A TEST FOR WEAK STATIONARITY IN THE SPECTRAL DOMAIN

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Abstract. We examine tests for stability of the dynamics of a time series against alternatives that cover both local-stationarity and break points. One key feature of the tests is that the asymptotic distribution are functionals of the standard Brownian Bridge sheet in $[0,1]^2$. The tests have nontrivial power against local alternatives converging to the null hypothesis at a $T^{-1/2}$ rate, where T is the sample size. We examine an easy-to-implement bootstrap analogue and present the finite-sample performance in Monte-Carlo experiment. Finally, we implement the methodology to assess the stability of the inflation dynamics in the United States and on a set of neuroscience tremor data.

1. INTRODUCTION

Weak stationarity – the property that the structure of the data in its first two moments is independent of time – plays an important and key role when invoking asymptotic arguments, making inferences on a time series sequence or accurate predictions of future values. However, the assumption of weak stationarity could be difficult to justify a priori and it is possible that some sequences show nonstationary behaviour. In economics, a well-known example is the Lucas's (1976) critique. The justification being the belief that the parameters of macroeconometric models might depend implicitly on agents' expectations and so are unlikely to remain stable as policymakers change their behaviour. The possibility of data exhibiting nonstationary behaviour is not constrained to economic data sets, see examples in Paparoditis (2009) or Dahlhaus (2009). Thus the purpose of this paper is to present easy-to-implement tests for weak stationarity. We are not concerned with the situation when the change on the dynamics is due to a random variable as in $SETAR$, Threshold or Markov switching models, as the latter models are regarded as nonlinear, and within this type of models one is often more concerned with testing for linearity.

Testing for weak stationarity is not a new endeavour. There are two main approaches or lines of research. A first one, focused on change point alternatives, assumes that the practitioner knows the family of parametric model that generates the data. See for example Picard (1985), Davis et al. (1995) and the surveys by Perron (2006) or Aue and Horvath (2013). A second and more recent approach describes testing procedures when the practitioner has not a parametric model in mind or she is not confident in a particular one. See Paparoditis (2009) or Preuß et al. (2013), under Gaussianity, in the context of the so-called local-stationary models or evolutionary spectra introduced by Priestley (1965). The approaches pursued by Paparoditis (2009) and Preuß et al. (2013) are quite different. The former employs a direct comparison of two different nonparametric fits of the spectral density function, one fit being under the null hypothesis and the other under the alternative. The methodology is similar to that in Härdle and Mammen (1993) for model specification. On the other hand, the approach adopted in Preuß et al. (2013) is based on the empirical function similar to Dahlhaus and Polonik (2009).

In this paper we are interested in testing procedures when the practitioner has not a parametric model in mind. Our tests parallels recent work by Dwivedi and Subba Rao (2011) or Jentsch and Subba Rao (2015). More specifically we use, among other properties, that under weak stationarity the periodogram at two different Fourier frequencies are asymptotically independent,

Date: May 8th, 2017.
Key words and phrases. Welch's estimator. Time-varying spectral density. Bootstrap methods.

whereas Dwivedi and Subba Rao (2011) relies on a similar result but for the the discrete Fourier transform. Although the implementation of their test differs quite substantially from ours (see Section 2 for details) both approaches share the feature that their asymptotic distribution does not depend on the second order dependence of the sequence, contrary to those mentioned in the previous paragraph.

However a major difference between our approach and those described above is that our tests are easier to implement requiring the choice of only one smoothing parameter, which is a minimal requirement due to the non-specification of the model even under the null hypothesis. On the other hand, tests proposed in Dwivedi and Subba Rao (2011) or Paparoditis (2009) require the choice of 3 or even 4 different bandwidths for their implementation. So our methodology reduces the sensitivity of the test and makes its implementation easier. Second, our tests detect local alternatives of order $T^{-1/2}$, being T the sample size and so comparable to the test of Preuß et al. (2013), although we do not need to assume Gaussianity which rules out from the outset many examples with real time series data. So our tests are more efficient than those which only detect local alternatives of order $T^{-\alpha}$ for some $\alpha < 1/2$ as is the case using Härdle and Mammen's (1993) methodology.

Finally, as our Monte-Carlo experiment suggests that the asymptotic distribution does not provide a good approximation for the finite-sample one, we present a valid bootstrap to the tests which does not require the choice of any additional bandwidth parameter, in contrast to the methodology suggested in the aforementioned work, as in Preuß et al. (2013) or Dwivedi and Subba Rao (2011). Our bootstrap algorithm echoes the approach in Hidalgo (2007) and Hidalgo and Seo (2015) for data collected in a lattice model and that exhibit long memory dependence which is well-known not to be strong mixing, see Ibragimov and Rozanov (1978). As a by-product, we present a very simple estimator of the fourth cumulant, although we do not pursue its comparison to those in Grenander and Rosenblatt (1957) or its time domain analogue in Fragkeskou and Paparoditis (2016). This is beyond the scope of this paper.

We now describe our null hypothesis. Let $\{x_{t,T}\}_{t=1}^{T}$, $t \in \mathbb{N}$, denote a sequence of zero mean random variables

$$x_{t,T} = \sum_{j=0}^{\infty} \beta_{t,T}(j)\,\varepsilon_{t-j}; \qquad \beta_{t,T}(0) = 1, \tag{1.1}$$

to be more specific in Condition $C1$ below. Model (1.1) allows for local stationarity as well as breaks in some of the coefficients $\beta_{t,T}(j)$, say $\beta_{t,T}(j) = \delta(j)\,\mathcal{I}(t < t_0) + \beta(j)$ with $\delta(j) \neq 0$. We define our null hypothesis as

$$H_0: \quad \beta_{t,T}(j) = \beta(j) \quad \text{for all } j \in \mathbb{N} \text{ and } t = 1, ..., T,\ T \in \mathbb{N},$$

being the alternative hypothesis the negation of the null.

We alternatively formulate our hypothesis in the spectral domain as follows. If $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a stationary sequence, there exists a Cràmer-representation

$$\varepsilon_t = \frac{1}{(2\pi)^{1/2}} \int_{-\pi}^{\pi} \exp(i\lambda t)\, d\xi(\lambda),$$

where $\xi(\lambda)$ has zero mean and orthogonal increments, see Brillinger (1981). So the (time-varying) spectral representation of (1.1)

$$x_{t,T} = \int_{-\pi}^{\pi} \mathcal{B}_{t,T}(\lambda) \exp(i\lambda t)\, d\xi(\lambda), \tag{1.2}$$

see Dahlhaus (1997), where $\mathcal{B}_{t,T}(\lambda)$ denotes the transfer function

$$\mathcal{B}_{t,T}(\lambda) = \overline{\mathcal{B}_{t,T}(-\lambda)} = \sum_{j=0}^{\infty} \beta_{t,T}(j)\, e^{-ij\lambda}. \tag{1.3}$$

Under Condition $C2$ below, we can approximate (1.3) by

$$\mathcal{B}(u;\lambda) = \sum_{j=0}^{\infty} \beta(u;j)\, e^{-it\lambda}, \quad u \in [0,1]\,;\ \lambda \in [0,\pi]\,, \tag{1.4}$$

in the sense that

$$\sup_{\lambda\in[0,\pi],1\le t\le T} \left| \mathcal{B}_{t,T}(\lambda) - \mathcal{B}\left(\frac{t}{T};\lambda\right) \right| = O\left(\frac{1}{T}\right), \tag{1.5}$$

and hence we denote the "time varying" spectral density function of $\{x_{t,T}\}_{t=1}^{T}$, $T \in \mathbb{N}$, by

$$f(u;\lambda) = \frac{\sigma_\varepsilon^2}{2\pi} |\mathcal{B}(u;\lambda)|^2 \quad u \in [0,1]\,; \quad \lambda \in [0,\pi]\,. \tag{1.6}$$

As an example we have the general $tvARMA$ model

$$x_{t,T} - \sum_{p=1}^{P} \alpha_p\left(\frac{t}{T}\right) x_{t-p,T} = \sum_{q=0}^{Q} \beta_q\left(\frac{t}{T}\right) \varepsilon_{t-q}$$

which satisfies

$$\mathcal{B}(u;\lambda) = \frac{1}{(2\pi)^{1/2}} \frac{\sum_{q=0}^{Q} \beta_q(u) \exp(-i\lambda q)}{1 - \sum_{p=1}^{P} \alpha_j(u) \exp(-i\lambda p)}.$$

A second example is when there is an abrupt change on the values of the parameters at some particular time t_0 and $x_{t,T} = \alpha(t)\, x_{t-1,T} + \varepsilon_t$, where

$$\alpha(t) = \alpha_1 \mathcal{I}(t < t_0) + \alpha_2 \mathcal{I}(t \ge t_0)\,; \quad \alpha_1 \ne \alpha_2$$

and $|\alpha_1|, |\alpha_2| < 1$. Then $\mathcal{B}(u;\lambda) =: (2\pi)^{-1/2} (1 - \alpha_1 \exp(-i\lambda))^{-1}$ when $u = t/T < u_0 = t_0/T$ and $=: (2\pi)^{-1/2} (1 - \alpha_2 \exp(-i\lambda))^{-1}$ when $u \ge u_0$.

Given (1.6) we can alternatively write H_0 as

$$H_0: \quad f(u;\lambda) = f(\lambda) \quad \text{for all } u \in [0,1] \tag{1.7}$$

a.e. in $[0,\pi]$, being the alternative hypothesis the negation of the null. That is, denoting by $\mu(\cdot)$ the Lebesgue measure,

$$H_a: \ \mu(\mathcal{U},\Lambda) > 0, \tag{1.8}$$

where $(\mathcal{U},\Lambda) = \{u \in [0,1]\,; \lambda \in [0,\pi] : f(u;\lambda) \ne f(\lambda)\}$. At the end of Section 2.3 we comment that our tests are consistent against heteroscedastic alternatives, say $x_{t,T} = \sigma_t \varepsilon_t$, with $\sigma_t = \sigma(t/T)$.

The remainder of the paper is organized as follows. The next section describes and examines a test for H_0 in (1.7), as well as a modification such that the asymptotic behaviour does not depend on any unknown quantity and in particular on the unknown $f(\lambda) =: f(u,\lambda)$. We also discuss regularity conditions and the type of local alternatives for which the test has no trivial power. Section 3 presents a valid bootstrap algorithm for our hypothesis testing. Section 4 presents a Monte-Carlo experiment to shed light on the finite-sample performance of the test. We show that our test has a more robust performance (both in terms of size and power) across data-generating processes than other comparable test such as Preuß et al. (2013). We also apply our test to two real data sets. Section 5 concludes, whereas the proofs are confined to the Appendix.

2. THE TEST AND REGULARITY CONDITIONS

We first describe and discuss the motivation of the tests for H_0 in (1.7). Given a stretch of data $\{x_{t,T}\}_{t=1}^{T}$, where T denotes the sample size, we split it into $\mathbf{B}$ blocks, each of them of length n, assuming without loss of generality that $\mathbf{B} = T/n$. Thus, the $b-th$ block is based on the observations $\left\{x_{t+(b-1)n,T}\right\}_{t=1}^{n}$, and we denote its periodogram by

$$I_{x,b}(j) = \frac{1}{n}\left|\sum_{t=1}^{n} x_{t+(b-1)n,T}e^{-it\lambda_j}\right|^2, \; b = 1,...,\mathbf{B}, \tag{2.1}$$

where $\lambda_j = 2\pi j/n$, $j = 1,...,[n/2] =: \tilde{n}$, and where we abbreviate in what follows $g(\lambda_j)$ by $g(j)$ for a generic function $g(\lambda)$. Similarly, the periodogram of $\left\{\varepsilon_{t+(b-1)n}\right\}_{t=1}^{n}$, $b = 1,...,\mathbf{B}$, is given by $I_{\varepsilon,b}(j) = n^{-1}\left|\sum_{t=1}^{n}\varepsilon_{t+(b-1)n}e^{-it\lambda_j}\right|^2$.

Our definition of the periodogram in (2.1) is similar to that in Dahlhaus (1997), i.e.

$$I_{x,b}(j) = \frac{1}{n}\left|\sum_{t=-\tilde{n}+1}^{\tilde{n}} x_{t+(b-1)n,T}e^{-it\lambda_j}\right|^2, \tag{2.2}$$

which has the interpretation of being the periodogram over a segment of length n with midpoint at $t+(b-1)n$. There is no difference with ours from an asymptotic point of view, and we prefer (2.1) for notational simplicity. Alternatively we might have employed the so-called "pre-periodogram", see (3.7) in Neumann and Von Sachs (1997), given by

$$I_{t,T}(\lambda) = \sum_{s:1\leq[t-s/2],[t+s/2]\leq T} x_{[t-s/2],T}x_{[t+s/2],T}e^{-is\lambda}$$

which is regarded as a closer counterpart of the evolutionary spectra function

$$f_T(u,\lambda) = \sum_{s=-\infty}^{\infty} Cov\left(x_{[uT-s/2],T}; x_{[uT+s/2],T}\right)e^{-is\lambda},$$

which, as noticed by Dahlhaus (1997), satisfies that $\lim_{T\to\infty} f_T(u,\lambda) = f(u,\lambda)$ defined in (1.6). However it appears to have worst finite sample properties when compared to that in (2.1) or (2.2), so that we have decided to use the latter.

We now describe the tests. Suppose that we were interested in the null hypothesis H_0 given in (1.7) but only at some frequency λ_j, $j = 1,...,\tilde{n}$. Because $I_{x,b}(j)/f(j) \simeq I_{\varepsilon,b}(j)/\mathcal{E}\left(\varepsilon_t^2\right)$ using Bartlett's decomposition, see e.g. Brockwell and Davis (1991), and for all $j = 1,..,\tilde{n}$, $\left\{\left(I_{\varepsilon,b}(j)/\mathcal{E}\left(\varepsilon_t^2\right)\right)-1\right\}_{b=1}^{\mathbf{B}}$ behaves as a sequence of uncorrelated centered χ_2^2 random variables, we suggest the CUSUM-type of statistic

$$\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};j\right) = \frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\frac{I_{x,b}(j)}{\widehat{f}(j)} - 1\right\}, \;\; \mathbf{b}^* = 2,...,\mathbf{B}$$

where

$$\widehat{f}(j) = \frac{1}{\mathbf{B}}\sum_{v=1}^{\mathbf{B}} I_{x,v}(j) \tag{2.3}$$

is the estimator of the spectral density function $f(j)$ proposed by Welch (1967), which appears more natural in our context than the standard smooth periodogram estimator.

The previous arguments were given for a particular frequency λ_j. However since $Cov(I_{\varepsilon,b_1}(j); I_{\varepsilon,b_2}(k)) = 0$ if $j \neq k$ for all $b_1, b_2 = 1,...,\mathbf{B}$, then extending the argument to

$[0,\pi]$, our test for H_0 is based on

$$\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)=\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};j\right),\quad \mathbf{j}^*=1,...,\tilde{n}. \tag{2.4}$$

As we pointed out in the introduction, our statistic (2.4) draws some similarities to those given in Dwivedi and Subba Rao (2011) see also Jentsch and Subba Rao (2015) and Bandyopadhy et al. (2017). More specifically, denote $\omega_k=2\pi k/T$ and $\mathcal{J}_x(\omega_k)=T^{-1/2}\sum_{t=1}^T x_t e^{-it\omega_k}$, $k=1,...T$, and compute

$$\widehat{c}_T(r)=\frac{1}{T}\sum_{k=1}^T\frac{\mathcal{J}_x(\omega_k)\,\mathcal{J}_x(\omega_{k+r})}{\widehat{f}^{1/2}(\omega_k)\,\widehat{f}^{1/2}(\omega_{k+r})},\quad r=1,...,m,$$

for some finite chosen m with $\widehat{f}(\lambda_k)$ being the weighted average periodogram estimator. Then using the fact that under the null hypothesis $\left\{T^{1/2}\widehat{c}_T(r)\right\}_{r=1}^{[T/2]}$ behaves as a sequence of independent random variables, they propose $\mathcal{DSR}_T(m)=T\sum_{r=1}^m\left\{\left|\widehat{c}_T(r)\right|^2/\left(1+\widehat{\kappa}_4(\omega_r)\right)^{1/2}\right\}$, where $1+\widehat{\kappa}_4(\omega_r)$ is an estimator of the second moments of $T\left|\widehat{c}_T(r)\right|^2$. So, drawing similarities with Box-Pierce statistic, we can regard $\mathcal{DSR}_T(m)$ as a Portmanteau-type of test, see also the test proposed in Lee et al. (2003). However our tests lie into the goodness-of-fit type of tests, and in this sense we look at all frequencies in $[0,\pi]$, i.e. letting $m=T/2$, instead of a fixed and finite number of them as is the case with $\mathcal{DSR}_T(m)$. The latter suggests that showing the validity of our test is technically more challenging.

Theorem 1 indicates that the asymptotic distribution of $\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ depends on the fourth cumulant of the innovations $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ and hence it is not pivotal. So it might be of interest to examine if we can provide a modification of $\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ with pivotal asymptotic distribution. To that end, we borrow ideas from Anderson and Walker (1964), who observed that the asymptotic distribution of the estimator of the correlation coefficient depends only on the first two moments, compared with the dependence on the fourth cumulant when examining the estimator of the covariance. Then we propose the following statistic

$$\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)=\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};j\right),\quad \mathbf{j}^*=1,...,\tilde{n} \tag{2.5}$$

where

$$\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};j\right)=\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\frac{I_{x,b}(j)/\widehat{\sigma}_\varepsilon^2(b)}{\frac{1}{\mathbf{B}}\sum_{v=1}^{\mathbf{B}}\left(I_{x,v}(j)/\widehat{\sigma}_\varepsilon^2(v)\right)}-1\right\},\quad \mathbf{b}^*=2,...,\mathbf{B}$$

and $\widehat{\sigma}_\varepsilon^2(b)$, $b=1,...,\mathbf{B}$, is given in (2.23) below. We now introduce regularity conditions.

Condition C1: $\{x_{t,T}\}_{t=1}^T$, $T\in\mathbb{N}$, is a sequence of random variables defined as

$$x_{t,T}=\sum_{j=0}^{\infty}\beta_{t,T}(j)\,\varepsilon_{t-j},\ \text{with}\ \beta_{t,T}(0)=1, \tag{2.6}$$

such that $\sup_t|\beta_{t,T}(j)|<\upsilon(j)$, $\sum_{j=0}^{\infty}j\upsilon(j)<\infty$, where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is an independent and identically distributed sequence with $\mathcal{E}(\varepsilon_t)=0$, $\mathcal{E}\left(\varepsilon_t^2\right)=\sigma_\varepsilon^2$ and $\sup_t\mathcal{E}\left(|\varepsilon_t|^\ell\right)=\mu_\ell<\infty$ for some $\ell>8$. In addition, $|\mathcal{B}_{t,T}(\lambda)|$ is bounded away from zero for all $\lambda\in[0,\pi]$. Finally, we denote the fourth cumulant of $\varepsilon_t/\sigma_\varepsilon$ as κ_4.

Condition $C1$ is standard and very mild. It entails to weak dependence of the sequence, although not necessarily stationary as $\beta_{t,T}(j)$ may depend on t. The condition allows for many models such as when with jumps and smooth transitions as local-stationary sequences used to model nonlinearities with time series data. It appears that we can relax Condition $C1$ to allow the conditions stated in Dalla, Giraitis and Hidalgo (2005) or the independence assumption of the innovations $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ to be only martingale differences in its first and second moments at the expense of complicating the technical apparatus. In this way, the condition would become very similar to Assumption A1 in Dwivedi and Subba Rao's (2011), which plays a central role in their results. In addition, as we comment below, we can allow the fourth moments of the innovations to depend on time, relaxing the condition of stationarity among the first four moments needed in Dwivedi and Subba Rao (2011).

It is worth pointing out that $C1$ implies that the sequences $\{x_{t,T}\}_{t=1}^{T}$ have also an autoregression representation as Dahlhaus (1996) showed. Replacing (2.6) by

$$x_{t,T} = \sum_{j=0}^{\infty} \beta\left(\frac{t}{T}; j\right) \varepsilon_{t-j}; \qquad b\left(\frac{t}{T}; 0\right) = 1 \tag{2.7}$$

then, as was first shown by Dahlhaus (1996), the $tvAR$ model

$$x_{t,T} = \sum_{p=1}^{P} \alpha\left(\frac{t}{T}; p\right) x_{t-p,T} + \varepsilon_t$$

under standard regularity conditions on $\{\alpha(u;j)\}_{j\geq 0}$ for all $u \in [0,1]$, cannot be represented as in (2.7). Under H_0 the latter model collapses to the standard $AR(P)$.

***Condition C2*:** $\beta_{t,T}(j)$ satisfies that

$$\sup_{1\leq t\leq T} \left|\beta_{t,T}(j) - \beta\left(\frac{t}{T}; j\right)\right| \leq \frac{C}{T} \upsilon(j) \qquad \sum_{j=0}^{\infty} j\upsilon(j) < \infty. \tag{2.8}$$

Condition $C2$ indicates that $\beta_{t,T}(j)$ can be well approximated (locally) by a smooth function $\beta(u;j)$ and that observations which are close in time are regarded as stationary. The bound sequence $\upsilon(j)$ does not need to be the same as that in Condition $C1$. However we keep it for notational simplicity as both satisfies the upper bound $\upsilon(j) = O\left(j^{-2-\delta}\right)$ for some $\delta > 0$. Another implication of Condition $C2$ is that $\mathcal{B}_{t,T}(\lambda)$ and $\mathcal{B}(u;\lambda)$ given in $(1.3)-(1.4)$ satisfies (1.5). Under H_0, (2.8) holds trivially.

Thus, under H_0, Condition $C1$ implies that the sequence has a spectral density function

$$f(\lambda) = \sigma_\varepsilon^2 \left|\mathcal{B}(\lambda)\right|^2, \tag{2.9}$$

where $\mathcal{B}(z) = \sum_{j=0}^{\infty} \beta(j) e^{-ijz}$. Also, using the autoregressive representation we can write the spectral density function as $f(\lambda) = \sigma_\varepsilon^2 \left|\mathcal{A}(\lambda)\right|^{-2}$, where $\mathcal{A}(z) = 1 - \sum_{j=1}^{\infty} \alpha(j) e^{-ijz}$.

***Condition C3*:** n is such that as T increases to infinity, $\frac{T}{n^2} + \frac{n^3}{T^2} \to 0$ and $\mathbf{B} = T/n$.

We finish this section with a couple of comments. First, it is worth noticing that in view of the similarity of $\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}}; j\right)$ with the CUSUM tests, we could have employed a recursive version of the statistic,

$$\frac{1}{\mathbf{B}} \sum_{b=2}^{\mathbf{b}^*} \left\{ \frac{I_{x,b}(j)}{(\mathbf{b}^*-1)^{-1} \sum_{v=1}^{\mathbf{b}^*-1} I_{x,v}(j)} - 1 \right\} \qquad \mathbf{b}^* = 2, ..., \mathbf{B}.$$

The motivation of this modification, as given in Brown et al. (1975) when testing for constancy of the parameters, is to avoid the dependence of the distribution of the test on the estimator of the parameters of the model under the null hypothesis. This modification is most often known as Khamaladze's transformation (1981). Notice that we can regard $I_{x,b}(j)/\widehat{f}(j)$ as an estimator of the standardized mean $f(b/\mathbf{B};j)/\mathbf{B}^{-1}\sum_{b=1}^{\mathbf{B}} f(b/\mathbf{B};\mathbf{j})$. To simplify the exposition we focus on $\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}}; j\right)$ to test for H_0.

Second, if we were interested in testing the null hypothesis that some specific parametric model explains the dynamics of the sequence, say $f(\lambda, \theta_0)$, then a test for this specification is easily implemented as

$$\begin{aligned}\mathcal{T}^{spec}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right) &= \frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\mathcal{T}^{spec}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};j\right);\ \mathbf{j}^*=1,...,\tilde{n}\\ \mathcal{T}^{spec}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};j\right) &= \frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\frac{I_{x,b}(j)}{f\left(j,\widehat{\theta}\right)}-1\right\},\ \mathbf{b}^*=2,...,\mathbf{B}\end{aligned}$$

for some estimator $\widehat{\theta}$ of θ_0, say the Whittle estimator.

2.1. Asymptotic properties of (2.4).

For reasons which will become clear, it is convenient and useful to examine first the behaviour of

$$\breve{\mathcal{T}}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)=\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\frac{I_{x,b}(j)/\left|\mathcal{B}\left(\frac{nb}{T};j\right)\right|^2}{\mathbf{B}^{-1}\sum_{v=1}^{\mathbf{B}}\left\{I_{x,v}(j)/\left|\mathcal{B}\left(\frac{nv}{T};j\right)\right|^2\right\}}-1\right\}$$

for $\mathbf{b}^*=2,...,\mathbf{B}$ and $\mathbf{j}^*=1,...,\tilde{n}$. Observe that under H_0, $\breve{\mathcal{T}}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)=:\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ as $\left|\mathcal{B}\left(\frac{nb}{T};j\right)\right|^2=|\mathcal{B}(j)|^2$. We have the following result.

Theorem 1. *Assuming C1 to C3, we have that as $T\to\infty$,*

$$[T/2]^{1/2}\,\breve{\mathcal{T}}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)\overset{weakly}{\Rightarrow}\mathcal{BS}\left([0,1]^2\right),$$

where $\mathcal{BS}\left([0,1]^2\right)$ is a Gaussian process in $[0,1]^2$ with covariance structure

$$\mathcal{C}\left(\boldsymbol{\omega}_1^*,\boldsymbol{\omega}_2^*;\boldsymbol{v}_1^*,\boldsymbol{v}_2^*\right)=\boldsymbol{\omega}_1^*\left(1-\boldsymbol{\omega}_2^*\right)\left[\boldsymbol{v}_1^*+\frac{1}{2}\boldsymbol{v}_1^*\boldsymbol{v}_2^*\kappa_4\right],\ 0\le\boldsymbol{\omega}_1^*\le\boldsymbol{\omega}_2^*\le1,\ 0\le\boldsymbol{v}_1^*\le\boldsymbol{v}_2^*\le1. \tag{2.10}$$

Proof. The proof of this theorem, or any other result, is given in the appendix. ∎

We have the following corollary.

Corollary 1. *Let $\varphi(\cdot,\cdot)$ be a continuous functional in $[0,1]^2\to\mathbb{R}^+$. Then, under H_0 and assuming Conditions C1 and C3, we have that*

$$\text{(a)}\ \ [T/2]^{1/2}\,\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)\overset{weakly}{\Rightarrow}\mathcal{BS}\left([0,1]^2\right)$$

$$\text{(b)}\ \ \varphi\left([T/2]^{1/2}\,\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)\right)\overset{d}{\Rightarrow}\varphi\left(\mathcal{BS}\left([0,1]^2\right)\right).$$

Standard functionals $\varphi(\cdot,\cdot)$ are the Kolmogorov-Smirnov and the Cràmer von Mises given respectively as

$$\mathcal{KS}_{n,\mathbf{B}}=\max_{\mathbf{j}^*=1,...,\tilde{n};\mathbf{b}^*=1,...\mathbf{B}}\left|[T/2]^{1/2}\,\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)\right| \tag{2.11}$$

$$\mathcal{C}v\mathcal{M}_{n,\mathbf{B}}=\frac{1}{[T/2]}\sum_{\mathbf{j}^*=1}^{\tilde{n}}\sum_{\mathbf{b}^*=1}^{\mathbf{B}}\left|[T/2]^{1/2}\,\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)\right|^2. \tag{2.12}$$

The first conclusion that we draw from Corollary 1 is that when $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ are Gaussian, we have that

$$[T/2]^{1/2}\,\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)\overset{weakly}{\Rightarrow}\mathcal{BS}\left([0,1]^2\right),$$

where (2.10) becomes $\mathcal{C}\left(\boldsymbol{\omega}_1^*,\boldsymbol{\omega}_2^*;\boldsymbol{\upsilon}_1^*,\boldsymbol{\upsilon}_2^*\right)=\boldsymbol{\omega}_1^*\left(1-\boldsymbol{\omega}_2^*\right)\boldsymbol{\upsilon}_1^*$, where $\boldsymbol{\omega}_1^*\leq\boldsymbol{\omega}_2^*$ and $\boldsymbol{\upsilon}_1^*\leq\boldsymbol{\upsilon}_2^*$, which can be regarded as the covariance structure of the "product" of a standard Brownian Bridge and a Brownian motion. The implication is that the asymptotic distribution of $\varphi\left([T/2]^{1/2}\,\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)\right)$ is pivotal, and valid critical values can be easily computed regardless of the true underlying dependence structure of the sequence $\{x_t\}_{t\in\mathbb{Z}}$. This is a major difference compared to the test proposed in Dette et al. (2011). In addition, as we show below, our tests have nontrivial power when the alternative converges to the null at the rate $T^{1/2}$, which is faster than the $T^{1/4}$ obtained elsewhere in Dette et al. (2011) or using nonparametric fits, although the same rate as that in Preuß et al. (2013). However, the latter depends on the assumption of the innovation sequence $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ being Gaussian and it needs bootstrap procedures for its implementation.

2.2. **Asymptotic properties of** (2.5).

Theorem 1 shows that when the Gaussianity assumption is dropped, the asymptotic covariance structure of $[T/2]^{1/2}\,\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ depends, in a nontrivial way, on κ_4. So, to implement the test, one route is to provide a consistent estimator of κ_4 and from here to compute the (asymptotic) critical values. This could be achieved by simulating the critical values of the statistics given in (2.11) or (2.12) for a mesh, $\mathcal{M}$, of possible values of κ_4, denote that by $cr\left(\kappa_4\right)$. Then, given a particular data set, we use as critical values $cr\left(\widehat{\kappa}_4\right)$, where $\widehat{\kappa}_4$ is a consistent estimator of κ_4. A second route is to see if $\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ in expression (2.5) does not depend on κ_4. A third route is to compute valid asymptotic critical values via bootstrap algorithms. In this section we show that indeed the asymptotic distribution of $\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ does not depend on κ_4, presenting and examining a valid bootstrap in Section 3.

Because the implementation of $\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ in expression (2.5) requires to obtain $\{\varepsilon_t\}_{t\in\mathbb{Z}}$, which are also needed to compute our estimator of κ_4 or the bootstrap, we first provide a simple method to obtain the innovation sequence. For this purpose, given a generic sequence $\{z_t\}_{t=1}^{T}$, denote the *discrete Fourier transform* (*DFT*) of $\left\{z_{t+(b-1)n}\right\}_{t=1}^{n}$ by

$$\mathcal{J}_{z,b}\left(j\right)=\frac{1}{n^{1/2}}\sum_{t=1}^{n}z_{t+(b-1)n}e^{-it\lambda_j},\qquad j=1,\ldots,\tilde{n},\;\; b=1,...,\mathbf{B}. \tag{2.13}$$

It is well known, see expression (10.3.12) of Brockwell and Davis's (1991), that under the null hypothesis H_0 and Condition $C1$, the *DFT*s of $\left\{\varepsilon_{t+(b-1)n}\right\}_{t=1}^{n}$ and $\left\{x_{t+(b-1)n,T}\right\}_{t=1}^{n}$ satisfy the relation

$$\mathcal{J}_{x,b}\left(j\right)=\mathcal{B}\left(-j\right)\mathcal{J}_{\varepsilon,b}\left(j\right)+\Upsilon_{n,b}\left(j;0\right),\qquad b=1,...,\mathbf{B}, \tag{2.14}$$

where $\mathcal{B}\left(-j\right)=:\mathcal{B}\left(e^{-i\lambda_j}\right)$ and

$$\Upsilon_{n,b}\left(j;a\right)=\sum_{\ell=a}^{\infty}\beta\left(\ell\right)e^{-i\ell\lambda_j}\left(\frac{1}{n^{1/2}}\left\{\sum_{t=1-\ell}^{n-\ell}-\sum_{t=1}^{n}\right\}\varepsilon_{t+(b-1)n}e^{-it\lambda_j}\right). \tag{2.15}$$

Under $C1$, (2.14) and (2.15) become

$$\mathcal{J}_{x,b}\left(j\right)=\mathcal{B}\left(\frac{n\left(b-1\right)}{T};-j\right)\mathcal{J}_{\varepsilon,b}\left(j\right)+\check{\Upsilon}_{n,b}\left(j;0\right)+\ddot{\Upsilon}_{n,b}\left(j\right), \tag{2.16}$$

where $\mathcal{B}\left(\frac{n(b-1)}{T};j\right) =: \mathcal{B}\left(\frac{n(b-1)}{T};e^{i\lambda_j}\right)$ and

$$\check{\mathrm{Y}}_{n,b+1}(j;a) = \sum_{\ell=a}^{\infty}\beta\left(\frac{nb}{T};\ell\right)e^{-i\ell\lambda_j}\left(\frac{1}{n^{1/2}}\left\{\sum_{t=1-\ell}^{n-\ell}-\sum_{t=1}^{n}\right\}\varepsilon_{t+bn}e^{-it\lambda_j}\right), \tag{2.17}$$

$$\begin{aligned}\ddot{\mathrm{Y}}_{n,b+1}(j) &= \frac{1}{n^{1/2}}\sum_{t=1}^{n}\left(\sum_{\ell=0}^{\infty}\left(\beta\left(\frac{t+nb}{T};\ell\right)-\beta\left(\frac{nb}{T};\ell\right)\right)\varepsilon_{t+bn-\ell}\right)e^{it\lambda_j}\\ &\quad+\frac{1}{n^{1/2}}\sum_{t=1}^{n}\left(\sum_{\ell=0}^{\infty}\left(\beta_{t+bn,T}(\ell)-\beta\left(\frac{t+nb}{T};\ell\right)\right)\varepsilon_{t+bn-\ell}\right)e^{it\lambda_j}.\end{aligned} \tag{2.18}$$

Using the inverse transformation of the DFT,

$$z_{t+(b-1)n} = \frac{1}{n^{1/2}}\sum_{j=1}^{n}\mathcal{J}_{z,b}(j)\,e^{it\lambda_j}, \qquad t=1,\ldots,n,$$

and noting that $\mathrm{Y}_{n,b}(j;a)$ is negligible compared to $\mathcal{B}(-j)\,\mathcal{J}_{\varepsilon,b}(j)$ in (2.14), we obtain $\{\varepsilon_t\}_{t=1}^{T}$ as

$$\varepsilon_{t+(b-1)n} \simeq \frac{1}{n^{1/2}}\sum_{j=1}^{n}e^{it\lambda_j}\mathcal{A}(-j)\,\mathcal{J}_{x,b}(j)\,,\ \ t=1,...,n;\ \ b=1,...,\mathbf{B},$$

where "≈" should be read as "approximately" and by definition we have that $\mathcal{A}(j) =: \mathcal{A}(\lambda_j) =: \mathcal{B}^{-1}(\lambda_j) = \mathcal{B}^{-1}(j)$. Similarly we should expect that

$$\varepsilon_{t+(b-1)n} \simeq \frac{1}{n^{1/2}}\sum_{j=1}^{n}e^{it\lambda_j}\mathcal{B}^{-1}\left(\frac{n(b-1)}{T};-j\right)\mathcal{J}_{x,b}(j)\,.$$

Thus, under H_0, the problem to obtain $\left\{\widehat{\varepsilon}_{t+(b-1)n}\right\}_{t=1}^{n}$ becomes a problem of how to compute an estimator of $\mathcal{A}(j)$. To that end, using (2.3) we compute $\left\{\widehat{\varepsilon}_{t+(b-1)n}\right\}_{t=1}^{n}$ as

$$\widehat{\varepsilon}_{t+(b-1)n} = \frac{1}{n^{1/2}}\sum_{j=1}^{n-1}e^{it\lambda_j}\widehat{\mathcal{A}}(-j)\,\mathcal{J}_{x,b}(j)\,, \quad b=1,...,\mathbf{B}, \tag{2.19}$$

where

$$\begin{aligned}\widehat{\mathcal{A}}(j) &= \exp\left\{-\sum_{r=1}^{[\tilde{n}/2]}\widehat{c}_r e^{ir\lambda_j}\right\}, \quad j=1,...,\tilde{n} \\ \widehat{\mathcal{A}}(j) &= \overline{\widehat{\mathcal{A}}}(n-j)\,, \qquad\qquad j=\tilde{n}+1,...,n-1\end{aligned} \tag{2.20}$$

$$\widehat{c}_r = \frac{1}{\tilde{n}}\sum_{\ell=1}^{\tilde{n}}\log\widehat{f}(\ell)\cos r\lambda_\ell, \quad r=0,...,[\tilde{n}/2]\,. \tag{2.21}$$

Note that $\widehat{\sigma}_{\varepsilon}^2 = \exp(\widehat{c}_0)$ and $\widehat{\mathcal{A}}(\lambda) = \exp\left\{-\sum_{r=1}^{[\tilde{n}/2]}\widehat{c}_r e^{ir\lambda}\right\}$ is an estimator of $\mathcal{A}(\lambda) = \exp\left\{-\sum_{r=1}^{\infty}c_r e^{ir\lambda}\right\}$ with

$$c_r = \frac{1}{\pi}\int_0^{\pi}\log f(\lambda)\cos(r\lambda)\,d\lambda, \qquad r=0,1,.... \tag{2.22}$$

Observe that $f(\lambda) = \widehat{\sigma}_{\varepsilon}^2\left|\mathcal{A}(\lambda)\right|^{-2}$ and the motivation to estimate $\mathcal{A}(\lambda)$ by $\widehat{\mathcal{A}}(j)$ comes from the canonical spectral decomposition of $f(\lambda)$, see Brillinger (1981, $p.$ 78 − 79) or Hannan (1970).

Moreover, denoting

$$\widehat{a}_\ell = \frac{1}{n}\sum_{j=-\tilde{n}+1}^{\tilde{n}-1} \widehat{\mathcal{A}}(-j)\, e^{i\ell\lambda_j}, \quad \ell = 1,...,\tilde{n},$$

we could have estimated $\mathcal{A}(j)$ as $\widehat{\mathcal{A}}(j) = 1 + \widehat{a}_1 e^{-i\lambda_j} + ... + \widehat{a}_{\tilde{n}} e^{-i\tilde{n}\lambda_j}$. This is because $f(\lambda) = \left|\exp\left\{\frac{1}{2}c_0 + \sum_{r=1}^{\infty} c_r e^{-ir\lambda}\right\}\right|^2$ and a_ℓ is the ℓth Fourier coefficient of $\exp\left\{\sum_{r=1}^{\infty} c_r e^{-ir\lambda}\right\}$. In fact, one implication of the canonical decomposition is that $\exp\left\{\sum_{r=1}^{\infty} c_r e^{-ir\lambda}\right\} = 1 - \sum_{j=1}^{\infty} a_j e^{ij\lambda}$.

Given $\{\widehat{\varepsilon}_t\}_{t=1}^{T}$ in (2.19), we compute our estimator of κ_4 as

$$\widehat{\kappa}_4 = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{\widehat{\varepsilon}_t^4}{\widehat{\sigma}_\varepsilon^4} - 3\right),$$

where either $\widehat{\sigma}_\varepsilon^2 = \exp(\widehat{c}_0)$ or $\widehat{\sigma}_\varepsilon^2 = \mathbf{B}^{-1}\sum_{b=1}^{\mathbf{B}} \widehat{\sigma}_\varepsilon^2(b)$, with

$$\widehat{\sigma}_\varepsilon^2(b) = \frac{1}{n}\sum_{t=1}^{n} \widehat{\varepsilon}^2_{t+(b-1)n}, \quad b = 1,,,\mathbf{B}. \tag{2.23}$$

Notice that $\widehat{\sigma}_\varepsilon^2 = T^{-1}\sum_{t=1}^{T} \widehat{\varepsilon}_t^2$.

Corollary 2. *Under H_0 and assuming $C1$ and $C3$, we have that $\widehat{\kappa}_4 \to_P \kappa_4$.*

Our estimator of κ_4 is an alternative to that in Grenander and Rosenblatt (1957), or more recently the estimator in Paparoditis (2009), or its time domain analogue in Fragkeskou and Paparoditis (2016). One potential drawback of the latter estimators is that they need at least the choice of two additional bandwidths parameters as their computation depends on the covariance structure of the sequences $\left\{x^2_{t+(b-1)n}\right\}_{t=1}^{n}$ and $\left\{x_{t+(b-1)n}\right\}_{t=1}^{n}$. Another estimator might be based on the sieve estimator. However, once again the methodology involves the choice of an additional smoothing parameter, i.e. the degree p_T of the AR polynomial approximation. On the contrary, $\widehat{\kappa}_4$ does not require the choice of any additional bandwidth.

Once we obtain $\widehat{\kappa}_4$, the critical values of $\varphi\left([T/2]^{1/2}\,\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)\right)$ are computed as suggested above. This method might introduce substantial noise in small samples, so that it is desirable to see if our statistic $\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ would no longer depend on κ_4. This is confirmed in the following result.

Theorem 2. *Under H_0 and assuming $C1$ and $C3$, we have that*

$$[T/2]^{1/2}\,\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right) \overset{weakly}{\Rightarrow} \mathcal{WB}\left([0,1]^2\right),$$

where $\mathcal{WB}\left([0,1]^2\right)$ is a Brownian Bridge sheet in $[0,1]^2$ with covariance structure

$$\mathcal{C}\left(\boldsymbol{\omega}_1^*,\boldsymbol{\omega}_2^*;\boldsymbol{\upsilon}_1^*,\boldsymbol{\upsilon}_2^*\right) = \boldsymbol{\omega}_1^*\left(1-\boldsymbol{\omega}_2^*\right)\boldsymbol{\upsilon}_1^*\left(1-\boldsymbol{\upsilon}_2^*\right). \tag{2.24}$$

As before, we can apply the Kolmogorov-Smirnov and the Cràmer von Mises type of test using (2.11) and (2.12) but replacing $\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ by $\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$.

2.3. **Local Alternatives and Consistency.**

We finish describing the local alternatives for which our tests have no trivial power. For that purpose, we show that $\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ will have a mean different than zero under the alternative hypothesis. Indeed, under suitable regularity conditions, say $C1 - C3$,

$$\widehat{f}(j) =: \frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}} I_{x,b}(j) \overset{P}{\to} \lim_{\mathbf{B}(T)\to\infty}\frac{1}{\mathbf{B}(T)}\sum_{b=1}^{\mathbf{B}(T)} f\left(\frac{b}{\mathbf{B}(T)};j\right) \simeq \int_0^1 f(u;\lambda(j))\,du,$$

where $\lambda(j) = \lim_{n(T)\to\infty} \lambda_j$. So as the set $(\mathcal{U}, \Lambda)$ given after (1.8) has positive Lebesgue measure, the last displayed expression suggests that

$$\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right) \simeq \frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\frac{\acute{I}_{x,b}(j)}{\frac{1}{\mathbf{B}}\sum_{v=1}^{\mathbf{B}}\acute{I}_{x,v}(j)}-1\right\} + \frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\frac{f\left(\frac{b}{\mathbf{B}};j\right)}{\int_0^1 f(u;\lambda(j))\,du}-1\right\}(1+o_p(1)),$$

where $\acute{I}_{x,b}(j) = I_{x,b}(j) - f\left(\frac{b}{\mathbf{B}};j\right)$. The first term on the right of the last displayed expression is $o_p(1)$ proceeding similarly as with the proof of Theorem 1, whereas the second term on the right develops a "mean" different than zero since $f\left(\frac{b}{\mathbf{B}};j\right)/\int_0^1 f(u;\lambda(j))\,du \neq 1$.

We now examine the behaviour of the tests under local alternatives, say

$$H_l: \quad f(u;\lambda) = f(\lambda)\left(1+\frac{1}{[T/2]^{1/2}}g(u;\lambda)\right),$$

where $g\left(\frac{t}{T};\lambda\right)$ is different than zero in the set $(\mathcal{U},\Lambda)$. It is worth observing that when the local alternative corresponds to an abrupt change at some point in time, t_0, we have that $g\left(\frac{t}{T};\lambda\right) = g(\lambda)$ if $t > t_0$ and we could allow $C/T^{1/2} < t_0/T < 1 - C/T^{1/2}$ for some finite positive constant C. Introduce the function $d(\cdot;\cdot)$ defined as

$$d(\boldsymbol{\omega}^*;\pi\boldsymbol{v}) = \int_0^{\boldsymbol{\omega}^*} g(v;\boldsymbol{v})\,dv - \boldsymbol{\omega}^*\int_0^{\mathbf{1}} g(v;\boldsymbol{v})\,dv, \quad \boldsymbol{\omega}^* \in [0,1]; \ \boldsymbol{v} \in [0,1]$$

It is obvious that $d(\boldsymbol{\omega}^*;\pi\boldsymbol{v})$ is different than zero unless $f(\boldsymbol{\omega}^*;\boldsymbol{v}) =: f(\boldsymbol{v})$ a.e. in $\boldsymbol{v} \in [0,\pi]$ and $\boldsymbol{\omega}^* \in [0,1]$. That is, given $\boldsymbol{v}$, $g(\boldsymbol{\omega}^*;\pi\boldsymbol{v}) = 0$ for all $\boldsymbol{\omega}^* \in [0,1]$ if $f(\boldsymbol{\omega}^*;\boldsymbol{v}) =: f(\boldsymbol{v})$, i.e. a constant in the argument $\boldsymbol{\omega}^* \in [0,1]$.

Proposition 1. *Under H_l and assuming $C1$ to $C3$, we have that*

$$(\mathbf{a}) \quad [T/2]^{1/2}\,\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right) \overset{weakly}{\Rightarrow} D(\boldsymbol{\omega}^*;\boldsymbol{v}^*) + \mathcal{BS}\left([0,1]^2\right),$$

$$(\mathbf{a}) \quad [T/2]^{1/2}\,\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right) \overset{weakly}{\Rightarrow} \frac{1}{\sigma_\varepsilon^2}D(\boldsymbol{\omega}^*;\boldsymbol{v}^*) + \mathcal{WB}\left([0,1]^2\right),$$

where $\boldsymbol{v}^* = \lim_{n=n(T)\to\infty}\frac{\mathbf{j}^*}{\tilde{n}}$ and $\boldsymbol{\omega}^* = \lim_{\mathbf{B}=\mathbf{B}(T)\to\infty}\frac{\mathbf{b}^*}{\mathbf{B}}$, $\mathbf{j}^* = 1,...,\tilde{n}$; $\mathbf{b}^* = 1,...,\mathbf{B}$ and $D(\boldsymbol{\omega}^*;\boldsymbol{v}^*) = \int_0^{\boldsymbol{v}^*} d(\boldsymbol{\omega}^*;\pi\boldsymbol{v})\,d\boldsymbol{v}$.

The conclusion from the previous proposition is that the test has power comparable to parametric counterparts. The consistency of the tests is standard as the "drift" function $D(\boldsymbol{\omega}^*;\boldsymbol{v}^*)$ is non-zero everywhere.

We conclude the section indicating that our tests are able to detect departures from weak stationarity due to heteroscedasticity. Indeed, for illustration purposes we consider the example given in Dwivedi and Subba Rao (2011), that is $x_{t,T} = \sigma_t\varepsilon_t$, where $\sigma_t = \sigma(t/T)$. In this case, standard algebra implies that $EI_{x,b}(j) = \frac{1}{n}\sum_{t=1}^{n}\sigma^2((t+bn)/T)$ and

$$\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}} I_{x,b}(j) \overset{P}{\to} \lim_{T\to\infty}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\frac{1}{n}\sum_{t=1}^{n}\sigma^2((t+bn)/T)$$

$$= \lim_{T\to\infty}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\sigma^2(b/\mathbf{B})\left(1+O\left(\mathbf{B}^{-1}\right)\right) = \int_0^1 \sigma^2(v)\,dv$$

because under continuous differentiability of $\sigma^2(\cdot)$,

$$\sigma^2\left((t+bn)/T\right)-\sigma^2\left(b/\mathbf{B}\right)=O\left(\mathbf{B}^{-1}\right),\ b=1,...,\mathbf{B}. \tag{2.25}$$

From here it is standard to conclude that $\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};j\right)$ will have a mean given by

$$\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\frac{\frac{1}{n}\sum_{t=1}^{n}\sigma^2\left((t+bn)/T\right)}{\int_0^1\sigma^2(v)\,dv}-1\right\}\simeq\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\frac{\sigma^2\left(b/\mathbf{B}\right)}{\int_0^1\sigma^2(v)\,dv}-1\qquad\mathbf{b}^*=2,...,\mathbf{B}$$

which is clearly different than zero unless $\sigma_t^2=\sigma^2$, for all $t=1,...,T$. So our tests are consistent against this type of alternatives. It is worth remarking that in view of (2.25), we can regard $I_{x,b}(j)$ as an unbiased estimator of $\sigma^2(b/\mathbf{B})$ for all $j=1,...,\tilde{n}$.

3. BOOTSTRAP

We now focus on the bootstrap approach for our testing procedure based on either $[T/2]^{1/2}\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ or $[T/2]^{1/2}\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$. Although the asymptotic distribution of $[T/2]^{1/2}\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ is pivotal under Gaussianity or it only depends on the fourth cumulant of ε_t, our Monte Carlo experiment suggests that the asymptotic critical values do not provide a good approximation for the finite sample ones. In any case, as Gaussianity appears as a rather restrictive assumption with many real data sets, bootstrap algorithms might be a useful tool when making inferences. In those circumstances the practitioner hopes that bootstrap algorithms provides better finite-sample approximations. So, the main aim of this section is to present the bootstrap algorithm and examine its validity. As usual $\mathcal{E}^*$ or $\Pr^*\{\cdot\}$ indicate the expectation or the probability in bootstrap sense.

We now describe the bootstrap. To that end, the key is to recall that the asymptotic distribution of $[T/2]^{1/2}\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ is independent of the underlying dependence of x_t. That is, the statistical behaviour of $[T/2]^{1/2}\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ and that of $[T/2]^{1/2}\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$, is exactly the same (asymptotically) as if we were using ε_t instead of x_t in its computation. So, our bootstrap algorithm is based on *2 STEPS*.

***STEP 1*:** We compute $\left\{\widehat{\varepsilon}_{t+(b-1)n}\right\}_{t=1}^{n}$, $b=1,...,\mathbf{B}$, as in (2.19). We then obtain $\left\{\widetilde{\varepsilon}_t=\left(\widehat{\varepsilon}_t-\overline{\widehat{\varepsilon}}\right)/\widehat{\sigma}_\varepsilon\right\}_{t=1}^{T}$, where

$$\overline{\widehat{\varepsilon}}=\frac{1}{T}\sum_{t=1}^{T}\widehat{\varepsilon}_t;\quad\widehat{\sigma}_\varepsilon^2=\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\varepsilon}_t-\overline{\widehat{\varepsilon}}\right)^2.$$

***STEP 2*:** Obtain a random sample of size T from the empirical distribution of $\{\widetilde{\varepsilon}_t\}_{t=1}^{T}$. Denote the sample as $\{\varepsilon_t^*\}_{t=1}^{T}$ and compute the bootstrap statistics

$$(a)\quad\mathcal{T}^*_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)=\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left(\frac{I_{\varepsilon^*,b}(j)}{\mathbf{B}^{-1}\sum_{v=1}^{\mathbf{B}}I_{\varepsilon^*,v}(j)}-1\right),$$

$$(b)\quad\mathcal{TP}^*_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)=\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left(\frac{I_{\varepsilon^*,b}(j)/\widehat{\sigma}^2_{\varepsilon^*}(b)}{\mathbf{B}^{-1}\sum_{v=1}^{\mathbf{B}}\left\{I_{\varepsilon^*,v}(j)/\widehat{\sigma}^2_{\varepsilon^*}(v)\right\}}-1\right),$$

where $\mathbf{j}^*=1,\ldots,\tilde{n}$ and

$$I_{\varepsilon^*,b}(j)=\left|\mathcal{J}_{\varepsilon^*,b}(j)\right|^2,\quad\widehat{\sigma}^2_{\varepsilon^*}(b)=\frac{1}{n}\sum_{t=1}^{n}\varepsilon^{*2}_{t+(b-1)n};\quad b=1,...,\mathbf{B}$$

and $\mathcal{J}_{\varepsilon^*,b}(j)$ as was defined in (2.13) with z_t being replaced by ε_t^* there. Then compute the bootstrap analogues of (2.11) and (2.12), replacing $\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ by $\mathcal{T}^*_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ or $\mathcal{TP}^*_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}},\frac{\mathbf{j}^*}{\tilde{n}}\right)$ there.

Remark 1. (a) *We can replace* $\widetilde{\varepsilon}_t$ *by* $\widehat{\varepsilon}_t$ *in* STEP 2. *The reason being that* $\sum_{t=1}^{n}\overline{\widehat{\varepsilon}}e^{it\lambda_j}=0$ *and* $I_{\varepsilon^*,b}(j)/\mathbf{B}^{-1}\sum_{b=1}^{\mathbf{B}}I_{\varepsilon^*,b}(j)$ *is invariant to multiplicative constants.*

(b) *We may also compute* $\left\{\widetilde{\varepsilon}_{t+(b-1)n}=\left(\widehat{\varepsilon}_{t+(b-1)n}-\overline{\widehat{\varepsilon}}_b\right)/\widehat{\sigma}_{\varepsilon,b}\right\}_{t=1}^{n}$ *with*

$$\overline{\widehat{\varepsilon}}_b=\frac{1}{n}\sum_{t=1}^{n}\widehat{\varepsilon}_{t+(b-1)n};\quad \widehat{\sigma}^2_{\varepsilon,b}=\frac{1}{n}\sum_{t=1}^{n}\left(\widehat{\varepsilon}_{t+(b-1)n}-\overline{\widehat{\varepsilon}}_b\right)^2,\ b=1,...,\mathbf{B},$$

and obtain $\{\varepsilon_t^*\}_{t=1}^{T}=\left\{\left\{\varepsilon^*_{t+(b-1)n}\right\}_{t=1}^{n};\ b=1,...,\mathbf{B}\right\}$, *where* $\left\{\varepsilon^*_{t+(b-1)n}\right\}_{t=1}^{n}$ *is a random sample from the empirical distribution of* $\left\{\widetilde{\varepsilon}_{t+(b-1)n}\right\}_{t=1}^{n}$ *as in* STEP 2, *but random sampling in each "block"* $b=1,..,\mathbf{B}$. *Under stationarity in fourth moments, both procedures are valid. However, in view of our comments at the end of the previous section, we should expect that this bootstrap would be valid even when we do not assume that* κ_4 *is constant, but it depends on time, i.e.* $\kappa_4(u)$, $u\in[0,1]$.

We now have the following result on the validity of the bootstrap.

Theorem 3. *Assuming* $C1$ *and* $C3$, *we have that*

$$\text{(a)}\quad [T/2]^{1/2}\,\mathcal{T}^*_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)\overset{weakly}{\Rightarrow}\mathcal{BS}\left([0,1]^2\right)\qquad\text{(in probability).}$$

$$\text{(b)}\quad [T/2]^{1/2}\,\mathcal{TP}^*_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)\overset{weakly}{\Rightarrow}\mathcal{WB}\left([0,1]^2\right)\qquad\text{(in probability).}$$

4. MONTE-CARLO EXPERIMENT: EMPIRICAL EXAMPLES

The purpose of this section is to present a Monte Carlo experiment to shed some light on the finite-sample performance of the tests. We considered sample sizes of $T=256$, 512 and 1024. For each combination of T, n and models considered in the experiment, we have conducted 1,000 simulation runs. To save computational time, for each run we compute only one bootstrap counterpart. The bootstrapped distribution is obtained by stacking those statistics across iterations, which is then used to construct critical values and confidence regions at the desired levels. This is the idea behind the WARP algorithm of Giacomini et al. (2013).

4.1. Level considerations and choice of block sizes.

We study the nominal level of the bootstrapped modified statistic, given in equation (b) of Theorem 3, with the following $ARMA(2,1)$ model. We simulate

$$x_{t,T}=\phi_1x_{t-1,T}+\phi_2x_{t-2,T}+\varepsilon_{t,T}+\theta\varepsilon_{t-1,T}\tag{4.1}$$

for several combinations of the parameter vector (ϕ_1,ϕ_2,θ). In all specifications, $\varepsilon_t\sim\mathcal{NID}(0,1)$. Model (4.1) nests the models (4.2)-(4.4) considered in Dette et al. (2011) and also implemented as models (4.2)-(4.3) is Preuß et al. (2013). For the sake of comparison, their tests are also replicated here.

The $AR(1)$ model corresponds to $\phi_2=\theta=0$. Results for various ϕ_1 are reported in Table 1.[1] Even at small sample sizes, such as $T=256$, the rejection probabilities are, as expected, close to the 5% and 10% values. As T grows, size distortion are very small especially when $|\phi_1|$ is close to zero. The behaviour of the Cràmer von Mises ($\mathcal{C}v\mathcal{M}$) and Komolgorov-Smirnov ($\mathcal{KS}$)

[1]Table 4 in the appendix presents rejection probabilities as below for the non-bootstrapped statistics.

functionals appear similar. While no statistic clearly dominates the other, the latter emerged as slightly more robust at small parameter values.[2]

Since the choice of block size is an inherent aspect of the test,[3] we experiment with the performance of the statistic at different values of n. We choose combinations of T and n to approximately minimize condition $C3$, with states that

$$\frac{T}{n^2} + \frac{n^3}{T^2} \tag{4.2}$$

converges to zero as T grows. We consider three block sizes for every sample size.[4] In this sense, the optimal block size for $T = 256$ is $n = 32$; for $T = 512$, $n = 32$; and, finally, for $T = 1024$, $n = 128$. For the sake of clarity, in the tables that follow we mark those pairs with the "$\triangleright$" sign. We note that size distortions of both statistics are relatively small and invariant to the choice of block sizes. As expected, best performance was achieved at choices of n such that criteria (4.2) is approximately satisfied at most parameter values. We thus suggest this choice as a practical implementation rule.

Table 2 reports rejection probabilities at alternative parameter values of the data-generating process (4.1). Assuming $\phi_1 = \phi_2 = 0.3$ and $\theta = 0$, we obtain an autoregressive process with real roots; the process at parameters $\phi_1 = 0.4$, $\phi_2 = -0.3$ and $\theta = 0$ exhibits complex roots. The remaining cases contemplate two $ARMA(1,1)$ and a $ARMA(2,1)$ process. Again, size is again well-approximated at small sample size, block sizes and parameters values.

Overall, Monte-Carlo results show that size distortions for the $\mathcal{CvM}$ and $\mathcal{KS}$ are limited at small sample sizes and have comparable performances. We observe, however, very significant size distortions in Preuß et al. (2013) at the set of parameters $(\phi_1, \phi_2, \theta) = (0.4, -0.7, 0)$, $(0.5, 0, 0.5)$ and $(0.3, 0.3, 0.5)$.

4.2. **Power considerations.**

We study the power performance of the test with recourse to the following five DGPs

$$x_{t,T} = \begin{cases} 0.2x_{t-1,T} + e_{t,T}, & t = 2, \ldots, \frac{T}{2} \\ 0.7x_{t-1,T} + e_{t,T}, & t = \frac{T}{2} + 1, \ldots, T \end{cases} \tag{4.3}$$

$$x_{t,T} = \begin{cases} 0.4x_{t-1,T} - 0.7x_{t-2,T} + e_{t,T}, & t = 2, \ldots, \frac{T}{2} \\ 0.3x_{t-1,T} + 0.3x_{t-2,T} + e_{t,T}, & t = \frac{T}{2} + 1, \ldots, T \end{cases} \tag{4.4}$$

$$x_{t,T} = \begin{cases} 0.3x_{t-1,T} + 0.3x_{t-2,T} + e_{t,T}, & t = 2, \ldots, \frac{T}{2} \\ 0.8x_{t-1,T} + e_{t,T}, & t = \frac{T}{2} + 1, \ldots, T \end{cases} \tag{4.5}$$

$$x_{t,T} = 0.6 \sin(4\pi t/T)\, x_{t-1,T} + e_{t,T} \tag{4.6}$$

$$x_{t,T} = 1.1 \cos(1.5 - \cos(4\pi t/T))\, e_{t-1,T} + e_{t,T} \tag{4.7}$$

The first DGP is a simple break of the AR(1) coefficient. The break in the second DGP is such that roots switch from complex to real. The third model considers a change in the order of the autoregressive model. Finally, the fourth and fifth models have changing coefficients with t and originate from Dette et al. (2011), equations (4.6) and (4.7).

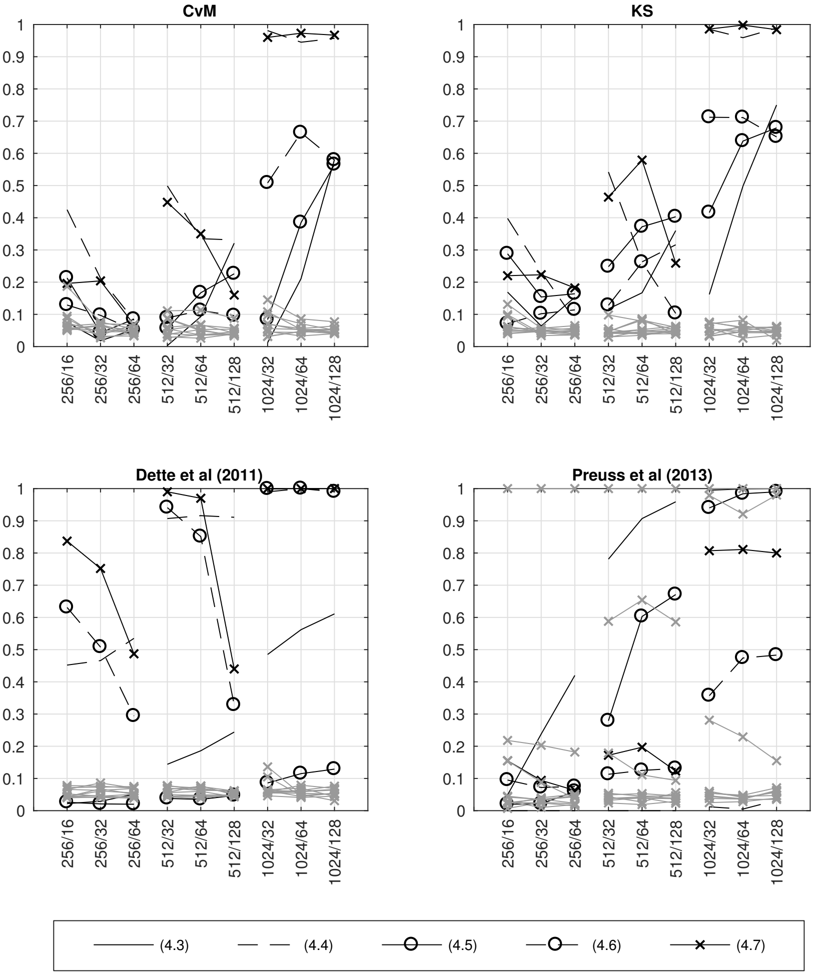
From Table 3 we observe that deviation from the null hypothesis is detected at a reasonable frequency which increases quickly as T grows. For $T = 2048$ rejection probabilities are either very close to one or rejection of the null was obtained at every simulation run. At smaller

[2] We also present the distribution of the non-bootstrapped statistics in Table 4. As previously mentioned, at small sample sizes the asymptotic distribution does not provide a reasonable approximation.

[3] For example, Dette et al. (2011, p. 1118) ponder that "(...) any statistical inference in locally stationary process depend on the choice of [the parameters] in the definition of the local periodogram."

[4] In all cases, we limit ourselves to sample and block sizes with length of powers of 2. This allows more efficient computation of Fast Fourier Transform algorithms.

FIGURE 1. Rejection probabilities for $\mathcal{C}v\mathcal{M}$, $\mathcal{KS}$, Dette et al. (2011) and Preuss et al. (2013) statistics



Note: Rejection probabilities for several combinations of T and n, represented by the notation T/n. DGPs implemented for size assessments are grayed out and DGPs (4.3)-(4.7) are individually labelled.

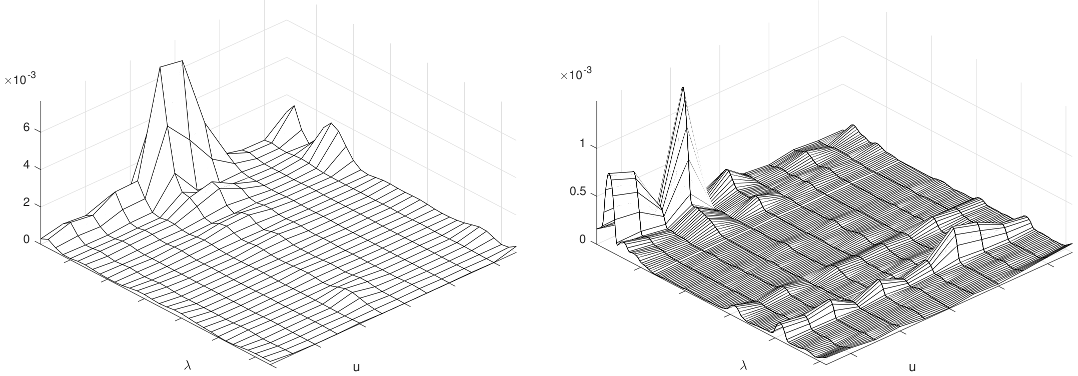
sample sizes, power is naturally higher for those DGPs that imposed a large change in the spectral density functions, particularly (4.4). $\mathcal{KS}$ functional showed slightly higher power than $\mathcal{C}v\mathcal{M}$ in small sample sizes, particularly DGPs (4.3) and (4.5).

For data generated according to DGP (4.3), Dette et al. (2011) achieved comparable power relative to $\mathcal{C}v\mathcal{M}$, followed by $\mathcal{KS}$ functional; Preuß et al. (2013) obtained higher power at small sample sizes. A similar pattern is observed in DGP (4.5). The ordering, however, appears inverted for DGP (4.4) and approximately so for (4.6) and (4.7). We note that our test achieved reasonable power for all data generating processes and more stable results compared to the tests above.

We summarize the size and power simulation results in Figure 1. The complete set of results is shown in appendix Tables 1-3.

4.3. **Empirical examples.** We employ the proposed tests to check the constancy of the dynamics for two real data sets.

Figure 2. Estimates of local spectral density for Consumer Price Index (left) and tremor data (right)



Note: Consumer Price Index, sourced from the Bureau of Labor Statistics (available at `https://www.bls.gov/cpi/`). Tremor data from Dette et al. (2011). We present time-series plots of both series in Figure 3 in Appendix I.

The choice of a model for the inflation dynamics is a necessity prior to empirical analysis. On our first application, we present evidence that inflation dynamics is not constant over time in the United States. We use data on the baseline Consumer Price Index (CPI), produced by the U.S. Bureau of Labour Statistics, between September 1959 and August 2015, totalling 768 data points.

Our second application replicates the neuroscience example of Dette et al. (2011), also used in von Sachs and Neumann (2000) and Paparoditis (2009).[5] The authors analyzed a data set of tremor activity recorded in the Cognitive Neuroscience Laboratory of the University of Quebec at Montreal of subjects with Parkinson's Disease. The objective is to compare different regions of brain activity of a patient with Parkinson's disease. The data are composed of $3,072$ observations.

To ensure stationarity, for both series the first-difference $\Delta x_{t,T} = x_{t,T} - x_{t-1,T}$ is analyzed. Figure 2 presents the smoothed spectral density estimate

$$\hat{f}(u;\lambda) \quad = \quad \frac{2\pi}{n}\sum_{j=1}^{n}\frac{1}{k}K\left(\frac{\lambda-\lambda_j}{k}\right)I_{x,b}(j) \tag{4.8}$$

where $K(\cdot)$ represents the Bartlett-Priestley kernel. A similar approach was introduced in Dette et al. (2011) and Paparoditis (2009). For the CPI application, $n = 64$ and $k = 0.01$ were chosen. For the neuroscience application, block sizes $n = 256$ with bandwidth $k = 0.18$ were employed.[6] From the figures, it is apparent that spectral densities tend to strong vary across time, especially at lower frequencies.

In Table 5, we show the Cràmer-von-Mises ($\mathcal{C}v\mathcal{M}$) and Kolmogorov-Smirnov's ($\mathcal{KS}$) test statistics for the CPI data, along with the bootstrapped 10%, 5% and 1% critical values.[7] The block sizes considered are in line with simulations presented in the previous subsection. In all the cases the null hypothesis of model stability is rejected at the 1% level. We have that the outcome of the test is not very sensitive to the choice of the bandwidth parameter and clearly the p-value is smaller than 1%.

[5]We thank Efstathios Paparoditis for sharing data.

[6]Again, following Dette et al. (2011) and Paparoditis (2009).

[7]We use 500 iterations in each case.

For the neuroscience application, in line with Dette et al. (2011) and others, we reject the null hypothesis of model stability at 1% level, with the expectation of a singular block size of the $\mathcal{C}v\mathcal{M}$ test where significance is achieved at 5% level only. Bootstrapped critical values and test statistics are presented in Table 6.

5. CONCLUSION

In this paper, we described and examined a simple test for the hypothesis of stability of the dynamics without assuming any parametric family under the null hypothesis. One interesting aspect of the test is that, even without knowledge of the spectral density function under the null hypothesis, there is no need to choose any bandwidth or smoothing parameter for its implementation, besides the choice of the length of the block size n. A second interesting aspect of the test is that its asymptotic distribution only depends on the fourth cumulant κ_4 of the innovation sequence. We suggest a very simple estimator of κ_4 based on the canonical decomposition of the spectral density function as given in Whittle (1963), see also Hannan (1970) or Brillinger (1981). We also present and investigate a modification of the test such that its asymptotic distribution becomes pivotal. For the implementation, we do not need any type of "bias" adjustments, and we are able to detect local alternatives converging to the null at the parametric rate $T^{-1/2}$.

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APPENDIX I: TABLES AND FIGURES

Table 1. Rejection probabilities, model (4.1) with $\phi_2 = \theta = 0$

			$\phi_1 = -0.5$		$\phi_1 = -0.25$		$\phi_1 = 0$		$\phi_1 = 0.25$		$\phi_1 = 0.5$	
	T	n	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
							$\mathcal{CvM}$					
	256	16	0.064	0.117	0.050	0.124	0.053	0.136	0.054	0.133	0.073	0.130
▷	256	32	0.071	0.123	0.068	0.114	0.064	0.124	0.051	0.104	0.046	0.078
	256	64	0.068	0.112	0.042	0.095	0.050	0.109	0.048	0.109	0.046	0.095
▷	512	32	0.088	0.140	0.052	0.095	0.055	0.094	0.060	0.116	0.038	0.077
	512	64	0.052	0.098	0.066	0.115	0.036	0.071	0.068	0.109	0.053	0.091
	512	128	0.033	0.080	0.057	0.095	0.046	0.105	0.042	0.091	0.054	0.099
	1024	32	0.108	0.176	0.071	0.125	0.045	0.081	0.069	0.134	0.031	0.081
▷	1024	64	0.053	0.103	0.050	0.108	0.044	0.103	0.056	0.104	0.047	0.076
	1024	128	0.042	0.091	0.046	0.083	0.053	0.094	0.064	0.107	0.041	0.081
							$\mathcal{KS}$					
	256	16	0.040	0.106	0.054	0.096	0.061	0.101	0.051	0.107	0.063	0.111
▷	256	32	0.047	0.098	0.058	0.112	0.050	0.106	0.048	0.108	0.043	0.102
	256	64	0.050	0.128	0.052	0.115	0.057	0.114	0.048	0.091	0.036	0.077
▷	512	32	0.047	0.092	0.042	0.079	0.047	0.109	0.055	0.091	0.042	0.088
	512	64	0.086	0.137	0.045	0.109	0.082	0.145	0.039	0.072	0.052	0.099
	512	128	0.038	0.079	0.050	0.093	0.060	0.119	0.040	0.092	0.040	0.090
	1024	32	0.054	0.108	0.032	0.086	0.047	0.086	0.075	0.104	0.033	0.073
▷	1024	64	0.050	0.105	0.046	0.098	0.060	0.101	0.060	0.104	0.048	0.080
	1024	128	0.063	0.114	0.047	0.087	0.060	0.091	0.058	0.121	0.044	0.084
							Dette et al. (2011)					
	256	16	0.040	0.143	0.043	0.147	0.052	0.105	0.044	0.086	0.066	0.108
	256	32	0.042	0.102	0.041	0.110	0.045	0.088	0.047	0.091	0.063	0.103
	256	64	0.044	0.111	0.039	0.086	0.051	0.090	0.039	0.079	0.069	0.123
	512	32	0.043	0.132	0.050	0.126	0.043	0.107	0.056	0.098	0.057	0.105
	512	64	0.038	0.092	0.047	0.085	0.049	0.102	0.060	0.109	0.077	0.126
	512	128	0.051	0.109	0.049	0.088	0.048	0.092	0.043	0.093	0.049	0.108
	1024	32	0.136	0.276	0.105	0.223	0.065	0.144	0.046	0.102	0.055	0.092
	1024	64	0.051	0.129	0.047	0.099	0.070	0.116	0.041	0.083	0.054	0.093
	1024	128	0.031	0.084	0.054	0.096	0.044	0.094	0.045	0.084	0.051	0.089
							Preuß et al. (2013)					
	256	16	0.007	0.049	0.045	0.077	0.042	0.092	0.025	0.081	0.029	0.055
	256	32	0.026	0.050	0.034	0.074	0.035	0.077	0.036	0.079	0.010	0.057
	256	64	0.021	0.064	0.043	0.078	0.046	0.094	0.023	0.062	0.019	0.050
	512	32	0.032	0.096	0.043	0.079	0.038	0.089	0.054	0.094	0.024	0.057
	512	64	0.040	0.071	0.027	0.085	0.039	0.088	0.045	0.092	0.019	0.063
	512	128	0.023	0.068	0.056	0.108	0.037	0.082	0.046	0.109	0.029	0.088
	1024	32	0.055	0.097	0.038	0.104	0.046	0.075	0.044	0.081	0.025	0.075
	1024	64	0.045	0.098	0.036	0.106	0.037	0.086	0.035	0.081	0.028	0.064
	1024	128	0.047	0.095	0.063	0.109	0.060	0.107	0.035	0.078	0.039	0.086

TABLE 2. Rejection probabilities, model (4.1), various parameters

			$\phi_1 = 0.3$		$\phi_1 = 0.4$		$\phi_1 = 0.5$		$\phi_1 = -0.5$		$\phi_1 = 0.3$	
			$\phi_2 = 0.3$		$\phi_2 = -0.7$		$\phi_2 = 0$		$\phi_2 = 0$		$\phi_2 = 0.3$	
			$\theta = 0$		$\theta = 0$		$\theta = 0.5$		$\theta = 0.5$		$\theta = 0.5$	
	T	n	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
							$\mathcal{CvM}$					
	256	16	0.094	0.152	0.187	0.325	0.061	0.112	0.057	0.113	0.087	0.140
▷	256	32	0.027	0.081	0.076	0.140	0.036	0.095	0.065	0.114	0.032	0.078
	256	64	0.042	0.089	0.055	0.108	0.068	0.126	0.035	0.082	0.033	0.092
▷	512	32	0.028	0.070	0.111	0.162	0.030	0.063	0.086	0.143	0.040	0.088
	512	64	0.037	0.082	0.113	0.171	0.026	0.093	0.049	0.106	0.035	0.077
	512	128	0.040	0.083	0.088	0.155	0.039	0.080	0.036	0.094	0.038	0.092
	1024	32	0.047	0.096	0.145	0.232	0.032	0.069	0.056	0.102	0.098	0.133
▷	1024	64	0.033	0.067	0.086	0.164	0.047	0.096	0.053	0.095	0.071	0.133
	1024	128	0.040	0.087	0.077	0.129	0.054	0.094	0.053	0.105	0.051	0.092
							$\mathcal{KS}$					
	256	16	0.099	0.167	0.131	0.211	0.041	0.084	0.047	0.091	0.094	0.147
▷	256	32	0.041	0.086	0.057	0.122	0.049	0.097	0.050	0.099	0.034	0.086
	256	64	0.047	0.107	0.065	0.109	0.039	0.088	0.040	0.097	0.036	0.069
▷	512	32	0.041	0.084	0.098	0.166	0.030	0.071	0.029	0.069	0.044	0.078
	512	64	0.040	0.085	0.079	0.124	0.035	0.072	0.048	0.097	0.053	0.109
	512	128	0.047	0.104	0.051	0.118	0.049	0.103	0.056	0.106	0.063	0.121
	1024	32	0.055	0.095	0.076	0.117	0.057	0.087	0.036	0.066	0.071	0.112
▷	1024	64	0.043	0.092	0.058	0.115	0.026	0.071	0.065	0.121	0.082	0.092
	1024	128	0.043	0.091	0.046	0.096	0.036	0.080	0.049	0.088	0.020	0.112
							Dette et al. (2011)					
	256	16	0.043	0.076	0.078	0.116	0.072	0.130	0.044	0.102	0.066	0.110
	256	32	0.068	0.099	0.076	0.126	0.073	0.122	0.061	0.113	0.086	0.131
	256	64	0.051	0.097	0.070	0.107	0.075	0.125	0.055	0.095	0.070	0.121
	512	32	0.070	0.101	0.066	0.115	0.076	0.133	0.045	0.101	0.078	0.127
	512	64	0.063	0.106	0.075	0.110	0.070	0.117	0.044	0.107	0.065	0.112
	512	128	0.062	0.109	0.061	0.116	0.055	0.103	0.045	0.104	0.061	0.098
	1024	32	0.058	0.093	0.058	0.099	0.063	0.112	0.062	0.134	0.057	0.099
	1024	64	0.063	0.096	0.079	0.129	0.060	0.100	0.044	0.100	0.059	0.104
	1024	128	0.076	0.124	0.069	0.131	0.066	0.099	0.054	0.111	0.064	0.111
							Preuß et al. (2013)					
	256	16	0.026	0.067	0.218	0.384	1.000	1.000	0.021	0.066	0.155	0.317
	256	32	0.024	0.064	0.203	0.443	1.000	1.000	0.040	0.073	0.088	0.216
	256	64	0.015	0.061	0.182	0.367	1.000	1.000	0.051	0.097	0.039	0.104
	512	32	0.049	0.081	0.588	0.082	1.000	1.000	0.038	0.084	0.179	0.293
	512	64	0.053	0.098	0.654	0.868	1.000	1.000	0.043	0.088	0.111	0.244
	512	128	0.046	0.091	0.586	0.764	1.000	1.000	0.039	0.080	0.094	0.238
	1024	32	0.054	0.124	0.979	1.000	1.000	1.000	0.060	0.114	0.281	0.453
	1024	64	0.048	0.095	0.921	0.996	1.000	1.000	0.041	0.070	0.229	0.352
	1024	128	0.071	0.131	0.980	1.000	1.000	1.000	0.057	0.102	0.155	0.247

Table 3. Rejection probabilities, models (4.3)- (4.7)

			(4.3)		(4.4)		(4.5)		(4.6)		(4.7)	
	T	n	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
							$\mathcal{C}v\mathcal{M}$					
	256	16	0.075	0.164	0.425	0.571	0.213	0.297	0.129	0.192	0.196	0.303
▷	256	32	0.018	0.053	0.212	0.311	0.043	0.127	0.097	0.167	0.204	0.359
	256	64	0.053	0.112	0.082	0.149	0.085	0.188	0.051	0.121	0.068	0.125
▷	512	32	0.003	0.050	0.499	0.683	0.056	0.150	0.089	0.193	0.448	0.628
	512	64	0.093	0.227	0.335	0.539	0.167	0.297	0.112	0.216	0.350	0.563
	512	128	0.321	0.488	0.331	0.573	0.226	0.421	0.096	0.223	0.160	0.361
	1024	32	0.012	0.136	0.981	0.991	0.084	0.326	0.508	0.700	0.960	0.987
▷	1024	64	0.211	0.396	0.945	0.996	0.385	0.591	0.664	0.770	0.973	0.997
	1024	128	0.581	0.827	0.957	0.998	0.565	0.762	0.579	0.775	0.967	0.992
▷	2048	64	0.691	0.937	1.000	1.000	0.642	0.844	0.995	0.998	1.000	1.000
	2048	128	0.972	0.997	1.000	1.000	0.772	0.964	0.986	1.000	1.000	1.000
	2048	256	0.997	1.000	1.000	1.000	0.974	0.996	0.987	0.998	1.000	1.000
							$\mathcal{KS}$					
	256	16	0.169	0.252	0.398	0.585	0.288	0.415	0.071	0.204	0.220	0.424
▷	256	32	0.064	0.118	0.232	0.378	0.154	0.252	0.102	0.206	0.223	0.441
	256	64	0.153	0.241	0.082	0.156	0.164	0.266	0.114	0.246	0.182	0.350
▷	512	32	0.117	0.232	0.542	0.736	0.248	0.374	0.128	0.288	0.464	0.665
	512	64	0.167	0.349	0.263	0.551	0.372	0.530	0.262	0.394	0.579	0.714
	512	128	0.360	0.551	0.316	0.549	0.403	0.584	0.103	0.254	0.259	0.496
	1024	32	0.162	0.441	0.985	1.000	0.416	0.651	0.712	0.881	0.986	0.997
▷	1024	64	0.498	0.718	0.959	0.998	0.638	0.800	0.711	0.903	0.998	1.000
	1024	128	0.750	0.907	0.989	1.000	0.679	0.831	0.651	0.797	0.984	0.998
▷	2048	64	0.799	0.948	1.000	1.000	0.978	0.994	0.995	1.000	1.000	1.000
	2048	128	0.978	1.000	1.000	1.000	0.982	1.000	1.000	1.000	1.000	1.000
	2048	256	0.998	1.000	1.000	1.000	0.990	1.000	1.000	1.000	1.000	1.000
							Dette et al. (2011)					
	256	16	0.022	0.107	0.452	0.710	0.026	0.085	0.631	0.817	0.837	0.919
	256	32	0.029	0.125	0.466	0.736	0.021	0.066	0.508	0.714	0.752	0.862
	256	64	0.051	0.172	0.535	0.759	0.020	0.071	0.294	0.489	0.487	0.675
	512	32	0.144	0.354	0.907	0.976	0.038	0.142	0.941	0.973	0.990	0.998
	512	64	0.186	0.388	0.916	0.997	0.035	0.119	0.851	0.932	0.970	0.986
	512	128	0.244	0.460	0.911	0.978	0.047	0.147	0.328	0.514	0.440	0.600
	1024	32	0.485	0.705	0.990	1.000	0.086	0.227	0.999	1.000	1.000	1.000
	1024	64	0.562	0.772	1.000	1.000	0.115	0.285	1.000	1.000	1.000	1.000
	1024	128	0.611	0.796	0.996	1.000	0.129	0.294	0.990	1.000	1.000	1.000
							Preuß et al. (2013)					
	256	16	0.049	0.160	0.000	0.020	0.020	0.028	0.095	0.187	0.155	0.210
	256	32	0.238	0.433	0.000	0.000	0.021	0.154	0.072	0.134	0.094	0.177
	256	64	0.420	0.590	0.000	0.000	0.060	0.190	0.075	0.126	0.063	0.138
	512	32	0.781	0.890	0.000	0.060	0.279	0.583	0.113	0.234	0.172	0.307
	512	64	0.907	0.949	0.000	0.000	0.603	0.754	0.125	0.203	0.197	0.310
	512	128	0.959	0.984	0.000	0.012	0.671	0.853	0.131	0.225	0.123	0.219
	1024	32	0.995	0.999	0.012	0.243	0.940	0.972	0.357	0.539	0.807	0.884
	1024	64	0.999	0.999	0.005	0.178	0.984	0.995	0.474	0.626	0.811	0.931
	1024	128	1.000	1.000	0.032	0.224	0.990	0.994	0.483	0.739	0.800	0.900

Table 4. Rejection probabilities, non-bootstrapped $\mathcal{CvM}$ and $\mathcal{KS}$ statistics, model (4.1) with $\phi_2 = \theta = 0$

		$\phi_1 = -0.5$		$\phi_1 = -0.25$		$\phi_1 = 0$		$\phi_1 = 0.25$		$\phi_1 = 0.5$	
T	n	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
						$\mathcal{CvM}$					
256	32	0.029	0.045	0.030	0.040	0.039	0.049	0.029	0.045	0.035	0.047
256	64	0.030	0.040	0.017	0.022	0.021	0.026	0.018	0.023	0.020	0.026
256	128	0.008	0.017	0.014	0.021	0.012	0.016	0.014	0.021	0.016	0.020
512	64	0.069	0.098	0.099	0.119	0.077	0.104	0.092	0.122	0.092	0.125
512	128	0.056	0.081	0.053	0.069	0.071	0.087	0.054	0.069	0.053	0.070
512	256	0.034	0.047	0.037	0.043	0.030	0.043	0.038	0.043	0.039	0.046
1024	128	0.167	0.213	0.184	0.232	0.167	0.197	0.183	0.234	0.186	0.236
1024	256	0.132	0.163	0.135	0.162	0.133	0.162	0.138	0.159	0.136	0.167
1024	512	0.079	0.097	0.083	0.099	0.094	0.111	0.086	0.103	0.086	0.101
						$\mathcal{KS}$					
256	16	0.057	0.118	0.046	0.098	0.046	0.092	0.048	0.098	0.060	0.114
256	32	0.023	0.049	0.021	0.043	0.027	0.061	0.020	0.039	0.020	0.047
256	64	0.012	0.035	0.014	0.031	0.013	0.024	0.015	0.031	0.018	0.033
512	32	0.066	0.117	0.057	0.107	0.045	0.090	0.063	0.112	0.064	0.123
512	64	0.025	0.065	0.025	0.062	0.026	0.061	0.023	0.058	0.026	0.058
512	128	0.008	0.028	0.008	0.029	0.007	0.026	0.011	0.032	0.013	0.033
1024	64	0.070	0.159	0.059	0.141	0.062	0.128	0.063	0.136	0.074	0.143
1024	128	0.051	0.105	0.048	0.095	0.054	0.107	0.044	0.106	0.052	0.110
1024	256	0.033	0.073	0.035	0.071	0.031	0.067	0.037	0.070	0.031	0.069

Figure 3. CPI (left) and tremor data (right)

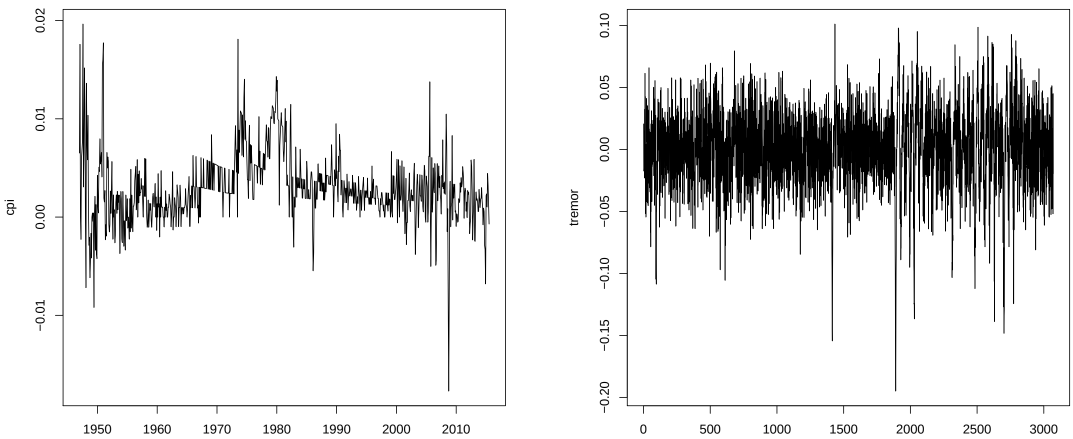

Table 5. U.S. Inflation Rate

	n	TS	p-value	10%	5%	1%
$\mathcal{C}v\mathcal{M}$	32	2.958	< 0.002	1.043	1.147	1.340
$\mathcal{C}v\mathcal{M}$	64	3.363	< 0.002	1.137	1.246	1.501
$\mathcal{C}v\mathcal{M}$	128	3.264	< 0.002	1.057	1.144	1.315
$\mathcal{KS}$	32	29.055	< 0.002	2.510	2.958	5.281
$\mathcal{KS}$	64	39.683	< 0.002	3.227	4.361	6.802
$\mathcal{KS}$	128	35.676	< 0.002	3.023	3.904	6.067

Table 6. Neuroscience Data

	n	TS	p-value	10%	5%	1%
$\mathcal{C}v\mathcal{M}$	64	2.059	< 0.002	1.279	1.345	1.597
$\mathcal{C}v\mathcal{M}$	128	1.980	0.012	1.556	1.712	1.985
$\mathcal{C}v\mathcal{M}$	256	1.830	< 0.002	1.224	1.329	1.550
$\mathcal{C}v\mathcal{M}$	512	1.368	0.008	0.986	1.145	1.332
$\mathcal{KS}$	64	23.547	< 0.002	7.505	8.966	12.824
$\mathcal{KS}$	128	20.775	< 0.002	11.276	14.389	18.939
$\mathcal{KS}$	256	19.390	< 0.002	6.390	7.482	10.234
$\mathcal{KS}$	512	14.781	< 0.002	6.651	8.557	10.397

APPENDIX II: PROOF OF MAIN RESULTS

We shall introduce some notation. In what follows we denote

$$\mathring{I}_{x,b}(j) = \frac{I_{x,b}(j)}{\left|\mathcal{B}\left(\frac{n(b-1)}{T};j\right)\right|^2}; \quad \overline{\mathring{I}}_x(j) = \frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}} \mathring{I}_{x,b}(j)$$

$$R_{n,b}(j) = \mathring{I}_{x,b}(j) - I_{\varepsilon,b}(j); \ \overline{R}_n(j) = \frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}} R_{n,b}(j)$$

$$\mathring{I}_{\varepsilon,b}(j) = I_{\varepsilon,b}(j) - 1; \quad \overline{\mathring{I}}_\varepsilon(j) = \frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}} \mathring{I}_{\varepsilon,b}(j) \tag{5.1}$$

$$\breve{R}_{n,b}(j) = R_{n,b}(j) - \mathcal{E}(R_{n,b}(j)); \quad \overline{\breve{R}}_n(j) = \frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}} \breve{R}_{n,b}(j).$$

Observe that under H_0 we have that $\mathring{I}_{x,b}(j) = f^{-1}(j)\, I_{x,b}(j)$. In addition for notational simplicity we assume that $\sigma_\varepsilon^2 = 1$ without loss of generality.

We also introduce the following definition: We say that a process $\mathrm{X}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ satisfies Condition BW if

$$\sup_{\mathbf{j}^*=1,\ldots,\tilde{n};\mathbf{b}^*=1,\ldots,\mathbf{B}} \left|\mathrm{X}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)\right| = o_p\left(T^{-1/2}\right).$$

Recall that by Bickel and Wichura (1972), a sufficient condition for BW is that for some $\alpha \geq 1$ and $\delta > 0$,

$$\mathcal{E}\left|\mathrm{X}_{n,\mathbf{B}}\left(\frac{\mathbf{b}_2^*}{\mathbf{B}};\frac{\mathbf{j}_2^*}{\tilde{n}}\right) - \mathrm{X}_{n,\mathbf{B}}\left(\frac{\mathbf{b}_1^*}{\mathbf{B}};\frac{\mathbf{j}_2^*}{\tilde{n}}\right) - \mathrm{X}_{n,\mathbf{B}}\left(\frac{\mathbf{b}_2^*}{\mathbf{B}};\frac{\mathbf{j}_1^*}{\tilde{n}}\right) + \mathrm{X}_{n,\mathbf{B}}\left(\frac{\mathbf{b}_1^*}{\mathbf{B}};\frac{\mathbf{j}_1^*}{\tilde{n}}\right)\right|^\alpha$$

$$= o\left(\frac{1}{T^{\alpha/2}}\left(\frac{\mathbf{j}_2^* - \mathbf{j}_1^*}{\tilde{n}}\right)^{1+\delta}\left(\frac{\mathbf{b}_2^* - \mathbf{b}_1^*}{\mathbf{B}}\right)^{1+\delta}\right). \tag{5.2}$$

Finally recall that $\breve{\mathcal{T}}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right) \equiv \mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ under H_0.

5.1. **Proof of Theorem 1.**

Using Taylor's expansion around 1 of $\overline{\mathring{I}}_x(j)^{-1}$, we obtain the following decomposition for $\breve{\mathcal{T}}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$,

$$\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\left\{\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\mathring{I}_{x,b}(j) - \overline{\mathring{I}}_x(j)\right\}\sum_{k=0}^{2}\frac{(-1)^k}{k!}\left(\overline{\mathring{I}}_x(j) - 1\right)^k\right\}$$

$$+\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\left\{\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\mathring{I}_{x,b}(j) - \overline{\mathring{I}}_x(j)\right\}\frac{\left(\overline{\mathring{I}}_x(j) - 1\right)^3}{(1-\alpha) + \alpha\overline{\mathring{I}}_x(j)}\right\}, \tag{5.3}$$

where $\alpha =: \alpha(j) \in (0,1)$. Notice that Lemma 4 and well known inequalities $\sup_j\left|\overline{\mathring{I}}_\varepsilon(j)\right| = O_p\left(n^{1/4}\mathbf{B}^{-1/2}\right)$ and so by $C3$, it implies that $\left(\inf_j\left|\overline{\mathring{I}}_x(j)\right|\right)^{-1} < C$ for some finite positive constant C and hence

$$\sup_j\left((1-\alpha) + \alpha\overline{\mathring{I}}_x(j)\right)^{-1} = O_p(1). \tag{5.4}$$

We first examine the second term of (5.3), and in particular

$$
\begin{aligned}
(5.5)\qquad & \frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\mathring{I}_{x,b}(j)-1\right\}\frac{\left(\overline{\mathring{I}}_x(j)-1\right)^3}{(1-\alpha)+\alpha\overline{\mathring{I}}_x(j)} \\
= \; & \frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\left(\sum_{b=1}^{\mathbf{b}^*}R_{n,b}(j)+\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon,b}(j)\right)\frac{\left(\overline{\mathring{I}}_x(j)-1\right)^3}{(1-\alpha)+\alpha\overline{\mathring{I}}_x(j)}.
\end{aligned}
$$

Now because (5.4) we have that the contribution due to $\sum_{b=1}^{\mathbf{b}^*}R_{n,b}(j)$ in the right of (5.5) is

$$
\begin{aligned}
& \left|\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\left(\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}R_{n,b}(j)\right)\frac{\left(\overline{R}_n(j)\right)^3}{(1-\alpha)+\alpha\overline{\mathring{I}}_x(j)}\right| \\
= \; & O_p(1)\left(\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\left|\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}R_{n,b}(j)\right|\left|\overline{R}_n(j)\right|^3\right)
\end{aligned}
$$

and hence Lemma 4 and Condition $C3$ imply that the second factor of the right side of last displayed equality satisfies (5.2) with $\alpha=1$ there and hence Condition BW. Similarly the contribution due to $\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon,b}(j)$ in the right of (5.5) is

$$
O_p(1)\left(\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\left|\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon,b}(j)\right|\left|\overline{\mathring{I}}_{\varepsilon,b}(j)\right|^3\right)
$$

which satisfies (5.2) and hence Condition BW because

$$
\begin{aligned}
& \frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\left|\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(j)\right|\left|\overline{\mathring{I}}_{\varepsilon,b}(j)\right|^3 \\
\leq \; & 4\frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\left|\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(j)\right|^4+4\frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\left|\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(j)\right|\left|\frac{1}{\mathbf{B}}\sum_{b\neq\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(j)\right|^3.
\end{aligned}
$$

From here the proof is standard after observing that $\mathring{I}_{\varepsilon,b}(j)$ and $\mathring{I}_{\varepsilon,v}(j)$ are independent if $b\neq v$ and $C3$ implies that $\mathbf{B}^{-2}=o\left(\mathbf{B}^{-1/2}T^{-1/2}\right)$.

Next, the first term of (5.3), which is

$$
\begin{aligned}
(5.6)\qquad & \frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\breve{R}_{n,b}(j)-\overline{\breve{R}}_n(j)\right\}\sum_{k=0}^{2}\frac{(-1)^k}{k!}\left(\overline{\mathring{I}}_x(j)-1\right)^k \\
& +\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\mathring{I}_{\varepsilon,b}(j)-\overline{\mathring{I}}_\varepsilon(j)\right\}\sum_{k=1}^{2}\frac{(-1)^k}{k!}\left(\overline{\mathring{I}}_x(j)-1\right)^k \\
& +\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\mathring{I}_{\varepsilon,b}(j)-\overline{\mathring{I}}_\varepsilon(j)\right\}
\end{aligned}
$$

since $\mathcal{E}\left(\overline{\mathring{I}}_x(j)\right)=\mathcal{E}\left(\mathring{I}_{x,b}(j)\right)$ and $\mathring{I}_{x,b}(j)-\mathcal{E}\left(\mathring{I}_{x,b}(j)\right)=\breve{R}_{n,b}(j)+\mathring{I}_{\varepsilon,b}(j)$.

We first show the first term of (5.6) satisfies Condition BW. Indeed, by Lemmas 4 and 5, we have that the contribution due to $\sum_{k=1}^{2}\overline{\breve{R}}_n^k(j)$ into the term satisfies Condition BW. So, noticing that $\overline{\overset{\circ}{I}}_x(j) - 1 = \overline{\breve{R}}_n(j) + \mathcal{E}(R_{n,b}(j)) + \overline{\overset{\circ}{I}}_\varepsilon(j)$, it suffices to show that

$$\frac{1}{\tilde{n}} \sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*} \left(\frac{1}{\mathbf{B}} \sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} \breve{R}_{n,b}(j) \right) \sum_{k=0}^{2} \frac{(-1)^k}{k!} \overline{\overset{\circ}{I}}{}^k_\varepsilon(j), \tag{5.7}$$

satisfies (5.2). When $k=0$ it is a direct consequence of Lemma 5. Next because $C1$ implies that $\mathcal{E}\left(\overset{\circ}{I}_\varepsilon(j)^{2k}\right) = O\left(\mathbf{B}^{-k}\right)$, Cauchy-Schwarz inequality, and then Lemma 5, yields that the contribution of the first absolute moment of the terms due to $k=2$ in (5.7) is bounded by

$$\frac{1}{\mathbf{B}\tilde{n}} \sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*} \left(\mathcal{E} \left(\frac{1}{\mathbf{B}} \sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} \breve{R}_{n,b}(j) \right)^2 \right)^{1/2} = \frac{C}{T^{1/2}\mathbf{B}} \left(\frac{\mathbf{j}_2^* - \mathbf{j}_1^*}{\tilde{n}} \right) \left(\frac{\mathbf{b}_2^* - \mathbf{b}_1^*}{\mathbf{B}} \right)^{1/2}.$$

So, it satisfies (5.2) because $C2$ implies that for some $\delta > 0$, $\mathbf{B}^{-1/2} \le T^{-\delta}\tilde{n}^{-\delta}$.

To finish that the first term of (5.6) satisfies Condition BW, it remains to do so for (5.7) when $k=1$, that is

$$\frac{1}{\tilde{n}} \sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*} \left(\frac{1}{\mathbf{B}} \sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} \breve{R}_{n,b}(j) \right) \overline{\overset{\circ}{I}}_\varepsilon(j).$$

To that end, using (5.29) but with (2.16), (2.17) and (2.18) instead of (2.14) and (2.15) there, it suffices to examine that

$$\frac{1}{\tilde{n}} \sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*} \overline{\overset{\circ}{I}}_\varepsilon(j) \frac{1}{\mathbf{B}} \sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} \left(\left| \dot{\Upsilon}_{n,b}(j;0) \right|^2 - \mathcal{E} \left| \dot{\Upsilon}_{n,b}(j;0) \right|^2 \right) \tag{5.8}$$

$$\frac{1}{\tilde{n}} \sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*} \overline{\overset{\circ}{I}}_\varepsilon(j) \overline{\tilde{I}}_\varepsilon\left(j, n^{1/2}\right) + \frac{1}{\tilde{n}} \sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*} \overline{\overset{\circ}{I}}_\varepsilon(j) \left(\overline{\tilde{I}}_\varepsilon(j,0) - \overline{\tilde{I}}_\varepsilon\left(j, n^{1/2}\right) \right), \tag{5.9}$$

satisfy Condition BW, where $\dot{\Upsilon}_{n,b}(j;0) = \check{\Upsilon}_{n,b}(j;0) + \ddot{\Upsilon}_{n,b}(j)$ and

$$\overline{\tilde{I}}_\varepsilon(j;q) = \frac{1}{\mathbf{B}} \sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} \left(\mathcal{J}_{\varepsilon,b}(j) \dot{\Upsilon}_{n,b}(-j;q) - \mathcal{E}\left(\mathcal{J}_{\varepsilon,b}(j) \dot{\Upsilon}_{n,b}(-j;q) \right) \right).$$

Because

$$\begin{aligned} \left| \dot{\beta}_{t+bn,T}(\ell) \right| &= \left| \beta_{t+bn,T}(\ell) - \beta\left(\frac{nb}{T};\ell\right) \right| = O\left(T^{-1}\left|\upsilon(\ell)\right|\right) \\ \left| \ddot{\beta}\left(\frac{t+nb}{T};\ell\right) \right| &= \left| \beta\left(\frac{t+nb}{T};\ell\right) - \beta\left(\frac{nb}{T};\ell\right) \right| \le C\upsilon(\ell)/n^{-1/2} \end{aligned} \tag{5.10}$$

it standard to conclude that the contribution into (5.8) or (5.9) of $\ddot{\Upsilon}_{n,b}(j)$ satisfy the sufficient condition (5.2). So, it suffices to examine the behaviour of (5.8) or (5.9) with $\dot{\Upsilon}_{n,b}(j;0)$ replaced by $\check{\Upsilon}_{n,b}(j;0)$. Now, standard inequalities yield that the first absolute moment of (5.8) is bounded by

$$\begin{aligned} &\frac{1}{\tilde{n}} \sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*} \left(\mathcal{E}\left(\overline{\overset{\circ}{I}}{}^2_\varepsilon(j) \right) \mathcal{E} \left| \frac{1}{\mathbf{B}} \sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} \left| \check{\Upsilon}_{n,b}(j;0) \right|^2 - \mathcal{E}\left| \check{\Upsilon}_{n,b}(j;0) \right|^2 \right|^2 \right)^{1/2} \\ = \; &O\left(\frac{1}{\mathbf{B}^{1/2}\tilde{n}} \left(\frac{\mathbf{j}_2^* - \mathbf{j}_1^*}{\tilde{n}} \right) \left(\frac{\mathbf{b}_2^* - \mathbf{b}_1^*}{\mathbf{B}} \right) \right), \end{aligned}$$

see (5.34) and (5.35) in Lemma 4. So Conditions $C2$ and $C3$ imply that (5.8) satisfies Condition BW.

Next (5.9), with $\dot{\Upsilon}_{n,b}(j;0)$ replaced by $\check{\Upsilon}_{n,b}(j;0)$. The second moment of the first term is

$$\frac{1}{\tilde{n}^2}\sum_{j,k=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathcal{E}\left(\overline{\mathring{I}}_{\varepsilon}(j)\overline{\mathring{I}}_{\varepsilon}(k)\overline{\overline{I}}_{\varepsilon}\left(j,n^{1/2}\right)\overline{\overline{I}}_{\varepsilon}\left(k,n^{1/2}\right)\right)=o\left(\frac{1}{T}\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)^2\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^2\right)$$

using that $\sum_{\ell=n^{1/2}}^{\infty}|\beta(u;\ell)|=o\left(n^{-1/2}\right)$ by $C1$ and independence of the sequence $\{\varepsilon_t\}_{t\in\mathbb{Z}}$. So the first term of (5.9) satisfies Condition BW. Finally the second term of (5.9). Using the definition of $\check{\Upsilon}_{n,b}(j;a)$ in (2.17), it suffices to consider

$$\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\overline{\mathring{I}}_{\varepsilon}(j)\mathcal{B}\left(u;e^{-i\lambda_j}\right)\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\left(\mathcal{J}_{\varepsilon,b}(j)\acute{\Upsilon}_{n,b}\left(-j;n^{1/2}\right)-\mathcal{E}\mathcal{J}_{\varepsilon,b}(j)\acute{\Upsilon}_{n,b}\left(-j;n^{1/2}\right)\right),$$

where $\acute{\Upsilon}_{n,b}\left(-j;n^{1/2}\right)=n^{-1/2}\sum_{\ell=1}^{n^{1/2}}\ell^{1/2}\beta(u;\ell)e^{-i\ell\lambda_j}\left(\ell^{-1/2}\sum_{t=n-\ell}^{n}\varepsilon_{t+(b-1)n}e^{-it\lambda_j}\right)$. But $\mathcal{E}\left|\acute{\Upsilon}_{n,b}\left(-j;n^{1/2}\right)\right|^2=o\left(n^{-1}\right)$, so the second moment of the last displayed expression is $O\left(\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{n}\right)^2\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)/\mathbf{B}^2n\right)$ and hence it satisfies Condition BW by Condition $C3$. This completes the proof that the first term of (5.6) satisfies Condition BW.

Next the second term of (5.6), i.e.

$$\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\mathring{I}_{\varepsilon,b}(j)-\overline{\mathring{I}}_{\varepsilon}(j)\right\}\sum_{k=1}^{2}\frac{(-1)^k}{k!}\left(\overline{\breve{R}}_n(j)+\mathcal{E}\left(R_{n,b}(j)\right)+\overline{\mathring{I}}_{\varepsilon}(j)\right)^k.$$

Proceeding as with the first term of (5.6), the contribution due to $\overline{\breve{R}}_n(j)+\mathcal{E}\left(R_{n,b}(j)\right)$ satisfies Condition BW. So, we only need to examine

$$\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\left(\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon,b}(j)-\frac{\mathbf{b}^*}{\mathbf{B}}\overline{\mathring{I}}_{\varepsilon}(j)\right)\sum_{k=1}^{2}\frac{(-1)^k}{k!}\overline{\mathring{I}}_{\varepsilon}^k(j).\tag{5.11}$$

The contribution due to $k=2$ is $o_p\left(T^{-1/2}\right)$ uniformly in $\mathbf{j}^*$ and $\mathbf{b}^*$, because $\sup_{\mathbf{j}^*}\left|\sum_{j=1}^{\mathbf{j}^*}a_j\right|\leq\sum_{j=1}^{\tilde{n}}|a_j|$, $\mathcal{E}\left(\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(j)\right)^{2k}=O\left(\left(\mathbf{b}_2^*-\mathbf{b}_1^*\right)^k\right)$ and then Condition $C3$ implies that

$$\begin{aligned}&\mathcal{E}\left(\sup_{\mathbf{j}^*}\frac{1}{\tilde{n}}\left|\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(j)\overline{\mathring{I}}_{\varepsilon}^2(j)\right|\right)\\ \leq\;&\frac{1}{\tilde{n}}\sum_{j=1}^{\tilde{n}}\mathcal{E}\left(\left|\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(j)\overline{\mathring{I}}_{\varepsilon}^2(j)\right|\right)=o\left(\frac{1}{T^{1/2}}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^{1+\delta}\right).\end{aligned}$$

Next, the contribution due to $k=1$ in (5.11), which is

$$\begin{aligned}&\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\left(\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon,b}(j)\overline{\mathring{I}}_{\varepsilon}(j)-\frac{\mathbf{b}^*}{\mathbf{B}}\left(1+\frac{\kappa_4}{n}\right)\right)\\ &-\frac{\mathbf{b}^*}{\mathbf{B}}\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\left\{\overline{\mathring{I}}_{\varepsilon}^2(j)-\frac{1}{\mathbf{B}}\left(1+\frac{\kappa_4}{n}\right)\right\}\end{aligned}$$

after we realize that $\mathcal{E}\left(\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon,b}(j)\,\overline{\mathring{I}}_{\varepsilon}(j)\right) = \mathbf{b}^*\mathcal{E}\left(\overline{\mathring{I}}_{\varepsilon}^{2}(j)\right) = \left(1+\frac{\kappa_4}{n}\right)\mathbf{b}^*/\mathbf{B}$. So, it suffices to examine the behaviour of

$$\mathcal{E}\left(\frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\left\{\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(j)\,\overline{\mathring{I}}_{\varepsilon}(j) - \frac{(\mathbf{b}_2^*-\mathbf{b}_1^*)}{\mathbf{B}^2}\left(1+\frac{\kappa_4}{n}\right)\right\}\right)^2$$

which is

$$\begin{aligned}
&\frac{1}{\tilde{n}^2}\left\{\sum_{j,k=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathcal{E}\left(\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(j)\,\overline{\mathring{I}}_{\varepsilon}(k)\right)\mathcal{E}\left(\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(k)\,\overline{\mathring{I}}_{\varepsilon}(j)\right)\right.\\
&+\sum_{j,k=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathcal{E}\left(\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(j)\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(k)\right)\mathcal{E}\left(\overline{\mathring{I}}_{\varepsilon}(j)\,\overline{\mathring{I}}_{\varepsilon}(k)\right)\\
&\left.+\sum_{j,k=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\frac{1}{\mathbf{B}^2}\sum_{b,v=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}Cum\left(\mathring{I}_{\varepsilon,b}(j);\mathring{I}_{\varepsilon,v}(k);\overline{\mathring{I}}_{\varepsilon}(j);\overline{\mathring{I}}_{\varepsilon}(k)\right)\right\}\\
=\;&O\left((\mathbf{j}_2^*-\mathbf{j}_1^*)(\mathbf{b}_2^*-\mathbf{b}_1^*)/\mathbf{B}^3n^2\right)
\end{aligned}$$

using Brillinger's (1980) Theorems 2.3.2 and 4.3.1., and in particular expressions in (2.3.7) and (4.3.15)), because (5.55) and that

$$\sum_{j,k=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}Cum\left(I_{\varepsilon,b}(j);I_{\varepsilon,b}(j);I_{\varepsilon,b}(k);I_{\varepsilon,b}(k)\right) = O\left((\mathbf{b}_2^*-\mathbf{b}_1^*)(\mathbf{j}_2^*-\mathbf{j}_1^*)\right).$$

Recall that $C1$ implies that $\mathring{I}_{\varepsilon,b_1}(j)$ and $\mathring{I}_{\varepsilon,b_2}(k)$ are independent for all j,k if $b_1\neq b_2$.

So, we conclude that uniformly in $\mathbf{j}^*$ and $\mathbf{b}^*$, the first and second terms of (5.6) satisfy Condition BW and hence (5.3) is

$$\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{p=1}^{\mathbf{b}^*}\left\{\mathring{I}_{\varepsilon,b}(j)-\overline{\mathring{I}}_{\varepsilon}(j)\right\}+o_p\left(\frac{1}{T^{1/2}}\right),$$

as we showed above that the second term of (5.3) satisfied (5.2). So the proof is completed if we show that the first term of the last displayed expression

$$\frac{1}{(\tilde{n}\mathbf{B})^{1/2}}\left\{\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon,b}(j)-\left(\frac{\mathbf{b}^*}{\mathbf{B}}\right)\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{B}}\mathring{I}_{\varepsilon,b}(j)\right\}\Rightarrow\mathcal{BS}\left([0,1]^2\right). \tag{5.12}$$

To that end, it is standard to show that

$$\frac{1}{\tilde{n}\mathbf{B}}\mathcal{E}\left(\sum_{j=1}^{\mathbf{j}_1^*}\sum_{b=1}^{\mathbf{b}_1^*}\mathring{I}_{\varepsilon,b}(j)\sum_{k=1}^{\mathbf{j}_2^*}\sum_{v=1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,v}(k)\right)=\frac{\mathbf{b}_1^*}{\mathbf{B}}\left[\left(\frac{\mathbf{j}_1^*}{\tilde{n}}\right)+\left(\frac{\mathbf{j}_1^*}{\tilde{n}}\right)\left(\frac{\mathbf{j}_2^*}{\tilde{n}}\right)\kappa_4\right],$$

where we have assumed without loss of generality that $\mathbf{j}_1^*\le\mathbf{j}_2^*$ and $\mathbf{b}_1^*\le\mathbf{b}_2^*$ and independence of $\mathring{I}_{\varepsilon,b}(j)$ and $\mathring{I}_{\varepsilon,v}(j)$ for $b\neq v$ by Condition $C1$. So, the covariance structure of (5.12) is, after standard algebra, given by (2.10).

From here the proof concludes by standard arguments if we show that

$$\frac{1}{(\tilde{n}\mathbf{B})^{1/2}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon,b}(j)=\frac{1}{(\tilde{n}\mathbf{B})^{1/2}}\sum_{b=1}^{\mathbf{b}^*}\sum_{j=1}^{\mathbf{j}^*}\mathring{I}_{\varepsilon,b}(j)$$

converges in distribution to a normal random variable. But this is the case as $\tilde{n}^{-1/2}\sum_{j=1}^{\mathbf{j}^*}\mathring{I}_{\varepsilon,b}(j)$ is a triangular array of independent identically distributed random variables with finite second moments. This completes the proof of the theorem. ■

5.2. **Proof of Corollary 1.**

Part (a) follows from Theorem 1 because under H_0, Condition $C2$ holds trivially and $\left|\mathcal{B}\left(\frac{nb}{T};j\right)\right|^2 = |\mathcal{B}(j)|^2 \equiv f(j)/\sigma_\varepsilon^2$, so that for all $\mathbf{j}^* = 1, ..., \tilde{n}$ and $\mathbf{b}^* = 2, ..., \mathbf{B}$, $\breve{\mathcal{T}}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right) \equiv \mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$. The proof of part (b) is omitted as it follows by standard arguments. ■

5.3. **Proof of Corollary 2.**

We begin showing the statistical properties of $\widehat{\sigma}_\varepsilon^2$. Recall that is $\widehat{\sigma}_\varepsilon^2 = \mathbf{B}^{-1}\sum_{b=1}^{\mathbf{B}}\widehat{\sigma}_\varepsilon^2(b)$, where $\widehat{\sigma}_\varepsilon^2(b)$ is given in (2.23). Now we have that

$$\widehat{\sigma}_\varepsilon^2(b) - 1 =: \frac{1}{n}\sum_{t=1}^{n}\left(\varepsilon_{t+(b-1)n}^2 - 1\right) + \left(\widehat{\sigma}_\varepsilon^2(b) - \widetilde{\sigma}_\varepsilon^2(b)\right), \tag{5.13}$$

recall that we assumed $\sigma_\varepsilon^2 = 1$ for notational simplicity, where

$$\widetilde{\sigma}_\varepsilon^2(b) = \frac{1}{n}\sum_{t=1}^{n}\varepsilon_{t+(b-1)n}^2; \quad b = 1, ..., \mathbf{B}. \tag{5.14}$$

From here and Lemma 7 is obvious that $\widehat{\sigma}_\varepsilon^2 - \sigma_\varepsilon^2 = o_p(1)$. Also

$$\frac{1}{T}\sum_{t=1}^{T}\widehat{\varepsilon}_t^4 = \frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\frac{1}{n}\sum_{t=1}^{n}\left(\widehat{\varepsilon}_{t+(b-1)n}^4 - \varepsilon_{t+(b-1)n}^4\right) + \frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\frac{1}{n}\sum_{t=1}^{n}\varepsilon_{t+(b-1)n}^4.$$

Now the first term on the right of the last displayed expression converges to zero in probability by Lemma 8, whereas by Condition $C1$ and weak law of large numbers, the second term converges to $3\sigma_\varepsilon^4 + \kappa_4$ in probability. Now standard arguments conclude that $\widehat{\kappa}_4 - \kappa_4 = o_p(1)$. ■

5.4. **Proof of Theorem 2.**

It suffices to examine the difference

$$\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\left\{\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};j\right) - \mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};j\right)\right\}. \tag{5.15}$$

To that end, we first examine

$$\frac{1}{\tilde{n}\mathbf{B}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}\frac{I_{x,b}(j)}{f(j)}\left\{\frac{1}{\widehat{\sigma}_\varepsilon^2(b)} - 1\right\} = \sum_{\ell=1}^{3}\Phi_{n,\ell}(\mathbf{j}^*,\mathbf{b}^*) + o_p\left(T^{-1/2}\right), \tag{5.16}$$

where the right side is due to Taylor's expansion because Theorem 2 and $C3$ imply that $\mathcal{E}\left\{\widehat{\sigma}_\varepsilon^2(b) - 1\right\}^4 = O\left(\mathbf{B}^{-2}\right) = o\left(T^{-1/2}\right)$ and $\sup_{b=1,...,\mathbf{B}}\left|\widehat{\sigma}_\varepsilon^2(b) - 1\right|^2 = o_p(\mathbf{B}/n)$ then yields that $o_p\left(T^{-1/2}\right)$ is uniformly in $\mathbf{j}^*$ and $\mathbf{b}^*$, and where

$$\Phi_{n,1}(\mathbf{j}^*,\mathbf{b}^*) = \frac{\mathbf{j}^*}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\sum_{k=1}^{3}\frac{(-1)^k}{k!}\left\{\widehat{\sigma}_\varepsilon^2(b) - 1\right\}^k \tag{5.17}$$

$$\Phi_{n,2}(\mathbf{j}^*,\mathbf{b}^*) = \frac{1}{\tilde{n}\mathbf{B}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}R_{n,b}(j)\sum_{k=1}^{3}\frac{(-1)^k}{k!}\left\{\widehat{\sigma}_\varepsilon^2(b) - 1\right\}^k \tag{5.18}$$

$$\Phi_{n,3}(\mathbf{j}^*,\mathbf{b}^*) = \frac{1}{\tilde{n}\mathbf{B}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon,b}(j)\sum_{k=1}^{3}\frac{(-1)^k}{k!}\left\{\widehat{\sigma}_\varepsilon^2(b) - 1\right\}^k. \tag{5.19}$$

First we examine (5.19). The contribution due to the terms when $k=2,3$ is easily shown that they satisfy (5.2) and hence Condition BW, so we will only handle $k=1$, which is

$$\frac{1}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\widetilde{\sigma}_{\varepsilon}^2(b)-\widehat{\sigma}_{\varepsilon}^2(b)\right\}\sum_{j=1}^{\mathbf{j}^*}\mathring{I}_{\varepsilon,b}(j)-\frac{1}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\widetilde{\sigma}_{\varepsilon}^2(b)-1\right\}\sum_{j=1}^{\mathbf{j}^*}\mathring{I}_{\varepsilon,b}(j) \tag{5.20}$$

using the notation in (5.13).

Next because $\mathcal{E}\left(\widetilde{\sigma}_{\varepsilon}^2(b)-1\right)^4=O\left(n^{-2}\right)$, the second term of (5.20) satisfies Condition BW because

$$\begin{aligned}
&\mathcal{E}\left|\frac{1}{\tilde{n}\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\left\{\widetilde{\sigma}_{\varepsilon}^2(b)-1\right\}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,b}(j)\right|^2\\
\leq\ & \frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}^2n^2}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathcal{E}\left|\left\{\widetilde{\sigma}_{\varepsilon}^2(b)-1\right\}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,b}(j)\right|^2\\
=\ & o_p\left(\frac{1}{T}\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)^{1+\delta}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^2\right),
\end{aligned}$$

by Cauchy-Schwarz's inequality and then Condition $C3$.

Now due to Lemma 7, the first term of (5.20) is

$$\frac{d_n}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left(\frac{1}{n}\sum_{p=1}^{n}\breve{\psi}_{b,n}(p)+\frac{1}{\mathbf{B}}\right)\sum_{j=1}^{\mathbf{j}^*}\mathring{I}_{\varepsilon,b}(j)+\frac{\Psi_{n,1}}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\Psi_{n,2}(b)\left|\sum_{j=1}^{\mathbf{j}^*}\mathring{I}_{\varepsilon,b}(j)\right|, \tag{5.21}$$

where using notation before Lemma 6,

$$\breve{\psi}_{b,n}(p)=\mathcal{A}^{-1}(p)\,\psi_{1,n}(p)\,\mathring{I}_{\varepsilon,b}(p). \tag{5.22}$$

Because $\mathcal{E}\Psi_{n,2}^2(b)=O\left(\mathbf{B}^{-3}+n^{-2}\right)$,

$$\sup_{\mathbf{j}^*,\mathbf{b}^*}\left|\frac{1}{(\tilde{n}\mathbf{B})^{1/2}}\sum_{b=1}^{\mathbf{b}^*}\sum_{j=1}^{\mathbf{j}^*}\mathring{I}_{\varepsilon,b}(j)\right|=O_p(1);\quad \sup_{\mathbf{j}^*}\left|\frac{1}{\tilde{n}^{1/2}}\sum_{j=1}^{\mathbf{j}^*}\mathring{I}_{\varepsilon,b}(j)\right|=O_p(1) \tag{5.23}$$

we can conclude then by $C3$, that the first term of (5.20) satisfies condition BW if the first term of (5.21) does. So, we need to examine the behaviour of

$$\frac{1}{n}\sum_{p=1}^{\tilde{n}}\mathcal{A}^{-1}(p)\,\xi^{(1)}(p,j)+\frac{1}{n}\sum_{p=1}^{\tilde{n}}\mathcal{A}^{-1}(p)\,\xi^{(2)}(p,j), \tag{5.24}$$

where

$$\begin{aligned}
\xi^{(1)}(p,j) &= \sum_{\ell=1}^{n}\varsigma_{\ell p}\left(\frac{1}{\mathbf{B}}\sum_{v=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,v}(\ell)\right)\left\{\frac{1}{\tilde{n}\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(p)\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,b}(j)\right\}\\
\xi^{(2)}(p,j) &= \sum_{\ell=1}^{n}\varsigma_{\ell p}\left(\frac{1}{\mathbf{B}}\sum_{v\neq\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,v}(\ell)\right)\left\{\frac{1}{\tilde{n}\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(p)\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,b}(j)\right\}.
\end{aligned}$$

The second moment of $\xi^{(2)}(p,j)$ is

$$\sum_{\ell_1,\ell_2=1}^{n} \varsigma_{\ell_1 p}\varsigma_{\ell_2 p}\left\{\frac{1}{\mathbf{B}^2}\sum_{v\neq \mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathcal{E}\left(\mathring{I}_{\varepsilon,v}(\ell_1)\,\mathring{I}_{\varepsilon,v}(\ell_2)\right)\right\}$$
$$\times\frac{1}{(\tilde{n}\mathbf{B})^2}\mathcal{E}\left\{\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(p)\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,b}(j)\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon,b}(p)\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,b}(j)\right\}$$
$$= O\left(\frac{\log^2 n}{n\mathbf{B}^2}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^2\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)\right)$$

by $C1$ and using (5.55) in Lemma 7. On the other hand, $\xi^{(1)}(p,j)$ is

$$\frac{1}{\tilde{n}\mathbf{B}^2}\sum_{v=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\left(\sum_{\ell=1}^{n}\varsigma_{\ell p}\mathring{I}_{\varepsilon,v}(\ell)\,\mathring{I}_{\varepsilon,v}(p)\right)\left\{\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,v}(j)\right\}$$
$$+\frac{1}{\tilde{n}\mathbf{B}^2}\sum_{v\neq b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\left(\sum_{\ell=1}^{n}\varsigma_{\ell p}\mathring{I}_{\varepsilon,v}(\ell)\,\mathring{I}_{\varepsilon,b}(p)\right)\left\{\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,b}(j)\right\}.$$

The first term of the last displayed expression satisfies Condition BW by a routine used of the Cauchy-Schwarz's inequality, whereas Condition $C1$ implies that the second moment of the second term is

$$\frac{1}{\tilde{n}^2\mathbf{B}^4}\sum_{\ell_1,\ell_2=1}^{n}\varsigma_{\ell_1 p}\varsigma_{\ell_2 p}\left\{\sum_{v=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathcal{E}\left(\mathring{I}_{\varepsilon,v}(\ell_1)\,\mathring{I}_{\varepsilon,v}(\ell_2)\right)\right.$$
$$\left.\times\left(\sum_{\substack{b_1,b_2=\mathbf{b}_1^*+1\\ b_1,b_2\neq v}}^{\mathbf{b}_2^*}\mathcal{E}\left(\mathring{I}_{\varepsilon,b_1}(p)\,\mathring{I}_{\varepsilon,b_2}(p)\left\{\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,b_1}(j)\right\}\left\{\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,b_2}(j)\right\}\right)\right)\right\}$$
$$=\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\tilde{n}^2\mathbf{B}^4}\left\{\sum_{\ell_1,\ell_2=1}^{n}\varsigma_{\ell_1 p}\varsigma_{\ell_2 p}\left(\mathcal{I}(\ell_1=\ell_2)+\frac{\kappa_4}{n}\right)\times\right.$$
$$\left\{\left(\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathcal{E}\left(\mathring{I}_{\varepsilon,b}(p)\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,b}(j)\right)\right)^2+\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathcal{E}\left(\mathring{I}^2_{\varepsilon,b}(p)\right)\mathcal{E}\left(\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\mathring{I}_{\varepsilon,b}(j)\right)^2\right.$$
$$\left.+\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\sum_{j,k=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}cum\left(\mathring{I}_{\varepsilon,b}(p);\mathring{I}_{\varepsilon,b}(p);\mathring{I}_{\varepsilon,b}(j);\mathring{I}_{\varepsilon,b}(k)\right)\right\}.$$

From here and a standard used of (5.55) of Lemma 7 it follows that it satisfies Condition BW. So, this concludes that (5.24) and hence the first term of (5.20) satisfies (5.2), i.e. Condition BW.

Next (5.18). As with (5.19) the contribution due to the terms when $k=2,3$ satisfies (5.2) with $\alpha=1$ there and hence Condition BW. So we will examine

$$\frac{1}{\tilde{n}\mathbf{B}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}R_{n,b}(j)\left(\widehat{\sigma}^2_{\varepsilon}(b)-1\right).$$

Using (5.29) and definition in (5.31), we have that it suffices to show (5.2) for

$$\frac{1}{\tilde{n}\mathbf{B}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}\mathrm{Z}_{n,b}^{(1)}(-j)\left(\widehat{\sigma}_{\varepsilon}^{2}(b)-1\right).$$

because $\mathcal{E}\left|\mathrm{Y}_{n,b}(j;0)\right|^4+E\left|\mathrm{Z}_{n,b}^{(2)}(-j)\right|^2=O\left(n^{-2}\right)$ and then Theorem 2 and Condition $C3$. Now because $\mathcal{E}\mathrm{Z}_{n,b}^{(1)}(-j)=O\left(n^{-1}\right)$, it implies that it suffices to show that

$$\frac{1}{\tilde{n}\mathbf{B}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}\left(\mathrm{Z}_{n,b}^{(1)}(-j)-\mathcal{E}\mathrm{Z}_{n,b}^{(1)}(-j)\right)\left(\widehat{\sigma}_{\varepsilon}^{2}(b)-\widetilde{\sigma}_{\varepsilon}^{2}(b)\right)$$
$$+\frac{1}{\tilde{n}\mathbf{B}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}\left(\mathrm{Z}_{n,b}^{(1)}(-j)-\mathcal{E}\mathrm{Z}_{n,b}^{(1)}(-j)\right)\left(\widetilde{\sigma}_{\varepsilon}^{2}(b)-1\right)$$

satisfies (5.2) and hence Condition BW. Clearly the second term of the last displayed expression satisfies (5.2) using Lemma 1 part (**a**) and that $\mathcal{E}\left(\widetilde{\sigma}_{\varepsilon}^{2}(b)-1\right)^2=O\left(n^{-1}\right)$, whereas the first term proceeding as with (5.21) is

$$\frac{1}{\tilde{n}\mathbf{B}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}\left(\mathrm{Z}_{n,b}^{(1)}(-j)-\mathcal{E}\mathrm{Z}_{n,b}^{(1)}(-j)\right)\frac{1}{n}\sum_{p=1}^{n}\breve{\psi}_{b,n}(p)+o_p\left(T^{-1/2}\right)$$

uniformly in $\mathbf{j}^*$ and $\mathbf{b}^*$. Then proceed step by step as with (5.24) but with $\mathring{I}_{\varepsilon,b}(j)$ replaced by $\mathrm{Z}_{n,b}^{(1)}(-j)-\mathcal{E}\mathrm{Z}_{n,b}^{(1)}(-j)$.

So, it remains to examine the behaviour of (5.17), which is

$$\frac{\mathbf{j}^*}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\widehat{\sigma}_{\varepsilon}^{2}(b)-1\right\}+\frac{\mathbf{j}^*}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\sum_{k=2}^{3}\frac{(-1)^k}{k!}\left\{\widehat{\sigma}_{\varepsilon}^{2}(b)-1\right\}^k \tag{5.25}$$

We first examine the second term of (5.25). The contribution due to $\widetilde{\sigma}_{\varepsilon}^{2}(b)-1$ is $o_p\left(T^{-1/2}\right)$, uniformly in $\mathbf{j}^*$ and $\mathbf{b}^*$ because

$$\begin{aligned}\mathcal{E}\sup_{\mathbf{j}^*;\mathbf{b}^*}\left|\frac{\mathbf{j}^*}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\sum_{k=2}^{3}\left\{\widetilde{\sigma}_{\varepsilon}^{2}(b)-1\right\}^k\right| &\leq \frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\sum_{k=2}^{3}\mathcal{E}\left|\widetilde{\sigma}_{\varepsilon}^{2}(b)-1\right|^k\\ &= O\left(n^{-1}\right)=o\left(T^{-1/2}\right),\end{aligned}$$

so it is the contribution due to $\frac{d_{2,n}}{\mathbf{B}}+\Psi_{n,1}\Psi_{n,2}(b)$ because by Lemma 7, $\mathcal{E}\left|\frac{d_{2,n}}{\mathbf{B}}+\Psi_{n,2}(b)\right|^k=o_p\left(T^{-1/2}\right)$ and $\Psi_{n,1}=O_p(1)$. Next the contribution due $\frac{1}{n}\sum_{p=1}^{n}\breve{\psi}_{b,n}(p)$, that is

$$\frac{\mathbf{j}^*}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\sum_{k=2}^{3}\frac{(-1)^k}{k!}\left(\frac{1}{n}\sum_{p=1}^{n}\mathcal{A}^{-1}(p)\,\psi_{1,n}(p)\,\mathring{I}_{\varepsilon,b}(p)\right)^k.$$

Now,

$$
\begin{aligned}
&\mathcal{E}\sup_{\mathbf{j}^*;\mathbf{b}^*}\left|\frac{\mathbf{j}^*}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\sum_{k=2}^{3}\left\{\frac{1}{n}\sum_{p=1}^{n}\mathcal{A}^{-1}(p)\,\psi_{1,n}(p)\,\mathring{I}_{\varepsilon,b}(p)\right\}^k\right| \\
=\;& \frac{C}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\left\{\mathcal{E}\left(\frac{1}{n}\sum_{p=1}^{n}\mathcal{A}^{-1}(p)\,\psi_{1,n}(p)\,\mathring{I}_{\varepsilon,b}(p)\right)^2+\frac{1}{n}\sum_{p=1}^{n}\mathcal{E}\left(|\psi_{1,n}(p)|^3\left|\mathring{I}_{\varepsilon,b}(p)\right|^3\right)\right\} \\
\le\;& \frac{C}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\left\{\mathcal{E}\left(\frac{1}{n\mathbf{B}}\sum_{p=1}^{n}\mathcal{A}^{-1}(p)\,\mathring{I}_{\varepsilon,b}(p)\sum_{\ell=1}^{n}\varsigma_{\ell p}\sum_{v\neq b}\mathring{I}_{\varepsilon,v}(\ell)\right)^2\right\} \\
&+\frac{C}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\left\{\mathcal{E}\left(\frac{1}{n\mathbf{B}}\sum_{p=1}^{n}\mathcal{A}^{-1}(p)\,\mathring{I}_{\varepsilon,b}(p)\sum_{\ell=1}^{n}\varsigma_{\ell p}\mathring{I}_{\varepsilon,b}(\ell)\right)^2\right\}+O\left(\frac{1}{\mathbf{B}^{3/2}}\right) \\
=\;& o_p\left(T^{-1/2}\right)
\end{aligned}
$$

because $\mathbf{B}^{-3/2}=o\left(T^{-1/2}\right)$ by $C3$ and using (5.55). So, we have that the second term of (5.25) satisfies Condition BW.

Next the first term of (5.25), which is using (5.22)

$$
\frac{\mathbf{j}^*}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\widetilde{\sigma}_{\varepsilon}^2(b)-1\right\}+\frac{\mathbf{j}^*}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\frac{1}{n}\sum_{j=1}^{n}\breve{\psi}_{b,n}(j)+\frac{\mathbf{j}^*\mathbf{b}^*}{\tilde{n}\mathbf{B}}\left(\frac{d_{2,n}}{\mathbf{B}}\right)+o_p\left(\frac{1}{T^{1/2}}\right).
$$

by Lemma 7. The second term is

$$
\begin{aligned}
&\frac{\mathbf{j}^*}{\tilde{n}\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\frac{1}{n}\sum_{j=1}^{n}\sum_{p=1}^{n}\varsigma_{jp}\left(\frac{1}{\mathbf{B}}\sum_{v\neq b}\mathring{I}_{\varepsilon,v}(p)\right)\mathring{I}_{\varepsilon,b}(j) \\
&+\frac{\mathbf{j}^*}{\tilde{n}\mathbf{B}}\sum_{p=1}^{n}\varsigma_{jp}\frac{1}{n\mathbf{B}}\sum_{j=1}^{n}\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon,b}(j)\,\mathring{I}_{\varepsilon,b}(p).
\end{aligned}
$$

Again, because

$$
\left(\frac{1}{n\mathbf{B}}\sum_{j=1}^{n}\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon,b}(j)\,\mathring{I}_{\varepsilon,b}(p)\right)^2=O\left(T^{-1/2}\right),
$$

it suffices to examine

$$
\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\frac{1}{n}\sum_{j=1}^{n}\sum_{p=1}^{n}\varsigma_{jp}\left(\frac{1}{\mathbf{B}}\sum_{v\neq b}\mathring{I}_{\varepsilon,v}(p)\right)\mathring{I}_{\varepsilon,b}(j)
$$

from our comments made after (5.19). The second moments are

$$
\frac{1}{\mathbf{B}^4}\frac{1}{n^2}\sum_{j_1,j_2=1}^{n}\sum_{p_1,p_2=1}^{n}\varsigma_{j_1p_1}\varsigma_{j_2p_2}\sum_{b=1}^{\mathbf{b}^*}\left\{\mathcal{E}\left(\mathring{I}_{\varepsilon,b}(j_1)\,\mathring{I}_{\varepsilon,b}(j_2)\right)\sum_{v\neq b}\mathcal{E}\left(\mathring{I}_{\varepsilon,v}(p_1)\,\mathring{I}_{\varepsilon,v}(p_2)\right)\right\}.
$$

Now use (5.55) to conclude that it is $o\left(T^{-1/2}\right)$.

So, we have obtained that, uniformly in $\mathbf{b}^*$ and $\mathbf{j}^*$, (5.16) is

$$
\frac{\mathbf{j}^*}{\tilde{n}}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left(\widetilde{\sigma}_{\varepsilon}^2(b)-1\right)+\frac{\mathbf{j}^*\mathbf{b}^*}{\tilde{n}\mathbf{B}}\left(\frac{d_{2,n}}{\mathbf{B}}\right)+o_p\left(\frac{1}{T^{1/2}}\right) \tag{5.26}
$$

and hence (5.15) becomes

$$\frac{\mathbf{j}^*}{\tilde{n}}\left\{\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left(\widetilde{\sigma}_{\varepsilon}^2(b)-1\right)-\frac{\mathbf{b}^*}{\mathbf{B}}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\left(\widetilde{\sigma}_{\varepsilon}^2(b)-1\right)\right\}+o_p\left(\frac{1}{T^{1/2}}\right)$$

proceeding as in the proof of Theorem 1 and because the second term of (5.26) is independent of b. From here we then conclude that

$$\begin{aligned}[T/2]\,\mathcal{TP}_{n,\mathbf{B}}\left(\frac{\mathbf{j}^*}{\tilde{n}},\frac{\mathbf{b}^*}{\mathbf{B}}\right) &= \frac{1}{(\tilde{n}\mathbf{B})^{1/2}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}\left\{\mathring{I}_{\varepsilon,b}(j)-\frac{1}{n}\sum_{t=1}^{n}\left(\varepsilon_{t+(b-1)n}^2-1\right)\right.\\ &\qquad\left.-\left(\overline{\mathring{I}}_{\varepsilon}(j)-\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_t^2-1\right)\right)\right\}+o_p(1)\end{aligned}$$

using the proof of Theorem 1.

But, standard algebra gives that

$$\begin{aligned}&\mathcal{E}\left(\mathring{I}_{\varepsilon,b}(j)-\frac{1}{n}\sum_{t=1}^{n}\left(\varepsilon_{t+(b-1)n}^2-1\right);\mathring{I}_{\varepsilon,b}(-k)-\frac{1}{n}\sum_{t=1}^{n}\left(\varepsilon_{t+(b-1)n}^2-1\right)\right)\\ =&\ \frac{1}{n^2}\sum_{t_1\neq s_1;t_2\neq s_2}\mathcal{E}\left(\varepsilon_{t_1}\varepsilon_{s_1}\varepsilon_{t_2}\varepsilon_{s_2}\right)e^{i(t_1-s_1)\lambda_j-i(t_2-s_2)\lambda_k}\\ =&\ \mathcal{I}(j=k)-\frac{2}{n}.\end{aligned}$$

The proof now follows by routine arguments, and so they are omitted. ■

5.5. **Proof of Proposition 1.**

We shall look at part (a), part (b) follows identically using Theorem 2 instead of Theorem 1 when needed and that $\sigma_{\varepsilon}^2(b)=\sigma_{\varepsilon}^2$. The proof is similar to that of Theorem 1 but we employ Lemmas 4 and 5 instead of Lemmas 2 and 3 when needed. Abbreviating $\acute{f}\left(\frac{bn}{T};j\right)/\overline{f}_{\mathbf{B}}(j)$ as $\ddot{f}\left(\frac{bn}{T};j\right)$, where $\acute{f}\left(\frac{bn}{T};j\right)=\left(1+g\left(bn/T;j\right)/[T/2]^{1/2}\right)$ and $\overline{f}_{\mathbf{B}}(j)=\mathbf{B}^{-1}\sum_{b=1}^{\mathbf{B}}\acute{f}\left(\frac{bn}{T};j\right)$, we easily see, proceeding as in Theorem 1, that $\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ is

$$\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\left\{\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left(\frac{\ddot{f}\left(\frac{bn}{T};j\right)\mathring{I}_{\varepsilon,b}(j)+\ddot{f}\left(\frac{bn}{T};j\right)}{\mathbf{B}^{-1}\sum_{b=1}^{\mathbf{B}}\ddot{f}\left(\frac{bn}{T};j\right)\mathring{I}_{\varepsilon,b}(j)+1}-1\right)\right\}1+o_p\left(T^{-1/2}\right).$$

Now, using Taylor's expansion of x^{-1} around 1 and the arguments in the proof of Theorem 1, we have that

$$\frac{1}{\mathbf{B}^{-1}\sum_{b=1}^{\mathbf{B}}\ddot{f}\left(\frac{bn}{T};j\right)\mathring{I}_{\varepsilon,b}(j)+1}\overset{asym}{\simeq}1-\upsilon_n(j)+\upsilon_n^2(j),$$

where $\upsilon_n(j)=\mathbf{B}^{-1}\sum_{b=1}^{\mathbf{B}}\ddot{f}\left(\frac{bn}{T};j\right)\mathring{I}_{\varepsilon,b}(j)$ and "$\overset{asym}{\simeq}$" denotes that the left- and right hand sides are asymptotically equivalent.

So, the asymptotic behaviour of $[T/2]^{1/2}\mathcal{T}_{n,\mathbf{B}}\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$ is governed by

$$\begin{aligned}&[T/2]^{1/2}\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\left\{\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\left\{\left(\ddot{f}\left(\frac{bn}{T};j\right)\mathring{I}_{\varepsilon,b}(j)-\upsilon_n(j)\right)+\left(\ddot{f}\left(\frac{bn}{T};j\right)-1\right)\right\}\right.\\ &\left.\times\left(1-\upsilon_n(j)+\upsilon_n^2(j)\right)\right\}.\end{aligned}$$

Now, except terms of smaller order of magnitude, the expectation of the last displayed expression is

$$
\begin{aligned}
& [T/2]^{1/2} \frac{1}{\tilde{n}} \sum_{j=1}^{\mathbf{j}^*} \frac{1}{\mathbf{B}} \sum_{b=1}^{\mathbf{b}^*} \left(\ddot{f}\left(\frac{bn}{T};j\right) - 1 \right) \\
= \; & \frac{1}{\tilde{n}} \sum_{j=1}^{\mathbf{j}^*} \frac{1}{\mathbf{B}} \sum_{b=1}^{\mathbf{b}^*} \left\{ g\left(\frac{bn}{T};j\right) - \frac{1}{\mathbf{B}} \sum_{b=1}^{\mathbf{B}} g\left(\frac{bn}{T};j\right) \right\} \underset{T,n \nearrow \infty}{\to} d\left(\boldsymbol{\omega}^*;\boldsymbol{\upsilon}^*\right)
\end{aligned}
$$

using the definition of $\ddot{f}\left(\frac{bn}{T};j\right)$ under H_l. Now the proof of the proposition proceeds as that of Theorem 1 and so it is omitted. ■

Let's introduce some notation. In what follows, we denote

$$
\begin{aligned}
\mathring{I}_{\varepsilon^*,b}(j) &= I_{\varepsilon^*,b}(j) - 1;\ \overline{I}_{\varepsilon^*}(j) = \frac{1}{\mathbf{B}} \sum_{b=1}^{\mathbf{B}} I_{\varepsilon^*,b}(j) \\
\overline{\mathring{I}}_{\varepsilon^*}(j) &= \frac{1}{\mathbf{B}} \sum_{b=1}^{\mathbf{B}} \mathring{I}_{\varepsilon^*,b}(j).
\end{aligned}
$$

Notice that $\mathcal{E}^*\left(\mathring{I}_{\varepsilon^*,b}(j)\right) = 0$. Also $\{H_n\}_{n\geq 1}$ is a sequence of strictly positive $O_p(1)$ random variables.

5.6. **Proof of Theorem 3.**

We shall handled only part (a), part (b) follows similarly. We need to show that

$$
[T/2]^{1/2}\, \mathcal{T}_{n,\mathbf{B}}^*\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right) \overset{weakly}{\Rightarrow} \mathcal{BS}\left([0,1]^2\right) \qquad \textit{(in probability)},
$$

Now, using Taylor's expansion of $\overline{I}_{\varepsilon^*}^{-1}(j)$ around 1, we obtain the following decomposition of $\mathcal{T}_{n,\mathbf{B}}^*\left(\frac{\mathbf{j}^*}{\tilde{n}},\frac{\mathbf{b}^*}{\mathbf{B}}\right)$

$$
\begin{aligned}
& \frac{1}{\tilde{n}} \sum_{j=1}^{\mathbf{j}^*} \frac{1}{\mathbf{B}} \sum_{b=1}^{\mathbf{b}^*} \left(\mathring{I}_{\varepsilon^*,b}(j) - \overline{\mathring{I}}_{\varepsilon^*}(j) \right) \overline{\mathring{I}}_{\varepsilon^*}^{3}(j)\, O_{p^*}(1) \\
& + \frac{1}{\tilde{n}} \sum_{j=1}^{\mathbf{j}^*} \left(\frac{1}{\mathbf{B}} \sum_{b=1}^{\mathbf{b}^*} \left\{ \mathring{I}_{\varepsilon^*,b}(j) - \overline{\mathring{I}}_{\varepsilon^*}(j) \right\} \right) \sum_{k=0}^{2} \frac{(-1)^k}{k!} \overline{\mathring{I}}_{\varepsilon^*}^{k}(j).
\end{aligned}
$$

Notice that Lemma 9 yields that $\left(\sup_j \overline{\mathring{I}}_{\varepsilon^*}(j)\right)^4 \leq \sum_{j=1}^{\tilde{n}} \overline{\mathring{I}}_{\varepsilon^*}^{4}(j) = o_{p^*}(1)$ by $C1$, so that $\overline{I}_{\varepsilon^*}^{-1}(j) < H_n$. So, proceeding as in the proof of Theorem 1 but using now Lemma 9, we easily conclude that $\mathcal{T}_{n,\mathbf{B}}^*\left(\frac{\mathbf{j}^*}{\tilde{n}},\frac{\mathbf{b}^*}{\mathbf{B}}\right)$ is governed by

$$
\frac{1}{\tilde{n}\mathbf{B}} \sum_{j=1}^{\mathbf{j}^*} \sum_{b=1}^{\mathbf{b}^*} \left\{ \mathring{I}_{\varepsilon^*,b}(j) - \overline{\mathring{I}}_{\varepsilon^*}(j) \right\} \sum_{k=0}^{2} \frac{(-1)^k}{k!} \overline{\mathring{I}}_{\varepsilon^*}^{k}(j). \tag{5.27}
$$

Next, we examine the contribution due to $k = 1,2$ in (5.27), and in particular

$$
\frac{1}{\tilde{n}} \sum_{j=1}^{\mathbf{j}^*} \left(\frac{1}{\mathbf{B}} \sum_{b=1}^{\mathbf{b}^*} \mathring{I}_{\varepsilon^*,b}(j) \right) \sum_{k=1}^{2} \overline{\mathring{I}}_{\varepsilon^*}^{k}(j). \tag{5.28}
$$

To that end, we write

$$\begin{aligned}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\mathring{I}_{\varepsilon^*,b}(j) &= \frac{1}{\mathbf{B}}\left\{\sum_{b=1}^{\mathbf{b}_1^*}+\sum_{b=\mathbf{b}_2^*+1}^{\mathbf{B}}\right\}\mathring{I}_{\varepsilon^*,b}(j)+\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathring{I}_{\varepsilon^*,b}(j)\\ &=: \overline{\dot{I}}_{\varepsilon^*}(j)+\overline{\ddot{I}}_{\varepsilon^*}(j).\end{aligned}$$

Now, because $\mathring{I}_{\varepsilon^*,b_1}(j)$ and $\mathring{I}_{\varepsilon^*,b_2}(k)$ are independent for all j,k when $b_1\neq b_2$, we have that

$$\begin{aligned}\mathcal{E}^*\left(\mathring{I}_{\varepsilon^*,b_1}(j)\,\mathring{I}_{\varepsilon^*,b_2}(k)\right) &= 0 \quad\text{if } b_1\neq b_2\\ \mathcal{E}^*\left(\mathring{I}_{\varepsilon^*,b}(j)\,\mathring{I}_{\varepsilon^*,b}(k)\right) &= \widehat{\sigma}_{\varepsilon}^2(b)\,\mathcal{I}(j=k)+\frac{1}{n}Cum^*\left(\varepsilon_t^*;\varepsilon_t^*;\varepsilon_t^*;\varepsilon_t^*\right),\end{aligned}$$

by standard arguments. So, the latter two displayed expressions imply that $\overline{\dot{I}}_{\varepsilon^*}(j)$ and $\overline{\ddot{I}}_{\varepsilon^*}(k)$ are independent and hence

$$\begin{aligned}&\mathcal{E}^*\left(\frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\overline{\ddot{I}}_{\varepsilon^*}(j)\sum_{k=1}^{2}\overline{\dot{I}}^k_{\varepsilon^*}(j)\right)^2\\ &= \frac{1}{\tilde{n}^2}\sum_{j,k=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\left(\frac{1}{\mathbf{B}^2}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\mathcal{E}^*\left(\mathring{I}_{\varepsilon^*,b}(j)\,\mathring{I}_{\varepsilon^*,b}(k)\right)\right)\mathcal{E}^*\left(\sum_{k=1}^{2}\overline{\dot{I}}^k_{\varepsilon^*}(j)\right)\left(\sum_{k=1}^{2}\overline{\dot{I}}^k_{\varepsilon^*}(k)\right)\\ &= o\left(T^{-1}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^2\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)^2\right)H_n.\end{aligned}$$

Also we have that $\mathcal{E}^*\left(\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\overline{\ddot{I}}^2_{\varepsilon^*}(j)\,\overline{\dot{I}}_{\varepsilon^*}(j)\right)^2=0$.

Next, to finish that the contribution due to $k=1,2$ in (5.27), we need to examine

$$\begin{aligned}&\mathcal{E}^*\left(\frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\overline{\ddot{I}}_{\varepsilon^*}(j)\sum_{k=1}^{2}\overline{\ddot{I}}^k_{\varepsilon^*}(j)\right)^2\\ &= \mathcal{E}^*\left(\sum_{k=1}^{2}\frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\overline{\ddot{I}}^{k+1}_{\varepsilon^*}(j)\right)^2\\ &= \mathcal{E}^*\left(\frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\overline{\ddot{I}}^2_{\varepsilon^*}(j)\right)^2+o\left(T^{-1}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^2\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)^2\right)H_n\end{aligned}$$

because by Condition $C3$, $\mathbf{B}^{-3}=o\left(T^{-1}\right)$ and Lemma 9. But by Lemma 9 and standard arguments the right side of the last displayed expression is

$$\frac{1}{\mathbf{B}^2}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^2\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)^2\mathcal{E}^*I^2_{\varepsilon^*}(j)+o\left(T^{-1}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^2\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)^2\right)H_n.$$

Now, proceeding similarly with

$$\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\left(\frac{\mathbf{b}^*}{\mathbf{B}}\overline{\mathring{I}}_{\varepsilon^*,b}(j)\right)\sum_{k=1}^{2}\overline{\mathring{I}}^k_{\varepsilon^*}(j)$$

we have that its second moments are

$$\frac{1}{\mathbf{B}^2}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^2\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)^2\mathcal{E}^* I_{\varepsilon^*}^2(j)+o\left(T^{-1}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^2\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)^2\right)H_n$$

and hence (5.27) is

$$\frac{1}{\tilde{n}\mathbf{B}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}\left\{\mathring{I}_{\varepsilon^*,b}(j)-\overline{\mathring{I}}_{\varepsilon^*}(j)\right\}+o\left(T^{-1}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^2\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)^2\right)H_n,$$

which implies that

$$[T/2]^{1/2}\mathcal{T}_{n,\mathbf{B}}^*\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)=\frac{1}{[T/2]^{1/2}}\sum_{j=1}^{\mathbf{j}^*}\sum_{b=1}^{\mathbf{b}^*}\left\{\mathring{I}_{\varepsilon^*,b}(j)-\overline{\mathring{I}}_{\varepsilon^*}(j)\right\}+o_{p^*}(1).$$

The proof is then completed if

$$\breve{\mathcal{T}}_{n,\mathbf{B}}^*\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)=\frac{1}{\tilde{n}}\sum_{j=1}^{\mathbf{j}^*}\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{b}^*}\mathring{I}_{\varepsilon^*,b}(j)\overset{weakly}{\Longrightarrow}\mathcal{BW}\left([0,1]^2\right)\qquad \textit{(in probability).}$$

Now $\mathcal{E}^*\breve{\mathcal{T}}_{n,\mathbf{B}}^*\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)=0$, whereas by independence of the sequence $\left\{\mathring{I}_{\varepsilon^*,b}(j)\right\}_{b=1}^{\mathbf{B}}$,

$$\begin{aligned}&[T/2]\,\mathcal{E}^*\left(\breve{\mathcal{T}}_{n,\mathbf{B}}^*\left(\frac{\mathbf{b}_1^*}{\mathbf{B}};\frac{\mathbf{j}_1^*}{\tilde{n}}\right)\breve{\mathcal{T}}_{n,\mathbf{B}}^*\left(\frac{\mathbf{b}_2^*}{\mathbf{B}};\frac{\mathbf{j}_2^*}{\tilde{n}}\right)\right)\\ =\;&\frac{(\mathbf{b}_1^*\wedge\mathbf{b}_2^*)}{[T/2]}\sum_{j=1}^{\mathbf{j}_1^*}\sum_{k=1}^{\mathbf{j}_2^*}\left(\mathcal{E}^*\left(\varepsilon_t^{*2}\right)\mathcal{I}(j=k)+\frac{1}{n}Cum^*\left(\varepsilon_t^*;\varepsilon_t^*;\varepsilon_t^*;\varepsilon_t^*\right)\right)\\ &\overset{P}{\to}(\boldsymbol{\omega}_1^*\wedge\boldsymbol{\omega}_2^*)\left(\boldsymbol{v}_1^*+\frac{1}{2}\boldsymbol{v}_1^*\boldsymbol{v}_2^*\kappa_4\right).\end{aligned}$$

Finally the tightness of $[T/2]^{1/2}\breve{\mathcal{T}}_{n,\mathbf{B}}^*\left(\frac{\mathbf{b}^*}{\mathbf{B}};\frac{\mathbf{j}^*}{\tilde{n}}\right)$, for which a sufficient condition is that

$$[T/2]^2\,\mathcal{E}^*\left(\breve{\mathcal{T}}_{n,\mathbf{B}}^{*4}\left(\frac{(\mathbf{j}_2^*-\mathbf{j}_1^*)}{\tilde{n}},\frac{(\mathbf{b}_2^*-\mathbf{b}_1^*)}{\mathbf{B}}\right)\right)=(\boldsymbol{\omega}_1^*-\boldsymbol{\omega}_2^*)^{1+\delta}(\boldsymbol{v}_1^*-\boldsymbol{v}_2^*)^{1+\delta}H_n.$$

But this proceeds by Lemma 7 in a standard way. ■

APPENDIX III: AUXILIARY LEMMAS

Before we present our lemmas, it is useful to introduce some notation. First from (2.14) and (2.15) we have that

(5.29) $$R_{n,b}(j)=\frac{\mathcal{B}(-j)}{|\mathcal{B}(-j)|^2}\mathcal{J}_{\varepsilon,b}(j)\,\mathrm{Y}_{n,b}(-j;0)+\frac{\mathcal{B}(j)}{|\mathcal{B}(j)|^2}\mathcal{J}_{\varepsilon,b}(-j)\,\mathrm{Y}_{n,b}(j;0)+|\mathrm{Y}_{n,b}(j;0)|^2,$$

where

(5.30) $$\mathrm{Y}_{n,b}(j;0)=\mathrm{Y}_{n,b}^{(1)}(j)+\mathrm{Y}_{n,b}^{(2)}(j)$$

with $\mathcal{U}_{n\ell,b}(j)=\left\{\sum_{t=1-\ell}^{n-\ell}-\sum_{t=1}^{n}\right\}\varepsilon_{t+(b-1)n}e^{it\lambda_j}$,

$$\mathrm{Y}_{n,b}^{(1)}(j)=\frac{1}{n^{1/2}}\sum_{\ell=0}^{n}\beta(\ell)\,e^{-i\ell\lambda_j}\mathcal{U}_{n\ell,b}(j);\quad \mathrm{Y}_{n,b}^{(2)}(j)=\frac{1}{n^{1/2}}\sum_{\ell=n+1}^{\infty}\beta(\ell)\,e^{-i\ell\lambda_j}\mathcal{U}_{n\ell,b}(j).$$

Also, we shall denote

(5.31) $$\mathrm{Z}_{n,b}^{(k)}(-j)=\mathcal{J}_{\varepsilon,b}(j)\,\mathrm{Y}_{n,b}^{(k)}(-j),\quad k=1,2.$$

Lemma 1. *Assuming C1 and C3, under the null hypothesis we have that*

$$\begin{aligned}
(\mathbf{a}) \quad \left|\mathcal{E}\left(\prod_{q=1}^{4} \mathrm{Z}_{n,b}^{(1)}(j_q)\right)\right| &= \frac{d}{n^2}\left\{\frac{1}{n^2}+\mathcal{I}(j_1=j_2)\,\mathcal{I}(j_3=j_4)\right\};\ d<\infty \\
(\mathbf{b}) \quad \left|\mathcal{E}\left(\mathrm{Z}_{n,b}^{(1)}(j_1)\,\mathrm{Z}_{n,b}^{(1)}(j_2)\right)\right| &= \frac{d}{n}\left\{\frac{1}{n}+\mathcal{I}(j_1=j_2)\right\} \\
(\mathbf{c}) \quad \left|\mathcal{E}\left(\prod_{q=1}^{2p} \mathrm{Z}_{n,b}^{(2)}(j_q)\right)\right| &= o\left(n^{-2p}\right), \qquad p=1,2.
\end{aligned}$$

Proof. We begin with part (**c**). We shall look at $p=2$ being the case for $p=1$ similarly handled. Because $\sum_{\ell=n+1}^{\infty}|\beta(\ell)| = o\left(n^{-1}\right)$ by $C1$, and definition of $\mathrm{Z}_{n,b}^{(2)}(-j)$, we have that the left side of the expression is bounded by

$$\begin{aligned}
&\sum_{\ell_1,\ldots,\ell_4=n+1}^{\infty}\left|\prod_{q=1}^{4}\beta(\ell_q)\right|\left|\mathcal{E}\left(\prod_{q=1}^{4}\mathcal{J}_{\varepsilon,b}(j_q)\,\frac{1}{n^{1/2}}\mathcal{U}_{n\ell_1,b}(j_q)\right)\right| \\
&= o\left(n^{-4}\right)\mathcal{E}\left|\prod_{q=1}^{4}\mathcal{J}_{\varepsilon,b}(j_q)\,\frac{1}{n^{1/2}}\left\{\sum_{t=1-\ell}^{n-\ell}-\sum_{t=1}^{n}\right\}\varepsilon_{t+(b-1)n}e^{it\lambda_{j_q}}\right| \\
&= o\left(n^{-4}\right),
\end{aligned}$$

because $\sum_{\ell=0}^{n}\ell\,|\beta(\ell)|<\infty$ by Condition 1. Next part (**a**), which by definition is

$$\begin{aligned}
&\frac{1}{n^2}\sum_{\ell_1,\ldots,\ell_4=0}^{n}\left|\prod_{q=1}^{4}\beta(\ell_q)\right|\left|E\left(\prod_{q=1}^{4}\mathcal{U}_{n\ell_1,b}(j_q)\,\mathcal{J}_{\varepsilon,b}(j_q)\right)\right| \\
&= \frac{d}{n^2}\sum_{\ell_1,\ldots,\ell_4=0}^{n}\left|\prod_{q=1}^{4}\beta(\ell_q)\,\ell_q\right|\left\{\frac{1}{n^2}+\left|E\left(\prod_{q=1}^{4}\mathcal{J}_{\varepsilon,b}(j_q)\right)\right|\right\},
\end{aligned}$$

since $\mathcal{U}_{n\ell,b}(j)=\left\{\sum_{t=1-\ell}^{0}-\sum_{t=n-\ell+1}^{n}\right\}\varepsilon_{t+(b-1)n}e^{it\lambda_j}$ when $\ell\le n$, so that $\mathcal{E}\left(\mathcal{U}_{n\ell,b}(j_q)\,\mathcal{J}_{\varepsilon,b}(j_q)\right) = O\left(\ell/n^{1/2}\right)$. Now conclude because $\sum_{\ell=0}^{n}\ell\,|\beta(\ell)|<\infty$. Finally the proof of part (**b**) proceeds similarly. ∎

Lemma 2. *Assuming C1 and C3, under the null hypothesis we have that,* $q=1,2$,

$$\mathcal{E}\left(\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}R_{n,b}(j)\right)^{2q} = O\left(\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}^2 n}\right)^{q}+\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}n}\right)^{2q}\right) \tag{5.32}$$

$$\mathcal{E}\sup_{j}\left|\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}R_{n,b}(j)\right| = O\left(\frac{(\mathbf{b}_2^*-\mathbf{b}_1^*)^{\frac{1}{2}}}{\mathbf{B}}+\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}n}\right) \tag{5.33}$$

Proof. Because Brockwell and Davis's (1991) Theorem 10.3.1 and then $C1$ and H_0 imply that $\mathcal{E}\left(R_{n,b}(j)\right)=O\left(n^{-1}\right)$ and $\mathcal{E}\left|\mathrm{Y}_{n,b}(j;0)\right|^{4q}=O\left(n^{-2q}\right)$, (5.32) and (5.33) hold true if

$$\mathcal{E}\left(\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\left(\mathrm{Z}_{n,b}^{(k)}(-j)-\mathcal{E}\mathrm{Z}_{n,b}^{(k)}(-j)\right)\right)^{2q} = O\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}^2 n}\right)^{q} \quad k=1,2;\ \ q=1,2 \tag{5.34}$$

$$\sup_{1\le j\le \tilde n}\left|\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\left(\mathrm{Z}_{n,b}^{(k)}(-j)-\mathcal{E}\mathrm{Z}_{n,b}^{(k)}(-j)\right)\right| = O_p\left(\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}^2}\right)^{1/2}\right),\quad k=1,2. \tag{5.35}$$

Recall that $C^{-1} < |\mathcal{B}(\lambda)|^2 < C$ for some positive finite constant C. But (5.34) follows by Lemma 1 and that $\mathrm{Z}_{n,b_1}^{(1)}(j)$ and $\mathrm{Z}_{n,b_2}^{(1)}(-k)$ are independent if $b_1 \neq b_2$ by Condition $C1$.

Next we examine (5.35) which follows easily because its second moment is bounded by

$$\sum_{j=1}^{\tilde n}\mathcal{E}\left|\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\left(\mathrm{Z}_{n,b}^{(k)}(-j;0)-\mathcal{E}\mathrm{Z}_{n,b}^{(k)}(-j;0)\right)\right|^2 \qquad k=1,2.$$

This completes the proof of the lemma. ∎

Lemma 3. *Assuming C1 and C3, under the null hypothesis we have that for* $q=1,2$*,*

$$\mathcal{E}\left(\frac{1}{\tilde n}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\breve{R}_{n,b}(j)\right)^{2q} = O\left(\frac{1}{n^{2q}}\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde n}\right)^{1+\delta}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^{1+\delta}\right). \tag{5.36}$$

Proof. We examine $q=1$, the proof for $q=2$ is similarly handled. By (5.29), (5.36) holds true if it is so for the second moments of

$$\frac{1}{\tilde n}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\left(|\mathrm{Y}_{n,b}(j;0)|^2-\mathcal{E}|\mathrm{Y}_{n,b}(j;0)|^2\right) \tag{5.37}$$

$$\frac{1}{\tilde n}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\left(\mathrm{Z}_{n,b}^{(k)}(-j;0)-\mathcal{E}\mathrm{Z}_{n,b}^{(k)}(-j;0)\right) \tag{5.38}$$

for $k=1,2$. Following Brockwell and Davis's (1991) Theorem 10.3.2., the second moment of (5.37) satisfies the right side of (5.36) with $\delta=1$ there. Next (5.38) when $k=2$. Because $\sum_{\ell=n}^{\infty}|\beta(\ell)| = o(n^{-1})$ by Condition $C1$ and that, say

$$\frac{1}{(\tilde n\mathbf{B})^{1/2}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\left(\mathcal{J}_{\varepsilon,b}(j)\frac{1}{n^{1/2}}\sum_{t=1}^{n}\varepsilon_{t+(b-1)n}e^{-it\lambda_j}-\mathcal{E}(\cdot)\right)$$

converge to a Gaussian process, we have that (5.38) satisfies the right side of (5.36). Observe that the sequence is uniform integrable, by Serfling (1980), we have that the second moment of the sequence converges to that of the limiting distribution.

Finally (5.37) when $k=1$. Because $\sum_{\ell=1}^{\infty}\ell|\beta(\ell)| < C$ it suffices to show

$$\mathcal{E}\left(\frac{1}{\tilde n^{3/2}\mathbf{B}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\left(\mathcal{J}_{\varepsilon,b}(j)\widetilde{\mathcal{J}}_{\varepsilon,b}(j,\ell)-\mathcal{E}\left(\mathcal{J}_{\varepsilon,b}(j)\widetilde{\mathcal{J}}_{\varepsilon,b}(j,\ell)\right)\right)\right)^2,$$

where $\widetilde{\mathcal{J}}_{\varepsilon,b}(j,\ell) = \ell^{-1}\sum_{t=n-\ell}^{n}\varepsilon_{t+(b-1)n}e^{-it\lambda_j}$ satisfies the right side of (5.36). But using Lemma 1 part (**b**), we have that it is

$$\begin{aligned} & O\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde n^3}\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}^2\ell}\right)+O\left(\frac{(\mathbf{j}_2^*-\mathbf{j}_1^*)^2}{\tilde n^4}\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}^2}\right) \\ = \;& O\left(\frac{1}{n^2}\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde n}\right)^{1+\delta}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^{1+\delta}\right),\end{aligned}$$

for some $\delta > 0$. ∎

Lemma 4. *Assuming C1 − C3, we have that,* $q = 1, 2$,

$$\mathcal{E}\left(\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} R_{n,b}(j)\right)^{2q} = O\left(\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}^2 n}\right)^{q} + \left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}n}\right)^{2q}\right) \tag{5.39}$$

$$\sup_{1\le j\le \tilde{n}}\left|\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} R_{n,b}(j)\right| = O\left(\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}^2}\right)^{1/2} + \frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}n}\right) \tag{5.40}$$

Proof. We examine $q = 1$, the proof for $q = 2$ is similarly handled. First recall our decomposition in (2.16), that is

$$\mathcal{J}_{x,b}(j) = \mathcal{B}\left(\frac{n(b-1)}{T}; -j\right)\mathcal{J}_{\varepsilon,b}(j) + \check{\Upsilon}_{n,b}(j;0) + \ddot{\Upsilon}_{n,b}(j). \tag{5.41}$$

Now by definition, i.e. (2.18), and using (5.10) we have that

$$\ddot{\Upsilon}_{n,b+1}(j) = \frac{1}{n^{1/2}}\sum_{t=1}^{n}\sum_{\ell=0}^{\infty}\left(\dot{\beta}_{t+bn,T}(\ell) + \ddot{\beta}\left(\frac{t+nb}{T};\ell\right)\right)\varepsilon_{t+bn-\ell}e^{it\lambda_j},$$

and so its contribution into (5.39) and (5.40) satisfies their right sides.

Proceeding as in the proof of Lemma 2 but with $f(j)$ replaced by $\left|\mathcal{B}\left(\frac{n(b-1)}{T};j\right)\right|^2$ and $\mathcal{B}(u;j)$ given in (1.4), we have that the contribution due to the second term on the right of (5.41) satisfies the statement of the lemma. Notice that there is no difference whether we have that the MA representation of the process has weights $\beta(u;\ell)$ or $\beta(\ell)$ as both sequences satisfy the same qualitative condition $\sum_{\ell=0}^{\infty}\ell\left|\beta(u;\ell)\right| < \infty$. ∎

Lemma 5. *Assuming C1 − C3, we have that for* $q = 1, 2$,

$$\mathcal{E}\left(\frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\breve{R}_{n,b}(j)\right)^{2q} = O\left(\frac{1}{n^{2q}}\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)^{1+\delta}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^{1+\delta}\right).$$

Proof. We examine $q = 1$, the proof for $q = 2$ is similarly handled. In view of (5.41) and comments in Lemma 4, it suffices to show

$$\mathcal{E}\left(\frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\ddot{R}_{n,b}(j;0,\infty)\right)^{2} = O\left(\frac{1}{n^{2}}\left(\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}}\right)^{1+\delta}\left(\frac{\mathbf{b}_2^*-\mathbf{b}_1^*}{\mathbf{B}}\right)^{1+\delta}\right), \tag{5.42}$$

where

$$\begin{aligned}\ddot{R}_{n,b}(j;q_1,q_2) &= \left|\frac{1}{n^{1/2}}\sum_{t=1}^{n}\left(\sum_{\ell=q_1}^{q_2}\dot{\beta}_{t,T}(\ell)\varepsilon_{t+(b-1)n-\ell}\right)e^{it\lambda_j}\right|^2 \\ &\quad -\mathcal{E}\left|\frac{1}{n^{1/2}}\sum_{t=1}^{n}\left(\sum_{\ell=q_1}^{q_2}\dot{\beta}_{t,T}(\ell)\varepsilon_{t+(b-1)n-\ell}\right)e^{it\lambda_j}\right|^2.\end{aligned}$$

By standard inequalities, the left side of (5.42) is bounded by

$$\begin{aligned}&\mathcal{E}\left(\frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\ddot{R}_{n,b}(j;0,n)\right)^{2} + \mathcal{E}\left(\frac{1}{\tilde{n}}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\ddot{R}_{n,b}(j;n,\infty)\right)^{2}. \\ \le\ &\frac{\mathbf{j}_2^*-\mathbf{j}_1^*}{\tilde{n}^2}\sum_{j=\mathbf{j}_1^*+1}^{\mathbf{j}_2^*}\left\{\mathcal{E}\left(\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\ddot{R}_{n,b}(j;0,n)\right)^{2} + \mathcal{E}\left(\frac{1}{\mathbf{B}}\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*}\ddot{R}_{n,b}(j;n,\infty)\right)^{2}\right\}.\end{aligned}$$

Now the proof proceeds straightforwardly after noticing that if $b_1 \neq b_2$ we have that $\mathcal{E}\left(\ddot{R}_{n,b_1}(j;0,n)\,\ddot{R}_{n,b_2}(j;0,n)\right)=0$, $\sum_{\ell>n}\upsilon(\ell)<Cn^{-1}$ and $\left|\dot{\beta}_{t,T}(\ell)\right|\leq C\upsilon(\ell)/n^{-1/2}$ by (5.10). Details are omitted. ∎

Let $|p|_+=\max\{1,|p|\}$ and denote

$$\begin{aligned}\psi_{k,n}(j) &=: \sum_{p=1}^{\tilde n}\varsigma_{pj}\overline{\mathring{I}}_{\varepsilon}^{k}(p),\quad \varphi_{k,n}(j)=:\sum_{p=1}^{\tilde n}\varsigma_{pj}\overline{R}_n^{k}(p)\quad k=1,2,3\\ \Xi_n(j) &= \sum_{k=1}^{3}\frac{(-1)^k}{k!}\left(\psi_{k,n}(j)+\varphi_{k,n}(j)\right)+\Phi_{n,1}\Phi_{n,2}(j),\end{aligned}$$

where $\varsigma_{pj}=\left(|p-j|_+^{-1}+|p+j|^{-1}\right)$, $\Phi_{n,1}$ is a sequence of $O_p(1)$ r.v. independent of $\mathbf{j}^*$ and $\mathbf{b}^*$ and $\mathcal{E}\Phi_{n,2}^2(j)=O\left(\mathbf{B}^{-2}\right)$. Also,

$$\mathcal{A}_n(j)=:\exp\left\{\sum_{\ell=1}^{\tilde n}c_{\ell,n}e^{-i\ell\lambda_j}\right\};\quad c_{\ell,n}=\frac{1}{\tilde n}\sum_{p=1}^{\tilde n}\log f(p)\cos(\ell\lambda_p).$$

Lemma 6. *Assuming C1 and C3, under H_0 we have that*

$$\text{(a)}\ \widehat{\mathcal{A}}(j)-\mathcal{A}_n(j) = \mathcal{A}(j)\,\Xi_n(j)+\frac{1}{2}|\mathcal{A}(j)|^2(\Xi_n(j))^2 \tag{5.43}$$

$$\text{(b)}\ \mathcal{A}_n(j)-\mathcal{A}(j) = \mathcal{A}(j)\frac{\log f(0)}{\tilde n}\sum_{\ell=1}^{\tilde n}e^{-i\ell\lambda_j}+O\left(n^{-2}\right). \tag{5.44}$$

Proof. First because Taylor's expansion of $\log z$ yields that

$$\begin{aligned}\widehat{c}_\ell-c_{\ell,n} &= \frac{1}{\tilde n}\sum_{p=1}^{\tilde n}\sum_{k=1}^{3}\frac{(-1)^k}{k!}\left(\frac{\widehat{f}(p)-f(p)}{f(p)}\right)^k\cos(\ell\lambda_p)\\ &\quad+\frac{1}{4!\tilde n}\sum_{p=1}^{\tilde n}\left(\frac{\widehat{f}(p)-f(p)}{\varkappa f(p)+(1-\varkappa)\widehat{f}(p)}\right)^4\cos(\ell\lambda_p),\end{aligned}\tag{5.45}$$

where $\varkappa=:\varkappa(p)\in(0,1)$, so that

$$\begin{aligned}\log\left(\widehat{\mathcal{A}}(j)/\mathcal{A}_n(j)\right) &= \sum_{k=1}^{3}\frac{(-1)^k}{k!}\sum_{p=1}^{\tilde n}\varsigma_{pj}\left(\frac{\widehat{f}(p)-f(p)}{f(p)}\right)^k\\ &\quad+\frac{1}{4!}\sum_{p=1}^{\tilde n}\left(\frac{\widehat{f}(p)-f(p)}{\varkappa f(p)+(1-\varkappa)\widehat{f}(p)}\right)^4\end{aligned}\tag{5.46}$$

because $\sum_{\ell=1}^{\tilde n}\cos(\ell\lambda_p)e^{-i\ell\lambda_j}=\tilde n\varsigma_{pj}$. The second term on the left of (5.46) is $\Phi_{n,1}\Phi_{n,2}(j)$, where $\Phi_{n,1}=O_p(1)$ and $\mathcal{E}\left|\Phi_{n,2}(j)\right|^2=O\left(\mathbf{B}^{-3}\right)$ uniformly in j. Indeed Lemma 2 and $\sup_{p=1,\ldots,\tilde n}|a_p|\leq\left(\sum_{p=1}^{\tilde n}|a_p|^q\right)^{1/q}$ imply that

$$\begin{aligned}\mathcal{E}\sup_{p=1,\ldots,\tilde n}\left|\frac{\widehat{f}(p)-f(p)}{f(p)}\right| &\leq \left(\sum_{p=1}^{\tilde n}\mathcal{E}\left|\overline{R}_n(p)\right|^4\right)^{1/4}+\left(\sum_{p=1}^{\tilde n}\mathcal{E}\left(\overline{\mathring{I}}_{\varepsilon,b}(p)\right)^4\right)^{1/4}\\ &= O\left(\mathbf{B}^{-1/2}n^{1/4}\right)=o(1)\end{aligned}\tag{5.47}$$

by Condition $C3$ and $\mathcal{E}\left(f^{-1}(p)\widehat{f}(p)-1\right)^{4}=O\left(\mathbf{B}^{-2}\right)$ by standard arguments. Next regarding the first term of (**??**), we have that because, say, $\mathcal{E}\left|\overline{\mathring{I}}_{\varepsilon}(p)\overline{R}_{n}(p)\right|^{2}=O\left(\mathbf{B}^{-2}T^{-1/2}\right)=o\left(\mathbf{B}^{-3}\right)$ by Lemma 2 and $C3$ and that $\sup_{p}\left\{\left|\overline{\mathring{I}}_{\varepsilon}(p)\right|+\left|\overline{R}_{n}(p)\right|\right\}=o_{p}(1)$, we obtain that it is $\Xi_{n}(j)$. Now we conclude the proof of part (**a**) by Taylor's expansion of $\exp z$.

Next part (**b**). To that end, because $\log f(\lambda)$ is three times continuously differentiable, exercise 1.7.14 part (*b*) in Brillinger (1981) implies that $c_{\ell,n}-c_{\ell}=\frac{\log f(0)}{\tilde{n}}+O\left(n^{-3}\right)$, and then we conclude that, uniformly in j,

$$\begin{aligned}\log\left(\mathcal{A}_{n}(j)/\mathcal{A}(j)\right) &= \sum_{\ell=1}^{\tilde{n}}\left(c_{\ell,n}-c_{\ell}\right)e^{-i\ell\lambda_{j}}-\sum_{\ell=\tilde{n}+1}^{\tilde{n}}c_{\ell}e^{-i\ell\lambda_{j}}\\ &= \frac{\log f(0)}{\tilde{n}}\sum_{\ell=1}^{\tilde{n}}e^{-i\ell\lambda_{j}}+O\left(n^{-2}\right).\end{aligned}\tag{5.48}$$

Now using (5.48) instead of (**??**), we obtain part (**b**). This concludes the proof. ∎

Lemma 7. *Assuming C1 and C2, we have that under H_0 for all $b=1,...,\mathbf{B}$,*

$$\widehat{\sigma}_{\varepsilon}^{2}(b)-\widetilde{\sigma}_{\varepsilon}^{2}(b)=\frac{d_{1,n}}{n}\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\psi_{1,n}(j)\mathring{I}_{\varepsilon,b}(j)+\frac{d_{2,n}}{\mathbf{B}}+\Psi_{n,1}\Psi_{n,2}(b),\tag{5.49}$$

where $\Psi_{n,1}$ $d_{1,n}$ and $d_{2,n}$ are independent of b such that $\Psi_{n,1}=O_{p}(1)$, $\mathcal{E}\left(d_{2,n}^{2}\right)<C$ and $\mathcal{E}\left|\Psi_{n,2}(b)\right|^{2}=O\left(\mathbf{B}^{-3}+n^{-2}\right)$ with $\widetilde{\sigma}_{\varepsilon}^{2}(b)$ given in (5.14)*.*

Proof. First by standard algebra, we have that

$$\widehat{\sigma}_{\varepsilon}^{2}(b)-\widetilde{\sigma}_{\varepsilon}^{2}(b)=\frac{1}{n}\sum_{t=1}^{n}\upsilon_{t,b}^{2}+\frac{2}{n}\sum_{t=1}^{n}\varepsilon_{t+(b-1)n}\upsilon_{t,b},\tag{5.50}$$

where $\upsilon_{t,b}=:\widehat{\varepsilon}_{t+(b-1)n}-\varepsilon_{t+(b-1)n}$, and it is

$$\begin{aligned}\upsilon_{t,b} &= \frac{1}{n^{1/2}}\sum_{j=1}^{n}e^{it\lambda_{j}}\left(\widehat{\mathcal{A}}(j)\mathcal{A}^{-1}(j)-1\right)\mathcal{A}(j)\Upsilon_{n,b}(j;0)\\ &\quad+\frac{1}{n^{1/2}}\sum_{j=1}^{n}e^{it\lambda_{j}}\left(\widehat{\mathcal{A}}(j)\mathcal{A}^{-1}(j)-1\right)\mathcal{J}_{\varepsilon,b}(j)\\ &\quad+\frac{1}{n}\sum_{j=1}^{n}e^{it\lambda_{j}}\mathcal{A}(j)\sum_{s=1}^{n}x_{s+(b-1)n}e^{-is\lambda_{j}}-\varepsilon_{t+(b-1)n}.\end{aligned}\tag{5.51}$$

Using (2.14) and because $\mathcal{A}(j)=\sum_{q=0}^{\infty}\alpha(q)e^{-iq\lambda_{j}}$ and $\sum_{j=1}^{n}e^{-i\ell\lambda_{j}}=n\mathcal{I}(\ell=0,n,...)$, we get that the third term of (5.51), with $b=1$ for notational simplicity, is

$$\begin{aligned}&\sum_{q=0}^{\infty}\alpha(q)\sum_{s=1}^{n}x_{s}\frac{1}{n}\sum_{j=1}^{n}e^{i(t-q-s)\lambda_{j}}-\varepsilon_{t}\\ &=\sum_{\ell=1}^{\infty}\sum_{q=1}^{t-1}\alpha(q+\ell n)x_{t-q}+\left\{\sum_{q=1}^{t-1}\alpha(q)x_{t-q}-\varepsilon_{t}\right\},\end{aligned}\tag{5.52}$$

whose second moment is $o\left((t\log(t+1))^{-2}\right)$. So the contribution due to the third term of (5.51) into (5.50) is such that its second moment is $O\left(n^{-2}\right)$.

Now the contribution due to the first two terms on the right of (5.51) into $\widehat{\sigma}_{\varepsilon}^{2}(b)-\widetilde{\sigma}_{\varepsilon}^{2}(b)$ is

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^{n}\left(\widehat{\mathcal{A}}(j)-\mathcal{A}(j)\right)^{2}\left|\mathrm{Y}_{n,b}(j;0)\right|^{2}\\&+\frac{2}{n}\sum_{j=1}^{n}\left(\widehat{\mathcal{A}}(j)-\mathcal{A}(j)\right)\left(\mathcal{A}^{-1}(j)+\frac{1}{2}\right)\mathcal{J}_{\varepsilon,b}(-j)\,\mathrm{Y}_{n,b}(j;0)\\&+\frac{1}{n}\sum_{j=1}^{n}\left\{\frac{\left(\widehat{\mathcal{A}}(j)-\mathcal{A}(j)\right)^{2}}{\left|\mathcal{A}(j)\right|^{2}}+\left(\widehat{\mathcal{A}}(j)-\mathcal{A}(j)\right)\mathcal{A}^{-1}(j)\right\}I_{\varepsilon,b}(j).\end{aligned}\tag{5.53}$$

The contribution due $\Phi_{n,1}\Phi_{n,2}(j)$ into the first term of (5.53) is

$$\Phi_{n,1}^{2}\frac{1}{n}\sum_{j=1}^{n}\Phi_{n,2}^{2}(j)\left|\mathrm{Y}_{n,b}(j;0)\right|^{2}.$$

Now we identify $\Phi_{n,1}^{2}\sup_{j}\Phi_{n,2}(j)$ with to $\Psi_{n,1}$ and

$$\Psi_{n,2}(b)=:\frac{1}{n}\sum_{j=1}^{n}\left|\Phi_{n,2}(j)\right|\left|\mathrm{Y}_{n,b}(j;0)\right|^{2}$$

noticing that $\mathcal{E}\Psi_{n,2}^{2}(b)=O\left(T^{-2}\right)=o\left(n^{-2}+\mathbf{B}^{-3}\right)$ because $\mathcal{E}\left|\mathrm{Y}_{n,b}(j;0)\right|^{2p}=O\left(n^{-p}\right)$ and Cauchy-Schwarz's inequality. Next the contribution due to $\psi_{k,n}(j)+\varphi_{k,n}(j)$, for $k=1,2,3$. Now since $(a+b)^{4}\leq 8\left(a^{4}+b^{4}\right)$ and $\mathcal{E}\left|\psi_{k,n}(j)\right|^{2}+\mathcal{E}\left|\varphi_{k,n}(j)\right|^{2}=O\left(\mathbf{B}^{-k}\right)$, we have that this contribution is also $\Psi_{n,1}\Psi_{n,2}(b)$. Recall again that $\sup_{j}\left|\psi_{k,n}(j)\right|=o_{p}(1)$ and $\sup_{j}\left|\varphi_{k,n}(j)\right|=o_{p}(1)$.

Next we examine the behaviour of the second term of (5.53). To that end and using (5.31), we first notice that Condition $C1$ implies that $\mathcal{E}\left|\mathrm{Z}_{n,b}^{(k)}(-j)\right|^{2}=O\left(n^{-1}\right)$, for $k=1,2$, and hence that

$$\mathcal{E}\left|\frac{\log f(0)}{\tilde{n}}\frac{1}{n}\sum_{j=1}^{n}\mathrm{Z}_{n,b}^{(k)}(-j)\sum_{\ell=1}^{\tilde{n}}e^{-i\ell\lambda_{j}}\right|^{2}=O\left(\frac{\log^{2}n}{n^{3}}\right)$$

by standard arguments. Next because $\sup_{j}\left|\psi_{k,n}(j)\right|=o_{p}(1)$ and $\sup_{j}\left|\varphi_{k,n}(j)\right|=o_{p}(1)$, we have that $\psi_{k,n}^{4}(j)=\xi_{n,1}\xi_{n,2}$, where $\xi_{n,1}=O_{p}(1)$ and $\mathcal{E}\left(\xi_{n,2}\right)^{2}=O\left(\mathbf{B}^{-3}\right)$, we have then that the second term of (5.53), except multiplicative constants, is

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^{n}\psi_{1,n}(j)\left(\mathrm{Z}_{n,b}^{(1)}(-j)+\mathrm{Z}_{n,b}^{(2)}(-j)\right)+\Psi_{n,1}\Psi_{n,2}(b)\\=\;&\frac{1}{n}\sum_{j=1}^{n}\psi_{1,n}(j)\,\mathrm{Z}_{n,b}^{(1)}(-j)+\Psi_{n,1}\Psi_{n,2}(b),\end{aligned}\tag{5.54}$$

as $\mathcal{E}\left(\varphi_{k,n}^{2}(j)\right)=O\left(T^{-k}\right)$. Now proceeding as with the proof of Lemma 2 and using the definition of $\psi_{1,n}(j)$, it suffices to examine the behaviour of

$$\begin{aligned}&\frac{1}{n^{1/2}\mathbf{B}}\frac{1}{n}\sum_{j=1}^{n}\mathring{I}_{\varepsilon,b}(j)\,\mathcal{J}_{\varepsilon,b}(-j)\frac{1}{\ell^{1/2}}\sum_{t=n-\ell+1}^{n}\varepsilon_{t+(b-1)n}e^{it\lambda_{j}}\\&+\frac{1}{n^{1/2}}\frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{\mathbf{B}}\sum_{b_{1}\neq b}^{\mathbf{B}}\mathring{I}_{\varepsilon,b_{1}}(j)\right)\left(\mathcal{J}_{\varepsilon,b}(-j)\frac{1}{\ell^{1/2}}\sum_{t=n-\ell+1}^{n}\varepsilon_{t+(b-1)n}e^{it\lambda_{j}}\right).\end{aligned}$$

The second moment of the first term of the last displayed expression is clearly $O\left(n^{-1}\mathbf{B}^{-2}\right)$, whereas the second term is $O\left(n^{-2}\mathbf{B}^{-1}\right)$ because the first factor in parenthesis is independent of the second one and

$$\mathcal{E}\left(\mathring{I}_{\varepsilon,b_1}(j)\,\mathring{I}_{\varepsilon,b_2}(k)\right)=\mathcal{I}(b_1=b_2)\left(\mathcal{I}(j=k)+n^{-1}\kappa_4\right). \tag{5.55}$$

To finish the proof of the lemma, we shall now examine the third term of (5.53). First using (5.43) and that $\mathcal{E}\left|\psi_{k,n}(j)\right|^{2p}+\mathcal{E}\left|\varphi_{k,n}(j)\right|^{2p}=O\left(\mathbf{B}^{-3}\right)$ when $p+k\geq 3$, the third term of (5.53) is

$$\frac{1}{n}\sum_{j=1}^{n}\left\{\sum_{k=1}^{2}\frac{d_{k,n}}{\mathcal{A}(j)}\left(\psi_{k,n}(j)+\varphi_{k,n}(j)\right)+d\left(\psi_{1,n}^{2}(j)+\varphi_{1,n}^{2}(j)\right)\right\}I_{\varepsilon,b}(j)+\Psi_n, \tag{5.56}$$

$d_{1,n}$ and $d_{2,n}$ independent of b and finite second moments and $d\geq\left|\mathcal{A}(j)\right|^{-2}$ finite. Now, because $\mathcal{E}\left|\mathbf{B}\psi_{2,n}(j)\right|=O(1)$, we have that

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\frac{1}{\mathcal{A}(j)}\psi_{2,n}(j)\,I_{\varepsilon,b}(j) &= \frac{d_{2,n}}{\mathbf{B}}+\frac{1}{n}\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\,\psi_{2,n}(j)\,\mathring{I}_{\varepsilon,b}(j)\\ &= \frac{d_{2,n}}{\mathbf{B}}+\Psi_n,\end{aligned} \tag{5.57}$$

as we now show. Indeed, the second term on the right of (5.57) is

$$\begin{aligned}&\frac{1}{\mathbf{B}^2 n}\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\sum_{p=1}^{\tilde{n}}\varsigma_{pj}\mathring{I}_{\varepsilon,b}^{2}(p)\,\mathring{I}_{\varepsilon,b}(j)\\ &+\frac{1}{n}\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\sum_{p=1}^{\tilde{n}}\varsigma_{pj}\left(\frac{1}{\mathbf{B}}\sum_{b_1\neq b}^{\mathbf{B}}\mathring{I}_{\varepsilon,b_1}(p)\right)^{2}\mathring{I}_{\varepsilon,b}(j)\\ &+\frac{2}{\mathbf{B}n}\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\sum_{p=1}^{\tilde{n}}\varsigma_{pj}\left(\frac{1}{\mathbf{B}}\sum_{b_1\neq b}^{\mathbf{B}}\mathring{I}_{\varepsilon,b_1}(p)\right)\mathring{I}_{\varepsilon,b}(p)\,\mathring{I}_{\varepsilon,b}(j).\end{aligned} \tag{5.58}$$

The second moments of first term of (5.58) is $O\left(\mathbf{B}^{-4}\right)$, whereas the second moment of the third term is $O\left(\mathbf{B}^{-3}\right)$, because $\mathcal{E}\left(\mathring{I}_{\varepsilon,b_1}(p)\,\mathring{I}_{\varepsilon,b}(j)\right)=0$ for all $b_1\neq b$. So, we are left to examine the second term of (5.58). But its second moment is clearly $O\left(\mathbf{B}^{-3}\right)$, since by independence of $\mathring{I}_{\varepsilon,b}(j)$ and $\left(\frac{1}{\mathbf{B}}\sum_{b_1\neq b}^{\mathbf{B}}\mathring{I}_{\varepsilon,b_1}(p)\right)^2$ and (5.55), we have that the second moment is

$$\frac{d}{\mathbf{B}^2 n^2}\sum_{j=1}^{n}\sum_{p=1}^{\tilde{n}}\varsigma_{pj}^{2}\left|\mathcal{A}(j)\right|^{-2}=o\left(\mathbf{B}^{-3}\right)$$

by Condition $C3$.

Next we examine the contribution into (5.56), i.e. the third term of (5.53), due to

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\,\varphi_{2,n}(j)\,I_{\varepsilon,b}(j) &= \frac{d_{2,n}}{\mathbf{B}}+\frac{1}{n}\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\,\varphi_{2,n}(j)\,\mathring{I}_{\varepsilon,b}(j)\\ &= \frac{d_{2,n}}{\mathbf{B}}+\Psi_n,\end{aligned} \tag{5.59}$$

because $\mathcal{E}\varphi_{2,n}^2(j) = O\left(T^{-1}\right) = o\left(\mathbf{B}^{-2}\right)$ by Condition $C3$. Regarding Ψ_n, by definition of $\varphi_{2,n}(j)$, we need to examine

$$\sum_{\ell=1}^{2}\frac{1}{n}\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\,\mathring{I}_{\varepsilon,b}(j)\sum_{p=1}^{\tilde{n}}\varsigma_{pj}\left(\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\mathrm{Z}_{n,b}^{(\ell)}(-p)\right)^2$$
$$+\frac{1}{n}\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\,\mathring{I}_{\varepsilon,b}(j)\sum_{p=1}^{\tilde{n}}\varsigma_{pj}\left(\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\left|\mathrm{Y}_{n,b}(p;0)\right|^2\right)^2.$$

But it is clear that the second moment is $O\left(\mathbf{B}^{-3}+n^{-2}\right)$ because $\mathcal{E}\left|\mathrm{Y}_{n,b}(j;0)\right|^4 = O\left(n^{-2}\right)$,

$$\mathcal{E}\left(\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\mathrm{Z}_{n,b}^{(2)}(-j)\right)^4 = O\left(n^{-4}\right);\quad \mathcal{E}\left(\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\mathrm{Z}_{n,b}^{(1)}(-j)\right)^4 = O\left(n^{-2}\right)$$

by simple inspection of the definition of $\mathrm{Y}_{n,b}(j;0)$, $\mathrm{Z}_{n,b}^{(1)}(j)$ and $\mathrm{Z}_{n,b}^{(2)}(j)$ respectively.

Next, we examine the contribution into (5.56), i.e. the third term of (5.53), due to

$$\frac{1}{n}\sum_{j=1}^{n}\frac{\psi_{1,n}^2(j)+\varphi_{1,n}^2(j)}{\left|\mathcal{A}(j)\right|^2}\mathring{I}_{\varepsilon,b}(j) = \frac{1}{n}\sum_{j=1}^{n}\frac{\mathring{I}_{\varepsilon,b}(j)}{\left|\mathcal{A}(j)\right|^2}\left(\sum_{p=1}^{\tilde{n}}\varsigma_{pj}\overline{\mathring{I}}_{\varepsilon}(p)\right)^2$$
$$+\frac{1}{n}\sum_{j=1}^{n}\frac{\mathring{I}_{\varepsilon,b}(j)}{\left|\mathcal{A}(j)\right|^2}\left(\sum_{p=1}^{\tilde{n}}\varsigma_{pj}\overline{R}_n(p)\right)^2. \tag{5.60}$$

The first term on the right of (5.60) is

$$\frac{1}{n}\sum_{j=1}^{n}\frac{\mathring{I}_{\varepsilon,b}(j)}{\left|\mathcal{A}(j)\right|^2}\left(\sum_{p=1}^{\tilde{n}}\varsigma_{pj}\frac{1}{\mathbf{B}}\sum_{b_1\neq b}^{\mathbf{B}}\mathring{I}_{\varepsilon,b_1}(p)\right)^2$$
$$+\frac{1}{\mathbf{B}^2 n}\sum_{j=1}^{n}\frac{\mathring{I}_{\varepsilon,b}(j)}{\left|\mathcal{A}(j)\right|^2}\left(\sum_{p=1}^{\tilde{n}}\varsigma_{pj}\mathring{I}_{\varepsilon,b}(p)\right)^2$$
$$+\frac{2}{\mathbf{B}n}\sum_{j=1}^{n}\frac{\mathring{I}_{\varepsilon,b}(j)}{\left|\mathcal{A}(j)\right|^2}\sum_{p_1,p_2=1}^{\tilde{n}}\varsigma_{p_1 j}\varsigma_{p_2 j}\mathring{I}_{\varepsilon,b}(p_2)\left(\frac{1}{\mathbf{B}}\sum_{b_1\neq b}^{\mathbf{B}}\mathring{I}_{\varepsilon,b_1}(p_1)\right).$$

Clearly the second moment of the second and third terms are $O\left(\mathbf{B}^{-4}+\mathbf{B}^{-3}\right)$, whereas the second moment of the first term is, by independence of $\mathring{I}_{\varepsilon,b_1}(p)$ and $\mathring{I}_{\varepsilon,b}(p)$ if $b_1 \neq b$, and (5.55) is easy to observe that is $O\left(\mathbf{B}^{-2}n^{-1}\right)$. Next the second term on the right of (5.60) also satisfies that its second moment is $O\left(\mathbf{B}^{-3}+n^{-2}\right)$ using Lemma 3. To complete the proof it remains to examine the behaviour of

$$\frac{1}{n}\sum_{j=1}^{n}\varphi_{1,n}(j)\,\frac{\mathring{I}_{\varepsilon,b}(j)}{\mathcal{A}(j)} \tag{5.61}$$

$$\frac{1}{n}\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\left(\psi_{1,n}(j)+\varphi_{1,n}(j)\right). \tag{5.62}$$

Now (5.61) is

$$\frac{1}{n}\sum_{p=1}^{\tilde{n}}\left(\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\mathrm{Z}_{n,b}^{(1)}(-j)\right)\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\varsigma_{pj}\mathring{I}_{\varepsilon,b}(j)$$
$$+\frac{1}{n}\sum_{p=1}^{\tilde{n}}\left(\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\mathrm{Z}_{n,b}^{(2)}(-j)\right)\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\varsigma_{pj}\mathring{I}_{\varepsilon,b}(j)$$
$$+\frac{1}{n}\sum_{p=1}^{\tilde{n}}\left(\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\left|\mathrm{Y}_{n,b}(j;0)\right|^{2}\right)\sum_{j=1}^{n}\mathcal{A}^{-1}(j)\varsigma_{pj}\mathring{I}_{\varepsilon,b}(j).$$

The second and third terms of the last displayed expression has second moments $O\left(n^{-2}\right)$, whereas the first term proceeding similarly as with (5.54) has a second moment $O\left(\mathbf{B}^{-3}+n^{-2}\right)$. Finally (5.62) which is

$$\frac{1}{n}\sum_{p=1}^{\tilde{n}}\left(\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}\mathring{I}_{\varepsilon,b}(p)\right)+\left(\frac{1}{\mathbf{B}}\sum_{b=1}^{\mathbf{B}}R_{n,b}(p)\right)\sum_{j=1}^{n}\varsigma_{pj}\mathcal{A}^{-1}(j).$$

By (5.55), the first term has second moment proportional to

$$\frac{1}{\mathbf{B}n^{2}}\sum_{p=1}^{\tilde{n}}\sum_{j_1,j_2=1}^{n}\left|\varsigma_{pj_1}\varsigma_{pj_2}\right|=\mathcal{O}\left(\frac{\log^{2}n}{T}\right)=O\left(\mathbf{B}^{-2}\right),$$

whereas by Lemma 2 and Condition $C3$, the second term is also $O\left(\mathbf{B}^{-2}\right)$. This concludes the proof of the lemma. ∎

Lemma 8. *Assuming C1 – C3, we have that for all* $b=1,...,\mathbf{B}$ *and uniformly in* t, $\upsilon_{t,b}=O_p\left((t\log(t+1))^{-1}+n^{1/2}\mathbf{B}^{-1}\right)$.

Proof. By (5.51) and (5.52), it suffices to examine

$$\frac{1}{n^{1/2}}\sum_{j=1}^{n}e^{it\lambda_j}\left(\widehat{\mathcal{A}}(j)\mathcal{A}^{-1}(j)-1\right)\mathcal{A}(j)\mathrm{Y}_{n,b}(j;0)$$
$$+\frac{1}{n^{1/2}}\sum_{j=1}^{n}e^{it\lambda_j}\left(\widehat{\mathcal{A}}(j)\mathcal{A}^{-1}(j)-1\right)\mathcal{J}_{\varepsilon,b}(j).$$

(5.43) and (5.44) imply that the first term of the last displayed expression is $O_p\left(\mathbf{B}^{-1/2}\right)$ uniformly in t as $\mathcal{E}\left(f^{-1}(\ell)\widehat{f}(\ell)-1\right)^{2}=O\left(\mathbf{B}^{-1}\right)$ and Cauchy-Schwarz's inequality, whereas the second term is

$$\frac{1}{n^{1/2}}\sum_{j=1}^{n}e^{it\lambda_j}\psi_{1,n}(j)\mathcal{J}_{\varepsilon,b}(j)+O_p\left(n^{1/2}\mathbf{B}^{-1}\right)$$
$$=\frac{1}{n^{1/2}}\sum_{j=1}^{n}e^{it\lambda_j}\overline{\mathring{I}}_{\varepsilon}(j)\mathcal{J}_{\varepsilon,b}(j)+O_p\left(n^{1/2}\mathbf{B}^{-1}+\mathbf{B}^{-1/2}\right)$$

again uniformly in t, proceeding with arguments in Lemma 7 and Lemma 2. Now, the first term on the right is

$$\frac{1}{n^{1/2}}\sum_{j=1}^{n}e^{it\lambda_j}\left(\frac{1}{\mathbf{B}}\sum_{b_1\neq b}^{\mathbf{B}}\mathring{I}_{\varepsilon,b_1}(j)\right)\mathcal{J}_{\varepsilon,b}(j)+\frac{1}{\mathbf{B}}\frac{1}{n^{1/2}}\sum_{j=1}^{n}e^{it\lambda_j}\mathring{I}_{\varepsilon,b}(j)\mathcal{J}_{\varepsilon,b}(j).$$

But it is easy to see that the fourth moment of the first term is $O\left(\mathbf{B}^{-2}\right)$, whereas the second term has a second moment of order $O\left(\mathbf{B}^{-2}\right)$. So, using that $\sup_{\ell=1,\ldots,\tilde{n}} |a_\ell|^q \leq \sum_\ell |a_\ell|^q$, for $q \geq 1$, we conclude that the last displayed expression is $O\left(n\mathbf{B}^{-2}\right)$ uniformly in t. ∎

Lemma 9. *Assuming C1 − C3, we have that*

$$\mathcal{E}^*\left(\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} \mathring{I}_{\varepsilon^*,b}(j)\right)^{2\ell} = \left(\mathbf{b}_2^* - \mathbf{b}_1^*\right)^\ell H_n, \quad \ell \geq 1, \tag{5.63}$$

$\{H_n\}_{n\geq 1}$ *being a sequence of strictly positive* $O_p(1)$ *random variables*

Proof. Because $\mathring{I}_{\varepsilon^*,b_1}(j)$ and $\mathring{I}_{\varepsilon^*,b_2}(j)$ are independent for $b_1 \neq b_2$, we have that the left side of (5.63) is bounded by

$$\sum_{b=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} \mathcal{E}^*\left(\mathring{I}_{\varepsilon^*,b}(j)\right)^{2\ell} + \binom{2\ell}{2} \sum_{b_1\neq b_2=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} \mathcal{E}^*\left(\mathring{I}_{\varepsilon^*,b_1}(j)\right)^2 \mathcal{E}^*\left(\mathring{I}_{\varepsilon^*,b_2}(j)\right)^{2\ell-2}$$
$$+\ldots+\binom{2\ell}{\ell} \sum_{b_1\neq\ldots\neq b_\ell=\mathbf{b}_1^*+1}^{\mathbf{b}_2^*} \left(\prod_{p=1}^{\ell} \mathcal{E}^*\left(\mathring{I}_{\varepsilon^*,b_p}(j)\right)^2\right).$$

But $\mathcal{E}^*\left(\mathring{I}_{\varepsilon^*,b}(j)\right)^{2\chi} = H_n$ because for all integers $\chi \geq 1$,

$$\mathcal{E}^*\left(\frac{1}{n}\sum_{t=1}^{n} \varepsilon^{*\chi}_{t+(n-1)b}\right) = \frac{1}{T}\sum_{t=1}^{n} \widehat{\varepsilon}_t^\chi$$

and by Theorem 2 and then $C1$, we have that

$$\frac{1}{T}\sum_{t=1}^{n}\left(\widehat{\varepsilon}_t^\chi - \varepsilon_t^\chi\right) = o_p(1); \quad \frac{1}{T}\sum_{t=1}^{n}\left(\varepsilon_t^\chi - \mathcal{E}\varepsilon_t^\chi\right) = O_p\left(\frac{1}{T^{1/2}}\right).$$

This completes the proof of the lemma. ∎

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