Globe-hopping

Dmitry Chistikov Mike Paterson

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Abstract

We consider the grasshopper problem [1] on the sphere. We find a lawn (subset L of the spherical surface) with the following properties:

- its area is half of the area of the sphere,
- it is antipodal, i.e., every point of the sphere is in L if and only if the opposite point is not in L, and
- the probability that a grasshopper's jump stays in L is strictly greater than that for the lawn S consisting of the southern hemisphere.

Let the sphere have radius 1 and the jump size be θ. All distances here are spherical (determined by the length of great circle arcs). Assuming $\theta \in (0, \pi/2)$, our results hold if θ/π is irrational or if $\theta/\pi = p/q$ where $p, q \in \mathbb{N}$, the fraction is irreducible, and p is even. If p is odd, our results may or may not hold depending on the specific values of p and q; in particular, jumps of size π/q, $q \in \mathbb{N}$, present an open problem.

1 Construction of the lawn

We now show how to construct the lawn L. Given θ, take a point on the equator of the sphere and take a sequence of consecutive jumps of size θ round and round the equator. Take a second sequence starting from the antipodal point and continuing with the sequence of antipodal points to the first sequence. Let $n = n(\theta)$ be the number of points in each sequence. Draw circles of sufficiently small radius $r = r(\theta)$ around these $2n$ points such that there are no overlaps. Our lawn consists of the southern hemisphere with these circles added for the first sequence and removed for the second sequence. In other words, we modify the hemisphere to fill semi-circular *caps*

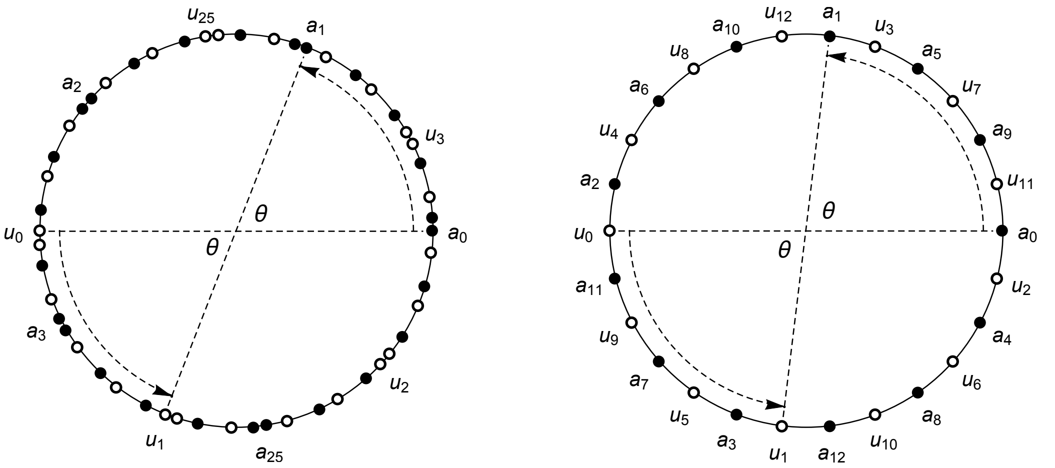

Figure 1: Caps and cups for $\theta = 1.2$ and $\theta = 6\pi/13$

above the equator and remove semi-circular *cups* below the equator, creating antipodal cogs akin to the construction in [1].

The choice of the number of pairs of points $n = n(\theta)$ and the radius $r = r(\theta)$ depends on whether the number θ/π is rational or irrational, and we consider several scenarios. Figure 1 shows the positions on the equator of caps and cups for two values of θ.

If θ/π is irrational, we pick n sufficiently large so that $\cos\theta < 1 - 1/n$. If θ/π is rational and has the form p/q where p is even and the fraction is irreducible, we pick $n = q$. In both cases the two sequences of points satisfy the following conditions:

(i) all points in the two sequences are distinct,

(ii) in each sequence, the distance between every pair of successive points is θ, and

(iii) no jump of length θ can link two points of different sequences.

If $\theta/\pi = p/q$ with p even, these sequences form two cycles of n points each. If θ/π is irrational then, in each sequence, the two jumps of length θ away from the sequence for the first and last point go to a point on the equator that does not belong to either sequence.

Note that although the value of θ is the distance between two consecutive jumps in the sequence, caps and cups may well be closer to each other than θ because the sequence can wrap around the equator. However, once we have fixed these sequences of points, we can pick $r > 0$ small enough that

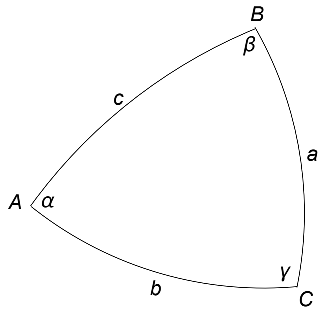

Figure 2: Spherical triangle

- circles of radius r around these $2n$ points do not overlap,
- in these two sequences of circles (for caps and cups), there is no jump of length θ that would join any two points belonging to circles from different sequences, and
- every jump of length θ from a circle either goes to another circle in the same sequence, or does not touch any of the circles.

We can then make $r > 0$ even smaller as needed, assuming $r \to 0$ when convenient.

2 Preliminaries

We recall some basic formulae in spherical geometry. Using the notation in Figure 2, where a, b, c are (great-circle) lengths and α, β, γ are angles, we have the *sine rule*:

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}, \tag{1}$$

the *cosine rule*:

$$\cos a = \cos b \cos c + \sin b \sin c \cdot \cos \alpha, \tag{2}$$

and, in particular,

$$\text{if } \alpha = \pi/2 \text{ then } \cos a = \cos b \cos c. \tag{3}$$

The next formula follows easily:

$$\text{if } \alpha = \pi/2 \text{ then } \tan a \cos \beta = \tan c. \tag{4}$$

For points on the sphere, it is convenient to use pairs of angles (ϕ, ψ) based on (longitude, latitude), where ϕ is the azimuthal angle and ψ is the *elevation* or *copolar* angle. We take the radius of the sphere to be unity, so a typical point on the surface with spherical coordinates (ϕ, ψ) has cartesian coordinates $(x, y, z) = (\sin\phi\cos\psi, \cos\phi\cos\psi, \sin\psi)$, where the z-axis passes through the poles.

For B and C points on the sphere, the dot-product of their cartesian coordinates gives their angular distance. In Figure 2,

$$B \cdot C = \cos a. \tag{5}$$

Finally, we recall the notation for the cosecant function: $\csc a = 1/\sin a$.

3 Analysis of the construction

Our main result is the following theorem.

Theorem 1 *Suppose $\theta \in (0, \pi/2)$ and θ/π is either irrational or equal to p/q where $p, q \in \mathbb{N}$, the fraction is irreducible, and p is even. Then the probability that a grasshopper's jump stays in the lawn L is strictly greater than that for the lawn S consisting of the southern hemisphere.*

We analyse the difference between jumps on a hemisphere and jumps on our lawn L. Denote by A, U and S, the c**a**ps, the c**u**ps and the **s**outhern hemisphere respectively. Note that L consists of S with A added and U taken away. The set of successful jumps from L to L can be classified as jumps from S to S plus jumps from A to S and vice versa, plus jumps from A to A, but minus jumps to or from U. These latter are jumps from U to $S \setminus U$ and vice versa since our construction ensures that no jump is possible between A and U. We account for jumps involving U as jumps U to S plus jumps S to U minus jumps U to U, since these last were counted twice. In symbols we may express this as:

$$\overset{\frown}{LL} \;=\; \overset{\frown}{SS} + \overset{\frown}{AS} + \overset{\frown}{SA} + \overset{\frown}{AA} - \;(\overset{\frown}{US} + \overset{\frown}{SU} - \overset{\frown}{UU}).$$

For subsets X, Y of the spherical surface, we denote by $\overset{\frown}{XY}$ the probability that the grasshopper starts at a point in X and ends up at a point in Y. For a sequence of caps corresponding to distance θ and a corresponding antipodal sequence, we define the following quantities: $\overset{\frown}{aa}$ is the probability of a jump from one particular cap to another particular cap distance θ from the first, $\overset{\frown}{uu}$ for the corresponding probability for cups, $\overset{\frown}{aS}$ for the jump probability between one cap and S, $\overset{\curvearrowright}{uS}$ for the corresponding probability for a cup, $\overset{\curvearrowright}{SS}$ for the probability of a jump from S to S, and finally $\overset{\curvearrowright}{aN}$ for a jump from a cap to the northern hemisphere. We define $\overset{\curvearrowright}{Sa}$ and $\overset{\curvearrowright}{Su}$ similarly. It is easy to show that, for all X, Y,

$$\overset{\curvearrowright}{XY} = \overset{\curvearrowright}{YX}. \tag{6}$$

By symmetry we find that

$$\overset{\curvearrowright}{aa} = \overset{\curvearrowright}{uu} \text{ and } \overset{\curvearrowright}{uS} = \overset{\curvearrowright}{aN}. \tag{7}$$

	A	U	S
A	$(2n-2)\cdot\overset{\curvearrowright}{aa}$	0	$n\cdot\overset{\curvearrowright}{aS}$
U	0	$(2n-2)\cdot\overset{\curvearrowright}{uu}$	$n\cdot\overset{\curvearrowright}{uS}$
S	$n\cdot\overset{\curvearrowright}{Sa}$	$n\cdot\overset{\curvearrowright}{Su}$	$\overset{\curvearrowright}{SS}$

Table 1: Classification of jumps

We summarise the total probabilities in Table 3, where n is the number of cap—cup pairs. If the sequences of caps and cups form two independent cycles (i.e., if $\theta/\pi = p/q$ with p even), then $2n-2$ is replaced by $2n$. Using equations (6) and (7), we have

$$\begin{aligned}
\overset{\curvearrowright}{LL} - \overset{\curvearrowright}{SS} &= \overset{\curvearrowright}{AS} + \overset{\curvearrowright}{SA} + \overset{\curvearrowright}{AA} - \overset{\curvearrowright}{US} - \overset{\curvearrowright}{SU} + \overset{\curvearrowright}{UU} \\
&\geq 2n\cdot\overset{\curvearrowright}{aS} + (2n-2)\cdot\overset{\curvearrowright}{aa} - 2n\cdot\overset{\curvearrowright}{uS} + (2n-2)\cdot\overset{\curvearrowright}{uu} \\
&= 4(n-1)\cdot\overset{\curvearrowright}{aa} + 2n\cdot(\overset{\curvearrowright}{aS} - \overset{\curvearrowright}{aN}).
\end{aligned}$$

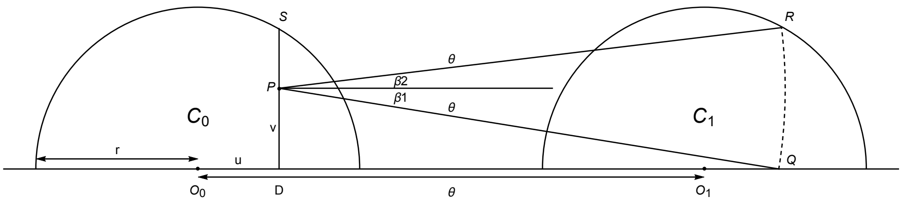

Figure 3: Jump geometry

In Figure 3, we show two successive caps C_0, C_1 of radius r, whose centres O_0, O_1 are at distance θ from each other, and a sample point P in C_0. For jumps from P towards C_1, β_1 and β_2 are the angles between the latitude through P and the direction of jumps to the equator at Q and to the circumference of C_1 at R, respectively. We see that $\beta_1 \geq 0$ always, but it is possible that $\beta_2 < 0$, for example, if P is close enough to the point S in Figure 3.

Lemma 2 *The circle of possible jump destinations from P intersects the upper semicircle of radius r centred at O_1 exactly once, so R is well-defined.*

Proof We first note that the point Q always lies within the diameter of C_1. If Q were to the left of this diameter then the point on the equator at distance θ to the left of Q is to the left of C_0. Since $r < \theta$ (i.e., the radius of the jump circle is greater than the radius of C_0), the point Q has distance less than θ from any point in C_0, contradicting the location of P. The case where Q is to the right of C_1 yields a similar contradiction more easily.

Since the jump circle around P intersects the equator exactly once to the right of P and this intersection is within C_1, the jump circle intersects the circumference of C_1 exactly once, and so R is well-defined. □

We see that

$$\begin{aligned}
\overset{\frown}{aa} &= \int_C (\beta_1 + \beta_2)\, ds, \\
\overset{\frown}{aS} &= \int_C (\pi - 2\beta_1)\, ds, \\
\overset{\frown}{aN} &= \int_C (\pi + 2\beta_1)\, ds,
\end{aligned}$$

where ds is a surface element of a cap C. So,

$$\begin{aligned}
\overset{\frown}{LL} - \overset{\frown}{SS} &\geq 4(n-1)\cdot \overset{\frown}{aa} - 2n \cdot (\overset{\frown}{aN} - \overset{\frown}{aS}) \\
&= \int_C 4(n-1)(\beta_1 + \beta_2) - 8n\beta_1\, ds.
\end{aligned}$$

To prepare estimates for these integrals we first find probabilities for a sample point P in a cap. In the following lemma we use the notation given by Figure 3, i.e., u and v are the azimuth and elevation (longitude and latitude) of P *relative to the centre of* C_0 and r is the radius of the cogs.

In the next lemma and everywhere below, the constants in our $O(\cdot)$ notation depend on θ, but not on n, and we let $r \to 0$.

Lemma 3 *For* $0 < \theta < \pi/2$,

$$\begin{aligned}
&(i) & \beta_1 &= v \cot\theta + O(r^3), \\
&(ii) & \beta_1 + \beta_2 &= \sqrt{r^2 - u^2}\, \csc\theta + O(r^2).
\end{aligned}$$

Proof Let us first prove that β_1 and β_2 are both $O(r)$. Observe that u, v are $O(r)$. Notice that $\pi/2 - \beta_1$ is an angle in the right-angled spherical triangle PDQ. Hence by equation (4), $\sin\beta_1 = O(r)/\tan\theta = O(r)$, and so $\beta_1 = O(r)$. Next, by the sine rule (1) for the spherical triangle PRQ, we have $\sin RQ/\sin(\beta_1+\beta_2) = \sin\theta/\sin\angle PRQ$, and so $\sin(\beta_1+\beta_2) = O(r)$, since the points R and Q are both inside a circle of radius r. Therefore, $\beta_1+\beta_2 = O(r)$ and $\beta_2 = O(r)$.

We now prove parts (i) and (ii) of the lemma. We take as origin for spherical coordinates the point O_1 in Figure 3. Then P has spherical coordinates $(-\theta+u, v)$ and cartesian coordinates $(-\sin(\theta-u)\cos v, \cos(\theta-u)\cos v, \sin v)$. Let Q be the point with spherical coordinates $(q, 0)$ on the equator inside C_1 at distance θ from P. Then $QD = q+\theta-u$ and from equation (4) we derive

$$\sin\beta_1 = \cos\angle QPD = \tan v\cot\theta = v\cot\theta + O(r^3),$$

which implies part (i) of the lemma.

Let R be the point with spherical coordinates (ψ, ϕ) on the circumference of C_1 at distance θ from P. Then

$$\begin{aligned} P.R &= \cos\theta, & (8)\\ \cos r &= \cos\psi\cos\phi, & (9) \end{aligned}$$

from equations (5) and (3).

We observe that ψ and ϕ are $O(r)$. In Proposition 5 below, we solve equations (8) and (9) to find that

$$\psi = u + O(r^2)$$

and, using the spherical law of cosines (2) for the triangle PQR,

$$\beta_1+\beta_2 = \sqrt{r^2-u^2}\csc\theta + O(r^2),$$

which establishes part (ii). □

We can now compare the probabilities $\overset{\frown}{LL}$ and $\overset{\frown}{SS}$.

Lemma 4 *For* $0 < \theta < \pi/2$,

$$\overset{\frown}{SS} - \overset{\frown}{LL} > 0 \quad \textit{if} \quad n > 1/(1-\cos\theta)$$

and r is sufficiently small.

Proof To make comparison easier, it is convenient to combine the contributions from a pair of points. For any point P we define its *mate* P'. When P has coordinates (u, v) relative to the centre of C_0, P' has coordinates $(u, v') = (u, SD - v)$, where $\cos SD \cos u = \cos r$ (see Figure 3 and equation (3)). If β_1' and β_2' are the angles corresponding to P', then

$$\beta_1' = (SD - v)\cot\theta + O(r^2) \text{ and } \beta_1' + \beta_2' = \beta_1 + \beta_2 \; + O(r^2),$$

from Lemma 3.

We show in Proposition 7 below that the equation $\cos SD \cos u = \cos r$ implies

$$SD = \sqrt{r^2 - u^2} + O(r^2).$$

For any integrand β,

$$\int_C \beta \, ds = \iint_{C'} \beta \cos v \, du \, dv,$$

since the area of an element ds is $\cos v \, du \, dv$, and where

$$C' = \{(u, v) \mid \cos u \cos v \geq \cos r, -r \leq u \leq r \text{ and } 0 \leq v \leq r\}.$$

Note however that $\cos v = 1 - O(r^2)$.

When we combine the integral for a sample point P with the integral for its mate P' we get

$$\begin{aligned} 2\widehat{aa} &= \int_C (\beta_1 + \beta_2)\, ds + \int_C (\beta_1' + \beta_2')\, ds \\ &= \iint_{C'} (\beta_1 + \beta_2)\cos v + (\beta_1' + \beta_2')\cos v' \, du \, dv \\ &= \iint_{C'} \left(2\sqrt{r^2 - u^2} \cdot \csc\theta + O(r^2)\right) du \, dv. \end{aligned}$$

Similarly,

$$\begin{aligned}
\overset{\frown}{aN} - \overset{\frown}{aS} &= \int\limits_C 2\beta_1\, ds + \int\limits_C 2\beta_1'\, ds \\
&= \iint\limits_{C'} 2\beta_1 \cos v + 2\beta_1' \cos v'\, du\, dv \\
&= \iint\limits_{C'} \left(2\sqrt{r^2-u^2} \cdot \cot\theta + O(r^2)\right) du\, dv.
\end{aligned}$$

As it is easy to check that $W = \iint\limits_{C'} \sqrt{r^2-u^2} du\, dv = \Theta(r^3)$, meaning that $W = O(r^3)$ and $r^3 = O(W)$, we conclude that

$$\begin{aligned}
\overset{\frown}{LL} - \overset{\frown}{SS} &= 2(n-1)(2W \csc\theta + O(r^4)) - 2n(2W\cot\theta + O(r^4)) \\
&= 4((n-1)\csc\theta - n\cot\theta)(W + O(r^4)).
\end{aligned}$$

and so

$$\overset{\frown}{LL} - \overset{\frown}{SS} > 0$$

for sufficiently small r, provided that

$$(n-1)\csc\theta - n\cot\theta > 0, \qquad \text{i.e.,} \qquad \cos\theta < 1 - 1/n.$$

□

Theorem 1 follows from Lemma 4.

Remark The calculations above can be performed for any θ; the reason the case $\theta/\pi = p/q$, p odd, presents a challenge is that our construction does not produce a valid (antipodal) lawn in that case.

Auxiliary propositions and proofs

Proposition 5 $\psi = u + O(r^2)$.

Proof Apart from the fact that $\psi, u \to 0$ as $r \to 0$, we will only rely on equations (8) and (9) from page 7 and the expressions for cartesian coordinates of P and R:

$$\begin{aligned}
R &= (\sin\psi\cos\phi, \cos\psi\cos\phi, \sin\phi) \quad \text{and} \\
P &= (-\sin(\theta-u)\cos v, \cos(\theta-u)\cos v, \sin v).
\end{aligned}$$

Substituting the latter into equation (8) gives

$$-\sin(\theta-u)\cos v\cdot\sin\psi\cos\phi+\cos(\theta-u)\cos v\cdot\cos\psi\cos\phi+\sin v\cdot\sin\phi=\cos\theta.$$

With the help of equation (9) this can be rewritten as

$$\begin{aligned} &F\cdot\tan\psi+G\cdot\sin\phi=H \\ &\qquad\text{where}\quad F=-\sin(\theta-u)\cos v\cos r=-\sin\theta+O(r), \\ &\qquad\qquad\quad\;\; G=\sin v,\quad\text{and} \\ &\qquad\qquad\quad\;\; H=\cos\theta-\cos(\theta-u)\cos v\cos r=-u\sin\theta+O(r^2). \end{aligned}$$

Rearranging the terms and squaring gives

$$G^2\sin^2\phi=(H-F\cdot\tan\psi)^2.$$

We can now express $\sin^2\phi$ in terms of $\tan\psi$ using equation (9) and the fact that $1/\cos^2\psi=1+\tan^2\psi$:

$$(F^2+G^2\cos^2 r)\cdot\tan^2\psi-2FH\cdot\tan\psi+(H^2-G^2\sin^2 r)=0.$$

It can be verified that the discriminant of this quadratic (in $\tan\psi$) equation is $O(r^4)$, and the two possible roots are

$$(\tan\psi)_\pm=\frac{u\sin^2\theta+O(r^2)\pm\sqrt{O(r^4)}}{\sin^2\theta+O(r)},$$

both satisfying $\tan\psi=u+O(r^2)$. Since $u,\psi\to 0$ as $r\to 0$, we conclude that $\psi=u+O(r^2)$. $\square$

Proposition 6 *Let y and z satisfy $y\geq 0$, $y+z\geq 0$, and $z=O(r^{2d})$ for some $d>0$ as $r\to 0$. Then*

$$\sqrt{y+z}=\sqrt{y}+O(r^d).$$

Proof Assume without loss of generality that $z\geq 0$; then

$$(\sqrt{y+z}-\sqrt{y})^2\leq(\sqrt{y+z}-\sqrt{y})(\sqrt{y+z}+\sqrt{y})=z,$$

and therefore $\sqrt{y+z}-\sqrt{y}\leq\sqrt{z}=O(r^d)$. $\square$

Proposition 7 $SD=\sqrt{r^2-u^2}+O(r^2)$.

Proof The equation $\cos SD\cos u=\cos r$ implies

$$1-\cos^2 SD=\frac{1}{\cos^2 u}\cdot(\cos^2 u-\cos^2 r)=r^2-u^2+O(r^4).$$

We now apply Proposition 6 with $d=2$ to conclude that $\sin SD$ and SD are both $\sqrt{r^2-u^2}+O(r^2)$. $\square$

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References

[1] Olga Goulko and Adrian Kent. The grasshopper problem. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 473(2207), 2017.

[2] Sabine Hossenfelder. Pure nerd fun: The grasshopper problem. 17 January 2018. Available from: `http://backreaction.blogspot.com/2018/01/pure-nerd-fun-grasshopper-problem.html`.