ISOTROPIC HARMONIC MAPS TO KÄHLER MANIFOLDS AND RELATED PROPERTIES

by

James F. Glazebrook

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ABSTRACT

This thesis presents a detailed account of how a class of harmonic maps of Riemann surfaces into certain homogeneous Kähler manifolds, with possibly an indefinite metric, may be constructed and classified in terms of holomorphic maps.

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INTRODUCTION.

Taking complex projective space of n dimensions, $\mathbb{C}P^n$, with its Fubini-Study metric, Eells and Wood in [35] describe in detail, a bijective correspondence between full, holomorphic maps $f:M \to \mathbb{C}P^n$ where M is a Riemann surface (open or closed) and isotropic harmonic maps $\phi:M \to \mathbb{C}P^n$ (see below). Their main result, broadly stated, is as follows:

Let $L \to \mathbb{C}P^n$ be the universal line bundle; we can define a universal lift Φ of a smooth map $\phi:M \to \mathbb{C}P^n$, as a section of the bundle $\mathrm{Hom}(\phi^{-1}L, \underline{\mathbb{C}}^{n+1}) \to M$ (here $\underline{\mathbb{C}}^{n+1}$ denotes the trivial $(n+1)$-plane bundle). If we take D to denote covariant differentiation in this bundle, then D splits into complex types D' and D''. The harmonicity equation for ϕ is

$$D''D'\Phi + |D'\Phi|^2\Phi = 0 \quad .$$

With respect to the Hermitian inner product $\langle , \rangle$, we say that ϕ is isotropic if

$$\langle D'^{\alpha}\Phi, D''^{\beta}\Phi\rangle \equiv 0 \qquad \text{for all } \alpha,\beta \text{ with } \alpha+\beta \geq 1 \ .$$

We say that a map into $\mathbb{C}P^n$ is full, if its image lies in no proper projective subspace.

THE EELLS-WOOD CLASSIFICATION THEOREM [34] [35]

Given an integer r $(0 \le r \le n)$ and a full holomorphic map $f:M \to \mathbb{C}P^n$, then the map $\phi:M \to \mathbb{C}P^n$ defined by $\phi(x) = f^{\perp}_{r-1}(x) \cap f_r(x)$, for all $x \in M$, is a full, isotropic harmonic map. Here f_k denotes the kth associated curve of f (Definition 2.3).

The correspondence $(f,r) \to \phi$ is bijective. The inverse assignment is given by setting r equal to the maximum dimension of the subspace spanned by $\{D''^{\beta}\phi\}$ and f is obtained from ϕ by a similar procedure to that defining ϕ in terms of f.

Let us briefly review the historical background to this theorem. The theorem, first announced in [34], had its origins in an analogous theorem due to Calabi [18] [19] for maps into $\mathbb{R}P^{2r}$ and S^{2r}. Calabi's theorem was further developed by Chern [25] [26] and Barbosa [4]. In his development of Calabi's ideas, Chern also reintroduced the methods of an original study made by Boruvka in 1933 [13].

These endeavours have not only been confined to the realm of pure mathematics. Some thirteen years after, Calabi's theorem was rediscovered by the physicists Borchers and Garber [9] whilst working within the context of non-linear σ-models (see e.g. [45] for a concise account). Shortly after, Din and Zakrzewski [28] [29], and Glaser and Stora [40], also working in this area, provided a foundation for the above classification theorem for $\mathbb{C}P^n$, in the case where $M = S^2$; Burns in [17] outlined the underlying principles from a mathematician's viewpoint. Independently,

Eells and Wood proceeded to develop and generalize the techniques within a more sophisticated framework. Their main result is more fully described in Chapter II. Wolfson, in his Berkeley thesis [90], has interpreted the Eells-Wood theorem from the viewpoint of moving frames.

In [39], Erdem and Wood generalised the methods of [35], to cover the case of the complex Grassmannian $G_k(\mathbb{C}^N)$; Din and Zakrzewski had previously generalised their own methods to certain cases [27]. Following these, Ramananathan [75] undertook a rigorous study of harmonic maps $S^2 \to G_2(\mathbb{C}^4)$. Quite recently, Rawnsley [77] has described these constructions in their full generality from the point of view of a generalised twistor space setting (see below).

This thesis presents an account of how the techniques of [35] and [39] may be extended and modified to construct and classify analogous (full, isotropic) harmonic maps from M to classes of classical symmetric spaces, possibly endowed with an indefinite metric. We also include results for maps to the quaternionic projective space $\mathbb{H}P^n$ and to the quaternionic Grassmannian $G_k(\mathbb{H}^N)$.

Here is a brief description of the contents of each chapter. Chapter I serves as an introduction to harmonic maps, Kähler manifolds and the composition principles employed in the thesis. In Chapter II, following a discussion of holomorphic curves in $\mathbb{C}P^n$, we describe the Eells-Wood construction and its restriction to $\mathbb{R}P^{2r}$ and S^{2r} . Chapter III deals with the generalisation of the methods in [35] to the case of the complex

Grassmannian $G_k(\mathbb{C}^N)$; this is essentially an exposition of the main result of Erdem and Wood [39], but we also demonstrate how their results can be extended to other compact (irreducible) Hermitian symmetric spaces (Types I-IV in Cartan's classification [20]). The case of the real Grassmannian (oriented and non-oriented), is also included.

Chapter IV is devoted to the quaternionic Grassmannian $G_k(\mathbb{H}^N)$. Again, we extend the methods of [39] to obtain a classification theorem for full, isotropic harmonic maps $M \to G_k(\mathbb{H}^N)$; this correspondence is structured around the homogeneous fibration

$$R_{k,N-k} \cong Sp(N)/Sp(k)\times U(N-k) \to G_k(\mathbb{H}^N) \cong Sp(N)/Sp(k)\times Sp(N-k) .$$

In brief, the main result of this chapter is: (Theorem 4.3) <u>There exists a bijective correspondence between full, isotropic harmonic maps $\phi: M \to G_k(\mathbb{H}^N)$ and certain holomorphic subbundles of rank</u> (N-k) <u>of the trivial 2N-plane bundle $\underline{\mathbb{C}}^{2N}$ on</u> M .

A key fact which is exploited, concerns the totally geodesic embedding of $G_k(\mathbb{H}^N)$ in $G_{2k}(\mathbb{C}^{2N})$. This is described in some detail for the special case of $\mathbb{H}P^n \subset G_2(\mathbb{C}^{2n+2})$, in the appendix, where we also include the Type III Hermitian symmetric space $Sp(n)/U(n)$ which is totally geodesic in $G_n(\mathbb{C}^{2n})$.

Harmonic maps to the indefinite metric analogues of the spaces in Chapters II, III and IV, are studied in Chapters V and VI. Chapter V is devoted to a detailed description of a class of harmonic maps from M

to the indefinite complex hyperbolic space $\mathbb{C}H^n_p$. We also mention the indefinite complex projective space $\mathbb{C}P^n_p$ and discuss this case in Chapter VI. The real analogues of these indefinite space forms were treated explicitly by Erdem [36] [37] who also made an independent study of the complex case in his Leeds thesis. The fruits of our respective research has constituted a joint paper [38] the main result of which (see also Theorem 5.31 in Chapter V) we shall now state, and refer to Chapter V for notation and terminology:

THE CLASSIFICATION THEOREM FOR $\mathbb{C}H^n_p$ [38]

Let Q_p *be an indefinite Hermitian inner product of signature* (p,q) (see (5.1)) *and let* M *be as above. There is a bijective correspondence between the pairs* (f,r) *where* $f:M \to \mathbb{C}P^n$ *is a full, holomorphic map with a non-empty index set* $\Lambda_p^{(0,1)}(f)$ *and* $r \in \Lambda_p^{(0,1)}(f)$, *and full* Q_p*-isotropic harmonic maps* $\phi:M \to \mathbb{C}H^n_p$ *where* Q_p *is non-degenerate on the subspaces* $\phi''_{(r)}(x)$ *for all* $x \in M$. *The map* ϕ *is defined by*

$$\phi(x) = f^{\perp_p}_{r-1}(x) \cap f_r(x) .$$

The map f *is recovered by an inverse transformation similar to the above construction for that of* ϕ .

The study presented in this thesis entails keeping track of the signature Q_p on certain subspaces. We also discuss the special case of $\mathbb{C}H^n_n = \mathbb{C}H^n$, the complex hyperbolic space with positive definite

Bergman metric (5.12) and its compact quotients $\mathbb{C}H^n/\Gamma$, where Γ is a certain discrete group.

In Chapter VI we look at analogous maps to the open orbits of a given Lie group action on the (irreducible) compact Hermitian symmetric spaces (Types I-IV). The general open orbit in each of the four types is endowed with an indefinite Kähler metric [87]; exactly one, the non-compact Hermitian space dual, is endowed with a positive definite Bergman metric (these metrics for each of Types I-IV, are described in [58]). We also look at the indefinite real and quaternionic Grassmannians.

Chapter VII is essentially an independent section of the thesis. Here, we establish a generalised Riemann-Hurwitz formula for a smooth branched covering map $\phi:M \to N$ where M and N are equi-dimensional, smooth, compact orientable manifolds. If N_1 is a closed, connected orientable submanifold of N of codimension 2, and if ℓ is taken to denote the degree of ϕ , then we establish the formula

$$X(M) = \ell.X(N) - (\ell-1)X(N_1) .$$

The methods are based on those of Ngô Van Quê [70]. We demonstrate, by introducing certain generalities, how his results are simultaneously true for certain characteristic classes in the smooth category.

Various <u>twistor spaces</u> play a key role in this thesis. Let us now give a brief idea as to what they are, <u>since we shall use this term freely</u> throughout the thesis without embarking on a detailed discussion in the text.

The original twistor fibering $\mathbb{C}P^3 \to S^4$ was introduced by Penrose and later developed by Atiyah, Hitchin and Singer, [2] to transform solutions of certain field equations on S^4 into holomorphic vector bundles on (the twistor space) $\mathbb{C}P^3$ (see [1] for a general survey). In fact the underlying geometric principle is based on a line-geometric correspondence due to Klein; this is described in [26]. Bryant in [15] has exploited this line geometry to show that every compact Riemann surface may be conformally and minimally immersed in S^4, by constructing maps with non-zero Jacobian, from certain holomorphic maps to $\mathbb{C}P^3$. More generally, one takes an even dimensional manifold N and considers the bundle Z (<u>the twistor space</u>) of almost complex structures on TN. For each linear connection on N, there exists a natural almost complex structure on Z. The problem of integrability of this almost complex structure has been studied in [6] and [71].

The homogeneous fibrations which will play a leading part in our work, are all examples of twistor fibrations. We do not pursue a detailed account of this matter, but refer to [16], [77] and [96] for generalisations and further examples.

CHAPTER I

PRELIMINARIES

1.1 Introduction to Chapter I.

Throughout this chapter, a nodding acquaintance with the basic principles of Riemannian geometry and the theory of fibre bundles over a smooth manifold, is assumed. With this in mind, I have omitted many elementary details and suggest [56] [57] and [52] respectively, as sources of foundational material. Some familiarity with the basic ideas of complex differential geometry is also assumed. References [24] and [86] are more than adequate for this purpose, as are the relevant sections of [42]; a very digestible account of the basic groundwork is provided by [63]. For Lie group theory, [48] is recommended, along with the relevant sections of [56] and [57].

In the next section, we shall introduce the notion of a harmonic map of Riemannian manifolds. Throughout this thesis we refer to the excellent report [31], as well as to [32], for further details.

1.2 Harmonic maps of Riemannian manifolds.

Let (M,g) and (N,h) denote two smooth (i.e. C^∞) Riemannian manifolds M and N (without boundary), with respective Riemannian metrics g and h. Both M and N are endowed with their respective Levi-Civita connections [56]. Let m and n denote their respective dimensions.

If

(1.1) $\quad V \to M$

is a smooth vector bundle on M, then we shall denote by $C(V)$ (respectively $C_U(V)$), the space of smooth sections of V defined on M (respectively, on an open set U in M). A connection ∇^V on V, and a metric g^F along the fibres, define <u>a Riemannian structure on V</u> if $\nabla^V g^F = 0$.

This implies

(1.2) $$Xg^F(\sigma_1,\sigma_2) = g^F(\nabla^V_X\sigma_1,\sigma_2) + g^F(\sigma_1,\nabla^V_X\sigma_2)$$

where $X \in C(TM)$ is a vector field on M, and $\sigma_1,\sigma_2 \in C(V)$.

<u>Remark</u>.

When g^F is an indefinite Riemannian metric [56], we will have defined an indefinite Riemannian structure. We shall discuss a class of indefinite (metric) Riemannian manifolds in Chapters V and VI.

<u>The curvature</u> of the connection ∇^V is the map $R^V : \Lambda^2 C(TM) \times C(V) \to C(V)$ defined by

(1.3) $$R^V(X,Y)\sigma = -\nabla^V_X\nabla^V_Y\sigma + \nabla^V_Y\nabla^V_X\sigma + \nabla^V_{[X,Y]}\sigma$$
$$= -R^V(Y,X)\sigma$$

for $X,Y \in C(TM)$. When $V = TM$, the tangent bundle of M , we usually write

$$(1.4) \qquad R(X,Y,Z,W) = g(R(X,Y)Z,W) \qquad W,X,Y,Z \in C(TM) .$$

Now let

$$(1.5) \qquad \phi : (M,g) \to (N,h)$$

be a smooth map. Henceforth, we will omit g and h in the notation such as (1.5), unless it is essential to include them.

We regard the differential of ϕ , $d\phi$, as a section of the bundle of 1-forms on M with values in the bundle $\phi^{-1}TN$, i.e.

$$(1.6) \qquad d\phi \in C(T^*M \otimes \phi^{-1}TN) .$$

We denote by $|d\phi|$ its norm at a point $x \in M$, induced by the metrics g and h , namely <u>the Hilbert-Schmidt norm</u> of the linear map $d\phi(x)$ [31].

<u>Definition 1.1</u>

<u>The energy density of</u> ϕ is the function given by

$$(1.7) \qquad e(\phi) = \tfrac{1}{2} |d\phi|^2 = \tfrac{1}{2} \operatorname{Trace} \phi^*h .$$

If M' is a compact domain in M , then <u>the Dirichlet</u> or <u>energy integral</u>

<u>of</u> ϕ <u>over</u> M' , $E(\phi,M')$, is given by

$$(1.8) \qquad E(\phi,M') = \int_{M'} e(\phi)dV_M$$

where dV_M denotes the Riemannian volume element of M .

Before introducing the notion of <u>the second fundamental form of</u> ϕ , we shall now give the variational definition of a harmonic map.

<u>Definition 1.2</u>

A smooth map $\phi:M \to N$ is <u>harmonic</u> if for each compact domain M' of M , ϕ is a critical point of (1.8) with respect to all one parameter variations of ϕ supported in M' .

Now $\phi^{-1}TN$ is equipped with a pull-back Riemannian structure with connection $\nabla^{\phi^{-1}TN}$, and let $T^*M \otimes \phi^{-1}TN$ be equipped with the tensor product structure with tensor product connection ∇^{ϕ} acting on sections:

$$(1.9) \qquad \nabla^{\phi} : C(T^*M \otimes \phi^{-1}TN) \to C(\odot^2 T^*M \otimes \phi^{-1}TN)$$

(here, $\odot^2$ denotes the second symmetric product).

<u>Definition 1.3</u>

With respect to (1.9) above, we shall call

$$\nabla^{\phi} d\phi \in C(\odot^2 T^*M \otimes \phi^{-1}TN)$$

<u>the second fundamental form of</u> ϕ , given by

$$(1.10) \qquad \nabla^{\phi} d\phi(X,Y) = \nabla_X^{\phi^{-1}TN} d\phi.Y - d\phi(\nabla_X^M Y)$$

where $X,Y \in C(TM)$.

<u>Definition 1.4</u>

Let $\tau(\phi)$ denote the trace of (1.10) i.e.

$$(1.11) \qquad \tau(\phi) = \text{Trace } \nabla^{\phi} d\phi = \sum_{i=1}^{m} \nabla^{\phi} d\phi(X_i, X_i)$$

where $\{X_i\}$ $i = 1,\ldots,m$, is a local orthonormal frame field for M . Then $\tau(\phi)$ is known as the <u>tension field of</u> ϕ .

<u>Proposition 1.5</u> (see [31], [32])

<u>Let</u> $\phi : M \to N$ <u>be a smooth map between the Riemannian manifolds</u> M <u>and</u> N . <u>Let</u> $\tau(\phi)$ <u>be the tension field of</u> ϕ . <u>Then</u> ϕ <u>is harmonic if and only if</u> $\tau(\phi) \equiv 0$.

1.3 <u>Complex vector bundles</u>.

Assume that the smooth vector bundle in (1.1) has fibres of even dimensions. <u>A complex structure</u> on V , is a smooth section J of $V^* \otimes V \to M$, that as an automorphism of the fibres, is such that $J^2 = -id$. We shall endow each fibre with a complex vector space structure via the assignment $iu = \sqrt{-1}u = Ju$, for $u \in V$. In this way, $V \to M$ becomes <u>a complex vector bundle of rank k</u> , where k is the dimension

of the complex vector space of each fibre. We write $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ for the complexification of V.

Given a complex structure J on V, we can extend it by complex linearity to an endomorphism of $V^{\mathbb{C}}$; we shall also denote this by J. Since $J^2 = -id$, we obtain eigenspaces V' and V'' corresponding to the eigenvalues $+i$ and $-i$ respectively. It is customary to write $V^{1,0}$ for V' and $V^{0,1}$ for V'', i.e. $V^{1,0} = \{Z \in V^{\mathbb{C}} : JZ = iZ\}$ and $V^{0,1} = \{Z \in V^{\mathbb{C}} : JZ = -iZ\}$.

<u>Proposition 1.6</u> [56]

<u>Let</u> $V^{1,0}$ <u>and</u> $V^{0,1}$ <u>be as above</u>. <u>Then</u>

i) $V^{1,0} = \{X-iJX; X \in V\}$ <u>and</u> $V^{0,1} = \{X+iJX; X \in V\}$.

ii) $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. (<u>complex vector space direct sum</u>).

iii) <u>The complex conjugation in</u> $V^{\mathbb{C}}$ <u>defines a real linear isomorphism between</u> $V^{1,0}$ <u>and</u> $V^{0,1}$.

Let $(V^*)^{\mathbb{C}}$ be the dual of $V^{\mathbb{C}}$. Then there is a similar decomposition which in turn induces decompositions of the various tensor product bundles, and the exterior product bundles, viz:

$$(1.12) \qquad \Lambda^r(V^*)^{\mathbb{C}} = \sum_{p+q=r} (V^*)^{p,q} .$$

When we take $V = TM$, the corresponding decomposition to ii) in the

above proposition, is written as

$$(1.13) \qquad T^{\mathbb{C}}M = T'M \oplus T''M = T^{1,0}M \oplus T^{0,1}M \ .$$

In this case, J is known as <u>an almost complex structure on M</u>, and we say that <u>M is an almost complex manifold</u>. The vanishing of the torsion tensor

$$T^J(X,Y) = [JX,JY] - [X,Y] - J([JX,Y]) - J([X,JY])$$

where $X,Y \in C(TM)$, is equivalent to the <u>integrability of the almost complex structure</u> [57]. When this is the case, then we say that M is a <u>complex manifold</u>: henceforth, we shall assume that M and N are complex manifolds.

Let $V \to M$ be a complex vector bundle; then in accordance with (1.12) we can decompose $\Lambda^r(T^*M)^{\mathbb{C}}$ to obtain

$$\Lambda^r(T^*M)^{\mathbb{C}} \otimes V = \sum_{p+q=r} \Lambda^{p,q}\, T^*M \otimes V$$

inducing the decomposition of <u>the vector space of r-forms</u> $A^r = \sum_{p+q=r} A^{p,q}$ where $A^{p,q} = C(\Lambda^{p,q}T^*M \otimes V)$ is <u>the vector space of (p,q)-forms on M with values in V</u>. Relative to ∇^V, we have the associated <u>exterior differential operator</u> $d:A^r \to A^{r+1}$ with its decomposition $d = \partial + \bar{\partial}$ where $\partial:A^{p,q} \to A^{p+1,q}$ and $\bar{\partial}:A^{p,q} \to A^{p,q+1}$ [86].

Let $\phi:M \to N$ be a smooth map and let J^M and J^N denote the

(integrable) almost complex structures of M and N respectively. Then we say that ϕ is <u>holomorphic</u> (+ <u>holomorphic</u>) if and only if

$$(1.14) \qquad d\phi \circ J^M = J^N \circ d\phi \quad .$$

On the other hand, if

$$(1.15) \qquad d\phi \circ J^M = -J^N \circ d\phi \quad ,$$

then we say that ϕ is <u>anti-holomorphic</u> (- <u>holomorphic</u>).

<u>An Hermitian inner product</u> on a complex vector space E is a function $\langle , \rangle : E \times E \to \mathbb{C}$ which for $v,v',w,w' \in E$ and $\lambda,\lambda' \in \mathbb{C}$, is linear in the first argument, i.e.

$$\langle \lambda v + \lambda' v', w\rangle = \lambda\langle v,w\rangle + \lambda'\langle v',w\rangle \quad ,$$

possesses the Hermitian symmetry property $\langle v,w\rangle = \overline{\langle w,v\rangle}$, and is anti-linear in the second argument, i.e.

$$\langle v, \lambda w + \lambda' w'\rangle = \bar{\lambda}\langle v,w\rangle + \bar{\lambda}'\langle v,w'\rangle \quad .$$

In most cases, we shall take $E \cong \mathbb{C}^k$ for some k ; for $v,w \in \mathbb{C}^k$,

$$\langle v,w\rangle = \sum_{i=1}^{k} v^i \bar{w}^i \quad .$$

We write $|v| = \sqrt{\langle v,v\rangle}$, and we say that two subspaces E and F of

$\mathbb{E}^k$ are <u>orthogonal</u> if $\langle v,w\rangle = 0$, for all $v \in E$, $w \in F$; for this we write $E \perp F$.

Let $V \to M$ be a complex vector bundle. <u>An Hermitian metric</u> g on V is an assignment of an Hermitian inner product $\langle , \rangle_z$ to each fibre V_z of V , $z \in M$, such that for any open set $U \subset M$ and $\xi,\eta \in C_U(V)$, the function $\langle \xi,\eta\rangle : U \to \mathbb{C}$, given by $\langle \xi,\eta\rangle(z) = \langle \xi(z),\eta(z)\rangle_z$, is smooth.

When V is equipped with an Hermitian metric, we say that V is an <u>Hermitian vector bundle</u>. When V admits a smooth connection

$$(1.16) \qquad D = D^V : C(T^{\mathbb{C}}M) \times C(V) \to C(V)$$

denoted by $(X,\sigma_1) \to D_X(\sigma_1)$, such that

$$(1.17) \qquad X\langle \sigma_1,\sigma_2\rangle = \langle D_X\sigma_1,\sigma_2\rangle + \langle \sigma_1,D_{\bar{X}}\sigma_2\rangle$$

where $X \in C(T^{\mathbb{C}}M)$, σ_1 and $\sigma_2 \in C(V^{\mathbb{C}})$, then we say that V is endowed with <u>an Hermitian connected structure</u> or V is <u>Hermitian connected</u>. In accordance with the rule in (1.17), D is said to be <u>compatible</u> with the Hermitian metric. When such a $V = TM$, then we say that M is an <u>Hermitian manifold</u>, with Hermitian metric g .

The decomposition in (1.13) induces a decomposition of D_X into (1,0) and (0,1)-parts by restricting X to $T'M$ and $T''M$ respectively. If W is a complex subbundle of an Hermitian connected bundle V , then

W inherits an Hermitian metric by restriction and an Hermitian connection by

$$(1.18) \qquad D_X^W : C_U(W) \xrightarrow{i} C_U(V) \xrightarrow{D_X^V} C_U(V) \xrightarrow{j} C_U(W)$$

where $X \in C_U(T^{\mathbb{C}}M)$. The first map i is induced by inclusion and the third map j , is induced by orthogonal projection.

<u>Remarks</u>.

1. If $\phi : M \to N$ is a smooth map and $E \to N$ is an Hermitian connected vector bundle, then we can define an Hermitian connected structure on $\phi^{-1}E \to M$ consisting of the pull-back metric and connection of E .

2. When the Hermitian fibre metric is indefinite (see Chapter V), then V as above has <u>an indefinite Hermitian connected structure</u>.

Assume now that $V \to M$ is a holomorphic vector bundle [86]. If V is equipped with an Hermitian metric, then V is known as an <u>Hermitian holomorphic vector bundle</u>. A (smooth) connection D on V is <u>compatible, with the holomorphic structure</u>, if the $(0,1)$-part of D coincides with the $\bar{\partial}$-operator of V [86]. Thus $\sigma \in C_U(V)$ is holomorphic if and only if $D_\zeta \sigma = 0$, for all $\zeta \in C(T''M)$. There exists a unique connection on any holomorphic vector bundle V with Hermitian metric, which is a metric connection [56] and which is compatible with the holomorphic structure; this is known as <u>the Hermitian connection of V</u> [86].

1.4 Kähler manifolds and harmonic maps.

Let M be an Hermitian manifold. The Kähler form ω^M of M, is defined by $\omega^M(\zeta,\eta) = \langle\zeta, J\eta\rangle$ for $\zeta,\eta \in T^{\mathbb{C}}M$. We say that M is a Kähler manifold when $d\omega^M = 0$.

Examples 1.7

1. The following are examples of Kähler manifolds appearing in this thesis: Complex projective space of n dimensions, $\mathbb{C}P^n$, with its Fubini-Study metric [57]; the complex n-ball B^n (or complex hyperbolic space $\mathbb{C}H^n$) with its Bergman metric [57]; the complex Euclidean space $\mathbb{C}^n$ with its Euclidean metric: the Grassmannian of complex k-planes in $\mathbb{C}^N$, $G_k(\mathbb{C}^N)$. Any algebraic variety of some $\mathbb{C}P^n$ is Kähler since as an embedded complex submanifold of $\mathbb{C}P^n$, it is endowed with a Kähler metric induced by that of $\mathbb{C}P^n$ (see 2. below). Any compact Riemann surface (or algebraic curve) with an Hermitian metric is Kähler. The classical bounded symmetric domains in $\mathbb{C}^n$ are endowed with a Bergman metric which is Kähler [48] [58], and this metric is preserved under the action of certain discrete groups [59].

2. Any complex submanifold S of a Kähler manifold N, is Kähler with metric induced by that of N.

3. If N_1 and N_2 are Kähler, then $N_1 \times N_2$ is Kähler with respect to the product metric.

Definition 1.8

Let M be a complex manifold and let $v \in T_xM$, $x \in M$, be a tangent vector of unit length. The holomorphic sectional curvature determined by v, is defined to be

$$R(v,Jv,v,Jv) \tag{1.19}$$

where J is the (integrable) almost complex structure of M.

Theorem 1.9 [57]

A simply connected complete Kähler manifold of constant holomorphic sectional curvature c, is holomorphically isometric to $\mathbb{C}P^n$, $\mathbb{C}^n$ or $B^n(\mathbb{C}H^n)$ according to whether c is > 0, $c = 0$ or $c < 0$ respectively.

Having mentioned some examples of Kähler manifolds, let us now return to harmonic maps. Given a smooth map $\phi:M \to N$ of Hermitian manifolds M and N, let us consider the decomposition of their complexified tangent bundles $T^{\mathbb{C}}M = T'M \oplus T''M$ and $T^{\mathbb{C}}N = T'N \oplus T''N$, respectively. Then for such a map, there exists a decomposition of the $\mathbb{C}$-linear extension $d^{\mathbb{C}}\phi:T^{\mathbb{C}}M \to T^{\mathbb{C}}N$ of $d\phi:TM \to TN$, into 4 maps [31]:

$$\begin{aligned} &\partial\phi:T'M \to T'N, \qquad && \bar\partial\phi:T''M \to T'N \\ &\overline{\bar\partial\phi} = \partial\bar\phi:T'M \to T''N, \qquad && \overline{\partial\phi} = \overline{\partial\phi}:T''M \to T''N. \end{aligned} \tag{1.20}$$

We shall assume for the rest of this section that M is compact (otherwise the following results are true over a compact domain in M).

The energy density $e(\phi)(x) = \frac{1}{2}|d\phi(x)|^2$ decomposes into a sum of the (1,0) and (0,1) energy densities, $e'(\phi)(x) = |\partial\phi(x)|^2$ and $e''(\phi)(x) = |\bar{\partial}\phi(x)|^2$ respectively:

$$(1.21) \qquad e(\phi)(x) = e'(\phi)(x) + e''(\phi)(x) \quad .$$

Their integrals,

$$(1.22) \qquad E'(\phi) = \int_M |\partial\phi(x)|^2 v_M(x) \qquad E''(\phi) = \int_M |\bar{\partial}\phi(x)|^2 v_M(x)$$

are known as the (1,0) and (0,1) <u>energy integrals of</u> ϕ , respectively. With respect to (1.20), ϕ is holomorphic if and only if $\bar{\partial}\phi \equiv 0$. Similarly, ϕ is -holomorphic if and only if $\partial\phi \equiv 0$.

If ω^M and ω^N denote the Kähler forms of M and N respectively, then an easy calculation yields the relationship:

$$(1.23) \qquad \langle\omega^M, \phi^*\omega^N\rangle = e'(\phi) - e''(\phi) \quad .$$

(See e.g. [32].)

<u>Proposition 1.10</u> [64]

<u>Let</u> $\phi: M \to N$ <u>be a smooth map between Kähler manifolds</u> M <u>and</u> N . <u>Then</u>:

i) $E'(\phi) - E''(\phi)$ <u>depends only on the homotopy class of</u> ϕ .

ii) <u>A</u> ± <u>holomorphic map</u> $\phi : M \to N$, <u>is a harmonic map of minimum energy in its homotopy class</u>.

iii) <u>The map</u> ϕ <u>is harmonic if and only if</u> ϕ <u>is a critical point of</u> E' , <u>if and only if</u> ϕ <u>is a critical point of</u> E'' .

1.5 Bundles over a Riemann surface and maps from a Riemann surface.

In this section we take M to be a Riemann surface (i.e. a 1-dimensional connected complex manifold) which may be compact with genus p (written M_p) or non-compact (otherwise said, closed and open respectively). By a chart $U = (U,z)$ of M , we mean a non-empty open set U of M equipped with a complex co-ordinate $z : U \to \mathbb{C}$, written $x \to z(x)$, for $x \in M$.

Let E be a complex vector bundle on M with connection D . With respect to a chart $U = (U,z)$ of M , we shall represent the $(1,0)$ and $(0,1)$ parts of the connection D by $D'(=D_{\partial/\partial z})$ and $D''(=D_{\partial/\partial \bar{z}})$ respectively. We shall write D'^k for $D' \circ \ldots \circ D'$ (k times) and set D'^0 = identity map; similarly for D'' . We shall also set $\partial' = \partial/\partial z$ and $\partial'' = \partial/\partial \bar{z}$.

If E is an Hermitian connected bundle with connection D over M , then there exists a unique holomorphic structure on E compatible with D [60]. Then E is an Hermitian holomorphic bundle (as in Section 1.3), and thus $\sigma \in C_U(E)$ is holomorphic if and only if $D''\sigma = 0$.

We shall say that a subbundle F is <u>D''-closed</u>, if on all charts U, $D''(C_U(F)) \subset C_U(F)$. Similarly, we have the notion of F being D'-closed. The orthogonal complement of a D'-closed bundle is D''-closed. In the case that E is an Hermitian holomorphic bundle, <u>a D''-closed subbundle of E is simply a holomorphic subbundle</u>.

Now take N to be a Kähler manifold (possibly endowed with an indefinite metric), and let $\phi : M \to N$ be a smooth map. Taking $\partial\phi \in C(T^*M \otimes \phi^{-1}TN)$ we may then represent such a globally defined section by a locally defined section

$$(1.24) \qquad \partial'\phi = \partial\phi(\partial/\partial z) \in C_U(\phi^{-1}T'N) .$$

Similarly for $\bar{\partial}\phi$, we have

$$(1.25) \qquad \partial''\phi = \bar{\partial}\phi(\partial/\partial\bar{z}) \in C_U(\phi^{-1}T'N) .$$

Then we may state the following:

<u>Proposition 1.11</u> [91] [92]

<u>Let</u> M <u>be a Riemann surface and</u> N <u>a Kähler manifold</u>. <u>A smooth map</u> $\phi : M \to N$ <u>is harmonic if and only if on any chart of</u> M

$$(1.26) \qquad \nabla''\partial'\phi = 0 \quad \text{\underline{or equivalently}}, \quad \nabla'\partial''\phi = 0$$

<u>where</u> ∇ <u>is taken to be the connection on</u> $\phi^{-1}T'N$ <u>and</u> ∇', ∇'' <u>are the</u> $(1,0)$ <u>and</u> $(0,1)$ <u>parts of</u> ∇ <u>respectively</u>.

We write $\nabla'^{\alpha}\phi = \nabla'(\nabla'^{\alpha-1}\phi)$ for $\alpha = 2,3,\ldots$ and similarly, we define $\nabla''^{\alpha}\phi$. For $\alpha,\beta = 1,2,\ldots,$ set

$$(1.27) \qquad \eta_{\alpha,\beta} = \langle \nabla'^{\alpha}\phi, \nabla''^{\beta}\phi \rangle^{N}$$

where $\langle , \rangle^{N}$ denotes the metric on $\phi^{-1}T'N$.

Definition 1.12

Let $\phi : M \to N$ be a smooth map. By the order of isotropy of ϕ , we mean the greatest value $\gamma \in \{1,2,\ldots,\infty\}$ such that on each chart of an open cover of M , $\eta_{\alpha,\beta} \equiv 0$ for all integers $\alpha,\beta \geq 1$ with $\alpha + \beta \leq \gamma$.

A smooth map has order of isotropy of at least one. A map is weakly conformal if and only if its order of isotropy is at least two [91]. A smooth map with an infinite order of isotropy is known as an isotropic map. It is this last class that is of principal interest in this thesis.

1.6 Riemannian submersions.

In this section, we shall use freely some of the basic notions and terminology from the theory of Lie groups; we refer to [48] for further details. Firstly, we shall introduce the notion of a Riemannian submersion and then proceed to relate this to certain homogeneous fibrations.

Definition 1.13

Let $\pi : P \to N$ be a smooth map of maximal rank, of Riemannian manifolds

P and N, such that for all $y \in P$,

$$d\pi(y) : T_yP \to T_{\pi(y)}N \text{ maps the subspace } T_y^HP$$

of tangent vectors orthogonal to the fibre $\pi^{-1}\{\pi(y)\}$, isometrically onto $T_{\pi(y)}N$. Then such a map π is known as a <u>Riemannian submersion</u>.

The subspace T_y^HP is referred to as the <u>horizontal subspace</u> of π at y, and its orthogonal complement, the subspace of tangent vectors to the fibres $(= \ker d\pi(y))$, is referred to as the <u>vertical subspace</u>.

<u>Definition 1.14</u>

A <u>totally geodesic</u> map $i:P \to Q$ between Riemannian manifolds P and Q, is one for which $\nabla di \equiv 0$. Equivalently, i maps geodesics of P linearly into geodesics of Q. When P happens to be a submanifold of Q, then the above map i is regarded as the inclusion of the <u>totally geodesic submanifold</u> P in Q.

<u>Example 1.15</u>

The Hopf fibration $S^{2n+1} \to \mathbb{C}P^n$ is a Riemannian submersion with totally geodesic fibres. This fibration characterises the Fubini-Study metric on $\mathbb{C}P^n$ (see [57]). We shall discuss the analogous fibration for $\mathbb{C}H^n$ in Chapter V.

We now propose to discuss how a particular class of Riemannian submersions arises in the setting of homogeneous space fibrations. This

discussion is adapted from the paper of L. Bérard-Bergery [7].

Let G be a connected Lie group and H, K two closed subgroups with $H \subset K$. We shall denote their corresponding Lie algebras by $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$ respectively. We assume that $ad_G(K)$ is relatively compact in $GL(\mathfrak{g})$ which implies that $ad_G(H)$ is also relatively compact in $GL(\mathfrak{g})$. We have $\mathfrak{h}$ invariant under $ad_G(H)$ and $\mathfrak{k}$ invariant under $ad_G(K)$ (and also $ad_G(H)$).

Suppose that $\mathfrak{k}$ has an $ad_G(K)$-invariant summand $\mathfrak{p}$ in $\mathfrak{g}$, and $\mathfrak{h}$ has an $ad_G(H)$-invariant summand $\mathfrak{m}$ in $\mathfrak{k}$. Then $\mathfrak{b} = \mathfrak{p} \oplus \mathfrak{m}$ is an $ad_G(H)$-invariant summand for $\mathfrak{h}$ in $\mathfrak{g}$.

Let g_1 be a G-invariant metric on G/K; this can be described by an $ad_G(K)$-invariant metric on $\mathfrak{p}$ and $T_{eK}G/K \cong \mathfrak{p}$. Thus there corresponds to g_1, an $ad_G(K)$-invariant inner product $<,>_1$ on $\mathfrak{p}$. Similarly, let g_2 be a K-invariant metric on K/H, with a corresponding $ad_G(H)$-invariant inner product $<,>_2$ on $\mathfrak{m}$.

We construct the inner product $<,>_3$ on $\mathfrak{b}$ via the restrictions:

(1.28) $\quad <,>_3 = <,>_2$ on $\mathfrak{m}$ and

(1.29) $\quad <,>_3 = <,>_1$ on $\mathfrak{p}$,

with $\langle \mathfrak{p}, \mathfrak{m} \rangle_3 = 0$.

This gives an inner product $<,>_3$ that is $ad_G(H)$-invariant on $\mathfrak{b}$ with a corresponding G-invariant metric g_3 on G/H.

Proposition 1.16 [7]

For G, H, K, g_1, g_2 and g_3 as above, the canonical projection

(1.30) $\pi : G/H \to G/K$

is a Riemannian submersion with totally geodesic fibres $(K/H, g_2)$, of $(G/H, g_3)$ onto $(G/K, g_1)$.

Proof

We outline the general idea. Since the metrics g_1 and g_2 are G-invariant and π commutes with the action of G, it suffices to consider the identity coset eH of G/H where the tangent space of G/H may be identified with $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{m}$. Then $\mathfrak{m}$ may be identified with the vertical subspace of π and accordingly, $\mathfrak{q}$ may be identified with the horizontal subspace, since it is orthogonal to $\mathfrak{m}$ (with respect to the metric). But it can be seen that $d\pi$ at eH is identified with the projection of $\mathfrak{p}$ onto $\mathfrak{q}$ parallel to $\mathfrak{m}$.

We deduce that a typical fibre of (1.30) is $(K/H, g_2)$. The inclusion of K/H in G/H is totally geodesic by general considerations (see [57, p.234;]). □

Let us take a look at a particular example which will feature again in Chapter IV.

Example 1.17

We introduce the twistor fibration

(1.31) $\mathbb{C}P^3 \to S^4$

which lies at the heart of the Penrose transform theorey [1] [2]. It is known that the Lie group $Sp(2)$ is the group of automorphisms of this fibration and preserves the natural metrics on $\mathbb{C}P^3$ and S^4. Its action is transitive on $\mathbb{C}P^3$ and thus we can express $\mathbb{C}P^3$ in terms of the coset space representation $\mathbb{C}P^3 \cong Sp(2)/U(1) \times Sp(1)$. Also, we have $S^4 \cong SO(5)/SO(4) \cong Sp(2)/Sp(1) \times Sp(1)$, where the latter represents S^4 as <u>the quaternionic projective line</u> $\mathbb{H}P^1$. Thus (1.31) can be expressed as a homogeneous fibration as

(1.32) $Sp(2)/U(1) \times Sp(1) \to Sp(2)/Sp(1) \times Sp(1)$

with fibre $Sp(1)/U(1) \cong \mathbb{C}P^1$.

Generalising, let $\mathbb{H}P^n$ denote the <u>n-dimensional quaternionic projective space</u>. The canonical twistor fibration is

(1.33) $\mathbb{C}P^{2n+1} \to \mathbb{H}P^n$

which as a homogeneous fibration is

(1.34) $Sp(n+1)/U(1) \times Sp(n) \to Sp(n+1)/Sp(1) \times Sp(n)$

with fibre $Sp(1)/U(1) \cong \mathbb{C}P^1$. In accordance with Proposition 1.16, (1.34) is a Riemannian submersion and is such (up to a constant) with

respect to the Fubini-Study metric on $\mathbb{C}P^{2n+1}$ [80].

1.7 Composition principles for harmonic maps.

Generally, the composition of two harmonic maps does not yield a harmonic map, this being contrary to the case of holomorphic maps. However, there are cases when the composition of two harmonic maps is a harmonic map. We thus proceed to discuss two noteworthy composition principles for harmonic maps; these will play an important role in our work. The first relates to the Riemannian submersions which were discussed in the last section. Firstly then, a definition:

Definition 1.18

Let $\psi: M \to P$ and $\pi: P \to N$ be smooth maps of Riemannian manifolds M, N and P. Further, assume that for each $z \in M$, $d\psi(z)$ maps $T_z M$ into the horizontal subspace of π. Then we shall say that the map ψ is π-horizontal.

Lemma 1.19 (Smith's Lemma) [82]

Let M, N and P be Riemannian manifolds, $\psi: M \to P$ a harmonic map and $\pi: P \to N$ a Riemannian submersion. Furthermore, assume that ψ is π-horizontal, then $\phi = (\pi \circ \psi): M \to N$, is a harmonic map.

Proof

The tension field of the composed map $\phi = \pi \circ \psi$, is seen to be

$$(1.35) \qquad \tau(\phi) = d\pi\circ\tau(\psi) + \sum_{i=1}^{m} \nabla d\pi(d\psi(e_i), d\psi(e_i))$$

where $\{e_i\}$ is an orthonormal basis for T_xM $(x \in M)$. Clearly, the first term in the second member vanishes, since ψ is harmonic. Following [72], we have $\nabla d\pi(e',e') = 0$, $e' \in T^HP$. Since each vector $d\psi(e_i)$ is horizontal, then the second term also vanishes and the result is proved.

□

<u>Remark</u>

If we replace 'Riemannian submersion' by '$\nabla d\pi$ vanishes identically on horizontal vectors', then clearly, the same end result is obtained (see Lemma 1.21).

Prior to taking traces in (1.35), the following lemma is easily proved and this is our second composition principle:

<u>Lemma 1.20</u> [32]

<u>Let</u> M, N <u>and</u> P <u>be Riemannian manifolds</u> $\psi: M \to P$ <u>a smooth map and</u> $i: P \to N$ <u>a totally geodesic map</u>. <u>Then</u> $\phi = (i\circ\psi): M \to N$ <u>is harmonic if</u> ψ <u>is harmonic</u>.

Most of the constructions of harmonic maps that are described in this thesis, involve fibrations of the type (1.30). However for certain specified metrics, these fibrations may not always be Riemannian submersions, e.g. for Kähler metrics on flag manifolds [78] (Example 1.22 below is such a case). We will, however, be mapping into the space of horizontal vectors, and the following lemma is well worth recording:

Lemma 1.21 [78]

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let K be a closed subgroup with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. Suppose there is a summand $\mathfrak{p}$ for $\mathfrak{k}$ in $\mathfrak{g}$ with $\mathrm{Ad}K(\mathfrak{p}) \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. If H is a closed subgroup of K with an Ad-H invariant summand for its Lie algebra, and G/H and G/K are given indefinite Riemannian metrics such that $\mathfrak{p}$ is the space of horizontal vectors at the identity coset in G/H, to the natural fibering $\pi: G/H \to G/K$, then the second fundamental form of π, $\nabla d\pi$, vanishes on horizontal vectors.

Example 1.22

The $G/H \to G/K$ fibration appearing in [35], affords an example. Here $G = U(n+1)$, $K = U(1) \times U(n)$ and $H = U(1) \times U(r) \times U(s)$, with $r+s = n$ (see also (2.34)). With the above notation, $\mathfrak{p} \cong \mathbb{C}^n$ at the identity coset, and this is further identified with the space of horizontal vectors in the above fibration.

CHAPTER II

HARMONIC MAPS TO COMPLEX PROJECTIVE SPACE

2.1 Holomorphic curves in complex projective space.

Our discussion here follows from the relevant sections of [35] [62] [93]; we refer accordingly for further details. Henceforth,* M will always denote an open or closed Riemann surface. A holomorphic curve in $\mathbb{C}P^n$ is taken to be an abstract Riemann surface M together with a holomorphic map

$$(2.1) \qquad f:M \to \mathbb{C}P^n .$$

Definition 2.1

Let $f:M \to \mathbb{C}P^n$ be a holomorphic map. Then we shall say that f is full (or non-degenerate in [93]) if its image is not contained in a proper projective subspace of $\mathbb{C}P^n$.

Let ξ be a local lift of f over an open set U in M and consider the derivatives of ξ up to a certain order α say.

$$\begin{tikzcd} & \mathbb{C}^{n+1}-\{0\} \arrow[d, "\pi"] \\ M \supset U \arrow[ur, "\xi"] \arrow[r, "f"] & \mathbb{C}P^n \end{tikzcd}$$

---

* Up to Chapter VI inclusive, unless otherwise stated.

<u>Definition 2.2</u>

For $0 \le \alpha \le \infty$, the αth <u>order (augmented) osculating space</u> $\theta_\alpha(x) = \theta_\alpha(f)(x)$ of f at $x \in M$, is defined to be the

(2.2) $\quad \mathrm{span}\{\xi^\gamma(x) : 0 \le \gamma \le \alpha\}$ where $\xi^\gamma = \partial^\gamma \xi / \partial z^\gamma$.

This may be seen to be independent of the lift and the chart chosen. Following [35], we know that if f is full, then at some $x \in M$, we have $\dim \theta_n(x) = n+1$. Let $A = \{x \in M : \dim \theta_n(x) < n+1\}$. For any integer α $(0 \le \alpha \le n)$ we define

(2.3) $\quad f_\alpha : M-A \to G_{\alpha+1}(\mathbb{C}^{n+1})$

by $f_\alpha(x) = \theta_\alpha(x)$.

For $x \notin A$, we then consider the following multivector (or <u>Wronskian</u>)

(2.4) $\quad w : U \to \Lambda^{\alpha+1}\, \mathbb{C}^{n+1} - \{0\}$

given by the holomorphic map $w(x) = \xi(x) \wedge \xi^1(x) \wedge \ldots \wedge \xi^\alpha(x)$ $(\not\equiv 0)$.

<u>Lemma and definition 2.3</u> [35] [93]

<u>There is a unique extension of</u> f_α <u>in</u> (2.3) <u>to a holomorphic map</u>

(2.5) $\quad f_\alpha : M \to G_{\alpha+1}(\mathbb{C}^{n+1}) \subset \mathbb{P}(\Lambda^{\alpha+1}\, \mathbb{C}^{n+1})$.

<u>We shall call this map, the</u> α^{th} <u>associated curve of</u> f .
<u>We note that</u> $f_0 = f$ <u>and</u> f_{-1} <u>is set equal to the trivial map</u>.

<u>Remark</u>

The classical geometric meaning of $f_\alpha(x)$ is that of the unique α-plane in $\mathbb{C}P^n$ having contact of order at least $\alpha+1$ with $f(M)$ at x .

At this stage it will be worth noting that for $0 \le k \le N$ (k,N integers), there exists a bijective correspondence between smooth maps $\psi : M \to G_k(\mathbb{C}^N)$ and smooth subbundles of rank k of the trivial N-plane bundle $M \times \mathbb{C}^N$ over M ; we shall denote the latter by $\underline{\mathbb{C}}^N$. The correspondence is obtained by setting the fibre of the subbundle of each $x \in M$, equal to $\psi(x)$.

Note that when ψ is holomorphic, the induced bundle is a holomorphic vector bundle. With some abuse of notation we shall denote the subbundle by that same letter used for the map. Hence corresponding to the associated curve f_α in (2.5), we have the induced holomorphic subbundle f_α over M , which is a subbundle of rank $\alpha+1$ of $\underline{\mathbb{C}}^{n+1}$.

<u>Definition 2.4</u>

Given a full holomorphic map $f : M \to \mathbb{C}P^n$, we shall define an associated map $g = f^{\perp}_{n-1} : M \to \mathbb{C}P^n$ known as the <u>polar (curve) of f</u> . We may regard this as the composition

$$(2.6) \qquad M \xrightarrow{f_{n-1}} G_n(\mathbb{C}^{n+1}) \xrightarrow{\perp} G_1(\mathbb{C}^{n+1}) = \mathbb{C}P^n .$$

Lemma 2.5 [35]

Let $f:M \to \mathbb{C}P^n$ be a holomorphic map.

a) If f is full, then its polar g is full and is antiholomorphic (-holomorphic). If ξ and ζ are the local lifts to $\mathbb{C}^{n+1}-\{0\}$ of f and g respectively, then we have the following (isotropy) relations between f and g.

(2.7) $\quad \langle \partial'^{\alpha}\xi , \partial''^{\beta}\zeta \rangle = 0$ for all $\alpha,\beta \geq 0$, $\alpha+\beta \leq n-1$;

(2.8) $\quad \langle \partial'^{\alpha}\xi(x) , \partial''^{\beta}\zeta(x) \rangle \neq 0$ for some $x \in M$ for all $\alpha,\beta \geq 0$, $\alpha+\beta = n$, where $\langle , \rangle$ denotes the usual Hermitian inner product.

b) Conversely, if $g:M \to \mathbb{C}P^n$ is a smooth map satisfying (2.7) and (2.8) for some $\alpha,\beta \geq 0$, $\alpha+\beta = n$, then f and g are full, and g is the polar of f .

Remarks

1. The proof in [35] shows that the set of points x where $\dim \operatorname{span}\{\partial'^{\alpha}\xi(x)\} < n+1$ and $\dim \operatorname{span}\{\partial''^{\beta}\zeta(x)\} < n+1$, are the same set A , as previously described.

2. As we did for f , we may form the associated (-holomorphic) curves of g and obtain the corresponding (-holomorphic) sub-bundles of $\underline{\mathbb{C}}^{n+1}$ over M .

2.2 Some Riemannian geometry of holomorphic curves.

Let $N^n(c)$ be a complete, simply connected Kähler manifold of constant holomorphic sectional curvature c. We will take N to be one of $\mathbb{C}P^n$, $\mathbb{C}^n$ or $\mathbb{C}H^n$ in accordance with Theorem 1.9.

If $f:M \longrightarrow N$ is a holomorphic immersion, then in terms of a local co-ordinate $z = u+iv$ on M, we can express the metric induced by f as

$$(2.9) \qquad ds^2 = 2F|dz|^2$$

where F is a certain real analytic function [18].

In terms of the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

the Laplace-Beltrami operator of this metric may be written as

$$(2.10) \qquad \Delta = \frac{2}{F}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} ,$$

and the induced (Gaussian) curvature of this metric is

$$(2.11) \qquad K = -\frac{1}{F}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log F .$$

Lemma 2.6 [18] [19]

Let f be as above, and further assume that $f(M)$ is not contained

in any proper totally geodesic submanifold of N. Then a sequence of functions may be defined iteratively, by setting

$$(2.12) \qquad F_{k+1} = \frac{F_k^2}{F_{k-1}} \left\{ \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log F_k + \frac{(k+1)}{2} c F \right\} \quad \text{with } F_0 = 1,\ F_1 = F .$$

For $0 \le k \le n$, F_k is ≥ 0 and vanishes only at isolated points. The succeeding function F_{k+1} is defined by (2.12) away from these points and extends to a real analytic function on all of M. The function F_{n+1} is identically zero. Conversely given (2.9) and the above sequence of functions, then there exists a unique holomorphic immersion of M into N.

With this in mind we note the following interesting observation.

Proposition 2.7 [94]

Let M be a holomorphic curve in a Kähler manifold $N(c)$ as above with $c \ge 0$. Assume that M has constant curvature with respect to the induced metric. Then M has strictly positive curvature.

Proof

Let $ds^2 = 2F|dz|^2$ be the induced metric on M. If we assume that K is a negative constant, then on substituting (2.11) into (2.12), one sees that F_{n+1} cannot be zero, thus yielding a contradiction. □

Corollary 2.8 (Blaschke's theorem, see e.g. [41])

The Poincare metric $(1-|z|^2)^{-2}dzd\bar{z}$ on the unit disc (or hyperbolic

plane) cannot be obtained by an isometric embedding into some $\mathbb{E}P^n$.

Information concerning the higher order jets of the immersion is contained in the intrinsically defined functions F_k in (2.12). In fact following [62], we have for each k ,

$$(2.13) \qquad F_k = \frac{2|\xi\wedge\xi^1\wedge\ldots\wedge\xi^k|^2}{|\xi|^{2k+2}} \quad .$$

For each k , $1 \le k \le n$, we can define curvature functions

$$(2.14) \qquad K_k = \frac{F_{k+1}F_{k-1}}{F\,F_k} \quad , \quad K_0 = \frac{c}{2} \quad .$$

It may be seen that $K_k \ge 0$ and for $k < n$, $K_k = 0$ only at isolated points $(K_n \equiv 0)$. From (2.10) and (2.12) we obtain

$$(2.15) \qquad K_k = \tfrac{1}{4}(\Delta \log F_k + (k+1)c) \ ,$$

and the following recurrence relations

$$(2.16) \qquad \begin{cases} \tfrac{1}{4}\,\Delta \log K_k = K_{k+1} + K_{k-1} - 2K_k + K & \text{for } k = 1,\ldots,n-1 \\ \tfrac{1}{4}\,\Delta \log(K_1\ldots K_n) = (2n-1)K_0 - nK_1 - K_{n-1} & \\ K_1 = 2K_0 - K & \text{(the Gauss equation of the immersion).} \end{cases}$$

Example 2.9 [62]

The holomorphic curve $f:\mathbb{C}P^1 \to \mathbb{C}P^n$ given by

$$(2.17) \qquad (W_0,W_1) \to (W_0^n, \sqrt{n}\, W_0^{n-1}W_1, \ldots, \sqrt{\binom{n}{k}} W_0^{n-k}W_1^k, \ldots, W_1^n)$$

has constant Gaussian curvature $1/n$. The curvature functions are given by $K_k = \frac{(k+1)}{2n}.(n-k)$. The kth associated curve f_k has constant Gaussian curvature equal to $1/(k+1)(n-k)$. The above non-degenerate curve f is the only such one with these properties.

2.3 Ramification and Plücker formulae.

Definition 2.10

Assume M is compact with genus p (this we denote by M_p) and $f:M_p \to \mathbb{C}P^n$ is a full holomorphic map (or a non-degenerate holomorphic curve) expressed in terms of Euclidean co-ordinates by $f^1(z),\ldots,f^n(z)$.

The ramification index $\beta_0(z_0)$ of f at z_0 , is defined to be the order of vanishing of the Jacobian matrix $(\frac{\partial f^1}{\partial z},\ldots,\frac{\partial f^n}{\partial z})$ at z_0 i.e.

$$(2.18) \qquad \beta_0(z_0) = \min(\mathrm{ord}_{z_0}(\frac{\partial f^i}{\partial z})) \ .$$

Following the scheme outlined in [42], we may choose a suitable system of

co-ordinates in $\mathbb{C}^{n+1}$ such that in a neighbourhood of z_0, we can express the map f in the form :

$$(2.19)\qquad \begin{cases} \xi_0(z) = 1 + \dots \\ \xi_1(z) = z^{1+\alpha_1} + \dots \\ \vdots \\ \xi_n(z) = z^{n+\alpha_1+\dots+\alpha_n} + \dots . \end{cases}$$

This is known as the <u>normal form of f at</u> z_0 and the ramification index $\beta_k(z_0)$ of the kth associated curve at z_0 is given by

$$(2.20)\qquad \beta_k(z_0) = \alpha_{k+1} .$$

Accordingly, we define the total ramification of f_k $(0 \le k \le n\ ,\ f_0 = f)$ as

$$(2.21)\qquad \beta_k = \sum_{z_0 \in M} \beta_k(z_0) .$$

<u>Examples 2.11</u>

i) If $f:\mathbb{C}P^1 \to \mathbb{C}P^n$ is given by $f(z) = (1,z,z^2,\dots,z^n)$, then f and its associated curves are unramified.

ii) If $f:\mathbb{C}P^1 \to \mathbb{C}P^2$ is given by $f(z) = (1,(z\pm1)^{t-i},z^t)$ $i < t$ then we have $\beta_0 = i-1$.

Topologically, the kth associated curve f_k defines a homology class (cycle) in $G_{k+1}(\mathbb{E}^{n+1})$, that is homologous to a positive integral multiple of the fundamental 2-cycle of $G_{k+1}(\mathbb{E}^{n+1})$. This integer which we shall denote by d_k, is known as the <u>degree (or order) of</u> f_k. We shall let $d_0 = d = \deg f$. Geometrically, it is the number of points of the curve at which the osculating spaces of dimension k intersect with a generic linear subspace of dimension $n-k-1$ in $\mathbb{C}P^n$. Along with β_k, we can assign a differential - geometric meaning to d_k.

<u>Proposition 2.12</u> [62]

<u>Let</u> K_k <u>be the</u> k^{th} <u>curvature function as defined in</u> (2.15), <u>then</u>

$$d_k = \frac{1}{2\pi}\iint_M K_k \, dA \, . \tag{2.22}$$

$$\beta_k = -\frac{1}{4\pi}\iint_M \Delta \log K_k \, dA \, . \tag{2.23}$$

We are now in a position to establish the following classical formulae.

<u>Theorem 2.13</u> <u>The Plücker formulae</u> (see also [42] [62] and [93]

$$d_k - 2d_{k-1} + d_{k-2} = 2p-2-\beta_{k-1} \, . \qquad (d_{-1} = 0) \, . \tag{2.24}$$

<u>Proof</u>

We consider the first relationship of (2.16) and apply (2.22) and (2.23) along with the Gauss-Bonnet formula [57]. □

Remarks

1. The remarkable feature of (2.24) is that it relates a topological invariant of M, namely its Euler-characteristic $X(M) = 2-2p$, to quantities depending on the map f.

2. An analogous formula exists for holomorphic curves in a complex torus (see [61] and [73]).

2.4 The Eells-Wood construction.

The discussion of the last section provided examples of maps $M \to \mathbb{C}P^n$ that are harmonic by virtue of Proposition 1.10. However, we are particularly interested in a somewhat special class of maps, namely those which are harmonic but not $\pm$ holomorphic. The construction of Eells-Wood [34] [35] which we now describe in brief, provides a process of constructing such a class of maps. Before outlining their main result, we will establish some definitions.

Consider first of all, the universal line bundle

$$L \to \mathbb{C}P^n . \tag{2.25}$$

We have a resulting exact sequence of complex vector bundles over $\mathbb{C}P^n$.

$$\underline{0} \longrightarrow L \xrightarrow{i} \underline{\mathbb{C}}^{n+1} \xrightarrow{j} L^{\perp} \longrightarrow \underline{0} \tag{2.26}$$

where i is an inclusion and j is given by orthogonal projection along L. The line bundle L is holomorphic and $L^{\perp}$ is given a holomorphic structure via the isomorphism $L^{\perp} \cong \underline{\mathbb{C}}^{n+1}/L$. Both L and $L^{\perp}$

are Hermitian holomorphic bundles with Hermitian metrics and connections induced from $\mathbb{C}^{n+1}$.

Tensoring (2.26) with L^* and pulling back via a (smooth) map $\phi:M \to \mathbb{C}P^n$, yields the following exact sequence over M :

$$(2.27) \qquad \underline{0} \to \phi^{-1}(L^* \otimes L) \xrightarrow{i} \phi^{-1}(L^* \otimes \underline{\mathbb{C}}^{n+1}) \xrightarrow{j} \phi^{-1}(L^* \otimes L^{\perp}) \to \underline{0} .$$

We note that $\phi^{-1}(L^* \otimes L)$ is trivial with constant section $M \to M \times \mathbb{C}$ given by $x \to (x,1)$, which we denote by 1 . We also note that $\phi^{-1}(L^* \otimes L^{\perp})$ is isometrically isomorphic to $\phi^{-1}T'\mathbb{C}P^n$ i.e. the pull-back by ϕ, of the holomorphic tangent bundle of $\mathbb{C}P^n$.

<u>Definition 2.14</u>

We define the <u>universal lift</u> Φ of ϕ to be the canonical section $i(1) \in C(\mathrm{Hom}(\phi^{-1}L,\mathbb{C}^{n+1}))$, defined by $\Phi(x)$ = inclusion map of $L_{\phi(x)}$ into $\mathbb{C}^{n+1}$. If $\phi_U:U \to \mathbb{C}^{n+1}-\{0\}$ is a local lift of ϕ over an open set U in M , then ϕ_U may be written in the form $\Phi o \rho$, where $\rho \in C_U(\phi^{-1}L)$.

Let D denote covariant differentiation in the bundle $\mathrm{Hom}(\phi^{-1}L,\underline{\mathbb{C}}^{n+1})$. Then D has a splitting into $(1,0)$ and $(0,1)$ complex types, denoted by $D'(= D_{\partial/\partial z})$ and $D''(= D_{\partial/\partial \bar{z}})$ respectively. The harmonicity of the smooth map ϕ is given by [35, Proposition 4.6]:

$$(2.28) \qquad D''D'\Phi + |D'\Phi|^2\Phi = 0 \qquad (\text{or } D'D''\Phi + |D''\Phi|^2\Phi = 0) .$$

Let us outline the proof of this: we have a connection preserving isometry $h:\phi^{-1}T'\mathbb{C}P^n \to \mathrm{Hom}(\phi^{-1}L,\phi^{-1}L^{\perp})$. For $\partial'\phi$ as in (1.24), the relation $h(\partial'\phi) = D'\phi$ is obtained [35 Proposition 4.5]. Let E denote the bundle $\phi^{-1}T'\mathbb{C}P^n$ and H denote the bundle $\mathrm{Hom}(\phi^{-1}L,\mathbb{C}^{n+1})$. Now consider (1.18) with W and V replaced by E and H respectively. With this in mind, the following relationship is established,

$$h(D^{E''}\partial'\phi) = D''D'\phi = j(D''D'\phi)$$

where the j of (1.18) is induced in this case, by the j of (2.27). Following (1.26), ϕ is harmonic if and only if $(D^E)''\partial'\phi = 0$. But this is true if and only if $j(D''D'\phi) = 0$. By the exactness of (2.27), this is true if and only if $D''D'\phi = F\phi$ where F is a smooth function on U . This is found to be equal to $|D'\phi|^2$. The other equation is similarly obtained.

<u>Definition 2.15</u>

Using the Hermitian inner product $<,>$ we say that ϕ is <u>isotropic if</u>

(2.29) $\quad <D'^{\alpha}\phi, D''^{\beta}\phi> = 0 \qquad$ for all $1 \le \alpha+\beta$, $\alpha,\beta \ge 0$.

We are now in a position to give a statement of the main result of Eells and Wood.

<u>Theorem 2.16</u> <u>The Classification theorem</u> [34] [35]

<u>Given an integer</u> r $(0 \le r \le n)$ <u>and a full holomorphic map</u> $f:M \to \mathbb{C}P^n$

where M is a Riemann surface, then the map $\phi : M \to \mathbb{E}P^n$ defined by

$$(2.30) \qquad \phi(x) = f^{\perp}_{r-1}(x) \cap f_r(x)$$

$$(2.31) \qquad = (f_{r-1}(x) + g_{s-1}(x))^{\perp} \qquad x \in M \ , \ r+s = n$$

is a full, isotropic harmonic map.

The correspondence $(f,r) \to \phi$ is bijective. The inverse transformation is given by setting r equal to the maximum dimension of the subspace spanned by $D'^{\mathbb{C}}\phi$ and f is obtained from ϕ by a procedure similar to that giving ϕ in terms of f.

At this stage we do not intend giving a detailed description of the methods used in proving Theorem 2.16, but as mentioned in the introduction, the geometric nature of this work may be modified to construct and classify similar harmonic maps to indefinite complex hyperbolic space. This will be the subject of Chapter V, and we shall elucidate the underlying geometric principles. However, we will now describe one particular aspect of the construction that will be used several times in this thesis. In particular, we look at (2.31), i.e. the construction of ϕ from the $(r-1)^{th}$-associated curve of f and the $(s-1)^{th}$-associated curve of g.

Before making a further analysis of the construction of the map in (2.31), we shall need, for technical purposes, to look at the tangent space to a Grassmannian. For any $V \in G_k(\mathbb{E}^{n+1})$, the holomorphic tangent space $T'_V G_k(\mathbb{E}^{n+1})$ may be identified with $\mathrm{Hom}(V,V^{\perp})$ via a complex linear isometry

(2.32) $\quad h: T'_V G_k(\mathbb{E}^{n+1}) \to \mathrm{Hom}(V, V^\perp)$.

We describe h as follows: Let $w \in T'_V G_k(\mathbb{E}^{n+1})$ and let $v \in V$. For $v = 0$, we set $h(w)(v) = 0$. Otherwise let $\phi: U \to G_k(\mathbb{E}^{n+1})$ be a smooth map from a neighbourhood U of a point $x \in \mathbb{E}$ with $\phi(x) = V$ and $\partial'\phi(x) = w$. We extend v to a smooth map $v: U \to \mathbb{E}^{n+1} - \{0\}$ such that $v(x) \in \phi(x)$ for all $x \in U$. Then $h(w)(v)$ = projection of $\partial' v(x)$ onto $V^\perp$.

<u>Definition 2.17</u>

For $0 \le r, s \le n$ and $r+s = n$, define

(2.33) $$H_{r,s} = \{(V,W) \in G_r(\mathbb{E}^{n+1}) \times G_s(\mathbb{E}^{n+1}) : V \perp W\}$$
$$\cong U(n+1)/U(r) \times U(s) \times U(1) .$$

Including $U(r) \times U(s)$ in $U(n)$ in the standard way, we obtain the homogeneous fibration

(2.34) $$\pi: H_{r,s} \cong U(n+1)/U(r) \times U(s) \times U(1) \to \mathbb{C}P^n \cong U(n+1)/U(n) \times U(1) ,$$

with

(2.35) $$\pi(V,W) = (V+W)^\perp .$$

The following lemma from [35, Lemma 3.9] will prove to be important in this work and we will exhibit a proof of the analogous result for the indefinite case in Chapter V (Lemma 5.24).

(2.32) $\quad h: T'_V G_k(\mathbb{E}^{n+1}) \to \mathrm{Hom}(V, V^\perp)$.

We describe h as follows: Let $w \in T'_V G_k(\mathbb{E}^{n+1})$ and let $v \in V$. For $v = 0$, we set $h(w)(v) = 0$. Otherwise let $\phi: U \to G_k(\mathbb{E}^{n+1})$ be a smooth map from a neighbourhood U of a point $x \in \mathbb{E}$ with $\phi(x) = V$ and $\partial'\phi(x) = w$. We extend v to a smooth map $v: U \to \mathbb{E}^{n+1} - \{0\}$ such that $v(x) \in \phi(x)$ for all $x \in U$. Then $h(w)(v)$ = projection of $\partial' v(x)$ onto $V^\perp$.

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For $0 \le r, s \le n$ and $r+s = n$, define

(2.33) $$H_{r,s} = \{(V,W) \in G_r(\mathbb{E}^{n+1}) \times G_s(\mathbb{E}^{n+1}) : V \perp W\}$$
$$\cong U(n+1)/U(r) \times U(s) \times U(1) .$$

Including $U(r) \times U(s)$ in $U(n)$ in the standard way, we obtain the homogeneous fibration

(2.34) $$\pi: H_{r,s} \cong U(n+1)/U(r) \times U(s) \times U(1) \to \mathbb{C}P^n \cong U(n+1)/U(n) \times U(1) ,$$

with

(2.35) $$\pi(V,W) = (V+W)^\perp .$$

The following lemma from [35, Lemma 3.9] will prove to be important in this work and we will exhibit a proof of the analogous result for the indefinite case in Chapter V (Lemma 5.24).

Lemma 2.18 [35]

Let us define a map $\psi : M \to H_{r,s}$ by

(2.36) $\psi(x) = (f_{r-1}(x), g_{s-1}(x))$.

Then the map ψ is horizontal to (2.34) in the sense of Definition 1.18.

Having established this lemma we can now bring into play Lemma 1.19.

Proposition 2.19 [35]

Let $M \to \mathbb{C}P^n$ be a full holomorphic map and r an integer $(0 \le r \le n)$. Define $\phi : M \to \mathbb{C}P^n$ by

(2.37) $\phi(x) = f^{\perp}_{r-1}(x) \cap f_r(x)$

or equivalently,

(2.38) $\phi(x) = (f_{r-1}(x) + g_{s-1}(x))^{\perp}$,

where g is the polar of f and $r+s = n$. Then ϕ is an isotropic harmonic map.

Proof

We endow $H_{r,s}$ with the (real) submanifold metric induced from $G_r(\mathbb{C}^{n+1}) \times G_s(\mathbb{C}^{n+1})$, $r+s = n$. With this metric, (2.34) is a Riemannian submersion (see [39]).

Now the map ϕ is the composition $\phi = \pi o \psi$. The map ψ is harmonic, since as a map into $G_r(\mathbb{E}^{n+1}) \times G_s(\mathbb{E}^{n+1})$, its components are $\pm$ holomorphic. Also, by Lemma 2.18, ψ is π-horizontal. Hence by Lemma 1.19, ϕ is harmonic.

The isotropy of ϕ follows from the mutual orthogonality of f_{r-1} and g_{s-1}, determined by the isotropy relations (2.7) existing between f and g. □

<u>Remarks</u>

1. As a flag manifold [50], $H_{r,s}$ is endowed with a Kähler metric and thus we could appeal to Lemma 1.21 to conclude that ϕ is harmonic, given that ψ is π-horizontal.

2. Rank conditions relating to horizontal maps to homogeneous fibrations such as (2.34), have been investigated in [76].

Consider a smooth map $\phi: M_p \to \mathbb{C}P^n$. By the <u>degree of ϕ</u> (denoted deg ϕ), we mean the degree of the induced map in 2nd cohomology; setting $N = \mathbb{C}P^n$, it is given explicitly by

$$\deg \phi = \frac{c}{4\pi} \int_M \phi^* \omega^N$$

($c = 4$ in [35]).

From the main results of [35], we see that for $n \geq 2$:

i) there exist nontholomorphic, harmonic maps from M_p to $\mathbb{C}P^n$, of all degress greater than p ;

ii) there exist nontholomorphic, harmonic maps of M_0 (i.e. $\mathbb{C}P^1$ or S^2) and M_1 (the torus, or elliptic curve) to $\mathbb{C}P^n$ of all degrees ≥ 0 .

To see how $\deg \phi$ is computed from the construction in Proposition 2.19, the formula (2.24) is summed for $1 \leq k \leq r$. For M_p , we obtain

$$\deg \phi = d_r - d_{r-1} \tag{2.39}$$

$$= r(2p-2) - \sum_{\alpha=0}^{r-1} \beta_\alpha + d \ . \tag{2.40}$$

The resulting energy, $E(\phi)$, is given by

$$E(\phi) = \pi\{d_r + d_{r-1}\} \tag{2.41}$$

$$= \pi\{\deg \phi + 2(rd+r(r-1)(p-1) - \sum_{\alpha=0}^{r-2} (r-\alpha-1)\beta_\alpha)\} \ . \tag{2.42}$$

The expression in (2.42) is obtained by deducing from (2.24) that

$$d_k = (k+1)d + k(k+1)(p-1) - \sum_{\alpha=0}^{k-1} (k-d)\beta_\alpha \quad (0 \leq k \leq n \ , \ d_0 = d, \ d_n = 0) \ . \tag{2.43}$$

A wide range of examples may be constructed by maps arising from compositions of the following type. Consider the composition

$$M_p \xrightarrow{h} \mathbb{C}P^1 \xrightarrow{f'} \mathbb{C}P^n \qquad n \geq 2 \tag{2.44}$$

where h is a holomorphic map (and therefore a branched covering map of $\mathbb{C}P^1$) and f' is a full holomorphic map. If μ denotes the degree of h, then μ will be controlled by the Riemann-Hurwitz formula ([42], see also Chapter VII). The construction of h is by no means unique: for example a Riemann surface with $p = 8$ in $\mathbb{C}P^3$, can be expressed as a holomorphic branched covering map of $\mathbb{C}P^1$ of degree 5, in 14 distinct ways [42].

With the Eells-Wood construction in mind, we may take $f = f'oh$ as our 'generating' holomorphic map. The following results were also obtained independently in [23].

<u>Proposition 2.20</u>

<u>Regarding the composition of</u> (2.44) <u>above. let</u> β_k, β_k' <u>be the ramification indices of the</u> k^{th} <u>associated curve of</u> f <u>and</u> f' <u>respectively</u> $(0 \le k \le n)$. <u>Further let</u> ϕ <u>and</u> ϕ' <u>be the respective isotropic harmonic maps constructed by</u> (f,r) <u>and</u> (f',r) <u>according to</u> (2.30), <u>then we have the following relations:</u>

(2.45) i) $\beta_k = \mu\beta_k' + 2(\mu+p-1)$

(2.46) ii) $\deg \phi = \mu(\deg \phi' - 2rp)$

(2.47) iii) $E(\phi) = \pi\{\deg\phi+\mu(E(\phi')-\deg\phi')+(p-1)\{r^2(1-\mu)-2r+1-\mu\}-r(r-1)\}$.

Proof

i) We substitute $d_k = \mu d'_k$ into (2.24) and then apply (2.24) again for genus equal to 0 .

ii) Substituting i) into (2.40) for deg ϕ , we deduce that

$$\deg \phi = d-2r\mu-\mu\left(\sum_{k=0}^{r-1} \beta'_k\right) .$$

Then taking (2.40) for deg ϕ' and multiplying by μ , we obtain (2.46) by eliminating the summation of the β'_k . iii) This result from: a) taking (2.42) for $E(\phi')$ and multiplying by μ , b) substituting for $\mu\beta'_k$ by (2.45) and c) elimination of β_k on considering (2.42) for $E(\phi)$. □

From these relations we can conclude that the isotropic harmonic maps generated by holomorphic maps $f = f' \circ h^{-1}$ have arbitrarily high energy since f' may be chosen such that $E(\phi')$ itself is arbitrarily high.

Remarks

1. In [35] it is asserted that for $n \geq 2$, there exist harmonic, non-holomorphic maps $M_p \to \mathbb{C}P^n$ of all degrees $\mu \geq p+1$ as well as $d = 0$, by considering compositions such as (2.44). (This is false for $n = 1$, since any harmonic map $M_p \to \mathbb{C}P^1$ of degrees $\geq p$ is holomorphic [31].)

2. In principle, a study of the moduli space of holomorphic curves in $\mathbb{C}P^n$ will yield information concerning the harmonic maps manufactured by

the Eells-Wood theory. For example, if $p \geq (n+1)(p+n-d)$, then the family of full rational maps $f:M_p \to \mathbb{C}P^n$, depends on at least $((n+1)(2d-n+1)-1-2np)$ real parameters, up to global $SU(n+1)$ action [22]. Here the degree of f is the degree of any effective divisor on M_p [42]. A concise account of the moduli space is given in [46].

2.5 Some constructions from algebraic geometry.

To supplement those in [35, §8] we shall mention two further examples that have their origins in algebraic geometry.

Example 2.21

Let $\Lambda_1,\ldots,\Lambda_{2n}$ be $2n$ vectors in $\mathbb{C}^n$ that are linearly independent over $\mathbb{R}$, and let Λ be the lattice consisting of all linear combinations of $\{\Lambda_1,\ldots,\Lambda_{2n}\}$. The lattice Λ acts naturally on $\mathbb{C}^n$ by translation $z \to \lambda+z$, for $\lambda \in \Lambda$. We define the equivalence relation $z \sim w$ with respect to Λ if $z = w+\lambda$ for some $\lambda \in \Lambda$. Let $\mathbb{C}T^n = \mathbb{C}^n/\Lambda$ be the set of equivalence classes with respect to Λ. Endowed with the usual quotient topology, $\mathbb{C}T^n$ is a complex manifold with universal covering space $\mathbb{C}^n$. The space $\mathbb{C}T^n$ is known as the n-dimensional complex torus. It is known that all $\mathbb{C}T^n$ are Kähler [86].

Again, let M_p be a compact Riemann surface of genus p. Associated to M is a period matrix Ω which we shall proceed to describe. We firstly consider the basis $\delta_1,\ldots,\delta_{2p}$ of 1 cycles in M, for the group $H_1(M,\mathbb{Z})$. Correspondingly, let $\omega_1,\ldots,\omega_p$ be a basis for the space of holomorphic 1-forms on M i.e. the group $H^0(m,\Omega^1)$

(notation as in [42]). The matrix Ω is defined to be the $p \times 2p$ matrix

$$(\Omega_{ij}) = (\int_{\delta_i} \omega_j) \ , \quad 1 \le i \le 2p \ , \quad 1 \le j \le p \ .$$

The column vectors $\Pi_i = [\int_{\delta_i} \omega_1, \ldots, \int_{\delta_i} \omega_p]$ are known as the <u>periods</u> and the $2p$ periods generate a lattice

$$(2.48) \qquad \Lambda = \{m_1\Pi_1 + \ldots + m_{2p}\Pi_{2p}, \ m_i \in \mathbb{Z}\}$$

in $\mathbb{C}P^p$.

<u>The Jacobian variety</u> $J(M)$ (or <u>Albanese torus</u>) associated to M is the complex torus $\mathbb{C}^p/\Lambda$. Taking a base point x_0 in M , we obtain the associated holomorphic map $\mathrm{Jac}: M_p \to J(M)$ given by

$$(2.49) \qquad \mathrm{Jac}(x) = (\int_{x_0}^{x} \omega_1, \ldots, \int_{x_0}^{x} \omega_p) \ .$$

It is known that the corresponding <u>Gauss map</u>

$$(2.50) \qquad G: M_p \to G_1(\mathbb{C}^p) \cong \mathbb{C}P^{p-1}$$

coincides with <u>the canonical curve</u> $i_K: M_p \to \mathbb{C}P^{p-1}$ [42]. Here $G = d(\mathrm{Jac})$ is given by

$$(2.51) \qquad z \to (\omega_1/dz, \ldots, \omega_p/dz) \ .$$

The pull-back of translation invariant 1-forms by (2.49) establishes a bijective correspondence

$$\left\{\begin{array}{c}\text{translation invariant}\\ \text{1-forms on } J(M)\end{array}\right\} \xleftrightarrow{1-1} \left\{\begin{array}{c}\text{holomorphic differentials}\\ \omega \text{ on } M\end{array}\right\}$$

Thus commencing with a holomorphic map f given by (2.51), we can construct a particular class of isotropic harmonic maps $M_p \to \mathbb{C}P^{p-1}$ defined by (2.30).

Example 2.22

Now for our second example. This relates to the theory of elliptic functions; we refer to the discussions in [42] or [47] for details. Let Λ be a period lattice in $\mathbb{C}$. We can construct a non-degenerate holomorphic map $f:M_1 \to \mathbb{C}P^2$ as follows:

We introduce the Weierstrass $\wp$-function which is an elliptic function defined by

$$(2.52) \qquad \wp(z) = \frac{1}{z^2} + \sum_{w\in\Lambda'} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right) \quad \text{where } \Lambda' = \Lambda - \{0\} .$$

Its derivative $\wp'(z) = \sum_{w\in\Lambda} -\frac{2}{(z-w)^3}$ is also an elliptic function.

The map f is then given by $f(z) = [1, \wp(z), \wp'(z)]$, the image of which is a non-degenerate, non-singular cubic curve in $\mathbb{C}P^2$. Its polar curve g is given by $g(z) = [\overline{\wp(z)\wp''(z) - \wp'(z)}, \overline{-\wp''(z)}, \overline{\wp'(z)}]$. The resulting isotropic harmonic map $\phi:M_1 \to \mathbb{C}P^2$ constructed from (2.38), is given by

$$\phi(z) = (\overline{P'(z)}\, P''(z) - \overline{P(z)}P'(z)\ ,\ P'(z)(|P'(z)|^2 - 1)$$

$$-\overline{P'(z)}P(z)P''(z)\ ,\ -P''(z)(1 + |P(z)|^2) + \overline{P(z)}P'(z)^2)\ .$$

Since f is a non-singular curve of degree 3 $(\beta_0 = 0)$, we apply formula (2.40) for $p = 1$ and $r = 1$, whence we obtain $\deg \phi = 3$.

2.6 Total isotropy.

We consider a full holomorphic map

$$(2.53) \qquad f : M \to \mathbb{C}P^{N-1}$$

along with the standard $\mathbb{C}$-bilinear symmetric inner product $(\ ,\ )^{\mathbb{C}}$ on $\mathbb{C}^N$ i.e.

$$(2.54) \qquad (u,v)^{\mathbb{C}} = \sum_{i=1}^{N} u^i v^i \qquad u,v \in \mathbb{C}^N \quad (\langle u,v\rangle = (u,\bar{v})^{\mathbb{C}})\ .$$

Definition 2.23

We shall say that f in (2.53), is <u>totally isotropic</u> if f_α and $\bar{f}_\beta$ are $(\ ,\ )^{\mathbb{C}}$-orthogonal for all $\alpha,\beta \geq 0$, $\alpha+\beta \leq N-2$. In terms of a local lift ξ of f, this means that

$$(2.55) \qquad (\partial'^{\alpha}\xi, \partial'^{\beta}\xi)^{\mathbb{C}} \equiv 0 \quad \text{for } \alpha,\beta \geq 0,\ \alpha+\beta \leq N-2\ .$$

Hence we see that f is totally isotropic if and only if its polar curve is $\bar{f}$. A subspace E of $\mathbb{C}^N$ is said to be <u>totally isotropic</u> if it is $(\ ,\ )^{\mathbb{C}}$-orthogonal to its conjugate $\bar{E}$. There is a definite restriction on holomorphic maps of this type:

Lemma 2.24 [18] [35]

If N *is even, then there are no full, totally isotropic holomorphic maps* $f:M \to \mathbb{C}P^{N-1}$.

Example 2.25

We obtain from [26], an example of a full, totally isotropic holomorphic map $f:\mathbb{C}P^1 \to \mathbb{C}P^4$. For $z \in \mathbb{C}P^1$, this is defined by $f(z) = (\xi_1,\ldots,\xi_5)$ where $\xi_1 = 2\sqrt{3}(1+z^4)$, $\xi_2 = -12z^2$, $\xi_3 = 2\sqrt{3}i(1-z^4)$

$$\xi_4 = \sqrt{3}i(-4z-4z^3) \ , \ \xi_5 = \sqrt{3}(-4z+4z^3) \ .$$

On setting $N = 2r+1$, $r \geq 1$, we have:

Proposition 2.26 [35]

Let $f:M \to \mathbb{C}P^{2r}$ $(r \geq 1)$ *be a full, totally isotropic holomorphic map then*

$$(2.56) \qquad \phi(x) = (f_{r-1}(x) + \overline{f_{r-1}(x)})^{\perp} \qquad x \in M ,$$

defines a harmonic map $\phi:M \to \mathbb{R}P^{2r}$ *(the real projective space of 2r-dimensions).*

We note that the above manufactured harmonic map is isotropic in so far that the composition $M \xrightarrow{\phi} \mathbb{R}P^{2r} \longrightarrow \mathbb{C}P^{2r}$ is isotropic (see [35, Lemma 5.7]) where the last map is the (totally geodesic) inclusion of $\mathbb{R}P^{2r}$ as real points of $\mathbb{C}P^{2r}$. The above construction may be seen by taking $g = \bar{f}$ and $r = s$ in (2.38).

The space H_r, the totality of totally isotropic r-dimensional subspaces of $\mathbb{C}^{2r+1}$, has coset space representation $H_r \cong SO(2r+1)/U(r)$. It is regarded as a totally geodesic submanifold of $H_{r,r}$ via the inclusion $V \to (V,\bar{V})$. Accordingly, when we restrict (2.34), for $r = s$, i.e.

$$\pi : H_{r,r} \cong U(2r+1)/U(r)\times U(r)\times U(1) \to \mathbb{C}P^{2r} \cong U(2r+1)/U(2r)\times U(1)$$

restricted to H_r, we obtain the restricted fibration for which we shall retain π to denote the projection:

$$(2.57) \qquad \pi : H_r \cong SO(2r+1)/U(r) \to \mathbb{R}P^{2r} \cong SO(2r+1)/S(O(2r)\times O(1)) .$$

Choosing the metric on H_r as in Proposition 2.19, (2.57) is a Riemannian submersion. Thus, as observed in [35], Proposition 2.26 may be rephrased as follows:

<u>Proposition 2.27</u>

<u>Let</u> $f : M \to \mathbb{C}P^{2r}$ <u>be a full, totally isotropic holomorphic map, then for</u> π <u>as given by</u> (2.57), <u>the map defined by</u>

$$(2.58) \qquad \phi(x) = \pi \circ f_{r-1}(x) \qquad (x \in M,\ 0 \le r \le n)$$

<u>defines an isotropic harmonic map</u> $\phi : M \to \mathbb{R}P^{2r}$.

<u>Proof</u>

If f is totally isotropic, then following [35], the subspaces

$f_{r-1}(x)$, $x \in M$, are totally isotropic. The map of (2.58) may then be seen to be the composition

$$(2.59) \qquad M \xrightarrow{f_{r-1}} H_r \xrightarrow{\pi} \mathbb{R}P^{2r} \quad .$$

The map f_{r-1} is π-horizontal and the harmonicity of ϕ follows from the argument in Proposition 2.19. The isotropy of ϕ follows from the fact that, as a map into $\mathbb{C}P^{2r}$, ϕ is isotropic. □

To construct an isotropic harmonic map $\phi:M \to S^{2r}$, we replace π in (2.59) by

$$(2.60) \qquad \pi:H_r \cong SO(2r+1)/U(r) \to SO(2r+1)/SO(2r) \cong S^{2r} \quad .$$

Here we choose an oriented basis (e_i) of $\mathbb{R}^{2r+1}$ such that $W \in H_r$ is spanned by $\{e_1 + ie_2,\ldots,e_{2r-1} + ie_{2r}\}$ $(i = \sqrt{-1})$. Then π is given by $\pi(W) = e_{2r+1}$. We shall generalise this construction to the real oriented Grassmannian in section 3.4 of Chapter III.

Imposing the condition of total isotropy on the holomorphic curve f in Theorem 2.16, leads to a recovery of Calabi's theorem [18] [19]:

<u>Corollary 2.28</u>

<u>Theorem 2.16 restricts to give a bijective correspondence between full,* isotropic harmonic maps $\phi:M \to \mathbb{R}P^{2r}$ and full, totally isotropic holomorphic maps $f:M \to \mathbb{C}P^{2r}$.</u>

These harmonic maps all lift to harmonic maps into the double

---

* This is the fullness of the map into $\mathbb{C}P^{2r}$ on taking the totally geodesic inclusion.

covering S^{2r} of $\mathbb{R}P^{2r}$. Thus one also recovers the principal result obtained in [4] [18] [19] [25] and [26]:

Theorem 2.29

There exists a 1:1 correspondence between arbitrary pairs of full, isotropic harmonic maps $\pm\phi : M \to S^{2r}$, and full, totally isotropic holomorphic maps $f : M \to \mathbb{C}P^{2r}$.

2.7 Isotropic harmonic maps to S^{2r} inducing constant curvature.

The totally isotropic holomorphic curve given by Definition 2.23, was classically known as the 'directix curve'. In [26], Chern considers the case $r = 2$ explicitly, by constructing a class of minimal immersions of S^2 into S^4 (unit radius). Restricting his attention to a particular class of rational curves (of which Example 2.9 is an example), he constructs an example of a minimal immersion $S^2 \to S^4$ of constant Gaussian curvature 1/3 (recall that the Gaussian curvature is the curvature induced by the pull-back of the metric of the immersion). The isometric harmonic maps (i.e. isometric minimal immersions of S^2 into $S^{2r}(\tilde{c})$, where $\tilde{c}$ denotes the constant curvature, that induce constant Gaussian curvature c, are given by spherical harmonic polynomial maps [4] [13] [18]. These induce constant Gaussian curvature $c = 2\tilde{c}/r(r+1)$, as well as giving the minimum allowable energy, $2\pi r(r+1)$, when $\tilde{c} = 1$. Generally, the minimal spheres obtained by the previous constructions are of non-constant Gaussian curvature.

We now turn our attention to the case where M is taken to be the

hyperbolic plane with its Poincaré metric of constant negative curvature. The following result is partly suggested by Proposition 2.7 and Corollary 2.8. We shall make use of the fact that the Plücker embedding of $G_r(\mathbb{C}^{2r+1})$ into $\mathbb{C}P^K$, $K = \binom{2r+1}{r} - 1$, is an isometric embedding [41].

<u>Proposition 2.30</u>

<u>There exists no isometric embedding</u> $\phi : \mathbb{R}H^2 \to S^{2r}$ $(r > 1)$ <u>where</u> ϕ <u>is a full, isotropic harmonic map inducing constant negative curvature on</u> $\mathbb{R}H^2$.

<u>Proof</u>

Assume that such a map ϕ was given, then with respect to (2.60) $\phi = \pi \circ f_{r-1}$, where f_{r-1} is the $(r-1)^{th}$-associated curve of a full, totally isotropic, holomorphic map $f : \mathbb{R}H^2 \to \mathbb{C}P^{2r}$. Following [4], the maps ϕ and f_{r-1} are isometric and induce the same constant curvatures with respect to their pull-back metrics. Since ϕ is isometric, there exist no points $x \in \mathbb{R}H^2$, such that $d\phi(x) = 0$, and by the construction of ϕ, the same is true of f_{r-1}. In particular, f_{r-1} is an isometrically embedded holomorphic curve in $G_r(\mathbb{C}^{2r+1})$ and via the Plücker embedding, is an isometrically embedded holomorphic curve in $\mathbb{C}P^K$ where K is given as above. But by Proposition 2.7, such a holomorphic curve f_{r-1} (or any of the associated curves f_k, $0 \le k \le r-1$) inducing a metric of constant negative curvature, cannot exist. Hence ϕ cannot exist.

$\square$

CHAPTER III

HARMONIC MAPS TO GRASSMANNIANS

3.1 The Hermitian symmetric spaces.

We start this chapter by discussing the Hermitian symmetric spaces. We intend using freely, some notions and terminology from the theory of Lie groups; thus we refer to [48] and [57] for this purpose.

A Riemannian symmetric space is a connected Riemannian manifold N such that given $y \in N$, there exists a globally defined isometry s_y for which y is an isolated fixed point, and s_y has differential $-I$ on the tangent space T_yN at y. In this case, s_y is known as the symmetry to N at y. The symmetry s_y is unique.

Let N be an Hermitian manifold. We obtain a Riemannian manifold by taking the underlying real manifold and the real part of the Hermitian metric. If this Riemannian manifold is symmetric and if the symmetries are Hermitian isometries, then N with its Hermitian metric is called an Hermitian symmetric space (HSS). An HSS is always Kählerian [57].

Let N be an HSS, then N decomposes as a product

$$N = N_0 \times N_1 \times \dots \times N_k . \tag{3.1}$$

Here N_0 is the quotient of a $\mathbb{C}$-vector space with a translation

invariant metric by a discrete group of translations. Such an HSS is said to be of Euclidean type. Each N_i $(1 \le i \le k)$ is a non-Euclidean irreducible HSS (not locally isometric to a product of lower dimensional Riemannian manifolds).

Remark

If N is a simply connected (globally) Riemannian symmetric space, then this is well known (see for example [57, p.246]). In the case that N is an HSS, simply-connectedness is not required.

The factors N_i $(i \neq 0)$ in (3.1) which are compact are the irreducible compact symmetric Kähler manifolds as classified by E. Cartan in [20]. In fact they are all known to be simply connected, rational projective varieties (i.e. they admit embeddings as submanifolds of some $\mathbb{C}P^n$).

The non-compact factors are bounded symmetric domains in $\mathbb{C}^n$ for the appropriate value of n.

If there is no N_0 factor and all the N_i are compact (non-compact), then we say that N is a compact type (non-compact type, respectively). From now on we shall use Y to denote a compact type and D to denote a non-compact type.

In all, there are six classes (Types I to VI) of HSS. Each class has its compact form Y with corresponding dual, non-compact form D [21] [48]. For example, the non-compact dual Hermitian space to $\mathbb{C}P^n$, is $\mathbb{C}H^n$ (complex hyperbolic space). The dual form D is in fact one of a finite number of open orbits of a certain Lie group action on the

compact forms. The remaining open orbits are endowed with indefinite Kähler metrics, following the work of J.A. Wolf in [88] and [89]. We will briefly discuss this in Chapter VI.

For the remainder of this section we will state the compact form for each of the six classes, together with a brief description. The non-compact dual forms will be discussed in Chapter VI.

Let us commence with some general considerations. Let G be a connected simple Lie group with trivial centre. Given an involutive automorphism θ of G, let K denote the identity component of the fixed point set of θ. In order that the irreducible symmetric space G/K be Kählerian, it is necessary and sufficient that:

a) K be compact;

b) the centre of K is a circle group whose element of period 2 has θ as its adjoint representation on G.

We now take G to be compact and K of maximal rank. The group of all complex analytic transformations of the compact complex manifold $Y = G/K$, is the complex extension $G^{\mathbb{C}}$ of G, for which G is a maximal compact subgroup. The above discussion characterises the irreducible HSS of compact type. We shall follow Cartan's classification [20], and use Roman numerals to denote the type with the corresponding complex dimension indicated in the brackets following.

<u>Type $I_{p,q}$</u> (pq)

(3.2) $\quad Y^{I} \cong U(p+q)/U(p)\times U(q)$. This is holomorphically equivalent

to $G_p(\mathbb{C}^{p+q})$, the Grassmannian of complex p-planes in $\mathbb{C}^{p+q}$.

<u>Type II_n</u> ($\frac{1}{2}n(n-1)$)

Let J' be a fixed non-degenerate symmetric bilinear form on $\mathbb{C}^{2n}$ $(n \geq 1)$. Then

$$(3.3) \qquad Y^{II} = \{n\text{-planes } w \in \mathbb{C}^{2n} : J'(w,w) = 0\} \cong SO(2n)/U(n) .$$

<u>Type III_n</u> ($\frac{1}{2}n(n+1)$)

Let J'' be a fixed non-degenerate, anti-symmetric bilinear form on $\mathbb{C}^{2n}$. Then

$$(3.4) \qquad Y^{III} = \{n\text{-planes } w \in \mathbb{C}^{2n} : J''(w,w) = 0\} \cong Sp(n)/U(n) .$$

<u>Type IV_n</u> (n)

Let B be a fixed non-degenerate symmetric bilinear form on $\mathbb{C}^{n+2}$. Then

$$(3.5) \qquad Y^{IV} = \{v \in \mathbb{C}P^{n+1} : B(v,v) = 0\} \cong SO(n+2)/SO(2)\times SO(n) .$$

This is holomorphically equivalent to the non-singular n-dimensional complex quadric hypersurface $Q_n \subset \mathbb{C}P^{n+1}$, $n \geq 1$, and is real analytically isomorphic to the real Grassmannian $G_2^0(\mathbb{R}^{n+2})$ of real oriented 2 planes in $\mathbb{R}^{n+2}$.

Type V (16)

(3.6) $$Y^V \cong E_6/\mathrm{Spin}(10) \times SO(2) .$$

Type VI (27)

(3.7) $$Y^{VI} \cong E_7/E_6 \times SO(2) .$$

Remark

The following isomorphisms reduce the list of distinct ones. These are also valid for the corresponding non-compact types:

$$I_{p,q} \cong I_{q,p} , \; I_{1,1} \cong II_2 \cong III_1 \cong IV_1 , \; II_3 \cong I_{1,3}$$
$$IV_3 \cong III_2 , \; IV_4 \cong I_{2,2} , \; IV_6 \cong II_4 .$$

To simplify the notation, we shall remove the subscripts in each case. Types I to IV inclusive are known as 'classical'. The 'exceptional' types V and VI, do not lend themselves to a straightforward geometrical realisation and we shall put them aside. Types II and III are totally geodesic in their ambient Grassmannian $G_n(E^{2n})$; this is a consequence of the following more general result the proof of which will be outlined in the appendix:

Let (G,K,θ) represent a Riemannian symmetric space $N = G/K$ with involutive automorphism θ . Let (G',K',θ') represent a symmetric subspace of (G,K,θ) i.e. $N' = G'/K'$ is such that G' is a θ-invariant connected Lie subgroup of G , $K' = G' \cap K$ and $\theta' = \theta|_{G'}$.

Theorem 3.1 [57, Theorem 4.1, p.234]

Let (G,K,θ) *and* (G',K',θ') *be as above. Then* $N' = G'/K'$ *embedds naturally as a totally geodesic submanifold of* $N = G/K$ *with respect to the canonical connection of* G/K . *The canonical connection of* N *restricted to* N' *coincides with the canonical connection of* N' .

3.2 The construction of isotropic harmonic maps to a complex Grassmannian.

In this section we shall be interested in constructing a certain class of non-trivial harmonic maps.

$$(3.8) \qquad \phi : M \to G_k(\mathbb{C}^N)$$

where M is a Riemann surface (open or closed) and $0 \le k \le N$. The construction we are about to describe is due to Erdem and Wood [39] and generalises the construction discussed in the last chapter. We will later show that this construction may also be used to study analogous harmonic maps to Types II, III and IV compact HSS. The non-compact types will be discussed in Chapter VI, where we shall make several modifications to this construction. We will use freely some of the ideas and terminology of Chapter I and II and proceed to establish the necessary details and results from [39] that will be required.

Definition 3.2

A map $\phi : M \to G_k(\mathbb{C}^N)$ is said to be *full* if the only subspace of

$\mathbb{C}^N$ containing each subspace $\phi(x)$, for $x \in M$, is $\mathbb{C}^N$ itself.

Definition 3.3

Let (V,X) be a pair of holomorphic subbundles of $\underline{\mathbb{C}}^N$ with

a) $V \subset X$ b) $\text{rank}(X)-\text{rank}(V) = k$, c) $\partial' C(V) \subset C(X)$.

Then we call such a pair a ∂'-pair of $\underline{\mathbb{C}}^N$ of rank difference k .

If A is a vector subspace of $\mathbb{C}^N$ then a subbundle of the form $M \times A$, is called a constant subbundle of $\underline{\mathbb{C}}^N$. With this in mind we have:

Definition 3.4

A ∂'-pair is said to be full if

i) the only constant subbundle of $\underline{\mathbb{C}}^N$ containing X is $\underline{\mathbb{C}}^N$ itself;

ii) the only constant subbundle of $\underline{\mathbb{C}}^N$ contained in V, is the zero subbundle $\underline{0} = M \times \{0\}$.

Let $L \to G_k(\mathbb{C}^N)$ be the universal k-plane bundle. We have an exact sequence of complex vector bundles over $G_k(\mathbb{C}^N)$ which generalises that of (2.26):

$$\underline{0} \to L \to \underline{\mathbb{C}}^N \to L^{\perp} \to \underline{0} \ .$$

Given a smooth map $\phi : M \to G_k(\mathbb{C}^N)$, we may take its universal lift $\Phi : \phi^{-1}L \hookrightarrow \underline{\mathbb{C}}^N$ a section of the bundle $\text{Hom}(\phi^{-1}L, \underline{\mathbb{C}}^N)$ on M . If D denotes

covariant differentiation in this bundle, then D has a splitting into $(1,0)$ and $(0,1)$ types, denoted by D' and D'' respectively. We may consider the iterated derivatives, $D'^{\alpha}\phi$ and $D'^{\beta}\phi$ of all orders $\alpha,\beta \geq 0$; these are systematically derived from:

<u>Lemma 3.5</u> [39]

<u>For any</u> $V \in C_U(\mathrm{Hom}(\phi^{-1}L,\underline{E}^N))$, $\rho \in C_U(\phi^{-1}L)$,

$$(3.9) \qquad (D''V)(\rho) = \partial''(V(\rho)) - V(D''_{\phi}(\rho))$$

$$= \partial''(V(\rho)) - V(P(\partial''(\rho)))$$

<u>where</u> $P:\underline{E}^N \to \phi^{-1}L$ <u>denotes orthogonal projection and</u> D_{ϕ} <u>denotes the connection on</u> $\phi^{-1}L$. <u>A similar formula holds for</u> $D'V$.

Let U be an open set in M ; then taking the smooth section $W \in C_U(\mathrm{Hom}\phi^{-1}L,\underline{E}^N))$, we shall denote by $\mathrm{Im}(W_x)$, the image of the linear map

$$(3.10) \qquad W_x : L_{\phi(x)} \to E^N .$$

We shall write

$$(3.11) \qquad \mathrm{Im}\, W = \bigcup_{x\in U}(\{x\} \times \mathrm{Im}(W_x)) \subset U \times E^N .$$

<u>Lemma 3.6</u> [39]

<u>A smooth map</u> $\phi:M \to G_k(E^N)$ <u>is harmonic if and only if in any chart</u> U ,

$$\mathrm{Im}(D''D'\phi) \subset \mathrm{Im}(\phi) ,$$

<u>or equivalently</u>, $\mathrm{Im}(D'D''\phi) \subset \mathrm{Im}(\phi)$.

<u>Lemma and Definition 3.7</u> [39]

<u>A smooth map</u> $\phi : M \to G_k(\mathbb{C}^N)$ <u>is</u> (<u>strongly</u>) <u>isotropic if and only if</u>

$$\mathrm{Im}(D'^{\alpha}\phi) \perp \mathrm{Im}(D''^{\beta}\phi) \quad \forall\ \alpha,\beta \geq 0 , \quad \alpha+\beta \geq 1 .$$

For a smooth map $\phi : M \to G_k(\mathbb{C}^N)$, we have the (unique) <u>associated subbundles</u> [39], $\phi'_{(\alpha)}$, $\phi''_{(\alpha)}$ of $\underline{\mathbb{C}}^N$, for each $\alpha \geq 1$, such that for all $x \in M$

$$(3.12) \qquad (\phi'_{(\alpha)})_x = \mathrm{span}\{\mathrm{Im}(D'^{\gamma}\phi)_x : 1 \leq \gamma \leq \alpha\}$$

$$(3.13) \qquad (\phi''_{(\beta)})_x = \mathrm{span}\{\mathrm{Im}(D''^{\delta}\phi)_x : 1 \leq \delta \leq \beta\}$$

where the span takes its maximum dimension in each case. This span is independent of the chart chosen for the definition. The <u>augmented associated bundle</u> of $\phi'_{(\infty)}(\phi''_{(\infty)})$ is defined to be

$$(3.14) \qquad \tilde{\phi}'_{(\infty)} = \phi'_{(\infty)} + \phi \quad (\text{respectively, } \tilde{\phi}''_{(\infty)} = \phi''_{(\infty)} + \phi) .$$

<u>THEOREM 3.8</u> [39, Th. 1.1]

<u>There exists a bijective correspondence between full ∂'-pairs (V,X)</u>

<u>of rank difference</u> k , <u>on</u> M , <u>and full, isotropic harmonic maps</u> $\phi : M \to G_k(\mathbb{C}^N)$ <u>given by</u>

$$(3.15) \qquad \phi(x) = V_x^{\perp} \cap X_x$$

<u>for all</u> $x \in M$.

<u>Conversely, given such a</u> ϕ , <u>the pair</u> $(\phi''_{(=)}, \phi''_{(=)})$ <u>is a full</u> ∂'-<u>pair</u> <u>giving</u> (V, X).

<u>Example 3.9</u>

The following example will serve as a model for several constructions to follow.

Consider a full, holomorphic map $f : M \to \mathbb{C}P^{N-1}$ along with its associated curves f_{r-k} and f_r , $0 \le r \le N-1$. Recalling from Section 2.1, the bijective correspondence between holomorphic maps $M \to G_t(\mathbb{C}^N)$ and holomorphic subbundles of $\underline{\mathbb{C}}^N$ of rank t on M , and setting

$$(3.16) \qquad V = f_{r-k} \quad \text{and} \quad X = f_r \ , \quad k \ge 1 \ ,$$

we thus obtain a full ∂'-pair of rank difference k . For $x \in M$,

$$(3.17) \qquad \phi(x) = f_{r-k}^{\perp}(x) \cap f_r(x)$$

defines a full, isotropic harmonic map $\phi : M \to G_k(\mathbb{C}^N)$ in accordance with Theorem 3.8.

We view the construction of ϕ as follows:

Let $F^N_{n_1,n_2} = \{(V,W) \in G_{n_1}(\mathbb{C}^N) \times G_{n_2}(\mathbb{C}^N) : V \perp W\}$

$$\cong U(N) / U(n_1) \times U(n_2) \times U(n_3)$$

where $n_3 = N - (n_1+n_2)$ $(\dim_{\mathbb{C}} F^N_{n_1,n_2} = \frac{1}{2}(N^2 - \sum_{i=1}^{3} n_i^2))$.

We have a natural fibration $\pi : F^N_{n_1,n_2} \to G_{n_3}(\mathbb{C}^N)$ induced by the natural inclusion of $U(n_1) \times U(n_2)$ in $U(n_1+n_2)$. We shall set $n_1 = r-k+1$ $n_2 = N-r-1$ and $n_3 = k$. The map $\psi : M \to F^N_{n_1,n_2}$ given by $\psi(x) = (V_x, X_x^{\perp})$ can be shown to be harmonic and π-horizontal [39]. We can endow $F^N_{n_1,n_2}$ with its Kähler metric and then appeal to Lemma 1.21, followed by Lemma 1.19, to conclude that $\phi = \pi \circ \psi$ is harmonic where $\pi(V,X) = V^{\perp} \cap X$.

On taking $k = 1$ and $N = n+1$, we recover the construction outlined in Proposition 2.19.

Remark

$F^N_{n_1,n_2}$ could equally well be endowed with other metrics to ensure that ϕ is harmonic (see [39]).

3.3 Real Grassmannians.

In this section and in the next, we shall consider the case of the real Grassmannians (non-oriented and oriented). Many of our considerations

are straight forward generalisations of the real cases examined in [35] and [39].

<u>Corollary 3.10</u> [39]

<u>The assignment</u> (3.15) <u>restricts to give a bijective correspondence between the set of full</u> ∂'-pairs (V,X) <u>of</u> $\mathbb{C}^N$ <u>of rank difference</u> k <u>on</u> M , <u>satisfying the 'total isotropy' condition</u>

$$(3.18) \qquad \bar{X} = V^{\perp} ,$$

<u>and the set of full, isotropic harmonic maps</u> $M \to G_k(\mathbb{R}^N)$ (<u>non-oriented real</u> k-<u>planes in</u> $\mathbb{R}^N$) .

The fullness and isotropy of the map is that of the map into $G_k(\mathbb{C}^N)$ via the totally geodesic inclusion of $G_k(\mathbb{R}^N)$ into $G_k(\mathbb{C}^N)$.

Setting $N = 2r+1$ and $k = 1$, we recover Proposition 2.28. On taking a full, totally isotropic holomorphic map $M \to \mathbb{C}P^{2r}$, the corresponding ∂'-pair (V,X) in this case, is obtained by setting $V = f_{r-1}$ and $X = f_r$. We recall that the appropriate lift of ϕ to H_r was obtained by $V = f_{r-1}$ alone. This suggests that for certain values of N and $k(\geq 1)$, Corollary 3.10 could be re-phrased in terms of a suitable V satisfying certain conditions. This is indeed possible, but firstly, a lemma:

Lemma 3.11

Let (V,X) *be a* ∂'*-pair of holomorphic subbundles of* $\underline{\mathbb{C}}^N$ *on* M . *Assume that* V *and* X *satisfy the total isotropy condition* (3.18); *then* (*pointwise*) V *satisfies the conditions*

$$(3.19)\qquad \begin{cases} \bar{V} \subset V^{\perp} \\ \overline{\partial' C(V)} \subset C(V^{\perp}) \end{cases}$$

Proof

The first condition follows from condition a) in Definition 3.3 and from (3.18), i.e. $\bar{V} \subset \bar{X} = V^{\perp}$.

By condition c) of that same definition, we have $\partial' C(V) \subset C(X)$ and on taking (3.18) into account, we have $\overline{\partial' C(V)} \subset \overline{C(X)} = C(V^{\perp})$, which is the second condition. □

Since $(v,w)^{\mathbb{C}} = \langle v,\bar{w}\rangle$ for $v,w \in \mathbb{C}^N$, then the inclusion $\bar{V} \subset V^{\perp}$ is equivalent to V being *isotropic* for $(\ ,\ )^{\mathbb{C}}$, i.e. $(V,V)^{\mathbb{C}} = 0$. To make these conditions more explicit, we note that since V is a holomorphic subbundle of $\underline{\mathbb{C}}^N$ of rank t say, on M , it corresponds to a holomorphic map $\psi : M \to G_t(\mathbb{C}^N)$; in which case (3.19) is interpreted as $(\psi(x),\psi(x))^{\mathbb{C}} = 0$ and $(\psi(x),\partial'\psi(x))^{\mathbb{C}} = 0$ respectively, for $x \in M$.

Given a V satisfying the conditions in (3.19), we seek an extra condition to ensure that the ∂'-pair (V,X) satisfying (3.18), is full

in the sense of Definition 3.4. The condition is that V should satisfy condition ii) in Definition 3.4, namely, the only constant subbundle of $\underline{\mathbb{C}}^N$ contained in V, is the zero subbundle $\underline{0} = M \times \{0\}$. Since $\bar{X} = V^{\perp} \subset M \times \mathbb{C}^N$, it then follows that the only constant subbundle of $\underline{\mathbb{C}}^N$ containing X, is $\underline{\mathbb{C}}^N$ itself.

Observe that apart from the case k and $N-k$ <u>both odd</u>, any Grassmannian $G_k(\mathbb{C}^N)$ may be expressed as $G_k(\mathbb{C}^{2r+k})$, for some $r \geq 1$ (note that $G_k(\mathbb{C}^{2r+k}) \cong G_{2r}(\mathbb{C}^{2r+k})$). For reasons that will later become apparent, we shall restrict our attention to $G_k(\mathbb{C}^{2r+k})$.

Taking into account conditions (3.18) and (3.19), we see that in this case, the correct choice of ranks for V and X $(N = 2r+k)$, is

$$\begin{cases} \text{Rank } V = r \\ \text{Rank } X = r+k \\ (\text{Rank } X^{\perp} = r) \end{cases}.$$

Following the construction in Example 3.9, the map $\psi : M \to F^{2r+k}_{r,r}$ defined by $\psi(x) = (V_x, X_x^{\perp})$ $(= (V_x, \bar{V}_x))$, is harmonic and is π-horizontal to the fibration

$$(3.20) \qquad \pi : F^{2r+k}_{r,r} \to G_k(\mathbb{C}^{2r+k}),$$

induced by the inclusion of $U(r) \times U(r)$ in $U(2r)$. However, the conditions in (3.18) and (3.19) imply, that for $x \in M$, the subspaces V_x and

$X_x^{\perp}$ are <u>r-dimensional isotropic subspaces of $\mathbb{E}^{2r+k}$, with respect to</u> $(\ ,\ )^{\mathbb{E}}$. Their totality is a certain <u>quadric Grassmannian</u> [74], that we shall denote by $S_{k,r} \subset G_r(\mathbb{E}^{2r+k})$. This has coset space representation $O(2r+k)/U(r) \times O(k)$; in particular $S_{1,r} \cong H_r$. As in the case of H_r, the space $S_{k,r}$ is regarded as a subspace of $F_{r,r}^{2r+k}$ via the totally geodesic inclusion $V \to (V,\bar{V})$. The space $S_{k,r}$ will be the receiving space for our map ψ, as constructed above.

Retaining π to denote the restriction of (3.20) to $S_{k,r}$, we obtain the induced fibration

$$(3.21) \qquad \pi : S_{k,r} \to G_k(\mathbb{R}^{2r+k})$$

which as a homogeneous fibration, is given by $O(2r+k)/U(r) \times O(k) \to O(2r+k)/O(2r) \times O(k)$ for which the fibre is $O(2r)/U(r)$. The inclusion associated with (3.20), induces the inclusions $U(r) \hookrightarrow O(2r) \hookrightarrow O(2r) \times O(k)$. We can now establish the following:

<u>Proposition 3.12</u>

<u>Let</u> V <u>be a holomorphic subbundle of</u> $\mathbb{E}^{2r+k}$ <u>of rank</u> r $(r \geq 1)$ <u>on</u> M, <u>satisfying the conditions in</u> (3.19). <u>For</u> $x \in M$ <u>and</u> π <u>as in</u> (3.21), <u>the map</u> $\phi : M \to G_k(\mathbb{R}^{2r+k})$ <u>defined by</u>

$$(3.22) \qquad \phi(x) = \pi o V_x = (V_x + \nabla_x)^{\perp}$$

<u>is a harmonic map</u>.

Proof

The map $\psi:M \to F^{2r+k}_{r,r}$ defined by $\psi(x) = (V_x,\bar{V}_x)$ is harmonic and π-horizontal with respect to (3.20). Because of the conditions on V, this map has image in $S_{k,r}$ which is totally geodesic in $F^{2r+k}_{r,r}$, and is π-horizontal with respect to (3.21). Hence $\phi = \pi o V$, is harmonic in accordance with the composition principle of Lemma 1.19. □

Example 3.13

Here is an example for $r = 1$ and $k = 3$. We take V to be the holomorphic map $f:\mathbb{C}P^1 \to \mathbb{C}P^4$ as defined in Example 2.25. Regarded as conditions on the map f, both conditions in (3.19) are satisfied, as a straightforward calculation shows. We obtain a ∂'-pair (V,X) of rank difference 3, by setting $X = \bar{f}^{\perp} = f_3$, the 3rd-associated curve of f (Rank $X = 4$). The pair (V,X) satisfies the total isotropy condition in (3.18), and from (3.15), the map $\phi:M \to G_3(\mathbb{R}^5)$ defined by $\phi(x) = f^{\perp}(x) \cap f_3(x)$, is an isotropic harmonic map.

We now proceed to rephrase Corollary 3.10 for the case of isotropic harmonic maps to $G_k(\mathbb{R}^{2r+k})$.

Corollary 3.14

<u>There exists a bijective correspondence between full, isotropic harmonic maps</u> $\phi:M \to G_k(\mathbb{R}^{2r+k})$ <u>and holomorphic subbundles</u> V <u>of</u> $\underline{\mathbb{C}}^{2r+k}$ <u>of rank</u> r <u>on</u> M, <u>satisfying conditions</u> (3.19) <u>and the additional condition that the only constant subbundle contained in</u> V, <u>is the zero subbundle</u>.

Proof

The conditions satisfied by V, together with the relationship (3.18), imply the existence of a holomorphic subbundle X of $\underline{\mathbb{C}}^{2r+k}$ of rank $r+k$ on M, such that (V,X) is a full ∂'-pair of rank difference k.

The map ϕ is defined by (3.22), and in the inverse assignment, V is set equal to $\phi''_{(\infty)}$, once we have taken the inclusion into $G_k(\mathbb{C}^{2r+k})$, in accordance with Theorem 3.8. □

We shall now justify our restriction to the Grassmannian $G_k(\mathbb{C}^{2r+k}) \cong G_{2r}(\mathbb{C}^{2r+k})$.

Corollary 3.15

For p and q both odd, there exist no full, isotropic harmonic maps $\phi : M \to G_p(\mathbb{R}^{p+q})$ $(\cong G_q(\mathbb{R}^{p+q}))$.

Proof

Assume on the contrary that such a map exists. Then by imposing the condition (3.18) on the corresponding ∂'-pair (V,X) obtained in accordance with Theorem 3.8, we would have

$$\text{Rank } X = \text{Rank } \bar{X} = (p+q) - \text{Rank } V \quad \text{and}$$

$$\text{Rank } X - \text{Rank } V = p .$$

But for p and q both odd, this would lead to impossible data. □

Remark

The case $p = 1$, q odd, was settled by Calabi in [18] (see also [35]).

3.4 Real oriented Grassmannians.

The Grassmannian $G_k^O(\mathbb{R}^{2r+k})$ of oriented k-planes in $\mathbb{R}^{2r+k}$, is a simply-connected Riemannian symmetric space with coset space representation $SO(2r+k)/SO(2r) \times SO(k)$. It is the orientable double covering of $G_k(\mathbb{R}^{2r+k})$ ($\cong SO(2r+k)/S(O(2r) \times O(k))$), for which the covering projection (that forgets orientation)

$$(3.23) \qquad G_k^O(\mathbb{R}^{2r+k}) \to G_k(\mathbb{R}^{2r+k}),$$

is a local diffeomorphism.

Lemma 3.16

Let $\phi : M \to G_k(\mathbb{R}^{2r+k})$ be an isotropic harmonic map. Then ϕ lifts as a harmonic map to $G_k^O(\mathbb{R}^{2r+k})$.

Proof

We have seen that such a map ϕ has an associated lift $\psi : M \to S_{k,r}$ that is harmonic and π-horizontal with respect to (3.21). For $W = \psi(x) \in S_{k,r}$, we choose an oriented orthonormal basis (e_i) of $\mathbb{R}^{2r+k}$, such that W is spanned by $\{e_1+ie_2, \ldots, e_{2r-1}+ie_{2r}\}$ $(i = \sqrt{-1})$.

This choice of an oriented basis automatically sets W in the space $S^O_{k,r} \cong SO(2r+k)/U(r) \times SO(k)$. Once again we have a homogeneous fibration

$$(3.24) \qquad \tilde{\pi}: S^O_{k,r} \cong SO(2r+k)/U(r)\times SO(k) \to G^O_k(\mathbb{R}^{2r+k}) \cong SO(2r+k)/SO(2r)\times SO(k) ,$$

induced by the inclusions $U(r) \hookrightarrow SO(2r) \hookrightarrow SO(2r) \times SO(k)$; explicitly, $\tilde{\pi}(W) = e_{2r+1} \wedge \ldots \wedge e_{2r+k}$.

The map ψ remains harmonic and is $\tilde{\pi}$-horizontal with respect to (3.24). By the previous arguments, the map $\tilde{\phi} = \tilde{\pi} \circ \psi$ is a harmonic map from M to $G^O_k(\mathbb{R}^{2r+k})$. □

By choosing a basis with the opposite orientation to that of the first, we may repeat this procedure to produce another lift of ϕ . The two resulting maps $\tilde{\phi}_1$ and $\tilde{\phi}_2$ say, cover the map ϕ to $G_k(\mathbb{R}^{2r+k})$.

<u>Remark</u>

The twistor space $S^O_{k,r}$ associated with $G^O_k(\mathbb{R}^{2r+k})$ has also been listed in Bryant's classification [16].

We shall say that a map $\tilde{\phi}: M \to G^O_k(\mathbb{R}^{2r+k})$ is <u>isotropic</u> if $\phi: M \to G_k(\mathbb{R}^{2r+k})$ is isotropic, where we take $\tilde{\phi}$ to be the lift of ϕ . Similarly, we assign the notion of fullness of $\tilde{\phi}$, when ϕ itself is full. Thus combining Corollary 3.14 and Lemma 3.16, we have:

Corollary 3.17

Corollary 3.14 induces a 2:1 correspondence between full isotropic harmonic maps $\tilde{\phi}:M \to G_k^O(\mathbb{R}^{2r+k})$ and holomorphic subbundles V of $\mathbb{E}^{2r+k}$ of rank r on M, satisfying the indicated conditions in Corollary 3.14.

For $2r=n$ and $k=2$, the above corollary classifies full, isotropic harmonic maps $M \to Q_n \cong G_2^O(\mathbb{R}^{n+2})$. For n odd, we must take the equivalent representation $G_n^O(\mathbb{R}^{n+2})$, i.e. set $r = 1$ and $k = n$.

As a result of Corollary 3.15, we also have:

Corollary 3.18

For p and q both odd, there exist no full, isotropic harmonic maps $M \to G_p^O(\mathbb{R}^{p+q})$ $(\cong G_q^O(\mathbb{R}^{p+q}))$.

Remarks

1. Setting $k = 1$, we recover from Corollary 3.17, Theorem 2.29. Here, we regard V as being equal to the $(r-1)^{th}$-associated curve of a full, totally isotropic holomorphic map $f:M \to \mathbb{C}P^{2r}$.

2. For $p = 1$, the result of Corollary 3.18, was also obtained in [18] : there exist no full, isotropic harmonic maps to S^n for n odd. In fact, for p and q both odd, the associated twistor space is empty [16].

3.5 The Hermitian symmetric spaces of Types II and III.

We shall restrict the main part of this discussion to $Y^{II} \cong SO(2n)/U(n)$; that for Y^{III} is similar. We recall that Y^{II} is totally geodesic in $G_n(\mathbb{C}^{2n})$ (as is Y^{III}) , and thus we commence by considering ∂'-pairs (V,X) of holomorphic subbundles of $\underline{\mathbb{C}}^{2n}$ of rank difference n . For a working definition, let us say that a map $\phi:M \to Y^{II}$ is isotropic if the map $i\circ\phi:M \to G_n(\mathbb{C}^{2n})$ is isotropic, where i is the totally geodesic inclusion map. The fullness of ϕ is taken to be that of $i\circ\phi$. Likewise, we may apply this terminology to analogous maps into Y^{III} .

Typically, for $1 \le k \le n$, we consider ∂'-pairs (V,X) with rank $V = k$ and rank $X = n+k$ (and hence Rank $X^{\perp} = n-k$) . Assuming that we have such a ∂'-pair, we can construct a map $\psi:M \to F^{2n}_{k,n-k}$, defined by $\psi(x) = (V_x, X^{\perp}_x)$. The map ψ is harmonic and is π-horizontal with respect to the homogeneous fibration

$$\pi:F^{2n}_{k,n-k} \cong U(2n)/U(k)\times U(n-k)\times U(n) \to G_n(\mathbb{C}^{2n}) \cong U(2n)/U(n)\times U(n) ,$$

induced by the inclusion $U(k) \times U(n-k) \hookrightarrow U(n)$.

We thus look for conditions on V and $X^{\perp}$ such that the restriction of π to the receiving space R say, of the above map ψ , projects $(V,X^{\perp})$ from R into Y^{II} .

Now J' can be taken to be the symmetric $\mathbb{C}$-bilinear form as given by (2.54) (on setting $N = 2n$) , and thus for $v,w \in \mathbb{C}^{2n}$, we have

$J'(v,w) = \langle v,\bar{w}\rangle$. From (3.15), we have, for $x \in M$, $\phi(x) = V_x^{\perp} \cap X_x$, and in accordance with (3.3), we require $\phi(x)$ to be a (<u>maximally</u>) n-dimensional J'-isotropic subspace of $\mathbb{C}^{2n}$, i.e. $J'(\phi(x),\phi(x)) = 0$. Pointwise, this is equivalent to

$$\bar{V}^{\perp} \cap \bar{X} = (V^{\perp} \cap X)^{\perp} = V + X^{\perp} \quad ;$$

hence, $\bar{V} + \bar{X}^{\perp} = (V + X^{\perp})^{\perp}$. This last equality is equivalent to $V + X^{\perp}$ also being maximally J'-isotropic, i.e. the fibres of $V + X^{\perp}$ are n-dimensional J'-isotropic subspaces of $\mathbb{C}^{2n}$. We have rank $(V + X^{\perp})=n$, and the J'-isotropy of $V + X^{\perp}$ is equivalent to

a) $J'(V,V) = 0$

b) $J'(X^{\perp},X^{\perp}) = 0$

c) $J'(V,X^{\perp}) = 0$.

Conditions a) and b) say that V and $X^{\perp}$ are isotropic for J' respectively. Condition c) says that V and $X^{\perp}$ are J'-orthogonal to each other.

Similar conditions are obtained for J'' . Thus we pose the following:

<u>Problem</u>

Find full ∂'-pairs (V,X) of rank difference n , where V and X are holomorphic subbundles of $\underline{\mathbb{C}}^{2n}$ on M , such that conditions a), b) and c) are satisfied for either J' or J'' .

Having found such examples, we can therefore proceed to construct full, isotropic harmonic maps to Y^{II} or Y^{III} .

<u>Note</u> that in the case of J' , V and X cannot be realised as holomorphic subbundles of $\mathbb{C}^{2n}$ corresponding to the $(k-1)^{th}$ and $(n+k-1)^{th}$ associated curves respectively, of a full holomorphic map $f:M \to \mathbb{C}P^{2n-1}$ that is totally isotropic (i.e. $J'(f,f) = 0$) , since such maps f do not exist (Lemma 2.24).

CHAPTER IV

THE CONSTRUCTION OF A CLASS OF HARMONIC MAPS TO QUATERNIONIC PROJECTIVE SPACES AND GRASSMANNIANS

4.1 Quaternionic projective space.

Let $\mathbb{H}$ denote the division ring of quaternions, that is, the four dimensional real vector space with basis $(1,i,j,k)$ such that:

$$(4.1) \qquad i^2 = j^2 = k^2 = -1 \ , \quad ij = k = -ji \ , \quad \text{etc.}$$

Letting $q \in \mathbb{H}$ be such that $q = a + bj$, where $a,b \in \mathbb{C}$, we obtain the identification $\mathbb{H} \cong \mathbb{C} + \mathbb{C}j$. We generalise this to $(n+1)$ quaternionic dimensions to obtain the identification $\mathbb{H}^{n+1} \cong \mathbb{C}^{n+1} + \mathbb{C}^{n+1}j$. Let

$$(4.2) \qquad \sigma : \mathbb{C}^{2n+2} \to \mathbb{C}^{2n+2}$$

be the conjugate linear map induced by left multiplication by j . This is given by

$$(4.3) \qquad \sigma : v \to J\bar{v} = \begin{pmatrix} \hat{j} & & 0 \\ & \ddots & \\ 0 & & \hat{j} \end{pmatrix} \bar{v}$$

where $v \in \mathbb{C}^{2n+2}$, and J is a $(2n+2) \times (2n+2)$ symplectic matrix with $\hat{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Following [80], the n-dimensional quaternionic projective

space $\mathbb{H}P^n \cong Sp(n+1)/Sp(1) \times Sp(n)$, may be viewed as the parametrisation of complex projective lines in $\mathbb{C}P^{2n+1}$ that are 'real' with respect to the symplectic structure induced by left multiplication by j in $\mathbb{H}^{n+1}$. This is the underlying principle for obtaining the canonical twistor fibering

$$(4.4) \qquad \pi : \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$$

with fibre $\mathbb{C}P^1$.

If $v \in \mathbb{H}^{n+1} - \{0\}$, we shall let $v\mathbb{C}$ and $v\mathbb{H}$ denote the complex and quaternionic line respectively, spanned by v . We see that $v\mathbb{C} \subset v\mathbb{H}$, and (4.4) is given by the assignment $v\mathbb{C} \to v\mathbb{H}$. More explicitly, if $(z_0,\ldots,z_{2n+1})$ are homogeneous co-ordinates in $\mathbb{C}P^{2n+1}$, then (4.4) is given by

$$(4.5) \qquad (z_0,\ldots,z_{2n+1}) \longrightarrow (z_0+z_1 j,\ldots,z_{2n}+z_{2n+1} j) .$$

This construction generalises that for the formulation of the Penrose transform $\mathbb{C}P^3 \to S^4 (\cong \mathbb{H}P^1)$ (see e.g. [1]).

Let $\langle\ ,\ \rangle$ denote the natural Hermitian inner product on $\mathbb{C}^{2n+2}$. Then we define a non-degenerate anti-symmetric inner product S for vectors in $\mathbb{C}^{2n+2} \cong \mathbb{H}^{n+1}$, by

$$(4.6) \qquad S(x,y) = \langle x,\sigma y\rangle \quad \text{for } x,y \in \mathbb{C}^{2n+2}.$$

We may find a basis of $\mathbb{C}^{2n+2}$ such that S may be identified with J in (4.3).

The Grassmannian $G_2(\mathbb{C}^{2n+2})$ may be regarded as the complexification of $\mathbb{H}P^n$, since the former parametrises all the linear $\mathbb{C}P^1$'s in $\mathbb{C}P^{2n+1}$; projectively, this is written as $G_1(\mathbb{C}P^{2n+1}) \cong G_2(\mathbb{C}^{2n+2})$. Thus, by the above definition of $\mathbb{H}P^n$, we obtain a (totally geodesic) embedding of $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{2n+2})$, as those complex 2-planes that are 'real' or fixed under the action of σ in (4.3). On replacing $n+1$ by N and adjusting the above data accordingly, we may generalise this characterisation of $\mathbb{H}P^n$ to the case of the quaternionic Grassmannian $G_k(\mathbb{H}^N) \cong Sp(N)/Sp(k) \times Sp(N-k)$ characterised as complex $2k$-planes in $\mathbb{C}^{2N}$ that are fixed by σ. This defines an embedding of $G_k(\mathbb{H}^N)$ into $G_{2k}(\mathbb{C}^{2N})$, which is a totally geodesic embedding (in the appendix, we shall discuss this embedding in the light of Theorem 3.1).

4.2 Harmonic maps to quaternionic Grassmannians.

In this section we shall draw freely on the terminology established in Chapter III for maps into $G_k(\mathbb{C}^N)$. In what follows, we shall say that a map $\phi: M \to G_k(\mathbb{H}^N)$ is isotropic, if $i\circ\phi$ is isotropic into $G_{2k}(\mathbb{C}^{2N})$, where i is the totally geodesic inclusion map. If this inclusion yields a full map into $G_{2k}(\mathbb{C}^{2N})$, then we say that ϕ is full. For brevity, we shall write ϕ for $i\circ\phi$. We recall the considerations of the last section and replace $n+1$ by N.

Proposition 4.1

Let (V,X) *be a full* ∂'*-pair of holomorphic subbundles of* $\mathbb{C}^{2N}$ *of rank difference* $2k$, *on* M $(1 \le k \le N)$. *Further, assume that*

V <u>and</u> X <u>satisfy the condition</u>

(4.7) $\sigma X = V^{\perp}$ <u>or equivalently,</u> $X = \sigma V^{\perp}$

<u>where</u> σ <u>is the conjugate linear map defined by</u> (4.3). <u>Then the assignment</u> (3.15) <u>defines a full isotropic harmonic map</u> $\phi : M \to G_k(\mathbb{H}^N)$.

<u>Proof</u>

By Theorem 3.8, the assignment (3.15) defines a full isotropic harmonic map from M to $G_{2k}(\mathbb{E}^{2N})$. Imposing condition (4.7), we have $\phi(x) = \sigma X_x \cap X_x$, for $x \in M$. Thus $\phi(x)$ is 'real' with respect to the action of σ in (4.3), and thus ϕ has image in $G_k(\mathbb{H}^N)$. □

<u>Remark</u>

We observe the similarity between (4.7) and the 'total-isotropy' condition (3.18).

Again, for such a ∂'-pair (V,X) , we have $V \subset X$ and hence $\sigma V \subset \sigma X = V^{\perp}$. But from (4.6), this implies that $V \subset V^{\perp_S}$ where $\perp_S$ denotes orthogonality with respect to S . Thus V is isotropic with respect to S . Furthermore, by condition c) of Definition 3.3, we see that

$$\sigma\partial' C(V) \subset \sigma C(X) = C(V^{\perp}) \quad .$$

Thus Proposition 4.1 can be restated in terms of a holomorphic subbundle V of $\underline{\mathbb{E}}^N$ of rank $N-k$ on M , such that V satisfies the conditions

$$(4.8)\quad \begin{cases} V \subset V^{\perp S} \\ \sigma\partial' C(V) \subset C(V^{\perp}) \end{cases} .$$

Observe that these are analogous conditions to those in (3.19). For $x \in M$ and $\zeta \in C(V)$, the above conditions may be re-stated as $S(V_x, V_x) = 0$ and $S(\zeta(x), \partial'\zeta(x)) = 0$ respectively (cf. remarks following Lemma 3.11). We shall exemplify these conditions for the case $k = 1$ and $N = 2$, in Section 4.3.

Once again, if we assume that the only constant subbundle contained in V, is the zero subbundle, then on account of (4.7), we can construct a full ∂'-pair (V,X) of rank difference $2k$ (Rank $X = N + k$).

$$\text{Let } F^{2N}_{N-k,N-k} = \{(P,Q) \in G_{N-k}(\mathbb{E}^{2N}) \times G_{N-k}(\mathbb{E}^{2N}) : P \perp Q\}$$

$$\cong U(2N)/U(2k) \times U(N-k) \times U(N-k) .$$

Again, we obtain a natural fibration

$$(4.9)\qquad \pi : F^{2N}_{N-k,N-k} \longrightarrow G_{2k}(\mathbb{E}^{2N})$$

given by including $U(2k) \times U(N-k) \times U(N-k)$ into $U(2k) \times U(2N-2k)$ in the standard way; explicitly, $\pi(P,Q) = (P+Q)^{\perp}$.

Since we have in mind a map $\psi : M \to F^{2N}_{N-k,N-k}$ defined by $\psi(x) = (V_x, X_x^{\perp})$ with V_x and $X_x^{\perp}$ $(N-k)$-dimensional isotropic subspaces with respect to S, we seek to determine the receiving space contained in $F^{2N}_{N-k,N-k}$.

Lemma 4.2

The totality of $(N-k)$**-dimensional** S**-isotropic subspaces of** $\mathbb{C}^{2N}$ **, contained in** $F^{2N}_{N-k,N-k}$ **, has the coset space representation**

$$(4.10) \qquad Sp(N) \,/\, Sp(k) \times U(N-k) \; .$$

Proof

It suffices to consider the tangent space to the identity of $F^{2N}_{N-k,N-k}$, and commence from the Lie algebra of $Sp(N)$ given by [48]:

$$\left\{ \begin{bmatrix} X_1 & X_2 \\ -\bar{X}_2 & \bar{X}_1 \end{bmatrix} : X_1, X_2 \text{ are complex } N \times N \text{ matrices, with } X_1 \text{ skew Hermitian and } X_2 \text{ symmetric} \right\} .$$

The subalgebra fixing a complex $(N-k)$-plane $\cong \mathbb{C}^{N-k}$ is represented in block form by:

$$\begin{array}{c} \\ N-k \\ \\ \\ \end{array} \left[\begin{array}{c|c|c|c} \multicolumn{2}{c|}{} & \multicolumn{2}{c}{\overbrace{\qquad\qquad}^{N-k}} \\ \multicolumn{2}{c|}{A} & 0 \cdots 0 & 0 \\ & 0 & \vdots \quad \vdots & \vdots \\ \hline 0 \cdots 0 & a \;\}k & 0 \cdots 0 & b \\ \hline 0 \cdots 0 & 0 & & 0 \\ \vdots \quad \vdots & \vdots & \multicolumn{2}{c|}{\bar{A}} \\ & 0 & & 0 \\ \hline 0 \cdots 0 & -b & 0 \cdots 0 & \bar{a} \end{array} \right]$$

where $A \in \mathfrak{u}(N-k)$ and $a,b \in \mathbb{E}^k$. From this we deduce that the corresponding subgroup of $Sp(N)$ is $Sp(k) \times U(N-k)$. $\square$

We shall denote the space described in Lemma 4.2 by $R_{k,N-k}$. The space $R_{k,N-k}$ is totally geodesic in $F^{2N}_{N-k,N-k}$, and the restriction of π in (4.9) to $R_{k,N-k}$, yields the homogeneous fibration

$$(4.11) \quad \pi : R_{k,N-k} \cong Sp(N)/Sp(k)\times U(N-k) \to G_k(\mathbb{H}^N) \cong Sp(N)/Sp(k)\times Sp(N-k)$$

with fibre $Sp(N-k)/U(N-k)$. For $N = n+1$, $k = 1$, we obtain

$$(4.12) \quad R_{1,n} \cong Sp(n+1)/Sp(1)\times U(n) \to \mathbb{H}P^n \cong Sp(n+1)/Sp(1)\times Sp(n)$$

with fibre $Sp(n)/U(n)$, and for $N = n+1$, $k = n$, we obtain

$$(4.13) \quad R_{n,1} \cong Sp(n+1)/Sp(n)\times U(1) \to G_n(\mathbb{H}^{n+1}) \cong Sp(n+1)/Sp(n)\times Sp(1) \cong \mathbb{H}P^n$$

with fibre $\mathbb{C}P^1 \cong Sp(1)/U(1)$ and $R_{n,1} \cong \mathbb{C}P^{2n+1}$. For $n = 1$, these coincide and we recover the celebrated fibration $\mathbb{C}P^3 \to S^4 (\cong \mathbb{H}P^1)$.

Let us denote $R_{1,n}$ by T_n. The spaces T_n and $\mathbb{C}P^{2n+1}$ are two twistor spaces for $\mathbb{H}P^n$ and they will each apply to two different types of harmonic maps $M \to \mathbb{H}P^n$, as will be apparent later.

Theorem 4.3

There exists a bijective correspondence between full, isotropic harmonic maps $\phi : M \to G_k(\mathbb{H}^N)$ and holomorphic subbundles V of $\mathbb{E}^{2N}$ of rank $N-k$ on M, satisfying conditions (4.8) and the additional condition that the only constant subbundle contained in V, is the zero subbundle.

Proof

Given such a V, we can construct a full ∂'-pair (V,X) of rank difference $2k$, by virtue of the relationship in (4.7). Here, Rank $X = N+k$ and hence we obtain Rank $X^{\perp} = N-k$. Thus we can proceed to construct a π-horizontal harmonic map $\psi : M \to F^{2N}_{N-k,N-k}$ defined, as usual, by setting $\psi(x) = (V_x, X^{\perp}_x)$ for $x \in M$.

The conditions (4.8) ensure that the map ψ has image in $R_{k,N-k}$, since V and $X^{\perp}$ are now isotropic for S. But the map ψ to the receiving space $R_{k,N-k}$ can be provided by V alone, when we regard the latter as a holomorphic map $M \to G_{N-k}(\mathbb{E}^{2N})$ and take into account the conditions satisfied by V.

Choosing the natural homogeneous space metrics on $F^{2N}_{N-k,N-k}$ and $R_{k,N-k}$, π (in each case) is a Riemannian submersion. Hence for $x \in M$, $\phi(x) = \pi o V_x$ defines a harmonic map $\phi : M \to G_k(\mathbb{H}^N)$ that is full and isotropic into $G_{2k}(\mathbb{E}^{2N})$.

Conversely, given such a ϕ, the inverse assignment is achieved by setting $V = \phi''_{(\infty)}$ and $X = \hat{\phi}''_{(\infty)}$ in accordance with Theorem 3.8.

Since $\phi = \phi''^{\perp}_{(\infty)} \cap \gamma''_{(\infty)}$ is a map into $G_k(\mathbb{H}^N)$, ϕ is real with respect to the action of σ; we have $\sigma\gamma''_{(\infty)} \cap \gamma''_{(\infty)} = \phi$, and we deduce that $\sigma\gamma''_{(\infty)} = \phi''^{\perp}_{(\infty)}$. But this implies condition (4.7) on the full ∂'-pair (V,X) and hence implies the conditions (4.8) on V. □

Corollary 4.4

For $k = 1$ and $N = n+1$, Theorem 4.3 classifies full, isotropic harmonic maps $\phi: M \to \mathbb{H}P^n$ in terms of holomorphic subbundles V of $\mathbb{C}^{2n+2}$ of rank n, on M, satisfying the indicated conditions.

Note that in this last case, rank $\phi''_{(\infty)} = n$, whereas if we consider the case $k = n$, $N = n+1$, we obtain a holomorphic map to $R_{n,1} \cong \mathbb{C}P^{2n+1}$ determined by V, and the resulting harmonic map to $G_n(\mathbb{H}^{n+1}) \cong \mathbb{H}P^n$, is such that rank $\phi''_{(\infty)} = 1$ (since rank $V = 1$). Thus whereas T_n can be regarded as the natural twistor space for the isotropic harmonic maps in question, $\mathbb{C}P^{2n+1}$ is the appropriate twistor space for harmonic maps $\phi: M \to \mathbb{H}P^n$ satisfying the inclusive condition [33] [96], namely, that $\phi_*(T_xM)$ should be contained in a one-dimensional quaternionic subspace, for $x \in M$.

4.3 The special case of the quaternionic projective line.

We shall now set $k = 1$ and $N = 2$, whence T_1 and $\mathbb{C}P^3$ coincide and $\mathbb{H}P^1 \cong S^4$. We have then, the following system of fibrations:

$$
(4.14)\qquad
\begin{array}{ccccc}
\mathbb{C}P^3 \cong \dfrac{Sp(2)}{U(1)\times Sp(1)} & \hookrightarrow & F^4_{1,1} \cong \dfrac{U(4)}{U(1)\times U(1)\times U(2)} \\
\pi\downarrow & & \downarrow \\
S^4 \cong \mathbb{H}P^1 \cong \dfrac{Sp(2)}{Sp(1)\times Sp(1)} & \hookrightarrow & G_2(\mathbb{C}^4) \cong \dfrac{U(4)}{U(2)\times U(2)}
\end{array}
$$

where the horizontal maps are totally geodesic inclusion maps. Let $\mathbb{H}P^1$ be covered by open sets V_1, V_2 where the homogeneous quaternion coordinates (q_0,q_1) are respectively non-zero.

On $V_1(V_2)$ the quaternion q_1/q_0 (respectively, q_0/q_1) defines a local co-ordinate corresponding to a point $u \in \mathbb{H}P^1$. With respect to the standard basis for $\mathbb{H}$ ($\cong \mathbb{C} + \mathbb{C}j$), namely $\{1,i,j,k\}$, we have as before

$$u = y_1 1 + y_2 i + y_3 j + y_4 k \qquad y_i \in \mathbb{R} .$$

We express u as $u = u_1 + u_2 j$ where

$$u_1 = y_1 + y_2 i \quad , \quad u_2 = y_3 - y_4 i \quad .$$

Consider the point $z \in \mathbb{C}P^3$ with homogeneous co-ordinates (z_0,z_1,z_2,z_3), and as in (4.5), let z be mapped to the point in $\mathbb{H}P^1$ with homogeneous co-ordinates (q_0,q_1) where $q_0 = z_0 + z_1 j$, $q_2 = z_2 + z_3 j$. Let $\{U_i\}$, $i = 0,1,2,3$, be open sets giving the standard affine covering of $\mathbb{C}P^3$ where $z_i \neq 0$; then $U_i = (z_0/z_i,\ldots,\hat{z}_i/z_i,\ldots,z_3/z_i)$ where $\hat{}$ denotes omission.

We shall work in the co-ordinate system U_0 , and the end result, (4.24) may be seen to hold for the other cases. Let then $w_k = z_k/z_0$, $k = 1,2,3$. For $u = q_0/q_1 = u_1 + u_2 j$, we have

$$(4.15) \qquad u_1 = \frac{\bar{w}_2 + w_3\bar{w}_1}{|w_2|^2 + |w_3|^2} , \qquad u_2 = \frac{\bar{w}_2 w_1 - w_3}{|w_2|^2 + |w_3|^2} .$$

We now intend to determine explicitly, the conditions of π-horizontality for a holomorphic map $\psi : M \to \mathbb{E}P^3$ and show that these are none other than the conditions in (4.8).

<u>Proposition 4.5</u>

<u>The complex vertical distribution</u> $\mathcal{V}(= \ker \pi_*)$ <u>is spanned by the</u> (1,0) <u>vector field</u>

$$(4.16) \qquad (1 + w_1\bar{w}_1)\frac{\partial}{\partial w_1} + (w_2\bar{w}_1 - \bar{w}_3)\frac{\partial}{\partial w_2} + (w_3\bar{w}_1 + \bar{w}_2)\frac{\partial}{\partial w_3} .$$

<u>Proof</u>

In terms of complex tangent vectors of type (1,0) and (0,1) we have the canonical decomposition of $\mathcal{V}$:

$$\mathcal{V} = V^{1,0} \oplus V^{0,1} = V^{1,0} \oplus \overline{V^{1,0}} .$$

Let $\zeta \in \mathcal{V}$, then ζ may be written as

$$\zeta = a_1 \frac{\partial}{\partial w_1} + a_2 \frac{\partial}{\partial w_2} + a_3 \frac{\partial}{\partial w_3} + \overline{a_1} \frac{\partial}{\partial \bar{w}_1} + \overline{a_2} \frac{\partial}{\partial \bar{w}_2} + \overline{a_3} \frac{\partial}{\partial \bar{w}_3}$$

where $a_i \in \mathbb{C}$. Given the explicit description of π as in (4.15), we proceed to determine an element of $\ker \pi_*$ (the kernel of the differential) at a point z , say, in $\mathbb{CP}^3$. Thus from the system

$$(4.17)\qquad \pi_*(\zeta)_z = \begin{bmatrix} \frac{\partial u_1}{\partial w_1} & \frac{\partial u_1}{\partial w_2} & \frac{\partial u_1}{\partial w_3} & \frac{\partial u_1}{\partial \bar{w}_1} & \frac{\partial u_1}{\partial \bar{w}_2} & \frac{\partial u_1}{\partial \bar{w}_3} \\ \frac{\partial u_2}{\partial w_1} & \frac{\partial u_2}{\partial w_2} & \frac{\partial u_2}{\partial w_3} & \frac{\partial u_2}{\partial \bar{w}_1} & \frac{\partial u_2}{\partial \bar{w}_2} & \frac{\partial u_2}{\partial \bar{w}_3} \\ \frac{\partial \bar{u}_1}{\partial w_1} & \frac{\partial \bar{u}_1}{\partial w_2} & \frac{\partial \bar{u}_1}{\partial w_3} & \frac{\partial \bar{u}_1}{\partial \bar{w}_1} & \frac{\partial \bar{u}_1}{\partial \bar{w}_2} & \frac{\partial \bar{u}_1}{\partial \bar{w}_3} \\ \frac{\partial \bar{u}_2}{\partial w_1} & \frac{\partial \bar{u}_2}{\partial w_2} & \frac{\partial \bar{u}_2}{\partial w_3} & \frac{\partial \bar{u}_2}{\partial \bar{w}_1} & \frac{\partial \bar{u}_2}{\partial \bar{w}_2} & \frac{\partial \bar{u}_2}{\partial \bar{w}_3} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{bmatrix} = (0)$$

we obtain the following system of four equations:

$$(4.18)\qquad \begin{cases} w_3(\bar{a}_1-\bar{a}_2\bar{u}_2-\bar{a}_3u_1) - \bar{w}_2(a_2u_1-a_3\bar{u}_2) = 0 \\ w_3(\bar{a}_2\bar{u}_1-\bar{a}_3u_2) + \bar{w}_2(a_1-a_2u_2-a_3\bar{u}_1) = 0 \\ \bar{w}_3(a_1-a_2u_2-a_3\bar{u}_1) - w_2(\bar{a}_2\bar{u}_1-\bar{a}_3u_2) = 0 \\ \bar{w}_3(a_2u_1-a_3\bar{u}_2) + w_2(\bar{a}_1-\bar{a}_2\bar{u}_2-\bar{a}_3u_1) = 0 \end{cases}$$

Clearly, the last two are redundant. Thus letting $b_1 = a_1-a_2u_2-a_3\bar{u}_1$ and $b_2 = a_2u_1-a_3\bar{u}_2$, the system in (4.18) reduces to

$$(4.19)\quad \begin{cases} w_3\bar{b}_1 - \bar{w}_2 b_2 = 0 \\ w_3\bar{b}_2 + \bar{w}_2 b_1 = 0 \ . \end{cases}$$

Since $w_3 \neq 0$, we see that $b_1 = b_2 = 0$ and hence

$$(4.20)\quad \begin{cases} a_1 = a_2 u_2 + a_3 u_1 \\ a_2 u_1 = a_3 \bar{u}_2 \ . \end{cases}$$

Now we can easily deduce from (4.20), the relations:

$$(4.21)\qquad a_2 = \frac{\bar{u}_2}{k} a_1 \ , \qquad a_3 = \frac{u_1}{k} a_1$$

where $k = |u_1|^2 + |u_2|^2 = (1 + |w_1|^2)/(|w_2|^2 + |w_3|^2)$. Thus we may write

$$\zeta = a_1 k^{-1}(k \frac{\partial}{\partial w_1} + \bar{u}_2 \frac{\partial}{\partial w_2} + u_1 \frac{\partial}{\partial w_3}) + \bar{a}_1 k^{-1}(k \frac{\partial}{\partial \bar{w}_1} + u_2 \frac{\partial}{\partial \bar{w}_2} + \bar{u}_1 \frac{\partial}{\partial \bar{w}_3}) \ .$$

It is clear then, that V is the complex vector space spanned by (4.16) . □

The coefficients h_{ij} of the Fubini-Study metric on $\mathbb{C}P^3$ are computed by

$$h_{ij} = (1 + |w|^2)\delta_{ij} - \bar{w}_i w_j \qquad i,j = 1,2,3 \ .$$

On setting $\mu_1 = k$, $\mu_2 = \bar{u}_2$, $\mu_3 = u_1$, the dual (1,0)-form corresponding to (4.16), calculated by $\sum_j h_{ij}(dw^i)\bar{\mu}_j$ in this co-ordinate system, is given by

$$(4.22) \qquad (1 + |w|^2)^{-2}(dw_1 - w_3 dw_2 + w_2 dw_3) \ .$$

Let then $\psi = (\psi^0, \psi^1, \psi^2, \psi^3)$ be a holomorphic map $M \to \mathbb{C}P^3$. For $a = 0,1,2,3,$ we set

$$f^a = \psi^a/\psi^0 = w_a \qquad (w_0 = 1) \ .$$

Then from (4.22), ψ is π-horizontal when

$$(4.23) \qquad (Xf^1) - f^3(Xf^2) + f^2(Xf^3) = 0 \ , \quad X \in T'M \ .$$

Back substituting for f^a , (4.23) becomes

$$\frac{(X\psi^1)}{\psi^0} - \psi^1 \frac{(X\psi^0)}{(\psi^0)^2} - \frac{\psi^3}{\psi^0}\left(\frac{X\psi^2}{\psi^0} - \psi^2 \frac{(X\psi^0)}{(\psi^0)^2}\right)$$

$$+ \frac{\psi^2}{\psi^0}\left(\frac{X\psi^3}{\psi^0} - \psi^3 \frac{(X\psi^0)}{(\psi^0)^2}\right) = 0 \ .$$

$$(4.24) \qquad \psi^0(X\psi^1) - \psi^1(X\psi^0) + \psi^2(X\psi^3) - \psi^3(X\psi^2) = 0 \ .$$

We can, of course, regard ψ as a holomorphic line bundle; its components, namely the sections $\psi^0, \ldots, \psi^3$ of this line bundle, define

the holomorphic map into $\mathbb{C}P^3$. To see the relationship with the conditions in (4.8), we note that,

for $J = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, (4.24) can be written as

$$(4.25) \qquad {}^t\psi \, J \, \partial'\psi = 0$$

which shows that the second condition in (4.8) is satisfied for the case $n = 1$; the first condition in this case is automatic.

Given that M is connected, let ζ_1 and ζ_2 be meromorphic functions on M , with ζ_2 non-constant. Then a class of maps satisfying (4.25) are those for which

$$(4.26) \qquad \psi = (1, \zeta_1 - 1/2\, \zeta_2(d\zeta_1/d\zeta_2), \zeta_2, 1/2(d\zeta_1/d\zeta_2))$$

or

$$(4.27) \qquad \psi^0 = \psi^1 , \psi^2 = \psi^3 \quad \text{or} \quad \psi^0 = \psi^3 , \psi^1 = \psi^2 , \text{ etc.}$$

The latter clearly satisfy ${}^t\psi \, J \, \psi = 0$, and their images lie in (horizontal) linear $\mathbb{C}P^1$s in $\mathbb{C}P^3$.

To make our point then, let us look at the classical formulation of the Penrose transform. Let x,y be distinct points in $\mathbb{C}P^3$ with homogeneous co-ordinates $x_0,\ldots,x_3$, $y_0,\ldots,y_3$, respectively. The point $p \in \mathbb{C}P^5$ with homogeneous co-ordinates $P_{ij} = x_iy_j - x_jy_i$, $i < j$,

is uniquely determined by the line $\overrightarrow{xy}$ and satisfies the equation

$$P_{01}\,P_{23} - P_{02}\,P_{13} + P_{03}\,P_{12} = 0 \ .$$

Via the assignment $\overrightarrow{xy} \to p$, we obtain the Plücker embedding of $G_1(\mathbb{C}P^3) \cong G_2(\mathbb{E}^4)$ into $\mathbb{C}P^5$. Note that in accordance with the list of isomorphisms of the HSS, in section 3.1, we have $G_2(\mathbb{E}^4) \cong Q_4$. If we consider the homogeneous co-ordinates $x_0,\ldots,x_5$ of $\mathbb{C}P^5$ given by

$$x_0 = P_{01} + P_{23}\ ,\quad x_2 = P_{13} + P_{02}\ ,\quad x_4 = P_{03} - P_{12}$$

$$x_1 = P_{01} - P_{23}\ ,\quad x_3 = i(P_{13}-P_{02})\ ,\quad x_5 = i(P_{03} + P_{12})\ ,$$

then the equation for Q_4 is $x_0^2 = x_1^2 + \ldots + x_5^2$, and we obtain the identification

$$S^4 \cong Q_4 \cap \{x \in \mathbb{C}P^5 : x = \bar{x}\} \ .$$

Thus in the light of Corollary 4.4, applying the usual composition principles to ψ and (4.14), we may obtain a full isotropic harmonic map to $G_2(\mathbb{E}^4) \cong Q_4$, that has its image isotropic in $S^4 \cong \mathbb{H}P^1$. Conversely, given such a map ϕ , with rank $(\phi''_{(\infty)}) = 1$, we may recover the holomorphic map ψ to $\mathbb{C}P^3$.

Remarks

1. Proposition 4.5 was also obtained by Bryant in [15] and (4.26) is his result.

2. In [51], Hitchin has proved that S^4 and $\mathbb{C}P^2$ are the only two compact Riemannian 4-manifolds admitting Kähler'ian twistor spaces. The twistor spaces in question are $\mathbb{C}P^3$ and the flag manifold $F^3_{1,1} \cong U(3)/U(1) \times U(1) \times U(1)$ respectively.

4.4 Examples of horizontal curves.

Example 4.6

It is known that the π-horizontal holomorphic curves $M \to \mathbb{C}P^3$ (otherwise said, the 'integrals' of (4.22), in a more classical vein) are those for which the curve and its tangents lie on a linear line complex in $\mathbb{C}P^3$ [26] [53]; by this we mean a three-parameter family of lines in $\mathbb{C}P^3$ corresponding to the intersection $G_2(\mathbb{C}^4) \cap H$, where H is a hyperplane in $\mathbb{C}P^5$. This reflects the conditions (4.8) imposed.

Amongst these holomorphic curves are the rational cubic curves in $\mathbb{C}P^3$, and more generally, the W-curves $\psi:\mathbb{C}P^1 \to \mathbb{C}P^3$ of Klein and Lie (as discussed in [26] [53]). These are given by $\psi(z) = [1, az^m, bz^n, cz^p]$ $m,n,p \in \mathbb{Z}$ and $a,b,c \in \mathbb{C}$. When we set $m = n+p$ and $a(n+p) = bc(n-p)$, then the curve ψ is π-horizontal, in accordance with (4.23).

For example, the holomorphic curve $\psi:\mathbb{C}P^1 \to \mathbb{C}P^3$ given by $\psi(z) = [1, z^3, \sqrt{3}z^2, \sqrt{3}z]$, is easily seen to satisfy the above conditions and giveasrise to a full, isotropic harmonic map $\phi:\mathbb{C}P^1 \to \mathbb{H}P^1$ defined by $\phi(z) = \pi\circ\psi(z) = (1 + z^3 j, \sqrt{3}(z + z^2)_j)$ and whose image is the projection of the Veronese curve in $\mathbb{C}P^3$ [42].

<u>Example 4.7</u>

The above example can be generalised to give a holomorphic map $\psi:\mathbb{C}P^1 \to \mathbb{C}P^{2n+1}$ that is π-horizontal with respect to the fibration $\pi:\mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ in (4.4). The following example is taken from [96] to which we refer for terminology and further details. Most of the facts stated below, are proved in [80] to which we also refer for a detailed account of the twistor spaces of certain quaternionic manifolds.

Firstly some remarks on notation. We shall take $\mathfrak{sp}(k)$ and $\mathfrak{u}(p)$ to denote the complexification of the Lie algebras of the Lie groups $Sp(k)$ and $U(p)$ respectively, and take $S^r(\ )$ to denote the rth symmetric (tensor) product. Recall that the fibration in (4.4) is given as a homogeneous fibration by

$$(4.28) \qquad \pi: Sp(n+1)/U(1)\times Sp(n) \to Sp(n+1)/Sp(1)\times Sp(n) \ .$$

Let $L \cong \mathbb{C}$, $H \cong \mathbb{C}^2$ and $E \cong \mathbb{C}^{2n}$ denote the respective basic representations of the subgroups $U(1)$, $Sp(1)$ and $Sp(n)$ of $Sp(n+1)$, in (4.28). We have the following relationships [80]:

$$(4.29) \qquad \mathfrak{sp}(n+1) = S^2(E\otimes H) \ , \ \mathfrak{sp}(n) \cong S^2(E) \ , \ \mathfrak{sp}(1) \cong S^2(H) \ ;$$

note also that $H = L \oplus \bar{L}$ and $S^{2n+1}(\mathbb{C}^2) \cong \mathbb{C}^{2n+2} \cong \mathbb{H}^{n+1}$.

On $\mathbb{H}P^n$ we have globally defined bundles associated to E and H ; we shall retain E and H respectively to denote these. With the same

use of notation, the hyperplane section bundle on $\mathbb{C}P^{2n+1}$ is identified with L, and the restriction of L to each fibre, $\mathbb{C}P^1$, is the Hopf bundle; furthermore, $T'\mathbb{C}P^1 \cong L^2$. (Note, in our previous notation, we took $\mathcal{L}$ to denote the universal line bundle on $\mathbb{C}P^{2n+1}$ and hence $\mathcal{L}^{-1}$ would denote the hyperplane section bundle in that notation.)

Following [80], we have

$$(4.30) \qquad (T\,\mathbb{H}P^n)^{\mathbb{C}} \cong E \otimes_{\mathbb{C}} H \quad .$$

Setting $Z = \mathbb{C}P^{2n+1}$, for the sake of simplicity, and taking h and v to denote horizontal and vertical components respectively of $T'Z$, with respect to π, we deduce from (4.29) and (4.30) that:

$$(4.31) \qquad \begin{cases} (T'Z)^h \cong E \otimes L \\ (T'Z)^v \cong L^2 \quad . \end{cases}$$

Consider now another copy of $U(1)$ and let $\eta \cong \mathbb{C}$ be the basic representation of the subgroup $U(1)$ of $Sp(1)$. We define an inclusion $\nu : U(1) \hookrightarrow Sp(n+1)$ via the assignments

$$(4.32) \qquad L = \eta^{2n+1} \quad , \qquad E \otimes H = E \otimes L \otimes \bar{L} \cong S^{2n+1}(\eta \oplus \bar{\eta})$$

where $S^{2n+1}(\eta \oplus \bar{\eta}) = \eta^{2n+1} \oplus \eta^{2n-1} \oplus \ldots \oplus \bar{\eta}^{2n-1} \oplus \bar{\eta}^{2n+1}$.

Relative to the standard inclusion $i : U(1) \hookrightarrow Sp(1)$, the basic representation of $Sp(1)$ on $\mathbb{C}^2$ decomposes as $\eta \oplus \bar{\eta}$. The map ν is seen to factor through i to give a map

$$\psi : \mathbb{C}P^1 \cong Sp(1)/U(1) \to \mathbb{C}P^{2n+1} \cong Sp(n+1)/U(1) \times Sp(n) .$$

Now considering the complexified Lie algebras, we have $sp(1) = \mathfrak{u}(1) \oplus (\eta^2 \oplus \bar{\eta}^2)$, and decreeing $T'\mathbb{C}P^1 \cong \eta^2$, we have $\psi_*(\eta^2) \subset L \otimes E$. This implies that ψ is holomorphic and π-horizontal with respect to (4.4), and whose image is the (generalised) Veronese curve in $\mathbb{C}P^{2n+1}$.

Now consider the fibration $\pi : T_n \to \mathbb{H}P^n$ in (4.12). Recall that as a homogeneous fibration, this is:

$$(4.33) \qquad \pi : Sp(n+1)/Sp(1)\times U(n) \to Sp(n+1)/Sp(1)\times Sp(n) .$$

We regard L, E and H as before and take $K \cong \mathbb{C}^n$ to denote the basic representation of the subgroup $U(n)$ in (4.33). Note also that $E = K \oplus \bar{K}$. If we now set $Z = T_n$, we again deduce from (4.29) and (4.30) that

$$(4.34) \qquad \begin{cases} (T'Z)^h \cong K \otimes H \\ (T'Z)^v \cong S^2K \end{cases} .$$

In a similar way, the assignment

$$(4.35) \qquad \eta = L \quad , \qquad E \oplus H = K \oplus \bar{K} \oplus H \cong S^{2n+1}(\eta \oplus \bar{\eta})$$

defines a π-horizontal holomorphic map $\psi : \mathbb{C}P^1 \to T_n$. In both cases these maps project under π to define harmonic maps $\mathbb{C}P^1 \to \mathbb{H}P^n$.

CHAPTER V

A CLASS OF HARMONIC MAPS TO INDEFINITE COMPLEX HYPERBOLIC SPACES

5.1 Indefinite Hermitian inner products.

Let K be one of $\mathbb{R}, \mathbb{C}, \mathbb{H}$, the real, complex and quaternionic numbers respectively. On the right K-module K^{p+q}, we define an Hermitian inner product by

$$(5.1) \qquad Q_p(K)(u,v) = \sum_{i=1}^{p} \bar{v}^i u^i - \sum_{k=p+1}^{p+q} \bar{v}^k u^k \qquad u,v \in K^{p+q} .$$

We shall frequently abbreviate this to $Q_p(K)$ and say that $Q_p(K)$ is <u>an Hermitian inner product of signature</u> (p,q). Effectively, this means that we can take a vector space V over $\mathbb{R}$ with a right K-module structure, such that for an orthogonal basis $\{e_i\}$ $(i=1,\ldots,p+q)$ of V over K, we have

$$(5.2) \qquad Q_p(K)(e_i,e_i) = \begin{cases} 1 & 1 \le i \le p \\ -1 & p+1 \le i \le p+q \end{cases} .$$

Now for some points of convention and notation.

We shall say that $Q_p(K)$ is <u>positive</u> (respectively, <u>negative</u>) <u>definite</u> and write $Q_p(K) \gg 0$ (respectively $Q_p(K) \ll 0$) if $q = 0$ (respectively $p = 0$), and <u>indefinite</u> if $p > 0$ and $q > 0$. Note

that when $K = \mathbb{R}$ and $p,q > 0$ (respectively, $q = 0$), we obtain the indefinite (respectively, definite) symmetric inner product on $\mathbb{R}^{p+q}$. We shall write K_p^{p+q} for $(K^{p+q}, Q_p(K))$ i.e. K^{p+q} endowed with $Q_p(K)$ and $\underline{K}_p^{p+q}$ will denote the trivial (indefinite) $(p+q)$-plane bundle. If I_k denotes the unit matrix of order k, then we shall set

$$I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \quad \text{and} \quad E_{p,q} = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & -I_q \end{bmatrix}.$$

Following [48], we define the Lie groups $U(p,q)$, $SO(p,q)$ and $Sp(p,q)$ as follows:

1. $U(p,q)$: The group of matrices g in $GL(p+q,\mathbb{C})$ that preserve $Q_p(\mathbb{C})$ i.e. ${}^t g(I_{p,q})\bar{g} = I_{p,q}$.
2. $SO(p,q)$: The group of matrices g in $SL(p+q,\mathbb{R})$ that preserve $Q_p(\mathbb{R})$ i.e. ${}^t g(I_{p,q})g = I_{p,q}$.
3. $Sp(p,q)$: The group of matrices g in $Sp(p+q,\mathbb{C})$ that preserve $Q_p(\mathbb{H})$ i.e. ${}^t g(E_{p,q})\bar{g} = E_{p,q}$.

Orthogonality with respect to $Q_p(K)$ will be denoted by $\perp_p$. When $q = 0$, we shall simply write $\perp$. Since a large part of our study entails dealing with the particular case $K = \mathbb{C}$, we shall adopt the convention of writing simply Q_p for $Q_p(\mathbb{C})$ and restore $\mathbb{C}$ should any confusion arise.

5.2 Indefinite complex hyperbolic and projective spaces.

In what follows, we shall use freely the notation and terminology of the last section. For integers n,p with $0 \le p \le n$ and a fixed real number $c > 0$, we set

$$(5.3) \qquad H_{2p}^{2n+1}(-c/4) = \{z \in \mathbb{C}^{n+1} : Q_p(z,z) = -4/c\}$$

and

$$(5.4) \qquad S_{2p}^{2n+1}(c/4) = \{z \in \mathbb{C}^{n+1} : Q_{p+1}(z,z) = 4/c\} \quad .$$

For brevity, we shall drop the ratio in parenthesis, since no confusion will arise.

We define an equivalence relation $\sim$ such that $w \sim z$, if and only if $w = e^{i\theta}z$ for some $\theta \in \mathbb{R}$, and then set $\mathbb{C}H_p^n = H_{2p}^{2n+1}/\sim$ and $\mathbb{C}P_p^n = S_{2p}^{2n+1}/\sim$. We shall call these spaces <u>the indefinite complex hyperbolic space</u> and <u>the indefinite complex projective space of n dimensions</u>, respectively. We regard $\mathbb{C}H_p^n$ and $\mathbb{C}P_p^n$ as the space of lines in $\mathbb{C}^{n+1}$ for which Q_p has signature $(0,1)$ and $(1,0)$ respectively, i.e. negative definite lines and positive definite lines respectively.

Letting $p+q = n+1$, the coset space representation of $\mathbb{C}H_p^n$ is given by:

$$(5.5) \qquad \mathbb{C}H_p^n \cong U(p,q)/U(1) \times U(p,q-1) \quad .$$

Following [89] (see also [5]), we obtain $\mathbb{C}P_p^n$ by taking $\mathbb{C}H_{n-p}^n$ and reversing its metric. The coset space representation of $\mathbb{C}P_p^n$ is given by:

(5.6) $\mathbb{C}P^n_p \cong U(p+1,n-p)/U(p,n-p) \times U(1)$.

Following the descriptions of (5.3) and (5.4), the complex space forms (5.5) and (5.6) have constant holomorphic sectional curvature equal to $-c$ and c respectively.

Let us note that following the above representations, $\mathbb{C}H^n_n$ is complex hyperbolic space (with positive definite metric) and $\mathbb{C}P^n_n$ is the usual complex projective space. Let us focus our attention on $\mathbb{C}H^n$ for a moment. We have already mentioned that $\mathbb{C}H^n$ is the non-compact Hermitian space dual to $\mathbb{C}P^n$. In fact following [87], $\mathbb{C}H^n = \{z \in \mathbb{C}^n : |z| < 1\}$ occurs, as one of two open orbits of the action of the group $U(n,1)$ on $\mathbb{C}P^n$, as the orbit of negative definite lines with respect to Q_n . The other open orbit is that of positive definite lines with respect to Q_n , and this is the indefinite projective space $\mathbb{C}P^n_{n-1}$. There is one closed orbit, the orbit of isotropic lines with respect to Q_n (i.e. $\{v \in \mathbb{C}P^n : Q_n(v,v) = 0\}$) ; this is seen to be an S^{2n-1} . The space $\mathbb{C}P^n_{n-1}$ has the property of fibering holomorphically over $\mathbb{C}P^{n-1}$ as a unit disc bundle.

This last property is indeed common to all the spaces $\mathbb{C}P^n_p$, they are holomorphic deformation retracts onto a $\mathbb{C}P^p$. We may see this as follows: Let us take $\mathbb{C}P^p$ to be the subspace of $\mathbb{C}P^n_p \subset \mathbb{C}P^n$ defined by $\{z = (z_0,\ldots,z_n) \in \mathbb{C}^{n+1} : z_{p+1} = z_{p+2} = \ldots = z_n = 0\}$. Then the above mentioned deformation retract is induced by the map

(5.7) $F : \{z \in \mathbb{C}^{n+1} : Q_{p+1}(z,z) >> 0\} \times [0,1] \to \mathbb{C}^{n+1} - \{0\}$

given by $F(z,t) = (z_0,\ldots,z_p\ ,\ (1-t)z_{p+1},\ldots,(1-t)z_n)$, with $F(az,t) = a.F(z,t)$ for all $a \in \mathbb{C}^*$. We may conclude that $\mathbb{C}P^n_p$ is homotopically equivalent to $\mathbb{C}P^p$, and note that $\mathbb{C}P^n_p$ is therefore simply-connected. We shall return to $\mathbb{C}P^n_p$ in Section 6.6.

5.3 An indefinite Hopf fibration.

We shall now consider the structure of $\mathbb{C}H^n_p$. Taking H^{2n+1}_{2p} as in (5.3), we determine the tangent space at $z \in H^{2n+1}_{2p}$ to be

$$(5.8) \qquad T_z H^{2n+1}_{2p} = \{w \in \mathbb{C}^{n+1} : \mathrm{Re}\ Q_p(z,w) = 0\} \quad .$$

The horizontal tangent space at z is given by

$$(5.9) \qquad T^H_z H^{2n+1}_{2p} = \{w \in T_z H^{2n+1}_{2p} : \mathrm{Re}\ Q_p(iz,w) = 0\}$$

$$= \{w \in \mathbb{C}^{n+1} : Q_p(z,w) = 0\} \quad .$$

As we stated above, the identification space of the circle group acting freely on H^{2n+1}_{2p} is $\mathbb{C}H^n_p$. Recalling the positive constant c , we define

$$(5.10) \qquad \tilde{h}_p(\xi_1,\xi_2) = -\frac{4}{c}\,\mathrm{Re}\ Q_p(\xi_1,\xi_2)$$

where $\xi_1,\xi_2 \in T^H_z H^{2n+1}_{2p}$. This defines a metric $\tilde{h}$ for which the projection map

$$(5.11) \qquad \pi : H^{2n+1}_{2p} \to \mathbb{C}H^n_p$$

is an (indefinite) Riemannian submersion, and is the only such one with totally geodesic fibres [65].

Since i is a complex structure on $T_z^H H_{2p}^{2n+1}$, we may define an almost complex structure J on $T_{\pi(z)}\mathbb{C}H_p^n$ by $J(v) = \pi_*(iw)$ if $\pi_*(v) = w$, $v \in T_{\pi(z)}\mathbb{C}H_p^n$ and $w \in T_z^H H_{2p}^{2n+1}$. This almost complex structure is integrable, so that $\mathbb{C}H_p^n$ is a complex manifold. The metric $\tilde{h}_p(\ ,\ )$ in (5.10) is an indefinite Hermitian metric and hence $\mathbb{C}H_p^n$ is an n-dimensional indefinite Hermitian manifold. The complex structure J of $\mathbb{C}H_p^n \cong U(p,q)/U(1)\times U(p,q-1)$ may be seen to be parallel with respect to its canonical connection which is compatible with (5.10), (see [57] for a discussion of these points). The space $\mathbb{C}H_p^n$ is endowed with an indefinite Kähler metric and thus it is regarded as an indefinite Kähler manifold of constant holomorphic sectional curvature equal to the negative constant $-c$.

Similar considerations apply to $\mathbb{C}P_p^n$ by replacing (5.3) by (5.4) and reiterating the above construction. The case where $p = n$ i.e. $\mathbb{C}H_n^n = \mathbb{C}H^n$ is also discussed in [57 Chapter XI]. Here $\tilde{h}_n\Big|_{T_z^H H_{2n}^{2n+1}}$ is positive definite and defines on $\mathbb{C}H^n$, the Bergman (Kähler) metric of constant holomorphic curvature $-c$. In terms of the co-ordinate system $z_1,\ldots,z_n$, in $\mathbb{C}^n$, this is given by

$$(5.12)\qquad ds^2 = \frac{4}{c}\,\frac{(1-\sum_\alpha z_\alpha\bar{z}_\alpha)(\sum_\alpha dz_\alpha d\bar{z}_\alpha)-(\sum_\alpha \bar{z}_\alpha dz_\alpha)(\sum z_\alpha d\bar{z}_\alpha)}{(1-\sum_\alpha z_\alpha\bar{z}_\alpha)^2}$$

where the summation is taken over $\alpha = 1,2,\ldots,n$.

The above Riemannian submersion may be seen to be analogous to the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$ which characterises the Fubini-Study metric of constant holomorphic sectional curvature c on $\mathbb{C}P^n$. More generally, we may thus regard the indefinite Riemannian submersions $H^{2n+1}_{2p} \to \mathbb{C}H^n_p$ and $S^{2n+1}_{2p} \to \mathbb{C}P^n_p$ as <u>indefinite Hopf fibrations</u>. These also exist for the indefinite quaternionic hyperbolic and projective spaces. A generalised formulation of the Hopf fibrations over the projective spaces KP^n may be found, for example, in [8, Chapter 3].

5.4 <u>Some preliminaries</u>.

In this section we commence the process of constructing and classifying certain harmonic maps from a Riemann surface M to the spaces $\mathbb{C}H^n_p$ and $\mathbb{C}P^n_p$. We shall concentrate on $\mathbb{C}H^n_p$, since it will be possible to derive from $\mathbb{C}H^n_p$, the corresponding results for $\mathbb{C}P^n_p$; we shall discuss this at a later stage. The underlying idea is to modify the general geometric nature of the Eells-Wood construction [35], for these spaces. Firstly, some points concerning the particular cases of $\mathbb{C}H^n$ and $\mathbb{C}P^n_{n-1}$, occurring in $\mathbb{C}P^n$ as open orbits of the group $U(n,1)$. It is well known that $\mathbb{C}H^n$ is a contractible Stein manifold [10] and hence will only admit trivial holomorphic vector bundles. The construction of [35] would therefore degenerate somewhat under such a trivialisation, and the lifts of maps over local charts could be extended to global lifts throughout. On the other hand, $\mathbb{C}P^n_{n-1}$ admits no non-constant holomorphic functions but holomorphic line bundles over $\mathbb{C}P^n_{n-1}$ can be obtained by pulling back powers of the hyperplane section bundle over $\mathbb{C}P^{n-1}$. The

full generality of the Eells-Wood construction would apply in this case without such over-trivialisation. The same considerations apply to $\mathbb{C}H^n_p$ and $\mathbb{C}P^n_p$ respectively.

Let

$$(5.13) \qquad \mathcal{L} \longrightarrow \mathbb{C}P^n$$

be the universal line bundle on $\mathbb{C}P^n$. We shall take a line bundle over $\mathbb{C}H^n_p$ to be the restriction of $\mathcal{L}$ to $\mathbb{C}H^n_p$ which appears as an open orbit of $\mathbb{C}P^n$ under the action of the indefinite unitary group $U(p,q)$ $(p+q=n+1)$. We shall denote this restricted bundle by L i.e. $L = \mathcal{L}\big|_{\mathbb{C}H^n_p}$. Since L is a holomorphic subbundle, we can endow $L^{\perp_p}$ with an holomorphic structure via the identification $L^{\perp_p} \cong \underline{\mathbb{C}}^{n+1}_p/L$. These are both endowed with the indefinite Hermitian metrics induced from $\underline{\mathbb{C}}^{n+1}_p$. As remarked in Section 1.3 there exists a unique metric connection on L and $L^{\perp_p}$ which is compatible with the holomorphic structure; thus L and $L^{\perp_p}$ are regarded as indefinite Hermitian holomorphic vector bundles. (Generally, by replacing a definite metric by an indefinite metric, the relevant definitions in Chapter I take their indefinite metric form.)

Analogous to (2.32) we have a connection preserving biholomorphic isomorphism

$$(5.14) \qquad h : T'\mathbb{C}H^n_p \to \mathrm{Hom}(L, L^{\perp_p})$$

which sends the metric to its negative for $0 \le p \le n$ and becomes an

isometry for $p = n+1$. If $\phi : M \to \mathbb{C}H^n_p$ is a smooth map, then the bundles $\phi^{-1}T'\mathbb{C}H^n_p$ and $\phi^{-1}\,\mathrm{Hom}(L, L^{\perp_p})$ with their respective connections, admit unique holomorphic structures compatible with their connections. Hence they too are regarded as indefinite Hermitian holomorphic vector bundles on the Riemann surface M . The isomorphism h in (5.14), remains as such on pulling back by ϕ ; we shall retain the letter h for this case.

5.5 The orthogonal projector.

The following discussion is based on the notes in [79] concerning harmonic maps $\phi : M \to G_k(\mathbb{C}^{n+1})$ where M is any smooth Riemannian manifold (which we shall assume M to be, for this section only). Here, we shall replace $\mathbb{C}^{n+1}$ by $\mathbb{C}^{n+1}_p$ (i.e. $\mathbb{C}^{n+1}$ endowed with an indefinite Hermitian form Q_p , in accordance with the convention adopted in Section 5.1). We consider the case $k = 1$.

Let $\phi : M \to \mathbb{C}H^n_p$ be a smooth map. We may characterise ϕ in terms of an $(n+1) \times (n+1)$ matrix-valued function P on M , as follows:

Define:

$$(5.15) \qquad P : \mathbb{C}^{n+1}_p \to \phi^{-1}L$$

to be orthogonal projection from $\mathbb{C}^{n+1}_p$ onto the line $\phi(x) \subset \mathbb{C}^{n+1}_p$, $x \in M$, on which Q_p has signature $(0,1)$.

The projector P has kernel $\phi^{-1}L^{\perp_p}$ and the property that $P^* = P = P^2$, where $*$ denotes the adjoint with respect to Q_p , and

$\phi(x) = \text{Range } P(x)$. We may also regard P as a section of the bundle $\mathrm{Hom}(\underline{\mathbb{C}}_p^{n+1}, \underline{\mathbb{C}}_p^{n+1})$, and as such, its image is $\phi^{-1}L$ and $(1-P)$ has $\phi^{-1}L^{\perp p}$ as its image. Now the connection in $T'\mathbb{C}H_p^n$ coincides with the connection in $\mathrm{Hom}(L,L^{\perp p})$ arising from the connections in L and $L^{\perp p}$, by projecting the trivial connection in $\underline{\mathbb{C}}_p^{n+1} \cong L \oplus L^{\perp p}$. Furthermore, the pull-back of a connection obtained by projecting a connection, coincides with projecting the pull-back of the connection. Observing these facts, we see that the pull-back connection ∇ in $\phi^{-1}T'\mathbb{C}H_p^n \cong \mathrm{Hom}(\phi^{-1}L,\phi^{-1}L^{\perp p})$, is the connection in this bundle arising from the connections in $\phi^{-1}L$ and $\phi^{-1}L^{\perp p}$, which are obtained by orthogonal projection of the trivial connection in $\underline{\mathbb{C}}_p^{n+1}$. These are given by

$$(5.16) \qquad \nabla_X\rho = PX(\rho) \qquad \rho \in C(\phi^{-1}L) , \quad X \in C(TM)$$

$$(5.17) \qquad \nabla_X\sigma = (1-P)X(\sigma) \qquad \sigma \in C(\phi^{-1}L^{\perp p}) .$$

If $A \in C(\mathrm{Hom}(\phi^{-1}L,\phi^{-1}L^{\perp p}))$, we can extend A to an $(n+1) \times (n+1)$ matrix valued function by making it zero on $\phi^{-1}L^{\perp p}$; thus we have $A \in C(\mathrm{Hom}(\underline{\mathbb{C}}_p^{n+1},\underline{\mathbb{C}}_p^{n+1}))$ and $A = AP = (1-P)A$. For ρ and A as above, we have

$$\begin{aligned}(5.18) \qquad (\nabla_X A)\rho &= \nabla_X(A\rho)-A(\nabla_X\rho) = (1-P)X(A\rho)-APX(\rho) \\ &= (1-P)(X(A)\rho + AX(\rho)) - AX(\rho) \\ &= (1-P)X(A)\rho + AX(\rho) - AX(\rho) = (1-P)X(A)\rho \quad .\end{aligned}$$

Thus the formula for the pull-back of the connection on $\mathrm{Hom}(\phi^{-1}L,\phi^{-1}L^{\perp p})$

is given by

(5.19) $\nabla_X A = (1-P)X(A)P$.

Given a smooth map $\phi : M \to \mathbb{C}H^n_p$, we shall consider its lift ϕ_U over an open set U in M

(5.20)
$$\begin{tikzcd} & \mathbb{C}^{n+1}_p - \{0\} \arrow[d, "\pi"] \\ M \arrow[ur, "\phi_U"] \arrow[r, "\phi"] & \mathbb{C}H^n_p \end{tikzcd}$$

For $x \in M$, let us fix a non-zero $v \in \phi(x)$ and take as a suitable open set, the neighbourhood of x given by $U = \{y \in M : P(y)v \neq 0\}$ where $P(y)v$ is orthogonal projection of v onto $\phi(y)$. Thus we have for $y \in U$, $\phi_U(y) = P(y)v$, and for $X \in C(TM)$

$$d\phi(X) = \pi_* d\phi_U(X) = \pi_* X(\phi_U) = \pi_* X(P)v$$

and hence

(5.21) $d\phi(X) = (1-P)X(P)$.

<u>Lemma 5.1</u> [79]

<u>Let</u> $\phi : M \to \mathbb{C}H^n_p$ <u>be a smooth map, then the tension field of</u> ϕ <u>is given by</u>

(5.22) $\tau(\phi) = -(1-P)\Delta(P)P$.

<u>Proof</u>

We recall the expression (1.10) for the second fundamental form of ϕ . Letting $\{X_i\}$ $i = 1,2$ be an orthonormal basis for $C(T_xM)$, $x \in M$, and tracing (1.10), we obtain

$$(5.23) \qquad \tau(\phi) = \sum_i \nabla_{X_i}(d\phi(X_i)) - d\phi(\nabla_{X_i} X_i)$$

$$= \sum_i (1-P)X_i(d\phi(X_i))P-(1-P)(\nabla_{X_i} X_i)(P)P \quad .$$

On applying (5.21) this becomes

$$\sum_i (1-P)X_i\{(1-P)X_i(P)\}P-(1-P)(\nabla_{X_i} X_i)(P)P$$

$$= \sum_i (1-P)(1-P)X_i(X_i(P))P+(1-P)\ X_i(1-P)X_i(P)P-(1-P)(\nabla_{X_i} X_i)(P)P$$

$$= -(1-P)\Delta(P)P - \sum_i (1-P)X_i(P)X_i(P)P \quad .$$

From the relation $(1-P).P = 0$, we have $X_i(1-P)P + (1-P)X_i(P) = 0$.

Hence we obtain

$$(5.24) \qquad (1-P)X_i(P) = X_i(P)P \quad \text{and}$$

$$(5.25) \qquad PX_i(P)P = 0 \quad .$$

From (5.24) we deduce that $\sum_i (1-P)X_i(P)X_i(P)P = \sum_i X_i(P)PX_i(P)P$ and from (5.25) this is zero, and the result follows. □

Remarks

1. As a computational artifice, the projector P has also been used in [27] and [40] to study harmonic maps from a Riemann surface to $G_k(\mathbb{E}^{n+1})$.

2. Following [27] [79], we can form complex derivatives of P , iteratively, by setting

$$(5.26) \qquad P'_{k+1} = \frac{\partial P'_k}{\partial z} P \quad , \quad P''_{k+1} = \frac{\partial P''_k}{\partial \bar{z}} P \quad , \quad k = 0,1,\ldots,n \ ,$$

and defining $P'_0 = P = P''_0$ with $PP'_\alpha = 0 = PP''_\alpha$, for all $\alpha \geq 1$.

5.6 The harmonicity equation.

We now return to smooth maps $\phi : M \to \mathbb{E}H^n_p$ where M is a Riemann surface. As in Definition 2.14, we may define a universal lift Φ of ϕ and so regard Φ as a smooth section of the bundle $\mathrm{Hom}(\phi^{-1}L, \underline{\mathbb{E}}^{n+1}_p)$ with the analogous properties.

We take D to denote covariant differentiation in $\mathrm{Hom}(\phi^{-1}L, \underline{\mathbb{E}}^{n+1}_p)$. Let $V \in C_U(\mathrm{Hom}(\phi^{-1}L, \underline{\mathbb{E}}^{n+1}_p))$ and $0 \neq \rho \in C_U(\phi^{-1}L)$. Then on taking a chart $U = (U,z)$ of M , the formula for the $(0,1)$ part of D , D'' $(= D_{\partial/\partial\bar{z}})$ is given by

$$(5.27) \qquad (D''V)(\rho) = \partial''(V(\rho)) - V(P(\partial''(\rho)) \ .$$

where ∂'' denotes the $(0,1)$ part of the trivial connection ∂ on

$\mathbb{C}_p^{n+1}$, i.e. $\partial'' = \partial/\partial\bar{z}$. Similarly, we obtain an expression for the (1,0)-part of D , $D'(= D_{\partial/\partial z})$.

<u>Proposition 5.2</u>

<u>For any</u> $X \in C_U(T^{\mathbb{C}}M)$, <u>the local section</u> $D_X\phi \in C_U(\mathrm{Hom}(\phi^{-1}L, \mathbb{C}_p^{n+1}))$ <u>is 'horizontal' in the sense that it has image in</u> $\phi^{-1}L^{\perp_p}$. <u>Thus</u> $D_X\phi$ <u>is regarded as a local section of</u> $\phi^{-1}(\mathrm{Hom}(L, L^{\perp_p}))$ <u>under</u> (5.14) <u>and</u>

$$(5.28) \qquad h(\partial'\phi) = D'\phi \quad , \quad h(\partial''\phi) = D''\phi \quad ,$$

<u>where we recall from</u> (1.25) <u>that</u> $\partial'\phi = \partial\phi(\frac{\partial}{\partial z})$ <u>and</u> $\partial''\phi = \bar{\partial}\phi(\frac{\partial}{\partial \bar{z}})$.

<u>Proof</u>

Let $x \in U$ and $\rho \in (\phi^{-1}L)_x$. We need to show that $D_X\phi(\rho)$ lies in $(\phi^{-1}L^{\perp_p})_x$. If $\rho = 0$, this is trivial; otherwise, we extend ρ to be a smooth no-where zero section of $\phi^{-1}L$ on U . Using (5.27), we obtain

$$(5.29) \qquad D_X\phi(\rho) = \partial_X(\phi(\rho)) - \phi(P(\partial_X(\rho)))$$

$$= \text{horizontal component of } \partial_X(\phi(\rho)) \ .$$

Since $\phi(\rho)$ is a local lift of ϕ , it follows that $h(\partial'\phi(x))$ = horizontal component of $\partial'(\phi(\rho)) = D'\phi(\rho)$, and likewise for ∂'' . □

As an analogue to (2.26), we have the following exact sequence of vector bundles on $\mathbb{C}H_p^n$:

$$(5.30) \qquad 0 \to L \xrightarrow{i} \underline{\mathbb{C}}_p^{n+1} \xrightarrow{j} L^{\perp_p} \to 0$$

where i is the inclusion map and j is given by orthogonal projection, with respect to Q_p, along L. Tensoring with L^* and pulling back by ϕ (noting that $L^* \otimes L$ is trivial), we get an exact sequence over M:

$$(5.31) \qquad 0 \to \phi^{-1}(L^* \otimes L) \xrightarrow{i} \phi^{-1}(L^* \otimes \underline{\mathbb{C}}_p^{n+1}) \xrightarrow{j} \phi^{-1}(L^* \otimes L^{\perp_p}) \to 0 ,$$

where i and j are the pull-backs of those in (5.30).

We now proceed to establish the harmonicity equation for ϕ.

<u>Proposition 5.3</u>

<u>If</u> $\phi : M \to \mathbb{C}H_p^n$ <u>is a smooth map, then</u> ϕ <u>is harmonic, if and only if, in any chart</u> U

$$(5.32) \qquad D''D'\phi + \mu'\phi = 0$$

<u>or equivalently</u>,

$$(5.33) \qquad D'D''\phi + \mu''\phi = 0$$

<u>for smooth functions</u> μ', μ'' <u>on</u> U.

<u>Proof</u>

We shall prove (5.32), the proof of (5.33) is similar. Let D_ϕ

$$(5.30) \qquad 0 \to L \xrightarrow{i} \underline{\mathbb{C}}_p^{n+1} \xrightarrow{j} L^{\perp_p} \to 0$$

where i is the inclusion map and j is given by orthogonal projection, with respect to Q_p, along L. Tensoring with L^* and pulling back by ϕ (noting that $L^* \otimes L$ is trivial), we get an exact sequence over M:

$$(5.31) \qquad 0 \to \phi^{-1}(L^* \otimes L) \xrightarrow{i} \phi^{-1}(L^* \otimes \underline{\mathbb{C}}_p^{n+1}) \xrightarrow{j} \phi^{-1}(L^* \otimes L^{\perp_p}) \to 0 ,$$

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$$(5.33) \qquad D'D''\phi + \mu''\phi = 0$$

<u>for smooth functions</u> μ', μ'' <u>on</u> U.

<u>Proof</u>

We shall prove (5.32), the proof of (5.33) is similar. Let D_ϕ

denote the connection on $\phi^{-1}\mathrm{Hom}(L,L^{\perp_p})$ and let ∇^E denote that on $E = \phi^{-1}T'\mathbb{C}H^n_p$. By virtue of (5.28), we have $h(\partial'\phi) = D'\Phi$, and since h is connection preserving

$$(\nabla^E)''\partial'\phi = D''_\phi D'\Phi = j(D''D'\Phi) .$$

Now by Proposition 1.11, ϕ is harmonic if and only if $(\nabla^E)''\partial'\phi = 0$, hence if and only if $j(D''D'\Phi) = 0$. But by the exactness of (5.31), this holds if and only if $D''D'\Phi = \mu\Phi$, where μ is a smooth function. Setting $\mu' = -\mu$, the result follows. □

From [79], we deduce for this indefinite case, that $\mu' = 2e'(\phi)$; thus using (5.28), we have $\mu' = \tilde{Q}_p(D'\Phi,D'\Phi)$ where $\tilde{Q}_p$ denotes the metric on $\phi^{-1}\mathrm{Hom}(L,L^{\perp_p})$. Similarly, we obtain $\mu'' = \tilde{Q}_p(D''\Phi,D''\Phi)$.

The following is a general result (see e.g. [32]):

<u>Lemma 5.4</u>

<u>If</u> E_1 and E_2 <u>are vector bundles on</u> M <u>with respective connections</u> D_1 <u>and</u> D_2 <u>and curvatures</u> R_1, R_2, <u>then the curvature</u> R <u>of the connection</u> D <u>of the bundle</u> $\mathrm{Hom}(E_1,E_2)$ <u>is given by:</u>

$$(5.34) \qquad (R(\tfrac{\partial}{\partial z},\tfrac{\partial}{\partial \bar z})V)\rho = R_2(\tfrac{\partial}{\partial z},\tfrac{\partial}{\partial \bar z})(V(\rho))-V(R_1(\tfrac{\partial}{\partial z},\tfrac{\partial}{\partial \bar z})\rho)$$

<u>where</u> $V \in C(\mathrm{Hom}(E_1,E_2))$ <u>and</u> $\rho \in C(E_1)$.

<u>Lemma 5.5</u>

<u>For any chart</u> U, <u>and for</u> $V \in C_U(\mathrm{Hom}(\phi^{-1}L,\mathbb{C}^{n+1}_p))$, <u>we have</u>

(5.35) $\quad (D'D''-D''D')V = FV$

<u>where</u> $F:U \to \mathbb{R}$ <u>is a smooth function given by</u>

$$F(x) = \mathcal{Q}_p(D'\phi(x),D'\phi(x)) - \mathcal{Q}_p(D''\phi(x),D''\phi(x)) \ .$$

<u>Proof</u>

Following Lemma 5.4, we set $E_1 = \phi^{-1}L$ and $E_2 = \underline{\mathbb{C}}_p^{n+1}$. Since the latter is trivial with the flat connection [56], then $R_2 \equiv 0$. Using the fact that M is a Riemannian 2-manifold, we have

$$(D'D''-D''D')V = R(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}})V = -V(R_1(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar{z}})) \ .$$

Determining the second member of (5.35) follows by an argument similar to [35, Lemma 4.7]. □

<u>Lemma 5.6</u>

<u>Let</u> $\phi:M \to \mathbb{C}H_p^n$ <u>be harmonic</u>. <u>Then for any point</u> x <u>in any chart</u> U , <u>we have</u>

a) $D''(D'^{\alpha}\phi)(x)$ <u>has values in span</u> $\{D'^{\gamma}\phi(x):0 \le \gamma \le \alpha-1\}$ <u>for any</u> $\alpha \ge 1$.

b) $D'(D''^{\beta}\phi)(x)$ <u>has values in span</u> $\{D''^{\delta}\phi(x):0 \le \delta \le \beta-1\}$ <u>for any</u> $\beta \ge 1$.

<u>Proof</u>

We shall prove case a); case b) is similar. For $\alpha = 1$, this is

simply (5.32). For $\alpha \geq 1$,

$$D''(D'^{\alpha+1}\phi) = D'D''D'^{\alpha}\phi - FD'^{\alpha}\phi$$

where F is as in (5.35). The result follows by an induction argument similar to that of [35, Lemma 4.8].

Remark

From the relations (5.26), the following relations [79] are deduced

$$(5.36) \qquad P'_{\alpha} = D'^{\alpha}\phi \quad , \qquad P''_{\beta} = D''^{\beta}\phi \quad .$$

5.7 Osculating spaces of a smooth map.

The following definitions and lemmata are adapted from [35, Section 5], to the case of smooth maps $\phi : M \to \mathbb{C}H^n_p$. The proofs of the lemmata are essentially the same as those for smooth maps to $\mathbb{C}P^n$, and are not reproduced.

Definition 5.7

For any integer $\alpha (1 \leq \alpha \leq \infty)$ and $x \in M$, the αth-order reduced D'-osculating space of ϕ at x, is the subspace

$$(5.37) \qquad \theta'_{\alpha}(x) = \theta'_{\alpha}(\phi)(x) = \mathrm{span}\{D'^{\gamma}\phi(x) : 1 \leq \gamma \leq \alpha\}$$

Definition 5.8

For any integer $\alpha (1 \leq \alpha \leq \infty)$, the αth-order augmented D'-osculating

<u>space of ϕ at x</u>, is the subspace

(5.38) $\quad \mathcal{J}'_\alpha(x) = \mathcal{J}'_\alpha(\phi)(x) = \mathrm{span}\{D'^\gamma\phi(x) : 0 \le \gamma \le \alpha\}$.

Similar definitions hold for the D'' derivatives with the obvious notations θ''_β and $\mathcal{J}''_\beta$.

<u>Definition 5.9</u>

For any integer $\alpha (1 \le \alpha \le \infty)$ and for any local lift ϕ_U of ϕ, the <u>α^{th} augmented ∂'-osculating space of ϕ at x</u>, is the subspace $\mathrm{span}\{\partial'^\gamma\phi_U(x), 0 \le \gamma \le \alpha\}$, for $x \in M$.

<u>Lemma 5.10</u>

<u>The αth augmented ∂'-osculating space is the αth augmented D'-osculating space.</u>

<u>A similar statement holds for the ∂'' and D'' derivatives.</u>

<u>Definition 5.11</u>

Let q be an integer $(0 \le q \le \infty)$ and $x \in M$. The qth <u>order full complex osculating space of ϕ at x</u>, is the subspace

(5.39) $\quad \mathrm{span}\{\partial'^\alpha\partial''^\beta\phi_U(x) : 0 \le \alpha+\beta \le q\}$

where $\phi_U : U \to E_p^{n+1}-\{0\}$ is the local lift of ϕ over a chart U containing x . This is independent of the choice of lift.

<u>Lemma 5.12</u>

<u>If</u> $\phi : M \to \mathbb{C}H^n_p$ <u>is harmonic then for any</u> $q (0 \le q \le \infty)$ <u>and</u> $x \in M$,

(5.40) $\quad \mathrm{span}\{\partial'^{\alpha}\partial''^{\beta}\phi_U(x) : 0 \le \alpha+\beta \le q\} = \tilde{\theta}'_q(x) + \tilde{\theta}''_q(x)$.

<u>Definition 5.13</u>

Let ϕ be the universal lift of the smooth map $\phi : M \to \mathbb{C}H^n_p$. We shall say ϕ is (complex) Q_p-isotropic if

(5.41) $\quad Q_p((D'^{\alpha}\phi)(\rho),(D''^{\beta}\phi)(\rho)) \equiv 0$

for all $\alpha,\beta \ge 0$, $\alpha+\beta \ge 1$, where $\rho \in C(\phi^{-1}L)$.

<u>Remarks</u>

1. We see that ϕ is Q_p-isotropic if and only if, for all $x \in M$, the D' and D''-reduced osculating spaces at x and the complex line $\phi(x)$ are mutually Q_p-orthogonal.

2. In terms of P , (5.41) may be written as

(5.42) $\quad P''^{*}_{\alpha}P'_{\bar{\beta}} \equiv 0 \quad$ for all $\alpha,\beta \ge 0$, $\alpha+\beta \ge 1$.

At this stage we will introduce the <u>real indefinite hyperbolic space</u> $\mathbb{R}H^n_p$. Firstly, for $K = \mathbb{R}$ or $\mathbb{C}$, let

(5.43) $\quad Q'_p(K)(u,v) = \sum_{i=1}^{p} u_i v_i - \sum_{j=p+1}^{n+1} u_j v_j \quad$ where $u,v \in K^{n+1}$,

be the symmetric K-bilinear indefinite inner product. For integers n,p with $0 \le p \le n$, we set

$$(5.44) \qquad H_p^n = \{x \in \mathbb{R}_p^{n+1} : Q_p'(\mathbb{R})(x,x) = -1\} \ .$$

The real indefinite hyperbolic space, $\mathbb{R}H_p^n$, is obtained from (5.44) by identifying 'antipodal' points $(x,-x)$. For the case $p = n$, we get the usual (projectivised) <u>real hyperbolic space</u> $\mathbb{R}H_n^n = \mathbb{R}H^n$.

<u>Definition 5.14</u>

Let $\phi : M \to \mathbb{R}H_p^n$ be a smooth map and $\phi_U : U \to \mathbb{R}_p^{n+1} - \{0\} \to \mathbb{C}_p^{n+1} - \{0\}$ the local lift with $Q_p'(\phi_U, \phi_U) = -1$; then we say that ϕ is Q_p'-<u>isotropic</u> if

$$(5.45) \qquad Q_p'(\partial'^{\alpha}\phi_U, \partial'^{\beta}\phi_U) \equiv 0$$

for all $\alpha,\beta \ge 0$, $\alpha+\beta \ge 1$.

In fact we have:

<u>Lemma 5.15</u>

<u>A smooth map</u> $\phi : M \to \mathbb{R}H_p^n$ <u>is</u> Q_p'-<u>isotropic if and only if</u>

$$(5.46) \qquad i \circ \phi : M \xrightarrow{\phi} \mathbb{R}H_p^n \xrightarrow{i} \mathbb{C}H_p^n$$

<u>is</u> Q_p-<u>isotropic</u>.

Proof

This is similar to [35, Lemma 5.7].

If we let $\tilde{k}_p$ and ∇ denote the metric and the connection on the bundle $\phi^{-1}T'\mathbb{C}H^n_p$, pulled back from those on $T'\mathbb{C}H^n_p$, then we have:

Lemma 5.16

Let $\phi:M \to \mathbb{C}H^n_p$ be a smooth map, then ϕ is Q_p-isotropic if and only if

$$(5.47) \qquad \tilde{k}_p(\nabla'^{\alpha}\phi, \nabla''^{\beta}\phi) \equiv 0 \qquad \text{for all } \alpha,\beta \geq 1 .$$

Proof

This is similar to [35, Proposition 5.8].

5.8 Manufacturing harmonic maps from holomorphic maps for the indefinite case.

We recall the features of the holomorphic curves in $\mathbb{C}P^n$ and their associated curves, as discussed in Section 2.1. For constructing harmonic maps to $\mathbb{C}H^n_p$, we shall need to construct a different kind of polar curve.

Definition 5.17

Let $f:M \to \mathbb{C}P^n$ be a full holomorphic map with $U(p,q)$ acting on $\mathbb{C}P^n$ $(p+q=n+1)$. Assume that Q_p is non-degenerate with constant

signature, on the subspaces $f_{n-1}(x)$, for all $x \in M$. The Q_p-polar (curve) of f is defined to be

$$(5.48) \qquad g^{(p)} = f_{n-1}^{\perp_p} : M \xrightarrow{f_{n-1}} G_n(\mathbb{C}^{n+1}) \xrightarrow{\perp_p} \mathbb{C}P^n .$$

If Q_p does not satisfy the above conditions on $f_{n-1}(x)$, we may still define the Q_p-polar, but $g^p(x)$ may lie in $f_{n-1}(x)$ for some $x \in M$. Note that $g^{(n+1)}$ is the polar of Definition 2.4. Just as in the definite case, we may form the associated curves of $g^{(p)}$ to all admissible orders.

Definition 5.18

We say that a map $\phi : M \to \mathbb{C}H_p^n$ (respectively, $\mathbb{C}P_p^n$) is full if its image is not completely contained within a proper subspace of $\mathbb{C}H_p^n$ (respectively, $\mathbb{C}P_p^n$) .

Lemma 5.19

Let $f : M \to \mathbb{C}P^n$ *be a holomorphic map.*

i) *If* f *is full, then its* Q_p-polar $g^{(p)}$ *is antiholomorphic and full. Furthermore, for local lifts* ξ, η *of* f *and* $g^{(p)}$ *respectively:*

$$(5.49) \qquad Q_p(\partial'^{\alpha}\xi, \partial''^{\beta}\eta) \equiv 0$$

for all $\alpha, \beta \geq 0$, $\alpha+\beta \leq n-1$ *and*

(5.50) $\quad Q_p(\partial'^{\alpha}\xi(x),\partial''^{\beta}\eta(x)) \neq 0$ <u>for some</u> $x \in M$,

<u>for all</u> $\alpha,\beta \geq 0$, $\alpha+\beta = n$.

ii) <u>Conversely. if</u> $g^{(p)}:M \to \mathbb{C}P^n$ <u>is a smooth map satisfying</u> (5.49) <u>and</u> (5.50) <u>for some</u> $\alpha,\beta \geq 0$, $\alpha+\beta = n$, <u>then</u> f <u>and</u> $g^{(p)}$ <u>are full and</u> $g^{(p)}$ <u>is the</u> Q_p<u>-polar of</u> f .

<u>Proof</u>

This is similar to [35, Lemma 3.7].

<u>Remarks</u>

1. We shall call (5.49) and (5.50) the Q_p<u>-isotropy relations</u>.
2. For full holomorphic maps $f,g^{(p)}:M \to \mathbb{C}P^n$, (5.49) is equivalent to $g_{\beta}^{(p)} = f_{n-1-\beta}^{\perp_p}$.

<u>Definition 5.20</u>

<u>The</u> D'<u>-order of a map</u> $\phi:M \to \mathbb{C}H_p^n$ is defined to be the maximum dimension of the $\mathrm{span}\{D'^{\alpha}\phi(x):1 \leq \alpha \leq \infty\}$.

<u>The</u> D"<u>-order</u> is similarly defined. The map ϕ has D"-order (respectively, D'-order) zero, if and only if ϕ is holomorphic (respectively, anti-holomorphic).

In what follows, we set s = D'order of ϕ and r = D"-order of ϕ .

Lemma 5.21

Let $\phi : M \to \mathbb{C}H^n_p$ be a full Q_p-isotropic harmonic map. Then

i) $r + s = n$.

ii) At some point $x \in M$, the sth reduced D'-osculating space has dimension s .

iii) Let $B = \{x \in M : \dim \operatorname{span}\{D'^{\gamma}\phi(x) : 1 \le \gamma \le s\} < s\}$.

The map $\phi' : M - B \to G_s(\mathbb{C}^{n+1}_p)$ given by $\phi'_\alpha(x) = \alpha^{th}$ reduced D'-osculating space of ϕ at x , may be extended uniquely to a real analytic map.

$$(5.51) \qquad \phi'_\alpha : M \to G_\alpha(\mathbb{C}^{n+1}_p) \qquad 1 \le \alpha \le s ,$$

known as the αth reduced D'-associated curve of ϕ .

We refer to those proofs in [35, Lemmata 6.2, 6.3, and 6.4] which will apply here.

By including $\phi(x)$ in (5.51), we define the αth augmented D'-associated curve of ϕ :

$$(5.52) \qquad \tilde{\phi}'_\alpha = (\phi'_\alpha + \phi) : M \to G_{\alpha+1}(\mathbb{C}^{n+1}_p) \qquad 0 \le \alpha \le s .$$

Similarly, we have the corresponding associated curves for the $\{D''^{\beta}\phi\}$ derivatives, with the obvious notation ϕ''_β , $\tilde{\phi}''_\beta$ for the reduced and augmented curves respectively, where $1 \le \beta \le r$ and $r+s = n$.

We shall leave these particular curves aside for a while and return

to them in the next section when we approach the statement of a classification theorem, this being the principal result of this chapter. In order to make our classification theorem as sharp as possible we will insist on keeping track of the signature of Q_p . It will facilitate matters somewhat, if at this stage, we proceed with describing a certain homogeneous space construction which generalises, to an extent, that of Proposition 2.19.

<u>Definition 5.22</u>

Consider the complex Grassmannian of k-planes in $\mathbb{C}^N_p$, $0 \le k \le N$. For integers a_1, a_2 such that $0 \le a_1 \le p$, $0 \le a_2 \le N-p$ and $a_1+a_2 = k$, we define the <u>indefinite complex Grassmannian</u> $G_{a_1,a_2}(\mathbb{C}^N_p)$ by

$$(5.53) \qquad G_{a_1,a_2}(\mathbb{C}^N_p) = \{\alpha \in G_k(\mathbb{C}^N_p) : Q_p \text{ restricted to } \alpha, \text{ has signature } (a_1,a_2)\}$$

$$\cong U(p,q)/U(a_1,a_2)\times U(p-a_1,q-a_2)$$

where $0 \le p,q \le N$ and $p+q = N$.

<u>Definition 5.23</u>

For integers k_1, k_2, t_1, t_2 such that:

$$(5.54) \qquad \begin{cases} 0 \le k_1, k_2 \le q \quad , & k_1 + k_2 = q-a_2 \\ 0 \le t_1, t_2 \le p \quad , & t_1 + t_2 = p-a_1 \quad , \end{cases}$$

we define the space

$$(5.55)\qquad K^{a_1,a_2}_{p,q}(t_1,k_1;t_2,k_2) = \{(V,W) \in G_{t_1,k_1}(\mathbb{E}^N_p) \times G_{t_2,k_2}(\mathbb{E}^N_p) :$$

$V \perp_p W$ and Q_p is non-degenerate on V and $W\}$.

We have the (indefinite) homogeneous fibration

$$(5.56)\qquad \begin{array}{c} U(p,q)/U(a_1,a_2)\times U(t_1,k_1)\times U(t_2,k_2) \cong K^{a_1,a_2}_{p,q}(t_1,k_1;t_2,k_2) \\ \downarrow \pi \\ U(p,q)/U(a_1,a_2)\times U(p-a_1,q-a_2) \cong G_{a_1,a_2}(\mathbb{E}^N_p) \end{array} .$$

The projection map π is given by

$$(5.57)\qquad \pi(V,W) = (V+W)^{\perp_p} .$$

We now make similar considerations for (5.55) as we did for the space $H_{r,s}(\cong K^{1,0}_{n+1,0}(r,0;s,0)$ with $r+s = n)$. We induce a metric on the space $K^{a_1,a_2}_{p,q}(t_1,k_1;t_2,k_2)$ as a real submanifold of the product manifold $\mathcal{G} = G_{t_1,k_1}(\mathbb{E}^N_p) \times G_{t_2,k_2}(\mathbb{E}^N_p)$. By including $U(t_1,k_1) \times U(t_2,k_2)$ in $U(p-a_1,q-a_2)$ in the standard way, (5.56) is a Riemannian submersion for this metric.

<u>Lemma 5.24</u>

<u>Let</u> $f:M \to \mathbb{E}P^n$ <u>be a full holomorphic map</u>.

<u>Let</u> $p,q \in \mathbb{N}$, $p+q = n+1$, <u>we take</u> $g^{(p)}$ <u>to denote the</u>

Q_p-polar curve of f.

Let r be an integer $0 \le r \le n$, put $s = n - r$. Let (t_1,k_1), (t_2,k_2) be integers satisfying (5.54) for $a_1 = 0$, $a_2 = 1$. Suppose that for all $x \in M$, Q_p is non-degenerate and of signatures (t_1,k_1), (t_2,k_2) on $f_{r-1}(x)$, $g^{(p)}_{s-1}(x)$ respectively.

Then the map $\psi : M \to K^{0,1}_{p,q}(t_1,k_1;t_2,k_2)$ defined by

$$(5.58) \qquad \psi(x) = (f_{r-1}(x), g^{(p)}_{s-1}(x))$$

is horizontal with respect to (5.56).

Proof

This is adapted from [35, Lemma 3.9]. It suffices to prove that $\Psi : M \xrightarrow{\psi} K^{0,1}_{p,q}(t_1,k_1;t_2,k_2) \hookrightarrow \mathcal{E}$ is horizontal with respect to $\Pi : K \to \mathbb{C}H^n_p$ where $K = \{(V,W) \in \mathcal{E} : \dim(V+W) = n$ and Q_p is non-degenerate on $(V,W)\}$ and $\Pi(V,W) = (V+W)^{\perp_p}$. It further suffices to prove that $\partial'\psi$ and $\partial''\psi \in T'K$, are Π-horizontal, in which case we shall confine ourselves to proving this for $\partial'\psi$.

In accordance with (5.58), set $(V,W) = \psi(x)$. Since $g^{(p)}$ is -holomorphic, then $\partial'\psi(x) = (\partial' f_{r-1}(x), 0)$. We identify the vertical subspace of $T'_{(V,W)}K$ with $T'_V G_{t_1,k_1}(V+W) \subset T'_V G_{t_1,k_1}(\mathbb{C}^{n+1}_p) \cong \mathrm{Hom}(V, V^{\perp_p})$.

Let ξ (resp. η) be a holomorphic (resp. -holomorphic) lift of f (resp. $g^{(p)}$) on some neighbourhood of x. Define $A \subset M$ as in Section 2.1, and assume $x \notin A$. Then following [35], one can show that

$\{\partial'^{\alpha}\xi(x) : 0 \le \alpha \le r-1\}$ and $\{\partial''^{\beta}\eta(x) : 0 \le \beta \le s-1\}$ are linearly independent with the former spanning V.

We have $\partial'\partial'^{i-1}\xi(x) = \partial'^{i}\xi(x)$ for all $i < r$ and $h(\partial' f_{r-1}(x)) \in \mathrm{Hom}(V, V^{\perp_p})$, where h is the isomorphism $h: T'_V G_{t_1,k_1}(\mathbb{E}_p^{n+1}) \xrightarrow{\cong} \mathrm{Hom}(V, V^{\perp_p})$ (see (5.14)), has image in the projection of $\partial'^{r}\xi(x)$ onto $V^{\perp_p}$.

Now, $\partial'\psi(x)$ is horizontal if and only if $h(\partial' f_{r-1}(x)) \in \mathrm{Hom}(V, V^{\perp_p})$ is Q_p-perpendicular to $\mathrm{Hom}(V,W)$, i.e. if and only if the projection of $\partial'\xi(x)$ on $V^{\perp_p}$, is Q_p-perpendicular to W. The latter follows from the Q_p-orthogonality of V and W and the Q_p-isotropy relations (5.49) and (5.50), since $\{\partial''^{\beta}\eta(x) : 0 \le \beta \le s-1\}$ spans W. Horizontally at the isolated points follows by continuity. □

Consider now a full holomorphic map $f: M \to \mathbb{E}P^n$ and its associated curves f_α of all admissible orders α. We define the set

$$(5.59) \qquad \Lambda_p^{(a_1,a_2)}(f)$$

assigned to f, as follows:

An integer r $(0 \le r \le n)$ belongs to $\Lambda_p^{(a_1,a_2)}(f)$ if and only if Q_p is non degenerate with constant signature on $f_r(x)$ and has signature (a_1,a_2) on $f_{r-k}^{\perp_p}(x) \cap f_r(x)$ for $0 \le k \le r$, $0 \le a_1, a_2 \le k$, $a_1 + a_2 = k$.

We now proceed to a construction analogous to Proposition 2.19 for constructing isotropic harmonic maps to $\mathbb{C}P^n$.

Proposition 5.25

Under the hypotheses of Lemma 5.24, for $r \in \Lambda_p^{0,1}(f)$, the map $\phi : M \to \mathbb{C}H_p^n$ defined by

$$(5.60) \qquad \phi(x) = (f_{r-1}(x) + g_{s-1}^{(p)}(x))^{\perp_p}$$

$$= f_{r-1}^{\perp_p}(x) \cap f_r(x) ,$$

for $x \in M$, is a Q_p-isotropic harmonic map.

Proof

Again, we see ϕ as the composition $\pi \circ \psi$, where π and ψ are given by (5.57) and (5.58) respectively, and the indicated metric on $K_{p,q}^{0,1}(t_1,k_1;t_2,k_2)$ is chosen, such that π is a Riemannian submersion.

The map ψ is harmonic, since as a map into $\mathbb{C}$, its components are $\pm$ holomorphic, and ψ is π-horizontal by virtue of Lemma 5.24. By appealing to Lemma 1.19, we conclude that ϕ is harmonic. The Q_p-isotropy of ϕ follows from the Q_p-isotropy relations existing between f and $g^{(p)}$. □

Remark

As an indefinite flag manifold, $K_{p,q}^{0,1}(t_1,k_1;t_2,k_2)$ could also be

endowed with its Kähler metric and once again, we could appeal to Lemma 1.21 followed by Lemma 1.19 to conclude that ϕ is harmonic.

5.9 A classification theorem.

We will now take a more instructive look at the map ϕ constructed by Proposition 5.25. But firstly a definition:

Definition 5.26

For the range of non-negative integers k_1, k_2, t_1, t_2 satisfying (5.54), we shall say that the pair (t_1, k_1) and (t_2, k_2) is an admissible signature pair.

In this section we set $a_1 = 0$, $a_2 = 1$ as before.

Proposition 5.27

Let f and $g^{(p)}$ be as in Proposition 5.25. Let (t_1, k_1) and (t_2, k_2) be an admissible signature pair and let $\Lambda_p^{(0,1)}(f)$ be defined as in (5.59). Further, assume that for almost $x \in M$, Q_p has signature (t_1, k_1) and (t_2, k_2) on the subspaces $f_{r-1}(x)$ and $g_{s-1}^{(p)}(x)$ respectively, for $r+s = n$, and Q_p is non-degenerate on both these. Then, for $r \in \Lambda_p^{0,1}(f)$, the assignment

$$(5.61) \qquad \phi(x) = f_{r-1}^{\perp_p}(x) \cap f_r(x)$$
$$= (f_{r-1}(x) + g_{s-1}^{(p)}(x))^{\perp_p}$$

<u>defines a full</u> Q_p<u>-isotropic harmonic map</u> $\phi : M \to \mathbb{C}H^n_p$ <u>with</u> r <u>equal to the</u> D''<u>-order of</u> ϕ . <u>Furthermore,</u> Q_p <u>is non-degenerate on the spaces</u> $\tilde{\phi}''_r(x), \phi''_r(x)$ <u>and</u> $\phi'_s(x)$, <u>and has signature</u> (t_1, k_1) <u>and</u> (t_2, k_2) <u>on</u> $\phi''_r(x)$ <u>and</u> $\phi'_s(x)$ <u>respectively for all</u> $x \in M$.

<u>Proof</u>

We recall the correspondence between subbundles of $\underline{E}_p^{n+1}$ and the maps f_{r-1} , f_r , $g^{(p)}_{s-1}$, $g^{(p)}_s$. The first two are + holomorphic (∂''-closed) and the last two are -holomorphic (∂'-closed). Let $0 \neq \rho \in C(\phi^{-1}L)$, then $\phi(\rho) \in C_U(f_r)$. Following the ∂''-closure of f_r and Proposition 5.2, we deduce that $D''\phi(\rho) \in C_U(f_r)$. Also from Proposition 5.2, $D''\phi(\rho) \perp_p \phi(\rho)$; hence $D''\phi(\rho) \in C_U(f_r)$. Since $\partial' C_U(f_{r-1}) \subset f_r$, we see that $D'D''\phi(\rho) \in C_U(f_r)$. Similarly, one shows that $D''D'(\phi) \in C_U(g^{(p)}_s)$. By Lemma 5.5, the first of these can be written $D''D'\phi(\rho) \in C_U(f_r)$. Therefore we have $D''D'\phi(\rho) \in C_U(f_r \cap g^{(p)}_s) = C_U(\phi)$, and on appealing to (5.32), we conclude that ϕ is harmonic. (Note the equivalences: $f^{\perp_p}_{r-1}(x) \cap f_r(x) = f_r(x) \cap g^{(p)}_s(x) = (f_{r-1}(x) + g^{(p)}_{s-1}(x))^{\perp_p}$, for $x \in M$.)

As in [35], the following relations may be easily deduced

$$(5.62) \qquad f_{r-1} = \phi''_r \,,\; f_r = \tilde{\phi}''_r \,,\; f_\alpha = (\tilde{\phi}''_{r-\alpha-1})^{\perp_p} \cap \tilde{\phi}''_r \qquad (0 \le \alpha \le r)$$

$$(5.63) \qquad g^{(p)}_{s-1} = \phi'_s \,,\; g^{(p)}_s = \tilde{\phi}'_s \,,\; g^{(p)}_\beta = (\tilde{\phi}'_{s-\beta-1})^{\perp_p} \cap \tilde{\phi}'_s \qquad (0 \le \beta \le s) \,.$$

Thus the imposed conditions on $f_{r-1}(x)$, $g^{(p)}_{s-1}(x)$ carry through to

$\phi''_r(x)$ and $\phi'_s(x)$ respectively, for $x \in M$. Since $\dim \phi''_r(x) = r$ and $\gamma''_r = f_r$, it follows that $r = D''$-order of ϕ and non-degeneracy on $\gamma''_r(x)$ follows by the definition of $\Lambda_p^{(0,1)}(f)$.

The associated curves f_{r-1} and $g_{s-1}^{(p)}$ lie in Q_p-orthogonal subspaces of $\mathbb{E}_p^{n+1}$, for which Q_p has signature (t_1,k_1) and (t_2,k_2) respectively. We have also shown that $D''^{\beta}\phi(\rho) \in C_U(f_{r-1})$ for $\beta = 1$ and by ∂''-closure of f_{r-1}, this holds for all $\beta \geq 1$. Similarly, we see that $D'^{\alpha}\phi(\rho) \in C_U(g_{s-1}^{(p)})$ for all $\alpha \geq 1$. Since f_{r-1} and $g_{s-1}^{(p)}$ are mutually Q_p- orthogonal, the Q_p-isotropy of ϕ is proved. Fullness follows by similar considerations as for [35, Proposition 5.9]. □

As in the case of isotropic harmonic maps $M \to \mathbb{C}P^n$, we can reverse this procedure to obtain a holomorphic map into $\mathbb{C}P^n$, by the following:

<u>Proposition 5.28</u>

<u>Let</u> $\phi: M \to \mathbb{C}H_p^n$ <u>be a full</u>. Q_p-<u>isotropic harmonic map such that</u> Q_p <u>is non-degenerate on the subspaces</u> $\gamma''_r(x)$ $\phi''_r(x)$ <u>and</u> $\phi'_s(x)$, <u>and has signatures</u> (t_1,k_1) <u>and</u> (t_2,k_2) <u>on</u> $\phi''_r(x)$ <u>and</u> $\phi'_s(x)$ <u>respectively, for</u> $r+s = n$ <u>and</u> $x \in M$. <u>Here</u> r <u>is equal to the</u> D''-<u>order of</u> ϕ. <u>Let</u>

$$(5.64) \qquad f(x) = \gamma''^{\perp_p}_{r-1}(x) \cap \gamma''_r(x)$$
$$= (\gamma''_{r-1}(x) + \phi'_s(x))^{\perp_p}.$$

Then $f(x)$ is complex 1-dimensional and thus defines a mapping from M into $\mathbb{C}P^n$. This map is holomorphic and full. Furthermore Q_p is non-degenerate on the subspaces $f_r(x)$, $f_{r-1}(x)$ and $g_{s-1}^{(p)}(x)$ for all $x \in M$ and has signatures (t_1,k_1) and (t_2,k_2) on $f_{r-1}(x)$ and $g_{s-1}^{(p)}(x)$ respectively. Also, we have $\phi(x) = f_{r-1}^{\perp_p}(x) \cap f_r(x)$ and $r \in \Lambda_p^{(0,1)}(f)$.

Proof

The first assertion follows from the non-degeneracy of Q_p on $\phi_r''(x)$, for $x \in M$. We shall outline the proof of the holomorphicity of f which is essentially that for [34]:

We claim that the subbundle $\tilde{\phi}_r''$ of $\underline{E}_p^{n+1}$ is ∂''-closed. The subbundle corresponding to $\tilde{\phi}_{r-1}''$ has the property that $\partial'\rho \in C_U(\tilde{\phi}_{r-1}''+\phi)$ for all $\rho \in C_U(\tilde{\phi}_{r-1}'')$ (a consequence of harmonicity). Since ϕ is Q_p-orthogonal to $\tilde{\phi}_{r-1}''$, the induced connection $\tilde{D}$ on $\tilde{\phi}_r''$ has the property that $\tilde{D}'(\rho) \in C_U(\tilde{\phi}_{r-1}'')$ for all ρ, i.e. $\tilde{\phi}_{r-1}''$ is $\tilde{D}'$-closed.

Now it follows that the Q_p-orthogonal complement of $\tilde{\phi}_{r-1}''$ in $\tilde{\phi}_r''$, i.e. $\tilde{\phi}_{r-1}''^{\perp_p} \cap \tilde{\phi}_r''$ is D'-closed in $\tilde{\phi}_r''$. But since $\tilde{\phi}_r''$ is a holomorphic subbundle, this suffices to show that $\tilde{\phi}_{r-1}''^{\perp_p} \cap \tilde{\phi}_r''$ is a holomorphic subbundle. Thus f as defined by (5.64) is holomorphic. The non-degeneracy and signature conditions follow as before from (5.62) and (5.63). The fullness of f is proved in a similar fashion to that in [35, Proposition 6.7].

By the way we have defined f and from Lemma 5.10, we deduce that

$Q_p(\partial'^{\alpha}\xi, \partial''^{\beta}\phi_U) \equiv 0$, for all $\alpha,\beta \geq 0$, $\alpha+\beta \leq r-1$; so $\phi_U \in C_U(f_{r-1}^{\perp_p})$. But by fullness and holomorphicity of f , we have $f_r(x) = \overset{\gamma}{\phi}{}''_r(x)$, for all $x \in M$, in particular $\phi_U \in C_U(f_r)$; so $\phi(x) = f_{r-1}^{\perp_p}(x) \cap f_r(x)$, for all $x \in M$. As $\phi(x)$ has signature $(0,1)$, we also have $r \in \Lambda_p^{(0,1)}$. □

We now propose to combine Proposition 5.27 and Proposition 5.28, in order to establish a bijective correspondence between full, Q_p-isotropic harmonic maps $\phi: M \to \mathbb{C}H_p^n$ and full holomorphic maps $f: M \to \mathbb{C}P^n$, satisfying the indicated conditions in each case. To help streamline matters, we shall make two definitions:

<u>Definition 5.29</u>

For admissible signature pairs (t_1,k_1) and (t_2,k_2) related by (5.54), with $a_1 = 0$ and $a_2 = 1$, let $A^{(0,1)}_{p(t_1,k_1)(t_2,k_2)}$ = {pairs (f,r) where $f: M \to \mathbb{C}P^n$ is a full holomorphic map with Q_p-polar curve $g^{(p)}$ and $r \in \Lambda_p^{0,1}(f)$, along with the condition that Q_p has signature (t_1,k_1) and (t_2,k_2) on the subspaces $f_{r-1}(x)$ and $g^{(p)}_{s-1}(x)$ $(s = n-r)$ respectively, for $x \in M$, and Q_p is non-degenerate on these subspaces}.

<u>Definition 5.30</u>

For the above admissible signature pairs (t_1,k_1) and (t_2,k_2) , let

$B^{(0,1)}_{p(t_1,k_1)(t_2,k_2)}$ = {full, Q_p-isotropic harmonic maps $\phi: M \to \mathbb{C}H_p^n$ with r = D"-order of ϕ , $0 \leq r \leq n$, along with the condition that Q_p has signature (t_1,k_1) and (t_2,k_2) on the subspaces $\phi''_r(x)$ and

$\phi_s'(x)$ $(s = n-r)$ respectively, for $x \in M$, and Q_p is non-degenerate on these subspaces} .

Having established these definitions we now proceed to the main result of this chapter:

Theorem 5.31 (see also [38])

For each choice of admissible signature pairs (t_1,k_1) and (t_2,k_2) as defined by Definition 5.26, there exists a bijective correspondence between the sets $A^{(0,1)}_{p(t_1,k_1)(t_2,k_2)}$ and $B^{(0,1)}_{p(t_1,k_1)(t_2,k_2)}$ as defined by Definitions 5.29 and 5.30 respectively. The correspondences are given by (5.61) $\left(A^{(0,1)}_{p(t_1,k_1)(t_2,k_2)} \to B^{(0,1)}_{p(t_1,k_1)(t_2,k_2)}\right)$ and

(5.64) $\left(B^{(0,1)}_{p(t_1,k_1)(t_2,k_2)} \to A^{(0,1)}_{p(t_1,k_1)(t_2,k_2)}\right)$.

Proof

This results by combining Proposition 5.27 with Proposition 5.28. □

5.10 Real indefinite hyperbolic space.

In Corollary 2.28, we stated the classification theorem for full, isotropic harmonic maps $\phi:M \to \mathbb{R}P^{2r}$ in terms of full, totally isotropic holomorphic maps $f:M \to \mathbb{C}P^{2r}$. We regarded $\mathbb{R}P^{2r}$ as the real points of $\mathbb{C}P^{2r}$. Likewise, we regard the 2r-dimensional real indefinite hyperbolic space $\mathbb{R}H^{2r}_k$ as the real points of $\mathbb{C}H^{2r}_k$. To establish an

analogous result to Corollary 2.28, we need to impose an analogous condition on the holomorphic map f. Following [37], k must be taken to be even, $k = 2p$ say.

<u>Definition 5.32</u>

Recalling (5.43), we say that a full, holomorphic map $f:M \to \mathbb{C}P^{N-1}$ is <u>totally</u> Q_{2p}-<u>isotropic</u> if for any local lift ξ of f:

$$(5.65) \qquad Q'_{2p}(\partial'^{\alpha}\xi, \partial'^{\beta}\xi) \equiv 0 \quad \text{for all} \quad \alpha,\beta \geq 0 \ , \quad \alpha+\beta \leq N-2 \ .$$

Note that by similar considerations to [35, Lemma 3.14], only $(N-1)$ even is admissible. Thus taking $N = 2r+1$ say, it can be seen that a full, totally Q_{2p}-isotropic holomorphic map $f:M \to \mathbb{C}P^{2r}$, has $\bar{f}$ as its Q_{2p}-polar curve.

On replacing p by $2p$ and n by $2r$ in Definitions 5.29 and 5.30, we shall denote by $\bar{A}^{(0,1)}_{2p(t_1,k_1)(t_2,k_2)}$, the subset of $A^{(0,1)}_{2p(t_1,k_1)(t_2,k_2)}$ defined by full, holomorphic maps $f:M \to \mathbb{C}P^{2r}$ such that f is totally Q_{2p}-isotropic with Q_{2p}-polar curve $\bar{f}$ (note that $r = s$ in this case).

Accordingly, we shall denote by $\bar{B}^{(0,1)}_{2p(t_1,k_1)(t_2,k_2)}$, the subset of $B^{(0,1)}_{2p(t_1,k_1)(t_2,k_2)}$ defined, on setting $r = s$, by those maps which have image in $\mathbb{R}H^{2r}_{2p} \subset \mathbb{C}H^{2r}_{2p}$. We also recall Definitions 5.14 and 5.15.

<u>Corollary 5.33</u> (see also [37])

<u>There exists a bijective correspondence between the sets</u> $\bar{A}^{(0,1)}_{2p(t_1,k_1)(t_2,k_2)}$ <u>and</u> $\bar{B}^{(0,1)}_{2p(t_1,k_1)(t_2,k_2)}$ <u>for each choice of admissible signature pairs</u> (t_1,k_1) <u>and</u> (t_2,k_2) .

<u>Proof</u>

The condition that $\bar{f}$ is the Q_{2p}-polar curve of f implies that for $x \in M$, $f^{\perp_{2p}}_{r-1}(x) = \bar{f}_r(x)$, hence $\phi(x)$ is real and thus ϕ has image in $\mathbb{R}H^{2r}_{2p}$.

Conversely, when ϕ is real, we deduce from (5.63) and (5.64) that $g^{(p)} = f$, hence f is totally Q_{2p}-isotropic. □

<u>Remark</u>

For $p = r$, we recover the corresponding result for the real hyperbolic space (with positive definite metric) $\mathbb{R}H^{2r}_{2r} = \mathbb{R}H^{2r} \subset \mathbb{C}H^{2r}$. We shall discuss this case again in Section 5.13.

5.11 <u>Complex hyperbolic space and compact quotients</u>.

For the remainder of this chapter, we shall focus our attention on the special case of $\mathbb{C}H^n_n = \mathbb{C}H^n$.

<u>Corollary 5.34</u> (see also [38])

<u>For</u> $p = n$, <u>Theorem 5.31 classifies full</u>, Q_n-<u>isotropic harmonic</u>

<u>maps</u> $\phi : M \to \mathbb{C}H^n$, <u>such that for</u> $x \in M$ <u>and</u> $r = D''$-<u>order of</u> ϕ , Q_n <u>is non-degenerate on the subspaces</u> $\tilde{\phi}''_r(x)$, $\phi''_r(x)$ <u>and</u> $\phi'_s(x)$, $s = n-r$, <u>and is positive definite</u> ($>> 0$) <u>on</u> $\phi''_r(x)$ <u>and</u> $\phi'_s(x)$.

<u>Remark</u>

Observe that if Q_n also happens to be non-degenerate on the rank r subspace $\tilde{\phi}''_{r-1}(x)$, then by the inclusion of ϕ (on which Q_n has signature $(0,1)$) in this subspace, we would see that Q_n has signature $(r-1,1)$ on $\tilde{\phi}''_{r-1}$. We also have Q_n of signature $(s,0)$ on $\phi'_s(x)$. Then by the second relationship in (5.64), and recalling that $r+s = n$, we would see that on the common Q_n-orthogonal subspace, i.e. $f(x)$, Q_n would have signature $(1,0)$. Hence f would actually have image in the indefinite complex projective space $\mathbb{C}P^n_{n-1}$ in $\mathbb{C}P^n$. We shall remark on this observation again, at a later stage.

Let $\mathrm{Aut}(\mathbb{C}H^n)$ denote the group of holomorphic automorphisms of $\mathbb{C}H^n$. It is known that the Bergman metric is invariant under the action of $\mathrm{Aut}(\mathbb{C}H^n)$ and hence is invariant under the action of a discrete subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{C}H^n)$. The discrete group Γ may be chosen such that $\mathbb{C}H^n/\Gamma$ is a compact algebraic variety [59] (with constant negative holomorphic sectional curvature c).

The desired properties of such a Γ , are:

a) Γ is properly discontinuous, meaning that any compact set in D

intersects only a finite number of its images under Γ .

b) $\mathbb{C}H^n/\Gamma$ is compact.

c) Γ is torsion-free and acts freely i.e. only the identity element of Γ has fixed points.

The most suitable such Γ are the discrete (arithmetic) subgroups determined by the arithmetic method relating to the ring of units of an indefinite Hermitian form. The underlying theory supporting this technique, it outlined in its full generality in Borel's paper [12].

Without loss of generality, we shall assume that Γ is of this type. As a result, the quotient map

$$(5.66) \qquad \tau:\mathbb{C}H^n \to \mathbb{C}H^n/\Gamma$$

is a local isometry [56]. This implies that the harmonicity and Q_n-isotropy of the maps $\phi:M \to \mathbb{C}H^n$ constructed via Proposition 5.25 (for $p = n$) , is preserved. Thus the following is immediate:

<u>Proposition 5.35</u>

<u>Let</u> $f:M \to \mathbb{C}P^n$ <u>be a full, holomorphic map with</u> $U(n,1)$ <u>acting on</u> $\mathbb{C}P^n$. <u>Let</u> $g^{(n)}$ <u>denote the</u> Q_n<u>-polar of</u> f . <u>Furthermore, assume that on the subspaces</u> $f_{r-1}(x)$, $g^{(n)}_{s-1}(x)$, Q_n <u>is non-degenerate and</u> $>> 0$, <u>for</u> $x \in M$. <u>Then the map defined by</u>

(5.67) $\phi(x) = (f_{r-1}(x) + g^{(n)}_{s-1}(x))^{\perp n} \bmod \Gamma$,

<u>is a</u> Q_n<u>-isotropic harmonic map</u> $\phi : M \to \mathbb{C}H^n/\Gamma$.

<u>Example 5.36</u>

This example is partly adapted from that appearing in [36]. Let B^2 denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Then B^2 is conformally equivalent to $\mathbb{R}H^2$ with conformal map given by $\alpha(z) = \alpha(x_1, x_2) = (2x_1, 2x_2, (1+a))/(1-a)$ where $a = x_1^2 + x_2^2$, $x_1 x_2 \in \mathbb{R}$. Let U be the open set in M given by $U = \{z \in \mathbb{C} : |z| < 1/2^{\frac{1}{4}}\}$; we can define a map $\eta : U \to \mathbb{C}^3$, where for $z \in U$,

$$\eta(z) = \left(\frac{1}{\sqrt{2}}\left(\tfrac{1}{4} + z^4\right) , i\left(\frac{1}{\sqrt{2}}\left(\tfrac{1}{4} - z^4\right)\right), z^2\right) .$$

We also have a map $\lambda : B^2 \to U$ given by $\lambda(z) = z/2^{\frac{1}{4}}$. We then proceed to define a holomorphic map $f : \mathbb{R}H^2 \to \mathbb{C}P^2$ given by $f \circ \alpha = \pi \circ \eta \circ \lambda : B^2 \to \mathbb{C}P^2$ where in this case, π is the projection $\mathbb{C}^3 - \{0\} \to \mathbb{C}P^2$. The map f is a full holomorphic map such that $Q_2 >> 0$, and Q_2 is non-degenerate on f . The map $\phi' : \mathbb{R}H^2 \to \mathbb{C}H^2$ given by $\phi'(z) = (f(z) + g^{(2)}(z))^{\perp_2}$ is a Q_2-isotropic harmonic map. Here the $g^{(2)}$ polar is equal to $\bar{f}$ (f is therefore totally Q_2'-isotropic) and ϕ' has image in $\mathbb{R}H^2$.

We then compose this example with the totally geodesic inclusion map $i : \mathbb{R}H^2 \to \mathbb{C}H^2$. The resulting map, $i \circ \phi'$ is harmonic and is Q_2-isotropic into $\mathbb{C}H^2$ by virtue of (5.46). We could then allow a suitable

discrete subgroup Γ of $U(2,1)$ to act freely on $\mathbb{C}H^2$ and thus we would have constructed a Q_2-isotropic harmonic map $\phi:\mathbb{R}H^2 \to \mathbb{C}H^2/\Gamma$.

As in the case of isotropic harmonic maps $\phi:M \to \mathbb{C}P^n$, the Q_n-isotropic harmonic maps $\phi:M \to \mathbb{C}H^n/\Gamma$ are examples of branched minimal immersions in the sense of [43]. We say that $x \in M$ is a <u>branch point</u> if $d\phi(x) = 0$. The determination of branch points for the maps constructed by Proposition 5.35, is:

<u>Proposition 5.37</u>

<u>Let</u> $\phi:M \to \mathbb{C}H^n/\Gamma$ <u>be the</u> Q_n<u>-isotropic harmonic map constructed by</u> (5.67). <u>Then</u> $x \in M$ <u>is a branch point if and only if</u> $\partial' f_{r-1}(x) = 0$ <u>and</u> $\partial' f_r(x) = 0$. <u>In particular if</u> M <u>is compact then the number of branches is</u> $\min(\beta_{r-1}(f),\beta_r(f))$.

<u>Proof</u>

This is similar to [35, Proposition 8.13]. We shall use the fact that $\tau:\mathbb{C}H^n \to \mathbb{C}H^n/\Gamma$ is a local isometry. Thus x is a branch point of ϕ , if and only if x is a branch point of $\phi':M \to \mathbb{C}H^n$, whereby $\phi = \tau\circ\phi'$. But this will be the case if and only if $\partial'\phi'(x) = 0$ and $\partial''\phi'(x) = 0$. By the way ϕ' is constructed (i.e. $\phi'(x) = f_{r-1}^{\perp_n}(x) \cap f_r(x)$), this holds if and only if $\partial' f_{r-1}(x) = 0$ and $\partial' f_r(x) = 0$.

<u>Remarks</u>

The map ϕ will be a conformal immersion everywhere when $\partial' f_{r-1}$ and $\partial' f_r$ have no common zeros.

The potentially most interesting maps are those from a compact M_p to $\mathbb{C}H^n/\Gamma$ (recall that p denotes the genus).

For $p = 0$, any harmonic map $\phi: S^2 \to \mathbb{C}H^n/\Gamma$ is constant. For $p = 1$, a harmonic map into $\mathbb{C}H^n/\Gamma$ is either constant or has image a closed geodesic. Both of these facts are consequences of applying the Weitzenböck formula [32].

From a topological viewpoint, it is well known that $N = \mathbb{C}H^n/\Gamma$ is a $K(\Gamma,1)$ (Eilenberg-MacLane) space (its homotopy groups $\pi_i = 0$, for $i \geq 2$, [83]). Following [83, Th.11, p.428] any continuous map from M_1 to N is seen to be homotopic to a map whose image is a circle (here one uses the fact that $\pi_1(M) \cong \mathbb{Z} \times \mathbb{Z}$ and $\Gamma = \pi_1(N)$ contains no abelian subgroups apart from $\mathbb{Z}$ [95]). A holomorphic map homotopic to such a map, would be a constant.

For any smooth map $\phi: M_p \to \mathbb{C}H^n/\Gamma$, $p \geq 2$, Toledo, in [85], proved an inequality on the degree of ϕ. On setting $N = \mathbb{C}H^n/\Gamma$, we define $\deg \phi$ by

$$\deg \phi = \frac{c}{4\pi} \int_M \phi^* \omega^N$$

where ω^N denotes the cohomology class corresponding to the Kähler form of N [64].

Taking $c = -1$, he proves $\left|\int \phi^* \omega^N\right| \leq 4\pi(p-1)$, $p \geq 2$; alternatively, $|\deg \phi| \leq p-1$.

5.12 Some technical problems.

We now ask, if given a Q_n-isotropic harmonic map $\phi : M \to \mathbb{C}H^n/\Gamma$, can we reverse the construction of Proposition 5.36 to produce a holomorphic map $f : M \to \mathbb{C}P^n$? There are certain obstacles encountered in this case.

Recall that the line bundle $L \to \mathbb{C}H^n$ was taken to be the restriction to $\mathbb{C}H^n$, of the universal line bundle $\mathcal{L} \to \mathbb{C}P^n$. On dividing through by the action of Γ, we obtain the induced holomorphic line bundle $E \to \mathbb{C}H^n/\Gamma$. This line bundle E belongs to a well-known class of fibre bundles known as flat bundles, which arise from representations of the fundamental group $\pi_1(\mathbb{C}H^n/\Gamma) = \Gamma$ (see e.g. [54] for a detailed definition and related properties). These are not subbundles of the trivial $(n+1)$-plane bundle, as we would wish, but if U is taken to be a simply-connected open set of M, then $\phi^{-1}E|_U$ is holomorphically trivial [44]. Therefore we choose a certain open covering $\{U_i\}_{i \in I}$ of M (where I is an indexing set) by simply-connected open sets U_i for which the intersections $U_i \cap U_j$ are contractible, for all $i,j \in I$. Such a covering is known as an acyclic or Leray covering of M [42]. For any Riemannian manifold M, such a choice of covering is always possible, since each point $x \in M$ may have a fundamental system of convex normal balls [48] in which any finite intersection is again convex and contractible.

Taking such a covering of M, we have on each U_i, a Q_n-isotropic harmonic lift $\phi^{(i)}$ of ϕ to $\mathbb{C}H^n$. We can then apply the inverse pro-

cedure (Proposition 5.28) to each $\phi^{(i)}$ to produce a full, holomorphic map $f^{(i)}:U_i \to \mathbb{C}P^n$ with an associated integer r_i (= D"-order of $\phi^{(i)}$).

Thus as we run across the covering $\{U_i\}$, we obtain a family of pairs $\{f^{(i)}, r_i\}$. Besides the problem of 'glueing together' the maps $f^{(i)}$ to define a globally defined holomorphic map f, we recall that $f^{(i)}$ may have image in $\mathbb{C}P^n_{n-1} \cong U(n,1)/U(n-1,1) \times U(1)$. Now $\mathbb{C}P^n_{n-1}$ has a non-compact stability subgroup and Γ, whose action is respected throughout the inverse procedure, may not act freely on $\mathbb{C}P^n_{n-1}$ and hence the quotient space $\mathbb{C}P^n_{n-1}/\Gamma$ would not be a manifold (we refer to [69] for a discussion of these points). Thus it is unlikely that a classification theorem can be obtained for this particular case.

5.13 Real hyperbolic space.

For $p = r$, we obtain from Corollary 5.33 the corresponding result for full, Q_{2r}-isotropic harmonic maps $\phi:M \to \mathbb{R}H^{2r}$, satisfying the indicated conditions, in terms of full, totally Q_{2r}-isotropic holomorphic maps $f:M \to \mathbb{C}P^{2r}$ (see also [36]).

Let us return to Proposition 5.35 and take f to be totally Q_{2r}-isotropic. We shall consider a discrete subgroup Γ of $SO(2r,1)$ having the properties indicated previously. The arithmetic method in [12] also covers the real case to determine such a Γ.

Proposition 5.38

Let $f:M \to \mathbb{C}P^{2r}$ be a full, totally Q_{2r}-isotropic holomorphic map

(i.e. $g^{(2r)} = \bar{f}$) <u>with</u> $U(2r,1)$ <u>acting on</u> $\mathbb{C}P^{2r}$. <u>Furthermore, assume that on the subspaces</u> $f_{r-1}(x)$ <u>and</u> $\overline{f_{r-1}(x)}$, Q_{2r} <u>is non-degenerate and</u> $>> 0$, <u>for</u> $x \in M$. <u>Then the map defined by</u>

$$(5.68) \qquad \phi(x) = (f_{r-1}(x) + \overline{f_{r-1}(x)})^{\perp_{2r}} \bmod \Gamma \quad , \qquad x \in M$$

<u>is full,</u> Q_{2r}<u>-isotropic harmonic map</u> $\phi : M \to \mathbb{R}H^{2r}/\Gamma$, <u>where</u> Γ <u>is a discrete subgroup of</u> $SO(2r,1)$ <u>with the above-mentioned properties</u>.

<u>Proof</u>

The $(r-1)^{th}$ associated curve of f defines a holomorphic map into the space K_r of totally Q_{2r}-isotropic subspaces of $\mathbb{C}^{2r+1}_{2r}$ that are contained in $K_{r,r}$; here $K_r \cong SO(2r,1)/U(r)$; the proof of this is similar to that for $H_r = SO(2r+1)/U(r)$. The space K_r has a totally geodesic embedding in $K_{r,r} = K^{0,1}_{2r,1}(r,0;0,r) \cong U(2r,1)/U(r)\times U(r)\times U(1)$.

We construct the Q_{2r}-isotropic harmonic map $\phi : M \to \mathbb{R}H^{2r}$ via the composition

$$\begin{array}{ccccc} M & \xrightarrow{f_{r-1}} & K_r \cong SO(2r,1)/U(r) & \hookrightarrow & K_{r,r} \\ & & \downarrow \pi & & \downarrow \\ & & \mathbb{R}H^{2r} \cong SO(2r,1)/S(O(2r)\times O(1)) & \hookrightarrow & \mathbb{C}H^{2r} \end{array}$$

whence the previous horizontally arguments apply. Here π is taken to be the restriction of $K_{r,r} \to \mathbb{C}H^{2r}$, to K_r . Letting Γ act freely on $\mathbb{R}H^{2r}$ we thus obtain a Q_{2r}-isotropic harmonic map into $\mathbb{R}H^{2r}/\Gamma$. $\square$

<u>Remark</u>

With regards to Proposition 5.37, we see that in this case, $x \in M$ is a branch point of ϕ , if and only if $\partial' f_{r-1}(x) = 0$.

CHAPTER VI

HARMONIC MAPS TO INDEFINITE GRASSMANNIANS

6.1 Introduction.

In this chapter we shall discuss isotropic harmonic maps to several classes of indefinite Grassmannians. We have in mind the indefinite metric analogues of those spaces discussed in Chapters III and IV. In the last chapter we discussed how the results of Eells and Wood may be modified to deal with maps to the indefinite complex hyperbolic space $\mathbb{C}H^n_p$. Our approach entails applying these modifications to the generalised results of Erdem and Wood. Thus we shall take as our basic model, the indefinite complex Grassmannian $G_{a_1,a_2}(\mathbb{C}^N_p)$ $(N = p+q)$ as defined in (5.53). This space $G_{a_1,a_2}(\mathbb{C}^N_p)$ may be regarded as a general open orbit of the action of the indefinite unitary group $U(p,q)$ on $G_k(\mathbb{C}^N)$ [87], and is endowed with an indefinite Kähler metric. For a special case, we shall state the exact number of such open orbits, at a later stage. We shall commence by discussing maps to $G_{a_1,a_2}(\mathbb{C}^N_p)$ and derive further results analogous to those in Chapters III and IV.

6.2 Harmonic maps to indefinite complex Grassmannians.

As in Chapter III, if we take L to denote the universal k-plane bundle on $G_k(\mathbb{C}^N)$, then we shall denote its restriction to $G_{a_1,a_2}(\mathbb{C}^N_p)$

by L . We endow L with an indefinite Hermitian metric and connection as a subbundle of $\underline{\mathbb{C}}_p^N$. We induce a holomorphic structure and indefinite Hermitian metric on $L^{\perp_p}$ via the identification $L^{\perp_p} \cong \mathbb{C}_p^N/L$.

Just as in (5.30), we may obtain the following exact sequence over $G_{a_1,a_2}(\mathbb{C}_p^N)$

$$(6.1) \qquad \underline{0} \longrightarrow L \xrightarrow{i} \underline{\mathbb{C}}_p^N \xrightarrow{j} L^{\perp_p} \longrightarrow \underline{0}$$

where i and j operate as in (5.30). Given a smooth map $\phi : M \to G_{a_1,a_2}(\mathbb{C}_p^N)$, the above exact sequence induces an exact sequence over M :

$$(6.2) \qquad \underline{0} \to \mathrm{Hom}(\phi^{-1}L, \phi^{-1}L) \xrightarrow{i} \mathrm{Hom}(\phi^{-1}L, \underline{\mathbb{C}}_p^N) \xrightarrow{j} \mathrm{Hom}(\phi^{-1}L, \phi^{-1}L^{\perp_p})$$

$$\cong \phi^{-1}\gamma' G_{a_1,a_2}(\mathbb{C}_p^N) \to \underline{0} .$$

These bundles are endowed with the indefinite pull-back connections and metrics.

Generalising (5.15), we define the projection $P : \mathbb{C}_p^N \to \phi^{-1}L$ to be Q_p-orthogonal projection from $\mathbb{C}_p^N$ onto the k-dimensional subspace $\phi(x) \subset \mathbb{C}_p^N$, $x \in M$, on which Q_p has signature (a_1, a_2) .

As in Chapter III, we consider the 'universal lift' Φ of ϕ , and by similar considerations, we can view Φ as a section of the bundle $\mathrm{Hom}(\phi^{-1}L, \mathbb{C}_p^N)$. Again, using D to denote covariant differentiation in the bundle $\mathrm{Hom}(\phi^{-1}L, \mathbb{C}_p^N)$, we consider the usual splitting into (1,0)

and (0,1) parts, denoted by D' and D'' respectively. As in (5.27), we compute the D'' derivatives commencing from

$$(6.3) \qquad (D''V)(\rho) = \partial''(V(\rho)) - V(P(\partial''(\rho)))$$

for any $V \in C_U(\mathrm{Hom}(\phi^{-1}L, \mathbb{E}_p^N))$ and $\rho \in C_U(\phi^{-1}L)$. A similar expression holds for the D' derivatives.

We now take $\mathrm{Im}(V_x)$ to denote the image of the linear map $V_x : L_x \to \mathbb{E}_p^N$ and define $\mathrm{Im}(V)$ as in (3.11), on replacing $\mathbb{E}^N$ by $\mathbb{E}_p^N$.

<u>Lemma 6.1</u>

a) <u>We have</u> $\mathrm{Im}(D'\phi) \perp_p \mathrm{Im}(\phi)$, <u>and</u> $D'\phi$ <u>can be regarded as a local section of</u> $\mathrm{Hom}(\phi^{-1}L, \phi^{-1}L^{\perp_p})$.

b) <u>There exists an isomorphism between</u> $\mathrm{Hom}(\phi^{-1}L, \phi^{-1}L^{\perp_p})$ <u>and</u> $\phi^{-1}T' G_{a_1,a_2}(\mathbb{E}_p^N)$, <u>whereby the section</u> $D'\phi$ <u>corresponds to</u> $\partial'\phi$.

<u>Proof</u>

a) This follows directly from (6.3). For part b), the isomorphism is simply the generalisation of (5.14) (cf. (2.32)) and the last assertion follows by the same argument as for (5.28). □

<u>Lemma 6.2</u>

<u>For any</u> $V \in C_U(\mathrm{Hom}(\phi^{-1}L, \mathbb{E}_p^N))$ <u>we have</u>

$$(6.4) \qquad \mathrm{Im}((D'D'' - D''D')V) \subset \mathrm{Im}(V) .$$

Proof

This may be proved directly from (6.3) or by the same argument as for Lemma 5.5. □

Lemma 6.3

A smooth map $\phi : M \to G_{a_1,a_2}(\mathbb{E}_p^N)$ is harmonic if and only if in each chart U

(6.5) $\quad \mathrm{Im}(D''D'\phi) \subset \mathrm{Im}(\phi)$ or equivalently $\mathrm{Im}(D'D''\Xi) \subset \mathrm{Im}(\phi)$.

Proof

Let ∇ be the connection on $\phi^{-1}T'G_{a_1,a_2}(\mathbb{E}_p^N)$ and consider the (1,0) and (0,1) parts of ∇ , namely ∇' and ∇'' respectively. From part b) of Lemma 6.1, the section $j(D''D'\phi)$ corresponds to $\nabla''\partial'\phi$, where j is the map in (6.2) and $\partial'\phi$ is defined by (1.24). By (1.26), ϕ is harmonic if and only if $\nabla''\partial'\phi \equiv 0$. Using the exactness of the sequence in (6.2), the result follows by an argument similar to that for Proposition 5.3. □

Lemma 6.4

For a harmonic map $\phi : M \to G_{a_1,a_2}(\mathbb{E}_p^N)$, we have for each $x \in M$, and $\alpha = 1,2,\ldots$

(6.6) $\quad \mathrm{Im}(D''D'^{\alpha}\phi)_x \subset \mathrm{Im}(\phi)_x + \mathrm{Im}(D'\phi)_x + \ldots + \mathrm{Im}(D'^{\alpha-1}\phi)_x$.

Proof

This is deduced from Lemmas 6.2 and 6.3 and generalises Lemma 5.6 (see also [39, Lemma 3.5]). □

Definition 6.5

We shall say that the smooth map $\phi : M \to G_{a_1,a_2}(\mathbb{E}^N_p)$ is Q_p-isotropic if the images of the linear maps $(\nabla'^{\alpha}\phi)_x$ and $(\nabla''^{\beta}\phi)_x$ are Q_p-orthogonal for all $x \in M$ and $\alpha,\beta = 1,2,\ldots$.

Lemma 6.6

A smooth map $\phi : M \to G_{a_1,a_2}(\mathbb{E}^N)$ is Q_p-isotropic if and only if

$$\mathrm{Im}(D'^{\alpha}\phi) \perp_p \mathrm{Im}(D''^{\beta}\phi) \quad \forall\, \alpha,\beta \geq 0 \,,\ \alpha+\beta \geq 1 \,. \tag{6.7}$$

Proof

This is proved by induction on $\alpha+\beta$ (cf. [35, Proposition 5.8]). □

We shall now formulate a definition, that generalises Definition 3.3, in order to construct Q_p-isotropic harmonic maps from M to $G_{a_1,a_2}(\mathbb{E}^n_p)$. These maps, and the corresponding subbundles of $\mathbb{E}^N_p$ are regarded as full in the sense of Definitions 3.2 and 3.4 respectively, with $\mathbb{E}^N$ replaced by $\mathbb{E}^N_p$.

Definition 6.7

Let (V,X) be a pair of holomorphic subbundles of $\mathbb{E}^N_p$ such that:

a) $V \subset X$ b) $\mathrm{rank}(X) - \mathrm{rank}(V) = k$ c) $\partial' C(V) \subset C(X)$

d) Q_p is non-degenerate and of constant signature on X .

Then we shall call (V,X) a ∂'_p-pair of $\mathbb{C}^N_p$ of rank difference k .

Remark

Let us remind ourselves that we are taking $k = a_1 + a_2$, $a_1, a_2 \geq 0$.

Having collected together these ideas, we can now state an analogue of Theorem 1.1(a) in [39]:

Theorem 6.8

Let (V,X) be a full ∂'_p-pair of subbundles of $\mathbb{C}^N_p$ of rank difference k . Further assume that Q_p restricted to the subspace $V_x^{\perp_p} \cap X_x$ is non-degenerate with constant signature (a_1, a_2) . Then the map $\phi : M \to G_{a_1,a_2}(\mathbb{C}^N_p)$ defined by

$$(6.8) \qquad \phi(x) = V_x^{\perp_p} \cap X_x , \qquad x \in M$$

is a full, Q_p-isotropic harmonic map.

Proof

By construction, $\mathrm{Im}(\phi)$ lies in the holomorphic subbundle X . Following (6.3), this is also true of $\mathrm{Im}(D''\phi)$ and hence by Lemma 6.1,

$\mathrm{Im}(D''\phi)$ also lies in V. Now $\mathrm{Im}(D'D''\phi)$ lies in X since $\partial' C(V) \subset C(X)$, and hence following Lemma 6.2, we deduce that $\mathrm{Im}(D''D'\phi) \subset X$. Repeating this argument with (V,X) replaced by the pair (W,Y) where $W = X^{\perp_p}$ and $Y = V^{\perp_p}$ and interchanging the roles of ∂', ∂'', D' and D'', we find that $\mathrm{Im}(D'D''\phi) \subset Y$. Combining these we see that $\mathrm{Im}(D''D'\phi) \subset X \cap Y = \mathrm{Im}(\phi)$ and hence following (6.5), ϕ is harmonic.

To see that ϕ is Q_p-isotropic, we first of all see that $\mathrm{Im}(D''\phi)$ is contained in V, and claim that $\mathrm{Im}(D''^{\alpha}\phi)$ is contained in V for all $\alpha \geq 1$, since V is ∂''-closed. Similarly, $\mathrm{Im}(D'^{\beta}\phi)$ is contained in W. Since V, W and $\mathrm{Im}(\phi)$ are mutually Q_p-orthogonal, it follows that ϕ is isotropic. Fullness follows by similar considerations to those in [39]. $\square$

To construct a harmonic map $\phi : M \to G_{a_1,a_2}(\mathbb{C}^N_p)$ starting from a full holomorphic map $f : M \to \mathbb{C}P^{N-1}$ with $U(p,q)$ acting on $\mathbb{C}P^{N-1}$, one follows the same line of action in proving Proposition 5.24. Thus we recall the admissible signature pairs (t_1,k_1) and (t_2,k_2) of (5.54) and the fibration π in (5.56). We shall re-introudce the Q_p-polar curve of f as well as the indexing set $A_p^{(a_1,a_2)}f$ as defined prior to Proposition 5.24. Having re-established these, the proof of the following proposition is almost identical to that of Proposition 5.24.

<u>Proposition 6.9</u>

<u>Let</u> $f : M \to \mathbb{C}P^{N-1}$ <u>be a full holomorphic map</u>.

<u>Let</u> $p,q \in \mathbb{N}$, <u>with</u> $p+q = N$, <u>we let</u> $g^{(p)}$ <u>denote the</u>

Q_p-polar of f. Assume that for $r \in \Lambda_p^{(a_1,a_2)}(f)$ and $r+s = N-1$, Q_p has signatures (t_1,k_1) and (t_2,k_2) on the subspaces $f_{r-k}(x)$ and $g_{s-1}^{(p)}(x)$ respectively, for all $x \in M$ in accordance with (5.54), and Q_p is non-degenerate on these subspaces. Then the map defined by

$$(6.9) \qquad \phi(x) = (f_{r-k}(x) + g_{s-1}^{(p)}(x))^{\perp_p}$$
$$= f_{r-k}^{\perp_p}(x) \cap f_r(x)$$

is a Q_p-isotropic harmonic map $\phi : M \to G_{a_1,a_2}(E_p^N)$.

To accomplish an inverse transformation, we consider the associated bundles $\phi'_{(\alpha)}$, $\phi''_{(\beta)}$ of ϕ, which are defined in a similar fashion to (3.12) and (3.13) respectively. We shall again denote their respective limits by $\phi'_{(\infty)}$ and $\phi''_{(\infty)}$ and accordingly, we define the corresponding augmented associated bundles $\tilde{\phi}'_{(\infty)}$ and $\tilde{\phi}''_{(\infty)}$, as in (3.14). We shall impose the extra condition that Q_p is non-degenerate and of constant signature on $\tilde{\phi}''_{(\infty)}$. The bundles $\phi''_{(\infty)}$ and $\tilde{\phi}''_{(\infty)}$ are ∂''-closed and are thus holomorphic subbundles of E_p^N. By Lemma 6.4, we see that $\partial' C(\phi''_{(\infty)}) \subset C(\tilde{\phi}'_{(\infty)})$ and hence we deduce that $(\phi''_{(\infty)}, \tilde{\phi}''_{(\infty)})$ is a ∂'_p-pair.

With the above restrictions we note that $\phi''^{\perp_p}_{(\infty)} \cap \tilde{\phi}''_{(\infty)} = \phi$ and claim that the assignment $\phi \to (\phi''_{(\infty)}, \tilde{\phi}''_{(\infty)})$ is the inverse to (6.8). Moreover, the assignment (6.8) is surjective. (cf. [39, Lemma 4.2]). The following lemma indicates the conditions for which (6.8) is injective.

Lemma 6.10

Let $\phi : M \to G_{a_1,a_2}(E_p^N)$ be a Q_p-isotropic harmonic map with Q_p

<u>non-degenerate and of constant signature on</u> $\overset{\gamma}{\phi}''_{(\infty)}$. <u>Then</u>

a) <u>any</u> ∂'_p<u>-pair</u> (V,X) <u>which constructs</u> ϕ <u>by</u> (6.8), <u>must satisfy</u>

$\phi''_{(\infty)} \subset V \subset \overset{\gamma}{\phi}{}'^{\perp_p}_{(\infty)}$ <u>and</u> $\overset{\gamma}{\phi}''_{(\infty)} \subset X \subset \phi'^{\perp_p}_{(\infty)}$;

b) <u>if</u> ϕ <u>is full, there is precisely one</u> ∂'<u>-pair</u> (V,X) <u>giving</u> ϕ .

<u>Proof</u>

a) In the proof of Theorem 6.8, we have $\mathrm{Im}(D''^{\alpha}\phi) \subset V$ (for all $\alpha \geq 1$) and so $\phi''_{(\infty)} \subset V$. Thus $\overset{\gamma}{\phi}''_{(\infty)} \subset V + \phi = X$. Taking $W = X^{\perp_p}$ and $Y = V^{\perp_p}$, we see by a similar argument, that $\phi'_{(\infty)} \subset W$ and $\overset{\gamma}{\phi}'_{(\infty)} \subset Y$.

b) Following [39], we deduce that $\phi''_{(\infty)} + \overset{\gamma}{\phi}'_{(\infty)} = \overset{\gamma}{\phi}''_{(\infty)} + \phi'_{(\infty)}$, is both a ∂'- and ∂''-closed subbundle of $\underline{\mathbb{C}}^N_p$ and is therefore a constant subbundle. Since ϕ is full, this must equal $\underline{\mathbb{C}}^N_p$.

We deduce that the inclusions of part a) are, in fact, equalities. □

<u>Lemma 6.11</u>

<u>Let</u> (V,X) <u>be a</u> ∂'_p<u>-pair and</u> ϕ <u>the isotropic harmonic map defined by</u> (6.8). <u>Then</u> (V,X) <u>is full if and only if</u> ϕ <u>is full</u>.

<u>Proof</u>

This is similar to [39, Lemma 4.4], once we observe the condition that Q_p is non-degenerate and of constant signature on $\overset{\gamma}{\phi}''_{(\infty)}$.

Before stating our main result, let us remind ourselves that from (5.54), we have

$$\begin{cases} 0 \le k_1, k_2 \le q \ , & k_1 + k_2 = q - a_2 \\ 0 \le t_1, t_2 \le p \ , & t_1 + t_2 = p - a_1 \ . \end{cases}$$

<u>Theorem 6.12</u>

<u>Let</u> M <u>be a Riemann surface and let</u> $x \in M$. <u>For each choice of admissible signature pairs</u> (a_1, a_2) , (t_1, k_1) <u>and</u> (t_2, k_2) <u>satisfying</u> (5.54), <u>the assignment (6.8) in Theorem 6.8 defines a bijective correspondence between the set:</u>

(6.10) $A^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$ = {<u>full</u> ∂'_p-<u>pairs</u> (V,X) <u>of</u> $\mathbb{E}^N_p$ <u>of rank difference</u> $k = a_1 + a_2$, <u>such that</u> Q_p <u>restricted to the subspaces</u> V_x , $X_x^{\perp p}$ <u>and</u> $V_x^{\perp p} \cap X_x$ <u>is non-degenerate with signature</u> (t_1, k_1) , (t_2, k_2) <u>and</u> (a_1, a_2) <u>respectively, for</u> $x \in M$} ,

<u>and the set:</u>

(6.11) $B^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$ = {<u>full</u> Q_p-<u>isotropic harmonic maps</u> $\phi : M \to G_{a_1,a_2}(\mathbb{E}^N_p)$ <u>such that on the subspaces</u> $(\phi''_{(\beta)})_x$ <u>and</u> $(\phi'_{(\alpha)})_x$, Q_p <u>is non-degenerate with signature</u> (t_1, k_1) <u>and</u> (t_2, k_2) <u>respectively, for</u> $\alpha, \beta \ge 1$} .

Proof

We consider Theorem 6.8 supplemented by the above signature conditions. This is combined with the fact that $(\phi''_{(\infty)}, \tilde{\phi}''_{(\infty)})$ is a ∂'_p-pair such that $\phi = \phi''^{\perp_p}_{(\infty)} \cap \tilde{\phi}''_{(\infty)}$, and the results of Lemmas 6.10 and 6.11. □

6.3 Indefinite real Grassmannians.

Definition 6.13

Given a ∂'_p-pair (V,X) of holomorphic subbundles of $\underline{\mathbb{C}}^N_p$, we shall say that (V,X) satisfies the total Q_p-isotropy condition if

$$(6.12) \qquad \bar{X} = V^{\perp_p} .$$

Let us define the set $\bar{A}^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$ to be the subset of $A^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$ whereby the condition (6.12) satisfied. Accordingly, we shall define the set $\bar{B}^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$ to be the maps in $B^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$ that have their images in the indefinite real Grassmannian $G_{a_1,a_2}(\mathbb{R}^N_p) \cong SO(p,q)/S(O(a_1,a_2) \times O(p-a_1,q-a_2))$.

Although it is possible to state the following classification result in terms of V alone, as was the case in Section 3.3, we shall restrict

matters to a statement of the result in terms of ∂'_p-pairs (V,X), for the sake of much simplicity.

Corollary 6.14

When the condition (6.12) is satisfied by a full ∂'_p-pair (V,X) in (6.10), the bijection of Theorem 6.12 gives a bijective correspondence between the sets $\bar{A}^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$ and $\bar{B}^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$ for a choice of admissible signature pairs (a_1,a_2), (t_1,k_1) and (t_2,k_2) satisfying (5.54).

6.4 Open orbits in the Hermitian symmetric spaces.

Let Y be an HSS of compact type and D its non-compact dual [48]. As coset spaces of Lie groups, they can be represented as $Y = G^{\mathbb{C}}/P$ and $D = G_0/K$ respectively, where $G^{\mathbb{C}}$ and G_0 are the respective largest connected groups of complex analytic automorphisms. In fact, following [48] and [87]:

1. $G^{\mathbb{C}}$ as a complex Lie group is the complexification of G_0.
2. P is a parabolic subgroup of $G^{\mathbb{C}}$ and Y is a projective algebraic variety.
3. Since $G_0 \subset G^{\mathbb{C}}$, we have $K = G_0 \cap P$ and thus D has a natural embedding as an open G_0-orbit on Y. Every complex analytic automorphism of D extends to such an automorphism of Y.
4. There exists a natural complex Euclidean space $\mathfrak{M}^+ \subset Y$ (notation

of [87]), whose complement is a variety of lower dimension in Y such that $D \subset \mathfrak{M}^+ \subset Y$, and the inclusion of $D \subset \mathfrak{M}^+$ is a canonical realisation of D as a bounded symmetric domain. The classical example is

unit disc $\subset \mathbb{C} \subset \mathbb{C} \cup \{\infty\}$ (the Riemann sphere).

We have already seen in Chapter V that for $Y = \mathbb{C}P^n$ and $G_0 \cong U(n,1)$, there are two open G_0-orbits, namely $\mathbb{C}H^n$ $(= D)$ and $\mathbb{C}P^n_{n-1}$. The space $\mathbb{C}P^n_{n-1}$ fibres holomorphically over $\mathbb{C}P^{n-1}$. In general, each G_0 orbit $\neq D$, has the property of fibering holomorphically over its maximal compact subvariety. The general open orbit is endowed with an indefinite Kähler metric; D alone is endowed with a positive definite Kähler metric which is a Bergman metric.

For each of the four cases Y^A $(A = I,\ldots,IV)$ discussed in Chapter III, we shall state the exact number of open orbits and their general realisation. The corresponding non-compact dual D^A is known as <u>a classical domain</u>. Our brief discussion is extracted from the enumeration of the general theory [88], and we shall state freely the relevant facts.

In what follows we shall let $M_{a,b}$ denote the space of complex $a \times b$ matrices, and for $Z \in M_{a,b}$ we shall write Z^* for ${}^t\bar{Z}$.

<u>Type I</u>

Consider $Y^I \cong G_p(\mathbb{C}^{p+q})$ and let $w = \min\{p,q\}$. Here, $G_0 \cong U(p,q)/\{aI_{p+q} : |a| = 1\}$ and for $0 \le t \le w$, there are exactly $(w+1)$ open G_0-orbits $D^I_{p-t,t}$, where

$$D^{I}_{p-t,t} = \{Z \in M_{q,p} : I_p - Z^*Z \text{ has } (p-t) \text{ positive and } t \text{ negative eigenvalues}\}$$
$$\cong U(p,q)/U(p-t,t) \times U(t,q-t) .$$

For $t = 0$, $D^{I}_{p} = D$ is the classical domain which is the non-compact Hermitian space dual to $G_p(\mathbb{C}^{p+q})$. Full, Q_p-isotropic harmonic maps $M \to D^{I}_{p-t,t}$ satisfying the indicated conditions, are classified by Theorem 6.12, on setting $N = p+q$, $a_1 = p-t$ and $a_2 = t$.

<u>Type II</u>

Here $Y^{II} \subset G_n(\mathbb{C}^{2n})$ and $G_0 \cong SO^*(2n)/\{\pm I_{2n}\}$. For $w = [n/2]$, there are exactly $(w+1)$ open G_0-orbits $D^{II}_{n-2t,2t}$, $0 \le t \le w$, where

$$D^{II}_{n-2t,2t} = \{Z \in M_{n,n} : {}^tZ = -Z \text{ and } I_n - Z^*Z \text{ has } n-2t \text{ positive and } 2t \text{ negative eigenvalues}\}$$
$$\cong SO^*(2n)/U(n-2t,2t) .$$

<u>Type III</u>

Here $Y^{III} \subset G_n(\mathbb{C}^{2n})$ and $G_0 \cong Sp(n,\mathbb{R})/\{\pm I_{2n}\}$. There are exactly $(n+1)$ open G_0-orbits $D^{III}_{n-t,t}$, where

$$D^{III}_{n-t,t} \cong \{Z \in M_{n,n} : {}^tZ = Z \text{ and } I_n - Z^*Z \text{ has } n-t \text{ positive and } t \text{ negative eigenvalues}\}$$
$$\cong Sp(n,\mathbb{R})/U(n-t,t) .$$

The discussion of Section 3.5 equally applies to these indefinite cases which are associated to Y^{II} and Y^{III} respectively. We thus pose the analogous problem of finding full, ∂'_n-pairs (V,X) of rank difference n where V and X are holomorphic subbundles of $\underline{\mathbb{C}}^{2n}_n$, such that

a) V is isotropic for $J'(J'')$

b) $X^{\perp_n}$ is isotropic for $J'(J'')$

c) $J'(V,X^{\perp_n}) = 0$ $(J''(V,X^{\perp_n}) = 0)$.

<u>Type IV</u>

We have $Y^{IV} \cong G_2^0(\mathbb{R}^{n+2})$ and $G_0 \cong SO(n,2)$. There are exactly 3 open G_0-orbits: $D^{IV} = D^{IV}_{0,n} \cong SO(2,n)/SO(2) \times SO(n)$, $D^{IV}_{1,n-1} \cong SO(2,n)/SO(1,1) \times SO(1,n-1)$ and $D^{IV}_{2,n-2} \cong SO(2,n)/SO(2) \times SO(2,n-2)$. The union $D^{IV} \cup D^{IV}_{2,n-2}$ is realised as the subdomain of $\mathbb{C}^n$ $\{z \in \mathbb{C}^n : 1 + |{}^t z.z| - 2z^*z >> 0\}$ and similarly, $D^{IV}_{1,n-1}$ is realised as $\{z \in \mathbb{C}^n : 1 + |{}^t z.z| - 2z^*z << 0\}$. For $0 \le t \le 2$, we denote a general open orbit by $D^{IV}_{t,n-t}$.

Now let us replace $G_{a_1,a_2}(\mathbb{R}^N_p)$ in the set $\bar{B}^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$, by the indefinite real oriented Grassmannian $G^0_{a_1,a_2}(\mathbb{R}^N_p) \cong SO(p,q)/SO(a_1,a_2) \times SO(p-a_1,q-a_2)$ and denote the resulting set by $\overline{OB}^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$. Then by analogous considerations to those made in Section 3.4, we can extend Corollary 6.14 to yield a 2:1

correspondence between the sets $\overline{OB}^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$ and $\bar{A}^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$:

<u>Corollary 6.15</u>

<u>There exists a 2:1 correspondence between the sets</u>

$\overline{OB}^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$ <u>and</u> $\bar{A}^{(a_1,a_2)}_{p(t_1,k_1)(t_2,k_2)}$ <u>for each choice of admissible signature pairs</u> (a_1,a_2) (t_1,k_1) (t_2,k_2) <u>satisfying</u> (5.54).

Thus for $p = 2$, $q = n$, $a_1 = 2-t$ and $a_2 = t$, $0 \le t \le 2$, the above corollary classifies full, Q_2-isotropic harmonic maps $\phi : M \to D^{IV}_{t,n-t}$ satisfying the indicated conditions in the set

$\overline{OB}^{(2-t,t)}_{2(t_1,k_1)(t_2,k_2)}$.

6.5 <u>Indefinite quaternionic Grassmannians</u>.

We may define an indefinite quaternionic Hermitian form on $\mathbb{H}^N$ by

$$(6.13) \qquad Q_p(\mathbb{H})(u,v) = Q_{2\ell}(u,v) + S(u,v)j \qquad u,v \in \mathbb{C}^{2N}$$

where S is a fixed, non-degenerate, anti-symmetric bilinear form on $\mathbb{C}^{2N}$. This gives rise, in this case, to the identification $Sp(p,q) \cong Sp(N,\mathbb{C}) \cap U(2p,2q)$ where $p+q = N$. We shall therefore consider the embedding of $G_{a_1,a_2}(\mathbb{H}^N_p) \cong Sp(p,q)/Sp(a_1,a_2)\times Sp(p-a_1,q-a_2)$ in $G_{2a_1,2a_2}(\mathbb{C}^{2N}_{2p})$, and subject a full ∂'_{2p} pair (V,X) to the condition

$$(6.14) \qquad \sigma X = V^{\perp_{2p}}$$

where σ is defined in (4.3). Thus, let us define the set $\hat{A}^{(2a_1,2a_2)}_{2p(t_1,k_1)(t_2,k_2)}$ to be the subset of $A^{(2a_1,2a_2)}_{2p(t_1,k_1)(t_2,k_2)}$ as defined in accordance with (6.10) (<u>on replacing</u> N , p , a_1 <u>and</u> a_2 <u>by</u> $2N$, $2p$, $2a_1$ <u>and</u> $2a_2$ <u>respectively</u>). whereby the extra condition (6.14) is satisfied by a full ∂'_{2p}-pair (V,X) . Now let $\hat{B}^{(2a_1,2a_2)}_{2p(t_1,k_1)(t_2,k_2)}$ be the set of maps in $B^{(2a_1,2a_2)}_{2p(t_1,k_1)(t_2,k_2)}$ that have their image in the indefinite quaternionic Grassmannian $G_{a_1,a_2}(\mathbb{H}^N_p)$.

As a further corollary to Theorem 6.12, we have:

<u>Corollary 6.16</u>

<u>When condition (6.14) is satisfied by a full</u> ∂'_{2p}-pair (V,X) <u>in (6.10), the bijection of Theorem 6.12 gives a bijective correspondence between the sets</u> $\hat{A}^{(2a_1,2a_2)}_{2p(t_1,k_1)(t_2,k_2)}$ <u>and</u> $\hat{B}^{(2a_1,2a_2)}_{2p(t_1,k_1)(t_1,k_2)}$ <u>for each choice of admissible signature pairs</u> $(2a_1,2a_2)$ (t_1,k_1) <u>and</u> (t_2,k_2) <u>satisfying</u> (5.54).

Setting $a_1 = 0$, $a_2 = 1$, we recover the indefinite quaternionic hyperbolic space $\mathbb{HH}^{p+q-1}_p$ and setting $a_1 = 1$, $a_2 = 0$ we recover the indefinite quaternionic projective space $\mathbb{HP}^{p+q-1}_p$.

<u>Remark</u>

All of the indefinite Kähler manifolds discussed in this chapter are

examples of <u>locally isotropic manifolds</u>, the general theory of which is discussed in [89].

6.6 <u>The indefinite complex projective space</u>.

We now discuss the special case $G_{1,0}(\mathbb{E}^{n+1}_{n-p}) = \mathbb{C}P^n_{n-p}$, introduced in Section 5.2. Recall that $\mathbb{C}P^n_{n-p}$ can be obtained from $\mathbb{C}H^n_p$ by reversing the metric; the map in question $\varepsilon_p : \mathbb{C}H^n_p \to \mathbb{C}P^n_{n-p}$, is an affine diffeomorphism given by

$$(6.15) \qquad \varepsilon_p(z_1,\ldots,z_p,z_{p+1},\ldots z_{p+q}) = (z_{p+1},\ldots,z_{p+q},z_1,\ldots,z_p) .$$

Furthermore, $\phi : M \to \mathbb{C}H^n_p$ is harmonic if and only if $\varepsilon_p \circ \phi : M \to \mathbb{C}P^n_{n-p}$ is harmonic. The corresponding classification theorem for $\mathbb{C}P^n_p = G_{1,0}(\mathbb{E}^{n+1}_p)$ is as follows:

<u>Theorem 6.17</u> [38]

<u>There exists a bijective correspondence between the pairs</u> (f,r) <u>where</u> $f : M \to \mathbb{C}P^n$ <u>is a full holomorphic map with non-empty set</u> $\Lambda^{(1,0)}_{p+1}(f)$ <u>with</u> $r \in \Lambda^{(1,0)}_{p+1}(f)$, <u>and full</u> Q_{p+1} <u>-isotropic harmonic maps</u> $\phi : M \to \mathbb{C}P^n_p$ <u>such that</u> Q_{p+1} <u>is non-degenerate on the subspaces</u> $\phi''_r(x)$ <u>for all</u> $x \in M$, <u>where</u> r <u>is the</u> D'' <u>order of</u> ϕ. <u>This correspondence is given by</u>:

$$(6.16) \qquad \phi(x) = f^{\perp_{p+1}}_{r-1}(x) \cap f_r(x)$$

<u>and conversely, by</u>

$$(6.17) \qquad f(x) = \gamma''^{\perp_{p+1}}_{r-1}(x) \cap \gamma''_r(x) .$$

Remark

1. For $p = n$, we recover the theorem of Eells-Wood [35], as stated in Theorem 2.16.

2. In [38] there was less emphasis on keeping track of the signature of Q_{p+1} on the subspaces $\phi'_s(x)$ and $\phi''_r(x)$.

3. Harmonic maps to the real indefinite projective space are discussed in [37].

To round off this chapter, we shall briefly discuss some cases where Q_{p+1}-isotropy holds for harmonic maps to $\mathbb{C}P^n_p$. In the case of $\mathbb{C}P^n$ as dealt with in [35], the isotropy condition may be related to the degree of ϕ and the topological type of M. Note that in the case of $\mathbb{C}H^n_p$, we have no notion of degree since $\mathbb{C}H^n_p$ is non-compact and contractible. On the other hand, since $\mathbb{C}P^n_p$ was seen to be homotopically equivalent to $\mathbb{C}P^p$ by virtue of (5.7), we may define the degree of a smooth map to $\mathbb{C}P^n_p$, as <u>the degree of the subsequent smooth map to</u> $\mathbb{C}P^p$.

Proposition 6.18

a) <u>Every harmonic map</u> $\phi : S^2 \to \mathbb{C}P^n_p$ <u>is</u> Q_{p+1} -<u>isotropic</u>.

b) <u>Every harmonic map of non-zero degree</u> M_1 (<u>the torus</u>) $\to \mathbb{C}P^n_p$ <u>is</u> Q_{p+1} -<u>isotropic</u>.

Proof

a) Let ϕ be the universal lift of the harmonic map $\phi : M \to \mathbb{C}P^n_p$. On any chart U we set, for $x \in M$,

Remark

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Proof

a) Let ϕ be the universal lift of the harmonic map $\phi : M \to \mathbb{C}P^n_p$. On any chart U we set, for $x \in M$,

(6.18) $\eta_{\alpha,\beta}(x) = Q_{p+1}(D'^{\alpha}\phi(x), D''^{\beta}\phi(x))$

noting that $\eta_{0,1} \equiv \eta_{1,0}$.

By induction on $\alpha+\beta$, we will show that $\eta_{\alpha,\beta} \equiv 0$, for all $\alpha,\beta \geq 0$, $\alpha+\beta \geq 1$. If $\alpha+\beta = 1$, this is trivial. If $\eta_{\alpha,\beta} \equiv 0$ for $\alpha+\beta \leq \gamma+1$, for some $\gamma \geq 0$, then as in [35, Lemma 7.2], $\eta_{\gamma+1,1}\, dz^{\gamma+2}$ is a holomorphic section of the holomorphic line bundle $T^*(r+2,0)_S{}^2$. But such a holomorphic differential vanishes, since this line bundle is seen to have a negative degree (for the notion of degree of a line bundle, we refer to the relevant section of [42]). Hence we have $\eta_{\gamma+1,1} \equiv 0$. By shifting D' in the definition of $\eta_{\gamma+1,1}$ as in [35], it follows that $\eta_{\alpha,\beta} \equiv 0$, for all $\alpha+\beta = \gamma+2$, completing the induction step.

b) Here we observe that the torus M_1 is conformally equivalent to $\mathbb{C}/\Lambda$ for some lattice Λ . Each $\eta_{\alpha,\beta}$ is a well-defined global function on $\mathbb{C}/\Lambda$. We consider the induction hypothesis:

$$(H_\gamma): \eta_{\alpha,\beta} = 0 , \quad \text{for all } \alpha,\beta \geq 0 , \quad 1 \leq \alpha+\beta \leq \gamma$$

and $\eta_{\gamma,1}$ is constant.

Before proceeding, we consider the following result:

Let γ be an integer, $1 \leq \gamma < \infty$, and suppose that $\eta_{\alpha,\beta} \equiv 0$, for

all $\alpha,\beta \geq 0$ with $1 \leq \alpha+\beta \leq \gamma$. Then

$$(6.19) \qquad \partial''\eta_{\gamma+1,1}(x) = -(\gamma+1)F(x)\eta_{\gamma,1}(x)$$

for all $x \in U$, where $F(x)$ is defined as in Lemma 5.5, on replacing p by $p+1$. The proof is essentially that of [35, Lemma 7.2].

Thus, for $\gamma = 1$, (H_1) is true, since by the above result, $\partial''\eta_{1,1} \equiv 0$. This implies that $\eta_{1,1}$ is a holomorphic function on M_1 and is therefore constant.

Suppose now (H_γ) is true. Integrating (6.19) over M_1 , yields

$$0 = -(\gamma+1)\eta_{\gamma,1}\int_{M_1} F(x)\omega^M(x) .$$

The above integral may be seen to be a non-zero multiple of the degree of ϕ as characterised above. Thus if $\deg \phi \neq 0$, then we have $\eta_{\gamma,1} \equiv 0$. On shifting D' in (6.18), we see that $\eta_{\alpha,\beta} = 0$ for all $\alpha,\beta \geq 0$, $\alpha+\beta = \gamma+1$. But by the above result, $\partial''\eta_{\gamma+1,1} \equiv 0$, so that $\eta_{\gamma+1,1}$ is a holomorphic function on $M_1 \cong \mathbb{E}/\Lambda$. This is therefore constant, and thus $(H_{\gamma+1})$ is true, completing the induction argument.

□

CHAPTER VII

ON THE GENERALISED RIEMANN-HURWITZ FORMULA

7.1 Introduction.

This chapter is to be regarded as an independent section of the thesis.

Let M and N be compact Riemann surfaces and let $\phi : M \to N$ be a surjective holomorphic map, then the classical Riemann-Hurwitz formula [42] [93] says that

$$X(M) = \deg \phi . X(N) - r$$

where X denotes the Euler characteristic, $\deg \phi$ is the degree of ϕ (= no. of points in $\phi^{-1}(y)$, $y \in N$) and r is the ramification index (= the sum of the orders of the points of ramification). We refer to [42] for a detailed account (see also Section 7.4).

Our main objective is to establish, in the smooth category, a generalised statement of the above formula (Theorem 7.1, see also [70, Theorem 1]). Our study involves an exposition of the 'clutching' techniques as utilised by Ngô Van Quê in [70]. His approach to the main result entails working explicitly with Chern classes and his theorem is stated in terms of these. However, he does remark that his results

can also be stated for any characteristic class representable by curvature forms of a connection, and the purpose of this exposition is to show that this is indeed the case.

Nearly all of the technical constructions described are due to Ngô Van Quê. In order to put his results in a more general setting, we introduce

a) the notion of the characteristic ring of a principal G-bundle, and

b) the Gysin sequence.

We state the main result as a statement in the characteristic ring of two principal bundles E and F* whose structural group G is a compact connected Lie group, and show that the theorem is true for any of the Chern, Pontrjagin and Euler classes.

Henceforth, we assume some familiarity with the fundamental elements of algebraic topology; in particular, we grant ourselves free use of the basic terminology of homology and cohomology theory. References [50] [52] and [83] are suggested for this purpose; for the theory of characteristic classes and related topics, [24] [50] and [67] are suggested.

7.2 Preliminaries.

a) The 'clutching' technique

The 'clutching' technique is explained in detail in [3] [52] and

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* It is to be assumed that E and F are both with a connection.

[55]. We shall outline this technique following [55]:

Let $\{U_i\}$ be an open cover of a topological space M with $E_i \to U_i$ a (real or complex) vector bundle over each U_i. Let

$$(7.1) \qquad h_{ji} : E_i \,|_{U_i \cap U_j} \to E_j \,|_{U_i \cap U_j}$$

be isomorphisms satisfying the 'cocycle' condition $h_{ki}\,|_V = h'_{kj} \cdot h'_{ji}$ where $V = U_i \cap U_j \cap U_k$ and $h'_{kj} = h_{kj}\,|_V$, $h'_{ji} = h_{ji}\,|_V$. Then there exists a bundle E over M and isomorphisms $h_i : E_i \to E|_{U_i}$ such that the following diagram commutes

$$\begin{tikzcd}
E_i\,|_{U_i \cap U_j} \arrow[rr, "h_{ji}"] \arrow[dr, "h_i\,|_{U_i \cap U_j}"'] & & E_j\,|_{U_i \cap U_j} \arrow[dl, "h_j\,|_{U_i \cap U_j}"] \\
& E\,|_{U_i \cap U_j} &
\end{tikzcd}$$

The h_{ji} are usually referred to as '<u>clutching' functions</u>. Moreover, the above bundle is unique in the sense that if E' is another such bundle, and $h'_i : E_i \to E'|_{U_i}$ are isomorphisms for which the following commutes,

$$\begin{tikzcd}
E_i\,|_{U_i \cap U_j} \arrow[rr, "h_{ji}"] \arrow[dr, "h'_i\,|_{U_i \cap U_j}"'] & & E_j\,|_{U_i \cap U_j} \arrow[dl, "h'_j\,|_{U_i \cap U_j}"] \\
& E'\,|_{U_i \cap U_j} &
\end{tikzcd}$$

then there exists a unique isomorphism $\alpha : E \to E'$ such that the following diagram commutes:

$$\begin{tikzcd} & E_i \arrow[dl, "h_i"'] \arrow[dr, "h_i'"] & \\ E|_{U_i} \arrow[rr, "\alpha|_{U_i}"'] & & E'|_{U_i} \end{tikzcd} .$$

b) <u>Transgression in a principal bundle</u>

The following discussion originates from [11]:

Let M be a smooth, n-dimensional manifold and E a principal bundle on M with structural group G (otherwise said, E is a principal G-bundle). A principal G-bundle E is said to be <u>n-universal</u> (for G) if its homology (or homotopy) groups vanish up to dimension n (except for H_0) . These exist for any compact Lie group and any n , including $n = \infty$. We shall omit n in what follows.

The base space B_G of the universal bundle E_G for G , is called <u>a classifying space for G</u> . It is known that all classes of G-bundle structures with a given base space M , are in 1-1 correspondence with the homotopy classes of maps $f:M \to B_G$. Thus to any G-bundle over M , there exists a corresponding homomorphism

$$(7.2) \qquad f^* : H^*(B_G,\Lambda) \to H^*(M,\Lambda)$$

known as the <u>characteristic map</u> of the fibering, the image of which is the <u>characteristic ring</u>, where Λ denotes the ring of coefficients.

The characteristic classes of Chern, Pontrjagin and Stiefel-Whitney, appear as images under f^* of particular elements in $H^*(B_{U(n)}, \mathbb{Z})$, $H^*(B_{SO(n)}, \mathbb{Z})$ and $H^*(B_{O(n)}, \mathbb{Z}_2)$ respectively. We regard $H^*(B_G, \Lambda)$ as the rings of 'universal' characteristic classes for G-bundles.

Let E be a bundle with typical fibre E_z over M $(z \in M)$, with projection $p: E \to M$. Then from [11] [84], we have the following maps in cohomology:

$$(7.3) \qquad H^q(M,\Lambda) \xrightarrow{p^*} H^q(E,\Lambda) \xleftarrow{\delta} H^{q-1}(E_z,\Lambda), \qquad 0 \le q \le n .$$

We say that $x \in H^{q-1}(E_z,\Lambda)$ is <u>transgressive</u> in E if $\delta x \in \mathrm{Im}\, p^*$.

The cohomology class $y \in H^q(M,\Lambda)$ is determined by x modulo a certain subgroup $L(q)$ say, and the <u>transgression</u> is the map of the transgressive elements of $H^{q-1}(E_z,\Lambda)$ into $H^q(M,\Lambda)/L(q)$ derived from $x \to y$. We write $y = \tau(x)$ whenever y is obtained via transgression from x.

Let G be a compact connected Lie group. Then $x \in H^{q-1}(G,\Lambda)$ is <u>universally transgressive</u> if it is transgressive in E_G. The above classification theorem says that it is then transgressive in all G-bundles.

Let x_i, $i = 1,\ldots,n$ be a system of transgressive elements, and let $y_i = \tau(x_i)$; then generally we have

(7.4) $\quad H^*(B_G,\Lambda) = \Lambda[y_1,\ldots,y_n]$.

For example, $H^*(B_{U(n)}, \mathbb{Z})$ is the ring of polynomials in the Chern classes $c_i(E)$ and $H^*(U(n), \mathbb{Z}) = \mathbb{Z}[x_1,\ldots,x_n]$, where by transgression $\tau(x_i) = c_i$, i.e. $H^*(B_{\bar{G}}, \mathbb{Z}) = \mathbb{Z}[c_1,\ldots,c_n]$.

For a given E and corresponding polynomial $P_E \in \Lambda[y_1,\ldots,y_n]$, we shall be interested in the image f^*P_E in $H^*(M,\Lambda)$. We shall denote by $f^*P_E[M]$ (or sometimes, $\langle f^*P_E,[M]\rangle$, which is more common), the <u>evaluation</u> of the cohomology class f^*P_E on the fundamental homology class of M (in de Rham cohomology this is explicitly represented by the integration of a <u>representative</u> form over a representative cycle). This evaluation is known as <u>the characteristic number</u> associated with f^*P_E .

c) <u>The Weil homomorphism</u>

Let E be a complex vector bundle of rank n on a smooth compact orientable manifold M of dimension m . We shall denote by T_c^* the bundle of complex 1-forms on M . Let ∇ be an Hermitian connection on E , i.e. ∇ is a linear differential operator

$$\nabla : \Lambda^p T_c^* \otimes E \to \Lambda^{p+1} T_c^* \otimes E$$

(sections understood), given by

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla(s)$$

for every complex p-form ω and section s of E on M. The curvature operator

$$\Theta = \nabla o \nabla : E \to \Lambda^2 T_c^* \otimes E$$

is a section on M of $\Lambda^2 T_c^* \otimes \mathrm{End}(E)$. Regarding Θ as a matrix of 2-forms, we define following [24]:

$$P(\Theta) = \det(I + \frac{\sqrt{-1}}{2\pi}\Theta) = 1 + P_1(\frac{\sqrt{-1}}{2\pi}\Theta) + \ldots + P_n(\frac{\sqrt{-1}}{2\pi}\Theta)$$

where for $1 \le i \le n$, $P_i(\frac{\sqrt{-1}}{2\pi}\Theta)$ is an <u>adjoint invariant polynomial</u>. For each i, we set $c_i(\Theta) = P_i(\frac{\sqrt{-1}}{2\pi}\Theta)$; $c_i(\Theta)$ is a real closed 2i-form, known as the <u>i^{th} Chern form</u> of the curvature Θ in E, and whose corresponding cohomology class it represents, is the ith Chern class $c_i(E) \in H^{2i}(M, \mathbb{Z})$.

Generally, if G is a connected Lie group, then let $I(G)$ denote the ring of adjoint invariant polynomials. By setting $w(P) = \{P(\Theta)\}$, where the right hand side denotes the cohomology class represented by the closed form $P(\Theta)$, for some $P \in I(G)$, we define a mapping

$$w : I(G) \longrightarrow H^*_{dR}(M, \mathbb{R})$$

($H^*_{dR}(M, \mathbb{R})$ denotes the de Rham cohomology ring of M) which is a ring homomorphism known as the <u>Weil homomorphism</u> (we refer to [24] for further details.

When G is a compact connected Lie group, we have the following commutative diagram

$$\begin{tikzcd} I(G) \arrow[rr, "w"] \arrow[dr, "w_0"'] & & H^*(M, \mathbb{R}) \\ & H^*(B_G, \mathbb{R}) \arrow[ur, "f^*"'] & \end{tikzcd}$$

where w is the Weil homomorphism and w_0 is an isomorphism. This tells us that the invariant polynomials can be identified with the cohomology classes of the classifying spaces and the Weil homomorphism gives rise to the representatives of the characteristic classes in question, by closed differential forms constructed from the curvature forms of a connection.

7.3 The generalised Riemann-Hurwitz formula.

We now proceed to discuss the generalised Riemann-Hurwitz formula as proved in [70]. But rather than always work with a particular characteristic class, we draw on the generalities and terminology of the last section, to describe the results as statements in the characteristic ring.

Let G be a compact connected Lie group and let E and F be two principal G-bundles of the same rank over a smooth compact orientable manifold M of dimension m, together with a bundle morphism

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr] & & F \arrow[dl] \\ & M & \end{tikzcd}$$

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$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr] & & F \arrow[dl] \\ & M & \end{tikzcd}$$

Let M_1 be a smooth closed submanifold of codimension r in M, and assume that $\psi : E|_{M-M_1} \xrightarrow{\cong} F|_{M-M_1}$ is an isomorphism.

Let $B(M_1)$ be a closed tubular neighbourhood of M_1 in M. We form its 'double' $S(M_1)$ by taking two distinct copies $B_i(M_1)$, $i = 1,2$ of $B(M_1)$ and identifying them along the common boundary (cf. [49]), i.e.

$$(7.5) \qquad S(M_1) = B_1(M_1) \cup_{\partial B(M_1)} B_2(M_1) \quad .$$

$S(M_1)$ is a smooth compact manifold without boundary of dimension m, and by this construction, $S(M_1) \to M_1$ is an r-sphere bundle on M_1.

On $S(M_1)$ there exists an essentially unique bundle ξ, which we write as (E,ψ,F), constructed, by taking respectively on $B_1(M_1)$ and $B_2(M_1)$, the restriction of E and F to $B(M_1)$, and identifying them on $B_1(M_1) \cap B_2(M_1) = \partial B(M_1)$, via the transition isomorphism ψ which we regard as a 'clutching' function [3] (see also Section 7.2 a)).

Consider now the case where $G = U(n)$ and E and F are complex vector bundles of the same rank n. Let ∇_1 be an Hermitian connection on E and let $B'(M_1)$ be a tubular neighbourhood of M_1 contained in the interior of $B(M_1)$. By using a partition of unity, Ngô Van Quê defines an Hermitian connection ∇_2 on F, such that on $M-B'(M_1)$, there is the following commutative diagram of sheaves of sections (notation as in [70]):

$$\begin{tikzcd} E \arrow[r, "\nabla_1"] \arrow[d, "\psi"'] & T_c^* \otimes E \arrow[d, "Id\otimes\psi"] \\ F \arrow[r, "\nabla_2"] & T_c^* \otimes F \end{tikzcd} .$$

Letting Θ_1 and Θ_2 be the curvature forms of ∇_1 and ∇_2 respectively, he deduces that

$$c_i(\Theta_1)\Big|_{M-B(M_1)} = c_i(\Theta_2)\Big|_{M-B(M_1)} \qquad 1 \le i \le n .$$

With some abuse of notation, we have $c_i(E)\Big|_{M-B(M_1)} = c_i(F)\Big|_{M-B(M_1)}$, and in particular, $\langle c_i(E),[M-B(M_1)]\rangle = \langle c_i(F),[M-B(M_1)]\rangle$.

For the principal G-bundles E and F with which we are concerned, and satisfying the indicated conditions, the above construction (with the necessary modifications) can be similarly described for each case. Generally, if Θ_1 and Θ_2 are the curvature forms of the corresponding connections ∇_1 and ∇_2 respectively on E and F respectively, we have, for some $P_i \in I(G)$

$$P_i(\Theta_1)\Big|_{M-B(M_1)} = P_i(\Theta_2)\Big|_{M-B(M_1)} .$$

Thus if $y_i(E)$ and $y_i(F)$ denote the characteristic classes represented by $P_i(\Theta_1)$ and $P_i(\Theta_2)$ respectively, we have

$$(7.6) \qquad y_i(E)\Big|_{M-B(M_1)} = y_i(F)\Big|_{M-B(M_1)}$$

We now proceed to the main result:

Theorem 7.1 (The generalised Riemann-Hurwitz formula, see also [70])

Let G be a compact connected Lie group. Let M be a smooth compact orientable manifold and let M_1 be a smooth closed submanifold of codimension r in M. Let E and F be two principal G-bundles of the same rank on M, equipped with a bundle morphism $\Phi : E \to F$, such that restricted to $M-M_1$, Φ is an isomorphism. Let $S(M_1)$ be the smooth compact manifold obtained by (7.5). Then if $\xi = (E,\psi,F)$ is taken to denote the bundle obtained by the above clutching construction on $S(M_1)$, we have

$$(7.7) \qquad (f^*P_E - f^*P_F)[M] = f^*P_\xi[S(M_1)]$$

where f^*P_E and f^*P_F are defined as in Section 7.1(b).

Proof

For $y_i(E) \in f^*P_E$, $y_i(F) \in f^*P_F$, we have from (7.6)

$$y_i(E)\Big|_{M-B(M_1)} = y_i(F)\Big|_{M-B(M_1)} \qquad (1 \le i \le \text{Rank } E)$$

where $y_i(E)$ and $y_i(F)$ may be regarded as obtained, via transgression, from the transgressive elements $x_i(E) \in H^{i-1}(E_z,\Lambda)$, $x_i(F) \in H^{i-1}(F_z,\Lambda)$ respectively, for $z \in M$.

Since by the clutching construction, we have

$$\xi|_{B_1(M_1)} \cong E|_{B(M_1)} \quad \text{and} \quad \xi|_{B_2(M_1)} \cong F|_{B(M_1)}$$

then
$$x_i(\xi)|_{B_1(M_1)} = x_i(E)|_{B(M_1)}$$
$$x_i(\xi)|_{B_2(M_1)} = x_i(F)|_{B(M_1)} \quad .$$

Thus via transgression, we have

$$(7.8) \quad \begin{cases} y_i(\xi)|_{B_1(M_1)} = y_i(E)|_{B(M_1)} \\ y_i(\xi)|_{B_2(M_1)} = y_i(F)|_{B(M_1)} \end{cases} ,$$

for $y_i(\xi) \in f^*P_\xi$.

Now,

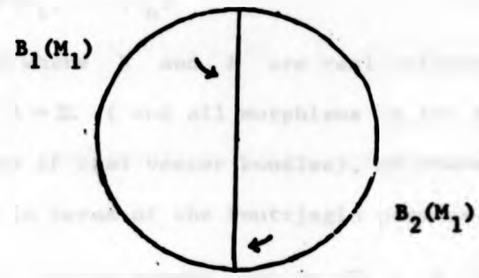
$$(f^*P_E - f^*P_F)[M] = f^*P_E[M-B(M_1)] + f^*P_E[B(M_1)]$$
$$- f^*P_F[M-B(M_1)] - f^*P_F[B(M_1)] \quad .$$

But by (7.6), we have $f^*P_E[M-B(M_1)] = f^*P_F[M-B(M_1)]$, hence

$$(f^*P_E - f^*P_F)[M] = f^*P_E[B(M_1)] - f^*P_F[B(M_1)]$$
$$= (f^*P_E - f^*P_F)[B(M_1)] \quad .$$

Following the restrictions of E and F to $B(M_1)$, the second member may be written as:

$$f^*P_E[B_1(M_1)] - f^*P_F[B_2(M_1)] = f^*P_E[B_1(M_1)] + f^*P_F[-B_2(M_1)] \ .$$



Now from (7.8) and by the way that the two copies of $B(M_1)$ are 'glued' together, the last expression can be seen to be none other than $f^*P_\xi[S(M_1)]$, where the minus sign arises from the orientation. □

<u>Example 7.2</u>

Let $G = U(n)$ and $\Lambda = \mathbb{Z}$. Let Θ_1 and Θ_2 be the respective curvature froms of the Hermitian connections ∇_1 and ∇_2 on E and F respectively; let $c_i(\Theta_1)$ and $c_i(\Theta_2)$ be the respective i^{th} Chern forms. Then as before, we have $c_i(\Theta_1) = c_i(\Theta_2)$ on $M-B(M_1)$.
Via the clutching construction on $S(M_1)$, an Hermitian connection ∇ is induced on $\xi = (E,\psi,F)$ from ∇_1 and ∇_2, such that

$$\text{on} \quad \xi|_{B_1(M_1)} \cong E|_{B(M_1)} \ , \quad \nabla = \nabla_1$$

$$\text{and on} \quad \xi|_{B_2(M_1)} \cong F|_{B(M_1)} \ , \quad \nabla = \nabla_2 \ .$$

In this case, (7.7) becomes

$$(7.9)\quad \int_M P(c_1(\nabla_1),\ldots,c_n(\nabla_1)) - \int_M P(c_1(\nabla_2),\ldots,c_n(\nabla_2)) = \int_{S(M_1)} P(c_1(\nabla),\ldots,c_n(\nabla))$$

where $P(\ ) \in \mathbb{Z}[c_1,\ldots,c_n]$.

In the case where E and F are real oriented bundles with $G = SO(n)$ and $\Lambda = \mathbb{Z}$ (and all morphisms in the above constructions are taken to be those of real vector bundles), an analogous result to (7.9) can be obtained in terms of the Pontrjagin classes of E and F .

As in [70], let us assume that on M_1 , ψ has constant rank, which implies that on M_1 , we have the following exact sequence of bundles

$$0 \to K_1 \to E|_{M_1} \xrightarrow{\psi} F|_{M_1} \to K_2 \to 0$$

where $K_1 \cong \ker \psi$ and $K_2 \cong \operatorname{coker} \psi$ (see e.g. [52]).

Let $L = \psi(E|_{M_1}) \subset F|_{M_1}$; then we have the isomorphisms

$$E|_{M_1} \cong K_1 \oplus L$$

$$F|_{M_1} \cong L \oplus K_2 \ .$$

Consider now the fibrations $\pi : S(M_1) \to M_1$ and $\pi|_{B_i(M_1)} = \pi_i : B_i(M_1) \to M_1$, $i = 1,2$. The submanifold M_1 is seen to be a deformation retract of $B(M_1)$ [70]; we thus have:

$$(7.10)\quad \begin{cases} E|_{B_1(M_1)} \cong \pi_1^* K_1 \oplus \pi_1^* L \\ F|_{B_2(M_1)} \cong \pi_2^* K_2 \oplus \pi_2^* L \end{cases}$$

(we use π^* to denote the pull-back by π) . The image under $\psi : E|_{B(M_1)} \to F|_{B(M_1)}$ of the subbundle $\pi_1^* L$ of $E|_{B(M_1)}$, is a subbundle of $F|_{B(M_1)}$ isomorphic to $\pi_2^* L$ and whose quotient bundle is isomorphic to $\pi_2^* K_2$.

We can assume that modulo the isomorphisms of (7.10), the isomorphism ψ is of the form

$$
(7.11) \qquad
\begin{array}{ccc}
\psi : E|_{\partial B(M_1)} & \longrightarrow & F|_{\partial B(M_1)} \\
\downarrow \mathfrak{n} & & \downarrow \mathfrak{n} \\
\pi_1^* K_1 \oplus \pi_1^* L & \longrightarrow & \pi_2^* K_2 \oplus \pi_2^* L \\
(x,y) & \longrightarrow & (\eta(x),y)
\end{array}
$$

where η is an isomorphism of vector bundles

$$
\eta : \pi_1^* K_1|_{\partial B_1(M_1)} \xrightarrow{\cong} \pi_2^* K_2|_{\partial B_2(M_1)} .
$$

We thus obtain the following lemma:

<u>Lemma 7.3</u> [70]

<u>On</u> $S(M_1)$ <u>there is a vector bundle isomorphism</u>

$$
(7.12) \qquad \xi = (E,\psi,F) \cong \pi^* L \oplus (\pi_1^* K_1, \eta, \pi_2^* K_2)
$$

<u>where</u> $\pi^* L$ <u>is the bundle induced by</u> π <u>on</u> $S(M_1)$ <u>by the fibration</u> $\pi : S(M_1) \to M_1$, <u>and</u> $(\pi_1^* K_1, \eta, \pi_2^* K_2)$ <u>is the bundle constructed on</u> $S(M_1) = B_1(M_1) \underset{\partial B(M_1)}{\cup} B_2(M_1)$ <u>by a certain transition function</u> η .

For simplicity, let us denote by K, the new 'clutched' bundle $(\pi_1^*K_1, \eta, \pi_2^*K_2)$. We wish to establish the relationship

$$(f^*P_K \cup f^*P_{\pi^*L})[S(M_1)] = k(f^*P_L)[M_1]$$

where M_1 in this case, is a <u>connected</u>, closed, orientable submanifold of M of codimension r, and where k is a constant. Before doing so, we introduce:

<u>Proposition 7.4</u> [50] [67] <u>The Gysin sequence</u>.

<u>Let</u> $\pi : V \to M$ <u>be an oriented sphere bundle with fibre</u> S^r. <u>Then there exists a long exact sequence</u>

$$(7.13) \qquad \ldots \to H^i(V) \xrightarrow{\pi_*} H^{i-r}(M) \xrightarrow{U e} H^{i+1}(M) \xrightarrow{\pi^*} H^{i+1}(V) \quad \ldots$$

<u>for which the maps</u> π_*, Ue <u>and</u> π^* <u>are integration along the fibre, cup product with the Euler class and the natural pull back respectively</u>.

<u>Proposition 7.5</u>

<u>Let</u> M_1 (<u>connected</u>), K <u>and</u> L <u>be as above, then</u>:

$$(7.14) \qquad (f^*P_K \cup f^*P_{\pi^*L})[S(M_1)] = k(f^*P_L)[M_1]$$

<u>where</u> k <u>is a constant</u>.

<u>Proof</u>

Let $\alpha \in f^*P_L$ and $\beta \in f^*P_K$. Then if π_* and π^* are the maps

in (7.13), we have, by fibre integration along $S(M_1)|_{x\in M_1}$, $\pi_*(\pi^*(\alpha)\cup\beta) = \alpha\cup\pi_*(\beta)$, which on integrating over M_1 yields:

$$\langle f^*P_L\langle f^*P_K,[S(M_1)_x]\rangle,[M_1]\rangle \quad .$$

Now $\langle f^*P_K,[S(M_1)_x]\rangle = f^*P_{K_x}[S(M_1)_x]$

where $K_x = (\pi_1^*K_1|_{B_1(M_1)_x}, \eta_x, \pi_2^*K_2|_{B_2(M_1)_x})$ is the bundle over $S(M_1)|_{x\in M_1}$ constructed by means of the transition function η_x this being the restriction of η to $\partial B(M_1)_x$, i.e. we have an isomorphism

$$\eta_x : \pi_1^*K_1|_{\partial B(M_1)_x} \xrightarrow{\cong} \pi_2^*K_2|_{\partial B(M_1)_x} \quad .$$

If $c(x,y)$ is a curve in M_1 joining two points $x,y \in M_1$, then $K_1|_{c(x,y)}$ and $K_2|_{c(x,y)}$ are trivial. We have the following diagram in which the vertical maps are isomorphisms:

$$\begin{array}{ccc}
\pi_1^*K_1|_{\partial B(M_1)_x} & \xrightarrow{\eta_x} & \pi_2^*K_2|_{\partial B(M_1)_x} \\
\downarrow \cong & & \downarrow \cong \\
\pi_1^*K_1|_{\partial B(M_1)_y} & \xrightarrow{\eta_y} & \pi_2^*K_2|_{\partial B(M_1)_y}
\end{array}$$

and modulo these isomorphisms, η_x and η_y are homotopic. Thus K_x and K_y, regarded as bundles on $S^r \cong S(M_1)|_x \cong S(M_1)|_y$, are isomorphic. By connectedness of M_1, this implies that $f^*P_{K_x}[S(M_1)_x]$ is a constant, k say, independent of x, whence (7.14) follows. □

Proposition 7.6

Under the hypotheses of Theorem 8.1 and Proposition 7.5, we have

$$(7.15) \qquad f^*P_E[M] - f^*P_F[M] = kf^*P_L[M_1] \quad .$$

Proof

This follows from the isomorphism in (7.12), and the combination of (7.6) with (7.14). □

Let us now consider some particular cases. Firstly, let E and F be taken to be complex $U(n)$-bundles on M, with $\Lambda = \mathbb{Z}$. Here we set $f^*P_E = c_n(E)$ the nth (top) Chern class of E, which is the same as the Euler class $\chi(E)$ of E [67]. When $m = 2n$, we write $\langle c_n(E),[M]\rangle = c_n(E)[M] = \chi(E)[M]$.

For this case, we obtain the generalised Riemann-Hurwitz formula in the form

$$(7.16) \qquad \chi(E)[M] - \chi(F)[M] = k\chi(L)[M_1] \quad ,$$

where the rank of L is half the dimension of M_1 (taken to be even dimensional) [70].

For $G = SO(2n)$, $\Lambda = \mathbb{Z}$, the same formula holds, but in this case, L is assumed orientable, and to have rank equal to the dimension of M_1 .

7.4 Smooth branched covering maps.

Consider a C^1 map $\phi : M \to N$ where M and N are two smooth compact

orientable manifolds of the same dimension m . <u>The degree of</u> ϕ (written $\deg \phi$) may be regarded as the number of pre-images $x \in M$ of a regular value of N counted positively or negatively according to whether the map ϕ preserves or reverses orientation at x (see e.g. [49]). Alternatively, if we consider the m dimensional integral homology of M and N , then this is canonically isomorphic to $\mathbb{Z}$ and we have an induced map ϕ_* in m-dimensional homology

$$\begin{array}{ccc} \phi_*: H_m(M) & \longrightarrow & H_m(N) \\ \cong & & \cong \\ \mathbb{Z} & & \mathbb{Z} \end{array} .$$

Since any map $\mathbb{Z}$ to $\mathbb{Z}$ may be characterised by multiplication by some integer, we define $\deg \phi$ as the integer associated to ϕ_* .

<u>Definition 7.7</u>

Let $\phi: M \to N$ be a smooth surjective map of two compact orientable manifolds M and N of the same dimension. Let N_1 be a closed connected orientable submanifold of N of codimension 2 . We say that ϕ is a <u>smooth branched covering</u> if (restrictions understood)

i) $\phi: M - \phi^{-1}N_1 \to N - N_1$

is a submersion, preserving orientation and $M - \phi^{-1}(N_1)$ is an ℓ-sheeted connected covering of $N - N_1$, where $\ell = \deg \phi$.

ii) $M_1 = \phi^{-1}(N_1)$ is a closed connected submanifold of M of codimension 2, and $\phi: M_1 \to N_1$ is a diffeomorphism.

One usually refers to N_1 as <u>the branch set of</u> ϕ (or $M_1 = \phi^{-1}(N_1)$ as <u>the ramification locus of</u> ϕ) .

Consider now the induced morphism

$$\begin{tikzcd} TM \arrow[rr, "d\phi"] \arrow[dr] & & \phi^{-1}TN \arrow[dl] \\ & M & \end{tikzcd}$$

of bundles on M . With reference to the construction of the last section, we follow [70] and set $E = TM$, $F = \phi^{-1}TN$ and $L = \phi^{-1}TN_1$. From (7.16) we obtain

$$(7.17) \qquad \langle X(TM),[M]\rangle - \langle X(\phi^{-1}TN),[M]\rangle$$

$$= k\langle X(\phi^{-1}TN_1),[\phi^{-1}N_1]\rangle$$

which, in terms of Euler characteristics, is written as

$$X(M) - \ell X(N) = kX(N_1) .$$

The constant k is found to be equal to $-(\ell-1)$ [70] (see also [14] and [81] in the holomorphic category). Thus in a more practical form, we obtain the formula

$$(7.18) \qquad X(M) = \ell X(N) - (\ell-1)X(N_1)$$

for a smooth branched covering map. For $m = 2$, (7.18) is the classical Riemann-Hurwitz formula. The formula (7.18) was also obtained in the holomorphic category by Pohl in [73].

Remark

When $M_1 = \phi^{-1}N_1$ is a disjoint union of connected components $B_1,\ldots,B_r$, then for $x \in B_i$, $\ell(x) = \deg\phi|_{B_i}$ is the <u>local degree of</u> ϕ <u>at</u> x. In this case we have $k = -(\sum_{x\in M_1} (\ell(x)-1))$ [14].

7.5 Examples

Example 7.8

Let N be a compact complex manifold. A <u>divisor</u> D on N is a finite formal linear combination $D = \Sigma a_i V_i$ of irreducible analytic hypersurfaces V_i of N. Canonically associated with D is a line bundle which we denote by $|D|$.

The following example, taken from [70], is very typical of such constructions in algebraic geometry: Let D be a non-singular algebraic curve of degree d in $\mathbb{C}P^2$. From the <u>adjunction formula</u> [47] for such a curve in $\mathbb{C}P^2$, we obtain the relationship

$$(7.19) \qquad \chi(D) = -d^2 + 3d .$$

Now $|D|$ is seen to define a line bundle on $\mathbb{C}P^2$ that is isomorphic to $H^d = H \otimes \ldots \otimes H$ (d times) where H is the Hopf bundle with projection $\pi : H \to \mathbb{C}P^2$. Consider now an open covering $\{U_i\}$ of $\mathbb{C}P^2$. Let $\psi_i = 0$ be the defining equation of $D \cap U_i$ in U_i. If ξ_i is taken to be the fibre co-ordinate of $H|_{U_i}$, then we have the trivialisation $H|_{U_i} \cong U_i \times \mathbb{C}$ given by $x \to (\pi(x), \xi_i(x))$. Then in $H|_{U_i}$,

the equation $\psi_i \circ \pi = \xi_i^d$ defines a subvariety $V_i \subseteq H|_{U_i}$. Running across the U_i, one defines, in this way, a non-singular (algebraic) variety V of H and V is a d-sheeted smooth branched covering of $\mathbb{C}P^2$, for which $D \subset \mathbb{C}P^2$ is the branch set. Hence from (7.18), we obtain

$$X(V) - dX(\mathbb{C}P^2) = -(d-1)X(D)$$

$$X(V) - 3d = -(d-1)(3d-d^2) ,$$

hence $$X(V) = d(6-4d+d^2) .$$

<u>Example 7.9</u>

Let M be a K-3 surface (a simply connected, compact 2-dimensional Kähler manifold with $c_1^2 = 0$ and $c_2 = 24$). According to [66], there exists a 2:1 smooth branched covering $M \to \mathbb{C}P^2$, branched over a non-singular curve of degree 6 in $\mathbb{C}P^2$. From (7.19), we see that $X(C) = -18$. Now $X(M) = 24$, and on setting $\ell = 2$, we see that (7.18) is satisfied by this data.

<u>Example 7.10</u>

In [61], Kuiper produced an example of a <u>non</u>-smooth, 2:1 branched covering map $\phi:\mathbb{C}P^2 \to N$, where N is the quotient space of $\mathbb{C}P^2$ under complex conjugation. The smoothness of ϕ is lost on the branch set. Now N is a smoothable 4-manifold, and by considering a particular embedding of $\mathbb{C}P^2$ in $\mathbb{C}^6$, it is shown that N can be realised as the 4-sphere with the branch set $\mathbb{R}P^2$ (non-orientable) suitably embedded.

The data $(\chi(\mathbb{C}P^2) = 3$, $\chi(S^4) = 2$, $\chi(\mathbb{R}P^2) = 1$ and $\ell = 2$) does appear to satisfy (7.18). This seems to suggest that some version of (7.18) may be obtained in the piecewise-smooth category.

7.6 Tangential homotopy type.

The following notion (originally due to Pontrjagin) is taken from [30]:

Definition 7.11

Let M and N be two smooth m-manifolds, then a smooth map $\phi: M \to N$ is said to be a tangential homotopy equivalence, if ϕ can be covered by a bundle map $\overset{\gamma}{\phi}$ of the principal $\tau_{GL(m,\mathbb{R})}$-bundles $P(\ )$ of their tangent bundles:

$$\begin{array}{ccc} P(M) & \xrightarrow{\overset{\gamma}{\phi}} & P(N) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & N \end{array} .$$

We say that M and N have the same tangential homotopy type if there exists a tangential homotopy equivalence $\phi: M \to N$.

Proposition 7.12

Let $\phi: \mathbb{C}P^2 \to N$ be a smooth branched covering map, where N is a compact, simply-connected 4-manifold, and the branch set N_1 of ϕ is a compact Riemann surface. Then N has the tangential homotopy type of $\mathbb{C}P^2$.

Proof

A simple application of the generalised Riemann-Hurwitz formula: As N is compact and simply connected, we have $X(N) = 2 + b_2(N)$, where $b_2(N)$ denotes the second Betti number of N. The branch set N_1 is a compact Riemann surface, and hence $X(N_1) = 2-2p$ where p is the genus of N_1. From (7.18)

$$X(\mathbb{C}P^2) = \ell X(N) - (\ell-1)X(N_1) ,$$

and putting in the above data, we obtain

$$3 = \ell(2 + b_2(N)) + 2(\ell-1)(p-1) .$$

Assuming $\ell = \deg \phi \geq 0$, we see that this last equation only makes sense when $\ell = 1$. Thus we conclude that $b_2(N) = 1$ and hence, $X(N) = X(\mathbb{C}P^2)$. In terms of Euler classes $e(\ )$, this implies

$$e(\phi^{-1}P(N)) = \phi^* e(N) = e(\mathbb{C}P^2) .$$

Following [30], we then have a bundle isomorphism $i: P(\mathbb{C}P^2) \to \phi^{-1}P(N)$ and the required covering bundle map is $\tilde{\phi} = \phi \circ i : P(\mathbb{C}P^2) \to P(N)$.

□

APPENDIX : TOTALLY GEODESIC SUBSPACES OF RIEMANNIAN SYMMETRIC SPACES.

The purpose of this appendix is to outline the proof of Theorem 3.1, and then apply this theorem to two representatives from the examples of totally geodesic embeddings appearing in the thesis.

Let us remind ourselves that we are considering a symmetric space (G,K,θ) and a symmetric subspace (G',K',θ') meaning that G' is a connected Lie group invariant under θ, $K' = G' \cap K$, and $\theta' = \theta|_{G'}$.

Let $(\mathfrak{g},\mathfrak{k},\theta)$ be the corresponding symmetric Lie algebra to (G,K,θ). Since θ is an involution, its eigenvalues as a linear transformation of $\mathfrak{g}$, are 1 and -1, and if $\mathfrak{k}$ and $\mathfrak{p}$ are the respective eigenspaces, then the decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

with $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, is called <u>the canonical decomposition of</u> $(\mathfrak{g},\mathfrak{k},\theta)$.

A word about the canonical connection of $N = G/K$. Theorem 2.1 in [57, p.191] asserts that there exists a bijective correspondence between the set of G-invariant affine connections on G/K, and the set of linear maps $\Lambda_{\mathfrak{p}}:\mathfrak{p} \to \mathfrak{gl}(n,\mathbb{R})$ such that

$$\Lambda_{\mathfrak{p}}(\mathrm{ad}\, k(X)) = \mathrm{ad}(\lambda(k))(\Lambda_{\mathfrak{p}}(X)) \quad \text{for } X \in \mathfrak{p},\ k \in K \text{ where}$$

$\lambda:K \to GL(n,\mathbb{R})$ is <u>the linear isotropy representation</u> (here, $T_0G/K = \mathbb{R}^n$ where 0 is the origin). The canonical connection of G/K is the invariant connection corresponding to $\Lambda_{\mathfrak{p}} = 0$, and coincides with the

natural torsion-free connection. Furthermore, the canonical connection is the only affine connection of G/K which is invariant under the symmetries of N [57, Theorem 3.1, p.230].

So now let us proceed to the proof of Theorem 3.1: Recall that we have denoted G/K and G'/K' by N and N' respectively; further, let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ be the canonical decompositions of $\mathfrak{g}$ and $\mathfrak{g}'$ respectively.

A geodesic of N tangent to N' at the origin O, is of the form $\zeta_t(O)$, where $\zeta_t = \exp tX$ with $X \in \mathfrak{p}' \subset \mathfrak{p}$, and $\exp: \mathfrak{g} \to G$ is the exponential mapping. Clearly this geodesic is contained in N'. Now given any point $x = g'(O)$ of N' with $g' \in G'$, a geodesic of N tangent to N' at x, is of the form $g'(\zeta_t(O))$, which is clearly contained in N', thus proving that N' is a totally geodesic submanifold of N.

Since the canonical connection of N is also invariant under the symmetries of N', then its restriction to N' is also invariant under the symmetries of N', and so the last part of Theorem 3.1 follows from the above assertion.

<u>Remarks</u>

1. The converse of this theorem is also true [57, Theorem 4.2, p.235].
2. Note that these theorems do not give rise to a bijective correspondence between the (complete) totally geodesic submanifolds N' through O of N, and the symmetric subspaces (G',K',θ') of (G,K,θ). Two different symmetric subspaces (G',K',θ') and (G'',K'',θ'') may give

rise to the same totally geodesic submanifold. However, a case where a bijective correspondence does arise, is in the following. Firstly, we establish a definition:

Definition A.1

A subspace $\mathfrak{m}$ of a Lie algebra $\mathfrak{g}$, satisfying

(A.1) $[[\mathfrak{m},\mathfrak{m}],\mathfrak{m}] \subset \mathfrak{m}$,

is known as a Lie triple system.

Theorem A.2. [57, Theorem 4.3 p.237].

Let (G,K,θ) be a symmetric space with canonical decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Then there exists a natural bijective correspondence between the set of linear subspaces $\mathfrak{p}'$ of $\mathfrak{p}$, satisfying (A.1), and the set of complete totally geodesic submanifolds N' through the origin O, of the affine symmetric space $N = G/K$. The correspondence is given by $\mathfrak{p}' = T_O(N')$, under the identification $\mathfrak{p} = T_O(N)$.

Example A.3

We now intend to show, in the light of Theorem 3.1, how the compact HSS $Y^{III} = Sp(n)/U(n)$ as defined in (3.4), is totally geodesic in its ambient Grassmannian $G_n(E^{2n}) = U(2n)/U(n) \times U(n)$. The case of $Y^{II} = SO(2n)/U(n)$, is similar. Firstly, in the case of $G_n(E^{2n})$, we can define the involutive automorphism θ, by $\theta(A) = SAS^{-1}$ where $A \in U(2n)$ and

$$S = \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix} .$$

We now recall the fixed non-degenerate, antisymmetric bilinear form J'' on $\mathbb{C}^{2n}$; a basis of $\mathbb{C}^{2n}$ may be chosen such that J'' may by written as $\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. The corresponding involutive automorphism θ', is defined by $\theta'(A) = S'AS'^{-1}$, where

$$S' = \pm \begin{bmatrix} iI_n & 0 \\ 0 & -iI_n \end{bmatrix}$$

(see e.g. [89, p.318]). It is clear that $\theta(A)|_{Y^{III}} = \theta'(A)$, for all $A \in U(2n)$.

Let us now set $G = U(2n)$ and $K = U(n) \times U(n)$; further, note that by definition $G' = Sp(n) = Sp(n,\mathbb{C}) \cap U(2n)$, hence $G' \subset G$. The isotropy subgroup $K' = U(n)$ here consists of all $\pm \begin{bmatrix} B & 0 \\ 0 & \bar{B} \end{bmatrix}$, such that $B \in U(n)$; we have therefore, $K' = G' \cap K = Sp(n,\mathbb{C}) \cap U(2n) \cap (U(n) \times U(n))$ Thus $Sp(n)/U(n)$ is totally geodesic in $G_n(\mathbb{C}^{2n})$.

<u>Example A.4</u>

We now consider the totally geodesic embedding of $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{2n+2})$, as a second representative example of Theorem 3.1. Firstly, taking $G_2(\mathbb{C}^{2n+2})$, we have $G = U(2n+2)$ and $K = U(2) \times U(2n)$, and the canonical decomposition is given by

$$\mathfrak{u}(2n+2) = \mathfrak{u}(2) + \mathfrak{u}(2n) + \mathfrak{p}$$

where $\mathfrak{u}(2) + \mathfrak{u}(2n) = \left\{ \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} : B \in \mathfrak{u}(2) \ , \ C \in \mathfrak{u}(2n) \right\}$

and $\mathfrak{p} = \left\{ \begin{bmatrix} 0 & -{}^t\bar{X} \\ X & 0 \end{bmatrix} : X \in M_{2,2n} \right\}$.

Here $\theta(A) = SAS^{-1}$ where $S = \begin{bmatrix} -I_2 & 0 \\ 0 & I_{2n} \end{bmatrix}$ and $A \in U(2n+2)$.

Turning now to $\mathbb{H}P^n$, we have here , $G' = Sp(n+1)$ and $K' = Sp(1)\times Sp(n)$; recall that

(A.2) $\quad Sp(n+1) = \{A \in GL(n+1,\mathbb{H}) : \langle Ax,Ay\rangle^{\mathbb{H}} = \langle x,y\rangle^{\mathbb{H}} x,y \in \mathbb{H}^{n+1}\}$

(A.3) $\quad = Sp(n+1,\mathbb{C}) \cap U(2n+2)$.

The canonical decomposition is given by

$$\mathfrak{sp}(n+1) = \mathfrak{sp}(1) + \mathfrak{sp}(n) + \mathfrak{p}'$$

where $\mathfrak{sp}(1) + \mathfrak{sp}(n) = \left\{ \begin{bmatrix} B' & 0 \\ 0 & C' \end{bmatrix} : B' \in \mathfrak{sp}(1) \ , \ C' \in \mathfrak{sp}(n) \right\}$

and $\mathfrak{p}' = \left\{ \begin{bmatrix} 0 & -{}^t\bar{\xi} \\ \xi & 0 \end{bmatrix} : \xi \in \mathbb{H}^n \right\}$.

$$\mathfrak{u}(2n+2) = \mathfrak{u}(2) + \mathfrak{u}(2n) + \mathfrak{p}$$

where $\mathfrak{u}(2) + \mathfrak{u}(2n) = \left\{ \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} : B \in \mathfrak{u}(2) \ , \ C \in \mathfrak{u}(2n) \right\}$

and $\mathfrak{p} = \left\{ \begin{bmatrix} 0 & -{}^t\bar{X} \\ X & 0 \end{bmatrix} : X \in M_{2,2n} \right\}$.

Here $\theta(A) = SAS^{-1}$ where $S = \begin{bmatrix} -I_2 & 0 \\ 0 & I_{2n} \end{bmatrix}$ and $A \in U(2n+2)$.

Turning now to $\mathbb{H}P^n$, we have here , $G' = Sp(n+1)$ and $K' = Sp(1)\times Sp(n)$; recall that

(A.2) $\quad Sp(n+1) = \{A \in GL(n+1, \mathbb{H}) : \langle Ax, Ay\rangle^{\mathbb{H}} = \langle x,y\rangle^{\mathbb{H}} x,y \in \mathbb{H}^{n+1}\}$

(A.3) $\quad = Sp(n+1,\mathbb{C}) \cap U(2n+2)$.

The canonical decomposition is given by

$$\mathfrak{sp}(n+1) = \mathfrak{sp}(1) + \mathfrak{sp}(n) + \mathfrak{p}'$$

where $\mathfrak{sp}(1) + \mathfrak{sp}(n) = \left\{ \begin{bmatrix} B' & 0 \\ 0 & C' \end{bmatrix} : B' \in \mathfrak{sp}(1) \ , \ C' \in \mathfrak{sp}(n) \right\}$

and $\mathfrak{p}' = \left\{ \begin{bmatrix} 0 & -{}^t\bar{\xi} \\ \xi & 0 \end{bmatrix} : \xi \in \mathbb{H}^n \right\}$.

With the representation (A.2) in mind, the involutive automorphism is given by $S'AS'^{-1}$, where

$$S' = \begin{bmatrix} -I & 0 \\ 0 & I_n \end{bmatrix} \quad \text{and} \quad A \in Sp(n+1) .$$

But on taking into account the identification $\mathbb{H} = \mathbb{C} + \mathbb{C}_j$, and in the light of the isomorphism (A.3), we take S' to be

$$\begin{bmatrix} -I_2 & 0 \\ 0 & I_{2n} \end{bmatrix} ,$$

and therefore, for $A \in Sp(n+1)$, we have $\theta(A)\big|_{\mathbb{H}P^n} = \theta'(A)$.

It is easy to see that $K' = G' \cap K$, and that $\mathfrak{p}'$ forms a Lie triple system.

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