Hereditary rings and rings of finite representation type

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Declaration

Except where it is stated otherwise, all the results of this thesis are, to the best of my knowledge, original and my own work.

List of frequently used symbols

Let R and T be rings, M and M' right R-modules, L a left R-module and B a T-R-bimodule. Let I be an ideal of R, P and Q prime ideals and X a clique of prime ideals of R.

Symbol	Meaning
$\mathbb{N}$	the set of natural numbers;
$M \subset M'$	M is strictly contained in M';
M_R	the right R-module M;
${}_RL$	the left R-module L;
${}_TB_R$	the T-R-bimodule B;
$\mathrm{Hom}(M_R, M'_R)$	the set of R-homomorphisms from M_R to M'_R;
$M_n(R)$	the ring of nxn matrices with entries from R;
$T_n(R)$	the ring of nxn upper triangular matrices from R;
$M^{(n)}$	the direct sum of n copies of the module M_R;
$\mathrm{End}(M_R)$	the set of endomorphisms of M_R;
$C(R)$ or C	the centre of R;
$Q(R)$	the quotient ring of R (whenever it exists);
$J(R)$ or J	the Jacobson radical of R;
$N(R)$ or N	the nilpotent radical of R;
$A(R)$	the Artin radical of R;
$\mathrm{ht}(P)$	the height of P;
$\mathrm{cl.K.dim}(R)$	the classical Krull dimension of R;
$\mathrm{K.dim}(R)$	the Krull dimension of R;
$\mathrm{gl.dim}(R)$	the global dimension of R;
$\mathrm{pd}(M_R)$	the projective dimension of M_R;
$\mathrm{u.dim}(M_R)$	the uniform dimension of M_R;
$\mathrm{l.ann}_T(B)$	the left annihilator of B;
$\mathrm{r.ann}_R(B)$	the right annihilator of B;
$\mathcal{C}(I)$	the set $\{r \in R \mid r + I$ is regular in $R/I\}$;
$P \rightsquigarrow Q$	P is (right) linked to Q;
R_X	the localisation of R at X;
$\equiv$	denotes.

Summary

This thesis is a study of Noetherian PI rings with the property that every proper Artinian homomorphic image is of finite representation type. Our main result is :

Theorem 5.1.12 *Let R be a Noetherian PI ring, which is an order in an Artinian ring. Suppose that every proper Artinian factor ring of R is of finite representation type. Then R is a direct sum of an Artinian ring of finite representation type and prime hereditary rings.*

As a special case we have that a prime Noetherian PI ring is hereditary if and only if all its proper Artinian homomorphic images are of finite representation type. The thesis is organized as follows.

In Chapter 1 we introduce the terminology and collect well known results on Noetherian rings that we shall use in later chapters. We also include some remarks on the relationship between the J-adic completion of a certain Noetherian prime PI ring and its centre that seem not appear in the literature. In Chapter 2 we present some known characterizations of hereditary rings. We introduce a discussion on rings of finite representation type and give motivation for our study.

In Chapter 3 we begin the proof of our main theorem. We adapt the proof of a theorem by S. Brenner [10], which shows that the 2×2 upper triangular matrix ring $T_2(\mathbb{Z}/p^4\mathbb{Z})$ is of infinite type, to the more general case of $T_2(D/d^nD)$, where D is a non-commutative local Dedekind prime PI ring and dD its maximal ideal. This result is crucial for the proof of Theorem 5.1.12. The reduction of our problem to this case is allowed by a theorem of M. Auslander (cf. Theorem 3.2.1) on trivial extension rings of Artin algebras. We describe this theorem and show that it holds also for Artinian PI rings. Then we analyze the graph of links between maximal ideals of R. We show that if R is a prime ring satisfying the hypothesis of Theorem 5.1.12 then all cliques of maximal ideals of R are finite. In the last section of this chapter we look at connections between Artinian serial rings and rings of finite representation type. We show that if R is a semiperfect local Noetherian PI ring which is not Artinian and such that $R/J(R)^2$ is of finite representation type, then R is a hereditary prime ring. Finally, we introduce some rings related to the ring R satisfying the hypothesis of Theorem 5.1.12, which inherit the property of the homomorphic images and that we shall use for the proof of the theorem.

In Chapter 4 we prove Theorem 5.1.12 under the additional assumption that every clique of maximal ideals of R is finite. This is done by analyzing in detail the

structure of the J-adic completion of the localisation of R at a clique of maximal ideals. Then from the results of Chapter 3 we deduce that Theorem 5.1.12 holds if R is semiprime.

In Chapter 5 we prove that cliques of maximal ideals of R are indeed finite. This finishes the proof of our result. Further, we prove that a Noetherian PI ring whose proper Artinian homomorphic images are all serial is an order in an Artinian ring.

In Chapter 6 we prove the analogue of Theorem 5.1.12 for a semiprime PI ring which is affine over a field. Then we give some examples of Noetherian PI rings of different global dimension to show that the assumption in Theorem 5.1.12 on the existence of an Artinian quotient ring for R is necessary.

For completeness of our study, some results are stated and proved in more generality than it is needed for the proof of our main theorem.

Introduction

In this thesis we give a new characterization of certain hereditary PI rings. A Noetherian prime ring whose right and left ideals are projective is called hereditary Noetherian prime (HNP). The class of HNP rings arises as one of the possible generalizations of Dedekind domains to the non-commutative setting. Its theory is well developed and a good account on the subject can be found for example in [37].

A particular subclass of HNP rings is the one of classical hereditary orders. A ring R is called a classical C'-order in a finite dimensional separable K'-algebra Q if C' is a Dedekind domain contained in the centre of R, with K' as its field of fractions, R is a finitely generated C'-module and $Q = RK'$. In particular, a classical C'-order is a PI ring. In the special case where C' is a complete discrete valuation ring (DVR), the structure of a hereditary order is completely described by Theorem 2.1.5, which appeared in [38]. The study of hereditary orders began in [3] where it is proved that maximal orders in the classical sense are hereditary (cf. Theorem 2.1.8). This result was then extended to prime Noetherian PI rings that are maximal orders and have Krull dimension one [37, Theorem 13.9.14]. In a famous paper [49], Robson and Small showed that, in fact, hereditary prime PI rings are classical hereditary orders over their centre and therefore the two theories coincide in this case.

In some classes of non-commutative rings, HNP rings are characterized by certain properties of their factor rings (see below). A well known theorem by Eisenbud and Griffith states that every factor ring of a HNP ring R is serial Artinian [19]. If R is also a maximal order, i.e. R is a Dedekind prime ring, then its factor rings are principal ideal rings [19]. Following these results, there have been attempts to find conditions for their converse to be true. This has been achieved for the class of bounded rings. In [28] it is shown that a Noetherian, bounded, prime ring whose factor rings are right principal ideal rings is a Dedekind prime ring. Further, in [56] it is proved that a bounded Noetherian prime ring of Krull dimension one whose factor rings are all serial is, indeed, hereditary. A different proof of this result, which uses localisation techniques, is given in [43]. Finally, in [35] a one-sided version of this theorem is proved, namely that a right Noetherian prime bounded ring that has only serial factor rings is right hereditary. In [60] Warfield characterizes a Noetherian ring that is finitely generated as a module over a central subring and whose Artinian factor rings are all serial, as a ring that is a direct sum of a serial Artinian ring and hereditary orders over Dedekind domains.

In this thesis, we consider a Noetherian ring R whose proper Artinian factor rings are all of finite representation type, i.e. they have only finitely many isomorphism classes of finitely generated indecomposable modules. This setting includes the case of a Noetherian ring whose proper Artinian factor rings are all serial. A theorem by Gustafson [27] states that if R is also a classical C''-order over a complete DVR in a finite dimensional separable algebra, then R is hereditary. A more technical proof of this theorem, under the additional assumption that C''/m is a perfect field, where m is the maximal ideal of C'', appeared in an earlier paper [25]. We generalize this result for a Noetherian prime PI ring and also prove that if the ring R is Noetherian PI, with Artin radical $A(R) = 0$, then R is hereditary (cf. Proposition 5.1.1).

The property for a given Artinian ring A to be of finite representation type is not easy to show. The Kronecker algebra $A = \begin{pmatrix} k & k \oplus k \\ 0 & k \end{pmatrix}$, where k is a field, is an example of a ring that, despite of its simple structure, is of infinite representation type. Note also that A is hereditary. There exists a method for proving that a given Artin algebra, i.e. an algebra that is finitely generated as a module over an Artinian central subring, is of finite representation type, by successively constructing all of its finitely generated indecomposable modules (cf. [4] and also [55] for recent generalizations to PI rings). However, it cannot be used for proving that an algebra is of infinite representation type. Further, for applying this method, it is necessary to know exactly the structure of the ring. If A is a basic finite dimensional algebra over an algebraically closed field, then it is isomorphic to the path algebra associated to its quiver factorized by an ideal generated by certain relations that depend on A [4, Theorem 1.9]. In this case there is a very fast method, using covering theory, for checking whether A is of infinite representation type [12]. As we do not assume our ring R to be an algebra over a field or its factor rings to be Artin algebras, these techniques are not available to us.

In the rather special situation of a hereditary Artinian PI ring A, the representation type of A can be observed on its associated quiver Γ. In this case, A is of finite representation type if and only if the underlying diagram of Γ is a disjoint union of Dynkin diagrams [17, Corollary 1.4]. In fact, if A is of finite representation type, then the category of finite length A-modules is equivalent to the category of representations of the species attached to A (cf. Definition 2.2.17). The connection between Dynkin diagrams and the representation theory of species is due to Gabriel [21]. Another special class of rings consists of Artinian PI rings with Jacobson radical square equal to zero. In fact, their representation type depends on that of certain hereditary Artinian PI rings [17].

For the proof of our result, we use the properties of these two special classes of rings. We relate the graph of links between maximal ideals of R to the quiver associated to some Artinian factor rings of R with Jacobson radical square equal to zero. This fact, together with the known results on the graph of links of a Noetherian prime PI ring (cf. Theorem 3.3.13) and the property of a Noetherian PI ring to have only finitely many idempotent ideals [50], allows us to prove directly that all cliques of maximal ideals of R are finite if R is prime (cf. Proposition 3.3.20).

Then we shift the problem to semilocal rings, by considering, rather than R, the localisation R_X of R at a finite clique X of maximal ideals (cf. Lemma 4.2.3). In fact, a theorem by Müller [43] asserts that the global dimension of a Noetherian PI ring whose cliques of maximal ideals are all finite, depends on that of the localised rings at each clique of maximal ideals.

For the proof of our main result we need a structure theorem that we can prove if R_X is semiperfect. So we consider the $J(R_X)$-adic completion $\hat{R}_X$ of R_X, which is semiperfect and also inherits the other assumptions on R (cf. Proposition 3.4.18). The J-adic completion of a Noetherian ring was also used by Warfield in the proof of the result quoted earlier.

There are several advantages of working with semiperfect rings. For instance, if $\hat{R}_X/J(\hat{R}_X)^2$ is serial then $\hat{R}_X$ itself is serial (cf. Lemma 3.4.1). Further, for any idempotent e of $\hat{R}_X$, the ring $e\hat{R}_Xe$ inherits from $\hat{R}_X$ the property of the Artinian factor rings to be of finite representation type (cf. Proposition 3.4.9). This turns out to be a key fact when e is primitive, as then the ring $e\hat{R}_Xe$ is scalar local. We show (cf. Theorem 4.2.5) that a local Noetherian PI ring whose Artinian homomorphic images are all of finite representation type, is either Artinian or hereditary and prime. For the proof of this, we use the fact that a local Artinian PI ring A with $J(A)^2 = 0$, that is of finite representation type, is always one sided-serial [17]. If A is also an Artin algebra, then A is two sided serial. The property mentioned above was used in [27] for orders, where the Artinian factor rings are Artin algebras. A structure theorem [45] allows the author to write the order as a matrix ring with diagonal entries from the same complete local Dedekind prime ring D and the other entries from some powers of the Jacobson radical M of D. In the case of a 2x2 matrix, he looks at the quiver associated to the ring modulo the Jacobson radical squared. Then he applies an easy induction argument to obtain a hereditary order in its standard form. In Proposition 4.1.2 we prove that if $\bar{R}_X$ is prime, then it is an order over a complete DVR in the classical sense. Thus we can apply Gustafson's Theorem (and hence the structure theorem) to conclude that $\bar{R}_X$ is hereditary. This result easily generalizes to the case where $\bar{R}_X$ is semiprime. Then we prove that $\bar{R}_X$ is indeed semiprime. Our argument is by contradiction. We reduce the problem to the case where $\bar{R}_X$ has just two primitive idempotents and we are able to give a detailed description of this ring (cf. Lemma 4.1.6). Then, a generalization of a theorem by S. Brenner [10] allows us to finish the proof. For this particular ring a countable family of finitely generated indecomposable modules, pairwise non-isomorphic, is detected. Once we have shown that $\bar{R}_X$ is a hereditary ring, and hence is serial, it is easy to prove that our original semilocal ring R_X is also hereditary.

The final step is to prove that every clique of maximal ideals of R is finite. This is true if the ring is prime, as we have already observed, and also if it is semiprime (cf. Lemma 4.2.7). More generally we reduce our problem to the case where R has no non-trivial idempotents and it is not prime. Again, an inspection of the graph of links G between maximal ideals of R yields that an infinite connected component of G must contain an infinite semiline with arrows all in the same direction, loops everywhere and no other arrows in G connecting these vertices, except the first one (cf. Lemma 5.1.7). Using this, together with the fact that our theorem holds for R modulo its nilpotent radical, we consider certain factor rings of R (cf. Lemma 5.1.8) and we are able to obtain rings of infinite representation type (cf. Lemma 5.1.10). This finishes the proof of our result.

When R is a semiprime PI ring, affine over a field, and all its Artinian homomorphic images are of finite representation type, we prove that R is also Noetherian (cf. Theorem 6.1.1). In fact, if R is prime, then R embeds in a central localisation S that is an Azumaya algebra. In particular, S is Noetherian and finitely generated as a module over its centre and our theorem applies for S. Then Schelter's formula [37, Theorem 13.10.12] on the classical Krull dimension of an affine PI ring, yields that R has classical Krull dimension one. So R itself is Noetherian and finitely generated as a module over its centre, by a well known result of Braun [5]. The case where R is not prime follows easily.

An equivalent way of stating our result in its general form is:

Let R be a Noetherian PI ring that is an order in an Artinian ring. Suppose that *every proper homomorphic image of R is of finite representation type. Then R is a (finite) direct sum of an Artinian ring of finite representation type and hereditary prime rings.*

In fact, the Artin radical of a Noetherian ring R that is an order in a Artinian ring is a direct summand of R [13]. On the other hand, a Noetherian ring with $A(R) = 0$ and Krull dimension one is an order in an Artinian ring [13].

The assumption $A(R) = 0$ is essential in our theorem. There are many examples of Noetherian PI rings of different global dimension and whose proper Artinian homomorphic images are all of finite representation type (cf. Section 6.2). Note that in the Artinian case, the representation type of the ring does not affect its global dimension.

However, if R is an indecomposable Noetherian PI ring with $A(R) \neq 0$, then $A(R/A(R)) = 0$ [13]. So our theorem, applied to $R/A(R)$, can still be a good test for the representation type of some factor rings of R. We use this to show that a Noetherian PI ring R, whose Artinian homomorphic images are all serial, is an order in an Artinian ring (cf. Theorem 5.2.5). In fact, for this ring, cliques of maximal ideals of the semiprime hereditary ring $R/A(R)$ all lift to cliques of maximal ideals of R. This result, which seems to be new, together with our main theorem, generalize Warfield's Theorem to Noetherian PI rings.

Chapter 1

Noetherian rings

In this chapter we list some well known results on Noetherian rings that we shall use throughout this thesis. In particular, we introduce the classical localisation at a clique and discuss its properties.

Then we focus on Polynomial Identity (PI) rings, which are the object of study of this thesis. We introduce the J-adic completion of a Noetherian semilocal PI ring R and show that it shares most of the features of the case when R is commutative.

Most of the results stated in Sections 1.1 and 1.2 will be used in the sequel without further mention.

1.1 Terminology

We list some standard definitions and notation (cf. [37] and [24]). Throughout, all rings will be associative with identity element and all modules will be unital. Let R be a ring. We denote by $\mathrm{Mod}(R)$ the category of right R-modules and by $\mathrm{mod}(R)$ the subcategory of $\mathrm{Mod}(R)$ consisting of finitely generated modules. We write $C(R)$, or simply C, for the centre of R. A right R-module M will be denoted by M_R, a left R-module by ${}_RM$ and a bimodule by the symbol ${}_TM_R$, where T is another ring. If M and M' are R-modules, then $\mathrm{Hom}(M, M')$ is the set of all R-homomorphisms from M to M'. In case we need to specify the ring R, we use the notations $\mathrm{Hom}(M_R, M'_R)$ and $\mathrm{Hom}({}_RM, {}_RM')$ for right and left R-modules respectively.

Let M be a right R-module. We say that M is *simple* if $M \neq 0$ and M has no submodules other than 0 and M. If M is a finite direct sum of simple modules, then M is called *semisimple*. The *socle* of M, written $\mathrm{socle}(M)$, is defined to be the submodule of M generated by all semisimple submodules of M. The module M

is *Noetherian* if it has the ascending chain condition (a.c.c.) on its submodules. If M has the descending chain condition (d.c.c.) on its submodules, then M is called *Artinian*. M has *finite length* if it is Artinian and Noetherian. In particular, if R_R (resp. ${}_RR$) has a.c.c. then R is called *right (left) Noetherian*. Similarly, R_R (resp. ${}_RR$) is *right (left) Artinian* if R_R (${}_RR$) has d.c.c. By saying that R is a *Noetherian (Artinian) ring* we mean that R is left and right Noetherian (Artinian).

The ring R is said to be *simple* if $R \neq 0$ and R has no two sided ideals other than 0 and R. If R is a finite direct sum of simple Artinian rings, then R is called *semisimple Artinian*.

R is *prime* if the product of two non-zero ideals of R is non-zero. An ideal N of R is *nilpotent* if $N^k = 0$ for some positive integer k. The ring R is *semiprime* if it has no non-zero nilpotent ideals. An ideal P of R is *prime (semiprime)* if R/P is a prime (semiprime) ring. The *prime spectrum* $\mathrm{Spec}(R)$ of R is the set of all prime ideals of R. It is partially ordered by set inclusion. Let P be a prime ideal of R and X a subset of R contained in P. Then P is said to be *minimal over* X if there is no prime ideal Q of R with $X \subseteq Q \subset P$. A *minimal prime ideal* of R is a prime ideal which does not properly contain any prime ideal of R. In a right Noetherian ring R there exist only finitely many minimal prime ideals. Their intersection is nilpotent and coincides with the *nilpotent radical* of R, which is the sum of all nilpotent right ideals of R, or, equivalently, the sum of all nilpotent left ideals of R. The nilpotent radical of a ring R will be denoted by $N(R)$ or simply by N.

R is called *indecomposable* if there do not exist non-zero ideals I_1 and I_2 of R, with $I_1 + I_2 = R$ and $I_1 \cap I_2 = 0$. Hence a prime ring is indecomposable.

Let S be a non-empty subset of M_R. We write $\mathrm{ann}_R(S) = \{x \in R \mid Sx = 0\}$ for the *annihilator* of S. If S is a submodule of M then $\mathrm{ann}_R(S)$ is a two sided ideal of R. If $M = R$ then we need to specify whether we are taking the annihilator on the right or on the left of S. In the first case we write $\mathrm{r.ann}_R(S)$, in the second case $\mathrm{l.ann}_R(S)$ for the set $\{x \in R \mid xS = 0\}$. A right R-module M is *faithful* if $\mathrm{ann}(M) = 0$.

When $S = \{s\}$, we use the notation $r(s)$ and $l(s)$, respectively, for the right and left annihilator of S. An element a of R is called *right (left) regular* if $r(a) = 0$ ($l(a) = 0$). Let I be an ideal of R. We write $\mathcal{C}(I)$ for the set of elements $r \in R$ such that $r + I$ is regular in the ring R/I. In particular, $\mathcal{C}(0)$ is the set of all regular elements of R.

Let M be a right R-module and I a non-zero ideal of R. By $\mathrm{ann}_M(I)$ we denote the submodule $\{m \in M \ |mI = 0\}$ of M, that we call *the annihilator in M of I*.

Suppose that R is a Noetherian ring and M is a finitely generated R-R-bimodule. A *right affiliated sub-bimodule* of M is a non-zero sub-bimodule of M of the form $\text{ann}_M(P)$, where P is an ideal of R, maximal amongst the annihilators of non-zero sub-bimodules of M. It can be shown that such ideals are prime. A *right affiliated series* for M is a chain $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ of sub-bimodules, where for each i with $1 \leq i < n$, M_i/M_{i-1} is a right affiliated sub-bimodule of M/M_{i-1}. If $P_i = \text{r.ann}_R(M_i/M_{i-1})$ then $\{P_1, P_2, \ldots, P_n\}$ is called the set of *right affiliated primes* of M, corresponding to a given affiliated series. It can be shown that $M_i = \text{ann}_M(P_i \ldots P_2 P_1)$, for all i (cf. [58, Lemma 1.2]). Further, any such M has a right affiliated series [24]. Left affiliated series and left affiliated primes are defined similarly.

Let M be a right R-module. Then M is *finite dimensional* if it does not contain an infinite direct sum of submodules. A submodule $U \neq 0$ of M is called *uniform* if every two non-zero submodules U_1 and U_2 of U have non-zero intersection. A submodule E of M is called *essential* in M if E has non-zero intersection with every non-zero submodule of M.

Theorem 1.1.1 [37, Lemma 2.2.8] *Let M_R be a finite dimensional module. Then there exist uniform submodules $U_1, \ldots, U_n$ of M such that $U_1 \oplus \ldots \oplus U_n$ is an essential submodule of M.*

It can be shown that the number n above is an invariant of the module M. This is called the *uniform dimension* of M.

Let R be a ring that is finite dimensional as right R-module and let c be a right regular element of R. Then cR is an essential right ideal of R [37, 2.3].

A ring R is called *right Goldie* if R_R is finite dimensional and R has the ascending chain condition on the right annihilators.

Proposition 1.1.2 [37, Proposition 2.3.5] *In a semiprime right Goldie ring every essential right ideal contains a regular element.*

Let R be a semiprime right Goldie ring and M a right R-module. The set $T = \{m \in M \mid mc = 0, \text{ for some } c \in \mathcal{C}(0)\}$ can be seen to be a submodule of M and is called *the torsion submodule of M*. A module M is called *torsion* if $T = M$ and *torsion-free* in case $T = 0$.

A ring R is called *right bounded* provided that every essential right ideal of R contains a non-zero ideal. A *left bounded* ring is defined similarly. By a *bounded* ring we understand a ring which is left and right bounded. A Noetherian ring R with the property that every prime factor is bounded is called a *fully bounded Noetherian (FBN)* ring. This is a remarkable class of non-commutative rings and includes PI rings. Important results have been obtained for this class.

An important tool in investigation of rings is the notion of *classical Krull dimension*, which is defined as follows:

Take $\mathrm{Spec}_{-1} = \emptyset$ and Spec_{α} be the set of prime ideals P in $\mathrm{Spec}(R)$ such that each prime that properly contains P is in Spec_{β}, for some ordinal $\beta < \alpha$. Thus $\mathrm{cl.K.dim}(R)$ is the smallest ordinal α such that $\mathrm{Spec}_{\alpha} R = \mathrm{Spec}(R)$.

When finite, the classical Krull dimension of R is the length of a maximal chain of prime ideals. Let P be a prime ideal of R. If $P_0 \subset P_1 \subset \ldots \subset P_n = P$ is a maximal chain of prime ideals contained in P, then n is called the *height* of P and it is denoted by $\mathrm{ht}(P)$.

The *(Gabriel-Rentschler) Krull dimension* of a right R-module M is defined as follows:

$\mathrm{K.dim}(0) = -1$ and, inductively, $\mathrm{K.dim}(M)$ is the smallest ordinal α such that for each descending chain $M = M_0 \supseteq M_1 \supseteq \ldots$ there exists an integer n such that $\mathrm{K.dim}(M_{i-1}/M_i) < \alpha$ for all $i > n$.

Theorem 1.1.3 [52, Proposition 3.5.51] *If R is an FBN ring then* $\mathrm{K.dim}(R) = \mathrm{cl.K.dim}(R)$.

Now we recall some concepts of homological algebra.

We say that a short exact sequence of right R-modules

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{1.1}$$

splits if there exists a R-module homomorphism ϕ from C to B such that $g\phi$ is the identity homomorphism of C. Equivalently, the sequence (1.1) splits if and only if A is a direct summand of B.

A right R-module P is *projective* if every short exact sequence as (1.1) with $C = P$ splits. It is easy to show that every finitely generated projective module P is isomorphic to a direct summand of a finitely generated free module. Projective modules are characterized by the following important property known as the Dual Basis Lemma.

Theorem 1.1.4 [37, Lemma 3.5.2] *Let P be a right R-module. Then P is projective if and only if there exist $\{a_\alpha\}$, $\{\phi_\alpha\}$, with $a_\alpha \in P$, $\phi_\alpha \in \mathrm{Hom}(P, R)$ such that for all $a \in P$, we have $a = \Sigma_\alpha a_\alpha(\phi_\alpha(a))$, where $\phi_\alpha(a) = 0$ for all except a finite number of α's.*

A direct sum $\bigoplus_i P_i$ of R-modules is projective if and only if each P_i is projective (cf. [1, Lemma 2.8.4]).

A right R-module M is said to have *finite projective dimension* if there exists an exact sequence

$$0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0 \tag{1.2}$$

where each P_i, $i = 0, \ldots, n$, is a projective module. We call (1.2) a *projective resolution* for M. The *projective dimension* of M is the least integer n for which a projective resolution as (1.2) exists. In fact, it can be shown, using Schanuel's Lemma [37, 7.1.2], that this number is independent of the particular exact sequence chosen.

The *right global dimension* of a ring R, denoted by $\mathrm{rgl.dim}(R)$, is the supremum of the projective dimensions of right R-modules, one from each isomorphism class. The *left global dimension* of R ($\mathrm{lgl.dim}(R)$) is defined analogously. For Noetherian rings $\mathrm{lgl.dim}(R) = \mathrm{rgl.dim}(R)$ [37, 7.1.11], so we use the notation $\mathrm{gl.dim}(R)$ in this case.

If R has right (left) global dimension equal to one then R is called *right (left) hereditary*.

Definition 1.1.5 Let F be a left R-module.

- F is *flat* if whenever $f : N_1 \to N_2$ is a monomorphism we have that $f \otimes_R 1_F : N_1 \otimes_R F \to N_2 \otimes_R F$ is a monomorphism.
- F is *faithfully flat* if F is flat and, for any N_R, $N \otimes_R F = 0$ implies $N = 0$.

In applications we shall use the following properties.

Theorem 1.1.6 [52, Theorem 2.11.14] *The following are equivalent for a left R-module F.*

1. *F is flat.*
2. *For any right ideal I of R the map $\phi : I \otimes_R F \to IF$ defined by $\phi(x \otimes f) = xf$ is an isomorphism.*

Lemma 1.1.7 [37, 7.2.3]

1. *A projective right R-module is flat.*
2. *A non-zero right free module is projective and faithfully flat.*

A right R-module E is called *injective* if for any homomorphism $f : M \to E$ and any monomorphism $g : M \to T$ there is a homomorphism $h : T \to E$ such that $f = hg$.

Theorem 1.1.8 (Baer) [52, Proposition 2.10.3'] *A right R-module E is injective if and only if for every right ideal I of R, each homomorphism $g : I \to E$ can be extended to a homomorphism $f : R \to E$.*

Let M be a right R-module. A *proper essential extension* of M is any module T such that $M \neq T$ and M is an essential submodule of T.

Thus a module E is injective if and only if it has no proper essential extensions [24, Proposition 4.6]. This characterization motivates the following definition.

An *injective hull (or injective envelope)* of M is an injective right R-module which is an essential extension of M. The injective hull is unique up to isomorphism and it is denoted by $E(M)$. In particular, if $E(R_R) = R$, then R is called *right selfinjective.*

Lemma 1.1.9 [24, Lemma 4.10] *A non-zero right R-module U is uniform if and only if $E(U)$ is indecomposable.*

1.2 Semilocal and semiperfect rings

An ideal P of a ring R is *right (left) primitive* if $P = \mathrm{ann}_R(S)$ for some simple right (left) R-module S. A *maximal right (left) ideal* of R is a right (left) ideal of R which is not properly contained in another proper right (left) ideal of R. Every right or left primitive ideal in a ring R is a prime ideal. Every maximal ideal of R is a right and left primitive ideal.

Proposition 1.2.1 [24, Proposition 8.4] *Let R be a right fully bounded Noetherian ring. If P is a right primitive ideal of R then R/P is a simple Artinian ring.*

The following corollary will be frequently used in the sequel.

Corollary 1.2.2 *Let R be an FBN ring and let $X = M_1M_2 \ldots M_n$ be a product of maximal ideals of R. Then the factor R/X is an Artinian ring. Further, every module of finite length is annihilated by a finite product of maximal ideals of R.*

Proof. By Proposition 1.2.1, each of the rings R/M_i is a simple Artinian ring for all i. Let us consider the chain $R \supseteq M_1 \supseteq M_1M_2 \supseteq \ldots \supseteq M_1M_2 \ldots M_n$. Each subfactor $M_1 \ldots M_i/M_1 \ldots M_iM_{i+1}$ is a finitely generated R/M_{i+1}-module, and so is semisimple. Hence R/X has finite length. The rest is clear.

□

The *Jacobson radical* of R, denoted by $J(R)$ or simply by J, is the intersection of all maximal right ideals of R. Equivalently, $J(R)$ is the intersection of all maximal left ideals of R. Hence, by Proposition 1.2.1, the Jacobson radical of an FBN ring is the intersection of its maximal ideals.

The following theorem is known as Nakayama's Lemma.

Theorem 1.2.3 [52, Proposition 2.5.24] *Let M be a non-zero finitely generated right R-module. Then:*

1. *$M \neq MJ$.*

2. *If N is a submodule of M and $M = MJ + N$ then $N = M$.*

Let M be a module of finite length over a semilocal ring. The *Loewy length* of M is defined to be the smallest integer n such that $MJ^n = 0$.

Definition 1.2.4 Let R be a ring.

- R is called *local* if R/J is a simple Artinian ring.

- If R/J is a division ring then R is called *scalar local*.

- R is a *semilocal ring* if R/J is a semisimple Artinian ring. If J is nilpotent then R is said to be *semiprimary*.

Thus in a local ring the Jacobson radical is the unique maximal ideal. If R is right Artinian ring then $J(R)$ is nilpotent.

Theorem 1.2.5 [52, Theorem 2.7.2] *A ring R is right Artinian if and only if R is right Noetherian and semiprimary.*

Theorem 1.2.6 [24, Theorem 8.12] *An FBN ring satisfies the Jacobson's conjecture, i.e. $\bigcap_{n=1}^{\infty} J^n = 0$.*

An ideal I of a ring R is said to have the *right Artin Rees property* if for every right ideal K of R there exists a positive integer n such that $K \cap I^n \subseteq KI$.

Proposition 1.2.7 [13, Corollary 11.3] *Let R be an FBN ring with Jacobson radical J. Suppose that R/J is Artinian. Then J has the right AR-property.*

An element e of R with the property $e^2 = e$ is called *idempotent*. The *trivial idempotents* of R are 0 and the unit element. Two idempotent elements e_1 and e_2 of R are *orthogonal* if $e_1e_2 = e_2e_1 = 0$. A *primitive* idempotent has the property that it cannot be written as a sum of two non-zero idempotents. The Jacobson radical of a ring R does not contain non-zero idempotents. If e is an idempotent of R then eRe is a subring of R having e as a unit element and $J(eRe) = eJe$ (cf. [52, Proposition 2.5.14]). Further, every ideal of eRe is equal to eIe, for some I ideal of R.

Lemma 1.2.8 *Let R be an FBN ring, e an idempotent of R and eIe a maximal ideal of eRe. Then $eIe = ePe$, for some P primitive ideal of R.*

Proof. By Proposition 1.2.1 all primitive ideals of R are maximal ideals. Suppose that I is not a maximal ideal of R. Then there exists a primitive ideal M/I of R/I which is the annihilator of a subfactor eR/K of eR/eI. Hence $eMe \subseteq Ke \neq eRe$ and so M is a maximal ideal of R with $eMe = eIe$.

□

Later, we shall see that the ring eRe inherits many properties from R.

An ideal $I \subseteq J(R)$ of a ring R is said to be *idempotent lifting* if every idempotent element of R/I can be written as $e + I$, with e an idempotent of R.

Lemma 1.2.9 [13, pp 79, Remark (1)] *A nilpotent ideal of a ring R is idempotent lifting.*

A *complete set of primitive idempotents* of a ring R is a set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents of R with $1 = \Sigma_{i=1}^n e_i$.

Definition 1.2.10 A semilocal ring is said to be *semiperfect* if $J(R)$ is idempotent lifting.

Right Artinian and scalar local rings are semiperfect.

Lemma 1.2.11 [1, Corollary 27.9] *A factor ring of a semiperfect ring is semiperfect.*

Proposition 1.2.12 [52, Proposition 2.7.21] *Suppose that R is a semiperfect ring and $R/J \simeq M_n(D)$, where D is a division ring. Then $R \simeq M_n(T)$, where T is a scalar local ring with $T/J(T) \simeq D$.*

The following straightforward corollary will be used in later chapters.

Corollary 1.2.13 *Let Q be an Artinian local ring. Suppose that e_iQe_i is a division ring, where e_i is a primitive idempotent of Q. Then Q is a simple Artinian ring.*

Proof. By Proposition 1.2.12, Q is isomorphic to a matrix ring over a scalar local ring T. Hence $e_iQe_i \simeq T$. It follows that T is a division ring and hence Q is simple Artinian.

□

Semiperfect rings share many properties with the Artinian ones. One is the following.

Proposition 1.2.14 [52, Proposition 2.7.20] *A semiperfect ring R has a complete set of primitive idempotents $e_1 \ldots, e_n$ such that the ring e_iRe_i is local for all i.*

Definition 1.2.15 A set $\{e_1, \ldots, e_k\}$ of primitive orthogonal idempotents of R is said to be *basic* if for every primitive idempotent f of R, we have $fR \simeq e_iR$, for exactly one e_i, $1 \leq i \leq k$. In this case $e = \Sigma_{i=1}^k e_i$ is called a *basic idempotent* and eRe is called a *basic subring* of R.

The following holds for semiperfect rings.

Theorem 1.2.16 [52, Theorem 2.7.30] *A semiperfect ring has a basic subring eRe with the property that $eRe/J(eRe)$ is a direct sum of division rings.*

Definition 1.2.17 Let M be a right R-module. A *projective cover* of M is an epimorphism $f : P \to M$ such that P is a projective module and $\ker f$ is "small", i.e. for every proper submodule T of M, we have $T + \ker f \neq M$.

Semiperfect rings are also characterized by the property that every finitely generated module over them has a projective cover [52, Theorem 2.8.40].

1.3 Quotient rings and the Artin radical

Definition 1.3.1 Let $\mathcal{S}$ be a multiplicatively closed subset of a right Noetherian ring R.

- $\mathcal{S}$ is called *right Ore set* if for any given $a \in R$ and $c \in \mathcal{S}$ there exist $b \in R$ and $d \in \mathcal{S}$ such that $ad = cb$. A *left Ore set* is defined dually.
- A ring $R_{\mathcal{S}}$ is called a *right localisation* for R if there exists a ring homomorphism $\phi : R \to R_{\mathcal{S}}$ such that:
  1. $\ker \phi = \{r \in R \mid rc = 0 \text{ for some } c \in \mathcal{S}\}$,
  2. $\phi(c)$ is a unit of $R_{\mathcal{S}}$, for all $c \in \mathcal{S}$ and
  3. every element of $R_{\mathcal{S}}$ has form $\phi(a)\phi(c)^{-1}$, with $a \in R$ and $c \in \mathcal{S}$.

  So R is a subring of $R_{\mathcal{S}}$ if and only if $\ker \phi = 0$.

The following is a generalization of Ore's Theorem.

Theorem 1.3.2 (Ore) [37, Chapter 2, Section 1] *The right localisation of R at $\mathcal{S}$ exists if and only if $\mathcal{S}$ is a right Ore set.*

Localisation on the left clearly requires $\mathcal{S}$ be a left Ore set. If both conditions hold, then the respective localised rings are isomorphic and in fact can be identified.

Definition 1.3.3 An ideal I of R is called *right localisable* if the set of elements of R which are right regular modulo I is a right Ore set.

In particular, if P is a prime right localisable ideal of R then the right localised ring R_P is a local ring with maximal ideal PR_P.

Definition 1.3.4 Suppose that $\mathcal{S} = \mathcal{C}(0)$ is a right Ore set. Then $R_{\mathcal{S}}$ is called the *right quotient ring* for R and it is usually denoted by $Q(R)$. The ring R is also referred to as *right order* in $Q(R)$.

The next result tells us when the quotient ring of R is Artinian.

Theorem 1.3.5 (Small) [37, Corollary 4.1.4] *A right Noetherian ring has a right Artinian right quotient ring if and only if* $\mathcal{C}(N) = \mathcal{C}(0)$.

In particular the following holds.

Theorem 1.3.6 (Goldie) [37, Theorem 2.3.6] *R has a semisimple Artinian right quotient ring if and only if R is a semiprime right Goldie ring.*

The *Artin radical* of a ring R, denoted by $A(R)$, is the sum of all Artinian right ideals of R. The following theorem summarizes some of the main properties of the Artin radical in a Noetherian ring.

Theorem 1.3.7 [13, Theorems 4.9 and 4.11] *Let R be a Noetherian ring and $A(R)$ its Artin radical. Then:*

1. *$A(R)$ is a two sided ideal of R and is also the sum of all the Artinian left ideals of R.*

2. *$R/A(R)$ has no non-zero Artinian one sided ideals.*

3. *$A(R) \subseteq N$ if and only if $A(R/N) = 0$.*

In Chapter 4 we shall use the following theorem.

Theorem 1.3.8 [13, Theorem 4.14] *Let R be a Noetherian ring which has an Artinian quotient ring, then $A(R)$ is a direct summand of R (as an ideal).*

There is also a converse in a special case.

Theorem 1.3.9 [13, Theorem 5.6] *Let R be a Noetherian ring of Krull dimension one and suppose that $A(R) = 0$, then R has an Artinian quotient ring.*

Lemma 1.3.10 [13, Lemma 5.7] *Let R be a Noetherian ring of Krull dimension one and suppose that $A(R) = 0$. Let K be a right ideal of R. Then R/K is an Artinian right R-module if and only if K contains a regular element.*

We shall need the following two results in Chapter 5.

Theorem 1.3.11 [58, Proposition 1.3] *Let R be a Noetherian ring. Choose a left and a right affiliated set of primes for R; say $\{P_i : 1 \leq i \leq n\}$ and $\{P'_j : 1 \leq j \leq m\}$ respectively. Then R has an Artinian quotient ring if and only if each P_i and P'_j is a minimal prime.*

Lemma 1.3.12 [58, Lemma 2.3] *Let R be a Noetherian ring. Then there exists a right affiliated series $R = B_m \supset \ldots \supset B_0$ such that $B_r = A(R)$, for some r and B_i/B_{i-1} is Artinian if and only if $i \leq r$.*

1.4 Localisation at clique

In a non-commutative ring localisation at a prime ideal is often impossible. When this happens we may attempt to localise at a certain collection of prime ideals which are "linked" by a property that we define below.

In this section, R denotes a Noetherian ring.

Definition 1.4.1 Let P and Q be prime ideals of R. We say that there is a *right second layer link* from P to Q, written $P \rightsquigarrow Q$, if there is an ideal A of R with $PQ \subseteq A \subset P \cap Q$ such that $P \cap Q/A$ is torsion-free as a right R/Q-module and as left R/P-module.

Definition 1.4.2 Let P be a prime ideal of R. A right R-module M is said to be *P-prime* if P is the right annihilator of every finitely generated submodule of M.

Links between prime ideals arise from the existence of some non-splitting sequences, as the following theorem shows.

Theorem 1.4.3 [30, 6.1.2, 6.1.3] *Let P and Q be prime ideals of R such that $P \not\subseteq Q$. Then $P \rightsquigarrow Q$ if and only if there is an exact sequence of right R-modules*

$$0 \to U \to M \to V \to 0$$

with U, M, V finitely generated uniform, such that U is Q-prime, V is P-prime and for all submodules M' of M not contained in U, $\mathrm{ann}(M') = \mathrm{ann}(M) = A$. Further, U is R/Q-torsionfree and V is R/P-torsionfree.

The reason for the name "right" link is due to the fact that in Theorem 1.4.3 only right modules are used.

Definition 1.4.4 The *graph of links* of R is a directed graph whose vertices are in one-to-one correspondence with the elements of $\mathrm{Spec}(R)$ and the arrows are given by the second layer links. A subset of $\mathrm{Spec}(R)$ which corresponds to the set of vertices of a connected component of this graph is called a *clique*.

So every prime ideal P of R belongs to just one clique, which is denoted by $\mathrm{Cl}(P)$.

Theorem 1.4.5 [61, Theorem 7] *Any clique of prime ideals of R is at most countable.*

The following theorem explains why links are an obstruction to localisation at a single prime ideal.

Theorem 1.4.6 [24, Lemma 12.17] *Let P and Q be prime ideals of R with $P \rightsquigarrow Q$ and $\mathcal{S}$ be a multiplicatively closed right Ore subset of R. Then $\mathcal{S} \subseteq \mathcal{C}(P)$ if $\mathcal{S} \subseteq \mathcal{C}(Q)$, where the equivalence holds if $\mathcal{S}$ is a (two sided) Ore set.*

Let X be a collection of prime ideals of R and set $\mathcal{C}(X) = \cap\{\mathcal{C}(P) \mid P \in X\}$. From Theorem 1.4.6 it follows that $\mathcal{C}(X)$ is the largest subset of $\mathcal{C}(P)$, for $P \in X$, we can consider as possible Ore set.

Definition 1.4.7 A clique X of prime ideals R is said to be *classically localisable* if:

- $\mathcal{C}(X)$ is a right and a left Ore set, so that the localisation R_X of R at $\mathcal{C}(X)$ exists;
- for every prime P, $P \in X$, the factor ring R_X/PR_X is isomorphic to the quotient ring of R/P;
- the prime ideals PR_X, $P \in X$, are precisely the primitive ideals of R_X;

- the Jacobson radical of R_X has the Artin Rees property.

The last condition implies that $\bigcap_{n=1}^{\infty} J(R_X)^n = 0$.

For FBN rings there is a sufficient condition for localisability at clique.

Theorem 1.4.8 [40, Theorem 5] *In an FBN ring any finite clique is classically localisable.*

In this case, the localised ring is Noetherian and semilocal. Further, the global dimension of an FBN ring depends just on the projective dimensions of the simple modules, as the following theorem shows.

Theorem 1.4.9 [43, Corollary 8] *Let R be a Noetherian FBN ring such that every clique X of maximal ideals is finite. Then:*

$$\text{gl.dim}(R) = \sup_X \{\text{gl.dim}(R_X)\}. \tag{1.3}$$

The graph of links of some FBN rings, namely Noetherian PI rings, will be discussed in greater detail in Chapter 3.

1.5 PI rings

The results presented in this section are taken from [37]. The reader is therefore referred to that book for more details.

A ring R is said to satisfy a *polynomial identity* if there is a non-trivial polynomial f in the free algebra $\mathbb{Z} < X_1, \ldots, X_n >$, for some n, such that $f(r_1, \ldots, r_n) = 0$, for all $r_i \in R$. Thus R is called a *PI ring*. For the free algebra $\mathbb{Z} < X_1, \ldots, X_n >$ we shall use the shorthand notation $\mathbb{Z} < X >$.

1.5.1 Some properties

Subrings and homomorphic images of PI rings are PI. Commutative rings are clearly PI rings since they satisfy the identity $X_1 X_2 = X_2 X_1$. In fact, PI ring theory, especially in the Noetherian case, can be thought of as a generalization of commutative rings. PI rings are closely related to their centres. Any ring which is finitely generated as a module over its centre is a PI ring [37, Corollary 13.1.13]. Further, prime and, more generally, semiprime PI rings embed in matrix rings over a commutative ring. As a consequence, every two sided ideal of a semiprime PI ring R contains a central element [37, Theorem 13.6.4]. A Noetherian PI ring is FBN.

A PI ring R is said to have *minimal degree* d if d is the least possible integer of a monic polynomial identity of R.

Theorem 1.5.1 (Kaplansky) [37, Theorem 13.3.8] *A primitive PI ring of minimal degree d is a central simple algebra of dimension $n^2 = (d/2)^2$ over its centre.*

Theorem 1.5.2 (Posner) [37, Theorem 13.6.5] *Let R be a prime PI ring with centre C. Let $S = C - \{0\}$, $Q = R_S$ and K the field of fractions of C. Then $Q = Q(R)$ is a simple central algebra with centre K, R is an order in Q and $Q = RK$.*

We say that R has *PI degree* n, abbreviated $\mathrm{PIdeg}(R) = n$, if n^2 is the dimension of $Q(R)$ over its centre. It can be shown that n is the minimal size of $n \times n$ matrices over a commutative ring in which R can be embedded [37, Corollary 13.6.7].

Theorem 1.5.3 [37, Theorem 136.10] *Let R be a prime PI ring with centre C and* $\mathrm{PIdeg}(R) = n$. *Then there is a C-submodule F of R which is free of rank n^2, together with an R-monomorphism $\phi : R_R \to R_R$ such that $\phi(R) \subseteq F$.*

Proof. The complete proof can be read in [37]. We just say that there exist $v_1, v_2, \ldots, v_t$ elements of R and $c \in C$ such that $cR \subseteq \bigoplus_{i=1}^{t} Cv_i$. So the monomorphism ϕ is the multiplication by c.

□

From the above the following useful fact can be deduced.

Proposition 1.5.4 [37, Proposition 13.6.11] *Let R be a prime PI ring with centre C. Then the following are equivalent.*

1. *C is Noetherian.*

2. *R is right Noetherian and is finitely generated as a module over its centre.*

Let k be a field. A ring R is said to be *affine* if it is finitely generated as an algebra over k. Affine PI rings have some nice properties that we summarize in the following theorem.

Theorem 1.5.5 [37, Theorem 13.10.3] *Let R be a semiprime affine PI k-algebra. Then:*

1. *R has finitely many minimal prime ideals.*

2. *Finite length R-modules are finite dimensional over k.*

3. *R has zero Jacobson radical.*

Let C be the centre of R. A polynomial $0 \neq g \in \mathbb{Z} < X >$ with the property $g(R) \subseteq C$ is called a *central polynomial* for R. Every prime PI ring has a central polynomial g_n, where $n = \text{PIdeg}(R)$.

There are some special prime PI rings called Azumaya algebras. For definition and details the reader is referred to [37, 13.7]. For our purpose we just need to know that these rings are Noetherian and finitely generated as modules over their centres and arise from a prime PI ring R in the following way.

Let $0 \neq a \in g_n(R)$ be not unit and set $\mathcal{S} = \{a^n \mid n \in \mathbb{N}\}$. Thus $\mathcal{S}$ is a multiplicatively closed subset of R, with respect to which R can be localised. We denote by $R[a^{-1}]$ the localised ring $R_{\mathcal{S}}$. It can be shown that $R[a^{-1}]$ is an Azumaya algebra (cf. [37, Theorems 13.7.14 and Proposition 13.7.4]). Such a ring turns out to be particularly useful in case R is a prime affine PI ring, as we shall see in Chapter 6.

1.5.2 Extensions of PI rings

Let R and S be rings, such that R is a subring of S with the same identity as S. Then S is called a *finite centralizing extension* of R if S_R is finitely generated and each element of the finite set of generators commutes with every $r \in R$.

Let R be a ring and C' be a central subring of R. An element r of R is said to be *integral over C'* if r satisfies a monic polynomial with coefficients in C'. If this holds for every $r \in R$, then R is said to be an *integral extension* of C'. In the particular case when R is the field of fractions of an integral domain C, the ring C is called *integrally closed* in R if the elements of R which are integral over C all belong to C.

Definition 1.5.6 Let R be a Noetherian prime PI ring. Then R is said to be *centrally integrally closed* if the elements of K, the field of fractions of the centre C of R, which satisfy a monic polynomial with coefficients in R all belong to C.

Theorem 1.5.7 [37, Theorem 13.8.14] *Let R be a PI ring and C' a central subring of R such that R is integral over C'. Then the following hold.*

1. *(Lying Over) For any $p \in \operatorname{Spec}(C')$ there exists $P \in \operatorname{Spec}(R)$ such that $P \cap C' = p$.*

2. *(Going Up) If $p, q \in \operatorname{Spec}(C')$ with $p \subseteq q$ and $P \in \operatorname{Spec}(R)$ with $P \cap C' = p$, then there exists $Q \in \operatorname{Spec}(R)$ with $P \subseteq Q$ and $Q \cap C' = q$.*

3. *(Incomparability) If $P, Q \in \operatorname{Spec}(R)$ with $P \subset Q$ then $P \cap C' \subset Q \cap C'$.*

It can be shown that Theorem 1.5.7 holds also for finite centralizing extensions [54, Theorem 1].

1.5.3 Completion of semilocal PI rings

We introduce the J-adic completion of a Noetherian semilocal PI ring R and show some of its properties. Furthermore, we study the relationship between the J-adic completion of a Noetherian prime PI ring and its centre by comparing it with the case where R is finitely generated as a module over its local centre. The J-adic completion of R is an important tool in the proof of the main result of this thesis, as our problem can be solved more easily for this ring.

For a general treatment on J-adic completions of Noetherian rings, the reader is referred to the book [36].

Let R be a Noetherian semilocal PI ring. For a finitely generated right R-module M we can define the *J-adic topology* by taking for any $m \in M$ the set $\{m + MJ^n\}_{n \in \mathbb{N}}$ as a basis of neighborhoods of m. In M the addition is a continuous map, hence M is a topological group. Further, the J-adic topology of M is Hausdorff. For all $m \in M$ and $n \in \mathbb{N}$, $m + MJ^n$ is an open set in the J-adic topology of M. So M/MJ^n is discrete in the quotient topology.

The *J-adic completion* of M, that is denoted by $\hat{M}$, is a topological subspace of $\Pi_n M/MJ^n$, considered with the product topology, and it is described as

$$\hat{M} = \left\{(m_1 + MJ, m_2 + MJ^2, \dots) \mid m_i \in M \text{ and } m_{i+1} - m_i \in MJ^i\right\}.$$

In the literature, the completion of M is also denoted by $\varprojlim M/MJ^n$ [52, pp 94]. The module $\hat{M}$ is complete with respect to the J-adic topology. A submodule H of $\hat{M}$ has its J-adic topology and the topology induced by $\hat{M}$, whose basis of neighborhoods of zero is $\{H \cap MJ^n\}_{n\in\mathbb{N}}$. By using the AR-property of J (cf. Proposition 1.2.7), it is easy to show that these two topologies are the same.

The J-adic completeness of M can also be expressed by the property that for every sequence $m_1, m_2, \ldots$ of elements of M such that $m_i - m_{i+1} \in MJ^i$, there exists a unique $m \in \hat{M}$ such that $m - m_i \in MJ^i$, for all i, i.e. every Cauchy sequence in $\hat{M}$ converges.

The *closure of* H *in* M is defined to be the submodule $\mathrm{cl}_M(H) = \bigcap_{n=1}^{\infty}(H + MJ^n)$ of M.

In particular, we consider $M = R$. In R the multiplication is also a continuous map. So R is a topological ring and $\hat{R}$ is a ring. Let $\psi : R \to \hat{R}$ be the natural homomorphism. Then $\ker \psi = \bigcap_{n=1}^{\infty} J^n = 0$ (cf. Theorem 1.2.6) and so $\hat{R}$ is an over ring of R. We therefore identify R with its image in $\hat{R}$. The map ψ is continuous and R is dense in $\hat{R}$. Thus R is complete if and only if ψ is an isomorphism. Some important properties of $\hat{R}$ are listed in the following theorems.

Theorem 1.5.8 [7] *Let R be a Noetherian semilocal ring with $\bigcap_{n=1}^{\infty} J^n = 0$. Then the following hold.*

1. *$\hat{R}$ is a semiperfect ring, with Jacobson radical $\hat{J}$ and $R/J^n \simeq \hat{R}/\hat{J}^n$, for all n.*
2. *If K is a right ideal of R, then $\mathrm{cl}_{\hat{R}}(K) = K\hat{R}$.*
3. *If I is an ideal of R, then $\hat{I} = \hat{R}I\hat{R} = I\hat{R}$.*
4. *$\widehat{(R/I)} \simeq \hat{R}/I\hat{R}$ as rings and the completion $\widehat{(R/I)}$ is Hausdorff.*

Theorem 1.5.9 [59] *Let R be a Noetherian semilocal PI ring. Then the following hold.*

1. *$\hat{R}$ is a Noetherian PI satisfying all the identities of R.*
2. *$\hat{R}$ is complete with respect to the $\hat{J}$-adic topology.*
3. *$\hat{R}$ is a faithfully flat extension of R.*
4. *$M \otimes_R \hat{R} \simeq \hat{M}$, for every finitely generated right R-module M.*

An useful corollary is the following.

Corollary 1.5.10 *If $c \in R$ is right regular, then c is right regular as an element of $\hat{R}$.*

Proof. Since c is right regular in R, it is easy to show that $c\hat{R} = \widehat{(cR)}$. By (4.) of Theorem 1.5.9 we have $\widehat{(cR)} \simeq cR \otimes_R \hat{R}$. As $R \simeq cR$, via the left multiplication map by c, from (3.) of Theorem 1.5.9 it follows that $\hat{R} \simeq cR \otimes_R \hat{R}$ via the map $c \otimes 1$. So c is right regular in $\hat{R}$.

□

Now suppose that C is Noetherian and local, with maximal ideal m. So $J \cap C = m$ and $m^n R \subseteq J^n$, for all n. It follows that $\bigcap_{n=1}^{\infty} m^n R = 0$. Further, as $R/m^n R$ is Artinian, $J^k \subseteq m^n R$ for some k. This shows that in R the J-adic topology is the same as the topology defined by $\{m^n R\}_{n \in \mathbb{N}}$. Let $\hat{C}$ be the m-adic completion of C. By Theorem 1.5.9 (4.), the m-adic completion $\hat{R}_m$ of R, considered as a C-module, is isomorphic to $R \otimes_C \hat{C}$. It is not difficult to show that $\hat{R}_m$ is a ring and it is isomorphic to the mR-adic completion of R. Therefore, $\hat{R}_m$ is also isomorphic to the J-adic completion of R. Hence $\hat{R}_m$ is a semiperfect ring. We also have

Proposition 1.5.11 [20, Lemme 3] $C(\hat{R}_m) \simeq C \otimes_C \hat{C} \simeq \hat{C}$.

In other words, if R is complete with respect to the J-adic topology and C is local Noetherian with maximal ideal m, then C is complete with respect to the m-adic topology.

As there are many Noetherian prime PI rings, which are not finitely generated as modules over their centre, we would like to investigate whether anything can be said about the centre of $\hat{R}$, if C is local. Suppose, further, that all maximal ideals of R are in the same clique, so that they have the same intersection with C (see [24]) and therefore $J \cap C = m$. In general, even when $R/m^n R$ is Artinian for all n, the topologies in C defined by $\{m^n R \cap C\}_{n \in \mathbb{N}}$ and $\{m^n\}_{n \in \mathbb{N}}$ are not necessarily the same. Also, even in the commutative case, the primeness of R is not usually inherited by $\hat{R}$. However, there is still a nice relationship between $\hat{R}$ and its centre that we now describe.

In Subsection 1.5.1 we have seen that a prime PI ring R embeds in a free C-submodule of R and a consequence is that if C is Noetherian then R is finitely generated as a C-module.

We assume that $R/m^k R$ are Artinian for all k, so that the J-adic and the mR-adic topologies of R are the same. All these assumptions are satisfied if, for example, R is integral over its centre. Let C^* be the closure of C in $\hat{R}$ and $v_1, \ldots, v_t$ elements of R as in the proof of Theorem 1.5.3. We have

Lemma 1.5.12 *1.* $C^* = \varprojlim C/(m^k\hat{R} \cap C)$;

2. $\mathrm{cl}_{\hat{R}}(\bigoplus_{i=1}^t Cv_i) = \sum_{i=1}^t \mathrm{cl}_{\hat{R}}(Cv_i)$, *and such a sum is direct.*

Proof. (1.) It can be proved literally in the same way as an argument of Braun [7, Lemma 4].

(2.) First we show that the sum of the RHS is direct. In fact, for all k we have

$$(Cv_i + m^k\hat{R}) \cap (Cv_1 + \ldots + Cv_{i-1} + Cv_{i+1} + \ldots + Cv_t + m^k\hat{R}) \subseteq m^k\hat{R}.$$

It is clear that the RHS is contained in the LHS of (2.).

Let $x \in \mathrm{cl}_{\hat{R}}(\bigoplus_{i=1}^t Cv_i)$. Then for all n there exists $c_{n1}v_1 + \ldots + c_{nt}v_t \in \bigoplus_{i=1}^t Cv_i$ such that $x \in c_{n1}v_1 + \ldots + c_{nt}v_t + m^n\hat{R}$. So $x \in (c_{n1}v_1 + m^n\hat{R}) + \ldots + (c_{nt}v_t + m^n\hat{R})$. Therefore $x \in \bigoplus_{i=1}^t \mathrm{cl}_{\hat{R}}(Cv_i), v_i \in R$.

□

Lemma 1.5.13 *There exists a monomorphism of right $\hat{R}$-modules $\hat{\phi} : \hat{R} \to \hat{R}$ such that $\hat{\phi}(\hat{R})$ is contained in a finitely generated free C^*-submodule of $\hat{R}$.*

Proof. Let c and $v_1, \ldots, v_t$ be elements of R as in Theorem 1.5.3. So $cR \subseteq \bigoplus_{i=1}^t Cv_i$. By Lemma 1.5.12, we have

$$c\hat{R} = \mathrm{cl}_{\hat{R}}(cR) \subseteq \mathrm{cl}_{\hat{R}}(\oplus_{i=1}^t Cv_i) = \oplus_{i=1}^t C^*v_i.$$

Since $c\hat{R} \simeq \hat{R}$ the result follows.

□

As C^* is contained in the centre $C(\hat{R})$ of $\hat{R}$, the above shows that in fact $\hat{R}$ embeds in a free $C(\hat{R})$-submodule. This property can be useful, for example in case of Noetherian $C(\hat{R})$, to conclude that $\hat{R}$ is finitely generated over $C(\hat{R})$. Indeed it can be shown that this is the situation for the ring described in Lemma 4.1.6 if the ring R of Proposition 4.1.3 is prime and integral over its centre.

We end this section by quoting an important result of Müller.

Theorem 1.5.14 [39, Theorem 4] *Let R be a Noetherian ring and $S = P_1 \cap \ldots \cap P_n$ a localisable semiprime ideal of R such that the Jacobson radical of the localised ring R_S has the Artin Rees property. Then there is a one-to-one correspondence between the central idempotents of $\hat{R}_S$ and the localisable subsets of $\{P_1, \ldots, P_n\}$.*

Morita equivalence and the ring eRe

Let M be a right R-module M and set $M^* = \mathrm{Hom}(M_R, R_R)$. Thus M is said to be a *generator* if $M^*M = R$. If M is also finitely generated projective then M is called a *progenerator*. Two rings R and S are said to be *Morita equivalent* if there is a progenerator M_R such that $S \simeq \mathrm{End} M_R$.

In this case, there is a functor $N_R \to (N \otimes_R M^*)_S$ which provides a category equivalence between right R-modules and right S-modules. A certain property is said to be *Morita invariant* if it is preserved under Morita equivalence.

Theorem 1.5.15 [37, Lemma 3.5.8] *The following properties of modules are Morita invariant:*

1. *being Artinian;*
2. *being Noetherian;*
3. *having uniform dimension k;*
4. *being indecomposable;*
5. *being projective;*
6. *being injective.*

Theorem 1.5.16 [37, Proposition 3.5.10] *The following properties of rings are Morita invariant:*

1. *being Artinian;*
2. *being Noetherian;*
3. *being local;*
4. *being semiperfect* (cf. [1, Corollary 27.8])*;*
5. *being prime;*
6. *being semiprime;*
7. *having right global dimension k.*

Proposition 1.5.17 [52, Proposition 4.1.18] *Let R and S Morita equivalent rings. Then there is an isomorphism ϕ between the lattice of ideals of R and the lattice of ideals of S, which preserves inclusion.*

Let R be any ring and e be an idempotent of R. In this thesis we often consider the subring eRe of R. So, for the convenience of the reader, we list some of the properties of R which are inherited by eRe.

Proposition 1.5.18 [52, Lemma 2.7.12] *The following properties pass from R to eRe:*

1. *being right Artinian;*
2. *being right Noetherian;*
3. *being prime;*
4. *being primitive;*
5. *being semiprime;*
6. *being semiperfect;*
7. *having right global dimension 1;*
8. *having an Artinian quotient ring. The quotient ring of eRe is $eQ(R)e$, where $Q(R)$ is the quotient ring of R.*
9. $N(eRe) = eNe$;
10. $J(eRe) = eJe$.

Proof. (8.) is proved in [57, Theorem 3].

$\square$

Chapter 2

Hereditary rings and rings of finite representation type

In this chapter we discuss hereditary rings and their properties. In particular, for PI rings we give some well known characterizations in terms of their factor rings, that are a motivation for our study.

2.1 Hereditary rings and orders

Let R be a Noetherian ring that has a quotient ring $Q(R)$. For an ideal I of R we define:

$$I^{\sharp} = \{q \in Q(R) \mid Iq \subseteq R\} \quad \text{and} \quad I^{*} = \{q \in Q(R) \mid qI \subseteq R\}.$$

The ideal I is said to be *invertible* in $Q(R)$ if $I^{*}I = II^{\sharp} = R$.

Theorem 2.1.1 [37, 5.2.7] *The following are equivalent for a commutative integral domain C.*

1. *Every ideal of C is invertible.*
2. *Every ideal of C is product of prime ideals.*
3. *C is hereditary.*
4. *C is hereditary Noetherian.*

A commutative ring C satisfying the above conditions is called a *Dedekind domain.* If C is also a local principal ideal domain, then C is called a *discrete valuation ring (DVR).*

Definition 2.1.2 A finite dimensional algebra Q over a field k is said to be *separable* if $Q \otimes_k k'$ is semisimple for every finite extension field k' of k.

Lemma 2.1.3 *A simple Artinian PI ring is a separable algebra over its centre.*

Proof. It follows from 1.5.2 and [52, Theorems 2.3.27 and 5.3.24]. □

Definition 2.1.4 Let C' be a DVR, with field of fractions K', and Q a finite dimensional K'-algebra. A *(classical) C'-order* in Q is a subring of Q, having the same unit element as Q, such that R is a finitely generated C'-module and $Q = RK'$.

The structure of hereditary orders over a complete discrete valuation ring is well known:

Theorem 2.1.5 [38, Theorem 6.1] *A Noetherian semiperfect prime ring R is hereditary if and only if R is isomorphic to the ring of $n \times n$ matrices of the following form:*

$$R \simeq \begin{pmatrix} D(m_1 \times m_1) & M(m_1 \times m_2) & M(m_1 \times m_3) & \dots & M(m_1 \times m_k) \\ D(m_2 \times m_1) & D(m_2 \times m_2) & M(m_2 \times m_3) & \dots & M(m_2 \times m_k) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \dots & \ddots & M(m_{k-1} \times m_k) \\ D(m_k \times m_1) & D(m_k \times m_2) & \dots & \dots & D(m_k \times m_k) \end{pmatrix}$$

where D is a scalar local principal ideal domain (not necessarily commutative) with maximal ideal $M \neq 0$ and $n = \Sigma_{j=1}^{k} m_i$, where $D(m_i \times m_j)$ and $M(m_i \times m_j)$ denote the set of all $m_i \times m_j$ matrices over D and M respectively.

Let R and S be orders in a given quotient ring $Q(R) = Q(S)$. Then R and S are *equivalent* if there are units $a, b, c, d \in Q$ such that $aRb \subseteq S$ and $cSd \subseteq R$. An order that it is maximal within its equivalence class is called a *maximal order.*

It can be shown that a prime ring R is a maximal order in $Q(R)$ if every non-zero ideal I of R is invertible [37, Proposition 5.2.6].

Theorem 2.1.6 [37, 5.4] *Let R be a Noetherian hereditary ring. Then:*

1. *R has an Artinian quotient ring;*

2. $\text{K.dim}(R) = 1$*;*

3. *R is a finite direct sum of Artinian hereditary rings and hereditary Noetherian prime rings.*

Thus the theorem above reduces the study of hereditary rings to that of hereditary Noetherian prime rings.

An ideal I of a ring R, with $I^2 = I$ is called an *idempotent ideal.*

Let I be an ideal of a Noetherian prime ring R with quotient ring $Q(R)$. We define

$$O_r(I) \equiv \{q \in Q \mid Iq \subseteq I\} \text{ and } O_l(I) \equiv \{q \in Q \mid qI \subseteq I\}.$$

Theorem 2.1.7 [37, 5.6] *Let R be a hereditary Noetherian prime ring. Then:*

1. *A maximal ideal of R is either idempotent or invertible.*

2. *Any invertible ideal is a commutative product of maximal invertible ideals (i.e. maximal amongst invertible ideals).*

3. *A maximal invertible ideal of R is either a maximal ideal or else it is an intersection of idempotent maximal ideals, $I = M_1 \cap \ldots \cap M_n$ such that*

$$O_r(M_1) = O_l(M_2), \quad O_r(M_2) = O_l(M_3), \ldots, O_r(M_n) = O_l(M_1). \tag{2.1}$$

A set of idempotent maximal ideals of a hereditary Noetherian prime ring R satisfying (2.1) is called a *cycle*.

If R is also bounded, a maximal ideal of R either is invertible and hence localisable (cf. [14, Theorem 1.3]), or idempotent and it belongs to just one cycle. In fact, the cycles described above are exactly the cliques of maximal ideals of R (cf. [43, Propositions 10 and 11]). Hence, the link components of a hereditary bounded Noetherian prime ring are finite and consist of circuits.

A hereditary Noetherian prime ring is called *Dedekind* if it has no non-trivial idempotent ideals.

Theorem 2.1.8 [37, Theorem 5.3.16] *Let C be a Dedekind domain and R a maximal C-order in $Q(R)$, then R is a Dedekind prime ring.*

The converse of the above theorem holds for PI rings and, more in general, the following holds.

Theorem 2.1.9 [37, Theorem 13.9.16] *Let R be a prime hereditary PI ring. Then R is Noetherian, the centre C of R is a Dedekind domain and R is a classical C-order.*

The next property of prime PI rings will be used in Chapter 4.

Theorem 2.1.10 [54, Theorem 3] *Let R be a Noetherian prime PI ring. Then the following are equivalent.*

1. *R is centrally integrally closed* (cf. Definition 1.5.6) *and has Krull dimension one.*

2. *R is finitely generated as a module over its centre which is a Dedekind domain.*

The structure of finitely generated modules over a hereditary Noetherian prime ring is completely described by the following proposition.

Proposition 2.1.11 [37, Lemma 5.7.4] *Let M be a finitely generated right module over a hereditary Noetherian prime ring R. Then:*

1. *$M \simeq \tau(M) \oplus M/\tau(M)$, where $\tau(M)$ is the torsion submodule of M, and hence has finite length, and $M/\tau(M)$ is projective.*

2. *If M is torsion-free then M is isomorphic to a direct sum of uniform right ideals of R. In particular, when R is Dedekind prime then $M \simeq R^n \oplus I$, for some n and a right ideal I of R.*

2.2 Representation theory

This section deals with the representation theory of factor rings of HNP rings.

2.2.1 Serial rings

Definition 2.2.1 Let R be a ring and M a right R-module.

- A right R-module is said to be *uniserial* if whenever A and B are submodules of M, either $A \subseteq B$ or $B \subseteq A$.
- R is called a *right serial* ring if there are orthogonal idempotents $e_1, \ldots, e_n$ of R such that $R = e_1R \oplus \ldots \oplus e_nR$, and e_iR is a uniserial R-module. A left serial ring is defined analogously. A ring R which is right and left serial is called *serial*.

Let J be the Jacobson radical of the right serial ring R. It can be shown (cf. [15, Lemma 1]) that if $\bigcap_{n=1}^{\infty} J^n = 0$ then R is right Noetherian. So, for any $k > 0$, $e_iR \supset e_iJ \supset e_iJ^2 \supset \ldots \supset e_iJ^k$ are all the submodules of e_iR containing e_iJ^k and every submodule of e_iR has form e_iJ^n, for some n.

Clearly, a submodule of a uniserial module is uniserial and a factor ring of a serial ring is serial.

Proposition 2.2.2 [60, Corollary 4.7] *Let R be a semiperfect, semiprime, left and right Goldie ring, then the following are equivalent.*

1. *R is left serial.*
2. *R is right serial.*

A Noetherian serial ring is not necessarily Artinian. Its structure is described in the following theorem.

Theorem 2.2.3 [60, Theorems 5.11 and 5.14] *Let R be a Noetherian serial ring, with Jacobson radical J. Then $\bigcap_{n=1}^{\infty} J^n = 0$ and R is a direct sum of an Artinian serial ring and prime hereditary serial rings. Further, R has an Artinian serial quotient ring.*

Remark 2.2.4 In [60, Theorem 5.14] it is shown that an indecomposable Noetherian serial ring, that is not Artinian, is Morita equivalent to a ring whose structure is as in Theorem 2.1.5. In particular, a Noetherian serial ring is semiperfect.

The following is an important characterization of Noetherian serial rings.

Theorem 2.2.5 [60, 1.3, 2.6, and 3.4] *A Noetherian ring R is (right and left) serial if and only if every finitely generated right R-module is a direct sum of uniserial modules.*

Definition 2.2.6 An Artinian ring A is said to be an *Artin algebra* if it is finitely generated as a module over an Artinian subring of its centre.

Definition 2.2.7 A ring A is of *finite representation type* if A is right Artinian and there are finitely many finitely generated indecomposable right A-modules up to isomorphism.

Theorem 2.2.8 [18, Theorem 1.2] *Let A be a right Artinian ring with only finitely many non-isomorphic finitely generated indecomposable right modules. Then the same statements hold when "right" is replaced by "left".*

Also,

Theorem 2.2.9 [47, Corollary 4.4] *Let A be of finite representation type. Then every right A-module is a direct sum of finitely generated modules. In particular, every indecomposable A-module is finitely generated.*

Theorem 2.2.10 [31] *Let R be a PI ring or a right Noetherian ring whose primitive factor rings are all right Artinian. If R has a finite number of isomorphism classes of finitely generated indecomposable right modules, then R is a right Artinian ring.*

In the Artinian case, there is an obvious relationship between serial rings and rings of finite representation type.

Proposition 2.2.11 *Let A be an Artinian ring. If A is serial then A is of finite representation type.*

Proof. Let $A = e_1A \oplus \ldots \oplus e_nA$ be a decomposition of A, where the e_iA are uniserial A-modules for all i. Let M be a finitely generated indecomposable right A-module. By Theorem 2.2.5 the module M is uniserial and so there are an index i and a projective cover $e_iA \to M$. Hence $M \simeq e_iA/e_iJ^k$ for some $k \geq 0$. Since each e_iA is Artinian, there are, up to isomorphism, at most finitely many modules M with this property.

□

But Artinian rings of finite representation type need not be serial (cf. Example 3.4.2).

A famous theorem states as follows.

Theorem 2.2.12 (Eisenbud and Griffith) [19, Corollary 3.2] *Let R be a hereditary Noetherian prime ring. Then every proper factor ring of R is a serial Artinian ring.*

Hence, by Proposition 2.2.11, every proper factor ring of a hereditary Noetherian prime ring is of finite representation type. We are interested in finding conditions for the converse of Theorem 2.2.12 to be true. In the situation where the (Artinian) factor rings are all serial there are results for some classes of rings. The first one is the following.

Theorem 2.2.13 [60, Theorem 6.6] *Let C' be a commutative Noetherian ring and R a C'-algebra that is finitely generated as a C'-module. Then the following properties are equivalent.*

1. *Every proper Artinian homomorphic image of R is serial.*
2. *R is a direct sum of an Artinian serial ring and a finite number of hereditary orders over Dedekind domains.*

The above theorem has been extended to the case of prime bounded rings.

Theorem 2.2.14 [56] *A bounded Noetherian prime ring of Krull dimension one is hereditary if and only if all its proper factor rings are serial.*

It is worth noting that $k[x]/(x)^2$ is an example of an Artinian serial ring of infinite global dimension. Nevertheless, by Theorem 2.2.3 and [60, Theorem 5.11] a Noetherian serial ring R with zero Artin radical is hereditary.

In this thesis we consider the more general case of a Noetherian prime PI ring whose Artinian factor rings are all of finite representation type. With this setting there is already a result for classical orders.

Theorem 2.2.15 (Gustafson) [27, Theorem 3.3] *Let C' be a complete discrete valuation ring, with field of fractions K', Q be a finite dimensional separable K'-algebra and R a C'-order in Q. Then R is hereditary if and only if all Artinian factor rings of R are of finite representation type.*

We extend the above result to semiperfect Noetherian semiprime PI rings in Chapter 4. For this, we need to know a little more about rings of finite representation type.

2.2.2 Quivers and hereditary rings of finite representation type

To any Artinian ring a directed graph, called a *quiver*, is associated (see below). In this subsection, we shall see that the representation type of two special classes of Artinian rings depends only on the quiver associated to them.

Definition 2.2.16 Let A be a basic Artinian ring and $\{e_1, \ldots, e_n\}$ a complete set of primitive orthogonal idempotents of A. Set $F_i = e_iAe_i/e_iJe_i$ and ${}_iH_j = e_iJe_j/e_iJ^2e_j$, for all i, j. The *(right) Gabriel valued quiver* of A is a quadruple $\Gamma = (V, E, d, d')$, where $V = \{1, \ldots, n\}$ is the set of vertices of Γ and $E = \{\alpha = (i,j) \in V \times V \mid {}_iH_j \neq 0\}$ the set of arrows. Let $\alpha = (i,j)$ be an arrow of Γ. Then the maps $d, d' : E \to \mathbb{N}$ are given by $d(\alpha) = \dim({}_iH_j)_{F_j}$ and $d'(\alpha) = \dim_{F_i}({}_iH_j)$.

We represent the valued arrow α of Γ as follows.

$$\underset{i}{\bullet} \xrightarrow{(d,d')} \underset{j}{\bullet}$$

Definition 2.2.17 The set of pairs $S = (F_i, \ {}_iH_j)$ with $\alpha = (i,j) \in E$ is called *species of the ring A*. A species S is said to be *quasi-Artin* if $d(\alpha)$ and $d'(\alpha)$ are finite for all $\alpha \in E$ and the division rings F_i, $i = 1, \ldots, n$ are finitely generated over their centres.

This happens, for example, when A is a PI ring, by Kaplansky's Theorem (cf. Theorem 1.5.1). More details on species and their representations can be found in [16] and [17].

When $(d, d') = (1, 1)$, the valuation (d, d') over the arrow is omitted. Set $T_i = Ae_i/Je_i$ and $S_i = e_iA/e_iJ$, for $i = 1, \ldots, n$. Thus $(T_i)_i$ and $(S_i)_i$ are complete irredundant sets of non-isomorphic simple left and right A-modules, respectively. Thus an arrow $i \to j$ exists in Γ if and only if one of the following equivalent conditions holds.

1. T_i is isomorphic to a direct summand of Je_j/J^2e_j.

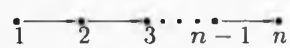
2. S_j is isomorphic to a direct summand of e_iJ/e_iJ^2.

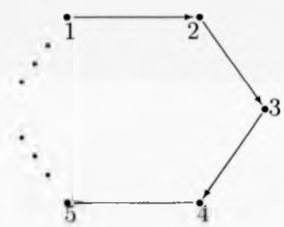
3. $\mathrm{Ext}^1(T_j, T_i) \neq 0$.

4. $\mathrm{Ext}^1(S_i, S_j) \neq 0$.

Theorem 2.2.18 [26, Lemma 3.1] *The ring A is indecomposable if and only if the quiver associated to it is connected.*

If A is serial, then the modules e_iJ/e_iJ^2 are simple for all i. So at most one arrow in Γ can end at the vertex i and at most one can start from i. Therefore, there are only two cases for a connected component of Γ:





For hereditary Artinian PI rings there is the following important result.

Theorem 2.2.19 [17, Corollary 1.4 and Theorem 1.1] *Let A be a basic Artinian hereditary PI ring with associated valued quiver Γ. Then A is of finite representation type if and only if Γ has underlying diagram which is a disjoint union of Dynkin diagrams. (See Figure 2.1 on page 32).*

If an Artinian PI ring A has Jacobson radical square equal to zero, then its representation type can also be seen from the associated quiver Γ. In this case, we make the following definition.

Definition 2.2.20 Let A be an Artinian PI ring and $\Gamma = (V, E)$ its associated quiver. The *separated quiver* of Γ is a quadruple $\Gamma' \equiv (V', E', d, d')$, where $V' = V \times \{0, 1\}$, and $E' = \{\alpha' = ((i, 0), (j, 1)) \mid \alpha = (i, j) \in E\}$. The values (d, d') are the same as in Γ.

Theorem 2.2.21 [17, Corollary 1.5] *Let A be a basic Artinian PI ring, such that $J^2 = 0$. Then A is of finite representation type if and only if the underlying diagram of the separated quiver Γ' associated to A is a disjoint union of Dynkin diagrams.*

This is because A is of finite representation type if and only if so is the hereditary ring

$$A' = \begin{pmatrix} A/J & J \\ 0 & A/J \end{pmatrix}.$$

In fact, the separated quiver Γ' of A is the associated quiver of A'. We present the proof of this result in Section 3.2.

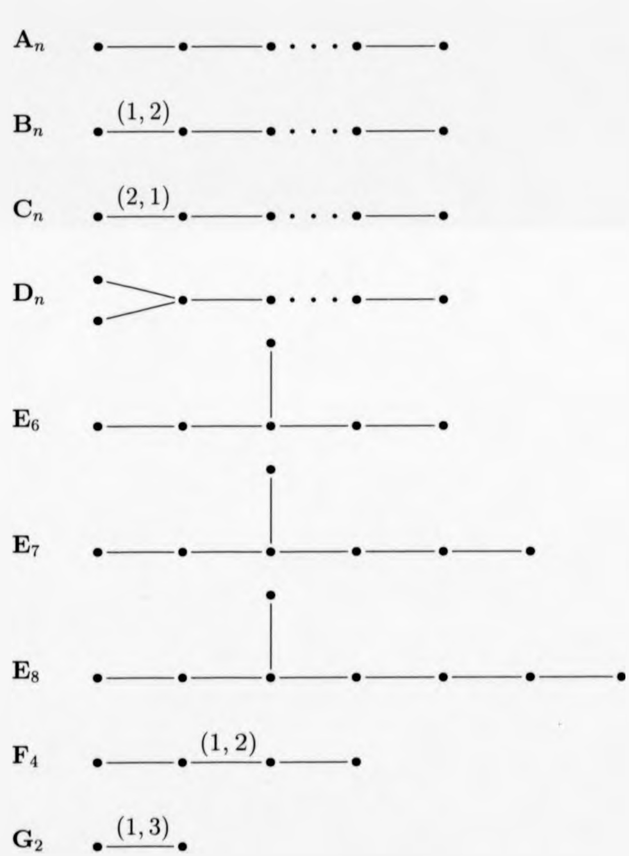


Figure 2.1: Dynkin diagrams

Chapter 3

Preliminaries

This chapter contains preparatory material and some technical results which will be used in the next chapters.

We start by presenting generalizations of a result by S. Brenner and a theorem by M. Auslander.

3.1 Formal triangular matrix rings

Let S and T be rings and V a S-T-bimodule. Then

$$R \equiv \begin{pmatrix} S & V \\ 0 & T \end{pmatrix} = \left\{ \begin{pmatrix} s & v \\ 0 & t \end{pmatrix} \mid s \in S, t \in T, v \in V \right\}$$

is a ring with the multiplication defined by

$$\begin{pmatrix} s & v \\ 0 & t \end{pmatrix} \begin{pmatrix} s' & v' \\ 0 & t' \end{pmatrix} = \begin{pmatrix} ss' & sv' + vt' \\ 0 & tt' \end{pmatrix}$$

for $s, s' \in S, t, t' \in T$ and $v, v' \in V$. The ring R is called a *formal triangular matrix ring*. It can be shown (cf. [24, pp 6]) that R is right Noetherian if and only if S and T are right Noetherian rings and V_T is finitely generated.

3.1.1 The module category

There is a convenient way of representing the modules over R. This is done as follows.

Let $\mathcal{C}$ be the category whose objects are triples (M_1, M_2, f) where M_1 is a right S-module, M_2 a right T-module and $f : M_1 \otimes_S V \to M_2$ a homomorphism of right

T-modules. A morphism from (M_1, M_2, f) to (M_1', M_2', f') is a pair (α, β) where $\alpha : M_1 \to M_1'$ is a homomorphism of right S-modules and $\beta : M_2 \to M_2'$ is a homomorphism of right T-modules, such that the following diagram commutes.

$$\begin{array}{ccc} M_1 \otimes_S V & \xrightarrow{f} & M_2 \\ \Big\downarrow{\scriptstyle \alpha\otimes 1} & & \Big\downarrow{\scriptstyle \beta} \\ M_1' \otimes_S V & \xrightarrow{f'} & M_2' \end{array} \tag{3.1}$$

There is a functor $F : \mathcal{C} \to \mathrm{Mod}(R)$ defined as follows.

Let (M_1, M_2, f) be an object of $\mathcal{C}$. Thus $M \equiv M_1 \oplus M_2$ is an abelian group. We define a right action of R on M by

$$(m_1, m_2) \begin{pmatrix} s & v \\ 0 & t \end{pmatrix} = (m_1 s, f(m_1 \otimes v) + m_2 t)$$

where $m_i \in M_i$, $i = 1, 2$, $s \in S, v \in V$ and $t \in T$. It is easy to check that M is a right R-module. We set $F(M_1, M_2, f) = M$. We note that if M_1 and M_2 are finitely generated modules then M is a finitely generated R-module. If $(\alpha, \beta) : (M_1, M_2, f) \to (M_1', M_2', f')$ is a morphism in $\mathcal{C}$, we set $F(\alpha, \beta) = \alpha \oplus \beta : M_1 \oplus M_2 \to M_1' \oplus M_2'$.

Conversely, let M be a right R-module and consider the idempotents $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ of R. So $M = Me_1 \oplus Me_2$ as abelian groups. Further, Me_1 is a right S-module and M_2 a right T-module.

Let V' be the ideal $\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}$ of R. Clearly $e_2 V' = V' e_1 = 0$, so $V' = e_1 V' = V' e_2$. We define a map $f' : Me_1 \times e_1 V' \to Me_2$ by $f(me_1, e_1 v') = me_1 v' = me_1 v' e_2$. It is easily seen that f is balanced and so it extends to a map $f : Me_1 \otimes_S V' \to Me_2$ which is also a homomorphism of right T-modules. Hence the triple (Me_1, Me_2, f) is an object of $\mathcal{C}$ and $F(Me_1, Me_2, f) = M$.

Now let $g : M \to N$ be a homomorphism of right R-modules. Then g induces homomorphisms $\alpha : Me_1 \to Ne_1$ and $\beta : Me_2 \to Ne_2$. It can be checked that $F(\alpha, \beta) = g$.

Suppose that R is right Noetherian. Hence Re_i is a finitely generated right-$e_i Re_i$-module, $i = 1, 2$. Let M be a finitely generated right R-module. There is an epimorphism $R^{(n)} \to M$, where n is the cardinality of a generating set of M, which restricts to epimorphisms $(Re_i)^{(n)} \to Me_i$, $i = 1, 2$ of right $e_i Re_i$-modules. This yields that Me_i is finitely generated. It can be shown (cf. [4, Proposition 2.2]) that, in fact, F induces an equivalence of categories $\mathcal{C}' \to \text{mod}(R)$, where $\mathcal{C}'$ is the full subcategory of $\mathcal{C}$ consisting of triples (M_1, M_2, f) with M_i, $i = 1, 2$ finitely generated modules.

There is a formula for computing the global dimension of a formal triangular matrix ring.

Proposition 3.1.1 [37, Proposition 7.5.1]

$$\sup\{\text{rgl.dim}S, \text{rgl.dim}T\} \leq \text{rgl.dim}R \leq \sup\{\text{rgl.dim}T, \text{rgl.dim}S + \text{pd}V_T + 1\},$$

where $\text{pd}V_T$ *is the projective dimension of* V_T.

A full description of the properties of formal triangular matrix rings can be found in [23, Chapter 4, Section A].

3.1.2 An example of a ring of infinite type

In [10, Proposition 1] it is shown that $T_2(\mathbb{Z}_{p^4})$, the ring of the upper triangular matrices over $\mathbb{Z}_{p^4}$, where p is a prime, is of infinite representation type. This is done by constructing directly a countable family of (pairwise) non-isomorphic finitely generated indecomposable modules.

We modify lines of that proof to show that similar modules occur for a certain factor ring of $T_2(D)$, where D is a non-commutative local Dedekind prime PI ring. Because of the non-commutativity of our domain we need to make explicit the homomorphism H (see below), for the description of a "typical" module of that family.

Theorem 3.1.2 *Let D be a (non-commutative) scalar local Dedekind prime PI ring. Then the ring $T_2(D)$ has a factor ring that is of infinite representation type.*

Proof. The ring D is finitely generated as a module over its centre C, which is a DVR (cf. Theorem 2.1.9). Let $m = yC$ be the maximal ideal of C. We show that the ring $R = T_2(D/y^4D)$ is of infinite type.

Let $A = (D/y^4D)^{(n)}$ be a free right D/y^4D-module of dimension n. We set $B = y^2A$. The ring D/y^4D is selfinjective, hence A is an injective D/y^4D-module.

By Baer's criterion every homomorphism from B to A can be extended to an endomorphism of A. Further, as A is projective, the functor $\mathrm{Hom}(A,-)$ is exact [52, Proposition 5.2.29]. So, by applying it to the exact sequence

$$0 \to y^2A \xrightarrow{1} A \xrightarrow{y^2} y^2A \to 0,$$

we obtain

$$0 \to \mathrm{Hom}(A, y^2A) \to \mathrm{Hom}(A, A) \to \mathrm{Hom}(A, y^2A) \to 0.$$

Hence every $f \in \mathrm{Hom}(A, B)$ can be thought of as equal to y^2g, with $g \in \mathrm{End}(A)$.

Set $M_1 = M_2 = A \oplus B$. So any $\phi : M_i \to M_j$, $i, j = 1, 2$, can be represented as follows

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ y^2\phi_{21} & \phi_{22} \end{pmatrix},$$

where $\phi_{ij} \in \mathrm{End}(A)$. Since A is free its endomorphism ring can be thought of as the ring of $n \times n$ matrices over D/y^4D. Further, $\mathrm{End}(B) \simeq M_n(D/y^2D)$.

With these identifications, let $H \equiv (h(i,j))$, $i, j = 1, \dots, n$ be the endomorphism of B defined by $h(i,j) = 1$ for $j = i, i+1$ and 0 otherwise.

Let $f : M_1 \to M_2$ be the homomorphism

$$f = \begin{pmatrix} yI & I \\ y^2I & yH \end{pmatrix},$$

where I is the identity of $\mathrm{End}(A)$. The triple $M \equiv (M_1, M_2, f)$ represents a finitely generated right R-module (cf. Subsection 3.1.1). The endomorphism ring of M consists of pairs (α, β), with $\alpha \in \mathrm{End}(M_1)$ and $\beta \in \mathrm{End}(M_2)$, which satisfy

$$f\alpha = \beta f. \tag{3.2}$$

Let

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ y^2\alpha_{21} & \alpha_{22} \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ y^2\beta_{21} & \beta_{22} \end{pmatrix},$$

be such a pair. We note that, since $y \in C$, the map yI commutes with every endomorphism of A. So

$$f\alpha = \begin{pmatrix} y\alpha_{11} + y^2\alpha_{21} & y\alpha_{12} + \alpha_{22} \\ y^2\alpha_{11} + y^3 H\alpha_{21} & yH\alpha_{22} + y^2\alpha_{12} \end{pmatrix},$$

and

$$\beta f = \begin{pmatrix} y\beta_{11} + y^2\beta_{12} & \beta_{11} + y\beta_{12}H \\ y^3\beta_{21} + y^2\beta_{22} & y^2\beta_{21} + y\beta_{22}H \end{pmatrix}.$$

From (3.2) we obtain the relations

$$\alpha_{11} = \beta_{11} = \alpha_{22} = \beta_{22}$$

and

$$\alpha_{11}H = H\alpha_{11}$$

that hold modulo $yI(\mathrm{End}(A))$, an ideal of $\mathrm{End}(A)$ isomorphic to $M_n(yD/y^4D)$.

Set $\alpha_{11} = (a(i,j))$, $i,j = 1,\dots,n$. By $(\alpha_{11}H)_{ij}$ we denote the (i,j)-entry of the matrix $\alpha_{11}H$. We have

$$(\alpha_{11}H)_{ij} = \begin{cases} a(i,j-1) + a(i,j) & \text{if} \quad j > 1, \\ a(i,j) & \text{if} \quad j = 1. \end{cases} \tag{3.3}$$

and

$$(H\alpha_{11})_{ij} = \begin{cases} a(i,j) + a(i+1,j) & \text{if} \quad i < n, \\ a(i,j) & \text{if} \quad i = n. \end{cases} \tag{3.4}$$

For $j = 1$ we get $a(i,1) = a(i,1) + a(i+1,1)$ and so $a(i+1,1) = 0$ for $i = 1,\dots,n-1$. For $i = n$ we have $a(n,j) = a(n,j-1) + a(n,j)$ and so $a(n,j-1) = 0$ for $j = 2,\dots,n$. For $j > 1$ and $i < n$ we have $a(i,j-1) = a(i+1,j)$.

Therefore, α_{11} is the matrix

$$\alpha_{11} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots & a_{1n} \\ 0 & a_{11} & a_{12} & a_{13} & \dots & a_{1n-1} \\ 0 & 0 & a_{11} & a_{12} & \dots & a_{1n-2} \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{12} \\ 0 & 0 & 0 & \dots & 0 & a_{11} \end{pmatrix}.$$

Now let us assume that (α, β) is an idempotent element of $\mathrm{End}(M)$. Then we have $\alpha^2 = \alpha$ and $\beta^2 = \beta$. In particular, α and β are idempotent modulo the ideal $yI\mathrm{End}(M_i)$ of $\mathrm{End}(M_i)$, $i = 1, 2$.

This gives $\alpha_{11}^2 = \alpha_{11}$ modulo $yI\mathrm{End}(A)$. Let us compute α_{11}^2. We have $\alpha_{11}^2(i,i) = a_{11}^2$, $\alpha_{11}^2(1,j) = \Sigma_{k=1}^{j} a(i,k)a(1, j-k+1)$ for $j > 1$, and $\alpha_{11}^2(i, j-1) = \alpha_{11}^2(i+1, j)$ for $1 < i < n$ and $j > 1$. Since D has no non-trivial idempotents, we have that either $a(1,1)^2 = 0$ and so $\alpha_{11} = 0$ or $a(1,1) = 1$ and in this case α_{11} is the identity matrix.

So, modulo the nilpotent ideal $(yI, yI)\mathrm{End}(M)$, the ring $\mathrm{End}(M)$ has no non-trivial idempotents. It follows that $\mathrm{End}(M)$ has no non-trivial idempotents and therefore M is an indecomposable R-module.

Now, n was arbitrary. Set $L_n = M$. By t_n we denote the cardinality of a minimal generating set for L_n. As M_1 has at least $2n$ generators, we have that $t_n \geq 2n$. Let us define a map $\phi : \mathbb{N} \to \mathbb{N}$ by $\phi(1) = t_1$ and $\phi(n+1) = t_{\phi(n)}$. We note that $\phi(n+1) = t_{\phi(n)} \geq 2\phi(n) > \phi(n)$ and so ϕ is an injective map. Therefore, for any two different values n_1 and n_2 of n we obtain two modules, $L_{\phi(n_1)}$ and $L_{\phi(n_2)}$, which are not isomorphic. Hence we have identified a countable number of finitely generated indecomposable non-isomorphic right R-modules.

□

Remark 3.1.3 By symmetry, one construct an infinite family of mutually non-isomorphic finitely generated, indecomposable left R-modules.

Remark 3.1.4 The same method can be used to show that the ring $T_2(D/y^{4h}D)$ is of infinite type for all h. This is done by replacing yI by y^hI in the proof of Theorem 3.1.2.

The following corollary will be needed in Chapter 4.

Corollary 3.1.5 *Let R and D be as in Theorem and let T be the ring*

$$T = \begin{pmatrix} D & D & D \\ 0 & D & D \\ 0 & D & D \end{pmatrix}.$$

Then T/y^4T is of infinite type.

Proof. Let $e = E_{11} + E_{22}$ be the idempotent of T, with E_{ii}, $i = 1,2$ the elementary matrices. We have $TeT = T$ and $R \simeq eTe$. So T is Morita equivalent to R via the bimodule ${}_ReT_T$.

Let us consider the factor ring T/y^4T. Using the same notation as in the proof of Theorem 3.1.2, we set $A' = A \otimes_{D/y^4D} (D/y^4D)^{(2)}$, $B' = B \otimes_{D/y^4D} (D/y^4D)^{(2)}$ and $M_2' = A' \oplus B'$. Let $f' : M_1 \otimes_{D/y^4D} (D/y^4D)^{(2)} \to M_2'$ be the homomorphism $f' = f \otimes 1$ and set $M' = (M_1, M_2', f')$. It is not difficult to show that M' is a right T-module with $M' \simeq M \otimes_{eTe} eT$. Therefore, M' is a finitely generated indecomposable right T-module and the rest of the proof of Theorem 3.1.2 applies.

□

3.2 On a theorem by M. Auslander

Let S be a ring and M an S-S-bimodule. We define a ring T, which we denote by $S \ltimes M$, as follows. The elements of T are pairs $(s, m) \in S \times M$. The addition is componentwise and the multiplication is given by

$$(s,m)(s',m') = (ss', sm' + ms')$$

for $(s, m), (s', m') \in S \times M$.

Thus T is called the *trivial extension ring of S by M*. Formal triangular matrix rings are trivial extensions.

A convenient way of representing T is

$$T = \begin{pmatrix} S & M \\ 0 & S \end{pmatrix}$$

where the lines mean that the entries $(1, 1)$ and $(2, 2)$ of every element of T are equal.

In fact, many examples of Noetherian PI rings which are not finitely generated as modules over their centre can be constructed by using trivial extension rings.

We say that an Artinian ring R has *index n* if n is the smallest integer such that $J^n = 0$.

We recall the following theorem proved by M. Auslander.

Theorem 3.2.1 [2, Proposition 4.8] *Let R be an Artin algebra with index n and J its Jacobson radical. If R is of finite representation type, then so are the trivial extension rings $R/J \ltimes J^i/J^{i+1}$, $i = 1, \dots, n-1$.*

In [2], this theorem is deduced from a more general result that we state below. We show that, in fact, the same argument applies also in the case where R is an Artinian PI ring. This is because R has the following two properties:

1. R/J is an Artin algebra (by Kaplansky's Theorem);

2. Every indecomposable injective R-module is finitely generated [51].

First we need some terminology.

Let S be the formal triangular matrix ring:

$$S = \begin{pmatrix} R/J & J^{n-1} \\ 0 & R/J \end{pmatrix}.$$

Let $M = (N_1, N_2, h')$ be a right S-module, where N_i is a right R/J-module $i = 1, 2$ and $h' : N_1 \otimes_{R/J} J^{n-1} \to N_2$ a homomorphism of right R/J-modules. In Subsection 3.1.1 we have observed that if R is Artinian, then M is of finite length if and only if so are the modules N_1 and N_2.

In [4, Proposition 2.4] the adjoint isomorphism

$$\phi : \mathrm{Hom}((N_1 \otimes_{R/J} J^{n-1})_{R/J}, (N_2)_{R/J}) \to \mathrm{Hom}((N_1)_{R/J}, \mathrm{Hom}(J^{n-1}, N_2)_{R/J})$$

is used to represent M by the triple (N_1, N_2, h), where h is the homomorphism $h = \phi(h') : (N_1)_{R/J} \to \mathrm{Hom}(J^{n-1}, N_2)_{R/J}$ of right R/J-modules.

The *Grassman category* $\mathrm{Gr}(R/J, J^{n-1})$ is the subcategory of $\mathrm{Mod}(S)$ whose objects are triples (N_1, N_2, h) with $h : (N_1)_{R/J} \to \mathrm{Hom}(J^{n-1}, N_2)_{R/J}$ a monomorphism of right R/J-modules.

Let (N_1, N_2, h) be a right S-module. Since R/J is semisimple, $\ker h$ is a direct summand of N_1, say $N_1 = \ker h \oplus X$. So

$$(N_1, N_2, h) = (X, N_2, h) \oplus (\ker h, 0, 0), \tag{3.5}$$

and (X, N_2, h) is an object of $\mathrm{Gr}(R/J, J^{n-1})$ (cf. [29, pp 175]).

Given a right R-module M and a two sided ideal I of R we denote by M_I the annihilator in M of I. Thus M_I is a right R/I-module. We note that

$$\mathrm{Hom}(J^{n-1}, M) = \mathrm{Hom}(J^{n-1}, \mathrm{socle}(M)),$$

for every right R-module M, since $J^n = 0$.

So there is a functor

$$F : \mathrm{Mod}(R) \to \mathrm{Gr}(R/J, J^{n-1})$$

defined by $F(M) = (M/M_{J^{n-1}}, \mathrm{socle}(M), f)$, where

$$f : M/M_{J^{n-1}} \to \mathrm{Hom}(J^{n-1}, \mathrm{socle}(M))$$

is given by $f(m + M_{J^{n-1}})(x) = mx$. Clearly, f is injective.

Now we state a theorem by M. Auslander in a form that is not the most general but is convenient for our purpose.

Theorem 3.2.2 (Auslander) [2, Theorem 3.1 A] *Let R be an Artinian ring with index n and Jacobson radical J.*

Let $\mathrm{Mod}(R)_{J^{n-1}}$ be the subcategory of $\mathrm{Mod}(R)$ whose objects are the R-modules such that $M_{J^{n-1}}$ is an injective R/J^{n-1}-module. Then the restriction L of F to $\mathrm{Mod}(R)_{J^{n-1}}$ is a representation equivalence, that is :

1. *For every object (N_1, N_2, h) of $\mathrm{Gr}(R/J, J^{n-1})$, there exists a right R-module M in $\mathrm{Mod}(R)_{J^{n-1}}$ such that $L(M) \simeq (N_1, N_2, h)$;*

2. *$L : \mathrm{Hom}_{\mathrm{Mod}(R)_{J^{n-1}}}(M, N) \to \mathrm{Hom}_{\mathrm{Gr}(R/J, J^{n-1})}(L(M), L(N))$ is surjective for all modules M, N in $\mathrm{Mod}(R)_{J^{n-1}}$;*

3. *If $f : M_R \to N_R$ is a homomorphism in $\mathrm{Mod}(R)_{J^{n-1}}$ for which $L(f)$ is an isomorphism, then f is an isomorphism.*

Let us consider the ring $R' = R/J \ltimes J^{n-1}$. We have that $J(R') = 0 \ltimes J^{n-1}$ and so $R'/J' \simeq R/J$. Since R/J is semisimple Artinian, every module over it is injective. Hence, by Theorem 3.2.2, $\mathrm{Mod}(R')$ and $\mathrm{Gr}(R/J, J^{n-1})$ have equivalent representations.

Now let us focus on the subcategories $\text{mod}(R)$, $\text{mod}(R)_{J^{n-1}}$ and $\text{mod}(R')$. Let M be a right R/J-module and let us consider the right R/J-module $\text{Hom}(J^{n-1}, M)$. Since R is Artinian and R/J is an Artin algebra, if M is finitely generated then so is $\text{Hom}(J^{n-1}, M)$ (cf. [4, pp 27]). Hence the triples (N_1, N_2, h) with N_1 and N_2 finitely generated form a full subcategory of $\text{Gr}(R/J, J^{n-1})$ that we denote by $gr(R/J, J^{n-1})$.

In the proof of Theorem 3.2.2 (1.), it is shown that, in fact, the right R-module M corresponding to (N_1, N_2, h) under L is isomorphic to a submodule of $E(N_2)$, the injective envelope of N_2 in $\text{Mod}(R)$. Since the injective envelopes of finitely generated modules over an Artinian PI ring are finitely generated, the restriction of L:

$$H : \text{mod}(R)_{J^{n-1}} \to \text{gr}(R/J, J^{n-1})$$

is a representation equivalence. Consequently, $\text{mod}(R)_{J^{n-1}}$ and $\text{mod}(R/J \ltimes J^{n-1})$ have equivalent representations.

Suppose that $\text{mod}(R)$ is of finite representation type. Then so is the subcategory $\text{mod}(R)_{J^{n-1}}$ and also $\text{mod}(R/J \ltimes J^{n-1})$. Since all R/J^i $i = 1, \ldots, n-1$ are of finite representation type, a similar argument shows that each of the rings $R/J \ltimes J^i/J^{i+1}$, $i = 1, \ldots, n-1$, is of finite representation type.

Further, from the observation (3.5) it follows that the hereditary rings

$$S_i \equiv \begin{pmatrix} R/J & J^i/J^{i+1} \\ 0 & R/J \end{pmatrix}$$

are also of finite representation type for all i.

So the following theorem holds.

Theorem 3.2.3 *Let R be an Artinian PI ring with index n and Jacobson radical J. If R is of finite representation type then so are the trivial extension rings $R/J \ltimes J^i/J^{i+1}$, $i = 1, \ldots, n-1$.*

3.3 Remarks on the graph of links of a Noetherian PI ring

In Section 1.4 we have defined the graph of links of a Noetherian ring and introduced the localisation at a clique.

Here we show that the graph of links between maximal ideals of a Noetherian PI ring R is "locally" the same as the quiver (without values) associated to certain Artinian factor rings of R. This fact imposes strong restrictions on the shape of the graph of links of R if its proper Artinian homomorphic images are all of finite representation type. In particular, it is used to show that if R is prime then all of its cliques of maximal ideals are finite.

3.3.1 Cliques of maximal ideals

Proposition 3.3.1 *The property for prime ideals to be linked is Morita invariant.*

Proof. Let R and S be Noetherian Morita equivalent rings. By Proposition 1.5.17 there is a semigroup isomorphism ϕ between the lattices of two sided ideals of R and of S, which preserves inclusion. It follows that $\phi(A \cap B) = \phi(A) \cap \phi(B)$ and $\phi(AB) = \phi(A)\phi(B)$, with A and B ideals of R.

Let P and Q be prime ideals of R with $P \rightsquigarrow Q$. Let A be an ideal of R such that $PQ \subseteq A \subset P \cap Q$ and $P \cap Q/A$ is a torsion free right R/Q-module and a torsion free left R/P-module.

$\phi(P)$ and $\phi(Q)$ are prime ideals of S. We have $\phi(P)\phi(Q) \subseteq \phi(A) \subset \phi(P) \cap \phi(Q)$. Let $T/\phi(A)$ be the torsion submodule of $\phi(P) \cap \phi(Q)/\phi(A)$ as a right $S/\phi(Q)$-module. Now T is a two sided ideal of S, and so $T = \phi(I)$ for an ideal I of R which contains A. We have $\phi(I)L \subseteq \phi(A)$ for some ideal L of S not contained in $\phi(Q)$. Again, $L = \phi(H)$ where H is an ideal of R that is not contained in Q. So $IH \subseteq A$, which is a contradiction.

Therefore $\phi(P) \cap \phi(Q)/\phi(A)$ is torsion free as right $S/\phi(Q)$-module. By symmetry $\phi(P) \cap \phi(Q)/\phi(A)$ is torsion free as left $S/\phi(P)$-module.

This shows that $\phi(P) \rightsquigarrow \phi(Q)$.

□

Most of the following results are due to Müller.

Lemma 3.3.2 [40, Remark 3] *Let R be a Noetherian PI ring and $P_1, \ldots, P_n$ prime ideals of R of the same height. Set $I = P_1 \cap \ldots \cap P_n$ and $\bar{R} = R/I^2$. Then $\mathcal{C}(\bar{P}_1 \cap \ldots \cap \bar{P}_n)$ is an Ore set in $\bar{R}$. The corresponding localisation A, which is a quotient ring, is Artinian, with maximal ideals $P_1A, \ldots, P_nA$ and $J(A)^2 = 0$.*

The following shows how links between prime ideals of R can be visualized in A.

Proposition 3.3.3 [40, Proposition 2] *In the situation of Lemma 3.3.2 the following are equivalent.*

1. *There is a short exact sequence* $0 \to U \to M \to V \to 0$ *of uniform right* R*-modules with* U P_j*-prime and* V P_i*-prime* (cf. Definition 1.4.2).

2. $\mathrm{Ext}^1_A(A/P_iA, A/P_jA) \neq 0$.

3. $P_iA \cap P_jA \supset P_iP_jA$.

By Theorem 1.4.3, these conditions yield a right link $P_i \rightsquigarrow P_j$. The reader should be warned that in [40] such a link is denoted by $P_j \rightsquigarrow P_i$. There, it is also observed that condition (3.) of Proposition 3.3.3 is left-right symmetric, as $IA = AI$ for any two sided ideal I of R. So (3.) is also equivalent to the left analogue of (1.) and (2.) with the role of P and Q interchanged, namely to the existence of a "left " link from P_j to P_i, which we denote by $P_j \rightsquigarrow_l P_i$.

Proposition 3.3.3 says that every chain $P_1 \rightsquigarrow P_2 \rightsquigarrow \ldots \rightsquigarrow P_n$ in the graph of links of R corresponds to a path in the quiver associated to A, where A is the quotient ring of R/I^2 and $I = P_1 \cap \ldots \cap P_n$.

Let P be a prime ideal of R. We denote by E_P the injective hull of a uniform right submodule of R/P. The injective hull of a uniform left submodule of R/P is denoted by ${}_PF$.

In [40] another definition of link is given.

Definition 3.3.4 We say that there is a *long right link* $P \leadsto\!\!\leadsto Q$ between two prime ideals P and Q of the same height of a Noetherian ring if there is a non-zero homomorphism $E_Q \to E_P$.

Clearly, for every prime ideal P of R, the identity map is always a non-zero endomorphism of E_P. This yields trivially a long selflink in P. From now on, we consider only non-trivial long links.

We use the following two results.

If $P \rightsquigarrow Q$ then $P \leadsto\!\!\leadsto Q$ ([40]).

For a Noetherian PI ring there is a long right link $P \leadsto\!\!\leadsto Q$ if and only if there is a long left link $Q \leadsto\!\!\leadsto_l P$, i.e. there is a non-zero homomorphism ${}_PF \to_Q F$ [41, Corollary 6].

In [42, Corollary 16] it is proved that for FBN rings long links coincide with the following definition.

Definition 3.3.5 We say that there is an *ideal link* from P to Q if there exist ideals A and B of R such that $A \subset B$ and B/A is torsion free left R/P-module and torsionfree right R/Q-module.

Lemma 3.3.6 [40, Lemma 3] *Let P and Q be prime ideals of the same height of a Noetherian PI ring. Then the following are equivalent.*

1. $P \rightsquigarrow\rightsquigarrow Q$.

2. *There exists $e \in E_Q$ with* $\mathrm{r.ann}_R(eR) \subseteq P$.

Short links and long links generate the same equivalence class.

Proposition 3.3.7 [40, Proposition 4] *Let P and Q be distinct prime ideals of the same height of a Noetherian PI ring R. Then a finite chain of links $P = P_1 \rightsquigarrow P_2 \rightsquigarrow \ldots \rightsquigarrow P_n = Q$ exists if and only if there is a finite chain of non-zero homomorphisms $E_Q \to E_{P'} \to \ldots \to E_P$.*

Theorem 3.3.8 [41, Theorem 7] *The graph of long links of a Noetherian PI ring is locally finite, i.e. each vertex meets only finitely many edges.*

So the graph of short links of a Noetherian PI ring is also locally finite.

Let N be the nilpotent radical of R. While a link $P/N \rightsquigarrow Q/N$ in R/N yields a link $P \rightsquigarrow Q$ in R, the converse is usually not true. We quote now a theorem by Müller which explains when links between prime ideals of R, whose images in R/N belong to different cliques of $\mathrm{Spec}(R/N)$, appear in $\mathrm{Spec}(R)$. This result is crucial for the description of a certain ring we give in Lemma 5.1.8.

Let bars denote images in R/N. We denote by $\mathrm{Cl}_{\bar{R}}(P)$ the set of prime ideals P' of R such that $\bar{P}'$ is in $\mathrm{Cl}_{\bar{R}}(\bar{P})$.

We write $\mathrm{Cl}_{\bar{R}}(P) \rightsquigarrow \mathrm{Cl}_{\bar{R}}(Q)$ if there exist $P' \in \mathrm{Cl}_{\bar{R}}(P)$ and $Q' \in \mathrm{Cl}_{\bar{R}}(Q)$ with $P' \rightsquigarrow Q'$.

Theorem 3.3.9 [40, Theorem 7] *Let R be a Noetherian PI ring with nilpotent radical N. With the notation as above suppose that* $\mathrm{K.dim}(R/P) = \mathrm{K.dim}(R/Q)$ *and that* $\mathrm{Cl}_{\bar{R}}(\bar{P}) \neq \mathrm{Cl}_{\bar{R}}(\bar{Q})$. *Then* $\mathrm{Cl}_{\bar{R}}(P) \rightsquigarrow \mathrm{Cl}_{\bar{R}}(Q)$ *if and only if there exist $P' \in \mathrm{Cl}_{\bar{R}}(P)$ and $Q' \in \mathrm{Cl}_{\bar{R}}(Q)$ such that* $\mathrm{r.ann}_R(N/P'N) \subseteq Q'$.

Remark 3.3.10 As we have already observed, $P \rightsquigarrow Q$ is equivalent to $Q \rightsquigarrow_l P$. So the left version of Theorem 3.3.9 reads as follows:

$\mathrm{Cl}_{\bar{R}}(Q) \rightsquigarrow_l \mathrm{Cl}_{\bar{R}}(P)$ *if and only if if there exist* $P' \in \mathrm{Cl}_{\bar{R}}(P)$ *and* $Q' \in \mathrm{Cl}_{\bar{R}}(Q)$ *such that* $\mathrm{l.ann}_R(N/NQ') \subseteq P'$.

In this thesis we are mainly interested in links between maximal ideals of a Noetherian PI ring. For every semiprime ideal $I = P_1 \cap \ldots \cap P_n$, P_i maximal, the factor ring R/I^2 is Artinian and therefore it is its own quotient ring. So, by Proposition 3.3.3, a link $P_i \rightsquigarrow P_j$ exists if and only if $P_i \cap P_j \supset P_iP_j$.

The following theorem states a useful property of Noetherian PI rings.

Theorem 3.3.11 [37, Proposition 13.7.15] *Let R be a PI ring with a.a.c. on ideals, then R has finitely many idempotent prime ideals.*

Translated into graphical terms, Theorem 3.3.11 asserts that in the graph of links of maximal ideals of a Noetherian PI ring there are only finitely many vertices without a loop.

We apply this property to Theorem 3.3.9 and deduce the following result, which will be used in Chapter 5.

Corollary 3.3.12 *Let R be a Noetherian PI ring. Suppose that the nilpotent radical N of R is Artinian and that R/N is a hereditary ring. Then cliques of maximal ideals of R are all finite.*

Proof. Clearly, we assume that N is not zero. Suppose that R has an infinite clique of maximal ideals and let $\mathcal{C}$ be its associated link component. By Theorem 3.3.11 almost all vertices of $\mathcal{C}$ have a loop and, as the same holds for the graph of links of R/N, the images in R/N of the prime ideals corresponding to these vertices are almost all invertible ideals (cf. Theorems 2.1.6 and 2.1.7).

Let I be the right annihilator of N. Using the (H)-condition of R (cf. [24, Theorem 8.9]), it is easy to show that R/I is right Artinian and hence, by Lenagan's Theorem (cf. [37, Theorem 4.1.6]), R/I is a (two sided) Artinian ring.

From Theorem 3.3.9 we deduce that I is contained in infinitely many maximal ideals of R, i.e. almost all those prime ideals which correspond to vertices of $\mathcal{C}$ with loops. Contradiction.

□

We want to study the shape of an infinite link component of a prime Noetherian PI ring. Braun and Warfield have proved that this consists of circuits of bounded length.

Proposition 3.3.13 [9, Corollary 7] *Let P and P' be prime ideals of a Noetherian prime PI ring with $P \rightsquigarrow P'$. Then there are prime ideals $P_1, \dots, P_r$ of R with $P' = P_1$, $P = P_r$, $P_i \rightsquigarrow P_{i+1}$ for all i, $i = 1, \dots, r-1$ and $r \leq \mathrm{PIdeg}(R)$.*

We observe the following.

Corollary 3.3.14 *Let R be a Noetherian prime PI ring of $\mathrm{PIdeg}(R) = n$ and $\mathcal{C}$ be an infinite link component associated to a clique of maximal ideals of R. Then $\mathcal{C}$ contains infinitely many circuits, of length at most n, where all the vertices have a loop (see Figure 3.1 on page 48).*

Proof. As we have already noted, $\mathcal{C}$ contains only finitely many vertices without a loop, say $v_1, \dots, v_h$.

Let V be the set of vertices of $\mathcal{C}$ and $w_1, w_2 \in V$. We say that w_1 has distance n from w_2 if the minimal undirected path connecting w_1 and w_2 consists of n edges.

Since $\mathcal{C}$ does not contain infinite stars (cf. Theorem 3.3.8), the diameter of V is infinite, i.e. there is no upper bound for the distances between vertices. In particular, for any finite subset M of V and integer m, there is a $v \in V$ whose distance from M is greater than m.

So we can choose a vertex $w \in V$ that has distance at least $n+2$ from $\{v_1, \dots, v_h\}$. This is connected to a vertex w' by an arrow, say $w \to w'$. By Proposition 3.3.13 there is a chain $w' = t_1 \to \dots \to t_{h-1} \to t_h = w$ in $\mathcal{C}$ with $h \leq n$. So this chain does not contain any of the vertices $v_1, \dots, v_h$.

With the same argument we can find successively distinct vertices $w_2, w_3, \dots$ and corresponding distinct circuits.

□

Next we look at links of prime ideals in finite ring extensions. Let $R \subseteq S$ be Noetherian PI rings, with S_R finitely generated. Then a generalization of the property lying over holds between R and S. That is, given a prime ideal P of R, there exists a prime ideal Q of S such that P is minimal over $Q \cap R$. Further, $\mathrm{cl.K.dim}(R/P) = \mathrm{cl.K.dim}(S/Q)$ (cf. [34, Corollary 4.4 and Theorem 4.8]).

The following result was proved by Letzter in the more general case of Noetherian rings with the second layer condition. We state it here for PI rings.

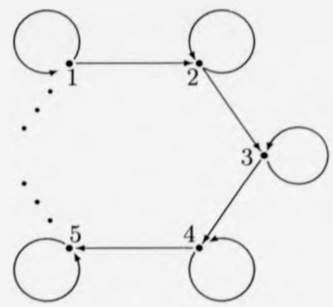


Figure 3.1: Polygon

Theorem 3.3.15 [34, Theorem 5.3] *Let $R \subseteq S$ be a PI rings as above and let P_1 and P_2 be prime ideals of R such that $P_1 \rightsquigarrow P_2$. Then :*

1. *There exist prime ideals Q_1 and Q_2, with Q_1 lying over P_1 and Q_2 lying over P_2 (in the sense mentioned above), such that either $Q_1 = Q_2$ or there exists a sequence of prime ideals $Q_1 = T_1, \ldots, T_l = Q_2$, with $l \geq 2$, such that $T_i \rightsquigarrow T_{i+1}$ for $1 \leq i \leq l-1$.*

2. *If the extension satisfies the property that $Q \cap R$ is semiprime for each prime ideal Q of S and $P_1 \neq P_2$, then we may choose Q_1 and Q_2 in (1.) such that $Q_1 \neq Q_2$.*

We use the above theorem to show a rather surprising fact.

Proposition 3.3.16 *Let $R \subseteq S$ be Noetherian PI rings as in Theorem 3.3.15 and such that the hypothesis of (2.) is satisfied. Suppose that S is hereditary. Then all cliques of maximal ideals of R are finite. Further, if R has infinitely many maximal ideals, then they are "almost all " localisable.*

Proof. We recall that a maximal ideal of a hereditary Noetherian semiprime PI ring is either invertible, and hence localisable, or idempotent, and cliques of prime ideals are all finite (cf. Theorems 2.1.7 and 2.1.6). Further, S has Krull dimension one and therefore so does R.

Let P_1, P_2, Q_1 and Q_2 be as in Theorem 3.3.15 and $P_1 \neq P_2$. So $Q_1 \neq Q_2$ and therefore Q_1 and Q_2 are idempotent ideals of S. So for every pair of distinct linked maximal ideals of R we find at least two distinct idempotent ideals of S.

Let Q be a maximal ideal of S. By assumption $Q \cap R$ is semiprime, say $Q \cap R = T_1 \cap \ldots \cap T_l$, for some T_i, $i = 1, \ldots, l$, prime ideals of R. Hence Q can appear at most l times in the process described above, namely as lying over for $T_1, \ldots, T_l$. So the existence of infinitely many links between distinct maximal ideals of R leads to the contradiction that S has infinitely many idempotent prime ideals.

$\square$

Note that in the proof of Proposition 3.3.16 we have only used the fact that S has finitely many idempotent ideals.

Links in semiperfect PI rings

Let R be a semiperfect Noetherian ring and suppose that R is basic. Let $R = e_1R \oplus \ldots \oplus e_nR$ be a decomposition of R as sum of indecomposable right ideals. Then there is a one-to-one correspondence between the set of the maximal ideals of R and $\{e_1, \ldots, e_n\}$, given by $e_i \to M_i = (1 - e_i)R + J$.

Clearly, the number of the maximal ideals is equal to the cardinality of a complete set of primitive idempotents of R. Hence, given a maximal ideal M_i of R, there is exactly one primitive idempotent e_i in $\{e_1, \ldots, e_n\}$ such that $e_i \notin M_i$. Therefore $R(1 - e_i)R + J \subseteq M_i$.

On the other hand, we have $M_i = e_iM_i + (1 - e_i)M_i \subseteq e_iJ + (1 - e_i)R \subseteq (1 - e_i)R + J$.

Proposition 3.3.17 [30, Proposition 5.3.13] *Let A be an Artinian ring that decomposes as a sum $A = e_1A \oplus \ldots \oplus e_nA$ of projective right ideals, such that e_i, $i = 1, \ldots, n$, commute modulo the Jacobson radical J of A. Let $P_i = A(1 - e_i)A + J$, $i = 1, \ldots, n$, be the prime ideals of A. Then :*

1. *$P_i \rightsquigarrow P_j$ if and only if $e_iJe_j \not\subseteq J^2$.*

2. *$P_i \leftrightsquigarrow P_j$ if and only if $e_iJe_j \neq 0$.*

From this we easily obtain the following generalization.

Lemma 3.3.18 *Let $R = e_1R \oplus \ldots \oplus e_nR$ be a semiperfect Noetherian PI ring, where e_i, $i = 1, \ldots, n$ are idempotent elements of R which commute modulo J. Let $P_i = R(1 - e_i)R + J$, $i = 1, \ldots, n$, be the corresponding maximal ideals of R. Then :*

1. *$P_i \rightsquigarrow P_j$ if and only if $e_iJe_j \not\subseteq J^2$.*

2. $P_i \rightsquigarrow\rightsquigarrow P_j$ *if and only if* $e_i J e_j \neq 0$.

Proof. (1.) It is straightforward from Proposition 3.3.17 since, if $P_i \rightsquigarrow P_j$, then $P_i/J^k \rightsquigarrow P_j/J^k$ for all $k > 1$ and R/J^k is Artinian.

(2.) It is not difficult to see that $E_{P_i/J^k} = \mathrm{ann}_{E_{P_i}}(J^k)$, for all $k \in \mathbb{N}$ and $i = 1, \ldots, n$.

Assume that $P_i \rightsquigarrow\rightsquigarrow P_j$ and let $e \in E_{P_j}$ such that $\mathrm{r.ann}_R(eR) \subset P_i$. Now eR is a module of finite length, so by Corollary 1.2.2 it is annihilated by a finite product of maximal ideals of R, say $eQ_1Q_2 \ldots Q_t = 0$, with $Q_1 = P_i$ and $Q_t = P_j$. Hence $eJ^k = 0$, for all $k \geq t$. Therefore $e \in E_{P_j/J^k}$, for all such k.

As $\mathrm{ann}_{R/J^k}(eR/J^k) \subset P_i/J^k$, we have $P_i/J^k \rightsquigarrow\rightsquigarrow P_j/J^k$, for all $k \geq t$ by Lemma 3.3.6. Hence Proposition 3.3.17, applied to the Artinian rings R/J^k, yields $e_i J e_j \not\subseteq J^k$, for all $k \geq n$. Since R is Noetherian PI, we have $\bigcap_{n=1}^{\infty} J^k = 0$ and so $e_i J e_j \neq 0$.

For the converse, the argument above can be repeated backwards.

□

3.3.2 Noetherian prime PI rings with finite cliques of maximal ideals

Proposition 3.3.19 *Let R be a Noetherian PI ring whose proper Artinian factor rings are all of finite representation type. Then for every semiprime ideal $I = P_1 \cap \ldots \cap P_n$, with P_i maximal for all i, the "separated" graph of links between prime ideals of R/I^2 is a disjoint union of Dynkin diagrams (See Figure 2.1 on page 32).*

Proof. The ring $A = R/I^2$ is Artinian by Corollary 1.2.2 and has Jacobson radical square equal to zero.

By Proposition 3.3.1 we may assume that A is basic. Let $e_1A \oplus \ldots \oplus e_nA$ be a decomposition of A as a sum of indecomposable modules, where the idempotents are numbered in such a way that $P_iA = (1 - e_i)A + I$, for $i = 1, \ldots, n$. By Lemma 3.3.18 and Definition 2.2.16, we have that $P_iA \rightsquigarrow P_jA$ if and only if there is an arrow $i \to j$ in the associated quiver Γ of A. So by Theorem 2.2.21 the separated quiver Γ' associated to A has underlying diagram that is a disjoint union of Dynkin diagrams.

□

Finally, we apply Lemma 3.3.18 to Corollary 3.3.14 and obtain the following result.

Proposition 3.3.20 *Let R be a Noetherian prime PI ring whose proper Artinian factor rings are all of finite representation type. Then the cliques of maximal ideals of R are finite.*

Proof. Suppose that $\mathrm{Spec}(R)$ has an infinite clique $\mathcal{C}$ of maximal ideals.

Let $\mathrm{PIdeg}(R) = n$. By Corollary 3.3.14 we can find ideals $P_1, \ldots, P_r$ in $\mathcal{C}$, $r \leq n$, such that $P_i \rightsquigarrow P_{i+1}$ for all $i = 1, \ldots, r-1$ and $P_r \rightsquigarrow P_1$. Further, $P_i \not\rightsquigarrow P_i$, for all i.

Set $I = P_1 \cap \ldots \cap P_r$. As R/I^2 is an Artinian ring, it is of finite representation type. The graph of links between images in R/I^2 of the P_is, $i = 1, \ldots, r$ is the subgraph of the graph of links associated to $\mathcal{C}$ which is obtained by removing all the vertices corresponding to the maximal ideals of R contained in $\mathcal{C} - \{P_1, \ldots, P_r\}$ and the edges connecting these vertices.

By Lemma 3.3.18 the quiver associated to R/I^2 has shape as in Figure 3.1. Hence if we compute the separated quiver of R/I^2, this has underlying diagram that is a closed line and therefore it cannot be a Dynkin diagram. This contradicts Theorem 2.2.21.

□

In [43, pp 354], an example of a Noetherian prime PI ring of Krull dimension two and with an infinite clique $\mathcal{C}$ of maximal ideals is given. The graph of links associated to $\mathcal{C}$ consists of infinitely many circuits as in Figure 3.1 with $n = 2$.

3.4 Some technical results

This section is an introduction to Noetherian PI rings with the property that every proper Artinian homomorphic image is of finite representation type. Let R be such a ring. After a discussion on connections between Artinian serial rings and PI rings of finite representation type, we prove that if R is semiperfect local and is not Artinian then it is a hereditary prime ring.

In the second half of this section we prove that the property of the factor rings is inherited by some other rings related to R. These rings will be used in later chapters.

3.4.1 Local PI rings

Lemma 3.4.1 *Let R be a right Noetherian semiperfect ring with Jacobson radical J and $\bigcap_{n=1}^{\infty} J^n = 0$. Then the following are equivalent.*

1. *R is right serial.*

2. *R/J^2 is right serial.*

Proof. (1.) $\Rightarrow$ (2.) is clear.
(2.) $\Rightarrow$ (1.). Let e be a primitive idempotent of R. We show that for every integer n the module eJ^n/eJ^{n+1} is simple. The proof of this statement for Artinian rings can be read in [1, Theorem 32.2]. We rewrite it here for completeness.

We proceed by induction on n. Let us assume the result true for a certain $n \geq 2$. So there exists a projective cover $f : P \to eJ^n$ with indecomposable module P. This gives rise to an epimorphism $g : PJ \to eJ^{n+1}/eJ^{n+2}$. The module PJ^2 is contained in the kernel of g and, by assumption, P/PJ^2 is uniserial. So $PJ^2 = \ker g$. Hence eJ^{n+1}/eJ^{n+2} is simple.

Now let H be a non-zero proper submodule of eR. Then $H \subseteq eJ$. Since $\bigcap_{k=1}^{\infty} J^k = 0$, we can find a k such that $H \subseteq eJ^k$ and $H \not\subseteq eJ^{k+1}$. As eJ^k/eJ^{k+1} is simple, we have $H + eJ^{k+1}/eJ^{k+1} = eJ^k/eJ^{k+1}$, and so $H + eJ^{k+1} = eJ^k$. As R is right Noetherian we can now apply Nakayama's Lemma to conclude that $H = eJ^k$. So the only submodules of eR are eJ^i, $i \in \mathbb{N}$. Hence R is right serial.

□

In Chapter 2 we have seen that a serial Artinian ring A is of finite representation type (cf. Proposition 2.2.11). There is a special case where these two concepts are equivalent, namely for a scalar local Artin algebra. Nevertheless, this fact is not true already for scalar local Artinian PI rings which are not Artin algebras, as the following example shows.

Example 3.4.2 Let k be a field. We define a homomorphism $\sigma : k(x) \to k(x)$ by $\sigma(a) = a$, if $a \in k$ and $\sigma(x) = x^2$. Then we can define a right module action of $k(x)$ over itself by $fg = f\sigma(g)$, for $f, g \in k(x)$. The symbol ${}_1k(x)_\sigma$ stands for the bimodule $k(x)$ over itself where the multiplication on the right is given via σ and on the left via the identity map.

Let us consider the trivial extension ring

$$A = \begin{pmatrix} k(x) & {}_1k(x)_\sigma \\ 0 & k(x) \end{pmatrix}.$$

Thus A is a scalar local Artinian PI ring with Jacobson radical $J \simeq_1 k(x)_\sigma$ and $J^2 = 0$. Now, A is left but not right serial, as

$$I = \begin{pmatrix} 0 & k(x^2) \\ 0 & 0 \end{pmatrix}$$

is a right ideal of A which is strictly contained in J. We have $d' = \dim_{k(x)} J = 1$ and $d = \dim J_{k(x)} = 2$. It is easy to see that the separated quiver of A is a Dynkin diagram of kind C_2. Hence it is of finite representation type by Theorem 2.2.19.

However, there are sufficient conditions for a local Artinian PI ring of finite representation type to be serial.

Theorem 3.4.3 [17, Theorem 1.6] *Let A be a local Artinian PI ring with Jacobson radical J. Set $F = R/J$, $d = \dim(J/J^2)_F$ and $d' = \dim_F(J/J^2)$. Then A is of finite representation type if and only if either $dd' = 1$ or $2 \leq dd' \leq 3$ and $J^2 = 0$.*

We note that if the ring A above is an Artin algebra the dimensions d and d' are the same and so $dd' = 1$, that is A is serial.

We deduce the following corollary.

Corollary 3.4.4 *Let R be a right Noetherian scalar local PI ring such that $J^2 \neq 0$ and $\bigcap_{n=1}^{\infty} J^n = 0$. Suppose that R/J^3 is of finite representation type. Then either R is an Artinian serial ring or R is a (two sided) hereditary prime ring.*

Proof. We note that in any right Noetherian scalar local PI ring R every J^i/J^{i+1} has finite length as right R-module. By Kaplansky's Theorem (cf. Theorem 1.5.1) the ring R/J is finite dimensional over its centre, so J^i/J^{i+1} has finite length as left R-module. Therefore, R/J^n is a (two sided) Artinian ring for all n.

The ring R/J^3 has finite representation type by assumption and $(J(R/J^3))^2 = J^2/J^3 \neq 0$ because otherwise $J^2 = J^3$ and so $J^2 = 0$ by Nakayama's Lemma. Hence, by Theorem 3.4.3, the factor ring R/J^2 is two sided serial. So, by Lemma 3.4.1, R/J^n is a serial ring for all n and R is a right serial ring. In particular, every two sided ideal of R is a power of J.

Now if $J^n = 0$ for some n, then $R = R/J^n$ is an Artinian serial ring. Suppose that R is not right Artinian. Therefore, J is not the only prime ideal of R and hence R must be a prime ring. So R is a left Goldie ring. By Proposition 2.2.2, R is also left serial. It follows that R is Noetherian and so is hereditary by Theorem 2.2.3.

□

Remark 3.4.5 From Theorem 1.5.15 and Proposition 1.5.17 it follows that the properties for a ring R to be PI, to be serial and to be of finite representation type are all Morita invariant. Therefore Corollary 3.4.4 holds also for semiperfect local Noetherian PI rings (cf. Proposition 1.2.12).

Hence we obtain the following result.

Proposition 3.4.6 *Let R be a Noetherian semiperfect local PI ring. Suppose that R is not Artinian. Then the following are equivalent.*

1. *R is hereditary and prime.*
2. *R/J^2 is serial.*
3. *R/J^2 is of finite representation type.*

Proof. We note that $\bigcap_{n=1}^{\infty} J^n = 0$.

(1.) $\Rightarrow$ (2.) $\Rightarrow$ (3.) follows from Theorem 2.2.12.

(3.) $\Rightarrow$ (1.). By Remark 3.4.5 we may assume that R is a scalar local ring. By Theorem 3.4.3, R/J^2 is one sided serial, and so is R by Lemma 3.4.1. Suppose that R_R is uniserial. Then by the proof of Corollary 3.4.4, R is hereditary and prime.

□

3.4.2 The property (P)

Definition 3.4.7 We say that a ring R has the *property* (P) if every proper Artinian factor ring of R is of finite representation type.

In the following we study how the property (P) is inherited from R by some other rings related to R.

Endomorphism rings of finitely generated projective modules

Lemma 3.4.8 [26, Proposition 7.1] *Let A be an Artinian ring and e an idempotent of A. If A is of finite representation type then so is the ring eAe.*

In [27], a version of the following proposition in the case of a classical order R in a finite dimensional separable algebra, is taken for granted. There, it is also implicitly assumed that, for any idempotent e of R, one has $eJ^2e = (eJe)^2$. This is not necessarily true, as we show in Example 3.4.11.

Proposition 3.4.9 *Let e be an idempotent element of a fully bounded Noetherian ring R. If R has the property (P), then so does the ring eRe.*

Proof. Every ideal of eRe has the form eIe, where I is an ideal of R.

Let I be a non-zero ideal of R such that eRe/eIe is Artinian. We need to find an ideal H of R such that R/H is Artinian and $eHe \subseteq eIe$.

We shall prove that there are maximal ideals $T_1, T_2 \dots, T_k$ of R, not necessarily distinct, such that $e(T_1T_2 \dots T_k)e \subseteq eIe$.

As R is a Noetherian ring, we have that eR is finitely generated as left eRe-module and so eR/eI is of finite length as left eRe/eIe-module. By Lenagan's Theorem (cf. [37, Theorem 4.1.6]), eR/eI is of finite length also as right R-module. By Corollary 1.2.2 there exist finitely many maximal ideals $T_1, T_2 \dots, T_k$ of R, not necessarily distinct, with $eT_1T_2 \dots T_k \subseteq eI$. Hence our claim is proved.

The ring $R/T_1 \dots T_k$ is Artinian (cf. Corollary 1.2.2) and therefore is of finite representation type. It follows from Lemma 3.4.8 that $eRe/eT_1 \dots T_ke$ is of finite representation type and hence so is its factor ring eRe/eIe.

□

Proposition 3.4.9 is a key fact for the proof of our main result.

We also obtain the following generalization.

Corollary 3.4.10 *Let R be as in Proposition 3.4.9 and P be a finitely generated projective right R-module. Then* $\mathrm{End}(P)$ *has the property (P).*

Proof. There exist a right R-module X and an integer n such that $P \oplus X \simeq R^{(n)}$. The ring $E = \mathrm{End}(R^{(n)}) \simeq M_n(R)$ is Morita equivalent to R and so has the same properties, specified above, as R. Let e be an idempotent of E such that $\mathrm{End}(P) = eEe$. The result now follows from Proposition 3.4.9.

□

It is worth noting that if in Proposition 3.4.9 we only assume that R/J^2 is of finite representation type then the ring eRe does not necessarily have the same property, as the following example shows.

Example 3.4.11 Let k be a field and

$$R \equiv \begin{pmatrix} k[x]_{(x)} & (x) & (x)^2 \\ k[x]_{(x)} & k[x]_{(x)} & (x) \\ k[x]_{(x)} & k[x]_{(x)} & k[x]_{(x)} \end{pmatrix},$$

where $k[x]_{(x)}$ is the ring $k[x]$ localised at the prime ideal (x). Then R is a Noetherian semiperfect prime PI ring of Krull dimension one. The associated quiver of R/J^2 is

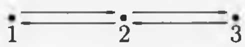


Figure 3.2: Two circuits

The separated quiver of R/J^2 is a disjoint union of two diagrams of kind A_2, hence by Theorem 2.2.21, the ring R/J^2 is of finite representation type. Let $e = E_{11} + E_{33}$, be the idempotent of R where E_{11} and E_{33} are elementary matrices. We note that $eJe = eJ^2e \supset (eJe)^2 = eJ^3e$. The quiver associated to $eRe/(eJe)^2$ is represented in Figure 3.3.

Figure 3.3: A circuit with loops

The separated quiver of $eRe/(eJe)^2$ is not a Dynkin diagram and so this ring is of infinite type by Theorem 2.2.21. Hence, by Proposition 3.4.9, the ring R does not have the property (P). We shall discuss more about the representation type of these rings in Chapter 6.

Split extension rings

Proposition 3.4.12 *Let C be a Noetherian local domain and S, T C-algebras, such that*

$$T = S \oplus H$$

where H is a S - S submodule of T. If T has the property (P) then so does S.

Proof. Let m be the maximal ideal of C. It is enough to show that S/m^kS is of finite representation type for all positive integers k.

We have that $m^kT = m^kS \oplus m^kH$, for all k. Hence

$$T/m^kT \simeq S/m^kS \oplus H/m^kH.$$

Since this property is true for Artinian rings (cf. [44]), the result follows.

□

Localised rings

We recall the following properties of localisations.

Lemma 3.4.13 *Let R be a Noetherian ring, $\mathcal{S}$ an Ore set in R and denote by $R_{\mathcal{S}}$ the localised ring. Let I be an ideal of R, with $IR_{\mathcal{S}} \cap R = I$ and M a right $R_{\mathcal{S}}$-module. Then:*

1. *The image $\bar{\mathcal{S}}$ of $\mathcal{S}$ in R/I is an Ore set in R/I and $R_{\mathcal{S}}/IR_{\mathcal{S}} \simeq (R/I)_{\bar{\mathcal{S}}}$.*
2. *$M \otimes_R R_{\mathcal{S}} \simeq M$ as right $R_{\mathcal{S}}$-modules.*

Proof. (1.) is [53, Proposition 3.2.34].

(2.) is [37, Proposition 7.4.2].

□

Proposition 3.4.14 *Let R be a Noetherian PI ring with the property (P) and X a classically localisable clique of maximal ideals of R. Then the ring R_X has the property (P).*

Proof. Set $T = \{r \in R \mid rx = 0 \text{ for some } x \in \mathcal{C}(X)\}$. Then T is an ideal of R.

Since the factor ring R/T has the same properties as R we may assume that $\mathcal{C}(X)$ consists of regular elements.

R_X is a Noetherian PI ring and is a flat extension of R (cf. [37, Proposition 2.1.16]). Let I be a non-zero ideal of R with $IR_X \cap R = I$ and such that R_X/IR_X is Artinian. So I contains a finite product of maximal ideals of R which belong to X (cf. Corollary 1.2.2 and Definition 1.4.7).

Let us say that $H = P_1P_2 \ldots P_n$ is such a product. By Corollary 1.2.2 the ring R/H is Artinian and so is of finite representation type. Hence so is R/I. Now, the prime ideals of R/I are the distinct members in the images of $P_1, \ldots, P_n$ in R/I and R/I is $\mathcal{C}(X)$-torsionfree. Further, for any $c \in \mathcal{C}(X)$, we have that $Rc + I = R$, i.e. $(R/I)c = (R/I)$. From this, it is easy to show that $(R/I)_X \simeq R/I$. Hence, by Lemma 3.4.13 the ring R_X/IR_X is of finite representation type.

□

Remark 3.4.15 With the notation as in Proposition 3.4.14, let M be a finitely generated indecomposable right R_X/IR_X-module. As R/I is of finite representation type, by Theorem 2.2.9 M might decompose as a right R/I- module. Let us say $M = \bigoplus_{i \in I} N_i$, with N_i finitely generated indecomposable right R/I-modules. Hence, by Lemma 3.4.13 (2.), we have

$$M \simeq M \otimes_R R_X = \bigoplus_{i \in I} (N_i \otimes_R R_X).$$

But M is indecomposable, so all but one of the above summands must be zero. Say $M \simeq N_1 \otimes_R R_X$. So, up to isomorphism, all finitely generated indecomposable R_X/IR_X-modules are obtained by tensoring over R the finitely generated indecomposable R/I-modules.

Remark 3.4.16 The same argument as in the proof of Proposition 3.4.14 shows that if R has the property that all its Artinian factor rings are serial, then so does R_X.

By applying a similar method, we also have:

Proposition 3.4.17 *Let R be a Noetherian prime PI ring which is affine over a field k and has the property (P). Let a be a non-zero central element of R. Then the ring $R[a^{-1}]$ has the property (P).*

Proof. $R[a^{-1}]$ is the localisation of R at the multiplicative set $\mathcal{S} = \{a^n \mid n \in \mathbb{N}\}$.

Let I be an ideal of R such that the ring $R_\mathcal{S}/IR_\mathcal{S}$ is Artinian and let M be a finitely generated right $R_\mathcal{S}/IR_\mathcal{S}$-module. Since $R_\mathcal{S}/IR_\mathcal{S}$ is a finite dimensional k vector space (cf. Theorem 1.5.5), from the natural homomorphism of k-algebras

$$f : R \to R_\mathcal{S}/IR_\mathcal{S}$$

we get that $R/\ker f = R/(IR_\mathcal{S} \cap R)$ is also finite dimensional over k. By assumption such a ring is of finite representation type. Now the proof proceeds as in Remark 3.4.15.

□

J-adic completion of Noetherian PI rings

Proposition 3.4.18 *Let R be a Noetherian semilocal PI ring and let $\hat{R}$ be the J-adic completion of R. If R has the property (P), then so does the ring $\hat{R}$.*

Proof. We have (cf. Theorem 1.5.8)

$$R/J^k \simeq \hat{R}/J^k\hat{R} = \hat{R}/\hat{J}^k$$

for all k. Each R/J^k is Artinian and so is of finite representation type.

□

Chapter 4

Prime PI rings

The main result of this chapter is the converse of Theorem 2.2.12 for PI rings. This is achieved by shifting the problem to semilocal rings and then applying Theorem 1.4.9.

4.1 Semiperfect PI rings

In this section we generalize Gustafson's Theorem (cf. Theorem 2.2.15) to a semiperfect Noetherian PI ring R with $A(R) = 0$.

The following is an easy observation.

Lemma 4.1.1 *Let R be a Noetherian PI ring which is an order in an Artinian ring. Let $R = e_1R \oplus \ldots \oplus e_nR$ be a decomposition of R with $e_1, \ldots, e_n$ idempotents. Then the following are equivalent.*

1. $A(R) = 0$.

2. $A(eRe) = 0$, *for every idempotent e of R which is sum of at most two idempotents in $\{e_1, \ldots, e_n\}$.*

Proof. (1.) $\Rightarrow$ (2.) Let e be any idempotent of R. We claim that $A(eRe) = 0$. Suppose not. So $A(eRe) = eAe$, with A a non-zero ideal of R.

Suppose first that R is a semiprime ring. Since eAe is Artinian, by Lemma 1.2.8 and Corollary 1.2.2 there are maximal ideals $M_1, M_2, \ldots, M_t$ of R, not necessarily distinct, such that $(eAe)(eM_1eM_2e \ldots eM_te) = 0$. So $(eAeM_1e \ldots eM_t)^2 = 0$ and hence, as R is assumed semiprime, $eAeM_1e \ldots eM_{t-1}eM_t = 0$. It follows that the right ideal $eAeM_1e\dots eM_{t-1}eR$ of R is a right R/M_t-module and so is Artinian. Hence it must be zero. Repeating the argument, finally we obtain the contradiction $eAeR = 0$.

Suppose now that R is not a semiprime ring. Then, by Theorem 1.3.7, $A(R/N) = 0$. As $e \notin N$, by the proof above we have $A(eRe/eNe) = 0$. So eRe is not an Artinian ring and $eAe \subseteq eNe$. By Proposition 1.5.18 eRe has an Artinian quotient ring. So by Theorem 1.3.8 eAe, and hence eNe, contains an idempotent of R, which is a contradiction.

(2.) $\Rightarrow$ (1.) For this we do not need the assumption that R has an Artinian quotient ring. Suppose that $A(R) \neq 0$. There exist idempotents e_i, e_j, with $i, j \in \{1, \dots, n\}$, such that $e_iA(R)e_j \neq 0$. Set $e = e_i + e_j$. Hence $0 \neq eA(R)e$ is an ideal of eRe which is Artinian by Proposition 1.5.18. This is a contradiction.

□

Proposition 4.1.2 *Let R be a semiprime semilocal Noetherian PI ring, which is complete with respect to the J-adic topology and with $A(R) = 0$. Suppose that R has the property (P). Then R is hereditary.*

Proof. R is a semiperfect ring (cf. Theorem 1.5.8). Let $R = e_1R \oplus \dots \oplus e_nR$ be a decomposition of R as a sum of indecomposable right ideals and let $Q(R)$ be the quotient ring of R. We also have $Q(R) = e_1Q(R) \oplus \dots \oplus e_nQ(R)$.

Each of the rings e_iRe_i $i = 1, \dots n$ is scalar local Noetherian semiprime PI with Jacobson radical e_iJe_i and has the property (P) by Proposition 3.4.9.

By Lemma 4.1.1, the ring e_iRe_i is not Artinian. Hence Proposition 3.4.6 tells us that e_iRe_i is a hereditary ring. So, being indecomposable, e_iRe_i is prime, for all i. By Theorem 2.1.9 the ring e_iRe_i is finitely generated as a module over its centre $C(e_iRe_i)$, which is a DVR. Further, as e_iRe_i is a scalar local ring, by Theorem 2.1.5 it is a maximal order in $e_iQ(R)e_i$.

Since $A(R) = 0$ by Theorem 1.3.7 no minimal prime ideal of R is maximal.

Suppose first that R is a prime ring. Let $0 \neq P$ and M be prime ideals of R with $P \subseteq M$. Hence $e_i \notin M$ for some i. We have that e_iPe_i and e_iMe_i, with $e_iPe_i \subseteq e_iMe_i$, are non-zero prime ideals of e_iRe_i as R is prime. Being hereditary, e_iRe_i has Krull dimension one. Therefore $e_iPe_i = e_iMe_i$. Hence $e_iMe_i \subseteq P$ and so $M = P$. This shows that $\mathrm{K.dim}(R) = 1$.

Let C be the centre of R. We claim that R is centrally integrally closed (cf. Definition 1.5.6). Let K be the field of fractions of C.

Each of the rings $C(e_iRe_i)$ is an integrally closed domain for all i. Let $0 \neq k \in K$ be integral over R. So k satisfies an equation

$$k^t + r_{t-1}k^{t-1} + \ldots + r_0 = 0$$

for $r_j \in R$. Clearly, e_ik is in the field of fractions of $C(e_iRe_i)$, for $i = 1, \ldots, n$. We have

$$0 = e_ik^t + e_ir_{t-1}e_ik^{t-1} + \ldots + e_ir_0e_i = (e_ik)^t + (e_ir_{t-1}e_i)(e_ik)^{t-1} + \ldots + e_ir_0e_i.$$

This means that e_ik is integral over $e_iC \subseteq C(e_iRe_i)$, for all i. We also note that, as R is a prime ring, $e_ik \neq 0$ and so the coefficients $e_ir_je_i \in e_iRe_i$ are not all zero.

So e_ik belongs to $C(e_iRe_i) \subseteq R$, for $i = 1, \ldots, n$. Hence

$$k = \sum_{i=1}^{n} e_ik \in R \cap K = C.$$

By Theorem 2.1.10, the ring R is finitely generated as a module over its centre which is a Dedekind domain. In particular, C is Noetherian (cf. Proposition 1.5.4).

The multiplication by e_i gives a ring isomorphism $C \simeq e_iC$. The ring e_iRe_i is finitely generated as a module over e_iC. Let p be a non-zero prime ideal of e_iC. By L.O. (cf. Theorem 1.5.7), p must be the intersection of e_iJe_i with e_iC. Thus e_iC, and hence C are local rings.

Let m be the maximal ideal of C. Now, R is also complete with respect to the mR-adic topology and C is complete with respect to the m-adic topology (cf. page 19). So R satisfies all the conditions of Theorem 2.2.15. It follows that R is a hereditary ring.

Suppose now that R is not a prime ring and let e be a primitive idempotent of R. We claim that the right annihilator P of eR is a prime ideal. For, let A and B be ideals of R such that $A \supseteq P$, $B \supseteq P$ and $AB \subseteq P$. Then $eAB = 0$. So $(eAe)(eBe) = 0$ and hence $eAe = 0$ or $eBe = 0$, as eRe is a prime ring. Since R is semiprime, we have that $eA = 0$ or $eB = 0$. Therefore $A \subseteq P$ or $B \subseteq P$.

So $P = \text{r.ann}(eR)$ is a non-zero prime ideal and it cannot be maximal because eR is not Artinian. Hence P is a minimal prime ideal of R.

So eR is an indecomposable R/P-module. The ring R/P is a Noetherian prime PI ring and has the property (P). It is not difficult to show that $J(R/P) = J + P/P$. So by Theorem 1.5.8 (4.), the ring R/P is complete with respect to the $J(R/P)$-adic topology. By the argument above, R/P is hereditary. Hence by Theorem 2.1.5 and Remark 2.2.4 R/P is a serial ring. It follows that eR is a uniserial R-module. Now e was arbitrary, so R is a direct sum of uniserial right ideals, that is R is a right serial ring. A similar argument shows that R is left serial. Hence R is a hereditary ring.

$\square$

We are now able to prove the following result.

Proposition 4.1.3 *Let R be a semilocal Noetherian PI ring which is complete with respect to the J-adic topology and with $A(R) = 0$. If R has the property (P) then R is a hereditary semiprime ring.*

Proof. R is a semiperfect ring. Assume, by contradiction, that the nilpotent radical N of R is not zero. By Theorem 1.3.7 $A(R/N) = 0$ and by Theorem 1.5.8 the ring R/N is complete with respect to its J/N-adic topology. So by Proposition 4.1.2 R/N is a hereditary ring. Therefore R has Krull dimension one and so by Theorem 1.3.9, R is an order in an Artinian ring.

For the rest of the proof the property of R to be complete with respect to the J-adic topology is not needed.

We may assume that R is basic. Let $e_1R \oplus \ldots \oplus e_nR$ be a decomposition of R as a sum of indecomposable right ideals.

By Lemma 4.1.1 no e_iRe_i $(i = 1, \ldots, n)$ is Artinian, and so each of these is a hereditary prime ring by Proposition 3.4.6.

For an easier reading we split now the proof into a series of lemmata.

The first observation is the following.

Lemma 4.1.4 *With the notation as above, the centre C of R is a semiprime ring.*

Proof. As e_iRe_i $(i = 1, \ldots, n)$ is a hereditary prime ring we conclude that e_iC is an integral domain for all i.

Let $f_i : C \to e_iC$ be the multiplication map. This gives a ring isomorphism $C/\ker f_i \simeq e_iC$. So $\ker f_i = \mathrm{ann}_C(e_i)$ is a prime ideal of C. Set $\ker f_i = p_i$. We have $\bigcap_{i=1}^n p_i = 0$.

□

The following lemma allows us to reduce the problem to the case where R has just two primitive idempotents.

Lemma 4.1.5 *Assume that R is not a semiprime ring. Then there exists an idempotent e of R that is a sum of two primitive idempotents, such that $eNe \neq 0$, $A(eRe) = 0$, and eRe has the property (P). Further, the ring eRe/eNe is hereditary.*

Proof. $R = e_1R \oplus \ldots e_nR$. So $N = (\sum_{i=1}^n e_i)N(\sum_{h=1}^n e_h)$. Since $N \neq 0$ and $e_jNe_j = 0$ for all j, we have that $e_iNe_j \neq 0$, for some $i, j \in \{1, \ldots, n\}$ and $i \neq j$. Set $e = e_i + e_j$. The ring eRe is semiperfect Noetherian PI, has an Artinian quotient ring by Proposition 1.5.18 and so $A(eRe) = 0$ by Lemma 4.1.1. Further, eRe has the property (P) by Proposition 3.4.9. As R/N is a hereditary ring, we conclude by Proposition 1.5.18 that the ring eRe/eNe is also hereditary.

□

Thus, if R is not a semiprime ring, we may replace it by the ring eRe as above and assume that 1 is a sum of two primitive orthogonal idempotents. We describe the structure of this ring.

Lemma 4.1.6 *Let $R = e_1R \oplus e_2R$ be as in Proposition 4.1.3 and $N \neq 0$. Then its centre C is an integral domain and R has the following matrix representation:*

$$R \simeq \begin{pmatrix} R/P_2 & N \\ 0 & R/P_1 \end{pmatrix},$$

where P_1 and P_2 are minimal prime ideals of R and $N = P_1 \cap P_2$. Each of the rings R/P_i, $i = 1, 2$ is a complete Dedekind prime ring and N is free as left R/P_2-module and as right R/P_1-module.

Proof. We have $R/N \simeq e_1R/e_1N \oplus e_2R/e_2N$ and the ring R/N is hereditary by Lemma 4.1.5. So either N is a prime ideal and the ring R/N is indecomposable or N is the intersection of two minimal primes.

We claim that $N = e_1Ne_2 \oplus e_2Ne_1$ is a direct sum of two ideals of R. Since $e_iNe_i = 0$ $i = 1, 2$ it is easy to check the required details. Further, $N^2 = 0$.

Suppose that N is a prime ideal. Let us consider $Q(R)$, the quotient ring of R. So $Q(R) = e_1Q(R) \oplus e_2Q(R)$ and $Q(R)$ is a local Artinian ring with Jacobson radical $NQ(R)$. Further, $e_iQ(R)e_i$ is a division ring, for $i = 1, 2$. By Corollary 1.2.13 we obtain $NQ(R) = 0$, in contradiction to the assumption $N \neq 0$.

So R/N is direct sum of two prime hereditary rings. Since N is a lifting idempotent ideal, we may assume that e_1 and e_2 are central modulo N. Hence $N = e_1Re_2 \oplus e_2Re_1$. Now, R has just two minimal primes $P_1 = e_1R + N = Re_1 + N$ and $P_2 = e_2R + N = Re_2 + N$, with $P_1 + P_2 = R$, $R/P_1 \simeq e_2Re_2$, $R/P_2 \simeq e_1Re_1$. Since $N^2 = 0$ it is easy to check that $P_i^2 = P_i$, $i = 1, 2$.

Since R/P_i $(i = 1, 2)$ is a scalar local ring, there are only two maximal ideals M_1, M_2 with $P_1 \subset M_1$, $P_2 \subset M_2$. Hence $\bigcap_{n=1}^{\infty} M_i^n \subseteq P_i$ and $Re_iR \subseteq \bigcap_{n=1}^{\infty} M_i^n$, $i = 1, 2$.

We have

$$P_i = P_i^2 = (Re_iR + N)^2 \subseteq Re_iR.$$

So $P_i = Re_iR = \bigcap_{n=1}^{\infty} M_i^n$, for $i = 1, 2$.

Now, since e_i is central modulo J, we have $M_i = e_iR + J = Re_i + J$, $i = 1, 2$. By Theorem 2.2.18, Proposition 3.3.17 applied to the Artinian ring R/J^2 and Lemma 3.3.18, the ideals M_1 and M_2 are linked. Suppose that $M_1 \rightsquigarrow M_2$. As $M_i^2 \neq M_i$, $i = 1, 2$, both maximal ideals are selflinked. The ring R/J^2 is of finite representation type and so, by Proposition 3.3.19, it must be $M_2 \not\rightsquigarrow M_1$, which gives $e_2Re_1 = 0$ by Lemma 3.3.18.

So $N = e_1Re_2$ and R has the required structure.

Since $NP_1 = 0$ and $P_2N = 0$, we have that N is finitely generated as left R/P_2-module and as right R/P_1-module. Because R/P_i, $i = 1, 2$ is a Dedekind prime ring and $A(R) = 0$, by Proposition 2.1.11 N is torsion free on both sides.

As $P_2P_1 = 0$, we conclude that $P_1 \rightsquigarrow P_2$ and consequently, by [24, Theorem 11.20], P_1 and P_2 have the same intersection with the centre. Since C is semiprime by Lemma 4.1.4, we conclude that it is an integral domain.

We also note that, by Proposition 3.1.1, we have $\mathrm{gl.dim}(R) = 2$.

$\square$

Lemma 4.1.7 *With the notation as in Lemma 4.1.6, we have that either $JN \subseteq NJ$ or $NJ \subseteq JN$.*

Proof. $R/J \simeq R/M_2 \oplus R/M_1$ is a basic Artinian PI ring. Set $F_1 = R/M_2$, $F_2 = R/M_1$, $F = F_1 \oplus F_2$ and ${}_iH_j = F_i(J/J^2)F_j$, $i, j = 1, 2$. Thus the species $S \equiv (F_i, {}_iH_j)$ is a quasi Artin species (cf. Definition 2.2.17).

Let us consider the ring R/J^2. An easy computation in the matrix representation of R gives:

$$R/J^2 \simeq \begin{pmatrix} R/M_2^2 & N/(M_2N + NM_1) \\ 0 & R/M_1^2 \end{pmatrix}.$$

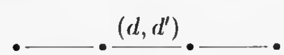
Hence

$$J/J^2 \simeq \begin{pmatrix} M_2/M_2^2 & N/(M_2N + NM_1) \\ 0 & M_1/M_1^2 \end{pmatrix}.$$

So ${}_1H_1 = M_2/M_2^2$, ${}_1H_2 = N/(M_2N + NM_1)$, ${}_2H_1 = 0$ and ${}_2H_2 = M_1/M_1^2$.

Let $d = \dim({}_1H_2)_{F_2}$ and $d' = \dim_{F_1}({}_1H_2)$. The quiver associated to R/J^2 is the following.

By hypothesis R/J^2 is of finite representation type, so by Theorem 2.2.21 the underlying diagram of the separated quiver Γ' associated to R/J^2 (cf. Definition 2.2.20) is a Dynkin diagram. We obtain that Γ' has the following underlying graph



An examination of the Dynkin diagrams (cf. Figure 2.1 on page 32) yields that this is either of kind A_4 or F_4. Therefore $(d, d') \in \{(1,1), (1,2)\}$.

By Theorem 3.2.3 all the trivial extension rings $B_n = F \ltimes J^n/J^{n+1}$, $n \in \mathbb{N}$ are of finite representation type. Let us compute J^2/J^3. We have

$$J^2/J^3 \simeq \begin{pmatrix} M_2^2/M_2^3 & (M_2N + NM_1)/(M_2^2N + M_2NM_1 + NM_1^2) \\ 0 & M_1^2/M_1^3 \end{pmatrix}.$$

We note that $M_2^k N = J^k N$ and $NM_1^k = NJ^k$, for all positive integers k.

Let us consider the ring $B_2 = F \ltimes J^2/J^3$. So $J(B_2) = 0 \ltimes J^2/J^3$ and $J(B_2)^2 = 0$. As before, by Theorem 2.2.21 the underlying diagram of the separated quiver of B_2 is a Dynkin diagram either of kind A_4 or F_4. Therefore, $\dim_{F_1}(F_1 J(B_2) F_2) \in \{1, 2\}$ and $\dim(F_1 J(B_2) F_2)_{F_2} = 1$.

So $F_1 J(B_2) F_2 \simeq (JN + NJ)/(J^2N + JNJ + NJ^2)$ is simple as right R-module. Hence the following cases can occur.

Either

$$JN + NJ = J^2N + JNJ + NJ^2 + NJ = J(JN + NJ) + NJ \tag{4.1}$$

and so, by Nakayama's Lemma, $JN \subseteq NJ$,

or

$$JN + NJ = J^2N + JNJ + NJ^2 + JN \tag{4.2}$$

and so $NJ \subseteq JN$, or both.

□

We shall use the following property of serial rings in the sequel.

Lemma 4.1.8 [60, Theorem 3.3] *Let R be a serial ring, P a finitely generated projective module, and M a finitely generated submodule of P. Then there is a decomposition $P = P_1 \oplus \ldots \oplus P_n$ of P into indecomposable projectives such that if $M_i = M \cap P_i$, then $M = M_1 \oplus \ldots \oplus M_n$.*

Finally, we get a contradiction.

Lemma 4.1.9 *The ring R of Lemma 4.1.6 does not satisfy the property (P).*

Proof. We prove that R can be embedded into a over ring T which does not satisfy the property (P). Then we show that an infinite family of pairwise non-isomorphic finitely generated indecomposable modules over a certain Artinian factor ring T/I of T is obtained by tensoring by T/I some finitely generated indecomposable modules over an Artinian factor ring of R.

For an easier reading we denote R/P_i by R_i and M_i/P_i by J_i, $i = 1, 2$.

We have two cases to examine.

<u>CASE 1</u> $JN \subsetneq NJ$.

By the proof of Lemma 4.1.7 $d = 1$, so ${}_1H_2 = N/NJ$ is simple as right R-module. Therefore $N_R \simeq R/P_1$. Hence N is uniserial as right R-module, say $N = nR$, with $n \in N$.

Let us consider the ring $E \equiv \mathrm{End}(N_R)$. Every element of E is determined by the image of n. So there is a ring homomorphism $\phi : E \to R_1$ which is defined as follows. Let $e \in E$. So $e(n) = nr$, for some $r \in R$. Set $\phi(e) = r + P_1$. It is easy to check that ϕ is bijective. We may also identify R_2 with the subring of E consisting of the left multiplication maps by the elements of R_2. In this sense, $R_2 \subsetneq E$.

Let us consider the map $f :_E E \to_E N$, defined by $f(e) = e(n)$. We have $f(J(E)) = J(E)N = NJ$.

It is easy to check that f is an isomorphism of E-modules and so also ${}_{R_2}E \simeq_{R_2} N$. Further, $f(R_2) = Rn$.

Now, R_2 is a direct summand of ${}_{R_2}E$. For, by Lemma 4.1.8, the ring R_2 is contained in a direct summand of ${}_{R_2}E$, say R_2e, with $e \in E$. If $R_2 \subsetneq J_2e$, then $R_2 \subsetneq J_2E \subsetneq J(E)$, and so $1 \in J(E)$, which is a contradiction.

As N_R is uniserial, the only right ideals of R contained in N are the ideals NJ^k, $k \in \mathbb{N}$. So $JN = NJ^h$, for some positive integer h.

Hence

$$I' \equiv \begin{pmatrix} J_2^{4t} & NJ^{4th} \\ 0 & J_1^{4th} \end{pmatrix}$$

is a two sided ideal of R for every $t \in \mathbb{N}$.

Now we consider the ring

$$T \equiv \begin{pmatrix} E & N \\ 0 & R_1 \end{pmatrix}.$$

Clearly, T is a semiperfect Noetherian PI ring and contains R. Further, R is a direct summand of ${}_RT$ and ${}_{R_2}E$ is free. It follows that ${}_RT$ is projective.

It is not difficult to check that the map $\psi : T \to T_2(R_1)$ defined by

$$\psi \begin{pmatrix} e & nr \\ 0 & s \end{pmatrix} = \begin{pmatrix} \phi(e) & r \\ 0 & s \end{pmatrix},$$

for $e \in E$, $r, s \in R_1$, is a ring isomorphism. In particular, T is finitely generated as a module over its centre.

From Theorem 3.1.2 we know that T does not have the property (P).

The map f induces a module isomorphism

$$E/J(E)^{nh} \simeq N/NJ^{nh} = N/J^n N, \tag{4.3}$$

for all $n \in \mathbb{N}$. So J_2^n is the left annihilator in R_2 of $E/J(E)^{nh}$. It follows that $J(E)^{nh} \cap R_2 = J_2^n$, for all $n \in \mathbb{N}$.

Let us consider the following ideal of T

$$I = \begin{pmatrix} J(E)^{4th} & NJ^{4th} \\ 0 & J_1^{4th} \end{pmatrix}.$$

So $I \cap R = I'$.

Now we compare the representation theories of the following factor rings of R and of T respectively,

$$R/I' = \begin{pmatrix} R_2/J_2^{4t} & N/NJ^{4th} \\ 0 & R_1/J_1^{4th} \end{pmatrix} \text{ and } T/I = \begin{pmatrix} E/J(E)^{4th} & N/NJ^{4th} \\ 0 & R_1/J_1^{4th} \end{pmatrix}.$$

By (4.3) $E/J(E)^{4th}$ is free as left R_2/J_2^{4t}-module and R_2/J_2^{4t} is a direct summand of ${}_{R_2}E/J(E)^{4th}$. So R/I' is a direct summand of ${}_RT/I$ and ${}_{R/I'}T/I$ is projective.

It is worth noting that T/I is not a separable extension of R/I' and the property for a ring to be of finite representation type is not usually preserved under these ring extensions. In our case this problem can be overcome as we show now.

Let C_1 be the centre of R_1 and yC_1, with $y \in C_1$, be its maximal ideal. So $yR_1 = J_1^t$, for some $t > 0$.

Let $n \geq 2$ and let $A_1 = (E/J(E)^{4th})^{(n)}$ be a finitely generated free right $E/J(E)^{4th}$-module and $B_1 = (J(E)^{2th}/J(E)^{4th})^{(n)}$. Let $A_2 = (R_1/J_1^{4th})^{(n)}$ be a finitely generated free right R_1/J_1^{4th}-module and $B_2 = (J_1^{2th}/J_1^{4th})^{(n)}$. Set $M_i = A_i \oplus B_i$, for $i = 1, 2$. Since N/NJ^{4th} is flat as a left $E/J(E)^{4th}$-module, by Theorem 1.1.6 we have

$$(J(E)^{2th}/J(E)^{4th})^{(n)} \otimes_{E/J(E)^{4th}} N/NJ^{4th} \simeq (NJ^{2th}/NJ^{4th})^{(n)}.$$

Hence

$$
\begin{aligned}
& M_1 \otimes_{E/J(E)^{4th}} N/NJ^{4th} \\
& \simeq (N/NJ^{4th})^{(n)} \oplus (NJ^{2th}/NJ^{4th})^{(n)} \simeq (R_1/J_1^{4th})^{(n)} \oplus (J_1^{2th}/J_1^{4th})^{(n)} \\
& = M_2,
\end{aligned}
$$

as right R_1-modules.

By Theorem 3.1.2 we can find a homomorphism $l : M_1 \otimes_{E/J(E)^{4th}} N/NJ^{4th} \to M_2$ such that the right T-module

$$L_n \equiv (M_1, M_2, l)$$

is indecomposable.

Set $A_1' = (R_2/J_2^{4t})^{(n)}$, $B_1' = (J_2^{2t}/J_2^{4t})^{(n)}$ and $M_1' = A_1' \oplus B_1'$.

Since $E/J(E)^{4th} \simeq N/NJ^{4th}$ is free as left R_2/J_2^{4t}-module we have

$$
\begin{aligned}
& M_1' \otimes_{R_2/J_2^{4t}} E/J(E)^{4th} \\
& = ((R_2/J_2^{4t})^{(n)} \oplus (J_2^{2t}/J_2^{4t})^{(n)}) \otimes_{R_2/J_2^{4t}} E/J(E)^{4th} \\
& \simeq (E/J(E)^{4th})^{(n)} \oplus (J(E)^{2th}/J(E)^{4th})^{(n)} \\
& = M_1.
\end{aligned}
$$

We consider the isomorphism above as an equality.

Let us call ϕ the isomorphism $\phi : N/NJ^{4th} \to E/J(E)^{4th} \otimes_{E/J(E)^{4th}} N/NJ^{4th}$. Hence

$$
\begin{aligned}
& M_1' \otimes_{R_2/J_2^{4t}} N/NJ^{4th} \\
& \simeq (M_1' \otimes_{R_2/J_2^{4t}} E/J(E)^{4th}) \otimes_{E/J(E)^{4th}} N/NJ^{4th} \\
& = M_1 \otimes_{E/J(E)^{4th}} N/NJ^{4th},
\end{aligned}
$$

via $\psi = 1_{M_1'} \otimes_{R_2/J_2^{4t}} \phi$.

Set $l' = l\psi$. So $l' : M_1' \otimes_{R_2/J_2^{4t}} N/NJ^{4th} \to M_2$ is a homomorphism of right R_1-modules and

$$L_n' \equiv (M_1', M_2, l')$$

is a finitely generated right R/I'-module. We have (cf. Appendix) :

$$L'_n \otimes_{R/I'} T/I \simeq (M'_1 \otimes_{R_2/J_2^{4t}} E/J(E)^{4th}, M_2 \otimes_{R_1/J_1^{4th}} R_1/J_1^{4th}, l) \simeq L_n.$$

It can be checked (cf. Appendix) that L'_n is an indecomposable module.

As $(L_n)_{n\in\mathbb{N}}$ contains a countable family of pairwise non-isomorphic finitely generated indecomposable right T/I-modules, so does $(L'_n)_{n\in\mathbb{N}}$ and hence R/I' cannot be of finite representation type.

CASE 2 $\quad NJ \subseteq JN$.

By the proof of Lemma 4.1.7, $(d,d') \in \{(1,1),(1,2)\}$. If $(d,d') = (1,1)$ then a similar argument will apply.

So let us assume that $(d,d') = (1,2)$. Hence ${}_1H_2 = N/JN$ is simple as right R-module and it is a sum of two simple left R-modules. Since N is free left R_2-module, we have ${}_RN \simeq R_2 \oplus R_2$, say $N = Rn_1 \oplus Rn_2$, with $n_1, n_2 \in N$. Let $F \equiv \mathrm{End}({}_RN)$. Thus N is a R-F-bimodule. Now, every element of F is of the form

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{ij} : Rn_i \to Rn_j$, $i,j = 1,2$ are homomorphisms of R-modules. Every f_{ij} is determined by the image of n_i. In a similar way as for Case 1 one can define a ring isomorphism between F and the ring of matrices $M_2(R_2)$. Further, we may identify R_1 with the subring of F consisting of the right multiplication maps by elements of R_1. In this sense, $R_1 \subseteq F$.

Let us consider the map $g : F_F \to N_F^{(2)}$, defined by $g(f) = (n_1 f, n_2 f)$. Thus g is a homomorphism of F-modules. The map g is surjective, for if $x = (r_1n_1 + r_2n_2, s_1n_1 + s_2n_2)$ is an element of $N^{(2)}$, define f_{11} by $n_1f_{11} = r_1n_1$, f_{12} by $n_1f_{12} = r_2n_2$, f_{21} by $n_2f_{21} = s_1n_1$ and f_{22} by $n_2f_{22} = s_2n_2$. Then $g(f) = x$. It is easy to see that g is also injective.

Hence also $F_{R_1} \simeq N_{R_1}^{(2)}$. It follows that F is a finitely generated free right R_1-module.

Now, $J(F) = \{f :_R N \to_R N \mid Nf \subseteq JN\}$. In fact, $NJ(F) = JN$.

Let $r + P_1 \in J_1$. So $Nr \subseteq NJ \subseteq JN$. Hence $r + P_1 \in J(F)$. Thus $J_1 \subseteq J(F)$ and so $J(F) \cap R_1 = J_1$, because R_1 is a local ring.

R_1 is a direct summand of F_{R_1}. For, by Lemma 4.1.8, the ring R_1 is contained in a direct summand of F_{R_1}, say fR_1, with $f \in F$. If $R_1 \subseteq fJ_1$, then $R_1 \subseteq FJ_1 \subseteq J(F)$, and so $1 \in J(F)$, which is a contradiction.

As $J^iN/J^{i+1}N$ is a simple right R-module for all i and $\bigcap_{n=1}^{\infty} J^nN = 0$, an easy application of Nakayama's Lemma gives that $NJ = J^hN$, for some integer h. So $NJ^n = J^{hn}N$, for all n. Now, $n_1F = N$. So $n_1J(F)^h = n_1FJ(F)^h = NJ(F)^h = J^hN = NJ$. Also, $n_1FJ_1 = NJ$ and hence $n_1J(F)^h = n_1FJ_1$. Analogousely, we obtain that $n_2J(F)^h = n_2FJ_1$. Now, by using the isomorphism g, it is not difficult to check that $FJ_1 = J(F)^h$ and so FJ_1 is a two-sided ideal of F. Also, the map g induces an isomorphism

$$F/J(F)^{hn} \simeq (N/J^{hn}N)^{(2)} = (N/NJ^n)^{(2)}. \tag{4.4}$$

So the annihilator in R_1 of $F/J(F)^{hn}$ is J_1^n. Hence $J(F)^{nh} \cap R_1 = J_1^n$ for all n.

Let us consider the ring

$$T \equiv \begin{pmatrix} R_2 & N \\ 0 & F \end{pmatrix}.$$

T is a Noetherian semiperfect PI ring and contains R. Further, R is a direct summand of T_R and F_{R_1} is free. Therefore, T_R is projective.

We note that T is Morita equivalent to the upper triangular matrix ring $T_2(R_2)$. In fact, we have

$$T \simeq \begin{pmatrix} R_2 & R_2 & R_2 \\ 0 & R_2 & R_2 \\ 0 & R_2 & R_2 \end{pmatrix}$$

via the homomorphism ϕ defined by

$$\phi \begin{pmatrix} s & \alpha n_1 + \beta n_2 \\ 0 & f \end{pmatrix} = \begin{pmatrix} s & \alpha & \beta \\ 0 & r_{11} & r_{12} \\ 0 & r_{21} & r_{22} \end{pmatrix},$$

for $s, \alpha, \beta \in R_2$ and $r_{ij} \in R_2$, $i, j = 1, 2$ such that $n_i f_{ij} = r_{ij} n_j$.

Let $e = E_{11} + E_{22}$ be an idempotent of T where E_{ii}, $i = 1, 2$ are the elementary matrices. Thus $eTe \simeq T_2(R_2)$ and $TeT = T$.

By Corollary 3.1.5, the ring T does not satisfy the property (P).

Let C_2 be the centre of R_2 and yC_2, with $y \in C_2$, its maximal ideal. So $yR_2 = J_2^t$, for some integer t. Let us consider the ideals of R and of T respectively

$$I' \equiv \begin{pmatrix} J_2^{4th} & NJ_1^{4t} \\ 0 & J_1^{4t} \end{pmatrix} \text{ and } I \equiv \begin{pmatrix} J_2^{4th} & J^{4th}N \\ 0 & J(F)^{4th} \end{pmatrix};$$

So $I \cap R = I'$.

Now we compare the representation type of the factor rings

$$R/I' = \begin{pmatrix} R_2/J_2^{4th} & N/NJ^{4t} \\ 0 & R_1/J_1^{4t} \end{pmatrix} \text{ and } T/I = \begin{pmatrix} R_2/J_2^{4th} & N/J^{4th}N \\ 0 & F/J(F)^{4th} \end{pmatrix}.$$

We first note that, from (4.4), $F/J(F)^{4th}$ is free as right R_1/J_1^{4t} and R_1/J_1^{4t} is a direct summand of $F/J(F)^{4th}$. Hence R/I' is a direct summand of $(T/I)_R$ and $(T/I)_{R/I'}$ is projective.

We construct a countable family of finitely generated indecomposable left R/I'-modules, pairwise non-isomorphic.

Let $n \geq 2$ and let $A_1 \equiv (R_2/J_2^{4th})^{(2n)}$ be a finitely generated free left R_2/J_2^{4th}-module and $B_1 \equiv (J_2^{2th}/J_2^{4th})^{(2n)}$. Let $A_2 \equiv (F/J(F)^{4th})^{(n)}$ be a finitely generated free left $(F/J(F)^{4th}$-module and $B_2 \equiv (J(F)^{2th}/J(F)^{4th})^{(n)}$.

Set $M_i = A_i \oplus B_i$, with $i = 1, 2$.

Since $N/J^{4th}N$ is flat as right $F/J(F)^{4th}$-module we have

$$\begin{aligned}
&N/J^{4th}N \otimes_{F/J(F)^{4ht}} M_2 \\
&\simeq \; (N/J^{4th}N \otimes_{F/J(F)^{4ht}} (F/J(F)^{4th})^{(n)}) \oplus (N/J^{4th}N \otimes_{F/J(F)^{4ht}} (J(F)^{2th}/J(F)^{4th})^{(n)}) \\
&\simeq \; (N/J^{4th}N)^{(n)} \oplus (J^{2th}N/J^{4th}N)^{(n)} \\
&\simeq \; (R_2/J_2^{4th})^{(2n)} \oplus (J_2^{2th}/J_2^{4th})^{(2n)} \\
&= \; M_1
\end{aligned}$$

as left R_2-modules.

So by Corollary 3.1.5 there is a homomorphism $l : (N/J^{4th}N) \otimes_{F/J(F)^{4th}} M_2 \to M_1$ such that the left T/I-module

$$L_n \equiv (M_1, M_2, l)$$

is indecomposable.

We set $A_2' = (R_1/J_1^{4t})^{(n)}$, $B_2' = (J_1^{2t}/J_1^{4t})^{(n)}$ and $M_2' = A_2' \oplus B_2'$.

Now $F/J(F)^{4th}$ is projective as right R_1/J_1^{4t}-module, hence

$$F/J(F)^{4ht} \otimes_{R_1/J_1^{4t}} (J_1^{2t}/J_1^{4t})^{(n)} \simeq (FJ_1^{2t}/J(F)^{4ht})^{(n)} = (J(F)^{2th}/J(F)^{4ht})^{(n)}.$$

So $F/J(F)^{4ht} \otimes_{R_1/J_1^{4t}} M_2' \simeq M_2$. We consider this isomorphism as an identity.

Let us call ϕ the isomorphism $\phi : N/NJ^{4th} \to N/NJ^{4th} \otimes_{F/J(F)^{4th}} F/J(F)^{4th}$. We have

$$\begin{aligned} & N/J^{4th}N \otimes_{R_1/J_1^{4t}} M_2' \\ \simeq\ & (N/J^{4th}N \otimes_{F/J(F)^{4ht}} F/J(F)^{4th}) \otimes_{R_1/J_1^{4t}} M_2' \\ =\ & N/J^{4th}N \otimes_{F/J(F)^{4th}} M_2, \end{aligned}$$

via $\psi = \phi \otimes_{R_1/J_1^{4t}} 1_{M_2'}$.

Set $l' = l\psi$. So $l' : N/J^{4th}N \otimes_{R_1/J_1^{4t}} M_2' \to M_1$ is a homomorphism of left R_2-modules and

$$L_n' \equiv (M_1, M_2', l')$$

is a finitely generated left R/I'-module. We have (cf. Appendix) :

$$T/I \otimes_{R/I'} L_n' \simeq (R_2/J_2^{4th} \otimes_{R_2/J_2^{4th}} M_1, F/J(F)^{4th} \otimes_{R_1/J_1^{4t}} M_2', l) \simeq L_n.$$

Further, L_n' is an indecomposable module (cf. Appendix). Since $(L_n)_{n\in\mathbb{N}}$ contains an infinite family of pairwise non-isomorphic finitely generated indecomposable left T/I-modules, the ring R/I' cannot be of finite representation type.

□

This contradiction forces the ring R, with the hypothesis of Proposition 4.1.3, to be semiprime and so R is hereditary by Proposition 4.1.2. This finishes the proof of Proposition 4.1.3.

□

Remark 4.1.10 Both cases examined in the proof of Lemma 4.1.9 can occur. In fact, an example of a Noetherian semiperfect PI ring R with $A(R) = 0$ and the property $J(R)N \subsetneq NJ(R)$ is

$$R \equiv \begin{pmatrix} k[x^2]_{(x^2)} & k[x]_{(x)} \\ 0 & k[x]_{(x)} \end{pmatrix},$$

where k is a field. Moreover, the ring

$$S \equiv \begin{pmatrix} k[x]_{(x)} & K[x]_{(x)} \\ 0 & K[x^2]_{(x^2)} \end{pmatrix},$$

where $K \supsetneq k$ are fields such that $[K : k] = 2$, is a Noetherian semiperfect PI ring with $A(S) = 0$ and satisfies $NJ(S) \subsetneq J(S)N$.

Remark 4.1.11 If the ring R of Proposition 4.1.3 is finitely generated as a module over its centre, it is possible to prove that the subring eRe of R of Lemma 4.1.9 admits a decomposition $eRe \simeq S \oplus M$, where

$$S = \begin{pmatrix} C & C \\ 0 & C \end{pmatrix},$$

C is the centre of eRe and M

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

is a S-S-submodule of eRe, with A, B and D finitely generated free C-submodules respectively of R_2, N and R_1. So by Proposition 3.4.12 S has the property (P). Thus a proof identical to that in [10] can be applied to obtain the final contradiction.

4.2 The main theorem

We need a few other lemmata.

Lemma 4.2.1 *Let R be a Noetherian PI ring which has an Artinian quotient ring $Q(R)$. Let $\mathcal{S}$ be an Ore set in R. Then $R_{\mathcal{S}}$ has an Artinian quotient ring.*

Proof. Set $T = \{r \in R \mid rs = 0 \text{ for some } s \in \mathcal{S}\}$. It is standard to show, by using the Ore condition on $\mathcal{S}$, that T is a two sided ideal of R and it is equal to $T' \equiv \{r \in R \mid sr = 0 \text{ for some } s \in \mathcal{S}\}$.

We claim that $Q(R)/TQ(R)$ is the quotient ring of R/T.

Since R is Noetherian we have that $Ts = 0$ for some $s \in \mathcal{S}$. We note that $TQ(R) \cap R = T$. For, clearly $T \subseteq TQ(R) \cap R$. On the other hand, $(TQ(R) \cap R)s = 0$, since every element of $TQ(R) \cap R$ can be written as $c^{-1}t$, with $t \in T$ and c a regular element of R. So $TQ(R) \cap R \subseteq T$ by the definition of T.

Hence there is a ring embedding $R/T \to Q(R)/TQ(R)$. It is straightforward to check that $\mathcal{C}_R(T) = \mathcal{C}_{Q(R)}(TQ(R)) \cap R$.

So $Q(R)/TQ(R)$ is the quotient ring of R_S.

□

Lemma 4.2.2 [13, Proposition 13.14] *Let R be a Noetherian FBN ring. Suppose that $A(R) = 0$. Then every maximal ideal of R contains a regular element.*

Lemma 4.2.3 *Let R be a Noetherian PI ring with $A(R) = 0$. Let X be a classically localisable clique of maximal ideals of R. Then $A(R_X) = 0$.*

Proof. Let $T = \{r \in R \mid rs = 0, \text{ for some } s \in \mathcal{C}(X)\}$.

We claim that $A(R/T) = 0$. Suppose that $A(R/T) \neq 0$. Let A be an ideal of R such that $A/T = A(R/T)$. Since A/T is an Artinian left R-module there exist maximal ideals $M_1, M_2, \ldots, M_t$ of R, not necessarily distinct, such that $M_1 M_2 \ldots M_t A \subseteq T$. By Lemma 4.2.2 there exists c a regular element inside $M_1 M_2 \ldots M_t$. So $cAs = 0$ for some $s \in \mathcal{C}(X)$. Hence $As = 0$ and therefore $A \subseteq T$.

Lemma 4.2.2 applied to R/T gives that every maximal ideal of R containing T intersects $\mathcal{C}(T)$. In particular, this holds for the maximal ideals of R which are in X.

Now R_X is an over ring of R/T, whose maximal ideals are MR_X, with $M \in X$ (cf. Definition 1.4.7). Hence MR_X contains a regular element of R_X, for all $M \in X$.

Suppose that R_X contains a minimal right ideal I. As R_X is a PI ring, $IMR_X = 0$, for some $M \in X$ and so $I = 0$. Contradiction.

□

Proposition 4.2.4 *Let R be a Noetherian PI ring with $A(R) = 0$. Suppose that every clique of maximal ideals of R is finite. Then the following are equivalent.*

1. *R has the property (P).*

2. *R is a hereditary semiprime ring.*

3. *Every proper Artinian factor ring of R is serial.*

Proof. (1.) $\Rightarrow$ (2.) Let X be a clique of maximal ideals of R and R_X be the localisation of R at $\mathcal{C}(X)$. So R_X is a semilocal Noetherian PI ring. By Lemma 4.2.3, we have $A(R_X) = 0$. Let $\hat{R}_X$ be the $J(R_X)$-adic completion of R_X. So $\hat{R}_X$ is a Noetherian semiperfect PI ring and is indecomposable by Theorem 1.5.14. By Proposition 3.4.18 the ring $\hat{R}_X$ has the property (P).

By Lemma 4.2.2 every maximal ideal of R_X contains a regular element. So $J(R_X)$ contains a regular element, c say. So $c \in \hat{J}$. By Corollary 1.5.10 c is regular in $\hat{R}_X$. It follows that $\hat{R}_X$ has no minimal right or left ideals and therefore $A(\hat{R}_X) = 0$.

By Proposition 4.1.3 $\hat{R}_X$ is a hereditary semiprime ring. So, being indecomposable, $\hat{R}_X$ is a prime ring. It follows that R_X is a prime ring and $R_X/J(R_X)^k \simeq \hat{R}_X/\hat{J}^k$ is serial for all k.

We claim that K.dim$(R_X) = 1$. For, let P a prime ideal of R_X that is not maximal. By Theorem 1.5.8 we have $(\widehat{R_X/P}) \simeq \hat{R}_X/P\hat{R}_X$. Now, $(\widehat{R_X/P})$ is the $J(R_X/P)$-adic completion of R_X/P. The ring R_X/P is a Noetherian semilocal prime PI ring and has the property (P), so by Proposition 4.1.3 $(\widehat{R_X/P})$ is a semiprime hereditary ring. It follows that $P\hat{R}_X$ is a semiprime ideal of $\hat{R}_X$. Since $\hat{R}_X$ has Krull dimension one and $A(R_X/P) = 0$ we have that $P\hat{R}_X = 0$ and therefore $P = 0$.

So by Theorem 2.2.14 R_X is a hereditary ring. Since this is true for all cliques of maximal ideals of R, the result follows by Theorem 1.4.9.

(2.) $\Rightarrow$ (3.) By Theorem 2.1.6, the ring R is a direct sum of hereditary Noetherian prime rings. Hence (3.) follows from Theorem 2.2.12.

(3.) $\Rightarrow$ (1.) This follows by Proposition 2.2.11.

$\square$

In Chapter 6 we show that the assumption that R has an Artinian quotient ring cannot be removed.

We can also improve Proposition 3.4.6 with the following Theorem.

Theorem 4.2.5 *Let R be a local Noetherian PI ring. Suppose that R is not Artinian and R/J^2 is of finite representation type. Then R is a hereditary prime ring.*

Proof. Let $\hat{R}$ be the J-adic completion of R. So $\hat{R}$ is a semiperfect Noetherian PI ring (cf. Theorems 1.5.8 and 1.5.9). By the proof of Proposition 3.4.18 the factor ring $\hat{R}/\hat{J}^2$ is of finite representation type. So $\hat{R}$ is a serial prime ring by Proposition 3.4.6. Hence R is a prime ring. Arguing as in the proof of Proposition 4.2.4, we show that R has Krull dimension one and so R is a hereditary ring by Theorem 2.2.14.

□

There is a class of Noetherian PI ring whose cliques of prime ideals are always finite, namely rings which are integral over their centre (cf. [40, Proposition 9]). Thus for these rings our result follows directly from Proposition 4.2.4.

Another case of finite cliques of maximal ideals is when the ring is prime and has Krull dimension one (cf. [40, Proposition 9]). Thus we have

Theorem 4.2.6 *Let R be a Noetherian PI ring. Suppose that R is not Artinian. Then the following are equivalent.*

1. *R is hereditary prime.*
2. *Every proper factor ring of R is of finite module type, i.e. has finitely many finitely generated indecomposable modules, up to isomorphism.*

Proof. (2.) $\Rightarrow$ (1.) By Theorem 2.2.10 every proper factor ring of R is Artinian, and so is of finite representation type. As R is not Artinian, it must be prime and has Krull dimension one. So every clique of maximal ideals of R is finite and so Proposition 4.2.4 applies.

□

Next we show that in the semiprime case the assumption above on "every" factor ring of R, in fact, is not necessary.

Lemma 4.2.7 *Let R be a semiprime Noetherian PI ring with $A(R) = 0$. Suppose that R has the property (P). Then all cliques of maximal ideals of R are finite.*

Proof. By Proposition 3.3.20 all cliques of maximal ideals are finite if R is prime and so Proposition 4.2.4 yields that R is hereditary.

Suppose now that R is not a prime ring. By the above, R has Krull dimension one and R/P is a hereditary ring for every minimal prime ideal P of R.

Let $P_1, \dots, P_n$ be the minimal prime ideals of R. Set $S = R/P_1 \oplus \dots \oplus R/P_n$. Thus S is a hereditary semiprime Noetherian PI ring. Let f be the natural ring monomorphism

$$f : R \to R/P_1 \oplus \ldots \oplus R/P_n.$$

We identify R with the subring Imf of S. Hence S is a ring extension of R and it is finitely generated as an R-module. Every maximal ideal of S has form $M_i' = R/P_1 \oplus \ldots \oplus M/P_i \oplus \ldots \oplus R/P_n$, where M is a maximal ideal of R containing P_i. So, for every maximal ideal M of R there are at most n maximal ideals of S lying over M, namely the M_i'. On the other hand, we have $M_i' \cap R = M$. Hence every maximal ideal of S lies over exactly one maximal ideal of R. So Proposition 3.3.16 applies and therefore the cliques of maximal ideals of R are finite.

□

Now we sum up Lemma 4.2.7 and Proposition 4.2.4 in the following theorem which is the main result of this chapter.

Theorem 4.2.8 *Let R be a Noetherian semiprime PI ring. Then R is hereditary if and only if R has the property (P).*

Proof.

□

Chapter 5

The main theorem

In this chapter we look at the general situation of a Noetherian PI ring R, which is an order in an Artinian ring and has the property (P). By Proposition 4.2.4 we only have to consider the case where R has an infinite clique of maximal ideals. Assuming this and reducing further the assumptions on R, we get a contradiction by detecting certain factor rings of infinite type and whose structure is very similar to that of the rings described in Lemma 4.1.6. The Jacobson radical of those rings is replaced here by a certain semimaximal ideal I of R. As I is not localisable, it does not have the AR-property. Therefore, some more technical work is needed to show that the factor rings R/I^n, $n \in \mathbb{N}$, have the required structure.

5.1 Orders in Artinian rings

We obtain the following result.

Proposition 5.1.1 *Let R be a Noetherian PI ring with the property (P). Suppose that $A(R) = 0$. Then R is a semiprime hereditary ring.*

Proof. By Theorem 4.2.8 the factor ring R/N is hereditary. So it is enough to prove that R is semiprime.

We have that $\mathrm{K.dim}(R) = 1$ and R has an Artinian quotient ring by Theorem 1.3.9. Let $e_1, \ldots, e_n$ be idempotents of R such that $R/N = \bar{e}_1 R/N \oplus \ldots \oplus \bar{e}_n R/N$ is a ring decomposition of R/N as a finite sum of prime hereditary rings. By Lemma 1.2.9 this lifts to a decomposition $R = e_1 R \oplus \ldots \oplus e_n R$ of R, with e_i idempotent for all i. There is a one-to-one correspondence between the set $\{e_1, \ldots, e_n\}$ and the minimal prime ideals $\{P_1, \ldots, P_n\}$ of R, given by $e_i \to P_i$ such that $e_i \notin P_i$.

Let e be any idempotent of R. By Lemma 4.1.1 we have $A(eRe) = 0$ and the ring eRe has the property (P) by Proposition 3.4.9.

Now we split the rest of the proof in a series of lemmata.

Lemma 5.1.2 *Suppose that e_iRe_i is a prime ring, for $i = 1, \ldots, n$. Then R is a semiprime ring.*

Proof. By Theorem 4.2.8 each of the rings e_iRe_i, $i = 1, \ldots, n$, is hereditary.

Let us suppose for a moment that R has just two minimal prime ideals. Therefore, $n = 2$.

By the Peirce decomposition we have

$$R \simeq \begin{pmatrix} e_1Re_1 & e_1Re_2 \\ e_2Re_1 & e_2Re_2 \end{pmatrix}.$$

Since $e_iNe_i = 0$, for $i = 1, 2$, and e_1 and e_2 commute modulo N, an easy computation yields that $N = e_1Re_2 \oplus e_2Re_1$. Further, e_1Re_2 and e_2Re_1 are two sided ideals of R.

Suppose that $N \neq 0$ and let us consider the factor ring R/e_2Re_1 of R and let A/e_2Re_1 be its Artin radical, with A an ideal of R. As $A(R/N) = 0$, by Theorem 1.3.7 (3.), applied to R/e_2Re_1, we have $A/e_2Re_1 \subsetneq N/e_2Re_1$ and so $A \subsetneq N$. It follows that $A = e_2Re_1 \oplus e_1Ae_2$ and $e_1Ae_2 = A \cap e_1Re_2$ is a two sided ideal of R. Hence $e_1Ae_2 \simeq A/e_2Re_1$ is an Artinian ideal of R and so it must be zero. Hence $A(R/e_2Re_1) = 0$. Similarly, $A(R/e_1Re_2) = 0$.

As R/e_2Re_1 has all the other properties of R, we may assume that $e_2Re_1 = 0$.

So $N = e_1Re_2$ and the right ideals of R contained in N are all the right e_2Re_2-submodules of N. Now, N is finitely generated as a right e_2Re_2-module and, since $A(R) = 0$, N must be torsion free (cf. Proposition 2.1.11). So by Proposition 3.1.1 and because $N \neq 0$, we obtain that gl.dim$(R) = 2$. It follows from [8, Theorem 5.13] that all cliques of maximal ideals of R are finite. This can also be seen by embedding R into a hereditary ring S (as in the proof of [8, Theorem 4.16]), which is Morita equivalent to e_2Re_2, and then applying Proposition 3.3.16. So Proposition 4.2.4 gives that $N \subsetneq e_2Re_1$ and so $N = 0$, that is a contradiction.

Now suppose that $n > 2$. We have $N = \bigoplus_{i \neq j} e_iRe_j$. By the above $e_iRe_j = 0$, for every $i \neq j$. Hence the result follows.

□

As announced at the beginning of this chapter, the final contradiction is due to the existence of some "bad" factor rings of R.

Lemma 5.1.3 *Let R be as in Proposition 5.1.1, we may assume that $N^2 = 0$.*

Proof. Let $Q \equiv Q(R)$ be the quotient ring of R and consider the ring $R/(N^2Q \cap R)$. We claim that $A(R/N^2Q \cap R) = 0$. To this end we note that the Artin radical of $R/(N^2Q \cap R)$ is $A/(N^2Q \cap R)$, with A an ideal of R contained in N (cf. Theorem 1.3.7 (3.)). By Lemma 4.2.2 every maximal ideal of R/N contains a regular element. So by Corollary 1.2.2 $A/(N^2Q \cap R)$ is torsion as an R/N-module. Therefore, $AQ = N^2Q$, and hence $A = N^2Q \cap R$.

Now $R/(N^2Q \cap R)$ has the same properties as R and nilpotent radical square equal to zero. So we may replace R with $R/(N^2Q \cap R)$.

□

Lemma 5.1.4 *We may assume that R has no non-trivial idempotents.*

Proof. Consider a fixed e_i and suppose that all cliques of maximal ideals of eRe are finite, for every primitive component e of e_i. Then by Proposition 4.2.4 each of the rings eRe is hereditary and prime. Since eRe is indecomposable as a right ideal, it is an integral domain. Therefore its quotient ring $eQ(R)e$ is a division ring.

The quotient ring $e_iQ(R)e_i$ of e_iRe_i is local Artinian, with Jacobson radical $e_iNQ(R)e_i$. By Corollary 1.2.13 we have $e_iNQ(R)e_i = 0$ and hence $e_iNe_i = 0$.

If this holds for all i, then by Lemma 5.1.2 the ring R is semiprime. Therefore we may assume that for at least one i and a primitive idempotent e, the ring eRe has an infinite clique of maximal ideals.

□

Thus, if the ring R with the hypothesis of Proposition 5.1.1 is not semiprime then we may replace it by eRe as above and assume that 1 is primitive in R and R has an infinite clique of maximal ideals.

Lemma 5.1.5 *In the situation of Proposition 5.1.1, of Lemma 5.1.3 and of Lemma 5.1.4, let us assume that N is a prime ideal. Let P and Q be maximal ideals of R that are not idempotent modulo N and with $P \rightsquigarrow Q$. Set $I = P \cap Q$ and let bars denote images in R/I^n. Then*

$$\bar{R} \simeq \begin{pmatrix} R/P^n & \bar{N} \\ 0 & R/Q^n \end{pmatrix}.$$

Proof. The ring R/N is hereditary prime and, being indecomposable, is uniform. By Proposition 2.1.11 (2.), every finitely generated torsion-free R/N-module is isomorphic to a direct sum of uniform right (left) ideals of R/N. In particular, this holds for N. Let M be a maximal ideal of R. By Theorem 2.1.7 either M/N is invertible or idempotent. So P and Q are invertible modulo N and hence $\bigcap_{k=1}^{\infty} P^k \subseteq N$ and $\bigcap_{k=1}^{\infty} Q^k \subseteq N$.

Now, $N \subseteq P$. Suppose there exists $t \in \mathbb{N}$ such that $N \subseteq P^t$ and $N \not\subseteq P^{t+1}$. So $0 \neq (P^{t+1}+N)/P^{t+1} \subseteq P^t/P^{t+1}$. By Corollary 3.4.4 and Remark 3.4.5 the Artinian local PI ring R/P^{t+1} is serial, hence we conclude that $P^{t+1} + N = P^t$. This leads to the contradiction $P^t/N \subseteq \bigcap_{k=1}^{\infty}(P/N)^k = 0$, as R/N is not Artinian.

Therefore, $\bigcap_{k=1}^{\infty} P^k = N$. Similarly, $\bigcap_{k=1}^{\infty} Q^k = N$ and so $\bigcap_{k=1}^{\infty} I^k \subseteq N$.

As P/N and Q/N are not linked in R/N, we have $I = PQ + N$.

Now, $N \not\subseteq I^2$. For otherwise $N \subseteq PQ$ and so $I = PQ$. Hence we would have the contradiction that $P \not\rightsquigarrow Q$. So $N \neq \bigcap_{k=1}^{\infty} I^k$.

The ring $\bar{R} = R/I^n$ is Artinian with two prime ideals $\bar{P}$ and $\bar{Q}$.

Let $\bar{R} = \bar{e}_1\bar{R} \oplus \bar{e}_2\bar{R}$ be a decomposition of $\bar{R}$, with $\bar{e}_i$, $i = 1, 2$, idempotents of $\bar{R}$, which are central modulo $\bar{I}$. Set $\bar{P} = \bar{e}_2\bar{R} + \bar{I}$ and $\bar{Q} = \bar{e}_1\bar{R} + \bar{I}$.

By Proposition 3.3.19 $Q \not\rightsquigarrow P$ and so $QP = I$. We prove that $\bar{R}$ has the required structure in three steps.

STEP 1

Clearly $\bar{Q} \not\rightsquigarrow \bar{P}$. So by Proposition 3.3.17, we have $\bar{e}_2\bar{R}\bar{e}_1 = \bar{0}$.

STEP 2

Assume, from now on, that $n \geq 2$. As $N \not\subseteq I^2$, we have that $\bar{N} \neq 0$.

Now $\bar{P} = \bar{e}_2\bar{R} + \bar{I} = \bar{R}\bar{e}_2 + \bar{I}$, so $\bar{e}_1\bar{P} = \bar{e}_1\bar{I}$ and $\bar{P}\bar{e}_1 = \bar{I}\bar{e}_1$. From the matrix representation of $\bar{R}$:

$$\bar{R} \simeq \begin{pmatrix} \bar{e}_1\bar{R}\bar{e}_1 & \bar{e}_1\bar{R}\bar{e}_2 \\ \bar{0} & \bar{e}_2\bar{R}\bar{e}_2 \end{pmatrix},$$

we easily obtain that $\bar{e}_1\bar{P}^k\bar{e}_1 = \bar{e}_1\bar{I}^k\bar{e}_1$, for all k.

So $\bar{e}_1\bar{N}\bar{e}_1 \subseteq \bar{e}_1\bar{P}^k\bar{e}_1$, for all k which gives $\bar{e}_1\bar{N}\bar{e}_1 = \bar{0}$, as $\bar{e}_1\bar{I}\bar{e}_1$ is nilpotent.

Similarly, $\bar{e}_2\bar{N}\bar{e}_2 = \bar{0}$. It follows that $\bar{e}_1\bar{N}\bar{e}_2 = \bar{N}$.

STEP 3

From $\bar{I} = \bar{P}\bar{Q} + \bar{N}$, $\bar{e}_1\bar{P} = \bar{e}_1\bar{I}$ and $\bar{Q}\bar{e}_2 = \bar{I}\bar{e}_2$, we get $\bar{e}_1\bar{I}\bar{e}_2 = \bar{e}_1\bar{I}^2\bar{e}_2 + \bar{e}_1\bar{N}\bar{e}_2$.

Now,

$$\bar{e}_1\bar{I}^2\bar{e}_2 = (\bar{e}_1\bar{I}\bar{e}_1)(\bar{e}_1\bar{R}\bar{e}_2) + (\bar{e}_1\bar{R}\bar{e}_2)(\bar{e}_2\bar{I}\bar{e}_2),$$

and so, recalling that $\bar{e}_1$ and $\bar{e}_2$ commute modulo $\bar{I}$, we get

$$\bar{e}_1\bar{R}\bar{e}_2 = \bar{e}_1\bar{I}\bar{e}_2 = (\bar{e}_1\bar{I}\bar{e}_1)(\bar{e}_1\bar{R}\bar{e}_2) + (\bar{e}_1\bar{R}\bar{e}_2)(\bar{e}_2\bar{I}\bar{e}_2) + \bar{e}_1\bar{N}\bar{e}_2.$$

Since $\bar{R}$ is Artinian, $\bar{e}_1\bar{R}\bar{e}_2$ is of finite length as right $\bar{e}_2\bar{R}\bar{e}_2$-module and as left $\bar{e}_1\bar{R}\bar{e}_1$-module. Hence Nakayama's Lemma applied twice, gives $\bar{e}_1\bar{N}\bar{e}_2 = \bar{e}_1\bar{R}\bar{e}_2$. It follows that $\bar{N} = \bar{e}_1\bar{R}\bar{e}_2$.

□

We need the following lemma.

Lemma 5.1.6 *Let G be an infinite connected locally finite graph. Then G contains an infinite semiline.*

Proof. We start with any vertex $k(0)$. Let $a_1, \ldots, a_n$ be the edges connecting $k(0)$ to the vertices $k(1), \ldots k(n)$ of G, with $k(i) \neq k(0)$ for all $i = 1, \ldots, n$.

Consider the subgraph G_1 of G that is obtained by removing $k(0)$ and $a_1, \ldots, a_n$ from G. Then G_1 has at least one infinite connected component that must contain at least one of the vertices $k(1), \ldots$ or $k(n)$ (this because any vertex in G_1 can be connected to $k(0)$ by a path in G, and so by a path in G_1, ending in one of the $k(i)$). So we choose one of these infinite components of G_1 that contains, say, $k(1)$. We repeat the argument on G_1 and $k(1)$, instead of G and $k(0)$, respectively. The existence of an infinite semiline follows from the axiom of choice (Zorn's Lemma).

□

Lemma 5.1.7 *Suppose that R has an infinite clique $\mathcal{C}$ of maximal ideals. Then the graph of links G associated to $\mathcal{C}$ contains an infinite subgraph G' as in Figure 5.2 on page 86. Let V' denote the set of vertices of G'. Then there are no other vertices in G connected to those of V' except the first vertex.*

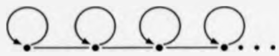
Proof. We recall that G has the following properties:

- G is locally finite, i.e. it does not contain infinite stars (cf. Theorem 3.3.8);

- There are only finitely many vertices in G without a loop (cf. Theorem 3.3.11);
- Every finite connected subgraph of G has a separated graph whose underlying diagram is a disjoint union of Dynkin diagrams (cf. Proposition 3.3.19).

Let us consider the subgraph of G obtained by removing all vertices without a loop with attached edges. This has still an infinite connected component, say H. So replacing G by H, we may assume that all vertices of G have a loop.

By Lemma 5.1.6 there is an infinite (undirected) semiline:



Suppose that the semiline contains one of the two subgraphs in Figure 5.1, where the directions of the rest are not yet specified.

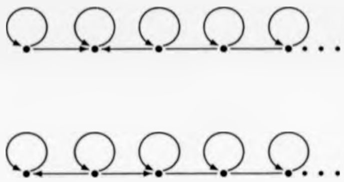


Figure 5.1: Two semilines

Let us consider the separated graph of the finite subgraph, containing the first six vertices and corresponding edges, of the first semiline in Figure 5.1. This is a Dynkin diagram of kind E_6. Adding sufficiently many arrows to the finite subgraph leads to a contradiction as the only 'long' Dynkin diagrams (without considering the values) are A_n and D_n.

A similar argument shows that G does not contain the second semiline of Figure 5.1 as a subgraph.

Hence the arrows of the semiline are all in the same direction as indicated in Figure 5.2.

Suppose that G contains something like Figure 5.3.
Then the separated graph would have a "long" line and a "branching" in between. This contradicts again the Dynkin diagram condition. In the case shown in Figure 5.4 (page 87) one can just remove the left vertex and consider the rest.

□

Figure 5.2: Semiray

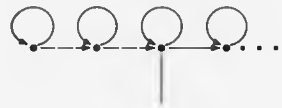


Figure 5.3: Semiray with a branch I

Then we detect factor rings of R whose structures are very similar to those described in Lemma 4.1.9.

Lemma 5.1.8 *In the situation of Lemma 5.1.5, there are prime ideals P and Q of R, such that $\bar{N} \simeq N/NQ^n$ or $\bar{N} \simeq N/P^nN$.*

Proof. We choose P', P, Q and Q', maximal ideals of R which correspond to four consecutive vertices of G' as shown in Figure 5.5.

As their images in R/N cannot belong to cycles, we conclude that they are all invertible ideals of R/N (cf. see discussion on page 25). By Theorem 2.1.7 the ideals P', P, Q and Q' of R commute modulo N.

Set $T = \text{l.ann}_R(N/NQ)$. So $N \subseteq T$. By Lenagan's Theorem (cf. [37, Theorem 4.1.6]) N/NQ is also Artinian as a left R-module. Hence, it is easy to see that ring R/T is Artinian. Now, P is the only maximal ideal of R, different from Q, with $P \rightsquigarrow Q$. So by Remark 3.3.10, we have $T \subseteq P$ and P is the only prime ideal of R, different from Q, which is minimal over T.

Let us consider the ring $R/NQ + QN$. Since $N/NQ + QN$ is an Artinian right R-module and $A(R/N) = 0$, by Theorem 1.3.7 (3.) applied to the ring $R/NQ + QN$ we have $A(R/NQ + QN) = N/NQ + QN$. Now $Q/NQ + QN$ is a finitely generated left R/N-module. Since R/N is hereditary, $N/NQ + QN$ is the torsion submodule of $Q/NQ + QN$. Hence $Q/NQ + QN = (N/NQ + QN) \oplus L$, where L is a left ideal of $R/NQ + QN$ which has no non-zero Artinian submodules (cf. Proposition 2.1.11). It follows that $Q^2 \cap N \subseteq NQ + QN$. Since $N \subseteq Q^2$, we conclude that

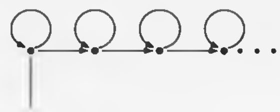


Figure 5.4: Semiray with a branch II

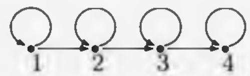


Figure 5.5: Subgraph

$NQ + QN = N$. Hence $QNQ + Q^2N = QN$ and so $NQ + Q^2N = NQ + QN = N$. An easy induction argument yields $NQ + Q^nN = N$, for all n. In other words, $(Q^n/N)(N/NQ) = N/NQ$.

Suppose that Q and P are both minimal over T. So $I/T = P \cap Q/T$ is the Jacobson radical of R/T. Since R/T is Artinian as left R/N-module, P^iQ^i is contained in T for some i. Hence

$$0 = (P^i/N)(Q^i/N)(N/NQ) = (P^i/N)(N/NQ)$$

and so $P^i \subseteq T \subseteq Q$. This yields $P \subseteq Q$ which is a contradiction.

So P is the only prime ideal of R which is minimal over T. Hence $P^n \subseteq T$, for some n. Since R/P^n is a serial local ring for all n, we have $T = P^i$, for some i, i.e. $\mathrm{l.ann}_R(N/NQ) = P^i$.

A similar argument (cf. Theorem 3.3.9 and Remark 3.3.10) yields

$$\mathrm{r.ann}_R(N/PN) = Q^j, \quad \mathrm{r.ann}_R(N/QN) = Q'^k, \quad \mathrm{l.ann}_R(N/NP) = P'^l, \tag{5.1}$$

for some j, k and $l \in \mathbb{N}$.

Set $I = P \cap Q$. As observed in the proof of Lemma 5.1.5, we have $\bigcap_{n=1}^{\infty} P^n = N = \bigcap_{n=1}^{\infty} Q^n$ and R/I^n has structure

$$\bar{R} \simeq \begin{pmatrix} R/P^n & \bar{N} \\ 0 & R/Q^n \end{pmatrix},$$

where $\bar{N} \simeq N/I^n \cap N$.

We note that, since P and Q commute modulo N, we have $I^k + N = P^k Q^k + N = Q^k P^k + N$ and so, as $N^2 = 0$,

$$I^k N + N I^k = P^k Q^k N + N P^k Q^k \tag{5.2}$$

for all $k \in \mathbb{N}$.

We shall prove that R/I^n is not of finite representation type for some n. To see this, we need to describe the bimodule $\bar{N}$.

For every $t \in \mathbb{N}$, from (5.1) it follows that $NP^tQ^t \supsetneq P'^{lt} P^{ti} N$ and so

$$P^tQ^tN + NP^tQ^t \supsetneq P^t(P^{ti-t}P'^{lt}N + Q^tN) = P^tN. \tag{5.3}$$

Similarly, $P^tQ^tN \supsetneq NQ^{tj}Q'^{tk}$, and so

$$P^tQ^tN + NP^tQ^t \supsetneq (NQ^{tj-t}Q'^{tk} + NP^t)Q^t = NQ^t. \tag{5.4}$$

Therefore, from (5.2), (5.3) and (5.4) it follows $I^tN + NI^t = P^tQ^tN + NP^tQ^t = P^tN + NQ^t$.

Now, by Theorem 1.3.7 (3.), the ideal $N/P^tN + NQ^t$ is the Artin radical of the ring $R/P^tN + NQ^t$. As R/N is hereditary, the right R/N-module $P^t/P^tN + NQ^t$ has a decomposition

$$P^t/P^tN + NQ^t = (N/P^tN + NQ^t) \oplus L, \tag{5.5}$$

where L is a right ideal of $R/P^tN + NQ^t$ which is torsion-free as a right R/N-module. From (5.5) we obtain

$$P^tQ^t \cap N \subsetneq P^tN + NQ^t \subsetneq P^tQ^t \cap N. \tag{5.6}$$

So in (5.6) we have equalities.

Now suppose that $P^tQ^t \cap N = P^{t+1}Q^{t+1} \cap N$, for some t. Hence $P^{t+1}N + NQ^{t+1} = P^tN + NQ^t$, and so $P^{t+1}N + NQ^t = P^tN + NQ^t$. We have $P^{t+1}N + NQ^t/NQ^t = P^tN + NQ^t/NQ^t$. As $\mathrm{l.ann}_R(N/NQ^t) \supseteq P^{it}$, with $i \geq 1$, $P^tN + NQ^t/NQ^t$ is a finitely generated left module over the serial ring R/P^{it}. Nakayama's Lemma gives $P^tN \subseteq NQ^t$. Hence we get

$$(NQ^t/P^{t+1}N)Q = P^{t+1}N + NQ^{t+1}/P^{t+1}N = NQ^t/P^{t+1}N. \tag{5.7}$$

As $\mathrm{r.ann}_R(NQ^t/P^{t+1}N) \supseteq \mathrm{r.ann}_R(N/P^{t+1}N) \supseteq Q^{jt}$ (cf. (5.1)), from (5.7) it follows that $NQ^t \subseteq P^{t+1}N$ and so $P^tN = P^{t+1}N$.

Now P/N is an invertible maximal ideal of the integral domain R/N and hence it is localisable. Let us denote the Ore set $\mathcal{C}(P/N)$ by $\mathcal{S}$. The ring $(R/N)_{\mathcal{S}}$ is a local principal ideal domain and, as N is torsion-free as left R/N-module, the localised module $(R/N)_{\mathcal{S}}N$ is $(R/N)_{\mathcal{S}}$-free. Therefore, the equality $(R/N)_{\mathcal{S}}P^{t+1}N = (R/N)_{\mathcal{S}}P^tN$ yields a contradiction.

We have just shown that the inclusions in the descending chain

$$N \supset PQ \cap N \supset P^2Q^2 \cap N \supset \ldots \supset P^tQ^t \cap N \supset \ldots$$

are all strict. Now, as $PQ \subseteq QP = I$, we have $P^tQ^t \subseteq I^t$, and so $P^tQ^t \cap N \subseteq I^t \cap N$. On the other hand, R/P^tQ^t is an Artinian ring with Jacobson radical I/P^tQ^t. Hence $I^k \subseteq P^tQ^t$, for some k. This shows that the chain $\{P^tQ^t \cap N\}_{t\in\mathbb{N}}$ is dense in $\{I^t \cap N\}_{t\in\mathbb{N}}$. Now, suppose that $I(I^t \cap N) + (I^t \cap N)I = I^t \cap N$, for some $t > 1$. Hence $I^t \cap N = I^{t+1} \cap N$. Therefore,

$$I^t \cap N = I(I^{t+1} \cap N) + (I^{t+1} \cap N)I \subseteq I^{t+2} \cap N.$$

Continuing this way, we get $I^t \cap N = I^{t+n} \cap N$, for all n, which is a contradiction by the above. Now $I^tN = I(I^{t-1}N) \subseteq I(I^t \cap N)$. So, as right R/N-modules,

$$I^t/(I(I^t \cap N) + (I^t \cap N)I) = [I^t \cap N/(I(I^t \cap N) + (I^t \cap N)I)] \oplus L$$

where L is a right ideal of $R/(I(I^t \cap N) + (I^t \cap N)I)$ with no non-zero Artinian submodules. Hence we obtain that

$$I^{t+1} \cap N = I(I^t \cap N) + (I^t \cap N)I. \tag{5.8}$$

The above holds for all $t \in \mathbb{N}$.

This shows that the inclusions in the following chain are all strict.

$$N = I \cap N \supset I^2 \cap N \supset \ldots \supset I^t \cap N \supset \ldots.$$

Now let us consider the ring R/I^3. We have $I^2 \cap N = IN + NI$ and so

$$I^3 \cap N = I(IN + NI) + (IN + NI)I = I^2N + INI + NI^2.$$

Hence

$$R/I^3 \simeq \begin{pmatrix} R/P^3 & N/I^2N + INI + NI^2 \\ 0 & R/Q^3 \end{pmatrix}.$$

By Theorem 3.2.3 the trivial extension ring $R/I \ltimes I^2/I^3$ is of finite representation type. Now, suppose that R/I^3 is basic. The separated quiver associated to $R/I \ltimes I^2/I^3$ is either of kind A_4 or F_4. In both cases, the ideal of R/I^3

$$IN + NI/I^2N + INI + NI^2 = PN + NQ/P^2N + PNQ + NQ^2$$

has no proper sub-bimodules. Hence the following two cases can occur.

1. $PN + NQ = P^2N + PNQ + NQ$.

2. $PN + NQ = PN + PNQ + NQ^2$.

Suppose that (1.) occurs. So $PN + NQ = NQ + P(PN + NQ)$. As $P^i = \text{l.ann}_R(N/NQ) \subseteq \text{l.ann}_R(PN + NQ/NQ)$, the above gives $NQ = PN + NQ$ and so $PN \subseteq NQ$. By considering the matrix representation of R/I^3 we get $I^3 \cap N = I^2N + INI + IN^2 = P^2N + PNQ + NQ^2 = NQ^2$. Now let $n > 3$. Using the relation (5.8) it is not difficult to show that $I^n \cap N = NQ^n$.

If case (2.) holds we have $PN + NQ = PN + (PN + NQ)Q$. Since $Q^j = \text{r.ann}_R(N/PN) \subseteq \text{r.ann}_R(PN + NQ/PN)$, we obtain $PN + NQ = PN$ and so $NQ \subseteq PN$. In a similar way as for case (1.), we show that $I^n \cap N = P^nN$, for all $n > 1$.

□

Remark 5.1.9 The final structure of the ring R/I^n in Lemma 5.1.8 follows easily if R is a trivial extension ring $R/N \ltimes N$, as in [40, Counterexample 1]. However, even in the case of a finite dimensional algebra, the existence of such a splitting is proved under an extra assumption on the base field (cf. [52, Wedderburn's Principal Theorem]).

We conclude with the following lemma.

Lemma 5.1.10 *For some n the ring R/I^n is not of finite representation type.*

Proof. There are two cases to consider.

CASE 1 $PN \subseteq NQ$.

As N is a torsion-free module over the hereditary integral domain R/N, we have

$$\bigcap_{n=1}^{\infty} NQ^n \subseteq (\bigcap_{n=1}^{\infty} NQ^n)(R/N)_S \subseteq \bigcap_{n=1}^{\infty} NQ^n(R/N)_S = 0,$$

where $(R/N)_S$ is the ring R/N localised at the maximal ideal Q/N.

Since PN is not zero, there exists an integer t such that $PN \subseteq NQ^t$ and $PN \not\subseteq NQ^{t+1}$. Now, NQ^t/NQ^{t+1} has no proper non-zero sub-bimodules, so $PN+NQ^{t+1} = NQ^t$. Using that $\text{r.ann}_R(N/PN) = Q^j$ (cf. Lemma 5.1.8, relations (5.1)), it is easy to show that $PN = NQ^t$. Therefore, $P^nN = NQ^{tn}$, for all n.

By Lemma 5.1.8 (case 1.) we have $I^{tn} \cap N = NQ^{tn}$, so

$$V_n = \begin{pmatrix} P^n/P^{tn} & 0 \\ 0 & 0 \end{pmatrix}$$

is a two sided ideal of R/I^{tn}. Let us consider the factor ring of R/I^{tn}

$$R/V_n \simeq \begin{pmatrix} R/P^n & N/NQ^{tn} \\ 0 & R/Q^{tn} \end{pmatrix}.$$

The bimodule N/NQ^{tn} is free as right R/Q^{tn}-module and as left R/P^n-module. As the separated quivers associated to the trivial extension rings $R/I \ltimes I^t/I^{t+1}$, $t \geq 1$, (assumed basic) all have underlying diagrams of kind A_4 or F_4 (see page 32), we conclude that N/NQ^{tn} has uniform dimension one as a right R-module. The method of Lemma 4.1.9 can now be applied to show that R/V_n can be embedded in a 2×2 triangular matrix ring whose entries are from the ring R/Q^{tn}. Now R/Q^{tn} can be thought of as factor ring of R/N localised at Q/N, which is a Dedekind prime ring. So the rest of the proof of Lemma 4.1.9 goes through to show that R/V_n is not of finite representation type for some n.

CASE 2 $\quad NQ \subsetneq PN$.

Since N is torsion-free as left R/N-module, arguing as for case (1.), we obtain that $\bigcap_{n=1}^{\infty} P^n N = 0$. As NQ is not zero, we find $t \in \mathbb{N}$ such that $NQ \subseteq P^t N$ and $NQ \not\subseteq P^{t+1}N$. Now $P^t N/P^{t+1}N$ has no non-zero proper sub-bimodules and so $NQ + P^{t+1}N = P^t N$. We have $NQ = P^t N$ because $\text{l.ann}_R(N/NQ) = P^t$. So $NQ^n = P^{tn}N$, for all n.

Let us consider the following two sided ideal of R/I^{tn}

$$H_n = \begin{pmatrix} 0 & 0 \\ 0 & Q^n/Q^{tn} \end{pmatrix},$$

and the factor ring

$$R/H_n \simeq \begin{pmatrix} R/P^{tn} & N/P^{tn}N \\ 0 & R/Q^n \end{pmatrix}.$$

The bimodule $N/P^{tn}N$ is free as right R/Q^n-module and as left R/P^{tn}-module. Furthermore, $N/P^{tn}N$ has uniform dimension at most two as left R-module. Suppose that it is equal to two. Then the method of Lemma 4.1.9 shows that R/H_n can be embedded in a 3×3 triangular matrix ring whose entries are from R/P^{tn}. The ring R/P^{tn} can be thought of as factor ring of R/N localised at P/N, which is a Dedekind prime ring. So the rest of the proof of Lemma 4.1.9 shows that for some n, the ring R/H_n is not of finite representation type.

□

We have shown that the assumption that R has an infinite clique of maximal ideals leads to a contradiction. Hence every clique of maximal ideals of R is finite and so Proposition 4.2.4 yields that R is a hereditary semiprime ring. This finishes the proof of Proposition 5.1.1.

□

We also deduce the following result.

Theorem 5.1.11 *Let R be an indecomposable Noetherian PI ring with the property (P). Then all cliques of maximal ideals of R are finite.*

Proof. Clearly, we assume that R is not Artinian and not semiprime. By Proposition 5.1.1 we need to consider just the case where $A(R) \neq 0$.

Suppose that R has an infinite clique of maximal ideals. We note that Lemma 5.1.7 holds without the assumption that $A(R) = 0$.

The ring $R/A(R)$ is a Noetherian PI ring and has the property (P). Further, by Theorem 1.3.7 (2.), we have $A(R/A(R)) = 0$. So by Proposition 5.1.1 the ring $R/A(R)$ is hereditary and semiprime. It follows that $N \subseteq A(R)$ and therefore N is an Artinian ideal. This is a contradiction by Corollary 3.3.12.

□

Now we state our main theorem in its final form.

Theorem 5.1.12 *Let R be a Noetherian PI ring, which is an order in an Artinian ring. Then the following are equivalent.*

1. *Every proper Artinian factor ring of R is of finite representation type.*
2. *$R = A \oplus S$, where A is an Artinian ring of finite type and S is a semiprime hereditary ring.*

Proof. (1.) $\Rightarrow$ (2.) As the Artin radical $A \equiv A(R)$ of R is a direct summand of R (cf. Theorem 1.3.8), we may assume that $A(R) = 0$. By Proposition 5.1.1 all cliques of maximal ideals of R are finite and so Proposition 4.2.4 applies.

(2.) $\Rightarrow$ (1.) This follows from Proposition 2.2.11 and Theorem 2.2.12.

□

5.2 A generalization of Warfield's Theorem

In this section we consider a Noetherian PI ring R whose proper Artinian factor rings are all serial. We show that, in this case, R is an order in an Artinian ring. This fact, together with Theorem 5.1.12, is then used to generalize Theorem 2.2.13.

Throughout, we assume that R is not an Artinian ring. We need the following fact.

Lemma 5.2.1 [24, Lemma 12.3] *Let S and R be Noetherian PI rings and Q be a prime ideal of R. Let M be a non-zero S-R-bimodule, finitely generated on each side,* *such that M_R is faithful. Then M has sub-bimodules M' and M'' with $M' \supset M''$ such that M'/M'' is torsionfree right R/Q-module as well as torsionfree left S/P-module for some prime P of S.*

Definition 5.2.2 We say that a ring R has the *property* $(\bar{P})$ if every proper Artinian factor ring of R is serial.

Lemma 5.2.3 *Let R be a Noetherian PI ring with the property $(\bar{P})$. Then all cliques of maximal ideals of R are finite and so localisable. For every such clique X the localised ring R_X is either an Artinian serial ring or a hereditary prime ring.*

Proof. Let P be a minimal prime ideal of R. By Proposition 2.2.11 and Theorem 4.2.8 the ring R/P is hereditary. It follows that R has Krull dimension one.

Now, the quiver associated to an indecomposable Artinian serial ring is either a circuit or a finite ray (cf. page 31). Hence, if the ring is not local, its associated quiver does not contain loops. Therefore, by the proof of Proposition 3.3.19 and by Theorem 3.3.11, all cliques of maximal ideals of R are finite.

Let X be one of them. By Remark 3.4.16 the localised ring R_X has the property $(\bar{P})$. Let us consider $\hat{R}_X$, the $J(R_X)$-adic completion of R_X. By Theorem 1.5.9 the ring $\hat{R}_X$ is semiperfect Noetherian PI and, by the proof of Proposition 3.4.18, $\hat{R}_X$ has the property $(\bar{P})$. Therefore $\hat{R}_X$ is a serial ring (cf. Lemma 3.4.1) and so, being indecomposable (cf. Theorem 1.5.14), is either an Artinian ring or a hereditary prime ring by Theorem 2.2.3.

Suppose that the first case holds. So R_X is an Artinian ring and hence a power of $J(R_X)$ is equal to zero. Set $T = \{r \in R \mid rc = 0, \text{ for some } c \in \mathcal{C}(X)\}$. As R/T is a subring of R_X, the ideal T contains a finite product of maximal ideals of R which belong to X (cf. Definition 1.4.7). It follows that R/T is an Artinian ring and so is serial by assumption. From this, it is easy to show that R_X is also a serial ring.

If $\hat{R}_X$ is a hereditary prime ring, then R_X is a prime ring and so is hereditary by Theorem 2.2.14.

□

Proposition 5.2.4 *Let R be an indecomposable Noetherian PI ring with the property $(\bar{P})$. Then $A(R) = 0$.*

Proof. Suppose that $A \equiv A(R) \neq 0$. By Lemma 5.2.3 every clique X of maximal ideals of R is finite. If the localised rings R_X are all hereditary, then, by Theorem

1.4.9, R is also a hereditary ring. As R is indecomposable, from Theorem 2.1.6 it follows that R is an Artinian ring, case which we have excluded.

So there exists a clique X of maximal ideals of R such that R_X is an Artinian serial ring. Set $T = \{r \in R \mid rc = 0, \text{ for some } c \in \mathcal{C}(X)\}$.

By Theorem 1.3.7 we have that $A(R/A) = 0$, and so R/A is a semiprime hereditary ring by Proposition 5.1.1. It follows that $A = P_1 \cap \ldots \cap P_h$, for some $P_1, \ldots, P_h$ minimal prime ideals of R. Since A is contained in every minimal prime ideal of R that is not maximal, we conclude that $P_1, \ldots, P_h$ are all those minimal prime ideals of R which are not maximal.

The link components corresponding to cliques of maximal ideals of R/A are circuits (cf. page 25), and so these cliques all lift to cliques of maximal ideals of R. This means that a maximal ideal of R which does not contain A cannot be linked to any maximal ideal of R containing A.

Suppose that $A \subseteq Q$, for some prime $Q \in X$. By the discussion above, we have that $A \subseteq Q'$, for every $Q' \in X$. So the image $\bar{X}$ of X in R/A is a clique of R/A. As the localised rings of R/A at cliques of maximal ideals are not Artinian (cf. Lemma 4.2.3), this is a contradiction. It follows that X consists entirely of maximal ideals of R that are also minimal primes.

By Lemma 1.3.12 we have

$$A = \mathrm{l.ann}_R(Q_1 \ldots Q_n), \tag{5.9}$$

for some right affiliated primes $Q_1, \ldots, Q_n$, with Q_i maximal for all $i = 1, \ldots, n$. As $P_1 \ldots P_h Q_1 \ldots Q_n = 0$, some of the Q_is are also minimal primes.

Now, by Theorem 1.4.3 the maximal ideals which are annihilators of subfactors of a uniserial module all belong to the same clique. So, as A is a serial module, we may re-enumerate the prime ideals that appear in (5.9) in such a way that $I = Q_1 \ldots Q_k$ is a product of maximal ideals of R which are also minimal primes and $H = Q_{k+1} \ldots Q_n$ is a product of maximal ideals containing A. Then $I + H = R$.

Set $A' = \mathrm{l.ann}_A(I)$ and $A'' = \mathrm{l.ann}_A(H)$. So $A = A' + A''$. Further, $A' \cap A'' = 0$ and hence the sum is direct. Note that $A' \neq 0$.

Now, let us consider the ring R/A''. We can construct a right affiliated series for R/A'', with right affiliated primes $Q_1, \ldots, Q_k, P_1, \ldots, P_h$. Let P be a left affiliated prime of A'. So P is a maximal ideal and is the annihilator of a subfactor $\bar{A}'$ of A'. By Lemma 5.2.1 we find a subfactor of A' that is torsionfree as left R/P-module and torsionfree as right R/Q'-module, where $Q' \in \{Q_1, \ldots, Q_k\}$. It follows that $P \rightsquigarrow\rightsquigarrow Q'$ (cf. Definition 3.3.5), and so P is a minimal prime ideal. So by Theorem 1.3.11 the ring R/A'' has an Artinian quotient ring. Hence, by Theorem 1.3.8 there exists an ideal L of R such that $A \cap L = A''$ and $A + L = R$. So $R = A' \oplus L$. As R is an indecomposable ring and is not Artinian, this is a contradiction.

□

In [43, Theorem 6] it was already observed that the link components of maximal ideals of a Noetherian prime bounded ring that has Krull dimension one, and whose factor rings are all serial, consist of circuits and so the corresponding cliques are finite.

In the proof of [43, Theorem 3] it is also shown that if an indecomposable FBN ring has a clique of maximal ideals that are also minimal primes, then R is Artinian. We have realized this after proving Proposition 5.2.4. However, the proof given in [43] is different and uses torsion theory.

We conclude this chapter with the following result.

Theorem 5.2.5 *Let R be a Noetherian PI ring. Then the following are equivalent.*

1. *Every proper Artinian factor ring of R is serial.*
2. *$R = A \oplus S$, where A is a serial Artinian ring and S is a semiprime hereditary ring.*

Proof. (1.) $\Rightarrow$ (2.) By Proposition 5.2.4 and Theorem 5.1.12 an indecomposable Noetherian PI ring with the property $(\bar{P})$ is either an Artinian serial ring or a hereditary prime ring. Hence the result follows.

(2.) $\Rightarrow$ (1.) This follows from Theorem 2.2.12.

□

Chapter 6

Affine rings

In this chapter we study PI rings which are affine over a field and have the property (P). We show that, in the semiprime case, the property of the ring to be Noetherian can be deduced from the assumptions.

6.1 Semiprime rings

We prove the following theorem.

Theorem 6.1.1 *Let R be a semiprime PI ring affine over a field k. Then R is hereditary if and only if every proper Artinian factor ring of R is of finite representation type.*

The crucial step in the proof of the above theorem is the following property of affine PI rings.

Theorem 6.1.2 [6, Lemma 1] *Let R be an affine semiprime PI ring such that* $\mathrm{cl.K.dim}(R) = 1$. *Then R is finitely generated as a module over its centre.*

Proof. (of Theorem 6.1.1) We note that, by Theorem 1.5.5, R has only finitely many minimal prime ideals.

Suppose that R has the property (P) and is a prime ring. Let a be a non-zero evaluation of a central polynomial for R. Since R is affine and PI, the Jacobson radical of R is zero (cf. Theorem 1.5.5) and so there exists a maximal ideal M of R such that $a \notin M$.

Let us consider the ring $R_S = R[a^{-1}]$. This is an Azumaya algebra (cf. page 16) and so is Noetherian and finitely generated as a module over its centre. The ring R_S has the property (P) by Proposition 3.4.17 and so, by Theorem 5.1.12, is a hereditary ring. In particular, R_S has Krull dimension one. It follows that $\mathrm{ht}(M) = 1$ and, by Schelter's formula [37, Theorem 13.10.12], we have

$$\mathrm{cl.K.dim}(R) = \mathrm{ht}(M) + \mathrm{cl.K.dim}(R/M) = 1.$$

So, by Theorem 6.1.2, the ring R is finitely generated as a module over its centre and is Noetherian (cf. [5, Theorem 1.6]). Hence Theorem 5.1.12 applies and therefore R is a hereditary ring.

Suppose now that R is not a prime ring. Let P be a minimal prime ideal of R. By the above R/P is a hereditary Noetherian ring and so R has classical Krull dimension one.

Let $P_1, \ldots, P_n$ be the minimal prime ideals of R. So $P_1 \cap \ldots \cap P_n = 0$ and therefore the natural homomorphism of right (and left) R-modules

$$R \to R/P_1 \oplus \ldots \oplus R/P_n \tag{6.1}$$

is injective. Hence R is Noetherian. By Theorem 6.1.2 the ring R is finitely generated as a module over its centre. So the result follows from Theorem 5.1.12.

Conversely, suppose that R is a hereditary ring.

Again, because of (6.1), it is enough to show that R/P_i is Noetherian, for $i = 1, \ldots, n$. This is due to the following two well known lemmata for which the assumption that R is affine is not essential.

The first one can be deduced from [37, Theorem 7.3.12], but we give here an easier and well known proof.

Lemma 6.1.3 *Let R be a ring and I an idempotent ideal of R. If R is right hereditary then so is R/I.*

Proof. First we note that if $f : I \to R$ is a homomorphism of right R-modules, we have $f(I) = f(I^2) = f(I)I \subseteq I$.

Let K/I be a right ideal of R/I, where K is a right ideal of R containing I. Since K is projective, by the Dual Basis Lemma there are $(x_\alpha, f_\alpha)_\alpha$, $x_\alpha \in K, f_\alpha \in \mathrm{Hom}(K, R)$ such that $x = \Sigma_\alpha x_\alpha f_\alpha(x)$, for every $x \in K$. As observed above, $f_\alpha(I) \subseteq$

I and so f_α induces a homomorphism $\bar{f}_\alpha : K/I \to R/I$ of right R/I-modules, for every α. Hence, by taking images in R/I, we have

$$\bar{x} = \Sigma_\alpha \bar{x}_\alpha \bar{f}_\alpha(\bar{x}),$$

for every $x \in K$. So $(\bar{x}_\alpha, \bar{f}_\alpha)_\alpha$ is a dual basis for K/I, which therefore is projective. So R/I is right hereditary.

□

Lemma 6.1.4 *Let R be a semiprime hereditary PI ring which has finitely many minimal prime ideals. Let P be a minimal prime ideal. So $P^2 = P$ and R/P is a hereditary ring.*

Proof. Let $0 = P_1 \cap \ldots \cap P_n$ be a non-ridondant intersection of minimal prime ideals of R and set $P = P_1$. So $P(P_2 \ldots P_n) = 0$. Now, since $P_i \not\subseteq P$, for $i \neq 1$ and so $P_2 \ldots P_n \neq 0$, we have $P_2 \ldots P_n \cap \mathcal{C}(P) \neq \emptyset$. Let c be an element of such an intersection. So $Pc = 0$.

Let $(x_\alpha, f_\alpha)_\alpha$ be a dual basis for P. So $f_\alpha(Pc) = f_\alpha(P)c = 0$. As $c \in \mathcal{C}(P)$, this yields $f_\alpha(P) \subseteq P$.

Hence $x = \Sigma_\alpha x_\alpha f_\alpha(x) \in P^2$, for all $x \in P$. Therefore, $P = P^2$ and so R/P is hereditary by Lemma 6.1.3.

□

So, by Lemma 6.1.3, each of the rings R/P_i, $1 \leq i \leq n$, is a HNP ring. Hence, by Theorem 2.1.9 the ring R is Noetherian and, being hereditary, is a finite direct sum of HNP rings. So by Theorem 2.2.12, R has the property (P). This finishes the proof of Theorem 6.1.1.

□

An alternative proof for a Noetherian prime ring that is finitely generated as a module over its centre

In the proof of Proposition 4.2.4, the most difficult part was to show that the J-completion of the localised ring is a semiprime ring. If we start with a prime Noetherian ring R, which is finitely generated as a module over its local centre C, we have, instead, a direct method.

Lemma 6.1.5 *Let R and C as above and let m be the maximal ideal of C. Suppose that R has the property (P) and let $\hat{R}$ be the mR-adic completion of R. Then $\hat{R}$ is a semiprime ring.*

Proof. We have that $\hat{R} \simeq R \otimes_C \hat{C}$ (cf. page 19). So $\hat{R}$ is a semiperfect Noetherian ring and is finitely generated as a module over its centre, which is isomorphic to $\hat{C}$ (cf. Proposition 1.5.11). Now, arguing as in the proof of Proposition 4.2.4 we obtain that the Artin radical of $\hat{R}$ is zero and by Proposition 4.1.2 the ring $\hat{R}/N(\hat{R})$ has Krull dimension one. It follows that $\text{K.dim}(\hat{R}) = 1$ and so Theorem 1.3.9 yields that $\hat{R}$ has an Artinian quotient ring. We obtain such a ring by inverting the non-zero central elements of R and show that this is a semisimple ring.

In Corollary 1.5.10 we observed that the regular element of R stay regular in $\hat{R}$ and by Lemma 4.1.4 the centre $\hat{C}$ of $\hat{R}$ is a semiprime ring.

Let us consider the ring $\hat{R}_{\mathcal{S}}$, where $\mathcal{S} = C - \{0\}$. We claim that this is a quotient ring for $\hat{R}$. It clearly contains $\hat{R}$. Let c be a regular element of $\hat{R}$. Since $\text{K.dim}(\hat{R}) = 1$, by Lemma 1.3.10 the module $\hat{R}/c\hat{R}$ is Artinian. So $c\hat{R}$ contains a certain power of the Jacobson radical $\hat{J}$ of $\hat{R}$. Since $J \subseteq \hat{J}$ and R is prime, we have that $c\hat{R}$ meets the centre C of R. So $cx = a$ for some $a \in C$ and $x \in \hat{R}$. Hence c is invertible in $\hat{R}_{\mathcal{S}}$. That $\hat{R}_{\mathcal{S}}$ is Artinian follows easily by observing that $\hat{R}/N$ is C-torsion free. Therefore, $N\hat{R}_{\mathcal{S}} \cap \hat{R} = N$ and hence the natural map $\hat{R}/N \to \hat{R}_{\mathcal{S}}/N\hat{R}_{\mathcal{S}}$ is a ring embedding.

So the centre $C(\hat{R}_{\mathcal{S}})$ of $\hat{R}_{\mathcal{S}}$ is Artinian and, as $C(\hat{R}_{\mathcal{S}}) \simeq C(\hat{R})_{\mathcal{S}} \simeq \hat{C}_{\mathcal{S}}$, it is the quotient ring of $\hat{C}$. Hence $C(\hat{R}_{\mathcal{S}})$ is a direct sum of fields.

Let $\{p_1, \ldots, p_n\}$ be the minimal prime ideals of $\hat{C}$ and set $k_i = \hat{C}_{\mathcal{S}}/p_i\hat{C}_{\mathcal{S}}$, $i = 1, \ldots n$. So $\hat{C}_{\mathcal{S}} = k_1 \times \ldots \times k_n$. As $p_i \cap C = 0$, every k_i is a field extension of the field of fractions k of C.

Clearly, the ring $\hat{R}_{\mathcal{S}}$ contains the quotient ring $Q(R)$ of R. There is a ring isomorphism

$$R \otimes_C \hat{C}_{\mathcal{S}} \simeq Q(R) \otimes_k \hat{C}_{\mathcal{S}}.$$

In fact, let $\psi : Q(R) \otimes_k \hat{C}_{\mathcal{S}} \to R \otimes_C \hat{C}_{\mathcal{S}}$ be defined by $\psi(x\lambda^{-1} \otimes_k \hat{y}\mu^{-1}) = (x \otimes_C \hat{y})(\lambda\mu)^{-1}$, with $x \in R, \hat{y} \in \hat{C}$ and $\lambda, \mu \in C$. Thus ψ is a ring homomorphism.

Let $\phi : R \otimes_C \hat{C}_{\mathcal{S}} \to Q(R) \otimes_k \hat{C}_{\mathcal{S}}$ be defined by $\phi(x \otimes_C \hat{y}\mu^{-1}) = x \otimes_C \hat{y}\mu^{-1}$, with $x \in R, \hat{y} \in \hat{C}$ and $\mu \in C$. So ϕ is a ring homomorphism and it can be checked that $\phi\psi = 1 = \psi\phi$.

Hence we have

$$\hat{R}_S \simeq R \otimes_C \hat{C}_S \simeq Q(R) \otimes_k \hat{C}_S \simeq Q(R) \otimes_k (\oplus_{i=1}^n k_i) \simeq \oplus_{i=1}^n (Q(R) \otimes_k k_i).$$

Since $Q(R)$ is separable over k [46, Theorem 7.6], $\hat{R}_S$ is a semisimple ring and by Goldie's Theorem $\hat{R}$ is a semiprime ring.

□

6.2 Examples

We give an example of an indecomposable semiperfect Noetherian non-Artinian PI ring with $A(R) \neq 0$, which satisfies the property (P) but is not hereditary. This shows, in particular, that the assumption $A(R) = 0$ in Proposition 5.1.1 cannot be dropped.

Example 6.2.1 Let R be the ring

$$R = \begin{pmatrix} k[x]_{(x)} & k \\ 0 & k[x]_{(x)} \end{pmatrix},$$

where $k[x]_{(x)}$ is the ring $k[x]$ localised at the maximal ideal $xk[x]$. Thus R is finitely generated as a module over its centre, which is isomorphic to $k[x]_{(x)}$, and $\mathrm{K.dim}(R) = 1$.

Let us consider the factor rings of R:

$$R_n \equiv \begin{pmatrix} k[x]/(x)^n & k \\ 0 & k[x]/(x)^n \end{pmatrix}.$$

For all $n \geq 1$, the ring R_n is Artinian and is well known be of finite representation type. We give here an elementary proof of this fact.

Let $M \equiv (A, B, f)$ be a finitely generated indecomposable right R_n-module, where A and B are finitely generated right $k[x]/(x)^n$-modules and $f : A \otimes_{k[x]/(x)^n} k \to B$ is a homomorphism.

Let us assume that A decomposes.

The proof now proceeds in steps.

<u>Step 1</u> Claim : the homomorphism f is injective.

Set $(x)^0/(x)^n = k[x]/(x)^n$. Now $A \simeq \bigoplus_{j=1}^{t} H_j$, where $H_j \in \{(x)^i/(x)^n \mid 0 \leq i < n\}$ and $t > 1$. Hence, $A \otimes k \simeq k^{(t)}$. Suppose that $\ker f \neq 0$. Then, as $k^{(t)}$ is semisimple, we have that $k^{(t)} = \ker f \oplus I$, with $I \simeq \mathrm{Im} f$. Let A'' be a direct summand of A such that $A'' \otimes k \simeq \ker f$ and A' the submodule of A isomorphic to A/A''. So $M = (A', B, f) \oplus (A'', 0, 0)$. Contradiction.

STEP 2 Claim : u.dim$(B) = t$.

We have that $\mathrm{Im} f \subseteq \mathrm{socle}(B)$. If the inclusion were strict, by a similar argument as in step 1 we would get a contradiction. Therefore $\mathrm{Im} f = \mathrm{socle}(B)$.

STEP 3 Claim : We may assume that $A \simeq (k[x]/(x^n))^{(t)}$.

Suppose that A is not free and let E be its injective hull. Then E is a free right $k[x]/(x^n)$-module of uniform dimension t. Set $L = (E, B, f)$. It is not difficult to check that L is a right R_n-module containing M. Further, M is essential in L. To see this, let $M' = (A', B', g)$ be a submodule of L with $M \cap M' = 0$. Then $B' = 0$, because otherwise $(0, b')$, $b' \neq 0$, would be a non-zero element of $M \cap M'$. So $M' = (A', 0, g)$. Now, as M' is a submodule of L and f is an isomorphism, we have that $g(A' \otimes k) \neq 0$, that is a contradiction.

Therefore, the module L is indecomposable and so we may replace M by L.

We have shown that $f : k^{(t)} \to k^{(t)} \simeq \mathrm{socle}(B)$ is an isomorphism. Let e be a non-trivial idempotent endomorphism of B such that the restriction of e at socle(B) is a non-trivial idempotent endomorphism of socle(B). Set $e' = f^{-1}ef$. So e' is a non-trivial idempotent endomorphism of $k^{(t)}$ with $fe' = ef$. We can find a non-trivial idempotent endomorphism α of A such that $\alpha \otimes 1 = e'$. This contradicts the indecomposability of M.

Hence $t = 1$. Thus also B must be indecomposable.

Therefore M is isomorphic to one of the following modules :

- $(k[x]/(x^n), 0, 0)$,
- $((x^i)/(x^n), 0, 0)$, $1 \leq i < n$,
- $(k[x]/(x)^n, k[x]/(x)^n, 1)$,
- $(k[x]/(x)^n, (x^i)/(x^n), 1)$, $1 \leq i < n$,
- $(0, k[x]/(x^n), 0)$,

- $(0, (x)^i/(x)^n, 0)$, $1 \leq i < n$,
- $((x)^j/(x)^n, (x)^i/(x)^n, 1)$, $1 \leq i, j < n$

As $\mathrm{pd}(k)_{k[x]} = 1$, from Proposition 3.1.1 we get $2 \leq \mathrm{gl.dim}(R) \leq 3$. Since $A(R)$ is nilpotent by [8, Theorem 2.3], R cannot be of global dimension two and so $\mathrm{gl.dim}(R) = 3$.

Now let us consider the following homomorphic images of R:

$$S \equiv \begin{pmatrix} k[x]_{(x)} & k \\ 0 & k \end{pmatrix} \text{ and } T \equiv \begin{pmatrix} k[x]_{(x)} & k \\ 0 & k[x]/(x)^2 \end{pmatrix}.$$

Then both S and T have the property (P). Again from Proposition 3.1.1, we get $\mathrm{gl.dim}(S) = 2$ and $\mathrm{gl.dim}(T) = \infty$, since $\mathrm{gl.dim}(k[x]/(x)^2) = \infty$.

The above examples show that when R is indecomposable and the Artin radical of R is not zero, the property (P) does not affect the global dimension of the ring. This is to be expected since the same holds for Artinian rings.

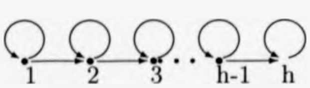
For the rest of this section, k will denote an algebraically closed field.

Remark 6.2.2 Let $h, n \in \mathbb{N}$ and consider the subring of $M_h(k[x]/(x)^n)$:

$$A_h \equiv \begin{pmatrix} k[x]/(x)^n & k & k & \cdots & k \\ 0 & k[x]/(x)^n & k & \cdots & \vdots \\ 0 & 0 & k[x]/(x)^n & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & k \\ 0 & \cdots & \cdots & 0 & k[x]/(x)^n \end{pmatrix}.$$

Using covering theory (see [22]), it can be shown that A_h is an algebra of finite representation type.

The associated quiver of A_h is



Definition 6.2.3 A finite dimensional k-algebra is said to be of *tame representation type* if there are infinitely many classes of finite length modules but these classes can be parametrised by a finite set of integers together with a polynomial irreducible over k.

Example 6.2.4 Let R be the ring of Example 3.4.11 and $n > 2$. The factor ring R/J^n has associated quiver Γ as in Figure 3.2 on page 56. Set

$$a = (1,2) \to \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + J^n \ ; \ b = (2,3) \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} + J^n \ ;$$

$$c = (3,2) \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + J^n \text{ and } d = (2,1) \to \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + J^n.$$

It can be shown (cf. [4, Theorem III.1.9]) that R/J^n is isomorphic to a factor ring of $k\Gamma/ < \rho >$, where $k\Gamma$ is the path algebra of Γ over k and $< \rho >$ is the ideal generated by the relation $cb = ad$ (see [4, pp 50, 58] for definitions and details). Now, this ring is of tame representation type (cf. [11, page 35]) and hence R/J^n is "at most" of tame representation type. As n was arbitrary and R is a semilocal ring, we conclude that R has the property that each of its proper homomorphic images is either of tame or of finite representation type.

In [33, Example 2.5] it is proved that $\mathrm{gl.dim}(R) = 2$. Now let us consider the subring eRe of R, where e is the idempotent of Example 3.4.11. Every finite length indecomposable right eRe-module M is isomorphic to He, where H is an indecomposable right R-module of finite length (see proof of [26, Proposition 7.1] or [32, Lemma 2] and by Proposition 3.4.9). So the ring eRe also has the property of the Artinian factor rings. However, it is well known that $\mathrm{gl.dim}(eRe) = \infty$ (cf. [48, Example 7.8]).

Further questions

- Is it possible to describe the structure of a Noetherian (non-Artinian) PI ring R, with the property (P) and such that $0 \neq A(R)$ is projective as left and as right R-module?

It is not difficult to show (see [8, Theorem 2.3]) that $A(R)^2 = A(R)$ and $A(R) = ReR$, for some idempotent e of R. Consequently, the ring eRe is Artinian and of finite representation type. Further, by Theorem 5.1.12 the ring $R/A(R)$ is hereditary and semiprime, whence $N \subset A(R)$.

By Theorem 5.1.11 all cliques of maximal ideals of R are finite, so one might start by assuming that R is a semilocal ring.

Appendix

This is an appendix to the proof of Proposition 4.1.3. With the same notation, we give rigorous proofs of the following two facts used at the end of Case 1, page 71 :

1. $L'_n \otimes_{R/I} T/I' \simeq (M'_1 \otimes_{R_2/J_2^{4t}} E/J(E)^{4th}, M_2, l)(\simeq L_n)$.

2. L'_n is an indecomposable module.

Similar claims, appearing at the end of Case 2, can be proved by using a similar argument.

Proof. (1.)
For simplicity we set $N \equiv N/NJ^{4th}$, $R_2 \equiv R_2/J_2^{4t}$ and $E \equiv E/J(E)^{4th}$.

So ϕ is the isomorphism $N \simeq E \otimes_E N$, given by $\phi(n) = 1 \otimes_E n$, for all $n \in N$; $\psi = 1_{M'_1} \otimes_{R_2} \phi$; $l : M_1 \otimes_E N \to M_2$, where $M_1 = M'_1 \otimes_{R_2} N$; $l' = l\psi$; $L_n = (M_1, M_2, l)$ and $L'_n = (M'_1, M_2, l')$.

For space reasons we write the elements of the modules in columns rather than in rows.

Let $f : L'_n \times T/I \to L_n$ be defined by

$$\left(\begin{pmatrix} m'_1 \\ m_2 \end{pmatrix}, \begin{pmatrix} e & n \\ 0 & r \end{pmatrix} \right) \to \begin{pmatrix} m'_1 \otimes_{R_2} e \\ l((m'_1 \otimes_{R_2} 1) \otimes_E n) + m_2 r \end{pmatrix},$$

for $m'_1 \in M'_1, m_2 \in M_2, e \in E, n \in N$, and $r \in R_1$.

Clearly, f is well defined. We note that $l'(m'_1 \otimes_{R_2} n) = l((m'_1 \otimes_{R_2} 1) \otimes_E n)$, for $m'_1 \in M'_1$ and $n \in N$, by the definition of l'. Using this, it is easy to check that f is balanced. So f extends uniquely to $\bar{f} : L'_n \otimes_{R/I'} T/I \to L_n$. It is also easy to show that $\bar{f}$ is a homomorphism of right T-modules.

Now, let $g : L_n \to L'_n \otimes_{R/I'} T/I$ be defined by

$$\begin{pmatrix} m'_1 \otimes_{R_2} e \\ m_2 \end{pmatrix} \to \begin{pmatrix} m'_1 \\ m_2 \end{pmatrix} \otimes_{R/I'} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix},$$

for $m'_1 \in M'_1, e \in E$ and $m_2 \in M_2$. By first representing the module $L'_n \otimes_{R/I'} T/I$ as a triple and then writing g as a pair (cf. page 34), it can be checked that g is well defined. We claim that g is a homomorphism of right T/I-modules.

Let

$$\begin{pmatrix} \bar{e} & \bar{n} \\ 0 & \bar{r} \end{pmatrix}$$

be an element of T/I. We have

$$\begin{pmatrix} m_1' \otimes_{R_2} e \\ m_2 \end{pmatrix} \begin{pmatrix} \bar{e} & \bar{n} \\ 0 & \bar{r} \end{pmatrix} = \begin{pmatrix} m_1' \otimes_{R_2} e\bar{e} \\ l(m_1' \otimes_{R_2} e \otimes_E \bar{n}) + m_2\bar{r} \end{pmatrix}.$$

Its image under g is

$$\begin{aligned}
&\begin{pmatrix} m_1' \\ l(m_1' \otimes_{R_2} e \otimes_E \bar{n}) + m_2\bar{r} \end{pmatrix} \otimes_{R/I'} \begin{pmatrix} e\bar{e} & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} m_1' \\ l'(m_1' \otimes_{R_2} e\bar{n}) + m_2\bar{r} \end{pmatrix} \otimes_{R/I'} \begin{pmatrix} e\bar{e} & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} m_1' \\ m_2 \end{pmatrix} \begin{pmatrix} 1 & e\bar{n} \\ 0 & \bar{r} \end{pmatrix} \otimes_{R/I'} \begin{pmatrix} e\bar{e} & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} m_1' \\ m_2 \end{pmatrix} \otimes_{R/I'} \begin{pmatrix} 1 & e\bar{n} \\ 0 & \bar{r} \end{pmatrix} \begin{pmatrix} e\bar{e} & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} m_1' \\ m_2 \end{pmatrix} \otimes_{R/I'} \begin{pmatrix} e\bar{e} & e\bar{n} \\ 0 & \bar{r} \end{pmatrix} \\
&= \begin{pmatrix} m_1' \\ m_2 \end{pmatrix} \otimes_{R/I'} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{e} & \bar{n} \\ 0 & \bar{r} \end{pmatrix},
\end{aligned}$$

which is equal to

$$g\left(\begin{pmatrix} m_1' \otimes_{R_2} e \\ m_2 \end{pmatrix}\right) \begin{pmatrix} \bar{e} & \bar{n} \\ 0 & \bar{r} \end{pmatrix}.$$

Similar computations show that $\bar{f}g = 1 = g\bar{f}$.

□

Proof. (2.)

Let (α, β) be an idempotent element of $\mathrm{End}(L_n')$. So $\alpha \in \mathrm{End}(M_1')$, $\beta \in \mathrm{End}(M_2)$ and

$$l'(\alpha \otimes_{R_2} 1_N) = \beta l'. \tag{6.2}$$

So we have the following commutative diagrams:

$$\begin{tikzcd}
M_1 \otimes_{R_2} N \arrow[r, "1_{M_1'} \otimes_{R_2} \phi^{-1}"] \arrow[d, "l"] & M_1' \otimes_{R_2} N \arrow[r, "\alpha \otimes_{R_2} 1_N"] \arrow[d, "l'"] & M_1' \otimes_{R_2} N \arrow[r, "1_{M_1'} \otimes_{R_2} \phi"] \arrow[d, "l'"] & M_1 \otimes_{R_2} N \arrow[d, "l"] \\
M_2 \arrow[r, "1_{M_2}"] & M_2 \arrow[r, "\beta"] & M_2 \arrow[r, "1_{M_2}"] & M_2
\end{tikzcd} \tag{6.3}$$

We rewrite (6.2) as

$$l(1_{M_1'} \otimes_{R_2} \phi)(\alpha \otimes_{R_2} 1_N) = \beta l(1_{M_1'} \otimes_{R_2} \phi),$$

which is equivalent to

$$l(1_{M_1'} \alpha \otimes_{R_2} \phi 1_N) = \beta l(1_{M_1'} \otimes_{R_2} \phi).$$

As $1_{M_1'} \otimes_{R_2} \phi$ is an isomorphism, we have

$$l(1_{M_1'} \alpha \otimes_{R_2} \phi 1_N)(1_{M_1'} \otimes_{R_2} \phi^{-1}) = \beta l$$

and so

$$l(\alpha \otimes_{R_2} \phi 1_N \phi^{-1}) = l(\alpha \otimes_{R_2} 1_{E \otimes_{R_2} N}) = \beta l,$$

that is

$$l(\alpha \otimes_{R_2} 1_E \otimes_E N) = \beta l. \tag{6.4}$$

Now, $\alpha \otimes_{R_2} 1_E$ is an idempotent element of $\mathrm{End}(M_1)$. Hence, from (6.4), $(\alpha \otimes_{R_2} 1_E, \beta) \in \mathrm{End}(L_n)$. As L_n is an indecomposable module, we have that either $(\alpha \otimes_{R_2} 1_E, \beta) = (0, 0)$ or $(\alpha \otimes_{R_2} 1_E, \beta) = (1_{M_1}, 1_{M_2})$.

As ${}_{R_2}E$ is faithfully flat module, the first case yields $(\alpha, \beta) = (0, 0)$ and the second case $(\alpha, \beta) = (1_{M_1'}, 1_{M_2})$.

□

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