FLOWS OF STOCHASTIC DYNAMICAL SYSTEMS:

ERGODIC THEORY OF STOCHASTIC FLOWS

ANDREW CARVERHILL

Thesis submitted for the degree of Ph.D at
the University of Warwick.

February, 1983

Mathematics Institute
University of Warwick
COVENTRY CV4 7AL
ENGLAND

CONTENTS

ACKNOWLEDGEMENT

First, I would like to thank my supervisor, Professor K.D. Elworthy, for the endless help, advice, and encouragement he has given me during my time at Warwick, and particularly in the preparation of this thesis.

Also, thanks to Professor L. Arnold and his group for allowing me to visit them in Bremen, West Germany, and for introducing me to the study of Ergodic Theory of Stochastic Systems on which this thesis is based.

Thanks also to the Science and Engineering Research Council of Great Britain for their financial support for two years of my studies at Warwick.

Finally, I would like to thank Terri Moss for an excellent typing job.

DECLARATION

I declare that no portion of this thesis has been previously submitted for any degree at any university or institute of learning. The contents are my own original work, except for material otherwise labelled.

Signed: Andrew Campbell

Date: 14 February 1983

SUMMARY

In this thesis we present results and examples concerning the asymptotic (large time)behaviour of the flow of a nondegenerate smooth stochastic dynamical system on a smooth compact manifold.

In Chapter 2 we prove a stochastic version of the Oseledec (Multiplicative Ergodic) Theorem for flows(theorem 2.1), in which we define the Lyapunov spectrum for the stochastic flow. Then we obtain stochastic analogues (Theorems 2.2.1, 2.2.2) of the Stable Manifold Theorems of Ruelle [16]. These theorems are proved by adapting Ruelle's techniques to our situation. Also we discuss the implications of 'Lyapunov stability', which we define to be the situation when the Lyapunov spectrum is strictly negative. In this situation the trajectories of the flow cluster in a certain way. (Proposition 2.3.3).

In Chapter 3 we give some examples of systems for which we can calculate the Lyapunov spectrum. We can choose our parameters such that these systems are Lyapunov stable, and in this case we can calculate the flows and their asymptotic behaviour completely.

In Chapter 4 we give a formula for the Lyapunov numbers which is analogous to that of Khas'minskii [9] for a linear system. Then we use this formula to prove a theorem on the preservation of Lyapunov stability under a stochastic perturbation.

1. INTRODUCTION

1.1 Aims

Our aim in this article is to present results and examples concerning the asymptotic (large time) behaviour of the flow of a nondegenerate smooth stochastic dynamical system (SDS) on a smooth compact manifold. The definition and existence of such (stochastic) flows is given in Carverhill and Elworthy [4], which we will use as our standard reference. See also Kunita [11], [12] Ikeda and Watanabe [8], Bismut [3], Elworthy [6].

In Chapter 2 we define the Lyapunov spectrum for the stochastic flow (Theorem 2.1) and obtain analogues (Theorems 2.2.1, 2.2.2) for the stochastic flow, of the stable manifold theorems of Ruelle [16]. These theorems are proved by adapting Ruelle's techniques to our situation. Also (Section 2.3) we discuss the implications of 'Lyapunov stability', which we define to be the situation when the Lyapunov spectrum is bounded above by a strictly negative constant. We see that in this situation, the trajectories of the flow cluster in a certain way. (Proposition 2.3.3).

In Chapter 3 we give some examples of SDS's for which we can calculate the Lyapunov spectrum. We can choose our parameters such that these systems are Lyapunov stable, and in fact in this case we can calculate the flows and their asymptotic behaviour completely.

In Chapter 4 we give a formula for the Lyapunov numbers which is analogous to that of Khas'minskii [9] for a linear system. Then we use this formula to prove a theorem on the preservation of Lyapunov stability under a stochastic perturbation. (For this we need an assumption about the invariant measure (see Section 1.2) which is studied by Ventsel and Freidlin [18].)

The extension of Ruelle's work to the case of a stochastic flow was suggested by Arnold and Kliemann [1], p. 81, though they have in mind real noise rather than white noise (Brownian motion). Also, their approach is to linearise the system about a stationary solution, whereas ours is to consider the derivative flow. Note that our work can be adapted to the case of real, time homogeneous noise.

The Lyapunov spectrum, and the asymptotic behaviour in the case of a linear SDS on $\mathbb{R}^m$ has been studied by Khas'minskii [9], [10] (see our Section 4.1). The linear case, and the linearisation of an SDS on $\mathbb{R}^m$ about an almost surely fixed point x_o of the flow, can be studied using the ideas of this article. (Essentially, we take the invariant measure for the system to be just δ_{x_o}. It does not matter in this case that the space $\mathbb{R}^m$ of the system is noncompact). See Arnold and Kliemann [1].

Note that in the case of a linear system, Lyapunov stability implies that the solution starting from any point tends a.s. to the trivial (zero) solution. In our situation (i.e. for a nondegenerate system) Lyapunov stability implies that the system. Then we use this formula to prove a theorem on the preservation of Lyapunov stability under a stochastic perturbation. (For this we need an assumption about the invariant measure (see Section 1.2) which is studied by Ventsel and Freidlin [18].)

The extension of Ruelle's work to the case of a stochastic flow was suggested by Arnold and Kliemann [1], p. 81, though they have in mind real noise rather than white noise (Brownian motion). Also, their approach is to linearise the system about a stationary solution, whereas ours is to consider the derivative flow. Note that our work can be adapted to the case of real, time homogeneous noise.

The Lyapunov spectrum, and the asymptotic behaviour in the case of a linear SDS on $\mathbb{R}^m$ has been studied by Khas'minskii [9], [10] (see our Section 4.1). The linear case, and the linearisation of an SDS on $\mathbb{R}^m$ about an almost surely fixed point x_o of the flow, can be studied using the ideas of this article. (Essentially, we take the invariant measure for the system to be just δ_{x_o}. It does not matter in this case that the space $\mathbb{R}^m$ of the system is noncompact). See Arnold and Kliemann [1].

Note that in the case of a linear system, Lyapunov stability implies that the solution starting from any point tends a.s. to the trivial (zero) solution. In our situation (i.e. for a nondegenerate system) Lyapunov stability implies that the trajectories of the flow cluster into groups (see Proposition 2.3.3), but the groups themselves move randomly.

We assume throughout that our system is nondegenerate (see Section 1.2).

1.2 Preliminary Results

In this section we establish some notation and summarise some standard results which we will need in the sequel. The notation is as in Carverhill and Elworthy [4], which is our standard reference.

We deal mostly with a Stratonovitch SDS (X,z) defined on a smooth, compact manifold M of dimension m, and driven by an n-dimensional Brownian motion B_t and a drift. When it is necessary to deal with an Itô system, we will denote it by $I(X,z)$. Thus, X is a bundle morphism $M \times \mathbb{R}^{n+1} \to TM$, i.e. a map such that $X(x) \in \mathbf{L}(\mathbb{R}^{n+1}, T_xM)$ for each $x \in M$. We will put $X(x) \circ dz_t = Y(x) \circ dB_t + A(x)dt$ (i.e. we will denote the noise by Y and drift by A), and we will assume throughout that (X,z) is nondegenerate, i.e. $Y(x) \in \mathbf{L}(\mathbb{R}^n, T_xM)$ is surjective for each $x \in M$.*

We must impose certain smoothness conditions on (X,z). These refer to the smoothness of X as a map $M \to \mathbf{L}(\mathbb{R}^{n+1}, TM)$. We are not concerned about the precise smoothness conditions required: all these conditions will be satisfied if X is C^∞.

The solution $\xi_t(\omega,x)$ of the SDS (X,z), starting from $x \in M$, is defined to be that stochastic process in M which

* Note (added by examiners)
All results hold also for degenerate systems, where ρ is any invariant measure. The existence of such a ρ follows by compactness.

satisfies the following equation in $\mathbb{R}$, where ϕ is any C^2 map $M \to \mathbb{R}$:

$$\phi\xi_t(\omega,x) = \phi x + \int_0^t d\phi X(\xi_s(\omega,x)) \circ dz_s(\omega).$$

(This is equivalent to Elworthy's [6] definition, in which ϕ is taken to be a chart map $U \to U_\phi \subset \mathbb{R}^m$ (U open in M) and the integral is taken up to time $t \wedge$(First exit time from U)). The solution exists and is unique up to equivalence if (X,z) is of class C^2. (See Elworthy [6], Ikeda and Watanabe [8], Kunita [12]).

The flow of (X,z) will be denoted by $\xi_t(\omega)$. It is defined to be such that a.s. $\xi_t(\omega)$ is a continuous map $M \to M$ for all t, and such that for each x, $\xi_t(\omega)x$ is a solution starting from x of (X,z). The existence and a.s. uniqueness of the flow if (X,z) is of class H^{s+2} $(s > (m/2)+1)$ is given in Carverhill and Elworthy [4]. See also [6], [8] and [12]. The flow is a.s. for all t an H^s diffeomorphism.

Denote the underlying probability space for B_t by (Ω,F,P), taking Ω to be the set of continuous paths in $\mathbb{R}^n$ starting from the origin, and denote the time shift by time s on Ω by θ_s (i.e. $(\theta_s(\omega))(t) = \omega(t+s) - \omega(s)$). The following proposition is immediate from the time homogeneity of the Brownian motion and the a.s. uniqueness of the flow.

Proposition 1.2.1

If the flow $\xi_t(\omega)$ of (X,z) exists, then for each $s > 0$ we have a.s.

$$\xi_t(\theta_s(\omega))\xi_s(\omega) = \xi_{s+t}(\omega) \text{ for all } t > 0. \qquad //$$

We will work extensively with invariant probability measures for (X,z) on M, i.e. measures ρ such that if $\{p_t(x,A) : A \in \mathcal{B}(M)\}$ denotes the transition probabilities for (X,z), then for any $B \in \mathcal{B}(M)$, we have $\rho(B) = \int_{x\in M} p_t(x,B)d\rho(x)$. Here, $\mathcal{B}(M)$ denotes the Borel σ- algebra over M. If (X,z) is nondegenerate, the solutions are metrically transitive, and there exists a unique invariant measure for (X,z). See Doob [5], Yosida [19] Chapter 13.

Denote the time shift by time s on $C(\mathbb{R}^{>0},M)$ by θ_s (thus if $f \in C(\mathbb{R}^{>0},M)$ then $(\theta_s f)(t) = f(t+s)$ for all $t > 0$). Now any invariant measure ρ for (X,z) on M induces a measure Q_ρ on $C(\mathbb{R}^{>0},M)$ (the Markov measure) which is invariant under θ_s for any $s > 0$. On cylinder sets, Q_ρ is given by

$$Q_\rho\{f \in C(\mathbb{R}^{>0},M) : f(t_i) \in B_i;\ i = 1,\dots,p;\ B \in \mathcal{B}(M)\}$$

$$= \int_{x_1\in B_1} \cdots \int_{x_p\in B_p} d\rho(x_1)p_{t_2-t_1}(x_1,dx_2)\dots p_{t_p-t_{p-1}}(x_{p-1},dx_p)$$

If (X,z) is nondegenerate, then for each $s > 0$, θ_s is ergodic with respect to the measure Q_ρ. (Doob [5]).

In Sections 1-5 of [16], Ruelle works with an 'abstract' measure space (M,Σ,ρ), measure preserving map $\tau: M \to M$, and measurable maps

$$T: M \to \mathbf{L}(\mathbb{R}^m, \mathbb{R}^m) \text{ and } F: M \to C^{r,\theta}(\bar{B}(1), 0; \mathbb{R}^m, 0).$$

In Section 6 he applies his results to the case when M is a smooth compact manifold, f is a $C^{r,\theta}$ diffeomorphism of M, $T = Tf$, $F = f$ over charts (essentially), and ρ is a measure on M which is preserved by f. In our Chapter 2, we apply the results of Ruelle's Sections 1-5, taking the measure space to be the product $(M \times \Omega\ , \mathcal{B}(M) \otimes F, \rho \otimes P)$, with the map ϕ_s (any $s > 0$) corresponding to Ruelle's τ, where
$\phi_s(x,\omega) = (\xi_s(\omega)x, \theta_s(\omega))$.

<u>Proposition 1.2.2</u>

For any $s > 0$, the map $\phi_s(x,\omega)$ preserves the measure $\rho \otimes P$ on $M \times \Omega$.

<u>Proof</u>

It suffices to show that for any $B \in \mathcal{B}(M)$, $\Lambda \in F$, the set $\phi_s^{-1}(B \times \Lambda)$ has measure $\rho(B).P(\Lambda)$. But

$$\rho \otimes P(\phi_s^{-1}(B \times \Lambda))$$

$$= \rho \otimes P\{(x,\omega) : \xi_s(\omega)x \in B\ ,\ \theta_s(\omega) \in \Lambda\}.$$

$$= \rho \otimes P\{(x,\omega) : \xi_s(\omega)x \in B\}.\ \rho \otimes P\{(x,\omega) : \theta_s(\omega) \in \Lambda \mid \xi_s(\omega)x \in B\}$$

$$= \int_{x \in M} p_s(x,B)\,d\rho(x).\ \rho \otimes P\{(x,\omega) : \theta_s(\omega) \in \Lambda\}$$

(N.B. The events of the conditional probability are independent).

$$= \rho(B).P(\Lambda). \quad //$$

The following result is a consequence of the ergodic theorem applied to the Markov process given by the solution of the SDS. See Yosida [19] Chapter 13, Doob [5].

<u>Proposition 1.2.3</u> (Strong law of large numbers (Special case))

Suppose (X,z) is nondegenerate with unique invariant measure ρ on M. Then for $\rho \otimes P$ - almost every $(x,\omega) \in M \times \Omega$ we have

$$1/T \int_0^T g(\xi_t(\omega,x))dt \to \int_{y \in M} g(y)d\rho(y) \text{ as } t \to \infty$$

for any $g \in C(M,\mathbb{R})$.

2. STABLE MANIFOLD THEOREMS FOR THE STOCHASTIC FLOW

In this chapter we apply the results of Ruelle [16], Sections 1-5, to obtain stochastic analogues of the results of his Section 6. We take the product measure space $(M \times \Omega,\ \mathcal{B}(M) \otimes F,\ \rho \otimes P)$ to correspond to Ruelle's 'abstract' measure space (M,Σ,ρ).

Throughout his Sections 1-6, Ruelle discusses the discrete time situation. However, our results apply to continuous time. Our technique is first to prove the results for discrete time increments $T > 0$, taking the map $\Phi_T: M \times \Omega \to M \times \Omega$ to correspond to Ruelle's τ, and then to proceed to continuous time in a similar way to Ruelle's Appendix B. Corresponding to the $C^{r,\theta}$ diffeomorphism f of Ruelle's Section 6 we take the stochastic flow $\xi_T(\omega)$ at time T.

2.1 The Oseledec (Multiplicative Ergodic) Theorem for the Derivative of the Stochastic Flow

Theorem 2.1 (Cf. Ruelle [16], Theorem 1.6, and introduction to Section 6).

Consider the nondegenerate H^{s+4} SDS (X,z) on the smooth compact manifold M, where $\dim M = m$ and $s \geq 3m/2 + 1$. Choose a version of the flow $\xi_t(\omega): M \to M$.

Then there exists a set $\Gamma \subset M \times \Omega$ of full $\rho \otimes P$-measure such that for each $(x,\omega) \in \Gamma$ we have a Lyapunov spectrum

$$\lambda^{(1)}_{(x,\omega)} < \lambda^{(2)}_{(x,\omega)} < \ldots < \lambda^{(r)}_{(x,\omega)}$$

and associated filtration $V^{(1)}_{(x,\omega)} < \ldots < V^{(r)}_{(x,\omega)}$ of T_xM. Thus, if $v \in V^{(i)}_{(x,\omega)} \setminus V^{(i-1)}_{(x,\omega)}$ then

$$1/t \log \|T\xi_t(\omega)v\| \to \lambda^{(i)}_{(x,\omega)} \text{ as } t \to \infty.$$

(Here, $\|\cdot\|$ denotes any norm on the tangent space coming from a Riemannian inner product . Also, we have taken $V^{(0)} = \{0\}$.)

Proof

(1) It is convenient to work in a flat space. Thus, we embed M in $\mathbb{R}^{\tilde{m}}$ for some $\tilde{m} > m$, and extend X to $\tilde{X}$ on $\mathbb{R}^{\tilde{m}}$ in the following way:

Choose $\tau > 0$ such that the set M_τ of points in $\mathbb{R}^{\tilde{m}}$ less than a distance τ from M is a tubular neighbourhood of M. Then for any $x \in M_\tau$, the nearest point $y \in M$ to x is unique and the line xy is perpendicular to M. Also for any other $z \in M$, if xz is perpendicular to M then $d(x,z) > \tau$. Take a smooth bump function $f: \mathbb{R} \to \mathbb{R}^{>0}$, supported on $[-\tau,\tau]$ and such that $f(0) = 1$. Now, for any $x \in \mathbb{R}^{\tilde{m}}$ if $x \notin M_T$, set $\tilde{X}(x) = 0$; otherwise take the nearest point y to x in M and put $\tilde{X}(x)e = f(|y-x|)X(y)e$.

Clearly, $\tilde{X}$ is just as smooth as X and is supported on a bounded domain say U in $\mathbb{R}^{\tilde{m}}$. Also, for any $x \in \mathbb{R}^{\tilde{m}}$, the solution to $(\tilde{X},z)$ starting from x remains at a constant distance from M a.s. (To see this, note that it is easily true for a deterministic system, and therefore for a piecewise linear approximation to our system, and consider Carverhill and Elworthy [4], Theorem 2.3). Denote the flow of $(\tilde{X},z)$ by $\tilde{\xi}_t(\omega)$ and let $\tilde{\phi}_t$ for $(\tilde{X},z)$ correspond to ϕ_t for (X,z). Now the choice of Riemannian metric on M is unimportant; we will take it to be that induced from $\mathbb{R}^{\tilde{m}}$ by the embedding.

(2) We prove our result first for discrete time increments, say of length T.

Denote by $G_o(x,\omega)$ the linear map $D\tilde{\xi}_T(\omega)x$, and put $G_n(x,\omega) = G_o(\tilde{\Phi}_{nT}(x,\omega))$, so that a.s. we have $G_n(x,\omega) = D[\tilde{\xi}_T(\theta_{nT}(\omega))]\tilde{\xi}_{nT}(\omega)x = D[\tilde{\xi}_{(n+1)T}(\omega).(\tilde{\xi}_{nT}(\omega))^{-1}]\tilde{\xi}_{nT}(\omega)x$. Also, put $G^n(x,\omega) = G_{n-1}(x,\omega) \circ \dots \circ G_o(x,\omega)$ so that a.s. we have $G^n(x,\omega) = D\tilde{\xi}_{nT}(\omega)x$.

From our Appendix A (Proposition A.1.2) we have that for each standard basis vector $e_1,\dots,e_{\tilde{m}}$ in $\mathbb{R}^{\tilde{m}}$, $\int_{\omega\in\Omega} \|G_o(x,\omega)e_i\|\, dP$ is bounded uniformly over $x \in M$. (N.B. by the Sobolev embedding theorem, since X is H^{s+4}, it is C^2). Now,

$$\|G_o(x,\omega)\| < \sum_{i=1}^{\tilde{m}} \|G_o(x,\omega)e_i\|,$$

therefore we can deduce that

$$\int_{x\in U}\int_{\omega\in\Omega} \log^+ \|G_o(x,\omega)\|\, d\rho(x)\, dP(\omega) < \infty .$$

($\|G_o(x,\omega)\|$ -operator norm on $L(\mathbb{R}^{\tilde{m}},\mathbb{R}^{\tilde{m}})$.

By $\log^+(x)$ we mean $\max\{0, \log x\}$. Here, ρ is regarded as a measure on U, supported on M).

For each $q = 1,\ldots,\tilde{m}$, consider $\|G_o(x,\omega)^{\wedge q}\|$. ($\wedge q$-qth exterior power). Denoting the eigenvalues of $\left[G_o(x,\omega)^* G_o(x,\omega)\right]^{1/2}$ by $t^{(1)}\le\ldots\le t^{(\tilde{m})}$, we have $\|G_o(x,\omega)^{\wedge q}\| = \prod_{p=\tilde{m}-q+1}^{\tilde{m}} t^{(p)}$, so that

$\log^+ \|G_o(x,\omega)^{\wedge q}\| < \sum_{p=\tilde{m}-q+1}^{\tilde{m}} \log^+ t^{(p)}$. Also, for each p,

$t^{(p)} < \|G_o(x,\omega)\|^2$. Therefore for each $q = 1,\ldots,\tilde{m}$, we have

$$\int_{x\in U}\int_{\omega\in\Omega} \log^+\|G_o(x,\omega)^{\wedge q}\|\, d\rho(x)dP(\omega) < \infty ,$$

and by the subadditive ergodic theorem (Ruelle, Theorem 1.1) applied to $\log \|G^n(x,\omega)^{\wedge q}\|$, $1/n \log \|G^n(x,\omega)^{\wedge q}\|$ tends to a limit a.s., which is invariant under the map Φ_T. Also, by the usual ergodic theorem, $1/n \sum_{i=0}^{n-1} \log^+ \|G_i(x,\omega)\|$ tends to a limit a.s., therefore $\limsup_{n\to\infty} 1/n \log \|G_n(x,\omega)\| < 0$ a.s..

These results allow us to apply Ruelle's Proposition 1.3 for each (x,ω) a.s., taking our $G_i(x,\omega)$ to correspond to

Ruelle's T_n, from which we deduce that there exists a.s. a spectrum $\tilde{\lambda}^{(1)}_{(x,\omega)} < \dots < \tilde{\lambda}^{(\tilde{s})}_{(x,\omega)}$, and filtration $\tilde{V}^{(1)}_{(x,\omega)} < \dots < \tilde{V}^{(\tilde{s})}_{(x,\omega)} \equiv \mathbb{R}^{\tilde{m}}$, such that if $v \in \tilde{V}^{(i)} \setminus \tilde{V}^{(i-1)}$, then $1/nT \log \|G^n(x,\omega)\| \to \tilde{\lambda}^{(i)}$. The discrete time T version of our theorem follows from this by interesecting each $\tilde{V}^{(\tilde{s})}_{(x,\omega)}$ with T_xM, using the fact that $T\tilde{\xi}_t(\omega):TU \to TU$ maps tangent spaces of M to tangent space a.s..

(3) To obtain the full (continuous time) result, we proceed as in Ruelle's Appendix B. For each $0 < s < t$, put $\tilde{\xi}_{st}(\omega) = \tilde{\xi}_t(\omega).\tilde{\xi}_s(\omega)^{-1}$. Then for fixed $s, \tilde{\xi}_{st}(\omega)$ is the 'nice' version of the flow of $(\tilde{X},z)$ from time s given in Carverhill and Elworthy [4] Lemma 6.1. Also, we have a.s. (independently of n,t) that

$$\begin{cases} \tilde{\xi}_t(\omega) = \tilde{\xi}_{nT,t}(\omega).\tilde{\xi}_{nT}(\omega), \\ \tilde{\xi}_{(n+1)T}(\omega) = \tilde{\xi}_{t,(n+1)T}(\omega).\tilde{\xi}_t(\omega), \end{cases}$$

for all n, and all $t \in [nT,(n+1)T]$. Hence we have a.s.,

$$\begin{cases} \log \|[D\tilde{\xi}_t(\omega)x]\ v\| < \log \|D[\tilde{\xi}_{nT,t}(\omega)]\tilde{\xi}_{nT}(\omega)x\| \\ \qquad\qquad + \log \|[D[\tilde{\xi}_{nT}(\omega)]x]v\|, \\ \log \|D[\tilde{\xi}_t(\omega)x]\ v\| > \log \|[D[\tilde{\xi}_{(n+1)T}(\omega)]x]v\| \\ \qquad\qquad - \log \|D[\tilde{\xi}_{t,(n+1)T}(\omega)]\tilde{\xi}_t(\omega)x\|, \end{cases}$$

for all $t \in [nT,(n+1)T]$, $x \in U$, $v \in \mathbb{R}^{\tilde{m}}$.

Thus, if we put

$$\begin{cases} \phi_1(x,\omega) = \sup\limits_{t\in[0,T]} \log \|D[\tilde{\xi}_t(\omega)]x\| , \\ \\ \phi_2(x,\omega) = \sup\limits_{t\in[0,T]} \log \|D[\tilde{\xi}_{t,T}(\omega)]\xi_t(\omega)x\| ; \end{cases}$$

then we have a.s., independently of x,v,n, that

$$\log \|[D[\tilde{\xi}_{(n+1)T}(\omega)]x]v\| - \phi_2(\tilde{\Phi}_{nT}(x,\omega))$$

$$< \log \|[D[\tilde{\xi}_t(\omega)]x]v\|$$

$$< \log \|[D[\tilde{\xi}_{nT}(\omega)]x]v\| + \phi_1(\tilde{\Phi}_{nT}(x,\omega))$$

for all $t \in [nT,(n+1)T]$.

(Here, ϕ_1, ϕ_2 correspond to the functions of (B.1), (B.2) in Ruelle's Appendix B).

Now, ϕ_1, ϕ_2 are $\rho \otimes P$-integrable. This follows from our Appendix, Proposition A.1.2 for ϕ_1, and Proposition A.2.3 for ϕ_2. (N.B. it is clear that $\phi_i > 0$ a.s.). Therefore by the ergodic theorem,

$$1/n \; \phi_i(\tilde{\Phi}_{nT}(x,\omega)) \to 0 \text{ as } n \to \infty \text{ for a.e. } (x,\omega).$$

For these (x,ω), $\lim\limits_{t\to\infty} 1/t \; \|[D[\tilde{\xi}_t(\omega)]x]v\| =$

$\lim\limits_{n\to\infty} 1/nT \log \|[D[\tilde{\xi}_{nT}(\omega)]x]v\|$ and the continuous time result follows. //

Notes

(i) From the proof we see that the Lyapunov spectrum is a.s. invariant under the map ϕ_t. The question arises of the nature of the dependence of the spectrum on (x,ω). In our Appendix B we discuss this question and show that to some extent the spectrum is constant, i.e. independent of (x,ω) a.s.. We conjecture that it is a.s. constant for all nondegenerate systems.

(ii) Ruelle's approach to dealing with a manifold, rather than a flat space, is different from ours. Rather than embed M in $\mathbb{R}^{\tilde{m}}$, he takes a finite Borel partition of M which trivialises TM, and uses this to express the derivatives as linear maps in $\mathbb{R}^m$, which are piecewise continuous in $x \in M$. (See Ruelle, proof of Theorem 6.1, and Appendix D, note (6)). Ruelle's approach in our situation would require more difficult $\rho \otimes P$-integrability estimates than ours.

2.2 The Stable Manifold Theorems

Theorem 2.2.1 (Local Stable Manifold Theorem. Cf. Ruelle [16] Theorem 6.1).

Consider the SDS (X,z) and flow $\xi_t(\omega)$ of Theorem 2.1, but suppose that X is of class H^{s+4}, $s > 3m/2+k$, $k \geq 2$. Take $\lambda < 0$, and assume λ is a.s. disjoint from the Lyapunov spectrum.

Then we have a set $\Gamma^{\lambda} \subset \Gamma$ of full $\rho \otimes P$-measure, and measurable functions $\alpha,\beta,\gamma:\Gamma^{\lambda} \to \mathbb{R}^{>0}$ such that if we denote by $V^{\lambda}_{(x,\omega)}(\alpha(x,\omega))$ the set

$$\{y \in \bar{B}(x,\alpha(x,\omega)): d(\xi_t(\omega)y,\xi_t(\omega)x) < \beta(x,\omega)e^{\lambda t}$$

for all $t > 0\}$ (Here, by $\bar{B}(x,\alpha)$ we mean the closed ball at x in M, of radius α), then:

(a) $V^{\lambda}_{(x,\omega)}(\alpha(x,\omega))$ is a C^k submanifold of $\bar{B}(x,\alpha(x,\omega))$ which is tangent at x to $V^{(i)}_{(x,\omega)}$, where i is such that $\lambda^{(i)}_{(x,\omega)} < \lambda < \lambda^{(i+1)}_{(x,\omega)}$.

(b) If $y,z \in V^{\lambda}_{(x,\omega)}(\alpha(x,\omega))$ then

$$d(\xi_t(\omega)y,\ \xi_t(\omega)z) < \gamma(x,\omega)\ d(y,z)e^{\lambda t}.$$ for $(x,\omega) \in \Gamma^{\lambda}$

(b') If $\lambda' < \lambda$ and $[\lambda',\lambda]$ is disjoint from the spectrum, Λ, then there exists a measurable map $\gamma': \Gamma^{\lambda} \to \mathbb{R}^{>0}$ such that if $y,z \in V^{\lambda}_{(x,\omega)}(\alpha(x,\omega))$ then $d(\xi_t(\omega)y,\xi_t(\omega)z) < \gamma'(x,\omega).d(y,z)e^{\lambda' t}$.

Proof

We retain the notation of the proof of Theorem 2.1.

(1) Embed M in $\mathbb{R}^{\tilde{m}}$ and extend (X,z) to $(\tilde{X},z)$ supported on $U \subset \mathbb{R}^{m}$, as in the proof of Theorem 2.1. Take the Riemannian metric on M to be that induced from $\mathbb{R}^{\tilde{m}}$ by the embedding. It is clear from the fact that solutions in U remain a.s. at a constant distance from M that if $x \in M$, $v \in T_xU \setminus T_xM$, then

Then we have a set $\Gamma^\lambda \subset \Gamma$ of full $\rho \otimes P$-measure, and measurable functions $\alpha, \beta, \gamma : \Gamma^\lambda \to \mathbb{R}^{>0}$ such that if we denote by $V^\lambda_{(x,\omega)}(\alpha(x,\omega))$ the set

$$\{y \in \bar{B}(x,\alpha(x,\omega)) : d(\xi_t(\omega)y, \xi_t(\omega)x) < \beta(x,\omega)e^{\lambda t}$$

for all $t > 0\}$ (Here, by $\bar{B}(x,\alpha)$ we mean the closed ball at x in M, of radius α), then:

(a) $V^\lambda_{(x,\omega)}(\alpha(x,\omega))$ is a C^k submanifold of $\bar{B}(x,\alpha(x,\omega))$ which is tangent at x to $V^{(i)}_{(x,\omega)}$, where i is such that $\lambda^{(i)}_{(x,\omega)} < \lambda < \lambda^{(i+1)}_{(x,\omega)}$.

(b) If $y,z \in V^\lambda_{(x,\omega)}(\alpha(x,\omega))$ then

$$d(\xi_t(\omega)y,\ \xi_t(\omega)z) < \gamma(x,\omega)\ d(y,z)e^{\lambda t}.$$

for $(x,\omega) \in \Gamma^\lambda$

(b') If $\lambda' < \lambda$ and $[\lambda',\lambda]$ is disjoint from the spectrum, Λ, then there exists a measurable map $\gamma' : \Gamma^\lambda \to \mathbb{R}^{>0}$ such that if $y,z \in V^\lambda_{(x,\omega)}(\alpha(x,\omega))$ then $d(\xi_t(\omega)y,\xi_t(\omega)z) < \gamma'(x,\omega).d(y,z)e^{\lambda' t}$.

<u>Proof</u>

We retain the notation of the proof of Theorem 2.1.

(1) Embed M in $\mathbb{R}^{\tilde{m}}$ and extend (X,z) to $(\tilde{X},z)$ supported on $U \subset \mathbb{R}^m$, as in the proof of Theorem 2.1. Take the Riemannian metric on M to be that induced from $\mathbb{R}^{\tilde{m}}$ by the embedding. It is clear from the fact that solutions in U remain a.s. at a constant distance from M that if $x \in M$, $v \in T_xU \setminus T_xM$, then

$\lim_{t\to\infty}$ $1/t \log \|[T\xi_t(\omega)x]v\| \geq 0$ a.s.. Therefore the strictly negative part of the Lyapunov spectrum is the same for $\xi_t(\omega)$ and $\tilde{\xi}_t(\omega)$.

(2) We prove our result first for discrete time increments of length T. Our technique follows Ruelle's for his Theorem 6.1.

Consider the map $\tilde{F}_o(x,\omega):\bar{B}(1) \to \mathbb{R}^m$ given by $F_o(x,\omega)y = \tilde{\xi}_T(\omega)(x+y) - \tilde{\xi}_T(\omega)x$. Thus, $F_o(x,\omega)$ represents $\tilde{\xi}_T(\omega)$ near x, and $F_o(x,\omega)(0) = 0$. Note that $G_o(x,\omega)$ of the proof of Theorem 2.1 is just $DF_o(x,\omega)(0)$. Put $F^n(x,\omega) = F_o(\tilde{\Phi}_{(n-1)T}(x,\omega)) \circ \ldots \circ F_o(x,\omega)$. Then $F^n(x,\omega)y = \xi_{nT}(\omega)(x+y) - \xi_{nT}(\omega)x$ a.s., so that $F^n(x,\omega)$ represents $\tilde{\xi}_{nT}(\omega)$ near x.

We will apply Ruelle's Theorem 5.1 to $F_o(x,\omega)$, with the map $\tilde{\Phi}_T$ corresponding to Ruelle's τ. For this we require the following regularity condition, which can be deduced from our Appendix A, Proposition A.2.1:

$$\int_{x\in U} \int_{\omega\in\Omega} \|F_o(x,\omega)\|_{C^k} \, dP(\omega)\, d\rho(x) < \infty,$$

where

$$\|F_o(x,\omega)\|_{C^k} = \sup_{y\in U} \left\{ \sum_{|\alpha|<k} \left\| \frac{\partial^{|\alpha|}}{\partial y^{\alpha}} F_o(x,\omega)y \right\| \right\}.$$

(This corresponds to Ruelle's (5.1)).

Also, we require that the Lyapunov spectrum of $G_o(x,\omega)$ (i.e. that of Theorem 2.1) does not contain $-\infty$. From Ruelle's Section 2, part 3, we see that this is ensured if we have

$$\iint \log^+ \|[G_0(x,\omega)]^{-1}\| \; d\rho \; dP < \infty .$$

For this estimate, see our Appendix, Proposition A.2.1. We obtain the discrete time increment T version of our result from Ruelle's Theorem 5.1, restricting attention to M as a subset of $U \subset \mathbb{R}^m$.

(3) To obtain the full (continuous time) result, consider the function $K(\omega) = \sup_{\substack{x \in M \\ t \in [0,T]}} \{ \|D[\tilde{\xi}_t(\omega)]x\| \}$. From our Appendix A, Proposition A.2.1 we see that $\mathbb{E}[\log K] < \infty$. (N.B. $K(\omega) \geq 1$ a.s.) Now, if $t \in [nT,(n+1)T]$, then for any $y, z \in M$, we have

$$d(\xi_t(\omega)y, \xi_t(\omega)z) \leq K(\theta_{nT}(\omega))d(\xi_{nT}(\omega)y, \xi_{nT}(\omega)z).$$

Also, $\limsup_{n \to \infty} 1/n \log K(\theta_{nT}(\omega)) = 0$ a.s. (by the ergodic theorem applied to $\log K(\omega)$), i.e. for any $\varepsilon > 0$, we have a.s. that $K(\theta_{nT}(\omega)) < e^{\varepsilon nT}$ for sufficiently large n.

Consider the measurable, ϕ_T-invariant partition $I_0, I_1, I_2, \ldots$ of Γ given by $I_m = \{(x,\omega) \in \Gamma$: {Largest $\lambda^i_{(x,\omega)} < \lambda$ in the spectrum at $(x,\omega)\} \in [\lambda - 1/m, \lambda - 1/m+1)\}$.
By part (b') of the discrete time version of the theorem applied to I_m, if we take $\lambda_m \in (\lambda - 1/m+1, \lambda]$ then there exists a function $\gamma_m : I_m \to \mathbb{R}^{>0}$ such that for all $y, z \in V^\lambda_{(x,\omega)}(\alpha(x,\omega))$ we have

$$d(\xi_{nT}(\omega)y,\xi_{nT}(\omega)z) < \gamma_{m}(x,\omega)\ d(y,z)e^{\lambda_{m}nT} .$$

Thus if $t \in [nT,(n+1)T]$, then

$$d(\xi_t(\omega)y,\xi_t(\omega)z) < K(\theta_{nT}(\omega))\ d(\xi_{nT}(\omega)y,\xi_{nT}(\omega)z)$$

$$< e^{(\lambda-\lambda_{m})nT}\ \gamma_n(x,\omega)\ d(y,z)e^{\lambda_{m}nT}$$

for sufficiently large n (taking $\varepsilon = \lambda-\lambda_{m}$)

$$\le e^{-\lambda T}\gamma_{m}(x,\omega)\ d(y,z)e^{\lambda t}.$$

Thus, we can extend parts (a), (b) to continuous time, replacing $\beta(x,\omega)$ by $e^{-\lambda T}\gamma_{m}(x,\omega)\ \alpha(x,\omega)$ and $\gamma(x,\omega)$ by $e^{-\lambda T}\gamma_{m}(x,\omega)$, if $(x,\omega) \in I_{m}$.

Part (b') for continuous time follows in a similar way. //

Note

Our approach to dealing with M rather than a flat space is again more convenient than Ruelle's. (See note (ii) after Theorem 2.1.) Ruelle's approach necessitates choosing a $\delta > 0$ as in the proof of his Theorem 6.1. In the stochastic situation this δ would have to be stochastic.

Theorem 2.2.2 (Full Stable Manifold Theorem. Cf. Ruelle, Theorem 6.3)

Consider the SDS of Theorem 2.2.1, but assume that the Lyapunov spectrum is a.s. constant. Let $\lambda^{(1)} < \ldots < \lambda^{(q)}$ be the strictly negative Lyapunov numbers.

Then we have a set Γ_1 in $\Gamma \subset M\times\Omega$ of full measure and such that for each $(x,\omega) \in \Gamma_1$ we have:

For each $p = 1,\dots,q$, the set $V^{(p)}_{(x,\omega)}$ defined to be $\{y \in M : \limsup_{t\to\infty} 1/t \log d(\xi_t(\omega)x, \xi_t(\omega)y) < \lambda^{(p)}\}$ is the image of $V^{(p)}_{(x,\omega)}$ by a C^{k-1} immersion which is tangent to the identity at x. Thus, $V^{(p)}_{(x,\omega)}$ is locally a C^{k-1} submanifold of M.

Proof

Again we follow Ruelle (Theorem (6.3) and we deal first with the case of discrete time increments T.

Take constants $\lambda_1,\dots,\lambda_q,\eta,\zeta$, such that

$$\lambda^{(1)} < \lambda_1 < \lambda^{(2)} < \dots < \lambda^{(q)} < \lambda_q < -\zeta < 0,\ 0 < \eta < -\zeta/5.$$

For each $p = 1,\dots,q$ define $\alpha \equiv \alpha_p : \Gamma^{\lambda p} \to \mathbb{R}^{>0}$ as in the discrete time increment T version of Theorem 2.2.1 for $\lambda = \lambda_p$, and put $\alpha = \min \alpha_p$. Then for each p we can define $V^{\lambda p}_{(x,\omega)}(\alpha(x,\omega))$ for (x,ω) in a set of full measure, as in Theorem 2.2.1.

To complete the proof in the discrete time case, we must show that $\alpha(\phi_{\ell T}(x,\omega))$ decreases at most like $e^{-\zeta\ell\eta}$ as $\ell \to \infty$. (See the proof of Ruelle's Theorem 6.3). This follows from Ruelle's Remark 5.2(c), when his Section 5 is applied to $F_o(x,\omega)$ of Part (2) of our proof of Theorem 2.2.1.

The result for continuous time is obtained as in Part (3) of the proof of Theorem 2.2.1. //

For each $p = 1,\ldots,q$, the set $V^{(p)}_{(x,\omega)}$ defined to be $\{y \in M : \limsup_{t\to\infty} 1/t \log d(\xi_t(\omega)x, \xi_t(\omega)y) < \lambda^{(p)}\}$ is the image of $V^{(p)}_{(x,\omega)}$ by a C^{k-1} immersion which is tangent to the identity at x. Thus, $V^{(p)}_{(x,\omega)}$ is locally a C^{k-1} submanifold of M.

Proof

Again we follow Ruelle (Theorem (6.3) and we deal first with the case of discrete time increments T.

Take constants $\lambda_1,\ldots,\lambda_q,\eta,\zeta$, such that

$\lambda^{(1)} < \lambda_1 < \lambda^{(2)} < \ldots < \lambda^{(q)} < \lambda_q < {}^{-}\zeta < 0,\ 0 < \eta < {}^{-}\zeta/5.$

For each $p = 1,\ldots,q$ define $\alpha \equiv \alpha_p : \Gamma^{\lambda p} \to \mathbb{R}^{>0}$ as in the discrete time increment T version of Theorem 2.2.1 for $\lambda = \lambda_p$, and put $\alpha = \min \alpha_p$. Then for each p we can define $V^{\lambda p}_{(x,\omega)}(\alpha(x,\omega))$ for (x,ω) in a set of full measure, as in Theorem 2.2.1.

To complete the proof in the discrete time case, we must show that $\alpha(\Phi_{\ell T}(x,\omega))$ decreases at most like $e^{-\zeta\ell\eta}$ as $\ell \to \infty$. (See the proof of Ruelle's Theorem 6.3). This follows from Ruelle's Remark 5.2(c), when his Section 5 is applied to $F_o(x,\omega)$ of Part (2) of our proof of Theorem 2.2.1.

The result for continuous time is obtained as in Part (3) of the proof of Theorem 2.2.1. //

2.3 Lyapunov Stability

Definition

We say that the flow of a nondegenerate SDS is *Lyapunov stable* if the Lyapunov spectrum is a.s. bounded below a strictly negative constant.

In the sequel, we will mostly be concerned with Lyapunov stable flows. In Chapter 3 we give examples in which Lyapunov stability occurs, and in Chapter 4 we show that certain stochastic perturbations of deterministic systems are Lyapunov stable. We work from Theorem 2.2.1 rather than 2.2.2; we are concerned with the clustering properties of the flow (see Proposition 2.3.3) rather than the smoothness properties of the stable manifolds themselves.

For a Lyapunov stable flow, let us denote by Γ^{o} the set Γ^{λ} of Theorem 2.2.1, where {Lyapunov spectrum} $< \lambda < 0$ a.s.. Any such choice of λ will give the same Γ^{o}, by Theorem 2.2.1(b'), and clearly Γ^{o} has full $\rho \otimes P$-measure.

Proposition 2.3.1

Suppose the flow of the SDS of Theorem 2.2.1 is Lyapunov stable. Then:

(i) For a.e. $\omega_o \in \Omega$, the set $\Gamma^{o}_{\omega_o}$ given by $\{x \in M:(x,\omega_o) \in \Gamma^{o}\}$ has full measure in M. Also, for each $T > 0$ we have ρ-a.s. that $\xi_T(\omega_o)\Gamma^{o}_{\omega_o} = \Gamma^{o}{}_{\theta_T(\omega_o)}$.

(ii) For a.e. $x_o \in M$, the set $\Gamma^o_{x_o}$ given by $\{\omega \in \Omega: (x_o,\omega) \in \Gamma^o\}$ has full measure in Ω.

<u>Proof</u>

Immediate from the fact that Γ^o has full measure, and that $\Phi_T: M \times \Omega \to M \times \Omega$ preserves that measure. //

<u>Proposition 2.3.2</u> (Cf. Ruelle, Corollary 6.2)

Suppose the flow of the SDS of Theorem 2.2.1 is Lyapunov stable. Then for a.e. ω_o, $\Gamma^o_{\omega_o}$ is a proper open subset of M.

<u>Proof</u>

Choose $\lambda < 0$ such that {Spectrum} $< \lambda < 0$ and choose ω_o such that $\Gamma^o_{\omega_o}$ has full measure in M. (See Proposition 2.3.1(i)). Consider the function $\alpha(-,\omega_o):\Gamma^o_{\omega_o} \to \mathbb{R}^{>0}$ given in Theorem 2.2.1. By part (a) of that theorem, for any $x \in \Gamma^o_{\omega_o}$, the local stable manifold $V^\lambda_{(x,\omega_o)}(\alpha(x,\omega_o))$ is a neighbourhood of x, and by part (b), we can define a local stable manifold at any point in the interior of $V^\lambda_{(x,\omega_o)}(\alpha(x,\omega_o))$. Thus, $\Gamma^o_{\omega_o}$ is an open subset of M.

Now, suppose $\Gamma^o_{\omega_o} = M$. Then by compactness, we can take a finite collection $x_1,\ldots,x_N$ such that the interiors of $V^\lambda_{(x_i,\omega_o)}(\alpha(x_i,\omega_o))$ cover M. Take $\beta = \underset{i=1,\ldots,N}{\text{Max}} \{\beta(x_i,\omega_o)\}$. ($\beta(x,\omega)$ defined in Theorem 2.2.1). Then by part (a) of

Theorem 2.2.1, diam $\{\xi_t(\omega_o)B(x_i,\omega_o)\} < \beta\, e^{\lambda t}$, which is impossible since $\xi_t(\omega_o)$ is a diffeomorphism for all t. //

<u>Note</u>

In Proposition 2.3.2, $\Gamma^o_{\omega_o}$ might be the complement of a Cantor set in M.

<u>Proposition 2.3.3</u> (Clustering property for Lyapunov stable flows)

Suppose the flow of Theorem 2.2.1 is Lyapunov stable. Then for a.e. $x_o \in M$, $P\{\omega : \text{diam}\,\{\xi_t(\omega)B(x_o,r)\} \to 0$ as $t \to \infty\} \to 1$ as $r \to 0$. (' diam $\{\xi_t(\omega)B(x_o,r)\} \to 0$ as $t \to \infty$' means that the trajectories starting in $B(x_o,r)$ of the flow $\xi_t(\omega)$ cluster together).

<u>Proof</u>

For this we must have $\Gamma^o_{x_o} \subset \Omega$ of full measure. This holds for a.e. $x_o \in M$. (Proposition 2.3.1(ii)). The result follows because $\alpha(x_o,-) : \Gamma^o_{x_o} \to \mathbb{R}^{>0}$ is strictly positive, P-measurable. //

3. <u>SOME EXAMPLES</u>

In this chapter we give some examples of stochastic flows, for which the Lyapunov spectrum can be calculated explicitly. We can choose parameters such that the flow is

Lyapunov stable (see Section 2.3) and in this case we can calculate explictly the flow itself and its asymptotic behaviour, and verify the assertions of Section 2.3.

3.1 An Example of the Circle (Noisy North-South Flow)

Example 3.1 (Noisy N.S. Flow)

(i) The (noise-free) N.S. flow is taken to be the stereographic projection from $\mathbb{R}$ to the unit circle S, via the north pole $x_N \in S$ and such that the south pole x_S sits at the origin in $\mathbb{R}$, of the flow η_t on $\mathbb{R}$, where $\eta_t(x) = x \exp(-t)$.

This flow clearly has a source at x_N and a sink at x_S. In this chapter we will denote this flow on S by ξ_t^o and the vector field for this flow by $A:S \to TS$.

(ii) The noisy N.S. flow (with parameter $\varepsilon > 0$) is taken to be the flow of the SDS (X^ε, z), where $X^\varepsilon(y) \circ dz_t = \varepsilon Y(y) \circ dB_t + A(y)d$ Here, B_t is Brownian motion in $\mathbb{R}$; A is as in part (i); and $Y(x) \in L(\mathbb{R}, T_xS)$ is given by $Y(x)(1) = \{$Tangent at x to S of unit length in the positive direction$\}$, so that the flow of the SDS (Y,B) is a stochastic rotation. We will denote this flow by $\xi_t^\varepsilon(\omega)$. Then putting $\varepsilon = 0$ gives the flow of part (i) and the flow for $\varepsilon > 0$ may be regarded as a stochastic perturbation of ξ_t^o.

Notes

(i) For $\varepsilon > 0$, (X^{ε}, z) is nondegenerate.

(ii) It is helpful to think of S as $\mathbb{R}/2\pi\mathbb{Z}$, and functions on S as periodic functions on $\mathbb{R}$, so that we can write derivatives explictly. If we do this then we can express the SDS as an Itô system. Since Y is constant over $S \equiv \mathbb{R}/2\pi\mathbb{Z}$, its derivative is zero, and so is the Stratonovitch correction term, therefore the SDS will look the same if it is written as an Itô system. //

By Theorem 2.1 there is just one Lyapunov number λ^{ε} for (X^{ε}, z), and it is given by $\lim_{t\to\infty} 1/t \log \|T\xi_t^{\varepsilon}(\omega)v\|$ for a.e. $(x,\omega) \in S \times \Omega$, where v is any non-zero vector in T_xS. In the following proposition we give a formula for λ^{ε}, and in Proposition 3.1.2 we show that for $\varepsilon > 0$ sufficiently small, we have Lyapunov stability.

Proposition 3.1.1

Denote the unique invariant measure on S for (X^{ε}, z) by ρ^{ε}. (See Section 1.2). Then the Lyapunov number λ^{ε} for (X^{ε}, z) is given by

$$\lambda^{\varepsilon} = \int_{y\in S} DA(y)\, d\rho^{\varepsilon}(y).$$

(Here, DA is the derivative of A, where we have identified $S \equiv \mathbb{R}/2\pi\mathbb{Z}$, so that A is a periodic map $\mathbb{R} \to \mathbb{R}$).

Proof

We can deduce from Carverhill and Elworthy [4], Remark 4.2, the following equation involving $D\xi_t^\varepsilon(\omega)$:

$$d[\xi_t^\varepsilon(\omega)x,\ D(\xi_t^\varepsilon(\omega)x)v]$$

$$= A(\xi_t^\varepsilon(\omega)x),\ [DA(\xi_t^\varepsilon(\omega))x][D\xi_t^\varepsilon(\omega)x]v]\ dt$$

$$+\ [\varepsilon Y(\xi_t^\varepsilon(\omega)x),\ \varepsilon[DY(\xi_t^\varepsilon(\omega))x][D\xi_t^\varepsilon(\omega)x]v]\ \circ\ dB_t(\omega).$$

(See also Proposition 4.2 , in which derivative flows are discussed generally).

Also, λ^ε is given a.s. by $\lim\limits_{t\to\infty} 1/t\ [D\xi_t^\varepsilon(\omega)x]v$.(Any $v \neq 0$).

Now, since Y is constant over $\mathbb{R}/2\pi\mathbb{Z}$, we have $DY = 0$, therefore $[D\xi_t^\varepsilon(\omega)x]v$ is just the solution $R_t^\varepsilon(\omega)$ of the linear equation

$$dR_t^\varepsilon(\omega) = DA(\xi_t^\varepsilon(\omega)x).R_t^\varepsilon(\omega)dt,$$

in which the coefficient $DA(\xi_t^\varepsilon(\omega)x)$ is stochastic, driven by $\xi_t^\varepsilon(\omega)x$ and is a.s. continuous in t. This equation has solution $R_t^\varepsilon(\omega) = R_o \exp \int_o^t DA(\xi_s^\varepsilon(\omega)x)ds$ for some R_o.

Therefore $\lambda^\varepsilon = \lim\limits_{t\to\infty} 1/t \int_o^t DA(\xi_s^\varepsilon(\omega)x)ds$ a.s., and the result follows using the Strong Law of Large Numbers (Proposition 1.2.3). //

For Proposition 3.1.2 we need the following lemma:

Lemma 3.1

The invariant measure ρ^{ε} for (X^{ε},z) tends weakly to δ_{x_s} (unit measure concentrated at x_s) as $\varepsilon \to 0$, i.e. for any continuous $f:S \to \mathbb{R}$, we have $\int_{y\in S} f(y)\, d\rho^{\varepsilon}(y) \to f(x_s)$ as $\varepsilon \to 0$.

Proof

This follows from the work of Ventsel and Freidlin [18]. See our Section 4.3. However, since our manifold is 1-dimensional, we can calculate ρ^{ε} explicitly and verify the result. This calculation is outlined in Khas'minskii [9]. //

Proposition 3.1.2

As $\varepsilon \to 0$, the Lyapunov number λ^{ε} tends to $DA(x_s) < 0$. Thus, for sufficiently small $\varepsilon > 0$, the flow $\xi_t^{\varepsilon}(\omega)$ is Lyapunov stable.

Proof

Note that the periodic function $DA:\mathbb{R} \to \mathbb{R}$ is continuous, and $DA(x_s) < 0$ since x_s is an attractor. The result follows from the formula of Proposition 3.1.1 for λ^{ε}, and Lemma 3.1. //

3.2 Asymptotic Behaviour of the Noisy North-South Flow

In this section we give a formula for the noisy N.S. flow, and from this we are able to deduce its asymptotic behaviour and describe the stable manifolds completely. We can attempt this analysis because (as we show in Lemma 3.2.1), the flow lies in a finite dimensional Lie group.

We will denote this Lie group by $\mathcal{D}_m(S)$: it is the group of Möbius diffeomorphisms on S, i.e. diffeomorphisms such that if we identify S with the one-point compactification $\mathbb{R}^*$ of $\mathbb{R}$ via the stereographic projection given in the definition of the noisy N.S. flow (Example 3.1), then we can write the diffeomorphism as $x \to \frac{ax + b}{cx + d}$, and represent it as a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that composition of Möbius diffeomorphisms corresponds to multiplication of these matrices. Also, the matrices $\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $\lambda \neq 0$ all represent the same diffeomorphism, and $\mathcal{D}_m(S)$ has dimension 3.

Throughout this section we will take $\varepsilon > 0$ fixed, such that $\lambda^{\varepsilon} < 0$, and we will suppress ε from our notation. Thus, in this section only, $\xi_t(\omega)$ denotes the noisy N.S. flow.

Lemma 3.2.1

Almost surely, for all $t > 0$ the diffeomorphism $\xi_t(\omega)$ lies in the Lie group $\mathcal{D}_m(S)$.

Proof

Restrict attention to a finite time interval $[0,U]$, and take a partition $\Pi \equiv \{0,V,2V,\ldots\}$ of this. Denote by (X,z^{π}) the piecewise linear approximation to (X,z) and denote by $\xi^{\pi}_t(\omega)$ the flow of (X,z^{π}). (See Carverhill and Elworthy [4] Section 2.3). Since $\mathcal{D}_m(S)$ is closed in the C^0 topology, and since $\xi^{\pi}_t(\omega)$ tends to $\xi_t(\omega)$ a.s. as mesh $(\Pi) \equiv V$ tends to zero, uniformly over $t \in [0,U]$ and in the C^0 topology (See [4], Proposition 4.2), it suffices to prove the result for $\xi^{\pi}_t(\omega)$. This we do below.

Now, for each $\omega \in \Omega$, $\xi^{\pi}_t(\omega)$ is a composition of flows of deterministic dynamical systems: if $t \in [qV, (q+1)V)$, then

$$\xi^{\pi}_t = \xi^{\pi}_{qV,t} \circ \xi^{\pi}_{(q-1)V,qV} \circ \ldots \circ \xi^{\pi}_{0,V},$$

where for $0 < i < q$, $\xi^{\pi}_{iV,(i+1)V}$ is the flow for time V of the vector field $\rho_i Y+A$, and $\xi^{\pi}_{qV,t}$ is the flow for time $t-qV$ of $\rho_q\, Y+A$. (Here,

$$\rho_i(\omega) = \frac{\varepsilon(B_{(i+1)V}(\omega) - B_{iV}(\omega).)}{V}\;)$$

Also, the flows of Y and A are Möbius diffeomorphisms: they are rotation and the N.S. flow respectively.

The result follows from the Lie-Trotter product formula (see for example Nelson [14] Chapter 4), which says that if ζ^1_t and ζ^2_t are flows of the vector fields Z^1 and Z^2, then the flow of $Z^1 + Z^2$ is just $\lim_{q\to\infty} (\zeta^1_{t/q} \circ \zeta^2_{t/q})^q$. (Here, the limit is in the C^0 topology). Thus, if the flows of Z^1 and Z^2 lie in a certain Lie group, then so does the flow of $Z^1 + Z^2$. //

<u>Note</u>

This result can also be obtained using the 'lift' $(\tilde{X},z)$ of (X,z) to $\mathcal{D}^s(S)$ (s sufficiently large) and showing that $\tilde{X}|_{\mathcal{D}_m(S)}$ is tangent to $\mathcal{D}_m(S)$. See Carverhill and Elworthy [4] for a description of $(\tilde{X},z)$, and Elworthy [6] Chapter 7, Section 3. //

Our technique for the analysis of this section is to compose each diffeomorphism of the flow $\xi_t(\omega)$ with a rotation, so that the composition fixes $x_s \in S$, and to get a formula for that. The rotations, and hence the compositions are still Möbious diffeomorphisms, and since the rotations are isometries, they do not affect the stable manifold structure. Note that a rotation through the angle θ has matrix $\begin{pmatrix} 1 & ,2q \\ -q/2, & 1 \end{pmatrix}$, where $q = 2 \tan \frac{1}{2}\theta$. Also, any Möbius diffeomorphism which fixes x_s has matrix $\begin{pmatrix} a, & 0 \\ c, & 1 \end{pmatrix}$.

In the following lemma we treat a discrete time analogue of our problem, and in Lemma 3.2.3 we let the discrete time increment tend to zero, to obtain the formula relating to the noisy N.S. flow itself.

Lemma 3.2.2

Choose $T > 0$, and take a stochastic process $y: \Omega \times \mathbb{R}^{>0} \to \mathbb{R}$. Consider the following stochastic flow with discrete time increment T, where $R_{iT,(i+1)T}(\omega)$ is a rotation through the angle $y_{(i+1)T}(\omega) - y_{iT}(\omega)$:

$$\eta^T_{rT}(\omega): [R_{(r-1)T,rT}(\omega) \circ \xi^o_T] \circ \ldots \circ [R_{0,T}(\omega) \circ \xi^o_T].$$

(Here, ξ^o_T is the noise-free N.S. flow at time T). Denote by $V^T_{rT}(\omega)$ the rotation which sends $\eta^T_{rT}(\omega)x_s$ back to x_s, and denote by θ_r the angle of V^T_{rT}.

(i) Then a.s., for all r, the flow $V^T_{rT}(\omega) \circ \eta^T_{rT}(\omega)$ has matrix $\begin{pmatrix} a^T_{rT}(\omega), & 0 \\ c^T_{rT}(\omega), & 1 \end{pmatrix}$, where

$$\alpha^T_{rT} = \alpha^T_0 . \alpha^T_T \ \ldots \ \alpha^T_{(r-1)T} \ ,$$

$$c^T_{rT} = \gamma^T_0 + \gamma^T_T . \ \alpha^T_0 + \gamma^T_T . \alpha^T_0 . \alpha^T_T + \ldots$$

$$+ \gamma^T_{(r-1)T} . \alpha^T_0 . \alpha^T_T \ldots \alpha^T_{(r-2)T} ,$$

$$\alpha^T_{iT} = \frac{1 + q_i^2}{q_i^2 e^{-T} + e^T} , \quad \gamma^T_{iT} = \frac{q_i(e^{2T}-1)}{2(q_i^2 + e^{2T})} ,$$

and $q_i = 2 \tan \tfrac{1}{2}\theta_i$.

The matrix $\begin{pmatrix} \alpha^T_{iT} , & 0 \\ \gamma^T_{iT} , & 1 \end{pmatrix}$ represents the diffeomorphism

$$\zeta^T_{iT} = V^T_{(i+1)T} \circ [R_{iT:(i+1)T} \circ \xi^o_T] \circ (V^T_{iT})^{-1}$$

(ii) If we identify $S \cong \mathbb{R}/2\pi\mathbb{Z}$, then a^T_{rT} is the derivative of $\eta^T_{rT}(\omega)$ at zero.

Note

If we take $y = \varepsilon B$ in this lemma, then $\eta^T_{nT}(\omega)$ is an approximation to the noisy N.S. flow.

Proof

(i) Clearly we have $V^T_{rT} \circ \eta^T_{rT} = \zeta^T_{(r-1)T} \circ \zeta^T_{(r-s)T} \circ \ldots \circ \zeta^T_0$, so that if ζ^T_{iT} has matrix $\begin{pmatrix} \alpha^T_{iT}, & 0 \\ \gamma^T_{iT}, & 1 \end{pmatrix}$, then

$$\begin{pmatrix} a^T_{rT}, & 0 \\ c^T_{rT}, & 1 \end{pmatrix} = \begin{pmatrix} \alpha^T_{(r-1)T}, & 0 \\ \gamma^T_{(r-1)T}, & 1 \end{pmatrix} \ \ldots \ \begin{pmatrix} \alpha^T_0, & 0 \\ \gamma^T_0, & 1 \end{pmatrix}$$

and the formulae for a^T_{rT}, c^T_{rT} in terms of α^T_{iT}, γ^T_{iT} follow. Therefore, it suffices to show that ζ^T_{iT} does have this matrix, as we do below.

The matrices for ξ^o_T and $(V^T_{iT})^{-1}$ are $\begin{pmatrix} e^{-T}, & 0 \\ 0, & 1 \end{pmatrix}$ and $\begin{pmatrix} 1, & -2q_i \\ q_i/2, & 1 \end{pmatrix}$ respectively. Also, ζ^T_{iT} fixes x_s, and is therefore $\xi^o_T \circ (V^T_{iT})^{-1}$ composed with a rotation which causes the whole to fix x_s. Therefore ζ^T_{iT} has matrix

$\begin{pmatrix} 1 & , 2p_i \\ -p_i/2, & 1 \end{pmatrix} \begin{pmatrix} e^{-T} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & , -2q_i \\ q_i/2, & 1 \end{pmatrix}$, where p_i is such that this matrix has the form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. Therefore $p_i = q_i e^{-T}$, and ζ^T_{iT} has matrix

$$\begin{pmatrix} e^{-T}\,(1+q_i^2) & , & 0 \\ q_i/2\,(1-e^{-2T}) & , & 1+q_i^2\,e^{-2T} \end{pmatrix}.$$

Dividing each term in this by $1 + q_i^2\,e^{-2T}$ gives the required formulae for α^T_{iT}, γ^T_{iT}.

(ii) For any Möbius diffeomorphism which fixes x_s and has matrix $\begin{pmatrix} a, & 0 \\ c, & 1 \end{pmatrix}$, the derivative (i.e. the enlargement factor) at x_s is a. To see this, just differentiate the map $x \to \dfrac{ax}{cx+1}$. Also, identifying $S \cong \mathbb{R}/2\pi\mathbb{Z}$, a rotation is identified with a translation and therefore it does not affect the enlargement factor. Thus, the derivative of $\eta^T_{rT}(\omega)$ at zero is equal to that of $V^T_{rT}(\omega) \circ \eta^T_{rT}(\omega)$, which is $a^T_{rT}(\omega)$. //

<u>Note</u>

α^T_{iT}, γ^T_{iT} depend only on θ_i and hence only or $\eta^T_{iT}(\omega)x_s$. Therefore a^T_{rT}, c^T_{rT} depend only on the stochastic process $\eta^T_{iT}(\omega)x_s$ for $i = 0,1,\ldots,r-1$.

<u>Lemma 3.2.3</u>

Denote by $V_t(\omega)$ the rotation which sends $\xi_t(\omega)x_s$ back to x_s ($\xi_t(\omega)$ - noisy N.S. flow).

Then a.s., the flow $V_t(\omega) \circ \xi_t(\omega)$ has matrix $\begin{pmatrix} a_t(\omega), & 0 \\ c_t(\omega), & 1 \end{pmatrix}$,

where $a_t(\omega) = \int_0^t DA(\xi_n(\omega)x_s)du$, $c_t(\omega) = \int_0^t C(\xi_u(\omega)x_s)a_n(\omega)du$,

and $C:S \to \mathbb{R}$ is defined as follows:

Take $x \in S$ and put θ = angle between x and x_s. Put $p = 2\tan\frac{1}{2}\theta$.

Define $D:\mathbb{R}^{>0} \times S \to \mathbb{R}$ by $D(t,x) = q\dfrac{(e^{2T}-1)}{2(q^2+e^{2T})}$.

(So D comes from γ of Lemma 3.2.2).

Take $C(x) = \frac{\partial}{\partial T} D(T,x)\Big|_{T=0}$.

(Here, we have identified $S \cong \mathbb{R}/2\pi\mathbb{Z}$ so that the vector field A is a periodic map $\mathbb{R} \to \mathbb{R}$).

<u>Proof</u>

The idea is to take $y_t = \varepsilon B_t$ in Lemma 3.2.2, so that η^T_{rT} is an approximation to the noisy N.S. flow, and to take limits as $T \to 0$. However, we must go via the piecewise linear approximation (see Carverhill and Elworthy [4] Section 2.3) to the SDS (X,z) for the noisy N.S. flow, because it is not clear that we have convergence if we go directly.

(1) So, restrict attention to a finite time interval $[0,U]$ and take a partition $\Pi \equiv \{0,V,2V,\dots\}$ of this, as in Lemma 3.2.1. Denote by (X,z^π) the piecewise linear approximation to (X,z), and denote by $\xi^\pi_t(\omega)$ the flow of (X,z^π). Take the stochastic process y of Lemma 3.2.2 to be εB^π, and denote by

$\xi^{T,\pi}_{nT}(\omega)$ the flow of $\eta^{T}_{nT}(\omega)$ with this choice of y. For t not a multiple of T, put $\xi^{T,\pi}_{t}(\omega) = \xi^{T,\pi}_{qT}(\omega)$, where $qT < t < (q+1)T$. We take T always to be a factor of V in this proof. From Carverhill and Elworthy [4] Proposition 4.2 we see that $\xi^{\pi}_{t}(\omega)$ tends to $\xi_{t}(\omega)$ a.s., uniformly over $t \in [0,U]$ and in the C^{o} topology, as mesh $(\Pi) \equiv V$ tends to zero. We show below that for each Π and each ω, $\xi^{T,\pi}_{t}(\omega)$ tends to $\xi^{\pi}_{t}(\omega)$ uniformly over t and in the C^{o} topology, as T tends to zero. (T a factor of V).

As we stated in Lemma 3.2.1, if $t \in [qV,(q+1)V)$ then

$$\xi^{\pi}_{t} = \xi^{\pi}_{qV,t} \circ \xi^{\pi}_{(q-1)V,qV} \circ \ldots \circ \xi^{\pi}_{0,V}.$$

Also, $\xi^{T,\pi}_{t} = \xi^{T,\pi}_{qV,t} \circ \xi^{\pi}_{(q-1)V,qV} \circ \ldots \circ \xi^{\pi}_{0,V}$,

where for $0 < i < q$, $\xi^{T,\pi}_{iV,(1+1)V}$ is $(R^{i}_{T} \circ \xi^{o}_{T})^{V/T}$, and $\xi^{T,\pi}_{qV,t}$ is $(R^{q}_{T} \circ \xi^{o}_{T})^{p}$.(p = largest integer $< \frac{t-qV}{T}$)

(Here, R^{i}_{T} is the flow of $\rho_{i}Y$ for time T and ξ^{o}_{T} is the noisy N.S. flow for time T).

By the Lie-Trotter product formula [14], for each i, $\xi^{T,\pi}_{iV,(i+1)V}(\omega)$ tends to $\xi^{\pi}_{iV,(i+1)V}(\omega)$ as T tends to zero, and the result follows.

Denote by $\begin{pmatrix} a^{T,\pi}_{t}(\omega) & 0 \\ c^{T,\pi}_{t}(\omega) & 1 \end{pmatrix}$ and $\begin{pmatrix} a^{\pi}_{t}(\omega), & 0 \\ c^{\pi}_{t}(\omega), & 1 \end{pmatrix}$

the matrices representing $V_t^{T,\pi}(\omega) \circ \xi_t^{T,\pi}(\omega)$ and $V_t^{\pi}(\omega) \circ \xi_t^{T}(\omega)$, where $V_t^{T,\pi}$ are rotations such that these compositions fix x_s. From the previous paragraph we see that

$$\left\{\begin{array}{l} a_t^{T,\pi}(\omega) \to a_t^{\pi}(\omega) \\ c_t^{T,\pi}(\omega) \to c_t^{\pi}(\omega) \end{array}\right\} \quad \begin{array}{l}\text{for each } \omega\text{, and uniformly over} \\ t \in [0,U]\text{, as } T \to 0,\end{array}$$

$$\left\{\begin{array}{l} a_t^{\pi} \to a_t \\ c_t^{\pi} \to c_t \end{array}\right\} \quad \begin{array}{l}\text{a.s., uniformly over } t \in [0,U]\text{, as} \\ \qquad \text{mesh } (\Pi) \equiv V \to 0.\end{array}$$

(2) As in Part (ii) of Lemma 3.2.2, we see that if we identify $S \cong \mathbb{R}/2\pi\mathbb{Z}$ then $a_t(\omega) = D\xi_t(\omega)x_s$. From the proof of Proposition 3.1.1 we see that this is $\exp \int_0^t DA(\xi_u(\omega)x_s)du$, as required.

(3) We now calculate c_t^{π} as $\lim_{T\to 0} c_t^{T,\pi}$, and then c_t as $\lim_{\text{Mesh}(\Pi)\to 0} c_t^{\pi}$.

For any choice of stochastic process y in Lemma 3.2.2, γ_{iT}^{T} depends only on T and θ_i, and we can write
$\gamma_{iT}^{T} = D(T,\eta_{iT}^{T}(\omega)x_s)$. (D as in the statement of the lemma).

Thus, from the formula of Lemma 3.2.2,

$$c_t^{T,\pi} = D(T,\xi_0^{T,\pi}(\omega)x_0) + D(T,\xi_T^{T,\pi}(\omega)x_0)\, a_T^{T,\pi}(\omega)$$

$$+ \ldots + D(T,\xi_{(r-1)T}^{T,\pi}(\omega)x_0).a_{(r-1)T}^{T,\pi}(\omega),$$

where r is the largest integer such that $rT < t$. Using the fact that D is smooth and that $a_t^{T,\pi}$ tends to a_t^T, we can deduce from this that

$$\lim_{T\to 0} c_t^{T,\pi}(\omega) \equiv c_t^{\pi}(\omega) = \int_0^t C(\xi^{\pi}(\omega)x_s)a_u^{\pi}(\omega)du.$$

Now, letting mesh $(\Pi) \equiv V$ tend to zero gives us the required formula for $c_t(\omega)$. //

Since the rotations $V_t(\omega)$ in Lemma 3.2.3 are isometries, we can study the stable manifolds of $\xi_t(\omega)$ by looking at the formula of Lemma 3.2.3 for $V_t(\omega) \circ \xi_t(\omega)$. Note that we can characterise a Möbius diffeomorphism which fixes x_s by the other fixed point, and the derivative (enlargement factor) at x_s. Also, for any $z \neq x_s$, there exists a Möbius diffeomorphism F which moves z to x_N, fixes x_s, and has derivative 1 at x_s. If we identify S with $\mathbb{R}^*$ via stereographic projection from x_s (*not* x_N), then F is given by $x \to x-$ {projection of z from x_s}. With respect to our usual projection from x_N, F has matrix

$$\begin{pmatrix} 1 & , \ 0 \\ \dfrac{-1}{\text{Projection of } z} & , \ 1 \end{pmatrix}.$$

In Lemma 3.2.4 we study the fixed point distinct from x_s for $V_t(\omega) \circ \xi_t(\omega)$, and to prove Proposition 3.2 we move this fixed point to x_N by conjugation with a suitable F, so that the flow has matrix $\begin{pmatrix} a_t(\omega) & , \ 0 \\ 0 & , \ 1 \end{pmatrix}$. For this conjugate we can easily describe the stable manifold structure.

Lemma 3.2.4

For each ω a.s., the flow $V_t(\omega) \circ \xi_t(\omega)$ of Lemma 3.2.3 has just two fixed points for each t, one of which is x_s, and the other will be denoted by $z_t(\omega)$. For each ω a.s., $z_t(\omega)$ tends to a limit $z_0(\omega)$ as t tends to ∞, and this limit is distinct from x_0.

Proof

Identifying S with $\mathbb{R}^*$ via the stereographic projection, $V_t(\omega) \circ \xi_t(\omega)$ is given a.s. by $x \to \dfrac{a_t(\omega)x}{c_t(\omega)x+1}$. Therefore the fixed points are the solutions to

$$c_t(\omega)x^2 + (1-a_t(\omega))x = 0, \text{ i.e. } x = 0,\ x = \frac{a_t(\omega)-1}{c_t(\omega)} .$$

Now x_s corresponds to 0; take $z_t(\omega)$ to correspond to the other solution. To show that $z_t(\omega)$ tends to a limit a.s., we will show that this other solution tends to a limit in $\mathbb{R}$ a.s..

By the formula of Lemma 3.2.3 and the Proposition 3.1.1, $1/t \log a_t(\omega) \to \lambda < 0$ a.s. (λ-Lyapunov number). Take $\mu \in (\lambda,0)$. Then for each ω a.s., there is a time $U(\omega)$ such that if $t > U(\omega)$ then $1/t \log a_t(\omega) < \mu$, i.e. $a_t(\omega) < \exp t\mu$. Also $a_t(\omega) > 0$ a.s.. Therefore $a_t(\omega) \to 0$ a.s. and it suffices to show that $c_t(\omega)$ tends to a limit a.s.. For this, observe that $C:S \to \mathbb{R}$ of Lemma 3.2.3 is bounded, and that if $t > r > U(\omega)$, then

$$|c_t(\omega) - c_r(\omega)| < \sup_{x \in S} \{|C(x)|\}. \int_r^t \exp \mu u \, du.$$

Therefore for each ω a.s., $c_t(\omega)$ is a Cauchy sequence.

To see that the limit of $z_t(\omega)$ is a.s. not equal to x_s, note that the limit of $\frac{a_t(\omega)-1}{c_t(\omega)}$ is a.s. not zero. //

Proposition 3.2

Take a version of the noisy N.S. flow $\xi_t(\omega)$. Then for a.e. $\omega \in \Omega$, $z_0(\omega)$ defined in Lemma 3.2.4 exists for $V_t(\omega) \circ \xi_t(\omega)$ and is distinct from x_s.

Also, for a.e. $\omega \in \Omega$, we have the following:

(i) For any $x \in S \setminus \{z_0(\omega)\}$ the Lyapunov spectrum exists at (x,ω) and is just $\{\lambda\}$. Thus, $1/t \log \|T\xi_t(\omega)v\| \to \lambda$ as $t \to \infty$, for any nonzero $v \in T_xS$.

(ii) The stable manifold $V^{\lambda}_{(x,\omega)}$ exists if $x \in S \setminus \{z_0(\omega)\}$ and is just $S \setminus \{z_0(\omega)\}$, i.e. for any $y \in S \setminus \{z_0(\omega)\}$,

$$\limsup_{t \to \infty} 1/t \log d(\xi_t(\omega)x, \xi_t(\omega)y) < \lambda.$$

(ii) For any closed set $K \subset S \setminus \{z_0(\omega)\}$ we have the clustering property:

$$\operatorname{diam} \{\xi_t(\omega)K\} \to 0 \text{ as } t \to \infty .$$

Proof

Since the rotations $V_t(\omega)$ of Lemma 3.2.3 are isometries, it suffices to prove the result for $V_t(\omega) \circ \xi_t(\omega)$. For a.e. $\omega \in \Omega$, the fixed point $z_t(\omega)$ tends to $z_0(\omega) \neq x_s$. Choose ω_0 such that this is so. Choose a closed segment L of S such that $z_0(\omega_0) \in L^o$, $x_s \notin L$, and choose a time $U(\omega_0)$ such that if $t \in U(\omega_0)$ then $z_t(\omega_0) \in L$. For $t > U(\omega_0)$ denote by $F_t(\omega_0)$ the Möbius map described before Lemma 3.2.4, which sends $z_t(\omega_0)$ to x_N. Note that the conjugate $F_t \circ [V_t \circ \xi_t] \circ F_t^{-1}$ has matrix $\begin{pmatrix} a_t(\omega), & 0 \\ 0 & , 1 \end{pmatrix}$. Also, since $z_t(\omega_0)$ is bounded away from x_s for $t > U(\omega_0)$, $F_t(\omega_0)$ and $F_t(\omega_0)^{-1}$ are both globally bi-Lipschitz, uniformly in $t > U(\omega_0)$. Therefore it suffices to obtain the result for $F_t(\omega_0) \circ [V_t(\omega_0) \circ \xi_t(\omega_0)] \circ F_t(\omega_0)^{-1}$, and for this the result follows easily from the fact that $1/t \log a_t(\omega) \to \lambda < 0$ a.s. as $t \to \infty$. //

3.3 A Further Example on the Circle (Noisy N.E.S.W. Flow)

In this section and the next, we construct some more complicated flows, using the noisy N.S. flow, and we deduce their asymptotic behaviour from that of the noisy N.S. flow itself.

Example 3.3 (Noisy N.E.S.W. Flow)

Consider the double covering map $E:S \to S$ given by $E(\theta) = 2\theta$, if S is parameterised by the angle θ from x_N. Induce from the maps A,Y of Example 3.1 the maps $\tilde{A},\tilde{Y}$ such that $\tilde{A}(x) = \frac{1}{2}A(Ex)$, $\tilde{Y}(x)(e) = \frac{1}{2}Y(Ex)(e)$ $(\equiv \frac{1}{2}Y(x)e)$. Then we take the noisy N.E.S.W. flow $\tilde{\xi}_t^{\varepsilon}(\omega)$ to be that of the SDS $(\tilde{X}^{\varepsilon},z)$, where $\tilde{X}^{\varepsilon}(y) \circ dz_t = \varepsilon\tilde{Y}(y) \circ dB_t + \tilde{A}(y)dt$.

Note

The deterministic flow $\tilde{\xi}_t^{o}$ has sources at x_N, x_S and sinks at x_E, x_W. //

The following proposition relates $\tilde{\xi}_t^{\varepsilon}(\omega)$ to $\xi_t^{\varepsilon}(\omega)$ and thus enables us to deduce the asymptotic behaviour of $\tilde{\xi}_t^{\varepsilon}(\omega)$ from that of $\xi_t^{\varepsilon}(\omega)$.

Proposition 3.3.1

For a.e. $\omega \in \Omega$ we have for all $t > 0$, $x \in E$ that $E\tilde{\xi}_t(\omega)x = \xi_t^{\varepsilon}(\omega)E_x$. ($E:S \to S$ given in Example 3.3).

Proof

This follows from the fact that

$TE\tilde{A}(x) = A(Ex)$, $TE\tilde{Y}(x)(e) = Y(Ex)(e)$, i.e. $\tilde{A},\tilde{Y}$ are the lifts of A,Y to the double cover.

We have

$$\tilde{\xi}_t^{\varepsilon}(\omega)x = x + \int_0^t \tilde{A}(\tilde{\xi}_s^{\varepsilon}(\omega)x)ds + \varepsilon\int_0^t \tilde{Y}(\tilde{\xi}_s(\omega)x) \circ dB_s.$$

Applying the Itô formula:

$$E\tilde{\xi}_t^{\varepsilon}(\omega)x = Ex + \int_0^t A(E\tilde{\xi}_s^{\varepsilon}(\omega)x)ds + \int_0^t Y(E\tilde{\xi}_s^{\varepsilon}(\omega)x) \circ dB_s.$$

Thus, $E\tilde{\xi}_t(\omega)x$ is a solution to (X^{ε},z) starting from Ex, and the result follows by a.s. uniqueness of flows. //

This result, and the fact that $\tilde{\xi}_t^{\varepsilon}(\omega)$ is continuous in time a.s., enables us to determine $\tilde{\xi}_t^{\varepsilon}(\omega)$ completely from $\xi_t^{\varepsilon}(\omega)$. We see that $\tilde{\xi}_t^{\varepsilon}(\omega)$ is antipodally symmetric a.s..

The following proposition gives the asymptotic behaviour of $\tilde{\xi}_t^{\varepsilon}(\omega)$, and is immediate from Proposition 3.3.1, and the corresponding facts for $\xi_t^{\varepsilon}(\omega)$ given in Propositions 3.1.2, 3.2.

<u>Proposition 3.3.2</u>

(i) For any $\varepsilon > 0$, the Lyapunov number $\tilde{\lambda}^{\varepsilon}$ for $\tilde{\xi}_t^{\varepsilon}(\omega)$ is the same as λ^{ε} for $\xi_t^{\varepsilon}(\omega)$. Thus, if $\varepsilon > 0$ is sufficiently small, then $\tilde{\lambda}^{\varepsilon} < 0$.

(ii) Choose ε such that $\tilde{\lambda}^{\varepsilon} < 0$, and take a version of $\tilde{\xi}_t^{\varepsilon}(\omega)$. Then for a.e. ω, $z_0(\omega)$ of Proposition 3.2 exists and is distinct from x_s. For such ω, take $z_1(\omega)$, $z_2(\omega)$ to be the inverse images of $z_0(\omega)$, under the map E of Example 3.3.

Then for a.e. $\omega \in \Omega$, the stable manifold $V^{\tilde{\lambda}^{\varepsilon}}_{(x,\omega)}$ exists for each $x \in S \setminus \{z_1(\omega), z_2(\omega)\}$, and is just the component of $S \setminus \{z_1(\omega), z_2(\omega)\}$ containing x.

3.4 A flow on the Torus (Double Noisy N.S. Flow)

The example of this section exhibits in a more complicated way than the noisy N.S. flow, the behaviour of Chapter 2.

Example 3.4 (Double Noisy N.S. Flow)

Take any $\varepsilon_1, \varepsilon_2 > 0$, and consider the independent pair $\xi_t^{\varepsilon_1}(\omega_1)$, $\xi_t^{\varepsilon_2}(\omega_2)$ of noisy N.S. flows. Then we take the double noisy N.S. flow to be the product $\xi_t^{\varepsilon_1}(\omega) \times \xi_t^{\varepsilon_2}(\omega)$ on $S \times S \equiv \mathbb{T}$.

The following obvious proposition tells us that the Lyapunov spectrum for this flow is $\{\lambda^{\varepsilon_1}, \lambda^{\varepsilon_2}\}$, and Proposition 3.4.2 tells us about the stable manifolds. Note that we can choose ε_1, ε_2 such that $\lambda^{\varepsilon_1} \neq \lambda^{\varepsilon_2}$. In fact by the explicit calculation of λ^{ε} outlined in Section 3.1, we see that λ^{ε} is a strictly increasing function of ε.

Proposition 3.4.1 (Cf. Theorem 2.1)

Choose $\varepsilon_1, \varepsilon_2 > 0$ and assume without loss of generality that $\lambda^{\varepsilon_1} < \lambda^{\varepsilon_2}$. For $(x_1, x_2) \in \mathbb{T}$, denote by $T^1_{(x_1,x_2)}\mathbb{T}$ the tangent to the curve $\{(x,x_2): x \in S\}$ in $\mathbb{T}$.

Then for a.e. $(\omega_1,\omega_2) \in \Omega \times \Omega$ we have the following:

The points $z_0(\omega_1)$, $z_0(\omega_2)$ of Proposition 3.2 exist, and we can define the filtration for the double noisy N.S. flow at the points $(x_1,x_2) \in [S\setminus\{z_0(\omega_1)\}] \times [S\setminus\{z_0(\omega_2)\}]$ in $\mathbb{T}$.

If $\lambda^{\varepsilon_1} < \lambda^{\varepsilon_2}$, then the spectrum is $\{\lambda^{\varepsilon_1}, \lambda^{\varepsilon_2}\}$, and the filtration is $T^1_{(x_1,x_2)}\mathbf{T} < T_{(x_1,x_2)}\mathbf{T}$. If $\lambda^{\varepsilon_1} = \lambda^{\varepsilon_2}$ then the spectrum is $\{\lambda^{\varepsilon_i}\}$, and the filtration is trivial.

<u>Proposition 3.4.2</u> (Cf. Theorem 2.2.2)

Choose ε_1, ε_2 as in Proposition 3.4.1, and assume that $\lambda^{\varepsilon_1} < \lambda^{\varepsilon_2}$.

Then for a.e. $(\omega_1,\omega_2) \in \Omega \times \Omega$, the conclusions of Proposition 3.4.1 hold and at those (x_1,x_2) for which the Lyapunov spectrum exists, the stable manifolds also exist.

If $\lambda^{\varepsilon_1} < \lambda^{\varepsilon_2}$, then

$$V^{\lambda^{\varepsilon_1}}_{(x_1,x_2,\omega_1,\omega_2)} = \{(x,x_2) : x \in S\setminus\{z_0(\omega_1)\}\},$$

$$V^{\lambda^{\varepsilon_2}}_{(x_1,x_2,\omega_1,\omega_2)} = \{(y_1,y_2) . y_i \in S\setminus\{z_0(\omega_i)\}\}.$$

If $\lambda^{\varepsilon_1} = \lambda^{\varepsilon_2}$, then

$$V^{\lambda^{\varepsilon_i}}_{(x_1,x_2,\omega_1,\omega_2)} = \{(y_1,y_2) : y_i \in S\setminus\{z_0(\omega_i)\}\}.$$

4. A FORMULA FOR THE LYAPUNOV NUMBERS. A PERTURBATION THEOREM

In [9], Khas'minskii gives a formula for the Lyapunov numbers in the case of a linear stochastic differential equation in $\mathbb{R}^m$. His technique is to project the solution onto the unit sphere S^{m-1} and to obtain an equation for that. In this chapter we give a formula which is analogous to his and which gives the Lyapunov numbers for an SDS on a smooth compact Riemannian manifold M. We work with the derivative flow and we take the radial projection onto the unit sphere bundle over M to correspond to Khas'minskii's projection onto S^{m-1}. In Section 4.1 we present Khas'minskii's work, adapted to our notation, and in Section 4.2 we give our analogue.

In Section 4.3 we give a perturbation theorem based on our formula: the theorem says roughly that if we have a deterministic dynamical system which is Lyapunov stable, then so is a small non-degenerate stochastic perturbation of it.

4.1 Khas'minskii's Formula [9]

Khas'minskii deals with a stochastic process on $\mathbb{R}^m$ which is the solution to a linear Itô stochastic equation: the work of this section will be essentially the same as his, except that we will express the equation in the Stratonovitch form as the SDS (X,z), where $X(y) \circ dz_t = A(y)dt + Y(y) \circ dB_t$, B_t being a Brownian motion in $\mathbb{R}^m$. ((X,z) being linear means that $A(y)$ and $Y(y)e$ (for any fixed $e \in \mathbb{R}^n$) are linear in y.)

Note

Suppose the solution to the linear SDS (X,z) starting from $x \neq 0$ is $\xi_t(\omega)x$. Then in this section we define the Lyapunov number $\lambda_{(x,\omega)}$ just to be $\limsup_{t\to\infty} 1/t \log \|\xi_t(\omega)x\|$. (Thus, we have a Lyapunov number rather than a Lyapunov spectrum at (x,ω)). If, for each x, the Lyapunov numbers are all negative for a.e. ω, then the solutions tend a.s. to the zero solution, and we say that the zero solution is asymptotically stable. See Arnold and Kliemann [1], Khas'minskii [10].

Lemma 4.1.1

(i) Denote by $G:\mathbb{R}^m\setminus\{0\} \to S^{m-1}$ the radial projection $G(x) = x/\|x\|$. Consider the map $DG:\mathbb{R}^m\setminus\{0\} \to L(\mathbb{R}^m,\mathbb{R}^m)$.
Then for any x,y, $DG(\beta x)(\beta y)$ is independent of $\beta > 0$.

(ii) Denote by $\psi:\mathbb{R}^m\setminus\{0\} \to \mathbb{R}$ the map $\psi(x) = \log \|x\|$. Then the derivative $D\psi:\mathbb{R}^m\setminus\{0\} \to L(\mathbb{R}^m,\mathbb{R})$ is given by $D\psi(x)y = \dfrac{\langle x,y\rangle}{\langle x,x\rangle}$.

From this we see that $D\psi(\beta x)(\beta y)$ is independent of $\beta > 0$.

Proof

(i) Immediate from the derivative:

$$DG(x)y = y/\|x\| - x\langle x,y\rangle/\|x\|^3 .$$

(ii) We have $\psi(x) = \frac{1}{2}\log\langle x,x\rangle$. The result follows from this using the chain rule. //

Lemma 4.1.2

Take any $x \in \mathbb{R}^m \setminus \{0\}$ and denote by $\xi_t(\omega)x$ the solution, starting from x, of the linear SDS (X,z) on $\mathbb{R}^m$.

(i) Then the stochastic process $G\xi_t(\omega)x$ (which we will denote by $\eta_t(\omega)x$) is the solution, starting from G(x), of the SDS $(\tilde{\tilde{X}},z)$ on S^{m-1}, where $\tilde{\tilde{X}}(x) = DG(x)X(x)$.

(ii) The stochastic process $\psi\xi_t(\omega)x$ in $\mathbb{R}$ is driven by $\eta_t(\omega)x$ according to the equation

$d(\psi\xi_t(\omega)x) = \langle\eta_t(\omega)x,\ X(\eta_t(\omega)x\rangle \circ dz_t(\omega)$, i.e.

$$\psi\xi_t(\omega)x = \psi(x) + \int_0^t \langle\eta_s(\omega)x,\ A(\eta_s(\omega)x)\rangle ds$$

$$+ \int_0^t \langle\eta_s(\omega)x,\ Y(\eta_s(\omega)x)\rangle \circ dB_s(\omega).$$

Converting to an Itô equation, and putting $\langle y,A(y)\rangle = \tilde{A}(y)$, $\langle y,Y(y)\rangle = \tilde{Y}(y)$, $\tilde{\tilde{Y}}(x) = DG(x)Y(x)$, we have

$$\psi\xi_t(\omega)x = \psi(x) + \int_0^t \tilde{A}(\eta_s(\omega)x)ds + \int_0^t \tilde{Y}(\eta_s(\omega)x)\ dB_s(\omega)$$

$$+ \frac{1}{2}\int_0^t \left[\sum_{i=1}^n D\tilde{Y}[\eta_s(\omega)x][\tilde{\tilde{Y}}(\eta_s(\omega)x)e_i]e_i\right]ds.$$

Proof

Note first that the (partial) flow $\xi_t(\omega)$ is a diffeomorphism whenever we can define it (see Carverhill and Elworthy [4]), and since (X,z) is linear, we have $\xi_t(\omega)0 = 0$ a.s. Therefore if $x \in \mathbb{R}^m \setminus \{0\}$, then $\xi_t(\omega)x \in \mathbb{R}^m \setminus \{0\}$ a.s. .

(i) Transforming via $G:\mathbb{R}^m \setminus \{0\} \to S^{m-1}$, using the Itô formula, we have

$$d[G\xi_t(\omega)x] = DG(\xi_t(\omega)x).X(\xi_t(\omega)x) \circ dz_t(\omega).$$

By Lemma 4.1.1 (i) and the fact that X is linear, this RHS is equal to

$$DG\left(\frac{\xi_t(\omega)x}{||\xi_t(\omega)x||}\right).X\left(\frac{\xi_t(\omega)x}{||\xi_t(\omega)x||}\right) \circ dz_t(\omega)$$

$$\equiv DG(\eta_t(\omega)x).X(\eta_t(\omega)x) \circ dz_t(\omega),$$

and the result follows.

(ii) Again by the Itô formula,

$$d[\psi\xi_t(\omega)x] = D\psi(\xi_t(\omega)x).X(\xi_t(\omega)x) \circ dz_t(\omega)$$

$$= \frac{\langle\xi_t(\omega)x,\ X(\xi_t(\omega)x)\rangle \circ dz_t(\omega)}{\langle\xi_t(\omega)x,\ \xi_t(\omega)x\rangle}$$

by Lemma 4.1.1 (ii).

Also, by linearity of X, this last expression is equal to $\langle \eta_t(\omega)x, X(\eta_t(\omega)x)\rangle \circ dz_t(\omega)$, and the result follows. //

Theorem 4.1 (Khas'minskii's Formula [9])

Consider the situation of Lemma 4.1.2.

(i) Then the Lyapunov number $\lambda_{(x,\omega)}$ $(\equiv \lim_{t\to\infty} 1/t \log \|\xi_t(\omega)x\|)$ is given by

$$\lim_{t\to\infty} 1/t \int_0^t \{\tilde{A}(\eta_s(\omega)x) + \frac{1}{2}\sum_{i=1}^{n} D\tilde{Y}[\eta_s(\omega)x][\tilde{\tilde{Y}}(\eta_s(\omega)x)e_i]e_i\}ds.$$

$(\tilde{A}, \tilde{Y}, \tilde{\tilde{Y}}$ as in Lemma 4.1.2)

(ii) Suppose (X,z) is non-degenerate on $\mathbb{R}^m$, i.e. if $x \neq 0$ then $Y(x) \in \mathbb{L}(\mathbb{R}^n,\mathbb{R}^m)$ is surjective. Then $(\tilde{\tilde{X}},z)$ is a non-degenerate SDS on S^{m-1} and there exists a unique invariant measure ρ on S^{m-1} for $(\tilde{\tilde{X}},z)$. Also,

$$\lambda_{(x,\omega)} = \int_{y\in S^{m-1}} \{\tilde{A}(y) + \frac{1}{2}\sum_{i=1}^{n} D\tilde{Y}(y)[\tilde{\tilde{Y}}(y)e_i]e_i\}\, d\rho(y),$$

for a.e. $(x,\omega) \in S^{m-1} \times \Omega$.

<u>Proof</u>

(i) Using the formula of Lemma 4.1.2(ii) for $\psi\xi_t(\omega)x \equiv \log \|\xi_t(\omega)x\|$, it suffices to prove that

$$\lim_{t\to\infty} 1/t \int_0^t \tilde{Y}(\eta_t(\omega)x)\, dB_s(\omega) = 0 \text{ a.s.}$$

For this, we will show that $\int_0^t \tilde{Y}(\eta_s(\omega)x)\, dB_s(\omega)$ ($\equiv \zeta_t(\omega)x$, say) is a time changed Brownian motion.

By the Itô formula,

$$(\zeta_t(\omega)x)^2 = 2\int_0^t [\zeta_s(\omega)x]\, \tilde{Y}(\eta_s(\omega)x)\, dB_s$$

$$+ \int_0^t \sum_{i=1}^n (\tilde{Y}(\eta_s(\omega)x)e_i)^2\, ds.$$

Define $\sigma:[0,\infty) \times \Omega \to [0,\infty]$ by

$$\sigma(t,\omega) = \text{Min } \{s \text{ such that} \int_0^s \sum_{i=1}^n \tilde{Y}(\eta_q(\omega)x)e_i)^2\, dq = t\}$$

if such an s exists, or ∞ otherwise. (Note that since the integrand is non-negative a.s., $\sigma(t,\omega)$ is non-decreasing a.s.).

Then $\int_0^{\sigma(t,\omega)} \sum_{i=1}^n \tilde{Y}(\eta_s(\omega)x)\, dB_s = t$, and $(\zeta_{\sigma(t,\omega)}(\omega)x)^2 - t$ is a martingale. Also, $\zeta_{\sigma(t,\omega)}(\omega)x$ is a martingale. Thus, by Levy's characterisation, $\zeta_{\sigma(t,\omega)}(\omega)x$ is a Brownian motion, $\tilde{B}_t(\omega)$ say. (See Elworthy [6], pp 80-84).

From this we deduce that

$$\zeta_t(\omega)x = \tilde{B}_{\rho(t,\omega)}(\omega), \text{ where } \rho(t,\omega) = \int_0^t \sum_{i=1}^n (\tilde{Y}(\eta_s(\omega)x)e_i)^2 ds.$$

Now, $\sum_{i=1}^n (\tilde{Y}(y)e_i)^2$ is continuous over $y \in S^{m-1}$, and therefore bounded, say by N. Therefore $\rho(t,\omega) < Nt$ a.s.. Also, $1/t\ \tilde{B}_{Nt}(\omega) \to 0$ a.s. as $t \to \infty$ (See Mckean [13], p.9), therefore $1/t\ \zeta_t(\omega)x$ ($\equiv 1/t\ B_{\rho(t,\omega)}$ a.s) $\to 0$ a.s. as $t \to \infty$, and we have the result.

(ii) That (X,z) non-degenerate implies $(\tilde{\tilde{X}},z)$ non-degenerate follows because for $x \neq 0$, $DG(x)$ is surjective. The formula for $\lambda_{(x,\omega)}$ follows from part (i), using the Strong Law of Large Numbers (Proposition 1.2.3). //

4.2 An Analogous Formula to Khas'minskii's for an SDS on a Smooth Compact Riemannian Manifold

In this section we give an analogous formula for the Lyapunov numbers (defined in Theorem 2.1) for a smooth SDS on a smooth compact Riemannian manifold M of dimension m. Our technique is analogous to that of Section 4.1 (i.e. Khas'minskii's), and we will use the same notation for some of the corresponding concepts.

Thus, we will denote our SDS on M by (X,z), $X(y) \circ dz = A(y)dt + Y(y) \circ dB$, where B is a Brownian motion on $\mathbb{R}^n$. Also, we will denote the flow of (X,z) by $\xi_t(\omega)$. Corresponding to the system of Section 4.1 we take the *derivative* SDS $(\delta X,z)$ on TM, whose flow is $T\xi_t(\omega)$ and which we define in Proposition 4.2. We define the Lyapunov number $\lambda_{(v,\omega)}$ for $v \in T_xM$ by $\lim_{t\to\infty} 1/t \, \|T\xi_t(\omega)v\|$, if this limit exists. (Thus, the Lyapunov spectrum at $(x,\omega) \in M \times \Omega$ is the collection of these limits for $v \in T_xM$.)

Note that in this section, (X,z) need not be non-degenerate. In fact, (X,z) could be a deterministic system.

<u>Proposition 4.2</u> (See Elworthy [6] Chapter 7, Section 8E)

For each $e \in \mathbb{R}^m$, $v \in T_xM$, put $\delta X(v)e = \alpha \circ TX(v)e$, where $\alpha : T^2M \to T^2M$ is given over charts by

$$\alpha : U \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to U \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m;$$

$(x,u,v,w) \mapsto (x,v,u,w)$.

Then $\delta X(\ )e : TM \to T^2M$ is a vector field over TM and $(\delta X,z)$ is an SDS over TM (called the *derivative* of (X,z)), whose flow is $T\xi_t(\omega)$.

Proof

Express δX over a chart $\begin{matrix} \phi : U & \to & V \\ \cap & & \cap \\ M & & \mathbb{R}^m \end{matrix}$.

Suppose that for each $e \in \mathbb{R}^n$, $X^e (\equiv X(\ )e)$ is given over the chart by $x \to (x, X^e_\phi(x))$. Then TX^e is given by $(x,v) \to (x, X^e_\phi(x), v, DX^e_\phi(x)v)$, and δX^e is given by $(x,v) \to (x, v, X^e_\phi(x), DX^e_\phi(x)v)$.

Denote by $\tilde{X}_\phi$ the extension of X_ϕ to all of $\mathbb{R}^m$, so that it is smooth and has bounded support. Denote by $\tilde{\xi}^\phi_t(\omega)$ the flow of $(\tilde{X}_\phi, z)$. Then for each $x \in U$, $\phi(\xi_t(\omega)x) = \tilde{\xi}^\phi_t(\omega)\phi(x)$ a.s. until the exit time $\tau_t(\omega)$ of $\xi_t(\omega)x$ from U. (This follows from Elworthy's definition of the solution to (X,z). To get it from ours, consider the coordinate functions of ϕ separately, and use Lemma 5.1 of Carverhill and Elworthy [4]).

Now, by [4] Remark 4.2, the solution to the SDS $(\delta\tilde{X}\ , z)$ $(\delta\tilde{X}^e_\phi : (x,v) \to (x, v, \tilde{X}^e_\phi(x), D\tilde{X}^e_\phi(x)v))$ is $D\tilde{\xi}^\phi_t(\omega)(x)v$. Also, until the exit time $\tau_t(\omega)$, this is just $T\phi . T\xi_t(\omega)u_y$, where $u_y \in T_yM$ and $T\phi(u_y) = (x,v)$.

Thus, until the exit time $T_t(\omega)$, the solution to $(\delta X, z)$ starting from $u_y \in T_yM$ is just $T\xi_t(\omega)(u_y)$, and the result follows. //

Lemma 4.2.1 (Cf. Lemma 4.1.1)

Put $T^oM = TM \setminus M \times \{0\}$ (i.e. TM without the zero section.) Denote by SM the unit sphere bundle over M. Take any $x \in M$, $v \in T_x^oM$, $y \in T_vT^oM$ and denote the horizontal and vertical components of y by y^H, y^h. Take y^h to lie in T_xM via the natural identification $T_v^hTM \cong T_xM$.

(i) Denote by $G : T^oM \to SM$ the radial projection

$G(v) = v/\|v\|$.

Then the map $TG: TT^oM \to TSM$ is given by $TG(y) = DG_x(v)(y^h) + Tm_{(1/\|v\|)}(y^H)$, where G_x is the radial projection $T_x^oM \to S_xM$ (i.e. the map of Lemma 4.1.1(i)) and $m_\beta : TM \to TM$ is multiplication in fibres by β.

(ii) Denote by $\Psi : T^oM \to \mathbb{R}$ the map $\Psi(v) = \log \|v\|$. Then $d\Psi : TT^oM \to \mathbb{R}$ is given by $d\Psi(y) = \dfrac{\langle v, y^h \rangle}{\langle v, v \rangle}$.

Proof

(i) Now, $y^h = \left.\dfrac{d\gamma^h(t)}{dt}\right|_{t=0}$, where γ^h is a vertical curve in T^oM, i.e. a curve which lies in T_x^oM, and where $\gamma^h(0) = v$. If y^h is identified with $u \in T_xM$, then we can take $\gamma^h(t) = v+tu$. Also, $G(v + tu) = G(v) + t\, DG_x(v)n + o(t)$. Since $TG(y^h)$ is defined by $\left.\dfrac{d}{dt}\, G(\gamma^h(t))\right|_{t=0}$, we see that $TG(y^h) = DG_x(v)y^h$.

Also, $y^H = \frac{d\gamma^H}{dt}(t)\Big|_{t=0}$, where γ^H is a horizontal curve in T^oM with $\gamma^H(0) = v$, i.e. a parallel translation of v along a curve in M. For such a curve, $\|\gamma^H(t)\| = \|v\|$, i.e. is constant, therefore $G(\gamma^H(t)) = \gamma^H(t)/\|v\| \equiv m_{(1/\|v\|)}\gamma^H(t)$ and $TG(y^H) = T_{m_{(1/\|v\|)}}(y^H)$.

(ii) Denote by $Q: T^oM \to \mathbb{R}$ the map $v \to \|v\|^2$. Then $dQ(y^V) = 2\langle y, y^V\rangle$. (To see this use the characterisation of y^V in part (i)). Also $dQ(y^H) = 0$, since length is preserved by parallel translation. Thus, $dQ(y) = 2\langle v, y^V\rangle$ and the result follows using the chain rule. //

<u>Lemma 4.2.2</u> (Cf. Lemma 4.1.2)

Take any $x \in M$, $v \in T^o_xM$ and consider the stochastic process $T\xi_t(\omega)v \in TM$.

(i) Then the stochastic process $GT\xi_t(\omega)v$ in SM (which we will denote by $\eta_t(\omega)v$) is the solution, starting from $G(v)$, of the SDS $(\tilde{X}, z)$ in SM, where $\tilde{X}(v) = TG.\delta X(v)$.

(ii) The stochastic process $\Psi.T\xi_t(\omega)v$ in $\mathbb{R}$ is driven by $\eta_t(\omega)v$ according to the equation

$$d[\Psi.T\xi_t(\omega)v] = [d\Psi.\delta X(\eta_t(\omega)v)] \circ dz_t,$$

i.e.

$$\Psi.T\xi_t(\omega)v = \Psi(v) + \int_0^t d\Psi.\delta A(\eta_s(\omega)v)ds$$

$$+ \int_0^t d\Psi.\delta Y(\eta_s(\omega)v) \circ dB_s(\omega).$$

Converting to an Itô equation, and putting $\tilde{A}(v) = \langle v,\delta X(v)^* \rangle$, $\tilde{Y}(v) = \langle v,\delta Y(v)^* \rangle$, $TG.\delta Y(v) = \tilde{\tilde{Y}}(v)$, we have

$$\Psi.T\xi_t(\omega)v = \Psi(v) + \int_0^t \tilde{A}(\eta_s(\omega)v)ds$$

$$+ \int_0^t \tilde{Y}(\eta_s(\omega)v)\ dB_s(\omega)$$

$$- \frac{1}{2}\int_0^t \{ \sum_{i=1}^n d\tilde{Y}[\tilde{\tilde{Y}}(\eta_s(\omega)v)e_i]e_i\}\ ds.$$

Proof

We prove this analogously to Lemma 4.1.2, by transforming the SDS $(\delta X,z)$ via G in part (i) and Ψ in part (ii).

(i) For this we need the standard result:

Suppose N_1, N_2 are (finite dimensional) smooth manifolds with C^2 SDS's (Y_1,z), (Y_2,z) defined on them. Denote the solutions to these by $\eta_t^i(\omega)y$. $(y \in N_i)$. Suppose that the following diagram commutes for each e, where $F:N_1 \to N_2$ is a smooth map:

$$\begin{array}{ccc} TN_1 & \xrightarrow{TF} & TN_2 \\ Y_1^e \uparrow & & \uparrow Y_2^e \\ N_1 & \xrightarrow[F]{} & N_2 \end{array}$$

Then for each $y \in N_1$, $F(\eta_t^1(\omega)y) = \eta_t^2(\omega)(F(y))$ a.s. .

To prove this, take any $g:N_2 \to \mathbb{R}$, and put $h = g \circ f:N_1 \to \mathbb{R}$. Then by definition of the solution to (Y_1,z), we have for each $y \in N_1$, that

$$h(\eta_t^1(\omega)y) = h(y) + \int_0^t dh \circ Y_1(\eta_s^1(\omega)y) \circ dz_s(\omega).$$

Also, since $h = g \circ f$ and the diagram commutes, we have

$$g(F(\eta_t^1(\omega)y) = g(F(y)) + \int_0^t dg \circ Y_2(F(\eta_t^1(\omega))) \circ dz_s(\omega).$$

This is so for any g, therefore $f(\eta_t^1(\omega)y)$ is a solution to (Y_2,z) starting from $f(y)$, and the result follows by the a.s. uniqueness of the solution.

For part (i), we will show that the following diagram commutes for each e:

$$\begin{tikzcd}
TT^{o}M \arrow[r, "TG"] & TSM \\
T^{o}M \arrow[r, "G"'] \arrow[u, "\delta X^{e}"] & SM \arrow[u, "TG.\delta X^{e}|_{SM}"']
\end{tikzcd} .$$

For this it suffices to show that $TG.\delta X.G^{-1}$ is well defined and equal to $TG.\delta X|_{SM}$, i.e. that $TG.\delta X(\beta v)$ is independent of $\beta > 0$.

From Lemma 4.2.1(i),

$$TG.\delta X(\beta v) = DG_x(\delta X(\beta v)^{h}) + T_{m_{(1/\|v\|)}}(\delta X(\beta v)^{\parallel}).$$

The first term in this RHS is independent of $\beta > 0$ by Lemma 4.1.1(i) (N.B. $\delta X(v)^{h}$ is linear over $v \in T_xM$). To deal with the other term, note that T_{m_β} expressed in coordinates is given by

$$(x, v, y_x, y_v) \to (x, \beta v, y_x, \beta y_x),$$

so that $T_{m_{(1/\|v\|)}}(\delta X(\beta v))$ is given by

$$(x, v) \to (x, \beta v/_{\|\beta v\|}, X(x), DX(x)\beta v/_{\|\beta v\|}),$$

which is independent of $\beta > 0$. The second term in the RHS being zero follows from this because T_{m_β} maps horizontal (vertical) to horizontal (vertical), so that $T_{m_\beta}(y^{\parallel}) = T_{m_\beta}(y)^{\parallel}$.

(ii) Transforming the SDS $(\delta X, z)$ for $T\xi_t(\omega)v$ via $\Psi: T^{o}M \to \mathbb{R}$, we have

$$\Psi.T\xi_t(\omega)v = \Psi(v) + \int_0^t d\Psi.\delta X(T\xi_t(\omega)v) \circ dz_s(\omega).$$

(In fact, this follows from our definition of the solution to $(\delta X, z)$.)

We must show that $d\Psi.\delta X.G^{-1}$ is well defined and equal to $d\Psi.\delta X|_{SM}$, i.e. that $d\Psi.\delta X(\beta v)$ is independent of $\beta > 0$. From Lemma 4.2.1 (ii) and the linearity of $\delta X(v)^{h}$ over fibres, we have

$$d\Psi.\delta X(\beta v) = \frac{\langle \beta v, \beta\, \delta X(v)^{h} \rangle}{\langle \beta v, \beta v \rangle}$$

and the result follows. //

<u>Sketch of an Alternative Proof</u>

The key facts for this lemma are that $TG.\delta X(\beta v)$ and $d\Psi.\delta X(\beta v)$ are independent of $\beta > 0$. We can prove these facts without using the horizontal-vertical decomposition, by expressing these things over charts for M. We do this below: this is more elementary but less geometrical than the above.

Suppose $\phi : \underset{\underset{M}{\cap}}{U} \to \underset{\underset{\mathbb{R}^m}{\cap}}{V}$ is a chart, over which X^e is given by $x \to (x, X^e_\phi(x))$. ($X^e \equiv X(-)e$.) Then over the corresponding chart $T\phi : \underset{\underset{TM}{\cap}}{\pi^{-1}(U)} \to V \times \mathbb{R}^m$, δX^e is given by

$$(x,v) \to (x, v, X^e_\phi(x),\ DX^e_\phi(x)v).$$

Now, from the Riemannian metric on M, $T\phi$ induces an inner product on the image of each fibre $T\phi(\pi^{-1}(y)) = \{\phi(y)\} \times \mathbb{R}^m$. (any $y \in U \subset M$.) For each $(x,v) \in V \times \mathbb{R}^m$, denote this by $\|v\|_x$.

Over the chart, G and Ψ are given by $G_\phi : (x,v) \to (x,\ v/\|v\|_x)$ and $\Psi_\phi : (x,v) \to \log \|v\|_x$. Therefore TG and $d\Psi$ are given by

$$TG_\phi : (x, v, y_x, y_v) \to (x, v/\|v\|_x,\ y_x,$$
$$D_xH(x,v)y_x + D_vH(x,v)y_v)$$

and

$$d\Psi_\phi : (x, v, y_x, y_v) \to \frac{\frac{1}{2} D_xE(x,v)y_x}{\langle v,v\rangle_x} + \frac{\langle v, y_v\rangle_x}{\langle v,v\rangle_x},$$

where $H(x,v) = v/\|v\|_x$ and $E(x,v) = \langle v,v\rangle_x$.

Thus, $TG.\delta X^e$ (in the chart) sends $(x, \beta v)$ to

$$(x,\ \beta v/_{\beta\|v\|_x},\ X^e_\phi(x)v,\ D_xH(x,\beta v)X^e_\phi(x)$$
$$+ D_vH(x,\beta v)DX^e_\phi(x)(\beta v)).$$

The term $D_v H(x,\beta v) DX^e_\phi(x)(\beta v)$ is essentially the same as that in the linear situation (Lemma 4.1.2(i)) and it is independent of $\beta > 0$ for the same reason. The term $D_x H(x,\beta v) X^e_\phi(x)$ is new, but since $H(x,\beta v)$ is independent of $\beta > 0$, this term is also.

Also, $d\phi.\delta X^e$ (in the chart) sends $(x,\beta v)$ to

$$\frac{1}{2}\,\frac{D_x E(x,\beta v) X^e_\phi(x)}{\langle\beta u,\ \beta v\rangle_x} + \frac{\langle\beta v,\ DX^e_\phi(x)(\beta v)\rangle_x}{\langle\beta u,\ \beta v\rangle_x} .$$

The second term in this expression is essentially the same as that in the linear situation (Lemma 4.1.2(ii)), and is independent of $\beta > 0$. The other term is new, but since $E(x,\beta v)$ is quadratic in β, $D_x E(x,\beta v) X^e_\phi(x)$ is also, and this new term is independent of $\beta > 0$.

So in the nonlinear case, we get some new terms in our expressions, but these new terms do not prevent the analysis from going through. In fact these terms are related to the curvature of M and can be expressed in terms of the Cristoffel symbols. The horizontal-vertical splitting corresponds to taking 'normal' coordinates about a point in M, in which the Cristoffel symbols vanish at that point. //

<u>Theorem 4.2</u> (Cf. Theorem 4.1)

Take any $x \in M$, $v \in T^o_x M$. Then the Lyapunov number $\lambda(v,w)$, if it exists, is just

$$\lim_{t\to\infty} 1/t \int_0^t \{\tilde{A}(\eta_s(\omega)v) + \frac{1}{2}\sum_{i=1}^{n} d\tilde{Y}[(\tilde{\tilde{Y}}(\eta_s(\omega)v)e_i]e_i\}\, ds.$$

($\tilde{A}$, $\tilde{Y}$, $\tilde{\tilde{Y}}$ as in Lemma 4.2.2)

Proof

Using the formula of Lemma 4.2.2(ii) for log $\|T\xi_t(\omega)v\|$, it suffices to prove that

$$\lim_{t\to\infty} 1/t \int_0^t \tilde{Y}(\eta_s(\omega)v)dB_s = 0 \quad \text{a.s.}\ .$$

For this, as in the proof of Theorem 4.1 (i), note that $\int_0^t \tilde{Y}(\eta_s(\omega)v)dB_s$ is a time changed Brownian motion $\tilde{B}_{\rho(\lambda,\omega)}(\omega)$, where $\rho(t,\omega) = \int_0^t \sum_{i=1}^{n} (\tilde{Y}(\eta_s(\omega)v)e_i)^2\, ds$. Also, since $\sum_{i=1}^{n} (\tilde{Y}(y)e_i)^2$ is continuous over $y \in SM$, it is bounded, say by N, and we have $\rho(t,\omega) < Nt$ a.s. //

Notes

(i) Even if the SDS (X,z) is nondegenerate, $(\tilde{\tilde{X}},z)$ may be degenerate, so we do not have a direct analogue of Theorem 4.1(ii). However, it might be possible to express the Lyapunov numbers in terms of invariant measures for $(\tilde{\tilde{X}},z)$ which are ergodic in the sense of Yosida [19] Chapter 13. See Appendix B Section 2. Note that this would actually give the existence of the Lyapunov numbers independently of our Chapter 2.

(ii) The vertical components $\delta A(v)^{v}$, $\delta Y(v)^{v}$ are just the covariant derivatives $\nabla A(x)v$, $\nabla Y(x)v$. $(v \in T_xM)$. To see this note that the connection map $K:TTM \to TM$ given by the Riemannian metric is just projection along the horizontal component onto the vertical component. Over a chart ϕ, K is given by

$$(x,v,y_x,y_v) \mapsto (x,y_v + \Gamma_\phi(x)(y_x,v)),$$

where Γ_ϕ is the local connector. Now, $\nabla A(x)$ is defined to be $K.T_xA$, so that over the chart ϕ we have $\nabla A_\phi(x)v = DA_\phi(x)v + \Gamma_\phi(x)(v,\phi(x))$. Expressing $\delta A(v)^{v} \equiv K.\delta A(v)$ over the chart gives the same expression, because $\Gamma_\phi(x)$ is symmetric.

4.3. Lyapunov Stability of Certain Stochastic Perturbations of Deterministic Dynamical Systems

In this section we consider a stochastic perturbation of a smooth deterministic dynamical system with vector field F, defined on a smooth compact manifold M. We take the perturbation to be the smooth nondegenerate SDS (X^ε, z), where

$$X^\varepsilon(y) \circ dz_t = \varepsilon Y(y) \circ dB_t + [\varepsilon A(y) + F(y)]dt$$

(B_t a Brownian motion in $\mathbb{R}^n$), and we will denote its flow by $\xi_t^\varepsilon(\omega)$ in this section. Also, we will denote the unique invariant measure for (X^ε, z) by ρ^ε.

We must assume that as ε tends to zero, the invariant measure ρ^ε concentrates on a finite set of hyperbolic attracting fixed points of the flow ξ_t^0. The question of when this assumption is valid is studied by Ventsel and Freidlin [18]. They show that it is valid for certain perturbations if the ω-limit sets of ξ_t^0 are all hyperbolic fixed points.

Note that at a hyperbolic stable fixed point x_0 of ξ_t^0 we can define the Lyapunov spectrum of Ruelle [16] and it is equal to the real part of the spectrum of $\nabla F(x_0)$, and is therefore strictly negative. To see this, observe that at the fixed point x_0, $F(x_0) = 0$, therefore over a chart ϕ, $\nabla F(x_0)$ is given by $DF_\phi(x_0)$.

The theorem of this section says that under the above assumption about ρ^ε, for sufficiently small $\varepsilon > 0$, the SDS (X^ε, z) is Lyapunov stable. Its proof relies on the formula of Theorem 4.2 for the Lyapunov numbers.

Lemma 4.3

Suppose all the eigenvalues of the linear map $L: \mathbb{R}^m \to \mathbb{R}^m$ have strictly negative real part. Then there exists an inner product $\langle\langle\ ,\ \rangle\rangle$ on $\mathbb{R}^m$ in which

$$\langle\langle Lv,v\rangle\rangle < 0 \text{ for all } v \in \mathbb{R}^m \setminus \{0\}.$$

Proof

This inner product is the Lyapunov function for L. See Arnold V.I. [2], p. 146. //

Theorem 4.3.

Consider the SDS (X^ε, z) defined above. Assume that as ε tends to zero, the invariant measure ρ^ε concentrates on a finite set $x_1,\ldots,x_p$ of hyperbolic attracting fixed points of ξ_t^0. (i.e. if D is a closed subset of M which does not contain $x_1,\ldots,x_p$, then $\rho^\varepsilon(D) \to 0$ as $\varepsilon \to 0$).

Then if $\varepsilon > 0$ is sufficiently small, the SDS (X^ε, z) is Lyapunov stable, i.e. for a.e. (x,ω), $\sup\limits_{v \in S_x M} \{\lambda^\varepsilon_{(v,\omega)}\}$ is strictly negative, where $\lambda^\varepsilon_{(v,\omega)} = \lim\limits_{t\to\infty} 1/t\ \|T\xi_t^\varepsilon(\omega)v\|$.

Proof

Applying Theorem 4.2 to (X^ε, z) gives the formula

$$\lambda^{\varepsilon}_{(v,\omega)} = \lim_{t\to\infty} 1/t \int_0^t \tilde{F}(\eta_s^{\varepsilon}(\omega)v)ds$$

$$+ \lim_{t\to\infty} \varepsilon/t \int_0^t \tilde{A}(\eta_s^{\varepsilon}(\omega)v)ds - \lim_{t\to\infty} \varepsilon/2t \int_0^t \sum_{i=1}^{n} d\tilde{Y}[\tilde{\tilde{Y}}^{\varepsilon}(\eta_s^{\varepsilon}(\omega)v)e_i]e_i \, ds,$$

for $v \in S_xM$. Here, $\tilde{F}(v) = \langle v, \delta F(v)^{\wedge}\rangle \equiv \langle v, \nabla F(v)\rangle$, and similarly for $\tilde{A}$ and $\tilde{Y}$, $\tilde{\tilde{Y}}(v) = TG.\delta Y(v)$, and $\eta_s^{\varepsilon}(\omega)v$ is the solution of $(\tilde{\tilde{X}}^{\varepsilon}, z)$ on SM.

Now, at the stable fixed points $x_1,\ldots,x_p$, the eigenvalues of $\nabla F(x)$ have strictly negative real part, therefore by Lemma 4.3 there exists an inner product on $T_{x_i}M$ in which $\tilde{F}(v) < 0$ for all $v \in S_{x_i}M$. We can assume that the Riemannian metric does give these inner products on $T_{x_i}M$. With this choice of Riemannian metric, define $C:M \to \mathbb{R}$ by $C(x) = \max_{v\in S_xM} \{\tilde{F}(v)\}$, and put $\max_{i=1,\ldots,p} \{C(x_i)\} = -q < 0$.

Now for $v \in S_xM$, $\pi\, \eta_t^{\varepsilon}(\omega)v = \xi_t^{\varepsilon}(\omega)x$, where $\pi:SM \to M$ is the bundle projection, therefore $\tilde{F}(\eta_t^{\varepsilon}(\omega)v) < C(\xi_t^{\varepsilon}(\omega)x)$. Thus,

$$\lim_{t\to\infty} \int_0^t \tilde{F}(\eta_s^{\varepsilon}(\omega)v)ds < \lim_{t\to\infty} \int_0^t C(\xi_s^{\varepsilon}(\omega)x)ds = \int_{x\in M} C(x)\, d\rho^{\varepsilon}(x).$$

Also, from the assumption on ρ^{ε}, we have

$$\lim_{\varepsilon\to 0} \int_{x\in M} C(x)\, d\rho^{\varepsilon}(x) < -q.$$

Therefore for any $\tau > 0$, if $\varepsilon > 0$ is sufficiently small (say $0 < \varepsilon < \varepsilon_1$) then the first term in the above formula for $\lambda^{\varepsilon}_{(v,\omega)}$ is less than $-q + \tau$.

To deal with the other terms, note that

$$\max_{\substack{\varepsilon \in [0,\varepsilon_1] \\ v \in SM}} \{ |\tilde{A}(v)| + \frac{1}{2} | \sum_{i=1}^{n} d\tilde{Y}[\tilde{\tilde{Y}}^{\varepsilon}(v)e_i]e_i | \}$$

(= K say) is finite. Therefore the sum of these terms is bounded by εK for $\varepsilon \in [0,\varepsilon_1]$, and choosing $\varepsilon > 0$ sufficiently small we can ensure that the sum is less than τ, and $\lambda^{\varepsilon}_{(v,\omega)} < -q + 2\tau$. //

Note

Propositions 3.1.1, 3.1.2 may be regarded as applications of Theorems 4.2, 4.3 to a situation in which the last two terms vanish in the formula for $\lambda^{\varepsilon}_{(v,\omega)}$ above.

APPENDIX A : REGULARITY ESTIMATES ON THE DERIVATIVES OF THE FLOW

Here we establish the estimates which are needed for Chapter 2. In Section A.1 we prove some general estimates on flow derivatives and in Section A.2 we give the specific results which we require.

Note that in Section A.1 we discuss Itô systems, although for Chapter 2 we need results about Stratonovitch systems. However, any Stratonovitch system can be converted into an Itô system with one degree of differentiability less, so that the H^{s+2} system of Carverhill and Elworthy [4] can be converted to an H^{s+1} Itô system. Also, since all the transformations involved are linear, the techniques of [4] Chapter 3 can be adapted to Itô equations and we can deduce that if the Itô system $I(X,z)$ is H^{s+1}, then the flow exists and is H^s. $(s > m/2 + 1)$. This is because the lifted SDS $\tilde{I}(X,z)$ is then C^1 and therefore has a C^0 solution.

A.1 Estimates for Partial Derivatives of a Stochastic Flow on a Bounded Domain in $\mathbb{R}^m$.

In this section we work with an H^{s+1} Itô SDS $I(X,z)$ $(s > m/2 + k)$, supported on a bounded open set $U \subset \mathbb{R}^m$. Using the techniques of [4] Chapter 3, we can deduce that the flow ξ_t exists and is H^s, and hence C^k. We will suppress ω from our notation in this section, and restrict attention to a finite time interval [0,T].

Our main result is Proposition A.1.2. Its proof is adapted from Ikeda and Watanabe [8], and works by induction on the order $|\alpha|$ of the partial derivative $\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \xi_t : U \to \mathbb{R}^m$, using a Stochastic equation for the derivative. The equation for the derivative is established in Lemma A.1.2 and Proposition A.1.1 enables us to make the inductive step. We also require some standard estimates, which we present in Lemma A.1.1.

Lemma A.1.1

Consider the set $\mathcal{B}$ of stochastic processes B_t on the time interval [0,T], which are continuous in the $\mathcal{L}^2$ norm, i.e.

$$\mathbb{E}[\,\|B_t\|^2] < \infty \text{ for } t \in [0,T] \quad \text{and}$$

$$\mathbb{E}[\,\|B_t - B_s\|^2] \to 0 \text{ as } s \to t.$$

Take z_t to be a Brownian motion on $\mathbb{R}^n$ + drift dt. Then

(i) $\exists\, \beta > 0$ such that for all $B \in \mathcal{B}$, $t \in [0,T]$, we have

$$\mathbb{E}[\,\|\int_0^t B_s\, dz_s\|^2] < \beta \int_0^t \mathbb{E}[\,\|B_s\|^2]\, ds.$$

(ii) $\exists\, \gamma > 0$ such that for all $B \in \mathcal{B}$, we have

$$\mathbb{E}[\sup_{t\in[0,T]} \|\int_0^t B_s\, dz_s\|\,] < \gamma \int_0^T \mathbb{E}[\,\|B_s\|^2]\, ds.$$

Proof

(i) See Elworthy [6], Chapter 3, Corollary 3.

(ii) Easily obtained from Elworthy [6], Chapter 4, Estimates 5C. //

Notes

(i) We must have Itô integrals and not Stratonovitch in Lemma A.1.1.

(ii) Throughout, we mean by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^m$; In [6], Elworthy takes $\|\cdot\|$ to mean the $\mathcal{L}^2$ norm, i.e. if $f \in \mathcal{L}(\Omega,\mathbb{R})$ then he puts $\|F\| = \sqrt{\int |F|^2 \, dP} \equiv \sqrt{\mathbb{E}[|F|^2]}$.

Proposition A.1.1

Suppose $\eta_t(x)$ is a stochastic process for each x in the bounded open set $U \subset \mathbb{R}^m$, and suppose

$$\mathbb{E}[\sup_{t\in[0,T]} \|\eta_t(x)\|] \text{ and } \sup_{t\in[0,T]} \mathbb{E}[\|\eta_t(x)\|^2]$$

are each bounded uniformly over $x \in U$.

For each $x \in U$, consider the stochastic equation in $\mathbb{R}^\ell$

$$\zeta_t(x) = \eta_t(x) + \int_0^t Y(\zeta_s(x)) \, dz_s,$$

where $Y:R^\ell \to \mathbb{L}(\mathbb{R}^{n+1}, \mathbb{R}^\ell)$ is Globally Lipschitz, i.e. there exists $L > 0$ such that $\|Y(x) - Y(y)\| < L \|x-y\|$ for all $x,y \in \mathbb{R}^\ell$.

Then the solution $\zeta_t(x)$ satisfies $\mathbb{E}[\sup_{t\in[0,T]} \|\zeta_t(x)\|]$, $\sup_{t\in[0,T]} \mathbb{E}[\|\zeta_t(x)\|^2]$ each bounded uniformly over $x \in U$.

Proof

This is an adaptation of the usual existence proof for the solution of a stochastic integral equation. See Elworthy [6], Chapter 5, Theorem 1C; Gikhman and Skorohod [7], §6, Theorem 1.

For each $x \in U$, define

$$\zeta_t^o(x) = \eta_t(x),\ \zeta_t^1(x) = \eta_t(x) + \int_0^t Y(\zeta_t^o(x))dz_s, \ldots,$$

$$\zeta_t^{p+1}(x) = \eta_t(x) + \int_0^t Y(\zeta_s^p(x))dz_s.$$

Put $\sup_{t\in[0,T]} \mathbb{E}[\|\zeta_t^1(x) - \zeta_t^o(x)\|^2] = K(x)$,

$$\mathbb{E}[\sup_{t\in[0,T]} \|\zeta_t^1(x) - \zeta_t^o(x)\|] = M(x).$$

Then by Lemma A.1.1, $K(x) < \beta \int_0^T \mathbb{E}[\|Y\eta_t(x)\|^2]dt$,
$M(x) < \gamma \int_0^T \mathbb{E}[\|Y\eta_t(x)\|^2]dt$.

Also, since Y is globally Lipschitz, we have L^1 such that $\|Y(x)\|^2 < L^1(\|x\|^2 + 1)$ for all $x \in \mathbb{R}^\ell$, and denoting $\max\{L,L^1\}$ again by L, we have $\mathbb{E}[\|Y(\eta_t(x))\|^2] < L^2\,\mathbb{E}[\|\eta_t(x)\|^2 + 1]$. Therefore $K(x)$ and $M(x)$ are both bounded uniformly over $x \in U$.

Now, $\mathbb{E}[\,\|\zeta_t^{n+1} - \zeta_t^n\|^2]$

$$= \mathbb{E}[\,\|\int_0^t [Y(\zeta_s^n) - Y(\zeta_s^{n-1})]\, dz_s\|^2]$$

$$\leq \beta\int_0^t \mathbb{E}[\,\|Y(\zeta_s^n) - Y(\zeta_s^{n-1})\|^2]\, ds$$

(Lemma A.1.1)

$$\leq \beta\, L^2 \int_0^t \mathbb{E}[\,\|\zeta_s^n - \zeta_s^{n-1}\|^2]\, ds,$$

and iterating, we obtain

$$\mathbb{E}[\,\|\zeta_t^{n+1} - \zeta_t^n\|^2] \leq (\beta L^2)^n \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, \mathbb{E}[\,\|\zeta_s^1 - \zeta_s^o\|^2]\, ds$$

$$\leq \frac{(\beta L^2)}{n!}\, T^n\, K(x).$$

This holds for each $n \geq 0$.

Also for $n \geq 1$, $\mathbb{E}[\sup_{t\in[0,T]} \|\zeta_t^{n+1} - \zeta_t^n\|]$

$$\leq \gamma\int_0^T \mathbb{E}[\,\|Y(\zeta_t^n) - Y(\zeta_t^{n-1})\|^2]\, dt$$

$$\leq \gamma\, L^2 \int_0^T \mathbb{E}[\,\|\zeta_t^n - \zeta_t^{n-1}\|^2]\, dt$$

$$\leq \frac{\gamma L^2 (\beta L^2)^{n-1}\, T^{n-1}\, K(x)}{(n-1)!}\quad .$$

From the convergence of $\Sigma\left[\frac{(\beta L^2T)^n}{n!}\right]^2$ we deduce that of

$\eta_t + \sum_{n=0}^{\infty} (\zeta_t^{n+1} - \zeta_t^n)$ to the solution $\zeta_t(x)$ of the equation,

in the norm $\sup_{t\in[0,T]} \mathbb{E}\left[\|\zeta_t\|^2\right]^{1/2}$ and we see that it is bounded

uniformly over $x \in U$. From the convergence of $\Sigma\frac{(\beta L^2T)^n}{n!}$ we

see that $\mathbb{E}\left[\sup_{t\in[0,T]} \|\zeta_t(x)\|\right]$ is bounded uniformly over $x \in U$. //

Lemma A.1.2

Consider the H^{s+1} Itô SDS $I(X,z)$ supported on a bounded open domain $U \subset \mathbb{R}^m$, where $s > m/2 + k$, and take $p < k$. Then the derivative $D^p\xi_t(x) \in \mathbf{L}(\mathbb{R}^m,\ldots,\mathbb{R}^m;\mathbb{R}^m)$ of the flow exists for each $x \in U$, and satisfies the equation

$$D^p\xi_t(x) = D^p\,\mathrm{Id}(x) + \int_0^t D^p(X\circ\xi_s)(x)\,dz_s.$$

Thus, for any $v_1,\ldots,v_p \in \mathbb{R}^m$, we have

$$D^p\xi_t(x)(v_1,\ldots,v_p) = D^p\,\mathrm{Id}(x)(v_1,\ldots,v_p)$$

$$+ \int_0^t \sum_{i=1}^{p} \sum_{\substack{\text{Partitions } U_1,\ldots,U_i \text{ of} \\ \{v_1,\ldots,v_p\},\text{ such that} \\ |U_k| \geq 1. \\ \text{Say } U_k = \{v_1^k,\ldots,v_{|n_k|}^k\}}} D^iX(\xi_s(x))\left[D^{|n_1|}\xi_s(x)(v_1^1,\ldots,v_{|n_1|}^1),\ldots,D^{|n_i|}\xi_s(x)(v_1^i,\ldots,v_{|n_i|}^i)\right] dz_s.$$

Proof

Consider the lift $I(\tilde{X},z)$ of $I(X,z)$ in $\mathcal{D}^s_X$, defined as in Carverhill and Elworthy [4]. (Except that our systems are Itô, not Stratonovitch). The flow ξ_s in $\mathcal{D}^s_X$ is a solution to the equation

$$\xi_t = \mathrm{Id} + \int_0^t \tilde{X}(\xi_t)\, dz_s.$$

Now, consider the differentiation map

$$\mathbb{D}^p : \mathcal{D}^s_X \to H^{s-p}_X(\mathbb{R}^m; \mathbb{L}(\underbrace{\mathbb{R}^m,\ldots,\mathbb{R}^m}_{p}; \mathbb{R}^m))$$

defined by $\mathbb{D}^p\xi : x \to D^p\xi(x)$. This is continuous and linear, therefore transforming ξ_t via $\mathbb{D}^p$, using the Itô formula, we have

$$\mathbb{D}^p\xi_t = \mathbb{D}^p\ \mathrm{Id} + \int_0^t \mathbb{D}^p(\tilde{X},\xi_s)\, dz_s.$$

Transforming again via the linear map

$$\mathrm{ev}_x : H^{s-p}_X(\mathbb{R}^m; \mathbb{L}(\mathbb{R}^m,\ldots,\mathbb{R}^m;\mathbb{R}^m))$$

$$\to \mathbb{L}(\mathbb{R}^m,\ldots,\mathbb{R}^m;\mathbb{R}^m)$$

gives

$$D^p\xi_t(x) = D^p\ \mathrm{Id}(x) + \int_0^t D^p(X\xi_s)(x)\, dz_s.$$

The second part follows by transforming again via

$$\mathrm{ev}_{(v_1,\ldots,v_p)} : \mathbb{L}(\mathbb{R}^m,\ldots,\mathbb{R}^m;\mathbb{R}^m) \to \mathbb{R}^m,$$

and applying the Leibniz formula. //

Note

Lemma A.1.2 works equally well for Itô and Stratonovitch systems.

Proposition A.1.2

Consider the H^{s+1} SDS $I(X,z)$ supported on the bounded open domain $U \subset \mathbb{R}^m$. $(s > m/2 + k)$, and with flow ξ_t.

Then for $\alpha = (\alpha_1,\ldots,\alpha_m)$, $|\alpha_i| > 0, |\alpha| < k$, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \xi_t(x) \right\| \right] \text{ and } \sup_{t\in[0,T]} \mathbb{E}\left[\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \xi_t(x) \right\|^2 \right]$$

both bounded uniformly over $x \in U$.

Proof

Induction on $|\alpha|$.

If $|\alpha| = 0$ the result is clear since a.s. $\xi_t(x)$ never leaves U.

Choose $p < k$ and assume the result for all α with $|\alpha| < p$. Now fix α with $|\alpha| = p$.

Substituting $(\overbrace{e_1,\ldots,e_1}^{\alpha_1}, \overbrace{e_2,\ldots,e_2}^{\alpha_2}, \ldots, \overbrace{e_m,\ldots,e_m}^{\alpha_m})$ for $(v_1,\ldots,v_p)$ in the equation of Lemma A.1.2 gives an equation for $\frac{\partial^{|\alpha|}}{\partial x^\alpha} \xi_t(x) \in \mathbb{R}^m$. Having made this substitution, take any term in the sum on the RHS, except that for $i = 1$. We can express this as

$$D^i X(\xi_s(x))\left[\frac{\partial^{|U_1|}}{\partial x^{V_1}}\xi_s(x),\ldots,\frac{\partial^{|U_i|}}{\partial x^{V_i}}\xi_s(x)\right],$$

where each V_k is a subset of $\{1,\ldots,m\}$ containing $|U_k|$ elements.

For this term,

$$\int_0^t \mathbb{E}\left[\left\|D^i X(\xi_s(x))\left[\frac{\partial^{|U_1|}}{\partial x^{V_1}}\xi_s(x),\ldots,\frac{\partial^{|U_i|}}{\partial x^{V_1}}\xi_s(x)\right]\right\|^2\right]ds$$

$$< \int_0^t \mathbb{E}\left[\|D^i X(\xi_s(x))\|^2\right]\cdot\mathbb{E}\left[\left\|\frac{\partial^{|U_1|}}{\partial x^{V_1}}\xi_s(x)\right\|^2\right]\cdots$$

$$\cdots\mathbb{E}\left[\left\|\frac{\partial^{|U_i|}}{\partial x^{V_1}}\xi_s(x)\right\|^2\right]ds.$$

Now, the first factor in the RHS of this inequality is bounded uniformly over $x \in U$, because X is C^{k+1}. The others are bounded uniformly by the inductive assumption. Thus, the LHS is bounded uniformly. So applying Lemma A.1.1 to each term in the RHS of the equation for $\frac{\partial^{|\alpha|}}{\partial x^\alpha}\xi_t(x)$ except that for $i = 1$, we have that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t D^i X(\xi_s(x))\left[\frac{\partial^{|U_1|}}{\partial x^{V_1}}\xi_s(x),\ldots,\frac{\partial^{|U_i|}}{\partial x^{V_i}}\xi_s(x)\right]dz_s\right\|\right]$$

and

$$\sup_{t\in[0,T]}\mathbb{E}\left[\left\|\int_0^t D^i X(\xi_s(x))\left[\frac{\partial^{|U_i|}}{\partial x^{V_1}}\xi_s(x),\ldots\right]dz_s\right\|^2\right]$$

are both uniformly bounded over $x \in U$. Thus,

$$\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \xi_t(x) = \eta_t(x) + \int_0^t DX(\xi_s(x)) \left(\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \xi_s(x)\right) dz_s,$$

where $\eta_t(x)$ satisfies the conditions of Proposition A.1.1.

Taking this last equation with that for $\xi_t(x)$ itself gives

$$\left[\xi_t(x), \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \xi_t(x)\right] = \left[x, \eta_t(x)\right]$$

$$+ \int_0^t [X(\xi_s(x)), DX(\xi_s(x))\left(\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}\right)\xi_s(x))]\ dz_s.$$

Since X is C^{k+1} and DX is linear, we see that this equation satisfies the conditions for Proposition A.1.1, and the result follows from that. //

Note

The conclusion of Proposition A.1.2 follows for an H^{s+2} Stratonovitch system, because such a system can be converted to an H^{s+1} Itô system.

A.2. Regularity Estimates for Chapter 2

In this section we deduce from Section A.1 the estimates which we need in the proofs of Chapter 2. Recall the definitions of the C^k and H^s_p norms on the space of (suitably smooth) maps

$U \to \mathbb{R}^m$ (U a bounded open domain in $\mathbb{R}^m$):

$$\|\zeta\|_{C^k} = \sum_{|\alpha|<k} \sup_{x\in U} \left\| \frac{\partial^{|\alpha|}}{\partial x^2} \zeta(x) \right\|$$

$$\|\zeta\|^p_{H^s_p} = \sum_{|\alpha|<k} \int_{x\in U} \left\| \frac{\partial^{|\alpha|}}{\partial x^2} \zeta(x) \right\|^p dx$$

(dx denotes Lebesgue measure.)

N.B. We have previously been working with the Hilbertian norm H^s, which corresponds to $\ell = 2$ in H^s_p here. We need to consider the H^s_1 norm in this section. Also, we need the following very general version of the Sobolev embedding theorem (See Palais [16], Section 9):

If $s > m/p + k$ then $H^s_p(U,\mathbb{R}^m)$ is continuously included in $C^k(U,\mathbb{R}^m)$.

We work with Stratonovitch systems in this section.

The first proposition gives the basic estimate for part (2) of the proof of Theorem 2.2.1. This estimate corresponds to Ruelle's condition 5.1.

<u>Proposition A.2.1</u>

Suppose the Stratonovitch SDS (X,z), supported on $U \subset \mathbb{R}^m$ is of class H^{s+4}_2 $(s > 3m/2 + k,\ k > 1.)$ Denote the flow by $\xi_t(\omega)$ and take $T > 0$.

Then

$$\int_{\omega\in\Omega} \sup_{t\in[0,T]} \|\xi_t(\omega)\|_{C^k} \, dP(\omega) < \infty .$$

Proof.

The flow $\xi_t(\omega)$ is H_2^{s+2} and hence C^r, where $r = s - m/2 + 1$, by the Sobolev embedding theorem with $p = 2$. By Proposition A.1.2, we have for $|\alpha| < r$ that

$$\int_{\omega\in\Omega} \sup_{t\in[0,T]} \left\| \frac{\partial^{|\alpha|}}{\partial x^2} \xi_t(\omega)x \right\| dP(\omega)$$

is bounded uniformly over $x \in U$.

Therefore

$$\int_{\omega\in\Omega} \sup_{t\in[0,T]} \|\xi_t(\omega)\|_{H_1^r} \, dP(\omega)$$

$$\equiv \int_{\omega\in\Omega} \sup_{t\in[0,T]} \left[\sum_{|\alpha|<r} \int_{x\in U} \left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \xi_t(\omega)x \right\| dx \right] dP(\omega)$$

$$< \sum_{|\alpha|<r} \int_{x\in U} \int_{\omega\in\Omega} \sup_{t\in[0,T]} \left[\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \xi_t(\omega)x \right\| \right] dP(\omega)dx < \infty .$$

By the Sobolev embedding theorem for $p = 1$, we have $H_1^r(U,\mathbb{R}^m)$ continuously embedded in $C^{r-m-1}(U,\mathbb{R}^m)$ and therefore in $C^k(U,\mathbb{R}^m)$, and the result follows. //

Then

$$\int_{\omega\in\Omega} \sup_{t\in[0,T]} \|\xi_t(\omega)\|_{C^k} \, dP(\omega) < \infty .$$

Proof.

The flow $\xi_t(\omega)$ is H_2^{s+2} and hence C^r, where $r = s - m/2 + 1$, by the Sobolev embedding theorem with $p = 2$. By Proposition A.1.2, we have for $|\alpha| < r$ that

$$\int_{\omega\in\Omega} \sup_{t\in[0,T]} \left\| \frac{\partial^{|\alpha|}}{\partial x^2} \xi_t(\omega)x \right\| dP(\omega)$$

is bounded uniformly over $x \in U$.

Therefore

$$\int_{\omega\in\Omega} \sup_{t\in[0,T]} \|\xi_t(\omega)\|_{H_1^r} \, dP(\omega)$$

$$\equiv \int_{\omega\in\Omega} \sup_{t\in[0,T]} \left[\sum_{|\alpha|<r} \int_{x\in U} \left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \xi_t(\omega)x \right\| dx \right] dP(\omega)$$

$$< \sum_{|\alpha|<r} \int_{x\in U} \int_{\omega\in\Omega} \sup_{t\in[0,T]} \left[\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \xi_t(\omega)x \right\| \right] dP(\omega)dx < \infty .$$

By the Sobolev embedding theorem for $p = 1$, we have $H_1^r(U,\mathbb{R}^m)$ continuously embedded in $C^{r-m-1}(U,\mathbb{R}^m)$ and therefore in $C^k(U,\mathbb{R}^m)$, and the result follows. //

The next proposition gives an estimate for the inverse of the derivative. This is needed in part (2) of the proof of Theorem 2.2.1, to ensure that the Lyapunov spectrum does not contain $-\infty$.

Proposition A.2.2.

Consider the situation of Proposition A.2.1. Denote by Inv the inversion map in $GL(\mathbb{R}^m)$. Then

$$\int_{x\in U} \int_{\omega\in\Omega} \| \mathrm{Inv}[D\xi_T(\omega)x] \|^2 \, dP(\omega) \, d\rho(x) < \infty .$$

Proof.

Denote by $\{\xi^V_t(\omega) : 0 < t < T\}$ the flow of the backward system (X, z^V) on $[0,T]$. (See Carverhill and Elworthy [4], Section 5.4). By [4] Lemma 5.4, $\xi^V_T(\omega)$ and $\xi_T(\omega)$ are inverses a.s., therefore a.s. we have for all $x \in M$ that

$\mathrm{Inv}[D\xi_T(\omega)x] = D[\xi^V_T(\omega)](\xi_T(\omega)x)$.

Now, the flow $\xi^V_T(\omega)$ is H^{s+2}_2 and therefore C^r $(r = s - m/2 + 1)$ and by the analogue of Proposition A.1.2 for (X, z^V), we have for each α with $|\alpha| < r$, that

$$\int_{\omega\in\Omega} \| \frac{\partial^{|\alpha|}}{\partial x^2} \xi^V_T(\omega)x \|^2 \, dP(\omega)$$

is uniformly bounded over $x \in U$.

Therefore $\int_{\omega\in\Omega} \| \xi_T^v(\omega) \|^2_{H_1^r} \, dP(\omega) < \infty$,

$$\int_{\omega\in\Omega} \| \xi_T^v(\omega) \|^2_{C^k} \, dP(\omega) < \infty ,$$

and hence

$$\int \sup_{y\in U} \| D\xi_T^v(\omega) y \|^2_{GL(\mathbb{R}^m)} \, dP(\omega) < \infty .$$

(N.B. $k > 1$).

The result follows by substituting $y = \xi_T(\omega)x$ in this last expression so that

$D\xi_T^v(\omega)y = \mathrm{Inv}[D\xi_T(\omega)x]$. //

The final proposition allows us to go from discrete time to continuous time in Theorem 2.1. (Part (3) of the proof.) The result is similar to that of Proposition A.2.2, except that we need the sup over $[0,T]$ of the integrand. For the proof we express $\mathrm{Inv}[D\xi_t(\omega)x]$ as a stochastic integral and use Lemma A.1.1.

<u>Proposition A.2.3</u>

Consider the situation of Proposition A.2.1.

Then

$$\int_{x\in U}\int_{\omega\in\Omega}\sup_{t\in[0,T]}\log\|D[\xi_{tT}(\omega)](\xi_t(\omega)x)\|_{GL(\mathbb{R}^n)}d\rho(x)dP(\omega)<\infty.$$

Here by $\xi_{tT}(\omega)$ we mean $\xi_T(\omega)\circ\xi_t(\omega)^{-1}$.

<u>Proof</u>

Note first that this integrand is a.s. not negative.

Now,

$$D[\xi_{tT}(\omega)](\xi_t(\omega)x) = [D\xi_T(\omega)x]\circ\mathrm{Inv}[D\xi_t(\omega)x],$$

so that

$$\sup_{t\in[0,T]}\log\|D[\xi_{tT}(\omega)x](\xi_t(\omega)x)\|$$

$$< \sup_{t\in[0,T]}\log^+\|D\xi_T(\omega)x\| + \sup_{t\in[0,T]}\log^+\|\mathrm{Inv}[D\xi_t(\omega)x]\|.$$

The first term on this RHS is $\rho\otimes P$-integrable by Proposition A.1.2, therefore it suffices to show that the other term is also. (This term is similar to that of Proposition A.2.2, except that we have $\sup\limits_{t\in[0,T]}$ in it.)

Now,

$$D\xi_t(\omega)x = \int_0^t DX(\xi_s(\omega)x)(D\xi_s(\omega)x)\circ dz_s(\omega).$$

(See [4], Remark 4.2).

Transforming this via Inv, we have

$$\mathrm{Inv}[D\xi_t(\omega)x] = \int_0^t D\ \mathrm{Inv}(D\xi_s(\omega)x).\ DX(\xi_o(\omega)x)(D\xi_s(\omega)x) \circ dz_s$$

$$= -\int_0^t \mathrm{Inv}[D\xi_s(\omega)x].DX(\xi_s(\omega)x).(\mathrm{Id}) \circ dz_s$$

(N.B. $D\ \mathrm{Inv}(S)T = -S^{-1} \circ T \circ S^{-1}$.)

Write this equation in Itô form:

$$\mathrm{Inv}[D\xi_t(\omega)x] = -\int_0^t \mathrm{Inv}[D\xi_s(\omega)x].DX(\xi_s(\omega)x) \circ dz_s + \int_0^t S(\omega,s)\ ds.$$

Here, $S(\omega,s)$ is the Stratonovitch correction term: it is a combination of the terms $\mathrm{Inv}[D\xi_s(\omega)x]$, $DX(\xi_s(\omega)x)$, $D^2X(\xi_s(\omega)x)$.

Now,

$$\int_{x\in U}\int_{\omega\in\Omega} \sup_{t\in[0,T]} \|\text{ First term on RHS}\|\ dP(\omega).d\rho(x)$$

$$< \sup_{x\in U} \|DX(x)\|.\gamma \int_{x\in U}\int_{s=0}^{T}\int_{\omega\in\Omega} \|\mathrm{Inv}[D\xi_s(\omega)x]\|^2\ dP(\omega)ds\ dx$$

(Lemma A.1.1(ii))

and this last expression is finite because X is C^1 and by Proposition A.2.2. (Actually, we need this result uniformly over $t \in [0,T]$.)

To deal with the term $\int_0^t S(\omega,s)ds$ is more elementary: for this we need the fact that X is C^2. //

APPENDIX B : INVARIANCE OF THE LYAPUNOV SPECTRUM

We saw from the proof of Theorem 2.1 that the Lyapunov spectrum is invariant under the map $\Phi_s : M \times \Omega \to M \times \Omega$. Also we conjectured that for nondegenerate systems the Lyapunov spectrum is constant for a.e. $(x,\omega) \in M \times \Omega$. In this Appendix we investigate the invariance of the spectrum. We give two approaches: the first is to study the map Φ_s and to look at ergodicity properties with respect to the measure $\rho \otimes P$ on $M \times \Omega$; the second is to study the limits given by the formula of Theorem 4.2 for the Lyapunov numbers.

B.1 Ergodicity of the Map Φ_s

Proposition B.1.

Consider the nondegenerate SDS (X,z) of Chapter 2, and take a version $\xi_t(\omega)$ of its flow. Choose any $s > 0$.

(i) Then for a.e. $(x,\omega) \in M \times \Omega$ we can define the maps

$$\Phi_s : M \times \Omega \to M \times \Omega; \ \Phi_s(x,\omega) = (\xi_s(\omega)x, \ \theta_s(\omega)),$$

$$\theta_s : C(\mathbb{R}^{\geq 0}, M) \to C(\mathbb{R}^{\geq 0}, M); \ \theta_s(f)(t) = f(s+t),$$

$$S : M \times \Omega \to C(\mathbb{R}^{\geq 0}, M); \ S(x,\omega) = \{\text{Path } t \to \xi_t(\omega)x\},$$

and the following diagram commutes:

$$\begin{tikzcd} M \times \Omega \arrow[r, "\Phi_s"] \arrow[d, "S"'] & M \times \Omega \arrow[d, "S"] \\ C(\mathbb{R}^{>0}, M) \arrow[r, "\Theta_s"'] & C(\mathbb{R}^{>0}, M) \end{tikzcd}$$

(Φ_s, Θ_s as in Chapter 1.)

(ii) Take any Q_ρ-measurable set A in $C(\mathbb{R}^{>0},M)$. (Q_ρ-Markov measure. See Section 1.2). Then $\rho \otimes P[S^{-1}(A)] = Q_\rho(A)$.

<u>Proof</u>

(i) Immediate from Proposition 1.2.1.

(ii) It suffices to prove this when A is a cylinder set. Thus suppose,

$$A = \{f \in C(\mathbb{R}^{>0},M) : f(t_i) \in B_i, \quad i = 1,\ldots,p\}.$$

Then

$$S^{-1}(A) = \{(x,\omega) : \xi_{t_i}(\omega)x \in B_i; \quad i = 1,\ldots,p\}$$

and we must show that this has $\rho \otimes$ P-measure equal to the expression of Section 1.2 for $Q_\rho(A)$.

By Fubini's theorem,

$$\rho \otimes P[S^{-1}(A)] = \int_{x\in M} C(x)\, d\rho(x), \text{ where}$$

$$C(x) = P\{\omega \cdot \xi_{t_i}(\omega)x \in B_i \ ; \ i = 1,\ldots,p\}.$$

In terms of transition densities,

$$C(x) = \int_{x_p\in B_p} \cdots \int_{x_2\in B_2} \int_{x_1\in B_1} p_{t_1}(x,dx_1)\, p_{t_2-t_1}(x_1,dx_2) \ldots$$

$$\ldots p_{t_p-t_{p-1}}(x_{p-1},dx_p).$$

To see that $\int_{x\in M} C(x)d\rho(x)$ is equal to the expression of Section 1.2, note that since ρ is invariant,

$$\int_{x\in M} p_{t_1}(x,dx_1)\, d\rho(x) = d\rho(x_1). \quad //$$

Since Θ_s is ergodic with respect to the measure Q_ρ, we see from Proposition B.1 that if S is a.s. injective, then Φ_s is ergodic with respect to $\rho \otimes P$, and the Lyapunov spectrum is constant. A reasonable conjecture is that S is a.s. injective if and only if the noise dimension n is equal to the dimension m of M. However, if this is so then the map $Y:M \times \mathbb{R}^n \to TM$ is a diffeomorphism and M is parallelisable. Thus for example on the 2-sphere we cannot have such an SDS.

B.2. Study of the Lyapunov Spectrum via the Formula of Theorem 4.2

The formula of Theorem 4.2 gives the Lyapunov number $\lambda_{(v,\omega)}$ as an ergodic average

$$\lambda_{(v,\omega)} = \lim_{t\to\infty} 1/t \int_0^t g(\eta_s(\omega)v)ds,$$

where $\eta_t(\omega)v$ is the solution to the SDS $(\tilde{\tilde{X}},z)$ on SM, starting from v, and $g:SM \to \mathbb{R}$ is a continuous function. Thus, a study of the ergodic properties of the Markov process given by $(\tilde{\tilde{X}},z)$ might tell us about the Lyapunov spectrum.

Yosida ([19] Chapter 13) gives a Krylov-Bogolioubov decomposition for Markov processes, which enables us to take disjoint subsets $\{U_\mu : \mu$ an ergodic measure for $(\tilde{\tilde{X}},z)$ on SM$\}$ of SM, such that $\bigcup_\mu U_\mu$ has full measure with respect to any invariant measure for $(\tilde{\tilde{X}},z)$, and for each μ, $\mu(U_\mu) = 1$. Also, we see from Yosida that for any ergodic measure μ on SM, we have for $\mu \otimes P$ a.e. $(v,\omega) \in SM \times \Omega$ that the ergodic average above is equal to $\int g(u)\, d\mu(u)$.

Thus, the formula of Theorem 4.2, and the work of Yosida, enable us to deduce the existence of the Lyapunov numbers $\lambda_{(v,\omega)}$ and their a.s. independence of $\omega \in \Omega$ for μ-a.e. $v \in SM$, where μ is any measure on SM which is invariant for $(\tilde{\tilde{X}},z)$.

The Lyapunov numbers are also a.s. invariant over the sets

$\{U_\mu,\ \mu$ an ergodic measure for $(\tilde{\tilde{X}},z)\}$.

To study their space invariance we might study these sets. With $(\tilde{\tilde{X}},z)$ we can associate a control system (See Strook and Varadhan [17]) and it is reasonable to look for a relationship between the control sets of this system and the sets U_μ. We conjecture that the control sets are the closures of these sets.

Note also that for any measure μ on SM which is invariant for $(\tilde{\tilde{X}},z)$, we have $\pi(\mu) = \rho$, where $\pi: SM \to M$ is the bundle projection and ρ is the invariant measure for (X,z). Therefore the assertions concerning $(\tilde{\tilde{X}},z)$-invariant measures tell us something about the spectrum for ρ-a.e. $x \in M$.

The Lyapunov spectrum at $(x,\omega) \in M \times \Omega$ is the set $\{\lambda_{(v,\omega)} : v \in S_xM\}$, therefore to study it we need results about *all* $v \in S_xM$ for almost all $x \in M$. It might be possible to get such results from the above by looking at the control sets and bearing in mind the logarithmic rule: If $\lambda_{(u,\omega)} > \lambda_{(v,\omega)}$ then for $\alpha \neq 0$, we have $\lambda_{(\alpha u+\beta v,\omega)} = \lambda_{(u,\omega)}$.

REFERENCES

[1] Arnold, L. and Kliemann, W. Qualitative Theory of Stochastic Systems. In Probabilistic Analysis and Related Topics, Vol. 3, A.T. Bharucha-Ried, Ed. New York : Academic Press, 1981.

[2] Arnold, V.I., Ordinary Differential Equations. M.I.T. Press 1973.

[3] Bismut, J.-M., Flots Stochastiques et Formula de Itô-Stratonovitch Généralisé, C.R. Acad. Sci. Paris t 290, 1980.

[4] Carverhill, A.P. and Elworthy, K.D., Flows of Stochastic Dynamical Systems : the Functional Analytic Approach. Preprint, University of Warwick, Coventry CV4 7AL, England, 1982, Submitted to Zeitschrift fur Wahrscheinlichkeitstheorie.

[5] Doob, J.L., Stochastic Processes. Wiley 1953.

[6] Elworthy, K.D., Stochastic Differential Equations on Manifolds. L.M.S. Lecture Notes. Cambridge University Press, 1982.

[7] Gikhman, I.I. and Skorohod, A.V., Stochastic Differential Equations. Berlin, Heidelberg, New York : Springer-Verlag 1972.

[8] Ikeda, N. and Watanabe, S., Stochastic Differential Equations and Diffusion Processes. Tokyo : Kodansha. Amsterdam, New York, Oxford : North Holland, 1981.

[9] Khas'minskii, R.Z., Necessary and Sufficient Conditions for the Asymptotic Stability of Linear Stochastic Systems. Th. Prob. Appl. 12 (1967) pp. 144-147.

[10] Khas'minskii, R.Z., Stochastic Stability of Differential Equations. Alphen aan den Rijn, the Netherlands : Sijthoff and Noordhoff, 1980. (Originally in Russian, Moscow : Nauka, 1969).

[11] Kunita, H., On Backward Stochastic Differential Equations, 1981. To appear in Stochastics.

[12] Kunita, H., Stochastic Differential Equations and Stochastic Flows of Homeomorphisms. Preprint: Dept. of Applied Science, Faculty of Engineering, Kyushu University, Fukuoka 812, Japan, 1982.

[13] Mckean, H.P. Jr., Stochastic Integrals. New York : Academic Press, 1969.

[14] Nelson, E., Topics in Dynamics 1 : Flows. Princeton University Press, 1970.

[15] Palais, R.S. Foundations of Global Nonlinear Analysis. New York : Benjamin, 1968.

[16] Ruelle, D., Ergodic Theory of Differentiable Dynamical Systems. Publications IHES, 1978.

[17] Strook, D.W. and Varadhan, S.R.S., On the Support of Diffusion Processes with Applications to the Strong Maximum Principle, in Proc. Sixth Berkeley Symposium, Vol. 3, pp. 333-359, 1972.

[18] Ventsel, A.D. and Freidlin, M.I., On Small Random Perturbations of Dynamical Systems. Russian Math. Surveys 25 (1970) no. 1, pp. 1-56.

[19] Yosida, K., Functional Analysis (3rd edition) Berlin, Heidelberg, New York : Springer-Verlag, 1971.