ON THE NORMAL SUBGROUPS OF THE GROUP OF VOLUME PRESERVING DIFFEOMORPHISMS OF $\mathbb{R}^n$ FOR $n \geqslant 3$

by Francisca Mascaro de Sivera

A thesis submitted for the degree of Ph.D. at the University of Warwick.

Mathematics Institute

University of Warwick

September 1982.

CONTENTS

to Rafael Sivera

ACKNOWLEDGEMENTS

I wish to express my deep gratitude to , Dr. D. McDuff , for introducing me to this subject and for her constant stimulus and encouragement. Also to my supervisor , Dr. C.P. Rourke , for his help in some difficult moments during the last years.

I would like to thank also the Ministerio de Universidades e Investigación of Spain for its financial support during my stay at Warwick and Vicky Aguilar for her careful typing of this manuscript .

SUMMARY

Let Ω be a volume element on $\mathbb{R}^n$. $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ is the group of Ω-preserving diffeomorphisms of $\mathbb{R}^n$. $\mathrm{Diff}^{\Omega}_{w}(\mathbb{R}^n)$ is the subgroup of all elements whose set of non-fixed points has finite Ω-volume. $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ is the subgroup of all elements whose support has finite Ω-volume. $\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ is the subgroup of all elements with compact suport. $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$ is the subgroup of all elements compactly Ω-isotopic to the identity.

We prove, in case $\mathrm{vol}_{\Omega}\mathbb{R}^n < \infty$ and for $n \geqslant 3$ that a subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$, N, is normal if and only if $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset N \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$. If $\mathrm{vol}_{\Omega}\mathbb{R}^n = \infty$ and for $n \geqslant 3$, there is no normal subgroup neither between $\mathrm{Diff}^{\Omega}_{w}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ nor between $\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$.

INTRODUCTION

The final goal of this dissertation is the study of the normal subgroups of the group of all smooth volume preserving diffeomorphisms of $\mathbb{R}^n$, $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$, for $n \geqslant 3$ and for any volume element Ω.

We were looking for similar results to the one on the group of smooth diffeomorphisms of $\mathbb{R}^n$, $\mathrm{Diff}(\mathbb{R}^n)$, got by Ling in [10] and McDuff in [14] saying that any non-trivial normal subgroup N of $\mathrm{Diff}(\mathbb{R}^n)$ satisfies

$$\mathrm{Diff}_{co}(\mathbb{R}^n) \subset N \subset \mathrm{Diff}_c(\mathbb{R}^n)$$

where $\mathrm{Diff}_c(\mathbb{R}^n)$ is the subgroup of all diffeomorphisms with compact support and $\mathrm{Diff}_{co}(\mathbb{R}^n)$ is the subgroup of all diffeomorphisms compactly isotopic to the identity.

Since the groups of diffeomorphisms of a manifold preserving equivalent volume elements are isomorphic we only have to study the group $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ for non-equivalent volume elements on $\mathbb{R}^n$. Using Moser [18] ,we were able to reduce it two essentialy different cases, the first one when Ω is a volume element on $\mathbb{R}^n$ with finite total volume and another one when Ω has infinite total volume.

In both cases we have the following chain of normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$

$$\{id\} \subset \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$$

where $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$ is the subgroup of all elements isotopic to the identity by an Ω-isotopy with compact support. $\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ is the subgroup of all elements with compact support. $\mathrm{Diff}_f^{\Omega}(\mathbb{R}^n)$ is the subgroup of all elements with support of finite Ω-volume . $\mathrm{Diff}_W^{\Omega}(\mathbb{R}^n)$ is the subgroup of all elements with set of non-fixed points of finite Ω-volume. Clearly, if Ω has finite total volume we have

$$\mathrm{Diff}_f^{\Omega}(\mathbb{R}^n) = \mathrm{Diff}^{\Omega}(\mathbb{R}^n) .$$

Now we are going to describe the contents of this disertation Chapter by Chapter.

Chapter 1 gives some results on volume elements on a smooth manifold including the one mentioned above.

Chapter 2 contains some general facts on the group $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$. In particular, we give a direct proof of the fact that two volume elements on $\mathbb{R}^n$ with the same total volume are equivalent (2.1) .This result can also be proved using [6] .

We also get a sufficient condition for a subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ to be normal, namely , any subgroup N of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that

$$\mathrm{Diff}_{co}^{\Omega}(\mathbb{R}^n) \subset N \subset \mathrm{Diff}_c^{\Omega}(\mathbb{R}^n)$$

is normal.

We end this Chapter giving some examples that prove that all the inclusions of the above chain are strict.

The aim of Chapter 3 is to decompose an element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ as a finite product of volume preserving diffeomorphisms each one with support in a strip. This method owes very much to Ling [10] who worked out the decomposition of a diffeomorphism of $\mathbb{R}^n$ in a finite product of diffeomorphisms each one with support in a locally finite union of disjoint cells. The modification has been necessary since two strips with the same Ω-volume are diffeomorphic by an element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ (3.4) while the same is not true for locally finite unions of disjoint cells.

Chapter 4 contains several technical results. We prove that the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ of all elements with support in a given strip is connected with respect to the compact-open C^{∞}-topology (4.10). The proof uses an extension to a smooth family of volume elements on $\mathbb{R}^n$ of a result of Greene and Shiohama [6] that is proved in the Appendix of this dissertation .

Following McDuff [15] we prove that the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ of all elements with support in a strip is perfect (4.7)

Another result that proves to be crucial is that for any element h of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that there is a disjoint union of cells $\coprod_{i\geq 1} C_i$ satisfying

$$\Big(\coprod_{i\geq 1} C_i\Big) \cap h\Big(\coprod_{i\geq 1} C_i\Big) = \phi$$

we find a strip ,V , and an element lying in the normal subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ generated by h , h' , such that $h'(V) \cap V = \phi$.

This enables us to get in Chapter 5 some results on the classification of the normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ when Ω has finite total volume.

We prove that for $n \geq 3$, there is no normal subgroup between $\mathrm{Diff}_c^{\Omega}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ (5.4) . Therefore, joining that theorem with a result of Chapter 2 and with Thurston [22] we get that a subgroup N

is normal if and only if

$$\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset N \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n) .$$

Chapter 6 is a complement of Chapter 4 , proving some additional results needed when Ω has infinite total volume .

We construct, for any volume preserving diffeomorphism , h , not lying in $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ a disjoint union of cells , $\coprod_{i \geqslant 1} C_i$, such that

$$\coprod_{i \geqslant 1} C_i \cap h(\coprod_{i \geqslant 1} C_i) = \phi$$

(6.2) .

Also, we prove the last of the decomposition results, namely, we see that any element of $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ with support in a strip of finite Ω-volume can be written as a finite product of elements of $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ each one having support in a strip of finite Ω-volume (6.4) and (6.6) .

As before, this enables us to get in Chapter 7 some results on the classification of the normal subgroups of $\mathrm{Diff}\,(\mathbb{R}^n)$ when Ω has infinite total volume .

We prove that, for $n \geqslant 3$, there is no normal subgroup neither between $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ (7.2) nor between $\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ (7.5) . Thus, joining the above theorems with Thurston [22] we get that the non-trivial subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$, N , must be either between $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ or between $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$.

To study those normal subgroups we have tried two methods that are explained in Chapter 8 .

The first one is taking the closures of the normal subgroups in the above chain with respect to the compact-open C^∞-topology and to the Whitney C^∞-topology .

We prove that $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$ is dense in $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with respect to the first of the topologies (8.1) and $\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ are both closed with respect to the second one (8.3) (8.4) .

A second one is studying different subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ between $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$. We construct an example of a subgroup normal in $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ but not in the whole group.

Notice that pages 3 to 9 and 32 to 35 have been deleted by indication of the examinators.

§1.- SOME PRELIMINARIES

This is an introductory chapter where we give the general definitions and some results on volume elements on a manifold needed in the following chapters.

If M is a connected n-dimensional smooth manifold we denote by $\Lambda^* T^* M$ the set of all differential forms of order n on M. We will say that M is orientable if there is an element of $\Lambda^* T^* M$ which does not vanish at any point of M. We denote by $\Gamma^* \subset \Lambda^* T^* M$ the subset of all differential forms of order n which do not vanish at any point of M.

If ω and θ are two elements of Γ^* we have $\theta = f\,\omega$ where f is a real valued function on M which does not vanish at any point of M and f is either positive for all points of M or negative for all points of M. So, we define ω and θ to be equivalent if $f > 0$ giving an equivalence relation in Γ^* with two equivalence classes.

A such class of Γ^* is called an orientation of M. An oriented manifold is a manifold with a chosen orientation. If M is assumed to be oriented, a diffeomorphism $\psi : M \to M$ is called orientation preserving if the induced map $\psi^* : \Lambda^* T^* M \to \Lambda^* T^* M$ sends any element of the chosen orientation of M, ω, to an element $\psi^*(\omega)$ equivalent to ω.

A volume element σ of an oriented manifold M is a differential form of order n belonging to the chosen orientation. Let A be a subset of M, we denote by $\mathrm{vol}_\sigma A$ the integral of the n-form σ along A (see [13]). A diffeomorphism $h: M \to M$ is σ-preserving or volume preserving if $h^*(\sigma) = \sigma$. We will say that two volume elements σ

and τ on M are equivalent if there is an orientation preserving transformation, $\psi: M \to M$, such that $\psi^*(\sigma) = \tau$.

Now, we state a result obtained by Moser in [18] about the equivalence of volume elements on compact manifolds.

1.3 THEOREM [18] - Let M be a compact connected n-dimensional manifold and let σ and τ be two volume elements on M such that $\mathrm{vol}_\sigma M = \mathrm{vol}_\tau M$. Then there is a diffeomorphism $h : M \to M$ such that $h^*(\tau) = \sigma$.

PAGES 3 – 8
ARE MISSING.

The folowing result give us a special extension of the above theorem of Moser for volume elemtns on a particular type of non-compact manifolds.

1.4 THEOREM [16] .- Let M be a non-compact smooth manifold. Let σ and τ be two volume elements on $M \times [-1, 1]$. Then, there is a diffeomorphism,

$$\Psi : M \times [-1, 1] \to M \times [-1, 1]$$

which equals the identity near $M \times \{-1, 1\}$ and on $M \times \{ 0 \}$ and such that $\Psi^*(\tau) = \sigma$ near $M \times \{ 0 \}$.

PROOF.- It suffices to prove the theorem when τ is a product $\tau' \wedge dt$ where τ' is some volume element on M, because if we have proved the theorem in the above case we have a diffeomorphism

$$\psi_1 : M \times [-1, 1] \to M \times [-1, 1]$$

which equals the identity near $M \times \{-1, 1\}$ and on $M \times \{0\}$ and $\psi_1^* (\tau' \wedge d\, t) = \sigma$ near $M \times \{0\}$. Also we have a diffeomorphism

$$\psi_2 : M \times [-1, 1] \to M \times [-1, 1]$$

which equals the identity near $M \times \{-1, 1\}$ and on $M \times \{0\}$ and $\psi_2^* (\tau' \wedge d\, t) = \tau$.

Therefore, $\Psi = \psi_2^{-1} \circ \psi_1$ satisfy the desired properties.

Thus, from now on we assume $\tau = \tau' \wedge d\,t$. We have $\sigma = f\tau$ where f is a smooth real valued function on $M \times [-1, 1]$.

We choose a smooth family of functions, $\rho(x,t)$, such that they are equal to the identity near $M \times \{-1, 1\}$ and for $|t|$ small they are defined by

$$\rho(x,t) = \int_0^t f(x,s)\, d\,s .$$

We define now,

$$\Psi\colon M \times [-1, 1] \to M \times [-1, 1] ,$$

by

$$\Psi\,(x,t) = (x, \rho(x,t)).$$

It is a diffeomorphism that it is the identity near $M \times \{-1, 1\}$ and on $M \times \{0\}$ and

$$\Psi^*(\tau) = \frac{\partial \rho}{\partial t}\,\tau$$

so, $\Psi^*(\tau) = \sigma$ near $M \times \{0\}$.

Along this dissertation we will need many times to extend embedding to a volume preserving diffeomorphisms.

We will use the following result of Krygin proved in [9]. The proof is not included because it uses very different techniques to the ones used in this dissertation like extension of vector fields and Hodge's theory.

1.5.- THEOREM [9] .- Let M be a connected orientable closed n-manifold. Let W be a n-dimensional submanifold with smooth boundary ∂W. We denote by by W_i the connected components of W and by N_i the connected components of M-W. Let σ be a volume element on M. And let $f_t\colon \partial W \to M$ be a family of embeddings such that f_0 equals the identity on ∂W, $\mathrm{vol}_\sigma W_i = \mathrm{vol}_\sigma \bar{f}_1(W_i)$ and $\mathrm{vol}_\sigma N_i = \mathrm{vol}_\sigma \bar{f}_1(N_i)$ where $\bar{f}_t$ is some extension of f_t. Then, there is a family of diffeomorphisms $F_t : M \to M$ such that $F_t^*(\sigma) = \sigma$, F_0 is the identity and F_1 equals f_1 on W.

Moreover, if f_t is defined on some components V of $M - W$ and if it preserves σ on V and preserves the total volume of the other components of $M - W$ either for all t or when $t = 1$, then, we may assume that $F_t = f_t$ on V for those values of t.

1.6 NOTE.- Some extension $\bar{F}_t$ exists according to [20].

§2.- GENERAL FACTS ON THE GROUP $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

Let Ω be any volume element on $\mathbb{R}^n$. We denote by $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ the group of smooth diffeomorphisms of $\mathbb{R}^n$ which preserve the given volume element Ω.

To study the group $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ one could expect to have a different group for any volume element considered on $\mathbb{R}^n$. But it is obvious that if Ω_1 and Ω_2 are two volume elements equivalent on $\mathbb{R}^n$ and if $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is the diffeomorphism such that $\psi^*(\Omega_1) = \Omega_2$ the map

$$\Psi : \mathrm{Diff}^{\Omega_2}(\mathbb{R}^n) \to \mathrm{Diff}^{\Omega_1}(\mathbb{R}^n)$$

given by $\Psi(h) = \psi \circ h \circ \psi^{-1}$ is an isomorphism.

Thus, it is very interesting to know when two volume elements are equivalent and the next theorem gives us a sufficient condition.

2.1. THEOREM.- Let Ω_0 and Ω_1 be two volume elements on $\mathbb{R}^n$ such that $\mathrm{vol}_{\Omega_0} \mathbb{R}^n = \mathrm{vol}_{\Omega_1} \mathbb{R}^n$. Then, there is a diffeomorphism,

$$\Psi : \mathbb{R}^n \to \mathbb{R}^n$$

such that

$$\Psi^*(\Omega_1) = \Omega_0 .$$

PROOF.- First of all we will reduce to prove this theorem for two volume elements Ω_0 and Ω_1' with the same total volume and such that

$$\mathrm{vol}_{\Omega_0} B_i = \mathrm{vol}_{\Omega_1'} B_i$$

for any $i \in \mathbb{N}$, where B_i is the closed ball of $\mathbb{R}^n$ of centre the origin and radius i.

There is a positive number λ_1 such that

$$\mathrm{vol}_{\Omega_0} B_1 = \mathrm{vol}_{\Omega_1} B_{\lambda_1}$$

Also, there is a positive number λ_2 such that $\lambda_1 < \lambda_2$ and $\mathrm{vol}_{\Omega_0} B_2 = \mathrm{vol}_{\Omega_1} B_{\lambda_2}$. Thus, inductively, we get $0 < \lambda_1 < \lambda_2 < \ldots$. satisfying $\mathrm{vol}_{\Omega_0} B_i = \mathrm{vol}_{\Omega_1} B_{\lambda_i}$.

Now we will construct a diffeomorphism of $\mathbb{R}^n$ sending B_i into B_{λ_i}. There is a positive number $\lambda_0 < \lambda_1$ such that $\lambda_0 < 1$ and a diffeomorphism, $f: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x) = x \quad \text{for} \quad x \leq \lambda_0 \quad \text{and}$$

$$f(i) = \lambda_i \quad \text{for any} \quad i \in \mathbb{N},$$

So, we can define a smooth function, $\psi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\psi(x_1,\dots,x_n) = \frac{f(\|(x_1,\dots,x_n)\|)}{\|(x_1,\dots,x_n)\|}\,(x_1,\dots,x_n)$$

Therefore, the volume elements, Ω_0 and $\psi^*(\Omega_1) = \Omega_1'$ satisfy

$$\mathrm{vol}_{\psi^*(\Omega_1)}\,\mathbb{R}^n = \mathrm{vol}_{\Omega_1}\,\psi(\mathbb{R}^n) = \mathrm{vol}_{\Omega_1}\,\mathbb{R}^n = \mathrm{vol}_{\Omega_0}\,\mathbb{R}^n$$

and

$$\mathrm{vol}_{\psi^*(\Omega_1)}\,B_i = \mathrm{vol}_{\Omega_1}\psi(B_i) = \mathrm{vol}_{\Omega_1}\,B_{\lambda_i} = \mathrm{vol}_{\Omega_0}\,B_i\ .$$

Thus, if we prove the theorem for Ω_0 and Ω_1' we will get a diffeomorphism $\Psi' : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Psi'^*(\Omega_1') = \Omega_0$ and the diffeomorphism $\Psi = \psi \cdot \Psi'$ satisfies the desired property.

Now, we will construct, inductively, the diffeomorphism Ψ' .

Let $\Omega_0 = f_1\,\Omega_1'$. We choose a smooth function $\bar{f}_1$ such that $\bar{f}_1$ equals f_1 on $B_{3/2}$, $\bar{f}_1$ equals 1 on a neighbourhood of ∂B_2 and

$$\mathrm{vol}_{\bar{f}_1 \cdot \Omega_1}\,B_2 = \mathrm{vol}_{\Omega_0}\,B_2\ .$$

We can apply 1.3 to the volume elements Ω_1' and $\bar{f}_1\,\Omega_1'$ and we get a diffeomorphism $g_1 : B_2 \to B_2$ such that g_1 is the identity on a neighbourhood of ∂B_2 and $g_1^*(\Omega_1') = \bar{f}_1\,\Omega_1'$. Therefore, $g_1^*(\Omega_1') = \Omega_0$ on a neighbourhood of B_1.

Now, we assume that we have a diffeomorphism $g_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $g_i^*(\Omega_1') = \Omega_0$ on a neighbourhood of B_i and

$$\mathrm{vol}_{g_i(\Omega_1')} B_{i+1} = \mathrm{vol}_{\Omega_0} B_{i+1} \qquad \text{and}$$

$$\mathrm{vol}_{g_i^*(\Omega_1')} B_{i+2} = \mathrm{vol}_{\Omega_0} B_{i+2} \ .$$

We will find a new diffeomorphism $g_i' : B_{i+2} \to B_{i+2}$ satisfying $g_i'^* \circ g_i^*(\Omega_1') = \Omega_0$ on a neighbourhood of B_{i+1} and g_i' equals the identity on B_i . Then we will define $g_{i+1} = g_i \circ g_i'$.

We call $\Omega_{i+1} = g_i^*(\Omega_1')$. Let Ω_0 be equal to $f_{i+1}\Omega_{i+1}$. We can choose a smooth function, $\bar{f}_{i+1}$, such that $\bar{f}_{i+1}$ equals f_{i+1} on $B_{i+(3/2)}$, $\bar{f}_{i+1}$ equals 1 on a neighbourhood of ∂B_{i+2} and

$$\mathrm{vol}_{\bar{f}_{i+1}\Omega_{i+1}} B_{i+2} = \mathrm{vol}_{\Omega_0} B_{i+2} \ .$$

We can apply 1.3. to the volume elements Ω_{i+1} and $\bar{f}_{i+1}\Omega_{i+1}$ getting a diffeomorphism

$$g_i' : B_{i+2} \to B_{i+2}$$

such that g_i' is the identity on B_i and on a neighbourhood of ∂B_{i+2} and $g_i'^*(\Omega_{i+1}) = \bar{f}_{i+1}\Omega_{i+1}$. Therefore, $g_i'^*(\Omega_{i+1}) =$ $= f_{i+1}\Omega_{i+1} = \Omega_0$ on a neighbourhood of B_{i+1} .

Thus, inductively we have defined a diffeomorphism of $\mathbb{R}^n$, $\psi' : \mathbb{R}^n \to \mathbb{R}^n$, satisfying $\psi'^*(\Omega_1') = \Omega_0$.

A generalization of this result is included in the work of Greene and Shiohama in [6].

2.2 REMARK.- A consequence of the above theorem is that to study the group $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ it is sufficient to consider two different cases only, namely $\mathrm{vol}_{\Omega} \mathbb{R}^n = \infty$ and $\mathrm{vol}_{\Omega} \mathbb{R}^n < \infty$. Furthermore, if $\mathrm{vol}_{\Omega} \mathbb{R}^n = \infty$ we can assume

$$\Omega = d x_1 \wedge \ldots \wedge d x_n$$

the standard volume element on $\mathbb{R}^n$ and otherwise

$$\Omega = \rho(\|x\|^2) \, d x_1 \wedge \ldots \wedge d x_n$$

for some non-vanishing smooth function ρ.

Recall that an isotopy on $\mathbb{R}^n$ is a smooth map

$$F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$$

such that, for any $t \in [0,1]$. the map $F_t : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$F_t(x) = F(x , t) ,$$

is a diffeomorphism.

Thus, inductively we have defined a diffeomorphism of $\mathbb{R}^n$, $\psi' : \mathbb{R}^n \to \mathbb{R}^n$, satisfying $\psi'^*(\Omega_1') = \Omega_0$.

A generalization of this result is included in the work of Greene and Shiohama in [6].

2.2 REMARK.- A consequence of the above theorem is that to study the group $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ it is sufficient to consider two different cases only, namely $\mathrm{vol}_{\Omega}\, \mathbb{R}^n = \infty$ and $\mathrm{vol}_{\Omega}\, \mathbb{R}^n < \infty$. Furthermore, if $\mathrm{vol}_{\Omega}\, \mathbb{R}^n = \infty$ we can assume

$$\Omega = d\,x_1 \wedge \ldots \wedge d\,x_n$$

the standard volume element on $\mathbb{R}^n$ and otherwise

$$\Omega = \rho(\|x\|^2)\, d\,x_1 \wedge \ldots \wedge d\,x_n$$

for some non-vanishing smooth function ρ.

Recall that an isotopy on $\mathbb{R}^n$ is a smooth map

$$F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$$

such that, for any $t \in [0,1]$. the map $F_t : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$F_t(x) = F(x , t) ,$$

is a diffeomorphism.

We call F_1 isotopic to F_0 and F an isotopy from F_0 to F_1.

We define an Ω-isotopy as an isotopy $F: \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that F_t preserves the volume element Ω for any $t \in [0,1]$.

In [8] it is proved that if we consider $\mathrm{Diff}(\mathbb{R}^n)$ with the compact-open C^∞-topology, to have a smooth path

$$\alpha : [0,1] \to \mathrm{Diff}(\mathbb{R}^n)$$

is equivalent to have an isotopy

$$F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$$

where

$$F(x,t) = \alpha(t)(x)$$

and viceversa.

Now we will prove the fact that $\mathrm{Diff}^\Omega(\mathbb{R}^n)$ is path connected with respect to the compact-open C^∞ - topology (see §8 for a description)

2.3. PROPOSITION.- [15].- Every element of $\mathrm{Diff}^\Omega(\mathbb{R}^n)$ is Ω-isotopic to the identity.

PROOF.- a) Case $\mathrm{vol}_\Omega\, \mathbb{R}^n = \infty$.

As we have seen we can assume that Ω is the standard volume element on $\mathbb{R}^n$. Let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ and let ψ be the translation that sends $h(0)$ to 0 (where 0 is the origin of $\mathbb{R}^n$). Obviously ψ is Ω-preserving. Then, the composition map $g = \psi \circ h$ fixes the origin. The standard isotopy

$$g_t(x) = (1/t)\, g(t x) \qquad \text{if} \quad 0 < t \leq 1$$

$$g_0(x) = \lim_{t \to 0} (1/t)\, g(t x)$$

is an Ω-isotopy from g to the linear map g_0.

As $SL(n, \mathbb{R})$ is path-connected we can join g_0 to the identity by a path in $SL(n, \mathbb{R})$. So, we have an Ω-isotopy from $\psi \circ h$ to the identity.

As the translation ψ is Ω-isotopic to the identity by the linear isotopy, h is Ω-isotopic to the identity.

b) Case $\mathrm{vol}_{\Omega}\, \mathbb{R}^n < \infty$.

In this case we can assume that Ω is spherically symmetric, that is

$$\Omega = \rho(\|x\|^2)\, d x_1 \wedge \ldots \wedge d x_n .$$

$(\mathbb{R}^n, \Omega)$ is diffeomorphic to (D, Ω_0) where D is an open disc in $\mathbb{R}^n$ and

$$\Omega_0 = d\,x_1 \wedge d\,x_2 \wedge \ldots \wedge d\,x_n$$

is the restriction to D of the standard volume element on $\mathbb{R}^n$

Let h be an element of Diff $^{\Omega_0}(D)$. So, 0 and h(0) lie in D . Thus, by restricting the isotopy of a) to a suitable neighbourhood U of 0 we get a path, g_t, of embeddings of U into D such that g_0 is the inclusion $U \subset D$, $g_1 = h|_U$ and g_t preserves Ω_U, for any t. By 1.5. we get an Ω_0 isotopy, $h_t : D \to D$ such that h_t equals g_t near 0 and h_0 is the identity.

Thus, $h_1^{-1} \circ h$ is an element of $\mathrm{Diff}^{\Omega_0}(D)$ such that it is the identity near 0 and it is Ω_0-isotopic to h .

There are two balls B_λ, B_μ of centre the origin and radius λ and μ respectively (assume $\lambda < \mu$) such that the subgroup of Ω_0-preserving diffeomorphisms of D which are the identity on a small disc of centre 0 can be identified with the subgroup of Ω_0-preserving diffeomorphisms of $B_\mu - \{0\}$ such that they are the identity on $B_\mu - B_\lambda$. Then any element f of the latter group is Ω_0-isotopic to the identity by the Ω_0-isotopy

$$f_t(x) = tf(x/t) \qquad \text{if } \|x\| < t\mu$$
$$f_t(x) = x \qquad \text{if } \|x\| > t\lambda \ .$$

So, we have that h is Ω_0-isotopic to the identity.

The aim of this dissertation is the study of the normal subgroups of the group $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$. Now we will define obvious subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ that we will consider .

For any diffeomorphism h of $\mathbb{R}^n$ we denote by W_h the set of non-fixed points that is

$$W_h = \{ x \in \mathbb{R}^n : h(x) \neq x \} .$$

Notice that the support of h is the closure of W_h .

Then, we denote by

$\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ whose elements h have the set W_h of finite Ω-volume

$\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ whose elements h have support of finite Ω-volume

$\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ whose elements have compact support

and

$\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$ the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ whose elements are isotopic to the identity by an Ω-isotopy of compact support.

2.4. PROPOSITION.- Every one of the subgroups considered above are normal in $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

The aim of this dissertation is the study of the normal subgroups of the group $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$. Now we will define obvious subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ that we will consider .

For any diffeomorphism h of $\mathbb{R}^n$ we denote by W_h the set of non-fixed points that is

$$W_h = \{ x \in \mathbb{R}^n : h(x) \neq x \} .$$

Notice that the support of h is the closure of W_h .

Then, we denote by

$\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ whose elements h have the set W_h of finite Ω-volume

$\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ whose elements h have support of finite Ω-volume

$\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ whose elements have compact support

and

$\mathrm{Diff}^{\Omega}_{c\,o}(\mathbb{R}^n)$ the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ whose elements are isotopic to the identity by an Ω-isotopy of compact support.

2.4. PROPOSITION.- Every one of the subgroups considered above are normal in $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

The proof is an immediate consequence of the fact that for any g and h, diffeomorphisms of $\mathbb{R}^n$, the support of the composition $g \circ h \circ g^{-1}$ is exactly the image by g of the support of h.

Thus, we have then the following chain of normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

$$\{id\} \subset \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$$

where $\{ id \}$ is the trivial subgroup.

Notice that if Ω is a volume element on $\mathbb{R}^n$ such that $\mathrm{vol}_{\Omega}\, \mathbb{R}^n < \infty$ we have

$$\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n) = \mathrm{Diff}^{\Omega}(\mathbb{R}^n) .$$

If G_1 and G_2 are groups, we denote by $[G_1, G_2]$ the group generated by the elements of the form $g_1\, g_2\, g_1^{-1}\, g_2^{-1}$ where g_1 lies in G_1 and g_2 lies in G_2. We also denote

$$[g_1, g_2] = g_1\, g_2\, g_1^{-1}\, g_2^{-1} .$$

We have

2.5 PROPOSITION.-

$$[\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n), \mathrm{Diff}^{\Omega}(\mathbb{R}^n)] \subset \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$$

PROOF.- Let g be an element of $\mathrm{Diff}_c^{\Omega}(\mathbb{R}^n)$ and let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$. By 2.3 there exists an Ω-isotopy h_t from h to the identity. Then, $F_t = [g, h_t]$ is an Ω-isotopy from $[g, h]$ to the identity. Furthermore, the support of that Ω-isotopy is compact since, for any t, we have

$$\begin{aligned} \operatorname{supp}\, [g, h_t] &\subset \operatorname{supp} g \cup h_t(\operatorname{supp} g) \subset \\ &\subset H((\operatorname{supp} g) \times [0,1]) \end{aligned}$$

where $H: \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ is given by $H(x,t) = h_t(x)$. Since the support of the Ω-isotopy F is included in the closure of

$$\bigcup_t \operatorname{supp}\, [g, h_t]$$

we have

$$\operatorname{supp} F \subset \mathrm{cl}\left(\bigcup_t \operatorname{supp}\, [g, h_t] \right) \subset H((\operatorname{supp} g) \times [0,1]).$$

So, since $\operatorname{supp} g$ is compact, F has compact support.

Therefore, any generator of $[\mathrm{Diff}_c^{\Omega}(\mathbb{R}^n), \mathrm{Diff}^{\Omega}(\mathbb{R}^n)]$ lies in $\mathrm{Diff}_{co}^{\Omega}(\mathbb{R}^n)$. Then, we have the desired inclusion.

As a corollary of this proposition we get a sufficient condition for a subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ to be normal.

2.6. COROLLARY.- Let N be a subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that

$$\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset N \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n).$$

Then, N is a normal subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

PROOF.- By 2.5 we have

$$[\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n), \mathrm{Diff}^{\Omega}(\mathbb{R}^n)] \subset \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n).$$

Therefore, we have

$$[N, \mathrm{Diff}^{\Omega}(\mathbb{R}^n)] \subset [\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n), \mathrm{Diff}^{\Omega}(\mathbb{R}^n)] \subset \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset N.$$

So, N is a normal subgroup.

Also, we have the following

2.7 COROLLARY.-

$$\frac{\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)}{\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)}$$

is an abelian group.

PROOF.- Let h_1, h_2 be two elements of $\mathrm{Diff}_c^{\Omega}(\mathbb{R}^n)$. We have, by 2.5 , that

$$h_1 \circ h_2 \circ (h_2 \circ h_1)^{-1} = [h_1, h_2]$$

lies in $\mathrm{Diff}_{co}^{\Omega}(\mathbb{R}^n)$. Thus the above group is abelian.

2.8. PROPOSITION.- The groups

$$\frac{\mathrm{Diff}_c^{\Omega}(\mathbb{R}^n)}{\mathrm{Diff}_{co}^{\Omega}(\mathbb{R}^n)} \quad \text{and} \quad \frac{\mathrm{Diff}_c(\mathbb{R}^n)}{\mathrm{Diff}_{co}(\mathbb{R}^n)}$$

are isomorphic.

PROOF.- Let

$$\psi : \frac{\mathrm{Diff}_c^{\Omega}(\mathbb{R}^n)}{\mathrm{Diff}_{co}^{\Omega}(\mathbb{R}^n)} \longrightarrow \frac{\mathrm{Diff}_c(\mathbb{R}^n)}{\mathrm{Diff}_{co}(\mathbb{R}^n)}$$

be the natural map $\psi[h] = [h]$.

Clearly it is well-defined and a homomorphism.

It is 1-1 , since if g and h are elements of $\mathrm{Diff}_c^{\Omega}(\mathbb{R}^n)$ such that $\psi[g] = \psi[h]$ we have an isotopy, of compact support, G_t , from $h^{-1} \circ g$ to the identity. Therefore, by [18] we get a compactly supported Ω-isotopy from $h^{-1} \circ g$ to the identity. Then, $h^{-1} \circ g$ lies in $\mathrm{Diff}_{co}^{\Omega}(\mathbb{R}^n)$ and $[g] = [h]$.

It is onto, since if g is any element of $\mathrm{Diff}_c(\mathbb{R}^n)$ we get by 1.3 a diffeomorphism, $\phi: \mathbb{R}^n \to \mathbb{R}^n$, witn compact support and compactly isotopic to the identity such that $\phi^* g^* \Omega = \Omega$. Therefore, $g \circ \phi$ is an element of $\mathrm{Diff}_c^{\Omega}(\mathbb{R}^n)$ that satisfy

$$\psi\,[g \circ \phi] = [g \circ \phi] = [g] .$$

Notice that joining 2.7 and 2.8 we get, by a different way, the result proved by Cerf in [3] that the group

$$\frac{\mathrm{Diff}_c(\mathbb{R}^n)}{\mathrm{Diff}_{co}(\mathbb{R}^n)}$$

is abelian.

To finish this chapter we will remark that the inclusions of the chain of normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ that we are considering are strict inclusions.

In the case that Ω is the standard volume element on $\mathbb{R}^n$, any translation in $\mathbb{R}^n$ is a volumen preserving diffeomorphisms lying in $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ but not in $\mathrm{Diff}_W^{\Omega}(\mathbb{R}^n)$. Thus,

$$\mathrm{Diff}_W^{\Omega}(\mathbb{R}^n) \subsetneq \mathrm{Diff}^{\Omega}(\mathbb{R}^n) .$$

Now, we will construct an element of $\mathrm{Diff}_W^{\Omega}(\mathbb{R}^n)$ not lying in $\mathrm{Diff}_f^{\Omega}(\mathbb{R}^n)$.

We denote by $B(r)$ the closed ball of $\mathbb{R}^n$ of centre the origin and radius r and by $S^{n-1}(r)$ its boundary, dropping r when $r = 1$. Recall that

a) $\mathrm{vol}_\Omega B(r) = r^n \mathrm{vol}_\Omega B$

b) $\mathrm{vol}_\Omega S^{n-1}(r) = r^{n-1} \mathrm{vol}_\Omega S^{n-1}$

c) $\mathrm{vol}_\Omega B = \frac{1}{n} \mathrm{vol}_\Omega S^{n-1}$

d) $\mathrm{vol}_\Omega S^{n-1}(r) = \frac{2}{((n-2)/2)!} \pi^{(n/2)} r^{n-1}$.

(See [2])

Thus, we have

$$\mathrm{vol}_\Omega B(R) - \mathrm{vol}_\Omega B(r) = (R^n - r^n) \mathrm{vol}_\Omega B =$$

$$= (R-r)(R^{n-1} + r R^{n-2}+\ldots+r^{n-1}) \mathrm{vol}_\Omega B =$$

$$= \frac{R-r}{n} (R^{n-1} +\ldots+ r^{n-1}) \mathrm{vol}_\Omega S^{n-1} =$$

$$= \frac{R-r}{n} \frac{2^{n-1}}{(R+r)^{n-1}} (R^{n-1}+\ldots+ r^{n-1}) \mathrm{vol}_\Omega S^{n-1}(\frac{R+r}{2}) \le$$

$$\le \frac{2^{n-1}}{n} (R-r) \mathrm{vol}_\Omega S^{n-1}(\frac{R+r}{2}) .$$

Therefore, roughly speaking, we will construct a sequence of disjoint annuli of finite total volume whose closure is $\mathbb{R}^n$ and then we will define a volume preserving diffeomorphism by rotating in a particular way each annulus, so, the set of non-fixed points will be the sequence of annuli.

Let $\{r_i\}_{i=1}^{\infty}$ be any ordering of the positive rational numbers greater than 1 and let us define

$$\ell_i = \frac{1}{i^2\, r_i^{n-1}}$$

We call I_1 the open interval of $\mathbb{R}$,

$$I_1 = \left(r_1 - \frac{\ell_1}{2} \;,\; r_1 + \frac{\ell_1}{2} \right)$$

and A_1 the closed annulus of $\mathbb{R}^n$

$$A_1 = \mathrm{cl}\left(B\left(r_1 + \frac{\ell_1}{2}\right) - B\left(r_1 - \frac{\ell_1}{2}\right)\right) .$$

Let n_2 be the smallest integer such that $r_{n_2} \notin \mathrm{cl}\, I_1$ and let $\ell_2' < \ell_{n_2}$ be a positive number such that

$$\left(r_{n_2} - \frac{\ell_2'}{2} \;,\; r_{n_2} + \frac{\ell_2'}{2}\right) \cap I_1 = \phi .$$

Then, we call

$$I_2 = \left(r_{n_2} - \frac{\ell_2'}{2} \;,\; r_{n_2} + \frac{\ell_2'}{2} \right)$$

we will define a volume preserving diffeomorphism by rotating in a particular way each annulus, so, the set of non-fixed points will be the sequence of annuli.

Let $\{r_i\}_{i=1}^{\infty}$ be any ordering of the positive rational numbers greater than 1 and let us define

$$\ell_i = \frac{1}{i^2 \, r_i^{n-1}}$$

We call I_1 the open interval of $\mathbb{R}$,

$$I_1 = \left(r_1 - \frac{\ell_1}{2} \, , \, r_1 + \frac{\ell_1}{2} \right)$$

and A_1 the closed annulus of $\mathbb{R}^n$

$$A_1 = \mathrm{cl}\left(B\left(r_1 + \frac{\ell_1}{2}\right) - B\left(r_1 - \frac{\ell_1}{2}\right) \right) .$$

Let n_2 be the smallest integer such that $r_{n_2} \notin \mathrm{cl}\, I_1$ and let $\ell_2' < \ell_{n_2}$ be a positive number such that

$$\left(r_{n_2} - \frac{\ell_2'}{2} \, , \, r_{n_2} + \frac{\ell_2'}{2}\right) \cap I_1 = \phi .$$

Then, we call

$$I_2 = \left(r_{n_2} - \frac{\ell_2'}{2} \, , \, r_{n_2} + \frac{\ell_2'}{2} \right)$$

and A_2 the corresponding annulus. Proceeding inductively we get a family of closed annuli, $\{A_i\}_{i \geq 1}$, satisfying

i) $\bigcup_{i \geq 1} A_i$ is a subset dense of $\mathbb{R}^n - B$

ii) $\mathrm{vol}_\Omega (\bigcup_{i \geq 1} A_i) < \infty$, since we have

$$\mathrm{vol}_\Omega (\bigcup_{i \geq 1} A_i) = \sum_{i \geq 1} \mathrm{vol}_\Omega A_i \leq \sum_{i \geq 1} \frac{2^{n-1}}{n} \ell_i' \,\mathrm{vol}_\Omega S^{n-1}(r_{n_i}) \leq$$

$$\leq \frac{2^{n-1}}{n} \sum_{i \geq 1} \ell_i \ \mathrm{vol}_\Omega S^{n-1}(r_i) =$$

$$= \frac{2^{n-1}}{n} \sum_{i \geq 1} \ell_i \frac{2}{((n-2)/2)!} \pi^{(n/2)} r_i^{n-1} \leq$$

$$\leq \frac{2^n \pi^{(n/2)}}{n((n-2)/2)!} \sum_{i \geq 1} \frac{1}{i^2} < \infty .$$

Thus, $\bigcup_{i \geq 1} A_i$ is a subset of $\mathbb{R}^n$ with finite Ω-volume whose closure has infinite Ω-volume.

Now, we will construct a volume preserving diffeomorphism h, such that $W_h = \bigcup_{i \geq 1} \mathrm{int}\, A_i$.

As $C = \mathbb{R} - \bigcup_{i \geq 1} I_i$ is a closed subset of $\mathbb{R}$, we have a smooth real valued function $\psi : \mathbb{R} \to [\, 0, \infty\,)$ such that $C = \psi^{-1}(0)$ (for the existence of ψ see [19]).

We can define, for any $r \in \mathbb{R}$ the matrix

$$M(r) = \begin{pmatrix} \cos \psi(r) & -\sin \psi(r) & 0 \\ \sin \psi(r) & \cos \psi(r) & \\ & & \\ & 0 & I \end{pmatrix}$$

and the diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ given by $h(x) = x\, M(\|x\|)$. Clearly h is smooth and volume preserving. Furthermore, $W_h = \bigcup_{i \geq 1} A_i$ and $\operatorname{supp} h = \mathbb{R}^n - B$. Therefore, h lies in $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ but not in $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$. Then, $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n) \subsetneq \mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$.

An example of a volume preserving diffeomorphism with support of finite volume which is not compact can be constructed following the same idea.

Let C_i be the open ball of $\mathbb{R}^n$ of centre $(i,0,\dots,0)$ and radius $1/i$. Then we have

$$\mathrm{vol}_{\Omega}\Big(\bigcup_{i \geq 1} C_i\Big) = \sum_{i \geq 1} \mathrm{vol}_{\Omega} C_i = \sum_{i \geq 1} \frac{1}{i^n} \mathrm{vol}_{\Omega} B \, .$$

So, if $n > 2$

We can define, for any $r \in \mathbb{R}$ the matrix

$$M(r) = \begin{pmatrix} \cos \psi(r) & -\sin \psi(r) & 0 \\ \sin \psi(r) & \cos \psi(r) & \\ & & \\ & 0 & I \end{pmatrix}$$

and the diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ given by $h(x) = x\, M(\|x\|)$. Clearly h is smooth and volume preserving. Furthermore, $W_h = \bigcup_{i \geq 1} A_i$ and $\operatorname{supp} h = \mathbb{R}^n - B$. Therefore, h lies in $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ but not in $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$. Then, $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n) \subsetneq \mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$.

An example of a volume preserving diffeomorphism with support of finite volume which is not compact can be constructed following the same idea.

Let C_i be the open ball of $\mathbb{R}^n$ of centre $(i, 0, \ldots, 0)$ and radius $1/i$. Then we have

$$\mathrm{vol}_{\Omega}\Big(\bigcup_{i \geq 1} C_i\Big) = \sum_{i \geq 1} \mathrm{vol}_{\Omega}\, C_i = \sum_{i \geq 1} \frac{1}{i^n}\, \mathrm{vol}_{\Omega}\, B \,.$$

So, if $n > 2$

$$\mathrm{vol}_\Omega\left(\bigcup_{i\geq 1} C_i\right) < \infty.$$

Repeating the construction above we get a volume preserving diffeomorphism whose support is

$$\bigcup_{i\geq 1} \mathrm{cl}\ C_i\ .$$

Therefore, $\mathrm{Diff}_c^\Omega(\mathbb{R}^n) \subsetneq \mathrm{Diff}_f^\Omega(\mathbb{R}^n)$.

Ling in [10] proved that $\mathrm{Diff}_{co}(\mathbb{R}^n) \neq \mathrm{Diff}_c(\mathbb{R}^n)$. Thus, by 2.8 we have that by any volume element Ω on $\mathbb{R}^n$,

$$\mathrm{Diff}_{co}^\Omega(\mathbb{R}^n) \subsetneq \mathrm{Diff}_c^\Omega(\mathbb{R}^n)\ .$$

Obviously,

$$\{id\} \subsetneq \mathrm{Diff}_{co}^\Omega(\mathbb{R}^n)\ .$$

Thus, throughout this dissertation we will consider the following chain of normal subgroups of $\mathrm{Diff}^\Omega(\mathbb{R}^n)$

$$\{id\} \subsetneq \mathrm{Diff}_{co}^\Omega(\mathbb{R}^n) \subsetneq \mathrm{Diff}_c^\Omega(\mathbb{R}^n) \subsetneq \mathrm{Diff}_f^\Omega(\mathbb{R}^n) \subset \mathrm{Diff}_W^\Omega(\mathbb{R}^n) \subset \mathrm{Diff}^\Omega(\mathbb{R}^n)$$

and the two last inclusions are also strict when Ω has in finite total volume.

$$\mathrm{vol}_\Omega\left(\bigcup_{i \geq 1} C_i\right) < \infty.$$

Repeating the construction above we get a volume preserving diffeomorphism whose support is

$$\bigcup_{i \geq 1} \mathrm{cl}\ C_i .$$

Therefore, $\mathrm{Diff}_c^\Omega(\mathbb{R}^n) \subsetneq \mathrm{Diff}_f^\Omega(\mathbb{R}^n)$.

Ling in [10] proved that $\mathrm{Diff}_{co}(\mathbb{R}^n) \neq \mathrm{Diff}_c(\mathbb{R}^n)$. Thus, by 2.8 we have that by any volume element Ω on $\mathbb{R}^n$,

$$\mathrm{Diff}_{co}^\Omega(\mathbb{R}^n) \subsetneq \mathrm{Diff}_c^\Omega(\mathbb{R}^n) .$$

Obviously,

$$\{id\} \subsetneq \mathrm{Diff}_{co}^\Omega(\mathbb{R}^n) .$$

Thus, throughout this dissertation we will consider the following chain of normal subgroups of $\mathrm{Diff}^\Omega(\mathbb{R}^n)$

$$\{id\} \subsetneq \mathrm{Diff}_{co}^\Omega(\mathbb{R}^n) \subsetneq \mathrm{Diff}_c^\Omega(\mathbb{R}^n) \subsetneq \mathrm{Diff}_f^\Omega(\mathbb{R}^n) \subset \mathrm{Diff}_W^\Omega(\mathbb{R}^n) \subset \mathrm{Diff}^\Omega(\mathbb{R}^n)$$

and the two last inclusions are also strict when Ω has finite total volume.

§3.- DECOMPOSITION THEOREMS

The aim of this chapter is to prove that if $n \geq 3$ we can factor each element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ as the product of five elements of the same group each one with support in a strip (Theorem 3.8). To do that we need several definitions

3.1. DEFINITION.- A straight strand in $\mathbb{R}^n$ is the line $\mathbb{R}^+ \times \{ x \}$ where x is a point in $\mathbb{R}^{n-1}$ and $\mathbb{R}^+ = [0, \infty)$. A strand is the image under an element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ of a straight strand. A tangle is a finite union of disjoint strands. A tangle, L, is said to be unknotted or trivial if there is an element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ which straightens all the strands in L simultaneously.

3.2. DEFINITION.- A strip in $\mathbb{R}^n$ is the image under some element of $\mathrm{Diff}(\mathbb{R}^n)$ of the standard tube

$$T = \{ x \in \mathbb{R}^n : \sum_{i \geq 2}^{n} x_i^2 \leq 1, \quad x_1 \geq 0 \}$$

Notice that a strip may have finite Ω-volume since the diffeomorphism used may not be volume preserving.

Now we state a result on trivial tangles proved by McDuff in Lemma 1.4 of [17] .

3.3 PROPOSITION [17].- Let s_1 and s_2 be two disjoint strands in $\mathbb{R}^n$. Then, there is a strand, s_0 , disjoint with both, such that the tangles $s_1 \cup s_0$ and $s_2 \cup s_0$ are both trivial.

§3.- DECOMPOSITION THEOREMS

The aim of this chapter is to prove that if $n \geq 3$ we can factor each element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ as the product of five elements of the same group each one with support in a strip (Theorem 3.8). To do that we need several definitions

3.1. DEFINITION.- A straight strand in $\mathbb{R}^n$ is the line $\mathbb{R}^+ \times \{ x \}$ where x is a point in $\mathbb{R}^{n-1}$ and $\mathbb{R}^+ = [0, \infty)$. A strand is the image under an element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ of a straight strand. A tangle is a finite union of disjoint strands. A tangle, L, is said to be unknotted or trivial if there is an element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ which straightens all the strands in L simultaneously.

3.2. DEFINITION.- A strip in $\mathbb{R}^n$ is the image under some element of $\mathrm{Diff}(\mathbb{R}^n)$ of the standard tube

$$T = \{ x \in \mathbb{R}^n : \sum_{i \geq 2}^{n} x_i^2 \leq 1, \ x_1 \geq 0 \}$$

Notice that a strip may have finite Ω-volume since the diffeomorphism used may not be volume preserving.

Now we state a result on trivial tangles proved by McDuff in Lemma 1.4 of [17] .

3.3 PROPOSITION [17].- Let s_1 and s_2 be two disjoint strands in $\mathbb{R}^n$. Then, there is a strand, s_0 , disjoint with both, such that the tangles $s_1 \cup s_0$ and $s_2 \cup s_0$ are both trivial.

PAGES 32 – 33
ARE MISSING.

Next proposition gives a condition in two strips, V_1 and V_2, to have a volume preserving diffeomorphism of $\mathbb{R}^n$ sending V_1 onto V_2.

3.4. PROPOSITION.- Let V_1 and V_2 be two strips with the same Ω-volume satisfying $\mathrm{vol}_\Omega(\mathbb{R}^n - V_1) = \mathrm{vol}_\Omega(\mathbb{R}^n - V_2)$ if $\mathrm{vol}_\Omega V_1 = \mathrm{vol}_\Omega V_2 = \infty$.

Then, there is an element , h , of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that $h(V_1) = V_2$.

PROOF.- Let be $V_1 = g_1(T)$ and $V_2 = g_2(T)$. Then, $g = g_2 \circ g_1^{-1}$ is a diffeomorphism of $\mathbb{R}^n$ such that $g(V_1) = V_2$. Now, we will modify g to be volume preserving.

First of all we modify g near ∂V_2 in such a way that the volume elements Ω and $\Omega' = g^* \Omega$ are equivalent near ∂V_2. Let be $M = \partial V_2$ and we identify $M \times [-1,1]$ with a small bicollar of M. Then, by 1.4 we find a diffeomorphism $\phi: M \times [-1,1] \to M \times [-1,1]$ that is the identity near $M \times \{-1,1\}$ and such that $\phi^* g^* \Omega = \Omega$ near $M \times \{0\}$. We denote also by ϕ its extension to $\mathbb{R}^n$ by the identity.

Now, we can apply Theorem 1 of the Appendix to the volume elements Ω and $\phi^* g^* \Omega$ on V_2 and also on $\mathbb{R}^n - V_2$ with the same volume elements. So,we get a diffeomorphism, $\psi : \mathbb{R}^n \to \mathbb{R}^n$ that is the identity near ∂V_2 and such that $\psi^* \phi^* g^* \Omega = \Omega$. Therefore, $h = g \circ \phi \circ \psi$ is the desired volume preserving diffeomorphism.

The following two propositions says us that we can modify a diffeomorphism of $\mathbb{R}^n$ to be volume preserving but leaving it fixed on a given strand.

3.5 PROPOSITION.- Let g be an element of $\mathrm{Diff}(\mathbb{R}^n)$ with support in a strip, V , containing a strand, s, in its interior. Then, there is a volume preserving diffeomorphism, h, with support in V which equals g on s.

PROOF.- Without loss of generality we can assume that V is a tubular neighbourhood of s .Then, there is an $(n-1)$-dimensional submanifold M, of V containing s . We can assume that M is a closed subspace of $\mathbb{R}^n$ without boundary. Thus, applying 1.4 to M with the restriction of the volume elements Ω and $g^*\Omega$ on $\mathbb{R}^n$ and identifying $M \times [-1,1]$ with a small bicollar neighbourhood of M we get a diffeomorphism

$$\phi : M \times [-1,1] \to M \times [-1,1] ,$$

that is the identity near $M \times \{-1,1\}$ and on $M \times \{0\}$ and is such that $\phi^* g^* \Omega = \Omega$ near $M \times \{0\}$. The extension to $\mathbb{R}^n$ of this map by the identity will also be denoted by ϕ .

Let Z be a small neighbourhood of s such that $\phi^* g^* \Omega = \Omega$ on it. Since,

$$\mathrm{vol}_{\Omega} (V-Z) = \mathrm{vol}_{\phi^*g^*\Omega} (V-Z)$$

we can apply theorem 1 of the Appendix to Ω and $\phi^*g^*\Omega$ to get a diffeomorphism, $\psi : V-Z \to V-Z$, such that $\psi^* \phi^* g^*\Omega = \Omega$ and it is the identity near $\partial(V-Z)$. Therefore, $h = g \circ \phi \circ \psi$ is a volume preserving diffeomorphism that equals g on s.

3.6. PROPOSITION.- Let g be as in 3.5 and such that g is volume preserving on a strip $V' \subset V$ containing s in its interior. Then, there is an element h of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with support in V which equals g on a strip $V'' \subset V'$.

PROOF.- We have, $g^*\Omega = \Omega$ on V', therefore, as in the proof of 3.5 we can apply theorem 1 of the Appendix to get a diffeomorphism

$$\psi : V - V' \to V - V'$$

such that $\psi^* g^* \Omega = \Omega$ and it is the identity near $\partial(V - V')$. Then, $h = g \circ \psi$ satisfy this proposition.

Notice that if in 3.6, $\mathrm{vol}_\Omega V' = \infty$ we can get the strip V'' also of infinite Ω-volume.

Now, we want to prove that if we have a volume preserving diffeomorphism of $\mathbb{R}^n$ that it is the identity on a strand, it can be modified to be the identity on a neighbourhood of the strand. Thus, we have

3.7. PROPOSITION.- Let s be a strand and let h be any element of $\mathrm{Diff}^\Omega(\mathbb{R}^n)$ that is the identity on s. Then, there is an element $h' \in \mathrm{Diff}^\Omega(\mathbb{R}^n)$ with support in a strip V' of finite Ω-volume and equal to h on a strip $V'' \subset V'$ containing s in its interior.

PROOF.- Let V_1 and V_2 be strips of finite Ω-volume containing s in its interior and such that $V_2 \cup h(V_2) \subset V_1$. Both V_2 and V_1 are tubular neighbourhoods of s.

Let $T = (E, \phi)$ be a tubular neighbourhood of s where E is a normed vector bundle on s, $\phi : E \to \mathbb{R}^n$ is an embedding that is the identity on s and $\phi(E) = V_2$. We apply the Uniqueness of Tubular Neighbourhood Theorem of Mather (see [12]) to the tubular neighbourhoods T and h_*T, where $h_*T = (h^*E, h^{-1}\circ\phi)$, and we get an isotopy, H $\mathbb{R}^n \times I \to \mathbb{R}^n$, with support in V_1 such that $(H_1^{-1})_* T$ and h_*T are equivalent that is the vector bundles $(H_1^{-1})^*E$ and h^*E are isomorphic and if we denote by $\Psi : (H_1^{-1})^* E \to h^* E$ the bijection between the total spaces of the corresponding vector bundles the following diagram

$$\begin{tikzcd}
 & (H_1^{-1})^*E \arrow[r, "\Psi"] \arrow[dl, "\cong"'] \arrow[ddd] & h^*E \arrow[dr, "\cong"] \arrow[ddd] & \\
E \arrow[d, "\phi"'] \arrow[rrr, dashed, bend left=60, "\overline{\Psi}"] & & & E \arrow[d, "\phi"] \\
\mathbf{R}^n & & & \mathbf{R}^n \\
 & \mathbf{R}^n \arrow[ul, "H_1^{-1}"] \arrow[r, "id"] & \mathbf{R}^n \arrow[ur, "h"'] &
\end{tikzcd}$$

is commutative. Therefore, $\overline{\Psi}: E \to E$ is an automorphism of vector bundles such that $\phi \circ \overline{\Psi} = h \circ H_1 \circ \phi$.

Since s is contractible the vector bundle E is trivial. So, the automorphism $\overline{\Psi}$ is isotopic to the identity. Let $\overline{\Psi}_t$ be such isotopy with $\overline{\Psi}_1 = \overline{\Psi}$ and $\overline{\Psi}_0$ is the identity.

Now, we will define a diffeomorphism $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ that equals $\overline{\Psi}$ on a neighbourhood of s as follows. Let $D' \subset E$ be the disc bundle of radius 2 and let D'' be the disc bundle of radius 1. We define

$$\Phi(x) = \phi \circ \overline{\Psi} \circ \phi^{-1}(x) \qquad \text{for} \quad x \in \phi(D'')$$

$$\Phi(x) = \phi \circ \overline{\Psi}_t \circ \phi^{-1}(x) \qquad \text{for} \quad x \in \phi(D') - \phi(D'')$$

$$\text{with} \quad t = 2 - \|\phi^{-1}(x)\|$$

$$\Phi(x) = \text{identity} \qquad \text{otherwise.}$$

It is diffeomorphism of $\mathbb{R}^n$ with support in V_2 and we have $\Phi \circ H_1^{-1}$ equals h on $\phi(D'') \subset V_2$.

Since h is volume preserving we can apply 3.6 to $\Phi \circ H_1^{-1}$ and we get an element h' of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with support in V_1 and equal to h on a strip $V'' \subset V_2$ containing s in its interior.

3.8. PROPOSITION.- Let s and h be as in 3.7 and let $\mathrm{vol}_\Omega \mathbb{R}^n = \infty$. Then, there is an element $h' \in \mathrm{Diff}^\Omega(\mathbb{R}^n)$ with support in a strip V' of infinite Ω-volume such that $\mathrm{vol}_\Omega(\mathbb{R}^n - V') = \infty$ and h' equal to h on a strip $V'' \subset V'$ also of infinite Ω-volume and containing s in its interior.

PROOF.- The proof goes as in 3.7 once we have constructed two strips of infinite Ω-volume, V_1 and V_2, containing s in its interior and such that $h(V_2) \cup V_2 \subset V_1$ and $\mathrm{vol}_\Omega(\mathbb{R}^n - V_1) = \infty$.

Now we will construct V_1 and V_2. Let x_1 be a point of s and let C_1 be a cell containing x_1 in its interior and $\mathrm{vol}_\Omega C_1 = 1$. Let B_{λ_1} be the ball of centre the origin, radius λ_1 and such that $C_1 \cup h(C_1) \subset \mathrm{int}\, B_{\lambda_1}$. Let x_2 be a point of s not lying in B_{λ_1}. Since $\mathrm{vol}_\Omega(\mathbb{R}^n - B_{\lambda_1}) = \infty$, there is a cell, $C_2 \subset \mathbb{R}^n - B_{\lambda_1}$, containing x_2 in its interior and $\mathrm{vol}_\Omega C_2 = 1$. Let B_{λ_2} be the ball of centre the origin, radius λ_2 with $\lambda_1 < \lambda_2$ and such that $C_2 \cup h(C_2) \subset \mathrm{int}\, B_{\lambda_2}$. Thus, inductively, we get a locally finite sequence of disjoint cells $\coprod_{i \geq 1} C_i$, such that

$$\mathrm{vol}_\Omega\left(\coprod_{i \geq 1} C_i\right) = \infty$$

and $s \cap \mathrm{int}\, C_i \neq \emptyset$ for any i. If

$$\mathrm{vol}_\Omega\left(\mathbb{R}^n - \left(\coprod_{i \geq 1} C_i \cup \coprod_{i \geq 1} h(C_i)\right)\right) < \infty$$

we consider the sequence $\coprod_{j=2i} C_j$.

Therefore, if V_2 is the strip obtained by joining C_j to C_{j+1} by a small bridge around s and V_i the strip obtained by joining C_j to $h(C_j)$ and $h(C_j)$ to C_{j+1} by small bridges around s, they satisfy the desired properties.

Now, we are able to prove the main factorization theorem

3.9. THEOREM.- Let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$. If $n \geq 3$ we can decompose h as the product of five elements of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$, h_1, h_2, h_3, h_4, h_5, where h_i has support in some strip V_i for any i.

PROOF.- Let s be a straight strand. By transversality [8] , there is a diffeomorphism, h_1', with support in an arbitrarily small strip, V_1, containing $h(s)$ in its interior and such that $h_1' \circ h(s) \cap s = \phi$. Then, applying 3.5 to h_1' we get a volume preserving diffeomorphism, h_1^{-1} , with support in V_1 and such that $h_1^{-1} \circ h(s) \cap s = \phi$.

By 3.3 there is a strand, t, disjoint from both s and $h_1^{-1} \circ h(s)$ and such that both tangles, $t \cup h_1^{-1} \circ h(s)$ and $t \cup s$ are unknotted.

Let M be a surface in $\mathbb{R}^n$ diffeomorphic to $\mathbb{R}^+ \times [0,1]$ and bounded by t and s. Let V_3 be a neighbourhood of M that is a strip of finite Ω-volume. There is a diffeomorphism of $\mathbb{R}^n$, h_3, with support in V_3 and sending s onto t. As above, we can assume, by 3.5 , that h_3 is volume preserving.

Repeating the same process with the trivial tangle $t \cup h_1^{-1} \circ h(s)$ we get an element, h_2, of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with support in a strip, V_2, sending t onto $h_1^{-1} \circ h(s)$. Furthermore, we can get h_2 such that $h_2 \circ h_3$ equal to $h_1^{-1} \circ h$ on s.

Let be $g = h_3^{-1} \circ h_2^{-1} \circ h_1^{-1} \circ h$, we have that g equals the identity on s. So, by 3.7 there is an element, $h_4 \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with support in a strip, V_4, such that h_4 equals g near s.

Let $h_5 = h_4^{-1} \circ h_3^{-1} \circ h_2^{-1} \circ h_1^{-1} \circ h$. Since it is the identity on a strip near s and the closure of the complement of a strip is contained in a strip, h_5 has support in a strip V_5.

Therefore, $h = h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1$ is the product of five volume preserving diffeomorphisms of the appropriate type.

3.10. REMARK.- Notice that in the proof of the Theorem above we can get the strips V_1, V_2 and V_3 of Ω-volume as small as we like and we can get also that the Ω-volume of V_4 is finite or $\mathrm{vol}_{\Omega}(\mathbb{R}^n - V_4) = \infty$ and $\mathrm{vol}_{\Omega}(\mathbb{R}^n - V_5) = \infty$.

Also, as an immediate consequence of the proof of 3.9 and since the set $W_{g_1 \circ g_2}$ of non-fixed points of the composition of the diffeomorphisms $g_1 \circ g_2$ is included in the union of W_{g_1} and W_{g_2}, we have the two following corollaries.

3.11. COROLLARY.- If $n \geq 3$, any element, $h \in \mathrm{Diff}_f^{\Omega}(\mathbb{R}^n)$ can be decomposed as the product of five elements of $\mathrm{Diff}_f^{\Omega}(\mathbb{R}^n)$, $h = h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1$, with support in strips V_i, for $i = 1,2,3,4,5$ and such that $\mathrm{vol}_{\Omega} V_i < \infty$ for $i \leq 4$.

3.12. COROLLARY.- If $n \geq 3$, any element, $h \in \mathrm{Diff}_W^{\Omega}(\mathbb{R}^n)$ can be decomposed as the product of five elements of $\mathrm{Diff}_W^{\Omega}(\mathbb{R}^n)$, $h = h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1$, with support in strips V_i, for $i=1,2,3,4,5$.

As a consequence of the following lemma we can prove the factorization theorem for volume preserving diffeomorphisms with support in a strip of finite Ω-volume.

3.13. LEMMA [15] .- If $n \geq 3$, any diffeomorphism $h \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with support in the interior of a cell C is the product, $h = h_1 \circ h_2 \circ h_3$ of three elements $h_i \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ which are supported in the interiors of cells E_i where $E_i \subset \mathrm{int}\ C_i$ and

$$\mathrm{vol}_{\Omega} E_i < \frac{2}{3} \mathrm{vol}_{\Omega} C .$$

PROOF.- Let M be a region $M \subset C$ bounded by a hyperplane intersected with C and such that

$$\mathrm{vol}_\Omega\, M = \frac{1}{3}\, \mathrm{vol}_\Omega\, C\ .$$

There is an open neighbourhood N of M in C such that

$$\mathrm{vol}_\Omega\, h(N) + \mathrm{vol}_\Omega\, N < \frac{2}{3}\, \mathrm{vol}_\Omega\, C\ .$$

Let p_t be an isotopy with support in N which shrinks M so close to ∂C that h is the identity on $p_1(M)$. Then, the isotopy $g_t = h \circ p_t^{-1} \circ h^{-1} \circ p_t$ has support in $N \cup h(N)$ and satisfies that g_0 is the identity and g_1 equals h on M.

Applying 1.5 we can assume that the isotopy g_t is an Ω-isotopy Thus, $g_1^{-1} \circ h$ is a volume preserving diffeomorphism with support in a cell $(C-M)$ of volume less than $(2/3)\mathrm{vol}_\Omega\, C$.

We will finish the proof by decomposing g_1 as the product of two factors of the appropriate type.

We apply Vitali Covering Lemma [21] to the covering of $(\mathrm{int}\, C)-(N \cup h(N))$ by all open balls and we get a finite number of disjoint balls, $B_1,\dots, B_m$, in $(\mathrm{int}\, C)-(N \cup h(N))$ such that

$$\sum_{i=1}^{m} \mathrm{vol}_\Omega\, B_i = \frac{1}{3}\, \mathrm{vol}_\Omega\, C\ .$$

We can join ∂C to B_1 by a path α_1 and ∂B_{i-1} to ∂B_i

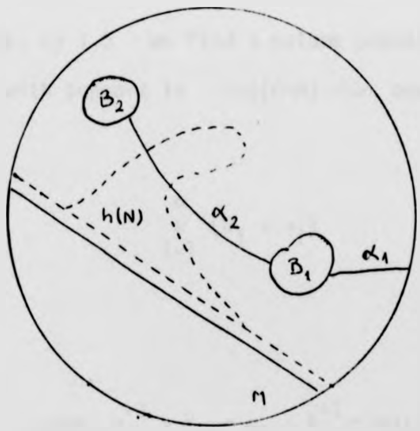
by a path α_i, for $1 < i \leq m$. Also we can construct such paths satisfying $\alpha_i \cap \alpha_j = \phi$ if $i \neq j$, $\alpha_i \cap \partial B_j = \phi$ if $j \neq i-1, i$, $\alpha_i \cap \partial B_{i-1} = \alpha_i(0)$, $\alpha_i \cap \partial B_i = \alpha_i(1)$

$$\bigcup_{i=1}^{m} \alpha_i \cap (M \cup \bigcup_{i=1}^{m} \operatorname{int} B_i) = \phi .$$

Therefore, the complement in C of a suitable neighbourhood of

$$\bigcup_{i=1}^{m} B_i \cup \bigcup_{i=1}^{m} \alpha_i$$

is a cell. Then we will modify g_1 to be the identity near $\cup \alpha_i$, (already we have that g_1 equals the identity near $\cup B_i$).



Let k_t be an isotopy with support in N which pushes the paths $g_1(\alpha_i)$ outside M for any $i=1,\ldots,m$. By 1.5 we can assume that k_t is volume preserving. Then, $k_1 \circ g_1 \circ k_1^{-1}$ is a volume preserving diffeomorphism with support in $N \cup h(N)$ such that the paths $k_1 \circ g_1 \circ k_1^{-1}(\alpha_i)$ lie outside M, for any i.

Let f be a diffeomorphism of $\mathbb{R}^n$ which is the identity near

$$\bigcup_{i=1}^{m} (B_i \cup \alpha_i \cup k_1 \circ g_1 \circ k_1^{-1}(\alpha_i)) \cup \partial C$$

and pushes the support of $k_1 \circ g_t \circ k_1^{-1}$ outside M. Thus $f \circ k_1 \circ g_t \circ k_1^{-1} \circ f^{-1}$ is an isotopy with support in $\mathrm{int}(C-M)$ such that $f \circ k_1 \circ g_1 \circ k_1^{-1} \circ f^{-1} = k_1 \circ g_1 \circ k_1^{-1}$ near

$$\bigcup_{i=1}^{m} (B_i \cup \alpha_i) .$$

Therefore, by 1.5 we find a volume preserving diffeomorphism of $\mathbb{R}^n$, q, with support in $\mathrm{int}(C-M)$ which equals $k_1 \circ g_1 \circ k_1^{-1}$ near

$$\bigcup_{i=1}^{m} (B_i \cup \alpha_i)$$

Then,

$$\mathrm{supp}\ q^{-1} \circ k_1 \circ g_1 \circ k_1^{-1} \subset \mathrm{int}(C-(\bigcup_{i=1}^{m} B_i \cup \bigcup_{i=1}^{m} \alpha_i))$$

and we have $k_1 \circ g_1 \circ k_1^{-1} = q \circ (q^{-1} \circ k_1 \circ g_1 \circ k_1^{-1})$. So,

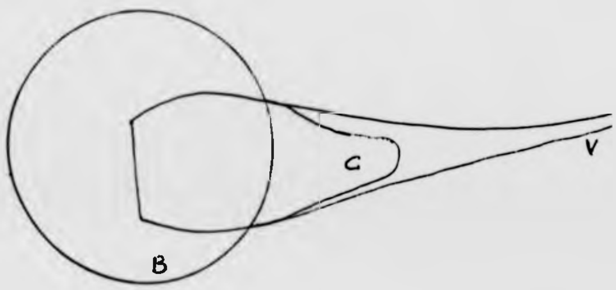
$g_1 = (k_1^{-1} \circ q \circ k_1) \circ (k_1^{-1} \circ q^{-1} \circ k_1 \circ g_1 \circ k_1^{-1} \circ k_1)$ is the product of two factors of the appropriate type.

3.14. THEOREM.- Let V be a strip of finite Ω-volume and let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ Ω-isotopic to the identity by an Ω-isotopy h_t, with support in V. Then, if $n \geq 3$ for any $\varepsilon > 0$ we can factor h as a finite product of volume preserving diffeomorphisms each one having support in strips of Ω-volume less than ε.

PROOF.- Let B be a closed ball in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{\Omega}(V-(\mathrm{supp}\, h \cap B)) < \varepsilon/2 .$$

There is a cell C, in $\mathbb{R}^n$ such that $h_t(B) \subset C$ for any $t \in [0,1]$ and $C \subset B \cup V$.



Applying 1.5 to $h_t|_B$, we get an element $f_1 \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with support in C and equal to h on B. Thus, by 3.13 we can

write f_1 as a finite product of volume preserving diffeomorphisms each one having support in a cell of Ω-volume less than ε. Therefore, since every cell is contained in a strip of Ω-volume as near to the Ω-volume of the cell as we like, we can decompose f_1 as a finite product of elements of the appropriate type.

Let us define $f_2 = f_1^{-1} \circ h$. It is a volume preserving diffeomorphism with support included in $V-B$ that is a strip of Ω-volume less than ε.

Thus, $h = f_1 \circ f_2$ satisfy the theorem.

§4.- TECHNICAL RESULTS

In this chapter we prove the main technical results needed in this dissertation. In particular we prove:

(4.7.) If $n \geq 2$, the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ of all elements with support in a fixed strip V is perfect.

and

(4.9.) If $n \geq 3$, for any element $h \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that there is a disjoint union of cells, $\coprod_{i \geq 1} C_i$, satisfying

$$(\coprod_{i \geq 1} C_i) \cap h(\coprod_{i \geq 1} C_i) = \phi ,$$

there is a strip V and a volume preserving diffeomorphism of $\mathbb{R}^n$, h', lying in the normal subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ generated by h and satisfying $h'(V) \cap V = \phi$.

Let X be any subset of $\mathbb{R}^n$. We denote by G_X the subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ of all elements with support in X.

Notice that in general, G_X is not a normal subgroup.

First of all let us prove

4.1. PROPOSITION.- Let V be a strip in $\mathbb{R}^n$. Then G_V is path-connected with respect to the compact-open C^{∞}-topology.

PROOF.- Let V be the image by $g \in \mathrm{Diff}(\mathbb{R}^n)$ of the standard tube of $\mathbb{R}^n$. Let h be any element of G_V.

We will construct an Ω-isotopy from h to the identity with support in V.

Let H_t be the standard isotopy given by

$$H_t(x) = (1/t)\, g^{-1} \circ h \circ g(t\, x_1) \quad \text{for} \quad t > 0$$

$$H_0(x) = x$$

where $x = (x_1, \ldots, x_n)$

So, $F_t = g \circ H_t \circ g^{-1}$ is an isotopy from h to the identity with support in V, but F_t is not an Ω-isotopy. Thus, $F_t^* \Omega = \sigma_t$ is a smooth family of volume elements on $\mathbb{R}^n$ such that $\sigma_0 = \Omega = \sigma_1$ and

$$\mathrm{vol}_{\sigma_t} V = \mathrm{vol}_\Omega V, \quad \text{for any} \quad t \in [0,1].$$

Therefore, by Theorem 2 proved in the Appendix of this dissertation we get a smooth isotopy, $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$, with support in V such that $\Phi_0 = \Phi_1$ equal to the identity and $\Phi_t^* \sigma_t = \Omega$, for any $t \in [0,1]$.

Then, $F_t \circ \Phi_t$ is an Ω-isotopy from h to the identity with support in V. So, G_V is path-connected.

4.2. REMARK.- The above Proposition proves that any element $h \in \mathrm{Diff}^\Omega(\mathbb{R}^n)$ with support in a strip V is Ω-isotopic to the identity by an Ω-isotopy

PROOF.- Let V be the image by $g \in \mathrm{Diff}(\mathbb{R}^n)$ of the standard tube of $\mathbb{R}^n$. Let h be any element of G_V.

We will construct an Ω-isotopy from h to the identity with support in V.

Let H_t be the standard isotopy given by

$$H_t(x) = (1/t)\, g^{-1} \circ h \circ g(t\, x_1) \quad \text{for} \quad t > 0$$

$$H_0(x) = x$$

where $x = (x_1, \ldots, x_n)$

So, $F_t = g \circ H_t \circ g^{-1}$ is an isotopy from h to the identity with support in V, but F_t is not an Ω-isotopy. Thus, $F_t^* \Omega = \sigma_t$ is a smooth family of volume elements on $\mathbb{R}^n$ such that $\sigma_0 = \Omega = \sigma_1$ and

$$\mathrm{vol}_{\sigma_t} V = \mathrm{vol}_{\Omega} V, \quad \text{for any} \quad t \in [0,1].$$

Therefore, by Theorem 2 proved in the Appendix of this dissertation we get a smooth isotopy, $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$, with support in V such that $\Phi_0 = \Phi_1$ equal to the identity and $\Phi_t^* \sigma_t = \Omega$, for any $t \in [0,1]$.

Then, $F_t \circ \Phi_t$ is an Ω-isotopy from h to the identity with support in V. So, G_V is path-connected.

4.2. REMARK.- The above Proposition proves that any element $h \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with support in a strip V is Ω-isotopic to the identity by an Ω-isotopy with support in V. Thus, in 3.14 the hypothesis that h must be Ω-isotopic to the identity by an Ω-isotopy with support in V is not necessary. We only need h having support in V.

4.3. REMARK.- As an immediate consequence of 3.9 , 3.14 and 4.2 we get that if $\mathrm{vol}_\Omega\ \mathbb{R}^n < \infty$ and $n \geq 3$, any element of $\mathrm{Diff}^\Omega(\mathbb{R}^n)$ can be decomposed, for any $\varepsilon > 0$, as a finite product of volume preserving diffeomorphisms each of them having support in a strip of Ω-volume less than ε .

Now, we want to prove that if V is a strip in $\mathbb{R}^n$, G_V is perfect. The proof is based on a modification of the proof that $\mathrm{Diff}^\Omega(\mathbb{R}^n)$ is perfect given by McDuff in [15] .

We say that an element $h \in \mathrm{Diff}^\Omega(\mathbb{R}^n)$ has a Ling factorization with p-factors if it can be decomposed as a product $h = h_1 \circ h_2 \circ \dots \circ h_p$, where, for any i, h_i is a volume preserving diffeomorphism with support in a locally finite union of disjoint cells. Usually we will call these unions a disjoint union of cells.

In the proof that G_V is perfect, the special factorization given in the next Lemma is crucial .

4.4. LEMMA.- If $n \geq 2$, any element $h \in \mathrm{Diff}^\Omega(\mathbb{R}^n)$ with support in a strip V has a Ling factorization with two factors, $h_1 \circ h_2 = h$, and such that $\mathrm{supp}\ h_i \subset V$, for any i.

PROOF.- Let V be the image by $g \in \mathrm{Diff}(\mathbb{R}^n)$ of the standard tube, T, of $\mathbb{R}^n$ and let h_t be an Ω-isotopy from h to the identity with support in V, it exists by 4.1.

There is a sequence of positive integers, $\lambda_1 < \lambda_2 < ,\ldots,$ such that for any $t \in [0,1]$ and any i

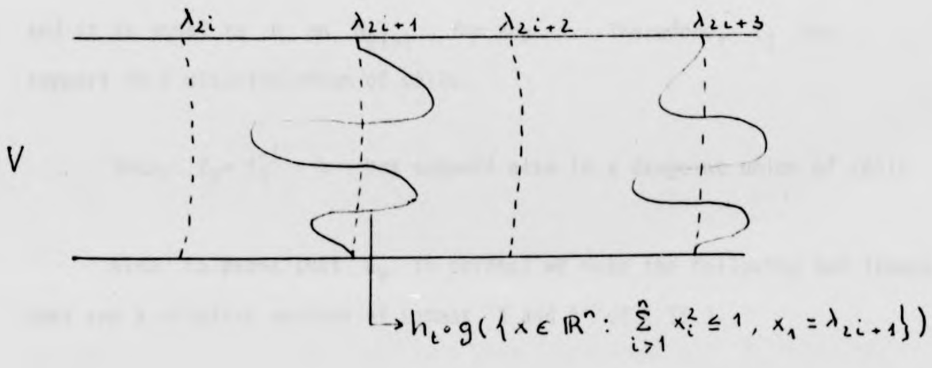
$$h_t \circ g(\{x\in \mathbb{R}^n : \sum_{i>1}^{n} x_i^2 \le 1 , \; 0 \le x_1 \le \lambda_{2i+1}\}) \subset$$

$$\subset g(\{x\in \mathbb{R}^n : \sum_{i>1}^{n} x_i^2 \le 1 , \; 0 \le x_1 < \lambda_{2i+2}\})$$

and

$$h_t \circ g(\{x\in \mathbb{R}^n : \sum_{i>1}^{n} x_i^2 \le 1, \; \lambda_{2i+1} \le x_1 < + \infty\}) \subset$$

$$\subset g(\{x\in \mathbb{R}^n : \sum_{i>1}^{n} x_i^2 \le 1, \; \lambda_{2i} < x_1 < + \infty\})$$



By continuity of the Ω-isotopy and using the fact that

$$g(\{x \in \mathbb{R}^n : \sum_{i>1}^{n} x_i^2 \leq 1 , x_1 = \lambda_{2i+1}\})$$

is a compact set for any i, there is a small neighbourhood, N_{2i+1}, of the above set such that, for any t and any i

$$h_t(N_{2i+1}) \subset g(\{x \in \mathbb{R}^n : \sum_{i>1}^{n} x_i^2 \leq 1, \lambda_{2i} < x_1 < \lambda_{2i+2}\}) .$$

Applying 1.5 to each compact

$$g(\{x \in \mathbb{R}^n : \sum_{i>1}^{n} x_i^2 \leq 1 , \lambda_{2i} \leq x_1 \leq \lambda_{2i+2}\})$$

we get a volume preserving diffeomorphism, f_1, such that is the identity for any i, near

$$g(\{x \in \mathbb{R}^n : \sum_{i>1}^{n} x_i^2 \leq 1, x_1 = \lambda_{2i} \}) \cup g(\{x \in \mathbb{R}^n : \sum_{i>1}^{n} x_i^2 \leq 1, x_1 = \lambda_{2i+2}\})$$

and it is equal to h on N_{2i+1}, for any i. Therefore, f_1 has support in a disjoint union of cells.

Thus, $f_2 = f_1^{-1} \circ h$ has support also in a disjoint union of cells.

Also to prove that G_V is perfect we need the following two lemmas that are a relative version of Lemmas 3 and 4 of [15].

4.5. LEMMA.- Let $n \geq 2$ and let X be a closed subset of $\mathbb{R}^n$ such that $\mathbb{R}^n - X$ is connected and X has only a locally finite set of connected components. Assume that $\coprod_{i \geq 1} C_i$ is a disjoint union of cells in $\mathbb{R}^n - X$ and that $\{\omega_i\}_{i \geq 1}$ is a sequence of positive numbers such that $\omega_i \geq \mathrm{vol}_\Omega\, C_i$, for any i. Assume also that

$$\sum_{i \geq 1} (\omega_i - \mathrm{vol}_\Omega\, C_i) \leq \mathrm{vol}_\Omega((\mathbb{R}^n - X) - \coprod_{i \geq 1} C_i)$$

where strict inequality holds if both sides are finite.

Then, there is a disjoint union of cells, $\coprod_{i \geq 1} D_i$, in $\mathbb{R}^n - X$ satisfying

a) $\mathrm{vol}_\Omega\, D_i = \omega_i$, for any i,

b) $C_i \subset D_i$, for any i,

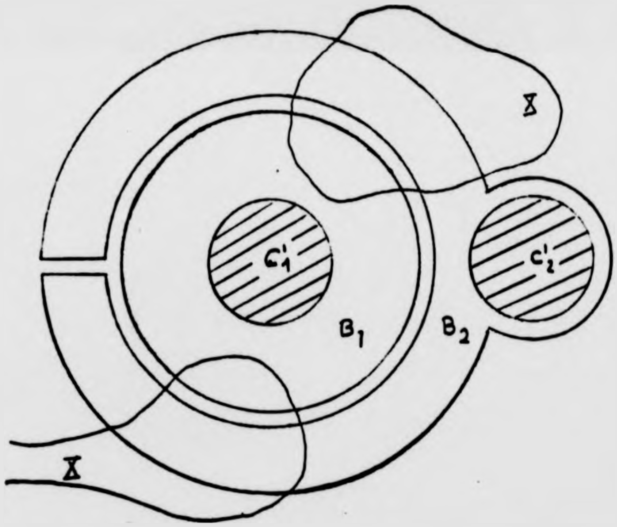
and

c) $\mathrm{vol}_\Omega((\mathbb{R}^n - X) - \coprod_{i \geq 1} D_i) = \infty$, if

$\mathrm{vol}_\Omega((\mathbb{R}^n - X) - \coprod_{i \geq 1} C_i) = \infty$

PROOF.- There is a locally finite union of balls, $\coprod_{i \geq 1} C_i'$, in $\mathbb{R}^n - X$ such that each ball C_i' has centre on the positive x_1-axis and a diffeomorphism, g of $\mathbb{R}^n$ such that sends each cell C_i onto the ball C_i' and it is the identity on X. Therefore, the problem reduces to find a locally finite union of cells, $\coprod_{i\geq 1} D_i'$, in $\mathbb{R}^n - X$ such that $D_i' \supset C_i'$ and whose $g^*\Omega$-volume satisfy the desired conditions.

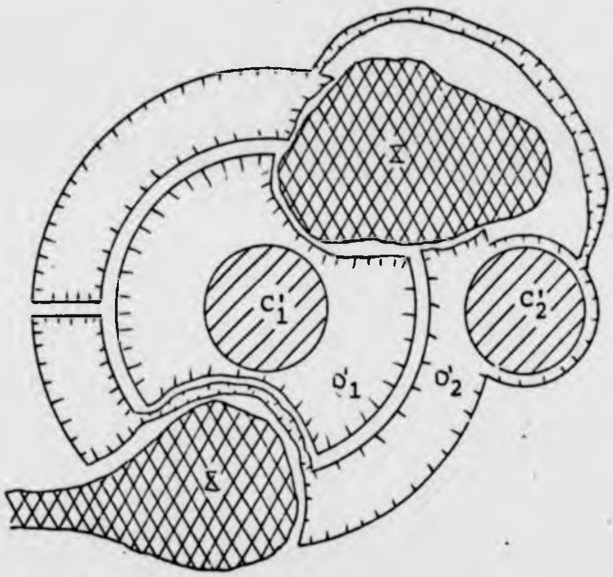
We can choose a locally finite union of cells $\coprod_{i\geq 1} B_i$, each one being a solid of revolution about the x_1-axis and having the shape of an annulus with a hole along the negative x_1-axis so that they are cells and they are distorted along the positive x_1-axis so that $C_i' \subset \mathrm{int}B_i$, for any i .In this way we can choose $\coprod_{i\geq 1} B_i$ to fill up as much or as little of $\mathbb{R}^n$ as necessary.



Now, we will modify the cells B_i in order to get the desired D_i' .

cl $(B_i - X)$ has a finite number of connected components. There is a cell in each one of such components with Ω-volume very close to the Ω-volume of the corresponding connected component. Then, since $\mathbb{R}^n - X$

is connected we can join, by a small bridge in $\mathbb{R}^n-X$, the connected components of $cl(B_i-X)$ getting a cell D_i'. Making inductively the above construction, we get a disjoint union of cells, $\coprod_{i \geq 1} D_i'$, with the desired properties.



4.6. LEMMA.- Let $n \geq 2$ and let X be closed subset of $\mathbb{R}^n$ such that $\mathbb{R}^n-X$ is connected. If $\coprod_{i \geq 1} C_i$ and $\coprod_{i \geq 1} D_i$, are disjoint unions of cells in $\mathbb{R}^n-X$ such that $\mathrm{vol}_\Omega C_i = \mathrm{vol}_\Omega D_i$, for any i, and $\mathrm{vol}_\Omega((\mathbb{R}^n-X)-\coprod_{i \geq 1} C_i) = \mathrm{vol}_\Omega((\mathbb{R}^n-X)-\coprod_{i \geq 1} D_i)$, there is an element $h \in \mathrm{Diff}^\Omega(\mathbb{R}^n)$ such that it is the identity on a neighbourhood of X and $h(C_i) = D_i$, for any i.

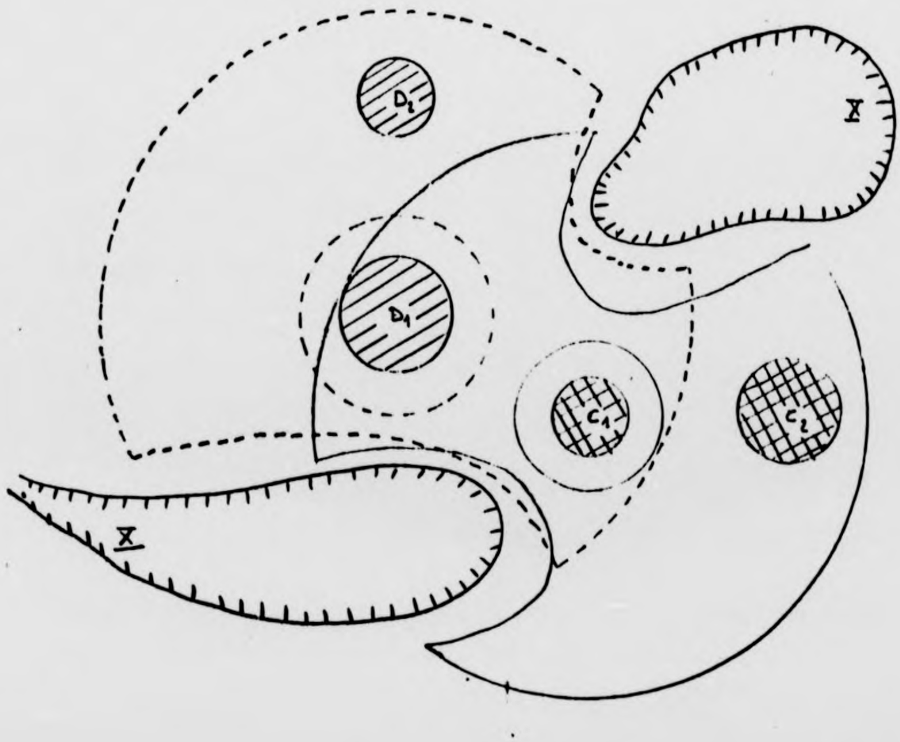
PROOF.- Without any difficulty we can construct an increasing sequence of cells , C'_i , in $\mathbb{R}^n - X$, such that,

$$C'_i \subset \text{int } C'_{i+1} \quad , \quad C_i \subset \text{int}(C_i - C'_{i-1}) \; , \quad \text{for any } i \; ,$$

and $\bigcup_{i \geq 1} C'_i = \mathbb{R}^n - X$.

We can construct also a similar sequence, D'_i , for the cells D_i. So, the sequence D'_i satisfy $D'_i \subset$ int D'_{i+1} , $D_i \subset \text{int}(D'_i - D'_{i-1})$ for any i, and $\bigcup_{i \geq 1} D'_i = \mathbb{R}^n - X$. Moreover, we can get

$\text{vol}_\Omega D'_i = \text{vol}_\Omega C'_i$, for any i.



There is an isotopy , g_t^1 , of $\mathbb{R}^n$ with support in $\mathbb{R}^n-X$ such that $g_t^1(C_1')= D_1'$. We can assume that g_t^1 is an Ω-isotopy by 1.5. Again by 1.5 we can extend $g_t^1|_{C_1'}$ to an Ω-isotopy of $\mathbb{R}^n$, g_t^2 , with support in $\mathbb{R}^n-X$ such that $g_1^2(C_2')= D_2'$. Thus, inductively we get an Ω-isotopy, g_t , of $\mathbb{R}^n$ with support in $\mathbb{R}^n-X$ such that $g_1(C_i')= D_i'$ for any i.

Now, applying again 1.5 to each set $(D_{i+1}'- D_i')$ we get an Ω-isotopy, g_t' , of $\mathbb{R}^n$ with support in $\coprod_{i\geq 1} \operatorname{int}(D_{i+1}'-D_i')-X$ and such that $g_1' \circ g_1(C_i)= D_i$ for any i. Therefore, $h= g_1' \circ g_1$ satisfies the required conditions.

Now,we are able to prove

4.7. THEOREM.- If $n \geq 2$ and V is a strip in $\mathbb{R}^n$, G_V is perfect, i.e. $G_V= [G_V, G_V]$ where by $[G_V, G_V]$ we denote the commutator subgroup of G_V .

PROOF.- a) case $\operatorname{vol}_\Omega V = \infty$.

Let h be any element of G_V , by 4.4 we can assume that h has support in a disjoint union of cells $\coprod_{i\geq 1} C_i \subset V$. Also, without loss of generality we can assume that $\operatorname{vol}_\Omega(V-\coprod_{i\geq 1} C_i)= \infty$, (if necessary, we consider separately the restrictions of h to $\coprod_{i \in J} C_i$ and $\coprod_{i \notin J} C_i$ for some subset J such that $\mathrm{vol}_\Omega(V-\coprod_{i \in J} C_i) = \infty$ and $\mathrm{vol}_\Omega(V-\coprod_{i \notin J} C_i) = \infty$.

We apply 4.5 with $\omega_1 = \mathrm{vol}_\Omega C_1$ and $\omega_i = \sup(\omega_{i-1}, \mathrm{vol}_\Omega C_i)$. So, we get a disjoint union of cells, $\coprod_{i \geq 1} D_i \subset V$, such that

$$\mathrm{vol}_\Omega D_i \leq \mathrm{vol}_\Omega D_{i+1} ,$$

$C_i \subset D_i$, for any i, and $\mathrm{vol}_\Omega(V-\coprod_{i \geq 1} D_i) = \infty$. By 4.6 there is a volume preserving diffeomorphism, f, with support in V and such that $f(D_i) \subset D_{i+1}$ for any i.

We define the following sequence of volume preserving diffeomorphisms

$$g^1(x) = h \circ (f \circ h \circ f^{-1})(x) \qquad \text{if } x \in D_1$$

$$g^1(x) = x . \qquad \text{otherwise}$$

$$\vdots$$

$$g^i(x) = h \circ (f \circ h \circ f^{-1}) \circ \dots \circ (f^i \circ h \circ f^{-i})(x) \quad \text{if } x \in D_i$$

$$g^i(x) = x \qquad \text{otherwise}$$

$$\vdots$$

They define a volume preserving diffeomorphism, g, with support in $\coprod_{i \geq 1} D_i$.

We have $[g, f] = g \circ f \circ g^{-1} \circ f^{-1}$. So, $\operatorname{supp}[g,f] \subset \operatorname{supp} g \cup$ $\cup \operatorname{supp} f \circ g^{-1} \circ f^{-1}$, but $\operatorname{supp} f \circ g \circ f^{-1} \subset \coprod_{i \geq 1} D_i$. Thus,

$\operatorname{supp}[g, f] \subset \coprod_{i \geq 1} D_i \subset V$.

Furthermore, $[g, f](x) = h(x)$ for any $x \in D_i$. Therefore, $[g,f] = h$ and $h \in [G_V, G_V]$.

b) Case $\operatorname{vol}_\Omega V < \infty$.

First of all we will prove that any element h of G_V can be decomposed as a finite product of volume preserving diffeomorphisms each one having support in a disjoint union of cells, $\coprod_{i \geq 1} C_i \subset V$, whose Ω-volumes, $v_i = \operatorname{vol}_\Omega C_i$ satisfy $(1/2)\ v_i \leq v_{i+1} \leq v_i$ for any i, and $\sum_{i \geq 1} v_i < (1/2) \operatorname{vol}_\Omega V$.

We can assume, by 4.4 that $h \in G_V$ has support in a disjoint union of cells, $\coprod_{i \geq 1} C_i' \subset V$. Applying 3.13 to each cell C_i' we can represent h as a finite product of volume preserving diffeomorphisms each one with support in a disjoint union of cells $\coprod_{i \geq 1} C_i''$ such that $C_i'' \subset C_i'$, $\operatorname{vol}_\Omega C_i'' < (2/3)^4 \operatorname{vol}_\Omega C_i' < (1/4) \operatorname{vol}_\Omega C_i'$. After renumbering we can assume

$$\operatorname{vol}_\Omega C_1'' \geq \operatorname{vol}_\Omega C_2'' \geq \operatorname{vol}_\Omega C_3'' \geq \dots .$$

Now, we define inductively

$$v_1 = \mathrm{vol}_\Omega\, C''_1$$

$$v_i = \mathrm{vol}_\Omega\, C''_i \qquad \text{if } \mathrm{vol}_\Omega C''_i \geq (1/2)v_{i-1}$$

$$v_i = \mathrm{vol}_\Omega\, C''_i + (1/2)v_{i-1} \quad \text{otherwise.}$$

We have

$$\sum_{i\geq 1} v_i \leq \sum_{i\geq 1} \mathrm{vol}_\Omega\, C''_i + (1/2) \sum_{i\geq 1} v_{i-1} \quad .$$

So,

$$\sum_{i\geq 1} v_i \leq 2 \sum_{i\geq 1} \mathrm{vol}_\Omega\, C''_i < 2 \sum_{i\geq 1} (1/4)\mathrm{vol}_\Omega C'_i \leq (1/2)\mathrm{vol}_\Omega V,$$

and

$$(1/2)\, v_{i-1} \leq v_i \leq v_{i-1} \qquad \text{for any } i.$$

Thus, applying 4.5 with $w_i = v_i$ we get a disjoint union of cells, $\coprod_{i\geq 1} C_i \subset V$, which satisfy the desired properties.

Now, we will prove that if $h \in G_V$ has support in a disjoint union of cells, $\coprod_{i\geq 1} C_i$, which satisfy the above properties, then $h \in [G_V, G_V]$.

Since $\sum_{i\geq 1} \mathrm{vol}_\Omega C_i < (1/2)\,\mathrm{vol}_\Omega V$ we can apply 4.5 with

$w_i = vol_\Omega C_{i-1}$ and we get , $\coprod_{i \geq 1} D_i \subset V$, such that $C_i \subset D_i$ and

$vol_\Omega D_i = vol_\Omega C_{i-1}$, for any $i \geq 2$.

By 4.6 there is an element $f \in Diff^\Omega(\mathbb{R}^n)$ such that $f(C_i)=D_{i+1}$ for any i and supp $f \subset V$. We can define inductively, the following volume preserving diffeomorphisms

$$g^1(x) = h(x) \qquad \text{if } x \in C_1$$

$$g^1(x) = x \qquad \text{otherwise,}$$

assume we have defined g^i with support in C_i and we will define g^{i+1} with support in C_{i+1} as follows:

Since supp $g^i \subset C_i$ and since $(2/3)^2 \, vol_\Omega C_i \leq vol_\Omega C_{i+1}$ we get, by 3.13 ,that

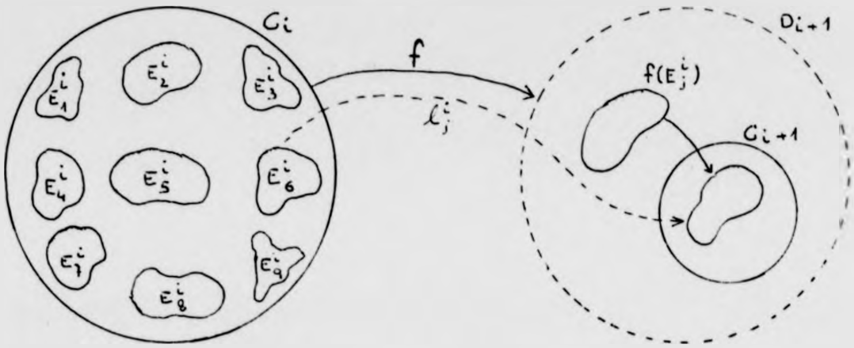
$$g^i = k_9^i \circ k_8^i \circ \ldots \circ k_1^i$$

with

$$k_j^i \in Diff^\Omega(\mathbb{R}^n)$$

and supp $k_j^i \subset E_j^i$ where $vol_\Omega E_j^i < vol_\Omega C_{i+1}$ for any $j=1,\ldots, 9$. There is an element ℓ_j^i of $Diff^\Omega(\mathbb{R}^n)$ with support in V such that it equals f outside C_i and $\ell_j^i(E_j^i) \subset \text{int } C_{i+1}$, for any $j=1,\ldots, 9$. We define

$$g^{i+1}(x) = h\circ(\ell_1^i \circ k_1^i \circ (\ell_1^i)^{-1}) \circ\dots\circ (\ell_9^i \circ k_9^i \circ (\ell_9^i)^{-1})$$



In fact we have defined $g, k_1, \dots, k_9, \ell_1, \dots, \ell_9$ elements of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that $\mathrm{supp}\, g \subset \coprod_{i \geq 1} C_i$, $\mathrm{supp}\, k_j \subset \coprod_{i \geq 1} E_j^i$, $\mathrm{supp}\, \ell_j \subset V$, ℓ_j equals f outside $\coprod_{i \geq 1} C_i$ and such that

$$g = k_9 \circ k_8 \circ\dots\circ k_1 = h\circ(\ell_1 \circ k_1 \circ \ell_1^{-1})\circ\dots\circ(\ell_9 \circ k_9 \circ \ell_9^{-1}) .$$

Thus, we have

$$h = k_9 \circ k_8 \circ\dots\circ k_1 \circ (\ell_9 \circ k_9 \circ \ell_9^{-1})^{-1} \circ\dots\circ (\ell_1 \circ k_1 \circ \ell_1^{-1})^{-1} =$$

$$= (k_9 \circ k_8 \circ\dots\circ k_1 \circ \ell_9 \circ k_9^{-1} \circ \ell_9^{-1}\, k_1^{-1} \circ k_2^{-1} \circ\dots\circ k_8^{-1}) \circ\dots\circ$$

$$\circ\dots\circ (k_1 \circ \ell_1 \circ k_1^{-1} \circ \ell_1^{-1}).$$

So, we have written h as the product of 9 elements of $[G_V, G_V]$.

The next three Lemmas are the main tool in the proof of the results included in Chapters 5 and 7. We use the following notation. If h is an element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ we denote by $N(h)$ the normal subgroup generated by h in $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

The idea of the next Lemma has been obtained from Epstein [5].

4.8. LEMMA.- Let X be a subset of $\mathbb{R}^n$ and let h be an element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ satisfying

a) $h(X) \cap X = \phi$

b) There is an element, $f \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that $f(X) \cap X = \phi$ and $h(X) \cap f(X) = \phi$.

Then, we have $[G_X, G_X] \subset N(h)$.

PROOF.- For any two elements, g_1 and g_2 of G_X we have

$$\mathrm{supp}\,[g_1, h] \subset \mathrm{supp}\, g_1 \cup h(\mathrm{supp}\, g_1) \subset X \cup h(X),$$

also,

$$\mathrm{supp}\,[g_2, f] \subset X \cup f(X).$$

Since on $X \cup h(X)$, $[g_2, f]$ equals g_2, we have

$\mathrm{supp}\,[[g_1, h], [g_2, f]] \subset X \cup h(X)$. Moreover,

$[[g_1, h], [g_2, f]]$ is the identity on $h(X)$. Thus,

$[[g_1, h]\ ,\ [g_2, f\ ]] = [g_1, g_2]$.

Since, $[g_1, h]$ lies obviously in $N(h)$, we have $[g_1, g_2] \in N(h)$. Therefore, $[G_X, G_X\ ] \subset N(h)$.

4.9. LEMMA.- Let $n \geq 3$ and let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that there is a disjoint union of cells, $\coprod_{i \geq 1} C_i$, satisfying:

$$h(\coprod_{i \geq 1} C_i) \cap (\coprod_{i \geq 1} C_i) = \phi,$$

$\mathrm{vol}_{\Omega}\ (\mathbb{R}^n - \coprod_{i \geq 1} C_i) = \infty$ if $\mathrm{vol}_{\Omega}\, \mathbb{R}^n = \infty$ and $\mathrm{vol}_{\Omega}(\coprod_{i \geq 1} C_i) < (1/4)\mathrm{vol}_{\Omega}\mathbb{R}^n$

if $\mathrm{vol}_{\Omega}\ \mathbb{R}^n < \infty$. Then, there is a strip, V, containing $\coprod_{i \geq 1} C_i$ in its interior and an element $h' \in N(h)$ such that $h'(V) \cap V = \phi$.

PROOF. Let s be a strand such that $s \cap \mathrm{int}\ C_i \neq \phi$, $s \cap C_i$ is connected, for any i, and $s \cap (\coprod_{i \geq 1} h(C_i)) = \phi$. Applying transversality [8] and 1.5 we get a volume preserving diffeomorphism, m, with support in a disjoint union of cells , $\coprod_{i \geq 1} D_i'$, and such that $m \circ h(s) \cap s = \phi$. Furthermore, we can choose the above disjoint union of cells satisfying

$[[g_1, h] \; , \; [g_2, f]] = [g_1, g_2]$.

Since, $[g_1, h]$ lies obviously in $N(h)$, we have $[g_1, g_2] \in N(h)$. Therefore, $[G_X, G_X] \subset N(h)$.

4.9. LEMMA.- Let $n \geq 3$ and let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that there is a disjoint union of cells, $\coprod_{i \geq 1} C_i$, satisfying:

$$h(\coprod_{i \geq 1} C_i) \cap (\coprod_{i \geq 1} C_i) = \phi,$$

$\mathrm{vol}_{\Omega}(\mathbb{R}^n - \coprod_{i \geq 1} C_i) = \infty$ if $\mathrm{vol}_{\Omega}\mathbb{R}^n = \infty$ and $\mathrm{vol}_{\Omega}(\coprod_{i \geq 1} C_i) < (1/4)\mathrm{vol}_{\Omega}\mathbb{R}^n$

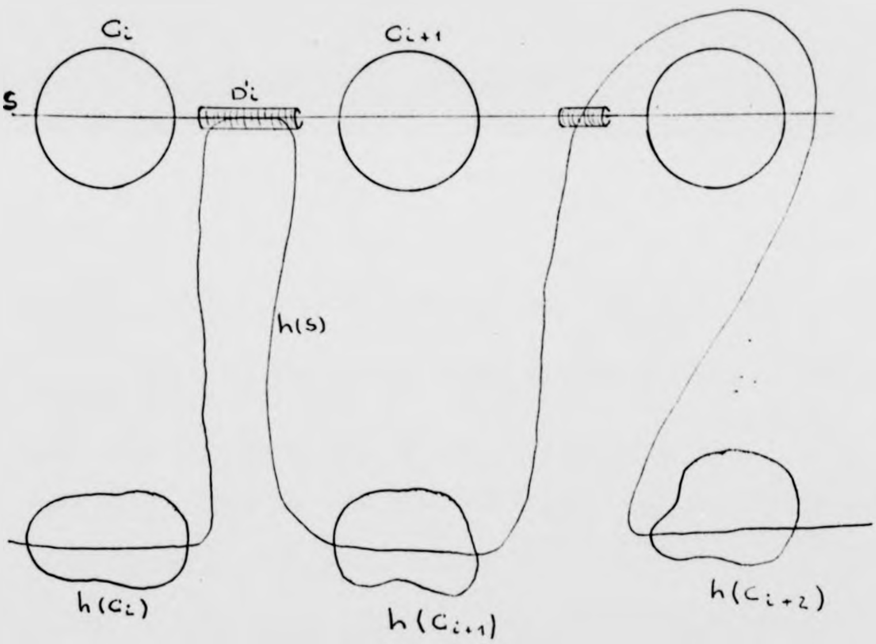
if $\mathrm{vol}_{\Omega}\mathbb{R}^n < \infty$. Then, there is a strip, V, containing $\coprod_{i \geq 1} C_i$ in its interior and an element $h' \in N(h)$ such that $h'(V) \cap V = \phi$.

PROOF. Let s be a strand such that $s \cap \mathrm{int}\, C_i \neq \phi$, $s \cap C_i$ is connected, for any i, and $s \cap (\coprod_{i \geq 1} h(C_i)) = \phi$. Applying transversality [8] and 1.5 we get a volume preserving diffeomorphism, m, with support in a disjoint union of cells , $\coprod_{i \geq 1} D_i'$, and such that $m \circ h(s) \cap s = \phi$. Furthermore, we can choose the above disjoint union of cells satisfying

a) vol $D_i' < (1/2)$ vol C_i , for any i

b) $(\coprod_{i \geq 1} D_i') \cap (\coprod_{i \geq 1} C_i) = \phi$

c) $(\coprod_{i \geq 1} D_i') \cap (\coprod_{i \geq 1} h(C_i)) = \phi$



In general, m is not an element of N(h). So, we will construct now an element, $m' \in N(h)$ such that it equals m on h(s).

Let us call $X = (\coprod_{i \geq 1} C_i) \cup (\coprod_{i \geq 1} h(C_i))$ If we apply 4.5

to $\coprod_{i \geq 1} D_i'$ with $w_i = \mathrm{vol}_\Omega C_i$, for any i, we get a disjoint union of cells, $\coprod_{i \geq 1} C_i'$, satisfying:

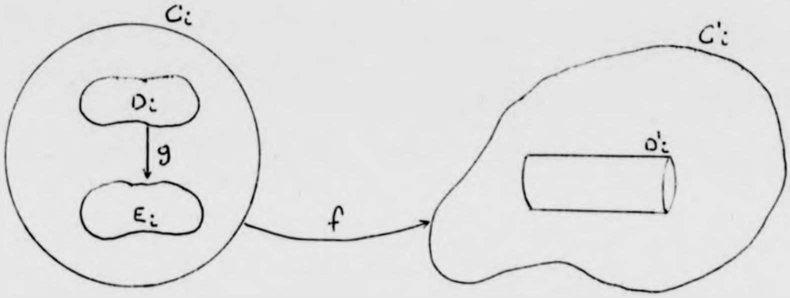
a') $D_i' \subset \mathrm{int}\ C_i$, for any i ,

b') $(\coprod_{i \geq 1} C_i') \cap (\coprod_{i \geq 1} C_i) = \phi$.

c') $(\coprod_{i \geq 1} C_i) \cap (\coprod_{i \geq 1} h(C_i)) = \phi$.

d') $\mathrm{vol}_\Omega C_i' = \mathrm{vol}_\Omega C_i$, for any i .

Also, if we apply 4.6 to $\coprod_{i \geq 1} C_i$ and $\coprod_{i \geq 1} C_i'$, with $X = \coprod_{i \geq 1} h(C_i)$, we get a volume preserving diffeomorphism, f, such that it is the identity near X and $f(C_i) = C_i'$, for any i. Let $D_i = f^{-1}(D_i')$. Since $D_i \subset \mathrm{int}\ C_i$ and $\mathrm{vol}_\Omega D_i = \mathrm{vol}_\Omega D_i' < (1/2)\mathrm{vol}_\Omega C_i$, we can construct, for any i, a new cell, $E_i \subset \mathrm{int}\ C_i$, such that $E_i \cap D_i = \phi$ and $\mathrm{vol}_\Omega E_i = \mathrm{vol}_\Omega D_i$. Thus, we have constructed a disjoint union of cells, $\coprod_{i \geq 1} E_i$. Now, applying again 4.6 to $\coprod_{i \geq 1} E_i$ and $\coprod_{i > 1} D_i$ with $X = \mathbb{R}^n - \coprod_{i \geq 1} \mathrm{int}\ C_i$, we get a volume preserving diffeomorphism, g, with support in $\coprod_{i \geq 1} C_i$ and such that $g(D_i) = E_i$ for any i.



Let $X = \coprod_{i \geq 1} C_i$ and let $\tilde{m} = f^{-1} \circ m \circ f$. By construction we have $g \in G_X$ and $\operatorname{supp} \tilde{m} = f^{-1}(\operatorname{supp} m) \subset f^{-1}(\coprod_{i \geq 1} D'_i) \subset \coprod_{i \geq 1} C_i$.

So, $\tilde{\tilde{m}} = [\tilde{m}, g]$ is an element of $[G_X, G_X]$. Furthermore, since $\operatorname{supp} m \subset \coprod_{i \geq 1} D'_i$ and $(\coprod_{i \geq 1} E_i) \cap (\coprod_{i \geq 1} D_i) = \phi$ we have that $\tilde{\tilde{m}}$ equals $\tilde{m}$ on $\coprod_{i \geq 1} D_i$. We have $h(X) \cap X = \phi$ by hypothesis, $f(X) \cap X = \phi$ by construction of f and b'), and $h(X) \cap f(X) = \phi$ by construction of f and c'). Thus, we can apply 4.8 to get $[G_X, G_X] \subset N(h)$.

Therefore, $\tilde{\tilde{m}}$ lies in $N(h)$. So, $m' = f \circ \tilde{\tilde{m}} \circ f^{-1}$ is also an element of $N(h)$ and since $\operatorname{supp} m' = f(\operatorname{supp} \tilde{\tilde{m}}) \subset \coprod_{i \geq 1} D'_i \cup \coprod_{i \geq 1} f(E_i)$ we have, m' equals m on $h(s)$.

To finish the proof of this lemma we call V the strip obtained from $\coprod_{i \geq 1} C_i$ by joining each cell C_i to C_{i+1} by a small bridge around s and we call $h' = m' \circ h$.

If G is any subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ we denote by $N(G)$ the normal subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ generated by G. And we have

4.10. LEMMA.- Let V be a strip and let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with support in a strip V' such that $\mathrm{vol}_{\Omega} V' \leq \mathrm{vol}_{\Omega} V$ and $\mathrm{vol}_{\Omega} (\mathbb{R}^n - V) = \mathrm{vol}_{\Omega} (\mathbb{R}^n - V) = \infty$ if $\mathrm{vol}_{\Omega} V' = \mathrm{vol}_{\Omega} V = \infty$. Then h is an element of $N(G_V)$.

PROOF.- By 3.4 there is an element, $f \in \mathrm{Diff}(\mathbb{R}^n)$, such that $f(V') \subset V$. Thus, $f \circ h \circ f^{-1}$ lies in G_V. Therefore, h is an element of $N(G_V)$.

4.11. COROLLARY.- For any strip V in $\mathbb{R}^n$ we have $\mathrm{Diff}^{\Omega}_c(\mathbb{R}^n) \subset N(G_V)$

PROOF.- Let h be any element of $\mathrm{Diff}^{\Omega}_c(\mathbb{R}^n)$. Since $\mathrm{supp}\, h$ is compact there is a cell C such that $\mathrm{supp}\, h \subset C$. Therefore, by 3.13 we can assume that h is the product of a finite number of volume preserving diffeomorphisms with support in cells of Ω-volume less or equal than $\mathrm{vol}_{\Omega} V$. Thus, we can apply 4.10 to each factor. So, h lies in $N(G_V)$.

§5.- CASE OF FINITE TOTAL VOLUME.

Throughout this chapter Ω will denote a volume element of $\mathbb{R}^n$ with finite total Ω-volume

We have the following chain of normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

$$\{id\} \subset \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}(\mathbb{R}^n).$$

where by $\{id\}$ we denote the trivial subgroup. Thurston in [22] proved that if $n \geq 3$ there is no normal subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ between $\{id\}$ and $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$, We will prove here that if $n \geq 3$ there is no normal subgroup between $\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ (5.4).

First of all we will prove a preliminary lemma.

5.1. LEMMA.- Let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with non-compact support. Then, there is a disjoint union of cells $\coprod_{i \geq 1} C_i$, such that $(\coprod_{i \geq 1} C_i) \cap (\coprod_{i \geq 1} h(C_i)) = \phi$.

PROOF.- We denote by, fix h, the set of points of $\mathbb{R}^n$ fixed by h, i.e. $\text{fix } h = \mathbb{R}^n - W_h$.

Let x_1 be any point of W_h. There is an open set, A_1, with compact closure such that $x_1 \cup h(x_1) \subset A_1$. Also there is a cell, C_1, in A_1 satisfying: $x_1 \in \operatorname{int} C_1$, $h(C_1) \subset A_1$ and $C_1 \cap h(C_1) = \phi$.

We define $V_1 = A_1 \cup h^{-1}(A_1)$. Since cl V_1 is compact we can find $x_2 \in \operatorname{supp} h - (V_1 \cup \operatorname{fix} h)$; an open set, A_2, with compact closure such that

$$x_2 \cup h(x_2) \cup V_1 \subset A_2$$

and a cell, C_2, in $A_2 - \mathrm{cl}\, A_1$ such that $x_2 \in \operatorname{int} C_2$, $h(C_2) \subset A_2 - \mathrm{cl} A_1$ and $C_2 \cap h(C_2) = \phi$. Thus, inductively, we may construct a sequence of points of $\mathbb{R}^n$, $\{x_i\}_{i \geq 1}$, a disjoint union of cells, $\coprod_{i \geq 1} C_i$, and a sequence of open sets with compact closure $\{A_i\}_{i \geq 1}$ satisfying:

a) $x_i \in \operatorname{int} C_i \subset A_i - \mathrm{cl}\, A_{i-1}$

b) $h(C_i) \subset A_i - \mathrm{cl} A_{i-1}$

c) $C_i \cap h(C_i) = \phi$

, for any i.

Clearly, by construction we have $(\coprod_{i \geq 1} C_i) \cap (\coprod_{i \geq 1} h(C_i)) = \phi$.

5.2. REMARK.- In the above lemma we can get $\coprod_{i \geq 1} C_i$ such that

$$\mathrm{vol}_\Omega(\coprod_{i \geq 1} C_i) < (1/4)\, \mathrm{vol}_\Omega \mathbb{R}^n$$

(If necessary we consider $\coprod_{i \in J} C_i$, for a suitable subset $J \subset \mathbb{N}$ instead of $\coprod_{i \geq 1} C_i$). Thus, by 4.9. we know that for any element,

$h \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$, with non-compact support there is a strip V and an element, $h' \in N(h)$, such that $h'(V) \cap V = \phi$ and $\mathrm{vol}_{\Omega} V < (1/4)\,\mathrm{vol}_{\Omega} \mathbb{R}^n$.

5.3. THEOREM.- Let $n \geq 3$ and let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with non-compact support. Then $N(h) = \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

PROOF.- Let V be the strip that we have by 5.2 . Then, we can decompose any element f of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ as a finite product, $f = f_1 \circ f_2 \circ \ldots \circ f_m$ where, for any i, $f_i \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ and $\mathrm{supp}\, f_i \subset V_i$ with V_i a strip such that $\mathrm{vol}_{\Omega} V_i < \mathrm{vol}_{\Omega} V$.

Therefore, by 4.10 $f_i \in N(G_V)$, for $i = 1, \ldots, m$. So, $f \in N(G_V)$.

The proof will be finished if we see that $N(G_V) \subset N(h)$.

Since $\mathrm{vol}_{\Omega} V < (1/4)\,\mathrm{vol}_{\Omega} \mathbb{R}^n$ we have room enough to construct a new strip, V', such that $V' \cap V = \phi$, $V' \cap h'(V) = \phi$ and $\mathrm{vol}_{\Omega} V = \mathrm{vol}_{\Omega} V'$. Thus, by 3.4 there is an element $g \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that $g(V) = V'$. Therefore, the hypothesis of 4.8 are satisfied . So, $[G_V, G_V] \subset N(h')$. Since G_V is perfect (4.7) we have $G_V = [G_V, G_V]$. Then, $N(G_V) \subset N(h') \subset N(h)$.

5.4. COROLLARY. If $n \geq 3$, there is no normal subgroup between $\mathrm{Diff}_c^{\Omega}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

5.5. REMARK.- If $n \geq 3$, we have the following chain of normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$

$$\{id\} \;\text{———}\; \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n) \;\text{———}\; \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$$

where ——— means that there is no normal subgroup in between.

Furthermore, the above result and 2.6 prove that for $n \geq 3$ a non-trivial subgroup, N, of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ is normal if and only if

$$\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset N \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n).$$

§6. EXTRA RESULTS FOR THE CASE OF INFINITE TOTAL VOLUME.

In chapter 4 we have proved some technical results as 4.7 and 4.9 valid for any volume element Ω on $\mathbb{R}^n$. In this one we will prove the extra results needed when the Ω-total volume of $\mathbb{R}^n$ is infinite. Thus, throughout this section Ω will be a volume element on $\mathbb{R}^n$ of infinite total volume. In particular we prove :

(6.3) If $n \geq 3$, for any volume preserving diffeomorphism of $\mathbb{R}^n$, h, such that $\mathrm{vol}_\Omega W_h = \infty$, there is a strip V of infinite Ω-volume and an element h' of the normal subgroup generated by h such that $h'(V) \cap V = \phi$.

(6.4), (6.6) If $n \geq 3$, we can decompose any element $h \in \mathrm{Diff}^\Omega_f(\mathbb{R}^n)$ with support in a strip of infinite Ω-volume as a finite product of elements of $\mathrm{Diff}^\Omega_f(\mathbb{R}^n)$ each one having support in a strip of finite Ω-volume.

6.1. LEMMA.- Let h be an element of $\mathrm{Diff}^\Omega(\mathbb{R}^n)$ and let X be any open subset of W_h with compact closure. Then, there is a finite number of disjoint cells, $C_1, \dots, C_m$, included in X satisfying:

a) $\left(\coprod_{i\geq 1}^{m} C_i\right) \cap \left(\coprod_{i\geq 1}^{m} h(C_i)\right) = \phi$

b) $\sum_{i\geq 1}^{m} \mathrm{vol}_\Omega C_i > (1/16)\,\mathrm{vol}_\Omega X$.

PROOF.- We define, for any $\varepsilon > 0$ the set $X_\varepsilon(h) = \{ x\in X : ||x-h(x)|| > \varepsilon\}$ It is open because it can be written as $\rho^{-1}(\varepsilon, \infty)$ where $\rho: \mathbb{R}^n \to \mathbb{R}^n$ is the continuous map defined by $\rho(x) = ||x-h(x)||$. Also, we have $X = \bigcup_{\varepsilon>0} X_\varepsilon(h)$. Therefore, there is some $\varepsilon' > 0$ such that

$$\mathrm{vol}_\Omega X_{\varepsilon'}(h) > (1/2)\, \mathrm{vol}_\Omega X.$$

Applying Vitali Covering Lemma [19] to the Vitali Covering of $X_{\varepsilon'}(h)$ given by the set of all open balls of radius $r < (\varepsilon'/2)$, we get a finite number of such balls, $B_1, \ldots, B_p$ pairwise disjoint and such that

$$\sum_{j=1}^{p} \mathrm{vol}_\Omega B_j > (1/2)\, \mathrm{vol}_\Omega X_{\varepsilon'}(h).$$

Notice that since each ball B_j has radius $r < (\varepsilon'/2)$ and any point lying in $X_{\varepsilon'}(h)$ satisfies $||x - h(x)|| > \varepsilon'$ we have

$$h(B_j) \cap B_j = \phi, \quad \text{for any } j .$$

Now, we will construct the set of disjoint cells, $\{C_i\}_{i=1}^{m}$ by induction on j as follows.

Let C_1 be a closed ball included in B_1 with

$$\mathrm{vol}_\Omega C_1 > (1/2)\, \mathrm{vol}_\Omega B_1 .$$

We define $Y_1 = h(C_1) \cup h^{-1}(C_1)$ and we have $\mathrm{vol}_\Omega Y_1 \leq 2\, \mathrm{vol}_\Omega C_1$. Applying Vitali Covering Lemma [19] to the covering of the open set $B_2 - Y_1$ given by the set of all open balls, we get, $C_2', \ldots, C_{n_2}'$, disjoint open balls in $B_2 - Y_1$ such that

$$\sum_{i=2}^{n_2} \mathrm{vol}_\Omega C_i' > (2/3)\, \mathrm{vol}_\Omega (B_2 - Y_1) .$$

Let C_i be a closed ball in C_i' such that $\mathrm{vol}_\Omega C_i > (3/4)\, \mathrm{vol}_\Omega C_i'$. So, we have $C_2, \ldots, C_{n_2}$, disjoint closed balls in $B_2 - Y_1$ satisfying

$$\sum_{i=2}^{n_2} \mathrm{vol}_\Omega C_i > (3/4) \sum_{i=2}^{n_2} \mathrm{vol}_\Omega C_i' > (1/2)\, \mathrm{vol}_\Omega (B_2 - Y_1) .$$

Now, we define

$$Y_2' = Y_1 \cup \Big(\coprod_{i=2}^{n_2} h(C_i) \Big) \cup \Big(\coprod_{i=2}^{n_2} h^{-1}(C_i) \Big)$$

and $Y_2 = Y_2' - Y_1$. So, we have $\mathrm{vol}_\Omega Y_2 \leq 2 \sum_{i=2}^{n_2} \mathrm{vol}_\Omega C_i$.

Thus, applying inductively Vitali Covering Lemma to $(B_j - Y_{j-1}')$ for any $j = 2, \ldots, p$, we get, $C_1, C_2, \ldots, C_{n_2}, C_{n_2+1}, \ldots, C_{n_3}, \ldots, C_{n_p} = C_m$, disjoint closed balls in $X_{\varepsilon'}(h)$ satisfying

$$\Big(\coprod_{i=1}^{m} C_i \Big) \cap \Big(\coprod_{i=1}^{m} h(C_i) \Big) = \phi \qquad \text{and}$$

$$\sum_{i=1}^{m} \mathrm{vol}_\Omega C_i > (1/2)\, \mathrm{vol}_\Omega B_1 + (1/2)\mathrm{vol}_\Omega(B_2 - Y_1) +$$

$$+ (1/2)\, \mathrm{vol}_\Omega(B_3 - Y_2') + \ldots + (1/2)\mathrm{vol}_\Omega(B_p - Y_{p-1}') =$$

$$= (1/2) \sum_{j=1}^{p} \mathrm{vol}_\Omega B_j - (1/2)\mathrm{vol}_\Omega(Y_1 \cap B_2) -$$

$$- (1/2) \sum_{j=3}^{p} \mathrm{vol}_\Omega(Y_{j-1}' \cap B_j) \geq$$

$$\geq (1/2) \sum_{j=1}^{p} \mathrm{vol}_\Omega B_j - (1/2)\Big(\sum_{j=2}^{p} \mathrm{vol}_\Omega(Y_1 \cap B_j) + \sum_{j=3}^{p} \mathrm{vol}_\Omega(Y_2 \cap B_j) + \ldots + \mathrm{vol}_\Omega(Y_{p-1} \cap B_p)\Big)$$

$$\geq (1/2) \sum_{j=1}^{p} \mathrm{vol}_\Omega B_j - (1/2) \sum_{j=1}^{p-1} \mathrm{vol}_\Omega Y_j >$$

$$> (1/2) \sum_{j=1}^{p} \mathrm{vol}_\Omega B_j - \sum_{i=1}^{m} \mathrm{vol}_\Omega C_i .$$

So,

$$\sum_{i=1}^{m} \mathrm{vol}_\Omega C_i > (1/4) \sum_{j=1}^{p} \mathrm{vol}_\Omega B_j > (1/8)\mathrm{vol}_\Omega X_{\varepsilon'}(h) > (1/16)\, \mathrm{vol}_\Omega X.$$

6.2. LEMMA.- Let h be any element of $\mathrm{Diff}^\Omega(\mathbb{R}^n)$ with $\mathrm{vol}_\Omega W_h = \infty$. Then, there is a disjoint union of balls $\coprod_{i \geq 1} D_i$, such that

$$\sum_{i \geq 1} \mathrm{vol}_\Omega D_i = \infty \quad \text{and} \quad \Big(\coprod_{i \geq 1} D_i\Big) \cap \Big(\coprod_{i \geq 1} h(D_i)\Big) = \phi.$$

$$\sum_{i=1}^{m} \mathrm{vol}_\Omega C_i > (1/2)\ \mathrm{vol}_\Omega B_1 + (1/2)\mathrm{vol}_\Omega (B_2 - Y_1)\ +$$

$$+(1/2)\ \mathrm{vol}_\Omega (B_3 - Y_2') + \ldots + (1/2)\mathrm{vol}_\Omega (B_p - Y_{p-1}') =$$

$$=(1/2) \sum_{j=1}^{p} \mathrm{vol}_\Omega B_j - (1/2)\mathrm{vol}_\Omega (Y_1 \cap B_2) -$$

$$-(1/2) \sum_{j=3}^{p} \mathrm{vol}_\Omega (Y_{j-1}' \cap B_j) \geq$$

$$\geq (1/2) \sum_{j=1}^{p} \mathrm{vol}_\Omega B_j - (1/2)\Big(\sum_{j=2}^{p} \mathrm{vol}_\Omega (Y_1 \cap B_j) + \sum_{j=3}^{p} \mathrm{vol}_\Omega (Y_2 \cap B_j) + \ldots + \mathrm{vol}_\Omega (Y_{p-1} \cap B_p)\Big)$$

$$\geq (1/2) \sum_{j=1}^{p} \mathrm{vol}_\Omega B_j - (1/2) \sum_{j=1}^{p-1} \mathrm{vol}_\Omega Y_j >$$

$$> (1/2) \sum_{j=1}^{p} \mathrm{vol}_\Omega B_j - \sum_{i=1}^{m} \mathrm{vol}_\Omega C_i .$$

So,

$$\sum_{i=1}^{m} \mathrm{vol}_\Omega C_i > (1/4) \sum_{j=1}^{p} \mathrm{vol}_\Omega B_j > (1/8)\mathrm{vol}_\Omega X_{\varepsilon'}(h) > (1/16)\ \mathrm{vol}_\Omega X.$$

6.2. LEMMA.- Let h be any element of $\mathrm{Diff}^\Omega(\mathbb{R}^n)$ with $\mathrm{vol}_\Omega W_h = \infty$. Then, there is a disjoint union of balls $\coprod_{i \geq 1} D_i$, such that

$$\sum_{i \geq 1} \mathrm{vol}_\Omega D_i = \infty \quad \text{and} \quad \Big(\coprod_{i \geq 1} D_i \Big) \cap \Big(\coprod_{i \geq 1} h(D_i) \Big) = \phi.$$

PROOF.- Since $\mathrm{vol}_\Omega W_h = \infty$ we can make the following construction of disjoint open sets in W_h. Let $X_1 \subset W_h$ be any open set with compact closure and $\mathrm{vol}_\Omega X_1 = 1$. We define now $C_1 = X_1 \cup h(X_1) \cup h^{-1}(X_1)$.

There is a closed ball, $B_1 = \rho_1 B$, where B is the unit closed ball in $\mathbb{R}^n$ and $\rho_1 > 0$, satisfying $\mathrm{cl}\, C_1 \subset \mathrm{int}\, B_1$. Let $X_2 \subset W_h - B_1$ be an open set with compact closure and $\mathrm{vol}_\Omega X_2 = 2$. We define C_2

$$C_2 = X_2 \cup h(X_2) \cup h^{-1}(X_2).$$

There is a closed ball $B_2 = \rho_2 B$ such that $\mathrm{cl}\, C_2 \subset \mathrm{int}\, B_2$ and $\rho_2 > \rho_1$. Thus, inductively, we get a locally finite sequence of disjoint open sets, $\{X_j\}_{j \geq 1}$, in W_h each one having compact closure and such that $\sum_{j \geq 1} \mathrm{vol}_\Omega X_j = \infty$, and

$$X_i \cap h(X_j) = \phi, \quad X_i \cap h^{-1}(X_j) = \phi, \quad \text{for any } i \neq j .$$

Applying 6.1 to X_j, for any j, we get a disjoint union of cells, $\coprod_{i \geq 1} D_i$, satisfying

$$\sum_{i \geq 1} \mathrm{vol}_\Omega D_i > (1/16) \sum_{j \geq 1} \mathrm{vol}_\Omega X_j = \infty .$$

Furthermore, by construction of $\{X_j\}_{j \geq 1}$ we have

$$\Big(\coprod_{i \geq 1} D_i\Big) \cap \Big(\coprod_{i \geq 1} h(D_i)\Big) = \phi .$$

6.3. REMARK.- From the lemma above and 4.9 we get that if $n \geq 3$, for any element h of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with $\mathrm{vol}_{\Omega} W_h = \infty$ there is a strip, V, of infinite Ω-volume and an element $h' \in N(h)$ such that $h'(V) \cap V = \phi$.

Now we will prove the last decomposition result. We will see separately the cases $n \geq 4$ and $n=3$.

6.4. THEOREM.- Let h be any element of $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ with support in a strip, V, of infinite Ω-volume. Let $n \geq 4$. Then, we can decompose h as $h = h_1 \circ h_2 \circ h_3 \circ h_4$ where h_i lies in $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ and has support in a strip, V_i, of finite Ω-volume, for any $i=1,\dots,4$.

PROOF.- Let us assume that $V = g(T)$ where T is the standard tube of $\mathbb{R}^n$ and g a diffeomorphism of $\mathbb{R}^n$. Let $A_i = \{x \in T: i < x_1 < i+1\}$ and $X_i = g(\mathrm{int}\, A_i)$-supp h. Applying Vitali Covering Lemma to the Vitali Covering of X_i given by all open balls included in X_i, we get, in each X_i, a finite number of disjoint open balls, $C^i_1,\dots, C^i_{n_i}$, such that

$$\mathrm{vol}_{\Omega}\,(X_i - \coprod_{j=1}^{n_i} C^i_j) = 1/2^i .$$

Let B^i_j be a closed ball included in C^i_j such that $\mathrm{vol}_{\Omega} B^i_j = \mathrm{vol}_{\Omega} C^i_j - \varepsilon_i$, where $\varepsilon_i < \dfrac{1}{n_i 2^i}$. So, doing that construction in each X_i we get a disjoint union of closed balls, $\coprod_{i \geq 1} B_i$, in int V-supp h such that

$$\mathrm{vol}_\Omega(V - \coprod_{i \geq 1} B_i) = \mathrm{vol}_\Omega V - \sum_{i \geq 1} \mathrm{vol}_\Omega B_i =$$

$$= \sum_{j\geq 1} \mathrm{vol}_\Omega (X_j - \coprod_k B_k^j) + \mathrm{vol}_\Omega \operatorname{supp} h =$$

$$= \sum_{j\geq 1} \mathrm{vol}_\Omega X_j - \sum_{j\geq 1} \sum_{k=1}^{n_j} \mathrm{vol}_\Omega B_k^j + \mathrm{vol}_\Omega \operatorname{supp} h =$$

$$= \sum_{j\geq 1} \mathrm{vol}_\Omega X_j - \sum_{j\geq 1} \sum_{k=1}^{n_j} (\mathrm{vol}_\Omega C_k^j - \varepsilon_j) + \mathrm{vol}_\Omega \operatorname{supp} h =$$

$$= \sum_{j\geq 1} \mathrm{vol}_\Omega X_j - \sum_{j\geq 1} \sum_{k=1}^{n_j} \mathrm{vol}_\Omega C_k^j + \sum_{j\geq 1} n_j \varepsilon_j + \mathrm{vol}_\Omega \operatorname{supp} h =$$

$$= \sum_{j\geq 1} \frac{1}{2^i} + \sum_{j\geq 1} n_j \varepsilon_j + \mathrm{vol}_\Omega \operatorname{supp} h < \infty .$$

We can join each ball B_i to ∂V by a smooth path, α_i, in V satisfying

a) The set $\{\alpha_i\}_{i \geq 1}$ is locally finite

b) $\alpha_i \cap \alpha_j = \phi$ if $i \neq j$

c) $\alpha_i \cap B_j = \phi$ if $i \neq j$ and $\alpha_i \cap B_i = \alpha_i(1)$.

By transversality [8] and 1.5 we get a volume preserving diffeomorphism, h_1^{-1}, with support in a disjoint union of cells, $\coprod_{i \geq 1} C_i \subset V - \coprod_{i \geq 1} B_i$, each one having Ω-volume as small as we like and such that $h_1^{-1} \circ h(\alpha_i) \cap \alpha_j = \phi$ for any $i \neq j$ and $h_1^{-1} \circ h(\alpha_i)$ only meets α_i on a connected neighbourhood of its end points.

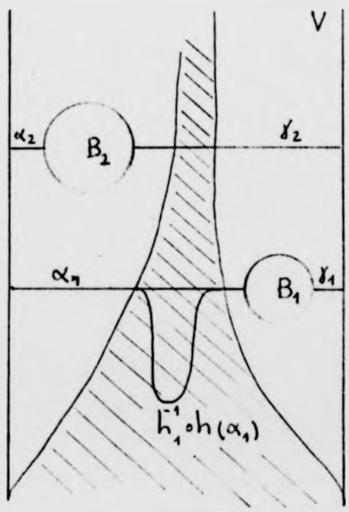
Clearly, joining each cell C_i to C_{i+1} by a small bridge in int $V - \coprod_{i \geq 1} B_i$ we can assume that h_1 has support in a strip V_1 of finite Ω-volume.

Since $V - \coprod_{i \geq 1} B_i - \coprod_{i \geq 1} \alpha_i$ is connected we can join, in that set, each ball B_i to ∂V by a new path, γ_i satisfying

a') The set $\{\gamma_i\}_{i \geq 1}$ is locally finite.

b') $\gamma_i \cap \gamma_j = \phi$ if $i \neq j$.

c') $\gamma_i \cap B_j = \phi$ if $i \neq j$ and $\gamma_i \cap B_i = \gamma_i(0)$.



We have to construct a volume preserving diffeomorphism h_2, such that it is the identity on a neighbourhood of $(\coprod_{i\geq 1} B_i) \cup (\coprod_{i\geq 1} \gamma_i)$ and equals $h_1^{-1} \circ h$ on $\coprod_{i\geq 1} \alpha_i$. So, h_2 will have support in a strip $V_2 \subset V - \coprod_{i\geq 1} B_i$.

To do that, let V' be some neighbourhood of $(\coprod_{i\geq 1} B_i) \cup (\coprod_{i\geq 1} \gamma_i)$ such that $V-V'$ is a strip. Since $n \geq 4$, $h_1^{-1} \circ h(\alpha_i) \cup \alpha_i$ is unknotted, for any i. So, there is a smooth family of smooth embeddings, $\theta_t^i : \alpha_i \to V-V'$, such that, θ_o^i is the inclusion, θ_t^i is the identity near $\alpha_i(o)$ and near $\alpha_i(1)$ and θ_1^i equals $h_1^{-1} \circ h$ on α_i. Let $\tilde{\alpha}_i$ be α_i minus some neighbourhood of $\alpha_i(o)$ and of $\alpha_i(1)$ on which θ_t^i is the identity. Then we have a smooth family of embeddings,

$$\theta_t : \coprod_{i\geq 1} \tilde{\alpha}_i \to V-V' .$$

By transversality [8], we can assume that if $n > 4$

$$\theta : \coprod_{i\geq 1} \tilde{\alpha}_i \times [0,1] \to V-V'$$

is a smooth embedding and if $n=4$, θ is a smooth immersion with transverse interior double points corresponding to different values of the parameter in $\coprod_{i\geq 1} \tilde{\alpha}_i$. Thus, each path, $\theta(x \times [0,1])$, with $x \in \coprod_{i\geq 1} \tilde{\alpha}_i$ meets at most one double point.

Let $E_i = \mathrm{cl}\ \theta(\tilde{\alpha}_i \times [0,1])$ and let U_i be a very small neighbourghood of E_i such that $U_i \cap U_j = \phi$ whenever $E_i \cap E_j = \phi$, all triple intersections $U_i \cap U_j \cap U_k$ are empty and each point of

$$\Big(\coprod_{i \geq 1} \alpha_i\Big) \cup \Big(\coprod_{i \geq 1} h_1^{-1} \circ h(\alpha_i)\Big)$$

lies in at most one U_i . By 1.5 we extend each θ_t^i to an Ω-isotopy, $\tilde{\theta}_t^i$, with support in U_i. Doing the construction of the $\{U_i\}_{i \in \mathbb{N}}$ inductively we can assume that if $x \in U_i \cap U_j$, $\theta_t^j(x)$ does not meet any U_k for $k > j$ and any t. Therefore, the volume preserving diffeomorphism

$$h_2^m = \tilde{\theta}_1^m \circ \tilde{\theta}_1^{m-1} \circ \ldots \circ \tilde{\theta}_1^1$$

is well defined when m tends to ∞. So, it defines a volume preserving diffeomorphism, h_2, with support in $\bigcup_{i \geq 1} U_i$. By construction we have that h_2 equals $h_1^{-1} \circ h$ on $\coprod_{i \geq 1} \alpha_i$ and joining U_i to U_{i+1} by a small bridge we can assume that h_2 has support in a strip $V_2 \subset V - V'$ of finite Ω-volume.

Since $h_1^{-1} \circ h$ is the identity on $\coprod_{i \geq 1} \alpha_i$ and on a neighbourhood of $\coprod_{i \geq 1} B_i$ we have, by 3.7 , a volume preserving diffeomorphism, h_3 , with support in a strip, V_3 of finite Ω-volume and such that it equals $h_2^{-1} \circ h_1^{-1} \circ h$ near $\coprod_{i \geq 1} \alpha_i$. Thus, $h_4 = h_3^{-1} \circ h_2^{-1} \circ h_1^{-1} \circ h$ is the identity near

$$\Big(\coprod_{i \geq 1} \alpha_i\Big) \cup \Big(\coprod_{i \geq 1} B_i\Big) .$$

Therefore, h_4 has support in a strip h_4 of finite Ω-volume. Then, $h = h_1 \circ h_2 \circ h_3 \circ h_4$.

6.5. REMARK.- By 3.11 and 6.4 we have that if $n \geq 4$ any element of $\mathrm{Diff}_f^\Omega(\mathbb{R}^n)$ can be decomposed as a product of eight elements of $\mathrm{Diff}_f^\Omega(\mathbb{R}^n)$ with supports in strips of finite Ω-volume.

Notice that the proof of 6.4 does not work for $n=3$ because $h_1^{-1} \circ h(\alpha_i) \cup \alpha_i$ could be knotted. For $n=3$ we have

6.6. THEOREM.- Let h be any element of $\mathrm{Diff}_f^\Omega(\mathbb{R}^3)$ with support in a strip V of infinite Ω-volume. Then, $h = h_1 \circ h_2 \circ \dots \circ h_6$ where h_i lies in $\mathrm{Diff}_f^\Omega(\mathbb{R}^3)$ and it has support in a strip, V_i, of finite Ω-volume, for any $i=1,\dots,6$.

To prove it we need some definitions and Lemmas about infinite links.

6.7. DEFINITION.- Let $\coprod_{i \geq 1} \alpha_i$, $\coprod_{i \geq 1} \beta_i$ be two locally finite sets of disjoint smooth paths in $\mathbb{R}^3$ such that $\alpha_i \cap \beta_j = \phi$ if $i \neq j$ and $\alpha_i \cap \beta_i = (\alpha_i(o) = \beta_i(o)) \cup (\alpha_i(1) = \beta_i(1))$.
Let $p: \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}$ given by $p(x, y, z)=(x, y, o)$ be the parallel projection.

We call a crossing of the link $L = (\coprod_{i \geq 1} \alpha_i) \cup (\coprod_{i \geq 1} \beta_i)$

the set of points $p^{-1}(c)$ where c is a multiple point of $P|_L$. When no confu sion is possible we also call a crossing the point c.

Since every differentiable knot is equivalent to one in regular position and since in L we have a locally finite sequence of differentiable paths, we can assume that all crossings are double. Let c be a double point of $P|_L$, we call c' the point of $p^{-1}(c)$ with larger z-coordinate and c'' the other one.

We have two different types of crossings

a) $p^{-1}(c) \subset \alpha_i \cup \alpha_j$ or $p^{-1}(c) \subset \beta_i \cup \beta_j$

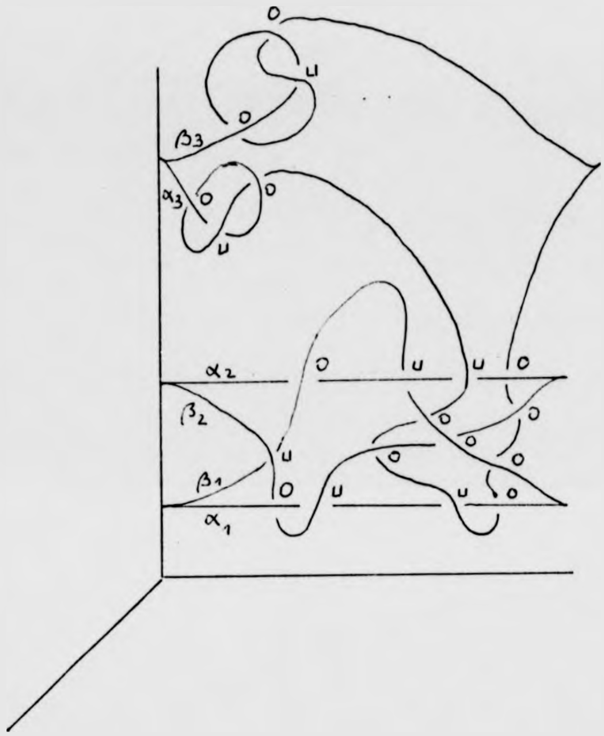
b) one point of $p^{-1}(c)$ lies in α_i and the other one in β_j.

6.8. DEFINITION.- A crossing, $p^{-1}(c)$, of type a) is an overcrossing if c' lies in α_i when $i < j$ or if we find c' first when α_i is traversed from $\alpha_i(o)$ to $\alpha_i(1)$ if $i=j$. Similarly if $p^{-1}(c) \subset \beta_i \cup \beta_j$.

Also, a crossing, $p^{-1}(c)$, of type b) is an overcrossing if c' lies in α_i when $i \leq j$ or in β_j when $j < i$.

In both case we denote the crossing by "O".

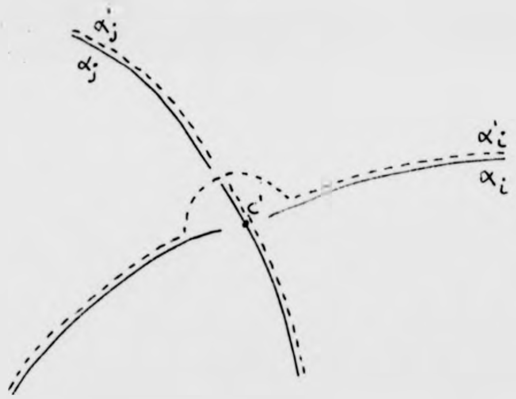
Otherwise, we call a crossing an undercrossing and we denote it by "U".



6.9. LEMMA.- Let L be as above. Then, there are smooth paths, $\coprod_{i \geq 1} \alpha_i'$, $\coprod_{i \geq 1} \beta_i'$, such that α_i' is very close to α_i, β_i' very close to β_i, $\alpha_i' \cap \beta_j' = \phi$ if $i \neq j$, $\alpha_i' \cap \beta_i' = (\alpha_i'(o) = \beta_i'(o)) \cup (\alpha_i'(1) = \beta_i'(1))$ and all crossings of $(\coprod_{i \geq 1} \alpha_i') \cup (\coprod_{i \geq 1} \beta_i')$ are overcrossings.

PROOF.- We define α_i', β_i', inductively on i. α_i', β_i' are different from α_i , β_i only in a chosen neighbour hood of each under-crossing $U = p^{-1}(c)$ where α_i' and β_i' are defined as follows:

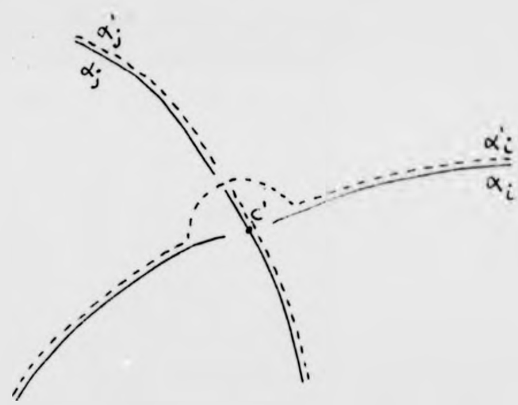
i) U is of type a). On a neghbour hood of c'' , α_i' (resp. β_i') goes vertically (in the z-direction) over α_i (resp. β_i) instead of under . On a neighbour hood of c' , α_j' (resp β_j') is the same as α_j (resp. β_j)



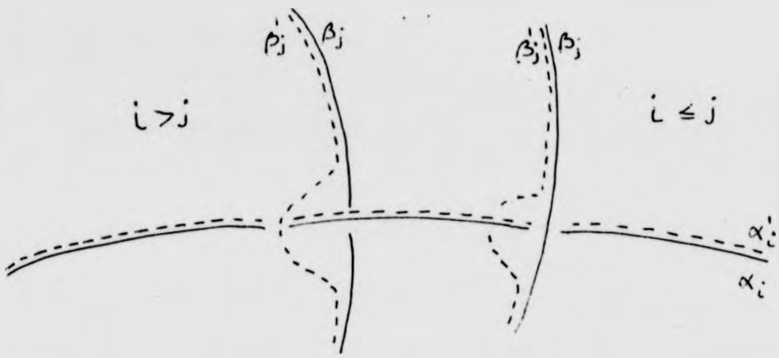
ii) U is cf type b). α_i' is α_i . On a neighbour hood of c', β_j' goes vertically (in the z-direction) under α_i instead of over it if $i \leq j$ and if $i > j$ on a neighbour hood of c'', β_j' goes vertically (also in the z-direction) over α_i instead of under.

different from α_i , β_i only in a chosen neighbour hood of each under-crossing $U = p^{-1}(c)$ where α_i' and β_i' are defined as follows:

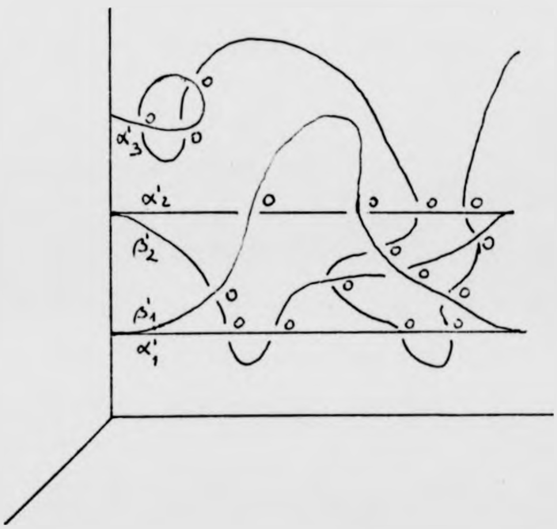
i) U is of type a). On a neghbour hood of c'' , α_i' (resp. β_i') goes vertically (in the z-direction) over α_i (resp. β_i) instead of under . On a neighbour hood of c' , α_j' (resp β_j') is the same as α_j (resp. β_j)



ii) U is of type b). α_i' is α_i . On a neighbour hood of c', β_j' goes vertically (in the z-direction) under α_i instead of over it if $i \leq j$ and if $i > j$ on a neighbour hood of c'', β_j' goes vertically (also in the z-direction) over α_i instead of under.



Clearly, all crossings of $\left(\coprod_{i \geq 1} \alpha_i'\right) \cup \left(\coprod_{i \geq 1} \beta_i'\right)$ are overcrossings,



6.10. REMARK.- Let L be a link as above, then, there are paths, $\coprod_{i \geq 1} \alpha_i'$, and $\coprod_{i \geq 1} \beta_i'$, such that $\left(\coprod_{i \geq 1} \alpha_i'\right) \cup \left(\coprod_{i \geq 1} \beta_i'\right)$ is untangled.

Furthermore, we know by McDuff [15] that $\alpha_i \cup \alpha'_i$ and $\beta_i \cup \beta'_i$ are both unknotted, for any i.

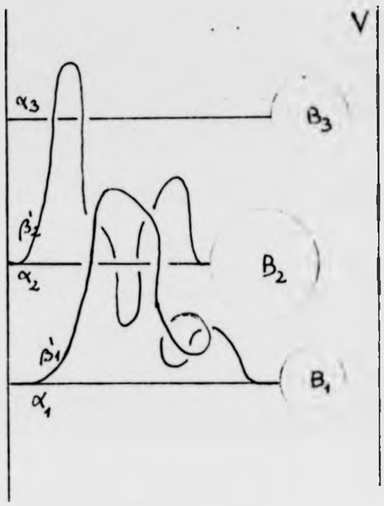
Now, we are able to prove 6.6

PROOF.- As in 6.4 we get a disjoint union of closed balls, $\coprod_{i \geq 1} B_i \subset \text{int } V - \text{supp } h$, such that $\text{vol}_\Omega (V - \coprod_{i \geq 1} B_i) < \infty$. Also. we join each ball B_i to ∂V by an unknotted smooth path, α_i, in V satisfying a), b) and c) of 6.4 And we get a volume preserving diffeomorphism, h_1, with support in a strip, V_1, of finite Ω-volume such that $h_1^{-1} \circ h(\alpha_i) \cap \alpha_j = \phi$, for any $i \neq j$ and $h_1^{-1} \circ h(\alpha_i)$ and α_i only meet on a connected neighbourhood of its end points.

We consider the infinite link $L = (\coprod_{i \geq 1} \alpha_i) \cup (\coprod_{i \geq 1} \beta_i)$ where $\beta_i = h_1^{-1} \circ h(\alpha_i)$ and we apply 6.9 to it. So, we get $\alpha'_i = \alpha_i$, for any i, because α_i never cross each other and we get also $\coprod_{i \geq 1} \beta'_i$, where β'_i is different from $h_1^{-1} \circ h(\alpha_i)$ only in a small neighbourhood of each undercrossing. We have

$$(\coprod_{i \geq 1} \alpha'_i) \cup (\coprod_{i \geq 1} \beta'_i)$$

untangled and $\alpha'_i \cup \beta'_i$, $\beta_i \cup \beta'_i$ unknotted, for any i.



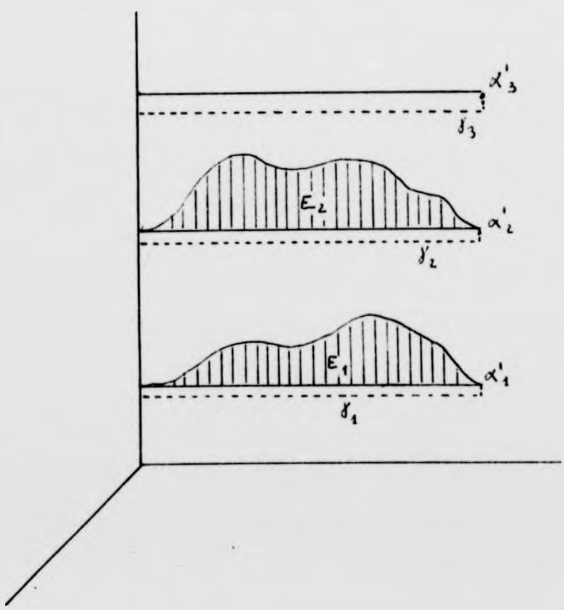
Now, we want a volume preserving diffeomorphisms with support in a strip of finite Ω-volume and such that sends β_i on β_i', for any i.

Since the change from an undercrossing to an overcrossing can happen inside a cell and we can choose these cells pairwise disjoint and as small as we like. We get, for the crossings of type a) a diffeomorphism of $\mathbb{R}^3$, h_6, with support in the cells containing a crossing of type a) (so, with support in a strip of finite Ω-volume), such that $h_6(\alpha_i) = h^{-1} \circ h_1(\beta_i')$, for any i, we can assume, by 1.5 that h_6 is volume preserving. For the crossings of type b) we get a volume preserving diffeomorphism of $\mathbb{R}^3$, h_2^{-1}, with support in a strip of finite Ω-volume such that $\beta_i' = h_2^{-1} \circ h_1^{-1} \circ h \circ h_6(\alpha_i)$. for any i.

Now, we can construct, inductively, pairwise disjoint embedded 2-dimensional open discs E_i such that the boundary of $\mathrm{cl}\, E_i$ is $\alpha_i \cup \beta_i'$, for any i. Also there are smooth unknotted paths, γ_i, in

$$V - \coprod_{i \geq 1} B_i - \coprod_{i \geq 1} \mathrm{cl}\, E_i$$

joining $\alpha_i(0)$ and $\alpha_i(1)$, near α_i and such that each crossing of $\coprod_{i \geq 1} \gamma_i \cup \coprod_{i \geq 1} \beta_i'$ is an overcrossing.



Thus, there are pairwise disjoint small neighbourhoods, U_i, of $\mathrm{cl}\, E_i$ in $V - \coprod_{i \geq 1} B_i - \coprod_{i \geq 1} \gamma_i$. So, there is an isotopy

$\theta : \coprod_{i \geq 1} \alpha_i \times [0,1] \to \coprod_{i \geq 1} U_i$ with θ_0 the identity and θ_1 equal to $h_2^{-1} \circ h_1^{-1} \circ h \circ h_6$. By 1.5. we get an Ω-isotopy, $\tilde{\theta}_t$, with support in $\coprod_{i \geq 1} U_i$ and $\tilde{\theta}_1$ equal to $h_2^{-1} \circ h_1^{-1} \circ h \circ h_6$ on $\coprod_{i \geq 1} \alpha_i$. Let $h_3 = \theta_1$. We have h_3 with support in a strip V_3 of finite Ω-volume and suth that $h_3^{-1} \circ h_2^{-1} \circ h_1^{-1} \circ h \circ h_6$ is the identity on $\coprod_{i \geq 1} \alpha_i$ and on a neighbourhood of $\coprod_{i \geq 1} B_i$.

Now the proof follows as in 6.4

6.11. REMARK.- By 3.11 and 6.6 we have that any element of $\mathrm{Diff}_f^{\Omega}(\mathbb{R}^3)$ can be decomposed as a product of ten elements of $\mathrm{Diff}_f^{\Omega}(\mathbb{R}^n)$ with supports in strips of finite Ω-volume.

§7.- CASE OF INFINITE TOTAL VOLUME.

Throughout this chapter Ω will denote a volume element of $\mathbb{R}^n$ of infinite total Ω-volume.

Then, we have the following chain of normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$

$$\{id\} \subset \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$$

where $\{id\}$ denotes the trivial subgroup. Thurston in [22] proved that if $n \geq 3$ there is no normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ between $\{id\}$ and $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$. We will prove here, also for $n \geq 3$, that there is no normal subgroup between $\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ (7.5). And, there is no normal subgroup between $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ (7.2).

7.1. THEOREM.- Let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with $\mathrm{vol}_{\Omega} W_h = \infty$. If $n \geq 3$ the normal subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ generated by h, $N(h)$, is the whole group.

PROOF.- By 6.3 there is a strip V with infinite Ω-volume and an element $h' \in N(h)$ such that $h'(V) \cap V = \phi$. Clearly, without loss of generality we can assume $\mathrm{vol}_{\Omega}(\mathbb{R}^n - (V \cup h'(V))) = \infty$.

We will prove $\mathrm{Diff}^{\Omega}(\mathbb{R}^n) \subset N(h)$. Let f be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$. We have, by 3.9 and 3.10 that $f = f_1 \circ f_2 \circ \ldots \circ f_5$ with $f_i \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ and f_i has support in a strip V_i such that

§7.- CASE OF INFINITE TOTAL VOLUME.

Throughout this chapter Ω will denote a volume element of $\mathbb{R}^n$ of infinite total Ω-volume.

Then, we have the following chain of normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$

$$\{id\} \subset \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$$

where $\{id\}$ denotes the trivial subgroup. Thurston in [22] proved that if $n \geq 3$ there is no normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ between $\{id\}$ and $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$. We will prove here, also for $n \geq 3$, that there is no normal subgroup between $\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ (7.5). And, there is no normal subgroup between $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ (7.2).

7.1. THEOREM.- Let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with $\mathrm{vol}_{\Omega} W_h = \infty$. If $n \geq 3$ the normal subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ generated by h, $N(h)$, is the whole group.

PROOF.- By 6.3 there is a strip V with finite Ω-volume and an element $h' \in N(h)$ such that $h'(V) \cap V = \phi$. Clearly, without loss of generality we can assume $\mathrm{vol}_{\Omega}(\mathbb{R}^n - (V \cup h'(V))) = \infty$.

We will prove $\mathrm{Diff}^{\Omega}(\mathbb{R}^n) \subset N(h)$. Let f be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$. We have, by 3.9 and 3.10 that $f = f_1 \circ f_2 \circ \ldots \circ f_5$ with $f_i \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ and f_i has support in a strip V_i such that

$vol_\Omega(\mathbb{R}^n - V_i) = \infty$, for any i. Therefore, by 4.10 , f_i is an element of $N(G_V)$, for any i. So, f lies in $N(G_V)$.

We will prove that $N(G_V) \subset N(h)$ using a very similar method to the one used in 5.3 Since $vol_\Omega(\mathbb{R}^n-(V \cup h'(V))) = \infty$ we have enough room to construct a new strip, V', in $\mathbb{R}^n-(V \cup h'(V))$ of infinite Ω-volume. Since V and V' are both of infinite Ω-volume we have, by 3.4 , a volume preserving diffeomorphism, g, such that $g(V) = V'$. We have, $g(V) \cap V = \phi$ and $g(V) \cap h'(V) = \phi$. So, by 4.8 we know that

$$[G_V , G_V] \subset N(h') .$$

As G_V is perfect (proved in 4.7) we have

$$G_V = [G_V, G_V] \subset N(h') \subset N(h) .$$

Therefore, $N(G_V) \subset N(h)$.

7.2. COROLLARY.- If $n \geq 3$ there is no normal subgroup between $Diff^\Omega_W(\mathbb{R}^n)$ and $Diff^\Omega(\mathbb{R}^n)$.

Similarly as in 5.1 we have

7.3. LEMMA.- Let h be an element of $Diff^\Omega(\mathbb{R}^n)$ with non-compact support. Then, there is a disjoint union of cells, $\coprod_{i \geq 1} C_i$, such that

$$(\coprod_{i \geq 1} C_i) \cap (\coprod_{i \geq 1} h(C_i)) = \phi$$

PROOF.- Let x_1 be any point of W_h. There is an open set $A_1 \subset \mathbb{R}^n$ with compact closure such that $x_1 \cup h(x_1) \subset A_1$. Also, there is a cell, $C_1 \subset A_1$, satisfying $x_1 \in \text{int } C_1$, $h(C_1) \subset A_1$ and $C_1 \cap h(C_1) = \phi$. Let be $V_1 = A_1 \cup h^{-1}(A_1)$. Since supp h is non-compact there is an element $x_2 \in W_h - V_1$.

Thus, inductively we get a locally finite sequence of disjoint cells $\coprod_{i \geq 1} C_i$ satisfying the desired property.

7.4. THEOREM.- Let h be an element of $\text{Diff}_f^\Omega(\mathbb{R}^n)$ with non-compact support. If $n \geq 3$ the normal subgroup generated by h is $\text{Diff}_f^\Omega(\mathbb{R}^n)$.

PROOF.- By 7.3 and 4.9 we know that there is a strip, V, and an element $h' \in N(h)$ such that $h'(V) \cap V = \phi$.

We will prove $\text{Diff}_f^\Omega(\mathbb{R}^n) \subset N(h)$. Let f be any element of $\text{Diff}_f^\Omega(\mathbb{R}^n)$. By 6.5 and 6.11 we have $f = f_1 \circ \ldots \circ f_{10}$ where $f_i \in \text{Diff}_f(\mathbb{R}^n)$ and has support in a strip of finite Ω-volume, for $i=1,\ldots, 10$. We can assume, by 3.14, that f is a finite product support. Then, there is a disjoint union of cells, $\coprod_{i \geq 1} C_i$, such that

$$(\coprod_{i \geq 1} C_i) \cap (\coprod_{i \geq 1} h(C_i)) = \phi$$

PROOF.- Let x_1 be any point of W_h. There is an open set $A_1 \subset \mathbb{R}^n$ with compact closure such that $x_1 \cup h(x_1) \subset A_1$. Also, there is a cell, $C_1 \subset A_1$, satisfying $x_1 \in \operatorname{int} C_1$, $h(C_1) \subset A_1$ and $C_1 \cap h(C_1) = \phi$. Let be $V_1 = A_1 \cup h^{-1}(A_1)$. Since supp h is non-compact there is an element $x_2 \in W_h - V_1$.

Thus, inductively we get a locally finite sequence of disjoint cells $\coprod_{i \geq 1} C_i$ satisfying the desired property.

7.4. THEOREM.- Let h be an element of $\mathrm{Diff}_f^{\Omega}(\mathbb{R}^n)$ with non-compact support. If $n \geq 3$ the normal subgroup generated by h is $\mathrm{Diff}_f^{\Omega}(\mathbb{R}^n)$.

PROOF.- By 7.3 and 4.9 we know that there is a strip, V, and an element $h' \in N(h)$ such that $h'(V) \cap V = \phi$.

We will prove $\mathrm{Diff}_f^{\Omega}(\mathbb{R}^n) \subset N(h)$. Let f be any element of $\mathrm{Diff}_f^{\Omega}(\mathbb{R}^n)$. By 6.5 and 6.11 we have $f = f_1 \circ \ldots \circ f_{10}$ where $f_i \in \mathrm{Diff}_f(\mathbb{R}^n)$ and has support in a strip of finite Ω-volume, for $i=1,\ldots, 10$. We can assume, by 3.14, that f is a finite product of elements of $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ each of which has support in a strip of Ω-volume less than $\mathrm{vol}_{\Omega}\, V$. Therefore each factor lies in $N(G_V)$ by 4.10 So, f is an element of $N(G_V)$.

As in proof of 7.1 we can see that

$$G_V \subset [G_V, G_V] \subset N(h') \subset N(h).$$

Therefore, $f \in N(G_V) \subset N(h)$.

7.5. COROLLARY.- If $n \geq 3$ there is no normal subgroup between $\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$.

7.6. REMARK.- If $n \geq 3$ we have the following chain of normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

$$\{id\} \text{ --- } \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n) \text{ --- } \mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n) \text{ --- } \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$$

where ——— means that there is no normal subgroup in between.

Also, we know by 2.6 that any subgroup N of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset N \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ is normal.

To obtain the same result for the normal subgroups of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ in the case of $\mathrm{vol}_{\Omega}\, \mathbb{R}^n = \infty$ as in the case $\mathrm{vol}_{\Omega}\, \mathbb{R}^n < \infty$

it remains to study the subgroups between $\mathrm{Diff}_f^{\Omega}(\mathbb{R}^n)$ and $\mathrm{Diff}_W^{\Omega}(\mathbb{R}^n)$.

The arguments used in this chapter do not work in this case because we know, by 3.12 , that any element h of $\mathrm{Diff}_W^{\Omega}(\mathbb{R}^n)$ can be decomposed as $h = h_5 \circ h_4 \circ \dots \circ h_1$ where $h_i \in \mathrm{Diff}_W^{\Omega}(\mathbb{R}^n)$, $\mathrm{supp}\, h_i \subset V_i$, for any $i=1,\dots, 5$. So, we can have one of the strips V_i of infinite Ω-volume. And on the other hand, given any element f of $\mathrm{Diff}_W^{\Omega}(\mathbb{R}^n)$ we do not know if there is a strip , V, of infinite Ω-volume and an element $f' \in N(f)$ such that $f'(V) \cap V = \phi$.

§8.- SOME ADDITIONAL FACTS

With the idea of studying the normal subgroups between $\mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ in the case that $\mathrm{vol}_{\Omega}\mathbb{R}^n = \infty$, we can consider $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ as a topological group with differents topologies and since the closure of any normal subgroup is itself a normal subgroup we can try to identify the closure of the normal subgroups in the chain

$$\{id\} \subset \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{f}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n) \subset \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$$

It is known that $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ is a topological group both with respect to the weak or compact-open C^{∞}-topology and with respect to the strong or Whitney C^{∞}-topology but not with respect to the uniform topology. We prove in (8.1) that $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$ is dense in $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with respect to the weak C^{∞}-topology. With respect to the Whitney C^{∞}-topology we prove that $\mathrm{Diff}^{\Omega}_{c}(\mathbb{R}^n)$ and $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$ are both closed (8.3) (8.4) .

Now we recall a description of the weak or compact-open C^{∞}-topology on $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$. Let f be an element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$, let K be a compact subset of $\mathbb{R}^n$ and let U be an open subset of $\mathbb{R}^n$ such that $f(K) \subset U$. For any $\varepsilon > 0$ we define

$$\begin{aligned} N^r(f; K,U,\varepsilon) = \{h \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n) : \ & h(K) \subset U , \\ & \| D^k(f)(x) - D^k(h)(x) \| < \varepsilon , \text{ for all } x \in K \\ & \text{and } k = 0,\dots,r \} . \end{aligned}$$

The sets $N^r(f; K,U,\varepsilon)$ for all possible K, U, ε form a subbase of neighbourhoods of f for the weak C^r-topology. We define the C^∞-topology as the union of the C^r-topologies for $r \geq 0$.

8.1.PROPOSITION.- $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$ is dense in $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with respect to the compact-open C^∞-topology.

PROOF. We will prove that the closure of $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$ is $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ by constructing an element h lying in $\mathrm{cl}\ \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$ but not in $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$. Then, by 7.2 we will have that

$$\mathrm{cl}\ \mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n) = \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$$

Let $\{C_i\}_{i\geq 1}$ be the family of closed balls of $\mathbb{R}^n$ of centre $(i,0,\dots,0)$ and radius $1/4$. Let $\psi_i : \mathbb{R} \to [0,1]$ be a bump function such that $\psi_i(r) = 0$ if either $-\infty < r \leq i - (1/4)$ or $i + (1/4) \leq r < +\infty$. For any $r \in \mathbb{R}$, we can define the matrix $M_i(r)$ as follows

$$M_i(r) = \begin{pmatrix} \cos \psi_i(r) & -\sin \psi_i(r) & 0 \\ \sin \psi_i(r) & \cos \psi_i(r) & \\ & & \\ & 0 & I \end{pmatrix}$$

Thus, the map $h_i : \mathbb{R}^n \to \mathbb{R}^n$ given by $h_i(x) = x.M_i(\|x\|)$ is a volume preserving diffeomorphism with support in C_i . Furthermore, there exists , $\psi_i^t : \mathbb{R} \to [0,1]$, a C^∞-family of bump functions such that for any t , $\psi_i^t(r) = 0$ if either $-\infty < r \leq i - (1/4)$ or

$i + (1/4) \leq r < +\infty$, $\psi_i^0(r) = 0$ for any $r \in \mathbb{R}$ and ψ_i^1 equals ψ_i ; the map $H_i : \mathbb{R}^n \times I \to \mathbb{R}^n$ given by $H_i(x,t) = x.M_i^t(\|x\|)$ is an Ω-isotopy from h_i to the identity with support in C_i . Therefore, h_i is an element of $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$.

Since , h_i has support in C_i for any i , we can define a new volume preserving diffeomorphism of $\mathbb{R}^n$, $h = \ldots \circ h_2 \circ h_1$. Clearly we have

$$W_h = \bigsqcup_{i \geq 1} (\operatorname{int} C_i - (i,0,\ldots,0)) .$$

So,

$$\mathrm{vol}_{\Omega} W_h = \mathrm{vol}_{\Omega} \bigsqcup_{i \geq 1} C_i = \sum_{i \geq 1} \mathrm{vol}_{\Omega} C_i = \infty .$$

Therefore, h does not lie in $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$.

On the other hand, h is the limit of the sequence $\{h_j \circ h_{j-1} \circ \ldots \circ h_1\}_{j \geq 1}$ with respect to the weak C^{∞}-topology. Since each element of the sequence lies in $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$, h lies in the closure of $\mathrm{Diff}^{\Omega}_{co}(\mathbb{R}^n)$ with respect to the weak C^{∞}-topology .

As an immediate consequence we have

8.2. COROLLARY.- The closure of any normal subgroup of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with respect to the compact-open C^{∞}-topology is the whole group $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

Now recall a description of the strong or Whitney C^{∞}-topology Let f be an element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$, let $\{U_i\}_{i \in \Lambda}$ be a locally finite set of open subsets of $\mathbb{R}^n$ and let $\{K_i\}_{i \in \Lambda}$ be a locally finite family of compact subsets of $\mathbb{R}^n$ such that $f(K_i) \subset U_i$ for any $i \in \Lambda$. For any family of positive numbers $\{\varepsilon_i\}_{i\in\Lambda}$ we define

$$N^r(f\ ;\ \{K_i\}_{i\in\Lambda},\{U_i\}_{i\in\Lambda},\{\varepsilon_i\}_{i\in\Lambda}) = \{h \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n) : \text{for all } i$$

$$h(K_i) \subset U_i\ , \| D^k(f)(x) - D^k(h)(x) \| < \varepsilon_i$$

$$\text{for all } x \in K_i \text{ and } k = 0,\dots,r\ \}\ .$$

The sets $N^r(f\ ;\ \{K_i\}_{i\in\Lambda},\{U_i\}_{i\in\Lambda},\{\varepsilon_i\}_{i\in\Lambda})$ for all possible families $\{K_i\}_{i\in\Lambda}$, $\{U_i\}_{i\in\Lambda}$, $\{\varepsilon_i\}_{i\in\Lambda}$, form a base of neigbourhoods of f for the strong C^r-topology. We define the Whitney C^∞-topology as the union of the C^r-topologies for $r \geqslant 0$.

8.3. PROPOSITION.- $\mathrm{Diff}_c^{\Omega}(\mathbb{R}^n)$ is closed in $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with respect to the Whitney C^∞-topology.

PROOF. Let h be any element of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ with non-compact support. We will construct a neighbourhood of h not intersecting $\mathrm{Diff}_c^{\Omega}(\mathbb{R}^n)$.

By 7.3 there is a disjoint union of cells, $\coprod_{i\geq 1} C_i$, such that

$$\Big(\coprod_{i\geq 1} C_i \Big) \cap \Big(\coprod_{i\geq 1} h(C_i) \Big) = \phi\ .$$

Without loss of generality we can assume that each cell C_i is a closed ball of centre x_i and radius r_i . Let C_i' be the closed ball of centre x_i and radius $r_i/2$.

We define

$$N(h\ ;\ \{\text{int } h(C_i)\},\{C_i'\},\{r_i/2\}\ ) = \{g \in \text{Diff}^{\Omega}(\mathbb{R}^n) : \text{for all } i$$

$$g(C_i') \subset \text{int } h(C_i)\ ,\| D^k(h)(x) - D^k(g)(x)\| < r_i/2$$

$$\text{for all } x \in C_i' \quad \text{and any} \quad k \geqslant 0\}$$

Obviously, it is a neigbourhood of h in $\text{Diff}^{\Omega}(\mathbb{R}^n)$ with the Whitney C^{∞}-toplogy.

It does not meet $\text{Diff}^{\Omega}_c(\mathbb{R}^n)$ since if f is an element of $\text{Diff}^{\Omega}_c(\mathbb{R}^n)$ there is some index $j \in \mathbb{N}$ such that $(\text{supp } f) \cap C_j = \phi$. Therefore, for any point $x \in C_j'$ we have $h(x) \neq x$ and $f(x) = x$, thus, $\| x - h(x)\| > r_j/2$. So, f is not an element of

$$N(\ h\ ;\ \{\text{int } h(C_i)\},\ \{C_i'\},\ \{r_i/2\}\ )\ .$$

8.4. PROPOSITION.- $\text{Diff}^{\Omega}_W(\mathbb{R}^n)$ is closed in $\text{Diff}^{\Omega}(\mathbb{R}^n)$ with respect to the Whitney C^{∞}-topology.

PROOF. We will use a similar argument to the one used before.

Let h be any element of $\text{Diff}^{\Omega}(\mathbb{R}^n)$ such that $\text{vol}_{\Omega}W_h = \infty$. By 6.2 there is a disjoint union of closed balls $\coprod_{i \geqslant 1} C_i$ such that

$$\text{vol}_{\Omega} \coprod_{i \geqslant 1} C_i = \infty \quad , \quad (\ \coprod_{i \geqslant 1} C_i\ ) \cap (\ \coprod_{i \geqslant 1} h(C_i)\ ) = \phi$$

and C_i is the ball of centre x_i and radius r_i. As above, let $\{C_i'\}$ be the family of closed balls of centre x_i and radius $r_i/2$.

Therefore, the subset

$$N(\ h\ ;\ \{\text{int } h(C_i)\},\ \{C_i'\}\ ,\ \{r_i/2\}\ )$$

defined as above, is a neighbourhood of h with respect to the Whitney

C^∞-topology. It does not meet $\mathrm{Diff}^\Omega_W(\mathbb{R}^n)$ since if f is an element of $\mathrm{Diff}^\Omega_W(\mathbb{R}^n)$ we have $\mathrm{vol}_\Omega W_f < \infty$; so, there is some $j \in \mathbb{N}$ and $x \in C_j'$ such that $f(x) = x$ .Thus, $\| h(x) - x \| > r_j/2$.

Then,

$$\mathrm{cl}\ \mathrm{Diff}^\Omega_W(\mathbb{R}^n) = \mathrm{Diff}^\Omega_W(\mathbb{R}^n) \quad .$$

Another way to study if there is any normal subgroup between $\mathrm{Diff}^\Omega_f(\mathbb{R}^n)$ and $\mathrm{Diff}^\Omega_W(\mathbb{R}^n)$ is to define a subset in between identifying the normal subgroup generated by it.In this sense we have been able to define a subgroup of $\mathrm{Diff}^\Omega(\mathbb{R}^n)$ that is normal in $\mathrm{Diff}^\Omega_W(\mathbb{R}^n)$ as follows.

8.5. PROPOSITION.- Let B_i be the closed ball of $\mathbb{R}^n$ of centre the origin and radius i . Then, the subset N of $\mathrm{Diff}^\Omega(\mathbb{R}^n)$ of all elements h $\mathrm{Diff}^\Omega_W(\mathbb{R}^n)$ such that the sequence

$$\left\{ \frac{\mathrm{vol}_\Omega((\mathrm{supp}\ h) \cap B_i)}{\mathrm{vol}_\Omega B_i} \right\} \quad i \in \mathbb{N}$$

tends to 0 as i grows to ∞ , is a normal subgroup of $\mathrm{Diff}^\Omega_W(\mathbb{R}^n)$ and

$$\mathrm{Diff}^\Omega_f(\mathbb{R}^n) \subset N \subset \mathrm{Diff}^\Omega_W(\mathbb{R}^n) \ .$$

PROOF. a) N is a group.

Let h and f be two elements of N . We have

$$\mathrm{supp}\,(f\circ h^{-1}) \subset \mathrm{supp}\, f \cup \mathrm{supp}\, h^{-1} = \mathrm{supp}\, f \cup \mathrm{supp}\, h \quad .$$

So,

$$\lim_{i\to\infty} \frac{\mathrm{vol}_\Omega((\mathrm{supp}\,(f\circ h^{-1}) \cap B_i)}{\mathrm{vol}_\Omega B_i} \leq$$

$$\leq \lim_{i\to\infty} \frac{\mathrm{vol}_\Omega((\mathrm{supp}\, f) \cap B_i)}{\mathrm{vol}_\Omega B_i} + \lim_{i\to\infty} \frac{\mathrm{vol}_\Omega((\mathrm{supp}\, h) \cap B_i)}{\mathrm{vol}_\Omega B_i} = 0 \quad .$$

Therefore, $f\circ h^{-1}$ lies in N .

b) N is normal in $\mathrm{Diff}_W^\Omega(\mathbb{R}^n)$.

Let h be any element of N and let g be any element of $\mathrm{Diff}_W^\Omega(\mathbb{R}^n)$. We have $\mathrm{supp}\,(g\circ h\circ g^{-1}) = g(\mathrm{supp}\, h)$. Also, we have,

$$g(\mathrm{supp}\, h) \cap B_i \subset (\mathrm{supp}\, h \cap B_i) \cup W_g \quad .$$

Then, since $\mathrm{vol}_\Omega W_g < \infty$ we have

$$\lim_{i\to\infty} \frac{\mathrm{vol}_\Omega(\mathrm{supp}\,(g\circ h\circ g^{-1}) \cap B_i)}{\mathrm{vol}_\Omega B_i} \leq$$

$$\leq \lim_{i\to\infty} \frac{\mathrm{vol}_\Omega((\mathrm{supp}\, h) \cap B_i)}{\mathrm{vol}_\Omega B_i} + \lim_{i\to\infty} \frac{\mathrm{vol}_\Omega W_g}{\mathrm{vol}_\Omega B_i} = 0 \quad .$$

Therefore, $g\circ h\circ g^{-1}$ lies in N . So, N is normal in $\mathrm{Diff}_W^\Omega(\mathbb{R}^n)$.

Clearly we have

$$\mathrm{Diff}_f^\Omega(\mathbb{R}^n) \subset N \subset \mathrm{Diff}_W^\Omega(\mathbb{R}^n) \quad .$$

8.6. PROPOSITION.- Let N be as above. Then N is not normal in $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$.

PROOF. We will construct an element h of N and we will find an element $f \in \mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that $f \circ h \circ f^{-1}$ does not lie in N .

Let T be the standard tube of $\mathbb{R}^n$. We will construct an element h of $\mathrm{Diff}^{\Omega}_W(\mathbb{R}^n)$ with support in T . The construction is similar to the one made in §2 .

Let $\{r_i\}_{i=1}^{\infty}$ be any ordering of the positive rational numbers and let be

$$\ell_i = \frac{1}{i^2}$$

we define I_1 the open interval of $\mathbb{R}$

$$I_1 = \left(r_1 - \frac{\ell_1}{2} \ , \ r_1 + \frac{\ell_1}{2} \right)$$

and A_1 the closed subset of T

$$A_1 = \{x \in T : \sum_{i\geq 2}^{n} x_i^2 \leq 1 \ , \ r_1 - \frac{\ell_1}{2} \leq x_1 \leq r_1 + \frac{\ell_1}{2} \}$$

Let n_2 be the smallest integer such that $r_{n_2} \notin \mathrm{cl}\, I_1$ and let $\ell'_2 < \ell_{n_2}$ be a positive number such that

$$\left(r_{n_2} - \frac{\ell'_2}{2} \ , \ r_{n_2} + \frac{\ell'_2}{2} \right) \cap I_1 = \phi \ ,$$

and we call

$$A_2 = \{x \in T : \sum_{i\geq 2}^{n} x_i^2 \leq 1 \ , \ r_{n_2} - \frac{\ell_2'}{2} \leq x_1 \leq r_{n_2} + \frac{\ell_2'}{2} \} \quad .$$

Inductively we get a family $\{A_i\}$ of closed subsets of T satisfying

a) $\coprod_{i\geq 1} A_i$ is dense in T .

b) $$\mathrm{vol}_\Omega(\coprod_{i\geq 1} A_i) = \sum_{i\geq 1} \mathrm{vol}_\Omega A_i = \sum_{i\geq 1} \ell_i' \mathrm{vol}_\Omega B^{n-1} \leq \mathrm{vol}_\Omega B^{n-1} \sum_{i\geq 1} \ell_i =$$

$$= \mathrm{vol}_\Omega B^{n-1} \sum_{i\geq 1} \frac{1}{i^2} < \infty \quad .$$

Also, as in the example that we get in §2 there is a smooth function $\psi\colon \mathbb{R} \to [0, \infty)$ such that

$$\psi^{-1}(0) = \mathbb{R} - \bigcup_{i\geq 1} I_i \quad .$$

Let $\phi\colon \mathbb{R} \to [0, 1]$ be a bump function such that $\phi(r) = 0$ for $-\infty < r \leq 0$ or $1 \leq r < +\infty$. We define, for any $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ the matrix

$$M(x) = \begin{pmatrix} I & & 0 \\ & \cos \phi(\sum_{i\geq 2} x_i^2)\psi(x_1) & -\sin \phi(\sum_{i\geq 2} x_i^2)\psi(x_1) \\ 0 & & \\ & \sin \phi(\sum_{i\geq 2} x_i^2)\psi(x_1) & \cos \phi(\sum_{i\geq 2} x_i^2)\psi(x_1) \end{pmatrix}$$

Then we define $h : \mathbb{R}^n \to \mathbb{R}^n$ the diffeomorphism given by $h(x) =$ $= x.M(x)$. It is clearly volume preserving and we have

$$W_h = \coprod_{i \geq 1} A_i$$

and supp $h = T$. Therefore, h lies in $\mathrm{Diff}^{\Omega}_{W}(\mathbb{R}^n)$.

Furthermore, h is an element of N since

$$\lim_{i\to\infty} \frac{\mathrm{vol}_\Omega((\mathrm{supp}\ h) \cap B_i)}{\mathrm{vol}_\Omega B_i} = \lim_{i\to\infty} \frac{\mathrm{vol}_\Omega(T \cap B_i)}{\mathrm{vol}_\Omega B_i} \leq \lim_{i\to\infty} \frac{(\mathrm{vol}_\Omega B^{n-1})i}{\mathrm{vol}_\Omega B_i} =$$

$$= \lim_{i\to\infty} \frac{i(\mathrm{vol}_\Omega B^{n-1})}{i^n(\mathrm{vol}_\Omega B^n)} = \lim_{i\to\infty} \frac{\mathrm{vol}_\Omega B^{n-1}}{i^{n-1}(\mathrm{vol}_\Omega B^n)} = 0$$

where B^n is the ball of centre the origin and radius 1 in $\mathbb{R}^n$ and B^{n-1} is the ball of centre the origin and radius 1 in $\mathbb{R}^{n-1}$.

Let V be the subset of $\mathbb{R}^n$

$$V = \{ x \in \mathbb{R}^n : x_1 \geq 0 \} .$$

There is, by 3.4 an element f of $\mathrm{Diff}^{\Omega}(\mathbb{R}^n)$ such that $f(T) = V$. Then, we have supp $f_\circ h_\circ f^{-1} = f(\mathrm{supp}\ h) = V$. Therefore,

$$\lim_{i\to\infty} \frac{\mathrm{vol}_\Omega((\mathrm{supp}\ f_\circ h_\circ f^{-1}) \cap B_i)}{\mathrm{vol}_\Omega B_i} = \lim_{i\to\infty} \frac{\mathrm{vol}_\Omega(V \cap B_i)}{\mathrm{vol}_\Omega B_i} =$$

$$\lim_{i\to\infty} \frac{(1/2)\mathrm{vol}_\Omega B_i}{\mathrm{vol}_\Omega B_i} = \frac{1}{2}$$

and $f_\circ h_\circ f^{-1}$ is not an element of N .

APPENDIX

In this appendix we prove an extension of the following theorem of Greene and Shiohama ([6]) .

A.1. THEOREM ([6]).- If M is a non-compact oriented manifold and if σ and τ are volume elements on M such that $\mathrm{vol}_\sigma M = \mathrm{vol}_\tau M$ and if each end of the manifold M has finite σ-volume if it has finite τ-volume and infinite σ-volume if it has infinite τ-volume, then there is a diffeomorphism $\psi : M \to M$ such that $\psi^*\sigma = \tau$.

The extension involves smooth families of volume elements on $\mathbf{R}^n$ as follows.

A.2.THEOREM.- Let V be a strip in $\mathbf{R}^n$ and let σ_t be a smooth family of volume elements on $\mathbf{R}^n$ such that, for any t , $\sigma_t = \sigma_0$ on $\mathbf{R}^n - \mathrm{int}\, V$, $\sigma_0 = \sigma_1$ and $\mathrm{vol}_{\sigma_t} V = \mathrm{vol}_{\sigma_0} V$. Then, there is an isotopy $\Psi_t : \mathbf{R}^n \to \mathbf{R}^n$, with support in V such that Ψ_0 and Ψ_1 are the identity and $\Psi_t^*\sigma_t = \sigma_0$ for any t .

The proof is based in the three following lemmas. The first one is an easy consequence of Moser [16] .

A.3.LEMMA.- Let σ_t be a smooth family of volume elements on $\mathbf{R}^n$. Let K be a compact subset of $\mathbf{R}^n$ with int K connected, such that all σ_t are equal on $\mathbf{R}^n - \mathrm{int}\, K$ and $\mathrm{vol}_{\sigma_t} K = \mathrm{vol}_{\sigma_0} K$, for any t .Then, there is a smooth isotopy $\psi_t : \mathbf{R}^n \to \mathbf{R}^n$ such that ψ_t is the identity outside K and $\psi_t^* \sigma_t = \sigma_0$, for any $t \in [0, 1]$. Furthermore, if $\sigma_1 = \sigma_0$ we can get ψ_0 and ψ_1 equal to the identity.

A.4.LEMMA.- Let σ_t be a smooth family of volume elements on $\mathbb{R}^n$. Let M be a connected compact submanifold of codimension one in $\mathbb{R}^n$ and let U be a tubular neighbourhood of M. Then, there is an isotopy $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$ such that ψ_t is the identity on $\mathbb{R}^n - U$, $\psi_t^* \sigma_t = \sigma_0$ on some neighbourhood of M in U, $\mathrm{vol}_{\psi_t^*\sigma_t} U_+ = \mathrm{vol}_{\sigma_0} U_+$ and $\mathrm{vol}_{\psi_t^*\sigma_t} U_- = \mathrm{vol}_{\sigma_0} U_-$ where U_+ and U_- are the connected components of $U - M$. Furthermore, if $\sigma_1 = \sigma_0$ we can get ψ_0 and ψ_1 equal to the identity.

PROOF. Let U' be a tubular neighbourhood of M with compact closure and $\mathrm{cl}\, U' \subset U$. There is a smooth function $G : \mathbb{R}^n \to \mathbb{R}$ and a smooth family of functions $F_t : \mathbb{R}^n \to \mathbb{R}$ with supports in U' satisfying :

a) G takes the value one on a neighbourhood of M and also F_t takes the value one on a neighbourhood of M.

b) $F_t(x) \leq 1$ for any $t \in [0, 1]$ and any $x \in \mathbb{R}^n$. Also, $G(x) \leq 1$ any $x \in \mathbb{R}^n$.

c) $\mathrm{vol}_{(1-G)\sigma_0 + F_t\sigma_t} U_+ \cap U' = \mathrm{vol}_{\sigma_0} U_+ \cap U'$

and

$$\mathrm{vol}_{(1-G)\sigma_0 + F_t\sigma_t} U_- \cap U' = \mathrm{vol}_{\sigma_0} U_- \cap U'$$

So, since $\mathrm{supp}\,(\sigma_0 - ((1 - G)\sigma_0 + F_t\sigma_t)) \subset U'$ we can apply A.3 to the smooth family of volume elements $(1 - G)\sigma_0 + F_t\sigma_t$

and we get an isotopy $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$ such that ψ_t is the identity on M - cl U' and $\psi_t^*\sigma_0 = (1 - G)\sigma_0 + F_t\sigma_t$. Therefore, $\psi_t^*\sigma_0 = \sigma_t$ near M .

Clearly, if we have $\sigma_1 = \sigma_0$, we get ψ_0 and ψ_1 equal to the identity.

A.5.LEMMA.- Let $\{K_i\}$ be a sequence of compact connected submanifolds of V with boundary such that $V = \bigcup_{i\geq 1} K_i$ and $K_i \cap K_j$ is either empty or a codimension one submanifold of V included in both boundaries. Let σ_t be a smooth family of volume elements on $\mathbb{R}^n$ such that all σ_t are equal on $\mathbb{R}^n$ - int V and $\mathrm{vol}_{\sigma_0} K_i = \mathrm{vol}_{\sigma_t} K_i$, for any i and $t \in [0 , 1]$. Then, there is an isotopy $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$ with support in V such that $\psi_t^*\sigma_t = \sigma_0$ for any $t \in [0 , 1]$ .Furthermore, if $\sigma_0 = \sigma_1$ we get $\psi_0 = \psi_1 = \mathrm{id}$.

PROOF. By A.4 there is a smooth isotopy $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ satisfying:

a) ϕ_t is the identity outside the union of disjoint tubular neighbourhoods of the connected components of the boundary of each K_i . So, the isotopy ϕ_t has support in V .

b) $\phi_t^*\sigma_t = \sigma_0$ on the union of some neighbourhoods of the boundary components.

c) $\mathrm{vol}_{\phi_t^*\sigma_t} K_i = \mathrm{vol}_{\sigma_0} K_i$ for any $t \in [0 , 1]$ and any i .

Applying A.3 to K_i we get an isotopy $\theta_t^i : K_i \to K_i$ such that θ_t^i is the identity on a neighbourhood of the boundary,

$\theta_t^{i*}\phi_t^*\sigma_t = \sigma_0$ on K_i for any $t \in [0, 1]$. We apply A.3 to each K_i.
Therefore, we have an isotopy $\theta_t : \mathbb{R}^n \to \mathbb{R}^n$ with support in V such that $\theta_t^*\phi_t^*\sigma_t = \sigma_0$ on K_i for any i.

Thus the isotopy $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\psi_t = \phi_t \circ \theta_t$ satisfy this lemma.

Notice that if $\sigma_0 = \sigma_1$ we get ψ_0 and ψ_1 equal to the identity.

PROOF OF A.2.- Let V be $g(T)$ where T is the standard tube of $\mathbb{R}^n$ and g an element of $\mathrm{Diff}(\mathbb{R}^n)$.

We define

$$K_i = g(\{x \in \mathbb{R}^n : \sum_{j=2}^{n} x_j^2 \leq 1 , i - 1 \leq x_1 \leq i\}) ,$$

we have $V = \bigcup_{i \geq 1} K_i$.

The proof is mainly an inductive modification of the smooth family of volume elements σ_t to satify the hypothesis of A.5.

There is an isotopy $\phi_t^1 : \mathbb{R}^n \to \mathbb{R}^n$, with compact support in V such that, for any t

$$\mathrm{vol}_{\sigma_0} K_1 = \mathrm{vol}_{\phi_t^{1*}\sigma_t} K_1 .$$

Since $\sigma_0 = \sigma_1$, we get that ϕ_0^1 and ϕ_1^1 are the identity.

Also, there is an isotopy $\phi_t^2 : \mathbb{R}^n \to \mathbb{R}^n$, with compact support in $\bigcup_{i=2}^{\infty} K_i$ such that, for any t

$$\mathrm{vol}_{\sigma_0} K_2 = \mathrm{vol}_{\phi_t^{2*}\circ\phi_t^{1*}\sigma_t} K_2 .$$

Thus, inductively we get an isotopy $\phi_t^k : \mathbb{R}^n \to \mathbb{R}^n$ with support in $\bigcup_{i=k}^{\infty} K_i$ such that, for any t

$$\mathrm{vol}_{\sigma_0} K_k = \mathrm{vol}_{\phi_t^{k*}\circ\cdots\circ\phi_t^{1*}\sigma_t} K_k \ .$$

Therefore we can define an isotopy with support in V $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$\phi_t|_{K_i} = \phi_t^1\circ\cdots\circ\phi_t^{i-1}\circ\phi_t^i \ .$$

We have that ϕ_1 and ϕ_0 are the identity and ϕ_t satisfy

$$\mathrm{vol}_{\phi_t^*\sigma_t} K_i = \mathrm{vol}_{\sigma_t} \phi_t(K_i) = \mathrm{vol}_{\sigma_t} \phi_t^1\circ\cdots\circ\phi_t^i(K_i) =$$

$$= \mathrm{vol}_{\phi_t^{i*}\circ\cdots\circ\phi_t^{1*}\sigma_t} K_i = \mathrm{vol}_{\sigma_0} K_i \ .$$

So, by A.5 we have an isotopy $\psi_t: \mathbb{R}^n \to \mathbb{R}^n$ with support in V such that for any $t \in [0 , 1]$

$$\psi_t^*\phi_t^*\sigma_t = \sigma_0$$

and ψ_1 and ψ_0 are the identity.

Then, $\Psi_t = \phi_t\circ\psi_t$ gives the isotopy we where looking for .

REFERENCES

1.- BANYAGA A. "Sur la structure du groupe des diffeomorphismes qui préservent une forme symplectique" Comm. Math. Helv. 53(1978) 174-227 .

2.- BERGER M. , GAUDUCHON P. , MAZET E. "Le spectre d'une variété Riemanniene" Lectures Notes in Math. 194(1971) Springer Verlag.

3.- CERF J. "Sur les diffeomorphismes de la sphere de dimension trois ($\Gamma_4 = 0$)" Lectures Notes in Math. 53(1968), Springer Verlag.

4.- CROWELL R.H. , FOX R.H. "Introduction to knot theory" Graduate Texts in Math. n°57 (1963) Sringer Verlag .

5.- EPSTEIN D.B.A. "The simplicity of certain groups of homeomorphisms" Compositio Math. 22(1970) 165-173 .

6.- GREENE R.E. , SHIOHAMA K. "Diffeomorphisms and volume preserving embeddings of non-compact manifolds" Trans. Amer. Math. Soc. 255(179) 403-414 .

7.- HALMOS P.R. "Measure theory" Graduate Texts in Math. n°18 (1974) Springer Verlag .

8.- HIRSCH M.W. "Differential topology" Graduate Texts in Math. n°33 (1976) Springer Verlag .

9.- KRYGIN A.B. "Continuation of diffeomorphisms preserving volume" Functional Analysis and its Applications 5(1971) 147-150 .

REFERENCES

1.- BANYAGA A. "Sur la structure du groupe des diffeomorphismes qui préservent une forme symplectique" Comm. Math. Helv. 53(1978) 174-227 .

2.- BERGER M. , GAUDUCHON P. , MAZET E. "Le spectre d'une variété Riemanniene" Lectures Notes in Math. 194(1971) Springer Verlag.

3.- CERF J. "Sur les diffeomorphismes de la sphere de dimension trois (Γ_4 = 0)" Lectures Notes in Math. 53(1968), Springer Verlag.

4.- CROWELL R.H. , FOX R.H. "Introduction to knot theory" Graduate Texts in Math. n°57 (1963) Sringer Verlag .

5.- EPSTEIN D.B.A. "The simplicity of certain groups of homeomorphisms" Compositio Math. 22(1970) 165-173 .

6.- GREENE R.E. , SHIOHAMA K. "Diffeomorphisms and volume preserving embeddings of non-compact manifolds" Trans. Amer. Math. Soc. 255(179) 403-414 .

7.- HALMOS P.R. "Measure theory" Graduate Texts in Math. n°18 (1974) Springer Verlag .

8.- HIRSCH M.W. "Differential topology" Graduate Texts in Math. n°33 (1976) Springer Verlag .

9.- KRYGIN A.B. "Continuation of diffeomorphisms preserving volume" Functional Analysis and its Applications 5(1971) 147-150 .

10.- LING W. "Simple and perfect groups of manifold automorphism germs" preprint .

11.- LING W. "Real analytic diffeomorphisms of $\mathbb{R}^n$, I" preprint .

12.- MATHER J. "Notes on topological stability" Harvard University (1970) .

13.- MATSUSHIMA Y. "Differentiable manifolds" Marcel Dekker (1972).

14.- McDUFF D. "The lattice of normal subgroups of the group of diffeomorphisms or homeomorphisms of an open manifold" J. London Math. Soc. (2), 18(1978) 353-364 .

15.- McDUFF D. "On the group of volume preserving diffeomorphisms of $\mathbb{R}^n$" Trans. Amer. Math. Soc. 261(1980) 103-113 .

16.- McDUFF D. "On the group of volume preserving diffeomorphisms and foliations with transverse volume form" Proc. London Math. Soc. 43 (1981) 295-320 .

17.- McDUFF D. "On tangle complexes and volume preserving diffeomorphisms of open 3-manifolds" Proc. London Math. Soc. 43(1981) 321-333 .

18.- MOSER J. "On the volume elements on a manifold" Trans. Amer. Math. Soc. 120(1965) 286-294 .

19.- NARASIMAN R. Analisis on real and complex manifolds" Advanced Studies in Pure Math.(1968) North-Holland .

20.- PALAIS R.S. "Local triviality of the restriction map for embeddings" Comm. Math. Helv. 34(1960) 305-312.

21.- SACKS S. "Theory of the integral" (1964) Dover.

22.- THURSTON W. "On the structure of the group of volume preserving diffeomorphisms" to appear .

20.- PALAIS R.S. "Local triviality of the restriction map for embeddings" Comm. Math. Helv. 34(1960) 305-312.

21.- SACKS S. "Theory of the integral" (1964) Dover.

22.- THURSTON W. "On the structure of the group of volume preserving diffeomorphisms" to appear .