ON IMMERSIONS AND DYER-LASHOF OPERATIONS

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SUMMARY

We study the bordism algebra of immersions of closed smooth manifolds in closed smooth manifolds. We give two sets of polynomial generators. We obtain a splitting in terms of bordism groups of vector bundles with structural group $\Sigma_r \int O(k)$ and a splitting in terms of bordism of covering spaces and vector bundles. We study the problem of reducing, modulo bordism, the structural group $\Sigma_r \int O(k)$ to the subgroup $\mathbb{Z}_r \int O(k)$ by calculating some characteristic numbers.

INTRODUCTION

In this thesis we study the bordism of immersions of closed smooth manifolds in closed smooth manifolds and its relation with the Dyer-Lashof operations. We shall describe its contents chapter by chapter.

In chapter 1 we give two sets of generators for the bordism of embeddings of closed smooth manifolds in closed smooth manifolds. One set of generators is given in terms of Milnor manifolds and the other one in terms of projective spaces. To do this we study, in the first section, the relation between bordism and homology and the multiplicative properties of the Thom class as a map of spectra.

Chapter 2 deals with the Dyer-Lashof operations. We begin with some details about the homology mod. 2 of the spaces $EG \underset{G}{\times} X^r$, where G is a subgroup of the symmetric group of degree r. We need these results, in the case $r=2$ to define the operations and in the general case for the calculations in chapter 7. We then give the definition of the operations Q_i in the homology mod. 2 of any $\mathcal{C}_\infty$-space, where $\mathcal{C}_\infty$ is the cubes operad, following [13 & 16]. In section 3 we define the operations $\widetilde{Q}_i$ in unoriented bordism $N_*(-)$. The general properties of these operations were studied in [28 & 39]. In section 4 we see that the operations in bordism correspond to the operations in homology under the Thom homomorphism $\mu: N_*(-) \to H_*(- ; \mathbb{Z}_2)$.

In chapter 3 we give two sets of polynomial generators for the bordism of immersions of closed smooth manifolds in closed smooth manifolds in codimension k that we denote by $I(*, k)$, $k > 0$. For this we use a theorem of P. Schweitzer [40] that gives an isomorphism of N_*-algebras $I(*, k) \cong N_*(Q\, MO_k)$ then we obtain the structure of $N_*(Q\, MO_k)$ in terms of the operations $\tilde{Q}_i$ using the calculations of [11 & 16] in homology, then we give an interpretation of the operations $\tilde{Q}_i$ in terms of immersions and finally we use the generators for the bordism of embeddings that we obtained in chapter 2.

In chapter 4 we use the results in [28] and [43] to get a splitting of the groups $I(n,k)$, for $n \geq 0$, $k > 0$ in terms of bordism groups of vector bundles with structural group $\Sigma_r \int O(k)$, where $\int$ denotes the wreath product. Using this splitting we can associate to a self-transverse representative in a bordism class of immersions the characteristic numbers associated to this bundles and then we obtain that two self-transverse immersions are bordant if and only if their characteristic numbers are the same.

In chapter 5 we use some results of A. Borel [5] on homogeneous spaces to show that any (rk)-vector bundle over a closed smooth manifold is bordant to a vector bundle with structural group $\Sigma_r \int O(k)$.

In chapter 6 we use the method of F.W. Roush for classifying transfers [1] to give an interpretation of the groups $N_*(E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r)$ in terms of bordism of pairs $(\tilde{M}, \xi)$ where $\tilde{M} \to M$ is an r-covering over a closed smooth manifold M and ξ is a vector bundle over $\tilde{M}$.

We see the relation of this with Atiyah's direct image construction and we get another interpretation for the splitting of the bordism of immersions.

Chapter 7 deals with the problem of reducing, modulo bordism, the structural group of a bundle with group $\Sigma_r \int O(k)$ to the subgroup $\mathbb{Z}_r \int O(k)$.

In section 1 we study the edge homomorphisms of a spectral sequence converging to $H^*(E\Sigma_r \underset{\Sigma_r}{\times} X^r; \mathbb{Z}_2)$, for any space X of finite type; in section 2 we give a brief account of the work of G. Segal [41] on the cohomology of topological groups and some other related results and in section 3 we use the results of the previous 2 sections to calculate some of the characteristic numbers defined in chapter 4 to give examples related to the problem of Cyclic reduction modulo bordism.

Finally, in chapter 8 we use the operations $\tilde{Q}_i$ to show that the Dyer-Lashof operations in the mod. 2 homology of an Eilenberg-Mac Lane space $K(A,n)$ are zero for any abelian group A. An algebraic proof, for the case $A = \mathbb{Z}_2$, can be found in [32].

Chapter 1 : Generators for the Bordism of embeddings

In this chapter we give 2 sets of generators for the bordism of manifolds in manifolds. To do this we first study the relation between bordism and homology.

1.1 The Thom class

We denote by MO the Thom spectrum for unoriented cobordism, its k-th space $MO_k = T(\gamma(k))$ is the Thom space for the universal k-vector bundle over $BO(k)$. We denote by $H\mathbb{Z}_2$ the Eilenberg-Mac Lane spectrum with coefficients $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$, its k-th space $(H\mathbb{Z}_2)_k = K(\mathbb{Z}_2, k)$ is the Eilenberg-Mac Lane space of type $(\mathbb{Z}_2, k)$. Both are ring spectra [49].

1.1) Proposition [10] . The Thom classes $t_{\gamma(k)} \in \tilde{H}^k(MO_k; \mathbb{Z}_2) = [MO_k, K(\mathbb{Z}_2, k)]$ define a map of ring spectra $t: MO \to H\mathbb{Z}_2$.

1.2) Definition.- Any map of spectra $f: E \to F$ defines a natural transformation $\bar{f}: E^*(\_) \to F^*(\_)$ between the cohomology theories defined by E and F, and a natural transformation $\underline{f}: E_*(\_) \to F_*(\_)$ between the homology theories. For details see [49]. In particular we have $\bar{t}: MO^*(\_) \to H^*(\_;\mathbb{Z}_2)$ and $\underline{t}: MO_*(\_) \to H_*(\_;\mathbb{Z}_2)$.

1.3) Definition. [49] . Let E be a ring spectrum and X, Y pointed spaces. then we have products $\tilde{E}^p(X) \otimes \tilde{E}_q(X \wedge Y) \xrightarrow{\backslash} \tilde{E}_{q-p}(Y)$ (slant product) $\tilde{E}_p(X) \otimes \tilde{E}_q(Y) \xrightarrow{\wedge} \tilde{E}_{p+q}(X \wedge Y)$ (smash product or cross product $(\times)$ in the unreduced case). And similarly for cohomology $\tilde{E}^p(X) \otimes E^q(Y) \xrightarrow{\wedge} \tilde{E}^{p+q}(X \wedge Y)$.

1.4) Proposition - Let $f\colon E \to F$ be a map of ring spectra then

a) $\underline{f}(a \wedge b) = \underline{f}(a) \wedge \underline{f}(b)$

b) $\underline{f}(a \setminus b) = \bar{f}(a) \setminus \underline{f}(b)$

c) $\bar{f}(x \wedge y) = \bar{f}(x) \wedge \bar{f}(y)$.

Proof.- We shall prove a) The proof of b) and c) are entirely analogous.

Let $\mu\colon E \wedge E \to E$ and $\eta\colon F \wedge F \to F$ be the product maps. Let $a \in \tilde{E}_n(X)$ and $b \in \tilde{E}_m(Y)$ be represented by $S^n \xrightarrow{g} E \wedge X$ and $S^m \xrightarrow{h} E \wedge Y$, where S^n denotes the suspension spectrum of the n-sphere. Then $\underline{f}(a \wedge b)$ is represented by $S^n \wedge S^m \xrightarrow{g \wedge h} (E \wedge X) \wedge (E \wedge Y) \xrightarrow{\tau_E} (E \wedge E) \wedge (X \wedge Y) \xrightarrow{\mu \wedge 1} E \wedge X \wedge Y \xrightarrow{f \wedge 1} F \wedge (X \wedge Y)$ and $\underline{f}(a) \wedge \underline{f}(b)$ is represented by

$$S^n \wedge S^m \xrightarrow[(f \wedge 1) \circ g \wedge (f \wedge 1) \circ h]{} (F \wedge X) \wedge (F \wedge Y) \xrightarrow[\tau_F]{} (F \wedge F) \wedge (X \wedge Y) \xrightarrow[\eta \wedge 1]{} F \wedge (X \wedge Y)$$

To see that they are homotopic we use 3 facts: 1) The smash product is a bifunctor on the homotopy category of spectra [49]. 2) τ_E and τ_F are natural equivalences on the same category [49]. 3) As f is a map of ring spectra then $\eta \circ (f \wedge f) \simeq f \circ \mu$. Then we have

$$(f \wedge 1) \circ (\mu \wedge 1) \circ \tau_E \circ (g \wedge h) \underset{1}{\simeq} (f \circ \mu) \wedge 1 \circ \tau_E \circ (g \wedge h) \underset{3}{\simeq}$$

$$\simeq \eta \circ (f \wedge f) \wedge 1 \circ \tau_E \circ (g \wedge h) \underset{1}{\simeq} (\eta \wedge 1) \circ (f \wedge f) \wedge 1 \circ \tau_E \circ (g \wedge h) \underset{2}{\simeq}$$

$$\simeq (\eta \circ 1) \circ \tau_F \circ (f \wedge 1) \wedge (f \wedge 1) \circ (g \wedge h) \underset{1}{\simeq} (\eta \wedge 1) \circ \tau \circ (f \wedge 1) \circ g \wedge (f \wedge 1) \circ h.$$

□

1.5) Corollary.- Let $f\colon E \to F$ be a map of ring spectra, then a) if X is an H-space $\underline{f}\colon E_*(X) \to F_*(X)$ preserves the Pontrjagin product. Let $\langle , \rangle$ denote the Kronecker product and $\cap$ the cap product, then b) $\langle \bar{f}(x), \underline{f}(y) \rangle = \underline{f} \langle x, y \rangle$; c) $\underline{f}(a \cap b) = \bar{f}(a) \cap \underline{f}(b)$.

Proof.- a) Let $m: X\times X \to X$ denote the product map. by prop. 1.4.a) $\underline{f}$ preserves the cross product, and as $\underline{f}$ is natural we have $\underline{f}(a\cdot b)= \underline{f}\, m_*(a\times b)= m_*\, \underline{f}(a\times b)= m_*(\underline{f}(a)\times \underline{f}(b))= \underline{f}(a)\cdot \underline{f}(b)$.

b) by 1.4.b) $\bar{f}$ preserves the slant product so $<\bar{f}(x), \underline{f}(y)> = =\bar{f}(x)\backslash \underline{f}(y)= \underline{f}(x\backslash y)= \underline{f}<x,y>$. c) by 1.4.b) and naturality we have $\underline{f}(a\cap b)= \underline{f}(a\backslash \Delta_*(b))= \bar{f}(a)\backslash \underline{f}\,\Delta_*(b)= \bar{f}(a)\backslash \Delta_*\, \underline{f}(b)= \bar{f}(a)\cap \underline{f}(b)$.

□

1.6) Remark.- The universal Thom class in cobordism is an element in $MO^\circ(MO)=[MO,MO]$ given by the identity [10] . Hence $\bar{t}: MO^\circ(MO)\to H^\circ(MO;\mathbb{Z}_2)$ sends the universal Thom class in cobordism to the universal Thom class in $\mathbb{Z}_2$-cohomology.

1.7) Definition.- Let T^2 denote the category of topological pairs (X,A), $A\subset X$. For a fixed pair (X,A) define a singular manifold in (X,A) to be a map $f:(M,\partial M)\to(X,A)$ where M is a compact C^∞ manifold of dimension n . Such a map is said to bord if there is a compact C^∞ manifold V of dimension $n+1$ and a map $F: V\to X$ such that i) there is an embedding $e: M\hookrightarrow \partial V$, ii) $F|\partial V\circ e = f$,iii) iii) $F(\partial V-e(M))\subset A$. Two singular manifolds $(M,\partial M,f)$ and $(N,\partial N,g)$ will be called bordant if their disjoint union $(M\sqcup N, \partial M\sqcup\partial N, f\sqcup g)$ bords. This is an equivalent relation and we write $\mathcal{N}_n(X,A)$ for the set of equivalence classes. $\mathcal{N}_*(\_)$ is a generalised homology theory on T^2 [15] .

The Thom-Pontrjagin construction defines a natural transformation of homology theories $\Phi:\mathcal{N}_*(\_)\to MO_*(\_)$ [15] . By Thom's theorem $\Phi: \mathcal{N}_*(pt)\to MO_*(pt)$ is an isomorphism so Φ is an equivalence on the category of finite C.W pairs. One can easily prove that $\mathcal{N}_*(\_)$ satisfies the wedge axiom and the weak homotopy equivalence axiom so Φ is an equivalence on T^2 [49].

1.8) Definition.- We define a natural transformation $\mu : \mathcal{N}_*(\ ) \to H_*(\ ;\mathbb{Z}_2)$ by $\mu[M,f] = f_*(\sigma(M))$ where $\sigma(M)$ is the fundamental class mod. 2 of M. This is well defined and Thom proved that it is surjective [50] .

1.9) Theorem [15] . The $\mathcal{N}_*$-module $\mathcal{N}_*(X)$ is free. A family $\{a_i\}_{i\in I}$ of homogeneous elements in $\mathcal{N}_*(X)$ is an $\mathcal{N}_*$-basis if and only if $\{\mu(a_i)\}_{i\in I}$ is a $\mathbb{Z}_2$-basis for $H_*(X;\mathbb{Z}_2)$.

□

To finish this section we shall prove that under the isomorphism ϕ, $\underline{t}$ and μ coincide. For this recall that if M is a closed smooth n-manifold then the fundamental class of M in geometric bordism, wich we denote $\mathring{\delta}(M)$, is given by $\mathring{\delta}(M) = [M,id] \in \mathcal{N}_n(M)$.

1.10) Lemma.- μ is the unique natural transformation that sends the fundamental class in bordism of a closed smooth manifold to its fundamental class mod. 2.

Proof.- $\mu(\mathring{\delta}(M)) = \mu[M,id] = id_*(\sigma(M)) = \sigma(M)$, so μ has this property. Now suppose θ is a natural transformation with this property and let $[M,f] \in \mathcal{N}_n(X)$, then we can write $[M,f] = f_*[M,id] = f_*(\mathring{\delta}(M))$. So $\theta[M,f] = \theta f_*(\mathring{\delta}(M)) = f_*\theta(\mathring{\delta}(M)) = f_*(\sigma(M)) = \mu[M,f]$.

□

1.11) Proposition.- The following diagram commutes

$$\begin{tikzcd} MO_*(X) \arrow[r, "\underline{t}"] & H_*(X:\mathbb{Z}_2) \\ \mathcal{N}_*(X) \arrow[u, "\phi", "\cong"'] \arrow[ur, "\mu"'] & \end{tikzcd}$$

Proof.- by 1.10) it is enough to show that $\underline{t}\circ\phi$ sends the fundamental class $\mathring{\sigma}(M)$ to $\sigma(M)$ for a closed smooth manifold M. To see this we use the following definition for the fundamental class [49] . Consider an embedding $M^n \hookrightarrow S^{n+k}$ with normal bundle ν, let Ψ denote the Thom isomorphism in $\mathbb{Z}_2$-homology, $c: S^{n+k} \to T(\nu)$ the collapsing map and $\iota \in \tilde{H}_{n+k}(S^{n+k}; \mathbb{Z}_2)$ the generator then $\sigma(M) = \Psi(c_*(\iota))$. Similarly $\phi(\mathring{\sigma}(M)) = \mathring{\Psi}(c_*(\mathring{\iota}))$ where $\mathring{\Psi}$ is the Thom isomorphism in $MO_*(\_)$ and $\mathring{\iota} \in \widetilde{MO}_{n+k}(S^{n+k})$. Now recall that the Thom isomorphism is given by $\Psi(x) = p_*(t(\nu) \cap x)$ where p is the projection of ν, and $\mathring{\Psi}(x) = p_*(\mathring{t}(\nu) \cap x)$. By 1.5 c) t preserves cap products and by 1.6 $\bar{t}(\mathring{t}(\nu)) = t(\nu)$, as t is a map of ring spectra $\underline{t}(\mathring{\iota}) = \iota$, so we have $\underline{t}\phi(\mathring{\sigma}(M)) = \underline{t}\,\mathring{\Psi}(c_*(\mathring{\iota})) = \Psi\,\underline{t}(c_*(\mathring{\iota})) = \Psi\, c_*\, \underline{t}(\mathring{\iota}) = \Psi\, c_*(\iota) = \sigma(M)$.

□

§1.2 The bordism of BO(k)

In this section we shall give generators for $\mathcal{N}_*(BO(k))$.

1.12) Definition.- Given a vector bundle ξ, we denote by $\mathring{e}(\xi)$ and $e(\xi)$, the Euler classes of ξ in cobordism and $\mathbb{Z}_2$-cohomology respectively. The element $\mathring{e}(\gamma(1)) \in MO^1(BO(1))$ satisfies the properties for the existence of generalised Stiefel-Whitney classes in cobordism [49] , these classes are called Conner-Floyd classes, and are denoted by $\mathring{w}_i(\_)$. For an n-vector bundle ξ, $\mathring{e}(\xi) = \mathring{w}_n(\xi)$.

1.13) Remark.- If ξ is a vector bundle over a space X, and $z: X \to T\xi$ is the inclusion of the zero section then by definition $\mathring{e}(\xi) = z^*(\mathring{t}(\xi))$ and $e(\xi) = z^*(t(\xi))$. As $\bar{t}$ is natural then by 1.6

we have that $\bar{t}(\overset{\circ}{e}(\xi))= e(\xi)$. The fact that $\bar{t}(\overset{\circ}{e}(\gamma(1))= e(\gamma(1))$ together with the multiplicative and naturality properties of $\bar{t}$ imply that $\bar{t}(\overset{\circ}{w}_i(\xi))= w_i(\xi)$.

1.14) Definition.- We have that $MO^*(BO(1))\cong MO^*(pt)\,[[e(\gamma(1))]]$ and $H^*(BO(1);\mathbb{Z}_2)\cong \mathbb{Z}_2\,[e(\gamma(1))]$ [49]. We then have unique elements $\overset{\circ}{\beta}_i\in \tilde{N}_i(BO(1))$ and $\beta_i\in \tilde{H}_i(BO(1);\mathbb{Z}_2)$ with the property $<\overset{\circ}{e}(\gamma(1))^i, \overset{\circ}{\beta}_j>=\delta_{ij}$ and $<e(\gamma(1))^i, \beta_j>=\delta_{ij}$. We define $\overset{\circ}{\beta}_0=[\{point\}\hookrightarrow BO(1)]$ and β_0 the generator of $H_0(BO(1);\mathbb{Z}_2)$.

1.15) Proposition.- Consider elements $\overset{\circ}{\beta}_i\in N_i(BO(1))$ and $\beta_i\in H_i(BO(1);\mathbb{Z}_2)$ as defined in 1.14) then $\mu(\overset{\circ}{\beta}_i)=\beta_i$.

Proof.- For simplicity let us write $\overset{\circ}{e}$ for $\overset{\circ}{e}(\gamma(1))$ and e for $e(\gamma(1))$. By 1.4) $\bar{t}(\overset{\circ}{e}^i)=\bar{t}(\overset{\circ}{e})^i$ and by 1.13) $\bar{t}(\overset{\circ}{e})=e$. So using the formula 1.5 b) we have $<e^i, \underline{t}(\overset{\circ}{\beta}_j)>=<\bar{t}(\overset{\circ}{e}^i), \underline{t}(\overset{\circ}{\beta}_j)>=\underline{t}<\overset{\circ}{e}^i,\overset{\circ}{\beta}_j>=$ $=\underline{t}(\delta_{ij})=\delta_{ij}$, and hence $\underline{t}(\overset{\circ}{\beta}_j)=\beta_j$, as we have identified $\underline{t}$ with μ (1.11) then $\mu(\overset{\circ}{\beta}_i)=\beta_i$.

□

In order to get the generators for $N_*(BO(k))$ we shall first give a basis for $H_*(BO(k);\mathbb{Z}_2)$.

For this recall that $H^*(BO(k);\mathbb{Z}_2)=\mathbb{Z}_2\,[w_1(\gamma(k)).\dots.w_k(\gamma(k))]$ [49]. Let $\xi_k=\gamma(1)\times\dots\times\gamma(1)$ the k-fold product of the bundle $\gamma(1)$. This is a k-vector bundle over $BO(1)\times\dots\times BO(1)$.
Let $m_k\colon BO(1)\times\dots\times BO(1)\to BO(k)$ be a classifying map for ξ_k.
Then one can show [49] that $m_k^*(w_i(\gamma(k)))=\sigma_i(u_1,u_2,\dots,u_k)$, where σ_i is the i-th symmetric polynomial in the variables $u_1,u_2,\dots,u_k$ where $u_j=p_j^*(e)$ and $p_j\colon BO(1)\times\dots\times BO(1)\to BO(1)$ is the projection

on the j-th coordinate. As m_k^* is a ring homomorphism and any element can be written uniquely as $\sum_\alpha d_\alpha w_1^{\alpha_1}\cdot\ldots\cdot w_k^{\alpha_k}$ then $m_k^*(\sum_\alpha d_\alpha w_1^{\alpha_1}\cdot\ldots\cdot w_k^{\alpha_k}) = \sum_\alpha d_\alpha \sigma_1^{\alpha_1}\cdot\ldots\cdot\sigma_k^{\alpha_k}$ so m_k^* is an isomorphism onto the symmetric sub-algebra $S \subset H^*(BO(1)\times,\ldots,\times BO(1);\mathbb{Z}_2)$.

Now for each sequence $0 \le i_1 \le \ldots \le i_k$ consider the monomial $u_1^{i_1} u_2^{i_2} \ldots u_k^{i_k}$ and let $S_{i_1,\ldots,i_k}$ be the smallest symetric polynomial containing this monomial, as $S = \mathbb{Z}_2[\sigma_1, \sigma_2,\ldots,\sigma_k]$ then it is clear that the polynomials $S_{i_1,\ldots,i_k}$ are a basis for S. As m_k^* is an isomorphism onto S then we get:

1.16) Proposition.- For each collection $0 \le i_1 \le i_2 \le,\ldots,\le i_k$ the elements $a_{i_1,\ldots,i_k} \in H^*(BO(k);\mathbb{Z}_2)$ such that $m_k^*(a_{i_1,\ldots,i_k}) = S_{i_1,\ldots,i_k}$ are a basis for $H^*(BO(k);\mathbb{Z}_2)$.

□

1.17) Proposition.- Consider the elements $\beta_i \in H_i(BO(1);\mathbb{Z}_2)$ and the map $m_k: BO(1)\times,\ldots,\times BO(1) \to BO(k)$ as above, then $m_{k*}(\beta_{i_1}\times\beta_{i_2}\times\ldots\times\beta_{i_k})$ for each collection $0 \le i_1 \le,\ldots,\le i_k$ forms a basis for $H_*(BO(k);\mathbb{Z}_2)$.

Proof.- All modules are finitely generated $\langle\, ,\, \rangle$ is non-singular and by 1.16) $\{a_{i_1,\ldots,i_k}\}$ for collection $0 \le i_1 \le,\ldots,\le i_k$ is a basis for $H^*(BO(k);\mathbb{Z}_2)$, so we are going to see that its dual basis is $\{m_{k*}(\beta_{i_1}\times,\ldots,\times\beta_{i_k})\}$. For this we use the following facts [49] : $\langle\, ,\, \rangle$ is bilinear and satisfies $\langle f^*(x), y\rangle = \langle x, f_*(y)\rangle$ and $\langle x\times y, a\times b\rangle = \langle x,a\rangle\langle y,b\rangle$. If p_j is the projection on the j-th factor then $u_j = p_j(e) = 1\times 1\times\ldots\times e\times 1\times\ldots\times 1$. If $\cup$ denotes the cup then $(x \cup x') \times (y \cup y') = (x \times y) \cup (x' \times y')$.

We can then write:

$$< a_{i_1,\dots,i_k}, m_{k*}(\beta_{j_1} \times \dots \times \beta_{j_k}) > = < m_k^*(a_{i_1,\dots,i_k}), \beta_{j_1} \times \dots \times \beta_{j_k} > =$$

$$= < S_{i_1,\dots,i_k}\ \beta_{j_1} \times \dots \times \beta_{j_k} > = < u_1^{i_1} u_2^{i_2} \dots u_k^{i_k} + \dots + \dots\ \beta_{j_1} \times \dots \times \beta_{j_k} > =$$

$$= < u_1^{i_1} u_2^{i_2} \dots u_k^{i_k}, \beta_{j_1} \times \dots \beta_{j_k} > + \text{ other terms } =$$

$$= < e^{i_1} \times e^{i_2} \times \dots \times e^{i_k}, \beta_{j_1} \times \dots \times \beta_{j_k} > + \text{ other terms } =$$

$$= < e^{i_1}, \beta_{j_1} > < e^{i_2}, \beta_{j_2} > \dots < e^{i_k}, \beta_{j_k} > + \text{ other terms } =$$

$$= \begin{cases} 1 & \text{if } i_1 = j_1,\ i_2 = j_2, \dots, i_k = j_k \\ 0 & \text{otherwise.} \end{cases}$$

1.18) Proposition.- An N_*-basis for $N_*(BO(k))$ is given by the elements $m_{k*}(\mathring{\beta}_{i_1} \times \dots \times \mathring{\beta}_{i_k})$ for each collection $0 \leq i \leq \dots \leq i_k$.

Proof.- μ is natural and by 1.4 a) it preserves products. By 1.15) $\mu(\mathring{\beta}_i) = \beta_i$ so we have $\mu\, m_{k*}(\mathring{\beta}_{i_1} \times \dots \times \mathring{\beta}_{i_k}) = m_{k*}\mu\,(\mathring{\beta}_{i_1} \times \dots \times \mathring{\beta}_{i_k}) =$
$= m_{k*}(\mu\mathring{\beta}_{i_1} \times \dots \times \mu\mathring{\beta}_{i_k}) = m_{k*}(\beta_{i_1} \times \dots \times \beta_{i_k})$.
By 1.17) these elements are a $\mathbb{Z}_2$-basis for $H_*(BO(k);\mathbb{Z}_2)$ so by 1.9) $\{m_{k*}(\mathring{\beta}_{i_1} \times \dots \times \mathring{\beta}_{i_k})\}$ is an N_*-basis.

□

We now give specific representatives for the elements $\mathring{\beta}_i \in \tilde{N}_i(BO(1))$.

We take as $BO(1)$ the infinite real projective space P^∞.

1.19) Definition.- Let P^m be the real projective m-space with coordinates $x = [x_0, \dots, x_m]$; for $1 \leq m \leq n$ let $H_{m,n} =$
$= \{([x], [y]) \; P^m \times P^n \mid \sum_{i=0}^{m} x_i y_i = 0\}$. H_{mn} is a closed smooth manifold of dimension $m+n-1$ called Milnor manifold. $H_{m,n}$ is fibred as $P^{n-1} \to H_{m,n} \to P^m$ where the projection is induced by the proiection $P^m \times P^n \to P^m$.

1.20) Proposition.- [10] Let γ_m be the restriction of $\gamma(1)$ to P^m, and $\gamma_m \otimes \gamma_n$ the external tensor product over $P^m \times P^n$, then $\overset{\circ}{e}(\gamma_m \otimes \gamma_n) \in MO'(P^m \times P^n)$ is the Poincaré dual of $[H_{m,n} \hookrightarrow P^m \times P^n]$.

1.21) Definition.- We define $b_k \in \mathcal{N}_k(P^\infty)$ to be the class of:
$H_{1,k} \hookrightarrow P^1 \times P^k \xrightarrow{p_k} P^k \overset{i_k}{\hookrightarrow} P^\infty$, where p_k is the projection on P^k and i_k the inclusion. We will show that these elements can be taken as $\overset{\circ}{\beta}_k$, for this we need the following 2 propositions.

1.22) Proposition.- With the notation as above $\overset{\circ}{e}(\gamma(1)) \cap b_k = b_{k-1}$.

Proof - It is clear that 1 and 2 commute and that 3 is a pull-back:

$$1 \quad \begin{tikzcd} P^1 \times P^k \arrow[r, "p_k"] & P^k \\ P^1 \times P^{k-1} \arrow[u, "id \times j_k"] \arrow[r, "p_{k-1}"'] & P^{k-1} \arrow[u, "j_k"'] \end{tikzcd} \qquad \begin{tikzcd} P^k \arrow[r, hook] & P^\infty \\ P^{k-1} \arrow[u, "j_k"] \arrow[ur, "i_{k-1}"'] & \end{tikzcd} \quad 2 \qquad \begin{tikzcd} \gamma_1 \otimes \gamma_{k-1} \arrow[r] \arrow[d] & \gamma_1 \otimes \gamma_k \arrow[d] \\ P^1 \times P^{k-1} \arrow[r, "id \times j_k"'] & P^1 \times P^k \end{tikzcd} \quad 3$$

let us write $\alpha_k = [H_{1,k} \hookrightarrow P^1 \times P^k]$. Then using the fact that $f_*(f^*(x) \cap y) = x \cap f_*(y)$ [49], we have:

$$\mathring{e}(\gamma(1))\cap b_k = \mathring{e}(\gamma(1))\cap (i_k\circ p_k)_*(\alpha_k)=(i_k\circ p_k)_*\left[(i_k\circ p_k)^*(\mathring{e}(\gamma(1))\cap \alpha_k\right] =$$

$$= (i_k\circ p_k)_*\left[(1\times \mathring{e}(\gamma_k))\cap \alpha_k\right] \qquad ⓐ$$

On the other hand by prop. 1.20) $\alpha_{k-1} = \mathring{e}(\gamma_1 \boxtimes \gamma_{k-1})\cap \sigma(P^1\times P^{k-1})$, and by 3 $\mathring{e}(\gamma_1\boxtimes\gamma_{k-1}) = (1\times j_k)^*(\mathring{e}(\gamma_1\boxtimes\gamma_k))$. Then we have:

$$b_{k-1} = (i_{k-1}\circ p_{k-1})_*(\alpha_{k-1}) \underset{2}{=} (i_k\circ j_k\circ p_{k-1})_*(\alpha_{k-1}) \underset{1}{=} (i_k\circ p_k)_*((id\times j_k)_*(\alpha_{k-1}))=$$

$$= (i_k\circ p_k)_*\left[(id\times j_k)_*((id\times j_k)^*\, \mathring{e}(\gamma_1\boxtimes\gamma_k)\cap\sigma(P^1\times P^{k-1}))\right] =$$

$$= (i_k\circ p_k)_*\left[\mathring{e}(\gamma_1\boxtimes\gamma_k)\cap (id\times j_k)_*\sigma(P^1\times P^{k-1})\right) \qquad ⓑ$$

But $\mathring{e}(\gamma_k)$ is Poincaré dual to $[P^{k-1} \xhookrightarrow{j_k} P^k]$ [10]. So $1\times\mathring{e}(\gamma_k)$ is Poincaré dual to $P^1\times P^{k-1} \xhookrightarrow[id\times jk]{} P^1\times P^k = (id\times j_k)_*\sigma(P^1\times P^{k-1})$ and α_k is Poincaré dual to $\mathring{e}(\gamma_1\boxtimes\gamma_k)$, hence their cap products are the same (recall that $(x\cup y)\cap z = x\cap(y\cap z)$ [49]), so ⓐ = ⓑ, i e., $\mathring{e}(\gamma(1))\cap b_k = b_{k-1}$.

□

1.23) Proposition $[H_{1,n}] = 0$ in $\mathcal{N}_n$ $(n\geq 1)$

Proof.- $H_{1,n+1} \cong P(\gamma_1\oplus\varepsilon^n)$, where $P(\_)$ is the projective bundle associated to $\gamma_1\oplus\varepsilon^n$ and ε^n is the trivial n-bundle [10].
For this we follow the method in [15] to compute the characteristic classes of the tangent bundle of a projective bundle.

Let ξ be a k-vector bundle over a closed n-manifold V. Let $w_1, w_2, \dots, w_n$ be the Stiefel-Whitney classes of V and $v_1, v_2, \dots, v_k$ those of ξ. Let $c\in H^1(P(\xi);\mathbb{Z}_2)$ be the euler class of the canonical line bundle over $P(\xi)$. Then Borel and Hirzebruch proved that [6]
$$w_j(P(\xi)) = \sum_{p+q+r=j}\binom{k-p}{q} p^*(w_r\, v_p)\, c^q .$$

In our case $\xi = \gamma_1 \oplus \varepsilon^n$ over P^1 so, $w_i=0, i>0$, $v_1=e=e(\gamma_1)$ and $v_i=0, i>1$ then $w_j(P(\gamma_1 \oplus \varepsilon^n)) = \binom{n+1}{j} c^j + \binom{n}{j-1} c^{j-1} p^*(e)$.

Let us write $w_j = w_j(P(\gamma_1 \oplus \varepsilon^n))$ and $\frac{j}{n+1}\binom{n+1}{j} = \binom{n}{j-1}$, then

$$w_j = \binom{n+1}{j} c^j + \frac{j}{n+1}\binom{n+1}{j} c^{j-1} p^*(e).$$

Now consider $j_1 + j_2 + \ldots + j_k = n+1$, then

$$w_{j_1} w_{j_2} \ldots w_{j_k} = \prod_{r=1}^{k} \binom{n+1}{j_r} c^{n+1} + \sum_{s=1}^{k} \frac{j_s}{n+1} \left[\prod_{r=1}^{k} \binom{n+1}{j_r} c^n p^*(e)) \right]$$

(the other terms are zero because $e^2 = 0$).

Now recall that for any vector bundle ξ, $H^*(P(\xi); \mathbb{Z}_2) \cong$ $\cong H^*(V)\{1, c, c^2, \ldots, c^{k-1}\}$ and that $c^k = p^*(w_k(\xi)) + \ldots + p^*(w_1(\xi))c^{k-1}$. [49]. So in our case $c^{n+1} = c^n p^*(e)$.

$$\text{Hence } w_{j_1} w_{j_2} \ldots w_{j_k} = \left[\prod_{r=1}^{k} \binom{n+1}{j_r} + \sum_{s=1}^{k} \frac{j_s}{n+1} \prod_{r=1}^{k} \binom{n+1}{j_r} \right] c^n p^*(e)$$

But $\sum_{s=1}^{k} j_s = n+1$ so both terms in the parenthesis are equal, hence $w_{j_1} w_{j_2} \ldots w_{j_k} = 0$, and by a theorem of Thom [50], $[H(1,n)] = 0$.

□

1.24 Corollary.- The elements $b_k \in \mathcal{N}_k(P^\infty)$ given by the Milnor manifolds are the duals of $\mathring{e}(\gamma_1)^k$.

Proof.- We have that $\langle x \cup y, z \rangle = \langle x, y \cap z \rangle$ [49]. So $\langle \mathring{e}(\gamma_1)^j, b_k \rangle =$ $= \langle 1, \mathring{e}(\gamma_1) \cap \ldots \cap \mathring{e}(\gamma_1) \cap b_k \rangle$ this is, by 1.22, equal to $\langle 1, b_{k-j} \rangle =$

$$= \begin{cases} <1, b_0> = 1 & \text{if } k=j \\ <1, b_{k-j}> = [H_{1,k-j}] = 0 & \text{if } k \neq j \text{ by } 1.23 \end{cases}$$

□

1.25) Corollary.- An N_*-basis for $N_*(BO(k))$ is given by the elements $(b_{i_1} x, \ldots, b_{i_k})$ for each collection $0 \le i_1 \le \ldots \le i_k$; where m_k classifies $\gamma(1)^k$.

Proof.- by 1.24, $b_k = \mathring{\beta}_k$, so the result follows from 1.18).

□

To finish this section we shall give a second N_*-basis for $N_*(BO(k))$.

1.26) Proposition.- The elements $[P^n \xhookrightarrow[g_n]{} P^\infty] \in N_n(P^\infty)$, $n \geq 0$ form an N_*-basis for $N_*(P^\infty)$.

Proof.- By 1.9 it is enough to show that the elements $\mu[P^n \hookrightarrow P^\infty]$, $n \geq 0$ are a $\mathbb{Z}_2$-basis for $H_*(P^\infty; \mathbb{Z}_2)$ For this consider

$$< e(\gamma(1))^i, \mu[P^j, g_j] > = < e(\gamma(1))^i, g_{j*}\sigma(P^j) > = < g_j^* e(\gamma(1))^i, \sigma(P^j) > =$$

$$= < e(\gamma_j)^i, \sigma(P^j) > .$$

Now if $i \neq j$ this product is clearly zero. If $i=j$, then we know $H^*(P^j; \mathbb{Z}_2) = \mathbb{Z}_2 [e(\gamma_j)] / e(\gamma_j)^{j+1}$ so $H^j(P^j) \cong \mathbb{Z}_2$ with generator $\sigma(P^j)$, as $<,>$ is non-singular then $<e(\gamma_j)^j, \sigma(P^j)> = 1$.
Hence $\mu[P^n, g_n] = \beta_n$.

□

1.27) Corollary .- An N_*-basis for $N_*(BO(k))$ is given by the elements $m_{k_*}([P^{i_1}, g_{i_1}] \times \ldots \times [P^{i_k}, g_{i_k}])$ for each collection $0 \le i_1 \le \ldots \le i_k$, where m_k classifies $\gamma(1)^k$.

Proof.- By 1.9 it is enough to show that

$\mu m_{k_*}([P^{i_1},g_{i_1}]\times\ldots\times[P^{i_k},g_{i_k}])$ are a $\mathbb{Z}_2$-basis for $H_*(BO(k);\mathbb{Z}_2)$.
μ is natural and multiplicative (1.4a)) and by 1.26 $\mu[P^n,g_n] = \beta_n$ so we have $\mu\, m_{k_*}([P^{i_1},g_{i_1}]\times\ldots\times[P^{i_k},g_{i_k}]) = m_{k_*}(\beta_{i_1}\times\ldots\times\beta_{i_k})$.
But by 1.17 these elements are a $\mathbb{Z}_2$-basis for $H_*(BO(k);\mathbb{Z}_2)$.

§1.3) Bordism of embeddings

In this section we shall give generators for the bordism of embeddings. We will use these generators in chapter 3 to study the bordism of immersions.

1.28) Definition.- Given 2 embeddings $f: M \to N$, $g: M' \to N'$, where M, M' are closed smooth n-manifolds and N,N' are closed smooth (n+k)-manifolds, we say that they are bordant if there exists an embedding $F: V \longrightarrow W$ such that i) V is a compact smooth (n+1)-manifold with a diffeomorphism $\partial V \cong M \amalg M'$ and W is a compact smooth (n+k+1)-manifold with a diffeomorphism $\partial W \cong N \amalg N'$. ii)The following diagrams commute:

$$\begin{array}{ccc} M & \hookrightarrow M\amalg M' \cong \partial V \subset & V \\ f\downarrow & & \downarrow F \\ N & \hookrightarrow N\amalg N' \cong \partial W \subset & W \end{array} \qquad \begin{array}{ccc} M' & \hookrightarrow M\amalg M' \cong \partial V \subset & V \\ g\downarrow & & \downarrow F \\ N' & \hookrightarrow N\amalg N' \cong \partial W \subset & W \end{array}$$

This is an equivalence relation and the set of equivalence classes is denoted by Emb(n,k). We denote by $[f:M \to N]$ the equivalence class of an embedding f. We can make Emb(n,k) into a group by considering disjoint union of embeddings. We can make Emb$(*,k)$ into an N_*-module by defining $[M'] \cdot [f:M \longrightarrow N] = [M'\times M \xrightarrow{id\times f} M'\times N]$.

1.29) Theorem [54].- The Thom-Pontrjagin construction defines an isomorphism of N_*-modules $\mathrm{Emb}(n,k) \cong N_{n+k}(MO_k)$.

□

To define the generators for $\mathrm{Emb}(*,k)$ we need 2 results.

1.30) Definition.- Let (X,x_0) be a pointed space and $f:(M, \partial M) \to (X, x_0)$ a map, where M is a smooth manifold, we define $\tilde{M} \xrightarrow{\tilde{f}} X$ as follows:

$\tilde{M} = M \cup_{\partial M} M$ is the double of M, which is well defined up to diffeomorphism [9] and $\tilde{f}$ is the map induced by the map $M \amalg M \xrightarrow{f \amalg c} X$ where $c(a) = x_0$, for all $a \in M$.

1.31) Proposition.- Let (X,x_0) be a pointed space and $i:\{x_0\} \hookrightarrow X$ the inclusion. Let $d: N_n(X,x_0) \to N_n(X)$ be defined by $d[M,f]=[\tilde{M},\tilde{f}]$, where $\tilde{M}$ is as above, then $\Psi : N_n \oplus N_n(X,x_0) \to N_n(X)$ given by $\Psi(a,b) = i_*(a) + d(b)$ is an isomorphism.

Proof.- Consider the exact sequence of the pair $(X,\{x_0\})$, and notice that the inclusion $i:\{x_0\} \hookrightarrow X$ has the constant map $p:X \to \{x_0\}$ as a left inverse so we get a short exact sequence :

$$0 \longrightarrow N_n\{x_0\} \xrightarrow{i_*} N_n(X) \xrightarrow{j_*} N_n(X,\{x_0\}) \longrightarrow 0.$$

One can see that $\phi : N_n(X) \to N_n \oplus N_n(X,x_0)$ given by $\phi(y)=(p_*(y),j_*(y))$ is an isomorphism and from this it is clear that j_* restricted to $\ker p_*$ is an isomorphism. We claim that $d: N_n(X,x_0) \to N_n(X)$ is the inverse. To see this let $[M,f] \in N_n(X,\{x_0\})$ and consider $j_* d[M,f]$, this element is represented by the map $(\tilde{M},\varnothing) \xrightarrow{\tilde{f}} (X, x_0)$. Define $F: \tilde{M} \times I \to X$ by $F(z, t) = \tilde{f}(z)$.

Then $\partial(\widetilde{M}\times I)=\widetilde{M}\times\{0\}\cup\widetilde{M}\times\{1\}$ and we can identify $\widetilde{M}$ with $\widetilde{M}\times\{0\}$ and embedd $M\hookrightarrow\widetilde{M}\times\{1\}$ in such a way that $\tilde{f}|M=f$; then $F|\widetilde{M}\times\{0\}=\tilde{f}$, $F|M\subset\widetilde{M}\times\{1\}=\tilde{f}|M=f$ and $F(\widetilde{M}\times\{1\}-M)=x_0$, so $(\widetilde{M}\times I, F)$ is a bordism between $(\widetilde{M},\tilde{f})$ and (M,f), i.e. $j_*d\,[M,f]=[M,f]$.
And now it is clear that $\Psi(a,b)=i_*(a)+d(b)$ satisfies $\phi\circ\Psi=id$ so Ψ is an isomorphism. □

1.32) Definition.- Let $\xi=(E,p,B)$ be a vector bundle over a space B. Let $N\xrightarrow{g}B$ be a map, where N is a manifold. Given a Riemannian metric on ξ, denote by $D(\xi)$, $S(\xi)$ the disc bundle and the sphere bundle respectively. Consider the pull-back $g^*(\xi)$ over N. If we give $g^*(\xi)$ the pull-back metric then the map between total spaces induces a map $\dot{g}$: $(D(g^*(\xi)), S(g^*\xi))\to(D(\xi).S(\xi))$. Let $q:(D(\xi);S(\xi))\to$ $\to(T(\xi),*)$ denote the identification map, then we get a map $q\circ\dot{g}$: $:(D(g^*\xi),S(g^*\xi))\longrightarrow(T(\xi),*)$, where $D(g^*\xi)$ is a manifold with boundary of dimension equal to $\dim N+\dim\xi$.

1.33) Proposition.- Let $\xi=(E,p,V)$ be a smooth n-vector bundle, then ϕ: $N_n(V)\to N_{n+k}(T\xi,*)$ given by $\phi[N,g]=[D(g^*\xi),\, q\circ\dot{g}]$ is an isomorphism.

Proof.- Let $\Psi:N_{n+k}(T\xi,*)\to N_n(V)$ be defined as follows: let $[M,f]\in N_{n+k}(T\xi,*)$ then we have a map $(M,\partial M)\xrightarrow{f}(T\xi,*)$, consider a map f_0 homotopic to f such that f_0 is smooth throughout $f_0^{-1}(T\xi-*)$, and is transverse to the zero cross-section, define $\Psi[M,f]=[f_0^{-1}(\text{zero section}),f_0|f_0^{-1}(\text{zero section})]$. Ψ is the Thom isomorphism in bordism [10]. As the map $\dot{g}$ is induced by a map of bundles it is clear that $\Psi\circ\phi[N,g]=[N,g]$ so ϕ is an isomorphism. □

1.34) Definition.- Let $H(1,n) \subset P^1\times P^n$ be a Milnor manifold, we denote by H_n the restriction to $H(1,n)$ of the line bundle $P^1\times\gamma_n \to P^1\times P^n$. For each collection $0\leq i_1 \leq\dots\leq i_k$, consider $D(H_{i_1}\widetilde{\times} H_{i_2}\times\dots\times H_{i_k})$, the double of the disc bundles, and take the embeddings $H(1,i_1)\times\dots\times H(1,i_k) \to D(H_{i_1}\widetilde{\times\dots\times} H_{i_k})$ as the zero section. Similarly we have $P^{i_1}\times\dots\times P^{i_k} \to D(\gamma_{i_1}\widetilde{\times\dots\times}\gamma_{i_k})$. We also consider the empty collection and to this one we associate the embedding $\emptyset \hookrightarrow \{\text{point}\}$. With this notation we have:

1.35) Theorem.- a) The embeddings $H(1,i_1)\times\dots\times H(1,i_k) \to D(H_{i_1}\widetilde{\times\dots\times} H_{i_k})$ for each collection $0\leq i_1\leq\dots\leq i_k$ are an N_*-basis for $\mathrm{Emb}(*,k)$.

b) The embeddings $P^{i_1}\times\dots\times P^{i_k} \to D(\gamma_{i_1}\widetilde{\times\dots\times}\gamma_{i_k})$ for each collection $0\leq i_1 \leq\dots\leq i_k$ are an N_*-basis for $\mathrm{Emb}(*,k)$.

Proof.- By 1.29 $\mathrm{Emb}(*,k) \cong N_*(MO_k)$. By 1.31) $\Psi: N_*\oplus N_*(MO_k,*) \to N_*(MO_k)$ given by $\Psi(a,b)= i_*(a)+d(b)$ where d is given by the double construction, is an isomorphism, and by 1.33) $\phi : N_*(BO(k)) \to N_*(MO_k,*)$ given by the disc bundle construction is an isomorphism. By 1.25) An N_*-basis for $N_*(BO(k))$ is given by the elements $m_{k*}(b_{i_1}\times\dots\times b_{i_k})$ for $0\leq i_1\leq\dots\leq i_k$, but we have a pull-back:

$$\begin{tikzcd}
H_{i_1}\times\dots\times H_{i_k} \arrow[r]\arrow[d] & (P^1\times\gamma_{i_1})\times\dots\times(P^1\times\gamma_{i_k}) \arrow[r]\arrow[d] & \gamma_{i_1}\times\dots\times\gamma_{i_k} \arrow[r]\arrow[d] & \gamma(1)\times\dots\times\gamma(1) \arrow[r]\arrow[d] & \gamma(k) \arrow[d] \\
H(1,i_1)\times\dots\times H(1,i_k) \arrow[r] & (P^1\times P^{i_1})\times\dots\times(P^1\times P^{i_k}) \arrow[r] & P^{i_1}\times\dots\times P^{i_k} \arrow[r] & P^\infty\times\dots\times P^\infty \arrow[r, "m_k"'] & BO(k)
\end{tikzcd}$$

By 1.27) an N_*-basis for $N_*(BO(k))$ is given by the elements

$m_{k_*}([P^{i_1}, g_{i_1}] \times \ldots \times [P^{i_k}, g_{i_k}])$ for $0 \le i_1 \le \ldots \le i_k$, and we have a pull-back:

$$\begin{array}{ccccc} \gamma_{i_1} \times \ldots \times \gamma_{i_k} & \longrightarrow & \gamma(1) \times \ldots \times \gamma(1) & \longrightarrow & \gamma(k) \\ \downarrow & & \downarrow & & \downarrow \\ P^{i_1} \times \ldots \times P^{i_k} & \xrightarrow[g_{i_1} \times \ldots \times g_{i_k}]{} & P^\infty \times \ldots \times P^\infty & \xrightarrow[m_k]{} & BO(k) \end{array}$$

Finally, it is clear that when we apply the Thom-Pontrjagin construction we get the embeddings stated in the Theorem. $\square$

Chapter 2.- The Dyer-Lashof Operations

In this chapter we define Dyer Lashof operations in homology and in bordism and study their relationship.

§2.1) The Homology mod.2 of $EG \underset{G}{\times} X^n$

In this section we give some results about $H_*(EG \underset{G}{\times} X^n; \mathbb{Z}_2)$ that we shall need later. To make the notation simpler we shall denote the mod.2 homology by $H_*(\_)$.

2.1) Definition.- Let $G \subset \Sigma_n$ be a subgroup of the symmetric group of degree n. Let EG be a contractible space on which G acts freely on the right. Let X be a space and define an action $X^n \times G \to X^n$ on the n-fold product of X by $(x_1,..,x_n)\cdot\sigma = (x_{\sigma(1)},\dots,x_{\sigma(n)})$; then G acts diagonally on $EG\times X^n$ and we denote by $EG \underset{G}{\times} X^n$ the quotient space. This construction is functorial, given a map

$f: X \to Y$, $id \times f^n$: $EG\times X^n \to EG\times Y^n$, induces a map

$id \underset{G}{\times} f^n: EG \underset{G}{\times} X^n \to EG \underset{G}{\times} Y^n$.

Notice that as the action on EG is free then the action on $EG\times X^n$ is also free and if X is Hausdorff the projection $p: EG\times X^n \to EG \underset{G}{\times} X^n$ is a covering projection [44].

Let $S_*(\ )$ denote the singular chain complex with $\mathbb{Z}_2$ coefficients and $\mathbb{Z}_2[G]$ the group ring of G with coefficients in $\mathbb{Z}_2$. The action of G on $EG \times X^n$ induces a $\mathbb{Z}_2[G]$-module structure on $S_*(EG \times X^n)$; if we give $\mathbb{Z}_2$ the trivial G action we have:

2.2) Lemma [30]: $S_*(EG \times X^n)$ is $\mathbb{Z}_2[G]$-free and we have a natural isomorphism $\phi: S_*(EG\times X^n) \underset{\mathbb{Z}_2[G]}{\otimes} \mathbb{Z}_2 \xrightarrow{\cong} S_*(EG \underset{G}{\times} X^n)$ given by $\phi(a\times 1)= p_\#(a)$. $\square$

The actions of G on EG and X^n also induce $\mathbb{Z}_2[G]$-module structures on $S_*(EG)$ and $S_*(X^n)$. We define a $\mathbb{Z}_2[G]$-module structure on $S_*(EG) \underset{\mathbb{Z}_2}{\otimes} S_*(X^n)$ by $(r\otimes t)\cdot\sigma = r\cdot\sigma \otimes t\cdot\sigma$

2.3) Lemma.- We have a G-equivariant chain equivalence $\alpha: S_*(EG\times X^n) \to S_*(EG)\otimes S_*(X^n)$ with a homotopy inverse β which is also G-equivariant. Furthermore the chain homotopies $\alpha\circ\beta\simeq 1$, $\beta\circ\alpha\simeq 1$ are also G-equivariant, and α and β are natural.

Proof.- By the Eilenberg-Zilber Theorem, we have a chain equivalence α of $\mathbb{Z}_2$-modules with homotopy inverse β. We will show that they are G-equivariant. For this recall that to define α you consider the category $T\times T$, where T is the category of spaces, and the functors $F(X,Y)= S_*(X\times Y)$ and $G(X,Y)= S_*(X)\otimes S_*(Y)$, and $\alpha: F \to G$ and $\beta: G \to F$ are natural, so in particular for $\sigma\in G$ we have homeomorphisms $\bar\sigma: EG \to EG$ and $\bar\sigma: X^n \to X^n$ given by $\bar\sigma(e)=e\cdot\sigma$ and $\bar\sigma(b)=b\cdot\sigma$, so we have a morphism in $T\times T$, $(\bar\sigma,\bar\sigma):(EG,X^n) \to (EG,X^n)$ and hence a commutative diagram

$$\begin{array}{ccc} S_*(EG\times X^n) & \xrightarrow{\alpha} & S_*(EG)\otimes S_*(X^n) \\ \Big\downarrow{\scriptstyle(\bar\sigma\times\bar\sigma)_\#} & & \Big\downarrow{\scriptstyle\bar\sigma_\#\times\bar\sigma_\#} \\ S_*(EG\times X^n) & \xrightarrow[\alpha]{} & S_*(EG)\otimes S_*(X^n) \end{array}$$

But the commutativity says precisely that α is G-equivariant. Similarly with β. Furthermore the theorem says that $\alpha\circ\beta$ and $\beta\circ\alpha$ are chain homotopic to the identity in a natural way so we can repeat the argument above to show that the chain homotopies are also G-equivariant. $\square$

2.4) Corollary.- We have a natural chain equivalence:

$$S_*(EG\times X^n)\underset{\mathbb{Z}_2[G]}{\otimes}\mathbb{Z}_2 \simeq S_*(EG)\otimes S_*(X^n)\underset{\mathbb{Z}_2[G]}{\otimes}\mathbb{Z}_2 \; . \qquad \square$$

2.5) Lemma.- $S_*(EG)\otimes S_*(X^n)\underset{\mathbb{Z}_2[G]}{\otimes}\mathbb{Z}_2$ and $S_*(EG)\underset{\mathbb{Z}_2[G]}{\otimes}S_*(X^n)$ are naturally isomorphic.

Proof.- One can easily verify that $a\underset{\mathbb{Z}_2}{\otimes}b\underset{G}{\otimes}1\mapsto a\underset{G}{\otimes}b$ is a natural isomorphism. $\square$

Let us denote by $S_*(X)^{\otimes n}$ the tensor product $S_*(X)\otimes\ldots\otimes S_*(X)$, n-times. We would like to put $S_*(X)^{\otimes n}$ instead of $S_*(X^n)$ in $S_*(EG)\underset{\mathbb{Z}_2[G]}{\otimes}S_*(X^n)$ but there is no equivariant chain equivalence between $S_*(X)^{\otimes n}$ and $S_*(X^n)$. (This would imply for $n=2$ that all Steenrod squares are zero), so we need a generalisation of the Theorem of acyclic models due to Dyer and Lashof.

2.6) Definition.- Let C be a category and G a finite group. We say that C is a G-category if for each $g\in G$ we have a functor $\bar{g}: C\to C$ such that i) $\bar{e}=\mathrm{id}$ (where e is the zero of the group); ii) $\overline{g_1 g_2}=\bar{g}_1\circ\bar{g}_2$. If $\mathcal{Y}$ denotes the category of chain complexes and $F: C\to\mathcal{Y}$ is a functor, we say that F is a G-functor if for each $g\in G$ there is a natural transformation $\alpha_g: F\to F\circ\bar{g}$ such that i) $\alpha_e=\mathrm{id}$. ii) $\alpha_{g_1g_2}=\alpha_{g_1}\circ\alpha_{g_2}$.

Let W be a chain complex, we denote by $W^{(n)}$ its n-skeleton, i.e., $W^{(n)}{}_r=\begin{cases} W_r & \text{if } r\leq n \\ 0 & \text{if } r>n\,. \end{cases}$

If W and V are any $\mathbb{Z}_2[G]$-chain complexes, and K and L G-functors we make $W \otimes K$ and $V \otimes L$ into G-functors by having G act on both factors.

2.7) Theorem [16].- Let K, L be G-functors, W and V $\mathbb{Z}_2[G]$-chain complexes and $f: K \to L$ a natural transformation. If W is G-free, if K is free and L is acyclic (for some set of models $\mathcal{M} \subset C$) and if f is equivariant in dimension zero; then given any G-equivariant chain map $t: W \to V$.

a) There exists a natural G-equivariant chain map $F: W \otimes K \to V \otimes L$, satisfying:

1) $F(W^{(n)} \otimes K(X)) \subset V^{(n)} \otimes L(X)$, all n.

2) $F(w \otimes a) = t(w) \otimes f(a)$, $w \in W$, $a \in K_0(X)$

b) If $t, t_1: W \to V$ are G-equivariantly chain homotopic, and F, F_1 are any two chain maps satisfying 1) and 2) above for t, t_1 respectively, then F and F_1 are G-equivariantly chain homotopic.

c) We may further choose F so that given any zero dimensional G-generator e_0 of W, $F(e_0 \otimes a) = t(e_0) \otimes f(a)$, $a \in K(X)$.

$\square$

2.8) Proposition.- We have a natural chain homotopy equivalence

$$S_*(EG) \underset{\mathbb{Z}_2[G]}{\otimes} S_*(X^n) \xrightarrow{\simeq} B_* \underset{\mathbb{Z}_2[G]}{\otimes} S_*(X)^{\otimes n}$$, where B_* is the normalized Bar resolution of $\mathbb{Z}_2$ over the group G.

Proof.- We apply theorem 2.7 to the following case. C the category $T^n = T \times \ldots \times T$ where T is the category of topological spaces. We can make C into a G-category by defining for $\sigma \in G \subset \Sigma_n$, $\bar{\sigma}: C \to C$ by

$\sigma(X_1,\dots,X_n)=(X_{\sigma^{-1}(1)},\dots,X_{\sigma^{-1}(n)})$ and in the obvious way in morphisms. The functors are $K(X_1,\dots,X_n)=S_*(X_1\times\dots\times X_n)$ and $L(X_1,\dots,X_n)=S_*(X)^{\otimes n}$. Both are G-functors because for each $\sigma\in G$ we have natural transformations $\alpha_\sigma: K\to K\circ\bar{\sigma}$ and $\beta_\sigma: L\to L\circ\bar{\sigma}$ given by $\alpha_\sigma(X_1,\dots,X_n)=h_{\sigma\#}$ where $h_\sigma(x_1,\dots,x_n)=(x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(n)})$ and $\beta_\sigma(X_1,\dots,X_n)(a_1\otimes\dots\otimes a_n)=a_{\sigma^{-1}(1)}\otimes\dots\otimes a_{\sigma^{-1}(n)}$ (as we are working over $\mathbb{Z}_2$ there is no need to introduce the usual change of signs).

Both functors are clearly free and acyclic on the usual models $\mathfrak{M}=\{(\Delta_{q_1},\dots,\Delta_{q_n})\}_{q_i\geq 0}$. We take as $f:S_*(X_1\times\dots\times X_n)\to S_*(X)^{\otimes n}$ the chain equivalence given by the Eilenberg-Zilber theorem which in dimension zero is G-equivariant because it sends $(x_1,\dots x_n)$ to $x_1\otimes\dots\otimes x_n$.

Finally, as EG is contractible and has a free G-action $S_*(EG)$ is a free $\mathbb{Z}_2[G]$ resolution of $\mathbb{Z}_2$. The Bar resolution B_* is also a free resolution of $\mathbb{Z}_2$ over G so we have a G-equivariant chain equivalence $t: S_*(EG)\xrightarrow{\simeq} B_*$ [21] .

Then by 2.7) We have a natural G-equivariant chain equivalence $S_*(EG)\otimes S_*(X^n)\xrightarrow{\simeq} B_*\otimes S_*(X)^{\otimes n}$ which gives a natural chain equivalence $F: S_*(EG)\underset{\mathbb{Z}_2[G]}{\otimes} S_*(X^n)\xrightarrow{\simeq} B_*\underset{\mathbb{Z}_2[G]}{\otimes} S_*(X)^{\otimes n}$.

□

2.9) Proposition.- $B_*\underset{\mathbb{Z}_2[G]}{\otimes} S_*(X)^{\otimes n}$ is chain homotopy equivalent to $B_*\underset{\mathbb{Z}_2[G]}{\otimes} H_*(X)^{\otimes n}$, where we consider $H_*(X)^{\otimes n}$ as a chain complex with trivial boundary.

$\sigma(X_1,\ldots,X_n)=(X_{\sigma^{-1}(1)},\ldots,X_{\sigma^{-1}(n)})$ and in the obvious way in morphisms. The functors are $K(X_1,\ldots,X_n)=S_*(X_1\times\ldots\times X_n)$ and $L(X_1,\ldots,X_n)=S_*(X)^{\otimes n}$. Both are G-functors because for each $\sigma\in G$ we have natural transformations $\alpha_\sigma: K\to K\circ\bar\sigma$ and $\beta_\sigma: L\to L\circ\bar\sigma$ given by $\alpha_\sigma(X_1,\ldots,X_n)=h_{\sigma\#}$ where $h_\sigma(x_1,\ldots,x_n)=(x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(n)})$ and $\beta_\sigma(X_1,\ldots,X_n)(a_1\otimes\ldots\otimes a_n)=a_{\sigma^{-1}(1)}\otimes\ldots\otimes a_{\sigma^{-1}(n)}$ (as we are working over $\mathbb{Z}_2$ there is no need to introduce the usual change of signs).

Both functors are clearly free and acyclic on the usual models $\mathfrak{M}=\{(\Delta_{q_1},\ldots,\Delta_{q_n})\}_{q_i\geq 0}$. We take as $f:S_*(X_1\times\ldots\times X_n)\to S_*(X)^{\otimes n}$ the chain equivalence given by the Eilenberg-Zilber theorem which in dimension zero is G-equivariant because it sends $(x_1,\ldots x_n)$ to $x_1\otimes\ldots\otimes x_n$.

Finally, as EG is contractible and has a free G-action $S_*(EG)$ is a free $\mathbb{Z}_2[G]$ resolution of $\mathbb{Z}_2$. The Bar resolution B_* is also a free resolution of $\mathbb{Z}_2$ over G so we have a G-equivariant chain equivalence $t: S_*(EG)\xrightarrow{\simeq} B_*$ [21] .

Then by 2.7) We have a natural G-equivariant chain equivalence $S_*(EG)\otimes S_*(X^n)\xrightarrow{\simeq} B_*\otimes S_*(X)^{\otimes n}$ which gives a natural chain equivalence $F: S_*(EG)\underset{\mathbb{Z}_2[G]}{\otimes} S_*(X^n)\xrightarrow{\simeq} B_*\underset{\mathbb{Z}_2[G]}{\otimes} S_*(X)^{\otimes n}$.

□

2.9) Proposition.- $B_*\underset{\mathbb{Z}_2[G]}{\otimes} S_*(X)^{\otimes n}$ is chain homotopy equivalent to $B_*\underset{\mathbb{Z}_2[G]}{\otimes} H_*(X)^{\otimes n}$, where we consider $H_*(X)^{\otimes n}$ as a chain complex with trivial boundary.

Proof.- Consider id: $H_*(S_*(X),\partial) \to H_*(H_*(X),0)$, as we are working over $\mathbb{Z}_2$, there exists a chain equivalence $\alpha : S_*(X) \to H_*(X)$ such that $\alpha_* = \mathrm{id}$. Recall that B_* is G-free, then the fact that $\mathrm{id}\otimes\alpha^{\otimes n}$ is a chain equivalence follows from lemma 5.2 of [45] which says that if 2 chain maps $f_0, f_1 : M \to N$ are chain homotopic and W is a G-free chain complex then $\mathrm{id}\otimes_G f_0^{\otimes n}, \mathrm{id}\otimes_G f_1^{\otimes n} : W\otimes_G M^{\otimes n} \to W\otimes_G N^{\otimes n}$ are chain homotopic.

□

2.10) Corollary.- There exists a natural isomorphism

$$H_*(EG \times_G X^n) \cong H_*(B_* \otimes_{\mathbb{Z}_2[G]} H_*(X)^{\otimes n}) .$$

Proof.- by 2.2, 2.4, 2.5 and 2.8 we have

$$S_*(EG \times_G X^n) \cong S_*(EG \times X^n) \otimes_{\mathbb{Z}_2[G]} \mathbb{Z}_2 \cong S_*(EG) \otimes_{\mathbb{Z}_2} S_*(X^n) \otimes_{\mathbb{Z}_2[G]} \mathbb{Z}_2 \cong$$

$$\cong S_*(EG) \otimes_{\mathbb{Z}_2[G]} S_*(X^n) \cong B_* \otimes_{\mathbb{Z}_2[G]} S_*(X)^{\otimes n}$$ which are all natural, and

by 2.9) $B_* \otimes_{\mathbb{Z}_2[G]} S_*(X)^{\otimes n} \simeq B_* \otimes_{\mathbb{Z}_2[G]} H_*(X)^{\otimes n}$, in this case if we have a map $f: X \to Y$ then

$$\begin{array}{ccc} S_*(X) & \xrightarrow{\alpha} & H_*(X) \\ \downarrow f_\# & & \downarrow f_* \\ S_*(Y) & \xrightarrow[\alpha]{} & H_*(Y) \end{array}$$

is chain homotopy commutative because the maps induced in homology $(f_* \circ \alpha)_* = f_* \circ \alpha_* = f_* = (\alpha \circ f_\#)_*$ are the same [30], hence $(\mathrm{id} \otimes_G f_*^{\otimes n}) \circ (\mathrm{id} \otimes_G \alpha^{\otimes n}) \simeq (\mathrm{id} \otimes_G f_\#^{\otimes n}) \circ (\mathrm{id} \otimes_G f_*^{\otimes n})$ so the isomorphism is natural.

Now we want to see that the isomorphism 2.10 is independent of the choice of EG.

2.11) Proposition.- Let $E'G$ be a contractible space with a free G-action and $\phi: EG \to E'G$ a G-equivariant map. Let $\phi \underset{G}{\times} id : EG \underset{G}{\times} X^n \to E'G \underset{G}{\times} X^n$ be the induced map, then the following diagram commutes:

$$\begin{tikzcd}
H_*(EG \underset{G}{\times} X^n) \arrow[r, "\cong"] \arrow[d, "(\phi \underset{G}{\times} id)_*"] & H_*(B_* \underset{\mathbb{Z}_2[G]}{\otimes} H_*(X)^{\otimes n}) \\
H_*(E'G \underset{G}{\times} X^n) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Proof.- Consider the following diagram:

$$\begin{tikzcd}[column sep=small]
S_*(EG \underset{G}{\times} X^n) \arrow[r, phantom, "\cong"] \arrow[d, "(\phi \underset{G}{\times} id)_\#"] & S_*(EG \times X^n) \underset{G}{\otimes} \mathbb{Z}_2 \arrow[r, phantom, "\cong"] \arrow[d, "(\phi \times id)_\# \underset{G}{\otimes} id"] & S_*(EG) \otimes S_*(X^n) \underset{G}{\otimes} \mathbb{Z}_2 \arrow[r, phantom, "\cong"] \arrow[d, "\phi_\# \otimes id \underset{G}{\otimes} id"] & S_*(EG) \underset{G}{\otimes} S_*(X^n) \arrow[r, phantom, "\simeq"] \arrow[d, "\phi_\# \otimes id"] & B_* \underset{G}{\otimes} S_*(X)^{\otimes n} \arrow[d, "id"] \\
S_*(E'G \underset{G}{\times} X^n) \arrow[r, phantom, "\cong"] & S_*(E'G \times X^n) \underset{G}{\otimes} \mathbb{Z}_2 \arrow[r, phantom, "\cong"] & S_*(E'G) \otimes S_*(X^n) \underset{G}{\otimes} \mathbb{Z}_2 \arrow[r, phantom, "\cong"] & S_*(E'G) \underset{G}{\otimes} S_*(X^n) \arrow[r, phantom, "\simeq"] & B_* \underset{G}{\otimes} S_*(X)^{\otimes n}
\end{tikzcd}$$

The first 3 squares are clearly commutative (the second one by naturality of the Eilenberg Zilber Theorem). We will show that the fourth one is chain homotopy commutative. For this we shall apply 2.7) part b).

We have

$$\begin{tikzcd}
S_*(EG) \arrow[r, "t"] \arrow[d, "\phi_\#"] & B_* \\
S_*(E'G) \arrow[ur, "t'"'] &
\end{tikzcd}$$

, as the 3 chain complexes are resolutions of $\mathbb{Z}_2$ over G, we have chain equivalences t and t'. If we consider t and $t' \circ \phi_\#$, then both commute with the augmentation and both lift $id: \mathbb{Z}_2 \to \mathbb{Z}_2$ hence $t' \circ \phi_\# \simeq t$.

Now we apply 2.7) as we did before to get chain equivalences

$F: S_*(EG \otimes_G S_*(X^n) \simeq B_* \otimes_G S_*(X)^{\otimes n}$ associated with t and

$F': S_*(E'G) \otimes_G S_*(X^n) \simeq B_* \otimes_G S_*(X)^{\otimes n}$ associated with t'. So now consider

$F' \circ \phi_\# \otimes_G id$. As $\phi_\# \otimes_G id$ preserves the filtration then so does $F' \circ \phi_\# \otimes_G id$

and $F' \circ \phi_\# \otimes_G id(w \otimes a) = F'(\phi_\#(w) \otimes a) = t'\phi_\#(w) \otimes f(a)$, for all $w \in S_*(EG)$,

$a \in S_0(X^n)$. Hence $F' \circ (\phi_\# \otimes_G id)$ satisfies i)and ii) of 2.7) with respect

to $t' \circ \phi_\#$, but we saw that $t' \circ \phi_\# \simeq t$ so by 2.7. b) $F' \circ (\phi_\# \otimes_G id) \simeq F$. □

§2.2 Dyer-Lashof operations in homology

Let $\mathscr{C}$ be an E_∞-operad and X a $\mathscr{C}$-space [34]. We can then define natural homomorphisms $Q_i: H_n(X; \mathbb{Z}_2) \to H_{2n+i}(X; \mathbb{Z}_2)$ for all $i, n \geq 0$ called Dyer-Lashof operations as follows. As X is a $\mathscr{C}$-space we have structure maps $\mathscr{C}(r) \times_{\Sigma_r} X^r \xrightarrow{\theta_r} X$, where the spaces $\mathscr{C}(r)$ are contractible with a free Σ_r-action. We are interested in the case $r=2$.

By 2.10) We have a natural isomorphism

$H_*(\mathscr{C}(2) \times_{\Sigma_2} X^2; \mathbb{Z}_2) \simeq H_*(B_* \otimes_{\mathbb{Z}_2[\Sigma_2]} H_*(X)^{\otimes 2})$ where B_* is the normalized Bar resolution for $\mathbb{Z}_2$ over Σ_2 so B_n is a free $\mathbb{Z}_2[\Sigma_2]$ - module in one generator e_n. If we denote $\Sigma_2 = \{1, T\}$ then $\partial(e_n) = (1+T)e_{n-1}$, with this notation we have:

2.12) Proposition [16]. Let $\{a_j\}_{j \in J}$ be an ordered basis for $H_*(X; \mathbb{Z}_2)$ then a $\mathbb{Z}_2$-basis for $H_*(B_* \otimes_{\mathbb{Z}_2[\Sigma_2]} H_*(X)^{\otimes 2})$ is given by the following elements

$$\begin{cases} e_r \underset{\Sigma_2}{\otimes} a_j \otimes a_j \ , & r \geq 0 \quad j \in J \\ e_0 \underset{\Sigma_2}{\otimes} a_j \otimes a_k \ , & j < k \end{cases}$$

□

2.13) Lemma.- We have a homomorphism $h_i : H_n(X;\mathbb{Z}_2) \to$ $\to H_{2n+i}(B_* \underset{\mathbb{Z}_2[\Sigma_2]}{\otimes} H_*(X)^{\otimes 2})$ given by $h_i(a) = e_i \underset{\Sigma_2}{\otimes} a \otimes a$.

Proof.- $h_i(a+b) = e_i \underset{\Sigma_2}{\otimes} (a+b)\otimes(a+b) = e_i \underset{\Sigma_2}{\otimes} a\otimes a + e_i \underset{\Sigma_2}{\otimes} b\otimes b + e_i \underset{\Sigma_2}{\otimes} (a\otimes b + b\otimes a)$,

but $\partial(e_{i+1} \underset{\Sigma_2}{\otimes} a \otimes b) = (1+T)e_{i+1} \underset{\Sigma_2}{\otimes} a\otimes b = e_i \underset{\Sigma_2}{\otimes} (a\otimes b + b\otimes a)$, hence

$h_i(a+b) = h_i(a) + h_i(b)$.

□

2.14) Definition.- Let $\mathcal{C}$ be an E_∞-operad and X a $\mathcal{C}$-space, then with the notation as above, we define homomorphisms $Q_i : H_n(X;\mathbb{Z}_2) \to$ $\to H_{2n+i}(X;\mathbb{Z}_2)$ by the composition $H_n(X) \xrightarrow{h_i} H_{2n+i}(B_* \underset{\mathbb{Z}_2[\Sigma_2]}{\otimes} H_*(X)^{\otimes 2}) \to$ $\xrightarrow{\cong} H_{2n+i}(\mathcal{C}(2) \underset{\Sigma_2}{\times} X^2) \xrightarrow{\theta_{2*}} H_{2n+i}(X)$.

2.15) Theorem [11,16,13]: Let $Q^s : H_n(X;\mathbb{Z}_2) \to H_{n+s}(X;\mathbb{Z}_2)$ be defined by $Q^s = Q_{s-n}$, where Q_{s-n} is as in 2.14, then the operations Q^s satisfy:

i) The Q^s are natural with respect to maps of $\mathcal{C}$-spaces.

ii) $Q^s(X) = 0$ if $\deg x > s$.

iii) $Q^s(X) = x^2$ if $\deg x = s$.

iv) Cartan formula: $Q^s(x\ y) = \sum_{i+j=s} Q^i(x)\ Q^j(y)$.

v) $Q^s\sigma = \sigma\, Q^s$, where $\sigma : \widetilde{H}_n(\Omega X) \to \widetilde{H}_{n+1}(X)$ is the homology suspension.

vi) Adem relations: if $r > 2s$ then $Q^r Q^s = \sum_i \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i$

vii) Nishida relations: Let S^r_{q*} be the dual of the Steenrod square S^r_q then $S^r_{q*} Q^{r+s} = \sum_i \binom{s}{r-2i} Q^{s+i} S^i_{q*}$.

□

2.16) Note.- If X is an infinite loop space, i.e., if we have spaces Y_r such that $X \cong \Omega^r Y_r$, $r \geq 1$: then X is a $\mathcal{C}_\infty$-space, where $\mathcal{C}_\infty$ is the cubes operad which is an E_∞-operad. The structure maps are defined as follows [34]:

$\theta_n : \mathcal{C}_r(n) \times (\Omega^r Y_r)^n \to \Omega^r Y_r$ is given by

$\theta_n((c_1,\dots,c_n),(\alpha_1\dots\alpha_n)):(I^r,\partial I^r) \to (Y_r, *)$ is the map sending any $x \in \operatorname{Im} c_i$ to $\alpha_i(c_i^{-1}(x))$ and any point outside $\bigcup_{i=1}^n \operatorname{Im} c_i$ to $*$.

These actions are compatible for different r giving an action of $\mathcal{C}_\infty = \lim_r \mathcal{C}_r$. So if X is an infinite loop space we have Dyer-Lashof operations $Q^s: H_n(X;\mathbb{Z}_2) \to H_{n+s}(X;\mathbb{Z}_2)$ for all $n,s \geq 0$.

§2.3) Dyer-Lashof operations in Bordism

In this section we shall define Dyer-Lashof operations in the bordism of a $\mathcal{C}_\infty$-space ($\mathcal{C}_\infty$, the cubes operad). For this we begin by defining functions $\tilde{q}^n_r: N_s(X) \to N_{2s+r}(S^n \times_{\Sigma_2} X \times X)$.

2.17) Definition.- Let (M,f) be a pair where $f:M \to X$ is a map from a closed smooth s-manifold to a space X, define the map $\psi^n_r(M,f)$ by $S^r \times_{\Sigma_2} M^2 \xrightarrow{\iota \times_{\Sigma_2} f^2} S^n \times_{\Sigma_2} X^2$, where $\iota : S^r \hookrightarrow S^n$ is the inclusion, $r \leq n \leq \infty$.

On disjoint unions ψ^n_r satisfies the following: $\psi^n_r(M \amalg N,\ f \amalg g) =$

$= S^r \times_{\Sigma_2} (M \amalg N)^2 \xrightarrow[\iota \times_{\Sigma_2} (f \amalg g)^2]{} S^n \times_{\Sigma_2} X^2$ which can be written as:

$(S^r \times_{\Sigma_2} M^2,\ \iota \times_{\Sigma_2} f^2) \amalg (S^r \times_{\Sigma_2} N^2, \iota \times_{\Sigma_2} g^2) \amalg (S^r \times M \times N,\ \phi)$ where

$\phi = p \circ \iota \times f \times g$, p the projection of the double covering.

2.18) Lemma.- Suppose (M,f) is bordant to (N,g) then $(S^r \times M \times N, \phi)$ bords in $S^n \times_{\Sigma_2} X^2$.

Proof.- As $(M,f) \sim (N,g)$ then we have a manifold V and a map $F: V \to X$ such that $\partial V \cong M \amalg N$ and $F|M = f$, $F|N = g$. Consider $M \times V \amalg V \times M$, then $\partial(M \times V) = M \times M \amalg M \times N$ and $\partial(V \times M) = M \times M \amalg N \times M$, taking $id: M \times M \to M \times M$ we can glue $M \times V$ to $V \times M$ [9] to form the manifold $M \times V \underset{id_{M \times M}}{\cup} V \times M$, whose boundary is $M \times N \amalg N \times M$, we have an action of Σ_2 on this manifold coming from the action on $M \times V \amalg V \times M$ given by $T(x,z) = (z,x)$.

We also have a map $M \times V \amalg V \times M \xrightarrow[f \times F \amalg F \times f]{} X \times X$, as $F|M = f$ this map passes to the quotient to give a Σ_2-equivariant map

$M \times V \underset{id_{M \times M}}{\cup} V \times M \xrightarrow[f \times F \underset{id_{M \times M}}{\cup} F \times f]{} X \times X$. So we can define

$$S^r \times_{\Sigma_2} (M \times V \underset{id_{M \times M}}{\cup} V \times M) \xrightarrow[\iota \times_{\Sigma_2} (f \times F \underset{id_{M \times M}}{\cup} F \times f)]{} S^n \times_{\Sigma_2} X \times X.$$

We clearly have $\partial(S^r \times_{\Sigma_2} (M \times V \underset{id_{M \times M}}{\cup} V \times M)) \cong S^r \times M \times N$ and as $F|N = g$ we have a commutative diagram:

On disjoint unions ψ^n_r satisfies the following: $\psi^n_r(M \amalg N,\ f \amalg g) =$

$= S^r \times_{\Sigma_2} (M \amalg N)^2 \xrightarrow[\iota \times_{\Sigma_2} (f \amalg g)^2]{} S^n \times_{\Sigma_2} X^2$ which can be written as:

$(S^r \times_{\Sigma_2} M^2,\ \iota \times_{\Sigma_2} f^2) \amalg (S^r \times_{\Sigma_2} N^2, \iota \times_{\Sigma_2} g^2) \amalg (S^r \times M \times N,\ \phi)$ where

$\phi = p \circ \iota \times f \times g$, p the projection of the double covering.

2.18) Lemma.- Suppose (M,f) is bordant to (N,g) then $(S^r \times M \times N, \phi)$ bords in $S^n \times_{\Sigma_2} X^2$.

Proof.- As $(M,f) \sim (N,g)$ then we have a manifold V and a map $F: V \to X$ such that $\partial V \cong M \amalg N$ and $F|M = f$, $F|N = g$. Consider $M \times V \amalg V \times M$, then $\partial(M \times V) = M \times M \amalg M \times N$ and $\partial(V \times M) = M \times M \amalg N \times M$, taking $id: M \times M \to M \times M$ we can glue $M \times V$ to $V \times M$ [9] to form the manifold $M \times V \underset{id_{M \times M}}{\cup} V \times M$, whose boundary is $M \times N \amalg N \times M$, we have an action of Σ_2 on this manifold coming from the action on $M \times V \amalg V \times M$ given by $T(x,z) = (z,x)$.

We also have a map $M \times V \amalg V \times M \xrightarrow[f \times F \amalg F \times f]{} X \times X$, as $F|M = f$ this map passes to the quotient to give a Σ_2-equivariant map

$M \times V \underset{id_{M \times M}}{\cup} V \times M \xrightarrow[f \times F \underset{id_{M \times M}}{\cup} F \times f]{} X \times X$. So we can define

$$S^r \times_{\Sigma_2} (M \times V \underset{id_{M \times M}}{\cup} V \times M) \xrightarrow[\iota \times_{\Sigma_2} (f \times F \underset{id_{M \times M}}{\cup} F \times f)]{} S^n \times_{\Sigma_2} X \times X.$$

We clearly have $\partial(S^r \times_{\Sigma_2} (M \times V \underset{id_{M \times M}}{\cup} V \times M)) \cong S^r \times M \times N$ and as $F|N = g$ we have

a commutative diagram:

$$\begin{tikzcd}
S^r \times_{\Sigma_2} (M\times V \cup_{id_{M\times M}} V\times M) \arrow[rr, "\iota \times_{\Sigma_2} (f\times F \cup_{id_{M\times M}} F\times f)"] & & S^n \times_{\Sigma_2} X\times X \\
S^r \times_{\Sigma_2} (M\times N \amalg N\times M) \arrow[u, "\iota"] \arrow[rr, "\cong"'] \arrow[urr, "\bar{\Phi}"] & & S^r \times M\times N \arrow[u, "\Phi"']
\end{tikzcd} \quad , \quad \bar{\Phi} = \iota \times_{\Sigma_2} (f\times g) \amalg (g\times f)$$

2.19) Remark.- Notice that if $r < n$, then $(S^r\times M\times N, \ )$ bords in $S^n \times_{\Sigma_2} X^2$ even without the asumption $(M,f) \sim (N,g)$. We only have to consider an extension $\bar{\iota}$:

$$\begin{tikzcd}
S^r \arrow[r, hook, "\iota"] \arrow[d, hook] & S^n \\
D^{r+1} \arrow[ur, dashed, "\bar{\iota}"']
\end{tikzcd}$$

and then

$$\begin{tikzcd}
S^r \times M\times N \arrow[r, "p\circ(\iota\times f\times g)"] \arrow[d] & S^n \times_{\Sigma_2} X\times X \\
D^{r+1} \times M\times N \arrow[ur, "p\circ(\bar{\iota}\times f\times g)"']
\end{tikzcd}$$

commutes. $\square$

2.20) Proposition.- If (M,f) bords in X then $\psi_r^n(M,f)$ bords in $S^n \times_{\Sigma_2} X\times X$.

Proof.- As (M,f) bords in X we have a manifold V and a map $F\colon V \to X$ such that $\partial V \cong M$ and $F|M = f$.

Let D_+^r and D_-^r be the upper and lower hemispheres of S^r respectively. We denote $\iota_+ = \iota|D_+^r$ and $\iota_- = \iota|D_-^r$.

By straightening the corners [9] we can define manifolds with boundary and maps: $D_+^r\times M\times V \xrightarrow{\iota_+\times f\times F} S^n \times X\times X$; $S^{r-1}\times V\times V \xrightarrow{\iota\times F\times F} S^n\times X\times X$; $D_-^r\times V\times M \xrightarrow{\iota_-\times F\times f} S^n\times X\times X$.

We have a Σ_2-action on the union of these manifolds given by $T(t,x,y) = (-t,y,x)$ sending $D_+^r\times M\times V$ to $D_-^r\times V\times M$ and $T(t,y_1,y_2) = (-t.y_2,y_1)$ sending $S^{r-1}\times V\times V$ to itself, the maps are clearly equivariant. The boundaries are as follows:

$\partial(D^r_+ \times M \times V) = S^{r-1}\times M\times V \underset{S^{r-1}\times M\times M}{\cup} D^r_+ \times M\times M$; $\partial(S^{r-1}\times V\times V) =$

$= S^{r-1}\times M\times V \underset{S^{r-1}\times M\times M}{\cup} S^{r-1}\times V\times M$; $\partial(D^r_- \times V\times M) = S^{r-1}\times V\times M \underset{S^{r-1}\times M\times M}{\cup} D^r_- \times M\times M.$

Now $S^{r-1}\times M\times V \subset \partial(D^r_+ \times M\times V)$ and $S^{r-1}\times M\times V \subset \partial(S^{r-1}\times V\times V)$, a collar of $S^{r-1}\times M$ in $D^r_+ \times M$ gives a collar for $S^{r-1}\times M\times V$ in $D^r_+ \times M\times V$ and a collar of $S^{r-1}\times M$ in $S^{r-1}\times V$ gives a collar for $S^{r-1}\times M\times V$ in $S^{r-1}\times V\times V$; the same applies for $S^{r-1}\times V\times M \subset \partial(S^{r-1}\times V\times V)$ and $S^{r-1}\times V\times M \subset \partial(D^r_- \times V\times M)$. Hence we can form the smooth manifold:

$D^r_+ \times M\times V \underset{S^{r-1}\times M\times V}{\cup} S^{r-1}\times V\times V \underset{S^{r-1}\times V\times M}{\cup} D^r_- \times V\times M$, whose boundary is

$D^r_+ \times M\times M \underset{S^{r-1}\times M\times M}{\cup} D^r_- \times M\times M \cong S^r\times M\times M$. This manifold has a free Σ_2-action coming from the one we gave above and we have a Σ_2-equivariant map to $S^n\times X\times X$ from the maps defined before. Passing to the quotient we get a manifold with boundary $S^r \underset{\Sigma_2}{\times} M\times M$ and a map to $S^n \underset{\Sigma_2}{\times} X\times X$ extending $\iota \underset{\Sigma_2}{\times} f\times f$. □

2.21) Definition.- We define $\tilde{q}^n_r : N_s(X) \to N_{2s+r}(S^n \underset{\Sigma_2}{\times} X\times X)$, $r \le n \le \infty$. by $\tilde{q}^n_r[M,f] = [\psi^n_r(M,f)] = [S^r \underset{\Sigma_2}{\times} M^2, \iota \underset{\Sigma_2}{\times} f^2]$. These operations were defined, when X is a closed manifold, in [51] .

2.22) Proposition.- $\tilde{q}^n_r$ is well defined and is natural.

Proof.- $[M,f] = [N,g]$ <=> $(M,f) \sim (N,g)$ <=> $(M \amalg N, f \amalg g)$ bords in X, by 2.20 $\psi^n_r(M \amalg N, f \amalg g)$ bords in $S^n \underset{\Sigma_2}{\times} X\times X$. But $\psi^n_r(M \amalg N, f \amalg g) =$ $= \psi^n_r(M,f) \amalg \psi^n_r(N,g) \amalg (S^r\times M\times N\ \phi)$ and by 2.18) $(S^r\times M\times N, \phi)$ bords in $S^n \underset{\Sigma_2}{\times} X\times X$. Hence $\psi^n_r(M,f) \sim \psi^n_r(N,g)$. The naturality is clear. □

2.23) Proposition.- If $r < n$, $\tilde{q}^n_r$ is a homomorphism.

Proof.- $\tilde{q}^n_r([M,f]+[N,g]) = \tilde{q}^n_r[M \amalg N, f \amalg g] = [\psi^n_r(M \amalg N, f \amalg g)] =$

$= [\psi^n_r(M,f) \amalg \psi^n_r(N,g) \amalg (S^r \times M \times N, \varphi)]$. As $r < n$ then, by 2.19, $(S^r \times M \times N, \phi)$ bords in $S^n \times_{\Sigma_2} X^2$, hence $\tilde{q}^n_r$ is a homomorphism.

□

2.24) Definition.- Let X be a $\mathscr{C}_\infty$-space then we define operations $\tilde{Q}_r : N_n(X) \to N_{2n+r}(X)$ as follows: $\mathscr{C}_\infty(2)$ is Σ_2-equivariantly homotopy equivalent to S^∞ [34], so we have a homotopy equivalence $S^\infty \times_{\Sigma_2} X \times X \simeq \mathscr{C}_\infty(2) \times_{\Sigma_2} X \times X$. $\tilde{Q}_r$ is the composition:

$$N_n(X) \xrightarrow{\tilde{q}^\infty_r} N_{2n+r}(S^\infty \times_{\Sigma_2} X \times X) \xrightarrow{\cong} N_{2n+r}(\mathscr{C}_\infty(2) \times_{\Sigma_2} X \times X) \xrightarrow{\theta_{2*}} N_{2n+r}(X).$$

We can use upper indices as we did with the operations in homology, we define $\tilde{Q}^r : N_n(X) \to N_{n+r}(X)$ by $\tilde{Q}^r = \tilde{Q}_{r-n}$.

2.25) Theorem [2,39].- Let X be a $\mathscr{C}_\infty$-space then the operations $\tilde{Q}^r : N_n(X) \to N_{n+r}(X)$ satisfy:

i) They are natural with respect to maps of $\mathscr{C}_\infty$-spaces.

ii) $\tilde{Q}^r(X) = 0$ if $\deg x > r$.

iii) $\tilde{Q}^r(X) = x^2$ if $\deg x = r$.

iv) Cartan formula: $\tilde{Q}_r(x \cdot y) = \sum_{0 \le i+j \le n} \left(\sum_{m_k} \left(\prod_k P^{2^k}_{m_k} \right) \right) \tilde{Q}_i(x)\, \tilde{Q}_j(y)$

$$\text{where } \sum_{k \ge 0} m_k 2^k = n-(i+j)$$

v) $\tilde{Q}^r \sigma = \sigma \tilde{Q}^r$ where $\sigma : \tilde{N}_n(\Omega X) \to \tilde{N}_{n+1}(X)$

vi) Adem relations: Let $a \in N_n(X)$ then in $N_{4n+z}(S^\infty \times_{\Sigma_2} (S^\infty \times_{\Sigma_2} X \times X)^2)$, there are relations modulo decomposable classes of the form:

$\tilde{q}_r^\infty \tilde{q}_s^\infty(a) = \sum_{i=0}^{p} \binom{k-i-1}{t-k-2i} \tilde{q}_{j+i}^\infty \tilde{q}_{j+t-i}^\infty(a)$ for each r,s such that $2s+r=z$ and j,t are respectively the smallest and largest integers ≥ 0 such that $3j + 2t = z$ and k is determined by $r = j+2k$, $s=j+t-k$ and if $t = 3p+q$, $q=0,1,2$ then $p < k \leq t$.

□

§2.4) Relation between the operations in bordism and in homology

To give this relation we study first $H_*(S^n \times_{\Sigma_2} X \times X; \mathbb{Z}_2)$, for this we only need 2 new propositions.

2.26) Proposition.- Let $B_*^{(n)}$ be the n-skeleton of the Bar resolution for $\mathbb{Z}_2$ over Σ_2, then we have a natural chain equivalence
$B_*^{(n)} \otimes_{\mathbb{Z}_2[\Sigma_2]} S_*(X)^{\otimes 2} \simeq S_*(S^n) \otimes_{\mathbb{Z}_2[\Sigma_2]} S_*(X^2)$.

Proof.- We define a Σ_2-equivariant chain map $\tilde{t}: B_*^{(n)} \to S_*(S^n)$ as follows: as $B_0^{(n)}$ is Σ_2-free with generator e_0, we define $\tilde{t}_0(e_0) = x_0$, $x_0 \in S^n$. As $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z}_2 & \text{if } i=n \\ 0 & \text{otherwise} \end{cases}$ we can define $\tilde{t}_1, \ldots, \tilde{t}_{n-1}$, and $\tilde{t}_n$ by sending $1\, e_n + Te_n$ to the generator of $H_n(S^n)$. We put $\tilde{t}_i = 0$ if $i > n$. Then $\tilde{t}$ is a chain map such that $H_i(B_*^{(n)}) \xrightarrow{\tilde{t}_*} H_i(S^n)$ is an isomorphism for all i. As both complexes are Σ_2-free then $\tilde{t}$ is a chain equivalence. Applying 2.7 as we did in 2.8 we get the natural chain equivalence $B^{(n)} \otimes_{\mathbb{Z}_2[\Sigma_2]} S_*(X)^{\otimes 2} \simeq S_*(S^n) \otimes_{\mathbb{Z}_2[\Sigma_2]} S_*(X^2)$.

2.27) Proposition.- We have a chain homotopy commutative diagram

$$\begin{tikzcd}
B_*^{(n)} \underset{\mathbb{Z}_2[\Sigma_2]}{\otimes} S_*(X)^{\otimes 2} \arrow[r, "\simeq"] \arrow[d, hook, "\iota \underset{\Sigma_2}{\otimes} \mathrm{id}"] & S_*(S^n) \underset{\mathbb{Z}_2[\Sigma_2]}{\otimes} S_*(X^2) \arrow[d, "i_\# \underset{\Sigma_2}{\otimes} \mathrm{id}"] \\
B_* \underset{\mathbb{Z}_2[\Sigma_2]}{\otimes} S_*(X)^{\otimes 2} \arrow[r, "\simeq"] & S_*(S^\infty) \underset{\mathbb{Z}_2[\Sigma_2]}{\otimes} S_*(X^2)
\end{tikzcd}$$

where $i: S^n \hookrightarrow S^\infty$, and the equivalences are the ones from 2.26 and 2.8.

Proof.- We want to apply 2.7 b). For this consider

$$\begin{tikzcd}
B_*^{(n)} \arrow[r, "\tilde{t}", "\simeq"'] \arrow[d, "\iota"'] & S_*(S^n) \arrow[d, "i"] \\
B_* \arrow[r, "t", "\simeq"'] & S_*(S^\infty)
\end{tikzcd}$$

$\tilde{t}$ was constructed in 2.26 and t in 2.8. Consider $i_\# \circ \tilde{t}$ and $t \circ \iota : B_*^{(n)} \to S_*(S^\infty)$, both compositions clearly lift $\mathrm{id}: \mathbb{Z}_2 \to \mathbb{Z}_2$, as $B_*^{(n)}$ is Σ_2-free and $S_*(S^\infty)$ is a resolution of $\mathbb{Z}_2$ then [30] $i_\# \circ \tilde{t} \simeq t \circ \iota$. Now apply 2.7 to the case $W = B_*^{(n)}$, $K(X) = S_*(X)^{\otimes 2}$ $V = S_*(S^\infty)$, $L = S_*(X^2)$. We have chain maps:

$$\begin{tikzcd}
B_*^{(n)} \otimes S_*(X)^{\otimes 2} \arrow[r, "G", "\simeq"'] \arrow[dr, "\iota \otimes \mathrm{id}"'] & S_*(S^n) \otimes S_*(X^2) \arrow[r, "i_\# \otimes \mathrm{id}"] & S_*(S^\infty) \otimes S_*(X^2) \\
& B_* \otimes S_*(X)^{\otimes 2} \arrow[ur, "\simeq", "H"'] &
\end{tikzcd}$$

Notice that by construction G is natural and satisfies 2.7) i), ii) with respect to $\tilde{t}$, as $i_\# \otimes \mathrm{id}$ clearly preserves the filtration so does $i_\# \otimes \mathrm{id} \circ G$; and if $w \in B_*^{(n)}$, $a \in S_0(X)^{\otimes 2}$ then $i_\# \otimes \mathrm{id} \circ G(w \otimes a) =$ $= i_\# \otimes \mathrm{id}(\tilde{t}(a) \otimes f(a)) = i_\# \tilde{t}(a) \otimes f(a)$, where $f: S_*(X)^{\otimes 2} \xrightarrow{\simeq} S_*(X^2)$, so $i_\# \otimes \mathrm{id} \circ G$ satisfies i), ii) with respect to $i_\# \circ \tilde{t}$. Similarly $H \circ \iota \otimes \mathrm{id}$ satisfies i,ii) with respect to $t \circ \iota$. Hence by 2.7 b) $(i_\# \underset{\Sigma_2}{\otimes} \mathrm{id}) \circ G \simeq H\, (\iota \underset{\Sigma_2}{\otimes} \mathrm{id})$. $\square$

2.28) Lemma.- We have a natural isomorphism

$$H_*(S^n \underset{\Sigma_2}{\times} X\times X) \cong H_*(B_*^{(n)} \underset{\mathbb{Z}_2[\Sigma_2]}{\otimes} H_*(X)^{\otimes 2}).$$

Proof.- The same as 2.10, except that instead of using 2.8, we use 2.26. □

2.29) Definition.- Define natural functions $q_r^n : H_s(X) \to H_{2s+r}(S^n \underset{\Sigma_2}{\times} X\times X)$, $0\le r\le n\le\infty$ by the composition: $H_s(X) \xrightarrow{h_r^n} H_{2s+r}(B_*^{(n)} \underset{\Sigma_2}{\otimes} H_*(X)^{\otimes 2}) \xrightarrow{\cong} H_{2s+r}(S^n \underset{\Sigma_2}{\times} X\times X)$ where $h_r^n(a) = e_r \underset{\Sigma_2}{\otimes} a\otimes a$. When $r < n$, then the h_r^n are homomorphisms, as in 2.13, and hence the q_r^n are also homomorphisms.

2.30) Proposition.- Let $i: S^n \hookrightarrow S^\infty$ be the inclusion, then we have a commutative diagram

$$\begin{tikzcd}
H_s(X) \arrow[r, "q_r^n"] \arrow[dr, "q_r^\infty"'] & H_{2s+r}(S^n \underset{\Sigma_2}{\times} X\times X) \arrow[d, "(i \underset{\Sigma_2}{\times} id)_*"] \\
& H_{2s+r}(S^\infty \underset{\Sigma_2}{\times} X\times X)
\end{tikzcd}$$

Proof.- The definition of q_r^n and q_r^∞ gives the following diagram

$$\begin{tikzcd}
H_s(X) \arrow[r, "h_r^n"] \arrow[dr, "h_r^\infty"'] & H_{2s+r}(B_*^{(n)} \underset{\Sigma_2}{\otimes} H_*(X)^{\otimes 2}) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "(i \underset{\Sigma_2}{\times} id)_*"] & H_{2s+r}(S^n \underset{\Sigma_2}{\times} X\times X) \arrow[d, "(i \underset{\Sigma_2}{\times} id)_*"] \\
& H_{2s+r}(B_* \underset{\Sigma_2}{\otimes} H_*(X)^{\times 2}) \arrow[r, "\cong"'] & H_{2s+r}(S^\infty \underset{\Sigma_2}{\times} X\times X)
\end{tikzcd}$$

The triangle is clearly commutative, one can easily check that the equivalences used in the definition of the isomorphisms commute, the only one that is not immediate follows from 2.27 .

□

2.31) Proposition.- Suppose M is a connected manifold of dimension m with fundamental class $\sigma(M)$, then $\sigma(S^n \underset{\Sigma_2}{\times} M\times M) = q_n^n(\sigma(M))$.

Proof.- Consider $q_n^n : H_m(M) \to H_{2m+n}(S^n \underset{\Sigma_2}{\times} M\times M)$. As M is connected then $S^n \underset{\Sigma_2}{\times} M\times M$ is also connected so that $H_{2m+n}(S^n \underset{\Sigma_2}{\times} M\times M) \cong \mathbb{Z}_2$. Consider the inclusion $i \underset{\Sigma_2}{\times} id : S^n \underset{\Sigma_2}{\times} M\times M \hookrightarrow S^\infty \underset{\Sigma_2}{\times} M\times M$.

By 2.30) $(i \underset{\Sigma_2}{\times} id)_* q_n^n = q_n^\infty$ hence $(i \underset{\Sigma_2}{\times} id)_* \circ q_n^n(\sigma(M)) = q_n(\sigma(M)) \neq 0$ because by 2.12 it belongs basis for the homology of $S^\infty \underset{\Sigma_2}{\times} M\times M$, hence $q_n^n(\sigma(M)) \neq 0$ so $q_n^n(\sigma(M)) = \sigma(S^n \underset{\Sigma_2}{\times} M\times M)$.

□

2.32) Proposition.-

$$\begin{tikzcd} N_s(X) \arrow[r, "\tilde{q}_r^\infty"] \arrow[d, "\mu"'] & N_{2s+r}(S^\infty \underset{\Sigma_2}{\times} X\times X) \arrow[d, "\mu"] \\ H_s(X) \arrow[r, "q_r^\infty"'] & H_{2s+r}(S^\infty \underset{\Sigma_2}{\times} X\times X) \end{tikzcd}$$

commutes.

Proof.- Notice that the elements $[M,f] \in N_s(X)$ with M connected generate, so we can assume M is connected. We have that $\tilde{q}_r^\infty[M,f] = [S^r \underset{\Sigma_2}{\times} M\times M,\ i \underset{\Sigma_2}{\times} f\times f]$, and we have a commutative diagram

$$\begin{tikzcd} S^r \underset{\Sigma_2}{\times} M\times M \arrow[r, "i \underset{\Sigma_2}{\times} f\times f"] \arrow[d, hook, "i \underset{\Sigma_2}{\times} id"] & S^\infty \underset{\Sigma_2}{\times} X\times X \\ S^\infty \underset{\Sigma_2}{\times} M\times M \arrow[ur, "id \underset{\Sigma_2}{\times} f\times f"'] & \end{tikzcd}$$

then $\mu\, \tilde{q}_r^\infty[M,f] = \mu[S^r \underset{\Sigma_2}{\times} M\times M,\ i \underset{\Sigma_2}{\times} f\times f] =$

$$= (i \underset{\Sigma_2}{\times} f\times f)_* \, \sigma(S^r \underset{\Sigma_2}{\times} M\times M)$$

$$= (i \underset{\Sigma_2}{\times} f \times f)_* \, q_r^r(\sigma(M)) \quad \text{by } 2.31$$

$$= (1 \underset{\Sigma_2}{\times} f \times f)_* (i \underset{\Sigma_2}{\times} id)_* \, q_r^r(\sigma(M)) \quad \text{by diag. above}$$

$$= (1 \underset{\Sigma_2}{\times} f \times f)_* \, q_r^\infty \, (\sigma(M)) \quad \text{by } 2.30$$

$$= q_r^\infty \, f_*(\sigma(M)) = q_r^\infty \, \mu[M,f] \quad \text{by naturality}$$

□

2.33) Corollary.- Let X be a $\mathscr{C}_\infty$-space then the following diagram commutes:

$$\begin{tikzcd}
N_n(X) \arrow[r, "\tilde{Q}_r"] \arrow[d, "\mu"'] & N_{2s+r}(X) \arrow[d, "\mu"] \\
H_n(X) \arrow[r, "Q_r"'] & H_{2n+r}(X)
\end{tikzcd}$$

Proof.- Consider the following diagram

$$\begin{tikzcd}
N_n(X) \arrow[r, "\tilde{q}_r^\infty"] \arrow[d, "\mu"'] & N_{2n+r}(S^\infty \underset{\Sigma_2}{\times} X \times X) \arrow[r, "\cong"] \arrow[d, "\mu"'] & N_{2n+r}(\mathscr{C}_\infty(2) \underset{\Sigma_2}{\times} X \times X) \arrow[r, "\theta_{2*}"] \arrow[d, "\mu"'] & N_{2n+r}(X) \arrow[d, "\mu"'] \\
H_n(X) \arrow[r, "q_r^\infty"'] & H_{2n+r}(S^\infty \underset{\Sigma_2}{\times} X \times X) \arrow[r, "\cong"'] & H_{2n+r}(\mathscr{C}_\infty(2) \underset{\Sigma_2}{\times} X \times X) \arrow[r, "\theta_{2*}"'] & H_{2n+r}(X)
\end{tikzcd}$$

The first square commutes by 2.32), the other two by naturality of μ. The composition at the top is the definition of $\tilde{Q}_r$. The composition at the bottom coincides with Q_r by 2.11 .

□

Chapter 3 Generators for the Bordism of immersions

§3.1) Bordism of immersions

All manifolds are compact and smooth. A smooth map $f: M \to N$ is an immersion if at each point $x \in M$ the differential $df_x: T_x M \to T_{f(x)}N$ has rank=dimension of M on the tangent space $T_x M$ of M at x.

3.1) Definition.- Given two immersions $f: M \to N$, $f': M' \to N'$, where M, M', N, N' are closed and $\dim M = \dim M' = n$, $\dim N = \dim N' = n+k$, we say that they are bordant if there exists an immersion $F: V \to W$ such that i) $\dim V = n+1$, and there is a diffeomorphism $\partial V \cong M \amalg M'$; $\dim W = n+k+1$ and there is a diffeomorphism $\partial W \cong N \amalg N'$. ii) the following diagrams commute:

$$\begin{array}{ccc} M \hookrightarrow M \amalg M' \cong \partial V \subset & V \\ f \downarrow & \downarrow F \\ N \hookrightarrow N \amalg N' \cong \partial W \subset & W \end{array} \qquad \begin{array}{ccc} M' \hookrightarrow M \amalg M' \cong \partial V \subset & V \\ f' \downarrow & \downarrow F \\ N' \hookrightarrow N \amalg N' \cong \partial W \subset & W \end{array}$$

It is convenient to assume that every bordism $F: V \to W$ satisfies that F is transverse to ∂W, and this can always be achieved by modifying F by a small homotopy. Under this assumption, the proof of the transitivity of the relation reduces to attaching the two bordisms along the common immersion. Consequently it is easy to see that bordism of immersions is an equivalence relation.

3.2) Definition.- We denote by $I(n,k)$ the set of equivalence classes, modulo bordism, of all immersions of closed manifolds $f: M \to N$ where $\dim M=n$, $\dim N= n+k$, $k, m \geq 0$. The equivalence class of $f: M \to N$ is denoted by $[f: M \to N]$.

3.3) Proposition.- [40] $I(*,k)$ is an N_*- algebra with the following products:

a) $[f: M \to N] + [f': M' \to N'] = [f \amalg f': M \amalg M' \to N \amalg N']$

b) $[M'] \cdot [f: M \to N] = [id \times f: M' \times M \to M' \times N]$

c) $[f: M \to N] \cdot [f': M' \to N'] = [f \times id \amalg id \times f': M \times N' \amalg N \times M' \to N \times N']$

3.4) Definition.- Given two immersions $f_0, f_1: M \to N$ we say that they are regularly homotopic if there exists a homotopy $H: M \times I \to N$ such that i) for each $t \in I$, $H(x,t)$ is an immersion, $H(x,0) = f_0(x)$, $H(x,1) = f_1(x)$. ii) The differentials $d\,H(x,t): TM \to TN$ form a homotopy. H is called a regular homotopy.

3.5) Proposition.- If two immersions $f_0, f_1: M \to N$ are regularly homotopic then they are bordant.

Proof.- Let H be a regular homotopy between f_0 and f_1, then we can approximate H by a regular homotopy H', between f_0 and f_1, such that H' is smooth [37], then the map $F: M \times I \to N \times I$ given by $F(x,t) = (H'(x,t),t)$ is an immersion and it is a bordism between f_0 and f_1.

□

3.6) Definition.- We define $\alpha_k: I(n,k) \longrightarrow N_{n+k}(Q\,MO_k)$ as follows: Let $[f: M \to N] \in I(n,k)$, we can find an embedding $f_0: M \to N \times \mathbb{R}^r$, (taking $r > n-k+1$) such that f_0 is regularly homotopic to $M \xrightarrow{f} N \hookrightarrow N \times \mathbb{R}^r$, as regular homotopy preserves normal bundles we have $\nu_{f_0} \cong \nu_f \oplus \varepsilon^r$. Consider a tubular neighborhood $D(\nu_f \oplus \varepsilon^r) \hookrightarrow N \times \mathbb{R}^r$ and let $e: D(\nu_f \oplus \varepsilon^r) \hookrightarrow N \times \mathbb{R}^r \hookrightarrow (N \times \mathbb{R})^* = S^r(N^+)$, we can now define a map $t_f: S^r(N^+) \to S^r T(\nu_f) = T(\nu_f \oplus \varepsilon^r)$ by

$$t_f(y) = \begin{cases} q\,e^{-1}(y) & \text{if } y \in e(D(\nu_f \oplus \varepsilon^r)) \\ * & \text{if } y \in S^r(N^+) - e(D(\nu_f \oplus \varepsilon^r)) \end{cases}$$

where $q : D(\nu_f \oplus \varepsilon^r) \to D(\nu_f \oplus \varepsilon^r)/S(\nu_f \oplus \varepsilon^r)$ is the identification map, $S^r(\_)$ is the r-th suspension and $X^+ = X \cup \{+\}$.

We have a pull-back
$$\begin{array}{ccc} \nu_f & \longrightarrow & \gamma(k) \\ \downarrow & & \downarrow \\ M & \longrightarrow & BO(k) \end{array}$$

that induces a map of Thom spaces $\tau_f: T(\nu_f) \to MO_k$, so we can take the composition: $S^r(N^+) \xrightarrow{t_f} S^r(T\nu_f) \xrightarrow{S^r\tau_f} S^rMO_k$, and taking the adjoint we get $N \subset N^+ \to \Omega^rS^rMO_k \subset \lim\limits_{r} \Omega^rS^rMO_k = Q\,MO_k$; then

$\alpha_k[f: M \to N] \in N_{n+k}(QMO_k)$ is the class of this map.

We can also define $\bar{\alpha}_k: N_{n+k}(Q\,MO_k) \to I(n,k)$ as follows: given $N \xrightarrow{\phi} QMO_k$ then as N is compact, ϕ factors through $\Omega^rS^rMO_k$ for some r so we have the adjoint $\phi : S^r(N^+) \to S^rMO_k$. We can then find a map homotopic to adj. ϕ such that it is differentiable on $N \times \mathbb{R}^r \longrightarrow D(\gamma(k) \oplus \varepsilon^r)$ and transverse to the zero section, taking the inverse image of the zero section gives an embedding $\phi_0: M \longrightarrow N \times \mathbb{R}^r$ whose normal bundle has the form $\xi^k \oplus \varepsilon^r$, for some bundle ξ. Then if $k > 0$ we can apply Hirsch's theorem [22] and obtain an immersion $\phi_1: M \to N$ regularly homotopic to ϕ_0. We define $\bar{\alpha}_k[N,\phi] = [\phi_1: M \to N]$

With these definitions we have:

3.7) Theorem [40].- $\alpha_k: I(*,k) \to N_*(Q\,MO_k)$, $k > 0$, is an isomorphism of N_*-algebras, with inverse $\bar{\alpha}_k$.

□

In the case of codimension $k=0$ we can obtain the following result.

3.8) Proposition.- We have an isomorphism of N_*-algebras $N_*(\coprod_{r \geq 0} B\Sigma_r) \xrightarrow{\cong} I(*,0)$, where the H-space structure on $\coprod_{r \geq 0} B\Sigma_r$

comes from the juxtaposition homomorphisms $\Sigma_r \times \Sigma_s \longrightarrow \Sigma_{r+s}$

Proof.- One can show that $N_*(B\Sigma_r)$ can be interpreted as follows (this is a particular case of something treated in chapter 6). $N_n(B\Sigma_r)$ is the group of bordism classes of r-coverings over closed n-manifolds, where two coverings $p: \tilde{M} \to M$ and $q: \tilde{N} \to N$ are bordant if there exists an r-covering $\bar{p}: \tilde{V} \to V$ such that i) $\partial V \cong M \amalg N$, ii) $\bar{p}|M \cong p$ and $\bar{p}|N \cong q$. We can then define homomorphisms $N_n(B\Sigma_r) \to I(n,0)$ by sending the class of each covering to itself considered as an immersion, when $r > 0$, when $r=0$ we send $[M]$ to $[\emptyset \hookrightarrow M]$; these homomorphisms define a homomorphism $F: \bigoplus_{r \geq 0} N_n(B\Sigma_r) \cong N_n(\coprod_{r\geq 0} B\Sigma_r) \to I(n,0)$. F is surjective because if $f: M \to N$ is an immersion in codimension zero and N is connected, then f is open and closed so $f(M)=N$ and f is locally trivial [17] so f is a covering. If N is not connected then it can be expresed as a disjoint union with each element in the image of F. As the multiplicity of a covering is constant in each connected component F is clearly injective. To see that F preserves the product notice that if we have coverings $p: \tilde{M} \to M$ and $q: \tilde{N} \to N$ classified by maps $\phi_p: M \to B\Sigma_r$ and $\phi_q: N \to B\Sigma_s$ then we have a pull-back:

$$\begin{tikzcd}
\tilde{M}\times N \amalg M\times\tilde{N} \arrow[r] \arrow[d, "p\times id \amalg id\times q"] & (E\Sigma_r \times E\Sigma_s) \times_{\Sigma_r\times\Sigma_s} \overline{r+s} \arrow[r] \arrow[d] & E\Sigma_{r+s} \times_{\Sigma_{r+s}} \overline{r+s} \arrow[d] \\
M \times N \arrow[r, "\phi_p \times \phi_q"'] & B\Sigma_r \times B\Sigma_s \arrow[r] & B\Sigma_{r+s}
\end{tikzcd}$$

where $\overline{r+s} = \{1,2,\ldots, r+s\}$.

□

§3.2) Calculation of $N_*(QX)$

3.9) Definition.- In chapter 2 we defined Dyer-Lashof operations $Q_i: H_n(X) \to H_{2n+i}(X)$ and $Q^j: H_n(X) \to H_{n+j}(X)$ such that $Q^j = Q_{j-n}$, for any $\mathscr{C}_\infty$-space X. Given a sequence $I=(i_1,\dots,i_r)$ we say that it is monotone if $0 < i_1 \leq i_2 \leq \dots \leq i_r$, $r \geq 0$, and we consider the iterated product $Q_I = Q_{i_1} Q_{i_2} \dots . Q_{i_r}$.

Given a sequence $J = (j_1,\dots,j_r)$ we define its excess $e(J) = j_1 - j_2 - \dots - j_r$ and we call J admissible if $j_t \leq 2j_{t+1}$ for $1 \leq t < r$, and we take the iterated product $Q^J = Q^{j_1} Q^{j_2} \dots Q^{j_r}$.

We have similar definitions for the operations $\tilde{Q}_i: N_n(X) \to N_{2n+i}(X)$ and $\tilde{Q}^j: N_n(X) \to N_{n+j}(X)$ where $\tilde{Q}^j = \tilde{Q}_{j-n}$.

For any pointed space X, $QX = \varinjlim_r \Omega^r S^r X$ is an infinite loop space, with deloopings $Q(SX)$, $Q(S^2X)$,..... Hence it is a $\mathscr{C}_\infty$-space (2.16) and we have Dyer-Lashof operations defined on $H_*(QX)$ and on $N_*(QX)$.

3.10) Lemma.- For any spectrum E, $\iota_*: E_*(X) \to E_*(QX)$ is a split monomorphism.

Proof.- Following [16] for any spectrum F we define a spectrum $\mathcal{Q}F$ by $(\mathcal{Q}F)_n = S^n \Omega^n F_n$ with structure maps $S S^n \Omega^n F_n \to S^{n+1}\Omega^{n+1} F_{n+1}$ given by $S^{n+1} \Omega^n \varepsilon_n$ where $\varepsilon_n: F_n \to \Omega F_{n+1}$. In particular we have $\mathcal{Q}S^\infty X$ and clearly $\tilde{E}_*(QX) \cong E_*(\mathcal{Q}S^\infty X)$; $\iota_n: X \to \Omega^n S^n X$ induces a map of spectra $\bar{\iota}: S^\infty X \to \mathcal{Q}S^\infty X$ given by $\bar{\iota}_n = S^n \iota_n$ and we have an evaluation map $v_n: S^n \Omega^n S^n X \to S^n X$ which also induces a map of spectra $v: \mathcal{Q}(S^\infty X) \to S^\infty X$, as $v_n \circ S^n \iota_n = id$ then $v \circ \bar{\iota} = id$.

□

§3.2) Calculation of $\mathcal{N}_*(QX)$

3.9) Definition.- In chapter 2 we defined Dyer-Lashof operations $Q_i: H_n(X) \to H_{2n+i}(X)$ and $Q^j: H_n(X) \to H_{n+j}(X)$ such that $Q^j = Q_{j-n}$, for any $\mathscr{C}_\infty$-space X. Given a sequence $I=(i_1,\dots,i_r)$ we say that it is monotone if $0 < i_1 \le i_2 \le \dots \le i_r$, $r \ge 0$, and we consider the iterated product $Q_I = Q_{i_1} Q_{i_2} \dots . Q_{i_r}$.
Given a sequence $J = (j_1,\dots,j_r)$ we define its excess $e(J) = j_1 - j_2 - \dots - j_r$ and we call J admissible if $j_t \le 2j_{t+1}$ for $1 \le t < r$, and we take the iterated product $Q^J = Q^{j_1} Q^{j_2} \dots Q^{j_r}$.
We have similar definitions for the operations $\tilde{Q}_i: \mathcal{N}_n(X) \to \mathcal{N}_{2n+i}(X)$ and $\tilde{Q}^j: \mathcal{N}_n(X) \to \mathcal{N}_{n+j}(X)$ where $\tilde{Q}^j = \tilde{Q}_{j-n}$.
For any pointed space X, $QX = \varinjlim_r \Omega^r S^r X$ is an infinite loop space, with deloopings $Q(SX)$, $Q(S^2X)$,..... Hence it is a $\mathscr{C}_\infty$-space (2.16) and we have Dyer-Lashof operations defined on $H_*(QX)$ and on $\mathcal{N}_*(QX)$.

3.10) Lemma.- For any spectrum E, $\iota_*: E_*(X) \to E_*(QX)$ is a split monomorphism.

Proof.- Following [16] for any spectrum F we define a spectrum $\mathcal{Q}F$ by $(\mathcal{Q}F)_n = S^n \Omega^n F_n$ with structure maps $S S^n \Omega^n F_n \to S^{n+1}\Omega^{n+1}F_{n+1}$ given by $S^{n+1}\Omega^n \varepsilon_n$ where $\varepsilon_n: F_n \to \Omega F_{n+1}$. In particular we have $\mathcal{Q}S^\infty X$ and clearly $\tilde{E}_*(QX) \cong E_*(\mathcal{Q}S^\infty X)$; $\iota_n: X \to \Omega^n S^n X$ induces a map of spectra $\bar{\iota}: S^\infty X \to \mathcal{Q}S^\infty X$ given by $\bar{\iota}_n = S^n \iota_n$ and we have an evaluation map $v_n: S^n \Omega^n S^n X \to S^n X$ which also induces a map of spectra $v: \mathcal{Q}(S^\infty X) \to S^\infty X$, as $v_n \circ S^n \iota_n = \mathrm{id}$ then $v \circ \bar{\iota} = \mathrm{id}$.

□

3.11) Theorem [11][16][13].- Let X be a connected space and let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a $\mathbb{Z}_2$-basis for $\tilde{H}_*(X) \subset H_*(QX)$ then we have an isomorphism of $\mathbb{Z}_2$-algebras

$$H_*(QX) \cong \mathbb{Z}_2 \left[Q_I(x_\alpha) \mid I \text{ is monotone, } \alpha\in\Lambda\right]$$

$$\cong \mathbb{Z}_2 \left[Q^J(x_\alpha) \mid J \text{ is admissible and } e(J)>\dim x_\alpha, \alpha\in\Lambda\right]$$

□

3.12) Theorem.- Let X be a connected space and let $\{y_\alpha\}_{\alpha\in\Lambda}$ be a N_*-basis for $\tilde{N}_*(X) \subset N_*(QX)$, then we have an isomorphism of N_*-algebras

$$N_*(QX) \cong N_*[\tilde{Q}_I(y_\alpha) \mid I \text{ is monotone , } \alpha\in\Lambda]$$

$$\cong N_* \tilde{Q}^J(y_\alpha) \mid J \text{ is admissible and } e(J)>\dim y_\alpha, \alpha\in\Lambda]$$

Proof.- Consider a monomial $\tilde{Q}_{I_1}(y_{\alpha_1})^{r_1}\tilde{Q}_{I_2}(y_{\alpha_2})^{r_2},\ldots,\tilde{Q}_{I_m}(y_{\alpha_m})^{r_m}$, each I_s monotone, then $\mu(\tilde{Q}_{I_1}(y_{\alpha_1})^{r_1}\tilde{Q}_{I_2}(y_{\alpha_2})^{r_2}\ldots\tilde{Q}_{I_m}(y_{\alpha_m})^{r_m})$

$= \mu(\tilde{Q}_{I_1}(y_{\alpha_1}))^{r_1}\,\mu(\tilde{Q}_{I_2}(y_{\alpha_2}))^{r_2}\ldots\mu(\tilde{Q}_{I_m}(y_{\alpha_m}))^{r_m}$ by 1.5

$= Q_{I_1}(\mu(y_{\alpha_1}))^{r_1}\, Q_{I_2}(\mu(y_{\alpha_2}))^{r_2}\ldots Q_{I_m}(\mu(y_{\alpha_m}))^{r_m}$ by 2.33

As $\{y_\alpha\}_{\alpha\in\Lambda}$ is an N_*-basis for $\tilde{N}_*(X)$, then by 1.9, the elements $\{\mu(y_\alpha)\}_{\alpha\in\Lambda}$ are a $\mathbb{Z}_2$-basis for $\tilde{H}_*(X)$ and hence by 3.11 $\mu(\tilde{Q}_{I_1}(y_{\alpha_1})^{r_1}\tilde{Q}_{I_2}(y_{\alpha_2})^{r_2}\ldots\tilde{Q}_{I_m}(y_{\alpha_m})^{r_m})$ is a $\mathbb{Z}_2$-basis for $H_*(QX)$, so by 1.9 , the monomials $\tilde{Q}_{I_1}(y_{\alpha_1})^{r_1}\tilde{Q}_{I_2}(y_{\alpha_2})^{r_2}\ldots Q_{I_m}(y_{\alpha_m})^{r_m}$ are an N_*-basis for $N_*(QX)$.

□

§3.3) Geometric interpretation of the operations $\tilde{Q}_r: N_*(QMO_k) \to N_*(QMO_k)$

3.13) Definition.- We define operations $\hat{Q}_r: I(n,k) \to I(2n+k+r,k)$, $k > 0$, by the commutativity of the following diagram:

$$\begin{tikzcd} I(n,k) \arrow[r, "\hat{Q}_r"] & I(2n+k+r,\ k) \\ N_{n+k}(Q\,MO_k) \arrow[u, leftrightarrow, "\cong"] \arrow[r, "\tilde{Q}_r"'] & N_{2(n+k)+r}(Q\,MO_k) \arrow[u, leftrightarrow, "\cong"'] \end{tikzcd}$$

The purpose of this section is to give a description of $\hat{Q}_r$. In order to do this we need to prove some results.

3.14) Proposition.- Let $f: M \to \tilde{N}$ be an immersion and $p: \tilde{N} \to N$ a covering space, consider the maps associated to f and p by the Thom-Pontrjagin construction $t_f: S^\infty(\tilde{N}^+) \to S^\infty(T\nu_f)$ and $t_p: S^\infty(N^+) \to S^\infty(\tilde{N}^+)$. Then the following diagram is homotopy commutative

$$\begin{tikzcd} S^\infty(N^+) \arrow[rr, "t_{p\circ f}"] \arrow[dr, "t_p"'] & & S^\infty(T\nu_f) \\ & S^\infty(\tilde{N}^+) \arrow[ur, "t_f"'] & \end{tikzcd}$$

Proof.- With r large enough we can find embeddings ϕ, Π such that

$$\begin{tikzcd} & \tilde{N} \times \mathbb{R}^r \arrow[d] \\ M \arrow[ur, "\phi"] \arrow[r, "f"'] & \tilde{N} \end{tikzcd} \qquad \begin{tikzcd} & N \times \mathbb{R}^r \arrow[d] \\ \tilde{N} \arrow[ur, hook, "\Pi"] \arrow[r, "p"'] & N \end{tikzcd} \qquad \text{commute.}$$

As the normal bundle of Π is trivial, we can find an embedding $\tilde{N} \times \mathbb{R}^r \xhookrightarrow{h} N \times \mathbb{R}^r$ whose restriction to the zero section is Π.

We have $\nu_\phi \cong \nu_f \oplus \varepsilon^r$, so take a tubular neighbourhood $\nu_f \oplus \varepsilon^r \xhookrightarrow[\bar\phi]{} \tilde N \times \mathbb{R}^r$; and for $p\circ f$ consider the composition $\nu_f \oplus \varepsilon^r \xhookrightarrow[\bar\phi]{} \tilde N \times \mathbb{R}^r \xrightarrow[h]{} N \times \mathbb{R}^r$. Then we have $t_p: S^r(N^+) \to S^r(\tilde N^+)$ given by $t_p(y,v)=\begin{cases} h^{-1}(y,v) & \text{if } (y,v)\in \operatorname{im} h \\ * & \text{otherwise}\end{cases}$ and $t_f: S^r(\tilde N^+) \to S^r(T\nu_f)$ given by $t_f(\tilde y,w)=\begin{cases} \phi^{-1}(\tilde y,w) & \text{if } (\tilde y,w)\in \operatorname{im}\bar\phi \\ * & \text{otherwise}\end{cases}$. On the other hand $t_{p\circ f}: S^r(N^+) \to S^r(T\,\nu_f)$ is given by $t_{p\circ f}(y,v)=\begin{cases} (h\circ\bar\phi)^{-1}(y,v) & \text{if } (y,v)\in \operatorname{im}(h\circ\phi) \\ * & \text{otherwise}\end{cases}$

So then if $(y,v)\in \operatorname{im}(h\circ\bar\phi)$ we have $t_{p\circ f}(y,v)=(h\circ\bar\phi)^{-1}(y,v)= =\bar\phi^{-1}\, h^{-1}(y,v)= t_f\circ t_p(y,v)$, and if $(y,v)\notin \operatorname{im}(h\circ\bar\phi)$ then $t_{p\circ f}(y,v)= *$ and if $(y,v)\in \operatorname{im} h$ then $t_f(t_p(y,v))=*$ or if $(y,v)\notin \operatorname{im} h$ then $t_f\, t_p(y,v)= *$.

□

3.15) Definition.- Let S be the category of C.W.-spectra and C the category of pointed C.W.-complexes we denote by $\Omega^\infty: S \to C$ the functor that associates to each spectrum the zeroth space of the associated Ω-spectrum [1].

3.16) Remark.- The functors Ω^∞ and $S^\infty: C \to S$, induce functors in the homotopy categories, which we denote with the same symbols. These induced functors are adjoint [1] . This means that for each pair $X\in C$, $E\in S$, we have bijection adj.: $[S^\infty X, E] \to [X, \Omega^\infty E]$ which is natural in X and E. The naturality in E implies that adj. $(g\circ f)\simeq \Omega^\infty g\circ \text{adj}.f$, and the naturality in X implies that adj. $(f\circ S^\infty h)\simeq \text{adj}.f\circ h$. [31].

3.17) Proposition.- Let $f: M \to N$ be an immersion and V a manifold and consider the immersion $\mathrm{id}\times f: V\times M \to V\times N$. If the Thom-Pontrjagin map of f is given by $S^s(N^+) \xrightarrow{t_f} S^s(T\nu_f) \xrightarrow{S^s t} S^s(MO_k)$ then the Thom-Pontrjagin map of $\mathrm{id}\times f$ is given by

$V^+\wedge S^s(N^+) \xrightarrow{\mathrm{id}\wedge t_f} V^+\wedge S^s(T\nu_f) \xrightarrow{\pi} S^s(T\nu_f) \xrightarrow{S^s t} S^s(MO_k)$ where π is the projection on the second factor.

Proof.- Consider an embedding $M \overset{\bar f}{\hookrightarrow} N\times \mathbb{R}^s$ such that

$$\begin{tikzcd} & N\times\mathbb{R}^s \arrow[d] \\ M \arrow[ur, "\bar f"] \arrow[r, "f"'] & N \end{tikzcd}$$

commutes.

Take the embedding $V\times M \overset{\mathrm{id}\times\bar f}{\hookrightarrow} V\times N\times\mathbb{R}^s$, then if $e:\nu_{\bar f} \hookrightarrow N\times\mathbb{R}^s$ is a tubular neighbourhood of M in $N\times\mathbb{R}^s$, $V\times\nu_{\bar f} \overset{1\times e}{\hookrightarrow} V\times N\times\mathbb{R}^s$ is a tubular neighbourhood of $V\times M$ in $V\times N\times\mathbb{R}^s$, therefore we get a map

$$\begin{tikzcd} (V\times N\times\mathbb{R}^s)^* \arrow[r] & T(V\times\nu_f) \\ V^+\wedge N^+\wedge S^s \arrow[r] \arrow[u, "\wr\|"] & V^+\wedge T(\nu_f) \arrow[u, "\wr\|"'] \\ V^+\wedge S^s(N^+) \arrow[r, "\mathrm{id}\wedge t_f"'] \arrow[u, "\wr\|"] & V^+\wedge S^s T\nu_f \arrow[u, "\wr\|"'] \end{tikzcd}$$

and we have a pull-back

$$\begin{tikzcd} V\times\nu_f \arrow[r] \arrow[d] & \nu_f \arrow[r] \arrow[d] & \gamma(k) \arrow[d] \\ V\times M \arrow[r] & M \arrow[r] & BO(k) \end{tikzcd}$$

where the first square is induced by the projection on the second factor. $\square$

We now recall the definition of the Kahn-Priddy transfer for coverings [1][27].

3.18) Definition.- Let $p: X \to Y$ be an n-covering (X,Y C.W. complexes) we want to define a transfer map $S^\infty(Y^+) \to S^\infty(X^+)$ for this we define the principal Σ_n-bundle associated to p as follows:

consider $\bar{X}=\{(x_1,\ldots,x_n)\ X^n \mid x_i \neq x_j$ if $i\neq j$ and $p(x_1)=\ldots=p(x_n)\}$, then we have a free Σ_n-action on $\bar{X}$ by permuting the coordinates. As the spaces $\mathcal{C}_\infty(n)$ are Σ_n-free and contractible they classify principal Σ_n-bundles so we have a pull-back

$$\begin{tikzcd} \bar{X} \arrow[r, "\lambda"] \arrow[d] & \mathcal{C}_\infty(n) \arrow[d] \\ \bar{X}/\Sigma_n \arrow[r] & \mathcal{C}_\infty(n)/\Sigma_n \end{tikzcd}$$

We define a map $\phi\colon Y\cong\bar{X}/\Sigma_n \to \mathcal{C}_\infty(n) \underset{\Sigma_n}{\times} X^n$ by

$\phi(x_1,\ldots,x_n)=[\lambda(x_1,\ldots,x_n),(x_1,\ldots,x_n)]$ and we call the unique map extending ϕ, $T\colon Y^+ \to \mathcal{C}_\infty(n) \underset{\Sigma_n}{\times} (X^+)^n$ the pretransfer.

Consider the inclusion $X^+ \overset{\imath}{\hookrightarrow} Q(X^+)$, and recall that as $Q(X^+)$ is an infinite loop space we have structure maps $\theta_n\colon \mathcal{C}_\infty(n)\underset{\Sigma_n}{\times} Q(X^+)\to Q(X^+)$, we call the following composition or its adjoint the transfer.

$$Y^+ \xrightarrow{T} \mathcal{C}_\infty(n) \underset{\Sigma_n}{\times} (X^+)^n \xrightarrow[\mathrm{id}\underset{\Sigma_n}{\times}\imath^n]{} \mathcal{C}_\infty(n)\underset{\Sigma_n}{\times} (Q\,X^+)^n \xrightarrow{\theta_n} Q(X^+)$$

Notice that as λ is defined up to equivariant homotopy, T is defined up to homotopy.

3.19) Proposition.- Let $p\colon \tilde{N} \to N$ be an n-covering, where $\tilde{N},N$ are closed smooth manifolds, then the Thom-Pontrjagin map associated to p, adj $t_p\colon N \to Q(\tilde{N}^+)$ and the transfer for p, $\tau(p)\colon N \to Q(\tilde{N}^+)$ are homotopic.

Proof.- Let $e\colon \tilde{N} \hookrightarrow N\times\mathring{I}^r$ be an embedding, where $\mathring{I}^r$ is the interior of the r-cube, r large enough, such that the following commutes

$$\begin{tikzcd} & N\times\mathring{I}^r \arrow[d] \\ \tilde{N} \arrow[ur, hook, "e"] \arrow[r, "p"'] & N \end{tikzcd}$$

as the normal bundle of e is trivial we can find an embedding

$\bar{e}\colon \tilde{N}\times\mathring{I}^r \hookrightarrow N\times\mathring{I}^r$ such that the following diagram commutes

$$\begin{tikzcd} \tilde{N}\times\mathring{I}^r \arrow[rr, hook, "\bar{e}"] \arrow[dr, "p\circ proj"'] & & N\times I^r \arrow[dl, "proj"] \\ & N & \end{tikzcd}$$

Observe that we can embed $\tilde{N} \xhookrightarrow[\tilde{e}]{} \tilde{N}\times\mathring{I}^r$ by $\tilde{e}(x)=(x,c)$, where c is the center of $\mathring{I}^r$, such that $\bar{e}\circ\tilde{e} = e$ so that $\bar{e}$ is a neighbourhood of $\tilde{N}$ in $N\times\mathring{I}^r$.

Now we can apply the Thom-Pontrjagin construction to get a map $S^r(N^+) \xrightarrow{t_p} S^r(\tilde{N}^+)$. We want to compare this map with the transfer for the cover $\tilde{N} \xrightarrow[p]{} N$, so to define $T\colon N^+ \cong (\tilde{N}/\Sigma_n{}^+) \to \mathscr{C}_\infty(n)\underset{\Sigma_n}{\times}(N^+)^n$ we need a pull-back

$$\begin{tikzcd} \tilde{N} \arrow[r, "\lambda"] \arrow[d] & \mathscr{C}_\infty(n) \arrow[d] \\ \tilde{N}/\Sigma_n \cong N \arrow[r] & \mathscr{C}_\infty(n)/\Sigma_n \end{tikzcd}$$

Define λ as follows: recall that $x\in\tilde{N}$ is given by $x=(x_1,\dots,x_n)$ such that $p(x_1)=\dots=p(x_n)$ then each x_i gives $\{x_i\}\times\mathring{I}^r \xhookrightarrow{\bar{e}|_i} \{y\}\times\mathring{I}^r$, we define $\lambda\colon \tilde{N} \to \mathscr{C}_r(n) \hookrightarrow \mathscr{C}_\infty(n)$ by $\lambda(x_1,\dots,x_n)=(\bar{e}|_1, \bar{e}|_2,\dots, \bar{e}|_n)$ we are going to see that adj. $t_p = \tau(p)$. For this take $y\in N$, then

$$\text{adj}.\, t_p(y)[t] = t_p(t,y) = \begin{cases} \bar{e}^{-1}(y,t) & \text{if } (y,t)\in \operatorname{im}\bar{e} \\ * & \text{otherwise} \end{cases}$$

For a fixed y, if $(y,t)\in\operatorname{im}\bar{e}$ then there exists x_i such that $p(x_i)=y$ and $\bar{e}(x_i,s)=(y,t)$. On the other hand given $y\in N$, $\tau(p)(y)$ is given by $\tau(p)(y) = \tau[x_1,x_2,\dots,x_n] = \theta_n\circ 1\underset{\Sigma_n}{\times}\iota^n\, T[x_1,\dots,x_n] =$

$= \theta_n\circ 1\underset{\Sigma_n}{\times}\iota^n[\lambda(x_1,\dots,x_n), x_1,\dots,x_n] = \theta_n[\lambda(x_1,\dots,x_n),\iota(x_1),\dots,\iota(x_n)] =$

$= \theta_n[\bar{e}|_1, \bar{e}|_2,\dots, \bar{e}|_n,\iota(x_1),\dots,\iota(x_n)]$.

Now given $t\in I^r$ we have $\theta_n[\bar{e}|_1,\dots,\bar{e}|_n, \iota(x_1),\dots,\iota(x_n)](t) = \begin{cases} \iota x_i(e|_i^{-1}(t) & \text{if } t\in\operatorname{im}\bar{e}|_i \\ * & \text{otherwise.} \end{cases}$

But $\bigcup_{i=1}^{n} \operatorname{im} \bar{e}|_i = \operatorname{im} \bar{e}$, for a fixed y, and $\iota\, x_i(\bar{e}|_i^{-1}(t)) = (x_i, s)$

where $\bar{e}|_i(s) = t$, i.e., $\bar{e}(x_i, s) = t$.

□

3.20) Theorem.- The operations $\hat{Q}_r : I(n,k) \to I(2n+k+r,\, k)$, $k > 0$, are given by $\hat{Q}_r\,[f: M \to N] = [\hat{f}: S^r \times N \times M \to S^r \times_{\Sigma_2} N \times N]$ where $\hat{f}(t,y,x) = [\,t, y, f(x)\,]$.

Proof.- We can write $\hat{f}$ as the composition:

$$S^r \times N \times M \xrightarrow{id_{S^r\times N} \times f} S^r \times N \times N \xrightarrow{p} S^r \times_{\Sigma_2} N \times N$$

By definition of $\hat{Q}_r$ we have a commutative diagram:

$$\begin{tikzcd}
I(n,k) \arrow[r, "\hat{Q}_r"] \arrow[d, leftrightarrow, "\cong"] & I(2n+k+r,k) \arrow[d, leftrightarrow, "\cong"] \\
N_{n+k}(Q\,MO_k) \arrow[r, "\tilde{Q}_r"'] & N_{2(n+k)+r}(Q\,MO_k)
\end{tikzcd}$$

By 3.7) The element of $N_{n+k}(Q\,MO_k)$ associated to $f: M \to N$ is given by the adjoint of $S^\infty(N^+) \xrightarrow{t_f} S^\infty(T\nu_f) \xrightarrow{S^\infty\tau} S^\infty(MO_k)$. Let $\phi = \operatorname{adj.}(S^\infty\tau \circ t_f)$ then $\phi : N \to QMO_k$ and $\tilde{Q}_r[\,N, \phi\,]$ is given by the composition

$$S^r \times_{\Sigma_2} N \times N \xrightarrow{1 \times_{\Sigma_2} \phi \times \phi} S^\infty \times_{\Sigma_2} Q\,MO_k \times QMO_k \xrightarrow[h \times_{\Sigma_2} id]{\simeq} \mathcal{C}_\infty(2) \times_{\Sigma_2} Q\,MO_k \times Q\,MO_k \xrightarrow{\theta_2} Q\,MO_k \qquad (*)$$

By 3.14 and 3.17 we have a homotopy commutative diagram

$$\begin{tikzcd}
S^\infty[(S^r \times_{\Sigma_2} N \times N)^+] \arrow[r, "t_{\hat{f}}"] \arrow[d, "t_p"'] & S^\infty(T\nu_f) \arrow[r, "S^\infty\tau"] & S^\infty MO_k \\
S^\infty[(S^r \times N \times N)^+] \arrow[d, phantom, "\wr"] & S^\infty((S^r \times N)^+ \wedge T\nu_f) \arrow[u, "S^\infty\pi"'] \arrow[d, phantom, "\wr"] & \\
(S^r \times N)^+ \wedge S^\infty(N^+) \arrow[r, "id \wedge t_f"'] & (S^r \times N)^+ \wedge S^\infty T\nu_f &
\end{tikzcd} \qquad (**)$$

To prove the theorem we are going to show that (*) and the adjoint of (**) are homotopic, thus representing the same element in $N_{2(n+k)+r}(Q\ MO_k)$.

To do this we are going to construct a transfer for the covering $S^r \times N \times N \xrightarrow{p} S^r \underset{\Sigma_2}{\times} N \times N$ as follows: consider the diagram:

$$\begin{array}{ccccccc} S^r \times N \times N & \longrightarrow & S^r & \overset{i}{\hookrightarrow} & S^\infty & \overset{h}{\longrightarrow} & \mathscr{C}_\infty(2) \\ {\scriptstyle p}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^r \underset{\Sigma_2}{\times} N \times N & \longrightarrow & P^r & \hookrightarrow & P^\infty & \longrightarrow & \mathscr{C}_\infty(2)/\Sigma_2 \end{array}$$

where the first square is induced by the projection, all three maps are equivariant so the composition is equivariant and then it is a pull-back. By 3.18 we have a pretransfer T which in this case has the form $T(t,y_1,y_2) = [h \circ i(t), (-t,y_2,y_1), (t,y_1,y_2)]$. This gives a transfer for p that we denote by $\tau(p)$.

Now the map associated to the immersion $\hat{Q}_r\, [f: M \to N]$ is given by the adjoint of (**) so by 3.16 it is homotopic to $Q\tau \circ Q\Pi \circ \Omega^\infty(id \wedge t_f) \circ$ adj. t_p

By 3.19 ad. t_p is homotopic to the transfer for p so in part-particular adj. $t_p \simeq \tau(p)$ where $\tau(p)$ is the transfer constructed above, in other words, the map associated to $\hat{Q}_r\, [f: M \to N]$ is homotopic to $Q\tau \circ Q\Pi \circ \Omega^\infty(id \wedge t_f) \circ \tau(p)$ (***)

Now consider the diagram on next page, the composition at the bottom is (***). The triangle commutes by definition of the transfer and the squares commute because all the maps are infinite loop maps, so the composition at the bottom and the one at the top are the same, but one can easily verify that this composition is precisely $\hat{Q}_r\, [N, f]$.

$\square$

Diagram

$$
\begin{tikzcd}[column sep=small]
S^r \underset{\Sigma_2}{\times} N\times N \arrow[r, "T"] \arrow[drr, "\tau(p)"'] & \mathcal{C}_\infty(2)\underset{\Sigma_2}{\times}(S^r\times N\times N^+)^2 \arrow[r, "id\underset{\Sigma_2}{\times}1\times 1"'] & \mathcal{C}_\infty(2)\underset{\Sigma_2}{\times} Q(S^r\times N\times N^+)^2 \arrow[r, "id\underset{\Sigma_2}{\times}\Omega^\infty(id\wedge t_f)^2"'] \arrow[d, "\theta_{Q(S^r\times N\times N^+)}"] & \mathcal{C}_\infty(2)\underset{\Sigma_2}{\times} Q((S^r\times N)^+\wedge T\nu_f)^2 \arrow[r, "id\underset{\Sigma_2}{\times}Q\pi^2"'] \arrow[d, "\theta_{Q(S^r\times N^+\wedge T\nu_f)}"] & \mathcal{C}_\infty(2)\underset{\Sigma_2}{\times}(QT\nu_f)^2 \arrow[r, "id\underset{\Sigma_2}{\times}Q\tau^2"'] \arrow[d, "\theta_{QT\nu_f}"] & \mathcal{C}_\infty(2)\underset{\Sigma_2}{\times}(QMO_k)^2 \arrow[d, "\theta_{QMO_k}"] \\
 & & Q((S^r\times N\times N)^+) \arrow[r, "\Omega^\infty(id\wedge t_f)"'] & Q((S^r\times N)^+\wedge T\nu_f) \arrow[r, "Q\pi"'] & Q(T\nu_f) \arrow[r, "Q\tau"'] & Q(MO_k)
\end{tikzcd}
$$

3.21) Proposition.- Let G be a compact topological group and $\Pi \subset \Sigma_r$ a subgroup. Suppose that we have continuous actions $X \times G \to X$, $Y \times \Pi \to Y$ on spaces X, Y then $Y \underset{\Pi}{\times} (X/G)^r \cong Y \times X^r / \Pi \int G$, where "$\int$" denotes the wreath product. (We work in Steenrod's category of spaces).

Proof.- The action $(Y \times X^r) \times \Pi \int G \to Y \times X^r$ is given by

$$(y, x_1, x_2, \dots, x_r) \cdot (\sigma, g_1, g_2, \dots, g_r) = (y \cdot \sigma, x_{\sigma(1)} \cdot g_1, x_{\sigma(2)} \cdot g_2, \dots, x_{\sigma(r)} \cdot g_r)$$

As G is compact, the quotients are Hausdorff and have the identification topology so they are also in our category. Define $f\colon Y \times X^r \to Y \times (X/G)^r$ by $f = id \times p^r$ where $p\colon X \to X/G$, then f is an identification [47] and one can easily verify that it induces a homeomorphism $Y \times X^r / \Pi \int G \xrightarrow{\cong} Y \underset{\Pi}{\times} (X/G)^r$. □

3.22) Theorem.-

a) The immersions $\hat{Q}_{i_1} \hat{Q}_{i_2} \dots \hat{Q}_{i_r} [H(1,\alpha_1) \times \dots \times H(1,\alpha_k) \hookrightarrow D(\widetilde{H\alpha_1 \times \dots \times H\alpha_k})]$ for each sequence $0 < i_1 \leq i_2 \leq \dots \leq i_r$, $r \geq 0$ and each sequence $0 \leq \alpha_1 \leq \dots \leq \alpha_k$ are polynomial generators over N_* for $I(*,k)$, $k > 0$.

b) The immersions $\hat{Q}_{i_1} \hat{Q}_{i_2} \dots \hat{Q}_{i_r} [P^{\alpha_1} \times \dots \times P^{\alpha_k} \hookrightarrow D(\widetilde{\gamma_{\alpha_1} \times \dots \times \gamma_{\alpha_k}})]$ for each sequence $0 < i_1 \leq i_2 \leq \dots \leq i_r$, $r \geq 0$ and each sequence $0 \leq \alpha_1 \leq \dots \leq \alpha_k$ are polynomial generators over N_* for $I(*,k)$, $k \geq 0$.

Proof.- By 3.7 we have an isomorphism of N_*-algebras $\alpha_k\colon I(*,k) \to N_*(Q\, MO_k)$ and by 3.12 an isomorphism of N_*-algebras $N_*(Q\, MO_k) \cong N_*[\widetilde{Q}_I(y_\alpha) \mid I \text{ is monotone}, \alpha \in \Lambda]$ where $\{y_\alpha\}_{\alpha \in \Lambda}$ is an N_*-basis for $\widetilde{N}_*(MO_k)$. The following diagram:

$$\begin{tikzcd} E(*,k) \arrow[r, "j"] \arrow[d, leftrightarrow, "\cong"'] & I(*,k) \arrow[d, leftrightarrow, "\cong"] \\ N_*(MO_k) \arrow[r, "\iota_*"] & N_*(Q\ MO_k). \end{tikzcd}$$

where $j\,[M \xhookrightarrow[f]{} N] = [M \xrightarrow[f]{} N\,]$, clearly commutes.

By 3.20 the following diagram commutes

$$\begin{tikzcd} I(*,\ k) \arrow[r, "\hat{Q}_r"] \arrow[d, leftrightarrow, "\cong"'] & I(*,\ k) \arrow[d, leftrightarrow, "\cong"] \\ N_*(Q\ MO_k) \arrow[r, "\bar{Q}_r"'] & N_*(Q\ MO_k) \end{tikzcd}$$

Hence a) follows from 1.35 a) and b) follows from 1.35 b).

□

3.23) Remark.- Let $f: M \to N$ be an immersion, then by 3.21, after n iterations of the operations Q_r on f we get an immersion between manifolds of the form $V/H_n \to W/G_n$, where V is a product of spheres, copies of N and a copy of M, W is a product of spheres and copies of N and $H_n = \mathbb{Z}_2 \times \mathbb{Z}_2 \wr \mathbb{Z}_2 \times \ldots \times \underbrace{\mathbb{Z}_2 \wr \ldots \wr \mathbb{Z}_2}_{(n-1)} \subset G_n = \underbrace{\mathbb{Z}_2 \wr \mathbb{Z}_2 \wr \ldots \wr \mathbb{Z}_2}_{n}$

(we define the inclusion $g_n: H_n \hookrightarrow G_n$ inductively by $g_1: 0 \to \mathbb{Z}_2$ and $g_{n+1}(a,b) = (id,\ g_n(a),b))$.

Chapter 4: Multiple points of immersions and characteristic numbers

In this chapter we show that $I(*,k)$ splits as the direct sum of certain bordism groups of bundles. We use this to define characteristic numbers for immersions.

§4.1) Multiple points of immersions

4.1) Definition.- Let X be a space, define the r-th. configuration space $F(X;r)$ of X by $F(X;r)=\{(x_1,\dots,x_r) \mid x_i \in X, x_i \neq x_j \text{ if } i \neq j\} \subset X^r$
We have a free action of Σ_r on $F(X;r)$ given by $(x_1,\dots,x_r)\cdot\sigma =$ $=(x_{\sigma(1)},\dots,x_{\sigma(r)})$.

4.2) Definition.- Let $f: M \to N$ be an immersion and let $f^r:(M)^r \to (N)^r$ be the r-fold product of f. We say that f is self-transverse if $f^r|F(M;r)$ is transverse to the diagonal submanifold $N \xhookrightarrow[\Delta]{} (N)^r$ for all r. This means that if $f(x_1)=\dots=f(x_r)=y$, $x_i \neq x_j$, then the vector spaces $\mathrm{im}(df_{x_1}),\dots,\mathrm{im}(df_{x_r})$ are in general position in TN_y.

4.3) Note: The set of self-transverse immersions is open and dense in the space $\mathrm{Imm}(M,N)$ of smooth immersions with the C^1-topology [23]. (As M is compact the weak and strong topologies coincide). The space $\mathrm{Imm}(M,N)$ is locally contractible [37], therefore any immersion is regularly homotopic to an immersion which is self-transverse, and then, by 3.5, given any class in $I(n,k)$ we can find a representative that is self-transverse.

4.4) Definition.- Given a self-transverse immersion $f: M \to N$ we define the manifold $\mu_r=(f^r| F(M;r))^{-1}(\Delta)$. Using the fact that f is an immersion one can show that μ_r is compact [38]. The free action of Σ_r on $F(M;r)$ restricts to μ_r and the quotient μ_r/Σ_r is a manifold [8] called the manifold of r-tuple points.
We can also define the manifold of based r-tuple points as μ_r/Σ_{r-1}, where Σ_{r-1}

acts by permuting the first $(r-1)$ coordinates.

We define maps $f_r\colon \mu_r/\Sigma_r \to N$ and $\phi_r\colon \mu_r/\Sigma_{r-1} \to M$ by $f_r[x_1,\dots,x_r] = f(x_1)(=f(x_2)=\dots=f(x_r))$, $\phi_r[x_1,\dots,x_r] = x_r$

We also define $N_r(f) = \{y \in N \mid \#(f^{-1}(y)) = r\}$ and

$$M_r(f) = f^{-1}\,(N_r(f)).$$

We denote by ν_f the normal bundle of the immersion f and by $(\nu_f)^r$ the r-fold product.

With all this notation we have:

4.5) Proposition [38].- a) f_r and ϕ_r are immersions with normal bundles $\nu_{f_r} = [(\nu_f)^r \,|\, \mu_r]/\Sigma_r$ and $\nu_{\phi_r} = [\,(\nu_f)^{r-1}\ \{0\} \,|\, \mu_r]/\Sigma_{r-1}$

b) $f_r^{-1}(N_r(f))$ and $\phi_r^{-1}(M_r(f))$ are open and dense in μ_r/Σ_r and μ_r/Σ_{r-1} respectively; f_r restricted to $f_r^{-1}(N_r(f))$ and ϕ_r restricted to $\phi_r^{-1}(M_r(f))$ are diffeomorphisms onto $N_r(f)$ and $M_r(f)$ respectively.

c) $f_r(\mu_r/\Sigma_r) = \overline{N_r(f)} = \bigcup\limits_{i \geq r} N_i(f)$

$$\phi_r(\mu_r/\Sigma_{r-1}) = \overline{M_r(f)} = \bigcup_{i \geq r} M_i(f).$$

□

4.6) Definition.- Let $f\colon M \to N$ be a self-transverse immersion of codimension k, and consider an embedding $(f,e)\colon M \hookrightarrow N \times \mathbb{R}^\infty$, and the pull-back

$$\begin{array}{ccc} \nu_f & \xrightarrow{\bar{\rho}} & \gamma(k) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow[\rho]{} & BO(k) \end{array}$$

Then we can define a morphism of bundles as follows:

$$\begin{tikzcd}
(\nu_f)^r\,|\mu_r \arrow[r, "\bar{\phi}"] \arrow[d] & F(\mathbb{R}^\infty, r)\times\gamma(k)^r \arrow[d, "id\times q^r"] \\
\mu_r \arrow[r, "\phi"'] & F(\mathbb{R}^\infty, r)\times BO(k)^r
\end{tikzcd}$$

Given by $\bar{\phi}(y_1,\dots,y_r)=(e\,p(y_1),\dots,ep(y_r),\ \bar{p}(y_1),\dots,\ \bar{p}(y_r))$, and

$$\phi(x_1,\dots,x_r)=(e(x_1),\dots,e(x_r),\ \rho(x_1),\dots,\rho(x_r))$$

We can easily verify that it is a pull-back square. The maps involved are Σ_r-equivariant, so we can take the quotient under the actions of Σ_r. By 4.5 a) $(\nu_f)^r\,|\mu_r/\Sigma_r \cong \nu_{f_r}$, therefore we have a pull-back

$$\begin{tikzcd}
\nu_{f_r} \arrow[r, "\bar{\phi}"] \arrow[d] & F(\mathbb{R}^\infty, r)\underset{\Sigma_r}{\times}\gamma(k)^r \arrow[d, "id\underset{\Sigma_r}{\times} q^r"] \\
\mu_r/\Sigma_r \arrow[r, "\phi"'] & F(\mathbb{R}^\infty; r)\underset{\Sigma_r}{\times} BO(k)^r
\end{tikzcd}$$

§ 4.2) $F(\mathbb{R}^\infty; r)\underset{\Sigma_r}{\times} BO(k)^r$ as a classifying space

4.7) Proposition.- $F(\mathbb{R}^\infty; r)\underset{\Sigma_r}{\times} BO(k)^r = B(\Sigma_r \int O(k))$

Proof.- We can write $F(\mathbb{R}^\infty; r)=\lim\limits_{n\to\infty} F(\mathbb{R}^n; r)$. This space is Σ_r-free and contractible [34] . Each $F(\mathbb{R}^n; r)$ is a manifold so it is normal and Hausdorff and it is closed in $F(\mathbb{R}^{n+1}; r)$ therefore $F(\mathbb{R}^\infty, r)$ is normal and Hausdorff [53] and hence it is completly regular. Let $EO(k)$ be the infinite Stiefel manifold of k-frames, this is also a limit of manifolds so the same argument shows that $EO(k)$ is completly regular. Hence $F(\mathbb{R}^\infty; r)\times EO(k)^r$ is a contractible completly regular space.

We define a free $\Sigma_r \int O(k)$-action on $F(\mathbb{R}^\infty; r) \times EO(k)^r$ by:

$(a,b_1,b_2,\ldots,b_r)\cdot(\sigma, A_1,A_2,\ldots,A_r)=(a\cdot\sigma, b_{\sigma(1)}\cdot A_1, b_{\sigma(2)}\cdot A_2,\ldots,b_{\sigma(r)}\cdot A_r)$.

The group $\Sigma_r \int O(k)$ is a compact Lie group so by a theorem of A. Gleason [7] the quotient map $F(\mathbb{R}^\infty; r)\times EO(k)^r \to F(\mathbb{R}^\infty; r)\times EO(k)^r/\Sigma_r \int O(k)$ is a principal $\Sigma_r \int O(k)$-bundle, and by 3.21,

$F(\mathbb{R}^\infty; r)\times EO(k)^r/\Sigma_r \int O(k) \cong F(\mathbb{R}^\infty; r)\underset{\Sigma_r}{\times} BO(k)^r$

□

4.8)Definition.- Let $O(k)\times \mathbb{R}^k \to \mathbb{R}^k$ be the canonical action. Using this action we define a linear action $\Sigma_r \int O(k) \times (\mathbb{R}^k)^r \to (\mathbb{R}^k)^r$ by:

$(\sigma, A_1,\ldots,A_r)\,(v_1,\ldots,v_r)=(A_{\sigma^{-1}(1)}\cdot v_{\sigma^{-1}(1)},\ldots,A_{\sigma^{-1}(r)}\cdot v_{\sigma^{-1}(r)})$,

where $v_i \in \mathbb{R}^r$. This action defines a representation $\Sigma_r \int O(k) \hookrightarrow O(kr)$.

We then have a universal (rk)-vector bundle with structural group

$\Sigma_r \int O(k)$: $F(\mathbb{R}^\infty;r)\times EO(k)^r \underset{\Sigma_r \int O(k)}{\times} (\mathbb{R}^k)^r \longrightarrow F(\mathbb{R}^\infty;r)\times EO(k)^r/\Sigma_r \int O(k)$

4.9) Proposition.- The universal vector bundle defined in 4.8 is isomorphic to

$$\begin{array}{c} F(\mathbb{R}^\infty;r) \underset{\Sigma_r}{\times} \gamma(k)^r \\ \downarrow \\ F(\mathbb{R}^\infty;r) \underset{\Sigma_r}{\times} BO(k)^r \end{array}$$

Proof.- By 3.21 we have homeomorphisms: $F(\mathbb{R}^\infty;r)\times EO(R)^r/\Sigma_r\, O(k) \cong$ $F(\mathbb{R}^\infty;r) \underset{\Sigma_r}{\times} BO(k)^r$, and $F(\mathbb{R}^\infty;r)\times(EO(k)\times\mathbb{R}^k)^r/\Sigma_r \int O(k) \cong$

$F(\mathbb{R}^\infty;r) \underset{\Sigma_r}{\times} (EO(k) \underset{O(k)}{\times} \mathbb{R}^k)^r = F(\mathbb{R}^\infty;r)\underset{\Sigma_r}{\times} \gamma(k)^r$.

We also have a homeomorphism

ϕ: $F(\mathbb{R}^\infty; r)\times EO(k)^r \underset{\Sigma_r \int O(k)}{\times} (\mathbb{R}^k)^r \to F(\mathbb{R}^\infty;r)\times (EO(k)\times\mathbb{R}^k)^r/\Sigma_r \int O(k)$

given by $\phi\,[a,(b_1,\ldots,b_r),(v_1,\ldots,v_r)] = [a,(b_1,v_1),\ldots,(b_r,v_r)]$.

One can easily verify that the composition of both homeomorphisms commutes with the projections and that it is an isomorphism in each fiber. □

§4.3) The splitting of $I(n,k)$

In this section we assume that all spaces are compactly generated and Hausdorff and the base points are non-degenerate.

4.10) Definition.- Let $\mathcal{C}_r$ be the r-cubes operad and C_r the monad associated, then for any pointed space $(X,*)$ we have a space $C_rX = \coprod_{j\geq 0} \mathcal{C}_r(j) \times_{\Sigma_j} X^j/\sim$. A point of C_rX is uniquely represented by a function $f\colon D \to X$ where D is a finite set of disjoint r-cubes in $\mathring{I}^r$ (interior of I^r) and $f(D) \subset X - \{*\}$. See [34] for details.

Let $\iota : X \to \Omega^r S^r X$ be the inclusion and denote by γ_r the composition $C_rX \xrightarrow{C_r(\iota)} C_r\, \Omega^r S^r X \xrightarrow{\phi_r} \Omega^r S^r X$, where ϕ_r is obtained by glueing the structure maps $\theta_j\colon \mathcal{C}_r(j) \times_{\Sigma_j} (\Omega^r S^r X)^j \to \Omega^r S^r X$, so we have that $\gamma_r\, [(c_1,\ldots,c_r),\ (x_1,\ldots,x_r)] : (I^r,\ \partial I^r) \to (S^rX,\ *)$ is given by

$$\begin{cases} c_i^{-1}(t),\ x_i) & \text{if } t \in \operatorname{im} c_i \\ * & \text{if } t \in \bigcup_{i=1}^{r} \operatorname{im} c_i \end{cases}$$

The γ_r are compatible and we get a map $\gamma_\infty : C_\infty X \to \Omega^\infty S^\infty X = QX$

4.11) Theorem [34].- If X is connected $\gamma_r\colon C_rX \to \Omega^r S^r X$, for all $r \leq \infty$, is a weak homotopy equivalence.

□

When X is a Thom space there is a geometric proof in [28] of the fact that γ_∞ is a weak homotopy equivalence.

4.12) Definition.- Using the spaces $F(\mathbb{R}^r; j)$ instead of $\mathcal{C}_r(j)$ we can also construct, for each pointed $(X,*)$, a space $C_{\mathbb{R}^r} X = \coprod_{j\geq 0} F(\mathbb{R}^r; j) \times_{\Sigma_j} X^j/\sim$. A point of $C_{\mathbb{R}^r} X$ is uniquely represented by a function $\psi: A \to X$ such that A is a finite subset of $\mathbb{R}^r$ and $\psi(A) \subset X - \{*\}$. In fact the operads and the configuration spaces are examples of coefficient systems and each coefficient system defines a functor from spaces to spaces as above [14].

4.13) Proposition [28].- There exists a homotopy equivalence d
$d: C_r X \xrightarrow{\simeq} C_{\mathbb{R}^r} X$, $r \leq \infty$. □

4.14) Corollary. If X is connected then $C_{\mathbb{R}^\infty} X$ is weak homotopy equivalent to QX.

Proof.- By 4.13 and 4.11 the composition $C_{\mathbb{R}^\infty}X \xrightarrow{\bar{d}} C_\infty X \to QX$, where $\bar{d}$ is a homotopy inverse for d, is a weak homotopy equivalence. □

4.15) Definition.- Let $p: \coprod_{j \geq 0} F(\mathbb{R}^\infty; j) \times X^j \to C_{\mathbb{R}^\infty}X$ be the projection, then we define a filtration for $C_{\mathbb{R}^\infty}X$ as follows:

$$F_r C_{\mathbb{R}^\infty}X = p \coprod_{j=0}^{r} F(\mathbb{R}^\infty; j) \times X^j$$

4.16) Proposition [14].- $F_{r-1} C_{\mathbb{R}^\infty}X \subset F_r C_{\mathbb{R}^\infty}X$ is a cofibration and
$F_r C_{\mathbb{R}^\infty}X / F_{r-1} C_{\mathbb{R}^\infty}X \cong F(\mathbb{R}^\infty; r)^+ \wedge_{\Sigma_r} (X \wedge \ldots \wedge X) \equiv D_r X$. □

4.17) Theorem.-[43][14].- There exist maps of spectra
$h_r: S^\infty C_{\mathbb{R}^\infty}X \to S^\infty D_r X$, $r \geq 1$ such that the induced map
$h: S^\infty C_{\mathbb{R}^\infty}X \to \bigvee_{r \geq 1} S^\infty D_r X$ is a homotopy equivalence. □

4.18) Definition.- We define the isomorphism
$\beta_k: I(n,k) \xrightarrow{\cong} N_{n+k}(C_{\mathbb{R}^\infty} MO_k)$, $k > 0$, by the commutativity of the following diagram:

$$\begin{tikzcd} N_{n+k}(C_{\mathbb{R}^\infty} MO_k) & N_{n+k}(C_\infty MO_k) \arrow[l, "d_*"', "\cong"] \arrow[r, "\gamma_{\infty *}", "\cong"'] & N_{n+k}(Q\, MO_k) \\ & I(n,k) \arrow[ul, "\beta_k"] \arrow[ur, "\cong\ \alpha_k"'] & \end{tikzcd}$$

The isomorphism α_k was defined in 3.6, γ_∞ in 4.10 and d in 4.13. Following [28] we can describe β_k as follows:

Let $f: M \to N$ be an immersion, for m large enough, we can find an embedding of the form $(f,e): M \hookrightarrow N \times \mathbb{R}^m$. Let ν_f be the normal bundle of f classified by a pull-back

$$\begin{tikzcd} \nu_f \arrow[r, "\bar{\phi}"] \arrow[d, "p"'] & \gamma(k) \arrow[d] \\ M \arrow[r, "\phi"'] & BO(k) \end{tikzcd}$$

Extend f to an immersion $\bar{f}: \nu_f \to N$ such that i) $\bar{f}^{-1}(y)$ is finite for all $y \in N$. ii) $(\bar{f}, e\circ p): \nu_f \to N \times \mathbb{R}^m$ is an embedding.

Define $\beta_k(f): N \to C_{\mathbb{R}^m} MO_k \subset C_{\mathbb{R}^\infty} MO_k$ by

$$\beta_k(f)(y) = \begin{cases} \{(\bar{\phi}(v), e\circ p(v)) \mid \bar{f}(v)=y\} & \text{if } y \in \operatorname{im} \bar{f} \\ * & \text{otherwise} \end{cases}$$

4.19) Proposition.- The description above makes the diagram in the definition 4.18 commute .

Proof.- Consider the composition:

$N \xrightarrow{\beta_k(f)} C_{\mathbb{R}^m} MO_k \xrightarrow{\bar{d}} C_m\, MO_k \xrightarrow{\gamma_m} \Omega^m S^m MO_k$, where $\bar{d}$ is a homotopy inverse for the equivalence d. It was proved in [28] that its adjoint $S^m(N^+) \to S^m\, MO_k$ corresponds to the Thom-Pontrjagin construction for the embedding $(f,e) : M \hookrightarrow N\times \mathbb{R}^m$ which is precisely the definition of α_k.

□

To give the splitting of $I(n,k)$ we need some other results.

4.20) Proposition.- Let $g: S^mX \longrightarrow S^mY$ be a map and $\tilde{g}: S^\infty X \longrightarrow S^\infty Y$ the stable map induced. Let E be a spectrum and consider the following diagrams:

$$\begin{array}{ccc}
E_n(S^\infty X) & \xrightarrow{\tilde{g}_*} & E_n(S^\infty Y) \\
\cong \updownarrow \sigma & & \cong \updownarrow \sigma \\
\tilde{E}_n(X) & \dashrightarrow & \tilde{E}_n(Y)
\end{array}
\qquad
\begin{array}{ccc}
\tilde{E}_{n+m}(S^mX) & \xrightarrow{g_*} & \tilde{E}_{n+m}(S^mY) \\
s^m \updownarrow \cong & & \cong \updownarrow s^m \\
\tilde{E}_n(X) & \dashrightarrow & \tilde{E}_n(Y)
\end{array}$$

Then the homomorphisms induced by both diagrams are the same.

Proof.- Consider the following diagram in the homotopy category of spectra:

$$\begin{array}{ccc}
E_n(S^\infty X)=[\Sigma^nS^\infty S^o, E\wedge S^\infty X] & \xrightarrow{(id\wedge\tilde{g})_* = \tilde{g}_*} & [\Sigma^nS^\infty S^o, E\, S^\infty Y] = E_n(S^\infty Y) \\
\uparrow \sigma & & \uparrow \sigma \\
\tilde{E}_n(X)=[\Sigma^nS^\infty S^o, E\wedge X] & & [\Sigma^nS^\infty S^o, E\wedge Y] = \tilde{E}_n(Y) \\
\downarrow \Sigma^m & & \downarrow \Sigma^m \\
[\Sigma^{n+m}S^\infty S^o, \Sigma^mE\wedge X] & & [\Sigma^{n+m}S^\infty S^o, \Sigma^m E\wedge Y] \\
\downarrow \ell & & \downarrow \ell \\
\tilde{E}_{n+m}(S^mX)=[\Sigma^{n+m}S^\infty S^o, E\wedge S^m\wedge X] & \xrightarrow[(id\wedge g)_* = g_*]{} & [\Sigma^{n+m}S^\infty S^o, E\wedge S^m\wedge Y] = \tilde{E}_{n+m}(S^mY)
\end{array}$$

The isomorphism σ is given by the natural equivalence $E\wedge X \simeq E\wedge S^\infty X$. The isomorphism ℓ is given by the natural equivalence $\Sigma E \simeq E\wedge S'$ and the composition $\ell\circ\Sigma^m$ is the definition of the suspension isomorphism s^m. We are going to show that the maps induced by $\tilde{g}_*$ and g_* from $\Sigma^mE\wedge X$ to $\Sigma^mE\wedge Y$ are homotopic, this clearly implies the result. For this consider the following diagram.

$$\begin{tikzcd}
\Sigma^m(E\wedge S^\infty X) \arrow[r, "\Sigma^m(\mathrm{id}\wedge\tilde{g})"] \arrow[d, "\simeq"'] & \Sigma^m(E\wedge S^\infty Y) \arrow[d, "\simeq"] \\
E\wedge\Sigma^m S^\infty X \arrow[r, "\mathrm{id}\wedge\Sigma^m\tilde{g}"] & E\wedge\Sigma^m S^\infty Y \\
E\wedge S^m X \arrow[r, "\mathrm{id}\wedge g"] \arrow[u, "\simeq"] & E\wedge S^m Y \arrow[u, "\simeq"'] \\
\Sigma^m E\wedge X \arrow[u, "\simeq"] \arrow[uuu, bend left=60, "\simeq"] & \Sigma^m E\wedge Y \arrow[u, "\simeq"'] \arrow[uuu, bend right=60, "\simeq"']
\end{tikzcd}$$

The first square is homotopy commutative by naturality of the equivalence $\Sigma(E\wedge F)\simeq E\wedge\Sigma F$; we have that $\Sigma^m\tilde{g}=S^\infty g$ and $\Sigma^m S^\infty X=S^\infty S^m X$ so the second square is homotopy commutative by naturality of the equivalence $E\wedge Z\simeq E\wedge S^\infty Z$. Finally the composition of the three equivalences is precisely the equivalence on the outer arrows.

□

4.21) Proposition.- Let (X, x_0) be a pointed space then the suspension isomorphism $s\colon N_n(X,x_0)\to N_{n+1}(SX,*)$ is given by $s\,[N,g]=$ $=[N\times I,\ p\circ g\times\mathrm{id}]$, where $p\colon X\times I\to SX$ is the identification map

Proof.- We use the following description of s [49] :consider the triple $(C(X), X, x_0)$ where $C(X)$ is the cone of X. As $C(X)$ is contractible then the boundary homomorphism $\partial\colon N_{n+1}(C(X),X)\to N_n(X,x_0)$ of the exact sequence of the triple is an isomorphism, let $q\colon(C(X),X)\to(SX,*)$ be the projection then s is given by the composition

$N_n(X,x_0)\xrightarrow[\cong]{\partial^{-1}}N_{n+1}(C(X),X)\xrightarrow{q_*}N_{n+1}(SX,*)$.

Let $[N,g]\in N_n(X,x_0)$, then we have $g\colon(N,\partial N)\to(X,x_0)$, consider $N\times I$, by straightening the corners [9] we get a manifold with boundary $\partial(N\times I)=\partial N\times I\underset{\partial N\times\partial I}{\cup}N\times\partial I$. Consider the composition $N\times I\xrightarrow{g\times\mathrm{id}}X\times I\xrightarrow{\rho}C(X)$

where ρ is the projection. Then we get a commutative diagram:

$$\begin{tikzcd} \partial(N\times I) \arrow[r, "\tilde{g}"] \arrow[d, hook] & X \arrow[d, hook] \\ N\times I \arrow[r, "\rho\circ g\times id"'] & C(X) \end{tikzcd}$$

where $X \hookrightarrow C(X)$ by $x \mapsto [x,0]$ and $\tilde{g}|N\times\{0\} = g$ and $\tilde{g}(\partial(N\times I)-N\times\{0\}) = x_0$. We claim that $\partial^{-1}[N,g] = [N\times I,\rho\circ g\times id]$. To prove this we have to show that $\partial[N\times I,\rho\circ g\times id] = [\partial(N\times I),\tilde{g}]$ is equal to $[N,g]$ in $N_n(X, x_0)$, so consider $\partial(N\times I)\times I \xrightarrow{F} X$ given by $F(y,t) = \tilde{g}(y)$, then we have that $\partial(\partial(N\times I)\times I) = \partial(N\times I)\times\{0\}\cup\partial(N\times I)\times\{1\}$, so we identify $\partial(N\times I)$ with $\partial(N\times I)\times\{0\}$ and we embed $N \hookrightarrow \partial(N\times I)\times\{1\}$ by $y \mapsto ([y,0],1)$. Then $F|N = \tilde{g}|N = g$, $F(\partial(N\times I)\times\{1\} - N) = x_0$ and $F|\partial(N\times I)\times\{0\} = \tilde{g}$, hence $[\partial(N\times I),\tilde{g}] = [N,g]$.

Therefore $s[N,g] = q_*[N\times I,\rho\circ g\times id]$, but $q\circ\rho = p$.

$\square$

4.22) Definition.- We define homomorphisms

$\bar{h}_r: N_m(C_{\mathbb{R}^\infty}MO_k, *) \to N_m(D_rMO_k, *)$, $r \geq 1$. by the commutativity of the following diagram:

$$\begin{tikzcd} N_m(C_{\mathbb{R}^\infty}MO_k,*) \arrow[r] \arrow[d, leftrightarrow, "\sigma"', "\cong"] & N_m(D_r MO_k,*) \arrow[d, leftrightarrow, "\sigma"', "\cong"] \\ MO_m(S^\infty C_{\mathbb{R}^\infty}MO_k) \arrow[r, "h_{r_*}"'] & MO_m(S^\infty D_rMO_k) \end{tikzcd}$$

where h_r is the map of spectra of 4.17 .

Let $f: N \to C_{\mathbb{R}^\infty}MO_k$ be a map where N is a closed $(n+k)$-manifold, and let $i:(N,\emptyset) \hookrightarrow (N^+,+)$ be the inclusion, we denote by $f^+:(N^+,+) \to (C_{\mathbb{R}^\infty}MO_k,*)$ the extension of f to N^+ and by $s^m: N_{n+k}(N^+,+) \to N_{n+k+m}(S^m(N^+),*)$ the suspension isomorphism.

With this notation we have:

4.23) Proposition.- $\bar{h}_r[N,f] = (s^m)^{-1} g_* \, s^m \, [N,i]$, where $g\colon S^m(N^+) \to S^m D_r \, MO_k$, for m large enough, is a map representing the stable map $h_r \circ S^\infty f^+$.

Proof.- We clearly have that $[N,f] = f^+_* [N,i]$ and as N is compact then the stable map $h_r \circ S^\infty f^+ \colon S^\infty(N^+) \to S^\infty D_r MO_k$ should be given by a map $g\colon S^m(N^+) \to S^m D_r MO_k$, for some m, and its suspensions. Consider now the following diagram:

$$\begin{tikzcd}
\mathcal{N}_{n+k+m}(S^m(N^+), *) \arrow[rr, "g_*"] & & \mathcal{N}_{n+k+m}(S^m D_r MO_k, *) \\
\mathcal{N}_{n+k}(N^+, +) \arrow[u, "s^m"', "\cong"] \arrow[r, "f^+_*"] \arrow[d, leftrightarrow, "\sigma"', "\cong"] & \mathcal{N}_{n+k}(C_{\mathbb{R}^\infty} MO_k, *) \arrow[r, "\bar{h}_r"] \arrow[d, leftrightarrow, "\cong"', "\sigma"] & \mathcal{N}_{n+k}(D_r \, MO_k, *) \arrow[u, "\cong"', "s^m"] \arrow[d, leftrightarrow, "\cong"', "\sigma"] \\
MO_{n+k}(S^\infty (N^+)) \arrow[r, "(S^\infty f^+)_*"'] & MO_{n+k}(S^\infty C_{\mathbb{R}^\infty} MO_k) \arrow[r, "h_{r*}"'] & MO_{n+k}(S^\infty D_r MO_k)
\end{tikzcd}$$

The 2 squares at the bottom commute, the first one by naturality of the equivalence $E \wedge X \simeq E \wedge S^\infty X$, and the second by definition of $\bar{h}_r$. As the stable map $h_r \circ S^\infty f^+$ is induced by g then by 4.20 the homomorphisms g_* and $(h_r \circ S^\infty f^+)_*$ correspond under s^m and σ, i.e., the square at the top commutes, hence $\bar{h}_r \, [N,f] =$
$= \bar{h}_r \, f^+_* \, [N,i] = (s^m)^{-1} \, g_* \, s^m \, [N,i]$. $\square$

4.24) Remark: Let $\xi=(E,p,B)$ be a vector bundle, with Thom space $T(\xi)$, then the Thom space of the vector bundle $F(\mathbb{R}^\infty;r) \times_{\Sigma_r} E^r \xrightarrow{id \times_{\Sigma_r} p^r} F(\mathbb{R}^\infty;r) \times_{\Sigma_r} B^r$ is given by $F(\mathbb{R}^\infty;r)^+ \wedge_{\Sigma_r} T(\xi) \wedge \ldots \wedge T(\xi) \equiv D_r(T(\xi))$.

4.25) Proposition.- The homomorphism given by the composition:

$$\begin{array}{ccccc}
I(n,k) & \xrightarrow[\cong]{\beta_k} & N_{n+k}(C_{\mathbb{R}^\infty}MO_k) & \xrightarrow{j_*} & N_{n+k}(C_{\mathbb{R}^\infty}MO_k,*) \\
 & & & & \downarrow \bar{h}_r \\
 & & & & N_{n+k}(D_r\,MO_k,*) \\
 & & & & \cong\downarrow\Phi \\
 & & & & N_{n-(r-1)k}(F(\mathbb{R}^\infty;r)\underset{\Sigma_r}{\times}BO(k)^r)
\end{array}$$

where $j\colon (C_{\mathbb{R}^\infty}MO_k,\emptyset)\hookrightarrow(C_{\mathbb{R}^\infty}MO_k,*)$ is the inclusion and Φ is the Thom isomorphism, sends the class of a self-transverse immersion $f\colon M\to N$ to the class $[\mu_r/\Sigma_r,\ \nu_{f_r}]$, where μ_r/Σ_r is the manifold of r-tuple points and ν_{f_r} classifies the normal bundle of the immersion $f_r\colon \mu_r/\Sigma_r\to N$. For $r=1$ it is $[M,\nu_f]$.

Proof.- Let us denote by $[\phi\colon N\to C_{\mathbb{R}^\infty}MO_k]$ the class of $j_*\beta_k\,[f\colon M\to N]$, to evaluate $\Phi\,\bar{h}_r\,[N,\phi]$ consider the following diagram:

$$\begin{array}{ccc}
N_{n+k+m}(S^m(N^+),*) & \xrightarrow{g_*} & N_{n+k+m}(S^mD_rMO_k,*) \\
\uparrow\cong & & \uparrow\cong \\
N_{n+k}(N^+,+) & & N_{n+k}(D_rMO_k,*) \\
 & & \cong\downarrow\Phi \\
 & & N_{n-(r-1)k}(F(\mathbb{R}^\infty;r)\underset{\Sigma_r}{\times}BO(k)^r)
\end{array}$$

where $g\colon S^m(N^+)\to S^mD_rMO_k$ is a map inducing the stable map $h_r\circ S^\infty\phi^+\colon S^\infty(N^+)\to S^\infty D_rMO_k$. By 4.23) $\bar{h}_r[N,\phi]=(s^m)^{-1}g_*\,s^m\,[N,i]$, now recall that $D_rMO_k=$ Thom space of $F(\mathbb{R}^\infty;r)\underset{\Sigma_r}{\times}\gamma(k)^r$, hence $S^mD_rMO_k$ is the Thom space of $(F(\mathbb{R}^\infty;r)\underset{\Sigma_r}{\times}\gamma(k)^r)\oplus\varepsilon^m$. From the geometric definition of the Thom isomorphism (1.33) it is clear that the composition $\Phi(s^m)^{-1}$ is the Thom isomorphism for the bundle

$(F(\mathbb{R}^\infty; r) \underset{\Sigma_r}{\times} \gamma(k)^r) \oplus \varepsilon^m$, we denote this isomorphism by $\bar{\Phi}$.

On the other hand by [28] the composition $N \xrightarrow{\phi} C_{\mathbb{R}^\infty} MO_k \xrightarrow{\text{adj } h_r} Q\, D_r\, MO_k$ represents, under the Thom-Pontrjagin construction, the immersion $f_r: \mu_r/\Sigma_r \to N$, but by 3.16 $\text{adj}^{-1}(\text{adj } h_r \circ \phi^+) \simeq h_r \circ S^\infty \phi^+$, hence the map $g: S^m N^+ \to S^m D_r MO_k$ inducing $h_r \circ S^\infty \phi^+$ is the Thom-Pontrjagin map for the embedding

$$\begin{tikzcd} \mu_r/\Sigma_r \arrow[r, hook] \arrow[dr, "f_r"'] & N \times \mathbb{R}^m \arrow[d] \\ & N \end{tikzcd}$$

Finally by 4.21 we have that $s^m [N,i] = [N\times I^m, p]$ where $p: N\times I^m \to S^m(N^+)$ is the identification, hence $\Phi\, \bar{h}_r\, [N,\phi] = \Phi(s^m)^{-1}\, g_* s^m\, [N,i] =$
$= \bar{\Phi}\, g_*\, [N \times I^m, p] = [\mu_r/\Sigma_r, \nu_{f_r}]$.

For $r=1$ we get $N \xrightarrow{\phi} C_{\mathbb{R}^\infty} MO_k \xrightarrow{e} Q\, MO_k$, where e is the equivalence of 4.14, so by 4.19 it is the map corresponding to $f: M \to N$ itself.

$\square$

4.26) Definition.- By 4.7, $F(\mathbb{R}^\infty; r) \underset{\Sigma_r}{\times} BO(k)^r = B(\Sigma_r \int O(k))$. Following [15] we can give the following interpretation for the groups $N_m(B(\Sigma_r \int O(k)))$. We consider (rk)-vector bundles ξ with structural group $\Sigma_r \int O(k)$ over closed smooth m-manifolds. We say that a bundle $\xi \to M$ bords if there is an $(r\,k)$-vector bundle $\zeta \to V$ with structural group $\Sigma_r \int O(k)$ such that i) V is a compact smooth $(m+1)$ manifold and there is a diffeomorphism $M \cong \partial V$, ii) The pull-back of ζ under the composition $M \cong \partial V \hookrightarrow V$ is isomorphic to ξ. Two bundles ξ_1, ξ_2 are bordant if their disjoint union $\xi_1 \amalg \xi_2$ bords. We denote by $N_m[\Sigma_r \int O(k)]$ the set of equivalence classes which is made into a group by considering disjoint union of bundles. Given a bundle $\xi \to M$ we denote its equivalence class by $[\xi \to M]$. We give $N_*[\Sigma_r \int O(k)]$ an N_*-module structure by defining $[N][\xi \xrightarrow{p} M] = [N\times\xi \xrightarrow{\text{id}\times p} N\times M]$.

4.27) Proposition [15].- We have an isomorphism of N_*-modules $N_*(B(\Sigma_r \wr O(k))) \xrightarrow{\cong} N_*[\Sigma_r \wr O(k)]$ given by $[M,f] \longmapsto [f^*(\gamma) \to M]$, where γ is the universal bundle over $B(\Sigma_r \wr O(k)))$.

□

4.28) Theorem.- There is an isomorphism

$I(n,k) \cong N_{n+k} \oplus N_n\ [O(k)] \oplus \bigoplus_{r\geq 2} N_{n-(r-1)k}[\Sigma_r \wr O(k)]$ given by

$[f: M\to N] \mapsto \left([N],[\nu_f \to M],\ \sum_{r\geq 2}[\nu_{f_r} \to \mu_r/\Sigma_r]\right)$, for $n \geq 0$, $k > 0$, where f is a self-transverse representative.

Proof.- By 4.18 We have an isomorphism $\beta_k: I(n,k) \xrightarrow{\cong} N_{n+k}(C_{\mathbb{R}^\infty}MO_k)$. Let j be the inclusion $(C_{\mathbb{R}^\infty}MO_k, \emptyset) \hookrightarrow (C_{\mathbb{R}^\infty}MO_k, *)$, then by 1.31 we have an isomorphism $N_{n+k}(C_{\mathbb{R}^\infty}MO_k) \cong N_{n+k} \oplus N_{n+k}(C_{\mathbb{R}^\infty}MO_k,*)$ given by $[N,\phi] \mapsto ([N], j_*[N,\phi])$.

$$\begin{aligned}
N_{n+k}(C_{\mathbb{R}^\infty}MO_k,*) &\cong MO_{n+k}(S^\infty C_{\mathbb{R}^\infty} MO_k) && \text{(by 1.7 and suspension isom.)}\\
&\cong MO_{n+k}\Big(\bigvee_{r\geq 1} S^\infty D_r MO_k\Big) && \text{(by 4.17)}\\
&\cong \bigoplus_{r\geq 1} MO_{n+k}(S^\infty D_r MO_k) && \text{(Because MO satisfies the wedge axiom).}\\
&\cong \bigoplus_{r\geq 1} N_{n+k}(D_r\ MO_k, *)\\
&\cong \bigoplus_{r\geq 1} N_{n-(r-1)k}(F(\mathbb{R}^\infty;r)\underset{\Sigma_r}{\times} BO(k)^r) && \text{(By the Thom isomorphism)}\\
&\cong \bigoplus_{r\geq 1} N_{n-(r-1)k}(B(\Sigma_r \wr O(k))) && \text{(by 4.7)}\\
&\cong \bigoplus_{r\geq 1} N_{n-(r-1)k}\ [\Sigma_r \wr O(k)] && \text{(by 4.27)}
\end{aligned}$$

The fact that the isomorphism is as stated in the theorem follows from 4.25 and the definition of the isomorphism in 4.27 .

□

4.29) Definition.- Let $[\xi \to M] \in N_m[\Sigma_r \wr O(k)]$, then following [15] we can define characteristic numbers for ξ as follows:

for each cohomology class $a \in H^j(B(\Sigma_r \int O(k); \mathbb{Z}_2)$ and each partition ρ of $m-j$ (i.e. a sequence $0 \leq i_1 \leq \ldots \leq i_s$ such that $i_1+\ldots+i_s = m-j$) there is associated a Stiefel-Whitney number of the form $< \omega_\rho(M)\ \phi_\xi^*(a), \sigma(M) >$, where $\omega_\rho(M) = \omega_{i_1}(M)\omega_{i_2}(M)\ldots\omega_{i_s}(M)$, and $\phi_\xi : M \to B(\Sigma_r \int O(k))$ is a classifying map for ξ. Notice that when $a=1$ we get the ordinary Stiefel-Whitney numbers of M.
By 4.28) We can associate to each self-transverse immersion $f: M \to N$ the characteristic numbers of N and of each of the normal bundles $\nu_f, \nu_{f_1}, \nu_{f_2}, \ldots$ which we call the characteristic numbers of f.

4.30) Proposition.- Let $f: M \to N$ and $g: M' \to N'$ be self-transverse immersions then f and g are bordant if and only if their characteristic numbers are equal.

Proof.- The homology groups $H_*(B(\Sigma_r \int O(k)); \mathbb{Z}_2)$ are of finite type so by [15] the characteristic numbers of each normal bundle determine its bordism class. Therefore the result follows from 4.28.

□

Note: The result stated in [15] is for finite complexes however one can easily see that it is also true for spaces of finite type.

Chapter 5: Reduction of the structural group modulo bordism

In this chapter we shall use some results of A. Borel on homogeneous spaces to prove that any (rk)-vector bundle is bordant to a vector bundle with structural group $\Sigma_r \int O(k)$.

5.1) Definition.- Let $p: E \to B$ be a fibration with fiber F. We assume that B and F are path connected. We denote by $\{E_r^{p,q}, d_r\}$ the spectral sequence associated to p such that $E_2^{p,q} \cong H^p(B; \underline{H^q(F; \mathbb{Z}_2)})$, where $\underline{H^q(F; \mathbb{Z}_2)}$ denotes local coefficients. This spectral sequence converges to $H^*(E; \mathbb{Z}_2)$ [52].

5.2)Definition.- We say that the fiber F of a fibration $p: E \to B$ is totally non-homologous to zero if the inclusion $i: F \hookrightarrow E$ induces a surjective homomorphism in cohomology.

5.3) Theorem.- [42] Let $p: E \to B$ be a fibration with fiber F such that F and B are path connected and $H^*(F; \mathbb{Z}_2)$ or $H^*(B; \mathbb{Z}_2)$ is of finite type. Then the spectral sequence $\{E_r^{p,q}, d_r\}$ collapses and the the coefficients of $E_2^{p,q}$ are simple if and only if F is totally non-homologous to zero. $\square$

5.4) Lemma.- Let $p: E \to B$ be a fibration with fiber F, B and F path connected. If the spectral sequence $\{E_r^{p,q}, d_r\}$ collapses then p^* is injective.

Proof.- We have a filtration of
$H^*(E; \mathbb{Z}_2)$: $0 \subset J^{p,0} \subset J^{p-1,1} \subset \ldots \subset J^{0,p} = H^p(E; \mathbb{Z}_2)$.
Consider the edge homomorphism $e: E_2^{p,0} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \ldots \twoheadrightarrow E_\infty^{p,0}$, then we have a commutative diagram [52]:

$$\begin{tikzcd}
E_2^{p,0} \arrow[r, two heads, "e"] \arrow[d, phantom, "\wr\|"] & E_\infty^{p,0} = J^{p,0} \subset H^p(E;\mathbb{Z}_2) \\
H^p(B;\underline{H^o(F;\mathbb{Z}_2)}) \arrow[d, phantom, "\wr\|"] & \\
H^p(B;\mathbb{Z}_2) \arrow[uur, "p^*"'] &
\end{tikzcd}$$

So if the spectral sequence collapses p^* is an isomorphism onto its image therefore p^* is injective. $\square$

5.5) Definition.- Let X be a space such that $H_*(X;\mathbb{Z}_2)$ is of finite type, then we define the Poincare series of X by $P(X,t)=\sum_i \dim H^i(X;\mathbb{Z}_2)t^i$. Given a first quadrant spectral sequence $\{E_r^{p,q}, d_r\}$ such that $E_r^{p,q}$ has finite dimension for all r,p,q, we can also define a Poincare series $P(E_r, t)$ by considering the graded vector space $\{{}^nE_r\}_n$ where ${}^nE_r=\bigoplus_{p+q=n} E_r^{p,q}$.

5.6) Proposition.- [5] Let $p: E \to B$ be a fibration with fiber F such that B and F are path connected and E,B and F are of finite type. Then $P(E,t)= P(B,t)\, P(F,t)$ if and only if the spectral sequence $\{E_r^{p,q}, d_r\}$ collapses and the coefficients of $E_2^{p,q}$ are simple.

Proof$\Rightarrow$)We first show that the coefficients are simple. Let $S^q(F)$ be the biggest subspace of $H^q(F;\mathbb{Z}_2)$ where $\pi_1(B)$ acts trivially, i.e., $S^q(F)= E_2^{0,q}= H^o(B;\underline{H^q(F;\mathbb{Z}_2)})$. We shall prove by induction that $S^q(F)= H^q(F;\mathbb{Z}_2)$. For $q=0$, as F is connected, we have $H^o(F;\mathbb{Z}_2)=\mathbb{Z}_2= S^o(F)$. Now assume it is true for all $q<k$, then $E_2^{p,q} = H^p(B;\underline{H^q(F;\mathbb{Z}_2)})= H^p(B;\mathbb{Z}_2)\otimes H^q(F;\mathbb{Z}_2)$ for $q<k$.

As $\dim H^o(B;\mathbb{Z}_2)=1$ then $\dim {}^kE_2 = \dim \bigoplus_{p+q=k} E_2^{p,q} = \dim H^k(B\times F;\mathbb{Z}_2) -$ $- \dim H^k(F;\mathbb{Z}_2) + \dim S^k(F;\mathbb{Z}_2)$. By hypothesis $P(E,t) = P(B,t)\,P(F,t) = P(B\times F,t)$, therefore $\dim H^k(B\times F;\mathbb{Z}_2) = \dim H^k(E;\mathbb{Z}_2) = \dim {}^kE_\infty$, hence $\dim {}^kE_2 = \dim {}^kE_\infty - \dim H^k(F;\mathbb{Z}_2) + \dim S^k(F)$.

But $\dim {}^kE_2 \geqslant \dim {}^kE_\infty$ so $\dim S^k(F) \geqslant \dim H^k(F;\mathbb{Z}_2)$, and as $S^k(F) \subset H^k(F;\mathbb{Z}_2)$ then $\dim S^k(F) = \dim H^k(F;\mathbb{Z}_2)$, i.e., $S^k(F) = H^k(F;\mathbb{Z}_2)$

Now that we know that the coefficients are simple we can write $E_2^{p,q} = H^p(B;\mathbb{Z}_2)\otimes H^q(F;\mathbb{Z}_2)$ so $P(E_2,t) = P(B,t)P(F,t) = P(E,t) = P(E_\infty,t)$, hence $P(E_2,t) = P(E_\infty,t)$. i.e., $\dim {}^nE_2 = \dim {}^nE_\infty$ for all $n \geq 0$, so $\sum_{i+j=n} \dim E_2^{i,j} = \sum_{i+j=n} \dim E_\infty^{i,j}$. But each $E_{r+1}^{i,j}$ is a subquotient of $E_r^{i,j}$ so $\dim E_{r+1}^{i,j} \leq \dim E_r^{i,j}$ and then $\dim E_\infty^{i,j} \leq \dim E_2^{i,j}$, therefore $\dim E_\infty^{i,j} = \dim E_2^{i,j}$, and hence $E_\infty^{i,j} \cong E_2^{i,j}$

$\Leftarrow)$ $E_\infty^{p,q} \cong E_2^{p,q} \cong H^p(B;H^q(F;\mathbb{Z}_2)) \cong H^p(B;\mathbb{Z}_2)\otimes H^q(F;\mathbb{Z}_2)$. Therefore $H^n(E;\mathbb{Z}_2) \cong \bigoplus_{p+q=n} E_\infty^{p,q} \cong \bigoplus_{p+q=n} H^p(B,\mathbb{Z}_2)\otimes H^q(F;\mathbb{Z}_2)$. Hence $P(E,t) = P(B,t)\,P(F,t)$. $\square$

5.7)Definition.- Let $p\colon E \to B$ be a numerable principal G-bundle, where G is a topological group, and let $\phi_p\colon B \to BG$ be a classifying map for p. We call the image of ϕ_p^*, $\phi_p^*\colon H^*(BG;\mathbb{Z}_2) \to H^*(B;\mathbb{Z}_2)$, the characteristic subalgebra of the bundle.

5.8) Definition.- Let G be a compact Lie group and $H \subset G$ a closed subgroup, we say that H has the same 2-rank of G if there is a common maximal abelian subgroup isomorphic to $\mathbb{Z}_2^n$, we denote this subgroup by $Q(n)$, and call n the 2-rank.

We say that G/H verifies Hirsch's formula if

$$P(G/H,\ t) = \frac{(1-t^{m_1}),\ldots,(1-t^{m_n})}{(1-t^{q_1}),\ldots,(1-t^{q_n})}$$

where $m_1,\ldots,m_n$ are the degrees of the generators of $H^*(BG;\mathbb{Z}_2)$ and $q_1,\ldots,q_n$ are the degrees of the generators of $H^*(BH;\mathbb{Z}_2)$.

5.9) Proposition [5]: G be a compact Lie group and $H \subset G$ a closed subgroup of the same 2-rank. If $G/Q(n)$ and $H/Q(n)$ verify Hirsch's formula and if $H^*(H/Q(m);\mathbb{Z}_2)$ is equal to its characteristic subalgebra then G/H verifies formula.

Proof.- Consider the following diagram:

$$\begin{tikzcd} H \arrow[r, hook] \arrow[d, "p"'] & G \arrow[r] \arrow[d, "q"] & EQ(n) \arrow[d] \\ H/Q(n) \arrow[r, "i"'] & G/Q(n) \arrow[r, "j"'] & BQ(n) \end{tikzcd}$$

where i is induced by the inclusion and j is a classifying map for q (p and q are principal bundles because G is a Lie group and H is a closed subgroup[46]). Hence $j \circ i$ classifies p so by Hypothesis $(j \circ i)^*$ is surjective so i^* is surjective. But this implies that the fiber of the fibration $H/Q(n) \hookrightarrow G/Q(n) \xrightarrow{\phi} G/H$ is totally non-homologous to zero so by 5.3 the spectral sequence of the fibration collapses and the coefficients of $E_2^{p,q}$ are simple, so by 5.6) $P(G/Q(n),\ t) = P(H/Q(n),t)P(G/H,t)$, as $G/Q(n)$ and $H/Q(n)$ verify Hirsch's formula, then clearly G/H verifies Hirsch's formula.

$\square$

5.10) Proposition [5]. $P(O(n)/Q(n),\ t) = \dfrac{(1-t)\ (1-t^2)\ldots(1-t^n)}{(1-t)^n}$ and

$H^*(O(n)/Q(n);\mathbb{Z}_2)$ is equal to its characteristic subalgebra.

$\square$

5.11) Proposition [5]. $P(BO(n),t) = (1-t)^{-1}(1-t^2)^{-1},\ldots,(1-t^n)^{-1}$. □

5.12) Theorem [5]: Consider natural numbers $n_1, n_2, \ldots, n_k$ such that $n_1 + n_2 + \ldots + n_k = n$. Let $i: O(n_1) \times O(n_2) \times \ldots \times O(n_k) \hookrightarrow O(n)$ be the inclusion, then $Bi: B(O(n_1) \times \ldots \times O(n_k)) \longrightarrow BO(n)$ induces a monomorphism in $\mathbb{Z}_2$-cohomology.

Proof.- Consider the infinite Stiefel manifold $EO(n)$, and define an action of $O(n_1) \times \ldots \times O(n_k)$ on $EO(n)$ by $e\cdot(A_1,\ldots,A_k) = e\cdot i(A_1,\ldots,A_k)$. In 4.7 we saw that $EO(n)$ is completly regular, as $O(n_1) \times \ldots \times O(n_k)$ is a compact Lie group then by Glason's Theorem [7] the quotient $EO(n)/O(n_1) \times \ldots \times O(n_k) = B(O(n_1) \times \ldots \times O(n_k))$. Let $p: EO(n)/O(n_1) \times \ldots \times O(n_k) \rightarrow EO(n)/O(n)$ be the projection, as this map is induced by an i-equivariant map then $p \simeq Bi$ Consider the fibration

$$^{O(n)}/_{O(n_1)\times\ldots\times O(n_k)} \hookrightarrow {}^{EO(n)}/_{O(n_1)\times\ldots\times O(n_k)} \xrightarrow{p} {}^{EO(n)}/_{O(n)}$$

We want to apply 5.6 and for this we consider $P(O(n)/O(n_1) \times \ldots \times O(n_k), t)$. Notice that $O(n_1) \times \ldots \times O(n_k)$ and $O(n)$ have the same 2-rank with $Q(n) \subset O(n_1) \times \ldots \times O(n_k) \subset O(n)$, also notice that $O(n_1) \times \ldots \times O(n_k)/_{Q(n)} \cong$

$$O(n_1)\times\ldots\times O(n_k)/_{Q(n_1)\times\ldots\times Q(n_k)} \cong {}^{O(n_1)}/_{Q(n_1)} \times\ldots\times {}^{O(n_k)}/_{Q(n_k)}$$

By 5.10) ${}^{O(n_i)}/_{Q(n_i)}$ satisfies Hirsch's formula, for $1 \le i \le k$, so clearly the product satisfies Hirsch's formula. Furthermore By 5.10 $H^*(O(n_i)/Q(n_i); \mathbb{Z}_2)$ is equal to its characteristic subalgebra so clearly $H^*(O(n_1)/Q(n_1) \times \ldots \times O(n_k)/Q(n_k))$ is also equal to its characteristic subalgebra so by 5.9, $O(n)/O(n_1) \times \ldots \times O(n_k)$ verifies Hirsch's formula so $P(O(n)/O(n_1) \times \ldots \times O(n_k), t) = \dfrac{(1-t)(1-t^2)\ldots(1-t^n)}{\prod_{i=1}^{k}(1-t)\ldots(1-t^{n_i})}$

By 5.11) $P(BO(n),t)=(1-t)^{-1}(1-t^2)^{-1} \quad (1-t^n)^{-1}$ and $P(BO(n_1)x\dots xBO(n_k),t)=\prod_{i=1}^{k} (1-t)^{-1}\dots(1-t^{n_i})^{-1}$. Therefore we get

$P(BO(n_1)x\dots xBO(n_k),t)= P(BO(n),t)P(O(n)/O(n_1)x\dots xO(n_k),t)$ hence by 5.6 the spectral sequence of the fibration p collapses and by 5.4 p^* is injective.

□

5.13) Theorem.- Let ξ be an rk-vector bundle over a closed smooth manifold then ξ is bordant to a vector bundle with structural group $\Sigma_r \wr O(k)$.

Proof.- We have a commutative diagram:

$$\begin{tikzcd} \mathcal{N}_m[\Sigma_r \wr O(k)] \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & \mathcal{N}_m[O(rk)] \arrow[d, leftrightarrow, "\cong"] \\ \mathcal{N}_m(B(\Sigma_r \wr O(k))) \arrow[r, "Bj_*"'] & \mathcal{N}_m(BO(rk)) \end{tikzcd}$$

The isomorphisms are those of 4.37, and $j: \Sigma_r \wr O(k) \hookrightarrow O(rk)$. We are going to show that Bj_* is surjective. For this notice that we have a commutative diagram.

$$\begin{tikzcd} \Sigma_r \wr O(k) \arrow[r, hook, "j"] & O(rk) \\ O(k)^r \arrow[u, "\ell"] \arrow[ur, hook, "i"'] & \end{tikzcd}$$

, where $\ell(A_1,\dots,A_r)=(id,A_1,\dots,A_r)$ and i is the inclusion of 5.12. Then $B_j \circ B\ell \simeq B(j\circ \ell)=Bi$ By 5.12 Bi^* is injective, as all the homology groups are of finite type, then Bi_* is surjective, hence Bj_* is surjective in mod. 2 homology. The naturality of the equivalence $\mathcal{N}_*(X)\cong H_*(X) \otimes_{\mathbb{Z}_2} \mathcal{N}_*$ [18] implies that Bj_* is surjective in bordism.

□

5.14) Remark If k=2 in 5.13, one can give a proof of the theorem using the transfer as follows. Consider the fibration

$O(2r)/\Sigma_r \int O(2) \hookrightarrow B(\Sigma_r \int O(2)) \xrightarrow{p} BO(2r)$. By a theorem of Hopf and Samelson [24] we have that $\chi(O(2n)/N(T))=1$, where $N(T)$ is the normalizer of a maximal torus T in $O(2n)$. But $N(T)= \Sigma_r \int O(2)$ [25] , so $\chi(O(2r)/\Sigma_r \int O(2))= 1$. We can use now Becker and Gottlieb's transfer for the fibration p [4]. This transfer was defined when the base is a finite complex, but it was generalised in [12] to include the case when the base is infinite dimensional.

If τ denotes the transfer for p then we get $p_* \circ \tau_* = id$ so p_* is surjective.

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Chapter 6: Another interpretation for the groups $N_*(F(\mathbb{R}^\infty;r)\underset{\Sigma_r}{\times} BO(k)^r)$.

In this chapter we give an interpretation of $N_*(F(\mathbb{R}^\infty;r)\underset{\Sigma_r}{\times} BO(k)^r)$ in terms of covering spaces and arbitrary vector bundles, and apply this interpretation to the bordism of immersions.

§6.1) The functor $[\_, E\Sigma_r \underset{\Sigma_r}{\times} Y^r]$

In this section we use the method of F.W. Roush [1] for classifying transfers to give an interpretation of the functor $[\_, E\Sigma_r \underset{\Sigma_r}{\times} Y^r]$.

6.1) Definition.- Let X and Y be spaces and consider pairs $(\tilde{X},[\phi])$ where $\tilde{X} \xrightarrow{p} X$ is an r-covering over X and $[\phi]\in[\tilde{X},Y]$. We define a relation $\cong$, between these pairs as follows: $(\tilde{X}, [\phi])\cong(\tilde{X}', [\phi'])$ if and only if i) There is a map of coverings
$$\begin{tikzcd} \tilde{X} \arrow[rr, "h"] \arrow[dr, "p"'] & & \tilde{X}' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$
(i.e. a continuous map h making the triangle commute and such that it is a bijection on each fiber), ii) $h^*[\phi'] = [\phi]$ (i.e. $\phi' \circ h \simeq \phi$).

As a map of coverings is a homeomorphism, this is clearly an equivalence relation. We denote by $F_{r,Y}(X)$ the set of equivalence classes of such pairs.

6.2) Definition.- Let T denote the category of paracompact spaces, we define a functor $F_{r,Y}: T \to \text{Sets}$ as follows: to each $X\in T$, we associate $F_{r,Y}(X)$ and given $f: X_1 \to X_2$, we define

$f^* : F_{r,Y}(X_2) \to F_{r,Y}(X_1)$ by $f^*[\tilde{X}_2,[\rho]]=[f^*(\tilde{X}_2),[\rho\circ\bar{f}]]$, where the maps are given by the following pull-back

$$\begin{tikzcd} f^*(\tilde{X}_2) \arrow[r, "\bar{f}"] \arrow[d] & \tilde{X}_2 \arrow[d] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$

6.3) Lemma.- The functor $F_{r,Y}(\_)$ is well defined

Proof.- Suppose $(\tilde{X}_2,[\rho])\cong(\tilde{X}'_2,[\rho'])$, then we have a map of coverings

$$\begin{tikzcd} \tilde{X}_2 \arrow[rr, "h"] \arrow[dr, "q"'] & & \tilde{X}'_2 \arrow[dl, "q'"] \\ & X_2 & \end{tikzcd}$$

such that $\rho'\circ h \underset{G}{\simeq} \rho$. If we apply f^* we get the following diagrams:

$$\begin{tikzcd} f^*(\tilde{X}_2) \arrow[r, "\bar{f}"] \arrow[d] & \tilde{X}_2 \arrow[r, "\rho"] \arrow[d, "q"] & Y \\ X_1 \arrow[r, "f"'] & X_2 & \end{tikzcd} \qquad \begin{tikzcd} f^*(\tilde{X}'_2) \arrow[r, "\bar{f}'"] \arrow[d] & \tilde{X}'_2 \arrow[r, "\rho'"] \arrow[d] & Y \\ X_1 \arrow[r, "f"'] & X_2 & \end{tikzcd}$$

We can then define $\bar{h}$: $f^*(\tilde{X}_2)\to f^*(\tilde{X}'_2)$ by $h(x_1, \tilde{x}_2)=(x_1,h(\tilde{x}_2))$ this is well defined since $(x_1,\tilde{x}_2)\in f^*(\tilde{X}_2)\Longrightarrow f(x_1)= q(\tilde{x}_2)= q'h(\tilde{x}_2)$. As h is a map of coverings then so does $\bar{h}$.

Finally define a homotopy H: $f^*(\tilde{X}_2)\times I \to Y$ by $H(x_1,\tilde{x}_2,t)=G(\tilde{x}_2,t)$.

Then $H(x_1,\tilde{x}_2,0)=G(\tilde{x}_2,0)= \rho'h(\tilde{x}_2)= \rho'\bar{f}'(x_1,h(\tilde{x}_2))= \rho'\circ \bar{f}'\circ \bar{h}(x_1,\tilde{x}_2)$

and $H(x_1,\tilde{x}_2,1)=G(\tilde{x}_2,1)=\rho(\tilde{x}_2)=\rho\bar{f}(x_1,\tilde{x}_2)$ therefore $\rho'\circ\bar{f}'\circ\bar{h}\simeq \rho\circ\bar{f}$,

i.e., $h^*[\rho'\circ\bar{f}']=[\rho\circ\bar{f}]$.

□

6.4) Definition.- We define Φ_X:$[X, E\Sigma_r\underset{\Sigma_r}{\times} Y^r] \to F_{r,Y}(X)$ as follows: let $\bar{r}=\{1,2,\ldots,r\}$, we have an action $\Sigma_r\times\bar{r}\to\bar{r}$ given by $\sigma\cdot i= \sigma(i)$ and a principal Σ_r-bundle $E\Sigma_r\times Y^r \to E\Sigma_r\underset{\Sigma_r}{\times} Y^r$ so we can consider the r-covering associated $E\Sigma_r\times Y^r\underset{\Sigma_r}{\times}\bar{r}\xrightarrow{\Pi} E\Sigma_r\underset{\Sigma_r}{\times} Y^r$. Let f: $Y\to E\Sigma_r\underset{\Sigma_r}{\times} Y^r$ and consider the following pull-back:

$$\begin{tikzcd} f^*(\Pi) \arrow[r, "\bar{f}"] \arrow[d] & E\Sigma_r\times Y^r\underset{\Sigma_r}{\times}\bar{r} \arrow[r, "\alpha"] \arrow[d, "\Pi"] & Y \\ X \arrow[r, "f"] & E\Sigma_r\underset{\Sigma_r}{\times} Y^r & \end{tikzcd}$$

where $\alpha[e,y_1,\ldots,y_r, i] = y_i$, then define $\Phi_X[f] = [f^*(\Pi),[\alpha \circ \bar{f}]]$

To prove that Φ_X is well defined we need a Lemma:

6.5) Lemma.- Let $p: E \to B$ be a covering and $f,g: X \to B$ such that $f \simeq g$ then we have a map of coverings

$$\begin{tikzcd} f^*(p) \arrow[rr, "h"] \arrow[dr, "p_f"'] & & g^*(p) \arrow[dl, "p_g"] \\ & X & \end{tikzcd}$$

such that $\bar{g} \circ h \simeq \bar{f}$

Proof.- Let $H: X \times I \to B$ be a homotopy such that $H_0 = f$ and $H_1 = g$ and consider the composition $H \circ (p_f \times id): f^*(p) \times I \to B$ then we have a commutative diagram

$$\begin{tikzcd} f^*(p) \arrow[r, "\bar{f}"] \arrow[d] & E \arrow[d, "p"] \\ f^*(p) \times I \arrow[r, "H\circ(p_f \times id)"'] \arrow[ur, dashed, "\bar{H}"] & B \end{tikzcd}$$

, so we can find a lifting $\bar{H}$ making the diagram commute. Define $h: f^*(p) \to X \times E$ by $h(x,e) = (p_f(x,e) = x, \bar{H}(x,e,1))$, notice that $p\bar{H}(x,e,1) = H\circ(p_f \times id)(x,e,1) = H(x,1) = g(x)$ hence $h: f^*(p) \to g^*(p)$. Also notice that when we change t in $\bar{H}(x,e,t)$ and we project under p we get $H(x,t)$, as for each x, $H(x,t)$ is a path from $f(x)$ to $g(x)$, then for each e over $f(x)$ we get a lifting of this path beginning in e and ending in a point over $g(x)$, i.e., h is a map of coverings. Finally $\bar{g}\circ h(x,e) = \bar{g}(x,H(x,e,1)) = \bar{H}(x,e,1) \simeq \bar{H}(x,e,0) = \bar{f}(x,e)$, i.e., $\bar{g}\circ h \simeq \bar{f}$. □

6.6) Lemma.- $\Phi_X : [X, E\Sigma_r \underset{\Sigma_r}{\times} Y^r] \to F_{r,Y}(X)$ is well defined and it is natural.

Proof.- Suppose $f,g: X \to E\Sigma_r \times_{\Sigma_r} Y^r$ are homotopic then by 6.5 we have a map of coverings

$$\begin{tikzcd} f^*(\Pi) \arrow[rr, "h"] \arrow[dr, "\Pi_f"'] & & g^*(\Pi) \arrow[dl, "\Pi_g"] \\ & X & \end{tikzcd}$$

such that $\bar{g}\circ h \simeq \bar{f}$ hence $\alpha\circ\bar{g}\circ h \simeq \alpha\circ\bar{f}$, i.e.,

$h^*[\alpha\circ\bar{g}]=[\alpha\circ\bar{f}]$ therefore $(f^*(\Pi),[\alpha\circ\bar{f}])\cong(g^*(\Pi),[\alpha\circ\bar{g}])$.

The naturality follows from the fact that $f^*g^*(\Pi)\cong(g\circ f)^*(\Pi)$.

□

6.7) Definition.- We define $\Psi_X : F_{r,Y}(X)\to[X, E\Sigma_r \underset{\Sigma_r}{\times} Y^r]$ as follows: given a pair $(\tilde{X},[\phi])$, consider the map $X\to E\Sigma_r\underset{\Sigma_r}{\times}\tilde{X}^r$ defined in 3.18, by a slight abuse of notation we shall denote this map by T and call it also pretransfer, consider the composition

$X \xrightarrow{T} E\Sigma_r\underset{\Sigma_r}{\times}\tilde{X}^r \xrightarrow{id\underset{\Sigma_r}{\times}\phi^r} E\Sigma_r\underset{\Sigma_r}{\times} Y^r$. T is defined up to homotopy and if

$\phi\simeq\phi'$ then $id\underset{\Sigma_r}{\times}\phi^r\simeq id\underset{\Sigma_r}{\times}\phi'^r$ therefore we can associate to $(\tilde{X},[\phi])$ the homotopy class of $(id\underset{\Sigma_r}{\times}\phi^r)\circ T$ and we define

$\Psi_X[X,[\phi]] = [id\underset{\Sigma_r}{\times}\phi^r\circ T]$.

6.8) Lemma.- $\Psi_X : F_{r,Y}(X)\to[X, E\Sigma_r\underset{\Sigma_r}{\times} Y^r]$ is well defined and it is natural.

Proof.- Suppose $(\tilde{X}_1,[\phi_1])\cong(\tilde{X}_2,[\phi_2])$ then we have a map of coverings

$$\begin{tikzcd} \tilde{X}_1 \arrow[r, "h"] \arrow[dr, "p_1"'] & \tilde{X}_2 \arrow[r, "\phi_2"] \arrow[d, "p_2"] & Y \\ & X & \end{tikzcd}$$

such that $\phi_2\circ h\simeq\phi_1$

To define T we have to consider the principal Σ_r-bundles associated. Consider the following map $\bar{\beta} : \bar{X}_1\to\bar{X}_2$ given by $\bar{\beta}(\tilde{x}_1,\ldots,\tilde{x}_r)=(h(\tilde{x}_1),\ldots,h(\tilde{x}_r))$, as h is a map coverings this is well defined and it is clearly Σ_r-equivariant, therefore we have a pull-back diagram:

$$\begin{tikzcd} \bar{X}_1 \arrow[r, "\bar{\beta}"] \arrow[d] & \bar{X}_2 \arrow[r, "\bar{\rho}_2"] \arrow[d] & E\Sigma_r \arrow[d] \\ \bar{X}_1/\Sigma_r \arrow[r, "\beta"'] & \bar{X}_2/\Sigma_r \arrow[r, "\rho_2"'] & B\Sigma_r \end{tikzcd}$$

where ρ_2 classifies the principal Σ_r-bundle $\bar{X}_2 \to \bar{X}_2/\Sigma_r$. Hence we can define the pretransfer T_1, for p_1 using this pull-back and we get a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\cong"] \arrow[dr, "\cong"'] & \bar{X}_1/\Sigma_r \arrow[r, "T_1"] \arrow[d, "\beta"] & E\Sigma_r \times_{\Sigma_r} \tilde{X}_1^r \arrow[d, "id \times_{\Sigma_r} h^r"] \\
 & \bar{X}_2/\Sigma_r \arrow[r, "T_2"] & E\Sigma_r \times_{\Sigma_r} \tilde{X}_2^r
\end{tikzcd}$$

and as $h \circ \phi_2 = \phi_1$ we get a homotopy commutative diagram:

$$\begin{tikzcd}
E\Sigma_r \times_{\Sigma_r} \tilde{X}_1^r \arrow[d, "id \times_{\Sigma_r} h^r"] \arrow[dr, "id \times_{\Sigma_r} \phi_1^r"] & \\
E\Sigma_r \times_{\Sigma_r} \tilde{X}_2^r \arrow[r, "id \times_{\Sigma_r} \phi_2^r"'] & Y
\end{tikzcd}$$

Combining these 2 diagrams we get that $\Psi_X [\tilde{X}_1, [\phi_1]] = \Psi_X [\tilde{X}_2, [\phi_2]]$. By the same method one can also show that Ψ_X is natural.

□

6.9) Theorem.- The functors $[\_, E\Sigma_r \times_{\Sigma_r} Y^r]$ and $F_{r,Y}(\_): T \to \text{Sets}$ are naturally equivalent. The equivalences are given by Ψ and Φ.

Proof.- We shall prove that $\Psi_X \; \Phi_X = id$. Let $f: X \to E\Sigma_r \times_{\Sigma_r} Y^r$, then $\Psi_X \circ \Phi_X [f] = \Psi_X [f^*(\Pi), [\alpha \circ f]]$. To obtain the pretransfer for $f^*(\Pi)$ notice that if we have an r-covering of the form $E \times_{\Sigma_r} \bar{r} \to E/\Sigma_r$ then the pretransfer $T: E/\Sigma_r \to E\Sigma_r \times_{\Sigma_r} (E \times_{\Sigma_r} \bar{r})^r$ is given by

$T[e] = [\bar{\rho}(e), [e,1], \ldots, [e,r]]$ where we have a pull-back

$$\begin{tikzcd}
E \arrow[r, "\bar{\rho}"] \arrow[d] & E\Sigma_r \arrow[d] \\
E/\Sigma_r \arrow[r, "\rho"'] & E\Sigma_r/\Sigma_r
\end{tikzcd}$$

Now let us apply this observation to the covering

$E\Sigma_r \times Y^r \underset{\Sigma_r}{\times} \bar{r} \xrightarrow[\Pi]{} E\Sigma_r \underset{\Sigma_r}{\times} Y^r$, the classifying square for Π is given by

$$\begin{tikzcd}
E\Sigma_r \times Y^r \arrow[r, "\text{proj.}"'] \arrow[d] & E\Sigma_r \arrow[d] \\
E\Sigma_r \underset{\Sigma_r}{\times} Y^r \arrow[r] & E\Sigma_r/\Sigma_r
\end{tikzcd}$$

So $T_\Pi : E\Sigma_r \underset{\Sigma_r}{\times} Y^r \to E\Sigma_r \underset{\Sigma_r}{\times} (E\Sigma_r \times Y^r \underset{\Sigma_r}{\times} \bar{r})^r$ is given by

$T_\Pi\, [e,y_1,\dots,y_r] = [e,[\,e,y_1,\dots,y_r,\,1]\,,\dots,[\,e,y_1,\dots,y_r,\,r]]$

Recall that we have a pull-back:

$$\begin{tikzcd}
f^*(\Pi) \arrow[r, "\bar{f}"] \arrow[d] & E\Sigma_r \times Y^r \underset{\Sigma_r}{\times} \bar{r} \arrow[r, "\alpha"] \arrow[d] & Y \\
X \arrow[r, "f"] & E\Sigma_r \underset{\Sigma_r}{\times} Y^r &
\end{tikzcd}$$

Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "T_{f^*\Pi}"] \arrow[d, "f"'] & E\Sigma_r \underset{\Sigma_r}{\times} f^*(\Pi)^r \arrow[rr, "\mathrm{id}\underset{\Sigma_r}{\times}(\alpha\circ\bar{f})^r"] \arrow[d, "\mathrm{id}_{\Sigma_r}\times \bar{f}^r"] & & E\Sigma_r \underset{\Sigma_r}{\times} Y^r \\
E\Sigma_r \underset{\Sigma_r}{\times} Y^r \arrow[r, "T_\Pi"'] & E\Sigma_r \underset{\Sigma_r}{\times} (E\Sigma_r \times Y^r \underset{\Sigma_r}{\times} \bar{r})^r \arrow[urr, "\mathrm{id}_{\Sigma_r}\times \alpha^r"'] & &
\end{tikzcd}$$

The composition at the top represents $\Psi_X\ \Phi_X[f]$. By choosing suitable classifying maps, as we have done before, one can prove that the square is homotopy commutative. The triangle clearly commutes so

$\Psi_X\ \ \Phi_X\ [f] = [\mathrm{id}\underset{\Sigma_r}{\times}\ \alpha^r \circ T_\Pi \circ f]$, but the composition $(\mathrm{id}_{\Sigma_r} \times \alpha^r)^{\circ}\ T_\Pi$ is

the identity, in effect, $(\mathrm{id}_{\Sigma_r} \times \alpha^r)^{\circ}\ T_\Pi[e,y_1,\dots,y_r] =$

$=(\mathrm{id}_{\Sigma_r}\times\alpha^r)[e,[e,y_1,\dots,y_r,1],\dots,[e,y_1,\dots,y_r,r]]=$

$=[e,\alpha[e,y_1,\dots,y_r,1],\dots,\alpha[e,y_1,\dots,y_r,r]]=[e,y_1,\dots,y_r]$. Therefore

$\Psi_X\circ\Phi_X[f]=[f]$.

To see that $\Phi_X\circ\Psi_X=\mathrm{id}$, let $[\tilde X[\phi]]\in F_{r,Y}(X)$, then $\Phi_X\circ\Psi_X[\tilde X,[\phi]]$ is given by the following pull back

$$\begin{tikzcd}
\tilde X\cong\tilde X_{\Sigma_r}\times\bar r \arrow[r, "\bar T\underset{\Sigma_r}{\times}\mathrm{id}"] \arrow[d] & E\Sigma_r\times\tilde X^r\underset{\Sigma_r}{\times}\bar r \arrow[r, "\mathrm{id}\times\phi^r\underset{\Sigma_r}{\times}\mathrm{id}"] \arrow[d] & E\Sigma_r\times Y^r\underset{\Sigma_r}{\times}\bar r \arrow[r, "\alpha"] \arrow[d] & Y \\
X\simeq\tilde X/\Sigma_r \arrow[r, "T"] & E\Sigma_r\underset{\Sigma_r}{\times}\tilde X^r \arrow[r, "\mathrm{id}_{\Sigma_r}\times\phi^r"] & E\Sigma_r\underset{\Sigma_r}{\times}Y^r &
\end{tikzcd}$$

One can easily verify that the composition at the top is precisely ϕ so $\Phi_X\circ\Psi_X[\tilde X,[\phi]]=[\tilde X,[\phi]]$. □

§6.2) The direct image of a vector bundle

6.10) Definition .- Let $p:\tilde X\to X$ be a r-covering and let $\xi\to\tilde X$ be a k-vector bundle then Atiyah [3] defined an (rk)-vector bundle $p_*(\xi)\to X$, called the direct image of ξ, whose fibers are given by $p_*(\xi)_x=\bigoplus_{\tilde x\in p^{-1}(x)}\xi_{\tilde x}$.

Let us denote $F_{r,BO(k)}(\_)\equiv F_{r,k}(\_)$ for simplicity, then the direct image construction defines a natural transformation $F_{r,k}(\_)\to\mathrm{Vect}_{kr}(\_)$, where $\mathrm{Vect}_{kr}(X)$ denotes the set of isomorphism classes of (kr)-vector bundles over X.

By 6.9 we have:

$$\begin{tikzcd}
F_{r,k}(_) \arrow[r] \arrow[d, "\text{direct image}"'] & {[_,E\Sigma_r\underset{\Sigma_r}{\times}BO(k)^r]} \arrow[d, dotted] \\
\mathrm{Vect}_{rk}(_) \arrow[r] & {[_,BO(kr)]}
\end{tikzcd}$$

We can choose a model for $E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r$ that is paracompact as follows: we can take Milnor's construction [35] for $E\Sigma_r$ and $EO(k)$ which are numerable C.W.-complexes, hence $E\Sigma_r \times EO(k)^r$ is also a C.W.-complex and therefore a paracompact space, as $\Sigma_r \int O(k)$ is compact then the projection map is closed and the quotient $E\Sigma_r \times EO(k)^r / \Sigma_r \int O(k) \cong$ $\cong E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r$ (3.21) is Hausdorff and therefore paracompact. Recall that $E\Sigma_r \times EO(k)^r$ is contractible and completly regular with a free $\Sigma_r \int O(k)$- action so by Gleason's theorem [7] we have
$E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r = B(\Sigma_r \int O(k))$.

By Yoneda's lemma we have a bijection between natural transformations $[\_, E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r] \to [\_, BO(kr)]$ and $[E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r, BO(kr)]$, so the direct image construction defines a homotopy class $d: E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r \to BO(kr)$.

6.11) Proposition.- Let $i: \Sigma_r \int O(k) \hookrightarrow O(r\,k)$ then $Bi \simeq d$.

Proof.- By 6.9 we have a bijection $\Psi_X: F_{r,k}(X) \to [X, E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r]$ and by [26] the direct image of a vector bundle $\xi \to \tilde{X} \longrightarrow X$ is classified by the map $X \xrightarrow{T} E\Sigma_r \underset{\Sigma_r}{\times} \tilde{X}^r \xrightarrow[id \underset{\Sigma_r}{\times} \phi^r]{} E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r =$

$= B(\Sigma_r \int O(k)) \xrightarrow[Bi]{} BO(kr)$ where $[\phi]$ classifies ξ. But $\Psi_X(\tilde{X}, [\phi]) =$ $= [id \underset{\Sigma_r}{\times} \phi^r, T]$, therefore $Bi_* \Psi_X(\tilde{X}, [\phi]) = d_* \Psi_X(\tilde{X}, [\phi])$, as Ψ_X is a bijection then $Bi_* = d_*$ for all X particular if $X = E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r$, so $Bi_*[id] = d_*[id] \Rightarrow [Bi] = [d]$, i.e., $Bi \simeq d$.

□

6.12) Corollary.- Let ξ be an (rk)-vector bundle over a paracompact space. Then the structural group of ξ has a reduction to $\Sigma_r \int O(k)$ if and only if ξ is isomorphic to the direct image of a k-vector bundle over an r-covering.

Proof.- By 6.9 and 6.11 we have a commutative diagram:

$$\begin{tikzcd} F_{r,k}(X) \arrow[r, dash] \arrow[d, "\text{direct image}"'] & {[X, E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r]} \arrow[d, "Bi_*"] \\ \mathrm{Vect}_{kr}(X) \arrow[r, leftrightarrow] & {[X, BO(kr)]} \end{tikzcd}$$

If we denote by $\mathrm{Vect}_{kr}^{\Sigma_r \int O(k)}(X)$ the set of isomorphism classes of (kr)-vector bundles with structural group $\Sigma_r \int O(k)$ then we get a commutative diagram:

$$\begin{tikzcd} \mathrm{Vect}_{kr}^{\Sigma_r \int O(k)}(X) \arrow[r, leftrightarrow] \arrow[d, "\text{forgetful functor}"'] & {[X, E\Sigma_r \underset{\Sigma_r}{\times} BO(k)^r]} \arrow[d, "Bi_*"] \\ \mathrm{Vect}_{kr}(X) \arrow[r, leftrightarrow] & {[X, BO(kr)]} \end{tikzcd}$$

The result follows from the commutativity of the following diagram:

$$\begin{tikzcd} F_{r,k}(X) \arrow[rr, dash] \arrow[dr, "\text{direct image}"'] & & \mathrm{Vect}_{rk}^{\Sigma_r \int O(k)}(X) \arrow[dl, "\text{Forgetful functor}"] \\ & \mathrm{Vect}_{rk}(X) & \end{tikzcd}$$

□

§ 6.3) Another interpretation for $N_*(E\Sigma_r \underset{\Sigma_r}{\times} Y^r)$

6.13) Definition.- We consider pairs $(\tilde{M},[\phi])$ where $\tilde{M} \xrightarrow[p]{} M$ is an r-covering over a closed smooth n-manifold M and $[\phi] \in [\tilde{M}, Y]$. We say that 2 pairs $(\tilde{M},[\phi])$ and $(\tilde{N},[\psi])$ are bordant, $(\tilde{M},[\phi]) \sim (\tilde{N},[\psi])$, if there exists a compact smooth (n+1)-manifold W, an r-covering $\tilde{W} \to W$ and a map $\tilde{W} \xrightarrow{h} Y$ such that i) there is a diffeomorphism $\partial W \cong M \amalg N$; ii) if we denote by $i_M : M \hookrightarrow M \amalg N \cong \partial V \hookrightarrow W$ and by $i_N : N \hookrightarrow M \amalg N \cong \partial V \hookrightarrow W$, then $(i_M{}^*(\widetilde{W}), [h \circ \bar{i}_M]) \cong (\widetilde{M},[\phi])$ and $(i_N{}^*(\widetilde{W}), [h \circ \bar{i}_N]) \cong (\widetilde{N},[\psi])$, where " $\cong$" is the equivalence relation defined in 6.1 .

6.14) Lemma.- The relation " $\sim$" defined above is an equivalence relation.

Proof.- i) $(M,[\phi]) \sim (M,[\phi])$, in effect, take

$$\begin{tikzcd} \widetilde{M}\times I \arrow[r, "proj."] \arrow[d, "p\times id"] & M \arrow[r, "\phi"] & Y \\ M \times I & & \end{tikzcd}$$

ii) It is obviously symmetric

iii) Suppose $(\widetilde{M},[\phi]) \sim (\widetilde{N},[\psi]) \sim (\widetilde{T},[\gamma])$. Then we have compact manifolds V, W and r-coverings $\widetilde{V}$, $\widetilde{W}$ and maps $\widetilde{V} \xrightarrow{h} Y$, $\widetilde{W} \xrightarrow{j} Y$ such that $\partial V \cong M \amalg N$, $\partial W \cong N \amalg T$ and commutative diagrams:

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\bar{i}_{M,V}"] \arrow[d] & \widetilde{V} \arrow[r, "h"] \arrow[d] & Y \\ M \arrow[r, "i_{M,V}"'] & V & \end{tikzcd} \qquad \begin{tikzcd} \widetilde{N} \arrow[r, "\bar{i}_{N,V}"] \arrow[d] & \widetilde{V} \arrow[r, "h"] \arrow[d] & Y \\ N \arrow[r, "i_{N,V}"'] & V & \end{tikzcd}$$

$$\begin{tikzcd} \widetilde{N} \arrow[r, "\bar{i}_{N,W}"] \arrow[d] & \widetilde{W} \arrow[r, "j"] \arrow[d] & Y \\ N \arrow[r, "i_{N,W}"'] & W & \end{tikzcd} \qquad \begin{tikzcd} \widetilde{T} \arrow[r, "\bar{i}_{T,W}"] \arrow[d] & \widetilde{W} \arrow[r, "j"] \arrow[d] & Y \\ T \arrow[r, "i_{J,W}"'] & W & \end{tikzcd}$$

The maps satisfy: $h \circ \bar{i}_{M,V} \simeq \phi$, $h \circ \bar{i}_{N,V} \simeq \psi$, $j \circ \bar{i}_{N,W} \simeq \psi$, $j \circ \bar{i}_{T,W} \simeq \gamma$. To prove that $(\widetilde{M},[\phi]) \sim (\widetilde{T},[\gamma])$, we glue the manifolds V and W using the embeddings $i_{N,V}$ and $i_{N,W}$, we call this manifold $V \underset{i}{\cup} W$ and $\partial(V \underset{i}{\cup} W) \cong M \amalg T$. Notice that if we have a finite covering over a manifold we can give the total space the structure of a manifold making the projection a local diffeomorphism. Furthermore as a manifold is locally path connected if the base is a compact manifold then the total space is also a compact manifold [33].

We can then use the embeddings $\bar{i}_{N,V}$ and $\bar{i}_{N,W}$ to form the manifold $\tilde{V} \underset{\bar{i}}{\cup} \tilde{W}$. Clearly $\tilde{V} \underset{\bar{i}}{\cup} \tilde{W}$ is an r-covering over $V \underset{i}{\cup} W$.

In order to define a map $\tilde{V} \underset{\bar{i}}{\cup} \tilde{W} \dashrightarrow Y$ we do the following: We have a closed submanifold $\bar{i}_{N,V}(\tilde{N}) \subset \tilde{V}$ so it has the homotopy extension property ($\tilde{V}$ is compact hence normal and a submanifold is a retract of a neighbourhood [23]). Consider the following diagram:

$$\begin{tikzcd}
\tilde{N} \arrow[r, hook] \arrow[d, "\bar{i}_{N,V}"] & \tilde{N}\times I \arrow[d, "\bar{i}_{N,V}\times id"] \arrow[dr, "H"] & \\
\tilde{V} \arrow[r, hook] \arrow[rr, bend right, "h"'] & \tilde{V}\times I \arrow[r, dashed, "G"] & Y
\end{tikzcd}$$

where H is the homotopy between $h\circ \bar{i}_{N,V}$ and $j\circ \bar{i}_{N,W}$ (recall that $h\circ \bar{i}_{N,V} \simeq \psi \simeq j\circ \bar{i}_{N,W}$), therefore we can find a homotopy G making the diagram commute. Consider $G_1\colon \tilde{V} \to Y$, then $G_1 \simeq G_0 = h$ and $G_1 \circ \bar{i}_{N,V} = j\circ \bar{i}_{N,W}$. Now we can define a map $\tilde{V} \underset{\bar{i}}{\cup} \tilde{W} \xrightarrow{\rho} Y$ by glueing G_1 and j because both coincide on $\tilde{N}$.

Therefore we have a covering $\tilde{V} \underset{\bar{i}}{\cup} \tilde{W} \to V \underset{i}{\cup} W$ which restricts to $\tilde{M}$ over M and to $\tilde{T}$ over T and a map ρ such that $\tilde{M} \hookrightarrow \tilde{V} \underset{\bar{i}}{\cup} \tilde{W} \xrightarrow{\rho} Y$ is $G_1\circ \bar{i}_{M,V} \simeq h\circ \bar{i}_{M,V} \simeq \phi$ and $\tilde{T} \hookrightarrow V \underset{\bar{i}}{\cup} W \underset{\rho}{\to} Y$ is $j\circ \bar{i}_{T,W} \simeq \gamma$, i.e. $(\tilde{M},[\phi]) \sim (\tilde{T},[\gamma])$.
□

6.15) Definition.- We denote the set of bordism classes of pairs $(\tilde{M},[\phi])$, where $\dim M = n$, by $Cov_n(r,Y)$. We can make $Cov_n(r,Y)$ into a group by defining $[\tilde{M},[\phi]] + [\tilde{N},[\psi]] = [\tilde{M} \amalg \tilde{N},[\phi \amalg \psi]]$. We can make $Cov_*(r,Y)$ into an N_*-module by defining $[M]\cdot[\tilde{N},[\phi]] = [M\times\tilde{N},[\phi\circ proj]]$.

6.16) Definition.- We define a function $G\colon Cov_n(r,Y) \to N_n(E\Sigma_r \underset{\Sigma_r}{\times} Y^r)$ as follows:

consider a pair $(M,[\phi])$, then by 6.9 $F_{r,Y}(M) \xrightarrow{\psi} [M, E\Sigma_r \underset{\Sigma_r}{\times} Y^r]$, so we define $G[M,[\phi]]=[M,\psi(\tilde{M},[\phi])] \in N_n(E\Sigma_r \underset{\Sigma_r}{\times} Y^r)$.

6.17) Lemma.- G is well defined and it is a homomorphism of N_*-modules.

Proof.- First notice that for each pair $(\tilde{M},[\phi])$ we have a well defined element in $N_n(E\Sigma_r \underset{\Sigma_r}{\times} Y^r)$ because 2 homotopic maps induce the same element. Now suppose $(\tilde{M},[\phi])\sim(\tilde{N},[\psi])$, then we have a manifold V such that $\partial V \cong M \amalg N$ and an r-covering $\tilde{V} \to V$ with a map $\tilde{V} \xrightarrow{h} Y$ such that: $(i_M^*(\tilde{V}),[h\circ\bar{i}_M])\cong(\tilde{M},[\phi])$ and $(i_N^*(\tilde{V}),[h\circ\bar{i}_N])\cong(\tilde{N},[\psi])$. The pair $(\tilde{V},[h])$ gives a map $V \xrightarrow{\psi(\tilde{V},[h])} E\Sigma_r \underset{\Sigma_r}{\times} Y^r$, and this map is a bordism between $(M,\psi(\tilde{V},[h])\circ i_M)$ and $(N,\psi(\tilde{V},[h])\circ i_N)$ so $[M,\psi(\tilde{V},[h])\circ i_M] = {}$
$=[N,\psi(\tilde{V},[h])\circ i_N]$ (*).

Now we compare $[M,\psi(\tilde{V},[h])\circ i_M]$ with $[M,\psi(\tilde{M},[\phi])]$. By hypothesis $(\tilde{M},[\phi])\cong(i_M^*(\tilde{V}),[h\circ i_M])= i_M^*(\tilde{V},[h])$ so as ψ is natural we have $\psi(\tilde{M},[\phi])=\psi(i_M^*(\tilde{V},[h])= i_M^*\,\psi(\tilde{V},[h])= \psi(\tilde{V},[h])\circ i_M$. Hence $[M,\psi(\tilde{V},[h])\circ i_M]=$
$=[M,\psi(\tilde{M},[\phi])]$. In the same way $[N,\psi(\tilde{V},[h])\circ i_N] = [N,\psi(\tilde{N},[\psi])]$. Combining this with (*) we get $[M,\psi(\tilde{M},[\phi])]=[N,\psi(\tilde{N},[\psi])]$ so G is well defined. The fact that it is an N_*-module homomorphism follows directly from the definitions.
□

6.18) Definition.- We define a function $H: N_n(E\Sigma_r \underset{\Sigma_r}{\times} Y^r) \to Cov_n(r,Y)$ as follows: consider $[M, f] \in N_n(E\Sigma_r \underset{\Sigma_r}{\times} Y^r)$, by 6.9 we have $\Phi: [M, E\Sigma_r \underset{\Sigma_r}{\times} Y^r] \longrightarrow F_{r,Y}(M)$, so we define $H[M,f] = \Phi[f]$.

6.19) Lemma.- H is well defined.

Proof.- First notice that given $f: M \to E\Sigma_r \underset{\Sigma_r}{\times} Y^r$, $\Phi[f]$ is defined up to isomorphism but clearly if 2 pairs are isomorphic then they are bordant.

Now of $[M,f]=[N,g]$ then there exists a compact manifold W such that $\partial W \cong M \amalg N$ and a map $h\colon W \to E\Sigma_r \times_{\Sigma_r} Y^r$ such that $h\circ i_M = f$ and $h\circ i_N = g$. If we take $\Phi[h] = (\tilde{W}, [\gamma])$, then we have: $h\circ i_M = f \Leftrightarrow i_M{}^*[h] = [f]$, as Φ is natural $\Phi[f] = \Phi i_M{}^*[h] = i_M{}^*\Phi[h]$. In the same way $\Phi[g] = i_N{}^*\Phi[h]$, therefore $\Phi[f]$ and $\Phi[g]$ are bordant so H is well defined.

□

6.20) Theorem.- We have an isomorphism of N_*-modules $N_*(E\Sigma_r \times_{\Sigma_r} Y^r) \cong \cong \mathrm{Cov}_*(r, Y)$.

Proof.- By 6.17 we have a homomorphism of N_*-modules $G\colon \mathrm{Cov}_*(r,y) \to N_*(E\Sigma_r \times_{\Sigma_r} Y^r)$ and it follows from the definition 6.18 and from 6.9 that $H\colon N_*(E\Sigma_r \times_{\Sigma_r} Y^r) \to \mathrm{Cov}_*(r,Y)$ is the inverse. □

§6.4) Application to the bordism of immersions

In 4.38 we gave a splitting of $I(n,k)$ in terms of vector bundles ν_{f_r} with structural group $\Sigma_r \int O(k)$, in this section we identify the covering spaces and vector bundles whose direct image are the bundles ν_{f_r}.

6.21) Definition.- Let $f\colon M^n \to N^{n+k}$ be a self-transverse immersion. In 4.4 we defined compact manifolds $\mu_r \subset (M)^r$ with a free Σ_r-action. We define a map $\pi_r\colon (M)^r \times \bar{r} \to M$ by $\pi_r(x_1,\dots,x_r,i) = x_i$, consider the composition $\mu_r \times \bar{r} \subset (M)^r \times \bar{r} \xrightarrow{\pi_r} M$, this map is invariant under the action of Σ_r and defines a map $\pi_r\colon \mu_r \times_{\Sigma_r} r \to M$. Let $\nu_f\colon M \to BO(k)$ be a classifying map for the normal bundle of the immersion f, then we can define a pair by :

$$\begin{array}{c}\mu_r \times_{\Sigma_r} \bar{r} \xrightarrow{\pi_r} M \xrightarrow{\nu_f} BO(k)\\ \downarrow \\ \mu_r/\Sigma_r\end{array}$$

Now we want to compare this covering with the covering defined with the manifold of based r-tuple points, namely $\mu_r/\Sigma_{r-1} \to \mu_r/\Sigma_r$

6.22) Proposition.- The coverings
$$\begin{array}{ccc} \mu_r \underset{\Sigma_r}{\times} \bar{r} & , & \mu_r/\Sigma_{r-1} \\ \downarrow & & \downarrow \\ \mu_r/\Sigma_r & & \mu_r/\Sigma_r \end{array}$$
are isomorphic.

Proof.- Define an action $\Sigma_r \times \Sigma_r/\Sigma_{r-1} \to \Sigma_r/\Sigma_{r-1}$ by $\sigma\cdot\bar{\tau} = \overline{\sigma\tau}$, where the bar denotes the right coset. Then we have an isomorphism

$$\begin{tikzcd} \mu_r \underset{\Sigma_r}{\times} \Sigma_r/\Sigma_{r-1} \arrow[rr, "h"] \arrow[dr] & & \mu_r/\Sigma_{r-1} \arrow[dl] \\ & \mu_r/\Sigma_r & \end{tikzcd}$$

given by $h\,[z,\bar{\sigma}] = [z\cdot\sigma]$, where $z\in\mu_r, \sigma\in\Sigma_r$. The inverse is given by $h^{-1}[z] = [z,\bar{id}]$.

To show that $\mu_r \underset{\Sigma_r}{\times} \Sigma_r/\Sigma_{r-1}$ and $\mu_r \underset{\Sigma_r}{\times} \bar{r}$ are isomorphic we need the following

Claim .- Let $\Sigma_r/\Sigma_{r-1} = \{\bar{c}, \bar{c}^2, \dots, \bar{c}^r = \bar{id}\}$, where $c = (1\ 2 \dots r)$ and define a bijection $f: \Sigma_r/\Sigma_{r-1} \to \bar{r}$ by $f(\bar{c}^i) = i$, then f is Σ_r-equivariant.

Proof of claim.- Define $\rho: \Sigma_r \to \Sigma_r$ by $\rho(\sigma)(i) = f(\sigma\cdot f^{-1}(i))$, we will show that $\rho = id$. For this recall that Σ_r is generated by $\{c,(1\ r)\}$, we also have that $\rho(\sigma)(i) = f(\sigma\cdot\bar{c}^i) = f(\overline{\sigma c}^i)$, hence f is equivariant $\iff \rho(\sigma) = \sigma \iff f(\sigma c^i) = \sigma(i)$, for all $i \iff \overline{\sigma c}^i = \bar{c}^{\sigma(i)}$ for all $i\in\bar{r}, \sigma\in\{c, (1\ r)\}$.

Case.- $\sigma = c$. In this case we have $\overline{\sigma c}^i = \overline{cc}^i = \bar{c}^{i+1} = \bar{c}^{c(i)}$ if $i<r$ and $\overline{cc}^r = \bar{c} = \bar{c}^{c(r)=1}$ if $i=r$.

Case.- $\sigma = (1\ r)$. Let us denote $t=(1\ r)$, so we want to show that $\overline{tc}^i = \bar{c}^{t(i)}$ for all $i\in\bar{r}$. If $i=1$ then $tc\in\Sigma_{r-1}$ so $\overline{tc} = \bar{c}^{t(1)=r}$.

If $i=r$, then as $tc\in\Sigma_{r-1}$ and $t^{-1} = t$ we get $t^{-1}c\in\Sigma_{r-1} \iff \bar{t} = \bar{c}$.

Finally if $1 < i < r$ then $(tc^i)^{-1} c^i = (c^i)^{-1} tc^i \in \Sigma_{r-1} \Rightarrow \overline{tc}^i = \bar{c}^{t(i)}$.

So we have proved that f is Σ_r-equivariant and we can then define an isomorphism

$$\begin{tikzcd} \mu_r \times_{\Sigma_r} \Sigma_r/\Sigma_{r-1} \arrow[rr, "id \times_{\Sigma_r} f", "\cong"'] \arrow[dr] & & \mu_r \times_{\Sigma_r} \bar{r} \arrow[dl] \\ & \mu_r/\Sigma_r & \end{tikzcd}$$

$\square$

6.23) Proposition.- Let $f: M^n \to N^{n+k}$ be a self-transverse immersion and let $a = (\mu_r \times_{\Sigma_r} \bar{r}, [\nu_f \circ \pi_r])$ be the pair defined in 6.21, then the map $\psi(a): \mu_r/\Sigma_r \to E\Sigma_r \times_{\Sigma_r} BO(k)^r$ classifies the normal bundle of the immersion $f_r: \mu_r/\Sigma_r \longrightarrow N$

Proof.- We take $E\Sigma_r = F(\mathbb{R}^\infty; r)$, then by 4.6 , the normal bundle of the immersion f_r is classified by a map $\nu_{f_r}: \mu_r/\Sigma_r \to F(\mathbb{R}^\infty;r) \times_{\Sigma_r} BO(k)^r$ given by $\nu_{f_r}[x_1,\ldots,x_r] = [e(x_1),\ldots,e(x_r), \nu_f(x_1),\ldots,\nu_f(x_r)]$ where $e: M \to \mathbb{R}^\infty$ is such that $(f,e): M \to N \times \mathbb{R}^\infty$ is an embedding and $\nu_f: M \to BO(k)$ classifies the normal bundle of f.

On the other hand $\psi(a)$ is given by the following composition (6.7):

$$\mu_r/\Sigma_r \xrightarrow{T} F(\mathbb{R}^\infty;r) \times_{\Sigma_r} (\mu_r \times_{\Sigma_r} \bar{r})^r \xrightarrow[id \times_{\Sigma_r} (\nu_f \circ \pi_r)^r]{} F(\mathbb{R}^\infty;r) \times_{\Sigma_r} BO(k)^r$$

In the proof of 6.9 we saw that the pretransfer T could be given by $T[x_1,\ldots,x_r] = [g(x_1,\ldots,x_r), [x_1,\ldots,x_r,1],\ldots,[x_1,\ldots,x_r,r]]$ where g is given by a pull-back

$$\begin{tikzcd} \mu_r \arrow[r, "g"] \arrow[d] & F(\mathbb{R}^\infty;r) \arrow[d] \\ \mu_r/\Sigma_r \arrow[r] & F(\mathbb{R}^\infty;r)/\Sigma_r \end{tikzcd}$$

We can define $g(x_1,\ldots,x_r)=(e(x_1),\ldots,e(x_r))$ and then if

$[x_1,\dots,x_r] \in \mu_r/\Sigma_r$ we have $id^{\times}_{\Sigma_r}\,(\nu_f \circ \pi_r)^r \circ T\,[x_1,\dots,x_r] =$

$= id^{\times}_{\Sigma_r}\,(\nu_f \circ \pi_r)^r\,[e(x_1),\dots,e(x_r),[x_1,\dots,x_r,1],\dots,[x_1,\dots,x_r,r]] =$

$= [e(x_1),\dots,e(x_r),\nu_f(x_1),\dots,\nu_f(x_r)] = \nu_{f_r}\,[x_1,\dots,x_r].$

□

6.24) Theorem.- There is an isomorphism

$I(n,k) \cong N_{n+k} \oplus N_n[O(k)] \oplus \bigoplus_{r\geq 2} \mathrm{Cov}_{n-(r-1)k}\,(r, BO(k))$ given by

$$[f\colon M \to N] \mapsto \left([N],[\nu_f \to M]\,,\ \sum_{r\geq 2} \left[\begin{array}{ccc} \mu_r \times_{\Sigma_r} \bar{r} & \xrightarrow{\nu_f \circ \pi_r} & BO(k) \\ \downarrow & & \\ \mu_r/\Sigma_r & & \end{array}\right]\right) \text{for } n\geq 0,\ k>0.$$

where f is a self-transverse representative.

Proof.- The result clearly follows from 4.28, 6.20 and 6.23 .

□

Chapter 7: Cyclic reduction modulo bordism

In this chapter we study the problem of reducing, modulo bordism, the structural group of a bundle with group $\Sigma_r \int O(k)$ to the subgroup $\mathbb{Z}_r \int O(k)$.

§7.1) On the cohomology mod. 2 of $EG \times_G X^r$

In this section we study the edge homomorphisms of a spectral sequence converging to $H^*(EG \times_G X^r; \mathbb{Z}_2)$.

7.1)Proposition.- Let X be space such that $H_*(X; \mathbb{Z}_2) \equiv H_*(X)$ is of finite type, then there is a natural isomorphism $H^*(EG \times_G X^r; \mathbb{Z}_2) \cong H^*(\mathrm{Hom}_G(B_*, H^*(X)^{\otimes r}))$, where B_* is the normalized Bar resolution and $H^*(X)^{\otimes r}$ is considered as a cochain complex with trivial boundary and G acting by permutation. For simplicity we denote $\mathbb{Z}_2[G] \equiv G$.

Proof.- By 2.10 , $S_*(EG \times_G X^r) \simeq B_* \otimes_G H_*(X)^{\otimes r}$, where $H_*(X)^{\otimes r}$ has trivial boundary. Therefore $\mathrm{Hom}_{\mathbb{Z}_2}(S_*(EG \times_G X^r); \mathbb{Z}_2) \simeq$ $\simeq \mathrm{Hom}_{\mathbb{Z}_2}(B_* \times_G H_*(X)^{\otimes r}, \mathbb{Z}_2)$. We can give to $\mathrm{Hom}_{\mathbb{Z}_2}(H_*(X)^{\otimes r}; \mathbb{Z}_2)$ a right G-action as follows: given $f: H_*(X)^{\otimes r} \to \mathbb{Z}_2$ we define $f \cdot \sigma = f \circ \bar{\sigma}$ where $\bar{\sigma}(a_1 \otimes \ldots \otimes a_r) =$ $= a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(r)}$.

Now we apply adjointness [21] to get $\mathrm{Hom}_{\mathbb{Z}_2}(B_* \otimes_G H_*(X)^{\otimes r}; \mathbb{Z}_2) \cong$ $\cong \mathrm{Hom}_G(B_*; \mathrm{Hom}_{\mathbb{Z}_2}(H_*(X)^{\otimes r}; \mathbb{Z}_2))$.

As $H_*(X)$ is of finite type then the homomorphism μ .

$\mu: \mathrm{Hom}_{\mathbb{Z}_2}(H_*(X); \mathbb{Z}_2)^{\otimes r} \to \mathrm{Hom}_{\mathbb{Z}_2}(H_*(X)^{\otimes r}; \mathbb{Z}_2)$ given by

$\mu(f_1 \otimes \ldots \otimes f_n)(a_1 \otimes \ldots \otimes a_r) = f_1(a_1)\cdot\ldots\cdot f_n(a_n)$ is an isomorphism [44]. If we give $\mathrm{Hom}_{\mathbb{Z}_2}(H_*(X); \mathbb{Z}_2)^{\otimes r}$ the usual right G-action by permuting the factors then μ is G-equivariant, in effect, we have $\mu((f_1 \otimes \ldots \otimes f_r)\cdot\sigma)(a_1 \otimes \ldots \otimes a_r) = f_{\sigma(1)}(a_1)\cdot\ldots\cdot f_{\sigma(r)}(a_r)$, on the other hand $\mu(f_1 \otimes \ldots \otimes f_r)\cdot\sigma(a_1 \otimes \ldots \otimes a_r) = \mu(f_1 \otimes \ldots \otimes f_r)\bar{\sigma}(a_1 \otimes \ldots \otimes a_r) =$
$= \mu(f_1 \otimes \ldots \otimes f_r)(a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(r)}) = f_{\sigma(1)}(a_1)\ldots f_{\sigma(r)}(a_r)$.

Therefore we have $\mathrm{Hom}_G(B_*; \mathrm{Hom}_{\mathbb{Z}_2}(H_*(X)^{\otimes r}; \mathbb{Z}_2)) \cong$
$\cong \mathrm{Hom}_G(B_*; \mathrm{Hom}_{\mathbb{Z}_2}(H_*(X); \mathbb{Z}_2)^{\otimes r})$.

Finally we have $\mathrm{Hom}_{\mathbb{Z}_2}(H_*(X); \mathbb{Z}_2) \cong H^*(X; \mathbb{Z}_2)$. Combining all the equivalences we get:

$$\mathrm{Hom}_{\mathbb{Z}_2}(S_*(EG \underset{G}{\times} X^r); \mathbb{Z}_2) \cong \mathrm{Hom}_{\mathbb{Z}_2}(B_* \underset{G}{\otimes} H_*(X)^{\otimes r}; \mathbb{Z}_2) \cong$$

$$\cong \mathrm{Hom}_G(B_*, \mathrm{Hom}_{\mathbb{Z}_2}(H_*(X)^{\otimes r}; \mathbb{Z}_2)) \cong \mathrm{Hom}_G(B_*; \mathrm{Hom}_{\mathbb{Z}_2}(H_*(X); \mathbb{Z}_2)^{\otimes r}) \cong$$

$\cong \mathrm{Hom}_G(B_*; H^*(X)^{\otimes r})$. This gives the isomorphism. To see that it is natural recall that in 2.10, we proved that if $f: X \to Y$ then we have a chain homotopy commutative diagram

$$\begin{tikzcd} S_*(EG \underset{G}{\times} X^r) \arrow[r, phantom, "\simeq"] \arrow[d, "(id \underset{G}{\times} f^r)_\#"] & B_* \underset{G}{\otimes} H_*(X)^{\otimes r} \arrow[d, "id \underset{G}{\otimes} f_*^{\otimes r}"] \\ S_*(EG \underset{G}{\times} Y^r) \arrow[r, phantom, "\simeq"] & B_* \underset{G}{\otimes} H_*(Y)^{\otimes r} \end{tikzcd}$$

The second isomorphism is given by adjointness which is natural. One can easily verify that the third isomorphism is also natural. For the last one recall that under Kronecker duality f^* corresponds to $\mathrm{Hom}_{\mathbb{Z}_2}(f_*)$.

□

7.2) Defintion.- To define the spectral sequence notice that $\mathrm{Hom}_G(B_*; H^*(X)^{\otimes r})$ is the total complex of the bicomplex of cochains

$L^{p,q} = \mathrm{Hom}_G(B_p,(H^*(X)^{\otimes r})^q)$, if we denote (B_*,∂) then the coboundaries are $\delta_1 : L^{p,q} \to L^{p+1,q}$ given by $\delta_1(f) = f \circ \partial$ and $\delta_2 : L^{p,q} \to L^{p,q+1}$, $\delta_2 = 0$, because $H^*(X)^{\otimes r}$ has trivial boundary.

We give a filtration for Tot $L^{p,q} = \mathrm{Hom}_G(B_*;H^*(X)\ \ ^r)$ as follows, the p.th filtration in degree n is given by $F^p(\mathrm{Tot}^*)^n = \bigoplus_{i \geq p} L^{i,n-i}$.

We have a spectral sequence associated to this filtration [30] which we denote by $\{E_r^{p,q}, d_r\}$.

7.3) Proposition.- The spectral sequence $\{E_r^{p,q}, dr\}$ converges to $H^*(\mathrm{Hom}_G(B_*; H^*(X)^{\otimes n})) \cong H^*(EG \times_G X^r; \mathbb{Z}_2)$ and $E_2^{p,q} \cong H^p(G;(H^*(X)^{\otimes r})^q)$.

Proof.- The filtration F of Tot $L^{p,q}$ satisfies, in each degree n, $0 \subset F^n(\mathrm{Tot}^*) \subset F^{n-1}(\mathrm{Tot}^*) \subset \ldots \subset F^0(\mathrm{Tot}^*) = \mathrm{Tot}^n$, i.e., it is the canonical filtration so by [30] the spectral sequence converges to $H^*(\mathrm{Tot}L^{p,q}) = H^*(\mathrm{Hom}_G(B_*; H^*(X)^{\otimes r}))$ which is isomorphic by 7.1 to $H^*(EG \times_G X^r; \mathbb{Z}_2)$.

The spectral sequence associated to a filtration satisfies $E_1^{p,q} = H^{p+q}(F^p(-)/F^{p+1}(-))$, and from the definition of our filtration it is clear that $[F^p(\mathrm{Tot}^*)/F^{p+1}(\mathrm{Tot}^*)]^{p+q} = L^{p,q}$; as $\delta_2 = 0$, then $E_1^{p,q} \cong H^q(\mathrm{Hom}_G(B_p, H^*(X)^{\otimes r})) = \mathrm{Hom}_G(B_p;(H^*(X)^{\otimes r})^q)$. One can show [30] that under this isomorphism the differential d_1 coincides with the coboundary δ_1 of $L^{p,q}$, therefore $E_2^{p,q} \cong H^p(G; (H^*(X)^{\otimes r})^q)$.

□

7.4) Proposition.- The spectral sequence $\{E_r^{p,q}, d_r\}$ collapses.

Proof.- We recall the definition of the spectral sequence. Let us denote $\mathrm{Tot}^* = C^*$. Let $Z_r^{p,q} = \{a \in F^p C^{p+q} \mid \delta(a) \in F^{p+r} C^{p+q+1}\}$ where $\delta = \delta_1 + \delta_2$, then $E_r^{p,q} = Z_r^{p,q}/Z_{r-1}^{p+1,q-1} + \delta Z_{r-1}^{p-r+1,q+r-2}$ and $d_r[x] = [\delta(x)]$ of degree $(r,1-r)$.

$L^{p,q} = \mathrm{Hom}_G(B_p,(H^*(X)^{\otimes r})^q)$, if we denote (B_*,∂) then the coboundaries are $\delta_1 : L^{p,q} \to L^{p+1,q}$ given by $\delta_1(f) = f \circ \partial$ and $\delta_2 : L^{p,q} \to L^{p,q+1}$, $\delta_2 = 0$, because $H^*(X)^{\otimes r}$ has trivial boundary.

We give a filtration for Tot $L^{p,q} = \mathrm{Hom}_G(B_*;H^*(X)^{\otimes r})$ as follows, the p.th filtration in degree n is given by $F^p(\mathrm{Tot}^*)^n = \bigoplus_{i \geq p} L^{i,n-i}$.

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Proof.- The filtration F of Tot $L^{p,q}$ satisfies, in each degree n, $0 \subset F^n(\mathrm{Tot}^*) \subset F^{n-1}(\mathrm{Tot}^*) \subset \ldots \subset F^0(\mathrm{Tot}^*) = \mathrm{Tot}^n$, i.e., it is the canonical filtration so by [30] the spectral sequence converges to $H^*(\mathrm{Tot}L^{p,q}) = H^*(\mathrm{Hom}_G(B_*;H^*(X)^{\otimes r}))$ which is isomorphic by 7.1 to $H^*(EG \times_G X^r; \mathbb{Z}_2)$.

The spectral sequence associated to a filtration satisfies $E_1^{p,q} = H^{p+q}(F^p(-)/F^{p+1}(-))$, and from the definition of our filtration it is clear that $[F^p(\mathrm{Tot}^*)/F^{p+1}(\mathrm{Tot}^*)]^{p+q} = L^{p,q}$; as $\delta_2 = 0$, then $E_1^{p,q} \cong H^q(\mathrm{Hom}_G(B_p,H^*(X)^{\otimes r})) = \mathrm{Hom}_G(B_p;(H^*(X)^{\otimes r})^q)$. One can show [30] that under this isomorphism the differential d_1 coincides with the coboundary δ_1 of $L^{p,q}$, therefore $E_2^{p,q} \cong H^p(G;(H^*(X)^{\otimes r})^q)$.

□

7.4) Proposition.- The spectral sequence $\{E_r^{p,q}, d_r\}$ collapses.

Proof.- We recall the definition of the spectral sequence. Let us denote $\mathrm{Tot}^* = C^*$. Let $Z_r^{p,q} = \{a \in F^p C^{p+q} \mid \delta(a) \in F^{p+r} C^{p+q+1}\}$ where $\delta = \delta_1 + \delta_2$, then $E_r^{p,q} = Z_r^{p,q}/Z_{r-1}^{p+1,q-1} + \delta Z_{r-1}^{p-r+1,q+r-2}$ and $d_r[x] = [\delta(x)]$ of degree $(r,1-r)$.

We shall show that $d_r\colon E_r^{p,q} \to E_r^{p+r,q+1-r}$ is zero if $r \geq 2$. For this let $a= (0,0,\dots,0, a_p, a_{p+1},a_{p+2},\dots,a_n)\in F^p C^{p+q=n}$, then

$d_r(0,0,\dots,0, a_p,a_{p+1},a_{p+2},\dots,a_n)= (0,0,\dots,0,\delta_2(a_p),\delta_1(a_p)+\delta_2(a_{p+1})$, $\delta_1(a_{p+1})+\delta_2(a_{p+2}),\dots,\delta_1(a_n))$. We have that $d_r\,[a]=[\,\delta(a)]$ and $\delta(a)\in F^{p+r} C^{p+q+1}$, so as $r\geq 2$, then $\delta_2(a_p)=0$, $\delta_1(a_p)+\delta_2(a_{p+1})=0$ and as $\delta_2= 0$, then $\delta_1(a_p)= 0$. Consider the element $b=(0,0,\dots,0, 0,a_{p+1},a_{p+2},\dots,a_n)$, then $b \in F^{p+1} C^{p+q}$ and we have $\delta(b)=(0,0,\dots,0, 0, \delta_2(a_{p+1}),\delta_1(a_{p+1})+\delta_2(a_{p+2}),\dots,\delta_1(a_n))$, as $\delta_2= 0$, then $\delta(b)= d_r\,[a]$, therefore $d_r\,[a] = 0$.

□

Now we want to show that the edge homomorphisms make the following diagrams commute:

$$\begin{array}{ccc}
E_2^{p,0} \longrightarrow E_\infty^{p,0} = J^{p,0} & \subset & H^p(\mathrm{Hom}_G(B_*;\, H^*(X)^{\otimes r})) \\
\| & & \wr\| \\
H^p(G)\cong H^p(BG;\, \mathbb{Z}_2) & \xrightarrow{\ p^*\ } & H^p(EG \times_G X^r;\, \mathbb{Z}_2)
\end{array}$$

$$\begin{array}{ccc}
H^q(\mathrm{Hom}_G(B_*;\, H^*(X)^{\otimes r}))= J^{0,q} \longrightarrow E_\infty^{0,q} \hookrightarrow E_2^{0,q} \subset E_1^{0,q} & = & (H^*(X)^{\otimes r})^q \\
\wr\| & & \wr\| \\
H^q(EG \times_G X^r;\, \mathbb{Z}_2) & \xrightarrow{\ i^*\ } & H^q(X^r)
\end{array}$$

where $p\colon EG\times_G X^r \to BG$ is the projection and $i\colon X^r \hookrightarrow EG\times_G X^r$ the inclusion, and we have $0 \subset J^{n,0} \subset J^{n-1,1} \subset\dots\subset J^{0,n}= H^n(\mathrm{Tot}^*)$; $J^{n,0}= E_\infty^{n,0}$ and $J^{p,q}/J^{p+1,q-1} \cong E_\infty^{p,q}$.

In order to do this we need to prove some preliminary lemmas.

7.5) Lemma.- Let $h: \mathbb{Z}_2 \otimes_{\mathbb{Z}_2} S_*(X^r) \to S_*(EG) \otimes_G S_*(X^r)$ be the chain map induced by the map of bicomplexes given by $1 \otimes_{\mathbb{Z}_2} a \to x_0 \otimes_G a$, where $x_0 \varepsilon\, S_0(EG)$. Let $\phi: S_*(EG \times_G X^r) \xrightarrow{\simeq} S_*(EG) \otimes_G S_*(X^r)$ be the equivalence given by the composition of the equivalences 2.2 , 2.4 , 2.5 . Then the following diagram is chain homotopy commutative

$$\begin{tikzcd}
S_*(X^r) \arrow[r, "\cong"] \arrow[d, "i_\#"'] & \mathbb{Z}_2 \otimes_{\mathbb{Z}_2} S_*(X^r) \arrow[d, "h"] \\
S_*(EG \times_G X^r) \arrow[r, "\simeq", "\phi"'] & S_*(EG) \otimes_G S_*(X^r)
\end{tikzcd}$$

Proof.- Consider the following diagram:

$$\begin{tikzcd}
S_*(X^r) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathbb{Z}_2 \otimes_{\mathbb{Z}_2} S_*(X^r) \arrow[d, "j"] \\
S_*(\{x_0\} \times X^r) \arrow[r, "\psi", "\simeq"'] \arrow[d, "(\iota \times id)_\#"] & S_*(x_0) \otimes_{\mathbb{Z}_2} S_*(X^r) \arrow[d, "\iota_\# \otimes_{\mathbb{Z}_2} id"] \\
S_*(EG \times X^r) \arrow[r, "\psi", "\simeq"'] \arrow[d, "p_\#"] & S_*(EG) \otimes_{\mathbb{Z}_2} S_*(X^r) \arrow[d, "\Pi"] \\
S_*(EG \times_G X^r) \arrow[r, "\phi", "\simeq"'] & S_*(EG) \otimes_G S_*(X^r)
\end{tikzcd}$$

where the ψ's are the equivalences given by the Eilenberg-Zilber theorem, and Π is the projection.

The composition on the left-hand side of the diagram is i and the one on the right-hand side is h . The second square commutes by the naturality of the Eilenberg-Zilber theorem. The definition of ϕ clearly implies that the third one also commutes.

To prove that the first square commutes consider the functors $S_*(-)$ and $S_*(x_0) \otimes_{\mathbb{Z}_2} S_*(\_)$ from the category of spaces to the category of augmented chain complexes, both are clearly free and acyclic on the usual models.

Now consider the chain maps $\psi\circ\equiv$ and $j\circ\equiv$, both are maps of augmented chain complexes and are natural so by the acyclic models theorem [19] they are chain homotopic, i.e., the first square is homotopy commutative.

□

7.6) Lemma.- We have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}_2 \underset{\mathbb{Z}_2}{\otimes} S_*(X^r) \arrow[r, "id \underset{\mathbb{Z}_2}{\otimes} f"] \arrow[d, "h"'] & \mathbb{Z}_2 \underset{\mathbb{Z}_2}{\otimes} S_*(X)^{\otimes r} \arrow[d, "\bar{h}"] \\
S_*(EG) \underset{G}{\otimes} S_*(X^r) \arrow[r, "F", "\cong"'] & B_* \underset{G}{\otimes} S_*(X)^{\otimes r}.
\end{tikzcd}$$

where h was defined in 7.5, f is given by the Eilenberg-Zilber theorem, F is the equivalence defined in 2.8, and $\bar{h}$ is given by $\bar{h}(1 \underset{\mathbb{Z}_2}{\otimes} y) =$ $= 1\ e \underset{G}{\otimes} y$, where $e \in B_0 = \mathbb{Z}_2[\ G\ ]$ is the zero of the group.

Proof.- Recall that to get the equivalence F we have to give an equivalence $t: S_*(EG) \underset{\simeq}{\to} B_*$, we can choose t as follows: given any $x_0 \in EG$ we can form a G-basis for $S_0(EG)$ by considering x_0 and one point from each of the other orbits under the action of G, we can define then $t_0: S_0(EG) \to B_0$ by sending this G-basis to $1e \in B_0 = \mathbb{Z}_2[G]$, and then define the other t_i for $i > 0$ to get an equivalence t:

$$\begin{tikzcd}
\cdots \arrow[r] & S_n(EG) \arrow[r] \arrow[d, "t_n"] & \cdots \arrow[r] & S_1(EG) \arrow[r] \arrow[d, "t_1"] & S_0(EG) \arrow[r, "\varepsilon"] \arrow[d, "t_0"] & \mathbb{Z}_2 \arrow[d, equal] \\
\arrow[r] & B_n \arrow[r] & \cdots \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r, "\varepsilon"] & \mathbb{Z}_2
\end{tikzcd}$$

By 2.7 (c), $F(x_0 \underset{G}{\otimes} a) = t(x_0) \underset{G}{\otimes} f(a) = 1e \underset{G}{\otimes} f(a)$, for all $a \in S_*(X^r)$ so the diagram commutes.

□

7.7) Lemma.- Consider $\mathbb{Z}_2$ as a chain complex concentrated in dimension zero and with trivial G-action, and let $\varepsilon : S_*(X^r) \to \mathbb{Z}_2$ be the augmentation, then the following diagram is chain homotopy commutative.

$$\begin{tikzcd}
S_*(EG \times_G X^r) \arrow[r, "\cong", "\phi"'] \arrow[d, "p_\#"'] & S_*(EG) \otimes_G S_*(X^r) \arrow[d, dash, "id \otimes_G \varepsilon"] \\
S_*(BG) & S_*(EG) \otimes_G \mathbb{Z}_2 \arrow[l, "\cong"', "\gamma"]
\end{tikzcd}$$

Proof.- Consider the following diagram:

$$\begin{tikzcd}
S_*(EG \times_G X^r) \arrow[d, "p_\#"'] & S_*(EG \times X^r) \otimes_G \mathbb{Z}_2 \arrow[l, "\gamma'"', "\cong"] \arrow[r, "\psi \otimes_G id", "\simeq"'] \arrow[d, "q_\# \otimes_G id"] & (S_*(EG) \otimes_{\mathbb{Z}_2} S_*(X^r)) \otimes_G \mathbb{Z}_2 \arrow[r, "\cong", "\ell"'] \arrow[d, "(id \otimes_{\mathbb{Z}_2} \varepsilon) \otimes_G id"] & S_*(EG) \otimes_G S_*(X^r) \\
S_*(BG) & S_*(EG) \otimes_G \mathbb{Z}_2 \arrow[l, "\cong", "\gamma"'] \arrow[r, dash, "\cong \otimes_G id"'] & (S_*(EG) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2) \otimes_G \mathbb{Z}_2 &
\end{tikzcd}$$

where $q: EG \times X^r \to EG$ is the projection on the first factor, and the composition at the top is Φ.

The following diagram clearly commutes

$$\begin{tikzcd}
EG \times X^r \arrow[r, "q"] \arrow[d, "\rho'"'] & EG \arrow[d, "\rho"] \\
EG \times_G X^r \arrow[r, "p"'] & BG
\end{tikzcd}$$

and γ and γ' are given by $\gamma(a \otimes_G 1) = \rho_\#(a)$, $\gamma'(b \otimes_G 1) = \rho'_\#(b)$, therefore the first square commutes.

Using the acyclic models theorem, as we did in 7.5, one can show that $id \otimes_{\mathbb{Z}_2} \varepsilon \circ \psi \simeq (\cong) \circ q_\#$, therefore $(id_{\mathbb{Z}_2} \varepsilon \circ \psi) \otimes_G id \simeq ((\cong) \circ q_\#) \otimes_G id$ so the second square is chain homotopy commutative.

Finally notice that $id \otimes_G \varepsilon = (\cong) \otimes_G id \circ (id \otimes_{\mathbb{Z}_2} \varepsilon) \otimes_G id \circ \ell^{-1}$, so the commutativity up to chain homotopy of the diagram above proves the lemma.

□

7.8) Lemma.- We have a commutative diagram

$$\begin{tikzcd}
S_*(EG)\otimes_G S_*(X^r) \arrow[r, "F", "\simeq"'] \arrow[d, "id\otimes_G \varepsilon"] & B_*\otimes_G S_*(X)^{\otimes r} \arrow[d, "id\otimes_G \varepsilon"] \\
S_*(EG)\otimes_G \mathbb{Z}_2 \arrow[r, "\simeq", "t\otimes_G id"'] & B_*\otimes_G \mathbb{Z}_2
\end{tikzcd}$$

where F is the equivalence defined in 2.8 , and induced by t.

Proof.- By 2.7 (2) $F(w\otimes_G a) = t(w)\otimes_G f(a)$ for all $w\in S_*(EG), a\in S_0(X^r)$, where $f\colon S_*(X^r) \xrightarrow{\simeq} S_*(X)^{\otimes r}$. As f is a chain map of augmented chain complexes then we have $(id\otimes_G\varepsilon)\,F(w\otimes_G a) = id\otimes_G\varepsilon(t(w)\otimes_G f(a)=t(w)\otimes_G \varepsilon f(a)=t(w)\otimes_G\varepsilon(a)$, on the other hand $(t\otimes_G id)\,(id\otimes_G\varepsilon)(w\otimes_G a)= t\otimes_G id(w\otimes_G\varepsilon(a))=t(w)\otimes_G \varepsilon(a)$. □

7.9) Corollary.- We have a chain homotopy commutative diagram

$$\begin{tikzcd}
S_*(X^r) \arrow[r, dash, "\simeq"] \arrow[d, "i_\#"'] & \mathbb{Z}_2\otimes_{\mathbb{Z}_2} H_*(X)^{\otimes r} \arrow[d, "h"] \\
S_*(EG\times_G X^r) \arrow[r, dash, "\simeq"'] & B_*\otimes_G H_*(X)^{\otimes r}
\end{tikzcd}$$

where $h(1\,{}_{\mathbb{Z}_2}\!\otimes z)= 1e\otimes_G z$.

Proof.- It follows from 7.5, 7.6 and the fact that the following diagram clearly commutes

$$\begin{tikzcd}
\mathbb{Z}_2\,{}_{\mathbb{Z}_2}\!\otimes S_*(X)^{\otimes r} \arrow[r, "id_{\mathbb{Z}_2}\otimes\alpha^{\otimes r}"] \arrow[d, "\bar{h}"'] & \mathbb{Z}_2\,{}_{\mathbb{Z}_2}\!\otimes H_*(X)^{\otimes r} \arrow[d, "h"] \\
B_*\otimes_G S_*(X)^{\otimes r} \arrow[r, "id_G\otimes\alpha^{\otimes r}"'] & B_*\otimes_G H_*(X)^{\otimes r}
\end{tikzcd}$$

where α is the equivalence defined in 2.9

□

7.10) Corollary.- We have a chain homotopy commutative diagram

$$\begin{tikzcd} S_*(EG\times_G X^r) \arrow[r, "\cong"] \arrow[d, "p_*"'] & B_* \otimes_G H_*(X)^{\otimes r} \arrow[d, "id \otimes_G \varepsilon"] \\ S_*(BG) \arrow[r, "\cong"'] & B_* \otimes_G \mathbb{Z}_2 \end{tikzcd}$$

Proof.- It follows from 7.7, 7.8, and the fact that the following diagram clearly commutes

$$\begin{tikzcd} B_* \otimes_G S_*(X)^{\otimes r} \arrow[rr, "id \otimes_G \alpha^{\otimes r}"] \arrow[dr, "id \otimes_G \varepsilon"'] & & B_* \otimes_G H_*(X)^{\otimes r} \arrow[dl, "id \otimes_G \varepsilon"] \\ & B_* \otimes_G \mathbb{Z}_2 & \end{tikzcd}$$

where α is the equivalence defined in 2.9 .

□

7.11) Proposition.- The horizontal edge homomorphism of the spectral sequence $\{E_r^{p,q}, d_r\}$ makes the following diagram commute:

$$\begin{tikzcd} H^q(\mathrm{Hom}_G(B_*;H^*(X)^{\otimes r})) = J^{0,q} \arrow[r] \arrow[d, phantom, "\wr\|"] & E_\infty^{0,q} \hookrightarrow E_2^{0,q} \subset E_1^{0,q} = (H^*(X)^{\otimes r})^q \arrow[d, phantom, "\wr\|"] \\ H^q(EG \times_G X^r; \mathbb{Z}_2) \arrow[r, "i^*"] & H^q(X^r) \end{tikzcd}$$

Proof.- Let $h : \mathbb{Z}_2 \otimes_{\mathbb{Z}_2} H_*(X)^{\otimes r} \to B_* \otimes_G H_*(X)^{\otimes r}$ be the chain map of 7.9 and consider the following diagram:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}_2}(B_* \otimes_G H_*(X)^{\otimes r}; \mathbb{Z}_2) \arrow[r, "\mathrm{Hom}_{\mathbb{Z}_2}(h, id)"] \arrow[d, leftrightarrow, "\cong"'] & \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2 \otimes_{\mathbb{Z}_2} H_*(X)^{\otimes r}; \mathbb{Z}_2) \arrow[d, leftrightarrow, "\cong"] \\ \mathrm{Hom}_G(B_*; H^*(X)^{\otimes r}) \arrow[r, dashed, "\bar{h}"'] & \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2; H^*(X)^{\otimes r}) \end{tikzcd}$$

The isomorphism on the left-hand side was defined in 7.1. The same arguments used in 7.1 give the isomorphism on the right-hand side. This diagram defines a chain map $\bar{h}$ and one can easily verify that it is given by $\bar{h}(\ell)=\ell\circ\iota$, where $\iota:\mathbb{Z}_2\to B_*$ is defined by $\iota(1)=1e\in B_0$.

Now consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}_2}(S_*(EG\underset{G}{\times}X^r);\mathbb{Z}_2) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}_2}(i_\#,\mathrm{id})"] & \mathrm{Hom}_{\mathbb{Z}_2}(B_*\underset{G}{\otimes}H_*(X)^{\otimes r};\mathbb{Z}_2) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}_2}(h,\mathrm{id})"] & \mathrm{Hom}_G(B_*,H^*(X)^{\otimes r}) \arrow[d, "\bar{h}"] \\
\mathrm{Hom}_{\mathbb{Z}_2}(S_*(X^r);\mathbb{Z}_2) \arrow[r, "\cong"'] & \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2\underset{\mathbb{Z}_2}{\otimes}H_*(X)^{\otimes r};\mathbb{Z}_2) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2;H^*(X)^{\otimes r})
\end{tikzcd} \qquad (*)$$

By 7.9 the first square chain homotopy commutative and the second one commutes by definition of $\bar{h}$.

Consider $\mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2;H^*(X)^{\otimes r})$ as the total complex of a bicomplex, then using the same filtration as in 7.2, we get a spectral sequence $\{'E_r^{p,q}, d_r'\}$ such that

$$'E_1^{p,q} = {}'E_2^{p,q} = \begin{cases} \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2;(H^*(X)^{\otimes r})^q) = (H^*(X)^{\otimes r})^q & \text{if } p=0 \\ 0 \text{ if } p\neq 0 \end{cases}$$

The map $\bar{h}$ is induced by a map of bicomplexes so it induces a morphism of spectral sequences $\tilde{h}_r: E_r^{p,q}\to {}'E_r^{p,q}$. Consider the following diagram

$$\begin{tikzcd}
H^q(EG\underset{G}{\times}X^r) \arrow[r, "\cong"] \arrow[d, "i^*"] & H^q(\mathrm{Hom}_G(B_*;H^*(X)^{\otimes r})) \arrow[r, two heads] \arrow[d, "\bar{h}^*"] & E_\infty^{0,q} \arrow[d, "\tilde{h}_\infty"'] & E_2^{0,q} \arrow[l, hook'] \arrow[d, "\tilde{h}_2"'] & E_1^{0,q} \arrow[l, hook'] \arrow[d, "\tilde{h}_1"] \arrow[r, phantom, "="] & \mathrm{Hom}_G(\mathbb{Z}_2[G];(H^*(X)^{\otimes r})^q) \arrow[r, phantom, "="] & (H^*(X)^{\otimes r})^q \arrow[d, "\mathrm{id}"] \\
H^q(X^r) \arrow[r, "\cong"'] & \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2;(H^*(X)^{\otimes r})^q) \arrow[r, phantom, "="] & 'E_\infty^{0,q} \arrow[r, phantom, "="] & 'E_2^{0,q} \arrow[r, phantom, "="] & 'E_1^{0,q} \arrow[r, phantom, "="] & \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2;(H^*(X)^{\otimes r})^q) \arrow[r, phantom, "="] & (H^*(X)^{\otimes r})^q
\end{tikzcd}$$

The commutativity of the first square follows from (*). The other squares commute because $\tilde{h}$ is a morphism of spectral sequences [30] . □

7.12) Proposition.- The vertical edge homomorphism of the spectral sequence $\{E_r^{p,q}, d_r\}$ makes the following diagram commute:

$$\begin{tikzcd}
 & E_2^{p,0} \arrow[r] & E_\infty^{p,0} = J^{p,0} \subset H^p(\mathrm{Hom}_G(B_*; H^*(X)^{\otimes r})) \arrow[d, "\cong"] \\
H^p(G) \cong H^p(BG; \mathbb{Z}_2) \arrow[ur, equal] \arrow[rr, "p^*"] & & H^p(EG \times_G X^n; \mathbb{Z}_2)
\end{tikzcd}$$

Proof.- Let $id \otimes_G \varepsilon \equiv g \colon B_* \otimes_G H_*(X)^{\otimes r} \to B_* \otimes_G \mathbb{Z}_2$ be the chain map of 7.10. Then, as in 7.11, we have a chain map $\bar{g}$ defined by the commutativity of the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}_2}(B_* \otimes_G \mathbb{Z}_2, \mathbb{Z}_2) \arrow[r, "\mathrm{Hom}_{\mathbb{Z}_2}(g,id)"] \arrow[d, leftrightarrow, "\cong"] & \mathrm{Hom}_{\mathbb{Z}_2}(B_* \otimes_G H_*(X)^{\otimes r}; \mathbb{Z}_2) \arrow[d, leftrightarrow, "\cong"] \\
\mathrm{Hom}_G(B_*; \mathbb{Z}_2) \arrow[r, dashed, "\bar{g}"'] & \mathrm{Hom}_G(B_*, H^*(X)^{\otimes r})
\end{tikzcd}$$

One can easily verify that $\bar{g}$ is given by $g(\ell) = j \circ \ell$ where $j \colon \mathbb{Z}_2 \to H^*(X)^{\otimes r}$ sends 1 to $1 \otimes 1 \otimes \ldots \otimes 1$ and $1 = [\varepsilon] \colon S_0(X) \to \mathbb{Z}_2$.

Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}_2}(S_*(BG); \mathbb{Z}_2) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}_2}(p_\#, id)"] & \mathrm{Hom}_{\mathbb{Z}_2}(B_* \otimes_G \mathbb{Z}_2; \mathbb{Z}_2) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}_2}(g,id)"] & \mathrm{Hom}_G(B_*; \mathbb{Z}_2) \arrow[d, "\bar{g}"] \\
\mathrm{Hom}_{\mathbb{Z}_2}(S_*(EG \times_G X^r); \mathbb{Z}_2) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{Z}_2}(B_* \otimes_G H_*(X)^{\otimes r}; \mathbb{Z}_2) \arrow[r, "\cong"'] & \mathrm{Hom}_G(B_*, H^*(X)^{\otimes r})
\end{tikzcd} \quad (*)$$

By 7.10 the first square is chain homotopy commutative and the second commutes by definition of $\bar{g}$.

Consider $\mathrm{Hom}_G(B_*, \mathbb{Z}_2)$ as the total complex of a bicomplex, then using the same filtration as in 7.2 we get a spectral sequence $\{''E_r^{p,q}, d_r''\}$ such that $''E_2^{p,q} = \begin{cases} H^p(G; \mathbb{Z}_2) & \text{if } q=0 \\ 0 & \text{if } q\neq 0 \end{cases}$

the map $\bar{g}$ induces a morphism of spectral sequences $\tilde{g}: ''E_r^{p,q} \to E_r^{p,q}$.

Consider the following diagram:

$$\begin{tikzcd}
''E_2^{p,0} = H^p(G;\mathbb{Z}_2) \arrow[r, "id"] \arrow[d, "\tilde{g}_2 = id"] & ''E_\infty^{p,0} \arrow[r, phantom, "="] \arrow[d, "\tilde{g}_\infty"] & H^p(G;\mathbb{Z}_2) \arrow[r, "\simeq"] \arrow[d, "\bar{g}^*"] & H^p(BG;\mathbb{Z}_2) \arrow[d, "p^*"] \\
E_2^{p,0} = H^p(G;\mathbb{Z}_2) \arrow[r] & E_\infty^{p,0} \arrow[r, hook] & H^p(\mathrm{Hom}_G(B_*;H^r(X)^{\times r})) \arrow[r, "\cong"'] & H^p(EG \times_G X^r;\mathbb{Z}_2)
\end{tikzcd}$$

The commutativity of the last square follows from (*). The other commute because $\tilde{g}$ is a morphism of spectral sequences. □

§7.2) Cohomology of topological groups

In this section we give some results on the cohomology of topological groups that we shall use in the next section.

7.13) Definition.- In [41] G. Segal showed how to define cohomology groups of a topological group G with coefficients in a topological abelian group A (on which G acts continuously) by a derived functor method analogous to the one for defining cohomology of discrete groups. More specifically, let G be a compactly generated Hausdorff topological group and denote by G-Topab the category of compactly generated Hausdorff topological abelian groups on which G acts continuously and continuous G-equivariant homomorphisms.

We have the functor Γ^G: G-Topab $\to$ Ab , where Ab is the category of abelian groups, which associates to $A \in$ G-Topab its G-invariant subgroup $\Gamma^G(A)$. Then using suitable resolutions he defines right derived functors $R^n\,\Gamma^G(-)$.

The relation of the groups $R^n\,\Gamma^G(A)$, when A is discrete, to the cohomology of BG is as follows. Consider $p: EG \to BG$, we define a sheaf of abelian group σA, on the space BG by $\sigma A(U) = \mathrm{Map}^G(p^{-1}(U), A)$, where $\mathrm{Map}^G(-,-)$ means G-equivariant maps. This is the sheaf of continuous sections of $EG \times_G A \to BG$. With this notation we have:

7.14) Theorem [41] .- If A is discrete then there is a natural isomorphism. $R^n\,\Gamma^G(A) \cong H^n(BG; \sigma A)$.

□

7.15) Corollary.- If A is discrete and the action $G \times A \to A$ is trivial then $R^n\,\Gamma^G(A) \cong H^n(BG; A)$.

Proof.- If the action is trivial we get the constant sheaf, and we can find a BG such that it is a C.W.-complex [49], so BG is paracompact, Hausdorff and locally contractible, therefore by [44],
$H^n(BG; A) \cong H^n(BG; \sigma A)$, and by 7.14, $H^n(BG; \sigma A) \cong R^n\,\Gamma^G(A)$.

□

7.16) Definition.- Let $A \in$ G-Topab, we denote by $\mathcal{H}om(G,A)$ the abelian group of crossed homomorphisms from G to A (i.e., continuous functions $f: G \to A$ such that $f(g_1 g_2) = f(g_1) + g_1 \cdot f(g_2)$) modulo the subgroup of principal crossed homomorphisms (i.e., those f of the form $f(g) = g \cdot a - a$ for some $a \in A$).

7.17) Theorem [41] .- There is a natural isomorphism
$R^1\,\Gamma^G(A) \cong \mathcal{H}om(G,A)$. □

7.18) Corollary.- If A is discrete and the action of G is trivial then there is a natural isomorphism $H^1(BG;A) \cong \mathcal{H}om(G;A)$ where $\mathcal{H}om(G,A)$ is the group of continuous homomorphisms.

Proof.- By 7.15, $H^1(BG;A) \cong R^1\Gamma^G(A)$ and by 7.17 $R^1\Gamma^G(A) \cong \mathcal{H}om(G,A)$; as the action of G is trivial then the crossed homomorphisms modulo the principal crossed homomorphisms is just the continuous homomorphisms. $\square$

7.19)Definition.- Let $A \in G$-Topab, a topological extension $0 \to A \xrightarrow{i} E \xrightarrow{p} G \to 0$ is an exact sequence of topological groups such that i is an embedding of A as a closed subgroup of E, p is a principal A-bundle inducing a topological isomorphism $E/A \cong G$, and the action of G on A induced by p coincides with the given action. We say that two extensions $0 \to A \xrightarrow{i} E \xrightarrow{p} G \to 0$ and $0 \to A \xrightarrow{i} E' \xrightarrow{p'} G \to 0$ are isomorphic if there is a continuous homomorphism $f: E \to E'$ making the following diagram commutative

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "p"] \arrow[d, "f"] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "i'"'] & E' \arrow[r, "p'"'] & G \arrow[r] & 0
\end{tikzcd}$$

In this case f is clearly a topological isomorphism. We denote by $\mathrm{Ext}(G;A)$ the set of isomorphism classes of topological extensions. We can make $\mathrm{Ext}(G;A)$ into a bifunctor as follows: given a continuous homomorphism $\phi: G \to G'$, $\phi^*: \mathrm{Ext}(G';A) \to \mathrm{Ext}(G,A)$ associates to an extension E, the pull-back extension:

$$\begin{tikzcd}
\phi^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\
G \arrow[r, "\phi"'] & G'
\end{tikzcd}$$

If $\psi: A \to A'$ is a continuous homomorphism then $\psi_*: \mathrm{Ext}(G;A) \to \mathrm{Ext}(G;A')$ associates to an extension E, the extension $\psi_*(E) = E \times_A A'$ where the action $A \times A' \to A'$ is given by $a \cdot a' = \psi(a)a'$.

We can make Ext (G;A) into an abelian group as follows: given two extensions E, E' consider their product $E \times E'$, let $\Delta: G \to G \times G$ be the diagonal, then the pull-back $\Delta^*(E \times E')$ is an extension over G with fiber $A \times A$, and as A is abelian, the product map $A \times A \overset{\mu}{\to} A$ is a continuous homomorphism, so we define $E + E' = \mu_* \Delta^*(E \times E')$.

7.20) Theorem [41].- There is a natural isomorphism $R^2\Gamma^G(A) \cong \mathrm{Ext}(G;A)$.

$\square$

7.21) Corollary.- If A is discrete and the action of G on A is trivial then there is a natural isomorphism $H^2(BG;\, A) \cong \mathrm{Ext}(G;A)$, where $\mathrm{Ext}(G;A)$ is the group of topological central extensions.

Proof.- By 7.15, $H^2(BG;\, A) \cong R^2\Gamma^G(A)$, and by 7.20, $R^2\Gamma^G(A) \cong \mathrm{Ext}(G;A)$, as the action of G on A is trivial then the topological extensions are central.

$\square$

In the case $G = O(k)$ and $A = \mathbb{Z}_2$, the Stiefel-Whitney classes are identified in terms of homomorphisms and extensions in the following theorem.

7.22) Theorem [29].- a) $w_1 \in H^1(BO(k);\, \mathbb{Z}_2) \cong \mathcal{H}om(O(k);\, \mathbb{Z}_2)$ corresponds to the determinant map $d: O(k) \to \mathbb{Z}_2$.

b) In $H^2(BO(k);\, \mathbb{Z}_2) \cong \mathrm{Ext}(O(k);\, \mathbb{Z}_2)$ the correspondence is as follows:

$0 \mapsto$ trivial extension; $w_1^2 \mapsto$ the pull-back under $d: O(k) \to \mathbb{Z}_2$ of the extension $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$; $w_1^2 + w_2 \mapsto 0 \to \mathbb{Z}_2 \to \mathrm{Pin}\,(k) \to O(k) \to 0$

We can obtain w_2 by considering the sum, defined in 7.19, of the extensions corresponding to w_1^2 and $w_1^2 + w_2$.

□

We finish this section with two results that we shall need later.

7.23) Proposition.- Let G be a finite group, then $H^n(BG; \mathbb{Z}_2[G]) = 0$ for $n > 0$.

Proof.- $H^n(BG; \mathbb{Z}_2[G]) = \mathrm{Ext}^n_{\mathbb{Z}_2[G]}(\mathbb{Z}_2, \mathbb{Z}_2[G])$. By [20] we have

$\mathrm{Ext}^n_{\mathbb{Z}_2[G]}(\mathbb{Z}_2, \mathbb{Z}_2[G]) \cong \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}_2[G])$.

Let $\ldots \to F_n \to \ldots \to F_1 \to F_0 \to \mathbb{Z} \to 0$ be a free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$ and consider: $\ldots \leftarrow \mathrm{Hom}_{\mathbb{Z}[G]}(F_n, \mathbb{Z}_2[G]) \leftarrow \ldots \leftarrow \mathrm{Hom}_{\mathbb{Z}[G]}(F_1, \mathbb{Z}_2[G]) \leftarrow$

$$\leftarrow \mathrm{Hom}_{\mathbb{Z}[G]}(F_0, \mathbb{Z}_2[G]) \leftarrow 0$$

Now, as G is finite, then $\mathbb{Z}_2[G] \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]\,;\, \mathbb{Z}_2)$, therefore, using adjointness, we have $\mathrm{Hom}_{\mathbb{Z}[G]}(F_*, \mathbb{Z}_2[G]) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(F_*, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \mathbb{Z}_2)) \cong$

$\cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} F_*, \mathbb{Z}_2) = \mathrm{Hom}_{\mathbb{Z}}(F_*, \mathbb{Z}_2)$.

We then have $H^n(BG; \mathbb{Z}_2[G]) \cong \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}_2[G]) \cong \mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_2) = 0$ for $n > 0$.

□

7.24) Definition.- Let G be a finite group and A a commutative ring, then $H^*(BG; A)$ is a graded ring with the cup product. Using the Bar resolution one can see that in terms of homomorphisms and extensions the cup product is given as follows:

If we denote the group operation in G multiplicatively, then given $\alpha, \beta \in \mathrm{Hom}(G,A) \cong H^1(BG; A)$, $\alpha \cup \beta \in \mathrm{Ext}(G,A) \cong H^2(BG;A)$ is the extension $0 \to A \to A \tilde{\times} G \to G \to 0$, where $A \tilde{\times} G$ is $A \times G$ as a set and the group operation is defined by : $(a_1, g_1)\cdot(a_2, g_2) = (a_1 + a_2 + \alpha(g_1)\beta(g_2), g_1 g_2)$, where $\alpha(g_1)\beta(g_2)$ is the product in the ring A.

7.25) Definition.- Given a group G and a representation $r: G \to O(n)$, we define the Stiefel-Whitney classes of r by $w_i(r) = Br^*(w_i)$, where $Br : BG \to BO(n)$ and $w_i = w_i(\gamma(n))$.

Using the definition of cup product given in 7.24 one can prove the following.

7.26) Proposition [29] .- Let $\sigma \in \Sigma_n$ and denote by $<\sigma>$ the subgroup generated by σ and by $\iota : <\sigma> \hookrightarrow \Sigma_n$ the inclusion. Let $\rho : \Sigma_n \to O(n)$ be the permutation representation then $w_1(\rho \circ \iota)^2 = 0$ if and only if $(-1)^{c_2 + c_6 + \cdots + c_{4k+2} + \cdots} = 1$, where c_i is the number of i-cycles in the decomposition of σ.

□

§ 7.3) Cyclic reduction modulo bordism

In order to study the cyclic reductions modulo bordism we need the following theorem.

7.27) Theorem [15].-Let $\phi : X \to Y$ be a map between spaces of finite type (i.e. their mod. 2 homology is of finite type). The necesary and sufficient condition that $[M, f \in N_n(Y)$ lie in the image of $\phi_* : N_n(X) \to N_n(Y)$ is that every characteristic number of $[M,f]$ associated with an element in the kernel of $\phi^* : H^*(Y, \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)$ must vanish.

□

Note: This theorem is stated for the case when X and Y are finite complexes but it is easy to see that all they have used is the assumption that each $H_n(X; \mathbb{Z}_2)$ and $H_n(Y, \mathbb{Z}_2)$ is finite dimensional over $\mathbb{Z}_2$.

7.28) Corollary.- Let X be a connected space of finite type then an element $[M,f] \in N_n(X)$ is contained in $N_n \subset N_n(X)$ if and only if every characteristic number of $[M,f]$ associated with a positive dimensional cohomology class is zero.

Proof.- Take $x_0 \in X$, and let $i:\{x_0\} \hookrightarrow X$ be the inclusion, then $i^* : H^n(X) \to H^n(\{x_0\})$ is zero if $n > 0$ and an isomorphism if $n=0$. Therefore the result follows by applying 7.27 to the map i.

□

7.29) Proposition.- Let r be odd, then an r-covering over a closed manifold has a cyclic reduction modulo bordism if and only if it is bordant to the trivial covering.

Proof.- By 6.20 , the bordism of m-coverings is isomorphic to $N_*(B\Sigma_r)$. Let $i: \mathbb{Z}_r \to \Sigma_r$ be the inclusion then an r-covering has a cyclic reduction modulo bordism if and only if it is in the image of $Bi_*: N_*(B\mathbb{Z}_r) \to N_*(B\Sigma_r)$. By [21] $H^n(B\mathbb{Z}_r; \mathbb{Z}_2) = 0$ if $n > 0$.

Therefore the characteristic numbers associated to any positive dimensional cohomology class of an element in $\mathrm{im}Bi_*$ are zero so by 7.28, $\mathrm{im}\, Bi_* = N_*$.

□

7.30) Proposition.- Let r be even and consider $\mathbb{Z}_r \subset S^1 \subset \mathbb{C}$ acting on S^1 by complex multiplication, then the cyclic covering $S^1 \to S^1/\mathbb{Z}_r$ is not bordant to the trivial covering.

Proof.- The map classifying the covering $S^1 \to S^1/\mathbb{Z}_r$ is given as follows. Let $\mathbb{Z}_r = \{\bar{0},\bar{1},\dots,\overline{r-1}\}$ and $\bar{r}= \{1,2,\dots,r\}$, consider the bijection $f: \mathbb{Z}_r \to \bar{r}$ given by $f(\bar{1}) = 1$, $f(\bar{2})= 2,\dots, f(\overline{r-1})= r-1$ and $f(\bar{0})= r$, then the action of $\mathbb{Z}_r$ on $\mathbb{Z}_r$, by translation corresponds, under f to the usual inclusion $i: \mathbb{Z}_r \hookrightarrow \Sigma_r$, therefore we have:

$$\begin{array}{ccc} E\mathbb{Z}_r \times_{\mathbb{Z}_r} \bar{r} \cong E\mathbb{Z}_r \times_{\mathbb{Z}_r} \mathbb{Z}_r \cong E\mathbb{Z}_r \\ \searrow \quad \swarrow \\ B\mathbb{Z}_r \end{array}$$

If we take as $E\mathbb{Z}_r = S^\infty$, then we have a pull-back:

$$\begin{array}{ccccccc} S^1 & \hookleftarrow & S^\infty \cong E\mathbb{Z}_r \times_{\mathbb{Z}_r} \bar{r} & \longrightarrow & E\Sigma_r \times_{\Sigma_r} \bar{r} \\ \downarrow & & \downarrow \qquad\qquad \downarrow & & \downarrow \\ S^1/\mathbb{Z}_r & \xhookrightarrow[\phi]{} & S^\infty/\mathbb{Z}_r = B\mathbb{Z} & \xrightarrow[Bi]{} & B\Sigma_r \end{array}$$

Therefore we have to consider $[S^1/\mathbb{Z}_r, Bi\circ\varphi] \in N_1(B\Sigma_r)$ and by 7.26 it is enough to find a non-zero characteristic number associated to a positive dimensional class in $H^*(B\Sigma_r; \mathbb{Z}_2)$.

Consider the permutation representation $\Sigma_r \xrightarrow{\rho} O(r)$, and take $w_1(\rho) \in H^1(B\Sigma_r; \mathbb{Z}_2)$, by 7.18 and 7.22. a), $w_1(\rho)$ corresponds to the composition $\Sigma_r \xrightarrow{\rho} O(r) \xrightarrow{d} \mathbb{Z}_2$, therefore $Bi^*(w_1(\rho))$ corresponds to $\mathbb{Z}_r \xrightarrow{i} \Sigma_r \xrightarrow{\rho} O(r) \xrightarrow{d} \mathbb{Z}_2$, as r is even, then the permutation $(1\ 2\ 3\ \dots\ r)$ is odd so $Bi^*(w_1(\rho)) \neq 0$.

To evaluate ϕ^* we have the following [52] : $H^1(B\mathbb{Z}_r ; \mathbb{Z}_2) \cong$ $\cong \lim_{n \geq 0} H^1(S^{2n+1}/\mathbb{Z}_r;\mathbb{Z}_2)$; $H_1(S^1/\mathbb{Z}_r; \mathbb{Z}) \cong \mathbb{Z}$; $H_1(S^{2n+1}/\mathbb{Z}_r; \mathbb{Z}) \cong \mathbb{Z}_r$ $(n \geq 1)$ and the homomorphism $H_q(S^{2k+1}/\mathbb{Z}_r; \mathbb{Z}) \to H_q(S^{2\ell+1}/\mathbb{Z}_r; \mathbb{Z})$ (for $k < \ell$) is an isomorphism for $q < 2k+1$ and an epimorphism for $q = 2k+1$.

By the universal coefficient theorem we have the following commutative diagram (for $k < \ell$):

$$\begin{array}{ccccccccc}
0 \to & H_1(S^{2k+1}/\mathbb{Z}_r;\mathbb{Z})\otimes\mathbb{Z}_2 & \to & H_1(S^{2k+1}/\mathbb{Z}_r;\mathbb{Z}_2) & \to & \mathrm{Tor}(H_0(S^{2k+1}/\mathbb{Z}_r;\mathbb{Z});\mathbb{Z}_2) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H_1(S^{2\ell+1}/\mathbb{Z}_r;\mathbb{Z})\otimes\mathbb{Z}_2 & \to & H_1(S^{2\ell+1}/\mathbb{Z}_r;\mathbb{Z}_2) & \to & \mathrm{Tor}(H_0(S^{2\ell+1}/\mathbb{Z}_r;\mathbb{Z});\mathbb{Z}_2) & \to 0
\end{array}$$

The Tor groups are zero so applying duality we get:

$$H^1(S^1/\mathbb{Z}_r;\mathbb{Z}_2) \xleftarrow{\cong} H^1(S^3/\mathbb{Z}_r;\mathbb{Z}_2) \xleftarrow{\cong} H^1(S^5/\mathbb{Z}_r;\mathbb{Z}_2) \xleftarrow{\cong} \ldots$$

Hence $\phi^*: H^1(B\mathbb{Z}_r;\mathbb{Z}_2) \to H^1(S^1/\mathbb{Z}_r;\mathbb{Z}_2)$ is an isomorphism, and then $\phi^*(Bi^*(w_1(\rho)) \neq 0$, as the Kronecker product is non-degenerate, then $< (B_i \circ \phi)^* (w_1(\rho)), \sigma(S^1/\mathbb{Z}_r) > \neq 0$.

$\square$

7.31) Proposition.- Let r be odd, then a vector bundle with structural group $\Sigma_r \int O(k)$, over a closed manifold, is bordant to a bundle with structural group $\mathbb{Z}_r \int O(k)$ if and only if it is bordant to a bundle decomposed as the sum of r, k-vector bundles.

Proof.- We have an exact sequence $0 \to O(k)^r \to \mathbb{Z}_r \int O(k) \to \mathbb{Z}_r \to 0$, therefore we have a covering $\mathbb{Z}_r \hookrightarrow B(O(k)^r) \xrightarrow{p} B(\mathbb{Z}_r \int O(k))$, let τ be the transfer for this covering, then the composition

$$H_*(B(\mathbb{Z}_r \int O(k));\mathbb{Z}_2) \xrightarrow{\tau_*} H_*(B(O(k)^r);\mathbb{Z}_2) \xrightarrow{p_*} H_*(B(\mathbb{Z}_r \int O(k));\mathbb{Z}_2)$$

is the identity hence p_* is surjective. By naturality of the isomorphism $N_*(X) \cong H_*(X;\mathbb{Z}_2) \underset{\mathbb{Z}_2}{\otimes} N_*$, we have that $p_*: N_*(B(O(k)^r)) \to N_*(B(\mathbb{Z}_r \int O(k))$ is surjective. The following diagram clearly commutes:

$$\begin{tikzcd}
\mathbb{Z}_r \int O(k) \arrow[rr, hook, "j"] & & \Sigma_r \int O(k) \\
 & O(k)^r \arrow[ul, hook, "i"] \arrow[ur, hook', "\ell"'] &
\end{tikzcd}$$

inducing a commutative diagram:

$$\begin{tikzcd}[column sep=small] \mathcal{N}_*(B(\mathbb{Z}_r \int O(k)) \arrow[rr, "Bj_*"] & & \mathcal{N}_*(B(\Sigma_r \int O(k)), \\ & \mathcal{N}_*(B(O(k)^r)) \arrow[ul, "p_*=Bi_*"] \arrow[ur, "B\ell_*"'] & \end{tikzcd}$$

as p_* is surjective then $\operatorname{im} Bj_* = \operatorname{im} B\ell_*$.

□

7.32) Proposition.- Let r be even then the vector bundle $\xi = S^1 \times_{\mathbb{Z}_r} (\mathbb{R}^k)^r \to S^1/\mathbb{Z}_r$, where the action of $\mathbb{Z}_r$ on $(\mathbb{R}^k)^r$ is given by cyclic permutation of the coordinates $(v_1,\dots,v_r)$, $v_i \in \mathbb{R}^k$ $(1 \le i \le r)$, satisfies:

a) ξ is not bordant, as $\Sigma_r \int O(k)$ bundle, to the trivial bundle.

b) ξ is not bordant to a bundle decomposed as the sum of r k-vector bundles.

Proof.- The bundle ξ is classified by the map:

$$S^1/\mathbb{Z}_r \xrightarrow{\phi} S^\infty/\mathbb{Z}_r = B\,\mathbb{Z}_r \xrightarrow{B\iota} B(\mathbb{Z}_r \int O(k)) \xrightarrow{Bj} B(\Sigma_r \int O(k))$$

To prove a) we need, by 7.28, a non-zero characteristic number associated to a positive dimensional class in $H^*(B(\Sigma_r \int O(k)); \mathbb{Z}_2)$. Consider the representation given by $\Sigma_r \int O(k) \xrightarrow{\pi} \Sigma_r \xrightarrow{\rho} O(r)$, where $\pi(\sigma, A_1,\dots,A_r)=\sigma$, and take $w_1(\rho\circ\pi) \in H^1(B(\Sigma_r \int O(k)); \mathbb{Z}_2)$, by 7.18 and 7.22 this class corresponds to the composition:

$\Sigma_r \int O(k) \xrightarrow{\pi} \Sigma_r \xrightarrow{\rho} O(r) \xrightarrow{d} \mathbb{Z}_2$, therefore $(B\iota^* \circ Bj^*)(w_1(\rho\,\pi))$ corresponds to the composition $\mathbb{Z}_r \xrightarrow{\iota} \mathbb{Z}_r \int O(k) \xrightarrow{j} \Sigma_r \int O(k) \xrightarrow{\pi} \Sigma_r \xrightarrow{\rho} O(r) \xrightarrow{d} \mathbb{Z}_2$ which is not zero because the permutation $(1\ 2\ \dots\ r)$ is odd, so $B\iota^*\, Bj^*(w_1(\rho\circ\pi)) \neq 0$. In the proof of 7.30 we saw that $\phi^*: H^1(B\,\mathbb{Z}_r; \mathbb{Z}_2) \xrightarrow{\cong} H^1(S^1/\mathbb{Z}_r; \mathbb{Z}_2)$ was an isomorphism, when r is even

so, as the product $<,>$ is non-degenerate,
$<(Bj\ B\iota\circ\phi)^*(w_1(\rho\circ\pi)),\ \sigma(S^1/\mathbb{Z}_r)> \neq 0$.

To prove b) we need, by 7.27, a non-zero characteristic number associated to a class in the kernel of $B\ell^*$: $H^*(B(\Sigma_r \int O(k));\mathbb{Z}_2) \to H^*(B(O(k)^r);\mathbb{Z}_2)$. Consider the class that we used in a), $w_1(\rho\circ\pi) \in H^1(B(\Sigma_r \int O(k));\mathbb{Z}_2)$, then $B\ell^*\ w_1(\rho\circ\pi)$ corresponds to the composition $O(k)^r \xhookrightarrow{\ell} \Sigma_r \int O(k) \xrightarrow{\pi} \Sigma_r \xrightarrow{\rho} O(r) \xrightarrow{d} \mathbb{Z}_2$, but $\operatorname{im}\ell = \ker \pi$ so $B\ell^*\ w_1(\rho\circ\pi) = 0$ and in a) we saw that $<(Bj\circ B\iota\circ\phi)^*(w_1(\rho\circ\pi)),\sigma(S^1/\mathbb{Z}_r)> = 0$.

□

We now give examples where there is no cyclic reduction.

7.33) Proposition.- Let $\Sigma_2 \xrightarrow{\alpha} \Sigma_r$ be given by $\alpha(T) = \begin{pmatrix} 1 & 2 & 3, \ldots, r \\ 2 & 1 & 3, \ldots, r \end{pmatrix}$
and consider the r-covering $S^n \times_{\Sigma_2} \bar{r} \to P^n$. If $r \neq 4m+2$ and n is even $(n \geq 2)$ then this covering has no cyclic reduction (modulo bordism).

Proof.- We have a pull-back diagram:

$$\begin{tikzcd}
S^n \times_{\Sigma_2} \bar{r} \arrow[r] \arrow[d] & S^\infty \times_{\Sigma_2} \bar{r} \arrow[r] \arrow[d] & E\Sigma_r \times_{\Sigma_r} \bar{r} \arrow[d] \\
P^n \arrow[r, hook, "\phi"'] & P^\infty = B\Sigma_2 \arrow[r, "B\alpha"'] & B\Sigma_r
\end{tikzcd}$$

By 7.27 it is enough to find a non-zero characteristic number associated to a class in the kernel of Bj^*: $H^*(B\Sigma_r;\mathbb{Z}_2) \to H^*(B\mathbb{Z}_r;\mathbb{Z}_2)$.

Consider the permutation representation $\rho : \Sigma_r \to O(r)$ and take $w_1(\rho)^2 \in H^2(B\Sigma_r;\mathbb{Z}_2)$.

The inclusion $j: \mathbb{Z}_r \hookrightarrow \Sigma_r$ sends the generator to the permutation $(1\ 2\ 3\ \ldots\ r)$ and as $r \neq 4m+2$, then by 7.26, $Bj^*(w_1(\rho)^2) = 0$.

On the other hand $H^*(B\Sigma_2;\mathbb{Z}_2) \cong \mathbb{Z}_2[e]$, where $e \in H^1(B\Sigma_2;\mathbb{Z}_2)$, and $B\alpha^*(w_1(\rho))$ corresponds to the composition $\Sigma_2 \xrightarrow{\alpha} \Sigma_r \xrightarrow{\rho} O(r) \xrightarrow{d} \mathbb{Z}_2$

which is non-zero, therefore $B\alpha^*(w_1(\rho)) = e$ so $B\alpha^*(w_1(\rho)^2) = e^2 \neq 0$.

We also have that [49] $\phi : P^n \hookrightarrow P^\infty = B\Sigma_2$ induces a homomorphism $\phi^* : H^i(B\Sigma_2; \mathbb{Z}_2) \to H^i(P^n; \mathbb{Z}_2)$ such that it is an isomorphism if $i \leqslant n$, so if we denote $\phi^*(e) = e_n$, then $\phi^* B\,\alpha^*(w_1(\rho)^2) = e_n^2$ and as n is even then $w_1(P^n) = e_n$ [36]. Hence $< w_1(P^n)^{n-2}(B\alpha\ \phi)^*(w_1(\rho)^2), \sigma(P^n) > =$
$= < e_n^{n-2}\ e_n^2, \sigma(P^n) > \; = \; < e_n^n, \sigma(P^n) > \neq 0$.

$\square$

In order to give the examples in codimension greater than zero we need the following.

7.34) Proposition.- Consider the exact sequence
$0 \to O(k)^r \xrightarrow{i} \mathbb{Z}_r \int O(k) \xrightarrow{\pi} \mathbb{Z}_r \to 0$ and define a section $s: \mathbb{Z}_r \to \mathbb{Z}_r \int O(k)$ by $s(a) = (a, I, \dots, I)$, where I is the identity $(k \times k)$-matrix, then a class $z \in H^2(B(\mathbb{Z}_r \int O(k)); \mathbb{Z}_2)$ is zero if and only if $Bi^*(z)$ and $Bs^*(z)$ are zero.

Proof.- We take $B(\mathbb{Z}_r \int O(k)) = E\mathbb{Z}_r \times_{\mathbb{Z}_r} BO(k)^r$. The homology mod. 2 of $BO(k)$ is of finite type so by 7.3 we have a spectral sequence $\{E_r^{p,q}, d_r\}$ such that $E_2^{p,q} \cong H^p(B\mathbb{Z}_r; (H^*(BO(k))^{\otimes r})^q)$ and converging to $H^*(E\mathbb{Z}_r \times_{\mathbb{Z}_r} BO(k)^r; \mathbb{Z}_2)$; furthermore, by 7.4, this spectral sequence collapses, so $E_2^{p,q} = E_\infty^{p,q}$ and, as we are working over a field, there is no extension problem therefore we have that

$$H^n(B(\mathbb{Z}_r \int O(k)); \mathbb{Z}_2) \cong \bigoplus_{p+q=n} E_2^{p,q} .$$

We have a filtration: $E_2^{n,0} = E_\infty^{n,0} = J^{n,0} \subset J^{n-1} \subset \dots \subset J^{0,n} = H^n(B\mathbb{Z}_r; H^*(BO(k))^r)$, and if $p: E\mathbb{Z}_r \times_{\mathbb{Z}_r} BO(k)^r \longrightarrow B\mathbb{Z}_r$ denotes the projection and

$\iota : BO(k)^r \hookrightarrow E\mathbb{Z}_r \underset{\mathbb{Z}_r}{\times} BO(k)^r$ the inclusion then by 7.11 and 7.12 we have commutative diagrams:

a)
$$\begin{array}{ccc} H^q(\mathrm{Hom}_{\mathbb{Z}_r}(B_*; H^*(BO(k))^{\otimes r})) \xrightarrow{\beta} E_\infty^{0,q} = E_2^{0,q} \subset E_1^{0,q} = (H^*(BO(k))^{\otimes r})^q \\ \wr\| \qquad\qquad\qquad\qquad\qquad\qquad \wr\| \\ H^q(E\mathbb{Z}_r \underset{\mathbb{Z}_r}{\times} BO(k)^r; \mathbb{Z}_2) \xrightarrow{\quad \iota^* \quad} H^q(BO(k)^r; \mathbb{Z}_2) \end{array}$$

b)
$$\begin{array}{ccc} H^p(\mathbb{Z}_r) = E_2^{p,0} = E_\infty^{p,0} = J^{p,0} \subset J^{p-1,1} \overset{\alpha}{\hookrightarrow} H^p(\mathrm{Hom}_{\mathbb{Z}_r}(B_*; H^*(BO(k))^{\otimes r}) \\ \wr\| \qquad\qquad\qquad\qquad\qquad\qquad \wr\| \\ H^p(B\mathbb{Z}_r; \mathbb{Z}_2) \xrightarrow{\quad p^* \quad} H^p(E\mathbb{Z}_r \underset{\mathbb{Z}_r}{\times} BO(k)^r; \mathbb{Z}_2) \end{array}$$

We have exact sequences $0 \to J^{p+1,q-1} \to J^{p,q} \to E_\infty^{p,q} \to 0$

In the case $n=2$ we have ① $0 \to J^{2,0} \to J^{1,1} \to E_\infty^{1,1} \to 0$

② $0 \to J^{1,1} \xrightarrow{\alpha} H^2(B\mathbb{Z}_r; H^*(BO(k))^{\otimes r}) \xrightarrow{\beta} E_\infty^{0,2} \to 0$

These groups are: $E_\infty^{2,0} = E_2^{2,0} = J^{2,0} = H^2(B\mathbb{Z}_r; \mathbb{Z}_2)$

$E_\infty^{1,1} = E_2^{1,1} = H^1(B\mathbb{Z}_r; (H^*(BO(k))^{\otimes r})')$

$E_\infty^{0,2} = E_2^{0,2} = [(H^*(BO(k))^{\otimes r})^2]^{\mathbb{Z}_r} \subset E^{0,2} = (H^*(BO(k))^{\otimes r})^2$

The coefficients of $E_2^{1,1}$ are

$(H^*(BO(k))^{\otimes r}) = \bigoplus_{i_1+i_2+\ldots+i_r=1} H^{i_1}(BO(k)) \otimes \ldots \otimes H^{i_r}(BO(k))$. This is a permutation module, i.e., we have a basis which is invariant under the action of $\mathbb{Z}_r$, in effect, let $a_0 \in H^0(BO(k)) \cong \mathbb{Z}_2$ and $a_1 \in H^1(BO(k)) \cong \mathbb{Z}_2$ be the generators, then we have a $\mathbb{Z}_2$-basis of the form $\{(a_1 \otimes a_0 \otimes \ldots \otimes a_0, 0, \ldots, 0), (0, a_0 \otimes a_1 \otimes \ldots \otimes a_0), \ldots, (0,0,\ldots,0, a_0 \otimes a_0 \otimes \ldots \otimes a_1)\} = S$; there is only one orbit under the action of $\mathbb{Z}_r$ on S and the action is free, therefore $(H^*(BO(k)^{\otimes r})^1 \cong \mathbb{Z}_2[\mathbb{Z}_r]$. Hence $E_2^{1,1} = H^1(B\mathbb{Z}_r;(H^*(BO(k))^{\otimes r})^1) \cong$ $\cong H^1(B\mathbb{Z}_r;\mathbb{Z}_2[\mathbb{Z}_r])$ and this is zero by 7.23. Then the exact sequence ① becomes $E_\infty^{2,0} = E_2^{2,0} = J^{2,0} = J^{1,1}$, putting this in ② we get an exact sequence $0 \to E_2^{2,0} \xrightarrow{\alpha} H^2(B\mathbb{Z}_r;H^*(BO(k))^{\otimes r}) \xrightarrow{\beta} E_2^{0,2} \to 0$.

Combining this sequence with a) and b) we get a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & E_2^{2,0} & \xrightarrow{\alpha} & H^2(B\mathbb{Z}_r;\ H^*(BO(k))^{\otimes r}) & \xrightarrow{\beta} & E_2^{0,2} & \to & 0 \\
 & & \cong\updownarrow & & \updownarrow\cong & & \updownarrow\cong & & \\
0 & \to & H^2(B\mathbb{Z}_r;\mathbb{Z}_2) & \xrightarrow[p^*]{} & H^2(E\mathbb{Z}_r \underset{\mathbb{Z}_r}{\times} BO(k)^r;\mathbb{Z}_2) & \xrightarrow[\iota^*]{} & \text{image } \iota^* & \to & 0
\end{array}$$

Now notice that the bundle $E\mathbb{Z}_r \underset{\mathbb{Z}_r}{\times} BO(k)^r \xrightarrow[p]{} B\mathbb{Z}_r$ has a section s defined as follows: we take a point $x \in BO(k)$ then $(x,x,\ldots x) \in BO(k)^r$ is a fixed point under the action of $\mathbb{Z}_r$ so it defines a section $S[a] = [a,(x,x,\ldots,x)]$.

We can then define an isomorphism $\phi: H^2(E\mathbb{Z}_r \underset{\mathbb{Z}_r}{\times} BO(k)^r;\mathbb{Z}_2) \xrightarrow{\cong}$ $\xrightarrow{\cong} H^2(B\mathbb{Z}_r;\mathbb{Z}_2) \oplus \text{image } \iota^*$ by $\phi(z) = (S^*(z), \iota^*(z))$ so a class $z \in H^2(E\mathbb{Z}_r \underset{\mathbb{Z}_r}{\times} BO(k)^r;\mathbb{Z}_2)$ is zero if and only if $S^*(z)$ and $\iota^*(z)$ are zero.

Finally we identify S and ι with the maps of classifying spaces as follows. We are taking $E(\mathbb{Z}_r \int O(k)) = E\mathbb{Z}_r \times EO(k)^r \longrightarrow E\mathbb{Z}_r \times EO(k)^r/\mathbb{Z}_r \int O(k) \cong$ $\cong E\mathbb{Z}_r \underset{\mathbb{Z}_r}{\times} BO(k)^r$. The map $S;\ B\mathbb{Z}_r \longrightarrow E\mathbb{Z}_r \underset{\mathbb{Z}_r}{} BO(k)^r$ is induced by the inclusion $E\mathbb{Z}_r \hookrightarrow E\mathbb{Z}_r \times EO(k)^r$ which is clearly s-equivariant so $Bs \simeq S$. The map $\iota : BO(k)^r \longrightarrow E\mathbb{Z}_r \underset{\mathbb{Z}_r}{\times} BO(k)^r$ is induced by the inclusion $EO(k)^r \hookrightarrow \mathbb{Z}_r \times EO(k)^r$ which is clearly i-equivariant so $Bi \simeq \iota$.

□

7.35) Proposition.- Consider the vector bundle $S^n \times_{\Sigma_2} (\mathbb{R}^k)^r \longrightarrow P^n$, where Σ_2 acts on $(\mathbb{R}^k)^r$ by permuting the first two coordinates in $(v_1, v_2, \ldots, v_r)$, $v_i \in \mathbb{R}^k$, $1 \le i \le r$. If n is even and $r \ne 4m+2$ then this bundle has no cyclic reduction (modulo bordism).

Proof.- The classifying map for the bundle $S^n \times_{\Sigma_2} (\mathbb{R}^k)^r \longrightarrow P^n$ is given by the composition $P^n \xhookrightarrow[\phi]{} P^\infty = B\Sigma_2 \xrightarrow[B\alpha]{} B\Sigma_r \xrightarrow[B\gamma]{} B(\Sigma_r \int O(k))$ where $\alpha : \Sigma_2 \to \Sigma_r$ is given by

$$\alpha\,(T) = \begin{pmatrix} 1 & 2 & 3,\ldots, & r \\ 2 & 1 & 3,\ldots, & r \end{pmatrix} \text{ and } \gamma\colon \Sigma_r \to \Sigma_r \int O(k) \text{ is given by}$$

$$\gamma(\sigma) = (\sigma, I, \ldots, I)$$

where I is the identity $(k \times k)$-matrix.

By 7.27 it is enough to find a non-zero characteristic number associated with a class in the kernel of $Bj^*\colon H^*(B(\Sigma_r \int O(k)); \mathbb{Z}_2) \to H^*(B(\mathbb{Z}_r \int O(k); \mathbb{Z}_2)$

Consider the representation $\Sigma_r \int O(k) \xrightarrow{\pi} \Sigma_r \xrightarrow{\rho} O(r)$ and take $w_1(\rho\circ\pi)^2 \in H^2(B(\Sigma_r \int O(k)); \mathbb{Z}_2)$; then by 7.18 and 7.22, $Bj^*(w_1(\rho\circ\pi))$ corresponds to the composition:

$\mathbb{Z}_r \int O(k) \xrightarrow{j} \Sigma_r \int O(k) \xrightarrow{\pi} \Sigma_r \xrightarrow{\rho} O(r) \xrightarrow{d} \mathbb{Z}_2$. Now if we take $Bs^*(Bj^*(w_1(\rho\circ\pi)))$ where $s\colon \mathbb{Z}_r \to \mathbb{Z}_r \int O(k)$, we get $\mathbb{Z}_r \hookrightarrow \Sigma_r \xrightarrow{\rho} O(r) \xrightarrow{d} \mathbb{Z}_2$, so as $r \ne 4m+2$ then by 7.2 , $Bs^*(Bj^*(w_1(\rho\circ\pi))^2) = 0$.

On the other hand if we take $Bi^*(Bj^*(w_1(\rho\circ\pi)))$ where $i\colon O(k)^r \hookrightarrow \mathbb{Z}_r \int O(k)$ we get the trivial homomorphism so $Bi^*(Bj^*(w_1(\rho\circ\pi))^2) = 0$. Therefore, by 7.34, $Bj^*(w_1(\rho\circ\pi)^2) = 0$.

Now we want to evaluate $(B\gamma\circ B\alpha\circ\phi)^*(w_1(\rho\circ\pi)^2)$. The element $B\alpha^*\, B\gamma^*(w_1(\rho\circ\pi))$ corresponds to the composition

$\Sigma_2 \xrightarrow{\alpha} \Sigma_r \xrightarrow{\gamma} \Sigma_r \int O(k) \xrightarrow{\pi} \Sigma_r \xrightarrow{\rho} O(r) \xrightarrow{d} \mathbb{Z}_2$, which is just

$\Sigma_2 \xrightarrow{\alpha} \Sigma_r \xrightarrow{\rho} O(r) \xrightarrow{d} \mathbb{Z}_2$, this composition is not zero so

$B\alpha^* B\gamma^*(w_1(\rho\circ\pi)) = e$ where $H^*(B\Sigma_2; \mathbb{Z}_2) \cong \mathbb{Z}_2[e]$ and, as in the proof of 7.33 we have that $(B\gamma^\circ B\alpha^\circ\phi)^*(w_1(\rho\circ\pi)^2) = e_n^2$ where e_n is the restriction of e to P^n. Finally, as n is even $w_1(P^n) = e_n$ so

$< w_1(P^n)^{n-2}(B\gamma^\circ B\alpha^\circ\phi)^*(w_1(\rho\circ\pi)^2), \sigma(P^n) > = < e_n^{n-2} e_n^2, \sigma(P^n)> =$

$=<e_n^n, \sigma(P^n)> \neq 0.$ $\square$

7.36) Remark.- In the proof of 7.35 we have to work with the second cohomology groups because of the following. From the calculations above one can easily obtain that $H^1(B(\Sigma_r \int O(k)); \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and

$$H^1(B(\mathbb{Z}_r \int O(k)); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } r \text{ is even} \\ \mathbb{Z}_2 & \text{if } r \text{ is odd.} \end{cases}$$

Let $f: \Sigma_r \int O(k) \to \mathbb{Z}_2$ be given by the composition

$\Sigma_r \int O(k) \xrightarrow{\pi} \Sigma_r \xrightarrow{\rho} O(r) \xrightarrow{d} \mathbb{Z}_2$ and g by the composition

$\Sigma_r \int O(k) \xrightarrow{id \int d} \Sigma_r \int \mathbb{Z}_2 \xrightarrow{\psi} \mathbb{Z}_2$, where $\psi(\sigma, t_1, \ldots, t_r) = t_1 \ldots t_r$,

then $\{f,g\}$ is a $\mathbb{Z}_2$-basis for $H^1(B(\Sigma_r \int O(k)); \mathbb{Z}_2)$.

Let $j: \mathbb{Z}_r \int O(k) \hookrightarrow \Sigma_r \int O(k)$ be the inclusion then

$$Bj^*(f) = \begin{cases} \text{not zero if } r \text{ is even,} \\ \text{zero} \quad \text{if } r \text{ is odd} \end{cases}$$

and $Bj^*(g) \neq 0$. If r is even $Bj^*(f) \neq Bj^*(g)$ and hence Bj^* is an isomorphism.

Chapter 8.- Dyer-Lashof operations on Eilenberg-MacLane spaces

In this chapter we show that the Dyer-Lashof operations in the homology mod. 2 of an Eilenberg-Mac Lane space $K(A,n)$ are zero, without using the cohomology of the spaces $K(A,n)$ and hence this is valid for any abelian group A. An algebraic proof for the case $A=\mathbb{Z}_2$ can be found in [32] .

8.1) Definition.- Let X be an abelian topological monoid, then we can make X a $\mathcal{C}_\infty$-space by defining $\mathcal{C}_\infty(k)\underset{\Sigma_k}{\times} X^k \xrightarrow{\theta_k} X$ by

$\theta_k\,[c,x_1,x_2,\dots,x_k] = x_1\,x_2\,\dots\,x_k$. Then, as we saw in chapter 2, we have Dyer-Lashof operations in bordism $\tilde{Q}_r$: $N_n(X)\to N_{2n+r}(X)$ and in homology Q_r: $H_n(X;\mathbb{Z}_2)\to H_{2n+r}(X,\mathbb{Z}_2)$. Recall that the operations $\tilde{Q}_i$ are given by the composition (2.24):

$$N_n(X)\xrightarrow{\tilde{q}_r^\infty} N_{2n+r}(S^\infty\underset{\Sigma_2}{\times} X\times X)\xrightarrow{\cong} N_{2n+r}(\mathcal{C}_\infty(2)\underset{\Sigma_2}{\times} X\times X)\xrightarrow{\theta_{2*}} N_{2n+r}(X)\ ,$$

where $\tilde{q}_r^\infty\,[M,f]=[S^r\underset{\Sigma_2}{\times} M\times M,\ \iota\underset{\Sigma_2}{\ } f\times f]$ and $\iota: S^r\hookrightarrow S^\infty$. If we denote by $m: X\times X\to X$ the product in X and by $\bar{m}: X\underset{\Sigma_2}{\times} X\to X$ the map induced by m, then it is clear from the definition of θ_2 that the operations $\tilde{Q}_r$ are given by the composition

$$N_n(X)\xrightarrow{\tilde{q}_r^\infty} N_{2n+r}(S^\infty\underset{\Sigma_2}{\times} X\times X)\xrightarrow{\Pi_*} N_{2n+r}(X\underset{\Sigma_2}{\times} X)\xrightarrow{m_*} N_{2n+r}(X),\quad \text{where}$$

$\Pi: S^\infty\underset{\Sigma_2}{\times} X\times X\to X\underset{\Sigma_2}{\times} X$ is the projection.

8.2) Definition.- Let M be a closed manifold, we denote by $P(TM\oplus\varepsilon^{r+1})$ the projective bundle associated to the bundle $TM\oplus\varepsilon^{r+1}$, the sum of the tangent bundle of M and a trivial $(r+1)$-bundle. Given a map $f: M\longrightarrow X$ consider the composition $P(TM\oplus\varepsilon^{r+1})\xrightarrow{p} M\xrightarrow{g} X\underset{\Sigma_2}{\times} X$, where p is the projection of the projective bundle and $g(a)=[\,f(a),\ f(a)\,]$.

8.3) Proposition.- The pairs $(S^r \underset{\Sigma_2}{\times} M\times M, \pi\circ(\iota \underset{\Sigma_2}{\times} f\times f))$ and $(P(TM \oplus \varepsilon^{r+1}), g\circ p)$ are bordant in $X \underset{\Sigma_2}{\times} X$.

Proof.- Consider the manifold $D^{r+1}\times M\times M$ and define a Σ_2-action by $(b,x_1,x_2)\cdot T = (-b, x_2, x_1)$, where $b\in D^{r+1}$, $x_i \in M$. Consider an inclusion $D^{r+1} \xhookrightarrow{j} S^\infty$ such that the following diagram commutes

$$\begin{tikzcd} S^r \arrow[r, hook, "\iota"] \arrow[d, hook] & S^\infty \\ D^{r+1} \arrow[ur, "j"'] & \end{tikzcd}$$

Then we have a Σ_2-equivariant map: $D^{r+1}\times M\times M \xrightarrow[j\times f\times f]{} S^\infty\times X\times X \xrightarrow[\text{proj.}]{} X\times X$

The action of Σ_2 on $D^{r+1}\times M\times M$ is not free, the fixed points are the image of the embedding $e\colon M \hookrightarrow D^{r+1}\times M\times M$ given by $e(a)=(0,a,a)$. Notice that this embedding is the composition of $M \underset{\Delta}{\hookrightarrow} M\times M \hookrightarrow D^{r+1}\times M\times M$, hence the normal bundle of e, ν_e satisfies $\nu_e \cong TM \oplus \varepsilon^{r+1}$. We can find a tubular neighbourhood $D \cong D(\nu_e)$ such that the action on D coincides with multiplication by (-1) on $D(\nu_e)$ [7]. Let $\mathring{D}$ denote the interior of D, then the manifold $D^{r+1}\times M\times M - \mathring{D}$ has a free Σ_2-action and $\partial(D^{r+1}\times M\times M-\mathring{D}) = S^r\times M\times M \perp S(TM\oplus\varepsilon^{r+1})$, the restriction of proj.$\circ$ $j\times f\times f$ gives an equivariant map to $X\times X$, which we denote by F.

By [48] we can find an equivariant homotopy H, leaving fixed $S^r\times M\times M$ such that $H_0 = F$; H_1 and F coincide on the fixed points and the value of H_1 on D is given by the composition:

$D\cong D(\nu_e) \xrightarrow{q} e(M) \xrightarrow{F|e(M)} X\times X$, where q is the projection of the disc bundle. As everything we have done is Σ_2-equivariant we can pass to the quotients and then $H_1/\Sigma_2\colon D^{r+1}\times M\times M-\mathring{D}/\Sigma_2 \to X \underset{\Sigma_2}{\times} X$ is a cobordism between ($S^r \times_{\Sigma_2} M \times M, \Pi \circ (1 \times_{\Sigma_2} f \times f)$) and

$(S(TM \oplus \varepsilon^{r+1})/\Sigma_2 = P(TM \oplus \varepsilon^{r+1}), g \circ p)$. □

8.4) Proposition.- If X is an abelian topological monoid then the operations Q_r are zero, $r \geq 0$.

Proof.- By 2.33 we have the following commutative diagram:

$$\begin{array}{ccc} N_n(X) & \xrightarrow{\widetilde{Q}_r} & N_{2n+r}(X) \\ \mu\downarrow & & \downarrow\mu \\ H_n(X, \mathbb{Z}_2) & \xrightarrow[Q_r]{} & H_{2n+r}(X; \mathbb{Z}_2) \end{array}$$

The homomorphisms μ are surjective so given $x \in H_n(X; \mathbb{Z}_2)$ take $[M,f] \in N_n(X)$ such that $\mu[M,f] = x$; by 8.1 $\widetilde{Q}_r[M,f] = \bar{m}_* \Pi_* \tilde{q}_r^\infty[M,f]$, but by 8.3 $\Pi_* \tilde{q}_r^\infty[M,f] = [P(TM \oplus \varepsilon^{r+1}), g \circ p]$. The image of the fundamental class $\sigma(P(TM \oplus \varepsilon^{r+1}))$ under $p_*: H_{2n+r}(P(TM \oplus \varepsilon^{r+1}); \mathbb{Z}_2) \to H_{2n+r}(M; \mathbb{Z}_2) = 0$ is zero, therefore $Q_r(x) = Q_r\mu[M,f] = \widetilde{Q}_r[M,f] = 0$. □

8.5) Proposition.- Let A be an abelian group and let $K(A,n)$ be an Eilenberg-Mac Lane space then the Dyer-Lashof operations on the homology mod. 2 of $K(A, n)$ are zero.

Proof.- The space $K(A,n)$ is an infinite loop space so it is a $\mathscr{C}_\infty$-space. By [27] there exists an abelian topological monoid K and a homotopy equivalence $\ell : K(A,n) \xrightarrow{\simeq} K$ such that ℓ is a map of $\mathscr{C}_\infty$-spaces where the $\mathscr{C}_\infty$-space structure for K is the one defined in 8.1 Therefore we have a commutative diagram:

$$\begin{tikzcd}
H_n(K(A,n);\mathbb{Z}_2) \arrow[r, "Q_r"] \arrow[d, "\ell_*"', "\cong"] & H_{2n+1}(K(A,n);\mathbb{Z}_2) \arrow[d, "\cong"', "\ell_*"] \\
H_n(K;\mathbb{Z}_2) \arrow[r, "Q_r"] & H_{2n+r}(K;\mathbb{Z}_2)
\end{tikzcd}$$

By 8.4 the operations on K are zero and hence the operations on $K(A,n)$ are also zero.

8.6) Remark.- The result above is not true for the operations $\widetilde{Q}_r$ in bordism. For example, if we take $P^\infty = K(\mathbb{Z}_2,1)$, and $[P^n, i] \in N_n(P^\infty)$, where $i: P^n \hookrightarrow P^\infty$, then if n is even, $\widetilde{Q}_0\,[P^n, i] \neq 0$ because $P^n \times P^n$ is not a boundary.

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