Integral forms for Weyl modules of GL(2,Q)

Perdita Emma Stevens

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University of Warwick
Mathematics Institute
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Contents

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Graham Weetman did not live to see this work written up. It is dedicated to his memory.

Declaration

I declare that this thesis was composed by myself, and the work contained in it is my own except where otherwise stated.

Abstract

In this thesis we determine the integral forms in Weyl modules for GL(2, Q). We work with the Schur algebra exclusively; we do not use the Lie algebra of GL(2, Q).

In Chapter 1 we give the necessary background. We begin to simplify the problem, using the known reduction of it to the problem of finding those integral forms which lie between certain limits X and V. Together with localisation at each prime p, this enables us to restrict our attention to the structure of X/V_p. We show that we can deduce the integral structure of any Weyl module from that of Weyl modules with highest weight $(r, 0)$ for an integer r. We describe a duality which arises on X/V_p. In Chapter 2 we prove a rather surprising number-theoretic result which allows us to simplify the problem further. In Chapter 3 we arrive at a very simple characterisation of the integral forms, namely that they can be represented as those integer labellings of a particular graph, the scoreable set lattice, which satisfy a certain criterion. We exploit this to prove various general results about the structure of X/V_p. We show how it is possible, using our methods, to describe the structure of X/V_p in arbitrarily complicated cases in terms of simpler structures. In Chapter 4, we discuss the relevance of our work to the theory of modular Weyl modules, and we explain how our work relates to that of others.

Chapter 1

Introduction and preliminaries

1.1 General background

In this section we shall give the background to this thesis, briefly and largely without proof. For more detail, see [Green 1], [Green 2]. A great deal of terminology and notation is necessarily introduced; the reader is assured that only a small proportion of it will reappear in later chapters. The rest is used to explain and justify a reformulation, in elementary terms, of the question which is answered by this thesis.

The starting point for this work is the study of finite dimensional polynomial representations of general linear groups. By a *representation* of $\mathrm{GL}_n(\mathbf{C})$ is meant a group homomorphism

$$R : \mathrm{GL}_n(\mathbf{C}) \longrightarrow \mathrm{GL}_N(\mathbf{C})$$

for a positive integer N. In his doctoral dissertation of 1901, [Schur], Issai Schur found, for each integer n, all finite-dimensional polynomial representations of $\mathrm{GL}_n(\mathbf{C})$, in the following sense.

Definition 1.1.

Consider the representation R of $\mathrm{GL}_n(\mathbf{C})$ defined by the set of equations

$$R(g) = (r_{\mu\nu}(g))_{1 \le \mu,\nu \le N}$$

as g runs over $\mathrm{GL}_n(\mathbf{C})$. R is *polynomial* if, for each pair (μ, ν), there is some complex polynomial in n^2 variables (one variable for each place in the matrix g) such that the coefficient $r_{\mu\nu}(g)$ is the evaluation at g of this polynomial. $\triangleleft$

Moreover, Schur showed that every finite-dimensional polynomial representation of $\mathrm{GL}_n(\mathbf{C})$ is equivalent to a direct sum of homogeneous ones, where a polynomial representation is *homogeneous of degree* r if each of the corresponding polynomials in n^2 variables is homogeneous of degree r.

A complex representation R of $\mathrm{GL}_n(\mathbf{C})$ may be obtained from a module for the group algebra $\mathbf{C}\mathrm{GL}_n(\mathbf{C})$. The problem is that this algebra has infinite

$\mathbb{C}$-dimension, and so is hard to deal with. Schur's break-through was to show that any polynomial representation which is homogeneous of degree r may be obtained from a module over a certain finite-dimensional complex algebra, now known as the Schur algebra, and often denoted $S(n,r)$. Schur considered this as a complex algebra. However, it is easy to extend the definition to give an algebra $S_K(n,r)$ for any field K. It turns out that for fixed values of n and r, the family of algebras $S_K(n,r)$ is *defined over* $\mathbb{Z}$, in the sense that

(i) there is a basis $\{\xi_{\mathbb{C}}\}$ of $S_{\mathbb{C}}(n,r)$ whose $\mathbb{Z}$-span $S_{\mathbb{Z}}(n,r)$ is multiplicatively closed, and which contains the identity element of $S_{\mathbb{C}}(n,r)$;

(ii) for any field K there is a K-algebra isomorphism

$$\begin{aligned} S_{\mathbb{Z}}(n,r)\otimes_{\mathbb{Z}} K &\longrightarrow S_K(n,r) \\ v_{\mathbb{Z}}\otimes 1_K &\longmapsto v_K \end{aligned}$$

We choose to define the rational version $S_{\mathbb{Q}}(n,r)$ of the Schur algebra. The body of this thesis will be concerned with the $\mathbb{Z}$-lattice $S_{\mathbb{Z}}(n,r)$.

There are several equivalent definitions of the Schur algebra; the one which will be useful for our purposes is approached as follows.

Let E be an n-dimensional $\mathrm{GL}_n(\mathbb{Q})$-space with basis $\{e_1,\ldots,e_n\}$. In order to investigate the polynomial representations of $\mathrm{GL}_n(\mathbb{Q})$ which are homogeneous of degree r, we consider the r-fold tensor product $E^{\otimes r}$ which has basis

$$\{e_{i_1}\otimes\ldots\otimes e_{i_r} \mid i=(i_1,\ldots,i_r)\in I(n,r)\}$$

where

$$I(n,r)=\mathbf{n}^{\mathbf{r}}=\{1,\ldots,n\}^{\{1,\ldots,r\}}.$$

We shall normally write $I(n,r)$ simply as I. The symmetric group of degree r, which we denote by P, acts on the right on I, and hence on $E^{\otimes r}$, by place permutation. That is,

$$i\pi=(i_1,\ldots,i_r)\pi=(i_{\pi(1)},\ldots,i_{\pi(r)})$$

so

$$(e_{i_1}\otimes\ldots\otimes e_{i_r})\pi=e_{i_{\pi(1)}}\otimes\ldots\otimes e_{i_{\pi(r)}}.$$

We shall denote by Π a transversal of the P-orbits of I.

Definition 1.2.

The P-orbits on I are the *weights*, denoted

$$(\lambda_1, \ldots, \lambda_n) = \lambda \in \Lambda(n,r)$$

where λ_μ is the number of occurrences of the natural number μ in any element i of I which is in the P-orbit λ. Then the weights form the set

$$\Lambda(n,r) = \{\lambda \in \mathbb{N}^n \mid \lambda_1 + \ldots \lambda_n = r\}.$$

The size of the orbit λ is written $|\lambda|$, and we see that

$$|\lambda| = \frac{r!}{\lambda_1! \ldots \lambda_n!}$$

◁

Definition 1.3.

The weight λ is *dominant* if

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.$$

The set of dominant weights is written $\Lambda^+(n,r)$. ◁

Example 1.4.

Let $n = 2$ and $r = 5$. Then the following are elements of $I(n,r)$:

$$i = (1,1,1,2,1)$$

$$j = (2,1,2,1,2).$$

Then i is in the weight $\alpha = (4,1)$, as is, for example, $i' = (2,1,1,1,1)$. Similarly $j \in (2,3) = \beta$. Notice that α is a dominant weight whilst β is not.

We have given the right-action on $I(n,r)$ of P, the symmetric group of degree r. We now define a left-action on I which is equally important.

Definition 1.5.

Let the symmetric group of degree n be W, so that W acts naturally on $\mathbf{n}$ and hence on the basis $\{e_1, \ldots, e_n\}$ of E. Then W acts on the left on $I(n,r)$ by

$$w(i) = w(i_1, \ldots, i_r) = (w(i_1), \ldots, w(i_r)),$$

and hence on $E^{\otimes r}$, by extending linearly the action on the basis of $E^{\otimes r}$ which is given by

$$\begin{aligned} w(e_i) &= w(e_{i_1} \otimes \ldots \otimes e_{i_r}) \\ &= w(e_{i_1}) \otimes \ldots \otimes w(e_{i_r}) \\ &= e_{w(i_1)} \otimes \ldots \otimes e_{w(i_r)} \\ &= e_{w(i)}. \end{aligned}$$

The actions of P and W on I commute. Therefore the following action of W on $\Lambda(n,r)$ is well-defined:

$$w(\alpha) = (\alpha_{w^{-1}(1)}, \ldots, \alpha_{w^{-1}(n)}).$$

◁

W is so-called because it is the Weyl group of GL_n considered as algebraic group.

Definition 1.6.

We define a partial order $\geq$ on $\Lambda(n,r)$ by saying that $\alpha \geq \beta$ if and only if, for all integers ρ,

$$\sum_{\nu=1}^{\nu=\rho} \alpha_\nu \geq \sum_{\nu=1}^{\nu=\rho} \beta_\nu.$$

If $\alpha \geq \beta$ and $\alpha \neq \beta$ we say that α is *higher* than β. ◁

Then it is easy to see that if α is dominant then $\alpha \geq w(\alpha)$ for every $w \in W$, and that every W-orbit on $\Lambda(n,r)$ contains exactly one dominant weight. Moreover, if $n = 2$ then $\geq$ is a total order on $\Lambda(n,r)$, and $\lambda = (r,0)$ is higher than any other weight.

Definition 1.7.

The *Schur algebra* $S_{\mathbf{Q}}(n,r)$ is the $\mathbf{Q}$-algebra of all P-invariant $\mathbf{Q}$-module endomorphisms of the r-fold tensor product of E:

$$S_{\mathbf{Q}}(n,r) = \mathrm{End}_{\mathbf{Q}P}(E^{\otimes r})$$

◁

Definition 1.8.

The action of P on I can be extended to an action of P on $I \times I$ by setting

$$(i,j)\pi = (i\pi, j\pi).$$

We write $(i,j) \sim (k,l)$ if (i,j) and (k,l) fall in the same P-orbit under this action, that is, if there is some element π of P such that $i\pi = k$ and $j\pi = l$. We shall use Ω to denote a transversal of the set of P-orbits on $I \times I$. ◁

We may define an element X of $\mathrm{End}_{\mathbf{Q}}(E^{\otimes r})$ by its matrix $(X_{ij})_{i,j \in I(n,r)}$ over $\mathbf{Q}$, with respect to the basis $\{e_i \mid i \in I(n,r)\}$ of $E^{\otimes r}$. Then the condition for X to be an element of $\mathrm{End}_{\mathbf{Q}P}(E^{\otimes r})$ is that for every π in P and i in $I(n,r)$,

$$(X(e_i))\pi = X(e_i\pi),$$

that is,

$$\sum_{\nu \in I(n,r)} X_{\nu i}\, e_{\nu\pi} = \sum_{\nu \in I(n,r)} X_{\nu,i\pi}\, e_{\nu}.$$

Summing the left-hand side over $\mu = \nu\pi^{-1}$ instead of over ν, and comparing coefficients, we see that this is equivalent to the simple condition that $X_{ij} = X_{kl}$ whenever $(i,j) \sim (k,l)$.

We use this to define a $\mathbf{Q}$-basis

$$\{\xi_{ij} \mid i,j \in I\}$$

of the Schur algebra $S_{\mathbf{Q}}(n,r)$ in which the element ξ_{ij} is most easily visualised as an $n^r \times n^r$ matrix (over $\mathbf{Q}$) in which the (k,l) coefficient is 1 if $(i,j) \sim (k,l)$ and 0 otherwise. From this we see that the elements ξ_{ij} and ξ_{kl} are equal if and only if $(i,j) \sim (k,l)$. Therefore ([Green 1] page 19) the dimension of the Schur algebra is

$$\binom{n^2+r-1}{r}.$$

We derive the multiplication rule for basis elements:

$$\xi_{ij}\xi_{kl} = \sum_{(p,q)\in\Omega} |\{\, s \mid ((i,j) \sim (p,s)) \wedge ((k,l) \sim (s,q))\,\}|\, \xi_{pq}$$

of which we shall use mostly the following special cases:

Lemma 1.9.

For any elements i and j of $I(n,r)$,

(i) $\xi_{ij}\xi_{kl} = 0$ unless $j \sim k$;
(ii) $\xi_{ij}\xi_{jj} = \xi_{ij} = \xi_{ii}\xi_{ij}$;
(iii) $\xi_{w(i)i}\xi_{ij} = \xi_{w(i)j}$ and $\xi_{ij}\xi_{jv(j)} = \xi_{iv(j)}$ where w and v are any elements of W.

Proof.

We prove only the first part of (iii). Suppose that (p,q) and s are such that $(w(i), i) \sim (p,s)$ and $(i,j) \sim (s,q)$. That is, there are elements π and ϕ of P such that

$$w i\pi = p$$
$$i\pi = s$$
$$i\phi = s$$
$$j\phi = q;$$

that is, using the fact that the actions of P and W commute,

$$wi\phi = ws = wi\pi = p$$

so $(p,q) = (w(i),j)\phi$ and $s = i\phi$. The result follows, since the summation in the multiplication rule is only over Ω, so it will include exactly one such pair (p,q). $\square$

The elements ξ_{ii} and ξ_{jj} are equal if and only if i and j occur in the same P-orbit, or weight, α. In this case we shall write the element as ξ_α. Then we see that

(i) for any weight α, $\xi_\alpha \xi_\alpha = \xi_\alpha$;
(ii) for any weights $\alpha \neq \beta$, $\xi_\alpha \xi_\beta = 0$,

giving an orthogonal idempotent decomposition

$$1_{S_{\mathbf{Q}}(n,r)} = \sum_{\alpha \in \Lambda(n,r)} \xi_\alpha$$

which is easy to see in matrix terms. This decomposition induces a decomposition of any left $S_{\mathbf{Q}}(n,r)$ module M,

$$M = \bigoplus_{\alpha \in \Lambda(n,r)} \xi_\alpha M.$$

In fact, since for any α we have $\xi_\alpha M = M^\alpha$, (see [Green 1] page 37) this decomposition coincides with the weight space decomposition of M.

We next define a map from $\mathrm{GL}_n(\mathrm{Q})$ to $\mathrm{End}_{\mathbf{Q}P}(E^{\otimes r})$ which will enable us to explain Schur's crucial result.

Definition 1.10.

Let $g \in \mathrm{GL}_n(\mathbf{Q})$. Then define $\mathcal{T}(g)$ by

$$\mathcal{T}(g) = \sum_{(i,j) \in \Omega} g_{i,j} \xi_{ij}$$

where $g_{i,j}$ is defined to be $g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_r j_r}$ and extend this linearly to give a map

$$\mathcal{T} : \mathbf{Q}\mathrm{GL}_n(\mathbf{Q}) \longrightarrow \mathrm{End}_{\mathbf{Q}P}(E^{\otimes r}).$$

◁

The map $\mathcal{T}$ turns out to be a surjective Q-algebra homomorphism. ([Green 1] page 23). Schur's result may then be stated:

Theorem 1.11.

The category of left $S_{\mathbf{Q}}(n,r)$-modules is equivalent to the category of polynomial representations of $\mathrm{GL}_n(\mathbf{Q})$ which are homogeneous of degree r. In fact if V is any object from either category, it may be turned into an object of the other by the rule

$$\kappa v = \mathcal{T}(\kappa)v$$

where $\kappa \in \mathbf{Q}\mathrm{GL}_n(\mathbf{Q})$.

Let w and v be elements of W. Using the natural action of W on the basis of E, we find a matrix n_w in $\mathrm{GL}_n(\mathbf{Q})$ for w, given by

$$(n_w)_{ij} = \delta_{i,w(j)}$$

and a corresponding element $1.n_w$ of the group algebra $\mathbf{Q}\mathrm{GL}_n(\mathbf{Q})$. Then define the element of $S_{\mathbf{Q}}(n,r)$

$$\Gamma_w = \mathcal{T}(1.n_w),$$

and define Γ_v similarly.

Lemma 1.12.

For any i and j in $I(n,r)$,

$$\Gamma_w \xi_{i,j} \Gamma_v^{-1} = \xi_{w(i),v(j)}.$$

Proof.

In fact, from the definition of $\mathcal{T}$,

$$\begin{aligned}
\Gamma_w &= \sum_{(i,j)\in\Omega} (n_w)_{i,j}\xi_{ij} \\
&= \sum_{(i,j)\in\Omega} (n_w)_{i_1 j_1}(n_w)_{i_2 j_2}\dots(n_w)_{i_r j_r}\xi_{ij} \\
&= \sum_{(i,j)\in\Omega} \delta_{i_1,w(j_1)}\delta_{i_2,w(j_2)}\dots\delta_{i_r,w(j_r)}\xi_{ij} \\
&= \sum_{j\in\Pi} \xi_{w(j),j}
\end{aligned}$$

Therefore

$$\begin{aligned}
\Gamma_w\xi_{i,j}\Gamma_v^{-1} &= \Gamma_w\xi_{i,j}\Gamma_{v^{-1}} \\
&= (\sum_{k\in\Pi}\xi_{w(k),k})\xi_{i,j}(\sum_{k\in\Pi}\xi_{v^{-1}(k),k}) \\
&= (\sum_{k\in\Pi}\xi_{w(k),k})\xi_{i,j}(\sum_{k\in\Pi}\xi_{k,v(k)}) \\
&= \xi_{w(k),k}\xi_{i,j}\xi_{k,v(k)}
\end{aligned}$$

where $h \sim i$ and $j \sim k$. Suppose $h\pi = i$ and $j = k\phi$. Then

$$\begin{aligned}\xi_{w(h),h}\,\xi_{i,j}\,\xi_{k,v(k)} &= \xi_{w(h)\pi,h\pi}\,\xi_{i,j}\,\xi_{k\phi,v(k)\phi} \\ &= \xi_{w(i),i}\,\xi_{i,j}\,\xi_{j,v(j)}\end{aligned}$$

since the actions of P and W commute. Then by Lemma 1.9,we get

$$\Gamma_w \xi_{i,j} \Gamma_v^{-1} = \xi_{w(i),v(j)}$$

as required. □

Corollary 1.13.

For any $w \in W$, $\alpha \in \Lambda(n,r)$ and $V \in \mathrm{mod} S_{\mathbf{Z}}(n,r)$,

$$\Gamma_w V^\alpha = V^{w(\alpha)}$$

Proof.

Pick any $i \in \alpha$. We have $\Gamma_w \xi_{i,i} = \xi_{w(i),w(i)} \Gamma_w$, so applying each side to V gives

$$\Gamma_w V^\alpha = \Gamma_w \xi_\alpha V = \Gamma_w \xi_{i,i} V = \xi_{w(i),w(i)} \Gamma_w V \subseteq \xi_{w(\alpha)} V = V^{w(\alpha)}$$

□

1.2 Weyl modules

In [CarterLusztig] R. Carter and G. Lusztig defined, for infinite fields K, the $S_K(n,r)$-modules $V_{\lambda,K}$ for dominant weights λ, calling them Weyl modules. These modules are submodules of $E^{\otimes r}$. They are important because each has a unique maximal $S_K(n,r)$-submodule $V_{\lambda,K}^{\max}$, such that $V_{\lambda,K}/V_{\lambda,K}^{\max} = L_\lambda$ is a simple $S_K(n,r)$-module of highest weight λ. As λ runs over the set Λ^+ of dominant weights every simple $S_K(n,r)$-module occurs once. Moreover, when K has characteristic 0, as in our case when we consider $K = \mathbf{Q}$, the Weyl modules are themselves irreducible.

Carter and Lusztig gave a basis $\{b_i \mid i \in I\}$ for each $V_{\lambda,\mathbf{Q}}$ which can be partitioned to give bases for the weight spaces

$$V_{\lambda,\mathbf{Q}} = \bigoplus_{\alpha \in \Lambda} V_{\lambda,\mathbf{Q}}^\alpha.$$

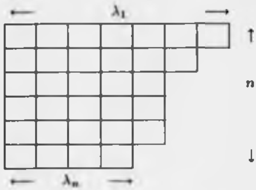
In order to explain what this basis is, we need to introduce some more machinery.

Definition 1.14.

Given a weight $\lambda \in \Lambda(n,r)$, the *shape* of λ, written $[\lambda]$, is the set of integer pairs

$$[\lambda] = \{(s,t) \mid 1 \leq s \leq n, 1 \leq t \leq \lambda_s\}.$$

This set may be regarded as the set of r squares in a diagram like:

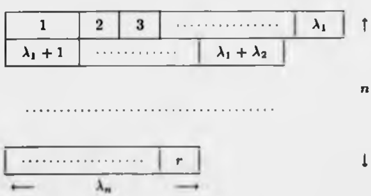


in which the i^{th} row has length λ_i, and the element (s,t) is in row s and column t. ◁

Definition 1.15.

A *λ-tableau* is a map from $[\lambda]$ to any set. The map may be used to label the squares of the diagram with elements of the set.

The *basic λ-tableau* $T^{(\lambda)}$ is an arbitrarily chosen bijection from $[\lambda]$ to $\mathbf{r}$. We shall normally omit the superscript. When we use T to label the squares of the diagram with the integers $1,\ldots,r$, each of the integers between 1 and r appears exactly once. From now on we shall assume that T is as shown:



◁

Definition 1.16.

The symmetric group of degree r, P, acts on the elements of $[\lambda]$ in a way determined by the choice of basic λ-tableau T:

$$(s,t)\pi \stackrel{\text{def}}{=} T^{-1}(T(s,t)\pi);$$

in words, the image, under π, of the square labelled with integer ν is the square labelled with $\nu\pi$.

Then the *column stabiliser* $C(T)$ is defined in the intuitive way, by

$$C(T) = \{\pi \in P \mid (\forall(s,t) \in [\lambda])(\exists u \in \mathbf{n})((s,t)\pi = (u,t))\}$$

and the *row stabiliser* $R(T)$ is defined analogously. ◁

Now, for any $i \in I(n,r)$ we form the composite λ-tableau T_i, a map from $[\lambda]$ to n, by:

$$[\lambda] \xrightarrow{T} \mathbf{r} \xrightarrow{i} \mathbf{n}$$

This may be seen as a labelling of the diagram with integers between 1 and n, in which the square (s,t) is labelled with $T_i(s,t)$.

Example 1.17.

Consider again the case $n = 2$ and $r = 5$, and take $\lambda = (4,1)$. A possible basic λ-tableau is

1	2	3	4
5			

Then if $i = (2,1,2,1,2)$ the λ-tableau T_i is

2	1	2	1
2			

Definition 1.18.

The λ-tableau T_i is ***standard*** if the entries in each row are weakly increasing from left to right and the entries in each column are strictly increasing from top to bottom. ◁

An important example of a standard tableau is T_l in the following definition.

Definition 1.19.

The element $l \in I(n,r)$ is that whose λ-tableau T_l is given by:

$$
\begin{array}{ll}
\longleftarrow \qquad \lambda_1 \qquad \longrightarrow & \\
\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & \cdots\cdots\cdots & 1 \\ \hline 2 & \multicolumn{3}{c|}{\cdots\cdots\cdots\quad 2} \\ \cline{1-4} \end{array} & \uparrow \\
 & n \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
\begin{array}{|c|c|c|}\hline n & \cdots & n \\ \hline \end{array} & \downarrow \\
\longleftarrow \quad \lambda_n \quad \longrightarrow &
\end{array}
$$

that is, such that for each (s,t), we have $l_{T(s,t)} = s$. ◁

Definition 1.20.

If Q is any subgroup of P, the element $\{Q\}$ of $\mathbb{Z}P$ is

$$\sum_{q\in Q} \operatorname{sign}(q)q.$$

◁

Definition 1.21.

Let

$$f_l = e_l\{C(T)\}$$

and for each $i \in I(n,r)$ let

$$b_i = \xi_{il} f_l.$$

◁

One can show ([Green 1] page 68) that $f_l \in V^{\lambda}_{\lambda,\mathbf{Q}}$. The same reference gives another expression for b_i, which can be more convenient:

$$b_i = \sum_{h\in iR(T)} e_h\{C(T)\}.$$

We can now finally state the Carter-Lusztig basis theorem.

Theorem 1.22.

The set

$$\{b_i \mid \text{the } \lambda\text{-tableau } T_i \text{ is standard}\}$$

is a basis for $V_{\lambda,\mathbf{Q}}$. Moreover, for each weight $\alpha \in \Lambda(n,r)$, the set

$$\{b_i \mid i \in \alpha \text{ and the } \lambda\text{-tableau } T_i \text{ is standard}\}$$

is a basis for $V^{\alpha}_{\lambda,\mathbf{Q}}$.

For a proof, see [CarterLusztig] or [Green 1] page 69.

Corollary 1.23.

(i) $V_{\lambda,\mathbf{Q}}$ is generated over $S_{\mathbf{Q}}(n,r)$ by f_l;

(ii) Since l is the only element of λ such that T_l is standard, $V^{\lambda}_{\lambda,\mathbf{Q}}$ is always one-dimensional, with basis

$$b_l = \xi_{ll} f_l = f_l$$

□

We now come to consider the $\mathbf{Z}$-version of these concepts, which are all defined over $\mathbf{Z}$ in the sense given earlier; see [Green 1] for details. Taking the $\mathbf{Z}$-span of the basis $\{e_i \mid i \in I(n,r)\}$ of $E_{\mathbf{Q}}^{\otimes r}$ gives a free $\mathbf{Z}$-lattice in $E_{\mathbf{Q}}^{\otimes r}$, which we call $E_{\mathbf{Z}}^{\otimes r}$. Taking the $\mathbf{Z}$-span of the basis for $S_{\mathbf{Q}}(n,r)$ gives a $\mathbf{Z}$-form in $S_{\mathbf{Q}}(n,r)$ which we denote by $S_{\mathbf{Z}}(n,r)$. We define the $S_{\mathbf{Z}}(n,r)$-module $V_{\lambda,\mathbf{Z}}$ to be $E_{\mathbf{Z}}^{\otimes r} \cap V_{\lambda,\mathbf{Q}}$, and it can be shown that the Carter-Lusztig basis for $V_{\lambda,\mathbf{Q}}$ is also a $\mathbf{Z}$-basis for $V_{\lambda,\mathbf{Z}}$; the partition into bases of the weight spaces is also preserved.

Definition 1.24.

The $S_{\mathbf{Z}}(n,r)$-submodule M of $V_{\lambda,\mathbf{Q}}$ is an *admissible $\mathbf{Z}$-lattice* if it is a free $\mathbf{Z}$-module with a $\mathbf{Z}$-basis which is also a $\mathbf{Q}$-basis of $V_{\lambda,\mathbf{Q}}$, so that

$$M \otimes_{\mathbf{Z}} \mathbf{Q} = V_{\lambda,\mathbf{Q}}.$$

◁

We have a weight space decomposition

$$M = \bigoplus_{\alpha \in \Lambda} M^{\alpha}$$

in which each M^{α} is contained in the corresponding $V^{\alpha}_{\lambda,\mathbf{Q}}$.

If M is an admissible $\mathbf{Z}$-lattice then so is any rational multiple of M. Therefore we introduce a normalisation condition

$$M^{\lambda} = \mathbf{Z} f_l$$

which implies that

$$M^{\lambda} = V^{\lambda}_{\lambda,\mathbf{Z}}.$$

We now state a theorem which greatly restricts the set of possibilities for normalised admissible $\mathbb{Z}$-lattices. The definition of the lattice $X_{\lambda,\mathbb{Z}}$ which appears in the statement will follow. For a proof of this theorem, see [Green 1] page 78.

Theorem 1.25.

Let M be any normalised admissible $\mathbb{Z}$-lattice. Then

$$V_{\lambda,\mathbb{Z}} \subseteq M \subseteq X_{\lambda,\mathbb{Z}}.$$

Definition 1.26.

The bilinear form $<,>$ on $E^{\otimes r}$ is defined on basis elements by

$$< e_i, e_j >= \delta_{ij}$$

◁

It is easy to show that $< x, y\pi >=< x\pi, y >$ for any x and y in $E^{\otimes r}$ and for any π in P. This enables the following definition to be made:

Definition 1.27.

The bilinear form $\langle\langle,\rangle\rangle$ on $E^{\otimes r}\{C(T)\}$ is defined by

$$\langle\langle x\{C(T)\}, y\{C(T)\}\rangle\rangle = \langle x\{C(T)\}, y\rangle = \langle x, y\{C(T)\}\rangle$$

◁

Finally we can define $X_{\lambda,\mathbb{Z}}$.

Definition 1.28.

$$X_{\lambda,\mathbb{Z}} = \{x \in V_{\lambda,\mathbb{Q}} \mid \langle\langle x, V_{\lambda,\mathbb{Z}}\rangle\rangle \subseteq \mathbb{Z}\}$$

◁

We have now introduced all of the general background material that we shall need. In the next section we introduce the special case with which this thesis is concerned, and show how the restrictions we shall impose affect the problem.

1.3 Specific background

In this thesis we show how to find all normalised admissible $\mathbb{Z}$-lattices for $n = 2$ and $\lambda = (r, 0)$. Thus we appear to have made two restrictions from the general case which we have so far been discussing. We have restricted to the case $n = 2$, and then further limited the cases we have to consider by setting $\lambda = (r, 0)$. In fact, only the restriction to $n = 2$ has any real substance. The next result deals with the case that $n = 2$, with no restriction on λ.

Lemma 1.29.

Let α and λ be in $\Lambda^+(2,r)$. Then

$$\dim(V^{\alpha}_{\lambda,\mathbb{Q}}) = \begin{cases} 1 & \text{if } \alpha \leq \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

Consider the possible standard tableaux T_i. The entries in such a tableau come from $\mathbf{n}$, so here the only possible entries are 1 and 2. Moreover, if T_i is standard then the entries in each column increase strictly; so any standard tableau looks like

$$\begin{array}{l}
\longleftarrow \qquad\qquad\qquad\qquad \lambda_1 \qquad\qquad\qquad\qquad \longrightarrow \\
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & \cdots\cdots\cdots & 1 & ? & \cdots\cdots\cdots & ? \\
\hline
2 & \cdots\cdots\cdots & 2 & \multicolumn{3}{c}{} \\
\cline{1-3}
\end{array} \\
\longleftarrow \qquad \lambda_2 \qquad \longrightarrow
\end{array}$$

where the first λ_2 entries in the first row are 1, and all entries in the second row are 2. Moreover, since the entries in each row must increase weakly, if any entry in the first row is 2 then every entry to its right is also 2. Therefore the possible number of 1s which occur in the standard tableau T_i – that is, α_1 where $i \in \alpha$ – must satisfy

$$\lambda_2 \leq \alpha_1 \leq \lambda_1$$

and conversely, for any α_1 in this range there is exactly one $i \in \alpha = (\alpha_1, \alpha_2)$ such that T_i is standard. The α which satisfy the condition are precisely those $\alpha \leq \lambda$, and the Carter-Lusztig basis theorem gives the result. $\square$

Remark 1.30.

Notice that the partial order $\geq$ on Λ is in fact a total order when $n = 2$. When $\alpha \leq \lambda$ we write the single basis element of $V^{\alpha}_{\lambda,\mathbb{Q}}$ as b_α.

Remark 1.31.

By combining Corollary 1.13 with Lemma 1.29 we get the dimension of any weight space V^{α}_{λ}, dropping the requirement that α be dominant.

Example 1.32.

Let $r = n = 2$ and consider $V_{(1,1)}$. By Lemma 1.29 the only non-zero weight space is $V^{(1,1)}_{(1,1)}$, and this has the single basis element

$$\begin{aligned} b_{(12)} &= \sum_{h \in (12)R(T)} e_h\{C(T)\} \\ &= e_{12} - e_{21} \end{aligned}$$

since here the row stabiliser is trivial and the column stabiliser consists of the identity and the single transposition. Next, consider the action of GL_2 on this Weyl module.

$$\begin{aligned}\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix} b_{12} &= \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix} e_{12} - \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix} e_{21} \\ &= \sum_{i \in I(2,2)} (g_{i_1 1} g_{i_2 2} - g_{i_1 2} g_{i_2 1}) e_i \\ &= (g_{11}g_{22} - g_{12}g_{21})e_{12} - (g_{11}g_{22} - g_{12}g_{21})e_{21} \\ &= \det(g) b_{12}.\end{aligned}$$

That is, any element of GL_2 acts on $V_{(1,1)}$ as multiplication by its determinant. $V_{(1,1)}$ is referred to as the *determinant representation* of GL_2.

Next we use this example to help show that the restriction that we wish to make, to considering weights $\lambda = (r, 0)$, is not a serious restriction.

Proposition 1.33.

Any normalised admissible $\mathbb{Z}$-lattice for $n = 2$ and $\lambda = (r, s)$ $(r \geq s)$ is equal to the tensor product of a normalised admissible $\mathbb{Z}$-lattice for $n = 2$ and $\lambda = (r - s, 0)$ with s copies of the *determinant representation*, namely the one-dimensional $\mathbb{Z}GL_2$-module which maps each element g of $GL_2(\mathbb{Z})$ to its determinant.

Proof.

In this result we find it convenient to use the basic λ-tableau T in which the labels increase down columns, then across rows.

Recall that a Weyl module V_α, where $\alpha \in \Lambda(n, r)$, is a submodule of $E^{\otimes r}$. Recall also (Lemma 1.29) that each non-zero weight space V^α of a Weyl module V for GL_2 is 1-dimensional, with basis b_i where i is the unique element of α such that T_i is standard. Suppose that $i \in \alpha \in \Lambda(2,2)$ and that $j \in \beta \in \Lambda(2, r+s)$. Then $b_i \in E^{\otimes 2}$ and $b_j \in E^{\otimes(r+s)}$. Then

$$b_i \otimes b_j \in E^{\otimes(r+s+2)}.$$

We define a bijection

$$\begin{aligned} M : I(2,2) \times I(2,r) &\longrightarrow I(2, r+2) \\ ((i_1, i_2), (j_1, \dots j_r)) &\longmapsto (i_1, i_2, j_1, \dots, j_r)\end{aligned}$$

and we consider the induced 'concatenation' of tableaux:

$$T_i^{(1,1)} \otimes T_j^{(r,s)} \longrightarrow T_{M(i,j)}^{(r+1,s+1)}$$

$$
\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \end{array}
\otimes
\begin{array}{|c|c|c|}\hline j_1 & \cdots\cdots\cdots & j_{r+s} \\ \hline j_2 & \cdots\cdots & \\ \cline{1-2} \end{array}
\longrightarrow
\begin{array}{|c|c|c|c|}\hline i_1 & j_1 & \cdots\cdots\cdots & j_{r+s} \\ \hline i_2 & j_2 & \cdots\cdots & \\ \cline{1-3} \end{array}
$$

The effect of M on an element of $I(2, r+s)$ is essentially to add 2 to the subscript of each element in the tuple $i = (i_1, \ldots, i_{r+s})$. This has a corresponding effect on the group P which permutes the places in such a tuple. We define a renumbering map

$$
\begin{aligned}
N : \{1, \ldots r+s\} &\longrightarrow \{3, \ldots r+s+2\} \\
\nu &\longmapsto \nu + 2
\end{aligned}
$$

and use the same symbol for the maps which it induces from $\mathrm{Sym}\{1, \ldots, r+s\}$ to $\mathrm{Sym}\{3, \ldots, r+s+2\}$ and from $\mathbb{Z}\mathrm{Sym}\{1, \ldots, r+s\}$ to $\mathbb{Z}\mathrm{Sym}\{3, \ldots, r+s+2\}$. Of course, applying this map has no effect on the structure of the Weyl module.

Notice that the tableau $T_{M(i,j)}$ is standard if and only if both T_i and T_j are standard, and that all standard $(r+1, s+1)$-tableaux are obtained in this way. If T_i is standard then of course $\alpha = (1,1)$ and $i = (1,2)$. If T_j is standard then $\beta \leq (r+1, s+1)$. We assume that this is the case, and show that

$$b_i \otimes b_N(j) = b_{M(i,j)}$$

from which it will follow that

$$V^{\alpha}_{(1,1)} \otimes V^{\beta}_{(r,s)} = V^{(\beta_1+1, \beta_2+1)}_{(r+1, s+1)}$$

hence

$$V_{(1,1)} \otimes V_{(r,s)} = V_{(r+1, s+1)}$$

and the result claimed follows by induction, using Example 1.32.

Now,

$$b_{M(i,j)} \stackrel{\text{def}}{=} \sum_k (e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{r+s+2}})\{C(T^{(r+1,s+1)})\}$$

where the sum is over k in the subset of $I(2, r+s+2)$ defined by

$$k \in M(i,j)R(T^{(r+1,s+1)}).$$

This is not as complicated as it looks. For we know, by the restriction to standard tableaux, that $(M(i,j))_1 = 1$ and that all entries in the bottom row of the composite tableau, including $(M(i,j))_2$, are 2. Therefore $k_2 = 2$ for any k occurring in the summation. Suppose that $k_1 = 2 = k_2$. Then

$$\begin{aligned} e_k\{C(T^{(r+1,s+1)})\} &= e_k(1-(12))N(C(T^{(r,s)})) \\ &= 0 \end{aligned}$$

The point is that such a k makes no contribution to the sum, so we deduce that the summation may be taken over k such that $k_1 = 1$ and $k_2 = 2$. Therefore the summation is over the subset of $I(2, r+s+2)$ defined by

$$k \in M(i,j)R(T^{(r,s)}).$$

Then the sum becomes

$$\begin{aligned} b_{M(i,j)} &= \sum_k (e_{k_1} \otimes e_{k_2})(1-(12)) \otimes (e_{k_3} \otimes e_{k_{r+s+2}})N(\{C(T^{(r,s)})\}) \\ &= \sum_k (e_{12} - e_{21}) \otimes (e_{N(k_1)} \otimes e_{N(k_{r+s})})N(\{C(T^{(r,s)})\}) \\ &= (e_{12} - e_{21}) \otimes b_N(j) \\ &= b_i \otimes b_N(j) \end{aligned}$$

as claimed. □

Next we start to show what simplifications are possible when we assume that $n = 2$ and that $\lambda = (r, 0)$. The proof of each part of the following lemma is immediate.

Lemma 1.34.

Let $\lambda = (r, 0)$. Then

(i) The column stabiliser $C(T)$ is trivial.

(ii) $f_I = e_I\{C(T)\} = e_I$

(iii) Let $\alpha \in \Lambda(n, r)$. Then

$$b_\alpha = \sum_{k \in \alpha} e_k.$$

(iv) The two forms $\langle , \rangle$ and $\langle\langle , \rangle\rangle$ are identical.

(v) $\langle b_\alpha, b_\beta \rangle = |\alpha| \delta_{\alpha\beta}$

□

Next, we describe the normalised admissible $\mathbb{Z}$-lattices $X_{\lambda,\mathbb{Z}}$ and $V_{\lambda,\mathbb{Z}}$, and show how we can make use of Theorem 1.25. We introduce the following piece of shorthand:

Definition 1.35.

The $S_{\mathbf{Z}}(n,r)$-submodule M of $V_{\lambda,\mathbf{Q}}$ is *valid* if and only if it is a normalised admissible $\mathbf{Z}$-lattice. ◁

Lemma 1.36.

Let $n = 2$ and let $\lambda = (r, 0)$. Then

(i)
$$V^{\alpha}_{\lambda,\mathbf{Z}} = b_{\alpha}\mathbf{Z}$$

(ii)
$$X^{\alpha}_{\lambda,\mathbf{Z}} = \frac{1}{|\alpha|} b_{\alpha}\mathbf{Z}$$

(iii) If M is any valid module then
$$M^{\alpha} = \frac{m_{\alpha}}{|\alpha|} b_{\alpha}\mathbf{Z}$$

for some positive integer m_{α}.

Proof.

We have already shown that (i) holds when $n = 2$, whatever the value of λ. Any element x of $V_{\lambda,\mathbf{Q}}$ can be written as
$$x = \sum_{\alpha\in\Lambda} x_{\alpha} b_{\alpha}$$
for some elements x_{α} of $\mathbf{Q}$, using the basis theorem. Using this, part (i) and Lemma 1.34 on the definition of $X_{\lambda,\mathbf{Z}}$, we see see that x is in $X_{\lambda,\mathbf{Z}}$ if and only if for every weight β the quantity
$$\begin{aligned}\left\langle\!\!\left\langle \sum_{\alpha\in\Lambda} x_{\alpha} b_{\alpha}, b_{\beta} \right\rangle\!\!\right\rangle &= \sum_{\alpha\in\Lambda} x_{\alpha} \langle\!\langle b_{\alpha}, b_{\beta} \rangle\!\rangle \\ &= \sum_{\alpha\in\Lambda} x_{\alpha}|\alpha|\delta_{\alpha\beta} \\ &= x_{\beta}|\beta|\end{aligned}$$
is in $\mathbf{Z}$. Therefore $X^{\alpha}_{\lambda,\mathbf{Z}}$ has the form claimed in part (ii). Part (iii) follows from these descriptions of X and V and Theorem 1.25. □

Therefore we may identify the valid module M by the tuple $\{m_{\alpha}\}_{\alpha\in\Lambda}$, and may investigate the conditions on this tuple which ensure that M is valid. We shall say that the tuple $\{m_{\alpha}\}_{\alpha\in\Lambda}$ corresponds to the module M. Notice that the tuple $(1, 1, \ldots, 1)$ corresponds to X and that the tuple $\{|\alpha|\}$ corresponds to V.

Now M is valid, if and only if for all i and j

$$\xi_{ij}M \subseteq M.$$

We can simplify this using the weight space decomposition of M and Lemma 1.9 (ii). For if $i \in \alpha$ and $j \in \beta$ we see that

$$\xi_{ij}V_{\mathbf{Q}}^{\beta} = \xi_{ij}\xi_{jj}V_{\mathbf{Q}} = \xi_{ii}\xi_{ij}V_{\mathbf{Q}} \subseteq \xi_{ii}V_{\mathbf{Q}} = V_{\mathbf{Q}}^{\alpha}$$

so the condition which we need to check is that for all weights α and β in Λ and for all elements i of α and j of β,

$$\xi_{ij}M^{\beta} \subseteq M^{\alpha},$$

which says that the $\mathbf{Z}$-lattice is not denser in some parts of M than in others. Checking this is greatly simplified in our case, since all the weight spaces have dimension 1. Rather surprisingly, it suffices to consider dominant weights α and β, as we shall now show.

Lemma 1.37.

Let the valid module $M \leq V_{(r,0)}$ be given by

$$M^{\alpha} = \frac{m_{\alpha}}{|\alpha|}b_{\alpha}\mathbf{Z}$$

where for each weight α, m_{α} is a positive integer, as previously explained. Then for any $w \in W$ and $\alpha \in \Lambda(2,r)$,

$$m_{\alpha} = m_{w(\alpha)}.$$

Proof.

Corollary 1.13 gives

$$\Gamma_w M^{\alpha} = M^{w(\alpha)},$$

that is,

$$\frac{m_{\alpha}}{|\alpha|}\Gamma_w b_{\alpha}\mathbf{Z} = \frac{m_{w(\alpha)}}{|w(\alpha)|}b_{w(\alpha)}\mathbf{Z}.$$

Now, $|w(\alpha)| = |\alpha|$ and

$$\Gamma_w b_{\alpha} = \Gamma_w \sum_{h\in\alpha} e_h = \sum_{h\in\alpha}\Gamma_w e_h = \sum_{h\in\alpha} e_{w(h)} = b_{w(h)}$$

so $m_{\alpha} = m_{w(\alpha)}$ as claimed. $\square$

Proposition 1.38.

Consider the conditions

(i)

$$\xi_{ij} M^{\beta} \subseteq M^{\alpha}$$

for all weights α and β in $\Lambda(n,r)$ and for all elements i of α and j of β;

(ii) for all weights α and all $w \in W$, $m_{\alpha} = m_{w(\alpha)}$;

(iii) for all weights α and β in $\Lambda^{+}(n,r)$ and for all elements i of α and j of β,

$$\xi_{ij} M^{\beta} \subseteq M^{\alpha}.$$

Condition (i) holds if and only if (ii) and (iii) both hold.

Proof.

Certainly (i) implies (iii), and we have shown in Lemma 1.37 that (i) implies (ii). Suppose that (ii) and (iii) both hold. Then take any α and β in $\Lambda(n,r)$ and any elements i of α and j of β. Pick w and v from W such that $w(\alpha)$ and $v(\beta)$ are in $\Lambda^{+}(n,r)$. Then $w(i) \in w(\alpha)$ and $v(j) \in v(\beta)$, and applying (iii) we get

$$\xi_{w(i),v(j)} M^{v(\beta)} \subseteq M^{w(\alpha)}.$$

Notice that the proof of Lemma 1.37 shows that condition (ii) implies that

$$\Gamma_w M^{\alpha} = M^{w(\alpha)} \qquad \text{and} \qquad \Gamma_v M^{\beta} = M^{v(\beta)}$$

so

$$\xi_{w(i),v(j)} M^{v(\beta)} = \Gamma_w \xi_{ij} \Gamma_{v^{-1}} M^{v(\beta)} = \Gamma_w \xi_{ij} M^{\beta}$$

so we have

$$\Gamma_w \xi_{ij} M^{\beta} \subseteq M^{w(\alpha)} = \Gamma_w M^{\alpha}$$

that is,

$$\xi_{ij} M^{\beta} \subseteq M^{\alpha}$$

so (i) holds, as required. □

Lemma 1.39.

Let $i \in \alpha$ and $j \in \beta$. Then

$$\xi_{ij} b_{\beta} = |P_i : P_i \cap P_j| b_{\alpha}$$

where by P_i we mean the stabiliser of i.

Proof.

$$\xi_{ij} b_\beta = \xi_{ij} \xi_{jl} e_l$$

Now

$$\xi_{ij}\xi_{jl} = \sum_{(p,q)\in\Omega} |\{\, s \mid ((i,j) \sim (p,s)) \wedge ((j,l) \sim (s,q)) \,\}|\, \xi_{pq}$$

and, for any $\phi \in P$ we have $l\phi = l$, so if the coefficient of ξ_{pq} is to be non-zero we must have $q = l$ as well as $p \sim i$. Since the summation is over Ω this means that it contains only one non-zero term, which we may assume to be when $(p,q) = (i,l)$. The coefficient of ξ_{il} in this case is

$$\begin{aligned} |\{\, s \mid ((i,j) \sim (i,s)) \wedge ((j,l) \sim (s,l)) \,\}| &= |\text{orbit of } j \text{ under action of } P_i| \\ &= |P_i : P_i \cap P_j| \end{aligned}$$

so that

$$\xi_{ij} b_\beta = |P_i : P_i \cap P_j| \xi_{il} e_l = |P_i : P_i \cap P_j| b_\alpha$$

as claimed. □

Next we calculate $|P_i : P_i \cap P_j|$.

Lemma 1.40.

If $i \in \alpha$ and $j \in \beta$ and α and β are both dominant, then

$$|P_i : P_i \cap P_j| = \frac{\alpha_1!\alpha_2!}{A!(\alpha_1 - A)!(\beta_1 - A)!(\alpha_2 - \beta_1 + A)} \qquad (*)$$

for some integer value of A such that

$$\beta_2 - \alpha_1 \leq A \leq \min(\alpha_1, \beta_1).$$

Moreover, for each value of A within this range there exists some pair (i,j) with $i \in \alpha$ and $j \in \beta$ such that $(*)$ holds.

Proof.

For any $\pi \in P$ we have

$$|P_i : P_i \cap P_j| = |P_{i\pi} : P_{i\pi} \cap P_{j\pi}|$$

so we may assume without loss of generality that i is such that T_i is standard. Then for any $\phi \in P_i$ we have

$$|P_i : P_i \cap P_j| = |P_i : P_i \cap P_{j\phi}|$$

so we may assume that the tableaux T_i and T_j look like

$\longleftarrow$	α_1				$\longrightarrow$	$\longleftarrow$		α_2			$\longrightarrow$	
1	$\cdots\cdots$				1	2	$\cdots\cdots$				2	T_i
1	$\cdots$	1	2	$\cdots$	2	1	$\cdots$	1	2	$\cdots$	2	T_j
$\longleftarrow$ A		$\longrightarrow\longleftarrow$	$(\alpha_1 - A)$		$\longrightarrow\longleftarrow$	$(\beta_1 - A)$		$\longrightarrow\longleftarrow$	$(\alpha_2 - \beta_1 + A)$		$\longrightarrow$	

and it is plain that to any value of A within the bounds given in the statement there corresponds such a diagram. Then we see that

$$|P_i| = \alpha_1!\alpha_2!$$

and that

$$|P_i \cap P_j| = A!(\alpha_1 - A)!(\beta_1 - A)!(\alpha_2 - \beta_1 + A)!$$

from which the result follows. $\square$

Finally we may use these results to show that necessary and sufficient conditions for the tuple $\{m_\alpha\}$ to correspond to a valid module are that

$$\frac{m_\beta}{m_\alpha}\frac{\beta_1!\,\beta_2!}{A!\,(\alpha_1 - A)!\,(\beta_1 - A)!\,(\beta_2 - \alpha_1 + A)!} \in \mathbf{Z}$$

for every pair (α, β) of dominant weights, and for every integer value of A satisfying

$$\alpha_1 - \beta_2 \leq A \leq \min(\alpha_1, \beta_1).$$

We shall refer to these conditions as the *validity conditions* $V(\alpha, \beta)$.

Setting α and β in turn to be λ, and remembering that $m_\lambda = 1$, shows that in this case we may rephrase the conditions as:

(i) For all weights β, $m_\beta \in \mathbf{Z}$;
(ii) For all weights α, m_α divides $|\alpha|$.

Therefore we shall restrict ourselves to begin with to such tuples, and shall need to consider the validity conditions $V(\alpha, \beta)$ only for α and β not λ.

Remark 1.41.

The condition that m_α should divide $|\alpha|$ is a consequence of our normalisation condition $m_\lambda = 1$. If (and when) we drop the normalisation condition, we may allow m_α not to divide $|\alpha|$.

Remark 1.42.

Of course, if M and N are valid modules then so are $M \cap N$ and $M + N$. If the tuple corresponding to M is $\{m_\alpha\}$ and that corresponding to N is $\{n_\alpha\}$, then

(i) the tuple corresponding to $M \cap N$ is $\{\mathrm{lcm}(m_\alpha, n_\alpha)\}$;
(ii) the tuple corresponding to $M + N$ is $\{\mathrm{hcf}(m_\alpha, n_\alpha)\}$;

Example 1.43.

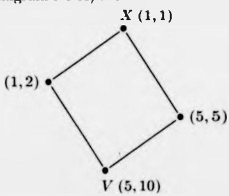
The easiest non-trivial example is that in which $r = 5$. In this case the dominant weights are $\lambda = (5,0)$, $\alpha = (4,1)$ and $\beta = (3,2)$, with sizes 1, 5 and 10 respectively. We insist that m_α be a positive integer dividing 5, i.e. 1 or 5, and that m_β be a positive integer dividing 10, i.e. 1, 2, 5 or 10. The only ordered pairs of weights to be considered are (α, β) and (β, α) and in either case the condition on A becomes $2 \leq A \leq 3$. Therefore the quantities which have to be integral are

$$\frac{3m_\beta}{m_\alpha}, \quad \frac{2m_\beta}{m_\alpha}, \quad \frac{6m_\alpha}{m_\beta} \quad \text{and} \quad \frac{4m_\alpha}{m_\beta}$$

By considering the two possibilities for m_α in turn one may see that the possibilities for the tuple (m_α, m_β) are

$$(1,1) \qquad (1,2) \qquad (5,5) \qquad \text{and} \qquad (5,10)$$

so that the structure diagram for X/V is



It is easy to see that there are very few examples small enough for manual calculation to be a feasible way of finding all valid modules.

1.4 Duality

It was significant that the structure diagram for X/V in the example just given had a degree of symmetry; such diagrams will always do so. We shall not really use this fact, but as it seems interesting we give it anyway.

Definition 1.44.

$\overline{m_\alpha} = |\alpha|/m_\alpha$ ◁

Lemma 1.45.

If $\bigoplus(m_\alpha b_\alpha/|\alpha|)\mathbf{Z}$ is a valid lattice then so is $\bigoplus(\overline{m_\alpha} b_\alpha/|\alpha|)\mathbf{Z}$.

Proof.

$$\begin{aligned}\frac{\overline{m_\beta}}{\overline{m_\alpha}}\binom{\beta_1}{A}\binom{\beta_2}{\alpha_1 - A} &= \frac{\overline{m_\beta}}{\overline{m_\alpha}}\frac{\beta_1!\,\beta_2!}{A!\,(\alpha_1 - A)!\,(\beta_1 - A)!\,(\beta_2 - \alpha_1 + A)!} \\ &= \frac{m_\alpha}{m_\beta}\frac{\alpha_1!\,\alpha_2!}{A!\,(\alpha_1 - A)!\,(\beta_1 - A)!\,(\beta_2 - \alpha_1 + A)!}\end{aligned}$$

□

This justifies

Definition 1.46.

If $M = \bigoplus m_\alpha b_\alpha/|\alpha|\mathbf{Z}$, then $\overline{M} = \bigoplus \overline{m_\alpha} b_\alpha/|\alpha|\mathbf{Z}$ ◁

Lemma 1.47.

This is a special case of the duality given by

$$M^* = \{\, v \in V_{\lambda,\mathbf{Q}} \mid \langle\langle v, M\rangle\rangle \subseteq \mathbf{Z} \,\}$$

Proof.

$$M^* = \left\{ v = \sum \lambda_\beta b_\beta, \lambda \in \mathbf{Q} \,\middle|\, \left\langle\left\langle \sum \lambda_\beta b_\beta, \frac{m_\alpha b_\alpha}{|\alpha|}\right\rangle\right\rangle \in \mathbf{Z}\ \forall\alpha \right\}$$

Recalling, from Lemma 1.34, that

(i) $C(T) = 1$ so $\langle\langle x, y\rangle\rangle = \langle x, y\rangle\ \ \forall x, y$ and

(ii) $\langle b_\alpha, b_\beta\rangle = |\alpha|\delta_{\alpha\beta}$,

we have

$$\begin{aligned}M^* &= \left\{ v = \sum \lambda_\beta b_\beta, \lambda \in \mathbf{Q} \mid m_\alpha\lambda_\alpha \in \mathbf{Z}\ \forall\alpha \right\} \\ &= \bigoplus_\alpha \frac{1}{m_\alpha} b_\alpha \mathbf{Z}\end{aligned}$$

as required. □

Notice that $\overline{X} = V$.

1.5 p-locality

This section, unlike the last, is essential to our treatment of the problem. We show that to find the valid modules for $\lambda = (r, 0)$ it suffices, in fact, to consider the primes no bigger than r one at a time.

If p is a prime number and x is an integer, we denote by x_p the p-part of x, that is, the largest power of p which divides x. By $\nu_p(x)$ we mean the exponent of p in x_p. That is,

$$x = x_p k = p^{\nu_p(x)} k$$

for some integer k such that p does not divide k.

Definition 1.48.

Let p be some fixed prime, and let M and N be valid modules with corresponding tuples $\{m_\alpha\}$ and $\{n_\alpha\}$. Then we write

$$M \sim N$$

if and only if for each weight α,

$$\nu_p(m_\alpha) = \nu_p(n_\alpha).$$

◁

This is an equivalence relation. We show that we may take as a set of representatives of the equivalence classes those M such that m_α is a power of p for all α; this entails showing that every valid module is equivalent to some valid module the entries in whose tuple are powers of p.

Definition 1.49.

Let M be a valid module with corresponding tuple $\{m_\alpha\}$. For some fixed prime p, let the *p-envelope* of M be M_p with corresponding tuple $\{(m_\alpha)_p\}$; that is, $M_p = \bigoplus(|m_\alpha|_p b_\alpha/|\alpha|)\mathbb{Z}$. ◁

Lemma 1.50.

If M is a valid lattice with basis $\{m_\alpha b_\alpha/|\alpha|\}$ and M_p with basis $\{m_{\alpha,p} b_\alpha/|\alpha|\}$ is its p-envelope, then M_p is a valid lattice.

Proof.

It is certainly a $\mathbb{Z}$-module. Because M is a valid lattice, we know that for all α, β,

$$\frac{m_\beta}{m_\alpha} \frac{\beta_1!\,\beta_2!}{A!(\alpha_1 - A)!(\beta_1 - A)!(\beta_2 - \alpha_1 + A)!} \in \mathbb{Z}$$

Taking p-parts, we see that

$$\frac{m_{\beta,p}}{m_{\alpha,p}}\left(\frac{\beta_1!\,\beta_2!}{A!\,(\alpha_1-A)!\,(\beta_1-A)!\,(\beta_2-\alpha_1+A)!}\right)_p \in \mathbb{Z}$$

so certainly

$$\frac{m_{\beta,p}}{m_{\alpha,p}}\frac{\beta_1!\,\beta_2!}{A!\,(\alpha_1-A)!\,(\beta_1-A)!\,(\beta_2-\alpha_1+A)!} \in \mathbb{Z}$$

as required. □

Lemma 1.51.

M_p can be characterised as the smallest valid lattice containing M such that X/M_p is a $\mathbb{Z}/p^\mu\mathbb{Z}$-module, where p^μ is the highest power of p dividing r.

Proof.

Let the tuple corresponding to M_p be $\{m_\alpha\}$, where each m_α is a power of p. Certainly $m_\alpha \mid r$, so for each α we have $m_\alpha \mid p^\mu$. We must show that X/M_p is a $\mathbb{Z}/p^\mu\mathbb{Z}$-module, in other words that $p^\mu\mathbb{Z}$ annihilates X/M_p. Now $p^\mu\mathbb{Z}X$ has basis $\{p^\mu b_\alpha/|\alpha|\}$ $p^\mu\mathbb{Z}X \subseteq M_p$ as required.

In fact, X/M is a $\mathbb{Z}/p^\mu\mathbb{Z}$-module if and only if $M = M_p$, that is, all coefficients m_α are powers of p. M_p is the smallest such module containing M.

□

Lemma 1.52.

M is the intersection over all primes of its p-envelopes.

Proof.

Let the tuple corresponding to M_p be $\{m_\alpha\}$. Let $\{n_\alpha\}$ be the tuple corresponding to $\bigcap_p M_p$. Then

$$\forall\alpha\ \forall p\ n_{\alpha,p} = m_{\alpha,p}$$

by Remark 1.42 □

Since Lemma 1.50 shows that any p-envelope is a valid lattice in its own right, any intersection of p-lattices is also a valid lattice. We have shown that any valid lattice is an intersection of p-envelopes. Therefore it suffices to study the possible p-envelopes for each prime p in turn, and from here on we shall do so. We shall fix a prime p, and shall assume that the m_α are powers of p. Therefore we shall often talk about the tuple $\{\nu_p(m_\alpha)\}$, rather than $\{m_\alpha\}$. We shall attempt to make it clear at each stage what is meant by 'the tuple'!

Notice that, for each p, $X = X_p$, but that $V \neq V_p$ except in trivial cases, and that for any valid module M we have

$$V_p \subseteq M_p \subseteq X.$$

Thus we shall be considering the inclusion diagram of valid modules which lie between X and V_p, which we refer to as the *p-diagram*.

We may combine the notions of p-locality and duality to give a duality on the p-diagram.

Definition 1.53.

Given a valid module M, with tuple $\{m_\alpha\}$, such that

$$V_p \subseteq M \subseteq X,$$

define $\widetilde{m_\alpha} = |\alpha|_p/m_p$, and write $\widetilde{M}_p$ for $\bigoplus_\alpha \widetilde{m_\alpha} b_\alpha/|\alpha|\mathbb{Z}$. Then $\widetilde{M} = (\overline{M})_p$ since $(|\alpha|/m_\alpha)_p = |\alpha|/(m_\alpha)_p$. ◁

Lemma 1.54.

$\widetilde{M}_p$ is a valid lattice.

Proof.

The maps $M \mapsto \overline{M}$ and $M \mapsto M_p$ each preserve validity, by Lemma 1.45 and Lemma 1.50. Therefore their composition $M \mapsto \widetilde{M}$ must also preserve validity. □

Notice that $\widetilde{X} = V_p$.

Lemma 1.55.

There is no valid lattice M such that $M = \widetilde{M}$, except in the trivial case where $X = V_p$.

It is possible to prove this directly using a small amount of number theory; however, in Chapter 3 we shall be able to give a very simple proof using the theory that we shall have developed, so we leave the proof until then.

Corollary 1.56.

Except in the trivial case where $X = V$, there is no valid lattice M such that $M = \overline{M}$.

Proof.

$M = \overline{M}$ if and only if $M = \widetilde{M}$ for every prime p. □

Chapter 2

Simplifying the problem.

In this chapter we shall greatly simplify the problem of finding all valid modules for given values of r, with the aid of a combinatorial result whose proof occupies most of the chapter. We shall take advantage of the results in the previous chapter, which showed that we may consider one prime p at a time, and from here on we shall do so. Thus all coefficients m_α are to be considered to be powers of p, for some arbitrary but locally fixed value of p.

The concept of 'point scoring' is important throughout this chapter:

2.1 Point scoring.

Definition 2.1.

If x and y have p-adic expansions

$$x = x_n p^n + \cdots + x_1 p + x_0$$
$$y = y_n p^n + \cdots + y_1 p + y_0$$

and i is an integer, we say that *y scores the i^{th} point in x* when

$$x_i p^i + \cdots + x_0 < y_i p^i + \cdots + y_0$$

in other words, when there is some integer $j \leq i$ such that $y_j > x_j$ and such that for all k with $j < k \leq i$, $y_k = x_k$. ◁

Remark 2.2.

We use the convention that if $i < 0$ then y does not score the i^{th} point in x, whatever the values of x and y, 'because the i^{th} point does not exist'. Notice that if $i > n$ then y scores the i^{th} point in x if and only if y scores the n^{th} point in x.

Definition 2.3.

We shall write

$$x_p(i) = x_i p^i + \cdots + x_0$$

and shall omit the subscript p when no confusion can result. ◁

We shall need some miscellaneous facts about p-adic expansions, which we collect here.

Lemma 2.4.

Whenever $0 \le y \le x$ and x, y and $x-y$ have p-adic expansions

$$\begin{aligned} x &= x_n p^n + \cdots + x_1 p + x_0 \\ y &= y_n p^n + \cdots + y_1 p + y_0 \\ (x-y) &= (x-y)_n p^n + \cdots + (x-y)_1 p + (x-y)_0 \end{aligned}$$

we have for each i

$$(x-y)_i + y_i = \begin{cases} x_i + p & \text{if } y \text{ scores the } i^{\text{th}} \text{ but not the } (i-1)^{\text{th}} \text{ point in } x; \\ x_i - 1 & \text{if } y \text{ scores the } (i-1)^{\text{th}} \text{ but not the } i^{\text{th}} \text{ point;} \\ x_i & \text{if } y \text{ scores neither point;} \\ x_i + p - 1 & \text{if } y \text{ scores both points.} \end{cases}$$

and

$$(x-y)(i) + y(i) = \begin{cases} x(i) + p^{i+1} & \text{if } y \text{ scores the } i^{\text{th}} \text{ point in } x; \\ x(i) & \text{otherwise.} \end{cases}$$

Definition 2.5.

For integers x and y where $y \le x$, define $\gamma_p(x,y)$ as the exponent of p in $\binom{x}{y}$. ◁

Lemma 2.6.

$\gamma_p(x,y)$ is the number of points scored by y in x.

Proof.

Consider the number of multiples of p^i between 1 and x inclusive, where $i \ge 0$. This is $\sum_{\nu \ge i} x_\nu p^{\nu - i}$; let us write it $n(i,x)$. Now

$$\begin{aligned} n(i, x-y) + n(i,y) &= \sum_{\nu \ge i} (x-y)_\nu p^{\nu-i} + \sum_{\nu \ge i} y_\nu p^{\nu - i} \\ &= \sum_{\nu \ge i} ((x-y)_\nu + y_\nu) p^{\nu - i} \\ &= \begin{cases} n(i,x) & \text{if } y \text{ scores the } (i-1)^{\text{th}} \text{ point in } x; \\ n(i,x) - 1 & \text{otherwise,} \end{cases} \end{aligned}$$

using Lemma 2.4 and noticing that all other points scored by y in x are irrelevant, since they either make no contribution to the expression or they make two cancelling contributions.

Now $\gamma_p(x,y)$ is the number of i such that the second case above holds. For

$$\gamma_p(x,y) = \nu_p(x!) - \nu_p(y!) - \nu_p((x-y)!)$$

and for any integer m,

$$\nu_p(m!) = \sum_{i\geq 1} n(i,m)$$

so

$$\begin{aligned}\gamma_p(x,y) &= \sum_{i\geq 1}(n(i,x) - n(i,y) - n(i,x-y)) \\ &= \sum_{i\geq 0}(n(i,x) - n(i,y) - n(i,x-y))\end{aligned}$$

since y cannot score the $(-1)^{\text{th}}$ point in x, because this point does not exist, or by inspection. □

Definition 2.7.

For integers x and y where $y \leq x$, define $\Gamma_p(x,y)$ to be the set of points scored by y in x, so that $i \in \Gamma_p(x,y)$ if and only if y scores the i^{th} point in x, and $\gamma_p(x,y) = |\Gamma_p(x,y)|$. ◁

Remark 2.8.

Lemma 2.6 implies that y and $x-y$ score the same points in x, that is, that $\Gamma_p(x,y) = \Gamma_p(x,x-y)$. In fact, it is easy to see that

$$y_i p^i + \cdots + y_0 > x_i p^i + \cdots + x_0$$

if and only if

$$(x-y)_i p^i + \cdots + (x-y)_0 > x_i p^i + \cdots + x_0.$$

In particular, when we have a weight $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_1 + \alpha_2 = r$, we see that α_1 scores the i^{th} point in r if and only if α_2 does. In this case, we shall often say simply 'α scores the i^{th} point in r', and write $i \in \Gamma_p(r,\alpha)$.

Lemma 2.9.

If $r = r_n p^n + \cdots r_1 p + r_0$ let

$$\gamma_p(r) = \max\{0, n-t\}$$

where t is the number of coefficients at the right-hand end of the p-adic expansion which are $p-1$, that is,

$$t = \begin{cases} \max\{\, s \mid (\forall u \leq s)(r_u = p-1)\,\} + 1 & \text{if } r_0 = p-1; \\ 0 & \text{otherwise,} \end{cases}$$

or more simply, $t = \nu_p(r+1)$. Then there is some weight $\alpha = (r-\imath, \imath)$ for which the exponent of p in $|\alpha|$, that is, $\gamma_p(r, \alpha_1) = \gamma_p(r, \alpha_2)$, is $\gamma_p(r)$, and this is the highest exponent of p in the size of any weight, i.e.

$$\forall \beta \in \Lambda \quad |\beta|_p \leq |\alpha|_p.$$

Proof.

By Lemma 2.6, $\nu_p(|\alpha|) = \gamma_p(r, \alpha_1)$ is the number of integers m such that $r(m) < \alpha_1(m)$. Any such m must be at least t. Moreover, n cannot be such an m, since $\alpha_1 < r$. Therefore, $\gamma_p(r, \alpha_1) \leq n - t$.

Now if $n - t \leq 0$ then $\gamma_p(r) = 0$, so, for example, $\alpha_2 = 1$ will do. Assume that $n - t > 0$, and set

$$(\alpha_1)_i = \begin{cases} p-1 & 0 \leq i < n; \\ r_n - 1 & i = n. \end{cases}$$

Then $\gamma_p(r, \alpha_1) = n - t$ as required. In fact, this value of α_1 is the largest which satisfies the condition. □

Informally, then, $\gamma_p(r)$ is the largest number of points which can be scored in r by any weight α. We write $\Gamma_p(r)$ for the set of all points which can be scored in r, so that $|\Gamma_p(r)| = \gamma_p(r)$.

2.2 A combinatorial result.

Take some fixed values of r and p. For every pair of weights, α and β, there is a certain set of points, that is, a certain subset of $\{0, \ldots, n\}$ where $r = r_n p^n + \cdots + r_0$, which turns out to have great importance.

Definition 2.10.

For any weights α and β, let $K_p(\alpha, \beta)$ be the set of points scored in r by β but not by α. That is,

$$\begin{aligned} K_p(\alpha, \beta) &= \Gamma_p(r, \beta) \setminus \Gamma_p(r, \alpha) \\ &= \{\, \imath \mid \alpha_1(\imath) \leq r(\imath) < \beta_1(\imath) \,\}. \end{aligned} \qquad (CF)$$

Denote the size of this set by $k_p(\alpha, \beta)$. As usual, we shall omit the subscript p when no confusion can result. ◁

Of course, the expression in (CF) is by no means unique; Lemma 2.4 shows us that either subscript 1, or both, may be replaced by 2 without altering the set.

The motivation for this definition is the following rather unlikely-looking combinatorial result, to whose proof and consequences the remainder of this chapter is devoted. Notice that only the size $k(\alpha,\beta)$ of $K(\alpha,\beta)$ occurs in the statement; however, the set $K(\alpha,\beta)$ will be important in the proof. Other sets of points will also appear.

Theorem 2.11.

Whenever $\alpha_1 - \beta_2 \leq A \leq \min(\alpha_1, \beta_1)$,

$$\begin{aligned} p(A) &\stackrel{\text{def}}{=} \nu_p\left(\frac{\alpha_1!\,\alpha_2!}{A!\,(\alpha_1 - A)!\,(\beta_1 - A)!\,(\beta_2 - \alpha_1 + A)!}\right) \\ &\geq k(\alpha,\beta) \end{aligned}$$

Moreover, there is some value of A within these limits such that

$$p(A) = k(\alpha,\beta).$$

Remark 2.12.

Notice that

$$p(A) = \nu_p\binom{\alpha_1}{A} + \nu_p\binom{\alpha_2}{\beta_1 - A}$$

and that

$$\binom{\alpha_2}{\beta_1 - A} = \binom{\alpha_2}{\alpha_2 - \beta_1 + A}.$$

By Lemma 2.6,

$$\begin{aligned} p(A) &= \gamma_p(\alpha_1, A) + \gamma_p(\alpha_2, \alpha_2 - \beta_1 + A) \\ &= \gamma_p(\alpha_1, A) + \gamma_p(\alpha_2, s) \end{aligned}$$

where we have set $s = \alpha_2 - \beta_1 + A = \beta_2 - \alpha_1 + A$. Then when A is $\alpha_1 - \beta_2$, its minimal value, s, is 0, and thereafter s increases by 1 whenever A does. We shall consider starting with $s = 0$ and increasing s to find the value of s and hence A which the lemma requires.

Lemma 2.13.

Whenever $\alpha_1 - \beta_2 \leq A \leq \min(\alpha_1, \beta_1)$,

(i)

$$p(A) \geq k(\alpha,\beta);$$

(ii)

$$K(\alpha,\beta) \subseteq \Gamma_p(\alpha_1, A) \cup \Gamma_p(\alpha_2, s).$$

Proof.

In view of Remark 2.12, we see that our claim is that for each A in the required range, and for the corresponding s,

$$k(\alpha,\beta) \leq \gamma_p(\alpha_1, A) + \gamma_p(\alpha_2, s).$$

We shall prove the apparently stronger set theoretic result, part (ii), that

$$K(\alpha,\beta) \subseteq \Gamma_p(\alpha_1, A) \cup \Gamma_p(\alpha_2, s);$$

then we shall have

$$\begin{aligned} k(\alpha,\beta) &= |K(\alpha,\beta)| \\ &\leq |\Gamma_p(\alpha_1, A) \cup \Gamma_p(\alpha_2, s)| \\ &\leq |\Gamma_p(\alpha_1, A)| + |\Gamma_p(\alpha_2, s)| \\ &= \gamma_p(\alpha_1, A) + \gamma_p(\alpha_2, s) \end{aligned}$$

which gives part (i), as required.

Suppose that $i \in K(\alpha,\beta)$, that is, that the i^{th} point is scored in r by β and not by α, and suppose that $i \notin \Gamma_p(\alpha_1, A)$, that is, that A does not score the i^{th} point in α_1. We show that in this case $i \in \Gamma_p(\alpha_2, s)$, that is, that s must score the i^{th} point in α_2.

We have

$$A(i) \leq \alpha_1(i) \leq r(i) < \beta_1(i). \qquad (*)$$

Then

$$\begin{aligned} s(i) &= (A - \beta_1 + \alpha_2)(i) \\ &= \begin{cases} (A+\alpha_2)(i) - \beta_1(i) + p^{i+1} & \text{if } \beta_1 \text{ scores the } i^{\text{th}} \text{ point in } A+\alpha_2 \\ (A+\alpha_2)(i) - \beta_1(i) & \text{otherwise} \end{cases} \end{aligned}$$

and, expanding further,

$$(A+\alpha_2)(i) = \begin{cases} A(i) + \alpha_2(i) - p^{i+1} & \text{if } \alpha_2 \text{ scores the } i^{\text{th}} \text{ point in } A+\alpha_2 \\ A(i) + \alpha_2(i) & \text{otherwise.} \end{cases}$$

We need to show that s scores the i^{th} point in α_2, which is so if and only if $s(i) - \alpha_2(i) > 0$. Now

$$s(i) - \alpha_2(i) = \begin{cases} A(i) - \beta_1(i) + p^{i+1} & \text{if } i \in \Gamma_p(A+\alpha_2, \beta_1) \setminus \Gamma_p(A+\alpha_2, \alpha_2); \\ A(i) - \beta_1(i) - p^{i+1} & \text{if } i \in \Gamma_p(A+\alpha_2, \alpha_2) \setminus \Gamma_p(A+\alpha_2, \beta_1); \\ A(i) - \beta_1(i) & \text{otherwise.} \end{cases}$$

Since by equation $(*)$ we know that

$$-p^{i+1} < A(i) - \beta_1(i) < 0$$

we need to show that the first case holds, that is, that

$$i \in \Gamma_p(A + \alpha_2, \beta_1) \setminus \Gamma_p(A + \alpha_2, \alpha_2).$$

We show first that $i \notin \Gamma_p(A + \alpha_2, \alpha_2)$, that is, that α_2 does not score the i^{th} point in $A + \alpha_2$. Now by equation $(*)$ $A(i) \leq \alpha_1(i)$ so

$$A(i) + \alpha_2(i) \leq \alpha_1(i) + \alpha_2(i) = r(i) < p^{i+1} \tag{$**$}$$

by the assumption that $i \notin \Gamma_p(r, \alpha)$. Therefore

$$\alpha_2(i) \leq A(i) + \alpha_2(i) = (A + \alpha_2)(i) \tag{$***$}$$

as required. To see that $i \in \Gamma_p(A + \alpha_2, \beta_1)$ notice that we have shown that

$$\begin{aligned} (A + \alpha_2)(i) &= A(i) + \alpha_2(i) && \text{by } (***) \\ &\leq r(i) && \text{by } (**) \\ &< \beta_1(i) && \text{by } (*). \end{aligned}$$

□

All that we have to do now, in order to prove Theorem 2.11, is to construct some value of A in the given range such that $p(A) = k(\alpha, \beta)$. In view of the set containment

$$K(\alpha, \beta) \subseteq \Gamma_p(\alpha_1, A) \cup \Gamma_p(\alpha_2, s)$$

which we have just established, this is equivalent to constructing an A in the given range such that

$$K(\alpha, \beta) = \Gamma_p(\alpha_1, A) \uplus \Gamma_p(\alpha_2, s). \tag{$\dagger$}$$

We must now introduce the next set of points.

Definition 2.14.

Denote by $i(\alpha, \beta)$ the initial value of

$$\nu_p\left(\frac{\alpha_1!\,\alpha_2!}{A!\,(\alpha_1 - A)!\,(\beta_1 - A)!\,(\beta_2 - \alpha_1 + A)!}\right)$$

that is, the value of this expression when $A = \alpha_1 - \beta_2$ and $s = 0$. Thus $i(\alpha, \beta) = p(\alpha_1 - \beta_2)$. By Remark 2.12,

$$i(\alpha, \beta) = \gamma_p(\alpha_1, \alpha_1 - \beta_2) + \gamma_p(\alpha_2, 0)$$

and so we may regard $i(\alpha,\beta)$ as $|I(\alpha,\beta)|$ where

$$I(\alpha,\beta) = \Gamma_p(\alpha_1, \alpha_1 - \beta_2) \uplus \Gamma_p(\alpha_2, 0),$$

the union being guaranteed to be disjoint since $\Gamma_p(\alpha_2, 0) = \emptyset$! ◁

The reason for writing $I(\alpha,\beta)$ in this apparently perverse way is that it shows that it is the set of points *initially scored* either by A in α_1 or by s in α_2. We may informally regard $\alpha_1 - \beta_2$, 0 and the set I as our first approximation to the required values of A, s and the set K. We have $p(\alpha_1 - \beta_2) = I$, we require $p(A) = K$. We also know, as a special case of Lemma 2.13 that $K \subseteq I$. If we are lucky enough to find that $K = I$, then $A = \alpha_1 - \beta_2$ (and hence $s = 0$) satisfy (†), and we are done.

Of course, we shall also require the more straightforward expressions for $I(\alpha,\beta)$, namely

$$I(\alpha,\beta) = \Gamma_p(\alpha_1, \alpha_1 - \beta_2) = \Gamma_p(\alpha_1, \beta_2).$$

Definition 2.15.

Denote by $RL(\alpha,\beta)$ the set $I(\alpha,\beta) \setminus K(\alpha,\beta)$, and let $rl(\alpha,\beta) = |RL(\alpha,\beta)|$. In other words, $i \in RL$ if and only if either

$$r(i) < \alpha_1(i) < \beta_2(i)$$

or

$$\alpha_1(i) < \beta_2(i) \leq r(i).$$

◁

This is the set of points which are *required losses*; for, since $K \subseteq I$,

$$I = K \uplus RL.$$

Notice that $RL = \emptyset$ if and only if $I = K$; that is, our initial value of A satisfies (†) if and only if there are no required losses. In general, we must tinker with our value of A (and hence s) in such a way as to produce a net loss of $rl(\alpha,\beta)$ points. In other words, if A satisfies (†) then

$$p(A) = p(\alpha_1 - \beta_2) - rl(\alpha,\beta).$$

Remark 2.16.

Notice that we cannot keep entirely to set notation because in general

$$\Gamma_p(\alpha_1, A) \cap \Gamma_p(\alpha_2, s) \neq \emptyset,$$

although we have seen that this intersection is empty both for $A = \alpha_1 - \beta_2$ and for any value of A which satisfies (†). There is no generally useful notion of a set $P(A)$ which would have size $p(A)$; the obvious candidate would have been

$$\Gamma_p(\alpha_1, A) \cup \Gamma_p(\alpha_2, s),$$

but this will have the right size ($p(A)$) only if

$$\Gamma_p(\alpha_1, A) \cap \Gamma_p(\alpha_2, s) = \emptyset.$$

Now let us consider how to construct a value of A which satisfies (†). The obvious approach, since $RL \subseteq I = \Gamma_p(\alpha_1, \alpha_1 - \beta_2)$ and $\Gamma_p(\alpha_2, 0) = \emptyset$, is to look for values of A and s such that

1) $\Gamma_p(\alpha_1, A) = I \setminus RL$, and
2) $\Gamma_p(\alpha_2, s) = \emptyset$.

Such values would have

$$\Gamma_p(\alpha_1, A) \uplus \Gamma_p(\alpha_2, s) = I \setminus RL = K$$

and would therefore satisfy (†). This approach has the merit of seeming to disturb our initial situation as little as possible. Unfortunately, however, such a value of A within the required limits will not always exist. We give a simple example.

Example 2.17.

Let $r = 6$ and $p = 2$, and consider the weights $\alpha = (4, 2)$ and $\beta = (5, 1)$. Then, writing in base 2,

$$r = 110$$
$$\alpha_1 = 100$$
$$\alpha_2 = 010$$
$$\beta_2 = 001$$

so we have $I = \{0, 1\}$, since β_2 scores the zeroth and first points in α_1, and $K = \{0\}$, since β scores the zeroth point in r and α scores no points in r; that is,

$$\Gamma_p(r, \beta) \setminus \Gamma_p(r, \alpha) = \{0\} \setminus \emptyset = \{0\}.$$

Therefore $RL = \{1\}$; in particular, it is not empty, so the initial value, 3, of A will not do. Now the range condition on A is

$$\alpha_1 - \beta_2 \leq A \leq \min\{\alpha_1, \beta_1\}$$

which here becomes $3 \leq A \leq 4$, so we can only consider $A = 4$. Then $\Gamma_p(\alpha_1, A) = \emptyset$ and $\Gamma_p(\alpha_2, s) = \{0\}$, so $p(4) = \{0\} = K$ as required (that is, this is not a counter-example to Theorem 2.11!) but not in the obvious way just described.

Therefore we must be slightly less ambitious. Notice that

$$\begin{aligned}\{0, \ldots, n-1\} &= I \uplus (\{0, \ldots, n-1\} \setminus I) \\ &= K \uplus RL \uplus (\{0, \ldots, n-1\} \setminus I).\end{aligned}$$

We shall ensure that for every point i, that is, for every $i \in \{0, \ldots, n-1\}$, exactly one of the following holds: (Conditions A)

A1) $i \in K$ and either $i \in \Gamma_p(\alpha_1, A)$ or $i \in \Gamma_p(\alpha_2, s)$, but not both;
A2) $i \in RL$ and $i \notin \Gamma_p(\alpha_1, A) \cup \Gamma_p(\alpha_2, s)$; or
A3) $i \in \{0, \ldots, n-1\} \setminus I$ and $i \notin \Gamma_p(\alpha_1, A) \cup \Gamma_p(\alpha_2, s)$.

Then we shall have ensured that

$$K(\alpha, \beta) = \Gamma_p(\alpha_1, A) \uplus \Gamma_p(\alpha_2, s),$$

which is (†), as required. Notice that with $A = \alpha_1 - \beta_2$ and $s = 0$, points in K satisfy A1 and points in $\{0, \ldots, n-1\} \setminus I$ satisfy A3; the problem is that points in RL do not satisfy A2.

Our procedure will be to increase s, and hence A, in an attempt to find values which satisfy Conditions A. We shall only check that $A \leq r$, not that $A \leq \min\{\alpha_1, \beta_1\}$. It so happens that A will then automatically lie in the correct range. It will be at least $\alpha_1 - \beta_2$ since it is found by adding s, at least 0, to $\alpha_1 - \beta_2$. Now, $A > \alpha_1$ if and only if A scores the n^{th} point in α_1 (recall that the n^{th} is the leading coefficient of r, and since we do insist that $A \leq r$ we see that $A(n) = A$). Similarly, $A > \beta_1$ if and only if $s = A - (\alpha_1 - \beta_2) > \alpha_2$, which is true if and only if s scores the n^{th} point in α_2. For either of these to be true, given that A satisfies the conditions above, the n^{th} point must be scored in r by β and not by α. But this implies that β_1 and β_2 are greater than r, which is not true.

In order to work with practical examples, we shall need to be able to identify the sets I, K, and RL. We shall need only limited extra information about the p-adic expansions of the integers involved. We know that $I = \Gamma_p(\alpha_1, \beta_2)$, that $K = \Gamma_p(r, \beta) \setminus \Gamma_p(r, \alpha)$, and that $RL = I \setminus K$. Therefore we shall need to be able to identify the three sets of points $\Gamma_p(\alpha_1, \beta_2)$, $\Gamma_p(r, \alpha)$, and $\Gamma_p(r, \beta)$. In other words, for every $i \in \{0, \ldots, n\}$ we need to be able to answer the following three questions:

1) Is $i \in \Gamma_p(\alpha_1, \beta_2)$, that is, is i scored by β_2 in α_1?
2) Is $i \in \Gamma_p(r, \alpha)$, that is, is i scored by α_1 in r?
3) Is $i \in \Gamma_p(r, \beta)$, that is, is i scored by β_2 in r?

We describe $\Gamma_p(r, \alpha)$ as the set of points scored by α_1 (rather than α_2) in r, and $\Gamma_p(r, \beta)$ as the set of points scored by β_2 (rather than β_1) in r in order to make it plain that we can deduce this set of information from the p-adic expansions of r, α_1 and β_2 alone. Recall that Lemma 2.4 showed that $\Gamma_p(r, \alpha_1) = \Gamma_p(r, \alpha_2)$ for any r, α and p, so this is legitimate.

We shall use the notation

$$\begin{array}{rl} & \checkmark \\ i: & \times \\ & \checkmark \end{array}$$

to indicate that β_2 scores the i^{th} point in α_1 (top $\checkmark$), α_1 does not score the i^{th} point in r (middle $\times$) and β_2 scores the i^{th} point in r (bottom $\checkmark$). Notice that the patterns

$$\begin{array}{rl} & \checkmark \\ i: & \checkmark \\ & \times \end{array} \quad \text{and} \quad \begin{array}{rl} & \times \\ i: & \times \\ & \checkmark \end{array}$$

cannot occur. For example, if the first of these held then we would have

$$\beta_2(i) > \alpha_1(i) \qquad \text{since } i \in \Gamma_p(\alpha_1, \beta_2) \tag{1}$$

$$\alpha_1(i) > r(i) \qquad \text{since } i \in \Gamma_p(r, \alpha) = \Gamma_p(r, \alpha_1) = \Gamma_p(r, \alpha_2) \tag{2}$$

$$\beta_2(i) \leq r(i) \qquad \text{since } i \in \Gamma_p(r, \beta) = \Gamma_p(r, \beta_1) = \Gamma_p(r, \beta_2) \tag{3}$$

These statements are inconsistent, for example, in that from (1) and (2) we may deduce that $\beta_2(i) > r(i)$, contradicting (3). The other case is similar.

It is easy to see that the patterns which show that $i \in RL$, that is, that the i^{th} point is a required loss, are

$$\begin{array}{rl} & \checkmark \\ i: & \checkmark \\ & \checkmark \end{array} \quad \text{and} \quad \begin{array}{rl} & \checkmark \\ i: & \times \\ & \times \end{array}$$

that the patterns which show that $i \notin I$, that is, that the i^{th} point is not scored initially are

$$\begin{array}{rl} & \times \\ i: & \times \\ & \times \end{array} \quad \begin{array}{rl} & \times \\ i: & \checkmark \\ & \times \end{array} \quad \text{and} \quad \begin{array}{rl} & \times \\ i: & \checkmark \\ & \checkmark \end{array}$$

and that the pattern

$$\begin{array}{rl} & \checkmark \\ i: & \times \\ & \checkmark \end{array}$$

shows that $i \in K$, that is, that the i^{th} point is initially scored, but is not a required loss.

2.3 Blocks.

We look at the p-adic expansion of $\alpha_1 - \beta_2$, our initial value of A, and split it into blocks where the left-hand end of each block (except the leftmost block, that which includes the n^{th} point) is a required loss, and the block contains no other required loss, though it may include points $i \in K$, where the pattern is

$$\begin{array}{rl} & \checkmark \\ i: & \times\,, \\ & \checkmark \end{array}$$

that is, where a point is initially scored, but is not a required loss. More formally:

Definition 2.18.

The subsequence $B = \{k, k-1, \ldots, j+1\} \subseteq \{n, \ldots, 0\}$ is a *block* if and only if

1) $B \cap RL = \{k\}$ unless $k = n$, in which case $B \cap RL = \emptyset$; and
2) $j = -1$ or $j \in RL$.

Plainly, then, the blocks partition $\{n, \ldots, 0\}$. Because the block which includes n is the only one which does not contain any required loss, that is, which has empty intersection with RL, we shall refer to this block as the *improper block*, and to any other blocks as *proper blocks*. ◃

Notice that if there is no proper block then $RL = \emptyset$, that is, the initial value $\alpha_1 - \beta_2$ of A itself satisfies (†). From now on, we shall assume that this is not the case; we shall assume that the improper block is not the first block.

We now introduce an example which we shall follow for the remainder of the proof of Theorem 2.11, since it illustrates most of the points that we shall discuss.

Example 2.19.

Let $p = 2$ and let $r = 42$. Consider the weights $\alpha = (33, 9)$ and $\beta = (36, 6)$. Then we need to know the p-adic expansions of r, of α_1 and of β_2. In base 2,

$i =$	5	4	3 2	1 0
$r =$	1	0	1 0	1 0
$\alpha_1 =$	1	0	0 0	0 1
$\beta_2 =$	0	0	0 1	1 0
Is $i \in \Gamma_p(\alpha_1, \beta_2)$?	×	✓	✓ ✓	✓ ×
Is $i \in \Gamma_p(r, \alpha)$?	×	×	× ×	× ✓
Is $i \in \Gamma_p(r, \beta)$?	×	×	× ✓	× ×

We show the information, which is easy to check, that

$$\begin{aligned} I &= \{4,3,2,1\} \\ RL &= \{4,3,1\} \\ K &= \{2\}. \end{aligned}$$

The vertical lines separate the blocks from one another; thus in this case the improper block is $\{5\}$ and there are three proper blocks, namely $\{4\}$, $\{3,2\}$ and $\{1,0\}$.

We shall deal with each block in turn, starting from the right hand end.

Definition 2.20.

We shall consider a block $B = \{k, \ldots j+1\}$ *dealt with* by coefficients $A_k, \ldots, A_0$ and $s_k, \ldots, s_0$ if when we define $A^{(B)}$ by

$$A_i^{(B)} = \begin{cases} A_i & \text{if } 0 \le i \le k \\ (\alpha_1 - \beta_2)_i & \text{if } k < i \le n \end{cases}$$

and $s^{(B)}$ by

$$s^{(B)}(i) = \begin{cases} A(i) - (\alpha_1 - \beta_2)(i) (\operatorname{mod} p^{i+1}) & \text{if } 0 \le i \le k \\ 0 & \text{if } k < i \le n \end{cases}$$

then for all i with $0 \le i \le k$ exactly one of Conditions A holds. ◁

Notice that, when the block B is dealt with, so, too, is every block to the right of it; this is implied by our definition of 'dealt with', as well as our declared intention. If we can successfully follow this procedure and deal with every block, when we have dealt with the improper block we shall have found a value of A to satisfy (†).

As the notation we have used suggests, we shall, in fact, never need to backtrack beyond the limits of the block with which we are currently dealing; once we have decided on coefficients of A and s in a particular block, we shall never need to alter our decision. More formally, let B_1 and B_2 be blocks, say

$B_1 = \{k_1, \ldots j_1 + 1\}$ and $B_2 = \{k_2, \ldots j_2 + 1\}$, where $k_2 < j_1 + 1$, that is, where B_2 lies to the right of B_1. Then our procedure will have the property that $A_j^{(B_1)} = A_j^{(B_2)}$ for all j such that $0 \leq j \leq k_2$.

We now have the vocabulary to describe an algorithm to give the required values of A and s. Given values of r, p, α and β,

1) Write out the p-adic expansions of r, α_1 and β_2.
2) Work out the tick/cross patterns for each point.
3) Find the partition into blocks.
4) Identify any block $B = \{k, \ldots j+1\}$ which satisfies the following conditions:

(a) $\beta_{2k} = r_k$;

(b) $k - 1 > j$ and the patterns in places k and $k-1$ are

$$k : \begin{array}{c} \checkmark \\ \times \\ \times \end{array} \qquad k-1 : \begin{array}{c} \times \\ \checkmark \\ \times \end{array}$$

and label any such blocks 'Problem Type 1'.

5) Identify any block $B = \{k, \ldots j+1\}$ which satisfies the following conditions:

(a) $\alpha_{1k} = 0$;

(b) $k - 1 > j$ and the patterns in places k and $k-1$ are

$$k : \begin{array}{c} \checkmark \\ \times \\ \times \end{array} \qquad k-1 : \begin{array}{c} \checkmark \\ \times \\ \checkmark \end{array}$$

and label any such blocks 'Problem Type 2'.

Notice that no block can be of both Problem Type 1 and Problem Type 2, and that the improper block is of neither problem type, since $n \notin RL$.

6) Write out the p-adic expansion of $\alpha_1 - \beta_2$.
7) Look at each block in turn from the right. Let the current block be $B = \{k, \ldots j+1\}$. Define the coefficients $A_k, \ldots A_{j+1}$ as follows, according to the type of the block.

7(i) If B is of neither problem type, then set

$$A_i = \begin{cases} (\alpha_1 - \beta_2)_i + 1 & \text{if } i = j+1 \neq k, \text{ or if } i = j+1 = k = n. \\ 0 & \text{if } i = k \neq n \\ (\alpha_1 - \beta_2)_i & \text{otherwise;} \end{cases}$$

7(ii) If B is of Problem Type 1, then

a) find the largest $\nu \leq k$ such that either $\beta_{2\nu} \neq r_\nu$, or the pattern is not

$$\begin{array}{cc} & \times \\ \nu - 1 : & \surd ; \\ & \times \end{array}$$

call this value $\nu(B)$. Notice that such a value of ν will always exist and will be in B, since $\nu = j + 1$ will always satisfy the second condition. For if $j = 0$ then there is no $(j-1)^{th}$ point, and if $j > 0$ then by the block definition $j \in RL$, and cannot have the given pattern.

b) Set

$$A_i = \begin{cases} (\alpha_1 - \beta_2)_i + 1 & \text{if } i = j + 1 \neq \nu \\ 0 & \text{if } \nu \leq i \leq k \\ (\alpha_1 - \beta_2)_i & \text{otherwise;} \end{cases}$$

7(iii) If B is of Problem Type 2, then

a) find the largest $\nu \leq k$ such that either $\alpha_{1\nu} \neq 0$, or the pattern is not

$$\begin{array}{cc} & \surd \\ \nu - 1 : & \times ; \\ & \surd \end{array}$$

call this value $\nu(B)$.

Notice, again, that such a value of ν will always exist and will be in B, since $\nu = j + 1$ will always satisfy the second condition. For if $j = 0$ then there is no $(j-1)^{th}$ point, and if $j > 0$ then by the block definition $j \in RL$, and cannot have the given pattern.

b) Set

$$A_i = \begin{cases} (\alpha_1 - \beta_2)_i + 1 & \text{if } i = j + 1 \neq \nu(B) \\ 0 & \text{if } \nu(B) \leq i \leq k \\ (\alpha_1 - \beta_2)_i & \text{otherwise.} \end{cases}$$

Then define the other coefficients of $A^{(B)}$ by setting any coefficients to the right of B to the values already found in previous iterations, and any to the left of B to the corresponding coefficients of $\alpha_1 - \beta_2$. That is, if B is the first block, $(j = -1)$, define

$$A_i^{(B)} = \begin{cases} A_i & \text{if } i \in B \\ (\alpha_1 - \beta_2)_i & \text{otherwise} \end{cases}.$$

Otherwise, recursively define

$$A_i^{(B)} = \begin{cases} A_i & \text{if } i \in B \\ A_i^{(B^-)} & \text{otherwise,} \end{cases}$$

where B^- is the block to the right of B.

Example 2.21.

We have already carried out steps 1 to 3 of the algorithm. We now demonstrate the rest.

4) The only block of Problem Type 1 is $\{1,0\}$.
5) The only block of Problem Type 2 is $\{3,2\}$.
6) $\alpha_1 - \beta_2 = 27 = 011011$ in base 2.
7) In the block $\{1,0\}$, we have $\nu(B) = 0$; in block $\{3,2\}$ we have $\nu(B) = 2$. Therefore the algorithm gives

$i =$	5	4	3	2	1	0
$r =$	1	0	1	0	1	0
$\alpha_1 =$	1	0	0	0	0	1
$\beta_2 =$	0	0	0	1	1	0
Is $i \in \Gamma_p(\alpha_1, \beta_2)$?	×	✓	✓	✓	✓	×
Is $i \in \Gamma_p(r, \alpha)$?	×	×	×	×	×	✓
Is $i \in \Gamma_p(r, \beta)$?	×	×	×	✓	×	×
$\alpha_1 - \beta_2 =$	0	1	1	0	1	1
$A^{(\{1,0\})} =$	0	1	1	0	0	0
$A^{(\{3,2\})} =$	0	1	0	0	0	0
$A^{(\{4\})} =$	0	0	0	0	0	0
$A = A^{(\{5\})} =$	1	0	0	0	0	0
$s =$	0	0	0	1	0	1

It is now easy to check that $\Gamma_p(\alpha_1, A) = \varnothing$ and $\Gamma_p(\alpha_2, s) = \{2\}$, so that

$$K(\alpha, \beta) = \Gamma_p(\alpha_1, A) \cup \Gamma_p(\alpha_2, s),$$

which is (†), as required.

The reader may care to try other examples. Observe that following the algorithm is a purely mechanical procedure, which can be done for any example; we have done what little was necessary to show it well-defined. It remains to prove that it works, that is, that the value of A which it produces does satisfy (†).

Our proof is by induction. We shall take as our induction hypothesis that every block (possibly there are none) to the right of the current block, B, has been dealt with by choosing coefficients according to the algorithm, and we shall show that applying the algorithm to B gives a value $A^{(B)}$ which renders B dealt with. Because we shall at every stage consider the possibility that the current block is the first, the same set of proofs will found the induction and build it.

2.3.1

The easiest part of the procedure is to show that once a block has been dealt with, it stays dealt with.

Lemma 2.22.

Suppose that the block $B = \{k, \ldots j+1\}$ has been dealt with by coefficients $A_k, \ldots, A_0$ and $s_k, \ldots, s_0$. Then, whatever the values of x_i in the definition

$$A_i{}' = \begin{cases} A_i & \text{if } 0 \leq i \leq k \\ x_i & \text{if } k < i \leq n \end{cases},$$

the two conditions in the definition of 'dealt with' still hold.

Proof.

The first condition refers only to the values of A_i and s_i where $0 \leq i \leq k$, so its truth is independent of higher coefficients. The same is true, albeit with a level of indirection, of the second condition, since for each of Conditions A one can check whether i satisfies it without referring to coefficients higher than i.

□

Since applying step 7 of the algorithm to a block does not involve altering any coefficient to the right of that block, it will result in the block being dealt with if we can show that each point i in the block satisfies the two conditions of the definition. There is no need to check that points to the right of the block still do so.

2.3.2

In order to prove the induction hypothesis, we now have to check, for every point in the current block, whether it is true that it satisfies exactly one of Conditions A with $A = A^{(H)}$, the value given by the algorithm. However, we may have changed only a few of the coefficients A_i in the block from their initial values, $(\alpha_1 - \beta_2)_i$. We can take advantage of this fact.

We separate the points to be checked into two sets; those i at the left-hand-end of the block such that A_i is set to 0 by the algorithm ($P(B)$), and the rest ($Q(B)$). More precisely,

(i) If B is proper and of neither Problem Type then $P(B) = \{k\}$, and $Q(B) = B \setminus P(B)$.

(ii) If B is of Problem Type 1 or 2 then $P(B) = \{k, \ldots, \nu(B)\}$ and $Q(B) = B \setminus P(B)$.

(iii) If B is improper then $P(B) = \emptyset$ and $Q(B) = B$

Of course, $Q(B)$ may be empty, provided that B is proper; indeed, since in this case $k \in P(B)$, we see that if the proper block B comprises just one point, then $Q(B)$ must be empty.

We may rephrase our definition of $A^{(B)}$ to take advantage of the new notation.

Whatever the problem type of B (Problem Type 1, 2 or neither), set

$$A_i^{(B)} = \begin{cases} (\alpha_1 - \beta_2)_i + 1 & \text{if } j+1 = i \in Q(B) \\ (\alpha_1 - \beta_2)_i & \text{if } j+1 \neq i \in Q(B) \\ 0 & \text{if } i \in P(B). \end{cases}$$

It is easy to check that this agrees with the previous version of the algorithm. The previous version is more convenient for practical purposes; however, this version is more convenient for the purposes of our proof, since it goes some way towards unifying the consideration of the problem types.

For the same reason of convenience, we shall set $\nu(B) = k$ in the case that B is a proper block of neither problem type. Then whenever B is a proper block, of whatever problem type, $\nu(B) = \min(P(B))$. Notice that there is still no value of ν defined for the improper block; nor will there be.

We shall require a couple of preliminary lemmas, which we place here.

Lemma 2.23.

Let $B = \{k, \ldots, j+1\}$ be a proper block, and let $A^{(B)}$ be the value given by the algorithm. Then

(i) $A^{(B)}(\nu(B)) < (\alpha_1 - \beta_2)(\nu(B))$;

(ii) $A^{(B)}(k) < (\alpha_1 - \beta_2)(k)$.

Proof.

We shall prove (i) by induction, in the same way that we prove the main result of this chapter. Therefore we take as induction hypothesis that if B is not the first block then all blocks to the right of B satisfy this result.

We see from the algorithm that $A_{\nu(B)}^{(B)} = 0$. Therefore if $(\alpha_1 - \beta_2)_{\nu(B)} > 0$ we are done. Suppose $(\alpha_1 - \beta_2)_{\nu(B)} = 0$. We consider the cases $\nu(B) \in I$ and $\nu(B) \notin I$ separately.

First, suppose that $\nu(B) \in I$. Then since $(\alpha_1 - \beta_2)_{\nu(B)} = 0$ we must have $\alpha_{1\nu(B)} = 0$ and $\nu(B) - 1 \in I$. Either $\nu - 1 \in RL$ or $\nu \in K$.

If $\nu - 1 \in RL$ then B is not the first block, so by the induction hypothesis

$$A^{(B)}(\nu(B) - 1) < (\alpha_1 - \beta_2)(\nu(B) - 1).$$

Therefore since $A^{(B)}_{\nu(B)} = 0$ we are done.

If $\nu - 1 \in K$ then since $\alpha_{1\nu(B)} = 0$ and $\nu - 1 \notin \Gamma_p(r, \alpha)$ we see that $\nu \notin \Gamma_p(r, \alpha)$. Therefore the pattern must be either

$$\begin{array}{cc} & \checkmark \\ \nu : & \times \\ & \checkmark \end{array} \quad \begin{array}{cc} & \checkmark \\ \nu - 1 : & \times \\ & \checkmark \end{array} \qquad \text{or} \qquad \begin{array}{cc} & \checkmark \\ \nu : & \times \\ & \times \end{array} \quad \begin{array}{cc} & \checkmark \\ \nu - 1 : & \times \\ & \checkmark \end{array}.$$

However, each of these is impossible. For in the first case B must be of Problem Type 2, and $\nu - 1 \in P(B)$, and in the second case by the pattern in place ν, B must be of neither problem type, but the situation is exactly that of Problem Type 2.

Next suppose that $\nu(B) \notin I$. Then B must be of Problem Type 1, so the pattern is

$$\begin{array}{cc} & \times \\ \nu : & \checkmark \\ & \times \end{array}.$$

Either $\nu - 1 \in I$ or $\nu - 1 \notin I$.

If $\nu - 1 \in I$ then

$$0 = (\alpha_1 - \beta_2)_\nu = \alpha_{1\nu} - \beta_{2\nu} - 1$$

so using the pattern

$$r_\nu \leq \alpha_{1\nu} = \beta_{2\nu} + 1 \leq r_\nu + 1$$

so either $r_\nu = \alpha_{1\nu}$ or $r_\nu = \beta_{2\nu}$ or both. Therefore recalling that the pattern

$$\begin{array}{cc} & \checkmark \\ \nu - 1 : & \checkmark \\ & \times \end{array}$$

is impossible, the pattern must be

$$\begin{array}{cc} & \times \\ \nu : & \checkmark \\ & \times \end{array} \quad \begin{array}{cc} & \checkmark \\ \nu - 1 : & \checkmark \\ & \checkmark \end{array} \qquad \text{or} \qquad \begin{array}{cc} & \times \\ \nu : & \checkmark \\ & \times \end{array} \quad \begin{array}{cc} & \checkmark \\ \nu - 1 : & \times \\ & \times \end{array}$$

In either case, since by the induction hypothesis

$$A^{(B)}(\nu(B)-1) < (\alpha_1 - \beta_2)(\nu(B)-1)$$

and since $A^{(B)}_{\nu(B)} = 0$ we are done.

If $\nu - 1 \notin I$ then

$$0 = (\alpha_1 - \beta_2)_\nu = \alpha_{1\nu} - \beta_{2\nu}$$

so using the pattern, $\alpha_{1\nu} = \beta_{2\nu} = r_\nu$ and the pattern is

$$\begin{array}{rl} & \times \\ \nu - 1: & \checkmark. \\ & \times \end{array}$$

But this is impossible, since it implies that $\nu - 1 \in P$.

Part (ii) follows immediately, remembering that

$$A_k = \ldots = A_\nu = 0.$$

□

Lemma 2.24.

Let $B = \{k, \ldots, j+1\}$ be a block, and let $s^{(B)}$ be the value given by the algorithm. Then for each $i \in Q(B)$, we have $s_i^{(B)} = 0$.

Proof.

We shall have to consider separately the cases

(i) B is a proper block and $|B| > 1$;

(ii) B is the improper block.

However, the proof will be almost identical in each case. Notice that we do not have to consider the case in which B is a proper block of size 1, since in this case $Q(B) = \varnothing$ and the proposition is vacuously true.

Consider case (i) first. It is equivalent to show that $s(\nu - 1) = s(j)$, where we define $s(j)$ as 0 if $j < 0$, that is, if B is the first block.

If B is the first block then by the algorithm

$$A(\nu - 1) = (\alpha_1 - \beta_2)(\nu - 1)$$

so

$$s(\nu - 1) = A(\nu - 1) - (\alpha_1 - \beta_2)(\nu - 1) = 0$$

as required.

Now suppose that B is not the first block. Then from the algorithm

$$A(\nu-1) = A(j) + \sum_{\mu=j+1}^{\nu-1} (\alpha_1 - \beta_2)_\mu p^\mu + p^{j+1}; \tag{1}$$

in particular

$$A(\nu-1) > (\alpha_1 - \beta_2)(\nu-1).$$

Also, by Lemma 2.23,

$$A(j) < (\alpha_1 - \beta_2)(j),$$

so

$$A(j) = s(j) + (\alpha_1 - \beta_2)(j) - p^{j+1}. \tag{2}$$

Therefore combining (1) and (2),

$$\begin{aligned} s(\nu-1) &= A(\nu-1) - (\alpha_1 - \beta_2)(\nu-1) \\ &= s(j) \end{aligned}$$

as required.

Now consider case (ii). It is equivalent to show that $s(n) = s(j)$. Recall that we may assume that B is not the first block. From the algorithm

$$A(n) = A(j) + \sum_{\mu=j+1}^{n} (\alpha_1 - \beta_2)_\mu p^\mu + p^{j+1}; \tag{3}$$

in particular

$$A(n) > (\alpha_1 - \beta_2)(n).$$

Also, by Lemma 2.23,

$$A(j) < (\alpha_1 - \beta_2)(j),$$

so

$$A(j) = s(j) + (\alpha_1 - \beta_2)(j) - p^{j+1}. \tag{4}$$

Therefore combining (3) and (4),

$$\begin{aligned} s(n) &= A(n) - (\alpha_1 - \beta_2)(n) \\ &= s(j) \end{aligned}$$

as required. $\square$

2.3.3 $Q(B)$

We shall consider points in $Q(B)$ first, and we may assume that $Q(B) \neq \varnothing$, or equivalently that $j+1 \notin P(B)$. This implies that either B is improper or $|B| > 1$.

We have remarked earlier that points not in RL satisfy exactly one of Conditions A when $A = \alpha_1 - \beta_2$; in particular, points in $Q(B)$ do so. Therefore, the following result will suffice to show that points in $Q(B)$ satisfy exactly one of Conditions A with $A = A^{(B)}$, the value given by the algorithm:

Lemma 2.25.

If $i \in Q(B)$ then

(i) $i \in \Gamma_p(\alpha_1, A)$ if and only if $i \in I = \Gamma_p(\alpha_1, \beta_2)$;

(ii) $i \notin \Gamma_p(\alpha_2, s)$;

(iii) i satisfies A1 (respectively A2, A3) with $A = A^{(B)}$ if and only if i satisfies A1 (respectively A2, A3) with $A = \alpha_1 - \beta_2$.

It is only part (iii) of this lemma that we require now; but reference to Conditions A makes it clear that part (iii) follows immediately from parts (i) and (ii), and we shall later require these parts.

We shall see that the case where $(\alpha_1 - \beta_2)_{j+1} = p-1$ is slightly tricky; we shall need the following lemma in order to deal with it, before we can prove Lemma 2.25. Notice that it applies to the improper block as well as to proper blocks.

Lemma 2.26.

Let $B = \{k, \dots j+1\}$ be a block other than the first, so that $j \geq 0$. If $(\alpha_1 - \beta_2)_{j+1} = p-1$ then $j+1 \in RL$, so $|B| = 1$ and $Q(B) = \varnothing$.

Proof.

By hypothesis the j^{th} point is a required loss. Thus $\alpha_1 - \beta_2$ scores the j^{th} point in α_1, since $j \in RL \subseteq I$. Then since $(\alpha_1 - \beta_2)_{j+1} = p-1$, $\alpha_1 - \beta_2$ cannot fail to score the $(j+1)^{\text{th}}$ point in α_1; so $j+1 \in I$. We must eliminate the possibility that $j+1 \in K$; that this point, although scored, is not a required loss. That is, we must show that the pattern cannot be

$$\begin{array}{rc} & \checkmark \\ j+1: & \times \\ & \checkmark \end{array}.$$

Now

$$(\alpha_1 - \beta_2)_{j+1} = \alpha_{1(j+1)} - \beta_{2(j+1)} + p - 1$$

which is $p-1$ if and only if $\alpha_{1(j+1)} = \beta_{2(j+1)}$. So the only way for $j+1$ to be in K is for r_{j+1} to be equal to $\alpha_{1(j+1)} = \beta_{2(j+1)}$, and for j to be in $\Gamma_p(r,\beta)$ but not in $\Gamma_p(r,\alpha)$. However, this is incompatible with the statement that $j \in RL$.
□

We now prove Lemma 2.25:

Proof.

In order to check whether a point i is in $\Gamma_p(\alpha_1, A)$ we need to know (at most) the coefficients α_{1i} and A_i and whether $i-1 \in \Gamma_p(\alpha_1, A)$. So if we have two different values of A, say A' and A'', but, for a given i, $A'_i = A''_i$ and $i-1$ is in both or neither of $\Gamma_p(\alpha_1, A')$ and $\Gamma_p(\alpha_1, A'')$, then $i \in \Gamma_p(\alpha_1, A')$ if and only if $i \in \Gamma_p(\alpha_1, A'')$

Similarly, in order to check whether a point i is in $\Gamma_p(\alpha_2, s)$ we need to know (at most) the coefficients α_{2i} and s_i and whether $i-1 \in \Gamma_p(\alpha_2, s)$.

(i) Suppose that B is the first block ($j+1=0$). Then for each $\mu \in Q(B)$ we have $A_\mu = (\alpha_1 - \beta_2)_\mu$. Suppose that i is the 'least criminal'; that is, that

$$\{i-1,\ldots,j+1\} \cap \Gamma_p(\alpha_1, A) = \{i-1,\ldots,j+1\} \cap \Gamma_p(\alpha_1, \beta_2).$$

Then none of the information needed to decide the question of whether $i \in \Gamma_p(\alpha_1, A)$ has changed with the replacement of $\alpha_1 - \beta_2$ by A; in particular $i-1 \in \Gamma_p(\alpha_1, A)$ if and only if $i-1 \in \Gamma_p(\alpha_1, \beta_2)$, so it is absurd to say that the answer to the question has changed. This is the required contradiction.
Now suppose that B is not the first block. If $Q(B) = \varnothing$ the claim is vacuously true, so suppose that $Q(B) \neq \varnothing$. By Lemma 2.26 $(\alpha_1 - \beta_2)_{j+1} \neq p-1$. Then either B is improper or $|B| > 1$. In either case, $A_{j+1} = (\alpha_1 - \beta_2)_{j+1} + 1$. For all $i \neq j+1 \in Q(B)$, $A_i = (\alpha_1 - \beta_2)_i$. Also $j \in I \setminus \Gamma_p(\alpha_1, A)$, by the induction hypothesis.
Suppose that $j+1 \in I$. Then $(\alpha_1 - \beta_2)_{j+1} \geq \alpha_{1(j+1)}$, so $A_{j+1} > \alpha_{1(j+1)}$, so $j+1 \in \Gamma_p(\alpha_1, A)$.
Conversely, suppose that $j+1 \in \Gamma_p(\alpha_1, A)$. Then since $j \notin \Gamma_p(\alpha_1, A)$, we must have that $A_{j+1} > \alpha_{1(j+1)}$. Therefore $(\alpha_1 - \beta_2)_{j+1} \geq \alpha_{1(j+1)}$, so since $j \in I$, $j+1 \in I$.
For all points higher than $j+1$, therefore, the same argument as for the first block applies; nothing germane to the question has altered.

(ii) Notice that, by Lemma 2.24, $s_i = 0$ for all $i \in Q(B)$, and $j \notin \Gamma_p(\alpha_2, s)$ by the induction hypothesis, so $\Gamma_p(\alpha_2, s) \cap Q(B) = \varnothing$.

(iii) This follows from parts (i) and (ii) and the definitions of Conditions A.

$\square$

This completes our proof that, given the induction hypothesis if B is not the first block, every point in $Q(B)$ satisfies exactly one of Conditions A when $A = A^{(B)}$.

2.3.4 $P(B)$

We now have to prove that every point in $P(B)$ satisfies exactly one of Conditions A. We may assume the result of Lemma 2.25, and that either B is the first block, or by the induction hypothesis all previous blocks have been dealt with by the algorithm.

We shall give the proof first with the assumption that B is of neither problem type, and then for each problem type. We shall need a preliminary lemma to express the duality between α and β, which follows.

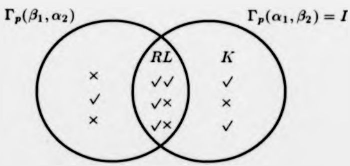
Lemma 2.27.

(i) $RL = I \cap \{\, i \mid \alpha_1(i) + \alpha_2(i) = \beta_1(i) + \beta_2(i) \,\}$;

(ii) $\Gamma_p(\beta_1, \alpha_2) \cap I = \Gamma_p(\beta_1, \alpha_2) \cap \Gamma_p(\alpha_1, \beta_2) = RL$;

(iii) $\Gamma_p(\beta_1, \alpha_2) \setminus \Gamma_p(\alpha_1, \beta_2) = \left\{\, i \mid i : \begin{matrix}\times \\ \surd \\ \times\end{matrix} \,\right\}$

This can be represented diagrammatically thus:



Proof.

(i) Notice that

$$\alpha_1(i) + \alpha_2(i) = \beta_1(i) + \beta_2(i)$$

if and only if either

$$i \in \Gamma_p(r, \alpha) \cap \Gamma_p(r, \beta)$$

in which case

$$\alpha_1(i) + \alpha_2(i) = \beta_1(i) + \beta_2(i) = r(i) + p^{i+1}$$

or

$$i \notin \Gamma_p(r, \alpha) \cup \Gamma_p(r, \beta)$$

in which case

$$\alpha_1(i) + \alpha_2(i) = \beta_1(i) + \beta_2(i) = r(i).$$

Therefore

$$i \in I \cap \{\, i \mid \alpha_1(i) + \alpha_2(i) = \beta_1(i) + \beta_2(i) \,\}$$

if and only if

$$\begin{aligned} i \in & (\Gamma_p(\alpha_1, \beta_2) \cap \Gamma_p(r, \alpha) \cap \Gamma_p(r, \beta)) \\ & \cup ((\Gamma_p(\alpha_1, \beta_2) \setminus \Gamma_p(r, \alpha)) \setminus \Gamma_p(r, \beta)) \end{aligned}$$

if and only if the pattern is

$$\begin{array}{ccccc} & \checkmark & & & \checkmark \\ i: & \checkmark & \quad\text{or}\quad & i: & \times \\ & \checkmark & & & \times \end{array}$$

if and only if $i \in RL$.

(ii) We show first that

$$\Gamma_p(\beta_1, \alpha_2) \cap \Gamma_p(\alpha_1, \beta_2) \subseteq RL.$$

Suppose that

$$i \in \Gamma_p(\beta_1, \alpha_2) \cap \Gamma_p(\alpha_1, \beta_2) = \Gamma_p(\beta_1, \alpha_2) \cap I.$$

Then certainly $i \in I$; we have to show that $i \notin K$. Suppose that $i \in K$. Then $i \in \Gamma_p(r, \beta)$. Therefore

$$\alpha_2(i) > \beta_1(i) > r(i)$$

so $i \in \Gamma_p(r, \alpha)$, which contradicts the assertion that $i \in K$. Therefore $i \in I \setminus K = RL$ as required. We must now show that

$$RL \subseteq \Gamma_p(\beta_1, \alpha_2) \cap \Gamma_p(\alpha_1, \beta_2).$$

Suppose that $i \in RL$. Then certainly $i \in I = \Gamma_p(\alpha_1, \beta_2)$, so

$$\beta_2(i) > \alpha_1(i). \tag{$*$}$$

We have to show that $i \in \Gamma_p(\beta_1, \alpha_2)$. By part (i),

$$\alpha_1(i) + \alpha_2(i) = \beta_1(i) + \beta_2(i),$$

so combining this with equation $(*)$ we have

$$\alpha_2(i) > \beta_1(i)$$

as required.

(iii) We show first that if the pattern is

$$\begin{array}{cc} & \times \\ i: & \checkmark \\ & \times \end{array}$$

then

$$i \in \Gamma_p(\beta_1, \alpha_2) \setminus \Gamma_p(\alpha_1, \beta_2).$$

It is immediate from the pattern that $i \notin \Gamma_p(\alpha_1, \beta_2)$. Also, since $i \in \Gamma_p(r, \alpha) \setminus \Gamma_p(r, \beta)$

$$\alpha_2(i) > r(i) \geq \beta_1(i)$$

so $i \in \Gamma_p(\beta_1, \alpha_2)$ as required. We must now show that if

$$i \in \Gamma_p(\beta_1, \alpha_2) \setminus \Gamma_p(\alpha_1, \beta_2)$$

then the pattern is

$$\begin{array}{cc} & \times \\ i: & \checkmark. \\ & \times \end{array}$$

It is immediate that the pattern is

$$\begin{array}{cc} & \times \\ i: & ?\,; \\ & ? \end{array}$$

moreover $\alpha_2(i) > \beta_1(i)$ and $\alpha_1(i) \geq \beta_2(i)$ so

$$\alpha_1(i) + \alpha_2(i) > \beta_1(i) + \beta_2(i),$$

so $i \in \Gamma_p(r, \alpha) \setminus \Gamma_p(r, \beta)$ and the pattern is as required. $\square$

Lemma 2.28.

If B is of neither problem type and $A^{(B)}$ is the value given by the algorithm, then

$$P(B) \cap \Gamma_p(\alpha_1, A^{(B)}) \subseteq K$$

Proof.

Since B is of neither problem type, $P(B) = \{k\}$. We know that $k \notin K$, so we have to prove that $k \notin \Gamma_p(\alpha_1, A^{(B)})$. Suppose that this assertion is false. Then $\alpha_1(k) < A^{(B)}(k)$. From the algorithm, $A_k^{(B)} = 0$, so this implies that $\alpha_{1k} = 0$ and $\alpha_1(k-1) < A^{(B)}(k-1)$. If $k - 1 = j$, that is, if $|B| = 1$, this contradicts the induction hypothesis, since it says that $k - 1 \in RL \cap \Gamma_p(\alpha_1, A)^{(B)}$, which should be empty. Suppose that $k - 1 > j$. If B is the first block, we have

$$\sum_{i=j+1}^{k-1} A_i p^i > \sum_{i=j+1}^{k-1} \alpha_{1i} p^i.$$

Otherwise, we use the induction hypothesis:

$$\begin{aligned}\sum_{i=j+1}^{k-1} A_i p^i + A(j) &> \sum_{i=j+1}^{k-1} \alpha_{1i} p^i + \alpha_1(j)\\ &\geq \sum_{i=j+1}^{k-1} \alpha_{1i} p^i + A(j)\end{aligned}$$

so

$$\sum_{i=j+1}^{k-1} A_i p^i > \sum_{i=j+1}^{k-1} \alpha_{1i} p^i.$$

Referring to the algorithm, we see that the left-hand side differs from

$$\sum_{i=j+1}^{k-1} (\alpha_1 - \beta_2)_i p^i$$

only in that if B is not the first block then

$$A_{j+1} = (\alpha_1 - \beta_2)_{j+1} + 1 \qquad \text{mod } p.$$

Therefore

$$\sum_{i=j+1}^{k-1} (\alpha_1 - \beta_2)_i p^i \geq \sum_{i=j+1}^{k-1} \alpha_{1i} p^i;$$

if B is the first block then in fact

$$\sum_{i=j+1}^{k-1} (\alpha_1 - \beta_2)_i p^i > \sum_{i=j+1}^{k-1} \alpha_{1i} p^i.$$

(Notice that the case $(\alpha_1 - \beta_2)_{j+1} = p - 1$ presents no difficulty here.) Now if B is not the first block then since $j \in RL$,

$$(\alpha_1 - \beta_2)(j) > \alpha_1(j).$$

Therefore whether or not B is the first block, we see by combining the sums that

$$(\alpha_1 - \beta_2)(k-1) > \alpha_1(k-1),$$

that is, that $k-1 \in I$. We are assuming that $k-1 > j$, so $k-1 \notin RL$. Therefore $k-1 \in K$.

In summary, whether B is the first block or not, $\alpha_{1k} = 0$ and

$$\begin{array}{cc} & \checkmark \\ k-1: & \times\,. \\ & \checkmark \end{array}$$

This implies that $k \notin \Gamma_p(r, \alpha)$, so since $k \in RL$ the pattern must be

$$\begin{array}{cccc} & \checkmark & & \checkmark \\ k: & \times & \qquad k-1: & \times\,. \\ & \times & & \checkmark \end{array}$$

But this is Problem Type 2, so we have the necessary contradiction. □

Lemma 2.29.

If B is of neither problem type and $A^{(B)}$ is the value given by the algorithm, then

$$P(B) \cap \Gamma_p(\alpha_2, s^{(B)}) = \varnothing$$

Proof.

Since B is of neither problem type, $P(B) = \{k\}$. We have to show that $k \notin \Gamma_p(\alpha_2, s^{(B)})$. Suppose that $k \in \Gamma_p(\alpha_2, s^{(B)})$. Then

$$s(k) = \sum_{j+1}^{k} s_i p^i + s(j) > \sum_{j+1}^{k} \alpha_{2i} p^i + \alpha_2(j) = \alpha_2(k).$$

By the induction hypothesis,

$$s(j) \leq \alpha_2(j)$$

and by Lemma 2.24 the only non-zero term of $\sum_{j+1}^{k} s_i p^i$ is $s_k p^k$. Moreover

$$s_k \leq p - (\alpha_1 - \beta_2)_k,$$

so $s(k) > \alpha_2(k)$ implies

$$p - (\alpha_1 - \beta_2)_k = p - (\beta_1 - \alpha_2)_k > \alpha_{2k}.$$

We know that $k \in RL = I \cap \Gamma_p(\beta_1, \alpha_2)$, so

$$p - (\beta_1 - \alpha_2)_k = \begin{cases} p - \beta_{1k} + \alpha_{2k} - p + 1 & \text{if } k - 1 \in \Gamma_p(\beta_1, \alpha_2) \\ p - \beta_{1k} + \alpha_{2k} - p & \text{otherwise} \end{cases}$$
$$= \begin{cases} \alpha_{2k} - \beta_{1k} + 1 & \text{if } k - 1 \in \Gamma_p(\beta_1, \alpha_2) \\ \alpha_{2k} - \beta_{1k} & \text{otherwise} \end{cases}$$

Therefore $s(k) > \alpha_2(k)$ implies that either

(i) $k - 1 \in \Gamma_p(\beta_1, \alpha_2)$ and $\beta_{1k} - 1 < 0$; or
(ii) $k - 1 \notin \Gamma_p(\beta_1, \alpha_2)$ and $\beta_{1k} < 0$.

The latter is impossible. The former implies that $\beta_{1k} = 0$, so since

$$k - 1 \notin RL = I \cap \Gamma_p(\beta_1, \alpha_2),$$

we must have

$$k - 1 \in \Gamma_p(\beta_1, \alpha_2) \setminus I,$$

so by Lemma 2.27(ii) the pattern must be

$$\begin{array}{rc} & \times \\ k-1 : & \checkmark \\ & \times \end{array}$$

Then since $k - 1 \notin \Gamma_p(r, \beta)$ and $\beta_{1k} = 0$, $k \notin \Gamma_p(r, \beta)$ so the pattern is

$$\begin{array}{rc} & \checkmark \\ k : & \times \\ & \times \end{array} \qquad \begin{array}{rc} & \times \\ k-1 : & \checkmark \\ & \times \end{array}$$

which is Problem Type 1. This is the required contradiction. □

We next consider the case that the current block B is of Problem Type 1.

Lemma 2.30.

If B is of Problem Type 1 and $A^{(B)}$ is the value given by the algorithm, then

$$P(B) \cap \Gamma_p(\alpha_1, A^{(B)}) \subseteq K.$$

Proof.

From the definition of Problem Type 1 we see that

$$P(B) \cap K = \varnothing$$

so we have to show that

$$P(B) \cap \Gamma_p(\alpha_1, A^{(B)}) = \varnothing.$$

Suppose that $i \in P(B)$, that is, that $k \geq i \geq \nu(B)$. Then

$$A_i = \ldots = A_\nu(B) = 0$$

so if $i \in \Gamma_p(\alpha_1, A^{(B)})$ then

$$\alpha_{1i} = \ldots = \alpha_{1\nu(B)} = 0$$

and $\nu(B) - 1 \in \Gamma_p(\alpha_1, A^{(B)})$. There are three possibilities:

(i) $\nu(B) - 1 \in Q(B)$;

(ii) $\nu(B) - 1 = j \geq 0$;

(iii) $\nu(B) - 1 = j = -1$.

In case (i) we have shown that $\nu(B) - 1 \in \Gamma_p(\alpha_1, A^{(B)})$ if and only if $\nu(B) - 1 \in K$. Since B is of Problem Type 1 this means that the pattern must be

$$\begin{array}{rcrc} & \times & & \checkmark \\ \nu(B): & \checkmark & \qquad \nu(B) - 1: & \times\,. \\ & \times & & \checkmark \end{array}$$

However, if $\alpha_{1\nu(B)} = 0$ and $\nu(B) - 1 \notin \Gamma_p(r, \alpha)$ then it is impossible that $\nu(B) \in \Gamma_p(r, \alpha)$, so we have the desired contradiction.

In case (ii) by the induction hypothesis $\nu(B) - 1 \notin \Gamma_p(\alpha_1, A^{(B)})$, so we have the desired contradiction. In case (iii) certainly $\nu(B) - 1 \notin \Gamma_p(\alpha_1, A^{(B)})$, since $\nu(B) - 1 < 0$. □

Lemma 2.31.

If B is of Problem Type 1 and $A^{(B)}$ is the value given by the algorithm, then

$$P(B) \cap \Gamma_p(\alpha_2, s^{(B)}) = \emptyset.$$

Proof.

Suppose that $i \in P(B)$, that is, that $k \geq i \geq \nu(B)$. We have to show that $i \notin \Gamma_p(\alpha_2, s^{(B)})$. Since using Lemma 2.23

$$A(i) = A(\nu(B)) < (\alpha_1 - \beta_2)(\nu(B)) \leq (\alpha_1 - \beta_2)(i)$$

we see that

$$\begin{aligned} s(i) &= A^{(B)}(i) - (\alpha_1 - \beta_2)(i) + p^{i+1} \\ &= A^{(B)}(\nu(B) - 1) - (\alpha_1 - \beta_2)(i) + p^{i+1}. \end{aligned} \qquad (*)$$

We consider separately the two cases

(i) $\nu(B) - 1 = j$ (possibly $j = -1$);

(ii) $\nu(B) - 1 > j$.

In case (i)

$$s(i) \leq \beta_1(j) - (\beta_1 - \alpha_2)(i) + p^{i+1}$$

using Lemma 2.23 and the fact that $\alpha_1 - \beta_2 = \beta_1 - \alpha_2$

$$\begin{aligned} &\leq \beta_1(i) - (\beta_1 - \alpha_2)(i) + p^{i+1} \\ &= \alpha_2(i) \end{aligned}$$

since $i \in \Gamma_p(\beta_1, \alpha_2)$ by Lemma 2.27. Therefore $i \notin \Gamma_p(\alpha_2, s^{(B)})$, as required.

In case (ii), recall that according to the algorithm

$$A_{j+1} \geq (\alpha_1 - \beta_2)_{j+1}$$

and that for all $j + 1 < \mu \leq \nu(B)$

$$A_\mu = (\alpha_1 - \beta_2)_\mu.$$

Therefore

$$s(\nu(B) - 1) = A^{(B)}(\nu(B) - 1) - (\alpha_1 - \beta_2)(\nu(B) - 1)$$

and since $\nu(B) - 1 \in Q(B)$, using Lemma 2.25

$$s(\nu(B) - 1) \leq \alpha_2(\nu(B) - 1)$$

so

$$\begin{aligned} A^{(B)}(\nu(B) - 1) &\leq \alpha_2(\nu(B) - 1) + (\beta_1 - \alpha_2)(\nu(B) - 1) \\ &= \begin{cases} \beta_1(\nu(B) - 1) + p^{\nu}(B) & \text{if } \nu(B) - 1 \in \Gamma_p(\beta_1, \alpha_2) \\ \beta_1(\nu(B) - 1) & \text{otherwise} \end{cases} \end{aligned}$$

If $\nu(B) - 1 \in \Gamma_p(\beta_1, \alpha_2)$ then since $\nu(B) - 1 \in Q(B)$ the pattern must be

$$\begin{array}{rl} & \times \\ \nu(B) - 1 : & \checkmark. \\ & \times \end{array}$$

Then by definition of $\nu(B)$, $\beta_{2\nu(B)} \neq r_\nu(B)$, and since $\nu(B) \notin \Gamma_p(r, \beta)$ this must mean that $\beta_{2\nu(B)} < r_\nu(B)$. Moreover, since $\nu(B) - 1 \notin \Gamma_p(r, \beta)$, $\beta_{1\nu(B)} + \beta_{2\nu(B)} \geq r_\nu(B)$, so $\beta_{1\nu(B)} > 0$. The point of this manipulation is to show that whether $\nu(B) - 1 \in \Gamma_p(\beta_1, \alpha_2)$ or not,

$$A^{(B)}(\nu(B) - 1) \leq \beta_1(\nu(B)) \leq \beta_1(i).$$

Therefore by equation $(*)$

$$\begin{aligned} s^{(B)}(i) &\leq \beta_1(i) - (\alpha_1 - \beta_2)(i) + p^{i+1} \\ &= \beta_1(i) - (\beta_1 - \alpha_2)(i) + p^{i+1} \\ &= \alpha_2(i) \end{aligned}$$

since $i \in \Gamma_p(\beta_1, \alpha_2)$. Thus $i \notin \Gamma_p(\alpha_2, s^{(B)})$, as required. $\square$

Finally, we consider the case that the current block B is of Problem Type 2.

Lemma 2.32.

If B is of Problem Type 2 and $A^{(B)}$ is the value given by the algorithm, then

$$P(B) \cap \Gamma_p(\alpha_1, A^{(B)}) = \emptyset.$$

Proof.

Suppose that $i \in P(B)$, that is, that $k \geq i \geq \nu(B)$. We have to show that $i \notin \Gamma_p(\alpha_1, A^{(B)})$. We suppose the contrary. From the algorithm,

$$A_i = \ldots = A_\nu(B) = 0$$

so if $i \in \Gamma_p(\alpha_1, A^{(B)})$ then

$$\alpha_{1i} = \ldots = \alpha_{1\nu(B)} = 0$$

and $\nu(B) - 1 \in \Gamma_p(\alpha_1, A^{(B)})$. There are three possibilities:

(i) $\nu(B) - 1 \in Q(B)$;

(ii) $\nu(B) - 1 = j \geq 0$;

(iii) $\nu(B) - 1 = j = -1$.

In case (i) we have shown that $\nu(B) - 1 \in \Gamma_p(\alpha_1, A^{(B)})$ if and only if $\nu(B) - 1 \in K$. Since we must also have $\alpha_{1\nu(B)} = 0$, this is impossible, for it contradicts the definition of $\nu(B)$.

In case (ii) by the induction hypothesis $\nu(B) - 1 \notin \Gamma_p(\alpha_1, A^{(B)})$, so we have the desired contradiction. In case (iii) certainly $\nu(B) - 1 \notin \Gamma_p(\alpha_1, A^{(B)})$, since $\nu(B) - 1 < 0$. $\square$

Lemma 2.33.

If B is of Problem Type 2 and $A^{(B)}$ is the value given by the algorithm, then

$$P(B) \cap \Gamma_p(\alpha_2, s^{(B)}) \subseteq K.$$

Proof.

Since $P(B) \cap K = \{k\}$, we only have to show that $k \notin \Gamma_p(\alpha_2, s^{(B)})$. By Lemma 2.23(ii),

$$s(k) = A(k) - (\alpha_1 - \beta_2)(k) + p^{k+1}$$

and

$$s(j) = A(j) - (\alpha_1 - \beta_2)(j) + p^{j+1}.$$

From the algorithm

$$\begin{aligned} A(k) &= A(j) + \sum_{\mu=j+1}^{\nu-1} (\alpha_1 - \beta_2)_\mu p^\mu + p^{j+1} \\ &= s(j) - \sum_{\mu=\nu}^{k} (\alpha_1 - \beta_2)_\mu p^\mu + p^{k+1} \\ &\leq \alpha_2(\nu-1) - (\beta_1 - \alpha_2)(k) + (\beta_1 - \alpha_2)(\nu - 1) + p^{k+1} \\ &= \alpha_2(\nu - 1) - \beta_1(k) + \alpha_2(k) + (\beta_1 - \alpha_2)(\nu - 1) \end{aligned}$$

since we know that $k \in \Gamma_p(\beta_1, \alpha_2)$

$$= \begin{cases} \alpha_2(k) - \beta_1(k) + \beta_1(\nu - 1) + p^\nu & \text{if } \nu - 1 \in \Gamma_p(\beta_1, \alpha_2) \\ \alpha_2(k) - \beta_1(k) + \beta_1(\nu - 1) & \text{otherwise.} \end{cases}$$

Thus the right-hand side is at most $\alpha_2(k)$ unless $\nu - 1 \in \Gamma_p(\beta_1, \alpha_2)$ and

$$\beta_{1k} = \ldots = \beta_{1\nu} = 0.$$

In this case, since $k - 1, \ldots \nu \in \Gamma_p(r, \beta)$, we must have

$$r_k = \ldots = r_\nu = 0,$$

and that $\nu - 1 \in \Gamma_p(r, \beta)$. By Lemma 2.27 this implies that the pattern must be

$$\begin{array}{rl} & \checkmark \\ \nu - 1: & \checkmark. \\ & \checkmark \end{array}$$

Then

$$\alpha_1(\nu) \geq \alpha_1(\nu - 1) > r(\nu - 1) = r(\nu)$$

so $\nu \in \Gamma_p(r, \alpha)$, which contradicts the definition of ν in Problem Type 2. Therefore in all cases $s(k) \leq \alpha_2(k)$ as required. □

Combining these results, we have proved that, whatever the problem type of the current block B, provided that either B is the first block or all previous blocks have been dealt with by the algorithm, applying the algorithm to B deals with B.

We promised to check that $A \leq r$. The leading coefficient of A is n (at most) so it suffices to check that $A(n) \leq r$. Now $n \in Q(B')$ where B' is the improper block, and since $\alpha_1 - \beta_2 \leq \alpha_1$, $n \notin I$. Therefore, by Lemma 2.25, $n \notin \Gamma_p(\alpha_1, A)$, so $A \leq \alpha_1 \leq r$ as required.

By induction, this proves Theorem 2.11.

We record some easy but important consequences of this result.

Corollary 2.34. For all weights α, β and γ,

(i) $k(\alpha,\beta) = k(\beta,\alpha) = 0$ if and only if α and β score the same points in r. In this case, m_α/m_β and m_β/m_α are both integers; that is, $m_\alpha = m_\beta$ for all valid modules.

(ii) $k(\beta,\alpha) = k(\alpha,\beta) + \nu_p(|\alpha|) - \nu_p(|\beta|)$

(iii) $k(\alpha,\gamma) \leq k(\alpha,\beta) + k(\beta,\gamma)$

(iv) If α scores all possible points in r, in the sense of Lemma 2.9 that is, $\gamma_p(r,\alpha_1) = \gamma_p(r)$, then $k(\alpha,\beta) = 0$ for all β.

□

We use this last remark to prove our first result on the structure of X/V_p.

Proposition 2.35.

Given r and a prime $p \leq r$, let

$$\mathcal{A} = \{\, \alpha \in \Lambda \mid \forall \beta \in \Lambda \quad |\beta|_p \leq |\alpha|_p \,\}.$$

(i) Then for any valid lattice M,

$$\alpha \in \mathcal{A} \Rightarrow \forall \beta \in \Lambda \quad m_\alpha \geq m_\beta.$$

(ii) This implies in particular that X/V_p has a unique maximal submodule $M_{\max}/V_p$ where the coefficients of $M_{\max}$ are

$$m_\alpha = \begin{cases} p, & \text{if } \alpha \in \mathcal{A}; \\ 1, & \text{otherwise.} \end{cases}$$

(iii) The highest weight α of the simple head of X/V_p is (α_1, α_2) where

$$(\alpha_1)_i = \begin{cases} p-1 & 0 \leq i < n; \\ r_n - 1 & i = n. \end{cases}$$

Proof.

(i) By Corollary 2.34

$$\alpha \in \mathcal{A} \Rightarrow \forall \beta \in \Lambda \quad k(\alpha,\beta) = 0.$$

Then m_α/m_β is an integer for each β, at each valid module. The first part follows.

(ii) We first check that $M_{\max}$ is a valid module. If it fails to be valid, there must be some α and β such that

$$\nu_p(m_\alpha) > \nu_p(m_\beta) + k(\beta,\alpha)$$

which can only happen if $m_\alpha = p$, $m_\beta = 1$ and $k(\beta,\alpha) = 0$, so $\alpha \in \mathcal{A}$, $\beta \notin \mathcal{A}$. By Corollary 2.34(iv), $k(\alpha,\beta) = 0$, so by Corollary 2.34(i), $k(\beta,\alpha) = 0$ implies $\beta \in \mathcal{A}$, which is the required contradiction. To show that $M_{\max}$ is the unique maximal submodule, it suffices to show that any non-zero valid module M is contained in $M_{\max}$. Pick some β such that $m_\beta^{(M)} > 1$. Then by part (i), $m_\alpha^{(M)} \geq m_\beta^{(M)} > 1$ for all $\alpha \in \mathcal{A}$, so in particular $M \subseteq M_{\max}$.

(iii) α_{1n} must be strictly less than r_n since α, which must score the $(n-1)^{\text{th}}$ point in r, would otherwise score the n^{th} point in r. This would imply that $\alpha_1 > r$, which is impossible. The rest is clear. □

Remark 2.36.

We may consider possible generalisations of this result. Plainly it is not true that

$$|\alpha|_p \geq |\beta|_p \Rightarrow m_\alpha \geq m_\beta,$$

but what about the property

$$|\alpha|_p > |\beta|_p \Rightarrow m_\alpha \geq m_\beta \ ?$$

In fact even this weaker assertion is false; a counter-example is $r = 30$, $p = 3$, $\alpha = (15,15)$, $\beta = (29,1)$, in which $\nu_p(|\alpha|) = 2$ and $\nu_p(|\beta|) = 1$, and there is a valid module in which $\nu_p(m_\alpha) = 0$ whilst $\nu_p(m_\beta) = 1$. In the next chapter we shall see why, and under what circumstances, this can happen.

The reader will see that we have now greatly simplified the problem of finding all valid modules for particular values of r and p. We began by describing this as the problem of finding all natural number solutions $\{m_\alpha\}$ to the validity conditions $V(\alpha,\beta)$:

$$\frac{m_\beta}{m_\alpha}\frac{\beta_1!\,\beta_2!}{A!\,(\alpha_1 - A)!\,(\beta_1 - A)!\,(\beta_2 - \alpha_1 + A)!} \in \mathbb{Z}$$

for every pair (α,β) of dominant weights, and for every integer value of A satisfying

$$\alpha_1 - \beta_2 \leq A \leq \min(\alpha_1,\beta_1),$$

and have reduced it to that of finding all integer solutions $\{\nu_p(m_\alpha)\}$ to the following two conditions.

V1) for all α and β,

$$\nu_p(m_\alpha) - \nu_p(m_\beta) + k(\alpha, \beta) \geq 0.$$

This is the validity condition, which ensures that M is an admissible lattice.

V2) for all α,

$$0 \leq \nu_p(m_\alpha) \leq \nu_p(|\alpha|)$$

This is the normalisation condition, which ensures that $X \geq M \geq V_p$.

Given the relative simplicity of calculating $k(\alpha, \beta)$, this set of inequalities is a great deal more tractable than the original. We are still forced to consider every pair of dominant weights in turn, and for all but the smallest problems this makes the task too large to be done by hand. However, anyone attempting to find all solutions by hand will notice that most of the inequalities are redundant, and will begin to suspect that the problem can be simplified still further. In the next chapter we shall show that this is so.

Chapter 3

The lattice of valid modules.

In this chapter we show that the information required to find the lattice of valid modules for particular values of r and p is encapsulated in a much more economical structure than the set of all pairs of dominant weights. We give a method for drawing the lattice that is computationally easy enough to enable comparatively large problems to be tackled by hand. Moreover, we use the knowledge we have gained to deduce some general facts about the structure of the lattice, and to give techniques which may be used to deduce others.

3.1 Weights and composition factors of X/V_p.

In this section we describe the composition factors of X/V_p in terms of the points scored by weights occurring in them, in the following sense.

Definition 3.1.

We say that a weight α occurs in a composition factor F of X/V_p if and only if the weight space $F^\alpha \neq 0$. ◁

Lemma 3.2.

If M and N are valid modules with $M > N$, and M/N is simple, then for each weight α, either $m_\alpha^{(N)} = m_\alpha^{(M)}$ or $m_\alpha^{(N)} = pm_\alpha^{(M)}$. The latter case occurs if and only if α is a weight of M/N.

Proof.

Certainly if M and N are valid modules then so is $pM + N$, and we have

$$M \geq pM + N \geq N.$$

Our claim is that if M/N is simple then $M \neq pM + N$, so that $N = pM + N$. Now the fact that M/N is non-trivial implies that there is some weight α such that $M^\alpha > N^\alpha$, so that $N^\alpha \leq pM^\alpha$. In any such case

$$(pM + N)^\alpha = pM^\alpha + N^\alpha = pM^\alpha < M^\alpha$$

and since any valid module is the direct sum of its weight spaces the result follows. □

We shall want to identify a composition factor by giving any weight which occurs in it, and later just by giving the set of points scored in r by such a weight. The result which allows us to do the first part is the following:

Theorem 3.3.

For any weight α' there is at most one isomorphism class of composition factors F of X/V_p such that $F^{\alpha'} \neq 0$.

We prove this result in several stages.

Consider A given by

$$m_\alpha^{(A)} = \begin{cases} p & \text{if } \nu_p(|\alpha|) > 0 \\ 1 & \text{if } \nu_p(|\alpha|) = 0. \end{cases}$$

Lemma 3.4.

A is a valid module.

Proof.

This is equivalent to saying that for all α and β,

$$\nu_p(m_\alpha) - \nu_p(m_\beta) + k(\alpha, \beta) \geq 0.$$

This is certainly true if $m_\alpha \geq m_\beta$, so we only need to check the case where $\nu_p(|\alpha|) = 0$ and $\nu_p(|\beta|) \neq 0$. In this case we need to show that $0 - 1 + k(\alpha, \beta) \geq 0$, that is, that $k(\alpha, \beta) > 0$. But

$$k(\alpha, \beta) = k(\beta, \alpha) + \nu_p(|\beta|) - \nu_p(|\alpha|) > 0$$

as required. □

In fact we may also see this, more easily, by noticing that $A = pX + V$.

Now consider any composition series of X/A. Because $m_\alpha^{(A)} \leq p$ for all α, there is at most one composition factor F of X/A such that $F^{\alpha'} \neq 0$. Also any weight α' for which $\nu_p(|\alpha'|) \neq 0$, that is, for which there is any composition factor F of X/V_p for which $F^{\alpha'} \neq 0$, does occur in some composition factor of X/A, since otherwise we could not have $m_{\alpha'}^{(A)} = p$.

We show that any composition factor of X/V_p is isomorphic to some composition factor of X/A. This will complete the proof of Theorem 3.3. It is possible to prove this in a more general context than we have here, using the Jordan-Hölder theorem. However, it is useful to demonstrate here some techniques that we shall need later. We need a further piece of notation.

Definition 3.5.

Suppose M/N is a composition factor of X/V_p. Let

$$J(M/N) = \left\{ \alpha \mid m_\alpha^{(N)} = p m_\alpha^{(M)} \right\},$$

the set of weights 'genuinely affected' by this composition factor. Then we may also describe $J(M/N)$ as the set of weights such that the weight space

$$(M/N)^\alpha = M^\alpha / N^\alpha \neq 0$$

$\triangleleft$

Notice that $J(M/N)$ determines the isomorphism class of M/N.

Lemma 3.6.

If α and β are in $J(M/N)$, then $m_\alpha^{(M)} = m_\beta^{(M)}$.

Proof.

Suppose not. Then let I be the proper subset of $J(M/N)$ defined by

$$I = \left\{ \alpha \in J(M/N) \mid m_\alpha^{(M)} \leq m_\beta^{(M)} \quad \forall \beta \in J(M/N) \right\}$$

and consider the module P given by

$$m_\alpha^{(P)} = \begin{cases} p m_\alpha^{(M)} & \text{if } \alpha \in I \\ m_\alpha^{(M)} & \text{otherwise} \end{cases}$$

so that $N < P < M$. We claim that P is a valid module, which will contradict the simplicity of M/N. By the validity of M and N, we need only check pairs (α, β) where $\beta \in I$ and $\alpha \in J(M/N) \setminus I$. Then

$$\nu_p(m_\alpha^{(P)}) - \nu_p(m_\beta^{(P)}) + k(\alpha, \beta) = \nu_p(m_\alpha^{(M)}) - \nu_p(m_\beta^{(M)}) - 1 + k(\alpha, \beta).$$

But by hypothesis $\nu_p(m_\alpha^{(M)}) > \nu_p(m_\beta^{(M)})$ so the right-hand side is at least zero, and so P is a valid module and we have the desired contradiction. $\square$

Now let B/C be any composition factor of X/A. Recall that

$$J(B/C) = \left\{ \alpha \mid m_\alpha^{(C)} = p m_\alpha^{(B)} \right\}$$

and note that if $\alpha \in J(B/C)$ then $m_\alpha^{(B)} = 1$ and $m_\alpha^{(C)} = p$.

Lemma 3.7.

If α and β are in $J(B/C)$ then $k(\alpha, \beta) = 0 = k(\beta, \alpha)$.

Proof.

If $|J(B/C)| = 1$ there is nothing to prove. We show first that $\nu_p(|\alpha|) = \nu_p(|\beta|)$ and so $k(\alpha,\beta) = k(\beta,\alpha)$. Suppose not. Then let I' be the proper subset of $J(B/C)$ defined by

$$I' = \{\, \alpha \in J(B/C) \mid \nu_p(|\alpha|) \geq \nu_p(|\beta|) \quad \forall \beta \in J(B/C) \,\}$$

and consider the module Q given by

$$m_\alpha^{(Q)} = \begin{cases} pm_\alpha^{(B)} & \text{if } \alpha \in I' \\ m_\alpha^{(B)} & \text{otherwise} \end{cases}$$

so that $C < Q < B$. We claim that Q is a valid module. By the validity of B and C, we need only check pairs (α,β) where $\beta \in I'$ and $\alpha \in J(B/C) \setminus I'$. Then

$$\nu_p(m_\alpha^{(Q)}) - \nu_p(m_\beta^{(Q)}) + k(\alpha,\beta) = 1 - 0 + k(\alpha,\beta)$$

which is at least zero, so Q is a valid module and we have the desired contradiction. Thus for all α and β in $J(B/C)$, $\nu_p(|\alpha|) = \nu_p(|\beta|)$. Now

$$k(\beta,\alpha) = k(\alpha,\beta) + \nu_p(|\alpha|) - \nu_p(|\beta|) = k(\alpha,\beta)$$

as claimed.

We now show that $k(\alpha,\beta) = k(\beta,\alpha) = 0$ using the fact that $k(\alpha,\beta)$ is the number of points in r scored by β and not by α.

We define a subset I'' of $J(B/C)$ for an arbitrary fixed element α of $J(B/C)$ by

$$I'' = \{\, \beta \in J(B/C) \mid k(\alpha,\beta) = 0 \,\}\,.$$

Then if β is in I'', γ is in $J(B/C)$ and $k(\beta,\gamma) = 0$ then by Corollary 2.34(iii)

$$k(\alpha,\gamma) \leq k(\alpha,\beta) + k(\beta,\gamma) = 0$$

so γ is in I''. Thus if β is in I'' and γ is in $J(B/C) \setminus I''$ then $k(\beta,\gamma) > 0$.

We would like to show that this situation cannot arise, that is, that

$$J(B/C) = I''.$$

First we show that $I'' \neq \varnothing$.

We have assumed $|J(B/C)| \neq 1$, so by simplicity of B/C the module D defined by

$$m_\beta^{(D)} = \begin{cases} pm_\beta^{(B)} & \text{if } \beta = \alpha \\ m_\beta^{(B)} & \text{otherwise} \end{cases}$$

must fail to be valid. By validity of B this means that there is some β in $J(B/C)$, $\beta \neq \alpha$, such that

$$\nu_p(m_\beta^{(D)}) - \nu_p(m_\alpha^{(D)}) + k(\beta,\alpha) < 0,$$

that is, such that $0 - 1 + k(\beta,\alpha) < 0$, so $k(\alpha,\beta) = k(\beta,\alpha) = 0$, so I'' is non-empty, as required.

Now consider the module E given by

$$m_\beta^{(E)} = \begin{cases} pm_\beta^{(B)} & \text{if } \beta \in I'' \\ m_\beta^{(B)} & \text{otherwise} \end{cases}$$

so that $C \leq E < B$. We claim that E is a valid module. For we need only check pairs (β,γ) where $\gamma \in I''$ and $\beta \in J(B/C) \setminus I''$. Then

$$\nu_p(m_\beta^{(E)}) - \nu_p(m_\gamma^{(E)}) + k(\beta,\gamma) = \nu_p(m_\beta^{(B)}) - \nu_p(m_\gamma^{(B)}) - 1 + k(\beta,\gamma)$$

which is $k(\beta,\gamma) - 1$ which is at least zero since $\gamma \in I''$ and $\beta \in J(B/C) \setminus I''$. Therefore E is a valid module as claimed, so by simplicity of B/C we have $E = C$, that is $I'' = J(B/C)$ as required. □

Corollary 3.8. By Corollary 2.34 (i), this implies that if M is any valid module and α and β are any weights in $J(B/C)$ then $m_\alpha^M = m_\beta^M$.

□

Now return to consideration of M/N and $J(M/N)$. Since every α lying in $J(M/N)$ must have non-zero weight space in some (unique) composition factor of X/A, we may fix an arbitrary composition series of X/A and choose the closest factor to X in which there is any α in $J(M/N)$ with non-zero weight space. Call this B/C. Then for every α in $J(M/N)$, by choice of B we have $m_\alpha^{(B)} = 1$ and so

$$\alpha \in J(B/C) \iff \left(m_\alpha^{(B)} = 1 \quad \text{and} \quad m_\alpha^{(C)} = p\right)$$

and, by choice of B/C, $J(M/N) \cap J(B/C) \neq \emptyset$. Therefore $J(B/C) \subseteq J(M/N)$, by Corollary 3.8, and we know that some weight in $J(B/C)$ satisfies the criterion for membership of $J(M/N)$ so, since this depends only on values of m_α at valid modules, all weights of $J(B/C)$ must satisfy the criterion.

Consider the module Z given by

$$m_\alpha^{(Z)} = \begin{cases} pm_\alpha^{(M)} & \text{if } \alpha \in J(B/C) \\ m_\alpha^{(M)} & \text{otherwise} \end{cases}$$

so that $N \leq Z < M$. We want to show that $Z = N$, which will show that $J(B/C) = J(M/N)$. Suppose not. Then by simplicity of M/N, Z cannot be a valid module, so there is some α in $J(B/C)$ and β in $J(M/N) \setminus J(B/C)$ such that

$$\nu_p(m_\beta^{(Z)}) - \nu_p(m_\alpha^{(Z)}) + k(\beta,\alpha) < 0,$$

that is,

$$\nu_p(m_\beta^{(M)}) - \nu_p(m_\alpha^{(M)}) - 1 + k(\beta,\alpha) < 0,$$

which implies $k(\beta,\alpha) = 0$ since we have proved that $\nu_p(m_\alpha^{(M)}) = \nu_p(m_\beta^{(M)})$. However, by choice of B/C,

$$m_\beta^{(B)} = m_\beta^{(C)} = 1$$

and $m_\alpha^{(C)} = p$, and by validity of C

$$\nu_p(m_\beta^{(C)}) - \nu_p(m_\alpha^{(C)}) + k(\beta,\alpha) \geq 0$$

that is, $k(\beta,\alpha) \geq 1$. This is the required contradiction, so $Z = N$, $J(B/C) = J(M/N)$ and so M/N is isomorphic to B/C.

This completes the proof of Theorem 3.3.

Lemma 3.9.

(i) If α satisfies $|\alpha|_p \neq 1$ then α occurs in exactly one of the isomorphism classes of composition factors of X/V_p. If $|\alpha|_p = 1$ then α does not occur in any of them.

(ii) If the weight α occurs in composition factor F, then composition factors isomorphic to F occur $\nu_p(|\alpha|)$ times in any composition series of X/V_p.

(iii) Two weights α and β occur in the same isomorphism class of composition factors of X/V_p only if $|\alpha|_p = |\beta|_p$, that is, only if they score the same number of points in r.

Proof.

(i) α occurs in at most one simple by Theorem 3.3. If $|\alpha|_p \neq 1$ then it must occur in at least one since $V_p^\alpha = |\alpha|_p X_p^\alpha$, that is, since the α-weight spaces of X and V_p are not equal. Similarly, if $|\alpha|_p = 1$ then the weight spaces are equal, so α cannot occur.

(ii) By Lemma 3.2, in any composition series, α must occur in a total of $\nu_p(|X^\alpha/V_p^\alpha|)$ composition factors, that is, in $\nu_p(|\alpha|)$ composition factors, which must all be isomorphic.

(iii) If α and β both occur in F then neither can occur in any other simple, so, using Lemma 3.2, $m_\alpha = m_\beta$ for all valid modules. In particular,

$$|\alpha|_p = m_\alpha^{(V_r)} = m_\beta^{(V_r)} = |\beta|_p.$$

Lemma 2.6 gives the result. □

The next result is the one, advertised earlier, which allows us to identify any composition factor F by giving only the set of points scored in r by some weight which occurs in F. This result extends Lemma 3.9.

Proposition 3.10.

Two weights α and β occur in the same composition factor if and only if where

$$r = r_n p^n + \cdots + r_0$$

$$\alpha_1 = (\alpha_1)_n p^n + \cdots + (\alpha_1)_0$$

$$\beta_1 = (\beta_1)_n p^n + \cdots + (\beta_1)_0$$

we have for all m, $0 \leq m \leq n-1$,

$$\alpha_1(m) > r(m)$$

if and only if

$$\beta_1(m) > r(m)$$

in other words, not only do α and β score the same number of points in r, they also score these points in the same positions in their p-adic expansions. That is, $\Gamma_p(r,\alpha_1) = \Gamma_p(r,\beta_1)$.

Proof.

If α and β score the same points then $k(\alpha,\beta) = k(\beta,\alpha) = 0$ and we have already remarked in Corollary 2.34 that in this case $m_\alpha = m_\beta$ at all valid modules, that is, the two weights occur in the same simple.

Conversely, suppose that α and β occur together, that is, $m_\alpha = m_\beta$ at every valid module. Suppose both occur in the composition factor M/N. We claim that if P is the module defined by

$$m_\nu^{(P)} = \begin{cases} pm_\nu^{(M)} & \text{if } \nu \text{ scores the same points in } r \text{ as } \alpha \\ m_\nu^{(M)} & \text{otherwise} \end{cases}$$

then P is a valid module, contradicting the hypothesis that M/N is simple. For if P is not a valid module, there must be some γ and δ such that

$$\nu_p(m_\delta^{(P)}) > \nu_p(m_\gamma^{(P)}) + k(\gamma,\delta)$$

so by validity of M, $\nu_p(m_\gamma^{(P)}) = \nu_p(m_\gamma^{(M)})$ and $\nu_p(m_\delta^{(P)}) = \nu_p(m_\delta^{(M)}) + 1$, that is, δ scores the same points in r as α whilst γ does not, and

$$\nu_p(m_\delta^{(M)}) = \nu_p(m_\gamma^{(M)}) + k(\gamma, \delta). \tag{$*$}$$

Now by validity of N,

$$\nu_p(m_\delta^{(N)}) \leq \nu_p(m_\gamma^{(N)}) + k(\gamma, \delta)$$

so since $\nu_p(m_\delta^{(N)}) = \nu_p(m_\delta^{(P)}) = \nu_p(m_\delta^{(M)}) + 1$ we have $\nu_p(m_\gamma^{(N)}) = \nu_p(m_\gamma^{(P)}) + 1 = \nu_p(m_\gamma^{(M)}) + 1$, that is, γ occurs in M/N together with α and δ, so γ scores the same number of points in r as α, although not in the same positions. But then $\nu_p(m_\gamma) = \nu_p(m_\delta)$ for all valid modules, so, by equation $(*)$, $k(\gamma, \delta) = 0$. Since γ and δ score the same number of points in r,

$$k(\delta, \gamma) = \nu_p(\gamma) - \nu_p(\delta) + k(\gamma, \delta) = 0$$

so γ and δ score the same points in r, which is a contradiction. $\square$

Corollary 3.11.

This implies that the weights α, mentioned in the statement of Proposition 2.35, which satisfy

$$\forall \beta \in \Lambda \quad |\beta|_p \leq |\alpha|_p$$

must all occur in the same composition factor, since they score all possible points in r. Of course, we have already proved this directly in Proposition 2.35! $\square$

To summarise, we have shown that we can classify composition factors according to the sets of points which are scored by weights occurring in the composition factors. Hereafter we shall be primarily concerned with this description of the composition factors.

Example 3.12.

Let $r = 10$ and $p = 2$. Then

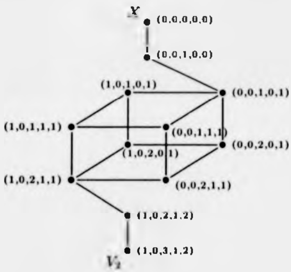
$$r = 10 = 1.2^3 + 0.2^2 + 1.2^1 + 0$$

and the possibilities for α_1 are

$$9 = 1.2^3 + 0.2^2 + 0.2^1 + 1$$
$$8 = 1.2^3 + 0.2^2 + 0.2^1 + 0$$
$$7 = 0.2^3 + 1.2^2 + 1.2^1 + 1$$
$$6 = 0.2^3 + 1.2^2 + 1.2^1 + 0$$
$$5 = 0.2^3 + 1.2^2 + 0.2^1 + 1$$

Now $\gamma_p(10,9) = 1$, and this point is scored in the units place of the p-adic expansion. Also $\gamma_p(10,6) = 1$, but in this case the point is scored in the 2^2 place of the p-adic expansion, so weights $(9,1)$ and $(6,4)$ occur in different isomorphism classes of composition factors of X/V_p. Since $\gamma_p(10,8) = 0$, $\gamma_p(10,7) = 3$ and $\gamma_p(10,5) = 2$, and no two of these are the same, in this case in each simple a unique weight occurs, and this weight is also the highest weight for the simple module. Any composition series for X/V_p must contain three copies of the composition factor with highest weight $(7,3)$, two copies of that with highest weight $(5,5)$, one copy of that with highest weight $(9,1)$ and one copy of that with highest weight $(6,4)$. In fact, calculation (by hand or using the program in Appendix A, output from which is in Appendix B) shows that the submodule lattice of X/V_p in this case is the following, where vertices represent valid modules and edges represent composition factors, in the obvious way. The tuple shown on the diagram adjacent to any vertex M is

$$\left(\nu_2(m^{(M)}_{(9,1)}), \nu_2(m^{(M)}_{(8,2)}), \nu_2(m^{(M)}_{(7,3)}), \nu_2(m^{(M)}_{(6,4)}), \nu_2(m^{(M)}_{(5,5)})\right).$$



3.2 The lattice of scoreable sets: definition and properties

We know that the isomorphism class of a composition factor is determined by the set of weights whose weight spaces in the composition factor are non-zero. We have shown that, since a given weight has a non-zero weight space in at most one isomorphism class of composition factors, the isomorphism class of a composition factor is determined by any one weight whose weight space in the composition factor is non-zero. In the previous section we showed that the isomorphism class, if any, of composition factors in which a given weight occurs is determined by the set of points which the weight scores in r. Therefore we may consider that we have defined an injective function from isomorphism classes of composition factors of X/V_p to subsets of $\{0,\ldots,n-1\}$. It seems reasonable to study the image of this function; and in this section we shall do so. In the next section, we shall show that this study is indeed profitable.

Recall the notation $\Gamma_p(x,y)$ for the set of points scored by y in x, with respect to the prime p, and $\gamma_p(x,y)$ for the size of this set. Recall also that $\Gamma_p(r,\alpha_1) = \Gamma_p(r,\alpha_2)$, and that we write this set $\Gamma_p(r,\alpha)$. Let r have the p-adic expansion $r = r_n p^n + \cdots + r_0$ where $r_n \neq 0$.

Lemma 3.13.

Let $A \subseteq \{0,\ldots,n-1\}$. Then there is some weight α such that $\Gamma_p(r,\alpha) = A$ if and only if both

(i) for each i such that $r_i = p-1$, either $i \notin A$ or both $i \in A$ and $i-1 \in A$; and

(ii) for each i such that $r_i = 0$, either $i \in A$ or both $i \notin A$ and $i-1 \notin A$.

When these conditions are satisfied, the highest weight α such that $\Gamma_p(r,\alpha) = A$ is given by

$$\alpha_{1i} = \begin{cases} p-1 & \text{if } i \in A \\ r_i - 1 & \text{if } i \notin A \text{ and } i-1 \in A \\ r_i & \text{if } i \notin A \text{ and } i-1 \notin A. \end{cases}$$

Proof.

We show first that if conditions (i) and (ii) are satisfied then the weight described in the statement does satisfy $\Gamma_p(r,\alpha) = A$. Notice that condition (ii) ensures that α_{1i} is not set to be $r_i - 1$ when $r_i = 0$! Informally, condition (i) ensures that α scores all the points it is required to score, that is, that $A \subseteq \Gamma_p(r,\alpha)$, whilst condition (ii) ensures that it scores no more, that is, that $\Gamma_p(r,\alpha) \subseteq A$. More formally, we use induction on the value of j such that, for all $k < j$, $k \in A$ if and only if $k \in \Gamma_p(r,\alpha)$. If $0 \in A$ then $\alpha_{10} = p-1$ so $0 \in \Gamma_p(r,\alpha)$ unless $r_0 = p-1$, which condition (i) forbids. Conversely, if $0 \notin A$ then $\alpha_{10} = r_0$, so $0 \notin \Gamma_p(r,\alpha)$. Now suppose that, for all $j < i$, $j \in A$ if and only if $j \in \Gamma_p(r,\alpha)$. If $i \in A$ then $\alpha_{1i} = p-1$ so if $r_i < p-1$ then $i \in \Gamma_p(r,\alpha)$ certainly. If $r_i = p-1$ then condition (i) applies, so $i-1 \in A$, hence $i-1 \in \Gamma_p(r,\alpha)$ by the induction hypothesis, so $i \in \Gamma_p(r,\alpha)$. Conversely, if $i \notin A$ then either $\alpha_{1i} = r_i - 1$, in which case certainly $i \notin \Gamma_p(r,\alpha)$, or $\alpha_{1i} = r_i$

and $i-1 \notin A$, in which case by induction $i-1 \notin \Gamma_p(r,\alpha)$, so $i \notin \Gamma_p(r,\alpha)$. This completes the inductive step, so $A = \Gamma_p(r,\alpha)$ as required.

Now suppose that there is a weight β, higher than α, satisfying $A = \Gamma_p(r,\beta)$. Then $\beta_1 > \alpha_1$, so in particular there is some i such that $\beta_{1i} > \alpha_{1i}$. Certainly $i \notin A$. Either $i-1 \in A$, so $i-1 \in \Gamma_p(r,\beta)$, and $\beta_{1i} \geq r_i$, or $i-1 \notin A$ and $\beta_{1i} > r_i$. In either case, $i \in \Gamma_p(r,\beta)$, contrary to the supposition. Hence α is the highest weight satisfying $\Gamma_p(r,\alpha) = A$, as claimed. □

Definition 3.14.

The subset A of $\{0,\ldots,n-1\}$ is a *scoreable set of points* when conditions (i) and (ii) of Lemma 3.13 hold. ◁

Corollary 3.15.

Let α and β be weights with $\Gamma_p(r,\alpha) = A$ and $\Gamma_p(r,\beta) = B$. Then

(i) There is a weight γ with $\Gamma_p(r,\gamma) = A \cup B$.
(ii) There is a weight δ with $\Gamma_p(r,\delta) = A \cap B$.
(iii) If $B \subset A$ then there is a weight ϵ with $B \subseteq \Gamma_p(r,\epsilon) \subset A$ and $\gamma_p(r,\epsilon) = |A| - 1$.

Proof.

For the first two parts, we need to show that if A and B satisfy conditions (i) and (ii) of Lemma 3.13, then so do $A \cup B$ and $A \cap B$. To show that if A and B satisfy (i) then so does $A \cup B$ notice that

$$\begin{aligned}&(i \notin A \vee (i \in A \wedge i-1 \in A)) \wedge (i \notin B \vee (i \in B \wedge i-1 \in B))\\ \Rightarrow &(i \notin A \cup B \vee (i \in A \cup B \wedge i-1 \in A \cup B));\end{aligned}$$

the other parts are similar.

We prove the third part by contradiction. Suppose that there is no $j \in A \setminus B$ such that there is a weight ϵ with $\Gamma_p(r,\epsilon) = A \setminus \{j\}$. That is, for every $j \in A \setminus B$ the set of points $A \setminus \{j\}$ fails either condition (i) or condition (ii) of Lemma 3.13. This implies

$$(\forall j \in A \setminus B)((j+1 \in A \wedge r_{j+1} = p-1) \vee (j-1 \in A \wedge r_j = 0)).$$

Let the elements of $A \setminus B$ be labelled $j_0, j_1, \ldots$ where $j_0 < j_1 < \ldots$. Now $j_0 - 1 \notin A \setminus B$. If $j_0 - 1 \in A \cap B$ and $r_{j_0} = 0$ then since $j_0 \notin A \cap B$ and by part (ii) $A \cap B$ is scoreable, we have a contradiction. Hence $j_0 - 1 \notin A$ so $j_0 + 1 \in A$. If $j_0 + 1 \in A \cap B$ and $r_{j_0+1} = p-1$ we again derive a contradiction from the fact that $j_0 \notin A \cap B$. Therefore $j_1 = j_0 + 1$, and $r_{j_1} = r_{j_0+1} = p-1$. Thus

$r_{j_1} \neq 0$ so $j_1 + 1 \in A \setminus B$, implying that $j_2 = j_1 + 1$, and $r_{j_2} = r_{j_1+1} = p - 1$. And so on; so $A \setminus B$ must be infinite! This is the required contradiction. $\square$

Remark 3.16.

The third part of Corollary 3.15 has the interesting consequence that if there is a unique subset A of $\{0, \ldots, n-1\}$ having size m and satisfying conditions (i) and (ii) of Lemma 3.13 (that is, if there is a unique scoreable set of size m, or, equivalently, a unique isomorphism class of simple modules occurring in X/V_p with multiplicity m) then any scoreable set B of size at least m must contain A. For repeated application of Corollary 3.15 (iii) shows that B must contain *some* subset of size m which satisfies conditions (i) and (ii) of Lemma 3.13, and by the uniqueness assumption this must be A.

Remark 3.17.

Any maximal contiguous subset of a scoreable set is scoreable. That is, if A is a scoreable set, and the subinterval $S = \{m, m+1, \ldots, m+r\}$ of $\{0, \ldots, n-1\}$ is contained in A, but neither $m-1$ nor $m+r+1$ is in A, then S must be a scoreable set. For if either of the conditions of Lemma 3.13 failed for such a maximal contiguous subset, it must fail for the original set too.

We may consider how to calculate the number of different sets of positions in which it is possible to score m points, that is, the number of scoreable sets of points of a given size m. One situation is particularly simple; it corresponds to Carter and Cline's *non-degenerate case* ([CarterCline],[Deriziotis]).

Lemma 3.18.

If no coefficient in the p-adic expansion of r is either 0 or $p-1$ then there are $\binom{n}{m}$ different ways of scoring m points in r, (recalling that the n^{th} is the highest non-zero coefficient in the p-adic expansion of r) and so there are $\binom{n}{m}$ isomorphism classes of simple modules occurring m times in any composition series for X/V_p.

Proof.

Conditions (i) and (ii) of Lemma 3.13 are vacuously satisfied by any subset A of $\{0, \ldots, n-1\}$. $\square$

We shall have to examine the inclusion diagram whose vertices are the scoreable sets of points. Notice that this diagram, which we shall refer to as the *scoreable set lattice for r and p*, or as $\mathcal{L}(r,p)$, has various elementary properties, as follows:

(i) It is a lattice; that is it is connected, and closed under meets and joins, by Corollary 3.15.

(ii) It is complete, for each scoreable set contains $\varnothing$ and is contained by $\Gamma_p(r)$.

(iii) It is a sublattice of the $\gamma_p(r)$-dimensional Boolean algebra which would be the scoreable set lattice if each of the points which can be scored in r were individually scoreable, so that every subset of this set of $\gamma_p(r)$ points were a scoreable set.

(iv) Adjacent vertices represent scoreable sets of points which differ in size only by 1, by Corollary 3.15.

Of course, Carter and Cline's non-degenerate case corresponds to the scoreable set lattice being the whole $\gamma_p(r)$-dimensional Boolean algebra, described in (iii). We shall return to the connection between their work and ours in Chapter 4.

It is worth recording an easy and systematic method for drawing the scoreable set lattice for any given values of r and p.

1) Identify the set of points which occur in some scoreable set. This is the set $\{t, \ldots, n-1\}$ in the notation of Lemma 2.9; recall that this is the set of points whose corresponding coefficients of r are neither the leading coefficient nor in the $p-1$-tail of r.

2) For each of these points k, identify the minimal scoreable set in which k occurs. Referring to Lemma 3.13, we see that this is well-defined as the set

$$\{j, \ldots, k, \ldots, l\}$$

where

$$j = \max\{i \leq k \mid r_i \neq p-1\}$$

and

$$l = \min\{i \geq k \mid r_{i+1} \neq 0\}$$

which may, of course, consist of k alone.

3) Form the lattice of intersections and unions of these minimal scoreable sets. This is the entire lattice of scoreable sets; for Corollary 3.15 showed that it is a sublattice of the lattice of scoreable sets, and any scoreable set is the union of the minimal scoreable sets containing each of its members.

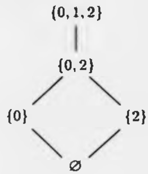
Remark 3.19.

This method exposes the fact that we can regard $\Gamma_p(r)$ as a topological space, by saying that a set is open if and only if it is scoreable. The set of minimal scoreable sets containing each scoreable point, described above, is then a base for the topology.

Example 3.20.

We illustrate this by continuing Example 3.12.

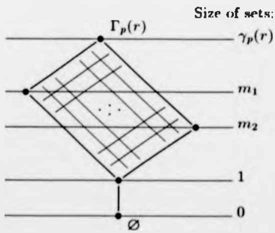
1) The points which occur are 0, 1 and 2.
2) The minimal scoreable set including 0 is $\{0\}$, that including 1 is $\{0,1,2\}$ and that including 2 is $\{2\}$.
3) Therefore the scoreable set lattice is



In general it seems to be quite hard to describe which sublattices of Boolean algebras can occur as scoreable set lattices, and in fact we shall not need such a general description. It is, however, convenient to give a simple description of the scoreable set lattices which arise if there is a unique scoreable singleton.

Lemma 3.21.

If there is a unique scoreable singleton then the scoreable set lattice has the form



for some values of m_1 and m_2, with m_1 and m_2 at least 1. Moreover, any such diagram is the scoreable set lattice for some values of r and p.

Proof.

Combining Remark 3.16 and Remark 3.17, we see that any scoreable set is a subinterval of $\{0, \ldots, n-1\}$ and includes the singleton, say $\{k\}$. The maximal scoreable set is $\{t, \ldots, n-1\}$ as described above; since there is a unique scoreable singleton we must have

$$r_t = \ldots = r_k = 0,$$

unless $t = k$, in which case there is no restriction on the value of r_k, and

$$r_{k+1} = \ldots = r_{n-1} = p - 1,$$

unless $n = k + 1$, in which case there is no restriction on the value of r_n. This is most easily seen by considering the values of r_{n-1} and r_t, and working inwards. Therefore any subinterval of $\{t, \ldots, n-1\}$ which includes k must be scoreable, and the result follows by setting $m_1 = n - k$ and $m_2 = k - t + 1$.

Conversely, consider

$$r = p^{m_1+m_2-1} + p^{m_2} \times (p^{m_1} - 1)$$

which has p-adic expansion

$$r = p^{m_1+m_2-1} + (p-1)p^{m_1+m_2-2} + \ldots + (p-1)p^{m_2} + 0.p^{m_2-1} + \ldots + 0.$$

Plainly this has the scoreable set lattice indicated; the scoreable singleton is $\{m_2 - 1\}$ and the scoreable sets are the subintervals of $\{0, \ldots, m_1 + m_2 - 2\}$ which include $m_2 - 1$. □

3.3 The lattice of scoreable sets: its importance

In this section we shall begin to use the scoreable set lattice for particular values of r and p to solve the original problem of finding all valid modules for those values. We shall be able to cease referring to weights almost entirely, since all the information we require is encapsulated in the scoreable set lattice.

Recall that $m_\alpha = m_\beta$ at all valid modules if $\Gamma_p(r, \alpha) = \Gamma_p(r, \beta)$, and that if also $\Gamma_p(r, \gamma) = \Gamma_p(r, \delta)$ then

$$k(\alpha, \gamma) = k(\beta, \delta) = |\Gamma_p(r, \gamma) \setminus \Gamma_p(r, \alpha)|.$$

Therefore we may rephrase the problem of finding all valid modules, which we described in Chapter 2 as that of finding all integer solutions $\{\nu_p(m_\alpha)\}_{\alpha \in \Lambda^+}$ to the conditions:

V1) (Validity) for all α and β,

$$\nu_p(m_\alpha) - \nu_p(m_\beta) + k(\alpha, \beta) \geq 0;$$

V2) (Normalisation) for all α,

$$0 \leq \nu_p(m_\alpha) \leq \nu_p(|\alpha|);$$

as that of finding all integer solutions $\{\nu_p(m_A)\}_{A \in \mathcal{L}(r,p)}$ solutions to the following (Conditions B):

B1) (Validity) for all scoreable sets of points A and B,

$$\nu_p(m_A) - \nu_p(m_B) + |B \setminus A| \geq 0;$$

B2) (Normalisation) for each scoreable set of points A,

$$0 \leq \nu_p(m_A) \leq |A|,$$

with the obvious notation. Having found all tuples satisfying these conditions, we obtain all tuples $\{m_\alpha\}$ corresponding to valid modules in the old sense simply by setting $m_\alpha = m_{\Gamma_p(r,\alpha)}$ for each weight α.

Plainly, every natural number tuple $\{\nu_p(m_\alpha)\}$ may be regarded as a labelling of the vertices of the scoreable set lattice, where the vertex A is labelled with $\nu_p(m_A)$.

Here we may reconsider the duality $M \mapsto \widetilde{M}$, considered in Chapter 1, with our new language. If the scoreable set A is labelled at M with $\nu_p(m_A)$, then at $\widetilde{M}$ it is labelled with $|A| - \nu_p(m_A)$. We give here our short proof of Lemma 1.55.

Proof.

Suppose that M is a valid module such that $M = \widetilde{M}$. That is, for every scoreable set A,

$$\nu_p(m_A) = |A| - \nu_p(m_A)$$

which implies that the size of every scoreable set is even. By Corollary 3.15 this is possible only if $\varnothing$ is the only scoreable set. $\square$

We next show that the tuples corresponding to valid modules are those which, regarded as labellings of the scoreable set lattice, satisfy a rather simple criterion.

First, a weak result.

Lemma 3.22.

If $D \subseteq C$ then at any valid module,

$$\nu_p(m_D) \leq \nu_p(m_C) \leq |C| - |D| + \nu_p(m_D).$$

Proof.

We have $k(C, D) = 0$ and $k(D, C) = |C| - |D|$. Rearranging

$$\nu_p(m_C) - \nu_p(m_D) + k(C, D) \geq 0$$

and

$$\nu_p(m_D) - \nu_p(m_C) + k(D, C) \geq 0$$

gives the result. □

In fact we can prove a much stronger result.

Theorem 3.23.

The natural number tuple $\{\nu_p(m_A)\}$ is a solution to Conditions B, and therefore describes a valid module, if and only if

C1) (Validity) for every $B \subseteq C$ with $|B| + 1 = |C|$ we have

$$\nu_p(m_B) \leq \nu_p(m_C) \leq 1 + \nu_p(m_B)$$

C2) (Normalisation) $\nu_p(m_\varnothing) = 0$

Proof.

From the previous Lemma we see that if $\{m_A\}$ is a solution then the condition is satisfied. Conversely, assume that the condition is satisfied for each such B and C, and consider any D and E such that $D \subseteq E$. Then by Corollary 3.15(iii) there is a chain of sets

$$D = D_0 \subset D_1 \ldots \subset D_m = E$$

such that each D_i is a scoreable set of points and, for each i, $|D_i| = |D_{i-1}| + 1$. Then

$$\nu_p(m_D) \leq \nu_p(m_{D_1}) \leq \ldots \leq \nu_p(m_E)$$

and

$$\nu_p(m_E) \leq \nu_p(m_{D_{m-1}}) + 1 \leq \ldots \leq \nu_p(m_D) + |E \setminus D|.$$

Finally consider any D and E, dropping the requirement that $D \subseteq E$. Now because $D \subseteq (D \cup E)$ and $E \subseteq (D \cup E)$, we have shown that

$$\nu_p(m_D) \leq \nu_p(m_{D \cup E}) \leq |(D \cup E) \setminus D| + \nu_p(m_D)$$

and that

$$\nu_p(m_E) \leq \nu_p(m_{D\cup E}) \leq |(D \cup E) \setminus E| + \nu_p(m_E).$$

Since $(D \cup E) \setminus D = E \setminus D$, we may deduce

$$\nu_p(m_E) \leq \nu_p(m_{D\cup E}) \leq |E \setminus D| + \nu_p(m_D)$$

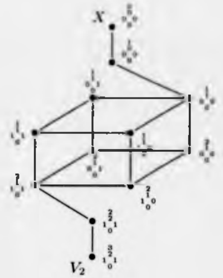
so Condition B1 holds as required. Since the scoreable set lattice is connected, $0 \leq \nu_p(m_A)$ for all A. Moreover, since adjacent vertices are scoreable sets that differ in size only by 1 (Corollary 3.15(iii)) C1 automatically ensures that $\nu_p(m_A) \leq |A|$ for all A. □

That is, those labellings of the scoreable set lattice which are valid are precisely those in which any two adjacent vertices either have the same labelling or have the higher vertex labelled with the natural number one greater than the natural number label on the lower vertex.

This provides us with an easy way to test whether a given module is valid. Moreover, we may use this result to draw the whole structure of X/V_p, as was our aim. For we may, in turn, label the vertices of the inclusion diagram of valid modules by labellings of the scoreable set lattice. X is labelled with the scoreable set lattice labelled entirely with 0s. Thereafter, given any valid module M one can tell what its maximal valid submodules are by finding the positions in the labelling of the scoreable set lattice at which it is possible to increase a label, without violating the condition on adjacent labels. These will be those vertices of the scoreable set lattice such that, if the vertex (say A) is labelled with $m \in \mathbb{N}$, then each of its superior neighbours is labelled with $m+1$ and each of its inferior neighbours is labelled with m. Increasing the label on such a scoreable set A by 1 yields a new valid labelling of the scoreable set lattice. This corresponds to finding a new valid module, say N, such that $J(M/N) = A$. An example may make this clearer.

Example 3.24.

We continue with the example already tackled as Example 3.12 and Example 3.20 We have found the scoreable set lattice for $r = 10$ and $p = 2$. Labelling it in the way just described, we get:



3.4 Consequences for structure graphs.

Although from the point of view of the reader who wished to know all integral Weyl modules for GL_2 of homogeneous degree r for some particular value of r we have done what we set out to do, there are other aims which require a different approach. In the next section we shall make some general observations about the high-level structure of X/V_p in certain special cases, and shall explain some techniques for exploring other cases.

Before we move on, however, we may extract a few interesting observations of the consequences of Conditions C for the structure of X/V_p. The reader may like to verify them in the particular case of the example we have been following, by referring to the lattice of valid modules shown in Example 3.24.

Although we are interested in the structure of X/V_p, the 'game' we play with integer labellings of the scoreable set lattice has much wider applications. For the rest of this section the only property of the scoreable set lattice $\mathcal{L}$ that we shall use is that it is a finite, connected, directed graph. Moreover, in playing the labelling game we shall not depend on the normalisation condition C2, but only on the validity condition C1. That is, we shall insist that no two labels on adjacent vertices differ by more than 1, and that if $A \to B$ is an edge then, at any valid module M, $\nu_p(m_A^{(M)}) \leq \nu_p(m_B^{(M)})$, but we shall put no other restriction on the labels. The reason for pointing out this generality is that we shall later want to use these results in the context of the modular Weyl modules M/pM, where pM does not satisfy the normalisation condition.

For a start, the Ext groups of the simple modules may be of interest. This corresponds to thinking about which pairs of scoreable sets in a given labelling of the scoreable set lattice can have their labels increased consecutively in one order but not in the other.

Lemma 3.25.

Suppose that $|\mathcal{L}| > 1$. Then there is no composition series in which two isomorphic composition factors occur adjacently.

Proof.

Suppose M, N and P are valid modules such that

$$M > N > P$$

and M/N and N/P are isomorphic and simple. Then the labellings of $\mathcal{L}$ which correspond to M, N and P differ in only one place; there is some $A \in \mathcal{L}$ such that $\nu_p(m_A^{(M)}) = m$, say, $\nu_p(m_A^{(N)}) = m - 1$ and $\nu_p(m_A^{(P)}) = m - 2$. Since $|\mathcal{L}| > 1$ and $\mathcal{L}$ is connected, there is some $B \in \mathcal{L}$ adjacent to A. It is labelled with some fixed value, and there are only two compatible values for A's label, namely the same value and a value differing by 1 in the appropriate direction. But we are hypothesising three different labels for A, all compatible with this fixed label for B; which is a contradiction. $\square$

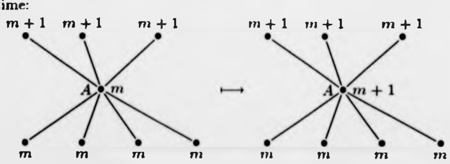
In fact we may easily prove a stronger result:

Lemma 3.26.

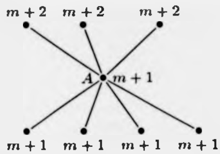
In any composition series, any two occurrences of the same composition factor, say that corresponding to scoreable set A, are separated by exactly one occurrence of the composition factor corresponding to each scoreable set adjacent to A in $\mathcal{L}$, as well as, perhaps, by composition factors corresponding to scoreable sets not adjacent to A.

Proof.

Consider the labels on $\mathcal{L}$ before and after increasing the label on A for the first time:



It will not be valid to increase the label on A again, to $m+2$, until this portion of the labelling has become



that is, until the label on each scoreable set adjacent to A has been increased by 1. □

Next we show that if composition factors correspond to adjacent vertices of $\mathcal{L}$ then any extension of one by the other must be split.

Lemma 3.27.

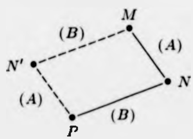
Let A and B be non-adjacent vertices of $\mathcal{L}$, and let M, N and P be valid modules such that

$$M > N > P$$

whilst M/N is simple, corresponding to A, and N/P is simple, corresponding to B. Then there is some valid module N' such that

$$M > N' > P$$

whilst M/N' is simple, corresponding to B, and N'/P is simple, corresponding to A, as illustrated by the following diagram:



Proof.

Consider the labelling of $\mathcal{L}$ that corresponds to M. Whether the label on B can be increased depends solely on the labels of B's neighbours, relative to B's label. If A is not adjacent to B, then this situation is not altered by increasing A's label. So if it is valid to increase B's label after having increased A's label, as hypothesised, it must have been valid to increase B's label in the first place. $\square$

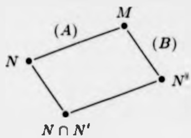
The next lemma may be regarded as a converse, since it shows that if there is a split extension of two composition factors, they must correspond to non-adjacent vertices of $\mathcal{L}$.

Lemma 3.28.

Let A and B be scoreable sets, and let M, N and N' be valid modules such that

$$M > N \qquad \text{and} \qquad M > N'$$

whilst M/N is simple, corresponding to A, and M/N' is simple, corresponding to B. Then A and B are not adjacent in $\mathcal{L}$. The following diagram illustrates:



Proof.

Suppose that A and B are adjacent in $\mathcal{L}$, and (without loss of generality) that $A \supset B$. Then if at M the label on A is m, then in order for it to be valid to increase A's label, the label on B at M must be m. Therefore it is invalid to increase the label on B, which contradicts the hypothesis. $\square$

Notice that it was not necessary to state these results only for single vertices of $\mathcal{L}$. Let $\mathcal{A}$ and $\mathcal{B}$ be sets of vertices of $\mathcal{L}$ such that $\mathcal{A}$ and $\mathcal{B}$ are disjoint, (but not necessarily connected). We say that $\mathcal{A}$ and $\mathcal{B}$ are ***adjacent*** if and only if there is some pair of scoreable sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that A and B are adjacent. Then we immediately get generalisations of the preceding results. The generalisation of Lemma 3.25 is

Lemma 3.29.

Let M, N and P be valid modules with

$$M > N > P,$$

and assume that there is some $A \in \mathcal{L}$ such that A's label is not increased between M and N; that is, that the composition factor corresponding to A does not occur in M/N. Then M/N is not isomorphic to N/P.

Proof.

The assumption provides us with an element $A \in \mathcal{L}$ which is not adjacent to the (multi)set of vertices $\mathcal{A}$ which corresponds to M/N. Given this, the previous proof stands. $\square$

Clearly Lemma 3.26 can be generalised similarly. The generalisation of Lemma 3.27 is

Lemma 3.30.

Let $\mathcal{A}$ and $\mathcal{B}$ be non-adjacent sets contained in $\mathcal{L}$, and let M, N and P be valid modules such that

$$M > N > P$$

whilst M/N corresponds to increasing by one the labels on vertices from $\mathcal{A} \subseteq \mathcal{L}$, and N/P corresponds to increasing by one the labels on vertices from $\mathcal{B} \subseteq \mathcal{L}$. Then there is some valid module N' such that

$$M > N' > P$$

whilst $M/N' \cong N/P$ and $N'/P \cong M/N$.

$\square$

Here $\mathcal{A}$ and $\mathcal{B}$ may again be permitted to be *multisets*; that is, they may contain vertices of $\mathcal{L}$ whose labels should be increased more than once. The generalisation of Lemma 3.28 reads

Lemma 3.31.

Let $\mathcal{A}$ and $\mathcal{B}$ be multisets of scoreable sets and let M, N and N' be valid modules such that

$$M > N \qquad \text{and} \qquad M > N'$$

whilst M/N corresponds to increasing by one each label on vertices from $\mathcal{A}$, and N/P corresponds to increasing by one each label on vertices from $\mathcal{B}$. Then $\mathcal{A}$ and $\mathcal{B}$ are not adjacent.

$\square$

To conclude this section, we show how to find the submodule generated by some set, and we return to our specific situation in which the vertices of $\mathcal{L}$ are scoreable sets, and so on, *except* that we do not reintroduce the normalisation condition. Then given $S_{\mathbb{Z}}(n,r)$-modules M and N, we may consider the submodule of M/N generated by some set of cosets of N in M. That is,

Lemma 3.32.

Suppose that we are given a tuple $\{\nu_p(m_\alpha^{(P)})\}$ such that, for each weight α, we have

$$\nu_p(m_\alpha^{(M)}) \leq \nu_p(m_\alpha^{(P)}) \leq \nu_p(m_\alpha^{(M)}),$$

so that P is a $\mathbf{Z}$-module between the $S_{\mathbf{Z}}(n,r)$-modules M and N, but such that the tuple $\{\nu_p(m_\alpha^{(P)})\}$ does not necessarily define an $S_{\mathbf{Z}}(n,r)$-module. Then we may find P', the unique $S_{\mathbf{Z}}(n,r)$-module generated by P, as follows:

1) For each scoreable set A, consider the given values $\nu_p(m_\alpha^{(P)})$ for each weight α such that $\Gamma_p(r,\alpha) = A$. Set the label on A, that is, $\nu_p(m_A)$, to the minimum of these values.
2) Consider the labelling of the scoreable set lattice so obtained. For each pair of adjacent vertices $A \subset B$ where $\nu_p(m_A) > \nu_p(m_B)$ (if any), decrease $\nu_p(m_A)$ to $\nu_p(m_B)$.
3) For each pair of adjacent vertices $A \subset B$ where $\nu_p(m_B) > \nu_p(m_A) + 1$ (if any), decrease $\nu_p(m_B)$ to $\nu_p(m_A) + 1$.
4) Repeat steps 2) and 3) until neither is applicable.

Proof.

This algorithm must terminate, since a finite number of steps will produce the labelling $\{\nu_p(m_\alpha^{(M)})\}$ which is known to be valid. Clearly when it does terminate, the final tuple is $\{\nu_p(m_\alpha^{(P')})\}$, the tuple defining the smallest $S_{\mathbf{Z}}(n,r)$-module containing P. □

3.5 How to use simple problems to solve complicated ones

Recall the module A defined in 'Weights and composition factors of X/V_p' by

$$m_\alpha^{(A)} = \begin{cases} p & \text{if } \nu_p(|\alpha|) > 0 \\ 1 & \text{if } \nu_p(|\alpha|) = 0, \end{cases}$$

and notice that

$$m_\alpha^{(A^*)} = \begin{cases} |\alpha|_p/p & \text{if } \nu_p(|\alpha|) > 0 \\ 1 & \text{if } \nu_p(|\alpha|) = 0. \end{cases}$$

In our new notation this becomes

$$\nu_p(m_B^{(A)}) = \begin{cases} 1 & \text{if } B \neq \varnothing \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nu_p(m_B^{(A^*)}) = \begin{cases} |B| - 1 & \text{if } B \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

Thus a module M is contained in A if and only if every label on the scoreable set lattice, bar the empty set's label, is at least 1, and contains A^* if and only if every label on the scoreable set lattice, bar the empty set's label, differs from its maximum value (the size of the scoreable set) by at least 1. Recall, from the same section, that every composition factor of X/V_p occurs exactly once in any composition series of X/A, and exactly once in every composition series of A^*/V. This is obvious from our new point of view; for example, the occurrence of a composition factor in a composition series of X/A corresponds to an increase by 1 in the label on the corresponding scoreable set. Since each label on a non-empty scoreable set reaches exactly 1 at A, and started at 0, each composition factor must have occurred exactly once.

Lemma 3.33.

$$X/A^* \cong A/V_p.$$

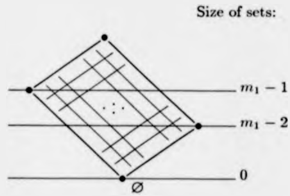
Moreover, if there is a unique scoreable singleton, so that there is a unique isomorphism class of composition factors of X/V_p which occur with multiplicity one in any composition series, then every valid module M is contained in A or contains A^*.

Proof.

It follows from the remarks above that a valid module M lies between X and A^* if and only if the label on every singleton is 0, and that it lies between A and V_p if and only if the label on every singleton is 1. Therefore to every valid module (say M) between X and A^* there corresponds one (say $f(M)$) between A and V_p which is obtained by adding 1 to the label on every non-empty scoreable set. Clearly this correspondence preserves validity and inclusion. Also, whenever M and N lie between X and A^* and are adjacent, such that $M/N \cong F$, then $f(M)/f(N) \cong F$; for the quotient is determined by which label(s) on the scoreable set lattice has (have) to be increased to move down the diagram from M to N, and this set of labels is unaffected by uniformly adding 1 to all labels. The reverse procedure is equally valid.

Moreover, if there is a unique scoreable singleton then at every valid module its label must be 0 (in which case the valid module contains A^*) or 1 (in which case the valid module is contained in A). □

Thus we have deduced some information about the lattice of valid modules in all cases in which there is a unique scoreable singleton; that it consists of two identical parts, the structure of each of which may be found by finding the set of all possible labellings of a rather smaller diagram than the original scoreable set lattice, namely



using the notation of Lemma 3.21. In fact it is easy to see that this lattice is itself a scoreable set lattice for some other problems; for example, it is the one that arises in the case ($p \neq 2$)

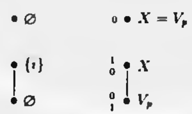
$$r = p^{m_1+m_2-2} + p^{m_2} \times (p^{m_1-2} - 1) + p^{m_2-1}$$

which has p-adic expansion

$$r = p^{m_1+m_2-2} + (p-1)p^{m_1+m_2-3} + \ldots + (p-1)p^{m_2} + 1\,p^{m_2-1} + 0\,p^{m_2-2} + \ldots + 0$$

This is a problem for which $\gamma_p(r)$ is one less than it was for the original, unique scoreable singleton problem; that is, the height of the scoreable set lattice has been decreased by 1 — we have chopped off the bottom of it! This may be useful. For it is easy to draw the lattice of valid modules for the simplest scoreable set lattices:

scoreable set lattices valid modules



and one may in this way proceed to build a list of all such diagrams, at each stage using a rather simple procedure to deduce the diagrams at the next layer of complexity from those diagrams which have already been found. Of course, we shall have to show how to build diagrams corresponding to scoreable set lattices with multiple scoreable singletons from those with unique scoreable singletons. Thus motivated, we give a couple of general results on deduction, of which we have just used a particular special case.

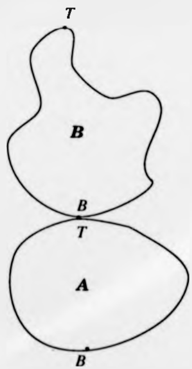
Remark 3.34.

Suppose that $\mathcal{C}$ is any finite connected directed graph with a bottom B, and that $\mathcal{A}$ and $\mathcal{B}$ are subgraphs each including B, such that any edge $(x \to y)$ of $\mathcal{C}$ occurs in at least one of $\mathcal{A}$ and $\mathcal{B}$. Consider the set of all integer labellings of $\mathcal{C}$ such that B is labelled with 0 and such that the vertices on any edge $(x \to y)$ are labelled $(n \to n)$ or $(n \to n+1)$ for some integer n. The set of all such labellings can be identified with the set of pairs of such labellings of $\mathcal{A}$ and $\mathcal{B}$ which agree on the intersection of $\mathcal{A}$ and $\mathcal{B}$.

(The condition that $\mathcal{A}$ and $\mathcal{B}$ include B is inessential; it is given solely in order to ensure that the sets of labellings we consider are finite.)

Remark 3.35.

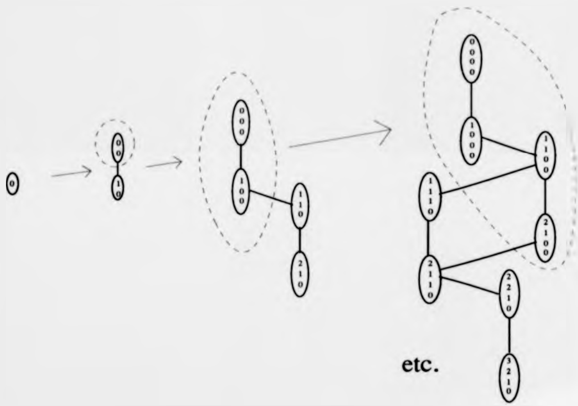
Suppose that $\mathcal{A}$ and $\mathcal{B}$ are finite connected directed graphs, each with a bottom B and a top T. Suppose that for each graph we know the set of valid labellings as described above. Then consider the composite graph in which the top of $\mathcal{A}$ is identified with the bottom of $\mathcal{B}$ thus:



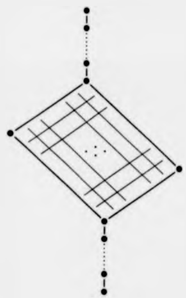
The set of its valid labellings may be identified with the set of pairs of valid labellings of $\mathcal{A}$ and $\mathcal{B}$ by adding to each of the labels on the vertices of $\mathcal{B}$ the label on the top of $\mathcal{A}$.

(These remarks may be considered as perpendicular or identical, according to taste.)

It will now be apparent that the case of the unique scoreable singleton which we considered at the beginning of the section was just a special case of the second of these Remarks. As a special case of *that*, we notice that it is particularly straightforward recursively to draw the lattices of valid modules for scoreable set lattices in which there is a unique scoreable set of any given size, up to some maximum size:



Then we may, for example, also find the set of all valid labellings of a given graph of the form



even though (remembering our analysis of scoreable set lattices with a unique scoreable singleton) it is apparent that graphs of this form are not in general scoreable set lattices. This provides a reasonably efficient way to use the first Remark to deal with an arbitrary scoreable set lattice in terms of simpler graphs, using, in particular, the knowledge of the structure of lattices of valid modules for scoreable set lattices which have unique scoreable singletons and height no greater than that of the general scoreable set lattice being considered. To conclude this section, we give an example of how to deal with a moderately complicated scoreable set lattice.

Example 3.36.

Consider the example touched on in Remark 2.36 in which $r = 30$ and $p = 3$. We have

$$r = 1.3^3 + 0.3^2 + 1.3 + 0$$

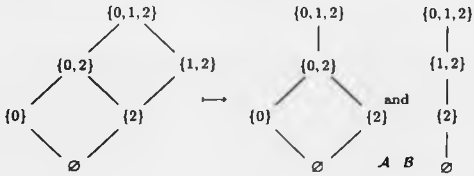
so the points which can be scored are 0, 1 and 2.

The minimal scoreable set including 0 is $\{0\}$.

The minimal scoreable set including 1 is $\{1,2\}$.

The minimal scoreable set including 2 is $\{2\}$.

Therefore the scoreable set lattice is

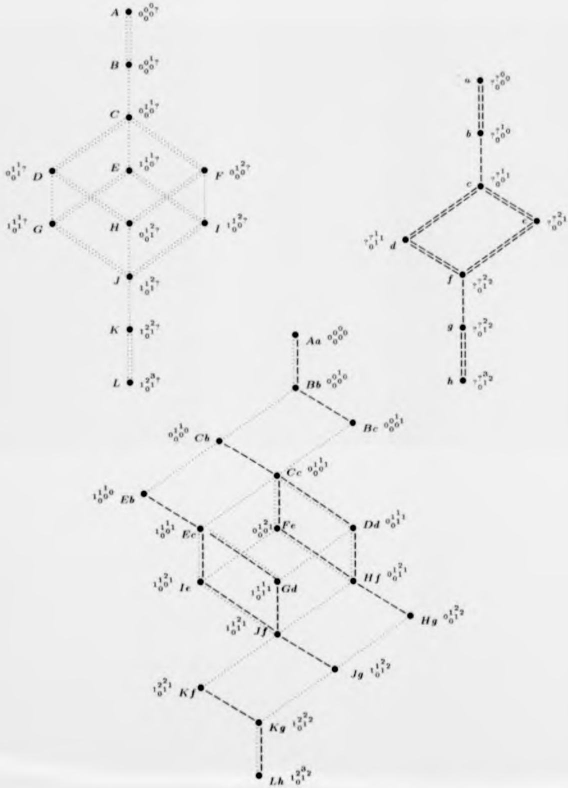


which we can split into two recognised bits in the manner of Remark 3.34 as shown. The first ($\mathcal{A}$) is that studied in Example 3.12, Example 3.20 and Example 3.24, so we know the corresponding lattice of valid modules, and the lattice of valid modules corresponding to the second is shown on page 91. In order to identify all pairs of vertices from the two lattices of valid modules ($\mathcal{U}$ and $\mathcal{V}$), we shall use the following procedure, which is easily seen to be correct, and to be less hard to follow than it looks in writing:

1) Draw the known lattices of valid partial labellings, one on each side of the page.
2) For convenience, identify the edges in $\mathcal{U}$ which correspond to increasing a label on a vertex of $\mathcal{A}$ which also occurs in $\mathcal{B}$; mark these edges with double lines in $\mathcal{U}$.
3) Do the same for $\mathcal{V}$.

Then we build up the required lattice of valid modules from the top. At each stage we consider what are the maximal valid submodules of the current module. We can do this either directly, considering the labelling of the whole scoreable set lattice at the current module, or by using the information in $\mathcal{U}$ and $\mathcal{V}$, as follows. Take the current pair of partial labellings, i.e. of vertices in $\mathcal{U}$ and $\mathcal{V}$, say (x, y). Consider the vertices below each of x and y. Wherever there is an ordinary edge $x \to u$ in $\mathcal{U}$, it corresponds immediately to an edge $(x, y) \to (u, y)$ in the new lattice; for this says that it is possible to increase the label on a vertex of $\mathcal{A}$ which does not occur in $\mathcal{B}$. The same applies to ordinary edges $y \to v$ in $\mathcal{V}$. Where there is a double edge $x \Rightarrow u$ in $\mathcal{U}$, it will only correspond to an edge in the new lattice if there is a corresponding double edge $y \Rightarrow v$ in $\mathcal{V}$, where the label which is being increased is the same in each case. For this says that it is possible to increase the label on a vertex of $\mathcal{A}$ which also occurs in $\mathcal{B}$. If we tried to draw an edge in the new lattice where there was a double arrow in one of $\mathcal{U}$ and $\mathcal{V}$ but no corresponding double arrow in the other, we would be trying to put in a pair of partial labellings which did not agree on their intersection, which would be wrong.

The resulting diagram is as follows, where dotted edges in the new lattice correspond to edges in $\mathcal{U}$, dashed ones to edges in $\mathcal{V}$. Notice that, naturally, if the new lattice is collapsed along edges which are dotted but not dashed (respectively dashed but not dotted) (which corresponds to ignoring labels on vertices of the scoreable set lattice which occur in $\mathcal{A}$ but not in $\mathcal{B}$ (respectively in $\mathcal{B}$ but not in $\mathcal{A}$)) the resulting diagram is $\mathcal{V}$ (respectively $\mathcal{U}$).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Jf
Jg
Kf
Kg
Lh

The techniques we have given for using simple problems to solve complicated ones become more valuable with even more complicated examples; however, the above should give an idea of their application, and of the fact that it is possible to get an idea of the structure of the solution to the complicated problem by looking at the solutions to the simple problems, even without calculating the whole solution. It is interesting to notice that, using the original method implemented (albeit simplistically) in Program 1 of Appendix A, even this calculation takes about 10 minutes to run on a computer. The output from that program is given in Appendix B, and will be found to agree with our calculation here.

This example was originally introduced to illustrate the existence of cases in which there is a valid module M and weights α and β for which, although $|\alpha|_p > |\beta|_p$, $m_\alpha < m_\beta$. Now that we have studied the scoreable set lattice, it is easy to see why this can happen; we are saying that the labelling

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & \\ & 0 & & & \end{array}$$

of the scoreable set lattice is valid, even though a label 0 occurs at a higher level in the diagram than the larger label 1.

Chapter 4

Applications to modular theory

Definition 4.1.

Let M be any valid module, and consider M/pM. This is the quotient of two $S_{\mathbf{Z}}(n,r)$ modules, so it is certainly an $S_{\mathbf{Z}}(n,r)$-module. Moreover it is annihilated by $p\mathbf{Z}$, so it is also an $S_{\mathbf{Z}/p\mathbf{Z}}(n,r)$-module. Such an M/pM is a *modular Weyl module* for GL_2. ◁

These modules have been extensively studied in the special case $M = V_p$ and in the context of SL_2 rather than GL_2, and their structure is known. The latter distinction is rather unimportant to us, since we have already restricted our attention to the Weyl modules $V_{(r,0)}$ and shown in Proposition 1.33 that any other Weyl module is the tensor product of one of these with a number of copies of the determinant representation. When we work over SL_2 the determinant representation is trivial.

4.1 Applying our theory

We first explain how to apply our methods. In order to consider the submodule structure of M/pM where M is a valid module, we have to modify our normalisation condition

$$M^{\lambda} = \mathbf{Z}f_l$$

to read

$$M^{\lambda} = p^i\mathbf{Z}f_l$$

where i may be 0 or 1. That is, rather than having

$$V_p \leq M \leq X$$

we now have

$$pV_p \leq M \leq X$$

and hence 'enough room' to consider pM for any valid module M.

What this means for the scoreable set lattice theory is that we are relaxing the normalisation condition, C2, that $\nu_p(m_\lambda) = 0$, that is, that the label on the empty set, $\nu_p(m_\varnothing^{(M)})$, in the scoreable set lattice be 0 at any M. We now insist only that $\nu_p(m_\varnothing^{(M)})$ be at most 1. Of course, the validity condition, C1, still applies. Therefore if M is an admissible lattice between X and V_p then $\nu_p(m_\varnothing^{(M)}) = 1$ only if $\nu_p(m_B^{(M)}) \geq 1$ for every scoreable set B which contains $\varnothing$ – that is, for every scoreable set. In summary, we shall say that M is an ***almost valid module*** if and only if

D1) (Validity) for every B and C in $\mathcal{L}(r,p)$ such that $B \subseteq C$ with $|B|+1 = |C|$, we have

$$\nu_p(m_B) \leq \nu_p(m_C) \leq 1 + \nu_p(m_B)$$

D2) (Normalisation) $\nu_p(m_\varnothing) \leq 1$

Then the structure diagram for X/pV_p is formed from two copies of the structure diagram for X/V_p joined together; that is, each of its vertices is M or pM for some valid module M. The labelling of the scoreable set lattice corresponding to pM is obtained from that corresponding to M by adding 1 to every label:

$$\nu_p(m_B^{(pM)}) = \nu_p(m_B^{(M)}) + 1$$

for every valid module M and scoreable set A.

Consider a composition series for M/pM. Every label on the scoreable set lattice must be increased precisely once; that is, each composition factor which occurs between X and V_p occurs exactly once, as does the simple module with highest weight λ. Recall (Lemma 3.13) that we have described the highest weight (α_1, α_2) of the composition factor corresponding to scoreable set A by giving the coefficients in the p-adic expansion of α_1:

$$\alpha_{1i} = \begin{cases} p-1 & \text{if } i \in A \\ r_i - 1 & \text{if } i \notin A \text{ and } i-1 \in A \\ r_i & \text{if } i \notin A \text{ and } i-1 \notin A \end{cases}$$

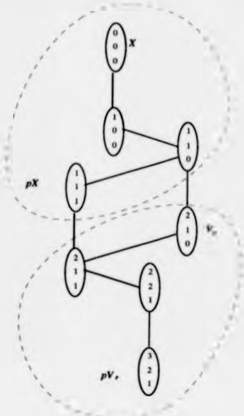
from which we deduce that the coefficients of α_2 are:

$$\alpha_{2i} = \begin{cases} 0 & \text{if } i \notin A \\ r_i + 1 & \text{if } i \in A \text{ and } i-1 \notin A \\ r_i & \text{if } i \in A \text{ and } i-1 \in A. \end{cases}$$

In order to draw the structure diagram for M/pM, one must first find the labelling of the scoreable set lattice corresponding to M (or pM), and then find the orders in which the labels can each be increased (or decreased) by 1, in exactly the same way as we have done.

Example 4.2.

Let $p = 3$ and let $r = 16 = 3^3 + 2 \times 3^3 + 1$. Then the scoreable sets are just $\emptyset$, $\{0\}$ and $\{0, 1\}$, so the inclusion lattice of almost valid modules is as shown. The sublattices showing the structures X/pX and V_p/pV_p are ringed. Of course the structure of M/pM can be read off the diagram for any valid module M.



We can make some general deductions about properties of the submodule structure diagram for M/pM; in particular, the results of Section 3.4 are all applicable, since their proofs did not depend on the presence of the normalisation condition.

Lemma 4.3.

Let M be any valid module. Then M/pM is indecomposable.

Proof.

The existence of a non-trivial decomposition of M/pM is equivalent to the existence of a partition of the set of all scoreable sets into two sets $\mathcal{A}$ and $\mathcal{B}$, neither of which is empty, such that either of

(i) increasing the label on every $A \in \mathcal{A}$; and

(ii) increasing the label on every $B \in \mathcal{B}$

is valid. Lemma 3.31 tells us that this implies that $\mathcal{A}$ and $\mathcal{B}$ are not adjacent. But this is impossible, since the scoreable set lattice is connected. $\square$

Suppose that the $S_{\mathbf{Z}}(n,r)$-module P'/pV_p is generated by

$$P/V_P = \bigoplus_{\alpha \in \Pi} (V_p/pV_p)^\alpha$$

which is not itself necessarily an $S_{\mathbf{Z}}(n,r)$-module. That is, P'/pV_p is generated by the weight spaces $(V_p/pV_p)^\alpha = V_p^\alpha/pV_p^\alpha$ for some subset Π of the set $\Lambda(n,r)$ of all weights, not necessarily dominant. The application of the algorithm in Lemma 3.32 becomes particularly easy in this case.

We may express this as follows. Suppose that we are given the tuple corresponding to P, say $\{\nu_p(m_\alpha^{(P)})\}$, so that, for each weight α, we have

$$\nu_p(m_\alpha^{(V_p)}) \le \nu_p(m_\alpha^{(P)}) \le \nu_p(m_\alpha^{(pV_p)}),$$

that is, such that

$$\gamma_p(r,\alpha) \le \nu_p(m_\alpha^{(P)}) \le \gamma_p(r,\alpha) + 1.$$

Then for a given α, we will have

$$\nu_p(m_\alpha^{(P)}) = \begin{cases} \gamma_p(r,\alpha) & \text{if } \alpha \in \Pi; \\ \gamma_p(r,\alpha)+1 & \text{otherwise.} \end{cases}$$

Then applying step 1 of the algorithm yields a labelling of the scoreable set lattice in which the label on A is either $|A|$ or $|A|+1$. The label will be $|A|$ if there is any weight $\alpha \in \Pi$ such that $\Gamma_p(r,\alpha) = A$, and $|A|+1$ otherwise. Therefore step 2 cannot be applicable. Consider step 3. Let us denote by Π' the set of vertices which are labelled with $|A|$; this set will change as we apply the algorithm. The first application of step 3 will cause us to decrease by one the label on any B such that

(i) B is labelled with $|B|+1$; and
(ii) there is some $A \subset B$, adjacent to B, which is labelled with $|A| = |B|-1$.

That is, it will add into the set Π' any B such that $B \notin \Pi'$ and B has an inferior neighbour $A \in \Pi'$. Repeated application of 3) will see us adding to Π' any set B which contains any set from Π'. The algorithm terminates when Π' contains each set which contains $\Gamma_p(r,\alpha)$ for some $\alpha \in \Pi$. Translating from labelling terminology back into weight space terminology, we get

$$P'/V_P = \bigoplus_{\beta \in \Sigma} (V_p/pV_p)^\beta$$

where $\beta \in \Sigma$ if and only if there is some $\alpha \in \Pi$ such that

$$\Gamma_p(r, \alpha) \subseteq \Gamma_p(r, \beta).$$

Lemma 4.4.

X/pX has both a unique maximal and a unique minimal submodule. The maximal submodule of X/pX corresponds to the scoreable set $\Gamma_p(r)$, and so has highest weight (α_1, α_2) where

$$(\alpha_1)_i = \begin{cases} p-1 & 0 \leq i < n; \\ r_n - 1 & i = n. \end{cases}$$

The minimal submodule of X/pX corresponds to the scoreable set $\varnothing$, and so has highest weight $(r, 0)$.

Moreover, there is a map d on the set of almost valid modules such that

(i) for any almost valid module M, $d^2(M) = M$;
(ii) $d(X) = pV_p$;
(iii) if $X > M > pX$ then $V_p > d(M) > pV_p$ and $X/M \cong d(M)/pV_p$.

Therefore V_p/pV_p also has a unique maximal and minimal submodule, its minimal submodule being isomorphic to the maximal submodule of X/pX and vice versa.

Proof.

The result is true because the scoreable set lattice has a top and a bottom, $\Gamma_p(r)$ and $\varnothing$ respectively. The labelling of the scoreable set lattice corresponding to X is all zeros, so when looking for a maximal submodule, that is, for a labelling of the scoreable set lattice which contains a unique label 1 among 0s, the only valid possibility is to increase $\nu_p(m_{\Gamma_p(r)})$ to 1. The labelling at pX is all 1s, and the last label to have been increased must have been $\nu_p(m_\varnothing)$.

Of course the maximal submodule of X/pX is also the maximal submodule of X/V_p, described in Proposition 2.35.

Define the map d by

$$\nu_p(m_B^{(d(M))}) = \begin{cases} |B| - \nu_p(m_B^{(M)}) + 1 & \text{if } V_p \leq M; \\ |B| - \nu_p(m_B^{(M)}) - 1 & \text{otherwise.} \end{cases}$$

That is, returning to the duality discussed in Chapter 1, if $X \geq M \geq pX$ then define $d(M)$ to be $p\widetilde{M}$ if $V_p \leq M$ and $p^{-1}\widetilde{M}$ otherwise. This is clearly a duality under which $d(X) = pV_p$; the remarks about V/pV_p follow. □

4.2 Connections with other work

Most of the work done in this area has dealt with polynomial representations of $\mathrm{SL}(2,\mathbf{Q})$, so we first explain how to translate into this language.

The maximal torus T of $\mathrm{SL}(2,\mathbf{Q})$ has dimension one, since it consists of the elements

$$\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$$

for $s \in \mathbf{Q}\setminus\{0\}$; therefore a weight, that is, the character of a representation of T is determined by the image of any one element of T. Since we are considering polynomial (that is, rational) representations, any weight has the form

$$\alpha : \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \longmapsto s^{\alpha}$$

for some integer α. The weight is *dominant* if $\alpha \geq 0$. We shall identify the weight α with the integer α, when no confusion can result.

In contrast, we may give another definition of the weights of GL_2 by describing them as weights of the two-dimensional maximal torus of GL_2,

$$\alpha : \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \longmapsto s^{\alpha_1} t^{\alpha_2}$$

where $\alpha = (\alpha_1, \alpha_2)$ for positive integers α_1 and α_2 as before, and s and t are non-zero elements of $\mathbf{Q}$.

Weights of GL_2 can be mapped to weights of SL_2 by mapping (α_1,α_2) to $\alpha_1 - \alpha_2$. If we fix r, the GL_2 weight (α_1,α_2) is determined by its SL_2 image $\alpha_1-\alpha_2$, since $\alpha_2 = (r-\alpha_1)/2$. Notice that this map preserves the ordering on weights, in the sense that $(\alpha_1,\alpha_2) > (\beta_1,\beta_2)$ if and only if $\alpha_1 > \beta_1$, if and only if $\alpha_1 - \alpha_2 > \beta_1 - \beta_2$. In particular, the GL_2 weight (α_1,α_2) is dominant if and only if its SL_2 image $\alpha_1-\alpha_2$ is dominant.

Remark 4.5.

For a given fixed r, the dominant SL_2 weights which occur as images of dominant GL_2 weights are $0,\ldots r$. Of course, any dominant SL_2 weight occurs as the image of some GL_2 weight in some (infinite number of) dimensions r.

The major work in this area is by R. Carter, E. Cline and D. Deriziotis, in [CarterCline] and [Deriziotis]. They consider the Weyl module V_p/pV_p. We give a brief summary, which is copied from [CarterCline] Section 1 except for minor changes of notation. We remark in passing that [CarterCline] uses the symbol m to denote the dimension of the Weyl module having highest weight

$m-1$, whereas [Deriziotis] uses m for the highest weight of the Weyl module, which therefore has dimension $m+1$. Since we have been concerned with Weyl modules with highest weight r, corresponding to $(r,0)$, which have dimension $r+1$, we continue to use this notation. That is, our r is Deriziotis' m and Carter and Cline's $m-1$.

Definition 4.6.

A *reflection* is a map

$$\rho_j : \mathbb{Z} \setminus \{0\} \longrightarrow \mathbb{Z}$$

defined by setting $\rho_j(r) = r - 2x$ where $r+1 = kp^j + x$, $k \geq 0$ and $0 \leq x < p^j$ ◁

Definition 4.7.

ρ_j is an *r-admissible reflection* if p does not divide k. ◁

Definition 4.8.

A strictly decreasing sequence of integers

$$r,\ \rho_{y_0}(r),\ \ldots,\ \rho_{y_s}\rho_{y_{s-1}}\cdots\rho_{y_0}(r)$$

is an *r-admissible sequence* if

(i) $0 < y_s < y_{s-1} \ldots < y_0$;
(ii) for each j, ρ_{y_j} is $\rho_{y_{j-1}}\rho_{y_{j-2}}\cdots\rho_{y_0}(r)$-admissible.

Let $\mathcal{V}(r;p)$ denote the set of integers z which appear in some r-admissible sequence. ◁

The first main theorem of [CarterCline] is

Theorem 4.9.

The weights which occur as highest weights of composition factors of V_p/pV_p are the elements of $\mathcal{V}(r;p)$.

We give the connection between this language and ours without the proof, which is by calculation and induction.

Lemma 4.10.

Let t be the length of the $(p-1)$-tail of r. If $(\alpha_1, \alpha_2) \in \Lambda^+(2,r)$ and

$$\alpha_1 - \alpha_2 = \rho_{y_s}\rho_{y_{s-1}}\cdots\rho_{y_0}(r)$$

is an element of an admissible sequence, then α is the highest weight such that $\Gamma_p(r,\alpha)$ is the set of points shown as ticks below:

$$\begin{array}{cccc} & & & \ldots 2\,1\,0 \\ & & (s \text{ even}) & \\ & & \checkmark\checkmark\cdots\checkmark\checkmark & \\ \times\times\times\times\checkmark\checkmark\cdots\cdots\checkmark\times\times\cdots\cdots\times & \cdots & \longleftarrow y_s \longrightarrow & \Big\} \times\times\times\cdots\times \\ \longleftarrow y_0 - y_1 \longrightarrow\longleftarrow y_1 - y_2 \longrightarrow & & \times\times\cdots\times\times & \longleftarrow t \longrightarrow \\ & & (s \text{ odd}) & \end{array}$$

that is, where $i \in \Gamma_p(r,\alpha)$ if and only if

$$y_{2k+1} \leq i - t \leq y_{2k}$$

for some natural number $k \leq s/2$, where for convenience we define y_{s+1} to be 0.

$\square$

We give some other correspondences between the notions in [CarterCline] and [Deriziotis] with ours. They follow, with some easy calculation, from the definitions.

Remark 4.11.

(i) x dominates y if and only if y scores no point in x.

(ii) If $k = \alpha_1 - \alpha_2$ and $m = r$ then $(m-k)/2 = \beta_2$.

(iii) Let $k \in V(m;p)$, so that (α_1, α_2) is the highest weight of some composition factor of V_p/pV_p, that corresponding to scoreable set $\Gamma_p(r,\alpha)$. The i^{th} coefficient of k is generically zero if and only if (α_1,α_2) does not score the i^{th} point in r; that is, if and only if $i \notin \Gamma_p(r,\alpha)$.

(iv) The partial order $\preceq$ on highest weights of composition factors is connected to our partial order by inclusion on scoreable sets, as follows. Suppose that k and l are highest weights of composition factors, that is, elements of $V(r,p)$, such that $k \preceq l$. That is, every generic zero of $k = \alpha_1 - \alpha_2$ corresponds to a generic zero of $l = \beta_1 - \beta_2$. This is true if and only if every point not scored by α in r is also not scored by β in r; that is, if and only if $\Gamma_p(r,\beta) \subseteq \Gamma_p(r,\alpha)$. $\square$

Translated according to these correspondences, Deriziotis' Theorem in section 2, the second main theorem of [CarterCline] becomes

Theorem 4.12.

Every submodule P'/pV_p of V_p/pV_p has the form

$$\sum_{\Gamma_p(r,\alpha)\in\mathcal{A}} V_p^\alpha/pV_p^\alpha$$

for some subset $\mathcal{A}$ of $\mathcal{L}(r,p)$. Moreover β occurs as the highest weight of a composition factor of L/pV_p if and only if $\Gamma_p(r,\beta) \supseteq \Gamma_p(r,\alpha)$ for some $\Gamma_p(r,\alpha) \in \mathcal{A}$.

$\square$

The first part is implied by our proof that valid modules correspond to labellings of the scoreable set lattice. We proved the second part on page 100, using the general result Lemma 3.32.

Other interesting work in this area has been done by P. W. Winter in [Winter], by S. Doty [Doty] and, more recently, by Z. Lin. Doty and Lin both work in the context of Lie algebras of the groups concerned. Lin gives a description of the submodule structure of the modular Weyl modules, our M/pM, in the case of SL_2, and he shows that they are indecomposable. Doty deals with part of the general problem of finding the structure of these modules in the case of SL_n. Both have concepts which turn out to correspond with our scoreable sets.

In section 2.3 of [Doty], Doty describes what he calls the *carry pattern* of the SL_n-weight β with respect to a fixed integer m, corresponding to our r. This can be seen to be the set $\Gamma_p(r,\beta)$, in the SL_2 case. He defines the set of all such sets, and a partial order on them, which correspond to our lattice of scoreable sets.

In section 3 of [Lin] there is a description of sets $\mathcal{S}(\lambda)$ which can be seen to be our scoreable sets. Lin deduces some of the properties that we have described, always in the modular SL_2 context. His Lemma 3.6 is the easy part of our Theorem 2.11, that is, our Lemma 2.13 (i).

In conclusion, our work has connections with other work which has been done, which is the more interesting as our approach has been quite different from that of other authors. In particular, the scoreable set lattice has been central to our work, and we have been able to demonstrate and exploit its properties in the more general context. We have showed that it can be used to determine completely the inclusion structure of all admissible lattices in Weyl modules for $GL_2(\mathbf{Q})$, which was previously unknown. It would be interesting to see to what extent these ideas could be generalised in the context of $GL_n(\mathbf{Q})$ for $n > 2$.

Appendix A

```
/*program to find normalised S(Z) modules between X(lambda,Q) and V(lambda,Q)
by giving their localised m_alphas for each relevant prime*/
/*PS: This was my first ever C program, so judge it not too harshly!*/
#include <stdio.h>
#define NO 0
#define YES 1
#define PMAX 50
#define RMAX 100

int r,primes[PMAX],*p;
int fact[RMAX][PMAX],m[RMAX/2][PMAX];
int y[RMAX/2][PMAX], wt[RMAX/2][PMAX],vmax[RMAX/2][RMAX/2][PMAX];

int rprimes();
void factors();
int weights();
int ffact();
void conditions();
void tuples();
void test();

main()                                                        main
{
int i,j,noprimes,wts,*q;
printf("When you give this program a positive integer r, it will, for\n");
printf("each relevant prime p, that is, for each prime p less than or\n");
printf("equal to r, calculate all  valid modules and list them, by\n");
printf("giving the tuples {nu_p(m_alpha^{(M)})}, \n");
printf("where alpha runs over the set of dominant\n");
printf("weights, for each valid module M in turn.\n\n");
printf("Please enter the value of r:\n");
scanf("%d", &r);
printf("r=%d\n",r);
```

```
p=primes;/*set pointer p to start of array primes*/
/*initialise fact,wt,y & m to 0s*/
for (i=0;i<PMAX;i++){
        for (j=0;j<RMAX;j++)
                fact[j][i]=0;
        for (j=0;j<RMAX/2;j++){
                wt[j][i]=0;
                y[j][i]=0;
                m[j][i]=0;
                }
        }
/*initialise primes to 100s -- can't remember why!*/
for (i=0;i<PMAX;i++)
        primes[i]=100;
noprimes=rprimes();/*here array p of primes gets set up*/
printf("The number of primes no bigger than %d is %d\n", r, noprimes);
printf("They are:\n");
for(i=0;i<noprimes;i++)
        printf("%d  ",p[i]);
printf("\n");
factors();/*here array fact of exps of primes in integers 2,..r set up*/
wts=weights(noprimes);/*no. dom weights not lambda. Arrays wt and y set up*/
printf("The prime exponents of sizes of dominant weights not lambda are:\n");
for(i=1;i<=wts;i++){
        printf("\n(%2d,%2d): ", r-i, i);
        for(q=p;q<p+noprimes;q++)
                printf("%d  ", wt[i][q-p]);
        }
printf("\n");
printf("giving the exponent of 2 first.\n");
conditions(wts,noprimes);/*array vmax of max (over admissible A) exponent of
each prime in denom expression for pairs of weights set up*/
printf("In what follows, each row represents a valid module localised\n");
printf("at the current prime p. The integers in the row are the exponents\n"
printf("of p in the values of m_alpha, where alpha runs over dominant\n");
printf("weights, highest first, as listed above.\n\n");
```

```
tuples(wts,noprimes);
}
```

rprimes

```
int rprimes()
/*find all primes leq r, put them in array p, return how many*/
{
int c,i,j,*q;
j=0;
for (i=2;i<=r;i++){
        c=NO;
        for (q=p;*q * *q<=i;q++)
                if (i%*q==0){
                        c=YES;
                        break;
                        }
        if (c==NO)
                p[j++]=i;
        }
return j;
}
```

factors

```
void factors()
/*fact[i][n] gets set to the exponent of the nth prime in i for i=2,...r*/
{
int i,*q,j;
for (i=2;i<=r;i++){
        j=i;
        for (q=p;j!=1;q++)
                while(j%*q==0){
                        j=j/(*q);
                        fact[i][q-p]++;
                        }
        }
}
```

noprimes

```
int weights(noprimes)
```

```
/*USES ffact*/
/*Returns no of dom weights not lambda. Also here arrays wt and y are set up.*/
{
int wts,j,*q;
wts=(r%2==0)?r/2:(r-1)/2;/*no of dominant weights not lambda*/
for(q=p;q<p+noprimes;q++){
        y[0][q-p]=ffact(r,q-p);
        for(j=1;j<=wts;j++){
                y[j][q-p]=y[j-1][q-p]+fact[j][q-p]-fact[r-j+1][q-p];
/*y[j][q-p]is exp of (q-p)th prime in (r-j)!j!*/
                wt[j][q-p]=y[0][q-p]-y[j][q-p];
/*wt[j][q-p]is exp of (q-p)th prime in size of dominant weight (r-j,j)*/
                }
        }
return wts;
}
```

ffact

```
int ffact(integ,reqprime)
/*CALLED BY weights and conditions*/
/*returns exponent of reqprime in integ-factorial*/
{
int j,s;
s=0;
for(j=1;j<=integ;j++)
        s+=fact[j][reqprime];
return s;
}
```

conditions

```
void conditions(wts,noprimes)
/*USES ffact*/
/*array vmax gets set; vmax[i][j][q-p] is max over admissible A of the
exponent of the q-pth prime in denom for weights (r-i,i),(r-j,j)*/
{
int i,j,a,b,c,d,v,maxv,*q,fa,fb,fc,ga,gb,gc,gd;
for(i=1;i<=wts;i++)/*alpha_2, so alpha_1=r-i*/
        for(j=1;j<i;j++)/*beta_2, leq alpha_2 wlog by symmetry*/
```

```
                for(q=p;q<p+noprimes;q++){
                        a=r-i-j;/*A is initially alpha1 - beta2 */
                        b=i;/*alpha1-A is initially beta2*/
                        c=j;/*beta1-A is initially alpha2*/
                        d=0;/*beta2 - alpha1 + A is initially 0*/
                        v=ffact(a,q-p)+ffact(b,q-p)+ffact(c,q-p);
                        maxv=v;
                        while(c>0){
                a++;/*Increase A by 1...*/
                d++;/*...so beta2-alpha1+A increases by 1...*/
                ga=fact[a][q-p];/*need to add in exps of NEW values of a,d*/
                gb=fact[b][q-p];/*and subtract out exps of OLD values of b,c*/
                gc=fact[c][q-p];
                gd=fact[d][q-p];
                b--;/*so only now does alpha1-A decrease by 1*/
                c--;/*and same for beta1-A*/
                v+=(ga-gb-gc+gd);/*giving new value of denom's p-part*/
                maxv=(v>maxv)?v:maxv;/*and keeping the most stringent*/
                }
                        vmax[i][j][q-p]=maxv;/*most stringent denom*/
                        vmax[j][i][q-p]=maxv;/*by symmetry in alpha, beta*/
                        }
}
```

tuples

```
void tuples(wts,noprimes)
/*USES test*/
/*we look at possible tuples in rev.lex.order, testing each*/
{
int level,i,*q;
for(q=p;q<p+noprimes;q++){
        printf("Here are possible tuples localised at current prime %d\n", *q
        for(i=1;i<=wts;i++)
                m[i][q-p]=0;
        for(;;){level=wts;
                while (m[level][q-p]==wt[level][q-p]&&level!=0)
                        level--;
```

```
                if (level==0)
                        break;
                for(i=level+1;i<=wts;i++)
                        m[i][q-p]=0;
                m[level][q-p]++;
                test(q-p,wts);
                }
        }
}

void test(pr,wts)                                                    test
/*CALLED BY tuples*/
/*tests present tuple (array m) until/unless finds pair of malphas not ok*/
{
int ok,i,j;
ok=YES;
for(i=1;i<=wts && ok==YES;i++)
        for(j=1;j<=wts;j++)
                if (y[j][pr]-vmax[i][j][pr]+m[j][pr]<m[i][pr]){
                        ok=NO;
                        break;
                        }
if (ok==YES){
        printf("(");
        for(i=1;i<=wts;i++)
                printf("%d   ", m[i][pr]);
        printf(")\n");
        }
}
```

Appendix B

Here is some output from the program in Appendix A, slightly doctored to get the symbols TEX ed.

When you give this program a positive integer r, it will, for
each relevant prime p, that is, for each prime p less than or
equal to r, calculate all valid modules and list them, by
giving the tuples $\nu_p(m_\alpha^{(M)})$,
where α runs over the set of dominant weights,
for each valid module M in turn.
Please enter the value of r:
10
r=10
The number of primes no bigger than 10 is 4
They are:
2 3 5 7
The prime exponents of sizes of dominant weights not lambda are:
(9, 1): 1 0 1 0
(8, 2): 0 2 1 0
(7, 3): 3 1 1 0
(6, 4): 1 1 1 1
(5, 5): 2 2 0 1
giving the exponent of 2 first.
In what follows, each row represents a valid module localised
at the current prime p. The integers in the row are the exponents
of p in the values of m_α, where α runs over dominant weights,
highest first, as listed above.
Here are possible tuples localised at current prime 2
(0 0 1 0 0)
(0 0 1 0 1)
(0 0 1 1 1)
(0 0 2 0 1)
(0 0 2 1 1)
(1 0 1 0 1)
(1 0 1 1 1)

(1 0 2 0 1)
(1 0 2 1 1)
(1 0 2 1 2)
(1 0 3 1 2)
Here are possible tuples localised at current prime 3
(0 1 0 0 1)
(0 1 1 1 1)
(0 2 1 1 2)
Here are possible tuples localised at current prime 5
(1 1 1 1 0)
Here are possible tuples localised at current prime 7
(0 0 0 1 1)

When you give this program a positive integer r, it will, for
each relevant prime p, that is, for each prime p less than or
equal to r, calculate all valid modules and list them, by
giving the tuples $\nu_p(m_\alpha^{(M)})$,
where α runs over the set of dominant weights,
for each valid module M in turn.
Please enter the value of r:
30
r=30
The number of primes no bigger than 30 is 10
They are:
2 3 5 7 11 13 17 19 23 29
The prime exponents of sizes of dominant weights not lambda are:
(29, 1): 1 1 1 0 0 0 0 0 0 0
(28, 2): 0 1 1 0 0 0 0 0 0 1
(27, 3): 2 0 1 1 0 0 0 0 0 1
(26, 4): 0 3 1 1 0 0 0 0 0 1
(25, 5): 1 3 0 1 0 1 0 0 0 1
(24, 6): 0 2 2 1 0 1 0 0 0 1
(23, 7): 3 3 2 0 0 1 0 0 0 1
(22, 8): 0 3 2 0 0 1 0 0 1 1
(21, 9): 1 1 2 0 1 1 0 0 1 1
(20,10): 0 2 1 1 1 1 0 0 1 1
(19,11): 2 2 2 1 0 1 0 0 1 1
(18,12): 0 1 2 1 0 1 0 1 1 1
(17,13): 1 3 2 1 0 0 0 1 1 1
(16,14): 0 3 2 0 0 0 1 1 1 1
(15,15): 4 2 1 0 0 0 1 1 1 1
giving the exponent of 2 first.
In what follows, each row represents a valid module localised
at the current prime p. The integers in the row are the exponents
of p in the values of m_α, where α runs over dominant weights,
highest first, as listed above.
Here are possible tuples localised at current prime 2
(0 0 0 0 0 0 0 0 0 0 0 0 0 0 1)
(0 0 0 0 0 0 1 0 0 0 0 0 0 0 1)
(0 0 0 0 0 0 1 0 0 0 0 0 0 0 2)

(0 0 1 0 0 0 1 0 0 0 1 0 0 0 1)
(0 0 1 0 0 0 1 0 0 0 1 0 0 0 2)
(0 0 1 0 0 0 2 0 0 0 1 0 0 0 2)
(0 0 1 0 0 0 2 0 0 0 1 0 0 0 3)
(1 0 1 0 1 0 1 0 1 0 1 0 1 0 1)
(1 0 1 0 1 0 1 0 1 0 1 0 1 0 2)
(1 0 1 0 1 0 2 0 1 0 1 0 1 0 2)
(1 0 1 0 1 0 2 0 1 0 1 0 1 0 3)
(1 0 2 0 1 0 2 0 1 0 2 0 1 0 2)
(1 0 2 0 1 0 2 0 1 0 2 0 1 0 3)
(1 0 2 0 1 0 3 0 1 0 2 0 1 0 3)
(1 0 2 0 1 0 3 0 1 0 2 0 1 0 4)
Here are possible tuples localised at current prime 3
(0 0 0 1 1 0 1 1 0 0 0 0 1 1 0)
(0 0 0 1 1 0 1 1 0 1 1 0 1 1 0)
(0 0 0 1 1 1 1 1 0 0 0 0 1 1 1)
(0 0 0 1 1 1 1 1 0 1 1 0 1 1 1)
(0 0 0 1 1 1 1 1 1 1 1 1 1 1 1)
(0 0 0 2 2 1 2 2 0 1 1 0 2 2 1)
(0 0 0 2 2 1 2 2 1 1 1 1 2 2 1)
(0 0 0 2 2 2 2 2 1 1 1 1 2 2 2)
(1 1 0 1 1 0 1 1 0 1 1 0 1 1 0)
(1 1 0 1 1 1 1 1 0 1 1 0 1 1 1)
(1 1 0 1 1 1 1 1 1 1 1 1 1 1 1)
(1 1 0 2 2 1 2 2 0 1 1 0 2 2 1)
(1 1 0 2 2 1 2 2 1 1 1 1 2 2 1)
(1 1 0 2 2 1 2 2 1 2 2 1 2 2 1)
(1 1 0 2 2 2 2 2 1 1 1 1 2 2 2)
(1 1 0 2 2 2 2 2 1 2 2 1 2 2 2)
(1 1 0 3 3 2 3 3 1 2 2 1 3 3 2)
Here are possible tuples localised at current prime 5
(0 0 0 0 0 1 1 1 1 0 1 1 1 1 0)
(0 0 0 0 0 1 1 1 1 1 1 1 1 1 1)
(1 1 1 1 0 1 1 1 1 0 1 1 1 1 0)
(1 1 1 1 0 1 1 1 1 1 1 1 1 1 1)
(1 1 1 1 0 2 2 2 2 1 2 2 2 2 1)
Here are possible tuples localised at current prime 7

(0 0 1 1 1 1 0 0 0 1 1 1 1 0 0)

Here are possible tuples localised at current prime 11

(0 0 0 0 0 0 0 0 1 1 0 0 0 0 0)

Here are possible tuples localised at current prime 13

(0 0 0 0 1 1 1 1 1 1 1 1 0 0 0)

Here are possible tuples localised at current prime 17

(0 0 0 0 0 0 0 0 0 0 0 0 0 1 1)

Here are possible tuples localised at current prime 19

(0 0 0 0 0 0 0 0 0 0 0 1 1 1 1)

Here are possible tuples localised at current prime 23

(0 0 0 0 0 0 0 1 1 1 1 1 1 1 1)

Here are possible tuples localised at current prime 29

(0 1 1 1 1 1 1 1 1 1 1 1 1 1 1)

References

[CarterCline]
R. Carter, E. Cline, *The submodule structure of Weyl modules for groups of type A_1*, Proc. Conf. Finite Groups (Univ. Utah, Park City, Utah, 1975) pp 303 – 311.

[CarterLusztig]
R. W. Carter, G. Lusztig, *On the modular representations of the general linear and symmetric groups*, Math. Z. 136 (1974), pp 193 – 242.

[Deriziotis]
D. I. Deriziotis, *The submodule structure of Weyl modules for groups of type A_1*, Communications in Algebra, 9(3) (1981) pp 247 – 265.

[Doty]
S. R. Doty, *The Submodule Structure of Certain Weyl Modules for Groups of type A_n*, Communications in Algebra, 9(3) (1981) pp 247 – 265.

[Green1]
J. A. Green, *Polynomial Representations of* GL(n), Lecture Notes in Mathematics no. 830, Springer-Verlag Berlin Heidelberg New York 1980.

[Green2]
J. A. Green, *Schur algebras and general linear groups*, Groups St Andrews 1989.

[Lin]
Z. Lin, *Filtrations of the modules for Chevalley groups arising from admissible lattices*, to appear.

[Schur]
I. Schur, *Uber eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen*, (1901) I. Schur, Gesammelte Abhandlungen I, pp 1 – 70, Springer-Verlag BerlinHeidelberg New York 1973.

[Winter]
P. W. Winter, *On the modular representation theory of the two-dimensional special linear group over an algebraically closed field*, J. London Math. Soc. (2), 16(1977), pp 237 – 252.

List of Notation

$\uplus$	disjoint union of sets	
$\mathbf{n}$	set $\{1, \ldots, n\}$	2
$I(n,r)$	set of r-tuples with entries from $\mathbf{n}$	2
P	symmetric group of degree r	2
P_i	stabiliser of i in P	20
Π	transversal of the P-orbits on $I(n,r)$	2
Ω	transversal of the P-orbits on $I(n,r) \times I(n,r)$	4
W	symmetric group of degree n, the Weyl group of $\mathrm{GL}(n)$	3
$\Lambda(n,r)$	set of weights $\lambda = (\lambda_1, \ldots, \lambda_n)$	3
$\Lambda^+(n,r)$	set of dominant weights	3
$\|\lambda\|$	size of weight (P-orbit on I) λ	3
$\alpha \geq \beta$	partial order on weights	4
$S_{\mathbf{Q}}(n,r)$	Schur algebra	4
T	map $\mathbf{Q}\mathrm{GL}_n(\mathbf{Q}) \longrightarrow S_{\mathbf{Q}}(n,r)$	6
M^α	α weight space of M	6
M_p	p-envelope of M	25
$V_{\lambda,K}$	Weyl module over K with highest weight λ	8
$[\lambda]$	shape of $\lambda \in \Lambda(n,r)$	9
T_i^λ	λ-tableau for $i \in I(n,r)$	9
$C(T)$	column stabiliser	10
$R(T)$	row stabiliser	10
$\{Q\}$	signed sum of elements of Q	11
b_i	basis element of Weyl module	11
$V_{\lambda,\mathbf{Z}}$	$\mathbf{Z}$-span of basis elements of $V_{\lambda,\mathbf{Q}}$	12
$X_{\lambda,\mathbf{Z}}$	dual to $V_{\lambda,\mathbf{Z}}$	13
$<,>$	bilinear form on $E^{\otimes r}$	13
$\langle\langle,\rangle\rangle$	bilinear form on $E^{\otimes r}\{C(T)\}$	13
$\mathsf{Sym}(A)$	symmetric group on A	16
m_α	element of tuple defining valid module	18
$\widetilde{M}$	dual to M	27
x_p	p-part of x	25
$\nu_p(x)$	exponent of p in x	25
$x_p(i)$	$x_i p^i + \cdots x_0$	28

$\gamma_p(x,y)$	points scored by y in x	29
$\Gamma_p(x,y)$	set of points scored by y in x	30
$\Gamma_p(r,\alpha)$	set of points scored by α_1 or equivalently α_2 in r	30
$\gamma_p(r)$	size of set of all scoreable points	30
$\Gamma_p(r)$	set of all scoreable points	31
$p(A)$	p-exponent of the factorial expression	32
$K_p(\alpha,\beta)$	$\Gamma_p(r,\beta) \setminus \Gamma_p(r,\alpha)$	31
$I(\alpha,\beta)$	set of points initially scored	34
$RL(\alpha,\beta)$	set of points which are required losses	35
B	block of points in r	39
$P(B)$	subset of block of points such that A_i is set to 0	44
$Q(B)$	$B \setminus P(B)$	44
$J(M/N)$	set of weights with non-zero weight spaces in M/N	66
$\mathcal{L}(r,p)$	lattice of scoreable sets	75
$k \preceq l$	partial ordering on weights	104

Index

This index gives the occurrences of all but minor definitions (in bold) and of major references to other topics.