ON THE CONSTRUCTION AND APPLICATION OF AUSLANDER-REITEN QUIVERS TO CERTAIN GROUP ALGEBRAS

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To my wife, Lindy.

CONTENTS.

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CHAPTER 1. Auslander-Reiten Theory.

Let A be an associative finite dimensional k-algebra, k an algebraically closed field, and denote by mod A (mod A^{op}) the category of all finite dimensional left (right) A-modules. Let $\mathrm{Ind}(A)$ be a set of representatives of the isomorphism classes of indecomposable objects of mod A ; we shall frequently ignore the distinction between a module and its isomorphism class and this will be reflected in the notation.

For M,N objects in mod A , $(M,N)_A$ will denote the space of A-morphisms from M to N and we shall frequently write this as (M,N) . A morphism $\theta \in (M,N)$ is <u>projective</u> if there exist a projective module P and morphisms θ' , θ'' such that the diagram

$$\begin{tikzcd} & P \arrow[dr, "\theta''"] & \\ M \arrow[ur, "\theta'"] \arrow[rr, "\theta"'] & & N \end{tikzcd}$$

commutes. Write $\mathcal{P}(M,N)$ for the space of projective morphisms from M to N . Let $\underline{(M,N)}$ denote the quotient space $(M,N)/\mathcal{P}(M,N)$ and let $\underline{\theta} = \theta + \mathcal{P}(M,N)$, $\theta \in (M,N)$.

Define the <u>stable category</u> $\underline{\mathrm{mod}}\ A$ to be the category whose objects are those in mod A and whose morphisms are $\underline{(M,N)}$. For A,B finite-dimensional k-algebras say mod A , mod B are stably equivalent if there exists an equivalence of categories

$$F : \underline{\text{mod}}\ A \to \underline{\text{mod}}\ B \ .$$

Let $\text{mod}_P A$ be the (full) subcategory of mod A whose objects are those in mod A having no projective direct summands. Let $\text{Ind}_P(A)$ be a full set of representatives of indecomposable objects in $\text{mod}_P A$. Clearly mod A , $\text{mod}_P A$ are stably equivalent.

For $M \in$ mod A let $\mathcal{R}M$ (ΣM) denote the radical (socle) of M and set $\text{Hd}(M) = M/\mathcal{R}M$. In addition let $P(M)$, $I(M)$ denote the projective cover, injective hull of M .

Finally, for $M,N \in$ mod A , we write $M|N$ if there exists a section $\theta \in (M,N)$ and a retraction $\psi \in (N,M)$ such that $\psi\circ\theta = 1_M$. That is, M is isomorphic to a direct summand of N .

1.1 Definition.

An A-map $\theta : V \to W$ is irreducible if

(i) θ is neither a section nor a retraction.

(ii) Given the diagram

$$\begin{tikzcd} & Z \arrow[dr, "\theta''"] & \\ V \arrow[ur, "\theta'"] \arrow[rr, "\theta"'] & & W \end{tikzcd}$$

Introduction.

Representation theory has seen some new developments recently with the application of Auslander-Reiten theory. Given a finite dimensional algebra A , one can construct a directed graph called the Auslander-Reiten (A-R) quiver whose vertices are the isomorphism classes of indecomposable A-modules. In addition there exists certain non-split short exact sequences called A-R sequences which are closely linked with the A-R quiver.

In Chapter 1 we outline this theory including a technique for constructing A-R sequences by J.A. Green [Gr 3]. In Chapter 2 we look at group representation theory concentrating on blocks of cyclic defect group. In particular we are able to construct the A-R quiver for such a block and the A-R sequence for the indecomposable modules.

In Chapter 3 we look at some of C. Riedtmann's work ([R1], [R2]) on abstract quivers and coverings of A-R quivers. In Chapter 4 we combine all these ideas to obtain results on blocks of cyclic defect group. In particular the composition factors (see [Ja] for example) for each indecomposable module are determined and a result concerning the Grothendieck group is obtained ([Bu2]).

A-R sequences for blocks of cyclic defect group have already been studied and we refer to [Re] and [GaR] for details concerning both the composition factors for modules and the A-R sequences.

In Chapter 5 we look at the group $SL(2,p^n)$. Recently there has been a lot of interest in this group, including work by K. Erdmann on certain filtrations of projective modules [E1] and periodic modules [E2]. We look at the simple periodic modules and construct the connected quiver components containing them, using certain pullback techniques. In this way we obtain infinite families of periodic modules of arbitrary large dimension.

Some work has been done on periodic modules in Auslander-Reiten quivers (see [H], [W]) and indeed the shapes of the quiver components containing periodic modules is predicted.

either θ' is a section or θ'' is a retraction.

It is an easy consequence of the definition (see [AR2] 2.6(a)) that an irreducible map is either a monomorphism or an epimorphism but never both. In particular if $\theta \in (V,W)$ is irreducible, $V \not\cong W$.

Let $V,W \in \text{mod } A$, $V \cong \coprod_{i=1}^{m} V_i$, $W \cong \coprod_{i=1}^{n} W_j$ with $V_i, W_j \in \text{Ind}(A)$. Given $\theta \in (V,W)$ we can express it as a matrix (θ_{ji}) , $\theta_{ji} \in (V_i, W_j)$. Define

(1) $R(V,W) = \{\theta \in (V,W) | \theta_{ji}$ is not an isomorphism for any $1 \le i \le m, 1 \le j \le n\}$;

and

(2) $$R^2(V,W) = \sum_{Z \in \text{Ind}(A)} R(Z,W)R(V,Z) \text{ .}$$

1.2 <u>Lemma</u> ([Ri] 120)

For $V,W \in \text{mod } A$, $\theta \in (V,W)$ irreducible implies that $\theta \in R(V,W) \backslash R^2(V,W)$. Furthermore if $V,W \in \text{Ind}(A)$ then θ is irreducible iff $\theta \in R(V,W) \backslash R^2(V,W)$. □

Define the <u>space of irreducible maps</u> to be

(3) $$\text{Irr}(V,W) = R(V,W)/R^2(V,W) \quad V,W \in \text{Ind}(A) \text{ .}$$

1.3 <u>Lemma</u> ([AR2] 528)

Let $F:\underline{\mathrm{mod}}\ A \to \underline{\mathrm{mod}}\ B$ be an equivalence of stable categories. Let $\theta \in (V,W)_A$ such that $\underline{\theta} \neq \underline{0}$ and $\psi \in (FV,FW)_B$ such that $F(\underline{\theta}) = \underline{\psi}$. Then θ is irreducible iff ψ is irreducible.

<u>Proof</u>.

We only need the result for $V,W \in \mathrm{Ind}_P(A)$ so we will prove this restricted case referring to the proof in [AR2] for the general case.

We claim that $P(V,W) \subseteq R^2(V,W)$. Let $\phi \in P(V,W)$. Certainly $\phi \notin (V,W)\backslash R(V,W)$ otherwise V or W would be projective. If $\phi \in R(V,W)\backslash R^2(V,W)$ then ϕ is irreducible by 1.2. Since $\phi \in P(V,W)$ there exists a projective module P such that the following diagram

$$\begin{tikzcd} & P \arrow[dr, "\phi_2"] & \\ V \arrow[ur, "\phi_1"] \arrow[rr, "\phi"'] & & W \end{tikzcd}$$

commutes for some $\phi_1 \in (V,P)$, $\phi_2 \in (P,W)$. But ϕ is irreducible so either ϕ_1 is a section or ϕ_2 is a retraction. In either case either $V|P$ or $W|P$ which implies V or W is projective, a contradiction which establishes our claim.

Consider θ as given in the lemma. Then $\theta \in R(V,W)\backslash R^2(V,W)$ by 1.2 and by the above remarks we can say that $\underline{\theta} \in \underline{R}(\underline{V,W})\backslash \underline{R}^2(\underline{V,W})$. Since we have an equivalence $F:\underline{\text{mod}}\ A \to \underline{\text{mod}}\ B$ it follows that $\psi \in R(\underline{FV,FW})\backslash R^2(\underline{FV,FW})$ and so $\psi \in R(FV,FW)\backslash R^2(FV,FW)$ since $P(V,W) \subseteq R^2(V,W)$. Therefore ψ is irreducible by 1.2. A similar proof gives the reverse implication.

□

1.4 Definition.

The Auslander-Reiten Quiver $Q(A)$, of an algebra A , is the directed graph whose vertices are the elements $V \in Ind(A)$. An arrow $V \to W$ is defined iff there exists an irreducible map $\theta:V \to W$. To each arrow we attach an integer $n_{V,W} = \dim_k Irr(V,W)$. In the case $n_{V,W} = 1$ we delete this number.

We remark that $Q(A)$ is finite iff A is representation finite. Also there are no loops $V \cdot \circlearrowleft$ in $Q(A)$ by the remarks following definition 1.1.

Define the stable quiver $Q(A)_S$ to be the directed subgraph of $Q(A)$ obtained by removing all projective vertices and their attaching arrows.

1.5 Proposition.

Let $F : \underline{\text{mod}}\ A \to \underline{\text{mod}}\ B$ be an equivalence of stable categories. Then $Q(A)_S \cong Q(B)_S$ and $n_{V,W} = n_{FV,FW}$ for all $V,W \in Ind_P(A)$.

Proof.

This follows immediately from 1.3.

□

1.6 <u>Definition</u>.

An <u>Auslander-Reiten (A-R) Sequence</u> is a short exact sequence (s.e.s.)

$$(M) \quad 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

satisfying the following conditions.

(a) $M, N \in \mathrm{Ind}(A)$.

(b) (M) is non-split.

(c) If $\ell \in (X,M)$ is not a retraction then there exists $h \in (X,E)$ such that $\ell = g \circ h$.

Condition (b) implies that M is non-projective and N is non-injective. Gabriel's expository paper [Ga1] (sections 1-3) and Auslander and Reiten's [AR1/2/3] contain many details on the theory and properties enjoyed by these sequences, a few of which we give below.

<u>Note</u>.

An A-R sequence is often referred to as <u>almost split</u>.

1.7 <u>Theorem</u> ([AR1] §4)

Let $M \in \mathrm{Ind}_p(A)$. Then there exists an A-R sequence

$$A(M) : 0 \to N \to E \to M \to 0$$

which is unique up to isomorphism. □

For such a sequence we denote by τM the module N, called the Auslander-Translate of M.

We will explain, briefly, how the A-R sequence is constructed introducing various 'machinery' from homological algebra as we go along. To start with, given $M \in \mathrm{mod}_P(A)$, let $\theta : P(M) \to M$ be a projective cover for M. Define the Heller Operator Ω by $\Omega M := \mathrm{Ker}(\theta)$. This is uniquely defined up to isomorphism and the s.e.s.

$$0 \to \Omega M \to P(M) \xrightarrow{\theta} M \to 0$$

is a minimal projective presentation (m.p.p.) of M. Define contravariant functors

$$(*), D : \mathrm{mod}\, A \to \mathrm{mod}\, A^{op}$$

by

$$M^* := (M,A)_A \quad , \quad DM := (M,k)_k \quad .$$

The right A-module action is given by

$$(fa)(m) = f(am)\ ,\ a \in A\ ,\ m \in M\ ,\ f \in DM$$

and $$(fa)(m) = f(m)a\ ,\ a \in A\ ,\ m \in M\ ,\ f \in M^*.$$

Let

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

be a m.p.p. for $M \in \mathrm{Ind}_P(A)$ and apply $(^*)$ which is left-exact to obtain

$$0 \to M^* \xrightarrow{p_0^*} P_0^* \xrightarrow{p_1^*} P_1^* \to \mathrm{Coker}(p_1^*) \to 0 \quad .$$

Applying the functor D which is exact we obtain the exact sequence

$$(4) \qquad 0 \to DTrM \to DP_1^* \xrightarrow{Dp_1^*} DP_0^* \xrightarrow{Dp_0^*} DM^* \to 0$$

where $TrM := \mathrm{Coker}(p_1^*)$. The module $DTrM$ is the Auslander-Translate M .

Suppose A is <u>symmetric</u>. That is there exists a linear map $\alpha : A \to k$ such that the bilinear form $\langle , \rangle : A \times A \to k$ given by $\langle a,b \rangle = \alpha(ab)$ is symmetric and non-degenerate. (In particular, $\alpha(J) \neq 0$ for any left (or right) ideal J of A) . It follows that $(^*)$, D are isomorphic as functors under the map $f \to \alpha \circ f$, $f \in M^*$. Therefore, (4) becomes the m.p.p.

$$0 \to \Omega^2 M \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

and we deduce the following.

1.8 Proposition.

Let $A = kG$, the group algebra for some finite group G . Then for $M \in \mathrm{Ind}_P(kG)$,

$$\tau M \cong \Omega^2 M \quad .$$

Proof.

The result follows from the above remarks since kG is a symmetric algebra. (Define $\alpha \in DkG$ by $\alpha(\sum_{g \in G} \lambda_g g) = \lambda_1$) . □

We now turn to the construction of A-R sequences. In [AR1] §4, the following condition is given for a s.e.s. to be almost split. Let

$$(M) \quad 0 \to \tau M \to E \to M \to 0$$

be a s.e.s. with $M \in \mathrm{Ind}_P(A)$ given by $x \in \mathrm{Ext}^1_A(M,\tau M)$. Then (M) is almost split if $0 \neq x \in \Sigma(\mathrm{Ext}^1_A(M,\tau M))$ considered as an (M,M)-module. So to determine (M) one has to compute the 'Ext' group, determine its socle and then construct the corresponding pushout. This is difficult to do in practice and alternative methods have been put forward by M. Butler

[Bul] and J.A. Green [GR3]. We look at the method described in [GR3] which involves looking at certain pullbacks. Recall the definition of a pullback; given $B,C,D \in \operatorname{mod} A$, $\theta \in (B,C)$, $\pi \in (D,C)$ the diagram

$$\begin{tikzcd} & B \arrow[d, "\theta"] \\ D \arrow[r, "\pi"'] & C \end{tikzcd}$$

can be completed thus:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\mu"] \arrow[d, equal] & E \arrow[r, "P_B"] \arrow[d, "P_D"] & B \arrow[d, "\theta"] \\ 0 \arrow[r] & K \arrow[r, "i"'] & D \arrow[r, "\pi"'] & C \end{tikzcd}$$

where

$$E = \{(b,d) \in B \oplus D \mid \theta(b) = \pi(d)\} \quad .$$

Here; $K = \operatorname{Ker}(\pi)$, P_B, P_D are the natural projections, $\mu(k) = (0,k)$ and i is the inclusion map. E is sometimes referred to as the pullback of (C,θ,π). Notice that if $\operatorname{Im}(\theta) \leq \operatorname{Im}(\pi)$ the top line of the diagram can be completed to form a s.e.s.

Consider the completed pullback

$$\begin{tikzcd}
0 \arrow[r] & \tau M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\theta"] & 0 & \\
0 \arrow[r] & \tau M \arrow[r] & DP_1^* \arrow[r, "Dp_1^*"'] & DP_0^* \arrow[r] & DM^* \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{Im}(\theta) \leq \mathrm{Im}(Dp_1^*)$. For such a $\theta \in (M, DP_0^*)$, Green defines a k-linear map

$$T_\theta : \mathrm{End}_A(M) \rightarrow k$$

such that the sequence (M) is almost split iff

(5) $\quad T_\theta \neq 0$

(6) $\quad T_\theta(R(\mathrm{End}_A(M))) = 0$.

Furthermore, such a θ can be chosen such that

(7) $\quad \mathrm{Ker}(\theta)$ is a maximal submodule of M $(\mathrm{Ker}(\theta) <\!\cdot\, M)$.

We describe this construction for A a symmetric algebra with associated k-map $\alpha : A \rightarrow k$. Recall that $DM^* \cong M$, $DP_0^* \cong P_0$ and $DP_1^* \cong P_1$ in this case.

Since P_0 is projective we may write

$$P_0 = \coprod_{i=1}^{m} Ae_i = As_1 \oplus \ldots \oplus As_m$$

where $s_i = (0,\ldots,e_i,\ldots,0)$, e_i a primitive idempotent. Given $\theta \in (M,P_0)$ define

$$(t_1,\ldots,t_m) \in \coprod_{i=1}^{m} (DM)e_i$$

by setting $t_i = \xi_i\theta$ where $\xi_i \in DP_0$ is given by $\xi_i(as_j) = \delta_{ij}\alpha(a)$. Finally, put $c_i = p_0(s_i)$ and define, for $\eta \in \mathrm{End}_A(M)$:

(8) $$T_\theta(\eta) = t_1(\eta(c_1)) + \ldots\ldots + t_m(\eta(c_m))$$

We now look at the connection between almost split sequences and A-R Quivers. Given an almost split sequence

$$A(M) \qquad 0 \to \tau M \xrightarrow{f} E \xrightarrow{g} M \to 0$$

we write $E \cong \coprod_{i=1}^{n} E_i^{a_i}$, $E_i \in \mathrm{Ind}(A)$ pairwise non-isomorphic and

$$f = (f_1^{(1)},\ldots\ldots,f_{a_1}^{(1)},\ldots\ldots\ldots,f_1^{(n)},\ldots\ldots,f_{a_n}^{(n)})^T$$

$$g = (g_1^{(1)}, \ldots\ldots, g_{a_1}^{(1)}, \ldots\ldots\ldots, g_1^{(n)}, \ldots\ldots, g_{a_n}^{(n)})$$

where $f_j^{(i)} \in (\tau M, E_i)$, $g_j^{(i)} \in (E_i, M)$ for $1 \le i \le n$, $1 \le j \le a_i$.

By [AR2] (2.15) f, g are irreducible and by [AR2] (2.5) $f_j^{(i)}$, $g_j^{(i)}$ are irreducible for all $1 \le i \le n$, $1 \le j \le a_i$. Therefore the almost split sequence $A(M)$ gives rise to the subquiver

$$\begin{tikzcd}
 & E_1 \arrow[dr, "a_1"] & \\
\tau M \arrow[ur, "a_1"] \arrow[r, "a_2"'] \arrow[dr, "a_n"'] & E_2 \arrow[r, "a_2"'] & M \\
 & \vdots & \\
 & E_n \arrow[uur, "a_n"'] &
\end{tikzcd}$$

where $a_i = n_{\tau M, E_i} = n_{E_i, M}$ by [Ri] (120).

Call such a subquiver a <u>mesh</u>. If, for some $1 \le i \le n$, E_i is projective, call the subquiver a <u>projective mesh</u>.

Conversely, given $M \in \mathrm{Ind}_p(A)$, define

(9) $\quad M^+ = \{V \in \mathrm{Ind}(A) \mid \text{there exists an arrow } M \to V\}$

and for $N \in \mathrm{Ind}(A)$ such that N is non injective, define

(10) $\quad N^- = \{U \in \mathrm{Ind}(A) \mid \text{there exists an arrow } U \to N\}$.

Then, see for example [Ri] (120),

$$(11) \qquad (\tau M)^+ = M^-$$

and there exist maps $f \in (\tau M, \coprod_{E\in M^-} E^{n_{E,M}})$, $g \in (\coprod_{E\in M^-} E^{n_{E,M}}, M)$

such that

$$0 \to \tau M \xrightarrow{f} \coprod_{E\in M^-} E^{n_{E,M}} \xrightarrow{g} M \to 0$$

is an almost split sequence.

In passing from $Q(A)$ to the stable quiver $Q(A)_s$ we observe that the non-projective meshes are preserved whilst the projective meshes lose their projective vertices and attaching arrows. One is frequently in the position where the stable quiver is known so, to recover the full quiver, one needs to place the projective modules in the correct mesh.

From now on we shall assume that A is a symmetric algebra. In particular, a module P is projective iff it is injective.

Let $P \in \mathrm{Ind}(A)$ be projective and simple. Suppose there exists an irreducible map $\theta:P \to M$. Since P is simple θ is a monomorphism and by the injectivity of P , θ is a section which contradicts 1.1(i). Similarly there is no irreducible map $\psi:N \to P$. Hence P occurs as an isolated vertex in $Q(A)$ and plays no significant role. We shall restrict ourselves, then, to non-simple projective modules.

1.9 Lemma.

Let $P \in \mathrm{Ind}(A)$ be projective and non-simple. The only irreducible maps to and from P are the natural maps $i : RP \to P$ and $\pi : P \to P/\Sigma P$.

Proof.

Suppose $\theta : M \to P$ is irreducible. Since P is projective θ is a monomorphism. If $M \not\cong RP$ then $\mathrm{Im}(\theta) < RP$ and we have a non-trivial factorisation

$$\begin{tikzcd} & RP \arrow[dr, "i"] & \\ M \arrow[ur, "\theta'"] \arrow[rr, "\theta"'] & & P \end{tikzcd}$$

Since neither θ' nor i are sections θ is not irreducible. Therefore, take $M = RP$ and suppose there exists a factorisation

$$\begin{tikzcd} & N \arrow[dr, "\theta''"] & \\ RP \arrow[ur, "\theta'"] \arrow[rr, "i"'] & & P \end{tikzcd}$$

such that θ' is not a section and θ'' is not a retraction. Since i is a monomorphism, so is θ', whilst θ'' is not surjective since P is projective. Hence $\mathrm{Im}(\theta'') = \mathrm{Im}(i) = RP$. Define $\psi : N \to RP$ by $\psi(n) = \theta''(n)$ for $n \in \mathrm{Im}(\theta')$ and $\psi(n) = 0$ otherwise. Then $\psi\theta' = 1_{RP}$ and so θ' is a section, a contradiction. Therefore i is irreducible as required.

The case $\pi : P \to P/\Sigma P$ is similar.

□

1.10 Theorem.

Let $P \in \mathrm{Ind}(A)$ be projective and non-simple and denote by $\{P\}$, the projective mesh in $Q(A)$ containing P. Then $\{P\}$ gives rise to the almost split sequence

$$0 \to RP \to P \oplus RP/\Sigma P \to P/\Sigma P \to 0 \quad .$$

Proof.

By 1.9 and earlier remarks, $\{P\}$ gives rise to an almost split sequence of the form

$$0 \to RP \to P \oplus X \to P/\Sigma P \to 0$$

for some $X \in \mathrm{mod}\, A$. Consider the pullback

$$\begin{tikzcd}
0 \arrow[r] & RP \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & P/\Sigma P \arrow[r] \arrow[d, "\theta"] & 0 & \\
0 \arrow[r] & RP \arrow[r, "\pi"'] & P \arrow[r] & P \arrow[r] & P/\Sigma P \arrow[r] & 0
\end{tikzcd}$$

where $\pi : P \to Hd(P)$ $(= P/RP)$ is the natural map and we identify $Hd(P)$ with $\Sigma P \leq P$. Referring to [GR3] a θ can be found such that the top line is an almost split sequence, and by (7) we may choose θ such that $\mathrm{Ker}(\theta) <\cdot P/\Sigma P$. Therefore $\mathrm{Ker}(\theta) = RP/\Sigma P$ and $\mathrm{Im}(\theta) = \Sigma P$. Let $P = Ae$ for some primitive idempotent in A. We may assume that:

$$\theta : P/\Sigma P \longrightarrow Hd(P)$$

$$e + \Sigma P \longrightarrow e + RP$$

and so $\pi(e) = \theta(e + \Sigma P)$. Therefore, $\pi(ae) = \theta(ae + \Sigma P)$ for all $a \in A$.

Consider the pullback

$$E = \{(p,q) \in P \oplus P/\Sigma P \mid \pi(p) = \theta(q)\}$$

which is generated as an A-module by $\mathrm{Ker}(\pi) \oplus \mathrm{Ker}(\theta)$ and $(e, e + \Sigma P)$. Consider the A-map

$$\psi : E \to P \quad , \quad \psi(p,q) = p \quad .$$

Since P is projective and ψ is surjective, ψ is a retraction; hence $E \cong P \oplus \mathrm{Ker}(\psi)$. However,

$$\mathrm{Ker}(\psi) = \{(0,q) \in P \oplus P/\Sigma P\} \cong \mathrm{Ker}(\theta) = RP/\Sigma P$$

as required.

$\square$

In the next chapter we look at the group algebra case, in particular at blocks with cyclic defect group, and apply some of the results of this chapter.

CHAPTER 2. Representation Theory.

Let G be a finite group and k an algebraically closed field of characteristic p such that $p \mid |G|$. Let $\mathbb{B}$ be a p-block of G with defect group D of order $p^d = q$. Finally let $H \leq G$ such that $N_G(D) \leq H$ and let B be the unique kH-block associated to $\mathbb{B}$ under the Brauer Correspondence.

We will want to take advantage of the Green Correspondence which exists between certain quotient categories of mod kH and mod kG . Here, however, we will only be interested in the situation which gives rise to an equivalence of stable categories and therefore we shall only outline the correspondence in this case. For further details we refer to [GR2] which contains a summary of the definitive paper [GR1].

2.1 Definition.

Let G be as above, $H \leq G$, let $U \in$ mod kH , $M \in$ mod kG and define $fM \in$ mod kH , $gU \in$ mod kG as follows. Write

(1) $M_H = fM \oplus P$

(2) $U^G = gU \oplus Q$

where P,Q are projective, fM,gU are projective free and are all uniquely determined up to isomorphism by the Krull-Schmidt Theorem.

From now on let D be cyclic and set $H = N_G(D_0) \geq N_G(D)$ where D_0 is the minimal subgroup of D. The following is then a corollary of the Green Correspondence.

2.2 Theorem.

Let $\mathbb{B}$, B, D be as above with D cyclic of order q and let $M \in \mathrm{Ind}_p(kG)$, $U \in \mathrm{Ind}_p(kH)$. Then:

(a) $fM \in \mathrm{Ind}\,(kH)$, $gU \in \mathrm{Ind}\,(kG)$ and

$g(fM) \cong M$, $f(gU) \cong U$.

(b) $U \in \mathrm{mod}\ B$ iff $gU \in \mathrm{mod}\ \mathbb{B}$

$M \in \mathrm{mod}\ \mathbb{B}$ iff $fM \in \mathrm{mod}\ B$.

(c) $\Omega(fM) \cong f(\Omega M)$, $\Omega(gU) \cong g(\Omega U)$.

2.3 Corollary.

There exists an equivalence of stable categories:

$$f: \underline{\mathrm{mod}}\ \mathbb{B} \to \underline{\mathrm{mod}}\ B .$$

Proof.

By 2.2 f gives a 1-1 correspondence of objects. For each $M \in \mathrm{mod}\ \mathbb{B}$, let

$$p_M : M_H \to fM \ , \quad i_M : fM \to M_H$$

be the natural retraction, section respectively. For $M,N \in \underline{\mathrm{mod}}\ \mathbb{B}$, $\theta \in (M,N)$, let $f\theta = p_N \circ \theta|_H \circ i_M$. This induces a natural k-isomorphism

$$\underline{(M,N)} \cong \underline{(fM,fN)} \ , \quad \underline{\theta} \to \underline{f\theta} \quad .$$

In a similar fashion, one can define

$$g : \underline{\mathrm{mod}}\ B \to \underline{\mathrm{mod}}\ \mathbb{B}$$

such that $f \circ g \cong 1_{\underline{\mathrm{mod}}\ B}$, $g \circ f \cong 1_{\underline{\mathrm{mod}}\ \mathbb{B}}$ as required. □

2.4 Definition ([P] 233)

Let H be a subgroup of G having a p-block b with defect group D of order q . We say that b is (q,e)-uniserial if:

(a) b contains, up to isomorphism, e simple modules S_i $(i \in I = \{0,1,\ldots,e-1\})$ where $e|q-1$.

(b) There exists a full set of projective indecomposable modules $\{T_i \mid i \in I\}$ such that:

(i) $T_i/R(T_i) \cong S_i$.

(ii) T_i has a unique composition series

$$T_i \underset{S_i}{\text{———}} \underset{S_{i+1}}{\text{———}} \underset{S_{i+2}}{\text{———}} \text{-----} \quad \underset{S_{i+q-1}}{\text{———}} 0$$

where, for $j \in \mathbb{Z}$, $S_j := S_i$ where $j = i+ae$, $i \in I$, $a \in \mathbb{Z}$.

(iii) There exists a full set

$$\{T_{i,\alpha} \mid i \in I \text{ , } \alpha = 1,\dots\dots,q\}$$

of indecomposable modules for mod b where each $T_{i,\alpha}$ has a unique composition series

$$T_{i,\alpha} \underset{S_i}{\text{———}} \underset{S_{i+1}}{\text{———}} \text{---------} \quad \underset{S_{i+\alpha-1}}{\text{———}} 0 \text{ .}$$

In particular, $T_i = T_{i,q}$, $S_i = T_{i,1}$ and

(3) $\qquad T_{i,\alpha}/R(T_{i,\alpha}) \cong S_i$.

(4) $\qquad \Sigma(T_{i,\alpha}) \cong S_{i+\alpha-1}$.

For $j \in \mathbb{Z}$, let $T_{j,\alpha} = T_{i,\alpha}$ where $j \equiv i \bmod(e)$.

The following can be found in Dade's paper [D].

2.5 Theorem.

Let B, $\mathbb{B}$, D be as in 2.2. Then B is (q,e)-uniserial where e is the inertial index of B and $e|p-1$.

Proof.

We sketch the proof referring to Dade's paper [D] and the later publication [GR2] for more details. Let T_i , S_i $(i \in I)$ be the projective indecomposable, simple modules respectively numbered such that $T_i/RT_i \cong S_i$. Dade shows that there exists $n \in kD$ such that $n^q = 0$ and for each $i \in I$,

(5) $$T_i > T_i.n > T_in^2 \;.........\; > T_i.n^{q-1} > 0$$

is the unique composition series for T_i . If we set $T_{i,\alpha} = T_i/T_i.n^\alpha$ then $\{T_{i,\alpha} | i \in I , \alpha = 1,..,q\}$ is a full set of indecomposable modules.

Green shows in [GR2] that for each $i \in I$,

(6) $$S_{i+a} \cong T_in^a/T_in^{a+1}$$

where the indexing is taken module e . It follows that the composition series (5) satisfies 2.4(ii) which proves the theorem.

□

Let $T^{(a)}_{i,\alpha} = T_{i,\alpha}.n^a$. By (6) it follows that for $\alpha = 1,..,q$:

(7) $\qquad T_{i,\alpha}^{(a)} \cong T_{i+a,\alpha-a} \qquad a = 0, \ldots, \alpha-1$;

(8) $\qquad T_{i,\alpha}/T_{i,\alpha}^{(a)} \cong T_{i,a} \qquad a = 1, \ldots, \alpha$.

If we consider the m.p.p.

$$0 \to T_i . n^a \to T_i \to T_{i,a} \to 0$$

we see that,

(9) $\qquad \Omega T_{i,a} \cong T_i^{(a)} \cong T_{i+a,q-a}$

which implies that $\Omega^2 T_{i,a} \cong T_{i+1,a}$ since $q \equiv 1 \bmod(e)$.

Finally let e_i be the primitive idempotent corresponding to T_i . That is $T_i = Be_i$. Then $e_{i,\alpha} = e_i + T_i n^\alpha \in T_{i,\alpha}$ is a generator for $T_{i,\alpha}$ as a B-module.

2.6 <u>Theorem</u>.

The A-R sequence $A(T_{i,\alpha})$ is given by:

$$0 \to T_{i+1,\alpha} \to T_{i,\alpha+1} \oplus T_{i+1,\alpha-1} \to T_{i,\alpha} \to 0$$

where, for $\alpha = 1$, $T_{i+1,\alpha-1}$ is the zero-module. Furthermore, there exist irreducible maps

$$\theta_1 : T_{i,\alpha} \to T_{i-1,\alpha+1} \qquad \alpha = 1,\dots,q-1$$

$$\theta_2 : T_{i,\alpha} \to T_{i,\alpha-1} \qquad \alpha = 2,\dots,q$$

where θ_1 is the inclusion map obtained by $T_{i,\alpha} \cong T^{(1)}_{i-1,\alpha+1}$ (7) and θ_2 is defined by $\theta_2(e_{i,\alpha}) = e_{i,\alpha} + T^{(\alpha-1)}_{i,\alpha}$ and the isomorphism given in (8).

<u>Proof</u>.

Consider the m.p.p.

$$0 \to \Omega^2 T_{i,\alpha} \to T_{i+\alpha} \to T_i \to T_{i,\alpha} \to 0$$

and the pullback

$$\text{(10')} \qquad \begin{array}{ccccccccc} 0 & \to & T_{i+1,\alpha} & \to & E(\theta) & \to & T_{i,\alpha} & \to & 0 \\ & & || & & \downarrow & & \downarrow\theta & & \\ 0 & \to & T_{i+1,\alpha} & \to & T_{i+\alpha} & \underset{\pi}{\to} & T_i & \to & T_{i,\alpha} \to 0 \end{array}$$

where $Im(\theta) \leq Im(\pi)$.

By earlier remarks (7) in Chapter 1 there exists a θ such that $\mathrm{Ker}(\theta) <\cdot\ T_{i,\alpha}$ and the resulting s.e.s. is almost split. Uniseriality of $T_{i,\alpha}$ forces $\mathrm{Ker}(\theta) = RT_{i,\alpha} = T_{i,\alpha}^{(1)}$. Given $\theta' \in (T_{i,\alpha},T_i)$ with $\mathrm{Ker}(\theta') = RT_{i,\alpha}$, it follows from Schur's Lemma that $\theta' = s.\theta$ for some $s \in k$ since $\mathrm{Im}(\theta)$ is simple. Thus the resulting s.e.s.

$$0 \to T_{i+1,\alpha} \to E(\theta') \to T_{i,\alpha} \to 0$$

is isomorphic to the one in (10') and is thus almost split. We may now assume that

$$\theta : T_{i,\alpha} \to T_i : e_{i,\alpha} \to e_i.n^{q-1} \quad .$$

Here $\mathrm{Ker}(\theta) = RT_{i,\alpha} \cong T_{i+1,\alpha-1}$ by (7) and $\mathrm{Im}(\theta) = T_i^{(q-1)} = \Sigma(T_i)$.

To determine $\pi : T_{i+\alpha} \to T_i$ we want such a map with $\mathrm{Ker}(\pi) \cong T_{i+1,\alpha}$. Let π be given by $\pi(e_{i+\alpha}) = e_i.n^{\alpha}$. Then $\mathrm{Ker}(\pi) = T_{i+\alpha}^{(q-\alpha)} \cong T_{i+1,\alpha}$ by (7) and the pullback is given by

$$E(\theta) = \{(u,v) \in T_{i,\alpha} \oplus T_{i+\alpha,q} \mid \theta(u) = \pi(v)\} \quad .$$

Now

$$\pi^{-1}(\mathrm{Im}(\theta)) = \pi^{-1}(T_i^{(q-1)}) = T_{i+\alpha}^{(q-\alpha+1)} \cong T_{i,\alpha+1} \quad \text{by (7)}$$

so $E(\theta)$ can be considered as the pullback

$$\begin{array}{ccc} E(\theta) & \to & T_{i,\alpha} \\ \downarrow & & \downarrow \theta \\ T_{i,\alpha+1} & \xrightarrow[\pi']{} & T_{i,q} \end{array}$$

where $\pi'(e_{i,\alpha+1}) = e_i.n^{q-1}$. Therefore:

$$E(\theta) \cong \{(u,v) \in T_{i,\alpha} \oplus T_{i,\alpha+1} | \theta(u) = \pi'(v)\} = E \text{ say.}$$

Let $\phi \in (T_{i,\alpha+1},T_{i,\alpha})$ be given by $\phi(e_{i,\alpha+1}) = e_{i,\alpha}$ whereby $\theta o \phi = \pi'$. Define a map

$$\Phi : E \to Ker(\theta) \oplus T_{i,\alpha+1}$$

$$(u,v) \to (u-\phi(v),v) .$$

Since $\theta(u-\phi(v)) = \theta(u)-\pi'(v) = 0$ we see that Φ is a well defined B-map which is clearly injective. To show that Φ is onto, consider $(e_{i,\alpha},e_{i,\alpha+1}) \in E$.

$$\Phi(e_{i,\alpha},e_{i,\alpha+1}) = (e_{i,\alpha}-\phi(e_{i,\alpha+1}),e_{i,\alpha+1}) = (0,e_{i,\alpha+1})$$

and so $\Phi(B(e_{i,\alpha},e_{i,\alpha+1})) = B(0,e_{i,\alpha+1}) = (0,T_{i,\alpha+1})$. Similarly, for

$(e_{i,\alpha}.n,0) \in E$,

$$\Phi(B(e_{i,\alpha}.n,0)) = B(e_{i,\alpha}.n,0) = (\mathrm{Ker}(\theta),0) .$$

Hence Φ is also onto and so

$$E(\theta) \cong E \cong \mathrm{Ker}(\theta) \oplus T_{i,\alpha+1} = T_{i+1,\alpha-1} \oplus T_{i,\alpha+1}$$

as required.

We now compute the irreducible maps θ_1,θ_2 . One can embed $T_{i+1,\alpha}$ in $T_{i,\alpha+1}$ by the map $e_{i+1,\alpha} \to e_{i,\alpha+1}.n$ so the monomorphism in the A-R sequence

$$0 \to T_{i+1,\alpha} \to E \to T_{i,\alpha} \to 0$$

is given by $e_{i+1,\alpha} \to (0,e_{i,\alpha+1}.n)$. But $\Phi(0,e_{i,\alpha+1}n) = (-e_{i,\alpha}.n,e_{i,\alpha+1}.n)$ $\in \mathrm{Ker}(\theta) \oplus T_{i,\alpha+1}$ which is mapped to the element $(-e_{i+1,\alpha-1},e_{i,\alpha}.n)$ $\in T_{i+1,\alpha-1} \oplus T_{i,\alpha+1}$. Therefore the map

$$f:T_{i+1,\alpha} \to T_{i+1,\alpha-1} \oplus T_{i,\alpha+1}$$

$$e_{i+1,\alpha} \to (-e_{i+1,\alpha-1},e_{i,\alpha+1}n)$$

is irreducible by [AR2] 2.15. Now $f = (f_1, f_2)^T$ where $f_1(e_{i+1,\alpha}) = -e_{i+1,\alpha-1}$ and $f_2(e_{i+1,\alpha}) = e_{i,\alpha+1}n$ and so f_1, f_2 are irreducible by [AR2] 2.5. Hence θ_1, θ_2 as given in the statement of the theorem are irreducible. The case $\theta_2 : T_{i,q} \to T_{i,q-1}$ is covered by 1.9 since $T_{i,q}$ is projective and $T_{i,q-1} \cong T_{i,q}/\Sigma(T_{i,q})$

□

2.7 Corollary.

$$\dim_k \mathrm{Irr}(T_{i,\alpha}, T_{i,\alpha-1}) = \dim_k \mathrm{Irr}(T_{i,\alpha}, T_{i-1,\alpha+1}) = 1 \ .$$

Proof.

Since $T_{i,\alpha+1}$, $T_{i+1,\alpha-1}$ occur with multiplicity one in the middle term of $A(T_{i,\alpha})$, the result follows from the remarks made in Chapter 1 (1.8-1.9).

□

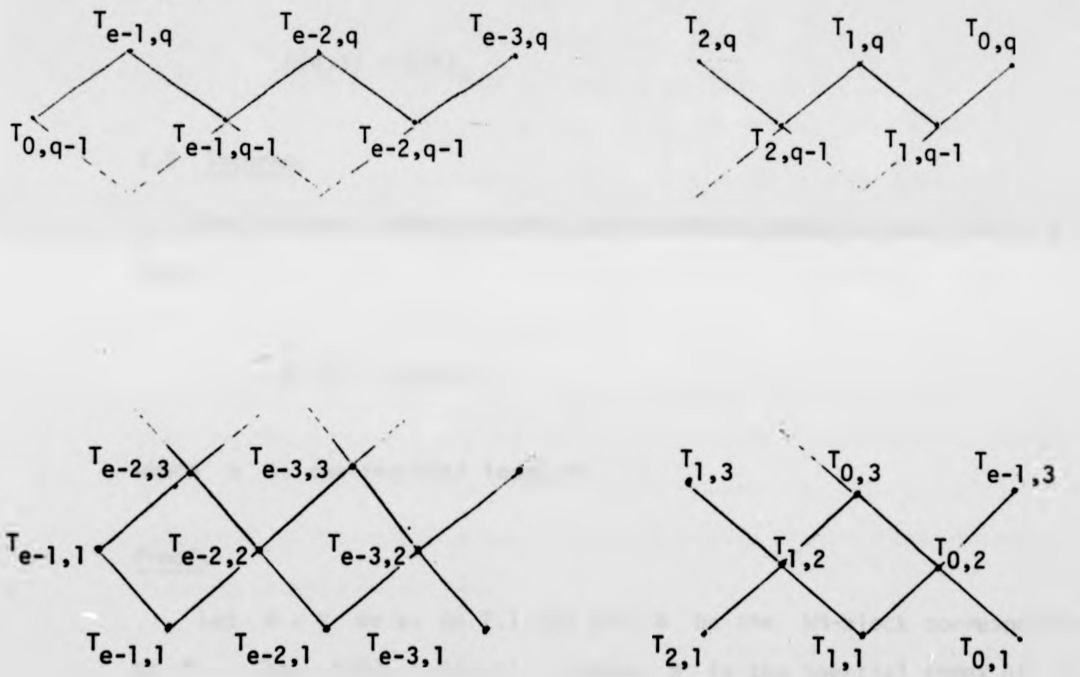
One also deduces that $A(T_{i,\alpha})$ gives rise to the mesh

$$\begin{tikzcd}[column sep=small] & T_{i,\alpha+1} \arrow[dr] & \\ T_{i+1,\alpha} \arrow[ur] \arrow[dr] & & T_{i,\alpha} \\ & T_{i+1,\alpha-1} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & T_{i,2} \arrow[dr] & \\ T_{i+1,1} \arrow[ur] & & T_{i,1} \end{tikzcd}$$

$\alpha = 2, \ldots, q-1$ $\qquad\qquad$ $\alpha = 1$.

The A-R quiver is then the union of all such meshes and is given below:

(10)



This is not quite the whole picture. Since we defined $T_{j,\alpha} = T_{i,\alpha}$ for $j \equiv i \bmod (e)$, $(j \in \mathbb{Z},\ i \in I)$ 2.6 implies that there exist irreducible maps $\theta_1 : T_{0,\alpha} \to T_{e-1,\alpha}$. Therefore we have to 'glue' the edges of the graph (10) together and the resulting cylinder is the full A-R quiver $Q(B)$. We shall discuss this 'glueing' in more detail in the next chapter.

The stable quiver $Q(B)_s$ is the shortened cylinder obtained by deleting the top row of vertices ($T_{i,q}$'s) and their attaching arrows. For such a (q,e)-uniserial block, write:

$$Q(q,e) = Q(B)_s \quad .$$

2.8 <u>Theorem</u>.

Let $\mathbb{B}$ be a kG-block with cyclic defect group D with $|D| = q$. Then,

$$Q(\mathbb{B})_s \cong Q(q,e)$$

where e is the inertial index of D .

<u>Proof</u>.

Let $H \le G$ be as in 2.1 and let B be the kH-block corresponding to $\mathbb{B}$. Then $Q(B)_s = Q(q,e)$, where e is the inertial index of D , by the above remarks. Since mod B , mod $\mathbb{B}$ are stably equivalent by 2.3, $Q(B)_s \cong Q(\mathbb{B})_s$, by 1.5 and the result follows. □

2.9 <u>Corollary</u>.

Let $gT_{i,\alpha} \in \mathrm{Ind}_p(\mathbb{B})$ be the Green Correspondent of $T_{i,\alpha}$, some

$\alpha \neq r_i$. Then $A(gT_{i,\alpha})$ is

$$0 \to gT_{i+1,\alpha} \to gT_{i,\alpha+1} \oplus gT_{i+1,\alpha-1} \oplus P \to gT_{i,\alpha} \to 0$$

where P is a (possibly zero) projective module.

$\square$

To recover the full quiver $Q(\mathbb{B})$ we need to know the position of each projective indecomposable module P . By 1.10 it is sufficient to know the position of $P/\Sigma P$. We show that it is sufficient to determine the position of each simple module.

We digress a little to discuss the Brauer Tree which plays a crucial role in this problem. In the next two chapters we show that a knowledge of the Brauer Tree is enough for one to recover the full quiver and to determine the irreducible maps. The following is an outline of results found in Green's paper [GR2].

Let $\{V_i | i \in I\}$ $(\{W_i | i \in I\})$ be full sets of simple (projective indecomposable) modules labelled such that $W_i/R(W_i) \cong V_i$ and let $\{\eta_i | i \in I\}$ be the corresponding projective modular characters. In [D] Dade shows that $\mathbb{B}$ has $e + (q-1)/e$ ordinary irreducible characters

$$X_1, \ldots, X_e, X_\lambda \quad (\lambda \in \Lambda) .$$

Furthermore, if one sets $X_{e+1} = \sum_{\lambda \in \Lambda} X_\lambda$ (and call this character exceptional), then for each $i \in I$ there exists a unique pair $i(1)$, $i(2) \in \{1,\dots,e+1\}$ such that:

$$\eta_i = X_{i(1)} + X_{i(2)} \quad . \tag{11}$$

Define the Brauer Tree Γ to be the graph with $e+1$ vertices corresponding to the set $\{X_1,\dots,X_{e+1}\}$ and with e edges corresponding to the η_i's ; two vertices X_ℓ , X_m being linked by an edge η_i iff $\{\ell,m\} = \{i(1),i(2)\}$.

In [GR2] Green shows that there exists a permutation $\delta: I \to I$ such that, renumbering the W_i , V_i if necessary, the following holds.

2.10 Theorem.

For all $i,j \in I$:

(a) $(fV_j,S_i) \cong (V_j,gS_i) \cong \begin{cases} k & i = j \\ 0 & i \neq j \end{cases}$

(b) $(S_i,fV_j) \cong (gS_i,V_j) \cong \begin{cases} k & \delta(i) = j \\ 0 & \delta(i) \neq j \end{cases}$.

(c) For each $i \in I$ there exist s.e.s.

$$0 \to \Omega gS_i \to W_{\delta(i)} \to gS_i \to 0$$
$$0 \to gS_{i+1} \to W_{i+1} \to \Omega gS_i \to 0 \quad .$$

□

Green goes on to show that if, for each $i \in I$, P_{2i}, P_{2i+1} denote the modular characters afforded by gS_i, ΩgS_i respectively then $P_j \in \{X_1,\dots,X_{e+1}\}$ for $j = 0,\dots,2e-1$ and so the equation (11) can be written as:

$$(12) \qquad \eta_i = P_{2i} + P_{2i-1}$$
$$= P_{2\delta^{-1}(i)} + P_{2\delta^{-1}(i)+1}$$

thus linking the Brauer Tree Γ with the permutation δ.

One can also deduce the Cartan matrix from the Brauer Tree, the following result being found in [Ja].

2.11 <u>Theorem</u>.

Let $C = (c_{ij})$ be the Cartan matrix for B. Then:

$$\begin{aligned}
c_{ij} &= 0 && \{i(1),i(2)\} \cap \{j(1),j(2)\} = \emptyset \\
&= 1 && \{i(1),i(2)\} \cap \{j(1),j(2)\} = \{\ell\} \neq \{e+1\} \\
&= (q-1)/e && \{i(1),i(2)\} \cap \{j(1),j(2)\} = \{e+1\} \\
c_{ii} &= 2 && e+1 \notin \{i(1),i(2)\} \\
&= (q-1)/e+1 && e+1 \in \{i(1),i(2)\}\ .
\end{aligned}$$

□

We return to the problem of recovering the full quiver $Q(\mathbb{B})$ from the stable one. For any $U \in \text{mod } B$, let $\ell(U)$ denote the composition length, in particular $\ell(T_{i,\alpha}) = \alpha$ for each $i \in I$, $1 \leq \alpha \leq q$. Define:

(13) $\qquad \lambda : I \to \{1,..,q-1\}$

by setting $\lambda(i) := \ell(fV_i)$. In addition, let $\lambda'(i) = q-\lambda(i)$.

2.12 Lemma.

For each $i \in I$ $\lambda(i) \equiv \delta^{-1}(i)-i+1 \mod (e)$.

Proof.

By 2.10(a), $Hd(fV_i) \cong S_i$ which implies $fV_i \cong T_{i,\alpha}$, some $\alpha \in \{1,...,q-1\}$. By (4) $\Sigma(fV_i) \cong S_{i+\alpha-1}$ whilst 2.10(b) gives $\Sigma(fV_i) \cong S_{\delta^{-1}(i)}$. Therefore $i+\alpha-1 \equiv \delta^{-1}(i) \mod(e)$ and so $\lambda(i) = \alpha \equiv \delta^{-1}(i)-i+1$ as required. $\square$

2.13 Theorem.

The (projective) mesh containing W_i corresponds to the A-R sequence stopping at $gT_{\delta^{-1}(i),\lambda'(i)}$.

<u>Proof</u>.

The A-R sequence whose middle term contains W_i is:

$$0 \to RW_i \to RW_i/\Sigma W_i \oplus W_i \to W_i/\Sigma W_i \to 0$$

by 1.10. Now $RW_i = \Omega V_i$ since

$$0 \to RW_i \to W_i \to V_i \to 0$$

is a m.p.p. for V_i. Hence $f(RW_i) \cong f(\Omega V_i) \cong \Omega(fV_i)$ (by 2.2(c)) $\cong \Omega T_{i,\lambda(i)} \cong T_{i+\lambda(i),\lambda'(i)} = T_{\delta^{-1}(i)+1,\lambda'(i)}$ by 2.12. Now $RW_i = \tau(W_i/\Sigma W_i) = \Omega^2(W_i/\Sigma W_i)$ by 1.8 which implies that $T_{\delta^{-1}(i)+1,\lambda'(i)} \cong \Omega^2 f(W_i/\Sigma W_i)$ and so

$$f(W_i/\Sigma W_i) \cong T_{\delta^{-1}(i),\lambda'(i)}$$

as required. □

2.14 <u>Corollary</u>.

$A(gT_{\delta^{-1}(i),\lambda'(i)})$ is:

$$0 \to gT_{\delta^{-1}(i)+1,\lambda'(i)} \longrightarrow gT_{\delta^{-1}(i),\lambda'(i)+1} \oplus gT_{\delta^{-1}(i)+1,\lambda'(i)-1} \oplus W_i$$
$$\to gT_{\delta^{-1}(i),\lambda'(i)} \longrightarrow 0\ ,$$

and in particular, $RW_i \cong gT_{\delta^{-1}(i)+1,\lambda'(i)}$, $W_i/\Sigma W_i \cong gT_{\delta^{-1}(i),\lambda'(i)}$ and $RW_i/\Sigma W_i \cong gT_{\delta^{-1}(i),\lambda'(i)+1} \oplus gT_{\delta^{-1}(i)+1,\lambda'(i)-1}$. □

CHAPTER 3. Riedtmann Quivers and Coverings.

In this chapter we introduce the abstract Riedtmann Quiver [R1] of which the A-R Quiver is an example.

3.1 Definition.

A quiver $Q = (Q_0, Q_1)$ is a directed graph with vertex set Q_0 and arrow set Q_1 which contains no loops or directed arrows.

For $x \in Q_0$ let x^+ denote the set of vertices which are endpoints for arrows starting at x and x^- the set of vertices which are starting points for arrows ending at x. We shall only consider quivers Q such that x^+, x^- are finite for all $x \in Q_0$.

3.2 Definition.

A Riedtmann Quiver $\Delta = (\Delta_0, \Delta_1, \tau)$ is a quiver (Δ_0, Δ_1) together with an injective function $\tau : \Delta_0' \to \Delta_0$ defined on a subset $\Delta_0' \subset \Delta_0$ such that $(\tau x)^+ = x^-$ for all $x \in \Delta_0'$.

Call Δ stable if $\Delta_0' = \Delta_0$ and τ is onto. Say $x \in \Delta_0'$ is periodic if $\tau^a x = x$ for some $a \in \mathbb{N}$.

3.3 Example.

Let $Q(A)$ be the A-R quiver of a symmetric algebra A. Then the triple (Q_0, Q_1, Ω^2) is a Riedtmann Quiver where $Q_0 = \text{Ind}(A)$, $Q_0' = \text{Ind}_P(A)$ and Q_1 is as defined in 1.4.

Notice that $Q(A)_s$ is also a Riedtmann Quiver and that $Q(A)_s$ is stable as defined above.

3.4 Definition.

A morphism of quivers $\theta: Q \to Q^*$ is a map such that if $x \xrightarrow{\alpha} y$ is an arrow in Q there exists an arrow $\theta(x) \xrightarrow{\theta(\alpha)} \theta(y)$ in Q^* .

Let $\Delta = (\Delta_0, \Delta_1, \tau)$, $\Delta^* = (\Delta_0^*, \Delta_1^*, \tau^*)$ be Riedtmann Quivers. A morphism of Riedtmann Quivers $\theta: \Delta \to \Delta^*$ is a morphism of quivers such that:

(1) $\tau^*\theta(x) = \theta\tau(x)$

for such $x \in \Delta_0$ where this is defined.

3.5 Definition.

A covering $\pi: \Delta \to \Delta^*$ is a surjective morphism of Riedtmann Quivers satisfying the following for all $x \in \Delta_0$.

(a) π maps x^- bijectively onto $(\pi x)^-$.

(b) π maps x^+ bijectively onto $(\pi x)^+$.

(c) τx is defined iff $\tau^*(\pi x)$ is defined.

3.6 Definition.

A directed tree $T = (T_0, T_1)$ is a quiver whose underlying undirected graph $\bar{T}$ is a tree. If there exists an arrow $x \xrightarrow{\alpha} y$ then there exists no arrow $y \xrightarrow{\beta} x$.

3.7 Example.

Let $T = (T_0, T_1)$ be a directed tree and define $\mathbb{Z}T$ to be the quiver with vertex set $\mathbb{Z}\times T_0$ and arrow set $\tilde{T}_1$ as follows. For each arrow $x \to y$ in T define arrows $(i,x) \to (i,y)$, $(i+1,y) \to (i,x)$ for all $i \in \mathbb{Z}$. Now define $\tau: \mathbb{Z}\times T_0 \to \mathbb{Z}\times T_0$ by $\tau(i,x) = (i+1,x)$. Clearly τ is bijective whilst $(\tau(i,x))^+ = \{(i+1,y) | y \in x^+\} \cup \{(i,z) | z \in x^-\} = (i,x)^-$ proving that the triple $\mathbb{Z}T = (\mathbb{Z}\times T_0, \tilde{T}_1, \tau)$ is a Riedtmann Quiver. Since τ is defined on $\mathbb{Z}\times T_0$ and is surjective, $\mathbb{Z}T$ is stable.

Riedtmann shows ([R1] 204) that given directed trees T, T^* then $\bar{T} \cong \bar{T}^*$ iff $\mathbb{Z}T \cong \mathbb{Z}T^*$. She also shows in [R2] (460) that if A is a self-injective algebra which is representation finite, then the stable quiver $Q(A)_s$ can be written as the disjoint union of connected components

$$Q(A)_s = \bigcup_{j\in J} q_j \quad , \quad J \text{ a finite set}$$

such that $q_j \cong \mathbb{Z}\Gamma_j/\Pi_j$. Here Γ_j is a directed tree, Π_j is a group of admissable automorphisms of $\mathbb{Z}\Gamma_j$ and $\bar{\Gamma}_j$ is a tree of type A_n, D_n, E_6, E_7 or E_8 for each $j \in J$. In particular there exists a covering

$$\pi' : \Delta_s = \bigcup_{j \in J} \mathbb{Z}\Gamma_j \to Q(A)_s \ .$$

Furthermore π' can be extended to a covering $\pi : \Delta \to Q(A)$ of Riedtmann Quivers such that

$$\begin{array}{ccc} \Delta_s & \xrightarrow{i_1} & \Delta \\ \pi' \downarrow & & \downarrow \pi \\ Q(A)_s & \xrightarrow[i_2]{} & Q(A) \end{array}$$

commutes, i_1, i_2 being the inclusion maps.

From now on let $A = \mathbb{B}$, a kG-block with cyclic defect group and set $Q := Q(\mathbb{B})$ so $Q_s \cong Q(q,e)$, some $q,e \in \mathbb{N}$, by 2.8. We shall construct a covering for Q_s and extend this to a covering for Q .

Let $\mathrm{Ind}_p(\mathbb{B}) = \{gT_{i,\alpha} | i \in I,\ \alpha = 1,..,q-1\}$ be a full set of non-projective indecomposable modules where $gT_{i,\alpha}$ is the Green Correspondent of $T_{i,\alpha}$.

Consider the linear tree A_n with the following arrows and vertices:

$$1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow \cdots\cdots \leftarrow n-1 \leftarrow n$$

3.8 Proposition.

The morphism of quivers

$$\pi' : \mathbb{Z}A_{q-1} \to Q_s : (i,\alpha) \to gT_{i,\alpha}$$

is a covering and induces an isomorphism

$$\mathbb{Z}A_{q-1}/\langle\tau^e\rangle \cong Q_s \quad .$$

Proof.

Recalling the construction of Q_s (and 2.9) it is clear that $\Omega^2\pi'(i,\alpha) = \pi'\tau(i,\alpha) = gT_{i+1,\alpha}$ so (1) is satisfied and π' is a morphism of Riedtmann Quivers. Now $(i,\alpha)^+ = \{(i-1,\alpha+1),(i,\alpha-1)\}$ and $(i,\alpha)^- = \{(i,\alpha+1),(i+1,\alpha-1)\}$ so by 2.9 it is easily seen that $\pi'((i,\alpha)^-) = (\pi'(i,\alpha))^-$ and $\pi'((i,\alpha)^+) = (\pi'(i,\alpha))^+$. Since $\mathbb{Z}A_{q-1}$, Q_s are stable quivers (3.7, 3.3) τ and Ω^2 are defined on the full vertex sets of $\mathbb{Z}A_{q-1}$ and Q_s respectively which shows that π' is a covering. Finally, the identity $\pi'\circ\tau^e = \pi'$ induces the desired isomorphism. $\square$

We now extend $\mathbb{Z}A_{q-1}$ to a quiver Δ and construct a covering $\pi:\Delta \to Q$ such that $\pi|_{\mathbb{Z}A_{q-1}} = \pi'$. This construction is essentially due to Riedtmann and can be found in [R2].

Recall the permutation $\delta:I \to I$ as defined in 2.10 and the maps $\lambda,\lambda':I \to \{1,\dots,q-1\}$ in §2 (13). We extend the domain to the integers as follows. For $j \in \mathbb{Z}$ write $j = i+ae$ where $i \in I$, $a \in \mathbb{Z}$ are uniquely determined and define $\delta:\mathbb{Z} \to \mathbb{Z}$, $\lambda,\lambda'\,\mathbb{Z} \to \{1,\dots,q-1\}$ as follows:

$$\delta(j) = \delta(i) + ae$$

$$\lambda(j) = \lambda(i)\ ,\ \lambda'(j) = \lambda'(i)\ .$$

Now for each $j \in \mathbb{Z}$ define a vertex w_j and arrows as shown:

(2)
$$\begin{tikzcd}[column sep=small]
 & {} \arrow[rr, phantom] & & {} \arrow[dr, dashed] & \\
(\delta^{-1}(j)+1,\lambda'(j)) \arrow[ur, dashed] \arrow[dr, dashed] \arrow[rr, no head] & & w_j \arrow[rr] & & (\delta^{-1}(j),\lambda'(j)) \\
 & {} & & {} \arrow[ur, dashed] &
\end{tikzcd}$$

Let Δ be the directed graph with vertex sets $\Delta_0 = \{(i,\alpha)|i\in\mathbb{Z},\ \alpha = 1,\dots,q-1\} \cup \{w_j|j \in \mathbb{Z}\}$ and having arrows as in $\mathbb{Z}A_{q-1}$ together with $\{(\delta^{-1}(j)+1,\lambda'(j)) \to w_j\ ,\ w_j \to (\delta^{-1}(j),\lambda'(j)\ |\ j \in \mathbb{Z}\}$.

Let $\tau:\mathbb{Z}A_{q-1} \cap \Delta_0 \to \Delta_0$ be as defined for the quiver $\mathbb{Z}A_{q-1}$.

By 3.7 and (2) we see that $(\tau x)^+ = x^-$ for all $x \in \mathbb{Z}A_{q-1} \cap \Delta_0$ so Δ is a Riedtmann Quiver. Now define $\pi : \Delta \to Q$ to be the map such that $\pi|_{\mathbb{Z}A_{q-1}} = \pi'$ and $\pi(w_j) = W_j$.

3.9 <u>Theorem</u>.

The map $\pi : \Delta \to Q$ as defined above is a covering of Riedtmann Quivers.

<u>Proof</u>.

Since $\pi|_{\mathbb{Z}A_{q-1}} = \pi'$ is a covering, it remains (see (2)) to show that $\pi(x^-) = (\pi x)^-$, $\pi(x^+) = (\pi x)^+$ for $x \in \{W_j, (\sigma^{-1}(j)+1, \lambda'(j)), (\delta^{-1}(j), \lambda'(j)) | j \in \mathbb{Z}\}$. By regarding the projective mesh corresponding to the A-R sequence in 2.14 it is easy to see that:

(i) $\pi(w_j^-) = (\pi w_j)^- = W_j/\Sigma W_j$

$\pi(w_j^+) = (\pi w_j)^+ = RW_j$

(ii) $\pi((\delta^{-1}(j), \lambda'(j))^-) = (W_j/\Sigma W_j)^- = (RW_j)^+$

$= \pi((\delta^{-1}(j)+1, \lambda'(j))^+) = \{gT_{\delta^{-1}(j),\lambda'(j)+1}\, gT_{\delta^{-1}(j)+1,\lambda'(j)-1}\, W_j\}$

as required.

□

CHAPTER 4. Functions on the quiver Δ .

Throughout this chapter Δ, Q, π are as in 3.9.

Let $\tilde{\mathcal{A}}$ be an abelian group and let $(\Delta,\tilde{\mathcal{A}})$ denote the space of all maps $f:\Delta \to \tilde{\mathcal{A}}$ which we regard as an abelian group viz : $(f+g)(x) = f(x) + g(x)$, $f,g \in (\Delta,\tilde{\mathcal{A}})$.

4.1 Definition.

Let $F \subset (\Delta,\tilde{\mathcal{A}})$ be the set of all maps which factor through π . That is, all $f \in (\Delta,\tilde{\mathcal{A}})$ such that $f(x) = f(y)$ whenever $\pi(x) = \pi(y)$. For such an f there exists a unique map $f':Q \to \tilde{\mathcal{A}}$ such that

$$\begin{tikzcd} \Delta \arrow[r, "f"] \arrow[d, "\pi"'] & \tilde{\mathcal{A}} \\ Q \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes. It also follows that $f(\tau^e x) = f(x)$ where this makes sense. Clearly F is a subgroup of $(\Delta,\tilde{\mathcal{A}})$ $(F \leq (\Delta,\tilde{\mathcal{A}}))$ and we shall frequently identify F with the set of maps $f:Q \to \tilde{\mathcal{A}}$ which we denote by $(Q,\tilde{\mathcal{A}})$.

Say $f \in F$ is <u>additive</u> if for a vertex $x \in \Delta_0 \cap \mathbb{Z}A_{q-1}$,

(1) $$f(x) + f(\tau x) = \sum_{y \in x^-} f(y)$$

and denote the space of all such maps by F^+ , noting that $F^+ \le F$. (Since Δ is a covering for Q and, for all $U,V \in \mathrm{Ind}(\mathbb{B})$, $n_{U,V} \le 1$ (1) does correspond to the usual definition of additivity -[W] 101).

4.2 Lemma.

The subgroup F^+ consists of all $f \in F$ which satisfy, for all $i \in \mathbb{Z}$, $1 \le \alpha \le q-1$,

$$(2)\quad (a)\quad f(i,\alpha) = f(i,\alpha-1)+f(i+1,\alpha-1)-f(i+1,\alpha-2)-f(W_{\delta(i)}) \quad \text{if } \alpha = \lambda'(\delta(i)) + 1$$

$$= f(i,\alpha-1) + f(i+1,\alpha-1) - f(i+1,\alpha-2) \quad \text{otherwise}$$

and for all $i \in \mathbb{Z}$,

$$(b)\quad f(i,q-1) = f(i+1,q-2)-f(i+1,q-1)+f(W_{\delta(i)}) \quad \text{if } \lambda'(\delta(i)) = q-1$$

$$= f(i+1,q-2) - f(i+1,q-1) \quad \text{otherwise.}$$

(We will make the convention that $f(i,\alpha) = 0$ whenever $i \in \mathbb{Z}$ and $\alpha \le 0$ or $\alpha \ge q$ for all $f \in (\Delta,\tilde{\Lambda})$. In particular (b) is case (a) with $\alpha = q$.)

Proof.

Recall that for each $i \in \mathbb{Z}$, $1 \leq \alpha \leq q-1$ there is a subquiver

$$\begin{tikzcd}
& (i,\alpha+1) \arrow[dr] & \\
(i+1,\alpha) \arrow[ur] \arrow[dr] \arrow[rr, dashed, "w_{\delta(i)}"] & & (i,\alpha) \\
& (i+1,\alpha-1) \arrow[ur] &
\end{tikzcd}$$

which contains $w_{\delta(i)}$ iff $\alpha = \lambda'(\delta(i))$. The result follows.

□

4.3 Definitions.

Let $F^{++} \leq F$ be the set of all maps satisfying (2). Clearly $F^{+} \leq F^{++} \leq F$ and any $f \in F^{++}$ is completely determined by the e-tuples

$$\underline{S}_f = (f(0,1),\ldots\ldots,f(e-1,1))$$

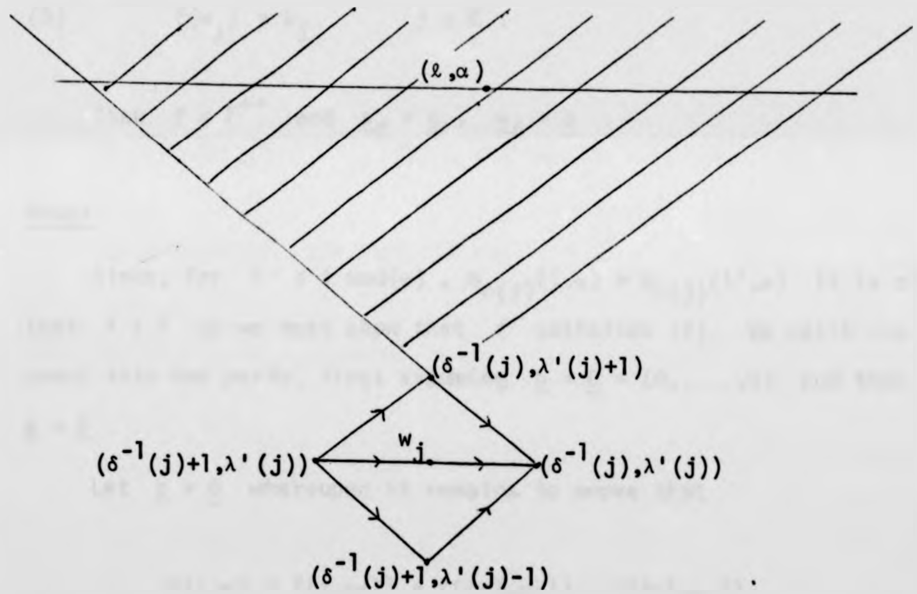
$$\underline{W}_f = (f(W_0),\ldots\ldots,f(W_{e-1})) \ .$$

For any $j \in \mathbb{Z}$ define a map

$$\text{(3)} \qquad \begin{aligned} m_{c(j)} &: \Delta \to \mathbb{Z} \\ (i,\alpha) &\to |[\delta^{-1}(j)-\alpha+\lambda'(j)+1,\delta^{-1}(j)] \cap i+e\mathbb{Z}| \\ w_i &\to 0 \end{aligned}$$

where $[a,b] = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$.

For $j' \equiv j \bmod(e)$ it is clear that $m_{c(j)} = m_{c(j')}$. In fact $m_{c(j)}(i,\alpha)$ is the number of vertices (ℓ,α) , such that $\ell \in i+e\mathbb{Z}$, which lie in the shaded area of the diagram below.



4.4 Theorem.

Given $\underline{a} = (a_0,\ldots\ldots,a_{e-1})$, $\underline{b} = (b_0,\ldots\ldots,b_{e-1}) \in \mathcal{X}^e$, then for $n \in \mathbb{Z}$, define $a_n, b_n \in \mathcal{X}$ such that $a_n = a_i$, $b_n = b_i$ for $n \equiv i \bmod(e)$, $i \in I$. Let $f \in (\Delta,\mathcal{X})$ be given by:

(4) $$f(i,\alpha) = \sum_{n=i}^{i+\alpha-1} a_n - \sum_{j \in I} m_{c(j)}(i,\alpha).b_j \quad , \quad (i,\alpha) \in \mathbb{Z}A_{q-1}$$

and

(5) $$f(w_j) = b_j \qquad j \in \mathbb{Z} \ .$$

Then $f \in F^{++}$ and $\underline{s}_f = \underline{a}$, $\underline{w}_f = \underline{b}$.

<u>Proof</u>.

Since, for $i' \equiv i \bmod(e)$, $m_{c(j)}(i,\alpha) = m_{c(j)}(i',\alpha)$ it is clear that $f \in F$ so we must show that f satisfies (2). We split the proof into two parts, first assuming $\underline{b} = \underline{0} = (0,\ldots.,0)$ and then $\underline{a} = \underline{0}$.

Let $\underline{b} = \underline{0}$ whereupon it remains to prove that

$$f(i,\alpha) = f(i,\alpha-1) + f(i+1,\alpha-1) - f(i+1,\alpha-2)$$

for all $(i,\alpha) \in \mathbb{Z}A_{q-1}$. Now the right-hand side is equal to

$$\sum_{n=i}^{i+\alpha-2} a_n + \sum_{n=i+1}^{i+\alpha-1} a_n - \sum_{n=i+1}^{i+\alpha-2} a_n$$

$$= \sum_{n=i}^{i+\alpha-2} a_n + a_{i+\alpha-1} = f(i,\alpha) \ ,$$

as required. We now assume $\underline{a} = \underline{0}$.

(Notation: To avoid confusion with the permutation δ , define $\varepsilon, \varepsilon_I : \mathbb{Z} \times \mathbb{Z} \to \{0,1\}$ by:

$$\varepsilon(i,j) = 1 \quad \text{iff} \quad i = j$$

$$\varepsilon_I(i,j) = 1 \quad \text{iff} \quad i \equiv j \bmod(e) \quad .)$$

It remains to show that for any $\underline{b} \in \mathbb{\lambda}^e$:

$$(6) \qquad - \sum_{j\in I} m_{c(j)}(i,\alpha).b_j =$$

$$- \sum_{j\in I} [m_{c(j)}(i,\alpha-1)+m_{c(j)}(i+1,\alpha-1)-m_{c(j)}(i+1,\alpha-2)].b_j$$

$$- \varepsilon(\alpha,\lambda'(\delta(i))+1).b_{\delta(i)} \ .$$

In fact we prove the stronger statement that for each $j \in I$:

$$(7) \qquad m_{c(j)}(i,\alpha) = m_{c(j)}(i,\alpha-1)+m_{c(j)}(i+1,\alpha-1)$$

$$- m_{c(j)}(i+1,\alpha-2) + \varepsilon(\alpha,\lambda'(j)+1).\varepsilon_I(\delta(i),j) \quad .$$

$$= \sum_{n=i}^{i+\alpha-2} a_n + a_{i+\alpha-1} = f(i,\alpha) .$$

as required. We now assume $\underline{a} = \underline{0}$.

(<u>Notation</u>: To avoid confusion with the permutation δ , define $\varepsilon, \varepsilon_I : \mathbb{Z} \times \mathbb{Z} \to \{0,1\}$ by:

$$\varepsilon(i,j) = 1 \quad \text{iff} \quad i = j$$

$$\varepsilon_I(i,j) = 1 \quad \text{iff} \quad i \equiv j \bmod(e) \quad .)$$

It remains to show that for any $\underline{b} \in \mathcal{R}^e$:

(6) $$- \sum_{j \in I} m_{c(j)}(i,\alpha).b_j =$$

$$- \sum_{j \in I} [m_{c(j)}(i,\alpha-1)+m_{c(j)}(i+1,\alpha-1)-m_{c(j)}(i+1,\alpha-2)].b_j$$

$$- \varepsilon(\alpha,\lambda'(\delta(i))+1).b_{\delta(i)} .$$

In fact we prove the stronger statement that for each $j \in I$:

(7) $$m_{c(j)}(i,\alpha) = m_{c(j)}(i,\alpha-1)+m_{c(j)}(i+1,\alpha-1)$$

$$- m_{c(j)}(i+1,\alpha-2) + \varepsilon(\alpha,\lambda'(j)+1).\varepsilon_I(\delta(i),j) \quad .$$

We divide the proof into three cases.

(a) <u>$\alpha \le \lambda'(j)$</u> : In this case $m_{c(j)}(i,\beta) = 0$ for all $\beta \le \alpha$ and so (7) is clearly satisfied.

(b) <u>$\alpha = \lambda'(j)+1$</u> : By part (a) it remains to show that $m_{c(j)}(i,\lambda'(j)+1) = \varepsilon_I(\delta(i),j)$. Now,

$$\begin{aligned} m_{c(j)}(i,\lambda'(j)+1) &= |[\delta^{-1}(j)-(\lambda'(j)+1)+\lambda'(j)+1,\delta^{-1}(j)] \cap i+e\mathbb{Z}| \\ &= |\{\delta^{-1}(j)\} \cap i+e\mathbb{Z}| \\ &= \varepsilon_I(\delta(i),j) \end{aligned}$$

as required.

(c) <u>$\alpha > \lambda'(j)+1$</u> : Let $a = \delta^{-1}(j)-\alpha + \lambda'(j)+1$, $b = \delta^{-1}(j)$.

Then:

$$\begin{aligned} & m_{c(j)}(i,\alpha-1) + m_{c(j)}(i+1,\alpha-1) - m_{c(j)}(i+1,\alpha-2) \\ & \qquad + \varepsilon(\alpha,\lambda'(j)+1).\varepsilon_I(\delta(i),j) \\ &= |[a+1,b]\cap i+e\mathbb{Z}| + |[a+1,b]\cap i+1+e\mathbb{Z}| - |[a+2,b]\cap i+1+e\mathbb{Z}| \\ &= |[a+1,b]\cap i+e\mathbb{Z}| + |[a,b-1]\cap i+e\mathbb{Z}| - |[a+1,b-1]\cap i+e\mathbb{Z}| \\ &= |[a+1,b]\cap i+e\mathbb{Z}| + |\{a\} \cap i+e\mathbb{Z}| \\ &= |[a,b] \cap i+e\mathbb{Z}| \\ &= m_{c(j)}(i,\alpha) \end{aligned}$$

as required proving that $f \in F^{++}$.

Finally, (5) implies $\underline{w}_f = \underline{b}$ and since $m_{c(j)}(i,1) = 0$ for all $j \in I$, $i \in \mathbb{Z}$, (4) implies that $f(i,1) = a_i$ and so $\underline{s}_f = \underline{a}$. □

4.5 Corollary.

Take any $f' \in F^{++}$ and let $\underline{a} = s_{f'}$, $\underline{b} = w_{f'}$. Then $f' = f$ where f is defined as in 4.4. □

Define a new map

$$\begin{aligned} (8) \qquad & m_j : \Delta \to \mathbb{Z} \\ & (i,\alpha) \to |[j-\alpha+1,j] \cap i+e\,\mathbb{Z}| \\ & w_i \to 0 \ . \end{aligned}$$

4.6 Lemma.

Let $\{a_i \mid i \in \mathbb{Z}\}$ be a set of elements of $\mathcal{K}$ such that $a_i = a_j$ for $i \equiv j \mod(e)$. Then for any $(i,\alpha) \in \mathbb{Z}A_{q-1}$:

$$(9) \qquad \sum_{n=i}^{i+\alpha-1} a_n = \sum_{j \in I} m_j(i,\alpha).a_j \ .$$

Proof.

We use induction. Since $m_j(i,1) = \varepsilon_I(i,j)$ (9) holds for $\alpha = 1$.

Now,

$$\sum_{n=i}^{i+\alpha-1} a_n = \sum_{n=i}^{i+\alpha-2} a_n + a_{i+\alpha-1}$$

(10) $$= \sum_{j\in I} m_j(i,\alpha-1).a_j + a_{i+\alpha-1}$$

by induction. Comparing (9) and (10) we need to show that $m_j(i,\alpha) = m_j(i,\alpha-1) + \varepsilon_I(j,i+\alpha-1)$ which is immediate from the definition (8). □

4.7 Definition.

Given $F \in (Q,\mathcal{A})$, say F is additive if, for $M \in \mathrm{Ind}_p(\mathbb{B})$,

$$F(M) + F(\tau M) = \sum_{E\in M^-} n_{E,M} F(E) = \sum_{E\in M^-} F(E)$$

by the remarks following (1).

4.8 Theorem.

Let $F \in (Q,\mathcal{A})$ be additive and let $M \cong gT_{i,\alpha} \in \mathrm{Ind}_p(\mathbb{B})$. Then

$$F(M) = \sum_{j\in I} \{m_j(i,\alpha).F(gS_j) - m_{c(j)}(i,\alpha).F(W_j)\} .$$

Proof.

Let $f = F\circ\pi \in (\Delta,\mathcal{A})$. Then $f \in F^+ \leq F^{++}$ since f is additive.

Therefore

$$F(M) = f(i,\alpha) = \sum_{n=i}^{i+\alpha-1} f(n,1) - \sum_{j\in I} m_{c(j)}(i,\alpha).f(w_j)$$

by 4.4, 4.5

$$= \sum_{j\in I} m_j(i,\alpha).f(j,1) - \sum_{j\in I} m_{c(j)}(i,\alpha).f(w_j)$$

by 4.6

$$= \sum_{j\in I} \{m_j(i,\alpha).F(gS_j) - m_{c(j)}(i,\alpha)F(W_j)\}$$

as required. □

We again note that the distinction between F and $(Q,\tilde{\lambda})$ will often be dropped and F , $f = F\circ\pi$ will be identified.

As a corollary we shall obtain a description of the composition factors for the non-projective indecomposable modules. Let $\mathbb{Z}(\mathbb{B})$ be the free abelian group with generators corresponding to $M \in \mathrm{Ind}(\mathbb{B})$ and define:

(11) $$S(\mathbb{B}) = \mathbb{Z}\mathrm{sp.}\{M-M'-M'' \mid 0 \to M' \to M \to M'' \to 0 \text{ is a s.e.s.}\}$$

The factor group $\mathbb{Z}(\mathbb{B})/S(\mathbb{B}) : = G_0(\mathbb{B})$ is the <u>Grothendieck Group</u> for $\mathbb{B}$

and for $M \in \mathrm{mod}\,\mathbb{B}$ let $[M] = M + S(\mathbb{B}) \in G_0(\mathbb{B})$. One can show ([CR] p.405) that $G_0(\mathbb{B}) = \underset{i \in I}{} \mathbb{Z}[V_i]$ and $[M] = \sum_{i \in I} r_i(M).[V_i]$ where $r_i(M)$ is the multiplicity of V_i as a composition factor of M .

4.9 Corollary.

Let $M \cong gT_{i,\alpha} \in \mathrm{Ind}_P(\mathbb{B})$. Then

$$[M] = \sum_{j \in I} \{m_j(i,\alpha).[gS_j] - m_{c(j)}(i,\alpha)[W_j]\} \quad .$$

Proof.

Let $F \in (Q, G_0(\mathbb{B}))$ be given by $F(M) = [M]$. By definition F is additive and the result follows by 4.8 □

To fully determine $[M]$ we need to know $[gS_j]$, $[W_j]$ for each $j \in I$. This has been done by Peacock in [P] but for the sake of completeness we give a proof here.

Recalling the Brauer Tree Γ as defined in Chapter 2 we follow Peacock's notation in [P] by defining $\rho: I \to I$, $\rho(i) \equiv \delta^{-1}(i)+1 \bmod (e)$. For any permutation $\gamma: I \to I$, $i \in I$ let $[\gamma(i)] = \{\gamma^a(i) | a \in \mathbb{Z}\}$. The following is true.

4.10 Proposition.

For all $i,j \in I$:

(a) $[gS_i] = [gS_j]$ iff $[\delta(i)] = [\delta(j)]$

(b) $[\Omega gS_i] = [\Omega gS_j]$ iff $[\rho(i)] = [\rho(j)]$.

Proof.

Recall that each vertex of the Brauer Tree can be labelled by some modular character P_{2i} or P_{2i+1} , $i \in I$. In fact, Green [GR2] shows that each vertex can be labelled in turn, at least once, in a 'walk' around Γ

$$P_0, P_1, \ldots\ldots\ldots, P_{2e-2}, P_{2e-1}, P_0 \quad .$$

By §2 (12) we see that the edges and vertices can be labelled

(12)
$$\begin{tikzcd}
P_{2i-1} \arrow[r, dash, "\eta_i"] \arrow[d, dash, dotted] & P_{2i} \arrow[d, dash, dotted] \\
P_{2j+1} \arrow[r, dash, "\eta_{\delta(j)}"] & P_{2j}
\end{tikzcd}$$

where $j = \delta^{-1}(i)$, with the resulting identification of characters indicated by the dotted lines.

Now there exists an isomorphism ([CR] 425) between $G_0(\mathbb{B})$ and the ring of modular (Brauer) characters given by $[M] \to \lambda(M)$ where $\lambda(M)$ is the modular character afforded by $M \in \text{mod}\,\mathbb{B}$. Together with (12) we deduce:

(13) $\qquad [gS_i] = [gS_{\delta^{-1}(i)}] \qquad\qquad i \in I$

(14) $\qquad [\Omega gS_{i-1}] = [\Omega gS_{\delta^{-1}(i)}] \quad . \quad i \in I$.

Part (a) follows from (13) whilst $[\Omega gS_{\delta(i-1)}] = [\Omega gS_i]$ by (14) and, since $\rho^{-1}(i) = \delta(i-1)$, part (b) follows.

□

We now have a one to one correspondence between $\{X_1,\ldots,X_{e+1}\}$ and $\{[\delta(i)],[\rho(i)] \mid i \in I\}$, X_j corresponding to $[\delta(i)]$ $([\rho(i)])$ iff $X_j = P_{2i}$ (P_{2i+1}) . We can now express 2.11 in terms of the permutation δ .

4.11 <u>Proposition</u>.

Let $C = (c_{ij})$ be the Cartan matrix for $\mathbb{B}$. Then

(a) $(i \neq j)$ $C_{ij} = 0$ iff $[\delta(i)] \neq [\delta(j)]$ and $[\rho(i)] \neq [\rho(j)]$

$= 1$ iff $[\delta(i)] = [\delta(j)]$ and $P_{2i} \neq X_{e+1}$

or $[\rho(i)] = [\rho(j)]$ and $P_{2i+1} \neq X_{e+1}$

$= (q-1)/e$ iff $[\delta(i)] = [\delta(j)]$ and $P_{2i} = X_{e+1}$

or $[\rho(i)] = [\rho(j)]$ and $P_{2i+1} = X_{e+1}$.

(b) $C_{ii} = 2$ iff $P_{2i} \neq X_{e+1} \neq P_{2i+1}$

$= (q-1)/e + 1$ iff $P_{2i} = X_{e+1}$ or $P_{2i+1} = X_{e+1}$. $\square$

From the s.e.s. in 2.10(c) we obtain the equations:

(15) $[W_{\delta(i)}] = [gS_i] + [\Omega gS_i]$;

(16) $[W_{i+1}] = [gS_{i+1}] + [\Omega gS_i]$.

4.12 <u>Proposition.</u>

For all $i \in I$ define $\alpha_i \in (Q\mathbb{Z})$ by:

$\alpha_i(M) = \dim_k(W_i, M)$.

Then for all $i,j \in I$

(a) $\alpha_i(gS_j) = 0$ iff $[\delta(i)] \neq [\delta(j)]$

$= 1$ iff $[\delta(i)] = [\delta(j)]$, $P_{2i} \neq X_{e+1}$

$= (q-1)/e$ iff $[\delta(i)] = [\delta(j)]$, $P_{2i} = X_{e+1}$.

(b) $\alpha_i(\Omega g S_j) = 0$ iff $[\rho(i)] \neq [\rho(j)]$

$= 1$ iff $[\rho(i)] = [\rho(j)]$, $P_{2i+1} \neq X_{e+1}$

$= (q-1)/e$ iff $[\rho(i)] = [\rho(j)]$, $P_{2i+1} = X_{e+1}$.

Proof.

We note that α_i is additive, $\alpha_i(W_j) = C_{ij}$ and that, for $M,N \in \text{mod } \mathbb{B}$, $[M] = [N]$ iff $\alpha_i(M) = \alpha_i(N)$ for all $i \in I$. The result now follows from 4.11 and (15), (16).

□

From the above and 4.9, 4.11 we have now determined the composition factors for each $M \in \text{Ind}_p(\mathbb{B})$. Notice that 4.9 relies on knowing $m_{c(j)} \in F$ which in turn relies on knowing $\lambda(j)$, for each $j \in I$. We calculate $\lambda(j)$ using 4.8 and the Brauer Tree. Since α_i is additive, by 4.8:

$$\alpha_i(\Omega g S_\ell) = \alpha_i(g T_{\ell+1,q-1})$$

$$= \sum_{t \in I} m_t(\ell+1,q-1).\alpha_i(gS_t) - \sum_{j \in I} m_{c(j)}(\ell+1,q-1).\alpha_i(W_j)$$

$$= (q-1)/e \sum_{t \in I} \alpha_i(gS_t) - \sum_{j \in I} m_{c(j)}(\ell+1,q-1).C_{ij}$$

since $m_t(\ell+1,q-1) = |[t+2-q,t] \cap \ell+1+e\mathbb{Z}| = (q-1)/e$ for all $\ell,t \in I$.

Therefore,

$$(17) \qquad \sum_{j\in I} c_{ij}.m_{c(j)}(\ell+1,q-1)$$

$$= (q-1)/e \sum_{t\in I} \alpha_i(gS_t) - \alpha_i(\Omega gS_\ell)$$

for all $i,\ell \in I$. Define $e{\times}e$ matrices $M' = (m'_{ij})$, $B = (b_{ij})$ by

$$m'_{i,j} = m_{c(i)}(j+1,q-1)$$

$$b_{ij} = (q-1)/e \sum_{t\in I} \alpha_i(gS_t) - \alpha_i(\Omega gS_\ell) .$$

By (17) we have

$$B = C.M'$$

and, by 4.11, 4.12, B and C are determined by the Brauer Tree. Since C is non-singular ([CR]) we can solve for M' . That is

$$(18) \qquad M' = C^{-1}.B \ .$$

Furthermore,

$$\sum_{j\in I} m'_{i,j} = \sum_{j\in I} m_{c(i)}(j+1,q-1)$$

$$= \sum_{j\in I} \left|[\delta^{-1}(i)-(q-1) + \lambda'(i)+1, \delta^{-1}(i)] \cap j+1+e\mathbb{Z}\right|$$

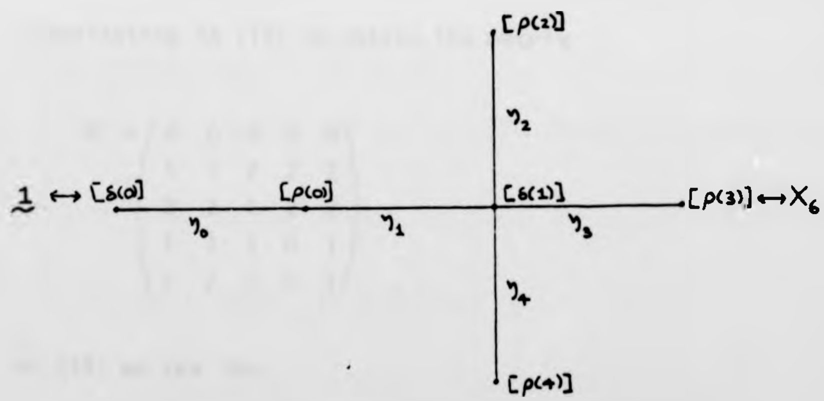
$$= \left|[\delta^{-1}(i) + 2 - \lambda(i), \delta^{-1}(i)]\right|$$

$$= \lambda(i)-1 \ .$$

Therefore,

(19) $$\sum_{j\in I} m'_{i,j} + 1 = \lambda(i) \ .$$

As an example we look at the principal 11-block for the Mathieu Group M_{11} . James ([J]) gives the Brauer Tree together with the exceptional vertex X_{e+1} and the trivial character $\underset{\sim}{1}$ as shown:



The 11-defect group is cyclic of order 11 and the inertial index $e = 5$. We may associate the trivial character with $[\delta(0)]$ where the permutation δ, ρ are:

$$\delta = (0)(1\ 2\ 3\ 4)$$

$$\rho = (01)(2)(3)(4)$$

whence the exceptional vertex X_6 is associated with $[\rho(3)]$. A few calculations give us the matrices B and C .

$$B = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 \\ 7 & 7 & 8 & 8 & 8 \\ 8 & 8 & 7 & 8 & 8 \\ 8 & 8 & 8 & 6 & 8 \\ 8 & 8 & 8 & 8 & 7 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix} .$$

Substituting in (18) we obtain the matrix

$$M' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 & 1 \end{pmatrix} .$$

By (19) we see that

$$(0) = 1,\ \lambda(1) = 9,\ \lambda(2) = 10,\ \lambda(3) = 5,\ \lambda(4) = 10$$

so that

$$fV_0 \cong T_{0,1}\ ,\ fV_1 \cong T_{1,q}\ ,\ fV_2 \cong T_{2,10}\ ,\ fV_3 \cong T_{3,5}$$

and $fV_4 \cong T_{4,10}$.

In the second half of the chapter we look at maps which are not necessarily additive and see if we can present them in the same way as we did for $f \in F^{++}$ in 4.8. It turns out that a very similar result holds true for these non-additive maps.

From now on let $\tilde{k} = \mathbb{Z}$.

Let Fun(B) (Fun*(B)) denote the category of contravariant (covariant) k-linear functors $F:\text{mod } B \to \text{mod } k$. To each $M \in \text{mod } \mathbb{B}$ we will associate an object S_M $({}_MS) \in$ Fun(B) (Fun*(B)) , the existence of such functors and further details being found in Gabriel's survey [Ga].

4.13 <u>Definitions</u>. ([Ga] §1)

Let $M \in \text{mod } \mathbb{B}$ and let $(\ ,M) \in \text{Fun}(\mathbb{B})$ be defined by

$$(\ ,M)(N) = (N,M)\ .$$

Let $R(\ ,M) \in \text{Fun}(\mathbb{B})$ be such that

$$R(\ ,M)(N) = R(N,M)$$

as defined in §1,(1). Now define

$$S_M = (\ ,M)/R(\ ,M) \in \mathrm{Fun}(\mathbb{B}) \ .$$

It follows from §1,(1) that, for $M,N \in \mathrm{Ind}(\mathbb{B})$

$$S_M(N) \cong \begin{cases} k & \text{if } M \cong N \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for $N \in \mathrm{mod}\,\mathbb{B}$, $S_M(N) \not\cong 0$ iff $M|N$ and then $\dim_k S_M(N)$ is the number of times M occurs as a direct summand of N . For $M \in \mathrm{mod}\,\mathbb{B}$, $M \cong \coprod_{j=1}^{n} M_j$, $M_j \in \mathrm{Ind}(\mathbb{B})$:

$$S_M = S_{M_1} \oplus \ldots. \oplus S_{M_n} \ .$$

In a similar way we can define ${}_MS = (M, \ )/R(M, \ ) \in \mathrm{Fun}^*(\mathbb{B})$.

For each $M \in \mathrm{Ind}(\mathbb{B})$ define $\gamma_M \in F$ by

$$\gamma_M(N) = \dim_k S_M(N) = \dim_k {}_MS(N) \ .$$

Let $A(M) : 0 \to \tau M \to E(M) \to M \to 0$ and $A(\tau^{-1}M) : 0 \to M \to E(\tau^{-1}M) \to \tau^{-1}M \to 0$ be the A-R sequences stopping and starting at M . Note also that for $M \in \mathrm{mod}(\mathbb{B})$, $(\ ,M)$ and $(M, \ )$ are projective objects in $\mathrm{Fun}(\mathbb{B})$ and $\mathrm{Fun}^*(\mathbb{B})$ respectively. The following is due to Auslander and Reiten.

4.14 <u>Theorem</u>. ([Ga] §1)

For each $M \in \mathrm{Ind}_P(\mathbb{B})$:

(a) S_M admits the m.p.p.

$$0 \to (\ ,\tau M) \to (\ ,E(M)) \to (\ ,M) \to S_M \to 0$$

in $\mathrm{Fun}(\mathbb{B})$;

(b) ${}_MS$ admits the m.p.p.

$$0 \to (\tau^{-1}M,\ ) \to (E(\tau^{-1}M),\ ) \to (M,\ ) \to {}_MS \to 0$$

in $\mathrm{Fun}^*(\mathbb{B})$.

For each projective module $P \in \mathrm{Ind}(\mathbb{B})$:

(c) S_P admits the m.p.p.

$$0 \to (\ ,RP) \to (\ ,P) \to S_P \to 0$$

in $\mathrm{Fun}(\mathbb{B})$;

(d) ${}_PS$ admits the m.p.p.

$$0 \to (P/\Sigma P,\ ) \to (P,\ ) \to {}_PS \to 0$$

in $\mathrm{Fun}^*(\mathbb{B})$. □

4.15 <u>Definition</u>.

For $X \in \mathrm{Ind}(\mathbb{B})$ define $\alpha_X, \beta_X \in F$ by

$$\alpha_X(M) = \dim_k(X,M)$$

$$\beta_X(M) = \dim_k(M,X)$$

where $M \in \mathrm{mod}\,\mathbb{B}$. We can extend this definition by $\mathbb{Z}$-linearity and consider

$$\alpha_X, \beta_X : \mathbb{Z}(\mathbb{B}) \to \mathbb{Z}$$

$$\alpha_X(y) = \sum_{Y \in \mathrm{Ind}(\mathbb{B})} a_Y \alpha_X(Y)$$

$$\beta_X(y) = \sum_{Y \in \mathrm{Ind}(\mathbb{B})} a_Y \beta_X(Y)$$

where $y = \sum_Y a_Y Y \in \mathbb{Z}(\mathbb{B})$. By $\mathbb{Z}$-linearity again, for

$z = \sum_{Z \in \mathrm{Ind}(\mathbb{B})} b_Z Z \in \mathbb{Z}(\mathbb{B})$, define

$$\alpha_z : \mathbb{Z}(\mathbb{B}) \to \mathbb{Z}$$

(20)
$$\alpha_z(y) = \sum_Z b_Z \alpha_Z(y) = \sum_Z \sum_Y a_Y b_Z \alpha_Z(Y) \quad .$$

Define β_z similarly.

For $M \in \mathrm{Ind}(\mathbb{B})$, define $A_M, B_M \in \mathbb{Z}(\mathbb{B})$ by:

$$\begin{aligned} A_M &= M + \tau M - E(M) && M \in \mathrm{Ind}_p(\mathbb{B}) \\ &= M - RM && M \text{ projective} \\ B_M &= M + \tau^{-1}M - E(\tau^{-1}(M) && M \in \mathrm{Ind}_p(\mathbb{B}) \\ &= M - M/\Sigma M && M \text{ projective.} \end{aligned}$$

The following is a direct consequence of 4.14.

4.16 Proposition. ([BP])

Let $M, X \in \mathrm{Ind}(\mathbb{B})$. Then:

(a) $\alpha_X(A_M) = \beta_X(B_M) = \gamma_M(X)$

(b) $\{\alpha_X | X \in \mathrm{Ind}(\mathbb{B})\}$, $\{\beta_X | X \in \mathrm{Ind}(\mathbb{B})\}$ are both $\mathbb{Z}$-bases for F .

((b) is false if $\mathbb{B}$ is of infinite representation type.)

Proof.

(a) Let $M \in \mathrm{Ind}_P(\mathbb{B})$, $X \in \mathrm{Ind}(\mathbb{B})$. Then:

$$\begin{aligned}\alpha_X(A_M) &= \alpha_X(M) + \alpha_X(\tau M) + \alpha_X(E(M)) \\ &= \dim_k(X,M) + \dim_k(X,\tau M) + \dim_k(X,E(M)) \\ &= \dim_k S_M(X) \qquad \text{by 4.14 (a)} \\ &= \gamma_M(X) .\end{aligned}$$

A similar proof, using 4.14(b), for M projective shows that $\alpha_X(A_M) = \gamma_M(X)$. Again, similar methods and 4.14(c),(d) show that $\beta_X(B_M) = \gamma_M(X)$.

(b) By part (a), for $M \in \mathrm{Ind}_P(\mathbb{B})$:

$$\gamma_M = \beta_M + \beta_{\tau M} - \beta_{E(M)} = \beta_M + \beta_{\tau M} - \sum_{N \in M^-} \beta_N$$

whilst for M projective,

$$\gamma_M = \beta_M - \beta_{RM} .$$

In particular we see that:

$$\mathbb{Z}\mathrm{sp}\{\beta_X | X \in \mathrm{Ind}(\mathbb{B})\} \geq \mathbb{Z}\mathrm{sp}\{\gamma_X | X \in \mathrm{Ind}(\mathbb{B})\} ,$$

the latter being the standard dual basis for F , and so $\mathbb{Z}\mathrm{sp}\{\beta_X | X \in \mathrm{Ind}(\mathbb{B})\} = F$ as required. A similar argument works for the α_X case. Part (a) shows that each spanning set is linearly independent and the proof is complete. □

Recalling (20) we deduce that any $f \in F$ can be written as α_y or β_z for unique elements $y,z \in \mathbb{Z}(\mathbb{B})$. Let

$$A(\mathbb{B}) = \sum^{\oplus}_{X \in \mathrm{Ind}_p(\mathbb{B})} \mathbb{Z}A_X = \sum^{\oplus}_{X \in \mathrm{Ind}_p(\mathbb{B})} \mathbb{Z}B_X \leq \mathbb{Z}(\mathbb{B}) ,$$

the second equality holding since $A_M = B_{\tau M}$ for all $M \in \mathrm{Ind}_p(\mathbb{B})$. In addition let

$$\alpha(\mathbb{P}) = \mathbb{Z}\mathrm{sp}\{\alpha_P | P \in \mathrm{Ind}(\mathbb{B}) , \ P \text{ projective}\}$$

$$\beta(\mathbb{P}) = \mathbb{Z}\mathrm{sp}\{\beta_P | P \in \mathrm{Ind}(\mathbb{B}) , \ P \text{ projective}\} .$$

4.17 Proposition.

(a) $\alpha(\mathbb{P}) = \beta(\mathbb{P}) = F^{+}$

(b) $F^{++} = \alpha(\mathbb{P}) \oplus \sum_{i\in I}^{\oplus} \mathbb{Z}\alpha_{\Omega g S_i} = \beta(\mathbb{P}) \oplus \sum_{i\in I}^{\oplus} \mathbb{Z}\beta_{\Omega g S_i}$

(c) $\mathbb{Z}(\mathbb{B}) = \sum_{X\in \mathrm{Ind}(\mathbb{B})}^{\oplus} \mathbb{Z}A_X = \sum_{X\in \mathrm{Ind}(\mathbb{B})}^{\oplus} \mathbb{Z}B_X$

(d) $A(\mathbb{B}) = S(\mathbb{B})$ as defined in (11) and in particular,

$\mathbb{Z}(\mathbb{B})/A(\mathbb{B}) = G_0(\mathbb{B})$.

Part (d) says that any s.e.s. is a 'linear combination' of A-R sequences. This has been proved in a more general context by M. Butler in [Bu2].

Proof.

(a) Since, for each $i \in I$, $(W_i, \ )$ and $(\ ,W_i)$ are exact functors it follows that $\alpha(\mathbb{P})$, $\beta(\mathbb{P}) \leq F^{+}$. Suppose $\alpha(\mathbb{P}) < F^{+}$ so there exists $f = \alpha_y \in F^{+}$ such that $y = \sum_{Y} a_Y Y$ and $a_M \neq 0$ for some $M \in \mathrm{Ind}_P(\mathbb{B})$. Since f is additive,

$$0 = f(A_M) = \sum_{Y} a_Y \alpha_Y(A_M) = a_M \quad \text{by 4.16(a),}$$

a contradiction. The proof for $\beta(\mathbb{P}) = F^{+}$ is similar.

(b) By part (a) $\alpha(\mathbb{P}) = F^{+} \leq F^{++}$ so we must show that for $X \in \mathrm{Ind}_P(\mathbb{B})$, $\alpha_X \in F^{++}$ iff $X \cong \Omega g S_i$, $i \in I$. Assume $X \cong \Omega g S_{j-1} \cong g T_{j,q-1}$ and let $f_X = \alpha_X \circ \pi$. It follows from 4.16(a) that for all $i \in I$, $\alpha = 1,\ldots,q-2$:

$$f_X(i,\alpha) + f_X(\tau(i,\alpha)) = \sum_{x \in (i,\alpha)^-} f_X(x)$$

which is the same as saying:

$$f_X(i,\alpha) + f_X(i+1,\alpha) = f_X(i,\alpha+1) + f_X(i+1,\alpha-1)$$
$$+ \varepsilon(\alpha,\lambda'(\delta(i)) + 1).f_X(W_{\delta(i)}) \quad .$$

But this is just condition (2) in 4.2 so $f_X \in F^{++}$. By the identification $\alpha_X \equiv f_X$, $\alpha_X \in F^{++}$ as required.

For $X \not\cong \Omega g S_i$ suppose $X \cong g T_{j,\alpha}$, some $j \in I$, $\alpha = 1,\ldots,q-2$. It follows from 4.16(a) that:

$$f_X(j,\alpha) + f_X(\tau(j,\alpha)) = \sum_{x \in (j,\alpha)} f_X(x) + 1$$

which implies that $f_X \notin F^{++}$.

(c) Let $K = \sum^{\oplus}_{X \in \mathrm{Ind}(\mathbb{B})} \mathbb{Z} A_X$. By 4.16(a) the A_X's are linearly independent and so $\mathrm{rank}(\mathbb{Z}(\mathbb{B})) = \mathrm{rank}(K) = q.e$ implying that $\mathbb{Z}(\mathbb{B})/K$ is finite. Suppose there exists $y \in \mathbb{Z}(\mathbb{B})\backslash K$. Then there exists $0 \neq n \in \mathbb{N}$ such that $n.y \in K$ implying that $n.y = \sum_{Y} a_Y A_Y$, $a_Y \in \mathbb{Z}$. For $M \in \mathrm{Ind}(\mathbb{B})$,

$$n.\alpha_M(y) = \alpha_M(ny) = \alpha_M(\sum_{Y} a_Y A_Y) = a_M$$

which implies that $n|a_M$ for all $M \in \mathrm{Ind}(\mathbb{B})$. Therefore:

$$y = \sum_{Y}(n^{-1}a_Y)A_Y \in K$$

which is a contradiction.

Similarly $\mathbb{Z}(\mathbb{B}) = \sum_{X}^{\oplus} \mathbb{Z}B_X$.

(d) First notice that $A(\mathbb{B}) \leq S(\mathbb{B})$ since $A_M \in S(\mathbb{B})$ for all $M \in \mathrm{Ind}_P(\mathbb{B})$. Since $G_0(\mathbb{B}) \cong \coprod_{i\in I} \mathbb{Z}[V_i]$, $\mathrm{rank}(G_0(\mathbb{B})) = e$ and so:

$$\begin{aligned}\mathrm{rank}(S(\mathbb{B})) &= \mathrm{rank}(\mathbb{Z}(\mathbb{B})) - \mathrm{rank}(G_0(\mathbb{B})) \\ &= q.e - e \\ &= \mathrm{rank}(A(\mathbb{B})) .\end{aligned}$$

Therefore $S(\mathbb{B})/A(\mathbb{B})$ is finite and a similar argument to that in part (c) completes the proof.

□

Let $F_S = \{f \in F | f(W_j) = 0 \;,\; j \in I\}$ which we shall identify with $(Q_S, \mathbb{Z})$. For each $j,\beta \in \mathbb{Z}$ define $m_{j,\beta} \in (Q_S,\mathbb{Z})$ by

$$m_{j,\beta} : gT_{i,\alpha} \to |[j+\beta-\alpha, j] \cap i+e\mathbb{Z}| \quad .$$

Notice that

(21) $$m_{c(j)}\Big|_{\mathbb{Z}A_{q-1}} = m_{\delta^{-1}(j),\lambda'(j)+1} \circ \pi \quad \text{by (3) ;}$$

(22) $$m_j\Big|_{\mathbb{Z}A_{q-1}} = m_{j,1} \circ \pi \quad \text{by (8).}$$

4.18 Theorem.

Let $X \cong gT_{j,\beta} \in \mathrm{Ind}_p(\mathbb{B})$. Then:

(a) $$\alpha_X\Big|_{Q_S} = \sum_{i \in I} \{\alpha_X(gS_i).m_{i,1} - \alpha_X(W_i).m_{\delta^{-1}(i),\lambda'(i)+1}\} - m_{j,\beta+1} \;;$$

(b) $$\beta_X\Big|_{Q_S} = \sum_{i \in I} \{\beta_X(gS_i).m_{i,1} - \beta_X(W_i).m_{\delta^{-1}(i),\lambda'(i)+1}\} - m_{j-1,\beta+1} \;.$$

Proof.

For X as above, define $g_X, h_X \in F$ by

(23) $$g_X = \alpha_X + m_{j,\beta+1}$$

Let $F_S = \{f \in F | f(W_j) = 0 \ , \ j \in I\}$ which we shall identify with $(Q_S, \mathbb{Z})$. For each $j,\beta \in \mathbb{Z}$ define $m_{j,\beta} \in (Q_S, \mathbb{Z})$ by

$$m_{j,\beta} : gT_{i,\alpha} \to |[j+\beta-\alpha, j] \cap i+e\mathbb{Z}| \quad .$$

Notice that

(21) $$m_{c(j)}\Big|_{\mathbb{Z}A_{q-1}} = m_{\delta^{-1}(j),\lambda'(j)+1} \circ \pi \quad \text{by (3) ;}$$

(22) $$m_j\Big|_{\mathbb{Z}A_{q-1}} = m_{j,1} \circ \pi \quad \text{by (8).}$$

4.18 <u>Theorem</u>.

Let $X \cong gT_{j,\beta} \in \mathrm{Ind}_p(\mathbb{B})$. Then:

(a) $$\alpha_X\Big|_{Q_S} = \sum_{i \in I} \{\alpha_X(gS_i).m_{i,1} - \alpha_X(W_i).m_{\delta^{-1}(i),\lambda'(i)+1}\} - m_{j,\beta+1} \quad ;$$

(b) $$\beta_X\Big|_{Q_S} = \sum_{i \in I} \{\beta_X(gS_i).m_{i,1} - \beta_X(W_i).m_{\delta^{-1}(i),\lambda'(i)+1}\} - m_{j-1,\beta+1} \quad .$$

<u>Proof</u>.

For X as above, define $g_X, h_X \in F$ by

(23) $$g_X = \alpha_X + m_{j,\beta+1}$$

(24) $$h_X = \beta_X + m_{j-1,\beta+1} \;.$$

We wish to show that $g_X, h_X \in F^{++}$ and by 4.17(b) it is enough to show that

$$g_X(A_Y) = h_X(B_Y) = 0$$

for all $Y \overset{\sim}{=} gT_{i,\alpha}$, $i \in I$, $\alpha = 1,\dots,q-2$. For such X,Y $\alpha_X(A_Y) = \beta_X(B_Y) = \varepsilon_I(i,j).\varepsilon(\alpha,\beta)$ and so we must prove:

(25) $$m_{j,\beta+1}(A_Y) = m_{j-1,\beta+1}(B_Y) = -\varepsilon_I(i,j).\varepsilon(\alpha,\beta) \;.$$

Now $m_{j,\beta+1}(A_Y) = m_{j,\beta+1}(A_{gT_{i,\alpha}}) = m_{j,\beta+1}(gT_{i,\alpha}) + m_{j,\beta+1}(gT_{i+1,\alpha}) - m_{j,\beta+1}(gT_{i,\alpha+1}) - m_{j,\beta+1}(gT_{i+1,\alpha-1})$. Setting $a = \alpha-\beta$:

$$m_{j,\beta+1}(gT_{i,\alpha}) = |[j-a+1,j] \cap i+e\,\mathbb{Z}| \quad ;$$

$$m_{j,\beta+1}(gT_{i+1,\alpha}) = |[j-a+1,j] \cap i+1+e\,\mathbb{Z}|$$

$$= |[j-a,j-1] \cap i+e\,\mathbb{Z}| \quad ;$$

$$m_{j,\beta+1}(gT_{i,\alpha+1}) = |[j-a,j] \cap i+e\,\mathbb{Z}| \quad ;$$

$$m_{j,\beta+1}(gT_{i+1,\alpha-1}) = |[j-a+2,j] \cap i+1+e\,\mathbb{Z}|$$

$$= |[j-a+1,j-1] \cap i+e\,\mathbb{Z}| \quad .$$

Comparing this with 4.4 we have an almost identical situation to (7). The proof is similar, taking case by case calculations with $a < 0$, $a = 0$, $a > 0$. Therefore $g_\chi \in F^{++}$. Now,

$$m_{j-1,\beta+1}(B_\gamma) = m_{j-1,\beta+1}(A_{\tau^{-1}\gamma}) = m_{j-1,\beta+1}(A_{gT_{i-1,\alpha}})$$

$= -\varepsilon_I(i-1,j-1).\varepsilon(\alpha,\beta) = -\varepsilon_I(i,j).\varepsilon(\alpha,\beta)$ which completes the proof of (25) and so $h_\chi \in F^{++}$. Since

$$g_\chi(W_i) = \alpha_\chi(W_i) + m_{j,\beta+1}(W_i) = \alpha_\chi(W_i) \quad ;$$

$$g_\chi(gS_i) = \alpha_\chi(gS_i) + m_{j,\beta+1}(gS_i) = \alpha_\chi(gS_i)$$

for all $i \in I$, by 4.8 :

$$g_\chi(gT_{i,\alpha}) = \sum_{\ell\in I}\{\alpha_\chi(gS_\ell).m_{\ell,1}(gT_{i,\alpha}) - \alpha_\chi(W_\ell).m_{\delta^{-1}(\ell),\lambda'(\ell)+1}\ldots$$
$$\ldots(gT_{i,\alpha})\}$$

and so

$$\alpha_\chi = g_\chi - m_{j,\beta+1}$$

$$= \sum_{i\in I}\{\alpha_\chi(gS_i).m_{i,1} - \alpha_\chi(W_i).m_{\delta^{-1}(i),\lambda'(i)+1}\} - m_{j,\beta+1}$$

which proves part (a).

Similarly, $h_X(W_i) = \beta_X(W_i)$, $h_X(gS_i) = \beta_X(gS_i)$ and part (b) is proven. □

We return briefly to the category $Fun(\mathbb{B})$. First consider the larger category $Fun(kG)$ of all contravariant functors

$$F : \text{mod } kG \to \text{mod } k \ .$$

For $F \in Fun(kG)$ define $F_H \in Fun(kH)$ by:

$$F_H(U) = F(U^G) \ , \quad U \in \text{mod } kH \ .$$

Let $e \in kH$ be the central primitive idempotent corresponding to the block B and, for $\tilde{F} \in Fun(kH)$, define $\tilde{F}^e \in Fun(kH)$ by

$$\tilde{F}^e(U) = \tilde{F}(eU) \ .$$

It follows that for $F \in Fun(kG)$, $F_H^e \in Fun(\mathbb{B})$. The following is due to J.A. Green.

4.19 <u>Lemma</u>.

Let $F = S_M \in Fun(\mathbb{B})$, $M \in Ind_P(\mathbb{B})$. Then $F_H^e \cong S_{fM}$.

Proof.

We have to show that for $U \in \mathrm{Ind}(kH)$, $F^e_H(U) = 0$ if $U \not\cong fM$ and $F^e_H(U) = k$ otherwise. If $U \notin \mathrm{mod}\ B$ $F^e_H(U) = 0$ so assume $U \in \mathrm{mod}\ B$. If U is projective, so is U^G and if $F^e_H(U) = (S_M)_H(eU) = S_M(U^G) \neq 0$ then $M|U^G$ by 4.13 which implies M is projective contradicting the fact that $M \in \mathrm{Ind}_{\mathcal{P}}(\mathbb{B})$. If U is not projective $F^e_H(U) \neq 0$ iff $M|U^G$. Now $U^G = gU \oplus Q$ with Q projective so $F^e_H(U) \neq 0$ iff $M \cong gU$ iff $fM \cong U$ (see 2.2) as required.

□

We use this lemma to prove the following.

4.20 Proposition.

Let $U, V \in \mathrm{Ind}_{\mathcal{P}}(\mathbb{B})$ be such that $\alpha_X(U) = \alpha_X(V)$ for $X \in \{W_i, gS_i, \Omega gS_i \,|\, i \in I\} = \mathcal{S}$. Then $U \cong V$.

Proof.

Suppose $U \cong gT_{i,\alpha}$, $V \cong gT_{j,\beta}$. Then it is enough to show that $\alpha = \beta$ and $\varepsilon_I(i,j) = 1$. Consider $U - V \in \mathbb{Z}(\mathbb{B})$ and let

$$(26) \qquad U-V = \sum_{Y \in \mathrm{Ind}(\mathbb{B})} a_Y A_Y \ , \ a_Y \in \mathbb{Z} \ ,$$

which is possible by 4.17(c). From the hypothesis, $\alpha_X(U-V) = 0$ for all $X \in \mathcal{S}$ and applying such α_X to both sides of (26) :

$$0 = \alpha_X(U-V) = \sum_{Y\in \mathrm{Ind}(\mathbb{B})} a_Y\alpha_X(A_Y) = a_X$$

by 4.16(a) and therefore

$$(27) \qquad U - V = \sum_{Y\in \mathrm{Ind}(\mathbb{B})\backslash S} a_Y A_Y \quad .$$

We now wish to restrict both sides of (27) to H and multiply by e, where e is the block idempotent corresponding to B. Since $f(gT_{i,\alpha}) \cong T_{i,\alpha}$ by 2.2(a) it follows that $(gT_{i,\alpha})|_H \cong T_{i,\alpha} \oplus P$ where P is projective and so

$$(28) \qquad (U-V)|_H \cong T_{i,\alpha} - T_{j,\beta} + \sum_{\ell\in I} a_\ell T_\ell \quad , \quad a_\ell \in \mathbb{Z} \quad .$$

Now recall that for $Y \in \mathrm{Ind}_P(\mathbb{B})$ the exact sequence

$$0 \to (\ ,\Omega^2Y) \to (\ ,E(Y)) \to (\ ,Y) \to S_Y \to 0$$

is a m.p.p. for S_Y (4.14(a)) and by 4.19 that $(S_Y)^e_H \cong S_{fY}$ as functors. Therefore

$$(29) \qquad 0 \to (\ ,\Omega^2 fY) \to (\ ,E(fY)) \to (\ ,fY) \to S_{fY} \to 0$$

$$(30) \qquad 0 \to (\ ,\Omega^2 Y)^e_H \to (\ ,E(\ Y\ ))^e_H \to (\ ,\ Y\ )^e_H \to S_{fY} \to 0$$

are both projective presentations of S_{fY}, (29) being a m.p.p. By

Frobenius Reciprocity ([CR] 232) and Schanuel's Lemma, (30) is isomorphic to

$$(31) \qquad 0 \to (\ ,\Omega^2 fY \oplus Q_1) \to (\ ,E(fY) \oplus Q_1 \oplus Q_2) \to (\ ,fY \oplus Q_2) \to S_{fY} \to 0 .$$

In particular, $e.(\Omega^2 Y_H) \cong \Omega^2 fY \oplus Q_1$, $e.(E(Y)_H) \cong E(fY) \oplus Q_1 \oplus Q_2$ and $e.Y_H \cong fY \oplus Q_2$ and therefore:

$$(32) \qquad e(A_Y)_H = A_{fY} .$$

Regarding (27) in the light of (28) and (32) -

$$(33) \qquad T_{i,\alpha} - T_{j,\beta} + \sum_{\ell \in I} a_\ell T_\ell = \sum_{Y \in Ind(B) \setminus S} a_Y A_{fY} .$$

Recall that for a (q,e)-uniserial block B , the projective module $T_{\ell,q}$ only occurs in the A-R sequence $(T_{\ell,q-1})$ where $T_{\ell,q-1} \cong \Omega S_{\ell-1}$. Since no such sequence occurs in the right-hand side of (33) it follows that $a_\ell = 0$ for all $\ell \in I$ and

$$(34) \qquad T_{i,\alpha} - T_{j,\beta} = \sum_{Y \in Ind(B) \setminus S} a_Y A_{fY} .,$$

Applying α_{T_ℓ} to both sides we see that

$$\alpha_{T_\ell}(T_{i,\alpha}-T_{j,\beta}) = \sum_{\gamma\in \mathrm{Ind}(\mathbb{B})\backslash S} a_\gamma \alpha_{T_\ell}(A_{f\gamma}) = 0 \quad .$$

This is true for all $\ell \in I$ and so $[T_{i,\alpha}] = [T_{j,\beta}]$, therefore $\alpha = \beta$. Similarly, $\alpha_{S_\ell}(T_{i,\alpha}-T_{j,\beta}) = 0$ (since $gS_\ell \in S$) for all $\ell \in I$ implying that $\Sigma(T_{i,\alpha}) \cong \Sigma(T_{j,\beta})$. That is, $i+\alpha-1 \equiv j+\beta-1 \mod(e)$. But $\alpha = \beta$ and so $i \equiv j \mod(e)$, that is $\epsilon_I(i,j) = 1$ as required. □

We conclude this chapter with a description of the irreducible maps in $\mathrm{mod}\,\mathbb{B}$. The projective irreducible maps are the inclusion/quotient maps $RP \to P \to P/\Sigma P$ as given in 1.9 where $P \in \mathrm{Ind}(\mathbb{B})$ is projective. Now let

$$\Psi(i,\alpha) : gT_{i,\alpha} \to gT_{i-1,\alpha+1} \qquad (\alpha \neq q-1)$$

$$\Phi(i,\alpha) : gT_{i,\alpha} \to gT_{i,\alpha-1} \qquad (\alpha \neq 1)$$

be irreducible maps with domain $gT_{i,\alpha}$. They are unique up to scalar multiplication modulo $R^2(gT_{i,\alpha},\ )$ since

$$\mathrm{Irr}(gT_{i,\alpha},gT_{i-1,\alpha+1}) \cong \mathrm{Irr}(gT_{i,\alpha},gT_{i,\alpha-1}) \cong k$$

by 2.7.

<u>Notation</u> (See [P])

For $M,N \in \mathrm{mod}\,\mathbb{B}$ denote by $N \circ M$ any extension of N by M so there exists a s.e.s.

$$0 \to M \to N \circ M \to N \to 0 \quad .$$

We note that $T_{i,\alpha} \cong S_i \circ T_{i+1,\alpha-1} \cong T_{i,\alpha-1} \circ S_{i+\alpha-1}$ where the extensions concerned are non-split.

The following two results, both due to Peacock, will be needed. We will make use of the obvious homological algebraic fact that

$$\mathrm{Ext}^1_B(U,V) \cong (\underline{\Omega U, V}) \quad \text{for} \quad U, V \in \text{mod } B \ .$$

4.21 Lemma [P] p.241.

Let $T_{i,\alpha} \circ T_{j,\beta}$ be an extension given by $\underline{\theta} \in (\underline{\Omega T_{i,\alpha}, T_{j,\beta}})$ and let $r(\theta)$ denote the length of $\mathrm{Im}(\theta)$. Then $\underline{\theta} \neq \underline{0}$ implies that $r(\theta) > \beta-\alpha$ and then there exists a non-split extension $T_{i,\alpha+r(\theta)} \oplus T_{j,\beta-r(\theta)}$. □

4.22 Lemma [P] 3.9

Let $M, N \in \mathrm{Ind}_p(\mathbb{B})$ and suppose $N \circ M \cong L \in \mathrm{Ind}_p(\mathbb{B})$. Then there exists a non-split extension $fN \circ fM$ such that $f(N \circ M) \oplus P \cong fN \circ fM$ where P is a (possibly zero) projective module. □

4.23 Lemma.

(a) The irreducible map $\Psi(i,\alpha)$ is either

(i) $\Psi_1(i,\alpha) : gT_{i,\alpha} \to gS_{i-1} \circ gT_{i,\alpha}$

or (ii) $\Psi_2(i,\alpha) : gT_{i,\alpha} \to gT_{i,\alpha}/\Omega gS_{i-1}$.

(b) The irreducible map $\Phi(i,\alpha)$ is either

(i) $\Phi_1(i,\alpha) : gT_{i,\alpha} \to gT_{i,\alpha}/gS_{i+\alpha-1}$

or (ii) $\Phi_2(i,\alpha) : gT_{i,\alpha} \to \Omega gS_{i+\alpha-2} \circ gT_{i,\alpha}$.

Proof.

Recall that an irreducible map is either a monomorphism or an epimorphism.

(a) (i) Suppose $\Psi(i,\alpha)$ is a monomorphism and let $\Psi_1(i,\alpha): gT_{i,\alpha} \to X \circ gT_{i,\alpha}$ be irreducible for some $X \in \mathrm{Ind}_p(\mathbb{B})$. Since $X \circ gT_{i,\alpha}$ is non-split and $f(X \circ gT_{i,\alpha}) \cong T_{i-1,\alpha+1} \cong S_{i-1} \circ T_{i,\alpha}$ we can apply 4.22. That is $X \cong gS_{i-1}$.

(ii) Suppose $\Psi(i,\alpha)$ is an epimorphism and let $\Psi_2(i,\alpha): gT_{i,\alpha} \to gT_{i,\alpha}/Y$ be irreducible, $Y \in \mathrm{Ind}_p(\mathrm{B})$. Then $gT_{i,\alpha} \cong gT_{i-1,\alpha+1} \circ Y$ and 4.22 again:

$$P \oplus T_{i,\alpha} \cong T_{i-1,\alpha+1} \circ fY$$

where $P \in \mathrm{mod}\ B$ is projective. Let $fY \cong T_{j,\beta}$ and let $T_{i-1,\alpha+1} \circ T_{j,\beta}$ be given by $\underline{\theta} \in \underline{(\Omega T_{i-1,\alpha+1}, T_{j,\beta})}$. By 4.21 $T_{i,\alpha} \oplus P \cong T_{i-1,\alpha+1+r(\theta)} \oplus T_{j,\beta-r(\theta)}$ which forces $j = i$, $\beta - r(\theta) = \alpha$ and $\alpha+1+r(\theta) = q$. Therefore

$r(\theta) = q-\alpha-1$ and $\beta = q-1$. It follows that $fY \cong T_{i,q-i} \cong \Omega S_{i-1}$ and so $Y \cong \Omega g S_{i-1}$.

Parts (b)(i), (ii) are proved similarly.

□

4.24 Theorem.

(a) $\Psi(i,\alpha)$ is a monomorphism iff $\alpha < \lambda'(\delta(i-1))$.

(b) $\Phi(i,\alpha)$ is an epimorphism iff $\alpha \le \lambda'(i+\alpha-1)$.

Proof.

We turn to the Grothendieck Group and consider the composition factors of each module as follows:

(a) By the previous lemma,

$$(35) \qquad [\text{Range } \Psi_1(i,\alpha)] - [gT_{i,\alpha}] = [gS_{i-1}]$$

$$\text{and} \quad [\text{Range } \Psi_2(i,\alpha)] - [gT_{i,\alpha}] = -[\Omega g S_{i-1}]$$

$$= [gS_{i-1}] - [W_{\delta(i-1)}] \quad \text{by (15).}$$

However, recalling 4.8

$$(36) \qquad [gT_{i-1,\alpha+1}] - [gT_{i,\alpha}]$$

$$= [gS_{i-1}] - \sum_{j\in I} \{m_{c(j)}(i-1,\alpha+1) - m_{c(j)}(i,\alpha)\}.[W_j] \quad .$$

Comparing (35) and (36) we see that:

(37) $$[gT_{i-1,\alpha+1}] = [\text{Range } \Psi_1(i,\alpha)] = [gS_{i-1} \circ gT_{i,\alpha}] \text{ if } m_{c(j)}(i,\alpha) = m_{c(j)}(i-1,\alpha+1) \text{ for all } j \in I$$

(38) and $$[gT_{i-1,\alpha+1}] = [\text{Range } \Psi_2(i,\alpha)] = [gT_{i,\alpha}/\Omega gS_{i-1}] \text{ if}$$

$$m_{c(j)}(i,\alpha) = m_{c(j)}(i-1,\alpha+1) \text{ for all } j \in I \backslash \delta(i-1) \text{ and}$$

$$m_{c(j)}(i,\alpha) + 1 = m_{c(j)}(i-1,\alpha+1) \text{ for } j = \delta(i-1) .$$

Setting $a = i-\alpha + \lambda'(\delta(i-1))$:

$$m_{c(\delta(i-1))}(i,\alpha) = |[a,i-1] \cap i+e\,\mathbb{Z}|$$

and $$m_{c(\delta(i-1))}(i-1,\alpha+1) = |[a,i] \cap i+e\,\mathbb{Z}| .$$

It follows that $m_{c(\delta(i-1))}(i,\alpha) = m_{c(\delta(i-1))}(i-1,\alpha+1)$ iff $a > i$. That is $\alpha < \lambda'(\delta(i-1))$ as required.

(b) As in part (a), comparing $\Phi_1(i,\alpha)$ with $\Phi_2(i,\alpha)$:

(39) $$[\text{Range } \Phi_2(i,\alpha)] - [gT_{i,\alpha}] = [\Omega gS_{i+\alpha-2}]$$

$$= [W_{i+\alpha-1}] - [gS_{i+\alpha-1}] \text{ by (16)}$$

and $$[\text{Range } \Phi_1(i,\alpha)] - [gT_{i,\alpha}] = - [gS_{i+\alpha-1}] .$$

Recalling 4.8,

(40) $[gT_{i,\alpha-1}] - [gT_{i,\alpha}]$

$$= -[gS_{i+\alpha-1}] + \sum_{j\in I} \{m_{c(j)}(i,\alpha)-m_{c(j)}(i,\alpha-1)\}.[W_j] \quad .$$

Comparing (39) with (40) :

(41) $[gT_{i,\alpha-1}] = [\text{Range } \Phi_1(i,\alpha)] = [gT_{i,\alpha}/gS_{i+\alpha-1}]$ if

$m_{c(j)}(i,\alpha) = m_{c(j)}(i,\alpha-1)$ for all $j \in I$, and

(42) $[gT_{i,\alpha-1}] = [\text{Range } \Phi_2(i,\alpha)] = [\Omega gS_{i+\alpha-2} \circ gT_{i,\alpha}]$ if

$m_{c(j)}(i,\alpha) = m_{c(j)}(i,\alpha-1)$ for all $j \not\equiv i+\alpha-1$ and

$m_{c(i+\alpha-1)}(i,\alpha) = m_{c(i+\alpha-1)}(i,\alpha-1) + 1$.

Setting $a = \delta^{-1}(i+\alpha-1) - \alpha + \lambda'(i+\alpha-1) + 1$:

$$m_{c(i+\alpha-1)}(i,\alpha) = |[a,\delta^{-1}(i+\alpha-1)] \cap i+e\,\mathbb{Z}|$$

and $m_{c(i+\alpha-1)}(i,\alpha-1) = |[a+1,\delta^{-1}(i+\alpha-1)] \cap i+e\,\mathbb{Z}|$.

However $a \equiv \delta^{-1}(i+\alpha-1) - \alpha + (i+\alpha-1) - \delta^{-1}(i+\alpha-1) + 1$ by 2.12

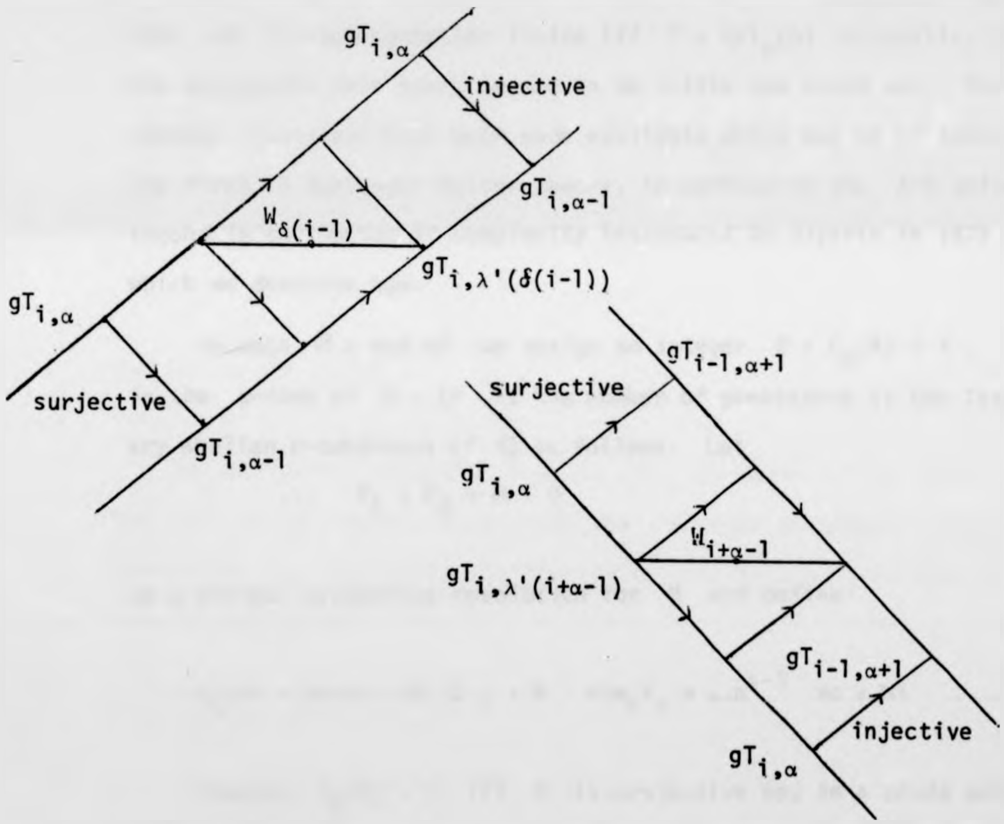
$= i$.

Therefore it follows that:

$$m_{c(i+\alpha-1)}(i,\alpha) = m_{c(i+\alpha-1)}(i,\alpha-1) \text{ iff } a > \delta^{-1}(i+\alpha-1) \ .$$

That is $\alpha < \lambda'(i+\alpha-1) + 1$ or $\alpha \leq \lambda'(i+\alpha-1)$ as required. □

The result can be better appreciated by the following diagram. In particular it is seen that an irreducible map is monomorphic/epimorphic depending on its position in Q relative to a certain projective mesh.



CHAPTER 5. Some Periodic $SL(2,p^n)$-modules.

As before, let k be an algebraically closed field of characteristic p, G a finite group. The classification of kG-modules seems an all but hopeless task. In 1954 ([Hi]) Higman showed that kG is representation finite iff $P \in Syl_p(G)$ is cyclic, but for the non-cyclic case there seemed to be little one could say. Recently, however, two tools have been made available which may be of some value. The first is Auslander-Reiten theory, in particular the A-R quiver. The second is the notion of complexity introduced by Alperin in 1975 ([Al]) which we describe now.

To each $M \in \text{mod } kG$ we assign an integer $0 \leq C_G(M) \leq r$, where r is the p-rank of G, (r is the number of generators of the largest elementary abelian p-subgroups of G) as follows. Let

$$\cdots \quad P_1 \to P_0 \to M \to 0$$

be a minimal projective resolution for M and define:

$$C_G(M) = \min\{s \in \mathbb{N} \mid \exists\ \mu \in \mathbb{R} : \dim_k P_n \leq \mu . n^{s-1} \quad \forall n \in \mathbb{N}\} \quad .$$

Clearly $C_G(M) = 0$ iff M is projective so, in a crude sense, the complexity of a module measures how much a module deviates from being projective. The following is quite easy to prove.

5.1 Proposition. ([Al] 779)

Let $H \leq G$, $L \in \text{mod } kH$, $M \in \text{mod } kG$ be such that $L|M_H$ and $M|L^G$. Then $C_G(M) = C_H(L)$.

□

5.2 Corollary.

Let $H \leq G$ be such that mod kH and mod kG are stably equivalent. Then

$$C_G(M) = C_H(fM)$$

for all $M \in \text{Ind}_p(kG)$.

□

We can further classify $M \in \text{Ind}\,(kG)$ by means of the A-R quiver. That is, for M above, we consider the connected component $q(M) \leq Q(kG)$ containing M. The following theorem by Webb relates this idea with that of complexity.

5.3 Theorem. ([W] p.99)

Let $M \in \text{Ind}_p(kG)$, $q(M) \leq Q(kG)$ be as above and let $N \in q(M) \cap \text{Ind}_p(kG)$. Then $C_G(M) = C_G(N)$.

□

We can see then that the A-R quiver refines the classification afforded by complexity.

5.4 Example. ([BP] 20)

Take $G = C_2 \times C_2$, $p = 2$. Then

$$Q(kG) = q(k_G) \mathbin{\dot{\cup}} \dot{\bigcup_{\lambda \in \mathbb{P}(k)}} q_\lambda$$

where the second term is parametrised by the projective line $\mathbb{P}(k)$. The component $q(k_G)$ contains modules of complexity two whilst each q_λ contains modules of complexity one. A covering for $Q(kG)_S$ is

$$\Delta = \mathbb{Z}A_\infty^\infty \mathbin{\dot{\cup}} \dot{\bigcup_{\lambda \in \mathbb{P}(k)}} \mathbb{Z}A_\infty$$

where A_∞ is the infinite tree

and A_∞^∞ is the 'doubly' infinite tree

5.5 Definition.

A kG-module M is said to be periodic if it is not projective and there exists $a \in \mathbb{N}$ such that $\Omega^a M \cong M$.

The following is due to Carlson.

5.6 Theorem. ([C])

If $M \in \mathrm{Ind}(kG)$ is periodic then $p^{r-1} | \dim_k M$ where r is the p-rank of G. Furthermore, M is periodic iff $C_G(M) = 1$. $\square$

From now on let $G = SL(2,p^n)$. In the rest of this chapter we want to look at some of the periodic kG-modules and, for such modules, construct the connected components containing them. We shall assume $n \geq 2$.

Recall that $G = SL(2,p^n)$ is the set of all 2×2 matrices of determinant one over the Galois Field $GF(p^n)$ - we shall assume that k contains $GF(p^n)$. Fix $P \in \mathrm{Syl}_p(G)$ to be the subgroup

$$\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in GF(p^n)\}$$

so P is elementary abelian and the p-rank of G is n. Let $B = N_G(P) = P.T$ where

$$T = \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in GF(p^n)^*\} \cong C_{p^n-1} .$$

Since $\mathrm{Syl}_p(G)$ is a T.I. set we can apply the Green Correspondence to get an equivalence

(1) $\quad f : \underline{\mathrm{mod}}\, kG \to \underline{\mathrm{mod}}\, kB$.

As a point of notation we will write

$$z(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in P \quad , \quad \underline{t} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in T \ .$$

For $b \in \mathbb{Z}_{p^n-1}$ define

$$\phi_b : B \to k^*$$

$$z(a).\underline{t} \to t^b$$

and let S_b be the corresponding irreducible kB-module. Also, set $U_b = P(S_b)$. Then $\{S_b | b \in \mathbb{Z}_{p^n-1}\}$, $\{U_b | b \in \mathbb{Z}_{p^n-1}\}$ are full sets of simple, projective indecomposable kB-modules.

Let $M \in \mathrm{mod}\ kB$, $b \in \mathbb{Z}_{p^n-1}$ and let $\{m_1,\ldots,m_t\}$ be a k-basis for M . We form the tensor product $M \otimes S_b$ which has k-basis $\{m_i \otimes s_b \mid i = 1,\ldots,t\}$, where $0 \neq s_b \in S_b$. For each $m \in M$ we shall write the corresponding element $m \otimes s_b \in M \otimes S_b$ as $m \otimes b$. We list a few properties which we shall need later.

5.7 <u>Proposition</u>.

Let $M \in \mathrm{mod}\ kB$, $b \in \mathbb{Z}_{p^n-1}$.

(a) $M_p \cong (M \otimes S_b)_p$;

(b) $P(M \otimes S_b) \cong P(M) \otimes S_b$ and in particular $U_a \otimes S_b \cong U_{a+b}$ since $S_a \otimes S_b \cong S_{a+b}$;

(c) $\Omega M \otimes S_b \cong \Omega(M \otimes S_b)$;

(d) Each $\theta : M \to N$ induces a map

$$\theta \otimes 1_b : M \otimes S_b \to N \otimes S_b$$
$$m \otimes b \to m \otimes b$$

satisfying

$$\mathrm{Ker}(\theta) \otimes S_b = \mathrm{Ker}(\theta \otimes 1_b)$$
$$\mathrm{Im}(\theta) \otimes S_b = \mathrm{Im}(\theta \otimes 1_b) \ .$$

□

We now look at the kG-modules. Let $V \in \mathrm{mod}\ kG$ be given by $\phi : G \to GL_m(k)$ and, for $i \in \mathbb{Z}$, let $\psi_i \in \mathrm{Aut}(G)$ be given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a^{p^i} & b^{p^i} \\ c^{p^i} & d^{p^i} \end{pmatrix} .$$

Define $V^{(i)}$ to be the kG-module given by $\phi \circ \psi_i$. Notice that $V^{(n)} = V$ since $\psi_n = 1_G$.

Let $R = k[X,Y]$ be the polynomial ring in two variables and define a G-action on R by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ f(X,Y) = f(aX + cY, bX + dY) .$$

We now describe the irreducible kG-modules. For $\mu \geq 0$ let $V(\mu)$ denote the $(\mu+1)$-dimensional subspace consisting of all homogenous polynomials $f \in R$ of degree μ. If, for each $0 \leq i \leq \mu$, we set

(2) $$U_i(\mu) = X^i\, Y^{\mu-i}$$

then clearly $\{U_i(\mu) \mid i = 0,\ldots,\mu\}$ is a basis for $V(\mu)$. Furthermore ([Br] §30), for $0 \leq \mu \leq p-1$, $V(\mu)$ is irreducible. We shall be interested in the restriction of such modules to the subgroup B; hence we describe the B-action.

(3) $$\begin{aligned} \underline{t} \circ U_i(\mu) &= (tX)^i (t^{-1}X)^{\mu-i} \\ &= t^{2i-\mu}\, U_i(\mu) \end{aligned}$$

(4) $$\begin{aligned} z(a) \circ U_i(\mu) &= X^i (aX + Y)^{\mu-i} \\ &= X^i \sum_{j=0}^{\mu-i} \binom{\mu-i}{j} a^j X^j Y^{\mu-i-j} \\ &= \sum_{j=0}^{\mu-i} \binom{\mu-i}{j} a^j U_{j+i}(\mu) \end{aligned}$$

Now the rest of the irreducible kG-modules can be described as follows. For $0 \leq \lambda \leq p^n-1$ we can write

(5) $$\lambda = \sum_{j=0}^{n-1} \lambda_j p^j$$

- the (unique) p-adic decomposition. With (5) in mind we define:

$$(6) \qquad L(\lambda) = \bigotimes_{j=0}^{n-1} V(\lambda_j)^{(j)} \quad .$$

As long ago as 1941 Brauer ([Br] §30) showed that $\{L(\lambda) \mid \lambda = 0,\ldots,p^n-1\}$ is a full set of irreducible kG-modules which fact can be deduced from the Steinberg Tensor-product theorem for algebraic groups. ([St]).

We note that

$$(7) \qquad \dim_k L(\lambda) = \prod_{j=0}^{n-1} (\lambda_j+1) \quad .$$

<u>Notation</u>.

For each $0 \le \lambda \le p^n-1$ define

(a) $I(\lambda) = \{\underline{i} = (i_0,\ldots,i_{n-1}) \mid 0 \le i_m \le \lambda_m\}$;

(b) $\underline{\lambda} = (\lambda_0,\ldots\ldots,\lambda_{n-1}) \in I(\lambda)$;

(c) $h: I(\lambda) \to \{0,\ldots,\lambda\}$

$$\underline{i} \to \sum_{j=0}^{n-1} i_j p^j \quad .$$

For each $\underline{i} \in I(\lambda)$ define:

$$(8) \qquad U_{\underline{i}}(\lambda) = U_{i_0}(\lambda_0)^{(0)} \otimes \ldots\ldots \otimes U_{i_{n-1}}(\lambda_{n-1})^{(n-1)} \in L(\lambda) \quad .$$

and $\{U_{\underline{i}}(\lambda) \mid \underline{i} \in I(\lambda)\}$ forms a k-basis for $L(\lambda)$.

We equip $I(\lambda)$ with a partial ordering - say $\underline{i} \leq \underline{j}$ iff $i_0 \leq j_0, \ldots\ldots, i_{n-1} \leq j_{n-1}$. Clearly $\underline{0}(\underline{\lambda})$ are minimal (maximal) elements of $I(\lambda)$ with respect to this ordering. Finally we note that $h(\underline{\lambda}) = \lambda$.

We now look at the B-action for $L(\lambda)$. For each $0 \leq \lambda \leq p^n - 1$, $\underline{i} \in I(\lambda)$

(9)
$$\begin{aligned}\underline{t} \circ U_{\underline{i}}(\lambda) &= \underline{t} \circ U_{i_0}(\lambda_0)^{(0)} \otimes \ldots\ldots \otimes \underline{t} \circ U_{i_{n-1}}(\lambda_{n-1})^{(n-1)} \\ &= t^{2i_0-\lambda_0} U_{i_0}(\lambda_0)^{(0)} \otimes \ldots\ldots\ldots \\ &\qquad \ldots\ldots \otimes t^{(2i_{n-1}-\lambda_{n-1})p^{n-1}} U_{i_{n-1}}(\lambda_{n-1})^{(n-1)}\end{aligned}$$

by (3)
$$= t^{2h(\underline{i})-\lambda} . U_{\underline{i}}(\lambda) .$$

(10)
$$\begin{aligned}z(a) \circ U_{\underline{i}}(\lambda) &= z(a) \circ U_{i_0}(\lambda_0)^{(0)} \otimes \ldots\ldots \otimes z(a)\, U_{i_{n-1}}(\lambda_{n-1})^{(n-1)} \\ &= \bigotimes_{m=0}^{n-1} \sum_{j_m=0}^{\lambda_m - i_m} \binom{\lambda_m - i_m}{j_m} a^{j_m p^m} U_{i_m + j_m}(\lambda_m)^{(m)}\end{aligned}$$

by (4)
$$= \sum_{\underline{0} \leq \underline{j} \leq \underline{\lambda} - \underline{i}} \left| \begin{matrix} \underline{\lambda} - \underline{i} \\ \underline{j} \end{matrix} \right| a^{h(\underline{j})} U_{\underline{i}+\underline{j}}(\lambda) ,$$

where
$$\left| \begin{matrix} \underline{\lambda} - \underline{i} \\ \underline{j} \end{matrix} \right| = \prod_{m=0}^{n-1} \binom{\lambda_m - i_m}{j_m} .$$

5.8 Lemma.

For each $0 \le \lambda < p^n-1$, $L(\lambda)_B$ is indecomposable. Furthermore $L(p^n-1)_B \cong U_0$.

Proof.

We divide the proof into two cases.

Case 1. $0 \le \lambda < p^n-1$.

Here $\underline{\lambda} = (\lambda_0,\ldots,\lambda_{n-1})$ where not all the λ_i's are equal to $p-1$. By (7) $\dim_k L(\lambda) < p^n$. Since the projective indecomposable kB-modules have dimension p^n we deduce that $fL(\lambda) = L(\lambda)_B$ and so the result follows by the Green Correspondence 2.2(a).

Case 2. $\lambda = p^n-1$.

Now $\underline{p^n-1} = (p-1,\ldots,p-1)$ so $\dim_k L(p^n-1) = p^n$ by (7) again. Since $L(p^n-1)$([5]) is projective it follows that $L(p^n-1)_B$ is and so $L(p^n-1)_B \cong U_a$ for some $0 \le a \le p^n-2$. By earlier remarks we know that

$$\Sigma(U_a) \cong Hd(U_a) \cong S_a$$

so it is sufficient to show that $\Sigma(L(p^n-1)) \cong S_0$. Consider $U_{\underline{p^n-1}}(p^n-1) \in L(p^n-1)$. By (9)

$$(11) \qquad \underline{t} \circ U_{\underline{p^n-1}}(p^n-1) = t^{2h(\underline{p^n-1})-p^n-1}U_{p^n-1}(p^n-1)$$

$$= t^{p^n-1}U_{\underline{p^n-1}}(p^n-1)$$

$$= t^0\, U_{\underline{p^n-1}}(p^n-1)$$

$$= U_{\underline{p^n-1}}(p^n-1)$$

and by (10)

$$(12) \qquad z(a) \circ U_{\underline{p^n-1}}(p^n-1) = \sum_{\underline{0} \le \underline{j} \le \underline{0}} \begin{bmatrix} \underline{0} \\ \underline{j} \end{bmatrix} a^{h(\underline{0})}\, U_{\underline{p^n-1}+\underline{j}}(p^n-1)$$

$$= U_{\underline{p^n-1}}(p^n-1) \ .$$

We deduce from (11) and (12) that

$$S_0 \cong k.U_{\underline{p^n-1}}(p^n-1) \le \Sigma(L(p^n-1)_B) \cong S_a$$

and the proof is complete. $\square$

From now on put $U_0 = L(p^n-1)_B$ and, for $a \in \mathbb{Z}_{p^n-1}$, $U_a = U_0 \otimes S_a$.

For each $V \in$ mod kB let $I(V)$ be its injective hull.

5.9 <u>Proposition.</u>

For $0 \leq \lambda \leq p^n-1$,

(a) $P(L(\lambda)_B) \cong U_{-\lambda}$

(b) $I(L(\lambda)_B) \cong U_{\lambda}$.

<u>Proof.</u>

(a) Define $\phi_1 : U_{-\lambda} \to L(\lambda)_B$

$$U_{\underline{i}}(p^n-1) \otimes -\lambda \to \begin{cases} \gamma(\underline{i}).U_{\underline{i}}(\lambda) & \underline{i} \leq \underline{\lambda} \\ 0 & \text{otherwise,} \end{cases}$$

where $\gamma(\underline{i}) = \left|\begin{matrix} \underline{\lambda} \\ \underline{i} \end{matrix}\right| \cdot \left|\begin{matrix} p^n-1 \\ \underline{i} \end{matrix}\right|^{-1}$.

Clearly ϕ_1 is a well-defined k-map which is onto. Now for $\underline{i} \leq \underline{\lambda}$:

$$\underline{t}\circ\phi_1(U_{\underline{i}}(p^n-1) \otimes -\lambda) = \underline{t}\circ\gamma(\underline{i}).U_{\underline{i}}(\lambda)$$

$$= \gamma(\underline{i}).t^{2h(\underline{i})-\lambda}U_{\underline{i}}(\lambda) \qquad \text{by (9)}$$

$$= t^{2h(\underline{i})-\lambda}\phi_1(U_{\underline{i}}(p^n-1) \otimes -\lambda)$$

whereas

$$\phi_1(\underline{t}\circ(U_{\underline{i}}(p^n-1) \otimes -\lambda))$$

$$= \phi_1(t^{2h(\underline{i})-(p^n-1)-\lambda}\ U_{\underline{i}}(p^n-1) \otimes -\lambda)$$

$$= t^{2h(\underline{i})-\lambda}\ \phi_1(U_{\underline{i}}(p^n-1) \otimes -\lambda)$$

as required.

Since $(U_\lambda)_{kP} \cong kP \cong L(p^n-1)_{kP}$ as left kP-modules for all $0 \leq \lambda \leq p^n-1$, we will drop the "$\otimes -\lambda$" in the next step. That is:

(13) $$z(a)\circ\phi_1(U_{\underline{i}}(p^n-1)) = z(a)\circ\gamma(\underline{i}).U_{\underline{i}}(\lambda)$$

$$= \gamma(\underline{i}) \sum_{0\ \underline{j}\ \lambda-\underline{i}} \left|\begin{matrix}\lambda-\underline{i} \\ \underline{j}\end{matrix}\right| a^{h(\underline{j})}\ U_{\underline{i}+\underline{j}}(\lambda)$$

by (10) whereas

(14) $$\phi_1(z(a)\circ U_{\underline{i}}(p^n-1)) = \sum_{0\leq\underline{j}\leq\underline{p^n-1}-\underline{i}} \left|\begin{matrix}\underline{p^n-1}-\underline{i} \\ \underline{j}\end{matrix}\right| a^{h(\underline{j})}\phi_1(U_{\underline{i}+\underline{j}}(p^n-1))$$

by (10)

$$= \sum_{0\leq\underline{j}\leq\underline{\lambda}-\underline{i}} \left|\begin{matrix}\underline{p^n-1}-\underline{i} \\ \underline{j}\end{matrix}\right| a^{h(\underline{j})}\gamma(\underline{i}+\underline{j}).U_{\underline{i}+\underline{j}}(\lambda) .$$

Comparing (13) and (14) we must show that

$$\gamma(\underline{i}) . \left|\begin{matrix}\underline{\lambda}-\underline{i} \\ \underline{j}\end{matrix}\right| = \gamma(\underline{i}+\underline{j}).\left|\begin{matrix}\underline{p^n-1}-\underline{i} \\ \underline{j}\end{matrix}\right|$$

for all $\underline{0} \le \underline{i} \le \underline{\lambda}$, $0 \le \underline{j} \le \underline{\lambda}-\underline{i}$; that is

$$\left|\begin{matrix}\underline{\lambda}\\ \underline{i}\end{matrix}\right| \left|\begin{matrix}\underline{p^n-1}\\ \underline{i}\end{matrix}\right|^{-1} \left|\begin{matrix}\underline{\lambda}-\underline{i}\\ \underline{j}\end{matrix}\right| = \left|\begin{matrix}\underline{\lambda}\\ \underline{i}+\underline{j}\end{matrix}\right| \left|\begin{matrix}\underline{p^n-1}\\ \underline{i}+\underline{j}\end{matrix}\right|^{-1} \left|\begin{matrix}\underline{p^n-1}-\underline{i}\\ \underline{j}\end{matrix}\right| .$$

In fact it is enough to show that

$$\binom{\lambda_m}{i_m}\binom{p-1}{i_m}^{-1}\binom{\lambda_m-i_m}{j_m} = \binom{\lambda_m}{i_m+j_m}\binom{p-1}{i_m+j_m}^{-1}\binom{p-1-i_m}{j_m}$$

for all $0 \le m \le n-1$ which is straightforward. Therefore $\phi_1 : U_{-\lambda} \to L(\lambda)_B$ is a surjective kB-map and since $U_{-\lambda}$ is indecomposable, $U_{-\lambda} \cong P(L(\lambda)_B)$ as required.

(b) Define

$$\phi_2 : L(\lambda)_B \to U_\lambda$$

$$U_{\underline{i}}(\lambda) \to U_{\underline{i}+\underline{p^n-1}-\underline{\lambda}}(p^n-1) \otimes \lambda \quad . \tag{15}$$

Clearly ϕ_2 is a well defined k-map which is injective. Furthermore

$$\underline{t}\circ\phi_2(U_{\underline{i}}(\lambda)) = \underline{t}\circ U_{\underline{i}+\underline{p^n-1}-\underline{\lambda}}(p^n-1) \otimes \lambda$$

$$= t^{2h(\underline{i}+\underline{p^n-1}-\underline{\lambda})-p^n-1+\lambda}U_{\underline{i}+\underline{p^n-1}-\underline{\lambda}}(p^n-1) \otimes \lambda$$

$$= t^{2h(\underline{i})-\lambda}\phi_2(U_{\underline{i}}(\lambda))$$

$$= \phi_2(t^{2h(\underline{i})-\lambda}U_{\underline{i}}(\lambda))$$

$$= \phi_2(\underline{t}\circ U_{\underline{i}}(\lambda))$$

as required. For the action of $z(a) \in P$ we again omit the "$\otimes \lambda$" .

$$z(a)\circ\phi_2(U_{\underline{i}}(\lambda)) = z(a)\circ U_{\underline{i}+\underline{p^n-1}-\underline{\lambda}}(p^n-1)$$

$$= \sum_{\underline{0}\leq\underline{j}\leq\underline{p^n-1}-(\underline{i}+\underline{p^n-1}-\underline{\lambda})} \left|\begin{matrix}\underline{p^n-1}-(\underline{i}+\underline{p^n-1}-\underline{\lambda}) \\ \underline{j}\end{matrix}\right| a^{h(\underline{j})}U_{\underline{i}+\underline{p^n-1}-\underline{\lambda}+\underline{j}}(p^n-1)$$

$$= \sum_{\underline{0}\leq\underline{j}\leq\underline{\lambda}-\underline{i}} \left|\begin{matrix}\underline{\lambda}-\underline{i} \\ \underline{j}\end{matrix}\right| a^{h(\underline{j})}U_{\underline{i}+\underline{p^n-1}-\underline{\lambda}+\underline{j}}(p^n-1)$$

$$= \sum_{\underline{0}\leq\underline{j}\leq\underline{\lambda}-\underline{i}} \left|\begin{matrix}\underline{\lambda}-\underline{i} \\ \underline{j}\end{matrix}\right| a^{h(\underline{j})}\phi_2(U_{\underline{i}+\underline{j}})(\lambda))$$

$$= \phi_2(z(a)\ U_{\underline{i}}(\lambda))$$

as required. This proves part (b).

□

In particular we see that $L(\lambda)_B$ is simple headed and has a simple socle since $\Sigma(U_\lambda) \cong Hd(U_\lambda) \cong S_\lambda$. As a consequence of 5.9,

(16) $$Hd(L(\lambda)_B) \cong S_{-\lambda} \;;$$

(17) $$\Sigma(L(\lambda)_B) \cong S_\lambda \;.$$

As a further corollary we see that:

(18) $$\Omega(L(\lambda)_B) \cong Ker(\phi_1)$$
$$= ksp\{U_{\underline{i}}(p^n-1) \otimes -\lambda \mid \underline{i} \nleq \underline{\lambda}\}$$

and

(19) $$\Omega^{-1}(L(\lambda)_B) \cong Coker(\phi_2)$$
$$= ksp\{\hat{U}_{\underline{i}}(p^n-1) \otimes \lambda \mid \underline{i} \in I(p^n-1)\}$$

where $\hat{U}_{\underline{i}}(p^n-1) \otimes \lambda = U_{\underline{i}}(p^n-1) \otimes \lambda + Im(\phi_2)$.

<u>Definition</u>.

For $0 \leq \lambda \leq p^n-1$ say λ is <u>almost-perfect</u> if $\lambda_i = p-1$ for all but one of the λ_i's . That is

$$\underline{\lambda} = (p-1,\ldots,\mu,p-1,\ldots,p-1)$$

where $\mu \neq p-1$.

For notational convenience we shall write

$$U(\underline{i}) = U_{\underline{i}}(p^n-1) \;,\quad \underline{i} \in I = I(p^n-1) \;.$$

5.10 Theorem.

For $0 \leq \lambda < p^n-1$ $L(\lambda)$ is periodic iff λ is almost perfect.

Proof.

Suppose $L(\lambda)$ is periodic. By 5.6 $p^{n-1} \mid \dim_k L(\lambda)$ since n is the p-rank of G . But by (7) $\dim_k L(\lambda) = \prod_{j=0}^{n-1}(\lambda_j+1)$ and we deduce that $\lambda_j = p-1$ for all but one of the j's . That is λ is almost perfect.

Conversely assume λ is almost-perfect so that $\underline{\lambda} = (p-1,\ldots,\mu,\ldots,p-1)$ and

$$h(\underline{\lambda}) = \sum_{\substack{j=0 \\ j\neq r}}^{n-1} p^j(p-1) + \mu p^r \;,\quad \text{some } \; 0 \leq r \leq n-1$$

$$= p^n-1 - (p-1-\mu)p^r \;.$$

Let $V(\mu,r) = L(\lambda)_B$. We show that $V(\mu,r)$, and hence $L(\lambda)$, is periodic. In fact we show that there exists an isomorphism.

(20) $$\Omega^2 V(\mu,r) \cong V(\mu,r) \otimes S_{2p^{r+1}}$$

so that in particular $V(\mu,r)$ is periodic with period p^n-1 .

Define a map

(21) $$\theta:\Omega^{-1}V(\mu,r)\otimes S_{2p^{r+1}} \to \Omega V(\mu,r)$$

$$\hat{U}(\underline{i}) \otimes \lambda \otimes 2p^{r+1}$$

$$\longrightarrow \begin{cases} \left|\dfrac{p^n-1-(\mu+1)p^r}{\underline{i}}\right| \left|\dfrac{p^n-1}{\underline{i}}\right|^{-1} . U(\underline{i}+(\underline{\mu+1})\underline{p^r}) \otimes -\lambda \, , & i_r < p-1-\mu \\ 0 & \text{otherwise.} \end{cases}$$

We must check this is well defined. Suppose $\hat{U}(\underline{i}) \otimes \lambda = 0$, that is $U(\underline{i}) \otimes \lambda \in \mathrm{Im}(\theta_2)$ as given in 5.9. By (15) $\underline{i} \geq (0,\ldots,p-1-\mu,0,\ldots,0)$ which implies that $i_r \geq p-1-\mu$ which means that $\theta(\hat{U}(\underline{i})\otimes\lambda\otimes 2p^{r+1}) = 0$. Also θ is onto. This follows because $U(\underline{i})\otimes-\lambda \in \Omega V(\mu,r)$ iff $\underline{i} \nleq \underline{\lambda}$ by (18) but $\underline{i} \nleq \underline{\lambda} = (p-1,\ldots,\mu,\ldots,p-1)$ which implies that $i_r \geq \mu+1$ and so θ is surjective. Clearly θ is injective so it is an isomorphism of k-spaces. We check the B-action noting that, since

$t^{p^n-1} = t^0$ $(t \in GF(p^n))$, $\underline{t}\circ U(\underline{j}) = t^{2h(\underline{j})}U(\underline{j})$ for all $\underline{j}$ I (9) .

(22) $\underline{t}\circ\theta(\hat{U}(\underline{i}) \otimes \lambda \otimes 2p^{r+1})$

$$= \left|\frac{p^n-1-(\mu+1)p^r}{\underline{i}}\right| \left|\frac{p^n-1}{\underline{i}}\right|^{-1} \underline{t}\circ U(\underline{i}+\underline{(\mu+1)p^r}) \otimes -\lambda$$

$$= \left|\frac{p^n-1-(\mu+1)p^r}{\underline{i}}\right| \left|\frac{p^n-1}{\underline{i}}\right|^{-1} t^{2h(\underline{i}+\underline{(\mu+1)p^r})-\lambda}U(\underline{i}+\underline{(\mu+1)p^r}) \otimes -\lambda$$

$$= t^{2h(\underline{i})+2(\mu+1)p^r-\lambda}\,\theta(\hat{U}(\underline{i}) \otimes \lambda \otimes 2p^{r+1})$$

whereas

(23) $\theta(\underline{t}\circ\hat{U}(\underline{i}) \otimes \lambda \otimes 2p^{r+1})$

$$= \theta(t^{2h(\underline{i})+\lambda+2p^{r+1}}\,\hat{U}(\underline{i}) \otimes \lambda \otimes 2p^{r+1}) \quad .$$

Comparing (22) with (23) it is enough to show that

(24) $\lambda + 2p^{r+1} \equiv 2(\mu+1)p^r - \lambda \mod(p^n-1)$.

Recall that $\lambda = p^n-1-(p-1-\mu)p^r \equiv -(p-1-\mu)p^r \bmod (p^n-1)$. The L.H.S. of (24) is $\lambda + 2p^{r+1} \equiv 2p^{r+1}-(p-1-\mu)p^r = (p+1+\mu)p^r = 2(\mu+1)p^r + (p-1-\mu)p^r \equiv 2(\mu+1)p^r-\lambda$ as required.

For the action of $z(a) \in P$ we ignore the tensor product as before and so we aim to show that

$$(25) \qquad z(a) \circ \theta(\hat{U}(\underline{i})) = \theta(z(a) \circ \hat{U}(\underline{i})) \ .$$

Now each U_λ is simple headed and is cyclic, being generated as a kB-module by $U(\underline{0}) \otimes \lambda$. Therefore $\Omega^{-1}(V(\mu,r))$ is generated by $\hat{U}(\underline{0}) \otimes \lambda$ (19) so it is enough to prove (25) for the case $\underline{i} = \underline{0}$.
Now,

$$z(a) \circ \theta(\hat{U}(\underline{0})) = z(a) \circ \left|\begin{matrix} \underline{p^n-1-(\mu+1)p^r} \\ \underline{0} \end{matrix}\right| \left|\begin{matrix} \underline{p^n-1} \\ \underline{0} \end{matrix}\right|^{-1} U((\underline{\mu+1})\underline{p^r})$$

$$= z(a) \circ U((\underline{\mu+1})\underline{p^r})$$

$$= \sum_{\underline{0} \le \underline{j} \le \underline{p^n-1-(\mu+1)p^r}} \left|\begin{matrix} \underline{p^n-1-(\mu+1)p^r} \\ \underline{j} \end{matrix}\right| a^{h(\underline{j})} U(\underline{j}+(\underline{\mu+1})\underline{p^r})$$

whilst

$$\theta(z(a) \circ \hat{U}(\underline{0})) = \sum_{\underline{0} \le \underline{j} \le \underline{p^n-1}} \left|\begin{matrix} \underline{p^n-1} \\ \underline{j} \end{matrix}\right| a^{h(\underline{j})} \theta(\hat{U}(\underline{j}))$$

$$= \sum_{\underline{0} \leq \underline{j} \leq \underline{p^n - 1} - (\underline{\mu+1})\underline{p^r}} \left| \frac{\underline{p^n-1}}{\underline{j}} \right| {}_a h(\underline{j}) \left| \frac{\underline{p^n-1} - (\underline{\mu+1})\underline{p^r}}{\underline{j}} \right| \left| \frac{\underline{p^n-1}}{\underline{j}} \right|^{-1} U(\underline{j} + (\underline{\mu+1})\underline{p^r})$$

$$= z(a) \circ \theta(\hat{U}(\underline{0}))$$

as required. Therefore $\Omega^{-1}V(\mu,r) \otimes S_{2p^{r+1}} \cong \Omega V(\mu,r)$ and applying Ω we see that

$$\Omega^2 V(\mu,r) \cong \Omega(\Omega^{-1}V(\mu,r) \otimes S_{2p^{r+1}})$$

$$\cong V(\mu,r) \otimes S_{2p^{r+1}} \quad \text{by 5.7(c)} .$$

$\square$

Consider the connected component containing $L(\lambda)$, λ almost-perfect. Since we have an isomorphism of stable quivers $Q(kG)_S \cong Q(kB)_S$ by (1) and 1.5 it will suffice to look at the connected component $q := q(V(\mu,r)) \leq Q(kB)_S$ where $\lambda = p^n - 1 - (p-1-\mu)p^r$. We need the following notation.

For each $\underline{i} \in I$, λ almost-perfect, let

(26) $$\underline{i}' = \underline{i} + \underline{p^n - 1} - \underline{\lambda}$$
$$= \underline{i} + \underline{(p-1-\mu)p^r} \; ; \quad (\text{if } i_r \leq \mu)$$

$$\underline{i}^* = \underline{i} + \underline{(\mu+1)p^r} \quad ; \qquad (\text{if } i_r+\mu+1 \le p-1)$$

$$\underline{i}_* = \underline{i} - \underline{(\mu+1)p^r} \quad ; \qquad (\text{if } i_r \ge \mu+1)$$

wherever this makes sense. Otherwise let $\underline{i}'$, $\underline{i}^*$, $\underline{i}_*$ equal zero.

5.11 Definition.

(a) For $m \in \mathbb{N}$ put $U(m) = \coprod_{\ell=1}^{m} U_{\lambda+2(\ell-1)p^{r+1}}$

(b) For $1 \le \ell \le m$, $\underline{i} \in I$ put

$$\underline{i}(\ell) = (0,\ldots,0,U(\underline{i})\otimes 2(\ell-1)p^{r+1}+\lambda,0,\ldots,0) \in U(m) \quad .$$

We aim to prove the following.

5.12 Theorem.

Let $q = q(V(\mu,r))$ be as above and let $\tau = \Omega^2$ be the Auslander-Translate.

(a) $q \cong \mathbb{Z}A_\infty/\langle\tau^{2^n-1}\rangle$, $p = 2$

$\mathbb{Z}A_\infty/\langle\tau^{(p^n-1)/2}\rangle$ p odd .

(b) The vertex set for q can be written

$$\{V_{a,m} | a = 0,\dots,2^n-1,\ m \in \mathbb{N}\}\ ,\quad p = 2$$

and $\{V_{a,m} | a = 0,\dots,(p^n-1)/2\ ,\ m \in \mathbb{N}\}$, p odd where, in both cases, $V_{0,1} \cong V(\mu,r)$, $V_{a,m} \cong V_{0,m} \otimes S_{2ap^{r+1}}$ and $\dim_k V_{a,m} = m.(\mu+1)p^{n-1}$.

(c) The almost split sequence ending in $V_{a,m}$ is

(i) $0 \to V_{a+1,1} \to V_{a,2} \to V_{a,1} \to 0$, $m = 1$

(ii) $0 \to V_{a+1,m} \to V_{a,m+1} \oplus V_{a+1,m-1} \to V_{a,m} \to 0$, $m > 1$.

(d) $V_{0,m} \subseteq U(m) = I(V_{0,m})$ and has a k-basis consisting of the following elements in $U(m)$:

$\underline{i}'(\ell)$ $\quad \underline{0} \neq \underline{i} \in I(\lambda)$, $\ell = 1,\dots,m$;

$w_s^m = \underline{0}'(s) + \gamma^{-1}.\underline{p^n-1}_*(s+1)$, $s = 1,\dots,m-1$;

$w_m^m = \underline{0}'(m)$

where $\gamma = \binom{p-1}{\mu+1}$.

<u>Proof.</u>

We use induction on m , constructing the almost-split sequence ending at $V_{0,m}$ to obtain $V_{0,m+1}$ as a direct summand of the middle term of $A(V_{0,m})$.

Set $V_{0,1} := I_m(\phi_2)$ as defined in (15) and let $V_{a,1} := \Omega^{2a}V_{0,1}$. Then $V_{0,1} \cong V(\mu,r)$ and $V_{a,1} \cong V_{0,1} \otimes S_{2ap^{r+1}}$ by (20). Notice that $V_{0,1}$ satisfies part (d) of the theorem by (15). Consider the pullback

$$\begin{tikzcd}
0 \arrow[r] & V_{1,1} \arrow[r] \arrow[d, equal] & E_1 \arrow[rr] \arrow[d] & & V_{0,1} \arrow[r] \arrow[d, "\theta_1"] & 0 & \\
0 \arrow[r] & V_{1,1} \arrow[r] & P(\Omega V_{0,1}) \arrow[rr, "\pi_1"'] & & P(V_{0,1}) \arrow[r] & V_{0,1} \arrow[r] & 0
\end{tikzcd}$$

recalling that $V_{1,1} \cong \Omega^2 V_{0,1}$. Since $V_{0,1}$ is simple headed by (16), $RV_{0,1}$ is the unique maximal submodule. To obtain an almost split sequence, J.A. Green's construction allows us to choose θ_1 such that $\mathrm{Ker}(\theta_1) = R(V_{0,1})$ and, by the argument used in §2 in the construction of almost split sequences in a (q,e)-uniserial block, we may use any θ_1 with this property. (See §1 and 2.6.) By considering the composition $V_{0,1} \to Hd(V_{0,1}) \xrightarrow{\cong} \Sigma(P(V_{0,1})) \xrightarrow{\cong} S_{-\lambda}$ (see (16)) we can define θ_1 explicitly by

(27)
$$\theta_1 : V_{0,1} \to U_{-\lambda}$$
$$\begin{aligned} U(\underline{i}')\otimes\lambda &\mapsto U(\underline{p^n-1})\otimes -\lambda, \quad \underline{i} = \underline{0} \\ &\ 0 \qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

We define π_1 as follows. Since $P(\Omega V_{0,1}) \cong P(\Omega^{-1}V_{0,1}) \otimes S_{2p^{r+1}}$ (5.10) and $P(\Omega^{-1}V_{0,1}) \cong U_{\lambda}$ by (19) we deduce that $P(\Omega V_{0,1}) \cong U_{\lambda+2p^{r+1}}$.

Now let π_1 be the composite $U_{\lambda} \otimes S_{2p^{r+1}} \rightarrow U_{\lambda}/V_{0,1} \otimes S_{2p^{r+1}}$ $\xrightarrow{\cong} \Omega^{-1}V_{0,1} \otimes S_{2p^{r+1}} \xrightarrow{\theta} \Omega V_{0,1} \xrightarrow{\text{incl.}} P(V_{0,1})$, noting that $\mathrm{Ker}(\pi_1) = V_{0,1} \otimes S_{2p^{r+1}} \cong V_{1,1}$. Explicitly -

$$(28) \qquad \pi_1 : P(\Omega V_{0,1}) \rightarrow P(V_{0,1})$$

$$U(\underline{i}) \otimes \lambda + 2p^r \rightarrow \gamma^*(\underline{i}).U(\underline{i}^*) \otimes -\lambda$$

where $\gamma^*(\underline{i}) = \left|\begin{matrix}\underline{p^n-1} \\ \underline{i}\end{matrix}\right|^{-1} \left|\begin{matrix}\underline{p^n-1_*} \\ \underline{i}\end{matrix}\right|$.

The middle term E_1 in the pullback is $\{(a,b) \in V_{0,1} \oplus P(\Omega V_{0,1}) | \theta_1(a) = = \pi_1(b)\}$ which, as a sum of k-spaces, is

$$\mathrm{Ker}(\theta_1) \oplus \mathrm{Ker}(\pi_1) \oplus k.(U(\underline{0}')\otimes\lambda, \gamma^{-1}.U(\underline{p^n-1_*})\otimes\lambda + 2p^{r+1})$$

(since $\gamma^*(\underline{p^n-1_*}) = \gamma$) which is equal to

$$(29) \qquad RV_{0,1} \oplus V_{1,1} \oplus k.w_1^2 \ .$$

Let $V_{0,2} := E_1$. By (29) we see that the k-basis for $V_{0,2}$ is as given in part (d) of the theorem. We will show later that $V_{0,2}$ is indecomposable.

Since $V_{a,1} \cong V_{0,1} \otimes S_{2ap^{r+1}}$ the above pullback construction shows that

$$0 \to V_{a+1,1} \to V_{0,2} \otimes S_{2ap^{r+1}} \to V_{a,1} \to 0$$

is the s.e.s. $A(V_{a,1})$. Consequently we define $V_{a,2} := V_{0,2} \otimes S_{2ap^{r+1}}$.

We now assume that Theorem 5.12 (c), (d) hold for all s , $1 \leq s \leq m$. That is : (1) $V_{0,s}$ has the desired k-basis as described in 5.12 (d);

(2) If the s.e.s.

$$0 \to V_{1,s-1} \to E_{s-1} \to V_{0,s-1} \to 0$$

is almost split then $E_{s-1} \cong V_{0,s} \oplus V_{1,s-2}$;

(3) There exist indecomposable modules $V_{a,s} \cong V_{0,s} \otimes S_{2ap^{r+1}}$ satisfying 5.12 (c).

Consider the pullback

$$(30) \qquad \begin{tikzcd} 0 \arrow[r] & V_{1,m} \arrow[r] \arrow[d, equal] & E_m \arrow[r] \arrow[d] & V_{0,m} \arrow[r] \arrow[d, "\theta_m"] & 0 & \\ 0 \arrow[r] & V_{1,m} \arrow[r] & P(\Omega V_{0,m}) \arrow[r, "\pi_m"'] & P(V_{0,m}) \arrow[r] & V_{0,m} \arrow[r] & 0 \end{tikzcd} .$$

We must show that $E_m \cong V_{0,m+1} \oplus V_{1,m-1}$ where $V_{0,m+1}$ is indecomposable and has the desired k-basis.

Recalling 5.8 and in particular (11), (12) it is easy to show that $\Sigma(V_{0,m}) = \sum_{j=1}^{m} {}^{\oplus} k.(\underline{p^n-1}(j) \cong \coprod_{j=0}^{m-1} S_{\lambda+2jp^{r+1}}$. From this we deduce that:

$$(31) \qquad P(\Omega^{-1}V_{0,m}) \cong I(V_{0,m}) \cong \coprod_{j=0}^{m-1} U_{\lambda+2jp^{r+1}} := U(m) .$$

$= \text{ksp } \{\underline{i}(\ell) \mid \underline{i} \in I , \ell = 1,..,m\}$. Therefore

$P(\Omega V_{0,m}) \cong P(\Omega^{-1}V_{0,m} \otimes S_{2p^{r+1}}) \cong P(\Omega^{-1}V_{0,m}) \otimes S_{2p^{r+1}} \cong U(m) \otimes S_{2p^{r+1}}$ by (21), 5.7(b) and (31).

Now let us make the identification

$$P(\Omega V_{0,m})/V_{1,m} = \Omega V_{0,m} \subseteq P(V_{0,m})$$

and define $\pi_m : P(\Omega V_{0,m}) \to P(V_{0,m})$ by

$$(32) \qquad \underline{i}(\ell) \otimes 2p^{r+1} \to \underline{i}(\ell) \otimes 2p^{r+1} + V_{1,m}$$

making $\text{Ker}(\pi_m) = V_{1,m}$. Now define

$$(33) \qquad \theta_m : V_{0,m} \to P(V_{0,m})$$

$$\underline{i}'(\ell) \to 0 \quad \underline{i} \neq 0 , \ell = 1,....,m ;$$

$$w_s^m \to 0 \qquad s = 2,\dots,m ;$$

$$w_1^m \to \gamma^{-1}\ \underline{p^n - 1}_*(1) \otimes 2p^{r+1} + V_{1,m} \ .$$

Notice that $\mathrm{Ker}(\theta_m) <\cdot V_{0,m}$, in fact we shall assume that θ_m induces the desired almost-split sequence via the pullback but shall prove this later. Now

$$E_m = \{(a,b) \in V_{0,m} \oplus P(\Omega V_{0,m}) | \theta_m(a) = \pi_m(b)\}$$

contains the submodule $\mathrm{Ker}(\theta_m) \oplus \mathrm{Ker}(\pi_m) \cong L \oplus V_{1,m}$ $(L = \mathrm{Ker}(\theta_m))$ and an element (w_1^m, b) . Here $b \in P(\Omega V_{0,m})$ is such that

$$\pi_m(b) = \theta_m(w_1^m) = \gamma^{-1}.\underline{p^n-1}_*(1) \otimes 2p^{r+1} + V_{1,m-1} \ ,$$

so we may let $b = \gamma^{-1}\underline{p^n-1}_*(1) \otimes 2p^{r+1}$. As a k-space, $E_m \cong L \oplus V_{1,m} \oplus k.(w_1^m, \gamma^{-1}\underline{p^n-1}_*(1) \otimes 2p^{r+1})$.

Define $\psi_1 : L \to RV_{0,1} \oplus V_{1,m-1}$

$$\underline{i}(s) \to (0, \underline{i}(s-1) \otimes 2p^{r+1}) \ ; \ s \neq 1 \tag{34}$$

$$\underline{i}(1) \to (U(\underline{i})\otimes\lambda, 0) \ .$$

A few calculations show that ψ_1 is a kB-isomorphism - it is instructive

$$w_s^m \to 0 \qquad s = 2,\ldots,m \ ;$$

$$w_1^m \to \gamma^{-1}\ \underline{p^n-1}_*(1) \otimes 2p^{r+1} + V_{1,m} \ .$$

Notice that $\mathrm{Ker}(\theta_m) <\cdot V_{0,m}$, in fact we shall assume that θ_m induces the desired almost-split sequence via the pullback but shall prove this later. Now

$$E_m = \{(a,b) \in V_{0,m} \oplus P(\Omega V_{0,m}) \mid \theta_m(a) = \pi_m(b)\}$$

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Define $\psi_1 : L \to RV_{0,1} \oplus V_{1,m-1}$

(34) $$\underline{i}(s) \to (0,\ \underline{i}(s-1) \otimes 2p^{r+1}) \ ; \quad s \neq 1$$

$$\underline{i}(1) \to (U(\underline{i})\otimes\lambda,\ 0) \ .$$

A few calculations show that ψ_1 is a kB-isomorphism - it is instructive to show that ψ_1 does not extend to a kB-isomorphism $\psi_1' : V_{0,m} \to V_{0,1} \oplus V_{1,m-1}$. Hence $E_m \cong RV_{0,1} \oplus V_{1,m-1} \oplus V_{1,m} \oplus k(w_1^m, \gamma^{-1}\underline{p^n-1}_*(1) \otimes 2p^{r+1})$ as a k-space. Define a map

$$(35) \qquad \psi_2 : k(w_1^m, \gamma^{-1}\underline{p^n-1}_*(1) \otimes 2p^{r+1}) \oplus RV_{0,1} \oplus V_{1,m} \to V_{0,m+1}$$

by

$$(w_1^m, \gamma^{-1}\underline{p^n-1}_*(1) \otimes 2p^{r+1}) \longrightarrow w_1^{m+1}$$

$$U(\underline{i}) \otimes \lambda \longrightarrow \underline{i}(1) \quad , \quad U(i) \otimes \lambda \in RV_{0,1} \quad ;$$

$$\underline{i}(s) \otimes 2p^{r+1} \longrightarrow \underline{i}(s+1) \quad , \quad \underline{i}(s) \otimes 2p^{r+1} \in V_{1,m} \quad .$$

Again, ψ_2 is a kB-isomorphism and combining (34) and (35) we see that

$$E_m \cong V_{0,m+1} \oplus V_{1,m-1}$$

and that $V_{0,m+1}$ is as defined in part (d). Since $V_{a,m} \cong V_{0,m} \otimes S_{2ap^{r+1}}$ the above pullback construction shows that:

$$0 \to V_{a+1,m} \to V_{0,m+1} \otimes S_{2ap^{r+1}} \oplus V_{a+1,m-1} \to V_{a,m} \to 0$$

is almost-split. Define $V_{a,m+1} := V_{0,m+1} \otimes S_{2ap^{r+1}}$. This proves parts (c) and (d) of the theorem. For parts (a), (b) we note that, for

$p = 2$, $\tau^a V_{0,m} \cong V_{0,m} \otimes S_{2a2^{r+1}} \cong V_{0,m}$ iff $2^n - 1 | a$ whilst for p odd, $\tau^a V_{0,m} \cong V_{0,m} \otimes S_{2ap^{r+1}}$ iff $(P^n - 1)/2 | a$. Finally, $\dim_k V_{a,1} = \dim_k V(\mu, r) = (\mu+1)p^{n-1}$ by (7) and $\dim_k V_{a,m} = m(\mu+1)p^{n-1}$ follows by induction.

It remains to show that each $V_{a,m}$ is indecomposable and that θ_m does induce an almost-split sequence. First we shall show that $\mathrm{End}_{kB}(V_{a,m})$ is uniserial, hence local, from which it follows that $V_{a,m}$ is indecomposable.

Let $E(m)$ denote the endomorphism algebra of the injective module $U(m) = I(V_{0,m})$. Since $U(m) \cong \coprod_{j=0}^{m-1} U_{\lambda + j2p^{r+1}}$, any $\eta \in E(m)$ is determined by the m-tuple $(\eta(\underline{O}(1)), \ldots, \eta(\underline{O}(m)))$. Let $\eta' \in \mathrm{End}_{kB}(V_{0,m})$. Since $U(m)$ is injective we can complete the diagram

$$\begin{tikzcd} & & U(m) \\ & V_{0,m} \arrow[ur, "i"] & \\ V_{0,m} \arrow[ur, "\eta'"] \arrow[rr, "i"'] & & U(m) \arrow[uu, dashed, "\eta"'] \end{tikzcd}$$

Hence each $\eta' \in \mathrm{End}_{kB}(V_{0,m})$ can be extended to an $\eta \in E(m)$ such that

(36) $\qquad \eta|_{V_{0,m}} = \eta'$.

Let $EP(m) = End_{kP}(U(m))$. We find it easier to work over kP and then restrict back to kB afterwards. For $\eta \in EP(m)$ define $C_{rs} \in k$, $1 \leq r,s \leq m$, such that

$$\eta(\underline{0}(s)) = \sum_{r=1}^{m} C_{rs}\underline{0}(r) \quad \text{modulo } R(U(m))..$$

Since the $\underline{0}(s)$ $(1 \leq s \leq m)$ are kP generators of $U(m)$ the C_{rs} $(1 \leq r,s \leq m)$ determine $\eta \in EP(m)$. To satisfy (36) we want an η such that $\eta(w_s^m) \in V_{0,m}$ (the w_s^m are kP generators of $V_{0,m}$) .

Notation.

For $\underline{i} \in I$ denote by $M(\underline{i}) \leq U(m)$ the submodule spanned by elements of the form $\{\underline{j}(r) \mid \underline{j} > \underline{i} , r = 1,\ldots,m\}$. In particular -

$$M(\underline{0}) = RU(m) \quad , \quad M(\underline{0}') = RV_{0,m} \ .$$

Omitting details, the following lemma is true.

5.13 Lemma.

Let $\eta \in EP(m)$, $C_{rs} \in k$ be given as above. Then $\eta(\underline{i}(s)) \equiv \sum_{r=1}^{m} C_{rs}.\underline{i}(r) \bmod M(\underline{i})$. □

Consider w_s^m , $1 \leq s \leq m$. For $1 \leq s < m$,

$$\eta(w_s^m) = \eta(\underline{0}'(s)) + \eta(\gamma^{-1}.\underline{p^n-1}_*(s+1))$$

$$\equiv \sum_{r=1}^{m} C_{rs}\ \underline{0}'(r) + M(\underline{0}') + \sum_{r=1}^{m} C_{r,s+1}.\underline{p^n-1}_*(r) + M(\underline{p^n-1}_*)$$

by 5.13. Since $\underline{p^n-1}_* \nleq \underline{0}'$ - implying $\underline{p^n-1}_*(r) \notin M(\underline{0}')$ - and $M(\underline{p^n-1}_*) \leq M(\underline{0}')$ it follows that

(37) $$\eta(w_s^m) \equiv \sum_{r=1}^{m} (C_{rs}.\underline{0}'(r) + C_{r,s+1}\gamma^{-1}.\underline{p^n-1}_*(r)) + M(\underline{0}') .$$

(38) For $s=m$ $$\eta(w_m^m) = \eta(\underline{0}'(m)) \equiv \sum_{r=1}^{m} C_{rm}\ \underline{0}'(r) + M(\underline{0}') .$$

However for η to satisfy (36), that is $\eta(V_{0,m}) \leq V_{0,m}$,

(39) $$(w_s^m) \equiv \sum_{r=1}^{m} b_{rs} w_r^m + M(\underline{0}')$$

$$\equiv \sum_{r=1}^{m} (b_{rs}\underline{0}'(r) + b_{rs}\gamma^{-1}\underline{p^n-1}_*(r)) + M(\underline{0}')$$

for $1 \leq s \leq m$. Comparing (37), (38) with (39) :

$$C_{rs} = C_{r+1,s+1} \qquad r,s < m ;$$

$$C_{rm} = 0 \qquad r < m .$$

Therefore each $\eta \in \mathrm{End}_{kP}(V_{0,m})$ is determined by a matrix $C = (C_{ij})$ of the form

$$\begin{pmatrix} C_1 & & & & \\ C_2 & C_1 & & 0 & \\ C_3 & C_2 & C_1 & & \\ & & & & \\ C_m & C_{m-1} & \cdots\cdots & C_2 & C_1 \end{pmatrix}$$

where $C_j = C_{j1}$. It is clear that $\mathrm{End}_{kP}(V_{0,m})$ is uniserial with a unique chain of ideals:

$$0 \lessdot I_m \lessdot I_{m-1} \lessdot \cdots\cdots \lessdot I_1 = \mathrm{End}_{kP}(V_{0,m})$$

where $I_S = \{(C_{ij}) \mid C_{ij} \in I_1 ,\ C_j = 0 \quad 1 \leq j \leq s-1\}$.

We can embed $\mathrm{End}_{kB}(V_{0,m})$ in $\mathrm{End}_{kP}(V_{0,m})$ and so $\mathrm{End}_{kB}(V_{0,m})$ is uniserial and local. Hence $V_{0,m}$ is indecomposable and thus $V_{a,m} \cong V_{0,m} \otimes S_{2ap^{r+1}}$ is indecomposable as required.

Finally we show that θ_m induces an almost-split sequence in the pullback (30). Recall from §1 that we must show:

(40) $\qquad T_{\theta_m}(1) \neq 0 \quad , \quad 1 = 1_{V_{0,m}} \in \mathrm{End}_{kB}(V_{0,m})$;

(41) $$T_{\theta_m}(R\ \mathrm{End}_{kB}(V_{0,m})) = 0 \ .$$

Now (40) is clear from the definition of T_{θ_m} and by the above remarks it is enough to show that $T_{\theta_m}(R\ \mathrm{End}_{kP}(V_{0,m})) = T_{\theta_m}(I_2) = 0$.

For $\eta \in I_2$:

$$\begin{aligned} T_{\theta_m}(\eta) &= t_1(\eta(w_1^m)) + \dots + t_m(\eta(w_m^m)) \\ &= \xi_1\theta_m\eta(w_1^m) + \dots + \xi_m\theta_m\eta(w_m^m) \ . \end{aligned}$$

Now $\mathrm{Ker}(\theta_m) <\cdot\ V_{0,m}$ and in particular $w_s^m \in \mathrm{Ker}(\theta_m)$ for $s = 2,\dots,m$. Since $\eta \in I_2$ $\eta(w_s^m) = \sum_{r=2}^{m} c_{rs}w_r^m$ and so $\theta_m(\eta(w_s^m)) = 0$ implying that $T_{\theta_m}(\eta) = 0$ as required.

$\square$

To conclude this chapter we give a few more details on the connected components and briefly describe their correspondents in $Q(kG)$.

Notice that, for $p = 2$, q contains $\Omega^a V(\mu,r)$ for all powers of Ω whereas for p odd q contains just the even powers. Let $\lambda, \lambda' \in \{1,\dots,p^n-1\}$ be almost-perfect such that $\lambda = p^n-1-(p-1-\ )p^r$, $\lambda' = p^n-1-(p-1-\nu)p^s$ and let $q_0(\lambda)$, $q_1(\lambda)$ be the connected components containing $L(\lambda)_B$, $\Omega L(\lambda)_B$ respectively.

5.14 Proposition.

(a) For $p = 2$, $q_0(\lambda) = q_1(\lambda')$ and $q_0(\lambda) = q_0(\lambda')$ iff $\lambda = \lambda'$.

(b) For p odd, $q_i(\lambda) = q_j(\lambda')$ iff $i = j$, $\lambda = \lambda'$ where $i,j \in \{0,1\}$.

Proof.

Assume $q_i(\lambda) = q_j(\lambda')$ and suppose $i \neq j$. We may assume that $i = 0$, $j = 1$ and so $L(\lambda)_B \in q_1(\lambda')$. Since $L(\lambda)_B$ $(= V(\mu,r))$ lies at the 'bottom' of $q_0(\lambda)$ we deduce that:

$$(42) \qquad L(\lambda)_B \cong \Omega^{2a+1}L(\lambda')_B \cong \Omega L(\lambda')_B \oplus S_{2ap^{s+1}}, \text{ some } a \in \mathbb{N} .$$

Taking dimensions of (42) using (7) it follows that $\dim_k L(\lambda)_B = p^n - \dim_k L(\lambda')_B$, implying that $(\mu+1)p^{n-1} = p^n-(\nu+1)p^{n-1} = p^{n-1}(p-\nu-1)$. By previous remarks we can assume p is odd and we deduce from the above that μ is even iff ν is even. But μ even implies that $\dim_k L(\lambda)_B$ is odd contradicting the fact that $\dim_k \Omega^{2a+1}L(\lambda')_B$ is even.

We may assume then that $i = j$ and so must show that $q_0(\lambda) = q_0(\lambda')$ implies that $\lambda = \lambda'$. Since $L(\lambda)_B \in q_0(\lambda) = q_0(\lambda')$ there exists $a \in \mathbb{N}$ such that

$$(43) \qquad L(\lambda)_B \cong \Omega^{2a}L(\lambda')_B \cong L(\lambda')_B \oplus S_{2ap^{s+1}} .$$

Taking dimensions (7) implies that $(\mu+1)p^{n-1} = (\nu+1)p^{n-1}$ forcing $\mu = \nu$. Suppose $r \neq s$. Applying $\tau = \Omega^2$ to both sides of (43) we obtain:

(44) $$L(\lambda)_B \otimes S_{2p^{r+1}} \cong \tau L(\lambda)_B \cong \tau L(\lambda')_B \otimes S_{2ap^{s+1}} \cong$$
$$L(\lambda')_B \otimes S_{2p^{s+1}} \otimes S_{2ap^{s+1}} .$$

Combining (43) and (44), $S_{2p^{r+1}} \cong S_{2p^{s+1}}$ which implies that $2p^{r+1} - 2p^{s+1} \equiv 0 \bmod p^n - 1$ and so $r = s$ as required. □

Finally we look at the full quivers $Q(kB)$, $Q(kG)$. To take advantage of the Green Correspondence between mod kB and mod kG we have had to look at the connected components of the stable quivers. For an almost-perfect λ let us look at $q(L(\lambda)) \leq Q(kG)_S$. Recalling 1.10 a projective module W only occurs in the almost-split sequence

(45) $$0 \to RW \to W \oplus RW/\Sigma W \to W/\Sigma W \to 0 .$$

If $W = P(L(\lambda))$ (45) can be written as

(46) $$0 \to \Omega L(\lambda) \to P(L(\lambda)) \oplus \Omega L(\lambda)/L(\lambda) \to \Omega^{-1}L(\lambda) \to 0 .$$

Thus, for p odd $q(L(\lambda))$ is the full component in $Q(kG)$ containing $L(\lambda)$ whilst for $p = 2$ the full component is $q(L(\lambda)) \cup \{P(L(\lambda))\}$ with the attaching arrows.

Taking dimensions (7) implies that $(\mu+1)p^{n-1} = (\nu+1)p^{n-1}$ forcing $\mu = \nu$. Suppose $r \neq s$. Applying $\tau = \Omega^2$ to both sides of (43) we obtain:

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$$L(\lambda')_B \otimes S_{2p^{s+1}} \otimes S_{2ap^{s+1}} \ .$$

Combining (43) and (44), $S_{2p^{r+1}} \cong S_{2p^{s+1}}$ which implies that $2p^{r+1} - 2p^{s+1} \equiv 0 \bmod p^n-1$ and so $r = s$ as required.

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(44) $$L(\lambda)_B \otimes S_{2p^{r+1}} \cong \tau L(\lambda)_B \cong \tau L(\lambda')_B \otimes S_{2ap^{s+1}} \cong$$
$$L(\lambda')_B \otimes S_{2p^{s+1}} \otimes S_{2ap^{s+1}} .$$

Combining (43) and (44), $S_{2p^{r+1}} \cong S_{2p^{s+1}}$ which implies that $2p^{r+1} - 2p^{s+1} \equiv 0 \bmod p^n-1$ and so $r = s$ as required. $\square$

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REFERENCES.

[A1] Alperin, J.L. : "Periodicity in groups." Illinois J. Math. 21, 776-783 (1977).

[AR1] Auslander, M., Reiten, I. : "Representation of Artin algebras III: Almost split sequences." Comm. Algebra 3, 239-294 (1975).

[AR2] Auslander, M., Reiten, I. : "Representations of Artin algebras IV : Invariants given by almost split sequences." Comm. Algebra 5, 443-518 (1977).

[AR3] Auslander, M., Reiten, I. : "Representations of Artin algebras V : Methods for computing almost split sequences and irreducible morphisms." Comm. Algebra 5, 519-544 (1977).

[BP] Benson, D., Parker, R.A. : "The Green ring of a finite group." Preprint (Aarhus) 1982.

[Br] Brauer, R., Nesbitt, C. : "On the modular characters of groups." Annals of Math. 42, 556-590 (1941).

[Bu1] Butler, M.C.R. : "The construction of almost split sequences I." Proc. London Math. Soc. 40, 72-86 (1980).

[Bu2] Butler, M.C.R. : "Grothendieck groups and almost split sequences." In: Integral representations and applications, Lecture notes in Mathematics 882, pp.357-368, Springer, Berlin-Heidelberg-New York, 1981.

[C] Carlson, J.F. : "The dimensions of periodic modules over modular group algebras." Illinois J. Math. 23, 295-306 (1979).

[CR] Curtis, C.W., Reiner, I. : "Methods of representation theory with applications to finite groups and orders." J. Wiley and Sons, New York, 1981.

[D] Dade, E.C. : "Blocks with cyclic defect groups." Annals of Math. 84, 20-48 (1966).

[E1] Erdmann, K. : "Projective modules in blocks with elementary abelian defect groups." Math. Z. 183, 177-215 (1983).

[E2] Erdmann, K. : "On periodic modules in p-blocks of type SL_2." Preprint.

[Ga] Gabriel, P. : "Auslander-Reiten sequences and representation-finite algebras." In: Representation theory I, Lecture notes in mathematics 831, pp.1-71, Springer, Berlin-Heidelberg-New York, 1980.

[Gr1] Green, J.A. : "Relative module categories for finite groups." J. Pure Appl. Algebra 2, 371-393, 1972.

[Gr2] Green, J.A. : "Walking around the Brauer tree." J. Austral. Math. Soc. 17, 197-213, 1974.

[Gr3] Green, J.A. : "Notes on almost split sequences, I." Preprint.

[H] Higman, D.G. : "Modules with a group of operators." Duke Math. J. 21, 369-376, 1954.

[J] James, G.D. : "The modular characters of Mathieu groups." J. Algebra 27, 57-111, 1973.

[Ja] Janusz, G. : "Indecomposable modules for finite groups." Annals of Math. 89, 209-241, 1969.

[P] Peacock, R.M. : "Blocks with a cyclic defect group." J. Algebra 34, 232-259, 1975.

[R1] Riedtmann, C. : "Algebren, Darstellungsköcher, Überlagerungen und zurück." Comment. Math. Helv. 55, 199-224, 1980.

[R2] Riedtmann, C. : "Representation-finite selfinjective algebras of class A_n." In: Representation theory II, Lecture notes in mathematics 832, pp.449-520, Springer, Berlin-Heidelberg-New York, 1980.

[Ri] Ringel, C.M. : "Report on the Brauer-Thrall conjectures: Roiter's theorem and the theorem of Nazarova and Roiter. In: Representation theory I. Lecture notes in mathematics 831, pp.104-136, Springer, Berlin-Heidelberg-New York, 1980.

[S] Steinberg, R. : "Representations of algebraic groups." Nagoya Math. 22, 33-56, 1963.

[W] Webb, P.J. : "The Auslander-Reitern Quiver of a Finite Group." Math. Z. 179, 97-121 (1982).

[Re] Reiten, I. : "Almost split sequences for group algebras of finite representation type." Trans Amer. Math. Soc. 233, 125-136 (1977).

[GaR] Gabriel, P., Riedtmann, C. : "Group representations without groups." Comment. Math. Helv. 54, 240-287 (1979).

[H] Happel, D., Preiser, U., Ringel, C.M. : "Vindberg's Characterization of Dynkin diagrams using subadditive functions with application to DTr-Periodic modules." In: Representation theory II, Lecture notes in mathematics 831, pp.280-294, Springer, Berlin-Heidelberg-New York, 1980.