

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\chi_\omega,(\zeta_{i+1n}(\alpha)\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))^{U_n(q)})_{U_n(q)}.$$

Since

$$\zeta_{i+1n}(\alpha)\zeta_{ai}(\beta)\zeta_{D_0}(\varphi_0)=(\xi_{i+1n}(\alpha)\xi_{ai}(\beta)\xi_{D_0}(\varphi_0))_{U_\omega(q)}$$

we have

$$(\zeta_{i+1n}(\alpha)\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))^{U_n(q)}=\xi_{i+1n}(\alpha)\xi_{ai}(\beta)\zeta_{bi+1}(\gamma,0)^{U_n(q)}\xi_{D_0}(\varphi_0).$$

The result follows because

$$\zeta_{bi+1}(\gamma,0)^{U_n(q)}=\xi_{bi+1}(\gamma)$$

(by corollary 5.2.5). ◆

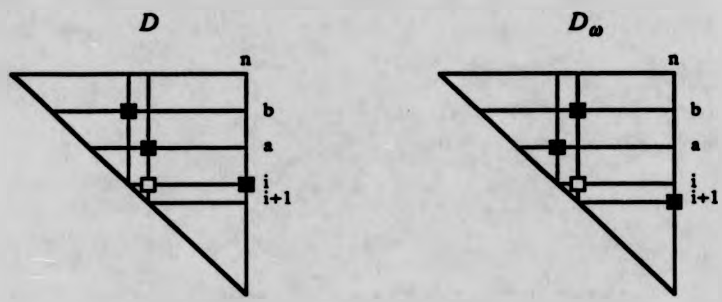
Lemma 5.2.16. *Suppose that $(i,i+1)\in S^{(c)}(D_\omega)$, $(i,i+1)\in S^{(r)}(D_\omega)$ and $(i,i+1)\in S^{(c)}(D)$. Suppose also that there exists $b\in\{1,\dots,i-1\}$ such that $(b,i)\in D$ and $(b,i)\in D_\omega$. Let $a\in\{1,\dots,i-1\}$ be such that $(a,i+1)\in D$ and assume that $b<a$. Then*

$$\chi(1)=\chi_\omega(1)$$

and

$$(\chi,\xi_D(\varphi))=q(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)).$$

Proof. In this lemma we are concerned with the following situation



As in the previous lemma we have

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{ai+1}(\beta)\xi_{bi}(\gamma)\xi_{D_0}(\varphi_0)$$

where $\alpha=\varphi(i,n)$, $\beta=\varphi(a,i+1)$, $\gamma=\varphi(b,i)$, $D_0=D\backslash\{(i,n),(a,i+1),(b,i)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ai}(\beta)\xi_{bi+1}(\gamma)\xi_{D_0}(\varphi_0)$$

because $D_\omega=\omega(D)$ and $\varphi_\omega=\varphi\omega$ (by theorem 5.1.7).

The same argument used in the first part of the proof of lemma 5.2.12 shows that the $U_n(K)$-orbit $O(f_\omega)$ is of the second kind with respect to $U_\omega(K)$. Therefore

$$\chi(1)=\chi_\omega(1) \quad \text{and} \quad \chi_\omega=(\theta^\omega)^{U_n(q)}$$

(by corollary 5.2.7 and by lemma 5.2.6 respectively).

Now we have

$$(\chi,\xi_D(\varphi))_{U_n(q)}=\sum_{\beta'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

We claim that

$$\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0)=\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,0)$$

for all $\beta'\in F_q$. This is clear if $\beta'=0$. Let $\beta'\in F_q{}^*$ and let O be the $U_\omega(K)$-orbit of the element

$$\beta' e_{ai+1}{}^*+\gamma e_{bi+1}{}^*+\beta e_{ai}{}^*\in U_\omega(K)^*.$$

Since $O_{bi+1}(\gamma)+O_{ai}(\beta)$ is the $U_n(K)$-orbit of $\beta' e_{ai+1}{}^*+\gamma e_{bi+1}{}^*+\beta e_{ai}{}^* \in \mathcal{U}_n(K)^*$ (see lemma 3.1.1) we have

$$\pi^{-1}(O)\cap(O_{bi+1}(\gamma)+O_{ai}(\beta))\neq\emptyset$$

where $\pi: \mathcal{U}_n(K)^* \to \mathcal{U}_\omega(K)^*$ is the usual projection. Therefore

$$((\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0))^{U_n(q)},\xi_{ai}(\beta)\xi_{bi+1}(\gamma))_{U_n(q)}\neq 0.$$

By Frobenius reciprocity we conclude that

$$(\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0),(\xi_{ai}(\beta)\xi_{bi+1}(\gamma))_{U_\omega(q)})_{U_\omega(q)}\neq 0.$$

Now (by lemma 5.2.4)

$$(\xi_{ai}(\beta)\xi_{bi+1}(\gamma))_{U_\omega(q)}=\sum_{\gamma'\in F_q}\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,\gamma').$$

For each $\gamma'\in \boldsymbol{F}_q$ let $O(\gamma')$ be the $U_\omega(K)$-orbit which corresponds to the irreducible character $\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,\gamma')$ of $U_\omega(q)$. Then $g(e_{bi})=\gamma'$ for all $g\in O$ and so

$$(\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0),\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,\gamma'))_{U_\omega(q)}=0$$

for all $\gamma'\in \boldsymbol{F}_q{}^*$. It follows that

$$(\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0),\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,0))_{U_\omega(q)}\neq 0,$$

i.e. the irreducible character $\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,0)$ of $U_\omega(q)$ is a component of $\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0)$. Finally we consider character degrees. On the one hand we have

$$(\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,0))(1)=q^{i-a-1}q^{i+1-b-2}=q^{i-a-1}q^{i-b-1}.$$

On the other hand

$$(\zeta_{ai+1}(\beta',\beta)\zeta_{bi+1}(\gamma,0))(1)=q^{i+1-a-2}q^{i+1-b-2}=q^{i-a-1}q^{i-b-1}.$$

The required equality follows. Therefore

$$(\chi,\xi_D(\varphi))_{U_n(q)}=q(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Now

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=\sum_{\gamma'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,\gamma')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Since the $U_\omega(K)$-orbit associated with the character θ contains the element $\pi(f_\omega)\in \mathcal{U}_\omega(K)^*$ ($\pi: \mathcal{U}_n(K)^* \to \mathcal{U}_\omega(K)^*$ is the usual projection) and

$$\Delta^D_{bi+1}(f_\omega)=0$$

we have

$$(\theta,\zeta_{in}(\alpha,0)\zeta_{ai+1}(\beta,0)\zeta_{bi+1}(\gamma',\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\gamma' \in \boldsymbol{F}_q{}^*$. Hence

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ai}(\beta)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}$$

and the lemma follows. ♦

Lemma 5.2.17. *Suppose that* $(i,i+1)\in S^{(c)}(D_\omega)\cap S^{(r)}(D_\omega)$ and $(i,i+1)\in S^{(c)}(D)$. *Then*

$$\chi(1)=\chi_\omega(1)$$

and

$$(\chi,\xi_D(\varphi))=q^{-1}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)).$$

Proof. In this lemma we are concerned with the following situation

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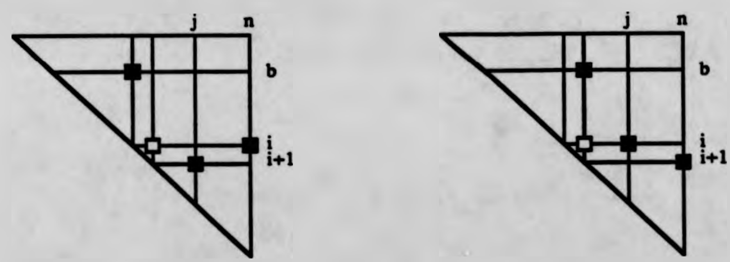


We follow the proof of lemma 5.2.12. The proof of the first assertion is precisely the same. For the second we have

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{i+1j}(\beta)\xi_{bi}(\gamma)\xi_{D_0}(\varphi_0)$$

where $j\in\{i+2,\dots,n-1\}$ is such that $(i+1,j)\in D$, $\alpha=\varphi(i,n)$, $\beta=\varphi(i+1,j)$, $\gamma=\varphi(b,i)$, $D_0=D\setminus\{(i,n),(i+1,j),(b,i)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ij}(\beta)\xi_{bi+1}(\gamma)\xi_{D_0}(\varphi_0).$$

By the usual kind of argument

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

On the other hand we have

$$\chi_\omega=(\theta^\omega)^{U_n(q)}$$

(by the first assertion of the lemma). Hence

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=\sum_{\beta',\gamma'\in \boldsymbol{F}_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')\zeta_{bi+1}(\gamma,\gamma')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Using the function $\Delta^D_{bi+1}:U_n(K)^*\to K$ we may conclude that

$$(\theta,\zeta_{in}(\alpha,0)\zeta_{ij}(\beta',\beta)\zeta_{bi+1}(\gamma',\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\gamma'\in \boldsymbol{F}_q{}^*$. Therefore

$$(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')\zeta_{bi+1}(\gamma,\gamma')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\gamma'\in \boldsymbol{F}_q{}^*$. It follows that

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=\sum_{\beta'\in \boldsymbol{F}_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

As in the proof of lemma 5.2.10 we have

$$\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')=\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)$$

for all $\beta'\in \boldsymbol{F}_q$. Therefore

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=q(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{bi+1}(\gamma,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}$$

and the proof of the lemma is complete. ♦

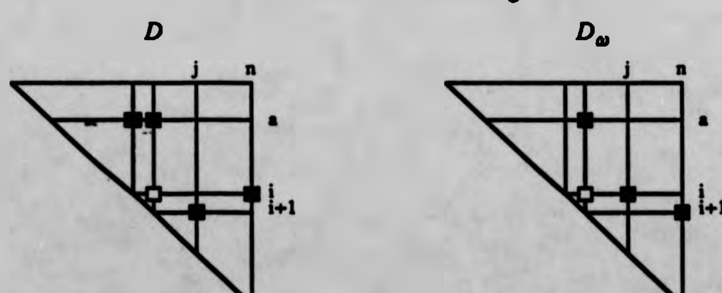
Lemma 5.2.18. *Suppose that $(i,i+1)\in S^{(c)}(D_\omega)\cap S^{(r)}(D_\omega)$ and $(i,i+1)\in S^{(c)}(D)$. Suppose also that $(b,i)\in D$ for all $b\in\{1,\ldots,i-1\}$. Then*

$$\chi(1)=\chi_\omega(1)$$

and

$$(\chi,\xi_D(\varphi))=q^{-1}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)).$$

Proof. In this lemma we are concerned with the following situation



We follow the proof of lemma 5.2.13. The first assertion is a repetition of the corresponding proof of that lemma. For the second we have

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{i+1j}(\beta)\xi_{ai+1}(\gamma)\xi_{D_0}(\varphi_0)$$

where $j\in\{i+2,\ldots,n-1\}$ is such that $(i+1,j)\in D$, $a\in\{1,\ldots,i-1\}$ is such that $(a,i+1)\in D$, $\alpha=\varphi(i,n)$, $\beta=\varphi(i+1,j)$, $\gamma=\varphi(a,i+1)$, $D_0=D\backslash\{(i,n),(i+1,j),(a,i+1)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ij}(\beta)\xi_{ai+1}(\mu)\xi_{D_0}(\varphi_0)$$

where $\mu=\varphi_\omega(a,i+1)$.

As in the proof of lemma 5.2.13

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai+1}(\mu,\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

On the other hand we have

$$\chi_\omega=(\theta^\omega)^{U_n(q)}$$

(by the first assertion of the lemma). Hence

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=\sum_{\beta',\mu'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')\zeta_{ai+1}(\mu,\mu')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Using the function $\Delta^D_{bi+1}:u_n(K)^*\to K$ we may conclude that

$$(\theta,\zeta_{in}(\alpha,0)\zeta_{ij}(\beta',\beta)\zeta_{ai+1}(\mu',\mu)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\mu'\in F_q\backslash\{\gamma\}$. Therefore

$$(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')\zeta_{ai+1}(\mu,\mu')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\mu'\in F_q\backslash\{\gamma\}$. It follows that

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=\sum_{\beta'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')\zeta_{ai+1}(\mu,\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

As in the proof of lemma 5.2.10 we have

$$\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')=\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)$$

for all $\beta'\in F_q$. Therefore

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=q(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai+1}(\mu,\gamma)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}$$

and the proof of the lemma is complete. ♦

Lemma 5.2.19. *Suppose that* $(i,i+1)\in S^{(c)}(D_\omega)\cap S^{(r)}(D_\omega)$ *and* $(i,i+1)\in S^{(c)}(D)$. *Suppose also that* $(b,l)\in D\cap D_\omega$ *for some* $b\in\{1,\ldots,i-1\}$. *Then*

$$\chi(1)=\chi_\omega(1)$$

and

$$(\chi,\xi_D(\varphi))=q^{-1}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)).$$

Proof. In this lemma we are concerned with the following situation

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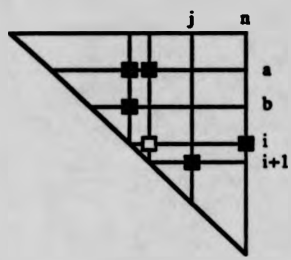

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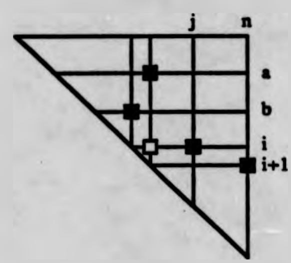

We follow the proof of lemma 5.2.14. The first assertion has the same proof. For the second we have

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{i+1j}(\beta)\xi_{ai+1}(\gamma)\xi_{bi}(\delta)\xi_{D_0}(\varphi_0)$$

where $j\in\{i+2,\dots,n-1\}$ is such that $(i+1,j)\in D$, $a\in\{1,\dots,i-1\}$ is such that $(a,i+1)\in D$, $\alpha=\varphi(i,n)$, $\beta=\varphi(i+1,j)$, $\gamma=\varphi(a,i+1)$, $\delta=\varphi(b,i)$, $D_0=D\backslash\{(i,n),(i+1,j),(a,i+1),(b,i)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ij}(\beta)\xi_{ai+1}(\mu)\xi_{bi}(\nu)\xi_{D_0}(\varphi_0)$$

where $\mu=\varphi_\omega(a,i+1)$ and $\nu=\varphi_\omega(b,i)$ (we note that $(a,i+1)\in D\cap D_\omega$.

As in the proof of lemma 5.2.14 we have

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai+1}(\mu,\gamma)\zeta_{bi+1}(\delta,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

To calculate the Frobenius product $(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}$ we first prove the equality

$$\xi_{D_\omega}(\varphi_\omega)=q(\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai+1}(\mu,\gamma)\zeta_{bi+1}(\delta,0)\zeta_{D_0}(\varphi_0))^{U_n(q)}.$$

By lemma 2.2.6 the character $\xi_{i+1n}(\alpha)\xi_{ij}(\beta)$ has q distinct irreducible components which are parametrized by the elements of the field $\boldsymbol{F}_q$. For each $\lambda\in\boldsymbol{F}_q$ let $\phi(\lambda)$ be the irreducible component of $\xi_{i+1n}(\alpha)\xi_{ij}(\beta)$ which corresponds to λ. Then

$$(\phi(\lambda),\xi_{i+1n}(\alpha)\xi_{ij}(\beta))=1$$

and

$$\xi_{i+1n}(\alpha)\xi_{ij}(\beta)=\sum_{\lambda\in\boldsymbol{F}_q}\phi(\lambda).$$

Therefore

$$\xi_{D_\omega}(\varphi_\omega)=\sum_{\lambda\in F_q}\phi(\lambda)\xi_{ai+1}(\mu)\xi_{bi}(\nu)\xi_{D_0}(\varphi_0).$$

The required equality will follow once we prove the equalities

$$\phi(\lambda)\xi_{ai+1}(\mu)\xi_{ai}(\nu)\xi_{D_0}(\varphi_0)=(\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai+1}(\mu,\gamma)\zeta_{bi+1}(\delta,0)\zeta_{D_0}(\varphi_0))^{U_n(q)}$$

where λ runs over F_q. Let $\lambda\in F_q$ be arbitrary. Since

$$\phi(\lambda)_{U_n(q)}=\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)$$

(see lemma 2.2.6) and

$$\xi_{D_0}(\varphi_0)_{U_n(q)}=\zeta_{D_0}(\varphi_0)$$

the rigth hand side of the equality above is equal to

$$\phi(\lambda)(\zeta_{ai+1}(\mu,\gamma)\zeta_{bi+1}(\delta,0))^{U_n(q)}\xi_{D_0}(\varphi_0).$$

As in the proof of lemma 5.2.14 we have

$$(\zeta_{ai+1}(\mu,\gamma)\zeta_{bi+1}(\delta,0))^{U_n(q)}=\xi_{ai+1}(\mu)\xi_{bi}(\nu).$$

The proof of the lemma now follows as in lemma 5.2.14. ◆

Lemma 5.2.20. *Suppose that $(i,i+1)\in S^{(c)}(D_\omega)\cap S^{(r)}(D_\omega)$ and $(i,i+1)\in S^{(c)}(D)$. Suppose also that there exists $b\in\{1,\ldots,i-1\}$ such that $(b,i)\in D$ and $(b,i)\in D_\omega$. Let $a\in\{1,\ldots,i-1\}$ be such that $(a,i+1)\in D$ and assume that $a<b$. Then*

$$(\chi,\xi_D(\varphi))=\begin{cases} q^{-1}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)) & \text{if } \chi(1)=q\chi_\omega(1)\\ q^{-2}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)) & \text{if } \chi(1)=\chi_\omega(1)\end{cases}.$$

Proof. In this lemma we are concerned with the following situation

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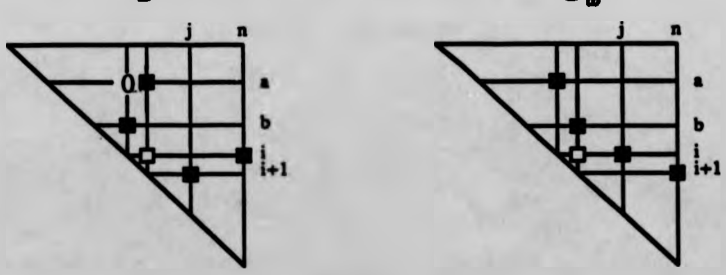


We follow the proof of lemma 5.2.15. We have

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{i+1j}(\beta)\xi_{ai+1}(\gamma)\xi_{bi}(\delta)\xi_{D_0}(\varphi_0)$$

where $j\in\{i+2,\dots,n-1\}$ is such that $(i+1,j)\in D$, $\alpha=\varphi(i,n)$, $\beta=\varphi(i+1,j)$, $\gamma=\varphi(a,i+1)$, $\delta=\varphi(b,i)$, $D_0=D\setminus\{(i,n),(i+1,j),(a,i+1),(b,i)\}$ and φ_0 is the restriction of φ to D_0. On the other hand

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ij}(\beta)\xi_{ai}(\gamma)\xi_{bi+1}(\delta)\xi_{D_0}(\varphi_0)$$

because $D_\omega=\omega(D)$.

As in the proof of lemma 5.2.15

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai}(\gamma)\zeta_{bi+1}(\delta,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Now suppose that $\chi(1)=\chi_\omega(1)$. Then

$$\chi_\omega=(\theta^\omega)^{U_n(q)}$$

so

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=\sum_{\beta',\delta'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')\zeta_{ai}(\gamma)\zeta_{bi+1}(\delta,\delta')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

Now we have

$$\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')=\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)$$

for all $\beta'\in F_q$ (see the proof of lemma 5.2.10). On the other hand

$$\zeta_{ai}(\gamma)\zeta_{bi+1}(\delta,\delta')=\zeta_{ai}(\gamma)\zeta_{bi+1}(\delta,0)$$

for all $\delta'\in F_q$ (see the proof of lemma 5.2.15). Therefore

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=q^2(\chi,\xi_D(\varphi))_{U_n(q)}$$

as required.

On the other hand suppose that $\chi(1)=q\chi_\omega(1)$. Then

$$(\chi_\omega)_{U_\omega(q)}=\theta^\omega$$

so

$$(\chi,\xi_D(\varphi))_{U_n(q)}=(\chi_\omega,(\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai}(\gamma)\zeta_{bi+1}(\delta,0)\zeta_{D_0}(\varphi_0))^{U_n(q)})_{U_n(q)}.$$

The result will follow once we prove the equality

$$\xi_{D_\omega}(\varphi_\omega)=q(\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai}(\gamma)\zeta_{bi+1}(\delta,0)\zeta_{D_0}(\varphi_0))^{U_n(q)}.$$

Since

$$\zeta_{ai}(\gamma)\zeta_{D_0}(\varphi_0)=(\xi_{ai}(\gamma)\xi_{D_0}(\varphi_0))_{U_n(q)},$$

the rigth hand side of the equality above is equal to

$$(\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{bi+1}(\delta,0))^{U_n(q)}\xi_{ai}(\gamma)\xi_{D_0}(\varphi_0).$$

By theorem 3.3.3 the character $\xi_{i+1n}(\alpha)\xi_{ij}(\beta)\xi_{bi+1}(\delta)$ has a unique irreducible component which appears with multiplicity q. We denote this component by ϕ and we claim that

$$\phi=(\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{bi+1}(\delta,0))^{U_n(q)}.$$

In fact ϕ corresponds to the $U_n(K)$-orbit of the element

$$\alpha e_{i+1n}{}^*+\beta e_{ij}{}^*+\delta e_{bi+1}{}^*\in u_n(K)^*.$$

The image of this element under the projection $\pi: u_n(K)^*\rightarrow u_\omega(K)^*$ lies in a $U_\omega(K)$-orbit which corresponds to an irreducible component of $\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{bi+1}(\delta,0)$. Therefore (by theorem 1.3.9)

$$(\phi,(\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{bi+1}(\delta,0))^{U_n(q)})_{U_n(q)}\neq 0.$$

Now we have

$$\phi(1)=q^{-1}q^{n-(i+1)-1}q^{j-i-1}q^{(i+1)-b-1}=q^{n+j-i-b-4}$$

whereas

$$(\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{bi+1}(\delta,0))^{U_n(q)}(1)=qq^{n-(i+1)-1}q^{j-i-2}q^{(i+1)-b-2}=q^{n+j-i-b-4}.$$

The desired equalities follow and the proof of the lemma is complete. ◆

Lemma 5.2.21. *Suppose that* $(i,i+1)\in S^{(c)}(D_\omega)\cap S^{(r)}(D_\omega)$ *and* $(i,i+1)\in S^{(c)}(D)$. *Suppose also that there exists* $b\in\{1,\ldots,i-1\}$ *such that* $(b,i)\in D$ *and* $(b,i)\in D_\omega$. *Let* $a\in\{1,\ldots,i-1\}$ *be such that* $(a,i+1)\in D$ *and assume that* $b<a$. *Then*

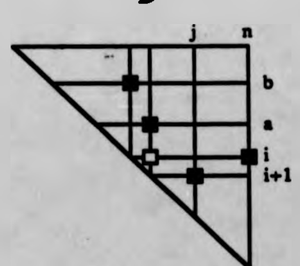
$$\chi(1)=\chi_\omega(1)$$

and

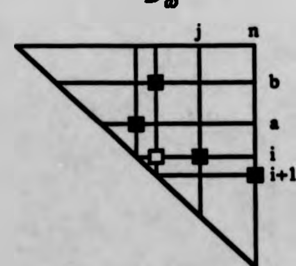
$$(\chi,\xi_D(\varphi))=(\chi_\omega,\xi_{D_\omega}(\varphi_\omega)).$$

Proof. In this lemma we are concerned with the following situation

D



D_ω



As in the previous lemma we have

$$\xi_D(\varphi)=\xi_{in}(\alpha)\xi_{i+1j}(\beta)\xi_{ai+1}(\gamma)\xi_{bi}(\delta)\xi_{D_0}(\varphi_0)$$

and

$$\xi_{D_\omega}(\varphi_\omega)=\xi_{i+1n}(\alpha)\xi_{ij}(\beta)\xi_{ai}(\gamma)\xi_{bi+1}(\delta)\xi_{D_0}(\varphi_0)$$

where $j\in\{i+2,\dots,n-1\}$ is such that $(i+1,j)\in D$, $\alpha=\varphi(i,n)$, $\beta=\varphi(i+1,j)$, $\gamma=\varphi(a,i+1)$, $\delta=\varphi(b,i)$, $D_0=D\setminus\{(i,n),(i+1,j),(a,i+1),(b,i)\}$ and φ_0 is the restriction of φ to D_0.

The first assertion is proved exactly as in lemma 5.2.16. In particular

$$\chi_\omega=(\theta^\omega)^{U_n(q)}.$$

For the second we have

$$(\chi,\xi_D(\varphi))_{U_n(q)}=\sum_{\gamma'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai+1}(\gamma',\gamma)\zeta_{bi+1}(\delta,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

As in the proof of lemma 5.2.16

$$\zeta_{ai+1}(\gamma',\gamma)\zeta_{bi+1}(\delta,0)=\zeta_{ai}(\gamma)\zeta_{bi+1}(\delta,0)$$

for all $\gamma'\in F_q$. Therefore

$$(\chi,\xi_D(\varphi))_{U_n(q)}=q(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai}(\gamma)\zeta_{bi+1}(\delta,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

On the other hand

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=\sum_{\beta',\delta'\in F_q}(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')\zeta_{ai}(\gamma)\zeta_{bi+1}(\delta,\delta')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}.$$

As in the previous cases

$$\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,\beta')=\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)$$

for all $\beta'\in F_q$. On the other hand (repeating the argument in the proof of lemma 5.2.16) we have

$$(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai}(\gamma)\zeta_{bi+1}(\delta,\delta')\zeta_{D_0}(\varphi_0))_{U_\omega(q)}=0$$

for all $\delta'\in F_q^*$. Hence

$$(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))_{U_n(q)}=q(\theta^\omega,\zeta_{i+1n}(\alpha)\zeta_{ij}(\beta,0)\zeta_{ai}(\gamma)\zeta_{bi+1}(\delta,0)\zeta_{D_0}(\varphi_0))_{U_\omega(q)}$$

and the lemma follows. ♦

Now we complete the proof of proposition 5.2.2. For $a\in\{1,2,\dots,14\}$, case (a) corresponds to the situation of lemma 5.2.(7+a).

Case 1. We have $\chi(1)=q\chi_\omega(1)$ and $(\chi,\xi_D(\varphi))=(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))$. Since

$$s(D)=s(D_\omega)+2 \quad\text{and}\quad l(D)=l(D_\omega)+1$$

we conclude that

$$\begin{aligned}\chi(1)&=q\chi_\omega(1)=qq^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))\\&=qq^{s(D)-2-l(D)+1}(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi)).\end{aligned}$$

Case 2. As in the previous case $\chi(1)=q\chi_\omega(1)$ and $(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))=(\chi,\xi_D(\varphi))$. Since

$$s(D)=s(D_\omega)+3 \quad\text{and}\quad l(D)=l(D_\omega)+2$$

we obtain

$$\begin{aligned}\chi(1)&=q\chi_\omega(1)=qq^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))\\&=qq^{s(D)-3-l(D)+2}(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi)).\end{aligned}$$

Case 3. In this case we have

$$s(D)=s(D_\omega)+1 \quad\text{and}\quad l(D)=l(D_\omega).$$

Therefore either

$$\begin{aligned}\chi(1)&=q\chi_\omega(1)=qq^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))\\&=qq^{s(D)-1-l(D)}(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi))\end{aligned}$$

or

$$\begin{aligned}\chi(1)&=\chi_\omega(1)=q^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))\\&=q^{s(D)-1-l(D)}q(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi)).\end{aligned}$$

Case 4. We have

$$s(D)=s(D_\omega)+2 \quad\text{and}\quad l(D)=l(D_\omega)+1.$$

Therefore either

$$\begin{aligned}\chi(1)&=q\chi_\omega(1)=qq^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))\\&=qq^{s(D)-2-l(D)+1}(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi))\end{aligned}$$

or

$$\begin{aligned}\chi(1)&=\chi_\omega(1)=q^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))\\&=q^{s(D)-2-l(D)+1}q(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi)).\end{aligned}$$

Case 5. In this case $\chi(1)=\chi_\omega(1)$ and $(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))=(\chi,\xi_D(\varphi))$. Proposition 5.2.2 follows immediately because

$$s(D)=s(D_\omega) \quad \text{and} \quad l(D)=l(D_\omega).$$

Case 6. We have $\chi(1)=\chi_\omega(1)$ and $(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))=(\chi,\xi_D(\varphi))$. Since

$$s(D)=s(D_\omega)+1 \text{ and } l(D)=l(D_\omega)+1$$

we obtain

$$\chi(1)=\chi_\omega(1)=q^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))$$
$$=q^{s(D)-1-l(D)+1}(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi)).$$

Case 7. This case has the same justification as the previous case.

Case 8. We have

$$s(D)=s(D_\omega)+2 \quad \text{and} \quad l(D)=l(D_\omega)+1.$$

Therefore either

$$\chi(1)=q\chi_\omega(1)=qq^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))$$
$$=qq^{s(D)-2-l(D)+1}(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi))$$

or

$$\chi(1)=\chi_\omega(1)=q^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))$$
$$=q^{s(D)-2-l(D)+1}\,q(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi)).$$

Case 9. We have $\chi(1)=\chi_\omega(1)$ and $(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))=q^{-1}(\chi,\xi_D(\varphi))$. Since

$$s(D)=s(D_\omega) \quad \text{and} \quad l(D)=l(D_\omega)+1$$

we conclude that

$$\chi(1)=\chi_\omega(1)=q^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))$$
$$=q^{s(D)-l(D)+1}\,q^{-1}(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi)).$$

Case 10. We have $\chi(1)=\chi_\omega(1)$ and $(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))=q(\chi,\xi_D(\varphi))$. Since

$$s(D)=s(D_\omega) \quad \text{and} \quad l(D)=l(D_\omega)-1$$

we conclude that

$$\chi(1)=\chi_\omega(1)=q^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))$$
$$=q^{s(D)-l(D)-1}\,q(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi)).$$

Case 11. In this case we have

$$s(D)=s(D_\omega)+1 \quad \text{and} \quad l(D)=l(D_\omega).$$

Therefore

$$\chi(1)=\chi_\omega(1)=q^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))$$
$$=q^{s(D)-1-l(D)}\,q(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi)).$$

Case 12. This case has the same justification as the previous one.

Case 13. We have

$$s(D)=s(D_\omega)+2 \quad \text{and} \quad l(D)=l(D_\omega).$$

Therefore either

$$\chi(1)=q\chi_\omega(1)=qq^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))$$
$$=qq^{s(D)-2-l(D)}q(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi))$$

or

$$\chi(1)=\chi_\omega(1)=q^{s(D_\omega)-l(D_\omega)}(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))$$
$$=q^{s(D)-2-l(D)}\,q^2(\chi,\xi_D(\varphi))=q^{s(D)-l(D)}(\chi,\xi_D(\varphi)).$$

Case 14. We have $\chi(1)=\chi_\omega(1)$ and $(\chi_\omega,\xi_{D_\omega}(\varphi_\omega))=(\chi,\xi_D(\varphi))$. Proposition 5.2.2 follows immediately because

$$s(D)=s(D_\omega) \quad \text{and} \quad l(D)=l(D_\omega).$$

The proof of proposition 5.2.2 is complete.

Proof of theorem 5.2.1. Let D be any basic subset of $\Phi(n)$. Then

$$\xi_D=\sum_{\chi\in I_D}(\chi,\xi_D)\chi$$

where I_D denotes the set of all irreducible components of ξ_D. Let $\chi\in I_D$ and let $\varphi\colon D\to \mathbf{F}_q^*$ be the unique map such that

$$(\chi,\xi_D(\varphi))\neq 0.$$

Then (by proposition 5.2.2)

$$(\chi,\xi_D)=(\chi,\xi_D(\varphi))=q^{l(D)-s(D)}\chi(1).$$

Hence

$$q^{s(D)-\ell(D)}\xi_D=\sum_{\chi\in I_D}\chi(1)\chi.$$

By theorem 2.2.1 we conclude that

$$\rho=\sum_{\chi\in Irr(U_n(q))}\chi(1)\chi=\sum_{\substack{D\subseteq\Phi(n)\\ D\ basic}}\sum_{\chi\in I_D}\chi(1)\chi=\sum_{\substack{D\subseteq\Phi(n)\\ D\ basic}}q^{s(D)-\ell(D)}\xi_D$$

as required. ♦

Corollary 5.2.22. *Let $x\in U_n(q)$. Then*

$$\sum_{\substack{D\subseteq\Phi(n)\\ D\ basic}}q^{s(D)-\ell(D)}\xi_D(x)=\delta_{x1}q^{n(n-1)/2}$$

where δ_{x1} is the Kronecker symbol.

Proof. This is an immediate consequence of theorem 5.2.1 and of the properties of the regular character ρ. ♦

Finally we note that proposition 3.2.15 is a consequence of the previous corollary. In fact

Corollary 5.2.23. *The following equality holds*

$$\sum_{\substack{D\subseteq\Phi(n)\\ D\ basic}}(q-1)^{|D|}q^{s(D)}=q^{n(n-1)/2}$$

Proof. This is clear from the previous corollary because

$$\xi_D(1)=\sum_{\varphi}\xi_D(\varphi)(1)=\sum_{\varphi}q^{\ell(D)}=(q-1)^{|D|}q^{\ell(D)}$$

where φ runs over all maps from D to $\boldsymbol{F}_q^*$. ♦

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