Regulator constants of integral representations,

together

with relative motives over Shimura varieties

by

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Contents

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All work contained in this thesis is my own, unless explicitly referenced. A slightly modified version of Chapter 2 is to appear in the *Mathematical Proceedings of the Cambridge Philosophical Society*. Some of the content of Chapters 3 and 4 may also appear separately at a later date.

Abstract

This thesis is split into three largely independent chapters. The first concerns the representation theory of $\mathbb{Z}_p[G]$-lattices. Specifically, we investigate regulator constants, due to Dokchitser–Dokchitser, which are isomorphism invariants of lattices whose extension of scalars to $\mathbb{Q}_p$ is self-dual. We first show that when G has cyclic Sylow p-subgroups then regulator constants are strong invariants of permutation modules in a way that can be made precise. Our main result is then that, subject to an additional technical hypothesis on G, this can be combined with existing work of Yakovlev to provide an explicit list of accessible invariants which completely determine, up to isomorphism, any $\mathbb{Z}_p[G]$-lattice whose extension to $\mathbb{Q}_p$ is self-dual.

The second chapter is an application of this result in the context of number fields. Given a Galois extension of number fields K/F with Galois group G, the extension of scalars to $\mathbb{Z}_p$ of the unit group of K modulo its torsion subgroup defines a $\mathbb{Z}_p[G]$-lattice. If we assume that G has cyclic Sylow p-subgroups and satisfies the aforementioned hypothesis, then the above result gives a list of invariants which determine the Galois module structure. The main result of this chapter is then that if p divides G at most once, we can explicate these invariants in terms of classical number theoretic objects. For example, in some cases this can be done in terms of capitulation of ideal classes and ramification information.

The final (unrelated) topic concerns relative motives over Shimura varieties. Given a Shimura datum $(G, \mathfrak{X})$ and neat open compact subgroup $K \leq G(\mathbb{A}_f)$, denote the corresponding Shimura variety $\mathrm{Sh}_K(G, \mathfrak{X})$ by S. The canonical construction described by Pink shows how to associate variations of Hodge structure on S^{an} to representations of G. It is expected that this should be motivic in nature, i.e. that there is a motive over S for every representation of G whose Hodge realisation is the variation of Hodge structure given by the canonical construction. Using mixed Shimura varieties, we show that this can be done functorially for representations with Hodge type $\{(-1,0),(0,-1)\}$ and that this is compatible with change of S. When $(G, \mathfrak{X})$ has a chosen PEL-datum, existing work of Ancona allows us to associate a motive over S to any representation of G. We then give results to show that in some cases this compatible with change of S and independent of the choice of PEL-datum.

Chapter 1

Introduction

This thesis is organised into three main chapters, each of which is designed to be read largely independently of any other. Chapters 2 and 4 are completely self-contained, whilst Chapter 3 is an application of a result from Chapter 2. Each makes use of very different techniques and has equally different aims and objectives. For these reasons, we have decided to defer anything other than a conceptual and organisational outline to more detailed self-contained introductions in the relevant chapters.

Chapter 2 is purely representation theoretic in nature. Specifically, it relates to the representation theory of $\mathbb{Z}_p[G]$-lattices. Here $\mathbb{Z}_p$ denotes the ring of p-adic integers, G is a finite group and by a $\mathbb{Z}_p[G]$-lattice we mean a $\mathbb{Z}_p[G]$-module which is free of finite rank as a $\mathbb{Z}_p$-module. When p divides $|G|$, the theory of such lattices is complex and poorly understood. For example, it is known that when the Sylow p-subgroups of G are anything other than cyclic of order $\leq p^2$, then there are infinitely many indecomposable $\mathbb{Z}_p[G]$-lattices. For only a handful of cases have all the $\mathbb{Z}_p[G]$-lattices been classified.

There are only a few known isomorphism invariants of $\mathbb{Z}_p[G]$-lattices that could be considered easily computable. For M a $\mathbb{Z}_p[G]$-lattice, one example is simply the isomorphism class of $M \otimes \mathbb{Q}_p$, where $\mathbb{Q}_p$ denotes the field of p-adic numbers. Another source of invariants comes from the cohomology of M. When G has cyclic Sylow p-subgroups, a result of Yakovlev shows that cohomology controls M in a strong and precise way (see Theorem 2.6.7. Within the theory of $\mathbb{Z}_p[G]$-lattices it is common to think of having cyclic Sylow p-subgroups as being the easiest case). However, there are still properties of M which are inaccessible to all of the above invariants.

More recently, Dokchitser–Dokchitser have defined a family of numerical isomorphism invariants of $\mathbb{Z}_p[G]$-lattices whose extension of scalars to $\mathbb{Q}_p$ is self-dual called *regulator constants* (see e.g. [DD09]). To date, there has been little work on the strength of these invariants or how they relate to the aforementioned ones. This is the aim of Chapter 2. We first show that if G has cyclic Sylow p-subgroups, then regulator constants are good invariants of permutation modules (see Theorem 2.4.1). Our main result is then that, for groups with cyclic Sylow p-subgroups satisfying an additional technical hypothesis, the isomorphism class of a $\mathbb{Z}_p[G]$-lattice M whose extension of scalars to $\mathbb{Q}_p$ is self-dual is determined by the isomorphism class of its extension of scalars to $\mathbb{Q}_p$, a cohomological invariant of Yakovlev and its regulator constants (Theorem 2.6.9). The self-duality condition is fairly mild and can be removed at minor expense (see Remark 2.6.10).

The focus of the third chapter is to apply this result in a number theoretic context. Let K/F be a Galois extension of number fields with Galois group G which is cyclic and satisfies the technical hypothesis mentioned above. Then, if $\mathcal{O}_K^\times/\mu_K$ denotes the unit group of K quotiented by its subgroup of roots of unity, then $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ defines a $\mathbb{Z}_p[G]$-lattice whose extension of scalars to $\mathbb{Q}_p$ is self-dual.

As a result, we immediately find that $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ is determined by its extension of scalars, cohomology and regulator constants. Really though, we wish to reinterpret the required information in terms of elementary invariants of number fields. For simplicity, assume that K does not contain the p^{th} roots of unity. The isomorphism class of $\mathcal{O}_K/\mu_K \otimes \mathbb{Q}_p$ is well understood as Artin's induction theorem shows it is a function of the signatures of the intermediate subfields of K/F (see Lemma 3.1.3). Furthermore, the regulator constants of K/F were calculated in terms of the class numbers of the intermediate subfields of K/F by Bartel [Bar12] (see Theorem 3.1.5).

The main part of Chapter 3, is to calculate the Yakovlev's cohomological invariant in the case of $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$. For this we restrict to the case of p dividing $|K : F|$ exactly once. We then find that the invariant can be calculated in terms of ramification information and capitulation of class groups. This is proven via a series of diagram chases using Arakelov class groups.

The final chapter is completely unrelated to the other two and concerns relative motives over Shimura varieties. Shimura varieties are obtained by quotienting certain symmetric spaces by the action of some subgroup of an algebraic group.

Shimura varieties are of importance to number theorists as they are expected to be moduli spaces for abelian varieties (or more generally abelian motives) with special properties. As such, studying Shimura varieties can lead to results which hold for whole families of abelian varieties (cf. Mazur's torsion bounds [Maz78]).

For any Shimura variety S, with corresponding algebraic group G, almost by definition there is a *canonical construction* which associates a variation of Hodge structure on S^{an} (see Definition 4.5.2) to every representation of G. In fact, S has a canonical algebraic model over some number field, known as its reflex field and there is also a parallel construction of a lisse ℓ-adic sheaf on S for any choice of ℓ. In particular, for every representation of G, there is an associated Galois representation for every point of S. It is natural to ask what sort of properties these ℓ-adic sheaves have. For example, do they arise from geometry, i.e. is their dual a subquotient of a higher direct image of some smooth projective variety over S? Furthermore, in either the Hodge or ℓ-adic cases, we might ask if the canonical construction is motivic, i.e. is there some construction of a relative Chow motive for every representation of G, whose realisation coincides with the canonical construction.

The answer to all these questions is expected to be yes. In Chapter 4, we show that this happens for the full subcategory of $\text{Rep}(G)$ consisting of representations whose Hodge type lies in $\{(-1,0),(0,-1)\}$. Moreover, this is compatible with change of the Shimura variety S. We show this using the formal properties of mixed Shimura varieties. We also use this language to give criteria to show that existing work of Ancona, which defines a motivic lift of the canonical construction in the case of Shimura varieties of "PEL-type", is independent of the choice of "PEL datum" and compatible with change of S.

Chapter 2

Regulator constants as invariants of lattices

2.1 Introduction

Let G be a finite group and p some prime. If $\mathcal{R}$ is a ring, then by an $\mathcal{R}[G]$-lattice we mean an $\mathcal{R}[G]$-module which is free of finite rank as an $\mathcal{R}$-module. In this section we are concerned with the study of $\mathbb{Z}_p[G]$-lattices, where here $\mathbb{Z}_p$ denotes the p-adic integers.

Let $\mathbb{Q}_p$ denote the field of p-adic numbers. If p does not divide the order of G, then two $\mathbb{Z}_p[G]$-lattices M, N are isomorphic if and only if $M \otimes \mathbb{Q}_p$ and $N \otimes \mathbb{Q}_p$ are [CR94, Thm. 30.16]. In other words, the study of $\mathbb{Z}_p[G]$-lattices reduces to character theory for primes not dividing $|G|$.

When p does divide the order of the group, things become much more complicated. For example, whenever the Sylow p-subgroups of G are not cyclic of order at most p^2, then there are infinitely many isomorphism classes of indecomposable $\mathbb{Z}_p[G]$-lattices [CR94, Sec. 33.7]. In this thesis, we will be principally concerned with the case of cyclic Sylow p-subgroups. In view of the previous paragraph, this may be viewed as the simplest non-trivial case.

A natural question to ask is, given two $\mathbb{Z}_p[G]$-lattices M, N, are they isomorphic? One way to tackle such questions is to describe some finite list of, hopefully computable and meaningful, isomorphism invariants of $\mathbb{Z}_p[G]$-lattices with the property that if each of the isomorphism invariants coincide for two given lattices, then the lattices are isomorphic. One might call such invariants a complete set of invariants of $\mathbb{Z}_p[G]$-lattices. The principal aim of this chapter is to find a computable complete set of invariants for groups with cyclic Sylow p-subgroups satisfying an additional technical hypothesis (see Theorem 2.1.4).

As remarked in Chapter 1, there are only a small number of known isomorphism invariants which can be considered computable. One such invariant is given by the isomorphism class of $M \otimes \mathbb{Q}_p$. As discussed previously, this is a complete invariant when $p \nmid |G|$. As a result, we shall attempt to supplement this invariant when p divides $|G|$.

Another family of invariants arise from cohomological constructions.

Example 2.1.1. If $G = C_2 = \langle \sigma \rangle$, then there are three indecomposable $\mathbb{Z}_2[C_2]$-lattices: $\mathbb{1}$ and ϵ which are of rank one with σ acting trivially and by multiplication by -1 respectively, and $\mathbb{Z}_2[C_2]$ acting on itself on the left. In this case, $(\mathbb{1} \oplus \epsilon) \otimes \mathbb{Q}_2 \cong \mathbb{Z}_2[C_2] \otimes \mathbb{Q}_2$. The lattices $\mathbb{1} \oplus \epsilon$ and $\mathbb{Z}_2[C_2]$ can be distinguished cohomologically:

$$\begin{aligned} H^1(C_2, \mathbb{1} \oplus \epsilon) &= H^1(C_2, \mathbb{1}) \oplus H^1(C_2, \epsilon) \\ &= 0 \oplus \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

by the explicit description of cohomology of cyclic groups (see e.g. [GS06, Ex. 3.2.9]). Whilst,

$$\begin{aligned} H^1(C_2, \mathbb{Z}_2[C_2]) &= H^1(\{1\}, \mathbb{1}) \\ &= 0 \end{aligned}$$

by Shapiro's lemma.

Suppose that G has a cyclic Sylow p-subgroup P and write $P_i \leq P$ for its subgroup of order p^i. One such cohomological invariant which is well understood consists of the following diagram

$$H^1(P_r, M) \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} H^1(P_{r-1}, M) \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} \quad \cdots \quad \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} H^1(P_0, M). \qquad (\star)$$

in which the horizontal rows are given by restriction and corestriction, and we consider each group as a module over the normaliser $N_G(P_i)$. Note that in the previous example, the diagram simplifies to consist just of $H^1(C_2, -)$ as an abelian group. Yakovlev showed that $(\star)$ determines the isomorphism class of M up to summands which are *trivial source*, i.e. summands of permutation modules [Yak96, Thm. 2.1] (see Theorem 2.6.7 for a precise statement). Thus, M would be completely determined if one could provide invariants which constrain the remaining trivial source summand of M. We refer to ($\star$) as the *Yakovlev diagram* of M.

We shall principally focus on a newer less well understood type of invariants, known as regulator constants. These are numerical invariants of representations of a finite group introduced by Dokchitser–Dokchitser (see for example [DD09]). We shall briefly recall some properties of regulator constants (cf. Section 2.2.5).

A Brauer relation in characteristic zero (resp. characteristic p) consists of a pair of G-sets for which the associated permutation modules over $\mathbb{Q}$ (resp. $\mathbb{F}_p$) are isomorphic. Characteristic zero (resp. p) relations form a free abelian group of finite rank, which we denote by $br_0(G)$ (resp. $br_p(G)$). All characteristic p relations are also characteristic zero relations so that $br_p(G) \subseteq br_0(G)$ (Lemma 2.2.7).

We call a $\mathbb{Z}_p[G]$-lattice *rationally self-dual* if its extension of scalars to $\mathbb{Q}_p$ is self-dual. Each characteristic zero Brauer relation θ defines a regulator constant $C_\theta(-)$ which assigns to a rationally self-dual $\mathbb{Z}_p[G]$-lattice M an element $C_\theta(M) \in \mathbb{Q}_p^\times/(\mathbb{Z}_p^\times)^2$. As will be made precise later, $C_\theta(M)$ measures the relative covolumes of certain fixed subspaces corresponding to the G-sets defining θ (see Definition 2.2.27).

In several number theoretic contexts, regulator constants have been found to both coincide with naturally occurring objects and to be computationally accessible. For example, if $K/\mathbb{Q}$ is a Galois extension of number fields with $G = \mathrm{Gal}(K/\mathbb{Q})$, $E/\mathbb{Q}$ is an elliptic curve and $M = E(K)/E(K)_{\mathrm{tors}}$, the torsion-free quotient of the Mordell-Weil group of E, then the regulator constants of $M \otimes \mathbb{Z}_p$ are closely related to the elliptic regulator of E [DD09]. Similarly, if $M = \mathcal{O}_K^\times/\mu_K$ is the unit group of K modulo roots of unity, then the regulator constants of $M \otimes \mathbb{Z}_p$ are closely related to Dirichlet's unit group regulator (see [Bar12] or Theorem 3.1.5 below).

The applications of regulator constants are dependent on showing that regulator constants are sufficiently good invariants of lattices. We now attempt to give meaning to such a statement.

Let $a(\mathbb{Z}_p[G])$ denote the representation ring of G over $\mathbb{Z}_p$. We denote the subring generated by $\mathbb{Z}_p[G]$-lattices which are rationally self-dual by $a(\mathbb{Z}_p[G], \mathrm{sd})$. Set $A(\mathbb{Z}_p[G]) = a(\mathbb{Z}_p[G]) \otimes_{\mathbb{Z}} \mathbb{Q}$, $A(\mathbb{Z}_p[G], \mathrm{sd}) = a(\mathbb{Z}_p[G], \mathrm{sd}) \otimes_{\mathbb{Z}} \mathbb{Q}$, $BR_0(G) = br_0(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $BR_p(G) = br_p(G) \otimes_{\mathbb{Z}} \mathbb{Q}$. Regulator constants are multiplicative in direct sums of lattices and also under summing Brauer relations. As such, if $v_p(-)$ denotes the

p-adic valuation, then there is a pairing

$$\begin{aligned} v_p(C_{(-)}(-))\colon BR_0(G) \times A(\mathbb{Z}_p[G], \mathrm{sd}) &\longrightarrow \mathbb{Q} \\ (\theta, M) \qquad\qquad &\longmapsto v_p(C_\theta(M)). \end{aligned}$$

The space $BR_0(G)$ is always finite dimensional, whilst $A(\mathbb{Z}_p[G], \mathrm{sd})$ will regularly be infinite dimensional. For formal reasons, elements of $BR_p(G)$ always lie in the kernel of $v_p(C_{(-)}(-))$. But, one might say that regulator constants are good invariants if the left kernel consists only of characteristic p relations. It is one of our main results that if G has cyclic Sylow p-subgroups, then this is always the case. But let us be more precise.

Outside of a few families of groups we do not have classifications of the $\mathbb{Z}_p[G]$-lattices for primes dividing the order of G. On the other hand, the isomorphism classes of permutation modules, that is $\mathbb{Z}_p[G]$-lattices on which G acts by permuting a choice of basis, are easy to enumerate and it is possible to give a formula for their regulator constants in terms of group theoretic information. For this reason we shall primarily restrict our attention from $A(\mathbb{Z}_p[G], \mathrm{sd})$ to $A(\mathbb{Z}_p[G], \mathrm{perm})$, the subspace generated by permutation modules.

Again, for trivial reasons $A(\mathbb{Z}_p[G], \mathrm{cyc})$, the subspace generated by the permutation modules $\mathbb{Z}_p[G/H]$ for $H \leq G$ cyclic, lies in the kernel of $v_p(C_{(-)}(-))$. We refer to the resulting pairing

$$\langle\ ,\ \rangle_{\mathrm{perm}}\colon BR_0(G)/BR_p(G) \times A(\mathbb{Z}_p[G], \mathrm{perm})/A(\mathbb{Z}_p[G], \mathrm{cyc}) \to \mathbb{Q}$$

as the *permutation pairing*. In fact, we shall see that both of $BR_0(G)/BR_p(G)$ and $A(\mathbb{Z}_p[G], \mathrm{perm})/A(\mathbb{Z}_p[G], \mathrm{cyc})$ are canonically isomorphic to the free $\mathbb{Q}$-vector space on the set of conjugacy classes of p-hypo-elementary subgroups. With respect to this identification, the pairing is symmetric. Prior to quotienting the spaces need not have the same dimension and there is no such identification (cf. Remark 2.3.7).

Theorem 2.1.2. *For a finite group G and prime p such that G has cyclic Sylow p-subgroups, the permutation pairing is non-degenerate.*

A formal consequence of Theorem 2.1.2 is that the isomorphism class of a permutation module M over $\mathbb{Z}_p$ is determined by its regulator constants and the isomorphism class of $M \otimes \mathbb{Q}_p$.

To show the theorem, we first reduce to p-hypo-elementary subgroups. Any p-hypo-elementary group with cyclic Sylow p-subgroups is of the form $C_{p^k} \rtimes C_n$

with $(p, n) = 1$. In this case, we find we are able to completely explicate the matrix representing the pairing, and showing invertibility becomes a combinatorial problem (Lemma 2.4.7).

For general G, the permutation pairing may be degenerate. For example, when $p = 3$, the group $C_3 \times C_3 \times S_3$ has a Brauer relation θ_Σ whose regulator constant is trivial on all permutation modules (see Section 2.7.8). I do not know if there are other lattices for which $C_{\theta_\Sigma}(-)$ does not vanish.

We do however provide a partial result for arbitrary G. For any group G, there is a canonical Brauer relation with leading term $[G]$ called the *Artin relation*, which we denote by θ_G (see Definition 2.2.14). Let $\mathbb{1}_G$ denote the trivial $\mathbb{Z}_p[G]$-module.

Theorem 2.1.3. *For any finite group G and prime p, we have $C_{\theta_G}(\mathbb{1}_G) \neq 0$ if and only if G is non-cyclic.*

The proof is group theoretic in nature and completely independent of that of Theorem 2.1.2.

The final aspect of the chapter, and the most important for our applications in Chapter 3, is to show that Theorem 2.1.2 can in some cases be combined with Yakovlev's results to give a complete set of invariants for $\mathbb{Z}_p[G]$-lattices.

Specifically, we must show that Theorem 2.1.2 can be used to determine the trivial source summand of a given M. Denote by $A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$ the subring of the representation ring $A(\mathbb{Z}_{(p)}[G])$ generated by trivial source lattices (see Definition 2.6.1). Note that extension of scalars defines an inclusion $A(\mathbb{Z}_{(p)}[G]) \hookrightarrow A(\mathbb{Z}_p[G])$ [Rei70, Thm. 5.6 *iii)*] and so an isomorphism of the subrings generated by permutation modules $A(\mathbb{Z}_{(p)}[G], \mathrm{perm}) = A(\mathbb{Z}_p[G], \mathrm{perm})$. When additionally $A(\mathbb{Z}_{(p)}[G], \mathrm{triv}) = A(\mathbb{Z}_p[G], \mathrm{perm})$ we are able to show:

Theorem 2.1.4. *Let G be a finite group and p a prime such that G has cyclic Sylow p-subgroups and such that $A(\mathbb{Z}_p[G], \mathrm{perm}) = A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$. Given two rationally self-dual $\mathbb{Z}_p[G]$-lattices M, N, then $M \cong N$ if and only if all the following conditions hold:*

i) $M \otimes \mathbb{Q}_p \cong N \otimes \mathbb{Q}_p$,

ii) for an explicit finite list of characteristic zero Brauer relations, the corresponding regulator constants of M, N are equal,

iii) M, N have isomorphic Yakovlev diagrams.

This is stated precisely as Theorem 2.6.9. It is relatively straightforward to obtain extensions of this result to arbitrary lattices (cf. Remark 2.6.10). If the conditions of the theorem hold for all p-hypo-elementary subgroups of G, then they hold for G. We also provide some explicit criteria for the condition $A(\mathbb{Z}_p[G], \mathrm{perm}) = A(\mathbb{Z}_p[G], \mathrm{triv})$ to be satisfied. Groups that satisfy the conditions include dihedral groups, abelian groups with cyclic Sylow p-subgroups and groups of order coprime to $(p-1)$.

In most cases representation theory over $\mathbb{Z}$ is much more difficult than over $\mathbb{Z}_p$, or even over $\mathbb{Z}_p$ for all primes p. However, for some groups, such as dihedral groups D_{2p} of order $2p$ for primes $p \leq 67$, the isomorphism class of a $\mathbb{Z}[G]$-lattice M is determined by its localisations at the primes dividing $|G|$. As a result, Theorem 2.1.4 may be applied at each prime to give data which determines the isomorphism class of M as a $\mathbb{Z}[G]$-lattice (cf. Remark 2.7.1).

It is possible to define regulator constants $C_\theta(M)$ of a $\mathbb{Z}[G]$-lattice M. Then $C_\theta(M)$ is the product of the p-part of $C_\theta(M \otimes \mathbb{Z}_p)$ for all p dividing $|G|$ (see Remark 2.2.32). As a result, for the purposes of regulator constants, confining ourselves to $\mathbb{Z}_p[G]$-lattices over $\mathbb{Z}[G]$-lattices is innocuous.

Outline: In Section 2.2, we set out notation and recall necessary background results on Brauer relations and regulator constants. In Section 2.3, we outline precise questions on pairings arising from regulator constants. We also show that these reduce to considering p-hypo-elementary groups and that whenever the permutation pairing is non-degenerate, then permutation modules are determined by regulator constants and extension of scalars. In Section 2.4, we prove Theorem 2.1.2 on the permutation pairing, and in Section 2.5, we prove Theorem 2.1.3. In Section 2.6, we apply Theorem 2.1.2 to prove Theorem 2.1.4 on determining lattices up to isomorphism. There we also provide criteria for groups to satisfy the conditions of Theorem 2.1.4. Finally, in Section 2.7, we provide examples and non-examples illustrating our results.

After reading Sections 2.2 and 2.3, the following Sections 2.4 and 2.5 may be read completely separately from each other, as may Section 2.6, which only requires the statement of Theorem 2.1.2.

2.2 Preliminaries

2.2.1 Notation

Throughout, G shall denote a finite group, p a prime and $\mathcal{R}$ any ring, but we will be most concerned with $\mathcal{R} = \mathbb{F}_p, \mathbb{Z}_{(p)}, \mathbb{Z}_p, \mathbb{Q}$ or $\mathbb{Q}_p$. Here $\mathbb{Z}_{(p)}$ denotes the localisation of $\mathbb{Z}$ at the prime ideal (p).

Notation 2.2.1. We fix the following notation:

- Let $\mathbb{1}_G$ or $\mathbb{1}$ denote the trivial $\mathcal{R}[G]$-module. Where the choice of ring requires emphasis we write $\mathbb{1}_{\mathcal{R},G}$.
- Given a subgroup $H \leq G$ and an $\mathcal{R}[G]$-module M, we shall denote the restriction of M to H by $M{\downarrow}^G_H$ or $M{\downarrow}_H$. Similarly, given an $\mathcal{R}[H]$-module N, we write $N{\uparrow}^G_H$ or $N{\uparrow}^G$ for the induction of N to G.
- We say that an $\mathcal{R}[G]$-module is an $\mathcal{R}[G]$-*lattice* if it is free of finite rank as an $\mathcal{R}$-module. Let $a(\mathcal{R}[G])$ denote the *representation ring* of G. As an abelian group, $a(\mathcal{R}[G])$ consists of formal $\mathbb{Z}$-linear combinations of isomorphism classes of $\mathcal{R}[G]$-lattices, subject to relations of the form $[M] + [N] = [M \oplus N]$. Here we use $[M]$, or just M, to denote the element of $a(\mathcal{R}[G])$ corresponding to an $\mathcal{R}[G]$-lattice M. The ring structure on $a(\mathcal{R}[G])$ is given by setting $[M] \cdot [N] = [M \otimes_{\mathcal{R}} N]$. Induction defines a group homomorphism $\mathrm{ind}\colon a(\mathcal{R}[H]) \to a(\mathcal{R}[G])$, whilst restriction defines a ring homomorphism $\mathrm{res}\colon a(\mathcal{R}[G]) \to a(\mathcal{R}[H])$.
- Let $A(\mathcal{R}[G])$ denote $a(\mathcal{R}[G]) \otimes \mathbb{Q}$. All our main results do not require the integral structure. As a result we frequently deal only with $A(\mathcal{R}[G])$ even though some intermediate results also hold integrally.
- Recall that a permutation module is a finite direct sum of modules of the form $\mathbb{1}{\uparrow}^G_H$ as H runs over subgroups of G. We denote the subgroup of $a(\mathcal{R}[G])$ spanned by such modules by $a(\mathcal{R}[G], \mathrm{perm})$. The equality $\mathbb{1}{\uparrow}^G_H \otimes_{\mathcal{R}} \mathbb{1}{\uparrow}^G_K = \mathbb{1}_H{\uparrow}^G{\downarrow}_K{\uparrow}^G$ and Mackey's formula show that $a(\mathcal{R}[G], \mathrm{perm}) \subseteq a(\mathcal{R}[G])$ is a subring, which we call the *permutation ring*. Both $\mathrm{res}, \mathrm{ind}$ restrict to maps of permutation rings, the former due to Mackey's formula. We set $A(\mathcal{R}[G], \mathrm{perm}) = a(\mathcal{R}[G], \mathrm{perm}) \otimes \mathbb{Q}$.
- Let $A(\mathcal{R}[G], \mathrm{cyc})$ be the $\mathbb{Q}$-subalgebra spanned by $\mathbb{1}{\uparrow}^G_H$ as H runs only over cyclic subgroups.
- Given a quotient $q\colon G \to G/N$ and an $\mathcal{R}[G/N]$-module M, we denote by

$\inf_{G/N}^G(M)$ the inflation of M to G. This defines a ring homomorphism $\inf\colon a(\mathcal{R}[G/N]) \to a(\mathcal{R}[G])$, which restricts to a map of permutation rings since, for $H \leq G/N$, $\inf_{G/N}^G(\mathbb{1}_H\!\uparrow^{G/N}) = \mathbb{1}\!\uparrow_{q^{-1}(H)}^G$.

- In the same notation, given a G-module M, we define its deflation to G/N, $\mathrm{defl}_{G/N}^G M$, to be the fixed submodule M^N with G/N-action. Restricting to permutation modules, $(\mathbb{1}\!\uparrow_H^G)^N \cong \mathbb{1}\!\uparrow_{NH}^G$, so that $\mathrm{defl}(\mathbb{1}\!\uparrow_H^G) \cong \mathbb{1}\!\uparrow_{q(H)}^{G/N}$. The composite $\mathrm{defl} \circ \inf$ is the identity on all of $a(\mathcal{R}[G/N])$.
- By $H \leq_G G$, we denote a conjugacy class of subgroups of G with representative H. When used in indices, the symbol $\leq_G$ denotes indexing over conjugacy classes of subgroups. Thus, $\sum_{H \leq_G G} 1$ is equal to the number of conjugacy classes of subgroups.

Remark 2.2.2. Recall that a module is called *indecomposable* if it can not be written as a direct sum of proper submodules. When $\mathcal{R} = \mathbb{Z}_p$ or $\mathbb{Q}$, every $\mathcal{R}[G]$-lattice admits a unique decomposition into direct sums of indecomposables [Rei70, Thm. 5.2], so that $a(\mathcal{R}[G])$ is free as a $\mathbb{Z}$-module with a basis given by isomorphism classes of indecomposable modules, but it need not have finite rank if $\mathcal{R} = \mathbb{Z}_p$ and p divides $|G|$ (see [CR94, Sec. 33]). Unique decomposition also ensures that for any two $\mathbb{Z}_p[G]$-lattices M, M', $[M] = [M'] \iff M \cong M'$.

If $\mathcal{R} = \mathbb{Z}_{(p)}$, then extension of scalars defines an inclusion $a(\mathbb{Z}_{(p)}[G]) \hookrightarrow a(\mathbb{Z}_p[G])$ [Rei70, Thm. 5.6 *iii)*] so we may again detect a lattice's isomorphism class from its class in the representation ring. However, $\mathbb{Z}_{(p)}[G]$-lattices need not admit unique decomposition into indecomposables [Ben06] and in general there is no obvious basis of $a(\mathbb{Z}_{(p)}[G])$. For simplicity, we shall often write $a(\mathbb{Z}_{(p)}[G]) \subseteq a(\mathbb{Z}_p[G])$.

2.2.2 Brauer relations

Definition 2.2.3. A G-set is a set with a left action of G. We define the *Burnside ring* $b(G)$ of G to be the free abelian group on isomorphism classes of finite G-sets, quotiented by relations of the form $[X \coprod Y] - [X] - [Y]$ where X, Y are any G-sets, and $[X], [Y]$ are the corresponding elements of the free group. The ring structure is then given by setting $[X] \cdot [Y] = [X \times Y]$.

By decomposing G-sets into their orbits, we find that $b(G)$ is a free $\mathbb{Z}$-module on isomorphism classes of transitive G-sets. Every transitive G-set is of the form G/H for some subgroup $H \leq G$, where H is unique up to conjugacy. We denote the element of $b(G)$ corresponding to G/H by $[H]$. Thus, $b(G)$ is free as a $\mathbb{Z}$-module on the set of symbols $[H]$ for $H \leq_G G$.

We write $B(G)$ for $b(G) \otimes \mathbb{Q}$.

Construction 2.2.4. For any ring $\mathcal{R}$, a G-set X canonically defines a permutation module and we obtain a surjective map

$$b_{\mathcal{R}} \colon b(G) \to a(\mathcal{R}[G], \mathrm{perm}),$$

which sends $[H]$ to $\mathbb{1}_{\mathcal{R},H}\uparrow^G$.

Definition 2.2.5. A *Brauer relation*, of a group G over a ring $\mathcal{R}$, is an element of $\ker b_{\mathcal{R}} \subseteq b(G)$. We shall refer to the ideal $\ker b_{\mathcal{R}}$ as the space of Brauer relations over $\mathcal{R}$ and shall denote it by $br_{\mathcal{R}}(G)$.

When $\mathcal{R} = \mathbb{Q}$ or $\mathbb{F}_p$, we call a Brauer relation over $\mathcal{R}$ a relation in characteristic zero or characteristic p, respectively, and denote $br_{\mathcal{R}}(G)$ by $br_0(G), br_p(G)$ respectively. In the literature it is common to refer to a characteristic zero relation as simply a Brauer relation, and we shall often do the same.

Example 2.2.6. If $G = S_3$, a characteristic zero Brauer relation is given by

$$\theta \colon [1] + 2[G] - [C_3] - 2[C_2].$$

We shall see that in fact, $br_0(S_3) = \theta \cdot \mathbb{Z}$.

All characteristic p relations are relations in characteristic zero also:

Lemma 2.2.7. *As subspaces of* $b(G)$, $br_p(G) = br_{\mathbb{Z}_{(p)}}(G) = br_{\mathbb{Z}_p}(G) \subseteq br_0(G)$.

Proof. Via the factorisation $\mathbb{Z}_{(p)} \to \mathbb{Z}_p \to \mathbb{F}_p$, the map $b_{\mathbb{F}_p}$ factors as

$$b(G) \to a(\mathbb{Z}_{(p)}[G], \mathrm{perm}) \overset{M \mapsto M \otimes \mathbb{Z}_p}{\to} a(\mathbb{Z}_p[G], \mathrm{perm}) \overset{M \mapsto M \otimes \mathbb{F}_p}{\to} a(\mathbb{F}_p[G], \mathrm{perm}).$$

The middle map is an isomorphism by [Rei70, Thm 5.6 iii)], as is the last map by [Ben98, 3.11.4 i)], so the kernels of $b_{\mathbb{F}_p}, b_{\mathbb{Z}_{(p)}}$ and $b_{\mathbb{Z}_p}$ agree. As $b_{\mathbb{Q}}$ factors through $b_{\mathbb{Z}_{(p)}}$, there is an inclusion $br_{\mathbb{Z}_{(p)}}(G) \subseteq br_{\mathbb{Q}}(G)$. ☐

Notation 2.2.8. Let G be a finite group and $H \leq G$ a subgroup.

- Given an H-set X, we let $X\uparrow_H^G$ denote the induced G-set $(G \times X)/H$; here the H-equivalence is by acting on G on the right and X on the left, whilst G

acts on the resulting set via its left action on G. For transitive G-sets (G/K) we have $(H/K)\uparrow^G = (G/K)$ and we shall regularly abuse notation by writing $[K]\uparrow^G$ simply as $[K]$, where now K is thought of as a subgroup of G.

- If Y is a G-set, we let $Y\downarrow^G_H$ denote its restriction to H. For a subgroup K of G, making good use of the above abuse of notation, Mackey's formula for G-sets states that

$$[K]\downarrow^G_H = \sum_{g\in K\backslash G/H} [K^g \cap H]. \tag{2.1}$$

If now $N \trianglelefteq G$ is a normal subgroup with quotient $q\colon G \to G/N$, then

- given a G/N-set X, we denote by $\inf^G_{G/N} X$ the inflated set X, on which elements of G act via their image in the quotient. For $H \leq G/N$, $\inf^G_{G/N}([H]) = [q^{-1}(H)]$,
- given a G-set Y, let $\mathrm{defl}^G_{G/N} Y$ denote its deflation, i.e. the set Y^N with its action of G/N. For a transitive G-set G/H, the fixed points under N is isomorphic to G/NH, which as a G/N-set is $(G/N)/q(H)$. In other words, $\mathrm{defl}^G_{G/N}([H]) = [q(H)]$, and thus $\mathrm{defl} \circ \inf$ is the identity map.

All of these operations induce group homomorphisms on Burnside rings, but only restriction and inflation will in general be ring homomorphisms. Each of ind, res, inf, defl commute with $b_{\mathcal{R}}$. As a result, each restricts to morphisms of $br_{\mathcal{R}}(-)$.

2.2.3 Relations in characteristic zero

Finding an explicit basis, for an arbitrary finite group G, of the $br_0(G)$ is a hard problem, which was recently completed by Bartel-Dokchitser [BD15, BD14]. On the other hand, in this section we recall that, a basis of the space $br_0(G) \otimes \mathbb{Q}$ is provided by Artin's induction theorem.

Notation 2.2.9. Let $\mathrm{cyc}(G) := \{H \leq_G G \mid H\text{–cyclic}\}$ denote a set of representatives of each conjugacy class of cyclic subgroups.

Theorem 2.2.10 (Artin's induction theorem [Sna94, Thm. 2.1.3])**.** *For any finite group G and $\mathbb{Q}[G]$-module M, there exists a unique $\alpha_H \in \mathbb{Q}$ for each cyclic $H \leq_G G$ such that*

$$M = \sum_{H\in \mathrm{cyc}(G)} \alpha_H \cdot \mathbb{1}_H\uparrow^G \in A(\mathbb{Q}[G]).$$

Definition 2.2.11. We say that an element $\theta \in B(G)$ is supported at some set S of conjugacy classes of subgroups of G if the only $[H]$ with non-zero coefficients lie in S.

Corollary 2.2.12. *For any finite group G,*

i) the rank of $br_0(G)$ is equal to the number of conjugacy classes of non-cyclic subgroups of G,

ii) there are no non-zero characteristic zero Brauer relations supported only at cyclic subgroups.

Proof. Immediate. □

Note that a group G is cyclic if and only if it has no non-trivial Brauer relations.

Definition 2.2.13. For any ring $\mathcal{R}$ and finite group G, let $b_{\mathcal{R},\mathbb{Q}}$ denote the base change of $b_{\mathcal{R}}$,

$$b_{\mathcal{R},\mathbb{Q}} \colon B(G) \to A(\mathcal{R}[G], \mathrm{perm}).$$

We shall also call an element of the kernel of $b_{\mathcal{R},\mathbb{Q}}$ a Brauer relation over $\mathcal{R}$ and refer to the kernel $BR_{\mathcal{R}}(G) := br_{\mathcal{R}}(G) \otimes \mathbb{Q}$ as the space of Brauer relations over $\mathcal{R}$. Where there is ambiguity, we shall refer to elements of the kernel of $b_{\mathcal{R}}$ as *integral Brauer relations* and of $b_{\mathcal{R},\mathbb{Q}}$ as *rational Brauer relations.*

Induction theorems of the form of Theorem 2.2.10 always give rise to a corresponding family of (possibly rational) Brauer relations.

Definition 2.2.14. For any group G, let

$$\mathbb{1}_G = \sum_{H \in \mathrm{cyc}(G)} \alpha_H \cdot \mathbb{1}_H \uparrow^G,$$

where the $\alpha_H \in \mathbb{Q}$ are given uniquely by Artin's induction theorem. Then

$$\theta_G = [G] - \sum_{H \in \mathrm{cyc}(G)} \alpha_H \cdot [H] \in B(G)$$

is a rational Brauer relation of G. We call θ_G the *Artin relation* of G. Note that if G is cyclic, then $\theta_G = 0 \in B(G)$, otherwise θ_G is non-zero and has $[G]$-coefficient 1. The uniqueness statement of Artin's induction theorem shows that θ_G is, up to scaling, the unique element of $BR_0(G)$ supported only at G and cyclic subgroups.

The following example will be returned to in Section 2.4.2.

Example 2.2.15. Let G be of the form $C_{p^r} \rtimes C_n$, with $p \nmid n$, and denote by $S \leq C_n$ the kernel of the action $C_n \to \mathrm{Aut}(C_{p^r})$. Writing s for $|S|$, we claim that

$$\theta_G = \frac{s}{n} \cdot [S] - [C_n] - \frac{s}{n} \cdot [C_{p^r} \times S] + [C_{p^r} \rtimes C_n],$$

which can be checked by direct calculation. If the action of C_n is not faithful, then S is a non-trivial subgroup of G and quotienting by S results in a group of the same form but with faithful action. The Artin relation of G is then the inflation of the Artin relation of G/S (using that the preimage of a cyclic subgroup of G/S is a cyclic subgroup of G).

Following Notation 2.2.8, when it is contextually clear we are referring to G-relations, for a subgroup $H \leq G$, we shall denote the G relation $\theta_H\uparrow^G$ simply by θ_H. Artin relations are well behaved under restriction:

Lemma 2.2.16. *Let G be a finite group and H, K subgroups. Then*

i) the restriction of the Artin relation of G to H is the Artin relation of H, i.e.

$$\theta_G\downarrow_H = \theta_H,$$

ii) more generally

$$\theta_H\uparrow_H^G\downarrow_K = \sum_{g \in H\backslash G/K} \theta_{H^g \cap K}\uparrow^K. \tag{2.2}$$

Proof. We prove *ii)*. Mackey's formula (2.1) states that

$$[H]\downarrow_K = \sum_{g \in H\backslash G/K} [H^g \cap K].$$

Also by Mackey, for any cyclic group $L \leq H$, $[L]\uparrow_H^G\downarrow_K$ is supported at cyclic subgroups. But then $\theta\uparrow_H^G\downarrow_K$ and $\sum_{g \in H\backslash G/K} \theta_{H^g \cap K}\uparrow^K$ are two relations whose coefficients agree at all non-cyclic subgroups and since there are no relations supported at cyclic subgroups (Corollary 2.2.12 *ii)*), they must therefore be equal. □

Lemma 2.2.17. *A basis of the space of rational Brauer relations $BR_0(G)_{\mathbb{Q}}$ is given by the set $\{\theta_H\}$ of Artin relations for non-cyclic $H \leq_G G$.*

Proof. The θ_H are linearly independent as each is zero on non-cyclic subgroups other than $[H]$ and must span by Corollary 2.2.12 *i)*. □

2.2.4 Relations in characteristic p

Definition 2.2.18. Let p be prime. A finite group G is called *p-hypo-elementary*, or simply p-hypo, if G has a normal Sylow p-subgroup P and G/P is cyclic, i.e. if G can be written in the form $P \rtimes C_n$ for P a p-group and $(p,n) = 1$.

Notation 2.2.19. We denote a set $\{H \leq_G G \mid H\ p\text{-hypo}\}$ of representatives of the conjugacy classes of p-hypo-elementary subgroups by $\mathrm{hyp}_p(G)$. Similarly, let $\mathrm{nchyp}_p(G) := \{H \leq_G G \mid H \text{ is } p\text{-hypo and non-cyclic}\}$.

Recall that characteristic p relations coincide with $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p$-relations (Lemma 2.2.7). Analogously to Corollary 2.2.12 we have:

Theorem 2.2.20. *For any finite group G,*

i) a basis of $A(\mathbb{Z}_p[G], \mathrm{perm})$ is given by $\{\mathbb{1}\uparrow_H^G\}_{H \in \mathrm{hyp}_p(G)}$,

ii) the rank of $br_p(G)$ is equal to the number of conjugacy classes of non-p-hypo-elementary subgroups of G,

iii) there are no non-zero characteristic p Brauer relations supported only at p-hypo-elementary subgroups.

Proof. The first statement is a consequence of Conlon's induction theorem as we show later in Theorem 2.6.21. Given *i)*, both *ii)* and *iii)* are automatic. □

The first part also holds with $\mathbb{Z}_p$ replaced by $\mathbb{F}_p$ or $\mathbb{Z}_{(p)}$.

Note that there are no non-zero characteristic p Brauer relations for p-hypo-elementary groups, and this is only true of such groups. As before, the theorem gives rise to privileged relations in characteristic p:

Definition 2.2.21. For any group G and prime p, write

$$\mathbb{1}_{\mathbb{Z}_p, G} = \sum_{H \in \mathrm{hyp}_p(G)} \alpha_H \cdot \mathbb{1}_{\mathbb{Z}_p, H}\uparrow_H^G$$

with $\alpha_H \in \mathbb{Q}$ uniquely by Theorem 2.2.20. Since $BR_{\mathbb{Z}_p}(G) = BR_p(G)$ (Lemma 2.2.7),

$$\theta_{\mathrm{Con},G} = [G] - \sum_{H \in \mathrm{hyp}_p(G)} \alpha_H \cdot [H]$$

is a rational Brauer relation in characteristic p, which we refer to as the *Conlon relation* of G. Note, $\theta_{\mathrm{Con},G}$ is identically zero if and only if G is p-hypo-elementary. The

Conlon relation is the unique p-relation supported only at G and p-hypo-elementary subgroups. However, the Conlon relation need not be unique amongst characteristic zero relations supported at these groups.

As before, when it is clear that we are referring to G-relations, for a subgroup $H \leq G$, we denote $\theta_{\mathrm{Con},H}\uparrow^G$ simply by $\theta_{\mathrm{Con},H}$. All characteristic p relations are rational linear combinations of Conlon relations:

Lemma 2.2.22. *Let G be a finite group and p a prime. Then*

i) a basis of $BR_p(G)$ is formed by the set $\{\theta_{Con,H}\}$ as H runs over conjugacy classes of non-p-hypo-elementary groups,

ii) this can be extended to a basis of $BR_0(G)$ by adding the Artin relations θ_H as H runs over conjugacy classes of non-cyclic p-hypo-elementary groups.

Proof. The proof of *i)* is as in Lemma 2.2.17. For *ii)*, in addition use Corollary 2.2.12, Theorem 2.2.20. □

Unlike in the characteristic zero case, a full classification of integral Brauer relations in characteristic p is unknown. However, there has been significant recent progress, see [BS17].

Example 2.2.23. Let $G = D_{2p} = C_p \rtimes C_2$ be the dihedral group of order $2p$ for p an odd prime. If ℓ is any prime, then

$$\{H \leq_G G \mid H \text{ is } \ell\text{-hypo-elementary}\} = \begin{cases} \{1\}, C_2, C_p & \ell \neq p \\ \{1\}, C_2, C_p, D_{2p} & \ell = p \end{cases}, \tag{2.3}$$

and so $\dim_{\mathbb{Q}} A(\mathbb{Z}_\ell[G], \mathrm{perm})$ is 3 unless $\ell = p$ when it is 4. A basis S of $A(\mathbb{Z}_p[G], \mathrm{perm})$ is formed by

$$S = \begin{cases} \mathbb{1}_{\{1\}}\uparrow^G, \mathbb{1}_{C_2}\uparrow^G, \mathbb{1}_{C_p}\uparrow^G & \ell \neq p \\ \mathbb{1}_{\{1\}}\uparrow^G, \mathbb{1}_{C_2}\uparrow^G, \mathbb{1}_{C_p}\uparrow^G, \mathbb{1}_G & \ell = p \end{cases}.$$

Since G has up to conjugacy four subgroups, of which three are cyclic, $\mathrm{rk}\, BR_0(G) = 1$. Let $\theta \in br_0(G)$ be the relation

$$2[G] - [C_p] - 2[C_2] + [1].$$

Then $\theta = 2\theta_G$. As θ is indivisible as an element of $b(G)$, we find $br_0(G) = \theta \cdot \mathbb{Z}$.

Since rk $br_\ell(G)$ is the number of conjugacy classes of non-p-hypo-elementary subgroups, by (2.3), rk $br_\ell(G)$ is also one unless $\ell = p$ when it is zero. Given that $br_\ell(G) \subseteq br_0(G)$ (Lemma 2.2.7), we find

$$br_\ell(G) = \begin{cases} \theta \cdot \mathbb{Z} & \text{if } \ell \neq p \\ 0 & \text{if } \ell = p \end{cases}.$$

The Conlon relation $\theta_{\mathrm{Con},G}$ is equal to the Artin relation unless $\ell = p$ when it is zero.

We conclude by giving a useful characterisation of Brauer relations.

Lemma 2.2.24. *Let G be a finite group and $\sum_i [H_i] - \sum_j [K_j] \in b(G)$. Then,*

i) $\sum_i [H_i] - \sum_j [K_j]$ is equal to zero in $b(G)$ if and only if, for all subgroups $T \leq G$, the number of fixed points of $\coprod_i (G/H_i)$ and $\coprod_j (G/K_j)$ under T are equal,

ii) $\sum_i [H_i] - \sum_j [K_j]$ is Brauer relation in characteristic zero if and only if, for all cyclic subgroups $T \leq G$, the number of fixed points of $\coprod_i (G/H_i)$ and $\coprod_j (G/K_j)$ under T are equal,

iii) $\sum_i [H_i] - \sum_j [K_j]$ is Brauer relation in characteristic p if and only if, for all p-hypo-elementary subgroups $T \leq G$, the number of fixed points of $\coprod_i (G/H_i)$ and $\coprod_j (G/K_j)$ under T are equal.

Proof. For *i)*, note that the fixed points of $\sum_i [H_i]$ and $\sum_j [K_j]$ under G are equal if and only if $[G]$ occurs an equal number of times on both sides. Similarly, by sequentially considering subgroups ordered by decreasing size, we find $\sum_i [H_i] = \sum_j [K_j]$ if and only if the fixed points under all subgroups are equal (cf. (2.1) p13).

For *iii)* consider the commutative diagram

$$\begin{array}{ccc} b(G) & \xrightarrow{b_{\mathbb{Z}_p}} & a(\mathbb{Z}_p[G], \text{perm}) \\ \downarrow & & \downarrow \\ \prod\limits_{H\ p\text{-hypo}} b(H) & \xrightarrow[\prod b_{\mathbb{Z}_p}]{} & \prod\limits_{H\ p\text{-hypo}} a(\mathbb{Z}_p[H], \text{perm}) \end{array}$$

where the horizontal arrows send G-sets to their permutation representations and the vertical maps are given by restriction. The lower arrow is injective as p-hypo-elementary groups admit no non-zero characteristic p Brauer relations (Theorem 2.2.20 *iii)*).

Claim. *The map* $a(\mathbb{Z}_p[G], \mathrm{perm}) \to \prod_{H \text{ } p\text{-hypo}} a(\mathbb{Z}_p[H], \mathrm{perm})$ *is injective.*

Proof of Claim. Write an arbitrary $\mathbb{Z}_p[G]$-permutation module as

$$M = \sum_{H \in \mathrm{hyp}_p(G)} \alpha_H \cdot \mathbb{1}\uparrow_H^G,$$

uniquely by Theorem 2.2.20 *i)*. Upon restriction to a p-hypo-elementary subgroup $K \leq G$, Mackey's formula shows that $\mathbb{1}\uparrow_H^G\downarrow_K$ contains a summand isomorphic to $\mathbb{1}$ if and only if K is contained in a conjugate of H. As a result, if K is a maximal p-hypo-elementary subgroup, then α_K can be read off from $M\downarrow_K$. More generally, we sequentially restrict to p-hypo-elementary subgroups of decreasing order, then M can be recovered from its restrictions. □

Brauer relations in characteristic p are precisely elements of $\ker(b_p \colon b(G) \to a(\mathbb{Z}_p[G], \mathrm{perm}))$ (Lemma 2.2.7) and thus are the elements of $b(G)$ which lie in the kernel of restriction $b(G) \to \prod_{H \in \mathrm{hyp}_p(G)} b(H)$. By *i)* these are the elements of $b(G)$ whose fixed points under all p-hypo-elementary subgroups cancel.

The proof of *ii)* is identical instead using Artin's induction theorem 2.2.10. □

The same line of reasoning yields the following corollary of Artin's induction theorem, which is useful in practice (e.g. in Chapter 3). Normally, this would be shown in the course of the proof of Artin's induction theorem.

Lemma 2.2.25. *Given two* $\mathbb{Q}[G]$*-representations* M, N*, we have that* $M \cong N$ *if and only if* $\dim_{\mathbb{Q}} M^H = \dim_{\mathbb{Q}} N^H$ *as* H *runs over cyclic subgroups of* G *up to conjugacy.*

Proof. In the course of the above proof we verified that

$$a(\mathbb{Q}[G]) \to \prod_{H \in \mathrm{cyc}(G)} a(\mathbb{Q}[H])$$

is injective (note that Artin's induction theorem states that $A(\mathbb{Q}[W], \mathrm{perm}) = A(\mathbb{Q}[W])$ in general). To conclude, we must show that the result holds for any cyclic group H. A basis of $A(\mathbb{Q}[H])$ consists of $\mathbb{1}\uparrow_C^H$ as C runs over all subgroups. Now for any $C, C' \leq H$, $\dim(\mathbb{1}_C\uparrow^H)^{C'} = |H|/\operatorname{lcm}(|C|, |C'|)$ by Mackey's formula. In particular, the functions $(-)^{C'}$ are linearly independent on $A(\mathbb{Q}[H])$ as C' ranges over all subgroups of H. Since the dimension of $A(\mathbb{Q}[H])$ is equal to the number of subgroups, any $\mathbb{Q}[H]$-representation is determined by its fixed points. □

2.2.5 Regulator constants

In this section, we recall how to associate to a characteristic zero Brauer relation, a function on (nice) $\mathcal{R}[G]$-lattices called its regulator constant.

We follow the construction given in [DD09] for an arbitrary PID $\mathcal{R}$ of characteristic zero. We are only ever concerned with $\mathcal{R} = \mathbb{Z}, \mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$. Let $\mathcal{K}$ denote the field of fractions of $\mathcal{R}$.

Definition 2.2.26. An $\mathcal{R}[G]$-lattice M is called *rationally self-dual* if $M \otimes \mathcal{K}$ is self-dual, i.e. $M \otimes \mathcal{K}$ is isomorphic to its linear dual $\mathrm{Hom}_{\mathcal{K}}(M \otimes \mathcal{K}, \mathcal{K})$ as $\mathcal{K}[G]$-modules. This is equivalent to the existence of a non-degenerate G-invariant inner product on $M \otimes \mathcal{K}$. If an inner product on $M \otimes \mathcal{K}$ exists, there is a restricted G-invariant inner product on M.

Rational self-duality is preserved under induction, restriction, inflation and deflation, as well as taking tensor products of two rationally self-dual lattices. We say that an element of $A(\mathcal{R}[G])$ is *rationally self-dual* if it can be written as a linear combination of self-dual lattices. We denote the subring of self-dual lattices by $a(\mathcal{R}[G], \mathrm{sd})$ and define $A(\mathcal{R}[G], \mathrm{sd})$ accordingly.

A rationally self-dual lattice M need not be linearly self-dual, i.e. a rationally self-dual M need not be isomorphic to $\mathrm{Hom}(M, \mathcal{R})$. If $\mathcal{R} = \mathbb{Z}$ or $\mathbb{Z}_{(p)}$, then, as all $\mathbb{Q}[G]$-modules are self-dual, all $\mathcal{R}[G]$-lattices are rationally self-dual.

Definition 2.2.27. Let G be a finite group and $\theta = \sum_i [H_i] - \sum_j [H'_j] \in br_0(G)$ be an integral characteristic zero Brauer relation of G. Given a rationally self-dual $\mathcal{R}[G]$-lattice M, fix a choice of non-degenerate G-invariant inner product $\langle\ ,\ \rangle$ on M. The *regulator constant* of θ evaluated at M is then

$$C_\theta(M) = \frac{\prod_i \det\left(\frac{1}{|H_i|}\langle\ ,\ \rangle|_{M^{H_i}}\right)}{\prod_j \det\left(\frac{1}{|H'_j|}\langle\ ,\ \rangle|_{M^{H'_j}}\right)} \in \mathcal{K}^\times/(\mathcal{R}^\times)^2.$$

This is independent of the choice of $\langle\ ,\ \rangle$ as an element of $\mathcal{K}^\times/(\mathcal{R}^\times)^2$ (see [DD09, Thm. 2.17]). For M a $\mathbb{Z}[G]$ or $\mathbb{Z}_{(p)}[G]$-lattice, we may take the pairing to be positive definite and so for all characteristic zero Brauer relations θ and modules M we have $C_\theta(M) > 0$.

Example 2.2.28. Let $G = S_3 = \langle \sigma, \tau \mid \sigma^3, \tau^2, \sigma\tau = \tau\sigma^2 \rangle$ and consider the Brauer relation $\theta = 2[G] - [C_3] - 2[C_2] + [1]$ of Example 2.2.23. There are two non-isomorphic rank 2 $\mathbb{Z}_{(3)}[G]$-lattices given by extending the $\mathbb{Z}[G]$-lattices defined by the following diagrams:

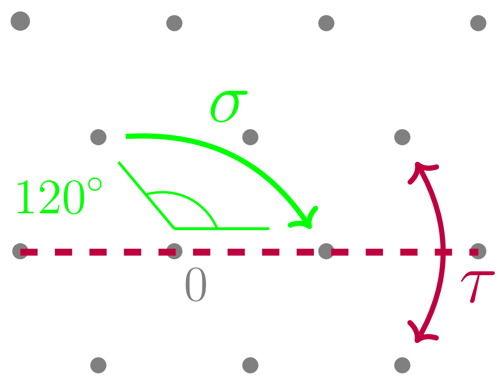


Figure 2.1: The lattice A.

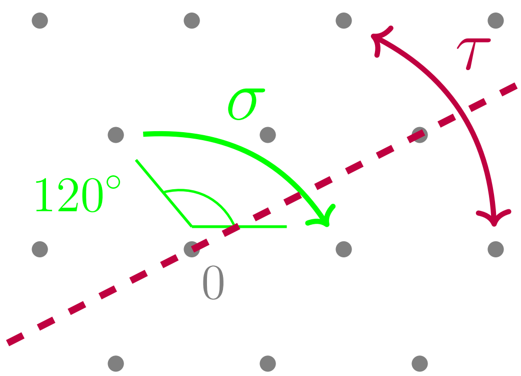
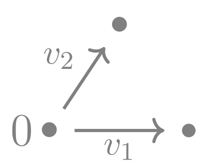


Figure 2.2: The lattice A'.

In both diagrams, σ acts by rotation by $120°$ and τ by reflection.

Note that $A \otimes \mathbb{Q} \cong A' \otimes \mathbb{Q}$. With respect to the basis



a G-invariant inner product $\langle -, - \rangle$ on both $A \otimes \mathbb{Q}$ and $A' \otimes \mathbb{Q}$ is represented by

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Directly applying Definition 2.2.27 we find

$$\begin{aligned}
C_\theta(A) &= \frac{\det\left(\frac{1}{6}\langle\ ,\ \rangle|_{A^G}\right)^2 \cdot \det\left(\frac{1}{1}\langle\ ,\ \rangle|_{A^1}\right)}{\det\left(\frac{1}{3}\langle\ ,\ \rangle|_{A^{C_3}}\right) \cdot \det\left(\frac{1}{2}\langle\ ,\ \rangle|_{A^{C_2}}\right)^2} \\
&= \frac{1^2 \cdot 3}{1^2 \cdot (\frac{2}{2})^2} \\
&= 3 \in \mathbb{Z}_{(3)}^\times/((\mathbb{Z}_{(3)})^\times)^2,
\end{aligned}$$

whilst

$$\begin{aligned}
C_\theta(A') &= \frac{\det\left(\frac{1}{6}\langle\ ,\ \rangle|_{A'^G}\right)^2 \cdot \det\left(\frac{1}{1}\langle\ ,\ \rangle|_{A'^1}\right)}{\det\left(\frac{1}{3}\langle\ ,\ \rangle|_{A'^{C_3}}\right) \cdot \det\left(\frac{1}{2}\langle\ ,\ \rangle|_{A'^{C_2}}\right)^2} \\
&= \frac{1^2 \cdot 3}{1^2 \cdot (\frac{6}{2})^2} \\
&= \frac{1}{3} \in \mathbb{Z}_{(3)}^\times/((\mathbb{Z}_{(3)})^\times)^2.
\end{aligned}$$

Since $3 \not\equiv \frac{1}{3} \in ((\mathbb{Z}_{(3)})^\times)^2$, this demonstrates that $A \not\cong A'$. On the other hand, $3 \equiv \frac{1}{3} \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$. As regulator constants are compatible with extension of scalars (see Lemma 2.2.30 *vi)*), this is a consequence of the fact that $A \otimes \mathbb{Q} \cong A' \otimes \mathbb{Q}$.

Remark 2.2.29. When evaluating regulator constants at the trivial module, the formula simplifies. For example, if $\theta = \sum_i [H_i] - \sum_j [H'_j]$, then

$$C_\theta(\mathbb{1}_G) = \frac{\prod_i \frac{1}{|H_i|}}{\prod_j \frac{1}{|H'_j|}} = \frac{\prod_j |H'_j|}{\prod_i |H_i|}. \tag{2.4}$$

This formula can be extended to permutation modules due to the formalism of regulator constants provided by the next lemma. That regulator constants of permutation modules can be made explicit in this way is crucial in the proof of Theorem 2.4.1.

Lemma 2.2.30 ([DD09]). *Let G be any finite group and H a subgroup and let $\mathcal{R}, \mathcal{K}$ be as above. Assume throughout that all modules are rationally self-dual. Then*

i) if M, N are two $\mathcal{R}[G]$-lattices, then for any Brauer relation θ of G, $C_\theta(M \oplus N) = C_\theta(M)C_\theta(N)$,

ii) if θ, θ' are two Brauer relations for G and M any $\mathcal{R}[G]$-lattice, then $C_{(\theta+\theta')}(M) = C_\theta(M)C_{\theta'}(M)$,

iii) if M is a $\mathcal{R}[G]$-lattice, then $C_{\theta\uparrow_H^G}(M) = C_\theta(M\downarrow_H^G)$, *iv) if M is a $\mathcal{R}[H]$-lattice, then $C_\theta(M\!\uparrow_H^G) = C_{\theta\downarrow_H^G}(M)$,*

v) if H is normal, then given a relation of G/H and a $\mathcal{R}[G]$-lattice M, $C_{\mathrm{inf}_{G/H}^G\theta}(M) = C_\theta(\mathrm{defl}_{G/H}^G M)$,

vi) for any inclusion $\mathcal{R} \hookrightarrow \mathcal{T}$, with $\mathcal{T}$ a PID, any relation θ, and $\mathcal{R}[G]$-lattice M, we have $C_\theta(M) = C_\theta(M \otimes \mathcal{T}) \in \mathcal{L}^\times/(\mathcal{T}^\times)^2$ where $\mathcal{L}$ denotes the field of fractions of $\mathcal{T}$.

Notation 2.2.31. By definition, regulator constants of rationally self-dual $\mathbb{Z}_p[G]$-modules take values in $\mathbb{Q}_p^\times/(\mathbb{Z}_p^\times)^2$. Let $v_p\colon \mathbb{Q}_p^\times \to \mathbb{Z}$ denote the usual p-adic valuation. This descends to a function $v_p\colon \mathbb{Q}_p^\times/(\mathbb{Z}_p^\times)^2 \to \mathbb{Z}$. For any prime $\ell \neq p$, we also have a "valuation at ℓ" function $\mathbb{Q}_p^\times/(\mathbb{Z}_p^\times)^2 \to \mathbb{Z}/2\mathbb{Z}$, which we also denote by v_ℓ.

Remark 2.2.32. Since the regulator constant of a $\mathbb{Z}[G]$-lattice is always a positive rational number (see Definition 2.2.27), Lemma 2.2.30 *vi)* shows that the regulator constant $C_\theta(M)$ of a $\mathbb{Z}[G]$-lattice M is a function of the values $v_p(C_\theta(M \otimes \mathbb{Z}_p))$ as p runs over all primes.

The following observation will be crucial:

Lemma 2.2.33 ([Bar12, Lem. 3.6])**.** *If G is a finite group and θ a relation in characteristic ℓ, then for any prime p (possibly equal to ℓ) and M any $\mathbb{Z}[G]$ or rationally self-dual $\mathbb{Z}_p[G]$-lattice we have*

$$v_\ell(C_\theta(M)) = 0.$$

Remark 2.2.34. If G is a finite group and p is a prime not dividing the order of G, then the p-hypo-elementary subgroups of G are the cyclic subgroups and so all characteristic zero relations are characteristic p relations (Lemma 2.2.22). Thus Lemma 2.2.33 shows that the only prime powers appearing in regulator constants divide the order of the group. If G itself is ℓ-hypo-elementary, then, for $p \neq \ell$, all its p-hypo-elementary subgroups are cyclic and so all its characteristic 0 relations are relations in characteristic p and its regulator constants are always ℓ^{th} powers.

2.3 Pairings from regulator constants

In this section, we remark that the construction of regulator constants canonically defines a pairing between Brauer relations and rationally self-dual $\mathbb{Z}_p[G]$-lattices. This pairing has obvious Brauer relations and lattices which must lie in the kernel, but it is unclear what the kernels should be in general. Sections 2.4 and 2.5 can be seen as partial results in this direction. Finally, we show that non-degeneracy of such pairings leads to methods to determine the isomorphism classes of permutation modules.

2.3.1 The regulator constant pairing

Construction 2.3.1. Let G be any finite group and p a prime. The map

$$\begin{aligned} v_p(C_{(-)}(-))\colon br_0(G) \times a(\mathbb{Z}_p[G], \mathrm{sd}) &\longrightarrow \mathbb{Z} \\ (\theta, M) \qquad\qquad &\longmapsto v_p(C_\theta(M)), \end{aligned}$$

is bi-additive (Lemma 2.2.30 *i),ii)*). Extending $\mathbb{Q}$-linearly we get a pairing,

$$v_p(C_{(-)}(-))\colon BR_0(G) \times A(\mathbb{Z}_p[G], \mathrm{sd}) \longrightarrow \mathbb{Q},$$

which we also denote by $v_p(C_{(-)}(-))$ and which we call the *regulator constant pairing*. By Lemma 2.2.33, this factors as

$$v_p(C_{(-)}(-))\colon BR_0(G)/BR_p(G) \times A(\mathbb{Z}_p[G], \mathrm{sd}) \longrightarrow \mathbb{Q}.$$

In Section 2.7.1, we calculate the full regulator constant pairing for dihedral groups D_{2p} with p odd, one of the few families of groups where a classification of all indecomposable lattices exists.

Remark 2.3.2. The pairing $v_p(C_{(-)}(-))$ is far from non-degenerate; $A(\mathbb{Z}_p[G], \mathrm{sd})$ is often infinite dimensional whilst $BR_0(G)$ is always finite dimensional. Explicit elements of the right kernel are given by taking a lattice $M{\uparrow}_H^G$ induced from a cyclic subgroup H. This pairs to zero with all relations since

$$v_p(C_\theta(M{\uparrow}_H^G)) = v_p(C_{\theta\downarrow_H^G}(M)) = 0,$$

where first equality is Lemma 2.2.30 *iv)* and the second is because cyclic groups have no non-zero Brauer relations (Corollary 2.2.12). The behaviour of the left kernel is less clear:

Question 2.3.3. Are there groups for which the left kernel of

$$v_p(C_{(-)}(-))\colon BR_0(G)/BR_p(G) \times A(\mathbb{Z}_p[G], \mathrm{sd}) \to \mathbb{Q}$$

is non-trivial?

It is also interesting to replace $\mathbb{Z}_p$ with other rings. For $\mathbb{Z}_{(p)}$ and so also for $\mathbb{Z}_p$, if G has cyclic Sylow p subgroups, then the left kernel is trivial (see Theorem 2.4.1). Outside of this case, things are less clear and there are some very small groups (e.g. $C_3 \times C_3 \times S_3$ when $p = 3$) for which we have been unable to determine the left kernel. There are groups with Brauer relations which pair trivially with all summands of permutation modules but are not relations in characteristic p (namely for $C_3 \times C_3 \times S_3$, see Section 2.7.3). On the other hand, the kernel is never all of $BR_0(G)$ (Theorem 2.5.1).

Lemma 2.3.4. *Let θ be a relation of a finite group G. For any prime p the following are equivalent,*

i) $v_p(C_\theta(-))\colon A(\mathbb{Z}_p[G], sd) \to \mathbb{Q}$ vanishes identically,

ii) $v_p(C_{\theta\downarrow_H}(-))\colon A(\mathbb{Z}_p[H], sd) \to \mathbb{Q}$ vanishes identically for all conjugacy classes of p-hypo-elementary subgroups $H \leq G$.

Proof. For the forward direction, use that $v_p(C_{\theta\downarrow_H}(M)) = v_p(C_\theta(M\uparrow^G)) = 0$ (Lemma 2.2.30 *iv)*). For the reverse, write $\mathbb{1} = \sum_{H\in \mathrm{hyp}_p(G)} \alpha_H \mathbb{1}_H\uparrow^G$ as in the Conlon relation. Then for any rationally self-dual $\mathbb{Z}_p[G]$-lattice

$$M = M \otimes \mathbb{1} = \sum_{H\in \mathrm{hyp}_p(G)} \alpha_H \cdot (M \otimes \mathbb{1}\uparrow_H^G) = \sum_{H\in \mathrm{hyp}_p(G)} \alpha_H \cdot M\downarrow_H\uparrow_H^G,$$

where the $M\downarrow_H$ are rationally self-dual. Then

$$v_p(C_\theta(M)) = \sum_{H\in \mathrm{hyp}_p(G)} \alpha_H v_p(C_\theta(M\downarrow_H\uparrow_H^G)) = \sum_{H\in \mathrm{hyp}_p(G)} \alpha_H v_p(C_{\theta\downarrow_H}(M\downarrow_H)) = 0.$$

□

Lemma 2.3.5. *For any finite group G, the following are equivalent:*

i) the left kernel of $v_p(C_{(-)}(-))\colon BR_0(G)/BR_p(G) \times A(\mathbb{Z}_p[G], sd) \to \mathbb{Q}$ is trivial,

ii) the left kernel of $v_p(C_{(-)}(-))\colon BR_0(G/N)/BR_p(G/N) \times A(\mathbb{Z}_p[G/N], sd) \to \mathbb{Q}$ is trivial for all normal subgroups N.

Moreover, both are implied by

iii) the left kernels of $v_p(C_{(-)}(-))\colon BR_0(H) \times A(\mathbb{Z}_p[H], sd) \to \mathbb{Q}$ are trivial for all isomorphism classes of p-hypo-elementary subgroups $H \leq G$.

Proof. To see *i)* $\implies$ *ii)*, suppose that the left kernel of the pairing for G is trivial and let θ be a relation for G/N which is not a relation in characteristic p. Since $\mathrm{defl} \circ \inf = \mathrm{id}$ and both take characteristic p relations to characteristic p relations, $\inf\theta$ must also not be a p-relation. So, by assumption, there exists an M for which $0 \neq v_p(C_{\inf\theta}(M))$. By Lemma 2.2.30 *v)*, $v_p(C_{\inf\theta}(M)) = v_p(C_\theta(\mathrm{defl}M)) \neq 0$ and $v_p(C_\theta(-))$ doesn't vanish identically. The reverse direction is automatic.

Now assume *iii)*. From Lemma 2.3.4 we get that $v_p(C_\theta(-))$ vanishes if and only if $v_p(C_{\theta\downarrow_H}(-))$ vanishes for all H. But if θ is not a relation in characteristic p, there exists a p-hypo-elementary subgroup H for which $\theta\downarrow_H \neq 0$ (use Lemma 2.2.22 *i)*) and so $v_p(C_{\theta\downarrow_H}(-))$ doesn't vanish. □

2.3.2 The permutation pairing

In this section, we study the restriction of the regulator constant pairing to permutation modules. Here we have a chance to be much more explicit as Theorem 2.2.20 describes a basis of $A(\mathbb{Z}_p[G], \mathrm{perm})$, and regulator constants of permutation modules are easy to calculate.

Notation 2.3.6. Let $P(G)$ denote the free $\mathbb{Q}$-vector space on the set of conjugacy classes of non-cyclic p-hypo-elementary subgroups.

Remark 2.3.7. As in Remark 2.3.2, if we restrict to $A(\mathbb{Z}_p[G], \mathrm{perm})$, then we find that $v_p(C_{(-)}(-))$ factors as

$$v_p(C_{(-)}(-))\colon BR_0(G)/BR_p(G) \times A(\mathbb{Z}_p[G], \mathrm{perm})/A(\mathbb{Z}_p[G], \mathrm{cyc}) \to \mathbb{Q}.$$

Lemma 2.2.22 *ii)* demonstrates that sending $H \to \theta_H$ canonically identifies $P(G)$ with $BR_0(G)/BR_p(G)$ (via $H \mapsto \theta_H$). On the other hand, Theorem 2.2.10 shows that $P(G)$ is canonically identified with $A(\mathbb{Z}_p[G], \mathrm{perm})/A(\mathbb{Z}_p[G], \mathrm{cyc})$ by sending H to $\mathbb{1}\uparrow_H^G$.

It is not true that the spaces can be identified before factoring v_p. Indeed, $BR_0(G)$ is of dimension equal to the number of non-cyclic subgroups, whereas $A(\mathbb{Z}_p[G], \mathrm{perm})$ is of dimension equal to the number of non-p-hypo-elementary subgroups.

Construction 2.3.8. Via these canonical identifications, we may consider the restricted pairing of Remark 2.3.7 as a pairing

$$\langle\ ,\ \rangle_{\mathrm{perm}}\colon P(G) \times P(G) \longrightarrow \mathbb{Q},$$

sending (H, K) to $v_p(C_{\theta_H}(\mathbb{1}\uparrow_K^G))$. We call $\langle\ ,\ \rangle_{\text{perm}}$ the *permutation pairing.*

Lemma 2.3.9. *For any finite group G and prime p, $\langle\ ,\ \rangle_{\text{perm}}\colon P(G) \times P(G) \longrightarrow \mathbb{Q}$ is symmetric.*

Proof. For any two subgroups H and K of G, Lemmas 2.2.16, 2.2.30 show that

$$\begin{aligned} C_{\theta_H\uparrow^G}(\mathbb{1}_K\uparrow^G) &= C_{\theta_H\uparrow^G\downarrow_K}(\mathbb{1}_K) \\ &= \prod_{g \in H\backslash G/K} C_{\theta_{H^g\cap K}}(\mathbb{1}_{H^g\cap K}). \end{aligned}$$

Whilst,

$$\begin{aligned} C_{\theta_K\uparrow^G}(\mathbb{1}_H\uparrow^G) &= C_{\theta_K}(\mathbb{1}_H\uparrow^G\downarrow_K) \\ &= C_{\theta_K}\left(\sum_{g\in H\backslash G/K} \mathbb{1}_{H^g\cap K}\uparrow^K\right) \\ &= \prod_{g \in H\backslash G/K} \left(C_{\theta_{H^g\cap K}}(\mathbb{1}_{H^g\cap K})\right). \end{aligned}$$

$\square$

Remark 2.3.10. Along the same lines, Lemma 2.2.30 and (2.2) show that, for $H, K \leq G$,

$$\begin{aligned} \langle H, K\rangle_{\text{perm}} := v_p(C_{\theta_H\uparrow^G}(\mathbb{1}_K\uparrow^G)) &= v_p(C_{\theta_H\uparrow^G\downarrow_K}(\mathbb{1}_K)) \\ &= \sum_{g\in H\backslash G/K} C_{\theta_{H^g}\downarrow_{H^g\cap K}\uparrow^K}(\mathbb{1}_K) \\ &= \sum_{g\in H\backslash G/K} C_{\theta_{H^g\cap K}\uparrow^K}(\mathbb{1}_K) \\ &= \sum_{g\in H\backslash G/K} v_p(C_{\theta_{H^g\cap K}}(\mathbb{1}_{H^g\cap K})). \end{aligned} \tag{2.5}$$

Combining this with (2.4) of p22 gives a formula for the permutation pairing.

It is tempting to ask if permutation pairing is non-degenerate for all groups G. This proves too naive, in Section 2.7.3, we exhibit a family of groups for which the permutation pairing is degenerate (e.g. $C_3 \times C_3 \times S_3$ when $p = 3$). Analogously to Lemmas 2.3.4, 2.3.5 we have:

Lemma 2.3.11. *Let θ be a relation of a finite group G. For any prime p the following are equivalent,* *i)* $v_p(C_\theta(-))\colon A(\mathbb{Z}_p[G], \mathrm{perm}) \to \mathbb{Q}$ *vanishes identically,*

ii) $v_p(C_{\theta\downarrow_H}(-))\colon A(\mathbb{Z}_p[H], \mathrm{perm}) \to \mathbb{Q}$ *vanishes identically for all conjugacy classes of p-hypo-elementary subgroups $H \leq G$.*

Lemma 2.3.12. *For any finite group G, the following are equivalent:*

i) the permutation pairing is non-degenerate,

ii) the permutation pairing of G/N is non-degenerate for all $N \trianglelefteq G$.

Moreover, both are implied by

iii) the permutation pairing of H is non-degenerate for all p-hypo-elementary subgroups H.

The proofs are identical to before. As a result, we see that infinitely many groups exist where the permutation pairing is degenerate, for example, when $p = 3$, all groups with a $C_3 \times C_3 \times S_3$ quotient. We prove two main results on the permutation pairing. Theorem 2.4.1 states that the permutation pairing is non-degenerate for all groups with cyclic Sylow p-subgroups. Whilst, for arbitrary G, Theorem 2.5.1 states that the permutation pairing is not the zero pairing (unless $P(G) = 0$).

This leaves many open questions. For example:

Question 2.3.13. Can one describe the groups for which the permutation pairing is degenerate?

It would also be interesting to know of the existence of a group with degenerate permutation pairing but for which the regulator constant pairing has trivial left kernel. There are also many hard problems which arise from considering these pairings integrally.

2.3.3 Regulator constants as invariants of permutation modules

The non-degeneracy of the permutation pairing is a measure of the strength of regulator constants as invariants of permutation modules. In this section, we show that the isomorphism class of an arbitrary $\mathbb{Z}_p[G]$-permutation module is determined by the isomorphism class of its extension of scalars to $\mathbb{Q}_p$ and regulator constants if and only if the permutation pairing is non-degenerate.

Construction 2.3.14. Let $P''(G)$ denote the free $\mathbb{Q}$-vector space on conjugacy classes of cyclic subgroups. Artin's induction theorem (Thm. 2.2.10) states that there is a canonical isomorphism $P''(G) \xrightarrow{\sim} A(\mathbb{Q}[G])$ sending $H \to \mathbb{1}_{\mathbb{Q},H}{\uparrow}^G$. In the same way, $A(\mathbb{Z}_p[G], \mathrm{cyc})$ is also canonically identified with $P''(G)$. Define a pairing

$$\begin{aligned}\langle\ ,\ \rangle_{\mathrm{char}} \colon P''(G) \times P''(G) &\longrightarrow \mathbb{Q}\\ (H, K) &\longmapsto \langle \mathbb{1}_{\mathbb{Z}_p,H}{\uparrow}^G \otimes \mathbb{Q}_p, \mathbb{1}_{\mathbb{Z}_p,K}{\uparrow}^G \otimes \mathbb{Q}_p\rangle_{\mathrm{char}},\end{aligned}$$

where the final inner product is the usual pairing given by character theory. Then $\langle\ ,\ \rangle_{\mathrm{char}}$ is symmetric and is non-degenerate by Artin's induction theorem 2.2.10.

Construction 2.3.15. Let $P'(G)$ denote the free $\mathbb{Q}$-vector space on conjugacy classes of p-hypo-elementary subgroups of G. We define the pairing

$$\begin{aligned}\langle\ ,\ \rangle_* \colon P'(G) \times P'(G) &\longrightarrow \mathbb{Q}\\ (H, K) &\longmapsto \begin{cases}\langle \mathbb{1}_{\mathbb{Z}_p,H}{\uparrow}^G \otimes \mathbb{Q}_p, \mathbb{1}_{\mathbb{Z}_p,K}{\uparrow}^G \otimes \mathbb{Q}_p\rangle_{\mathrm{char}} & \text{if } H \text{ is cyclic}\\ v_p(C_{\theta_H}(\mathbb{1}_{\mathbb{Z}_p,K}{\uparrow}^G)) & \text{if } H \text{ is non-cyclic}\end{cases}.\end{aligned}$$

This extends both $\langle\ ,\ \rangle_{\mathrm{perm}}$ and $\langle\ ,\ \rangle_{\mathrm{char}}$.

Remark 2.3.16. The pairing $\langle\ ,\ \rangle_*$ is chosen so that, via the identification $P'(G) \cong A(\mathbb{Z}_p[G], \mathrm{perm})$, in the second variable, the construction extends to a pairing $P'(G) \times A(\mathbb{Z}_p[G]) \to \mathbb{Q}$ on the full representation ring (cf. Remark 2.6.18).

Lemma 2.3.17. *For any finite group G, the following are equivalent,*

i) the permutation pairing of G is non-degenerate,

ii) the pairing $\langle\ ,\ \rangle_*$ is non-degenerate,

iii) the isomorphism class of an arbitrary permutation module over $\mathbb{Z}_p$ is determined by

a) the isomorphism class of $M \otimes \mathbb{Q}_p$, and

b) the valuations of the regulator constants $v_p(C_{\theta_H}(M))$ as H runs over elements of $\mathrm{nchyp}_p(H)$.

Proof. For equivalence of *i)* and *ii)*, note that, for any cyclic subgroup K, $v_p(C_{\theta_H}(\mathbb{1}{\uparrow}_K^G)) = 0$ (Remark 2.3.2). Thus, if we order the canonical basis of $P'(G)$ so that the cyclic subgroups come before the non-cyclic p-hypo-elementary subgroups, then the matrix representing $\langle\ ,\ \rangle_*$ is block upper triangular, with diagonal blocks given by the matrices representing $\langle\ ,\ \rangle_{\text{char}}$ and the permutation pairing respectively. The former is always invertible so $\langle\ ,\ \rangle_*$ is non-degenerate if and only if the permutation pairing is.

The equivalence of *ii)* and *iii)* is automatic. $\square$

Example 2.3.18. Let $G = D_{2p}$. Up to conjugacy, the p-hypo-elementary subgroups of G are $S = \{\{1\}, C_2, C_p, D_{2p}\}$. Applying (2.4) of p22 to θ_G as calculated in Example 2.2.15, we find $v_p(C_{\theta_G}(\mathbb{1}_G)) = -1/2$. Thus, the matrix representing $\langle\ ,\ \rangle_*$ with respect to the basis of $P'(G)$ given by S is:

$$\begin{array}{c|c} & \begin{array}{cccc} \{1\} & C_2 & C_p & D_{2p} \end{array} \\ \begin{array}{c} \{1\} \\ C_2 \\ C_p \\ D_{2p} \end{array} & \left(\begin{array}{ccc|c} 2p & p & 2 & 1 \\ p & (p+1)/2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ \hline 0 & 0 & 0 & -1/2 \end{array}\right) \end{array}$$

In Section 2.7.1, we extend this to allow arbitrary $\mathbb{Z}_p[D_{2p}]$-lattices.

2.4 Non-degeneracy of the permutation pairing for groups with cyclic Sylow p-subgroups

In this section we prove:

Theorem 2.4.1. *Let G be a finite group and p a prime such that G has cyclic Sylow p-subgroups. Then the permutation pairing*

$$v_p(C_{(-)}(-))\colon BR_0(G)/BR_p(G) \times A(\mathbb{Z}_p[G], \text{perm})/A(\mathbb{Z}_p[G], \text{cyc}) \to \mathbb{Q}$$

is non-degenerate.

As a result, for such groups G, the regulator constant pairing has trivial left kernel and we find:

Corollary 2.4.2. *Let G be a finite group and p a prime for which the Sylow p-subgroups of G are cyclic. Then the isomorphism class of an arbitrary $\mathbb{Z}_p[G]$-permutation module M is determined by,* *i) the isomorphism class of* $M \otimes \mathbb{Q}_p$*,*

ii) the valuations of the regulator constants $v_p(C_{\theta_H}(M))$ *as* H *runs over elements of* $\mathrm{nchyp}_p(H)$*.*

Proof. This follows by Lemma 2.3.17. □

The proof of Theorem 2.4.1 reduces to the case of G p-hypo-elementary. Since all p-hypo-elementary groups with cyclic Sylow p-subgroups are of the form $C_{p^k} \rtimes C_n$ with $(n,p) = 1$, we can then preform an explicit calculation.

2.4.1 GCD matrices

We first state and prove a purely combinatorial statement. Since this may be of limited independent interest this subsection is self contained.

Notation 2.4.3. For a natural number n and divisor s of n, denote by

- $D'(n)$ the set of divisors of n (ordered increasingly),
- $D(n,s) \subset D'(n)$ the set of divisors of n not dividing s,
- $N(n)$ the symmetric matrix with rows and columns indexed by elements of $D'(n)$ and $(d_1, d_2)^{\text{th}}$ entry given by $\gcd(d_1, d_2)$,
- $M(n,s)$ the symmetric matrix with rows and columns indexed by elements of $D(n,s)$ and $(d_1, d_2)^{\text{th}}$ entry given by $(\gcd(d_1, d_2) - \gcd(d_1, d_2, s))$.

Example 2.4.4. If $n = 12$ and $s = 2$ then $D(12, 2) = \{3, 4, 6, 12\}$ and

$$M(12,2) = \begin{array}{c c} & \begin{array}{cccc} 3 & 4 & 6 & 12 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 6 \\ 12 \end{array} & \begin{pmatrix} 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 2 & 4 & 10 \end{pmatrix} \end{array},$$

which has full rank.

Remark 2.4.5. Matrices of the form $N(n)$ are called GCD matrices and are always invertible (not necessarily integrally, see Lemma 2.4.6). Although matrices defined in a similar way to $M(n,s)$ have been studied (see [BL89, Beg10]), we have been unable to find results in the literature that directly cover matrices of the form

$M(n, s)$. For this reason, we have included a full calculation of their determinants and thus invertibility. First we recall the proof of the determinant formula for $N(n)$.

Lemma 2.4.6. *For any natural number n, the matrix $N(n)$ has determinant equal to $\prod_{d \in D'(n)} \phi(d)$, where ϕ denotes Euler's totient function, and thus is always of full rank.*

Proof. For $n = p^e$ a prime power,

$$N(p^e) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & p & \dots & p \\ \vdots & \vdots & & \vdots \\ 1 & p & \dots & p^e \end{pmatrix}.$$

If $e > 1$, expanding the final column shows that

$$\det(N(p^e)) = (p^e - p^{e-1}) \det(N(p^{e-1})) = \phi(p^e) \det(N(p^{e-1}))$$

as any $(e-1) \times (e-1)$ minor containing the first $(e-1)$ terms of the last two rows is not of full rank. Inductively this shows the determinant formula for prime powers.

Now let $s = rt$ with $(r, t) = 1$. Then using the bijection $D'(rt) \leftrightarrow D'(r) \times D'(t)$, after simultaneous permutation of rows and columns (which preserves the determinant) $N(s)$ is of the form:

$$\begin{array}{cc} & \begin{array}{cccc} u_1 & u_2 & \cdots & u_k \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_k \end{array} & \left(\begin{array}{c|c|c|c} \gcd(u_1,u_1)N(r) & \gcd(u_1,u_2)N(r) & \dots & \gcd(u_1,u_k)N(r) \\ \hline \gcd(u_2,u_1)N(r) & \gcd(u_2,u_2)N(r) & \dots & \gcd(u_2,u_k)N(r) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \gcd(u_k,u_1)N(r) & \gcd(u_k,u_2)N(r) & \dots & \gcd(u_k,u_k)N(r) \end{array}\right) \end{array} = N(r) \otimes N(t),$$

writing u_i for the elements of $D'(t)$. If A, B are matrices of dimension m, n respectively, then their tensor product satisfies the familiar formula

$$\det(A \otimes B) = \det(A)^n \det(B)^m.$$

Applying this inductively, using the bijection $D'(rt) \leftrightarrow D'(r) \times D'(t)$,

$$\begin{aligned}
\det(N(r) \otimes N(t)) &= \left(\prod_{d \in D'(r)} \phi(d) \right)^{|D'(t)|} \cdot \left(\prod_{l \in D'(t)} \phi(l) \right)^{|D'(r)|} \\
&= \prod_{d \in D'(r)} \left(\phi(d)^{|D'(t)|} \prod_{l \in D'(t)} \phi(l) \right) \\
&= \prod_{d \in D'(r)} \prod_{l \in D'(t)} \phi(d)\phi(l) \\
&= \prod_{w \in D'(rt)} \phi(w),
\end{aligned}$$

as required. $\square$

Lemma 2.4.7. *The matrix $M(n, s)$ has full rank for all natural numbers n and divisors s of n. Moreover, $\det(M(n, s)) = \prod_{d \in D(n,s)} \phi(d)$, where ϕ is the Euler totient function.*

Proof. We first prove the case when $s = 1$. Consider the matrix $N(n)$ (whose determinant equals $\prod_{d \in D'(n)} \phi(d)$ by Lemma 2.4.6). Within $N(n)$, the first row and column are constantly 1, and if we subtract the first column from all subsequent columns we get

$$\det(N(n)) = \det \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 1 & & & \\ \vdots & & M(n,1) & \\ 1 & & & \end{array} \right) = \det(M(n, 1)).$$

As $\phi(1) = 1$, this verifies the determinant formula in the case of $s = 1$.

We proceed by induction on the number of prime divisors of s. Assume that $M(n, s)$ has determinant

$$\det(M(n, s)) = \prod_{d \in D(n,s)} \phi(d),$$

and consider $M(p^r n, p^e s)$ with $p \nmid n, s$.

Let d be a divisor of $p^r n$, so d is of the form $p^k d'$ with $p \nmid d'$ and $0 \leq k \leq r$. Then $d \in D(p^r n, p^e s) \iff$ either $k \leq e$ and $d' \in D(n, s)$, or $k > e$ and $d' \in D'(n)$.

In other words, $D(p^r n, p^e s)$ can be partitioned as

$$D(p^r n, p^e s) = \left(\bigcup_{i=0}^{e} p^i D(n,s) \right) \cup \left(\bigcup_{i=p^{e+1}}^{r} p^i D'(n) \right).$$

Call $D_1 = \bigcup_{i=0}^{e} p^i D(n,s)$ and $D_2 = \bigcup_{i=e+1}^{r} p^i D'(n)$. Simultaneously reorder the rows and columns of $M(p^r n, p^e s)$ so that they respect this decomposition. Define A, B, C by

$$M(p^r n, p^e s) = \begin{array}{cc} & \begin{array}{cc} D_1 & D_2 \end{array} \\ \begin{array}{c} D_1 \\ D_2 \end{array} & \left(\begin{array}{c|c} A & C \\ \hline C^T & B \end{array} \right). \end{array}$$

For any two elements $p^{l_1} d_1, p^{l_2} d_2 \in D_1$, the corresponding entry of A is given by

$$\gcd(p^{l_1} d_1, p^{l_2} d_2) - \gcd(p^{l_1} d_1, p^{l_2} d_2, p^e s) = p^{\min\{l_1, l_2\}}(\gcd(d_1, d_2) - \gcd(d_1, d_2, s)).$$

So A is the tensor product

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & p & \dots & p \\ \vdots & \vdots & & \vdots \\ 1 & p & \dots & p^e \end{pmatrix} \otimes M(n,s) = N(p^e) \otimes M(n,s),$$

which has determinant

$$\det(A) = \det(N(p^e))^{|D(n,s)|} \cdot \det(M(n,s))^{e+1}.$$

By induction and Lemma 2.4.6,

$$\begin{aligned}
\det(A) &= \det(N(p^e))^{|D(n,s)|} \cdot \det(M(n,s))^{e+1} \\
&= \left(\prod_{k=0}^{e} \phi(p^k)^{|D(n,s)|}\right) \cdot \left(\prod_{d\in D(n,s)} \phi(d)^{e+1}\right) \\
&= \prod_{k=0}^{e} \left(\phi(p^k)^{|D(n,s)|} \cdot \prod_{d\in D(n,s)} \phi(d)\right) \\
&= \prod_{k=0}^{e} \prod_{d\in D(n,s)} \phi(p^k)\phi(n) \\
&= \prod_{d\in D_1} \phi(d).
\end{aligned}$$

We now row reduce to remove C^T. For $e < k \leq r$, let $v_{p^k d}$ denote the row vector with entries indexed by $D(n,s) = D_1 \cup D_2$ and whose t^{th} entry is defined by

$$(v_{p^k d})_t = \gcd(p^e d, t) - \gcd(p^e d, t, p^e s).$$

If $d \mid s$, then $\gcd(p^e d, t) = \gcd(p^e d, t, p^e s)$ and $v_{p^k d}$ is identically zero. If $d \nmid s$, then $p^e d \in D_1$ and $v_{p^k d}$ is the $(p^e d)^{\text{th}}$ row of the matrix $M(n,s)$. In either case, subtracting $v_{p^k d}$ from the $(p^k d)^{\text{th}}$ row is an elementary row operation and preserves the rank and determinant.

Call $M'(p^r n, p^e s)$ the matrix resulting from performing this reduction for all elements of D_2. The entries of the $p^k d^{\text{th}}$ row for $p^k d \in D_2$ now satisfy, for $p^{k'} d' \in D_1$,

$$\begin{aligned}
M'(p^r n, p^e s)_{p^k d, p^{k'} d'} &= \gcd(p^k d, p^{k'} d') - \gcd(p^k d, p^{k'} d', p^e s) - \gcd(p^e d, p^{k'} d') \\
&\quad + \gcd(p^e d, p^{k'} d', p^e s) \\
&= p^{k'} \gcd(d, d') - p^{k'} \gcd(d, d', s) - p^{k'} \gcd(d, d') \\
&\quad + p^{k'} \gcd(d, d', s) \\
&= 0,
\end{aligned}$$

and for $p^{k'} d' \in D_2$,

$$
\begin{aligned}
M'(p^r n, p^e s)_{p^k d, p^{k'} d'} &= \gcd(p^k d, p^{k'} d') - \gcd(p^k d, p^{k'} d', p^e s) - \gcd(p^e d, p^{k'} d') \\
&\quad + \gcd(p^e d, p^{k'} d', p^e s) \\
&= p^{\min\{k,k'\}} \gcd(d, d') - p^e \gcd(d, d', e) - p^e \gcd(d, d') \\
&\quad + p^e \gcd(d, d', s) \\
&= (p^{\min\{k,k'\}} - p^e) \gcd(d, d').
\end{aligned}
$$

Therefore, the row reduction results in a matrix of the form

$$
M'(p^r n, p^e s) = \begin{array}{c c} & \begin{array}{cc} D_1 & D_2 \end{array} \\ \begin{array}{c} D_1 \\ D_2 \end{array} & \left(\begin{array}{c|c} A & C \\ \hline 0 & B' \end{array}\right) \end{array}.
$$

where

$$
\begin{aligned}
B' &= N(n) \otimes \begin{pmatrix} p^{e+1} - p^e & p^{e+1} - p^e & \ldots & p^{e+1} - p^e \\ p^{e+1} - p^e & p^{e+2} - p^e & \ldots & p^{e+2} - p^e \\ \vdots & \vdots & & \vdots \\ p^{e+1} - p^e & p^{e+2} - p^e & \ldots & p^r - p^e \end{pmatrix} \\
&= N(n) \otimes M(p^r, p^e) \\
&= N(n) \otimes p^e M(p^{r-e}, 1).
\end{aligned}
$$

Since

$$
\det(M(p^r n, p^e s)) = \det(A) \cdot \det(B'),
$$

to complete the proof we must show that $\det(B') = \prod_{d \in D_2} \phi(d)$. Indeed,

$$
\begin{aligned}
\det(B') &= \det(N(n))^{r-e} \cdot \det(M(p^r, p^e))^{|D'(n)|} \\
&= \prod_{k=e+1}^{r} \phi(p^k)^{|D'(n)|} \det(N(n)) \\
&= \left(\prod_{k=e+1}^{r} \phi(p^k)^{|D'(n)|} \right) \cdot \left(\prod_{d \in D'(n)} \phi(d) \right) \\
&= \cdot \prod_{k=e+1}^{r} \prod_{d \in D'(n)} \phi(p^k d) \\
&= \prod_{d \in D_2} \phi(d).
\end{aligned}
$$

So we find

$$\det(M(p^k n, p^e s)) = \left(\prod_{d\in D_1} \phi(d)\right)\left(\prod_{d\in D_2} \phi(d)\right) = \prod_{d\in D} \phi(d).$$

This completes the proof of the determinant formula of $M(a,b)$ by induction on the number of prime factors of b. $\square$

2.4.2 Structure of $C_{p^k} \rtimes C_n$

We now perform an explicit calculation for p-hypo-elementary groups before deducing Theorem 2.4.1.

Lemma 2.4.8. *Let G be of the form $C_{p^r} \rtimes C_n$ with $p \nmid n$. Further, let S denote the kernel of the action of C_n on C_{p^r}. Then, for any two subgroups $H', K' \leq G$ of the form $H' = C_{p^e} \rtimes H, K' = C_{p^f} \rtimes K$ with $H, K \leq C_n \leq G$, as elements of the Burnside ring $B(K')$,*

$$\begin{aligned}\coprod_{g\in H'\backslash G/K'} [H'^g \cap K'] = {} & \frac{|C_n||H\cap K|}{|H||K|}[H'\cap K'] \\ & + \frac{p^{r-\max\{e,f\}}|C_n||H\cap K\cap S|}{|H||K|}[H'\cap K'\cap (C_{p^r}\times S)].\end{aligned}$$

Proof. First assume $e = f = 0$. Elements of $H\backslash G/K$ are in bijection with H-orbits of cosets gK. For such G, a set of coset representatives of G/K is given by elements $\sigma\tau_i$, where $\sigma \in C_{p^r}$ and $\{\tau_i\}$ are a set of coset representatives of C_n/K. The stabilizer of a right coset gK under the action of H is given by

$$\mathrm{Stab}_H(gK) = H \cap gKg^{-1}.$$

Using that C_n is abelian, for $k \in K$,

$$\begin{aligned}(\sigma\tau_i)k(\sigma\tau_i)^{-1} &= \sigma\tau_i k\tau_i^{-1}\sigma^{-1} = \sigma k\sigma^{-1} \\ &= \sigma k\sigma^{-1}k^{-1}k = \sigma\varphi(k)(\sigma^{-1})k,\end{aligned}$$

where $\varphi\colon C_n \to \mathrm{Aut}(C_{p^r})$ denotes the action of conjugation. Since the prime to p-part of $\mathrm{Aut}(C_{p^r})$ equals that of $\mathrm{Aut}(C_{p^e})$ for any non-trivial $C_{p^e} \leq C_{p^r}$, k acts trivially on $\sigma \neq e$ if and only if $k \in S$. Thus,

$$\sigma\varphi(k)(\sigma^{-1})k \in H \iff k \in H \text{ and } k \in \begin{cases} K & \text{if } \sigma = e \\ K \cap S & \text{if } \sigma \neq e \end{cases}.$$

In particular,

$$\mathrm{Stab}_H(gK) = \begin{cases} H \cap K & \text{if } g \in C_n \\ H \cap K \cap S & \text{if } g \notin C_n \end{cases}.$$

By orbit–stabiliser theorem, there are $\frac{|C_n||H\cap K|}{|H||K|}$ double cosets HgK of length $\frac{|H||K|}{|H\cap K|}$ and $(p^r-1)\frac{|C_n||H\cap K\cap S|}{|H||K|}$ double cosets of length $\frac{|H||K|}{|H\cap K\cap S|}$. Furthermore, as H has a unique subgroup of each order

$$H^g \cap K = \begin{cases} H \cap K & \text{if } g \in C_n \\ H \cap K \cap S & \text{else} \end{cases},$$

and applying Mackey's formula (2.1) p13 gives the desired formula in this case.

Now let $e, f \geq 0$. We first calculate the order of $H'\backslash G/K'$. As all p-subgroups of G are normal, there are canonical bijections

$$(C_{p^e} \rtimes H)\backslash G/(C_{p^f} \rtimes K) \leftrightarrow H\backslash G/(C_{p^{\max\{e,f\}}} \rtimes K) \leftrightarrow H\backslash((C_{p^r}/C_{p^{\max\{e,f\}}}) \rtimes C_n)/K.$$

So, from the first part, we find there are $\frac{|C_n||H\cap K|}{|H||K|}$ double cosets of length $\frac{|H||K|}{|H\cap K|}p^{\max\{e,f\}}$ and $(p^{\max\{e,f\}}-1)\frac{|C_n||H\cap K\cap S|}{|H||K|}$ double cosets of length $\frac{|H||K|}{|H\cap K\cap S|}p^{\max\{e,f\}}$. Taking preimages,

$$H'^g \cap K' = \begin{cases} H' \cap K' & \text{if } g \in C_n \\ H' \cap K' \cap (C_{p^r} \times S) & \text{else} \end{cases}.$$

Therefore, indeed

$$\coprod_{g \in H'\backslash G/K'} [H'^g \cap K'] = \frac{|C_n||H\cap K|}{|H||K|}[H' \cap K'] + \frac{p^{r-\max\{e,f\}}|C_n||H\cap K\cap S|}{|H||K|}[H' \cap K' \cap (C_{p^r} \times S)].$$

□

Notation 2.4.9. Given a group G and subgroup $H \leq G$, we denote by $N_G(H)$ the normaliser of H in G and by $Z_G(H)$ its centraliser.

Proof of Theorem 2.4.1. Lemma 2.3.12 states that the permutation pairing for G is non-degenerate if the pairing is non-degenerate for all p-hypo-elementary subgroups. So we shall assume that G is p-hypo-elementary, i.e. $G \cong C_{p^r} \rtimes C_n$ with $p \nmid n$.

For notational convenience, we make a fixed choice of subgroup of G isomorphic to C_n, which we also denote by C_n. Let S denote the kernel of the map $C_n \to \mathrm{Aut}(C_{p^r})$ defining the semi-direct product. Note that S is also the kernel of the map $C_n \to \mathrm{Aut}(C_{p^k})$ for all $1 \leq k \leq r$. Up to conjugacy, any subgroup of G is of the form $C_{p^k} \rtimes L$, with L contained in the fixed choice of C_n. Moreover, such a subgroup is cyclic and normal in G if and only if $L \leq S$.

Let H', K' be non-cyclic subgroups of G. We may assume, by replacing H', K' with conjugate subgroups if necessary, that $H' = C_{p^e} \rtimes H, K' = C_{p^f} \rtimes K$ with $H, K \leq C_n$. We first calculate $\langle H', K' \rangle_{\mathrm{perm}} = v_p(C_{\theta_{H'}}(\mathbb{1}_{K'}\uparrow^G)) = v_p(C_{\theta_{H'}\downarrow_{K'}}(\mathbb{1}_{K'}))$. The decomposition of $\theta_{H'}\downarrow_{K'}$ matches that of its leading term (Lemma 2.2.16), so applying Lemma 2.4.8 we find

$$\begin{aligned}\theta_{H'}\uparrow^G\downarrow_{K'} = &\left(\frac{|C_n||H \cap K|}{|H||K|}\right) \cdot \theta_{H' \cap K'}\uparrow^{K'} \\ &+ \left(\frac{p^{r-\max\{e,f\}}|C_n||H \cap K \cap S|}{|H||K|}\right) \cdot \theta_{H' \cap K' \cap (C_{p^r} \times S)}\uparrow^{K'}.\end{aligned}$$

But $H' \cap K' \cap (C_{p^r} \times S)$ is cyclic (so that $\theta_{H' \cap K' \cap (C_{p^r} \times S)} = 0$) and we find

$$v_p(C_{\theta_{H'}}(\mathbb{1}_{K'}\uparrow^G)) = \frac{|C_n||H \cap K|}{|H||K|} v_p(C_{\theta_{H' \cap K'}}(\mathbb{1}_{H' \cap K'})).$$

Let L' be an arbitrary non-cyclic subgroup of the form $C_{p^\ell} \rtimes L$ with $L \leq C_n$. Directly applying (2.4) of Remark 2.2.29 to the formula of Example 2.2.15, or by looking ahead to Example 2.5.17, we find that

$$\begin{aligned}v_p(C_{\theta_{L'}}(\mathbb{1}_{L'})) &= -\ell\left(1 - \frac{|Z_{L'}(C_{p^\ell})|}{|N_{L'}(C_{p^\ell})|}\right) \\ &= -\ell\left(1 - \frac{|L \cap S|}{|L|}\right).\end{aligned}$$

Concluding our calculation of $\langle H', K' \rangle_{\mathrm{perm}}$, we find

$$\begin{aligned}
\langle H', K' \rangle_{\mathrm{perm}} &= \frac{|C_n||H \cap K|}{|H||K|} v_p(C_{\theta_{H' \cap K'}}(\mathbb{1}_{H' \cap K'})) \\
&= \frac{|C_n||H \cap K|}{|H||K|} \min\{e, f\} \left(\frac{|H \cap K \cap S|}{|H \cap K|} - 1 \right) \\
&= \frac{|C_n| \min\{e, f\}}{|H||K|} (|H \cap K \cap S| - |H \cap K|). \qquad (2.6)
\end{aligned}$$

Let T be the matrix representing the pairing $\langle \ , \ \rangle_{\mathrm{perm}}$ with respect to the basis of $P(G)$ given by non-cyclic p-hypo-elementary subgroups ordered (totally in our case) by size. After a non-zero scaling of the rows and columns of T, we obtain a matrix T' with $(H', K')^{\mathrm{th}}$ entry

$$T'_{H',K'} = \min\{e, f\}(|H \cap K \cap S| - |H \cap K|).$$

Note T' remains symmetric and has the same rank as T. Since C_n is cyclic, $|H \cap K \cap S| = \gcd(|H|, |K|, |S|)$ and $|H \cap K| = \gcd(|H|, |K|)$. Thus, T' is the matrix with entries

$$T'_{H',K'} = \min\{e, f\} \left(\gcd(|H|, |K|, |S|) - \gcd(|H|, |K|) \right). \qquad (2.7)$$

Let $M(m, l)$ be as in Notation 2.4.3. If $Q(d)$ denotes the $d \times d$ matrix with $Q_{i,j} = \min\{i, j\}$, then, by (2.7), we may simultaneously permute the rows and columns of T' to get

$$T' \sim -Q(r) \otimes M(n, s),$$

where $|S| = s$. As $Q(r)$ is manifestly of full rank and Lemma 2.4.7 states that $M(n, s)$ is also, so the same is true for T and the permutation pairing for G is non-degenerate. $\square$

Example 2.4.10. Let $G = C_7 \rtimes C_{12}$. A set of representatives of the non-cyclic conjugacy classes of G is given by

$$S := \{C_7 \rtimes C_3, C_7 \rtimes C_4, C_7 \rtimes C_6, C_7 \rtimes C_{12}.\}$$

Applying (2.6), the matrix T representing the permutation pairing with respect to the basis given by S is given by

$$\begin{pmatrix} -8/3 & 0 & -4/3 & -2/3 \\ 0 & -3/2 & 0 & -1/2 \\ -4/3 & 0 & -4/3 & -2/3 \\ -2/3 & -1/2 & -2/3 & -5/6 \end{pmatrix}.$$

In the notation of the proof of Theorem 2.4.1, $n = 12$ and $s = 2$. After rescaling the rows and columns of T as in the proof, we obtain the matrix $M(12, 2)$ of Example 2.4.4.

2.5 Non-vanishing of the Artin regulator constant

In this section, we prove:

Theorem 2.5.1. *For any finite group G and prime p, $v_p(C_{\theta_G}(\mathbb{1}_G)) \neq 0$ if and only if G contains a non-cyclic p-hypo-elementary subgroup. If G does contain a non-cyclic p-hypo-elementary subgroup then $v_p(C_{\theta_G}(\mathbb{1}_G)) \leq -p/|G|$. Here, $\mathbb{1}_G$ denotes the trivial $\mathbb{Z}_p[G]$-module.*

The method of proof is of explicit group theoretic natured and is disjoint to that of Section 2.4. Moreover, Sections 2.6 and 2.7 have no dependency on this section.

Remark 2.5.2. The forward direction of 2.5.1 is formal: If G contains no non-cyclic p-hypo-elementary groups then all characteristic zero relations are relations in characteristic p (see Lemma 2.2.22). But the regulator constant of a characteristic p relation has trivial valuation at p when evaluated at any lattice (Lemma 2.2.33).

Remark 2.5.3. Let G be a p-hypo-elementary group. Then in terms of the permutation pairing of Construction 2.3.8, the theorem asserts that every entry in the row and column corresponding to G is strictly negative. By Lemma 2.3.5, the regulator constant pairing is non-degenerate whenever each p-hypo-elementary subgroup of G contains only cyclic proper subgroups, e.g. $G = S_4$. Under the same hypothesis, permutation modules over $\mathbb{Z}_p$ are determined by extension of scalars to $\mathbb{Q}_p$ and regulator constants (Lemma 2.3.17).

Corollary 2.5.4. *For any finite group G, as a function on $\mathbb{Z}[G]$-modules, the regulator constant associated to the Artin relation θ_H vanishes identically if and only if H is cyclic.*

Proof. For cyclic H, $\theta_H = 0$ so its regulator constant is trivial. For the converse, we first show:

Claim. *A finite group K is cyclic if and only if all its ℓ-hypo-elementary subgroups are cyclic for all ℓ.*

Proof of Claim. Suppose K is a group for which all its ℓ-hypo-elementary subgroups are cyclic for all ℓ. Then as the Sylow subgroups must be cyclic, the normaliser of every Sylow subgroup must be equal to its centraliser. Burnside's normal p-complement theorem then ensures that every Sylow p-subgroup normalises every Sylow ℓ-subgroup for $\ell \neq p$. As a result, K is a direct product of its (cyclic) Sylow subgroups for different p, and thus K is cyclic. $\square$

Now suppose $T \leq G$ is non-cyclic. By the claim, T has a non-cyclic ℓ-hypo-elementary subgroup L for some ℓ. Then

$$\begin{aligned} 0 &\overset{2.5.1}{>} v_\ell(C_{\theta_L}(\mathbb{1}_{\mathbb{Z}_\ell,L})) \\ &\overset{2.2.16\ i)}{=} v_\ell(C_{\theta_T\downarrow_L}(\mathbb{1}_{\mathbb{Z}_\ell,L})) \\ &\overset{2.2.30\ iv)}{=} v_\ell(C_{\theta_T}(\mathbb{1}_{\mathbb{Z}_\ell,L}\uparrow^T)) \\ &\overset{2.2.30\ vi)}{=} v_\ell(C_{\theta_T}(\mathbb{1}_{\mathbb{Z},L}\uparrow^T)). \end{aligned}$$ $\square$

Remark 2.5.5. By symmetry (Lemma 2.3.9), we find that a permutation module $\mathbb{1}\uparrow_H^G$ is trivial under all regulator constants if and only if H is cyclic.

2.5.1 Explicit Artin induction

The proof of Theorem 2.5.1 is made possible by Brauer's formula for explicit Artin induction.

Notation 2.5.6. Let $\mu(n)$ denote the Möbius function of a natural number n,

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ is squarefree and has } r \text{ distinct prime factors} \\ 0 & \text{if } n \text{ is not squarefree} \end{cases}.$$

Note that $\mu(1) = 1$.

Lemma 2.5.7 (Brauer, [Sna94, Thm. 2.1.3])**.** *If G is any finite group with Artin* *relation* $\theta_G = [G] - \sum_{\mathrm{cyc}(G)} \alpha_H[H]$, *then*

$$\alpha_H = \frac{1}{|N_G(H):H|}\sum_{C\geq H}\mu(|C:H|).$$

Here the sum runs over all cyclic overgroups of H *(not just up to conjugacy).*

Lemma 2.5.8. *Let* G *be a* p*-hypo-elementary group and* $\theta_G = [G]-\sum_{\substack{H\leq_G G\\ H\ cyclic}}\alpha_H[H]$. *Then* $\alpha_H \in \frac{p}{|G|}\cdot\mathbb{Z}$.

Proof. Let $G = P \rtimes C$ be non-cyclic and $H \leq G$ cyclic. By explicit Artin induction, $\alpha_H \in \frac{1}{|N_G(H):H|}\cdot\mathbb{Z}$, so there is only anything to prove when H is of order coprime to p and H is normalised by P (and so by G). Such an H must therefore lie in the kernel S of the action of C on P.

Let q be the quotient map $q\colon G \to G/H$. Then a subgroup $K \leq G$ is cyclic if and only if $q(K)$ is. So q defines an index preserving bijection between cyclic subgroups of G containing H and cyclic subgroups of G/H. As such, we may assume that $H = \{1\}$.

We shall show that the contributions to $\sum_{K\mathrm{cyclic}}\mu(|K|)$ from cyclic subgroups of order coprime to p, and of order divisible by p exactly once, cancel (recall that $\mu(|K|)$ vanishes for all other K). Let K be a cyclic subgroup of G of order coprime to p. We split into two cases: First assume K is normal. Any cyclic group containing K with index p is of the form $C_p \times K$ for some $C_p \leq P$. By (the general form of) Sylow's theorems there are 1 (mod p) such choices. Since $\mu(|C_p \times K|) = -\mu(|K|)$ the contributions of K and its overgroups cancel modulo p.

Now assume that K is not normal. In particular, K is not normalised by P and there are no cyclic subgroups isomorphic to $C_p \times K$. As P acts transitively on the non-singleton set of conjugates of K, orbit-stabiliser shows that the number of subgroups of G isomorphic to K is 0 (mod p). We have exhausted all cyclic subgroups and thus p divides $\sum_{\substack{C\leq G\\ C\mathrm{cyclic}}}\mu(|C|)$ and $\alpha_H \in \frac{p}{|G|}\cdot\mathbb{Z}$. □

Corollary 2.5.9. *For any non-cyclic* p*-hypo-elementary group* G *and module* M, $v_p(C_{\theta_G}(M)) \in \frac{p}{|G|}\cdot\mathbb{Z}$. *More generally, for any finite group* G*, given subgroups* H, K *and a* K *module* M, $v_p(C_{\theta_H}(M\uparrow_K^G)) \in \frac{p}{\gcd\{|H|,|K|\}}\cdot\mathbb{Z}$.

Proof. By definition the valuations of regulator constants of integral Brauer relations lie in $\mathbb{Z}$, so the first statement follows from the lemma and 2.2.30 *iii)*. For the second, the formalism of Lemma 2.2.30 and Mackey's formula gives

$$\begin{aligned}
v_p(C_{\theta_H\uparrow_H^G}(M\uparrow_K^G)) &= v_p(C_{\theta_H}(M\uparrow_K^G\downarrow_H)) \\
&= \sum_{g\in H\backslash G/K} v_p(C_{\theta_H}(M^g\downarrow_{K^g\cap H}\uparrow^H)) \\
&= \sum_{g\in H\backslash G/K} v_p(C_{\theta_H\downarrow_{K^g\cap H}}(M^g\downarrow_{K^g\cap H})).
\end{aligned}$$

But applying the first statement, each term of the sum lies in $\frac{p}{\gcd\{|H|,|K|\}}\cdot\mathbb{Z}$. $\square$

We shall use explicit Artin induction to provide a formula for $v_p(C_{\theta_G}(\mathbb{1}_G))$.

Notation 2.5.10. Recall that if two subgroups H_1, H_2 of G are conjugate, then $[H_1]$ and $[H_2]$ are isomorphic as G-sets. To make the Artin relation slightly more canonical, instead of writing

$$\theta_G = [G] - \sum_{H\leq_G G} \alpha_H[H],$$

we can choose to write θ_G uniquely as

$$\theta_G = [G] - \sum_{H\leq G} \alpha'_H[H],$$

subject to the stipulation that $\alpha'_{H_1} = \alpha'_{H_2}$ for conjugate H_1, H_2. Then $\alpha'_H = \frac{1}{|G:N_G(H)|}\cdot\alpha_H$, the α'_H are unique and the two notational choices denote identical elements of $B(G)$.

Notation 2.5.11. Fix a single prime p for the remainder of this section. Let $\mathcal{P}(G,k)$ denote the number of elements of a given finite group G whose order is divisible by p^k.

Lemma 2.5.12. *For any group G and prime p, if θ_G denotes the Artin relation, then*

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -v_p(|G|) + \frac{1}{|G|}\sum_{g\in G} v_p(|g|) + \frac{\mathcal{P}(G,1)}{|G|\cdot(p-1)}. \tag{2.8}$$

Proof. Running over all cyclic subgroups rather than their conjugacy classes, explicit Artin induction gives that

$$\theta_G = [G] - \sum_{\substack{H\leq G\\ H\text{-cyclic}}} [H]\cdot\frac{1}{|G:H|}\sum_{\substack{C\geq H\\ C\text{-cyclic}}} \mu(|C:H|).$$

Applying the formula (2.4) for regulator constants at the trivial module we find that

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -v_p(|G|) + \sum_{H \leq G} \frac{v_p(|H|)}{|G:H|} \sum_{C \geq H} \mu(|C:H|),$$

where from now on it is assumed that sums run only over all cyclic subgroups or overgroups. Changing the order of summation,

$$\begin{aligned} v_p(C_{\theta_G}(\mathbb{1}_G)) &= -v_p(|G|) + \sum_{C \leq G} \sum_{H \leq C} \frac{v_p(|H|)}{|G:H|} \mu(|C:H|) \\ &= -v_p(|G|) + \sum_{\substack{C \leq G \\ p \mid |C|}} \sum_{H \leq C} \frac{v_p(|H|)}{|G:H|} \mu(|C:H|) \end{aligned}$$

as only subgroups C for which p divides $|C|$ make any contribution. Within the second sum, by definition of the Möbius function, only the subgroups of squarefree index contribute. We separate into the sums over the subgroups H of C of index divisible by p, and subgroups H of index not divisible by p. There is a bijection between these two sets given by sending a subgroup H of index not divisible by p to pH. Thus,

$$\begin{aligned} &v_p(C_{\theta_G}(\mathbb{1}_G)) \\ &= -v_p(|G|) + \sum_{\substack{C \leq G \\ p \mid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \frac{v_p(|H|)}{|G:H|} \mu(|C:H|) + \sum_{\substack{C \leq G \\ p \mid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \frac{v_p(|pH|)}{|G:pH|} \mu(|C:pH|) \\ &= -v_p(|G|) + \sum_{\substack{C \leq G \\ p \mid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \frac{v_p(|H|)}{|G:H|} \mu(|C:H|) - \sum_{\substack{C \leq G \\ p \mid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \frac{v_p(|H|)-1}{p|G:H|} \mu(|C:H|) \\ &= -v_p(|G|) + \sum_{\substack{C \leq G \\ p \mid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \left(v_p(|H|) \cdot \frac{p-1}{p} \cdot \frac{\mu(|C:H|)}{|G:H|} + \frac{\mu(|C:H|)}{p|G:H|} \right) \\ &= -v_p(|G|) + \underbrace{\sum_{\substack{C \leq G \\ p \mid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} v_p(|H|) \cdot \frac{p-1}{p} \cdot \frac{\mu(|C:H|)}{|G:H|}}_{(\dagger)} + \underbrace{\sum_{\substack{C \leq G \\ p \mid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \frac{\mu(|C:H|)}{p|G:H|}}_{(\star)} \end{aligned}$$

We claim that $(\star)$ is equal to $\frac{\mathcal{P}(G,1)}{|G| \cdot (p-1)}$ and $(\dagger)$ is equal to $\frac{1}{|G|} \sum_{g \in G} v_p(|g|)$. To see this suppose that $f\colon G \to \mathbb{C}$ is any map of sets which is constant on elements $g \in G$ for which $v_p(|g|)$ is equal and for which $f(g) = 0$ when $v_p(|g|) = 0$. In this case, we have that

$$\sum_{\substack{C\leq G\\ p||C|}}\sum_{\substack{H\leq C\\ p\nmid|C:H|}}\frac{p-1}{p}\frac{\mu(|C:H|)}{|G:H|}f(h)=\frac{1}{|G|}\sum_{g\in G}f(g),$$

where on the left hand side h denotes any generator of H. This follows from the fact that for a cyclic group C

$$\sum_{H\leq C}\mu(|C:H|)|H|=|\{\text{generators of C}\}|.$$

Setting $f(g)=v_p(|g|)$ gives

$$\sum_{\substack{C\leq G\\ p||C|}}\sum_{\substack{H\leq C\\ p\nmid|C:H|}}v_p(|H|)\cdot\frac{p-1}{p}\cdot\frac{\mu(|C:H|)}{|G:H|}=\frac{1}{|G|}\sum_{g\in G}v_p(|g|),$$

whilst taking $f(g)=\begin{cases}\frac{1}{p-1} & p\mid|g|\\ 0 & p\nmid|g|\end{cases}$ shows that

$$\begin{aligned}\sum_{\substack{C\leq G\\ p||C|}}\sum_{\substack{H\leq C\\ p\nmid|C:H|}}\frac{\mu(|C:H|)}{p|G:H|}&=\frac{1}{|G|}\sum_{\substack{g\in G\\ v_p(|g|)\geq 1}}\frac{1}{p-1}\\ &=\frac{\mathcal{P}(G,1)}{|G|\cdot(p-1)}.\end{aligned}$$

In conclusion,

$$v_p(C_{\theta_G}(\mathbb{1}_G))=-v_p(|G|)+\frac{1}{|G|}\sum_{g\in G}v_p(|g|)+\frac{\mathcal{P}(G,1)}{|G|\cdot(p-1)}.$$

□

We shall see that the value of (2.8) is less than or equal to zero for all groups G. Thus, Theorem 2.5.1 gives a numerical characterisation of groups for which all p-hypo-elementary subgroups are cyclic:

Corollary 2.5.13. *Let G be any finite group and p a prime. Then G contains no non-cyclic p-hypo-elementary subgroups if and only if*

$$\frac{1}{|G|}\sum_{g\in G}v_p(|g|)+\frac{\mathcal{P}(G,1)}{|G|\cdot(p-1)}=v_p(|G|).$$

The reverse direction whilst a consequence of the argument given in Remark 2.5.2 is already somewhat non-obvious.

Remark 2.5.14. Suppose that G has non-cyclic Sylow p-subgroups. Let $d = v_p(|G|)$, as G contains no elements of order p^d, for any $g \in G$ $v_p(|g|) \leq d-1$ and we may crudely bound

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G} v_p(|g|) + \frac{\mathcal{P}(G,1)}{|G|\cdot(p-1)} &\leq \frac{1}{|G|}\sum_{g\in G}\left(d-1+\frac{1}{p-1}\right)\\ &< d.\end{aligned}$$

Applying (2.8) gives

$$v_p(C_{\theta_G}(\mathbb{1}_G)) < 0.$$

The case of cyclic Sylow p-subgroups requires considerably more care.

2.5.2 Average p-orders of elements of groups with cyclic Sylow subgroups

In this section, we complete the proof of Theorem 2.5.1 by explicit calculation of $v_p(C_{\theta_G}(\mathbb{1}_G))$ for groups with cyclic Sylow p-subgroups using (2.8). This requires an explicit calculation of $\mathcal{P}(G,k)$ in terms of elementary invariants:

Proposition 2.5.15. *Let G be any group with cyclic Sylow p-subgroups of order p^r. Then, for any $1 \leq k \leq r$,*

$$\mathcal{P}(G,k) = \left(\frac{p^{r-k+1}-1}{p^{r-k+1}}\right)\frac{|G||Z_G(Q)|}{|N_G(Q)|},$$

where Q denotes any choice of non-trivial p-subgroup of G. If $k=0$, then $\mathcal{P}(G,k) = |G|$ and if $k > r$, then $\mathcal{P}(G,k) = 0$.

Here, $Z_G(-)$ is as defined in Notation 2.4.9. We split the proof into four intermediate claims. Firstly, the ratio $|N_G(Q)|/|Z_G(Q)|$ is independent of the choice of Q:

Claim 1. *If G is any finite group and p a prime such that G has cyclic Sylow p-subgroups, then, as Q runs over non-trivial p-subgroups, $|N_G(Q)|/|Z_G(Q)|$ is constant.*

Proof. For such a group all p-subgroups of the same order are conjugate. If Q, Q' are conjugate p-subgroups their normalisers and centralisers are related by conjugation and so the above ratio is constant. Thus, we need just show that if P is a subgroup of order p^e, $e \geq 2$, and Q its unique subgroup of order p^{e-1}, then

$$\frac{|N_G(P)|}{|Z_G(P)|} = \frac{|N_G(Q)|}{|Z_G(Q)|}. \tag{2.9}$$

First note that $N_G(P) \cap Z_G(Q) = Z_G(P)$. This is because both sides contain P, but the coprime to p-part of $\mathrm{Aut}(P)$ is canonically isomorphic to the coprime to p-part of $\mathrm{Aut}(Q)$ (both are cyclic of order $p-1$). In other words, within $N_G(P)$, to centralise Q is to centralise P. As a result, there is an inclusion

$$N_G(P)/Z_G(P) \hookrightarrow N_G(Q)/Z_G(Q),$$

and to prove (2.9) we must show that $N_G(Q) = N_G(P)Z_G(Q)$. Indeed, as all terms are contained in $N_G(Q)$, we may assume that $Q \trianglelefteq G$. Each choice of subgroup of order p^e (i.e. conjugate of P) must centralise Q, its unique subgroup of order p^{e-1}, thus $\bigcup_{g \in G/N_G(P)} P^g \subseteq Z_G(Q)$. In particular, $Z_G(Q)$ contains a representative of each coset of $G/N_G(P)$ and so $N_G(P)Z_G(Q) = G$. And in general, $N_G(P)Z_G(Q) = N_G(Q)$. $\square$

Next, we show that to prove the formula for fixed k we may reduce to groups with a central C_{p^k} subgroup.

Claim 2. *For any finite group G, prime p and $k \geq 1$, all elements of G of order divisible by p^k are contained in $\bigcup_Q Z_G(Q)$ as Q runs over subgroups of G isomorphic to C_{p^k}. Moreover, if G has cyclic Sylow p-subgroups, then*

$$\mathcal{P}(G,k) = |G : N_G(Q)| \cdot \mathcal{P}(Z_G(Q),k),$$

for any choice of $Q \cong C_{p^k}$.

Proof. Let $g \in G$ and $v_p(|g|) \geq k$. Then g centralises the subgroup of $\langle g \rangle$ isomorphic to C_{p^k}. So g is contained in $\bigcup_Q Z_G(Q)$, the union of the centralisers of all C_{p^k}-subgroups of G. Now let G have cyclic Sylow p-subgroups and $Q \leq G$ be a choice of C_{p^k}-subgroup. Since G has cyclic Sylow p-subgroups, Q must be the unique C_{p^k}-subgroup of $Z_G(Q)$. As a result, if Q' is a distinct C_{p^k}-subgroup, then $Z_G(Q) \cap Z_G(Q')$ does not contain any C_{p^k}-subgroup, and so $\mathcal{P}(Z_G(Q) \cap Z_G(Q'),k) = 0$. Thus,

$$\mathcal{P}(G,k) = \sum_Q \mathcal{P}(Z_G(Q),k) = |G : N_G(Q)| \cdot \mathcal{P}(Z_G(Q),k). \qquad \square$$

As a basis for induction we show:

Claim 3. *Let G be any group and Q a subgroup of order p that is contained in the centre. Then*

$$\mathcal{P}(G,1) = \mathcal{P}(G/Q,1) + \frac{p-1}{p} \cdot |G|.$$

Proof. Consider the sequence

$$1 \to Q \to G \xrightarrow{q} G/Q \to 1,$$

and let h run over elements of G/Q. First assume that h has order not divisible by p. As $Q \leq Z(G)$, the preimage of $\langle h \rangle$ is isomorphic to $C_p \times C_{|h|}$ on which q is projection onto the second factor. Thus, $q^{-1}(h)$ contains precisely $p-1$ elements of order p.

Otherwise, if h has order divisible by p, then all elements of $q^{-1}(h)$ have order divisible by p. As a result

$$\mathcal{P}(G,1) = p \cdot \mathcal{P}(G/Q,1) + (p-1)(|G/Q| - \mathcal{P}(G/Q,1)),$$

giving the stated formula. □

The inductive step is given by:

Claim 4. *Let G be any group with cyclic Sylow p-subgroups and containing a central subgroup Q of order p^k with $k \geq 2$. Then*

$$\mathcal{P}(G,k) = p \cdot \mathcal{P}(G/\tilde{Q}, k-1),$$

where $\tilde{Q} \leq Q$ denotes the subgroup of order p

Proof. Consider the sequence

$$1 \to \tilde{Q} \to G \xrightarrow{q} G/\tilde{Q} \to 1.$$

Running over elements $h \in G/\tilde{Q}$, we find that if p^k divides $|h|$, then all p preimages have order divisible by p^k and conversely if $p^{k-1} \nmid |h|$, then none do.

Now assume that p^{k-1} is the maximal power of p dividing $|h|$. Then $H := q^{-1}(\langle h \rangle)$ is a subgroup of G with Sylow p-subgroups of order p^k. Thus, H must be of the form $C_{p^k} \times A$ with p not dividing the order of A. Via this description q is the quotient $C_{p^k} \times A \to C_{p^k}/C_p \times A$. So, as h has order divisible by p^{k-1}, all elements in the fibre of h have order divisible by p^k. In conclusion,

$$\mathcal{P}(G,k) = p \cdot \mathcal{P}(G/\tilde{Q}, k-1). \qquad \square$$

Proof of Prop. 2.5.15. We first show the formula when $k = 1$. The formula trivially holds when $r = 0$. If $r \geq 1$, we may apply Claim 2 to assume that G contains a central subgroup Q isomorphic to C_p. When $r = 1$, the formula is given by Claim 3. Now assume $r \geq 2$. We wish to show that

$$\mathcal{P}(G,1) = \left(\frac{p^r - 1}{p^r}\right) |G|.$$

Applying Claim 3 and the inductive hypothesis,

$$\begin{aligned}\mathcal{P}(G,1) &= \mathcal{P}(G/Q,1) + \left(\frac{p-1}{p}\right)|G| \\ &= \left(\frac{p^{r-1}-1}{p^{r-1}}\right)\frac{|G/Q|\cdot|Z_{G/Q}(Q')|}{|N_{G/Q}(Q')|} + \left(\frac{p-1}{p}\right)|G|,\end{aligned}$$

where Q' is a choice of C_p-subgroup of G/Q. Let P denote the preimage of Q' in G. Recall, for a chain of subgroups $A \geq B \geq C$ with $C \trianglelefteq A$, then $N_A(B)/C \cong N_{A/C}(B/C)$. Moreover if $C \subseteq Z(A)$, then $Z_A(B)/C = Z_{A/C}(B/C)$. Thus, $|G/Q : N_{G/Q}(Q')| = |G : N_G(P)|$ and $|Z_{G/Q}(Q')| = \frac{1}{p}|Z_G(P)|$. So

$$\begin{aligned}\mathcal{P}(G,1) &= \left(\frac{p^{r-1}-1}{p^{r-1}}\right)\frac{|G|\cdot|Z_G(P)|}{|N_G(P)|\cdot p} + \left(\frac{p-1}{p}\right)|G| \\ &= \left(\frac{p-1}{p} + \frac{p^{r-1}-1}{p^r}\right)|G| \\ &= \left(\frac{p^r-1}{p^r}\right)|G|\end{aligned}$$

as required, where we used the independence asserted in Claim 1 to show

$$\frac{|Z_G(P)|}{|N_G(P)|} = \frac{|Z_G(Q)|}{|N_G(Q)|} = 1.$$

Thus, the formula holds when $k = 1$.

Now assume $k > 1$ and that the formula holds for all groups and indices $\ell < k$. By Claim 2, we are reduced to verifying the formula for groups with a central subgroup Q isomorphic to C_{p^k}.

Fix a subgroup $\tilde{Q} \leq Q$ of order p. Applying Claim 4,

$$\begin{aligned}\mathcal{P}(G,k) &= p \cdot \mathcal{P}(G/\tilde{Q}, k-1) \\ &= p\left(\frac{p^{(r-1)-(k-1)+1}-1}{p^{(r-1)-(k-1)+1}}\right)\frac{|G/\tilde{Q}| \cdot |G/\tilde{Q}|}{|G/\tilde{Q}|} \\ &= \left(\frac{p^{r-k+1}-1}{p^{r-k+1}}\right)|G|\end{aligned}$$

which is the required formula. □

Proof of Theorem 2.5.1. By Remarks 2.5.2 and Corollary 2.5.9, we need only prove that if G has a non-cyclic p-hypo-elementary subgroup $v_p(C_{\theta_G}(\mathbb{1}_G)) < 0$. Whilst, by Remark 2.5.14, we may assume G has cyclic Sylow p-subgroups.

Applying Lemma 2.5.12, we want to show for such G that

$$\frac{1}{|G|}\sum_{g\in G} v_p(|g|) + \frac{\mathcal{P}(G,1)}{|G|\cdot(p-1)} \leq v_p(|G|),$$

with equality if and only if all p-hypo-elementary subgroups of G are cyclic. Proposition 2.5.15 shows that if G has Sylow p-subgroups of order p^r, then

$$\sum_{g\in G} v_p(|g|) = \sum_{k=1}^{r} \mathcal{P}(G,k) = \sum_{k=1}^{r}\left(\frac{p^{r-k+1}-1}{p^{r-k+1}}\right)\frac{|G||Z_G(Q)|}{|N_G(Q)|}$$

and

$$\frac{\mathcal{P}(G,1)}{(p-1)} = \left(\frac{p^r-1}{(p-1)p^r}\right)\cdot\frac{|G||Z_G(Q)|}{|N_G(Q)|},$$

where Q denotes any choice of subgroup of G isomorphic to C_p. Thus,

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G} v_p(|g|) + \frac{\mathcal{P}(G,1)}{|G|\cdot(p-1)} &= \left(\left(\sum_{i=1}^{r}\frac{p^{r-i+1}-1}{p^{r-i+1}}\right) + \frac{p^r-1}{(p-1)p^r}\right)\frac{|Z_G(Q)|}{|N_G(Q)|} \\ &= \left(\sum_{i=1}^{r}\frac{p^{r-i+1}-1}{p^{r-i+1}} + \sum_{i=1}^{r}\frac{1}{p^{r-i+1}}\right)\frac{|Z_G(Q)|}{|N_G(Q)|} \\ &= r\cdot\frac{|Z_G(Q)|}{|N_G(Q)|}.\end{aligned}$$

So that

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -r\cdot\left(1-\frac{|Z_G(Q)|}{|N_G(Q)|}\right)$$

whenever G has cyclic Sylow p-subgroups. Finally, note that a group with cyclic Sylow p-subgroups has no non-cyclic p-hypo-elementary groups if and only if all subgroups of order pq with q a prime distinct to p are isomorphic to $C_p \times C_q$. The latter holds if and only if the normaliser of each C_p-subgroup is equal to its centraliser. So indeed $v_p(C_{\theta_G}(\mathbb{1}_G)) < 0 \iff G$ contains a non-cyclic Sylow p-subgroup, otherwise it is zero. □

It is worth stating that during the proof we derived the following corollary:

Corollary 2.5.16. *For any finite group G and prime p such that the Sylow p-subgroups of G are cyclic,*

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -r \cdot \left(1 - \frac{|Z_G(Q)|}{|N_G(Q)|}\right).$$

Here Q denotes any choice of non-trivial p-subgroup of G unless $p \nmid |G|$ in which case $Q = \{1\}$ and $v_p(C_{\theta_G}(\mathbb{1}_G)) = 0$.

When G doesn't have cyclic Sylow p-subgroups, we are only able to say that $v_p(C_{\theta_G}(\mathbb{1}_G)) \leq -\frac{p}{|G|}$.

Example 2.5.17. Let G be a p-hypo-elementary group with a non-trivial cyclic Sylow p-subgroup. Then G is of the form $C_{p^r} \rtimes C_n$ with $(p, n) = 1$. Let S denote the kernel of the map $C_n \to \mathrm{Aut}(C_{p^r})$ defining the semi-direct product and $s = |S|$. Then

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -r(1 - \frac{s}{n}),$$

as the centraliser of C_{p^r} is $C_{p^r} \times S \leq G$ (the action is trivial) and $C_{p^r} \trianglelefteq G$.

We can also verify this directly. In Example, 2.2.15 we saw that the Artin relation of such a G is given by

$$\theta_G = [C_{p^r} \rtimes C_n] - [C_n] + \frac{s}{n}[S] - \frac{s}{n}[C_{p^r} \times S].$$

So applying formula (2.4)

$$C_{\theta_G}(\mathbb{1}_G) = \frac{\left(\frac{1}{|G|}\right) \cdot \left(\frac{1}{|S|}\right)^{\frac{s}{n}}}{\left(\frac{1}{|C_n|}\right)\left(\frac{1}{|C_{p^r} \times S|}\right)^{\frac{s}{n}}},$$

so that indeed

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -r + \frac{s}{n} \cdot r.$$

2.6 Yakovlev's theorem and the permutation pairing

If a group G has cyclic Sylow p-subgroups, then Theorem 2.4.1 shows that the permutation pairing is non-degenerate. As an application, if G is in addition abelian, dihedral, or more generally satisfies the conditions of Theorem 2.6.9, then we exhibit an explicit list of invariants which determine the isomorphism class of an arbitrary rationally self-dual $\mathbb{Z}_p[G]$-lattice. This requires also understanding the theory over $\mathbb{Z}_{(p)}$, which is somewhat less well behaved.

2.6.1 Trivial source modules

As we shall see, existing work reduces us to dealing with *trivial source modules*, which we now introduce.

Definition 2.6.1. For any finite G, we say that an $\mathcal{R}[G]$-module has *trivial source*, or is a *trivial source module*, if it is a direct summand of a permutation module. In other sources, trivial source modules may be referred to as any of relatively projective, permutation projective, p-permutation or invertible. For $\mathcal{R}$ any ring, we denote the subalgebra of $A(\mathcal{R}[G])$ generated by the trivial source modules by $A(\mathcal{R}[G], \mathrm{triv})$.

Our definition is slightly non-standard (cf. [Ben98, Def. 3.11.1]). When $\mathcal{R} = \mathbb{Z}_p$, it coincides with the usual definition [Ben98, Lem. 3.11.2], but when $\mathcal{R} = \mathbb{Z}_{(p)}$, due to the failure of Krull-Schmidt [Ben06], decomposition by vertices fails (see Example 2.6.20) and what we call an indecomposable trivial source module over $\mathbb{Z}_{(p)}$ need not have source which is trivial. However, in our definition, for M over $\mathbb{Z}_{(p)}$, M is a trivial source module if and only if $M \otimes \mathbb{Z}_p$ is a trivial source module.

Example 2.6.2. Let $\mathcal{R} = \mathbb{Z}_{(p)}$ and $G = C_p$. Up to isomorphism, there are 3 indecomposable $\mathbb{Z}_{(p)}[C_p]$-lattices

$$\mathbb{1}_G, I_G, \mathbb{Z}_{(p)}[C_p],$$

the trivial module, the augmentation ideal of $\mathbb{Z}_{(p)}[C_p]$, and the regular representation [HR62, Thm. 2.6]. The indecomposable trivial source modules are the summands of $\mathbb{1}_{\{1\}}\uparrow^{C_p} \cong \mathbb{Z}_{(p)}[C_p]$ and $\mathbb{1}_{C_p}\uparrow^{C_p} = \mathbb{1}_{C_p}$. So the trivial source indecomposables are precisely $\mathbb{1}_G$ and $\mathbb{Z}_{(p)}[C_p]$, and $A(\mathbb{Z}_{(p)}[G], \mathrm{triv}) = A(\mathbb{Z}_{(p)}[G], \mathrm{perm})$ is two dimensional. The same holds for $\mathcal{R} = \mathbb{Z}_p$.

Definition 2.6.3. Let M be any $\mathbb{Z}_p[G]$-lattice. We define M_{triv} to be the submodule generated by all indecomposable trivial source summands of M. We call M_{triv} the *trivial source part* of M and call the submodule M_{nt} generated by the indecomposable summands which are not trivial source summands the *non-trivial source part*. By the Krull-Schmidt property of $\mathbb{Z}_p[G]$-modules, we obtain the *trivial source decomposition* $M = M_{\mathrm{triv}} \oplus M_{\mathrm{nt}}$.

Remark 2.6.4. Over $\mathbb{Z}_{(p)}$, lattices do not admit unique decomposition [Ben06] and there is no general analogue of the trivial source decomposition. In particular, for M over $\mathbb{Z}_{(p)}$, the trivial source decomposition of $M \otimes \mathbb{Z}_p$ need not be defined over $\mathbb{Z}_{(p)}$. This is exhibited by the following example:

Example 2.6.5. Let $G = C_3 \times C_4$ and $p = 3$. Over the ring of integers $\mathcal{O}_K$ of $K = \mathbb{Q}_3(i)$, we have that $\mathbb{1}\uparrow_{C_3}^G$ decomposes as $\bigoplus_{i=0}^3 \chi^i$, where χ denotes the inflation of a faithful character of C_4. The trivial source modules over $\mathcal{O}_K$ are then summands of

$$\mathbb{1}\uparrow^G = \bigoplus_{i=0}^{3} \mathbb{1}\uparrow_{C_4}^G \otimes \chi^i,$$

$$\mathbb{1}\uparrow_{C_3}^G = \bigoplus_{i=0}^{3} \chi^i.$$

Let I_{C_3} denote the augmentation ideal of C_3. All non-trivial source modules are summands of

$$I_{C_3}\uparrow^G = \bigoplus_{i=0}^{3} I_{G/C_4} \otimes \chi^i$$

where I_{G/C_4} is the augmentation ideal of G/C_4, i.e. the kernel of the trace map $\mathcal{O}_K[G/C_4] \to \mathbb{1}_{\mathcal{O}_K,G}$. This decomposition can be checked via the isomorphism $\mathbb{1}\uparrow^G \cong \mathbb{1}\uparrow_{C_3}^G \otimes \mathbb{1}\uparrow_{C_4}^G$. Now consider

$$M := (I_{G/C_4} \otimes \chi) \oplus \chi \oplus (\mathbb{1}\uparrow_{C_4}^G \otimes \chi^3).$$

I claim that M is defined over $\mathbb{Z}_{(3)}$. By [Rei70, Prop. 5.7], we need only check that $M \otimes K$ is defined over $\mathbb{Q}$. But $(I_{G/C_4} \otimes \chi) \otimes K \cong (\mathbb{1}_{K,C_4}\uparrow^G \ominus \mathbb{1}_K) \otimes (\chi_K)$ (where $\ominus$ denotes quotienting by the image of $\mathbb{1}$ under a section of the trace map, and $\chi_K = \chi \otimes K$). Thus $M \otimes K \cong \mathbb{1}_{K,C_4}\uparrow^G \otimes (\chi_K \oplus \chi_K^3)$, which is defined over $\mathbb{Q}$.

It is clear that $M_{\mathrm{triv}} = \chi \oplus (\mathbb{1}\uparrow_{C_4}^{G} \otimes \chi^3)$ cannot be defined over $\mathbb{Z}_{(3)}$ as its character is not rational. Note that, $[M] \in A(\mathbb{Z}_{(p)}[G])$ and $[M_{\mathrm{triv}}] \in A(\mathbb{Z}_p[G], \mathrm{triv})$ but $[M_{\mathrm{triv}}] \notin A(\mathbb{Z}_{(p)}[G], \mathrm{triv}) \subset A(\mathbb{Z}_p[G], \mathrm{triv})$.

2.6.2 Yakovlev's result

Now assume that G has cyclic Sylow p-subgroups.

Notation 2.6.6. Let P be a choice of Sylow p-subgroup of G. Let r be such that $P \cong C_{p^r}$, and for $0 \leq i \leq r$, let $P_i \leq P$ denote the subgroup of order p^i.

Note that for a $\mathbb{Z}_p[G]$-lattice M, $H^1(P_i, M)$ is a $N_G(P_i)$-module.

Theorem 2.6.7 (Yakovlev [Yak96, Thm. 2.1])**.** *Let G be a finite group and p a prime such that G has cyclic Sylow p-subgroups. If M is a $\mathbb{Z}_p[G]$-lattice, then the isomorphism class of M_{nt} is determined by the following diagram,*

$$H^1(P_r, M) \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} H^1(P_{r-1}, M) \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} \quad \cdots \quad \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} H^1(P_0, M).$$

Figure 2.3: Yakovlev diagram

To be precise, when we say "determined by" we mean that, if M' is another $\mathbb{Z}_p[G]$-lattice for which there are $\mathbb{Z}_p[N_G(P_i)]$-module isomorphisms $\kappa_i : H^1(P_i, M) \to H^1(P_i, M'), 0 \leq i \leq n$ which commute with restriction and corestriction in the above diagram, then $M_{\mathrm{nt}} \cong M'_{\mathrm{nt}}$.

Construction 2.6.8. Call any diagram of the form

$$\bullet \underset{b_r}{\overset{a_r}{\rightleftarrows}} \bullet \underset{b_{r-1}}{\overset{a_{r-1}}{\rightleftarrows}} \quad \cdots \quad \underset{b_1}{\overset{a_1}{\rightleftarrows}} \bullet,$$

with the i^{th} term a finite $N_G(P_{r-i+1})$-module and a_i, b_i homomorphisms of abelian groups, a *Yakovlev diagram.* For any M, Figure 2.3 is of this form and we refer to it as the *Yakovlev diagram of M*.

There is an obvious notion of direct sum of such diagrams. Let $\mathcal{C}$ denote the free $\mathbb{Q}$-vector space on isomorphism classes of such diagrams subject to identifying addition of diagrams with addition of elements of $\mathcal{C}$. Taking Yakovlev diagrams defines a canonical map

$$\mathrm{Yak}\colon A(\mathbb{Z}_p[G]) \to \mathcal{C}.$$

Yakovlev's theorem is now the assertion that $\ker(\mathrm{Yak}) = A(\mathbb{Z}_p[G], \mathrm{triv})$. Yakovlev also gives a converse describing which Yakovlev diagrams arise as the cohomology of $\mathbb{Z}_p[G]$-lattices, but we do not need this.

Recall that $A(\mathbb{Z}_{(p)}[G], \mathrm{perm}) = A(\mathbb{Z}_p[G], \mathrm{perm})$ as subspaces of $A(\mathbb{Z}_p[G])$. We are now able to correctly formulate the theorem outlined in the introduction:

Theorem 2.6.9. *Let G be any finite group and p a prime such that G has cyclic Sylow p-subgroups and for which $A(\mathbb{Z}_p[G], \mathrm{perm}) = A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$. Then the isomorphism class of any rationally self-dual $\mathbb{Z}_p[G]$-lattice M is determined by*

i) the isomorphism class of $M \otimes \mathbb{Q}_p$ as a $\mathbb{Q}_p[G]$-module,

ii) the valuations $v_p(C_{\theta_H}(M))$ of the regulator constants of the Artin relations for $H \in \mathrm{nchyp}_p(G)$,

iii) the Yakovlev diagram

$$H^1(P_r, M) \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} H^1(P_{r-1}, M) \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} \quad \dots \quad \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} H^1(P_0, M).$$

Proof. By Theorem 2.6.7 the data of *iii)* determines the isomorphism class of M_{nt}. Now fix some trivial source $\mathbb{Z}_p[G]$-module M' such that $M_{\mathrm{nt}} \oplus M'$ is rationally self-dual. Such an M' exists as M_{triv} is an example. By linearity of regulator constants and extension of scalars, from *i),ii),iii)* we also obtain the regulator constants and isomorphism class of the extension of scalars of $t := [M] - [M_{\mathrm{nt}}] - [M']$ (note that t is rationally self-dual so that its regulator constants are defined). By construction $t \in \ker(\mathrm{Yak}\colon A(\mathbb{Z}_p[G]) \to \mathcal{C}) = A(\mathbb{Z}_p[G], \mathrm{triv})$. It suffices to show that any rationally self-dual element $t \in A(\mathbb{Z}_p[G], \mathrm{triv})$ is determined by the data of *i),ii)*.

Let M'' be any trivial source $\mathbb{Z}_p[G]$-module such that $t - M''$ has rational character, again such an M'' certainly exists. Now, any $\mathbb{Z}_p[G]$-lattice N for which $N \otimes \mathbb{Q}_p$ is defined over $\mathbb{Q}$ is the extension of scalars of a $\mathbb{Z}_{(p)}[G]$-lattice [Rei70, Prop. 5.7]. As a result, any element of $A(\mathbb{Z}_p[G])$ with rational character is contained within $A(\mathbb{Z}_{(p)}[G])$. In particular, $t - M'' \in A(\mathbb{Z}_p[G], \mathrm{triv}) \cap A(\mathbb{Z}_{(p)}[G])$. We claim that this space is precisely $A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$, so let W, V be trivial source $\mathbb{Z}_p[G]$-modules and assume that $[W]-[V]$ has rational character. If V is a summand of some permutation module T with complement V', then $[W \oplus V'] - [T] = [W] - [V]$ and $W \oplus V'$ is a trivial source module with rational character. So, by [Rei70, Prop. 5.7], $[W] - [V']$ lies in $A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$.

Thus, $t - M'' \in A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$ and so by assumption lies in $A(\mathbb{Z}_p[G], \mathrm{perm})$. As a result, it is determined by its regulator constants and extension of scalars (Corollary 2.4.2). Tracing back, we find that M is determined by the data of *i),ii),iii)*. $\square$

Since *i),ii),iii)* are isomorphism invariants, two $\mathbb{Z}_p[G]$-lattices are isomorphic if and only if *i),ii),iii)* are the same for both lattices.

Remark 2.6.10. The restriction on being rationally self-dual is a somewhat mild one. For example, if M_1, M_2 are any two $\mathbb{Z}_p[G]$-lattices, then $M_1 \cong M_2$ if and only if, *i),iii)* of Theorem 2.6.9 coincide for M_1, M_2 and

ii') there exists some $\mathbb{Z}_p[G]$-lattice N such that $M_1 \oplus N$ and $M_2 \oplus N$ are both rationally self-dual, and the valuations $v_p(C_{\theta_H}(M_i \oplus N))$ of the regulator constants of the Artin relations of $M_i \oplus N$ are equal for all $H \in \mathrm{nchyp}_p(G)$.

Note, it is easy to determine if there exists a $\mathbb{Z}_p[G]$-lattice N such that $M_1 \oplus N$ and $M_2 \oplus N$ are rationally self-dual using *i)*. If the $v_p(C_{\theta_H}(M_i \oplus N))$ are equal for one such N, then they are equal for all.

The condition that $A(\mathbb{Z}_p[G], \mathrm{perm}) = A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$ is investigated in the next subsection. For reference, we shall see that the equality can be checked on restriction to the p-hypo-elementary subgroups and that dihedral groups, abelian groups with cyclic Sylow p-subgroups and groups of order coprime to $p-1$ all have this property, but $C_p \rtimes C_{p-1}$ for $p \geq 5$ does not. In Section 2.7.1, we provide a worked example of Theorem 2.6.9 for dihedral groups of order $2p$ with p odd.

Remark 2.6.11. Theorem 2.6.9 is sharp in the following sense. If $A(\mathbb{Z}_p[G], \mathrm{perm}) \subsetneq A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$ or the permutation pairing is degenerate, then rationally self-dual $\mathbb{Z}_p[G]$-lattices are not determined by *i),ii),iii)*. The first case can be seen by comparing the dimension of $A(\mathbb{Z}_p[G], \mathrm{triv})$ and the maximum number of linear conditions on elements of $A(\mathbb{Z}_p[G], \mathrm{triv})$ we could possibly obtain from *i),ii)* using Lemmas 2.6.25, 2.6.26 and the formulae of Section 2.2.4. In the second case, not even all permutation lattices can be distinguished (see Lemma 2.3.17).

2.6.3 Species and trivial source modules over $\mathbb{Z}_{(p)}$

Now let G be any finite group.

Definition 2.6.12. A *species*[1] is a ring homomorphism $A(\mathbb{Z}_p[G], \mathrm{triv}) \to \mathbb{C}$.

Example 2.6.13. For any $g \in G$, $\mathrm{tr}(g \mid -)$ defines a ring homomorphism $A(\mathbb{Z}_p[G]) \to \mathbb{C}$ and so also a species.

Definition 2.6.14. For H a subgroup of G, we say that an indecomposable trivial source $\mathbb{Z}_p[G]$-lattice M has vertex H if M is a direct summand of $\mathbb{1}\!\uparrow_H^G$ but not of $\mathbb{1}\!\uparrow_{H'}^G$ for any $H' \lneq H$. The vertices of M form a conjugacy class of subgroups and only p-groups appear as vertices [Ben98, Prop. 3.10.2]. For an arbitrary $\mathbb{Z}_p[G]$-lattice M, we call the summand generated by the indecomposables with vertex P the *vertex* P summand of M, in this way we obtain a decomposition of M indexed by vertices.

Construction 2.6.15. Consider pairs (P, g), where $P \leq G$ is a p-group and g an element of $N_G(P)$ of order coprime to p, up to simultaneous conjugacy. Then $H := \langle P, g \rangle$ is p-hypo-elementary and to any such pair we may associate a species $t_{(P,g)}$ as follows. Consider the composite

$$A(\mathbb{Z}_p[G], \mathrm{triv}) \to A(\mathbb{Z}_p[H], \mathrm{triv}) \to A(\mathbb{Z}_p[H/P], \mathrm{triv}).$$

Here the first map is restriction. The second map sends a lattice M to its vertex P summand N, which, as M is trivial source, is inflated from H/P so can be considered as an H/P-module. We define $t_{(P,g)} \colon A(\mathbb{Z}_p[G], \mathrm{triv}) \to \mathbb{C}$ to be the postcomposition with $\mathrm{tr}(g \mid -)$, i.e. $t_{(P,g)}(M)$ is the trace of g acting on N.

Example 2.6.16. For any $g \in G$, the species defined by $\mathrm{tr}(g \mid -)$ is equal to $t_{(P, g^{|g|})}$, where $|P|$ is the Sylow p-subgroup of $\langle g \rangle$.

The $t_{(P,g)}$ need not be distinct, but all species arise in this way:

Theorem 2.6.17 (Conlon)**.** *For any finite group G and prime p, there is an inclusion*

$$\prod t_{(P,g)} \colon A(\mathbb{Z}_p[G], \mathrm{triv}) \to \prod_{(P,g)} \mathbb{C}.$$

Proof. This is usually stated for the ring of integers $\mathcal{O}_K$ of a sufficiently large extension $K/\mathbb{Q}_p$ (see [Ben98, Cor. 5.5.5]). The stated version then follows as, for $K/\mathbb{Q}_p$ and M, M' any $\mathbb{Z}_p[G]$-lattices, $M \cong M' \iff (M \otimes \mathcal{O}_K) \cong (M' \otimes \mathcal{O}_K)$ and that the action of Galois ensures vertices are preserved under base change by $\mathcal{O}_K/\mathbb{Z}_p$. □

[1] It is more common to define a species as a ring homomorphism from the trivial source ring over the ring of integers of a sufficiently large extension of $\mathbb{Q}_p$, but this is not necessary for our purposes.

Remark 2.6.18. It is worth remarking that although species are good invariants of trivial source modules they cannot be combined with Yakovlev diagrams to give results such as Theorem 2.6.9. This is because species cannot be canonically extended beyond $A(\mathbb{Z}_p[G], \mathrm{triv})$. On the other hand, regulator constants are defined for an arbitrary rationally self-dual lattice.

Lemma 2.6.19. *For a permutation module $\mathbb{1}_K{\uparrow}^G$, we find $t_{(P,g)}(\mathbb{1}_K{\uparrow}^G) = \#(G/K)^H$, where $H = \langle P, g\rangle$.*

Proof. By definition $t_{(P,g)}$ is a function of $\mathbb{1}_K{\uparrow}^G{\downarrow}_H = \bigoplus_{K\backslash G/H} \mathbb{1}{\uparrow}^H_{K^g\cap H}$. We claim that only the terms with $K^g \cap H = H$ have non-trivial species. Indeed, if $K^g \cap H \not\geq P$, then the vertex P summand of $\mathbb{1}{\uparrow}^H_{K^g\cap H}$ is zero, whilst if $K^g \cap H \geq P$, then $\mathbb{1}{\uparrow}^H_{K^g\cap H}$ is all of vertex P and is inflated from the quotient $\langle g\rangle$. It is then clear that $\mathrm{tr}(g \mid -)$ is zero if and only if $K^g \cap H \neq H$, else it is one.

Finally, the number of elements of $K\backslash G/H$ with $K^g \cap H = H$ is precisely the number of elements of G/K fixed under H. □

Example 2.6.20. It is not the case that the species of a trivial source lattice M over $\mathbb{Z}_{(p)}$ need take only rational values. This is made possible by the failure of Krull-Schmidt over $\mathbb{Z}_{(p)}$. For example, if $p \geq 5$ and $G = C_p \rtimes C_{p-1}$ with C_{p-1} acting faithfully, then a trivial source module with non-rational species is constructed as follows (cf. [Ben06]). Let χ denote the inflation of a faithful character of C_{p-1}. The trivial source $\mathbb{Z}_p[G]$-modules are then the summands of

$$\mathbb{1}{\uparrow}^G = \bigoplus_{i=0}^{p-2} \mathbb{1}{\uparrow}^G_{C_{p-1}} \otimes \chi^i,$$
$$\mathbb{1}{\uparrow}^G_{C_p} = \bigoplus_{i=0}^{p-2} \chi^i.$$

Then

$$M := \left(\bigoplus_{\substack{i=0 \\ i\neq 1}}^{p-2} \mathbb{1}{\uparrow}^G_{C_{p-1}} \otimes \chi^i \right) \oplus \chi$$

is defined over $\mathbb{Z}_{(p)}$, but as the only summand of vertex C_p is χ, the vertex decomposition is not defined over $\mathbb{Z}_{(p)}$, and the species of M are non-rational, as $t_{(C_p,\tau)}(M) = \chi(\tau)$ is a primitive $(p-1)^{\mathrm{st}}$ root of unity.

In particular, $A(\mathbb{Z}_p[G], \mathrm{perm}) \subsetneq A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$ as the species of permutation modules are integers (Lemma 2.6.19).

Theorem 2.6.21. *A basis of* $A(\mathbb{Z}_p[G], \mathrm{perm})$ *is given by* $\{\mathbb{1}\uparrow_H^G\}_{H \in \mathrm{hyp}_p(G)}$.

Proof. We first check linear independence. By Theorem 2.6.17, we need only check linear independence after taking species. By Lemma 2.6.19, the species $t_{(P,g)}(\mathbb{1}\uparrow_H^G)$ is zero whenever no conjugate of H contains $\langle P, g\rangle$, whilst $t_{(P,g)}(\mathbb{1}\uparrow_{\langle P,g\rangle}^G) \neq 0$. After (non-uniquely) ordering the elements of $\mathrm{hyp}_p(G)$ by increasing size, linear independence is now clear.

Now, let i be such that $\langle P, g\rangle = \langle P, g^i\rangle$ and let $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ raise the $|g|^{\mathrm{th}}$ roots of unity to the i^{th} power. Then, for any trivial source module M over $\mathbb{Z}_p$,

$$\begin{aligned} t_{(P,g^i)}(M) &= \mathrm{tr}(g^i \mid \text{vertex } P \text{ summand of } M\downarrow_{\langle P,g^i\rangle}) \\ &= \mathrm{tr}(g^i \mid \text{vertex } P \text{ summand of } M\downarrow_{\langle P,g\rangle}) \\ &= \mathrm{tr}(g \mid \text{vertex } P \text{ summand of } M\downarrow_{\langle P,g\rangle})^\sigma \\ &= t_{(P,g)}(M)^\sigma. \end{aligned}$$

But for permutation modules M, $t_{(P,g)}(M)$ is rational (Lemma 2.6.19), so $t_{(P,g)}(M)$ is constant on pairs (P, g) generating the same p-hypo-elementary subgroup up to conjugacy. Therefore,

$$\dim_{\mathbb{Q}} A(\mathbb{Z}_p[G], \mathrm{perm}) \leq \#\{\text{conjugacy classes of } p\text{-hypo-elementary groups}\}. \qquad \square$$

We used this in Theorem 2.2.20 to find a basis of the space of Brauer relations in characteristic p. Examining the proof of the theorem we find:

Corollary 2.6.22. *For any finite group G and prime p, within* $A(\mathbb{Z}_p[G], \mathrm{triv})$, $A(\mathbb{Z}_p[G], \mathrm{perm})$ *is precisely the subspace of elements whose species are all rational.*

This also follows from work of Fan Yun [Fan91].

Lemma 2.6.23. *Let G be a finite group and p a prime. If* $A(\mathbb{Z}_p[H], \mathrm{perm}) = A(\mathbb{Z}_{(p)}[H], \mathrm{triv})$ *for all* $H \in \mathrm{hyp}_p(G)$, *then* $A(\mathbb{Z}_p[G], \mathrm{perm}) = A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$.

Proof. Suppose that $a \in A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$ is such that, upon restriction to every p-hypo-elementary subgroup H, $a\downarrow_H$ is a permutation module. Then $a\downarrow_H$ has rational species (Lemma 2.6.19). But then a itself must have rational species as species are defined via restriction to the p-hypo-elementary subgroups. Applying Corollary 2.6.22 we find $a \in A(\mathbb{Z}_{(p)}[G], \mathrm{perm})$. $\square$

As a result, a group G satisfies the conditions of Theorem 2.6.9 if its p-hypo-elementary subgroups do.

Notation 2.6.24. Let $(P, g), (P', g')$ define two species and set $n = |g|$. We say $(P, g) \sim_p (P', g')$ if there exists an element $h \in G$ such that $(P')^h = P$ and $(g')^h = g^i$ for some $i \in (\mathbb{Z}/n\mathbb{Z})^\times$ with $i \equiv 1 \pmod{\gcd(n, p-1)}$.

Lemma 2.6.25. *For any finite group G and prime p,*

i) $\dim(A(\mathbb{Z}_p[G], \mathrm{triv})) = \#\left(\{\textit{species } (P, g)\}/\sim_p\right)$*,*

ii) $\dim(\mathrm{im}(A(\mathbb{Z}_p[G], \mathrm{triv}) \to A(\mathbb{Q}_p[G]))) = \#(\{\textit{species } (P, g) \mid \langle P, g\rangle \textit{ is cyclic}\}/\sim_p)$*.*

Proof. For any finite Galois extension $K/\mathbb{Q}_p$, the action of $\mathrm{Gal}(K/\mathbb{Q}_p)$ on lattices respects decompositions into vertices. As a result, $t_{(P,g)}(-) = t_{(P,g^i)}(-)$ as functions on $A(\mathbb{Z}_p[G], \mathrm{triv})$ for any integer i such that $(-)^i$ is an automorphism of $\langle g\rangle$ which acts trivially on the subgroup of order $m = \gcd(n, p-1)$, i.e. whenever $(P, g) \sim_p (P, g^i)$. Together with Theorem 2.6.17, this demonstrates the upper bound on $\dim A(\mathbb{Z}_p[G], \mathrm{triv})$. For the lower bound, use that the Green correspondence provides a distinct indecomposable trivial source module of vertex P for every projective indecomposable $\mathbb{Z}_{(p)}[N_G(P)/P]$-lattice (see e.g. [Ben98, Thm. 3.12.2]), of which there are $\#(\{\text{species } (Q, h) \mid Q = P\}/\sim_p)$.

We now show *ii)*. The dimension of $A(\mathbb{Q}_p[G])$ is equal to the number of distinct ring homomorphisms $\mathrm{tr}(g \mid -)\colon A(\mathbb{Q}_p[G]) \to \mathbb{C}$. The dimension of $\mathrm{im}(A(\mathbb{Z}_p[G], \mathrm{triv}) \to A(\mathbb{Q}_p[G]))$ is then the number of species up to $\sim_p$ which are of the form $\mathrm{tr}(g \mid -)$. By Example 2.6.16, we find that this is precisely to $\#\left(\{\text{species } (P, g) \mid \langle P, g\rangle \text{ is cyclic}\}/\sim_p\right)$. □

Lemma 2.6.26. *The following are equivalent*

i) $A(\mathbb{Z}_p[G], \mathrm{perm}) = A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$*,*

ii) the species of all trivial source $\mathbb{Z}_{(p)}[G]$-lattices are rational,

iii) there is an equality

$$\begin{aligned}\dim(A(\mathbb{Z}_p[G], \mathrm{perm})) = &\dim(A(\mathbb{Z}_p[G], \mathrm{triv})) \\ &- \dim(\mathrm{im}(A(\mathbb{Z}_p[G], \mathrm{triv}) \to A(\mathbb{Q}_p[G])) \\ &+ \dim(A(\mathbb{Q}[G])),\end{aligned}$$

iv) there is an equality

$$\begin{aligned}\#(\{p\text{-}hypo\text{-}elementary\ subgroups\}/\sim)\\ &= \#\left(\{species\ (P,g)\}/\sim_p\right)\\ &\quad - \#\left(\{species\ (P,g) \mid \langle P,g\rangle\ is\ cyclic\}/\sim_p\right)\\ &\quad + \#\left(\{cyclic\ subgroups\}/\sim\right),\end{aligned}$$

where $\sim$ denotes up to conjugacy.

Proof. The equivalence *i)* $\iff$ *ii)* is Corollary 2.6.22. For *i)* $\iff$ *iii)*, use that a trivial source $\mathbb{Z}_p[G]$-module is defined over $\mathbb{Z}_{(p)}$ if and only if it has rational character [Rei70, Prop. 5.7], together with the fact that $A(\mathbb{Z}_{(p)}[G], \mathrm{triv}) \to A(\mathbb{Q}[G])$ is surjective by Artin's induction theorem. For *iii)* $\iff$ *iv)*, combine Lemma 2.6.25, Theorem 2.6.21 and Artin's induction theorem. $\square$

We conclude this section by giving examples of groups which satisfy the condition $A(\mathbb{Z}_p[G], \mathrm{perm}) = A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$:

Example 2.6.27. If G is abelian with cyclic Sylow p-subgroups, then all p-hypo-elementary subgroups are cyclic and so, by Lemma 2.6.26 *iv)*, $A(\mathbb{Z}_p[G], \mathrm{perm}) = A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$ (note, there may be $\mathbb{Z}_{(p)}[G]$-lattices M for which $(M \otimes \mathbb{Z}_p)_{\mathrm{triv}}$ does not lie in $A(\mathbb{Z}_p[G], \mathrm{perm})$, cf. Example 2.6.5). When applying Theorem 2.6.9 for such G, the fact that there are no non-cyclic p-hypo-elementary subgroups makes the data of *ii)* empty.

Example 2.6.28. Let p be odd and $G = D_{2q}$ be the dihedral group of order $2q$ for any $q \geq 1$. Recall, that we need only check that the condition for all p-hypo-elementary subgroups. The only possible p-hypo-elementary subgroups of G are either cyclic, in which case they are covered by the previous example, or of the form D_{2p^r} for some $r \geq 1$. In that case, species up to $\sim_p$ are in bijection with subgroups and so Lemma 2.6.26 *iv)* holds.

Example 2.6.29. If $(|G|, p-1) = 1$, then all species of trivial source $\mathbb{Z}_p[G]$-lattices are rational and so by Corollary 2.6.22, $A(\mathbb{Z}_p[G], \mathrm{perm}) = A(\mathbb{Z}_p[G], \mathrm{triv})$. In particular, when $p = 2$ all groups with cyclic Sylow 2-subgroups satisfy the conditions of Theorem 2.6.9.

2.7 Examples

2.7.1 D_{2p}

Let $G = D_{2p} = C_p \rtimes C_2$ be the dihedral group of order $2p$ for p an odd prime. Then G satisfies the conditions of Theorem 2.6.9 for all primes ℓ, by Example 2.6.28. Since all $\mathbb{C}$ irreducible representations of D_{2p} are defined over $\mathbb{R}$, all $\mathbb{Z}_\ell[G]$-lattices are rationally self-dual for any ℓ.

In this section, we explicate how Theorem 2.6.9 distinguishes $\mathbb{Z}_\ell[D_{2p}]$-lattices. Broadly, there are three different cases, when $\ell = 2, p$ or when ℓ is coprime to the order of the group. In the latter case character theory applies (see the remarks on p4) and we shall for simplicity additionally assume that ℓ is a primitive element modulo p, so that all $\mathbb{Q}_\ell[G]$-representations are defined over $\mathbb{Q}[G]$.

It is only possible to be explicit in this case as D_{2p} is one of the few groups for which the isomorphism classes of all indecomposable $\mathbb{Z}_\ell[G]$-lattices have been classified for all ℓ dividing $|G|$.

By assumption all $\mathbb{Q}_\ell[G]$-representations are defined over $\mathbb{Q}$, so by [Rei70, Prop. 5.7] all $\mathbb{Z}_\ell[G]$-lattices are defined over $\mathbb{Z}_{(\ell)}[G]$.

In Example 2.6.28, we checked that $A(\mathbb{Z}_\ell[G], \mathrm{perm}) = A(\mathbb{Z}_{(\ell)}[G], \mathrm{triv})$ for all primes ℓ. So, as in Example 2.2.23, a basis of $A(\mathbb{Z}_\ell[G], \mathrm{perm}) = A(\mathbb{Z}_{(\ell)}[G], \mathrm{triv})$ is given by

$$S = \begin{cases} \mathbb{1}_{\{1\}}\uparrow^G, \mathbb{1}_{C_2}\uparrow^G, \mathbb{1}_{C_p}\uparrow^G & \ell \neq p \\ \mathbb{1}_{\{1\}}\uparrow^G, \mathbb{1}_{C_2}\uparrow^G, \mathbb{1}_{C_p}\uparrow^G, \mathbb{1}_G & \ell = p \end{cases}.$$

And so, this is also a basis of $A(\mathbb{Z}_\ell[G], \mathrm{triv})$. When $\ell \neq 2, p$, all $\mathbb{Z}_\ell[G]$-lattices are projective so S forms a basis of $A(\mathbb{Z}_\ell[G])$.

In the $\ell = 2, p$ cases, we can exhaust the non-trivial source modules via Yakovlev's Theorem 2.6.7. When $\ell = 2$, as $N_{D_{2p}}(C_2) = C_2$, the Yakovlev diagram for a module M simply consists of $H^1(C_2, M)$ as an abelian group. So any $\mathbb{Z}_2[D_{2p}]$-lattice M for which $H^1(C_2, M) \cong \mathbb{Z}/2\mathbb{Z}$ will extend S to a basis of $A(\mathbb{Z}_2[G])$. The sign representation ϵ, that is the non-trivial one dimensional irreducible lifted from $\mathbb{Z}_2[D_{2p}/C_p]$, is one such module.

When $\ell = p$, the Yakovlev diagram of a $\mathbb{Z}_p[G]$-lattice M consists of $H^1(C_p, M)$ as a $\mathbb{F}_p[D_{2p}/C_p]$-module. Since $\mathrm{char}(\mathbb{F}_p) \neq 2$, there are two irreducible $\mathbb{F}_p[D_{2p}/C_p]$-modules, both one dimensional, one with trivial action and one without. So any two lattices whose cohomology exhibits these modules will extend S to a basis of $A(\mathbb{Z}_p[G])$.

If ρ denotes the $(p-1)$-dimensional irreducible $\mathbb{Q}_p[G]$-representation, then there are two non-isomorphic $\mathbb{Z}_p[G]$-sublattices A, A' contained in ρ which are distinguished by the fact that both have $H^1(C_p, A), H^1(C_p, A') \cong \mathbb{Z}/p\mathbb{Z}$ as abelian groups, but the former having non-trivial D_{2p}/C_p action and D_{2p}/C_p acting trivially on the latter. In the case of $p = 3$, we described A and A' and calculated their regulator constants in Example 2.2.28. In general, these modules are explicitly constructed in [Lee64].

In conclusion,

$$\dim_{\mathbb{Q}}(A(\mathbb{Z}_\ell[G])) = \begin{cases} 3 & \ell \neq 2, p \\ 4 & \ell = 2 \\ 6 & \ell = p \end{cases}.$$

with a basis S' given by

$$S' = \begin{cases} \mathbb{1}_{\{1\}}\uparrow^G, \mathbb{1}_{C_2}\uparrow^G, \mathbb{1}_{C_p}\uparrow^G & \ell \neq 2, p \\ \mathbb{1}_{\{1\}}\uparrow^G, \mathbb{1}_{C_2}\uparrow^G, \mathbb{1}_{C_p}\uparrow^G, \epsilon & \ell = 2 \\ \mathbb{1}_{\{1\}}\uparrow^G, \mathbb{1}_{C_2}\uparrow^G, \mathbb{1}_{C_p}\uparrow^G, \mathbb{1}_G, A, A' & \ell = p \end{cases}.$$

Denote the extension of scalars map $A(\mathbb{Z}_\ell[G]) \to A(\mathbb{Q}_\ell[G])$ by a, and by b the map $A(\mathbb{Z}_\ell[G]) \to \bigoplus_{H \in \mathrm{nchyp}_\ell(G)} \mathbb{Q}$ which is defined by sending a lattice M to the vector $(v_p(C_{\theta_H}(M)))_{H \in \mathrm{nchyp}_\ell(G)}$. Then Theorem 2.6.9 states that $a \oplus b \oplus \mathrm{Yak}$ is injective (and so an isomorphism). The matrix representing $a \oplus b \oplus \mathrm{Yak}$ is given by:

$$\begin{array}{cc} & \begin{array}{ccc} \mathbb{1}\uparrow_{\{1\}}^G & \mathbb{1}\uparrow_{C_2}^G & \mathbb{1}\uparrow_{C_p}^G \end{array} \\ \begin{array}{c} \mathbb{1} \\ \epsilon \\ \rho \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \end{array} \qquad \text{if } \ell \neq 2, p \ ,$$

$$\begin{array}{rc} & \begin{array}{cccc} \mathbb{1}\uparrow_{\{1\}}^{G} & \mathbb{1}\uparrow_{C_2}^{G} & \mathbb{1}\uparrow_{C_p}^{G} & \epsilon \end{array} \\ \begin{array}{r} \mathbb{1} \\ \epsilon \\ \rho \\ \mathrm{Yak}(-) \end{array} & \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \end{array} \qquad \text{if } \ell = 2,$$

$$\begin{array}{rc} & \begin{array}{cccccc} \mathbb{1}\uparrow_{\{1\}}^{G} & \mathbb{1}\uparrow_{C_2}^{G} & \mathbb{1}\uparrow_{C_p}^{G} & \mathbb{1}_G & A & A' \end{array} \\ \begin{array}{r} \mathbb{1} \\ \epsilon \\ \rho \\ v_p(C_{\theta_G}(-)) \\ \mathrm{Yak}(-) \\ \\ \end{array} & \left(\begin{array}{ccc|c|cc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & -1/2 & 1/2 & -1/2 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} \quad \text{if } \ell = p.$$

Here, we take the basis $\mathbb{1}, \epsilon, \rho$ of $\mathbb{Q}_\ell[G]$, where $\mathbb{1}, \epsilon$ are the trivial and non-trivial one dimensional irreducibles and ρ the $(p-1)$ dimensional irreducible. When $\ell = 2$, the basis of $\mathcal{C}$ is taken to be $\mathbb{Z}/2\mathbb{Z}$, and when $\ell = p$, the basis is given by $\mathbb{Z}/p\mathbb{Z}$ with both its non-trivial and trivial $C_2 \cong D_{2p}/C_p$-actions respectively. The calculations of $v_p(C_{\theta_G}(A)), v_p(C_\theta(A'))$ can be found in [Bar12, Thm. 4.4].

Remark 2.7.1. For $p \leq 67$, a $\mathbb{Z}[D_{2p}]$-lattice is determined by its localisation at the primes $2, p$ (see [Bar12, Ex. 6.3]). So, by applying Theorem 2.6.9 at both primes we obtain a finite list of data which specifies the isomorphism class of an arbitrary $\mathbb{Z}[D_{2p}]$-lattice.

Remark 2.7.2. The above matrices can be seen to be block upper triangular. This was discussed in the proof of Lemma 2.3.17 and is a general phenomenon.

2.7.2 Hybrid group rings

In certain cases, the group ring $\mathbb{Z}_p[G]$ may split as a direct product of the group ring of a quotient and a maximal order. This yields alternative ways to determine $\mathbb{Z}_p[G]$-lattices up to isomorphism. In this section, we investigate how Theorem 2.6.9 relates to these results.

Definition 2.7.3. The following definition is due to Johnston–Nickel, see e.g. [JN16, Sec. 2.2]. Fix a prime p and a finite group G, and assume that N is a normal subgroup of G of order not divisible by p. We set $e_N = \frac{1}{N}\sum_{g\in N} g$. Then e_N is a central idempotent of $\mathbb{Z}_p[G]$. Let

$$\mathbb{Z}_p[G] \cong \mathbb{Z}_p[G/N] \times \mathcal{M},$$

be the corresponding decomposition. If $\mathcal{M}$ is a maximal order within $(1-e_N)\cdot\mathbb{Q}_p[G]$, then we say that G is N-*hybrid* at p and that $\mathbb{Z}_p[G]$ is a *hybrid group ring* for N.

Example 2.7.4. If $G = S_3$, then G is C_3-hybrid at 2. In fact,

$$\mathbb{Z}_2[S_3] \cong \mathbb{Z}_2[S_3/C_3] \times M_2(\mathbb{Z}_2).$$

This can be used to recover the classification given in Section 2.7.1, all indecomposable $\mathbb{Z}_2[S_3]$-lattices are either inflated from the $S_3/C_3 \cong C_2$ quotient (of which there are 3 isomorphism classes $\mathbb{1}, \epsilon, \mathbb{Z}_2[S_3/C_3]$) or isomorphic to a single two dimensional lattice ρ which has non-trivial C_3-action.

If $G = S_4$ and then G is $(C_2 \times C_2)$-hybrid at 3. In this case,

$$\mathbb{Z}_3[S_4] \cong \mathbb{Z}_3[S_4/(C_2 \times C_2)] \times M_3(\mathbb{Z}_3) \times M_3(\mathbb{Z}_3).$$

As a result, there are two rank three lattices with non-trivial $C_2\times C_2$ action together with the six isomorphism classes of lattices inflated from $S_4/(C_2\times C_2) \cong S_3$ detailed in Section 2.7.1.

Lemma 2.7.5. *Suppose that G is a finite group and p is a prime such that $\mathbb{Z}_p[G]$ is a hybrid group ring with normal subgroup N. Then the isomorphism class of any $\mathbb{Z}_p[G]$-lattice is determined by*

1) the isomorphism class of $M \otimes \mathbb{Q}_p$ as a $\mathbb{Q}_p[G]$-representation,

2) the isomorphism class of $\mathrm{defl}^G_{G/N}(M)$ as a $\mathbb{Z}_p[G/N]$-lattice.

Proof. Suppose $\mathbb{Z}_p[G] \cong \mathbb{Z}_p[G/N] \times \mathcal{M}$ and let M be some $\mathbb{Z}_p[G]$-lattice. We may uniquely write $M = M_1 \oplus M_2$ with M_1 inflated from G/N and the action of $\mathbb{Z}_p[G]$ on M_2 factoring through $\mathcal{M}$. As N acts non-trivially on all non-zero $\mathcal{M}$-modules, the isomorphism class of $M_2 \otimes \mathbb{Q}_p$ can be recovered from that of $M \otimes \mathbb{Q}_p$. Since $\mathcal{M}$ is a maximal order, it is hereditary and so all of its lattices are projective [CR94, Thm. 26.12]. As a result, the isomorphism class of M_2 is a function of $M_2 \otimes \mathbb{Q}_p$.

Finally, since N acts non-trivially on all $\mathcal{M}$-lattices, $\mathrm{defl}^G_{G/N}(M_2) = 0$. So the isomorphism class of M_1 is a function of $\mathrm{defl}^G_{G/N}(M)$. $\square$

Remark 2.7.6. Suppose that G is a group with cyclic Sylow p-subgroup and moreover that $\mathbb{Z}_p[G]$ is a hybrid group ring for some normal subgroup N. For any $\mathbb{Z}_p[G]$-lattice M, if we decompose $M = M_1 \oplus M_2$ as above, then the Yakovlev diagram M is isomorphic to the Yakovlev diagram of M_1. This is immediate given that M_2 is projective.

Example 2.7.7. For the groups of Example 2.7.4 we examine how Theorem 2.6.9 relates to hybrid group rings. Of course, *i)* of Theorem 2.6.9 coincides with 1) of Lemma 2.7.5. For S_3, we saw that the matrix for $\ell = 2$ of Section 2.7.1 representing $a \oplus b \oplus \mathrm{Yak}$ is invertible. This suggests there can be no redundancy amongst *i)-iii)*. As S_3 contains no non-cyclic 2-Sylow subgroups the data of *ii)* is empty, whilst the previous remark shows that the Yakovlev diagram is a function of the data of 2) in the lemma.

Now consider S_4 and $p = 3$. In this case, there is a section of the quotient map, allowing us to fix a subgroup $S_3 \leq S_4$. It is easy to repeat the proof of Lemma 2.7.5 replacing 2) by the isomorphism class of $M\downarrow_{S_3}$. The Yakovlev diagram of a $\mathbb{Z}_3[S_4]$-lattice M consists solely of the group $H^1(C_3, M)$ considered as an $N_G(C_3)$-module. But, $N_G(C_3) \cong S_3$, so the Yakovlev diagram is a function of the restriction to S_3. Similarly, there is only a single conjugacy class of non-cyclic p-hypo-elementary groups, namely the S_3 subgroups. As a result, the regulator constants are also a function only of the restriction to S_3 (as $C_{\theta_{S_3}\uparrow^{S_3}}(-) = C_{\theta_{S_3}}((-)\downarrow_{S_3})$).

It is not the case that all hybrid groups need be semi-direct products, so a section will not exist in general. The following example is due to David Watson. Let G be unique non-split extension of the extraspecial group of order 27 by S_3 and $N \leq G$ be its unique normal C_3-subgroup. Then G is N-hybrid for $p = 2$ as can be seen from the results of [JN16, Sec. 2], but G is not a semi-direct product.

2.7.3 Groups with degenerate permutation pairing

Example 2.7.8. Let $G = C_3 \times C_3 \times S_3$, a 3-hypo-elementary group. Up to conjugacy G has 17 subgroups, but the permutation pairing of Construction 2.3.8 is degenerate and has rank 16.

In this section, we define a canonical Brauer relation $\theta_{\Sigma,G}$ which is non-zero for any non-cyclic group G. When $G = C_3 \times C_3 \times S_3$, then $\theta_{\Sigma,G}$ generates the kernel of the permutation pairing.

Notation 2.7.9.

- Let G be any finite group and Σ denote the set of all subgroups of G, which is partially ordered with respect to containment. Let $\mu_\Sigma \colon \Sigma \to \mathbb{Z}$ denote the Möbius function on Σ, i.e. the unique function for which
$$\mu_\Sigma(G) = 1$$
and
$$\sum_{H' \geq H} \mu_\Sigma(H') = 0$$
for all $H \neq G$.
- Set
$$\theta_{\Sigma,G} = \sum_{H \in \Sigma} \frac{\mu_\Sigma(H)}{|G : H|}[H] \in B(G).$$
- For an element $\theta \in B(G)$ and $K \leq G$, let θ^K denote the number of fixed points of θ under K, i.e. if $\theta = \sum_H \alpha_H[H]$ then $\theta^K = \sum \alpha_H \#([H]^K)$.

Lemma 2.7.10. *For any $K \leq G$, $(\theta_{\Sigma,G})^K = \sum_{H \geq K} \mu_\Sigma(H)$.*

Proof. Since both $\#[H]^K$ and $\mu_\Sigma(H)$ are constant under replacing H with a conjugate, we have

$$\begin{aligned}
\sum_{H \leq G} \frac{\mu_\Sigma(H)}{|G : H|}\#([H]^K) &= \sum_{H \leq_G G} \frac{|H|\mu_\Sigma(H)}{|N_G(H)|}\#([H]^K) \\
&= \sum_{H \leq_G G} \frac{|H|\mu_\Sigma(H)}{|N_G(H)|}|\{g \in G/H \mid K^g \leq H\}| \\
&= \sum_{H \leq_G G} \frac{|H|\mu_\Sigma(H)}{|N_G(H)|}|\{g \in G/H \mid K \leq H^g\}| \\
&= \sum_{H \leq_G G} \frac{|H|\mu_\Sigma(H)}{|N_G(H)|} \cdot \frac{|N_G(H)|}{|H|}|\{g \in G/N_G(H) \mid K \leq H^g\}| \\
&= \sum_{H \geq K} \mu_\Sigma(H).
\end{aligned}$$

□

Corollary 2.7.11. *For any finite group G,*

i) $\theta_{\Sigma,G}$ is a Brauer relation in characteristic zero if and only if G is non-cyclic, *ii)* $\theta_{\Sigma,G}$ *is a Brauer relation in characteristic p if and only if G is non-p-hypo-elementary,*

iii) $\theta_{\Sigma,G}\downarrow_H$ *is zero for all proper subgroups H.*

Proof. We first check *i)*. An element of $B(G)$ is a relation in characteristic zero if and only if the number of its fixed points under all cyclic subgroups is zero (see for example the proof of [Ben98, Thm. 5.6.1]). By the lemma, $\theta_{\Sigma,G}$ is a relation in characteristic zero if and only if

$$\sum_{H \geq C} \mu_\Sigma(H) = 0$$

for all cyclic subgroups C. By the definition of μ_Σ, this is true if and only if G is not itself cyclic.

The argument for *ii)* is identical instead using that elements of $B(G)$ are relations in characteristic p if and only if the number of fixed points under all p-hypo-elementary subgroups is zero (repeat the proof of [Ben98, Thm. 5.6.1] but using species and Lemma 2.6.19).

For *iii)*, simply note that an element of $B(H)$ is zero if and only if its fixed points under all subgroups is zero (proven analogously to the previous cases). But $(\theta_{\Sigma,G}\downarrow_H)^K = (\theta_{\Sigma,G})^K$, which by the lemma vanishes for all proper subgroups $K < G$. □

Remark 2.7.12. By Lemma 2.2.30 *iv)*, $\theta_{\Sigma,G}$ automatically vanishes on all permutation modules other than possibly the trivial representation. Returning to Example 2.7.8, $C_3 \times C_3 \times S_3$ is a group for which in addition $\theta_{\Sigma,G}(\mathbb{1}_G) = 1$.

Example 2.7.13. Let $G = (C_p \times C_p) \rtimes C_q$ with p, q odd primes and $p = 2q + 1$ and where C_q acts diagonally on $C_p \times C_p$. Write $\alpha_H = \mu_\Sigma(H)/|G : H|$ so that $\theta_{\Sigma,G} = \sum_{H \leq G} \alpha_H [H]$. Then the α_H for each conjugacy class are given in the following table, which was found with the aid of Magma [Magma] but is also easy to verify by hand:

	1	C_q	C_p	C_p	C_p	C_p	$C_p \rtimes C_q$	$C_p \rtimes C_q$	$C_p \times C_p$	G
#conjugates	1	p^2	1	1	q	q	p	p	1	1
$\mu_\Sigma(H)$	$-p^2$	1	p	p	0	0	-1	-1	-1	1
α_H	$-1/q$	$1/p^2$	$1/q$	$1/q$	0	0	$-1/p$	$-1/p$	$-1/q$	1

And we find $\theta_{\Sigma,G}(\mathbb{1}_G) = 1$. Thus, the relation $\theta_{\Sigma,G}$ is trivial on all permutation representations. As G is p-hypo-elementary, $\theta_{\Sigma,G}$ is not a p-relation and the permutation pairing of Construction 2.3.8 is degenerate.

Remark 2.7.14. It is not clear to the author if, for any of the above groups, there exists a lattice M for which $\theta_{\Sigma,G}(M) \neq 1$. Necessarily, such an M must not be induced from a proper subgroup (Lemma 2.2.30). For $G = C_3 \times C_3 \times S_3$, $A(\mathbb{Z}_3[G], \mathrm{perm}) = A(\mathbb{Z}_{(3)}[G], \mathrm{triv}) = A(\mathbb{Z}_3[G], \mathrm{triv})$ since G has the same number of species as conjugacy classes of p-hypo-elementary subgroups (Lemma 2.6.26 *iv)*). Thus, such an M would also have to not have trivial source.

Chapter 3

Application to unit groups

3.1 Introduction

In this chapter, we provide an extended application of Theorem 2.6.9 in the context of unit groups of number fields. Let K/F be a Galois extension of number fields with Galois group $G = \mathrm{Gal}(K/F)$. Fix a prime p and assume that G has a cyclic Sylow p-subgroup P. We shall assume further that G satisfies the condition $A(\mathbb{Z}_p[G], \mathrm{perm}) = A(\mathbb{Z}_{(p)}[G], \mathrm{triv})$ defined in Section 2.6.2. For example, K/F may be a dihedral extension of order $2q$ for odd q (see Example 2.6.28).

Notation 3.1.1. Given a number field L,

- we write $\mathcal{O}_L$ for its ring of integers and $\mathcal{O}_L^\times$ for its unit group,
- we write μ_L for the group of roots of unity contained in L and $w(L)$ for $|\mu_L|$,
- we write $\mathrm{Cl}(L)$ for the class group of L and $h(L)$ for the class number $|\mathrm{Cl}(L)|$.

The units $\mathcal{O}_K^\times$ have the structure of a $\mathbb{Z}[G]$-module and $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ is a $\mathbb{Z}_p[G]$-lattice which is necessarily rationally self-dual in the sense of Definition 2.2.26. As remarked in the first two chapters, for general G, the study of $\mathbb{Z}[G]$-lattices can be very difficult and even basic questions can be intractable. This carries over to the study of lattices arising within number theory.

Even for fixed F and G, as K varies the module structure of $\mathcal{O}_K^\times$ may vary significantly and there are many open questions. For example, it is not known how the Galois module structure of $\mathcal{O}_K^\times$ relates to classical properties of the extension K/F, or if all $\mathbb{Z}[G]$-lattices M with $M \otimes \mathbb{Q} \cong \mathcal{O}_K^\times/\mu_K \otimes \mathbb{Q}$ can appear as $\mathcal{O}_{K'}^\times/\mu_{K'}$

for some Galois extension K'/F with Galois group G. Other interesting questions appear when considering the relative distributions of different module structures in families. For each of these questions, their analogue for $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ remains interesting but often is more approachable than for $\mathcal{O}_K^\times/\mu_K$ itself.

If we directly apply Theorem 2.6.9 to $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ we get:

Corollary 3.1.2. *The isomorphism class of $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ as a $\mathbb{Z}_p[G]$-lattice is determined by*

i) the isomorphism class of $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Q}$ as a $\mathbb{Q}[G]$-module,

ii) the valuations $v_p(C_{\theta_H}(\mathcal{O}_K^\times/\mu_K))$ of the regulator constants of Artin relations for $H \in \mathrm{nchyp}_p(G)$,

iii) the Yakovlev diagram

$$H^1(P_r, \mathcal{O}_K^\times/\mu_K) \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} H^1(P_{r-1}, \mathcal{O}_K^\times/\mu_K) \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} \ \dots \ \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} H^1(P_0, \mathcal{O}_K^\times/\mu_K).$$

where $P_i \leq P$ is the subgroup of order p^i and each P_i-term is considered as a module over the normaliser $N_G(P_i)$ (cf. Theorem 2.6.7).

Proof. Use that for any relation θ and $\mathbb{Z}[G]$-lattice M, $C_\theta(M) = C_\theta(M \otimes \mathbb{Z}_p)$ (Lemma 2.2.30 *vi)*) and that $H^i(P_k, \mathcal{O}_K^\times/\mu_K) \cong H^i(P_k, \mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p)$. Finally, note that the isomorphism class of $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Q}_p$ is determined by that of $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Q}$. □

Whilst each of the terms *i)–iii)* are amenable to computation, when written as above it unclear of how the isomorphism class of $\mathcal{O}_K^\times/\mu_K$ relates to classical invariants of K/F. The aim of this chapter is to reinterpret *i)–iii)* classically in some special cases.

The first two terms can be dealt with in maximal generality:

Lemma 3.1.3. *For any Galois extension K/F with Galois group G, the isomorphism class of $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Q}$ as a $\mathbb{Q}[G]$-module is determined by the signatures (r_1, r_2) of K^H as H ranges over cyclic subgroups of G up to conjugacy.*

Proof. It is a consequence of Artin's induction theorem (see Lemma 2.2.25) that a $\mathbb{Q}[G]$-module M is determined by the ranks M^H as runs over cyclic subgroups of G up to conjugacy. By Dirichlet's unit theorem, $(\dim \mathcal{O}_K^\times/\mu_K)^H = r_1 + r_2 - 1$ where (r_1, r_2) is the signature of K^H. □

Example 3.1.4. If p does not divide $|K:F|$, then, as discussed on p5, the $\mathbb{Z}_p[G]$-structure of $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ is determined by the isomorphism class of $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Q}_p$ as a $\mathbb{Q}_p[G]$-module, and so by the signatures of the intermediate subfields of K/F. In other words, $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ is arithmetically uninteresting for p not dividing $|K:F|$.

Theorem 3.1.5 (Bartel)**.** *Let K/F be any Galois extension and let $G = \mathrm{Gal}(K/F)$. For any subgroup H of G, write $\lambda(K^H)$ for the order of* $\mathrm{coker}(\mathcal{O}_{K^H}^\times/\mu_{K^H} \to (\mathcal{O}_K^\times/\mu_K)^H)$. *Then for any Brauer relation $\theta = \sum_i [H_i] - \sum_j [H'_j]$,*

$$C_\theta(\mathcal{O}_K^\times/\mu_K) = C_\theta(\mathbb{1}) \cdot \left(\frac{\prod_j h(K^{H'_j})}{\prod_i h(K^{H_i})} \frac{\prod_j \lambda(K^{H'_j})}{\prod_i \lambda(K^{H_i})} \frac{\prod_i w(K^{H_i})}{\prod_j w(K^{H'_j})} \right)^2. \tag{3.1}$$

Proof. Artin formalism combined with the analytic class number formula shows that

$$\prod_i \frac{h(K^{H_i}) \,\mathrm{Reg}(K^{H_i})}{w(K^{H_i})} = \prod_j \frac{h(K^{H_j}) \,\mathrm{Reg}(K^{H_j})}{w(K^{H_j})},$$

where, for a number field L, $\mathrm{Reg}(L)$ denotes Dirichlet's unit group regulator. The formula now follows from the following formula of Bartel [Bar12, Prop. 2.15]:

$$C_\theta(\mathcal{O}_K^\times/\mu_K) = C_\theta(\mathbb{1}) \cdot \left(\frac{\prod_i \mathrm{Reg}(K^{H_i})}{\prod_j \mathrm{Reg}(K^{H'_j})} \frac{\prod_j \lambda(K^{H'_j})}{\prod_i \lambda(K^{H_i})} \right)^2. \qquad \square$$

Remark 3.1.6. We are only concerned with the p-part of (3.1) for a single prime p. If K has no p^{th} roots of unity, then (3.1) simplifies to give

$$v_p(C_\theta(\mathcal{O}_K^\times/\mu_K)) = v_p(C_\theta(\mathbb{1})) + 2v_p \left(\frac{\prod_j h(K^{H'_j})}{\prod_i h(K^{H_i})} \right).$$

In this section we shall repeatedly have to do extra work to compensate for "erroneously" considering $C_\theta(\mathcal{O}_K^\times/\mu_K)$ instead of $C_\theta(\mathcal{O}_K^\times)$.

Remark 3.1.7. Note that Bartel's result shows that for any integral Brauer relation θ,

$$v_p(C_\theta(\mathcal{O}_K^\times/\mu_K)) \equiv v_p(C_\theta(\mathbb{1})) \pmod 2,$$

and in particular, it is independent of K/F modulo 2. We can see this directly as follows. For each infinite place v of F, fix a place $w \mid v$ and some decomposition group D_w of w. Then, $V := \bigoplus_{v|\infty} \mathbb{Q}[G/D_w]$ is the permutation module defined by the action of G on the infinite places of K. As such there is a map $\mathcal{O}_K^\times/\mu_K \hookrightarrow V$ defined as in the proof of Dirichlet's unit theorem. This is G-equivariant and its image is a lattice within a hyperplane I which is stable under the action of G. Moreover, V/I is the one dimensional trivial representation $\mathbb{1}$. Therefore, $(\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Q}) \oplus \mathbb{1} \cong V$. Applying Lemma 2.2.30 *i)* and *vi)*, we get

$$C_\theta(\mathcal{O}_K^\times/\mu_K) \cdot C_\theta(\mathbb{1}) \equiv C_\theta(V) \pmod{(\mathbb{Q}^\times)^2}.$$

But, the decomposition group of any infinite place is cyclic, so V is a direct sum of representations induced from cyclic subgroups and so must necessarily have vanishing regulator constant (cf. Remark 2.3.2).

We have now seen that both *i)* and *ii)* of Corollary 3.1.2 can be rewritten in terms of "elementary invariants". Reinterpreting *iii)*, even in simple cases, requires more work and is the main focus of this chapter.

As an aside, we remark that it is sometimes also possible to instead rewrite *ii)* in terms of cohomology.

Example 3.1.8. If G is a dihedral group of order $2q$, for q odd, then recent work of Caputo and Nuccio [CN18] proved the following formula

$$\frac{h(K)h(F)^2}{h(K^{C_q})h(K^{C_2})^2} = \frac{|\hat{H}^0(G, \mathcal{O}_K^\times \otimes \mathbb{Z}[\frac{1}{2}])|}{|\hat{H}^{-1}(G, \mathcal{O}_K^\times \otimes \mathbb{Z}[\frac{1}{2}])|}, \tag{3.2}$$

for the ratio of class numbers in terms of Tate cohomology of the units (here C_q is the subgroup of order q and C_2 denotes some choice of order two subgroup). Suppose further that K has no p^{th} roots of unity. Denote the Brauer relation $[1] - [C_q] - 2[C_2] + 2[G] = 2\theta_G$ by θ, then

$$C_\theta(\mathbb{1}) = \frac{\frac{1}{1} \cdot \left(\frac{1}{2q}\right)^2}{\frac{1}{q} \cdot \left(\frac{1}{2}\right)^2} = \frac{1}{q}.$$

So by (3.1),

$$C_\theta(\mathcal{O}_K^\times/\mu_K) = \frac{1}{q} \cdot \frac{h(K^{C_q})h(K^{C_2})^2}{h(K)h(F)^2}.$$

Combining this with (3.2) gives

$$v_p(C_\theta(\mathcal{O}_K^\times/\mu_K)) = -v_p(q) + 2v_p\left(\frac{|\hat{H}^{-1}(G, \mathcal{O}_K^\times \otimes \mathbb{Z}_p)|}{|\hat{H}^0(G, \mathcal{O}_K^\times \otimes \mathbb{Z}_p)|}\right).$$

So in the dihedral case, both *ii)* and *iii)* can be phrased in terms of the cohomology of units.

3.2 Calculation of the Yakovlev diagram

In this section, we shall further assume that K/F is of degree divisible by p exactly once (which, in view of Example 3.1.4, may be thought of as the simplest non-trivial case). On the other hand, the calculation of this section does not require G to satsify the condition on trivial source modules of Section 2.6.2. Since the Sylow subgroup is isomorphic to C_p, the Yakovlev diagram of Corollary 3.1.2 *iii)* simply becomes the isomorphism class of

$$H^1(P, \mathcal{O}_K^\times/\mu_K)$$

as an $N_G(P)/P$-module, where P is a choice of Sylow p-subgroup.

Notation 3.2.1. Let F' denote K^P. Consider the map $c\colon \mathrm{Cl}(F') \to \mathrm{Cl}(K)^P$ given by extension of ideals. Let $r(K/F')$ denote set of primes of F' which ramify in K. Set

$$W := \bigoplus_{\mathfrak{p} \in r(K/F')} \mathbb{F}_p.$$

This has an action of $N_G(P)/P$ via its action on $r(K/F')$, i.e. W is an $\mathbb{F}_p[N_G(P)/P]$-module.

There is an $N_G(P)/P$-equivariant map $\tau\colon W \to \mathrm{Cl}(K)^P/c(\mathrm{Cl}(F'))$ sending the $\mathfrak{p}^{\mathrm{th}}$ basis element to the class of $\mathfrak{q}$ where $\mathfrak{q}$ is the unique prime lying above $\mathfrak{p}$ in K (note that the exponent of $\mathrm{Cl}(K)^P/c(\mathrm{Cl}(F'))$ divides p as it is a quotient of the Tate cohomology group $\hat{H}^0(P, \mathrm{Cl}(K)) = \mathrm{Cl}(K)^P/N_{K/F'}\,\mathrm{Cl}(K))$.

Notation 3.2.2. Let H be a group of order coprime to p, then all finite dimensional $\mathbb{F}_p[H]$-modules are semi-simple and, as for any p, Krull-Schmidt holds. Let M be a $\mathbb{F}_p[H]$-module with a submodule isomorphic to N, then N is a summand and we denote by $M \ominus N$ a choice of complementary summand for N in M. The module $M \ominus N$ is well defined up to isomorphism.

Notation 3.2.3. For the remainder of this chapter, given a finite $\mathbb{F}_p[N_G(P)/P]$-module M, we denote by $M(1)$ the module $M \otimes H^2(P, \mathbb{Z})$. In this notation, as P is cyclic, for any $\mathbb{Z}[G]$-module N, cup product defines an $N_G(P)/P$-equivariant isomorphism $\hat{H}^{i+2}(P, N) \cong \hat{H}^i(P, N)(1)$ for all $i \in \mathbb{Z}$ (with $\hat{H}^i$ denoting Tate cohomology).

Proposition 3.2.4. *As $\mathbb{F}_p[N_G(P)/P]$-modules we have*

$$\begin{aligned} H^1(P, \mathcal{O}_K^\times/\mu_K) \cong \ker(\mathrm{Cl}(F') \xrightarrow{c} \mathrm{Cl}(K)) \oplus \ker(W \xrightarrow{\tau} \mathrm{Cl}(K)^P / \operatorname{im} \mathrm{Cl}(F')) \\ \oplus \operatorname{coker}(\mathcal{O}_{F'}^\times/\mu_{F'} \to (\mathcal{O}_K^\times/\mu_K)^P) \\ \oplus (\mu_{F'} \cap N_{K/F'}\mathcal{O}_K^\times)/N_{K/F'}\mu_K \ominus H^1(P, \mu_K). \end{aligned} \tag{3.3}$$

Remark 3.2.5. Note that $N_G(P)/P$ is of order coprime to p. If we allowed the Sylow subgroups to be cyclic of order $> p$, then, to apply Theorem 2.6.9, we would have to consider the action of $N_G(P_i)/P_i$ in cases where it may have order divisible by p. In which case, the existence of non-trivial extensions means that Proposition 3.2.4 need not hold in this setting. This is the main obstruction to allowing larger Sylow p-subgroups. One option to mitigate this would be to require that K/F is everywhere unramified so that the map b defined below in Figure 3.5 is bijective.

The proof of the proposition is a technical diagram chase involving Arakelov class groups and shall occupy several pages. If we assume that K has no p^{th} roots of unity, then this simplifies significantly.

3.2.1 Arakelov class groups

We briefly recall the definition and basic properties of Arakelov class groups following [Sch08] (cf. [Neu13, Sec. III.1]).

Definition 3.2.6. For any number field K, an *Arakelov divisor* is a finite formal sum $\sum_{\mathfrak{p}} n_{\mathfrak{p}} \cdot \mathfrak{p} + \sum_\eta x_\eta \cdot \eta$, where $\mathfrak{p}$ runs over prime ideals of K with coefficients $n_{\mathfrak{p}} \in \mathbb{Z}$ and η runs over the archimedean places of K with coefficients $x_\eta \in \mathbb{R}$. Denote the additive group of Arakelov divisors by $\mathrm{Div}(K)$.

Definition 3.2.7. To an element $x \in K^\times$ we associate the divisor $(x) = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \cdot \mathfrak{p} + \sum_\eta x_\eta \cdot \eta$ where $n_{\mathfrak{p}} = v_p(x)$, $x_\eta = -\log|\eta(x)|$. Divisors which arise in this way are said to be *principal*. Suppose $x \in K^\times$ lies in the kernel of the map $(-)\colon K^\times \to \mathrm{Div}(K)$. The conditions at the finite places ensure that $x \in \mathcal{O}_K^\times$ and so, by (the proof of) Dirichlet's unit theorem, x must in fact be a root of unity.

Definition 3.2.8. We define a degree map $\deg\colon \mathrm{Div}(K) \to \mathbb{R}$ by setting $\deg(1 \cdot \mathfrak{p}) = \log|\mathcal{O}_K/\mathfrak{p}|$ and $\deg(1 \cdot \eta)$ to be 1 or 2 if η is real or complex respectively. The product rule ensures that principal divisors have degree zero. Write $\mathrm{Div}^0(K)$ for the group of degree zero divisors and $\mathrm{Pic}^0(K)$ for the quotient $\mathrm{Div}^0(K)/\operatorname{im} K^\times$ of degree zero divisors modulo principal divisors. The group $\mathrm{Pic}^0(K)$ is called the *Arakelov class group* of K. By construction there is a short exact sequence

$$0 \to K^\times/\mu_K \to \mathrm{Div}^0(K) \to \mathrm{Pic}^0(K) \to 0.$$

Let $I(K)$ denote the group of fractional ideals of K. Projection onto the finite places defines a map $\mathrm{Div}(K) \to I(K)$, and this remains surjective on restriction to $\mathrm{Div}^0(K)$. There is a short exact sequence

$$0 \to V_K^0 \to \mathrm{Div}^0(K) \to I(K) \to 0,$$

where V_K^0 is an $r_1 + r_2 - 1$ dimensional $\mathbb{R}$-vector spaces and is identified with the hyperplane contained in $V_K := \left(\bigoplus_\eta \mathbb{R}\right)$ defined in the proof of Dirichlet's unit theorem.

The above can be neatly packaged via the following lemma

Lemma 3.2.9. *For any number field K, there is a commutative diagram of short exact sequences:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_K^\times/\mu_K \arrow[r] \arrow[d] & K^\times/\mu_K \arrow[r] \arrow[d] & P(K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V_K^0 \arrow[r] \arrow[d] & \mathrm{Div}^0(K) \arrow[r] \arrow[d] & I(K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T^0(K) \arrow[r] \arrow[d] & \mathrm{Pic}^0(K) \arrow[r] \arrow[d] & \mathrm{Cl}(K) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Figure 3.1: Defining diagram of Arakelov class groups

Here, $P(K)$ denotes the group of principal fractional ideals of K and $T^0(K)$ is defined to be the cokernel of the map $\mathcal{O}_K^\times/\mu_K \to V_K^0$ appearing in Dirichlet's unit theorem.

Proof. Exactness of the bottom row follows from exactness of the columns and the top two rows by the 9-lemma. □

Note that $T^0(K)$ is a divisible group isomorphic to $(\mathbb{R}/\mathbb{Z})^{r_1+r_2-1}$, where r_1 is the number of real embeddings of K and r_2 the number of pairs of complex conjugate embeddings.

Remark 3.2.10. For any finite extension of number fields K/F there is a corresponding map $\mathrm{Div}(F) \to \mathrm{Div}(K)$ which at the finite places is induced by extension of ideals and sends $1 \cdot \eta$ to $\sum_{\xi|\eta} e_\xi \cdot \xi$ where

$$e_\xi = \begin{cases} 2 & \text{if } \eta \text{ is real and } \xi \text{ is complex} \\ 1 & \text{otherwise} \end{cases}.$$

If K/F is Galois with group G, then G acts on $\mathrm{Div}(K)$ via its action on places and Figure 3.1 is equivariant for this action. There is an obvious map between Figure 3.1 for F and the G-fixed points of Figure 3.1 for K (which we may extend to long exact sequences of cohomology). We shall make repeated use segments of this construction without further explanation.

Remark 3.2.11. The Arakelov class group relates the class group of K to $T^0(K) = \mathcal{O}_K^\times/\mu_K \otimes \mathbb{R}/\mathbb{Z}$ (which is also the Pontryagin dual of $\mathrm{Hom}(\mathcal{O}_K^\times/\mu_K, \mathbb{Z})$). This forms the basis for our proof of Proposition 3.2.4. However, there are several other candidates that should fulfil a similar role of non-trivially linking class groups and units. For example, Caputo and Nuccio [CN18] make use of the idele class group and a corresponding 9-lemma diagram to prove related results, whilst it is possible to define a Selmer structure on $\mathcal{O}_K^\times$ so that the corresponding Selmer group has an inclusion of $\mathcal{O}_K^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p$ with its "Tate-Shafarevich group" given by $\mathrm{Cl}(K)$. In the context of abelian varieties, this Selmer method has been exploited in work of Burns-Macías Castillo-Wuthrich [BMCW15] to calculate analogous cohomology groups where the $\mathbb{Z}[G]$-module is the Mordell-Weil group of the abelian variety (or its Selmer group).

3.2.2 Proof of Proposition 3.2.4

Now assume that K/F is as at the start of Section 3.2.

Lemma 3.2.12. *There is an $N_G(P)/P$-equivariant isomorphism*

$$H^1(P, \mathcal{O}_K^\times/\mu_K) \cong T^0(K)^P/T^0(F').$$

Proof. Consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_F^\times/\mu_{F'} \arrow[r] \arrow[d] & V_{F'}^0 \arrow[r] \arrow[d, equal] & T^0(F') \arrow[r] \arrow[d, "a"] & 0 \\
0 \arrow[r] & (\mathcal{O}_K^\times/\mu_K)^P \arrow[r] & (V_K^0)^P \arrow[r] & T^0(K)^P \arrow[r] & H^1(P, \mathcal{O}_K^\times/\mu_K) \arrow[r] & 0
\end{tikzcd} \tag{3.4}$$

Applying snake lemma at coker a gives the desired isomorphism. That this is $N_G(P)$-equivariant is standard, see e.g. [NSW08, Prop. 1.5.2]. $\square$

It remains to describe $T^0(K)^P/T^0(F')$ (which we now know to be a finite group of exponent p). Note that $T^0(K)^P$ need not be divisible, but since $T^0(F')$ is of finite index and a divisible subgroup, $T^0(F')$ must be the maximal divisible subgroup. The basis for calculating $T^0(K)^P/T^0(F')$ is that $T^0(K)^P/T^0(F')$ appears as a cokernel in the kernel/cokernel sequence of the diagram:

$$\begin{tikzcd}
0 \arrow[r] & T^0(F') \arrow[r] \arrow[d, "a"] & \mathrm{Pic}^0(F') \arrow[r] \arrow[d, "b"] & \mathrm{Cl}(F') \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & T^0(K)^P \arrow[r] & \mathrm{Pic}^0(K)^P \arrow[r] & \mathrm{Cl}(K)^P \arrow[r] & H^1(P, T^0(K))
\end{tikzcd}$$

Here, both $a\colon T^0(F') \to T^0(K)^P$ and $b\colon \mathrm{Pic}^0(F') \to \mathrm{Pic}^0(K)^P$ are induced by the map $\mathrm{Div}(F) \to \mathrm{Div}(K)$ described in Remark 3.2.10.

Lemma 3.2.13. *The following is an exact sequence of finite $\mathbb{F}_p[N_G(P)/P]$-modules:*

$$0 \to \ker a \to \ker b \to \ker c \to T^0(K)^P/T^0(F') \to \ker(\operatorname{coker} b \xrightarrow{\gamma} \operatorname{coker} c) \to 0. \tag{3.5}$$

Proof. We must check that each term is a finite $\mathbb{F}_p[N_G(P)/P]$-module. For $\ker a$ and $\ker \gamma$, this will follow from checking the other terms, whilst for $T^0(K)^P/T^0(F')$, this follows from Lemma 3.2.12. For $\ker b$, consider following diagram obtained from the middle column of Figure 3.1:

$$\begin{tikzcd}
0 \arrow[r] & F'^\times/\mu_{F'} \arrow[r] \arrow[d] & \mathrm{Div}^0(F') \arrow[r] \arrow[d] & \mathrm{Pic}^0(F') \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & (K^\times/\mu_K)^P \arrow[r] & \mathrm{Div}^0(K)^P \arrow[r] & \mathrm{Pic}^0(K)^P \arrow[r] & H^1(P, K^\times/\mu_K)
\end{tikzcd} \tag{3.6}$$

As $\mathrm{Div}^0(F') \hookrightarrow \mathrm{Div}^0(K)$, we find that

$$\ker b \to \operatorname{coker}(F'^\times/\mu_{F'} \to (K^\times/\mu_K)^P)$$

is injective, but the latter is, by Hilbert Theorem 90, isomorphic to $H^1(P, \mu_K)$ and thus finite and of exponent at most p. For $\ker c$, finiteness is clear. By considering the right hand column of Figure 3.1, we find $\ker(\mathrm{Cl}(F') \to \mathrm{Cl}(K)) \hookrightarrow P(K)^G/P(F')$, which is a quotient of $\hat{H}^0(P, P(K))$ and so has exponent dividing p. □

In particular, in the notation of Notation 3.2.2,

$$T^0(K)^P/T^0(F') \cong \ker a \oplus \ker c \oplus \ker \gamma \ominus \ker b. \tag{3.7}$$

As such, to determine the isomorphism class of $T^0(K)^P/T^0(F')$ it suffices to describe the isomorphism class of each summand individually. For example, $\ker c = \ker(\mathrm{Cl}(F') \to \mathrm{Cl}(K))$ is already a classically studied invariant.

Lemma 3.2.14. $\ker a \cong \mathrm{coker}(\mathcal{O}_{F'}^\times/\mu_{F'} \to (\mathcal{O}_K^\times/\mu_K)^P)$ *as* $N_G(P)/P$*-modules.*

Proof. This is immediate from the kernel/cokernel sequence of (3.4). □

The calculation of $\ker \gamma$ requires more work.

Lemma 3.2.15. *There is a commutative diagram of finite* $\mathbb{F}_p[N_G(P)/P]$*-modules:*

$$\begin{tikzcd} 0 \arrow[r] & \ker b \arrow[r] \arrow[d] & H^1(P, \mu_K) \arrow[r, "d"] \arrow[d] & \mathrm{Div}^0(K)^P/\mathrm{Div}^0(F') \arrow[r] \arrow[d] & \mathrm{coker}\, b \arrow[r] \arrow[d, "\gamma"] & H^1(P, K^\times/\mu_K) \arrow[r] \arrow[d, "z"] & 0 \\ 0 \arrow[r] & \ker c \arrow[r] & P(K)^P/P(F') \arrow[r] & I(K)^P/I(F') \arrow[r] & \mathrm{coker}\, c \arrow[r] & H^1(P, P(K)) & \end{tikzcd} \tag{3.8}$$

with the horizontal rows being exact.

Proof. By Hilbert Theorem 90, $H^1(P, \mu_K) \cong \mathrm{coker}(F'^\times/\mu_{F'} \to (K^\times/\mu_K)^P)$. As a result, the kernel/cokernel sequence of (3.2.2) is given by

$$\begin{aligned} 0 \to \ker b \to H^1(P, \mu_K) \to \mathrm{Div}^0(K)^P/\mathrm{Div}^0(F') \to \mathrm{coker}\, b \\ \to H^1(P, K^\times/\mu_K) \to H^1(P, \mathrm{Div}^0(K)). \end{aligned} \tag{3.9}$$

Claim. $H^1(P, \mathrm{Div}^0(K)) = 0$.

Proof of Claim. By the degree map it suffices to show that $H^1(P, \mathrm{Div}(K)) = 0$. As a P-module, $\mathrm{Div}(K)$ decomposes as a direct sum of terms indexed by places of F'. Each of these is a permutation module and so has trivial cohomology in degree 1. As group cohomology commutes with arbitrary direct sums, the same is true of $\mathrm{Div}(K)$. □

Claim. *Every term of* (3.9) *is a finite* $\mathbb{F}_p[N_G(P)/P]$*-module.*

Proof of Claim. That each term has exponent dividing p is clear as all terms other than $\ker b$ are either P-cohomology groups or quotients of P-Tate cohomology groups. As $H^1(P, \mu_K)$ is finite, it suffices to check finiteness for $\mathrm{Div}^0(K)^P / \mathrm{Div}^0(F')$ and $\operatorname{coker} b$. By considering the degree map, $\mathrm{Div}^0(K)^P / \mathrm{Div}^0(F') \cong \mathrm{Div}(K)^P / \mathrm{Div}(F')$. The Galois stable elements of $\mathrm{Div}(K)$ are those divisors whose terms at all places w lying over each fixed place of F' are diagonal. Using this description we find that

$$\mathrm{Div}(K)^P / \mathrm{Div}(F') \cong \bigoplus_{\mathfrak{p} \in r(K/F')} \mathbb{F}_p =: W,$$

(in the notation of Notation 3.2.1) so that $\mathrm{Div}^0(K)/\mathrm{Div}^0(F')$ is finite. Finally, Hilbert Theorem 90 shows that $H^1(P, K^\times/\mu_K) \hookrightarrow H^2(P, \mu_K)$ so is finite. $\square$

Now consider:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P(F') & \longrightarrow & I(F') & \longrightarrow & \mathrm{Cl}(F') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle b} & & \\
0 & \longrightarrow & (K^\times/\mathcal{O}_K^\times)^P & \longrightarrow & I(K)^P & \longrightarrow & \mathrm{Cl}(K)^P & \longrightarrow & H^1(P, P(K))
\end{array}$$

Figure 3.2

This has kernel/cokernel sequence

$$0 \to \ker c \to P(K)^G/P(F') \to I(K)^P/I(F') \to \operatorname{coker} c \to H^1(P, P(K)). \qquad (3.10)$$

Claim. *Every term of* (3.10) *is a finite* $\mathbb{F}_p[N_G(P)/P]$*-module.*

Proof of Claim. As before, that each term has exponent p is automatic, as is finiteness of $\ker c$ and $\operatorname{coker} c$. Now finiteness of $I(K)^P/I(F')$, and so $P(K)^G/P(F')$ follows as it is isomorphic to $W = \bigoplus_{r(K/F')} \mathbb{F}_p$, whilst $H^1(P, P(K)) \hookrightarrow H^2(P, \mathcal{O}_K^\times)$ so $H^1(P, P(K))$ must be finite. $\square$

The maps between the middle column and right hand column of Figure 3.1 induce maps between Figure 3.9 and Figure 3.2 and so, by naturality of the snake lemma, of their associated kernel/cokernel sequences. Putting this all together gives (3.8). $\square$

Lemma 3.2.16. *We have that*

$$\begin{aligned}\ker\gamma \cong \ker(W \to \mathrm{Cl}(K)^P/\operatorname{im}\mathrm{Cl}(F')) \oplus \ker b \\ \oplus \ker(H^1(P, K^\times/\mu_K) \to H^1(P, P(K)) \ominus H^1(P, \mu_K)\end{aligned}$$

as $N_G(P)$-modules.

Proof. We argue from (3.8) of Lemma 3.2.15. After truncating we obtain a diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Div}^0(K)^P/\big(\mathrm{Div}^0(F')\cdot d(H^1(P,\mu_K))\big) \arrow[r] \arrow[d, two heads, "t"] & \operatorname{coker} b \arrow[r] \arrow[d, "\gamma"] & H^1(P, K^\times/\mu_K) \arrow[r] \arrow[d, "z"] & 0 \\ 0 \arrow[r] & I(K)^P/I(F')P(K)^P \arrow[r] & \operatorname{coker} c \arrow[r] & H^1(P, P(K)) & \end{tikzcd}$$

where the left hand arrow is surjective as $\mathrm{Div}^0(K)^P/\mathrm{Div}^0(F') \cong I(K)^P/I(F')$ ($\cong W$). As a result

$$\ker\gamma \cong \ker t \oplus \ker z.$$

Returning to (3.8),

$$d(H^1(P,\mu_K)) \subseteq \ker(\mathrm{Div}^0(K)^P/\mathrm{Div}^0(F') \to I(K)^P/I(F')P(K)^P)$$

so $\ker t$ can be rewritten as

$$\ker(\mathrm{Div}^0(K)^P/\mathrm{Div}^0(F') \to I(K)/I(F')P(K)) \ominus H^1(P,\mu_K) \oplus \ker b.$$

Now simply note that $I(K)/I(F')P(K) = \mathrm{Cl}(K)/c(\mathrm{Cl}(F'))$. $\square$

Proof of Proposition 3.2.4. By Lemmas 3.2.12, 3.2.13, we need only substitute our calculations into (3.7). Lemmas 3.2.14, 3.2.16 give

$$\begin{aligned} T^0(K)^P/T^0(F') &\cong \ker a \oplus \ker c \oplus \ker\gamma \ominus \ker b \\ &\cong \operatorname{coker}(\mathcal{O}_{F'}^\times/\mu_{F'} \to (\mathcal{O}_K^\times/\mu_K)^P) \oplus \ker(\mathrm{Cl}(F') \to \mathrm{Cl}(K)) \\ &\quad \oplus \ker(W \to \mathrm{Cl}(K)^P/c(\mathrm{Cl}(F'))) \oplus \ker b \\ &\quad \oplus \ker(H^1(P, K^\times/\mu_K) \to H^1(P, P(K))) \ominus H^1(P,\mu_K) \ominus \ker b \end{aligned}$$

Cancelling the $\ker b$ terms and applying Lemma 3.2.17 *i)*, we obtain the desired formula. $\square$

For later use, we give some alternative descriptions of the terms appearing in Proposition 3.2.4.

Lemma 3.2.17. *i) As $N_G(P)/P$-modules we have*

$$\ker(H^1(P, K^\times/\mu_K) \to H^1(P, P(K))) \cong ((\mu_{F'} \cap N_{K/F'}\mathcal{O}_K^\times)/N_{K/F'}\mu_K)(1),$$

(here $(-)(1)$ is as defined in Notation 3.2.3).

ii) As $N_G(P)/P$-modules we have

$$\hat{H}^{-1}(P, \mu_K) = \begin{cases} \mu_K[p](1) & \mu_K[p^\infty] = \mu_{F'}[p^\infty] \\ 0 & \mu_K[p^\infty] \neq \mu_{F'}[p^\infty] \text{ and } K \neq F'(\zeta_4) \\ \mathbb{Z}/2\mathbb{Z}(1) & K = F'(\zeta_4) \end{cases}, \qquad (3.11)$$

where for an abelian group A, $A[n]$ denotes the n-torsion subgroup and $A[p^\infty]$ denotes $\bigcup_n A[p^n]$.

Proof. For *i)*, consider the following diagram, which is exact by Hilbert Theorem 90:

$$\begin{tikzcd} 0 \arrow[r] & \hat{H}^{-1}(P, K^\times/\mu_K) \arrow[r] \arrow[d] & \hat{H}^0(P, \mu_K) \arrow[r] \arrow[d] & \hat{H}^0(P, K^\times) \arrow[d] \\ 0 \arrow[r] & \hat{H}^{-1}(P, K^\times/\mathcal{O}_K^\times) \arrow[r] & \hat{H}^0(P, \mathcal{O}_K^\times) \arrow[r] & \hat{H}^0(P, K^\times) \end{tikzcd}$$

Applying the definition of $\hat{H}^0$, this becomes:

$$\begin{tikzcd} 0 \arrow[r] & (\mu_{F'} \cap N_{K/F'}K^\times)/N_{K/F'}\mu_K \arrow[r] \arrow[d] & \mu_{F'}/N_{K/F'}\mu_K \arrow[r] \arrow[d] & F'^\times/N_{K/F'}K^\times \arrow[d] \\ 0 \arrow[r] & (\mathcal{O}_{F'}^\times \cap N_{K/F'}K^\times)/N_{K/F'}\mathcal{O}_{F'}^\times \arrow[r] & \mathcal{O}_{F'}^\times/N_{K/F'}\mathcal{O}_K^\times \arrow[r] & F'^\times/N_{K/F'}K^\times \end{tikzcd}$$

So that $\ker(\hat{H}^{-1}(P, K^\times/\mu_K) \to \hat{H}^{-1}(P, P(K))) \cong (\mu_{F'} \cap N_{K/F'}\mathcal{O}_K^\times)/N_{K/F'}\mu_K$. Twisting now gives the desired formula.

For *ii)*, consider

$$\hat{H}^{-1}(P, \mu_K) = \hat{H}^{-1}(P, \mu_K[p^\infty]) = \ker(N_{K/F'} \colon \mu_K[p^\infty] \to \mu_K[p^\infty])/(\sigma - 1)\mu_K[p^\infty],$$

where σ is a generator of P. We claim that

$$\ker(N_{K/F'}\colon \mu_K[p^\infty] \to \mu_K[p^\infty]) = \begin{cases} \mu_K[4] & \text{if } p = 2 \text{ and } K = F'(\zeta_4) \neq F' \\ \mu_K[p] & \text{otherwise} \end{cases}.$$

If $\mu_K[p^\infty] = \mu_{F'}[p^\infty]$, this is clear as $N_{K/F'}$ is raising to the power p. Otherwise, we may assume that, on a generator ζ_{p^n} of $\mu_K[p^\infty]$, σ acts by $\zeta_{p^n} \mapsto \zeta_{p^n}^{1+p^{n-1}}$. So

$$N_{K/F'}(\zeta_{p^n}) = \prod_{a \in \mathbb{Z}/p\mathbb{Z}} \zeta_{p^n}^{1+ap^{n-1}} = \zeta_{p^n}^{(p+p^{n-1}\cdot\sum_{a\in\mathbb{Z}/p\mathbb{Z}} a)}.$$

Since

$$\sum_{a\in\mathbb{Z}/p\mathbb{Z}} a = \begin{cases} 1 & \text{if } p = 2 \\ 0 & \text{if } p \geq 3 \end{cases},$$

we find the kernel of $N_{K/F'}$ is $\mu_K[p]$ unless $p = 2$ and $n = 2$, where ζ_4 also lies in the kernel.

It is also easy to see that

$$(\sigma - 1)\mu_K[p^\infty] = \begin{cases} \mu_K[p] & \mu_K[p^\infty] \neq \mu_{F'}[p^\infty] \\ 0 & \mu_K[p^\infty] = \mu_{F'}[p^\infty] \end{cases}.$$

Putting these together gives the final formula of the lemma. □

3.3 The main theorem

It remains to put together all our calculations:

Theorem 3.3.1. *Let K/F be a Galois extension of number fields with p dividing $|K\colon F|$ at exactly once and $G = \mathrm{Gal}(K/F)$ satisfying the condition of Theorem 2.6.9. Let P be a choice of Sylow p-subgroup and write $F' = F^P$. Then the isomorphism class of $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ as a $\mathbb{Z}_p[G]$-module is determined by*

1) the signatures of the intermediate subfields of K/F,

2) the class numbers $h(L)$ of the intermediate subfields of K/F,

3) the number of roots of unity $w(L)$ of the intermediate subfields of K/F,

4) the indices $\lambda(L)$ for the intermediate subfields of K/F, *5)* $\ker(\mathrm{Cl}(F') \to \mathrm{Cl}(K))$ *as an* $N_G(P)/P$*-module,*

6) $\ker(W \to \mathrm{Cl}(K)/c(\mathrm{Cl}(F')))$ *as an* $N_G(P)/P$*-module,*

7) $\mathrm{coker}(\mathcal{O}_{F'}^\times/\mu_{F'} \to (\mathcal{O}_K^\times/\mu_K)^P)$ *as an* $N_G(P)/P$*-module,*

8) $(\mu_{F'} \cap N_{K/F'}\mathcal{O}_K^\times)/N_{K/F'}\mu_K$ *as an* $N_G(P)/P$*-module,*

9) if $\mu_K[p^\infty] = \mu_{F'}[p^\infty]$*, then* $\mu_K[p]$ *as an* $N_G(P)/P$*-module.*

Proof. We must show that 1)–9) collectively allow us to calculate *i)–iii)* of Corollary 3.1.2. In Section 3.1 we saw that 1) and 2)–4) determine *i)* and *ii)* respectively. By (3.11), the isomorphism class of $\hat{H}^{-1}(P, \mu_K)$ is determined either by 9) or 3). Proposition 3.2.4 and Lemma 3.2.17 show that combined with 5)–8) this determines *iii)*. □

Recall that the case of p not dividing $|K : F|$ is discussed in Example 3.1.4.

In the next section, we provide four sample applications of Theorem 3.3.1. In certain cases, the required data for Theorem 3.3.1 simplifies significantly. For example, if we assume that K contains no p^{th} roots of unity, the theorem becomes:

Corollary 3.3.2. *Let* K/F *be a Galois extension of number fields with* p *dividing* $|K\colon F|$ *at most once,* K *containing no* p^{th} *roots of unity and* $G = \mathrm{Gal}(K/F)$ *satisfying the condition of Theorem 2.6.9. Let* P *be a choice of Sylow* p*-subgroup and write* $F' = F^P$*. Then the isomorphism class of* $\mathcal{O}_K^\times \otimes \mathbb{Z}_p$ *as a* $\mathbb{Z}_p[G]$*-module is determined by*

1) the signatures of the intermediate subfields of K/F*,*

2) the class numbers $h(L)$ *of the intermediate subfields of* K/F*,*

3) $\ker(\mathrm{Cl}(F') \to \mathrm{Cl}(K))$ *as a* $N_G(P)$*-module,*

4) $\ker(W \to \mathrm{Cl}(K)/c(\mathrm{Cl}(F')))$ *as a* $N_G(P)$*-module.*

3.4 Examples

3.4.1 Cyclic extensions of $\mathbb{Q}$ of prime degree

Let $K/\mathbb{Q}$ be a cyclic extension of prime degree with Galois group $G \cong C_p$. In this case, after tensoring by $\mathbb{Z}_p$, the representation theory is uninteresting, but we may still derive consequences of Theorem 3.3.1.

First assume that K is imaginary quadratic. In this case $\operatorname{rk}\mathcal{O}_K^\times = 0$ and so $\mathcal{O}_K^\times/\mu_K$ is the zero $\mathbb{Z}[C_2]$-lattice. In particular, $H^1(P, \mathcal{O}_K^\times/\mu_K) = 0$ so the right hand side of Formula 3.3 of Proposition 3.2.4 will be of dimension zero.

Consulting formula (3.11), we find that in all cases $\dim_{\mathbb{F}_2} H^1(P, \mu_K) = 1$. Therefore, the remainder of the right hand side of (3.3) must be of dimension one as well. Since $\mathrm{Cl}(\mathbb{Q}) = 0$, $\operatorname{rk}\mathcal{O}_K^\times = 0$ and so $(\mu_{F'} \cap N_{K/F'}\mathcal{O}_{K^H}^\times)/N_{K/F'}\mu_K = 0$, this simplifies to give

Lemma 3.4.1. *For any imaginary quadratic field $K/\mathbb{Q}$, $\dim_{\mathbb{F}_2}(\ker(W \to \mathrm{Cl}(K)))$ $= 1$.*

If we write $K = \mathbb{Q}(\sqrt{-d})$ with d squarefree, then the kernel must be generated by

$$\sum_{\mathfrak{p}|d} 1 \cdot \mathfrak{p} \tag{3.12}$$

as it certainly contains this element. Classically, this is a consequence of genus theory, which states that primes of K lying over the ramified primes generate $\mathrm{Cl}(K)[2]$ and asserts that (3.12) is the only relation amongst them.

For $K/\mathbb{Q}$ real quadratic, $\operatorname{rk}\mathcal{O}_K^\times = 1$ and $\mathcal{O}_K^\times/\mu_K$ is the unique non-trivial one dimensional $\mathbb{Z}[C_2]$-lattice ϵ.

Lemma 3.4.2. *For a real quadratic field $K/\mathbb{Q}$,*

$$\dim_{\mathbb{F}_2}(\ker(W \to \mathrm{Cl}(K))) + \dim_{\mathbb{F}_2} N_{K/\mathbb{Q}}\mathcal{O}_K^\times = 2.$$

In particular, $\dim_{\mathbb{F}_2}(\ker(W \to \mathrm{Cl}(K))) = 2$ if and only if K has a totally positive fundamental unit.

Proof. As before $H^1(G, \mu_K)$ is necessarily of dimension one, but now $H^1(G, \mathcal{O}_K^\times/\mu_K)$ $\cong H^1(G, \epsilon) \cong \mathbb{F}_2$ so that the rest of the right hand side of (3.3) is of dimension 2. For the last statement, simply note that a unit of K has norm -1 if and only if it is not totally positive or negative. $\square$

Finally when $G \cong C_p$ for $p \geq 3$, $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ is isomorphic to the augmentation ideal of $\mathbb{Z}_p[C_p]$ (i.e. the kernel of the degree map). This need not be the case over $\mathbb{Z}$, but in any case, we do have $H^1(G, \mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p) \cong \mathbb{F}_p$.

Lemma 3.4.3. *For $K/\mathbb{Q}$ a cyclic extension of degree $p \geq 3$, $\dim_{\mathbb{F}_p}(\ker(W \to \mathrm{Cl}(K)))$ $= 1$.*

Proof. In this case, $H^1(G, \mu_K)$ and $H^1(G, K^\times/\mu_K) \cong H^1(G, K^\times)$ are always trivial. As a result the right hand side of (3.3) simplifies to $\ker(W \to \mathrm{Cl}(K))$, so by Proposition 3.2.4 this must be dimension one. $\square$

Classically, this can be seen from the the ambiguous class number formula (see e.g. [Lem13, Thm. 1]).

3.4.2 Real quadratic extensions of real quadratic fields

Let $p = 2$ and $F = \mathbb{Q}(\sqrt{a})$ be a real quadratic field. Suppose that K varies amongst totally real quadratic extensions of F and set $G = \mathrm{Gal}(K/F) \cong C_2$. Then $\mathrm{rk}\,\mathcal{O}_F^\times = 1$ and $\mathrm{rk}\,\mathcal{O}_K^\times = 3$. As K varies, there are two possibilities for $\mathcal{O}_K^\times/\mu_K$ as a $\mathbb{Z}[C_2]$-module, namely,

$$\mathbb{1} \oplus \epsilon \oplus \epsilon \text{ or } \mathbb{Z}[C_2] \oplus \epsilon,$$

where $\mathbb{1}, \epsilon$ denote the trivial and non-trivial rank one $\mathbb{Z}[C_2]$-lattices and $\mathbb{Z}[C_2]$ is the regular representation. These two cases can still be distinguished over $\mathbb{Z}_2$ and, in particular, by the dimension of $H^1(G, \mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_2)$, which is 2 or 1 in the above cases respectively (see Example 2.1.1).

Lemma 3.4.4. *Let K/F be a totally real quadratic extension of a real quadratic field. Then*

$$\begin{array}{l} \dim_{\mathbb{F}_2}(\ker(\mathrm{Cl}(F) \to \mathrm{Cl}(K))) \\ \quad + \dim_{\mathbb{F}_2}(\ker(W \to \mathrm{Cl}(K)/\mathrm{Cl}(F))) \\ \quad + \dim_{\mathbb{F}_2}(\mathrm{coker}(\mathcal{O}_F^\times/\mu_F \to (\mathcal{O}_K^\times/\mu_K)^G)) \\ \quad + \dim_{\mathbb{F}_2}(\{\pm 1\} \cap N_{K/F}\mathcal{O}_K^\times) \end{array} = \begin{cases} 3 & \text{if } \mathcal{O}_K^\times/\mu_K \cong \mathbb{1} \oplus \epsilon \oplus \epsilon \\ 2 & \text{if } \mathcal{O}_K^\times/\mu_K \cong \mathbb{Z}[C_2] \oplus \epsilon \end{cases}.$$

Proof. We calculate $H^1(G, \mathcal{O}_K^\times/\mu_K)$ via Proposition 3.2.4. By Lemma 3.2.17 *ii)*, as $\mu_K = \mu_F$, the abelian group $H^1(G, \mu_K)$ is isomorphic to $\mu_K[2] = \{\pm 1\}$. Thus, the remainder of the terms of (3.3) must have total dimension 2 or 3 over $\mathbb{F}_2$, depending on whether or not $H^1(G, \mathcal{O}_K^\times/\mu_K)$ has dimension 1 or 2. Finally, note that $(\mu_F \cap N_{K/F}\mathcal{O}_K^\times)/N_{K/F}\mu_K \cong \{\pm 1\} \cap N_{K/F}\mathcal{O}_K^\times$ as $N_{K/F}$ annihilates $\mu_K = \{\pm 1\}$. $\square$

Note this only refers to 5)–9) of Theorem 3.3.1. This makes sense given that we have already prescribed the signatures of all subfields and that C_2 has no non-zero Brauer relations (Corollary 2.2.12 *ii)*).

Note also that, for fixed F, there is at most one choice of K for which $\mathrm{coker}(\mathcal{O}_F^\times/\mu_F \to (\mathcal{O}_K^\times/\mu_K)^G)$ is non-zero. To see this, note that there are only two possible quadratic extensions of F with this property, namely $F(\sqrt{u}), F(\sqrt{-u})$ for u a fundamental unit of $\mathcal{O}_F^\times$. Now, $F(\sqrt{u})$ will only be totally real if u is totally positive. For many choices of F there is no totally positive fundamental unit, e.g. $F = \mathbb{Q}(\sqrt{2})$, so $\mathcal{O}_F^\times/\mu_F \to (\mathcal{O}_K^\times/\mu_K)^G$ is always an isomorphism. In general, one may consider 4), 7) to be "generically zero", and can often be discounted in asymptotic calculations.

3.4.3 Complex D_{2p}-extensions

Assume that $K/\mathbb{Q}$ is a Galois extension with $G = \mathrm{Gal}(K/\mathbb{Q}) \cong D_{2p}$. Then $K/\mathbb{Q}$ satisfies the conditions of Theorem 3.3.1 at p. Let $\sigma, \tau \in D_{2p}$ be of order p and 2 respectively and label the intermediate subfields as in the following diagram:

$$\begin{tikzcd}[arrows=dash]
& & K \arrow[dll, "2"'] \arrow[dl] \arrow[ddr, "p"] & \\
L & \cdots L^{\sigma^{p-1}} \arrow[ddr] & & \\
& & & F' \arrow[dl, "2"] \\
& & \mathbb{Q} \arrow[from=uull, "p"'] &
\end{tikzcd}$$

In Section 2.7.1 we described the representation theory of D_{2p} over $\mathbb{Z}_p$, which we now briefly recall. The groups D_{2p} are one of the few infinite families of groups for which all integral lattices have been classified. Write $\mathbb{1}, \epsilon, \rho$ for the three irreducible $\mathbb{Q}_p[D_{2p}]$-representations, where $\mathbb{1}, \epsilon$ denote the trivial and non-trivial one dimensional representations, and ρ the $(p-1)$-dimensional irreducible. There are six indecomposable $\mathbb{Z}_p[D_{2p}]$-lattices:

$$\mathbb{1}, \epsilon, A, A', (A, \epsilon), (A', \mathbb{1}), \tag{3.13}$$

where $\mathbb{1}, \epsilon$ extend to the corresponding $\mathbb{Q}_p[D_{2p}]$-representations, A, A' are two non-isomorphic $\mathbb{Z}_p[D_{2p}]$-lattices within ρ, and $(A, \epsilon), (A', \mathbb{1})$ are non-trivial extensions of $\epsilon, \mathbb{1}$ by A, A' respectively (these are explicitly constructed in [Lee64]).

If K has no real embeddings, then $\mathrm{rk}\, \mathcal{O}_K^\times = 2$ and $\mathcal{O}_K^\times \otimes \mathbb{Q}_p \cong \rho$. Thus, $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ could potentially lie in one of two isomorphism classes of $\mathbb{Z}_p[G]$-lattices, namely,

$$A \text{ or } A'.$$

Referring to the matrix given on p65 for $\ell = p$, we find that these two cases are distinguished by the D_{2p}/C_p-isomorphism class of $H^1(P, \mathcal{O}_K^\times/\mu_K)$, which is a one dimensional $\mathbb{F}_p$-vector space with non-trivial or trivial C_2-action for A and A' respectively. This together with Corollary 3.3.2 and Proposition 3.2.4 gives the first part of:

Lemma 3.4.5. *Let $K/\mathbb{Q}$ be a D_{2p}-extension, for p odd, which has no real embeddings. Then*

i) if K does not contain any p^{th} roots of unity, then

$$\dim_{\mathbb{F}_p} \ker(\mathrm{Cl}(F') \to \mathrm{Cl}(K)) + \dim_{\mathbb{F}_p} \ker(W \to \mathrm{Cl}(K)/c(\mathrm{Cl}(F'))) = 1,$$

where W is as in Notation 3.2.1. Moreover, $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p \cong A'$ if and only if $\ker(W \to \mathrm{Cl}(K)/c(\mathrm{Cl}(F')))$ is one dimensional and acted on trivially by D_{2p}/C_p.

ii) if K does contain the p^{th} roots of unity (i.e. $p = 3$ and $F' = \mathbb{Q}(\zeta_3)$, where ζ_3 denotes a primitive third root of unity), then

$$\dim_{\mathbb{F}_p} \ker(W \to \mathrm{Cl}(K)/c(\mathrm{Cl}(F'))) + \dim_{\mathbb{F}_p}((N_{K/\mathbb{Q}(\zeta_3)}\mathcal{O}_K^\times)/\pm 1) = 2,$$

with

$$\left(\begin{array}{c} \dim_{\mathbb{F}_p} \ker(W \to \mathrm{Cl}(K)/c(\mathrm{Cl}(F')))^{\tau=1} \\ + \dim_{\mathbb{F}_p}((N_{K/\mathbb{Q}(\zeta_3)}\mathcal{O}_K^\times)/\pm 1)^{\tau=-1} \end{array} \right) = \begin{cases} 1 & \text{if } \mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p \cong A \\ 2 & \text{if } \mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p \cong A' \end{cases}.$$

Proof. For the last part if *i)*, we know that $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p \cong A'$ when the one dimensional space $\ker(\mathrm{Cl}(F') \to \mathrm{Cl}(K)) \oplus \ker(W \to \mathrm{Cl}(K)/c(\mathrm{Cl}(F')))$ is fixed by D_{2p}/C_p. But, τ acts by multiplication by -1 on $\mathrm{Cl}(F')$ (τ acts by the identity on primes of F' lying over ramified or inert primes and swaps the two primes lying over a split prime. In each case, τ takes it to its inverse in the class group) and so non-trivially on the $\mathbb{F}_3$-vector space $\ker(\mathrm{Cl}(F') \to \mathrm{Cl}(K))$ whenever it is non-zero. As a result, in the A' case it must be $\ker(W \to \mathrm{Cl}(K)/c(\mathrm{Cl}(F')))$ which is non-trivial.

In the case of *ii)*, again the representation theory of D_{2p} and Theorem 3.3.1 ensure that $H^1(P, \mathcal{O}_K^\times/\mu_K)$ is of rank one. Therefore, the right hand side of (3.3) is also of rank one. We may further simplify (3.3) in our case.

As $F' = \mathbb{Q}(\zeta_3)$ is imaginary quadratic, $\mathrm{rk}\, \mathcal{O}_{F'}^\times = 0$, so the summand given by $\mathrm{coker}(\mathcal{O}_{F'}^\times/\mu_{F'} \to (\mathcal{O}_K^\times/\mu_K)^P)$ is trivial.

On the other hand, by (3.11), $\hat{H}^{-1}(P, \mu_K) \cong \mu_K[p]$ is of rank one with non-trivial action of $N_G(P)/P$. Therefore, after twisting, $H^1(P, \mu_K)$ is of rank one with trivial action. For the $(\mu_{\mathbb{Q}(\zeta_3)} \cap N_{K/\mathbb{Q}(\zeta_3)}\mathcal{O}_K^\times)/N_{K/\mathbb{Q}(\zeta_3)}\mu_K$ term, note that $N_{K/\mathbb{Q}(\zeta_3)}\mathcal{O}_K^\times \subseteq \mu_{\mathbb{Q}(\zeta_3)} = \mathcal{O}_{\mathbb{Q}(\zeta_3)}^\times$, whilst $N_{K/\mathbb{Q}(\zeta_3)}\mu_K = \{\pm 1\}$. Finally, as $\mathrm{Cl}(\mathbb{Q}(\zeta_3)) = 0$, $\ker(\mathrm{Cl}(F') \to \mathrm{Cl}(K))$ is always trivial. $\square$

Remark 3.4.6. Our proof of Lemma 3.4.5 relies on the fact that we are able to describe the, relatively few, isomorphism classes of potential D_{2p}-lattices. It would be interesting to ask for a proof of, say, the formula of *i)* without appealing to classification results within integral representation theory. Another consequence of there being relatively few possibilities is that we are also able to distinguish the A and A' cases instead using regulator constants (this follows from Section 2.7.1 or directly from [Bar12]).

Example 3.4.7. Recall from Remark 2.7.1 that if $p \leq 67$ (or $p \leq 157$ assuming the generalised Riemann hypothesis), then a $\mathbb{Z}[D_{2p}]$-lattice M is determined by the isomorphism classes of $M \otimes \mathbb{Z}_\ell$ for $\ell = 2, p$ as a $\mathbb{Z}_\ell[G]$-lattice. If $K/\mathbb{Q}$ is a D_{2p}-extension and has no real embeddings, then $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_2$ as a $\mathbb{Z}_2[G]$-lattice is independent of K. So, if $p \leq 67$, by considering $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$, there are two candidate isomorphism classes for $\mathcal{O}_K^\times/\mu_K$ as a $\mathbb{Z}[G]$-lattice. We shall also denote these by A, A'. Moreover, Lemma 3.4.5 in fact determines the $\mathbb{Z}[G]$-lattice structure. If K/F is an arbitrary D_{2p}-extension, with F not necessarily $\mathbb{Q}$, then we may still calculate the isomorphism class of $\mathcal{O}_K^\times/\mu_K$ by applying Theorem 3.3.1 at both p and 2.

Example 3.4.8. In the $p = 3$ case, we applied Lemma 3.4.5 to create a Magma function [Magma] which returns the isomorphism class of $\mathcal{O}_K^\times/\mu_K$. Using a list of complex S_3-extensions obtained from the LMFDB [LMFDB], we calculated the total number of such fields with each isomorphism class:

A	A'	total
10480	8652	19132
$\approx 54.8\%$	$\approx 45.2\%$	

The largest discriminant amongst these fields is $\approx -2.4 \times 10^{15}$, but the list is not complete for all fields up to that discriminant (but did contain all complex S_3-extensions of $\mathbb{Q}$ contained in the LMFDB when accessed in May 2018). If we run only over extensions containing $\mathbb{Q}(\zeta_3)$, we find the relative proportions are

A	A'	total
334	1009	1343
$\approx 24.9\%$	$\approx 75.1\%$	

If we exclude fields containing $\mathbb{Q}(\zeta_3)$, the relative proportions are

A	A'	total
10146	7643	17789
$\approx 57.0\%$	$\approx 43.0\%$	

Such calculations are of interest in modern approaches to the study of Cohen–Lenstra heuristics. More specifically, it is expected that the distribution of possible Galois module structures of $\mathcal{O}_K^\times$ should link to the distribution and average sizes of class groups. When ordered by discriminant, a zero proportion of complex S_3-extensions of $\mathbb{Q}$ contain $\mathbb{Q}(\zeta_3)$, so for asymptotic calculations we may discount all such fields and only apply Lemma 3.4.5 *i)*.

Lemma 3.4.9. *Amongst complex S_3 extensions $K/\mathbb{Q}$ ordered by discriminant, a zero proportion have $\mathbb{Q}(\zeta_3)$ as their quadratic subfield.*

Proof. The idea of the following proof was suggested by Alex Bartel. We first show the following claim:

Claim. *All S_3-extensions $K/\mathbb{Q}$ containing $\mathbb{Q}(\zeta_3)$ are of the form $\mathbb{Q}(\sqrt[3]{a}, \zeta_3)$ for some cubefree $a \in \mathbb{Q}^\times$.*

Proof of Claim. By Kummer theory, C_3-extensions K of $F' := \mathbb{Q}(\zeta_3)$ must be of the form $F'(\sqrt[3]{b})$ for $b \in F'^\times/(F'^\times)^3$. If K is Galois over $\mathbb{Q}$, then $\langle b\rangle \subseteq F'^\times/(F'^\times)^3$ must be stable under the action of $\mathrm{Gal}(F'/\mathbb{Q})$. Let τ generate $\mathrm{Gal}(F'/\mathbb{Q})$. We now split into two cases, the first where $\tau(b) \equiv b \pmod{(F'^\times)^3}$ and the second where $\tau(b) \equiv b^2 \pmod{(F'^\times)^3}$.

In the first case, the fact that $\mathbb{Q}^\times/(\mathbb{Q}^\times)^3 = (F'^\times/(F'^\times)^3)^{\mathrm{Gal}(F'/\mathbb{Q})}$ (as $|F' : \mathbb{Q}| = 2$) ensures that we may assume that $b \in \mathbb{Q}^\times$. As a result, $K/\mathbb{Q}$ must be the S_3-extension of $\mathbb{Q}$ given by the splitting field of $(x^3 - b)$.

We claim that the second case does not result in any S_3-extensions. Suppose that $K/\mathbb{Q}$ as above is Galois with group G and fix an extension of τ to K, which we shall also denote τ. Fix a choice β of cube root of b and a primitive third root of unity ζ_3. Then G must act faithfully on the set

$$\Sigma := \{\beta, \zeta_3\beta, \zeta_3^2\beta, \tau(\beta), \zeta_3\tau(\beta), \zeta_3^2\tau(\beta)\}.$$

Let σ be the element of G of order 3 which sends $\beta \mapsto \zeta_3\beta$. We claim that $G = \langle \sigma, \tau \rangle$ acts on Σ in the following way:

$$\begin{tikzcd}
\beta \arrow[r, mapsto, "\sigma"] \arrow[d, leftrightarrow, "\tau"] & \zeta_3\beta \arrow[r, mapsto, "\sigma"] \arrow[dr, leftrightarrow, "\tau"] & \zeta_3^2\beta \arrow[ll, mapsto, bend right, "\sigma"'] \arrow[dl, leftrightarrow, "\tau"'] \\
\tau(\beta) \arrow[rr, mapsto, bend right, "\sigma"'] & \zeta_3\tau(\beta) \arrow[l, mapsto, "\sigma"'] & \zeta_3^2\tau(\beta) \arrow[l, mapsto, "\sigma"']
\end{tikzcd}$$

The action of σ on the top row is by definition and the action of τ is also given by assumption together with that fact that $\tau(\zeta_3) = \zeta_3^2$. The only non-obvious claim is the action of σ on $\tau(\beta)$. Using the assumption that $\tau(b) \equiv b^2 \pmod{(F'^{\times})^3}$, we find that $\tau(\beta) = \beta^2\lambda$ for some $\lambda \in \mathbb{Q}(\zeta_3)$ (as F' contains the third roots of unity). As a result:

$$\begin{aligned}
\sigma(\tau(\beta)) &= \sigma(\beta^2)\sigma(\lambda) \\
&= \zeta_3^2\beta^2\lambda \\
&= \zeta_3^2\tau(\beta)
\end{aligned}$$

Finally, note that this action on Σ is given by C_6 acting on itself faithfully, and so G cannot be isomorphic to S_3. $\square$

Without changing the extension, we may additionally assume that a is a cubefree integer and, given that -1 is a cube in $\mathbb{Q}(\zeta_3)$, that a is positive.

Claim. *For a cubefree integer a, the discriminant of $\mathbb{Q}(\sqrt[3]{a}, \zeta_3)$ is divisible by $(a')^4$, where a' is the largest squarefree divisor of a which is coprime to 3.*

Proof of Claim. The conductor-discriminant formula [Neu13, VII.11.9] gives that:

$$\Delta_{\mathbb{Q}(\sqrt[3]{a},\zeta_3)} = \mathfrak{f}(\mathbb{1})\mathfrak{f}(\epsilon)\mathfrak{f}(\rho)^2.$$

Here $\mathfrak{f}(-)$ denotes the Artin conductor, $\mathbb{1}$ and ϵ are the trivial and non-trivial one dimensional complex representations of $S_3 = \mathrm{Gal}(\mathbb{Q}(\sqrt[3]{a}, \zeta_3)/\mathbb{Q})$ and ρ its two dimensional representation. We claim that, for every prime $p \neq 3$ dividing a, we have that p^2 divides $\mathfrak{f}(\rho)$. As the Artin conductor is a product

$$\mathfrak{f}(\rho) = \prod_p p^{\mathfrak{f}_p(\rho)}$$

of local Artin conductors, it suffices to calculate $\mathfrak{f}_p(\rho)$ for p dividing a but coprime to 3. By definition,

$$\mathfrak{f}_p(\rho) = \sum_{i \geq 0} \frac{|G_i|}{|G_0|} \operatorname{codim} \rho^{G_i},$$

where G_i denotes the i^{th} ramification group of G at p. As $\mathbb{Q}(\zeta_3)$ is unramified away from 3, we find that G_0 is the C_3-subgroup of S_3. But, G_i is a p-group for all $i \geq 1$, and so must vanish. As a result

$$\mathfrak{f}_p(\rho) = \frac{|G_0|}{|G_0|} \operatorname{codim} \rho^{G_0} = 2.$$

This gives the desired result. □

We now wish to bound the number of extensions N_X of the form $\mathbb{Q}(\sqrt[3]{a}, \zeta_3)$ with discriminant at most X. By the above claim this is less than or equal to the number of cubefree integers a for which $a' \leq X^{1/4} =: Y$. By writing each such a as $3^d bc$ with b and c squarefree and not divisible by 3 we obtain the bound:

$$\begin{aligned} N_X &\leq 3 \sum_{1 \leq b \leq Y} \sum_{1 \leq c \leq Y/b} 1 \\ &\leq 3 \sum_{1 \leq b \leq Y} \frac{Y}{b} \end{aligned}$$

Here the sums run only over integers and the coefficient 3 corresponds to the number of possible choices of exponent of 3 in the prime decomposition of a. By breaking the summation into the first term and $2 \leq b \leq Y$-terms we obtain the bound:

$$\begin{aligned} N_X &\leq 3 \left(Y + \int_1^Y \frac{Y}{b} db \right) \\ &= 3Y(\log Y + 1). \end{aligned}$$

In other words, the number of S_3-extensions of $\mathbb{Q}$ containing $\mathbb{Q}(\zeta_3)$ of discriminant $\leq X$ is $O(X^{1/4} (\log X)^{1/4})$. In contrast, Bhargava and Wood have shown that the number of S_3-extensions of $\mathbb{Q}$ of discriminant at most X is asymptotic to $c \cdot X^{1/3}$

for some $c > 0$ [BW08, Thm. 2]. This concludes the proof. $\square$

3.4.4 Real D_{2p}-extensions

Now assume that $K/\mathbb{Q}$ is a Galois totally real D_{2p}-extension for some odd p. In this case $\operatorname{rk}\mathcal{O}_K^\times = 5$ and $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Q}_p \cong \epsilon \oplus \rho^{\oplus 2}$ (continuing the notation of Section 3.4.3). From (3.13), we find that $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ lies in one of the following five isomorphism classes:

$$\begin{aligned}
&A \oplus A \oplus \epsilon,\\
&A \oplus A' \oplus \epsilon,\\
&A' \oplus A' \oplus \epsilon,\\
&(A, \epsilon) \oplus A,\\
&(A, \epsilon) \oplus A'.
\end{aligned}$$

It can be checked that $(A, \epsilon) \oplus (A', \mathbb{1})$ is isomorphic to $\mathbb{1}\!\uparrow_{\{1\}}^G$, so $H^1(P, (A, \epsilon)) = 0$. As a result, the regulator constants and cohomology modules of the above lattices are given by:

	$A \oplus A \oplus \epsilon$	$A \oplus A' \oplus \epsilon$	$A' \oplus A' \oplus \epsilon$	$(A, \epsilon) \oplus A$	$(A, \epsilon) \oplus A'$
$C_{2\cdot\theta_{D_{2p}}}(-)$	p^3	p	$1/p$	p	$1/p$
$\dim_{\mathbb{F}_p} H^1(C_p, -)$	2	2	2	1	1
$\dim_{\mathbb{F}_p} H^1(C_p, -)^{D_{2p}/C_p}$	0	1	2	0	1

Figure 3.3

Note that since $2 \cdot \theta_{D_{2p}}$ is an integral Brauer relation and each of the candidates for $\mathcal{O}_K^\times/\mu_K$ lies within the same $\mathbb{Q}_p[G]$-representation, we get that the valuations of the regulator constants agree modulo 2 (cf. Remark 3.1.7).

Example 3.4.10. Applying Theorem 3.1.5, we find that

$$\left(\frac{h(K)}{h(F')h(L)^2}\right)^2 = C_{2\cdot\theta_{D_{2p}}}(\mathcal{O}_K^\times/\mu_K)^{-1} \cdot C_{2\cdot\theta_{D_{2p}}}(\mathbb{1}).$$

Since $C_{2\cdot\theta_{D_{2p}}}(\mathbb{1}) = 1/p$ we therefore get:

$$\frac{h(K)}{h(F')h(L)^2} = \begin{cases} 1/p^2 & \text{if } \mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p \cong A \oplus A \oplus \epsilon \\ 1/p & \text{if } \mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p \cong A \oplus A' \oplus \epsilon \text{ or } (A,\epsilon) \oplus A \\ 1 & \text{if } \mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p \cong A' \oplus A' \oplus \epsilon \text{ or } (A,\epsilon) \oplus A' \end{cases}.$$

This observation was made in [Bar12, Ex. 6.4].

Whilst any single invariant listed in Figure 3.3 gives only partial information, the five isomorphism classes can be distinguished by calculating any 2 of the 3 listed invariants. For example:

Lemma 3.4.11. *Let $K/\mathbb{Q}$ be a totally real D_{2p}-extension for p odd. Then the isomorphism class of $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$ as a $\mathbb{Z}_p[G]$-lattice is determined by $H^1(C_p, \mathcal{O}_K^\times/\mu_K)$ as a D_{2p}/C_p-module.*

Example 3.4.12. The module structure of $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_2$ is again independent of K, so that when $p \leq 67$, the isomorphism class of $\mathcal{O}_K^\times/\mu_K$ as a $\mathbb{Z}[D_{2p}]$-module is determined only by $\mathcal{O}_K^\times/\mu_K \otimes \mathbb{Z}_p$.

Example 3.4.13. When $p = 3$, using Magma, we also calculated the relative proportions of the five possibilities amongst all the totally real S_3-extensions stored in the LMFDB. The largest discriminant appearing is $\approx 2.44 \times 10^{11}$, but again the data is not complete up to such discriminants.

$A \oplus A \oplus \epsilon$	$A \oplus A' \oplus \epsilon$	$A' \oplus A' \oplus \epsilon$	$(A,\epsilon) \oplus A$	$(A,\epsilon) \oplus A'$	Total
22	526	2407	574	1832	5361
$\approx 0.41\%$	$\approx 9.8\%$	$\approx 44.9\%$	$\approx 10.7\%$	$\approx 34.2\%$	

Chapter 4

Relative motives over Shimura varieties

4.1 Background on classical motives

Prior to outlining the aims of this chapter, we give brief outline of the theory of motives as they were first conceived.

Classically, motives are associated to smooth projective varieties over a field k. Motives are an attempt to create an intermediate objects between geometry and cohomology, which are geometric in nature, but are linear and behave in a similar way to cohomology. In particular, the category of motives should be constructed without appealing to derived functors or injective resolutions, and yet capture cohomological behaviour.

We now very briefly sketch Grothendieck's original construction of motives for smooth projective varieties (for more details [Sch94] is a very good source). This case is referred to as the *pure* case in reference to the purity results for cohomology of smooth projective varieties. The basis for the construction is that morphisms should correspond to algebraic cycles. The intuition being that algebraic cycles canonically induce morphisms in any Weil cohomology theory. For this reason we must fix an "adequate equivalence relation" $\sim$ on algebraic cycles (for the definition, see e.g. [MNP13, Sec. 1.2]). Any such choice will then yield a different category $\mathcal{M}_\sim/k$ of motives with respect to $\sim$.

Let $X, Y/k$ be smooth projective varieties which need not be connected, but for simplicity we shall temporarily assume that X, Y are equidimensional of dimensions d_X, d_Y respectively. We define the group of degree p correspondences with respect to $\sim$ to be

$$\mathrm{Corr}^p(X, Y) = A^{d_X+p}_{\sim}(X \times Y),$$

where, for d an integer and Z a smooth projective variety, $A^d_{\sim}(Z)$ denotes the $\mathbb{Q}$-vector space of d-dimensional cycles up to equivalence by $\sim$. The objects of $\mathcal{M}_{\sim}/k$ consist of triples (X, e, n) where X is as above, e is an element of $\mathrm{Corr}^0(X, X)$ which is an idempotent with respect to composition of correspondences, and n is some integer. The morphisms $(X, e, n) \to (X', e', n')$ then consist of elements of

$$e\,\mathrm{Corr}^{n'-n}(X, X')e' \subseteq \mathrm{Corr}^{n'-n}(X, X'),$$

with composition given by composition of correspondences. It is formal to check that $\mathcal{M}_{\sim}/k$ is a $\mathbb{Q}$-linear additive category [Sch94, Thm. 1.6]. Given a smooth projective variety X, we call the triple $(X, \Delta_X, 0)$, with Δ_X the diagonal cycle of $X \times X$, the *motive associated to* X and denote it by $h(X)$. This extends to a functor

$$h\colon (\mathrm{SmProjVar}/k)^{\mathrm{op}} \to \mathcal{M}_{\sim}/k.$$

Now assume that k is a number field with a fixed embedding into $\mathbb{C}$. When $\sim$ is rational equivalence (resp. homological equivalence with respect to singular or equivalently ℓ-adic cohomology for any prime ℓ) we denote $\mathcal{M}_{\sim}/k$ by CHM/k (resp. HomM/k) which we refer to as the category of *Chow motives (resp. homological motives)* over k. Since rational equivalence is finer than homological equivalence, we obtain a forgetful functor $\mathrm{CHM}/k \to \mathrm{HomM}/k$.

It is expected that the category of homological motives is in some sense universal with respect "behaving like a cohomology theory". More precisely, if Grothendieck's standard conjecture D (see [MNP13, Conj. 1.2.19]) holds, then any Weil cohomology theory factors through HomM/k and homological equivalence is the coarsest adequate equivalence relation for which this is the case. This suggests that motives are important for the comparison of different cohomology theories. For example, proving results at the level of motives often leads to results for any cohomology theory e.g. that the cohomology of an abelian variety should form an exterior algebra (see [Sch94, Thm. 5.2]).

Unconditionally, we do know that ℓ-adic cohomology canonically factors through the construction of HomM$/k$. We denote the corresponding "realisation" functor HomM$/k \to \mathrm{Gr\acute{E}t}_{\mathbb{Q}_\ell}/k$ to graded ℓ-adic sheaves on k by $H^\bullet_\ell$ (it is equivalent to consider ℓ-adic sheaves on k or $\mathbb{Q}_\ell$-valued G_k-representations, specifically representations of the form $H^i_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$). The following is one of the most important conjectures pertaining to the theory of motives:

Conjecture 4.1.1 (Tate)**.** *Let k be a finitely generated field. Then, for any prime ℓ, the ℓ-adic realisation functor $H^\bullet_\ell \colon \mathrm{HomM}/k \to \mathrm{Gr\acute{E}t}/\mathbb{Q}_\ell$ is full.*

Note that $H^\bullet_\ell$ is faithful by construction. If the conjecture holds, any property of the motive associated to a variety should be recoverable from its étale cohomology. This to some extent justifies the prevalence of the study of Galois representations within number theory.

We call a motive an abelian motive if it is isomorphic to a triple (X, e, n) with X a union of abelian varieties.

Theorem 4.1.2 (Faltings [Fal83, Thm. 4])**.** *When k is a number field, the Tate conjecture holds upon restriction to the full subcategory of abelian motives.*

The analogous result for finite fields is due to Tate.

Example 4.1.3. One direct consequence of the conjecture is that if a homological motive M is such that $H^\bullet_\ell(M) = \bigoplus_i N_i$, then there should exist motives M_i with $M = \bigoplus_i M_i$ for which $H^\bullet_\ell(M_i) \cong N_i$.

For example, let E be an elliptic curve over a number field k. Then $\bigoplus_i H^i_{\text{ét}}(E_{\bar{k}}, \mathbb{Q}_\ell)$ is supported in degrees $0, 1$ and 2. Moreover, $H^1_{\text{ét}}(E_{\bar{k}}, \mathbb{Q}_\ell)^\vee \cong V_\ell E := T_\ell E \otimes \mathbb{Q}_\ell$, where $T_\ell E$ denotes the Tate module, which is absolutely irreducible if and only if E does not have complex multiplication defined over k.

This is indeed reflected at the level of motives. It can be shown that $h(E) \in$ HomM$/k$ decomposes as a sum of three motives $h^i(E)$ for $i \in \{0, 1, 2\}$ whose realisations are $H^i_{\text{ét}}(E_{\bar{k}}, \mathbb{Q}_\ell)$ respectively (this moreover holds for the Chow motive of E, see Theorem 4.4.8). It also makes sense to speak of motives with coefficients in general fields, as opposed to the definition we gave with coefficients in $\mathbb{Q}$ (see Definition 4.3.8). This allows us to say that a motive is absolutely irreducible if and only if it is irreducible with coefficients replaced by a sufficiently large finite extension. It is a fact that $h^1(E)$ is absolutely irreducible if and only if E does not have complex multiplication defined over k (the same also holds for Chow motives, cf. 4.4.6).

However, the process of extracting information from the Galois representations need not be practical. For most properties of varieties, how to do this is highly mysterious. We conclude this overview by giving two extended examples where it is possible to be more explicit.

Definition 4.1.4. A $\mathbb{Q}$-valued *pure Hodge structure* of weight n consists of a finite dimensional $\mathbb{Q}$-vector space V together with a decomposition of $\bigoplus_{p+q=n} V^{p,q}$ of $V_{\mathbb{C}} := V \otimes \mathbb{C}$ into $\mathbb{C}$-subspaces $V^{p,q}$ for $p, q \in \mathbb{Z}$ for which the action of complex conjugation on $V_{\mathbb{C}}$ interchanges $V^{p,q}$ and $V^{q,p}$. A morphism of Hodge structures $V \to V'$ is then a map of $\mathbb{Q}$-vector spaces which respects the grading. We call $\dim_{\mathbb{C}} V^{p,q}$ the $(p,q)^{\text{th}}$ *Hodge number* of V and denote it $h^{p,q}$. We call the set of (p,q) for which $h^{p,q}$ is non-zero the *Hodge type* of V.

Alternatively, the data of a pure Hodge structure of weight n is equivalent to a vector space V equipped with a descending filtration $F^\bullet$ of $\mathbb{C}$-subspaces of $V_{\mathbb{C}}$ for which, whenever $p + q = n + 1$, $F^p V_{\mathbb{C}}$ and $\overline{F^q V_{\mathbb{C}}}$ intersect trivially and span $V_{\mathbb{C}}$. The correspondence is given by setting $F^i V_{\mathbb{C}} = \bigoplus_{\substack{p+q=n \\ p \geq i}} V^{p,q}$.

There is an obvious notion of tensor products and duals of Hodge structures. Let $W = \mathbb{Q}$. Then W has a Hodge structure of weight -2 given by setting $W^{-1,-1} = W_{\mathbb{C}}$. We call W the *Tate Hodge structure* and denote it $\mathbb{Q}(1)$. We further set $\mathbb{Q}(n)$ for $n \in \mathbb{N}$ to be the n^{th} tensor power of $\mathbb{Q}(1)$ and $\mathbb{Q}(-n) = \mathbb{Q}(n)^\vee$. For an arbitrary Hodge structure V and $n \in \mathbb{Z}$, we let $V(n)$ denote its n^{th} *Tate twist* $V \otimes \mathbb{Q}(n)$.

The prototypical example of a pure Hodge structure of weight n comes from the degree n singular cohomology of a smooth projective variety defined over $\mathbb{C}$. Here, the Hodge structure arises from the canonical comparison isomorphism of singular cohomology with complex coefficients and de Rham cohomology. This results in a complex vector space with both a $\mathbb{Q}$-structure coming from that of singular cohomology and the Hodge decomposition of de Rham cohomology. Morphisms of varieties then induce morphisms of Hodge structures.

It is natural to ask if the Hodge numbers can be found directly from the étale cohomology of the variety. Tate was the first to suggest a conjectural answer to this and attempting to provide a proof became a motivational question within the field of p-adic Hodge theory. This was subsequently shown by Faltings:

Theorem 4.1.5 (Faltings [Fal88, Thm. III.4.1])**.** *Let k be a p-adic local field and G_k denote its absolute Galois group. For X/k a smooth proper variety, there is a*

G_k-equivariant isomorphism

$$\mathbb{C}_k \otimes_{\mathbb{Q}_p} H^n_{\acute{e}t}(X, \mathbb{Q}_p) \cong \bigoplus_{i+j=n} \mathbb{C}_k(-j) \otimes_k H^i(X, \Omega^j_{X/k}).$$

Here, $\Omega^q_{X/k}$ denotes the q^{th} exterior power of the sheaf of algebraic differentials on X, whilst $\mathbb{C}_k$ denotes the completion of the algebraic closure of k and $\mathbb{C}_k(-q)$ denotes the $(-q)^{th}$ Tate twist. The action on G_k on the left hand side is diagonal via its action on $H^n_{\acute{e}t}(X_{\bar{k}}, \mathbb{Q}_p)$ and $\mathbb{C}_k$, whilst on the right hand side is via its action on $\mathbb{C}_k(-q)$ (so the action on both sides is $\mathbb{C}_k$-semilinear).

In particular, when k is again a number field, the $(p,q)^{\text{th}}$ Hodge number of X can be recovered by the restriction of the associated Galois representations to any choice of decomposition group.

For the second example, we consider only abelian varieties over a number field k. In this case, it is well understood what information should be carried by the motive. Namely, $h(A) \cong h(A')$ if and only if A is isogenous to A' (cf. Theorem 4.4.10). In particular, the rank of A is a function of its motive. Consequently, Theorem 4.1.2 asserts that, for any prime p, the isomorphism class of V_pA as a Galois representation encodes the rank of A.

As part of the formalism of the Birch–Swinnerton-Dyer conjecture, it is expected that the rank of A should be a function of the p^∞-Selmer group of A/k. This is defined in terms of the p-power torsion on A, but also in terms of local conditions defined by the Kummer map. As a result, it is not clear, a priori, that the p^∞-Selmer rank is a function of the Galois representation and thus can be read off from the motive. This was resolved by work of Bloch–Kato, where they showed that the local conditions could in fact be written purely in terms of the Galois representation V_pA [BK90]. Consequently, if the p-part of the Tate-Shafarevich group is known to be finite (this is the Shafarevich conjecture for the prime p), then this yields a procedure to extract the rank of an abelian variety from its Galois representation.

4.2 Introduction

Let k be a number field with a fixed embedding into $\mathbb{C}$. Fix some smooth quasi-projective variety S over a number field k.

If we consider smooth quasi-projective varieties X equipped with smooth projective structure maps $X \xrightarrow{\pi} S$ (as opposed to the case of $S = \operatorname{Spec} k$ considered in the previous section), then the standard cohomology theories admit analogues in this context. For example, taking for any prime ℓ the higher direct images $R^i\pi_*(\mathbb{Q}_\ell)_X$ of the one dimensional constant ℓ-adic sheaf on X yields an ℓ-adic sheaf on S whose stalks calculate the ℓ-adic cohomology of the fibres in the traditional sense (as we may apply proper base change). Many results for ℓ-adic cohomology carry over to this relative setting and for simplicity we denote $R^i\pi_*(\mathbb{Q}_\ell)_X$ simply by $H^i_\ell(X)$. The Hodge structure defined by singular cohomology and de Rham cohomology also extends to give a "variation of Hodge structure" on S (this is made precise in Section 4.5).

It is also possible to generalise the classical construction of motives to define categories of *relative motives* CHM/S and HomM/S in this context (see Section 4.4) as well as relative realisations functors (see Section 4.5). Many questions about motives as defined in Section 4.1 carry over to the case of relative motives.

Throughout the theory of motives, we find ourselves needing to produce algebraic cycles. For example, the Tate Conjecture 4.1.1 is a statement about the existence of cycles lying over cohomology classes. Unfortunately, in general, it is very hard to demonstrate all conjectural algebraic cycles actually exist. One setting where we have additional tools is in the theory of Shimura varieties.

A Shimura variety is defined by a *Shimura datum* $(G, \mathfrak{X})$. Here G is a reductive algebraic group over $\mathbb{Q}$ and $\mathfrak{X}$ is a conjugacy class of maps $\mathrm{Hom}(\mathbb{S}, G_\mathbb{R})$, where $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ denotes the Deligne torus, such that G and $\mathfrak{X}$ satisfy various compatibility properties (see e.g. [Mil17, Def. 5.5]). It follows that $\mathfrak{X}$ is canonically a disjoint union of Hermitian symmetric domains. We shall often denote $\mathfrak{X}$ simply by a choice of representative element $h \in \mathfrak{X}$.

Definition 4.2.1. Let $\mathbb{A}_f$ denote the finite adeles of $\mathbb{Q}$ and fix an element $g = (g_p)_p \in \mathrm{GL}_n(\mathbb{A}_f)$. We denote by Γ_p the subgroup of $\overline{\mathbb{Q}}_p^\times$ generated by the eigenvalues of g_p. Then we say g is *neat* if after fixing some embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ for all primes p, $\bigcap_p(\overline{\mathbb{Q}}^\times \cap \Gamma_p) = \{1\}$. This is independent of the choice of embeddings. We say that $K \leq G(\mathbb{A}_f)$ is *neat* if for some (or equivalently for every) faithful representation $\rho\colon G \to \mathrm{GL}_{n,\mathbb{Q}}$ we have that $\rho(g)$ is neat for all $g \in K$ (see for example [Pin90, Sec. 0.6]).

For every neat compact open subgroup K of $G(\mathbb{A}_f)$, the space

$$\mathrm{Sh}_K(G, \mathfrak{X}) := G(\mathbb{Q})\backslash(\mathfrak{X} \times (G(\mathbb{A}_f)/K))$$

has the structure of a smooth quasi-projective complex algebraic variety. In fact, it has a canonical model over some number field lying within $\mathbb{C}$ known as its reflex field, which we also denote by $\mathrm{Sh}_K(G, \mathfrak{X})$. As such, $\mathrm{Sh}_K(G, \mathfrak{X})$ satisfies the requirements to be used as a choice of S in the above discussion.

One of the reasons to study Shimura varieties is that, in high generality, they are expected to be moduli spaces for abelian varieties with extra structure (or more accurately abelian motives). When this is the case, proving results about Shimura varieties can yield results for whole families of abelian varieties. Mazur's bounds on the size of the torsion subgroup of an elliptic curves over $\mathbb{Q}$ are one example of this in action [Maz78].

Example 4.2.2. Let $G = \mathrm{GL}_2$ and $\mathfrak{X}$ be the conjugacy class of the map $h \colon \mathbb{S} \to \mathrm{GL}_2$ which sends $a + bi \in \mathbb{C}^\times = \mathbb{S}(\mathbb{R})$ to $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Then $(G, \mathfrak{X})$ defines a Shimura datum and $\mathfrak{X}$ is isomorphic to $\mathcal{H} = \mathcal{H}^+ \coprod \mathcal{H}^-$, the union of the upper and lower half planes. The induced action of G on $\mathcal{H}$ is given by Möbius transformations and the $S := \mathrm{Sh}_K(G, \mathfrak{X})$ are nothing but modular curves. As such, for a $\mathbb{Q}$-algebra A, the A-points of S parametrise elliptic curves over S with a choice of level structure. The meaning of level structure is dependent on K, for example, in the case of

$$K_1(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{A}_f) \;\middle|\; c \equiv 0 \text{ and } d \equiv 1 \pmod{p} \right\}$$

the level structure consists of a marked A-point of order p.

This moduli interpretation automatically results in a universal elliptic curve $\mathcal{E} \to S$. The endomorphisms of $\mathcal{E}$ also give rise to families of cycles on the products $\mathcal{E} \times_S \ldots \times_S \mathcal{E}$.

Example 4.2.3. Let $F/\mathbb{Q}$ be an imaginary quadratic field and write T_F for the torus $\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$. Fix an isomorphism $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}$ and let $\mathfrak{X}$ consist of the single point defined by the corresponding isomorphism $h \colon (T_F)_{\mathbb{R}} \xrightarrow{\sim} \mathbb{S}$. Then (T_F, h) defines a Shimura datum. Moreover, a choice of $\mathbb{Q}$-basis of F defines an inclusion $\iota \colon T_F \hookrightarrow \mathrm{GL}_2$ for which the image of h lies in the subspace $\mathcal{H}$ defined in Example 4.2.2. Thus there is a morphism of Shimura data $(T_F, h) \hookrightarrow (\mathrm{GL}_2, \mathcal{H})$. The reflex field of (T_F, h) is F whilst the reflex field of $(\mathrm{GL}_2, \mathcal{H})$ is $\mathbb{Q}$. As such, for any choice of neat open compact subgroups K, K' of T_F and GL_2 respectively for which $\iota(K) \leq K'$, there is a closed immersion

$$\iota_* \colon \mathrm{Sh}_K(T_F, h) \hookrightarrow \mathrm{Sh}_{K'}(\mathrm{GL}_2, \mathcal{H})_F$$

of smooth algebraic varieties over F.

For any F-algebra A, the A-points of $\mathrm{Sh}_K(T_F, h)$ parametrise elliptic curves E/A with a choice of level structure and map $F \hookrightarrow \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ for which induced action on Ω^1_E is F-linear. The map ι_* is then the map given by forgetting the inclusion.

There is also a universal elliptic curve $\mathcal{E}' \to \mathrm{Sh}_K(T_F, h)$. The compatibility of the moduli intepretations then ensures that the following diagram is cartesian:

$$\begin{tikzcd}
\mathcal{E}' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}_F \arrow[d] \\
\mathrm{Sh}_K(T_F, h) \arrow[r, hook, "\iota_*"'] & \mathrm{Sh}_{K'}(\mathrm{GL}_2, \mathcal{H})_F
\end{tikzcd}$$

Just as in Example 4.1.3, it makes sense to consider relative motives with coefficients. Then the base change of the relative motive $h^1(\mathcal{E})_F$ is absolutely irreducible, whilst its pullback to $\mathrm{Sh}_K(T_F, h)$, i.e. $h^1(\mathcal{E}')$, is not (see Example 4.1.3). Correspondingly $\mathcal{E}' \times_{\mathrm{Sh}_K(T_F,h)} \mathcal{E}'$ has additional cycles.

Fix a Shimura datum $(G, \mathfrak{X})$. The definition of Shimura varieties ensures that, given a representation of G, there is a corresponding variation of Hodge structure on $\mathfrak{X}$. Indeed, the data of a pure Hodge structure on a $\mathbb{Q}$-vector space V is equivalent to a choice of representation $\mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$. But, given a representation $\rho\colon G \to \mathrm{GL}(V)$, for each $(h\colon \mathbb{S} \to G_{\mathbb{R}}) \in \mathfrak{X}$ we obtain a representation $\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \xrightarrow{\rho} \mathrm{GL}(V_{\mathbb{R}})$. These compatibly define a variation of Hodge structure on $\mathfrak{X}$ (see Definition 4.5.2).

These variations of Hodge structure on $\mathfrak{X}$ descend to give variations of Hodge structure on $S(\mathbb{C})$, the complex points of $\mathrm{Sh}_K(G, \mathfrak{X})$, considered as a complex manifold. This extends to a functor

$$\mu_G^{\mathcal{H}}\colon \mathrm{Rep}(G) \to \mathrm{VHS}/S(\mathbb{C}),$$

which we refer to as the *canonical construction*, here $\mathrm{Rep}(G)$ denotes the category of $\mathbb{Q}$-valued representations of G and $\mathrm{VHS}/S(\mathbb{C})$ the category of variations of Hodge structures on the complex manifold $S(\mathbb{C})$. There is also an analogous construction of lisse ℓ-adic sheaves on $\mathrm{Sh}_K(G, \mathfrak{X})$.

In general, there are very few methods to produce interesting variations of Hodge structure or ℓ-adic sheaves (even in the case when $S = \mathrm{Spec}\, k$, where ℓ-adic sheaves correspond to Galois representations). The canonical construction provides a large source of such objects, and it is natural to compare these to other examples occurring in nature. For example, do the sheaves produced by the canonical construction "arise from geometry", i.e. do they occur as a subquotient of a higher direct image of some smooth projective variety X/S up to Tate twists? It is expected that this is the case.

Example 4.2.4. We continue using the notation of Example 4.2.2. If $\mathbb{1}$ denotes the one dimensional trivial representation of GL_2, then $\mu_G^{\mathcal{H}}(\mathbb{1})$ will be the variation of Hodge structure $\mathbb{Q}(0)$, i.e. the constant one dimensional local system with its unique variation of Hodge structure of weight 0. This is the realisation of the trivial motive $(S, \Delta_S, 0)$. If now W is the one dimensional representation on which elements of GL_2 act via the determinant map, then $\mu_G^{\mathcal{H}}(W)$ is $\mathbb{Q}(1)$, which is the variation of Hodge structure associated to $(S, \Delta_S, 1)$. A more interesting example is when V is the standard representation of GL_2. We shall see that $\mu_G^{\mathcal{H}}(V)$ is the variation of Hodge structure $(R^1\pi_*\mathcal{E})^\vee$, where $\mathcal{E} \xrightarrow{\pi} S$ is the universal elliptic curve over S.

It is actually expected that variations of Hodge structure produced by the canonical construction should arise from geometry functorially. Specifically, there should be an $\mathbb{Q}$-linear tensor functor

$$\mu_G^{\mathrm{mot}} \colon \mathrm{Rep}(G) \to \mathrm{CHM}/S$$

which should lift the canonical construction, i.e. composing μ_G^{mot} with its Hodge realisation coincides with $\mu_G^{\mathcal{H}}$ up to a specified natural isomorphism (cf. Lemma 4.10.1). In other words, the canonical construction should be motivic. We would then hope that this lift μ_G^{mot} lifts not just the canonical construction of variations of Hodge structure, but also of ℓ-adic sheaves for all ℓ. As such, we may think of it as a "universal" canonical construction.

Finally, we also expect that μ_G^{mot} should be compatible with morphisms of Shimura data (such as in Example 4.2.3).

This aim of this chapter is to attempt to answer some of these questions. The key idea is to use "mixed Shimura varieties" to automate the construction of motives over traditional (or pure) Shimura varieties. We are able to define such a motivic lift μ_G^{mot} on the full subcategory $\mathrm{Rep}(G)^{\mathrm{AV}}$ of $\mathrm{Rep}(G)$ whose objects have Hodge type contained in $\{(-1,0),(0,-1)\}$ (Proposition 4.7.9). In Section 4.8, we check that this does lift the canonical construction and is compatible with arbitrary base change of Shimura data. In summary we obtain:

Theorem 4.2.5. *Let $(G, \mathfrak{X})$ be a Shimura datum and $K \leq G(\mathbb{A}_f)$ be a neat open compact subgroup. Write S for the corresponding Shimura variety $\mathrm{Sh}_K(G, \mathfrak{X})$. There is a canonical functor $\mu_G^{\mathrm{mot}} \colon \mathrm{Rep}(G) \to \mathrm{CHM}/S$ with the property that the following diagram*

$$\begin{tikzcd}
\mathrm{Rep}(G)^{\mathrm{AV}} \arrow[rr, "\mu_G^{\mathrm{mot}}"] \arrow[dr, "\mu_G^{\mathcal{H}}"'] & \arrow[d, phantom, "\Lleftarrow"] & \mathrm{CHM}/S \arrow[dl, "H_B^\bullet"] \\
& \mathrm{VHS}/S(\mathbb{C}) &
\end{tikzcd}$$

commutes up to a canonical natural isomorphism. Now let $f \colon (G', \mathfrak{X}') \to (G, \mathfrak{X})$ be a morphism of Shimura data and $K \leq G(\mathbb{A}_f), K' \leq G'(\mathbb{A}_f)$ neat open compact subgroups with $f(K') \leq K$. We also write $f \colon S' := \mathrm{Sh}'_K(G, \mathfrak{X}') \to S := \mathrm{Sh}_K(G, \mathfrak{X})$ for the induced map between the corresponding Shimura varieties. Then there is a commutative prism:

$$\begin{tikzcd}
\mathrm{Rep}(G)^{\mathrm{AV}} \arrow[rr, "\mu_G^{\mathrm{mot}}"] \arrow[dr, "\mu_G^{\mathcal{H}}"'] \arrow[dd, "f^*"'] & & \mathrm{CHM}/S \arrow[dl, "H_B^\bullet"] \arrow[dd, "f^*"] \\
& \mathrm{VHS}/S(\mathbb{C}) \arrow[dd, "f^*" near start] & \\
\mathrm{Rep}(G')^{\mathrm{AV}} \arrow[rr, "\mu_{G'}^{\mathrm{mot}}" near start] \arrow[dr, "\mu_{G'}^{\mathcal{H}}"'] & & \mathrm{CHM}/S' \arrow[dl, "H_B^\bullet"] \\
& \mathrm{VHS}/S'(\mathbb{C}) &
\end{tikzcd}$$

where the vertical maps are base change by f and $H_B^\bullet$ denotes the Hodge realisation (see Corollary 4.5.7).

This is stated more precisely as Theorem 4.8.4.

One type of Shimura varieties where it is possible to say more is that of PEL-type Shimura varieties. PEL-type Shimura data are Shimura data which admit an auxiliary choice of PEL-datum (see Definition 4.9.2). This extra structure ensures that Shimura varieties with a choice of PEL-datum satisfy a specific moduli problem in terms of abelian varieties. Many classical Shimura varieties admit PEL-data, including those given in Examples 4.2.2, 4.2.3.

The property of Shimura data $(G, \mathfrak{X})$ with a choice of PEL-datum we shall most use is that all representations of such a G are summands of some family of explicit representations depending on the PEL-datum (Proposition 4.9.6), and recent work of Ancona has described the maps between them [Anc15, Prop. 8.5]. Moreover, Ancona shows that all of these representations and morphisms can be canonically lifted to CHM/S (see Theorem 4.9.7).

It is worth remarking that a single Shimura datum may admit more than one PEL-datum, each with distinct moduli interpretations. For example, consider the Shimura datum for GL_2 described in Example 4.2.2, $(\mathrm{GL}_2, \mathcal{H})$, then each modular curve S is simultaneously a moduli space for elliptic curves with certain level structure and also two-dimensional abelian varieties which are products of the same elliptic curve with duplicated level structure. These correspond to two distinct choices of PEL-datum for $(\mathrm{GL}_2, \mathcal{H})$ (see Example 4.9.4). A priori, Ancona's construction of motivic lifts depends on the choice of PEL-datum for $(G, \mathfrak{X})$.

Let $f\colon (G', \mathfrak{X}') \to (G, \mathfrak{X})$ be a morphism of Shimura data. Assume that each Shimura datum has a fixed, but not necessarily related, choice of PEL-datum satisfying the following condition: If the choices of PEL-data have standard representations V, V' respectively, then

$$f^*V \text{ is isomorphic to a summand of } \bigoplus_{i=1}^{k} V',$$

for some k.

Assuming this, via the viewpoint of mixed Shimura varieties we are able to check that Ancona's construction commutes with base change. More specifically, that there is a prism analogous to that of Theorem 4.2.5 (see Theorem 4.10.8). In the case when $(G', \mathfrak{X}') = (G, \mathfrak{X})$, this can be used to demonstrate the independence of Ancona's construction from the choice of PEL-datum.

The condition on f^*V is required to ensure we are able to make use of the functoriality provided by mixed Shimura varieties. In practice, this condition appears to be often satisfied. For example, if $(G', \mathfrak{X}')$ is the Shimura datum corresponding to a Siegel modular variety with its usual PEL-datum, then any morphism $f\colon (G', \mathfrak{X}') \to (G, \mathfrak{X})$ will satisfy the conditions.

We also show all the above results also hold with the Hodge realisation replaced by the ℓ-adic realisations for any ℓ. It is also important to have analogues of all these results for motives with coefficients (cf. Example 4.1.3). For this reason we state all our results with coefficients lying in some arbitrary number field.

One application of results such as Theorem 4.2.5 is in the context of Euler systems. An Euler system consists of a compatible family of classes within the Galois cohomology of some Galois representation, which in the case of Shimura varieties would be given by étale cohomology. Euler systems for Shimura varieties have applications in proving partial cases of conjectures such as the Birch–Swinnerton-Dyer Conjecture.

For such Euler systems, it is also important to consider comparison with other cohomology theories, for example, in order to show that the classes do not vanish. For this reason, it is useful to use the language of motives and construct the classes motivically. There are relatively few sources of classes with which to build Euler systems on Shimura varieties and it is vital to be able to pushforward such classes at the motivic level. For an example of this in practice see [LSZ17].

4.3 Change of coefficients

Throughout this chapter we deal with many different $\mathbb{Q}$-linear categories, such as $\mathrm{CHM}/S, \mathrm{Rep}(G), \mathrm{VHS}/S(\mathbb{C})$. Let $L/\mathbb{Q}$ be a finite field extension. Some of these categories naturally generalise to allow coefficients in L. For example, for $\mathrm{Rep}(G)$ the obvious analogue is to take $\mathrm{Rep}_L(G)$, the category of L-linear representations of G_L. This is not always possible though. For example, the definition of a Hodge structure naturally extends to allow coefficients in L only when $L \subseteq \mathbb{R}$.

In this section, we introduce two ways of formally adjoining extra coefficients to an arbitrary F-linear additive category. We shall use these as a substitute when dealing with categories whose definition doesn't directly allow for additional coefficients.

The main work of this section is to show that the two formal constructions coincide and that, in cases which directly allow additional coefficients, they agree with the existing definition. The work of this section is well-known, the original sources being [SR72], [Del79, Sec. 2], [DM82, Pf. of Thm. 3.11], and for a good overview see [AK02, Sec. 5].

Notation 4.3.1. Let L/F be a finite extension of fields and $\mathcal{A}$ be a F-linear category.

Construction 4.3.2. We construct a category $\mathcal{A}_{(L)}$ as follows: Let the objects of $\mathcal{A}_{(L)}$ consist of pairs (M, σ) with M an object of $\mathcal{A}$ and $\sigma \colon L \hookrightarrow \mathrm{End}_{\mathcal{A}}(M)$

an F-linear inclusion. The morphisms $(M, \sigma) \to (M', \sigma)$ are then elements of $\mathrm{Hom}_{\mathcal{A}}(M, M')$ which are equivariant for the action of L. In other words, $\mathcal{A}_{(L)}$ consists of representations of L within $\mathcal{A}$. We think of $\mathcal{A}_{(L)}$ as extending the coefficients of $\mathcal{A}$ from F to L. This notation was first introduced by Saavedra Rivano in [SR72].

Example 4.3.3. Let Vec_F denote the category of finite dimensional F-vector spaces. Then, as an L-vector space is precisely an F-vector space with an F-linear L-structure, $(\mathrm{Vec}_F)_{(L)}$ is canonically equivalent to Vec_L. Let H be a finite abstract group and $\mathrm{Rep}_F(H)$ the category of F-representations of H. Then by the previous example $\mathrm{Rep}_F(H)_{(L)}$ is canonically equivalent to $\mathrm{Rep}_L(H)$ as the H-structure respects the action of L.

Construction 4.3.4. We construct another category $\mathcal{A}_L$ as follows: Let $\mathcal{A}_L$ have the same objects as $\mathcal{A}$ but set $\mathrm{Hom}_{\mathcal{A}_L}(M, N) = \mathrm{Hom}_{\mathcal{A}}(M, N) \otimes_F L$.

If $\mathcal{A}$ is abelian then $\mathcal{A}_{(L)}$ is also, but $\mathcal{A}_L$ need not be. For example, let $F = \mathbb{R}$, $L = \mathbb{C}$ and let $\mathcal{A}$ be $\mathrm{Rep}_{\mathbb{R}}(C_3)$, where C_3 is the cyclic group of order 3. Then the non-trivial irreducible representation W of C_3 over $\mathbb{R}$ is not absolutely indecomposable and $\mathrm{End}_{\mathcal{A}_L}(W)$ contains idempotents which do not correspond to a summand.

Definition 4.3.5. We say that an additive category $\mathcal{A}$ is *pseudo-abelian*, if for any object $A \in \mathcal{A}$ and idempotent $e \in \mathrm{End}_{\mathcal{A}}(A)$, e has an image and the map $\mathrm{Im}(e) \oplus \mathrm{Im}(1-e) \to A$ is an isomorphism. As a result, in a pseudo-abelian category, all idempotents admit kernels and cokernels.

Example 4.3.6. Any abelian category is pseudo-abelian. If $\mathcal{A}$ is pseudo-abelian, then so is $\mathcal{A}_{(L)}$. The categories of motives we have defined are pseudo-abelian. For example, let $(X, p, n) \in \mathrm{CHM}/S$ and $e \in \mathrm{End}_{\mathrm{CHM}/S}((X, p, n)) = pA^{d_X - d_S}_{\sim_{\mathrm{rat}}}(X \times_S X)p$ an idempotent. Then $\mathrm{Im}(e) = (X, pep, n)$ and we canonically get

$$\mathrm{Im}(e) \oplus \mathrm{Im}(1-e) = (X, pep, n) \oplus (X, p - pep, n) \cong (X, p, n).$$

Construction 4.3.7. Given an additive category $\mathcal{B}$, we define its pseudo-abelianisation $B^\natural$ to be the category whose objects are pairs (X, e) with X an object of $\mathcal{B}$ and $e \in \mathrm{End}_{\mathcal{B}}(X)$ an idempotent and for which

$$\mathrm{Hom}_{\mathcal{B}^\natural}((X, e), (X', e')) = e \circ \mathrm{Hom}_{\mathcal{B}}(X, X') \circ e'.$$

Using the same argument as for motives, it is easy to see that $\mathcal{B}^\natural$ is pseudo-abelian. There is an obvious fully faithful functor $\mathcal{B} \to \mathcal{B}^\natural$ which sends $B \mapsto (B, \mathrm{id}_B)$. If $\mathcal{B}$ is pseudo-abelian, then $\mathcal{B} \to \mathcal{B}^\natural$ is an equivalence of categories. If $\mathcal{B} \to \mathcal{C}$ is an additive functor from an additive category to a pseudo-abelian category, then there is a unique, up to canonical natural isomorphism, additive functor $\mathcal{B}^\natural \to \mathcal{C}$ extending $\mathcal{B} \to \mathcal{C}$ (see [CH00, Thm. 2.5]).

Since $\mathcal{A}_L$ is poorly behaved, we shall instead consider its pseudo-abelianisation $(\mathcal{A}_L)^\natural$.

Definition 4.3.8. Let $F/\mathbb{Q}$ be a finite field extension. Given an adequate equivalence relation $\sim$, we define the category $\mathcal{M}_{\sim,F}/k$ of *relative motives over k with coefficients in F with respect to $\sim$* to be $((\mathrm{CHM}/k)_F)^\natural$ (cf. [Del79, Sec. 2.1]). We define CHM_F/k, HomM_F/k analogously.

Remark 4.3.9. All the constructions for motives with coefficients in $\mathbb{Q}$ now carry over, for example, we obtain functors $h\colon (\mathrm{SmProjVar}/k)^{\mathrm{op}} \to \mathrm{CHM}_F/k$ by composition with the canonical functor $\mathrm{CHM}/k \to \mathrm{CHM}_F/k$.

Construction 4.3.10. Given a finite dimensional F-vector space V and an object $M \in \mathcal{A}$, we define $V \otimes_F M$ as the object of $\mathcal{A}$ which represents the functor which sends

$$N \longmapsto \mathrm{Hom}_F(V, \mathrm{Hom}_{\mathcal{A}}(M, N)).$$

Explicitly, such a $V \otimes_F M$ is given by $\bigoplus_i M$ where $\{x_1, ..., x_n\}$ is an F-basis for V. If we take $V = L$, then by functoriality we obtain a canonical map $\alpha_M\colon L \hookrightarrow \mathrm{End}_{\mathcal{A}}(L \otimes_F M)$ acting on the left. After choosing a basis, the action of $\lambda \in L$ is given by writing $\lambda = \sum_i a_i x_i$ with $a_i \in F$ and using the F-linearity of $\mathcal{A}$.

Lemma 4.3.11. *Sending $M \in \mathcal{A}$ to $(L \otimes_F M, \alpha_M) \in \mathcal{A}_{(L)}$, with α_M denoting acting by L on the left, defines a functor $\Phi_A\colon \mathcal{A} \to \mathcal{A}_{(L)}$. This canonically extends to a fully faithful functor $\mathcal{A}_L \to \mathcal{A}_{(L)}$.*

Proof. The first part is clear. For $M, N \in \mathcal{A}$, we must compatibly define isomorphisms

$$L \otimes_F \mathrm{Hom}_{\mathcal{A}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}_{(L)}}(L \otimes_F M, L \otimes_F N). \tag{4.1}$$

Firstly, we claim there are compatible isomorphisms

$$L \otimes_F \mathrm{Hom}_{\mathcal{A}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(M, L \otimes_F N). \tag{4.2}$$

In fact, in general, $V \otimes_F N$ also represents the functor $V \otimes_F \mathrm{Hom}_{\mathcal{A}}(-, N)$ (see [SR72, Prop. I.1.5.1], [DM82, Pf. of Thm. 2.11]). If we make a choice of F-basis of L, this can be seen explicitly by writing $L \otimes_F \mathrm{Hom}_{\mathcal{A}}(A, B) = (\bigoplus_i F) \otimes_F \mathrm{Hom}_{\mathcal{A}}(A, B)$ and $\mathrm{Hom}_{\mathcal{A}}(A, L \otimes_F B) = \mathrm{Hom}_{\mathcal{A}}(A, \bigoplus_i B)$.

We now claim that for any $(B, \sigma) \in \mathcal{A}_{(L)}$ and $A \in \mathcal{A}$, there are isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}_{(L)}}((L \otimes_F A, \alpha_A), (B, \sigma)).$$

Here $L \otimes_F A$ is given the left L-structure defined in Construction 4.3.10. In other words, Φ_A is left adjoint to the forgetful map $\mathcal{A}_{(L)} \to \mathcal{A}$. This is clear as if we fix a basis $x_1, ..., x_n$ of L over F, then an element $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ is sent to

$$\sum_i \sigma(x_i) \circ f \colon \bigoplus_i A \to B.$$

Setting $B = L \otimes_F N$ with its usual left action gives

$$\mathrm{Hom}_{\mathcal{A}}(M, L \otimes_F N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}_{(L)}}(L \otimes_F M, L \otimes_F N).$$

Composing (4.2) with this gives (4.1). $\square$

Lemma 4.3.12. *Let L/F be a finite field extension and $\mathcal{A}$ a pseudo-abelian F-linear category, then $\Phi_{\mathcal{A}}$ defines a canonical equivalence of categories*

$$(\mathcal{A}_L)^\natural \to \mathcal{A}_{(L)}.$$

Proof. That the map is fully faithful is automatic from Lemma 4.3.11. So we need only check essential surjectivity, i.e. that every element $(M, \sigma) \in \mathcal{A}_{(L)}$ is isomorphic to a summand of $(L \otimes_F N, \alpha_N)$ for some $N \in \mathcal{A}$. In fact, we may take $N = M$. Indeed, start with $L \otimes_F M$ considered as an element of $\mathcal{A}$ by L acting on the left. Then we have canonical isomorphisms

$$\begin{aligned} L \otimes_F M &\cong L \otimes_F (L \otimes_{L,\sigma} M) \\ &\cong (L \otimes_F L) \otimes_{L,\iota_2,\sigma} M, \end{aligned}$$

where $\otimes_{L,\iota_2,\sigma}$ denotes that the tensor is with respect to acting by L on the right of $L \otimes_F L$ and considering M as an object of $\mathcal{A}_{(L)}$ via σ. Now, $L \otimes_F L$ has a summand isomorphic to L which is the characterised as being the subspace on which the left and right actions of L agree. If we let e be the corresponding idempotent, then

$$(e \cdot (L \otimes_F L)) \otimes_{L,\iota_2,\sigma} M$$

is a summand of $L \otimes_F M$. But,

$$\begin{aligned}(e \cdot (L \otimes_F L)) \otimes_{L,\iota_2,\sigma} M &\cong L \otimes_{L,\iota_2,\sigma} M \\ &\cong L \otimes_{L,\sigma} M \\ &\cong M\end{aligned}$$

with L-acting via σ. Here, the middle isomorphism uses that acting on $L = e \cdot (L \otimes_F L)$ on the left and right agree. $\square$

We may define a canonical essential inverse to Φ_A in the following way. Given $(M, \sigma) \in \mathcal{A}_{(L)}$, consider $L \otimes_F M$ as an $L \otimes_F L$-module via the left action of L and the action by σ. Then project using the idempotent corresponding to the submodule of $L \otimes_F L$ on which the left and right L-actions agree.

Remark 4.3.13. In the case of motives with coefficients, it is helpful to be able to switch between the $((\mathrm{CHM}/k)_F)^\natural$ and $(\mathrm{CHM}/k)_{(F)}$ constructions freely. For example, the tensor structure on $((\mathrm{CHM}/k)_F)^\natural$ is given by tensor product of the underlying elements of CHM/k, whereas for $(\mathrm{CHM}/k)_{(F)}$, the tensor product of $((X, e, n), \sigma)$ and $((X', e', n'), \sigma)$ is defined to be the largest summand of $(X, e, n) \otimes (X', e', n')$ on which the two F-structures defined by σ, σ' agree. On the other hand, realisations for $(\mathrm{CHM}/k)_{(F)}$ are given simply by the realisations of the underlying motive with the additional action of σ.

Lemma 4.3.14. *Let G be an algebraic group over F. Then there is a canonical equivalence of categories $(\mathrm{Rep}_F(G))^\natural_L \to \mathrm{Rep}_L(G)$.*

Proof. A functor $(\mathrm{Rep}_F(G))_L \to \mathrm{Rep}_L(G)$ is given by sending an object $G \to \mathrm{GL}(V)$ of $\mathrm{Rep}_F(G)$ to $G_L \to \mathrm{GL}(V_L)$ and is defined on morphisms via the L-structure of $\mathrm{Rep}_L(G)$, mimicking the construction of $\Phi_{\mathcal{A}}$. This extends to a functor $(\mathrm{Rep}_F(G)_L)^\natural \to \mathrm{Rep}_L(G)$ since the latter is abelian. It is easy to see that this is fully faithful. We shall show that it is essentially surjective.

For T an S-scheme and X a T-scheme, let $\mathrm{Res}_{T/S}(X)$ denote Weil restriction, which exists whenever T, S, X are affine varieties. We then have unit and counit maps

$$X \to (\mathrm{Res}_{L/F} X)_L, \ \mathrm{Res}_{L/F}(Y_L) \to Y,$$

given by adjunction, as well as the counit-unit equation which says that

$$X_L \to (\mathrm{Res}_{L/F} X_L)_L \to X_L$$

is the identity map. We can be a bit more explicit about the counit map in this case. If L/F is Galois, then $(\mathrm{Res}_{L/F} X_L)_L$ is canonically a product of copies of X_L indexed by $\mathrm{Gal}(L/F)$. In general, it is given by $\mathrm{Res}_{L\otimes_F L/L} X_{L\otimes_F L}$, which necessarily decomposes in a way matching that of $L \otimes_F L$. In any case, the counit map is given by mapping onto the component for which the left and right L-structures agree (see [BLR90, p. 197]).

Now, given a representation $G_L \xrightarrow{\rho} \mathrm{GL}(W)$, consider the representation

$$G_L \to (\mathrm{Res}_{L/F} G_L)_L \xrightarrow{(\mathrm{Res}_{L/F}\rho)_L} (\mathrm{Res}_{L/F}\,\mathrm{GL}(W))_L \hookrightarrow \mathrm{GL}((\mathrm{Res}_{L/F} W)_L).$$

This representation is in the image of $\mathrm{Rep}_F(G)_L$, indeed it is the base change of

$$G \to \mathrm{GL}(\mathrm{Res}_{L/F} W)$$

with G acting via its action on W. By the above paragraph, the subspace of $(\mathrm{Res}_{L/F} W)_L$ on which the two L-actions agree is isomorphic to W. Moreover, since this operation is functorial, this subspace is stable for the action of G_L and the G_L-action is precisely given by ρ. So we have shown ρ isomorphic to a direct summand of a representation in the image of $\mathrm{Rep}_F(G)_L$ and so the functor $(\mathrm{Rep}_F(G)_L)^\natural \to \mathrm{Rep}_L(G)$ is essentially surjective. $\square$

This result motivates the definition of Tannakian categories in the case where the fibre functor is not defined over the base field. The proof can be reinterpreted as a consequence of the fact that every L-representation W lies within a canonical choice of representation which is defined over F. The same argument also carries over to representations of finite abstract groups, now using that any representation V is a summand of $L \otimes_F V$, which is realisable over F.

Remark 4.3.15. We can be a bit more explicit about how to canonically define the corresponding functor $\mathrm{Rep}_L(G_L) \to \mathrm{Rep}_F(G)_{(L)}$. It sends a representation $G_L \xrightarrow{\rho} \mathrm{GL}(W)$ to the representation

$$G \to \mathrm{Res}_{L/F} G_L \xrightarrow{\rho} \mathrm{Res}_{L/F}\,\mathrm{GL}(W) \hookrightarrow \mathrm{GL}(\mathrm{Res}_{L/F} W),$$

together with the L-structure given by its action on W and functoriality of restriction of scalars. Here, the map $G \to \mathrm{Res}_{L/F} G_L$ is given by adjunction.

Example 4.3.16. Let $F = \mathbb{R}, L = \mathbb{C}$ and $G = \mathbb{S}$ the Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$. Fix an isomorphism

$$\gamma\colon \mathbb{S}_{\mathbb{C}} \xrightarrow{\sim} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \tag{4.3}$$

such that projection onto the first component is the canonical map $\mathbb{S}_{\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$ corresponding to the subspace of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ on which the left and right $\mathbb{C}$-actions agree.

Consider the one dimensional representation $\rho\colon \mathbb{S}_{\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$ given by projection onto the first component. Now restrict the underlying vector space to $\mathbb{R}$. If we fix the $\mathbb{R}$-basis $1, i$ of $\mathbb{C}$, then this canonically decomposes as two copies of $\mathbb{A}^1_{\mathbb{R}}$. The $\mathbb{C}$-structure, which for simplicity we only give on the $\mathbb{R}$-points, is then given by

$$a + bi \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

We now wish to explicate the $\mathbb{S}$-action given by

$$\mathbb{S} \to \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{S}_{\mathbb{C}} \to \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_1 \to \mathrm{GL}_2 . \tag{4.4}$$

By our choice of (4.3), the map

$$\mathbb{S} \to \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{S}_{\mathbb{C}} \overset{\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\gamma}{\to} \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \times \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m = \mathbb{S} \times \mathbb{S} \overset{\mathrm{pr}_1}{\to} \mathbb{S}$$

is the identity map. As a result, (4.4) is simply given by

$$a + bi \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Together the $\mathbb{C}$-structure and $\mathbb{S}$-action define an element of $\mathrm{Rep}_{\mathbb{R}}(\mathbb{S})_{(\mathbb{C})}$.

Now, if we base change this back to $\mathbb{C}$, then the underlying space $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$ has two actions of $\mathbb{C}$, namely that given by the $\mathbb{C}$-structure as an element of $\mathrm{Rep}_{\mathbb{R}}(\mathbb{S})_{(\mathbb{C})}$:

$$a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

And the structure given by extension of scalars

$$a+bi \mapsto \begin{pmatrix} a+bi & 0 \\ 0 & a+bi \end{pmatrix}.$$

The subspace on which these agree is spanned by $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and this does indeed have the action of $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ given by projection onto the first component.

In subsequent sections we shall usually consider categories with scalars via their $\mathcal{A}_{(L)}$ type description.

4.4 Relative motives

Notation 4.4.1. Assume that k is a field of characteristic zero equipped with a fixed embedding into $\mathbb{C}$. Given a k-variety Z, we write $Z(\mathbb{C})$ for its complex points considered as a complex manifold.

In this section we shall fix S to be a smooth quasi-projective k-scheme. For simplicity, we shall assume that all components of S have the same dimension d_S.

Definition 4.4.2. Following [DM91, Sec. 1], fix an adequate equivalence relation $\sim$ on all k-varieties and let X, Y be smooth projective S-schemes. Assume for simplicity that X, Y are equidimensional of dimensions d_X, d_Y respectively. We define the group of degree p correspondences from X to Y, up to equivalence by $\sim$, to be

$$\mathrm{Corr}_S^p(X,Y) = A_\sim^{d_X - d_S + p}(X \times_S Y),$$

where $A_\sim^*(-)$ is as defined in Section 4.1. Proceeding as in the classical case we obtain the category $\mathcal{M}_\sim/S$ of *relative motives over S with respect to $\sim$*, whose objects are triples (X, e, n) consisting of a variety X, an idempotent $e \in \mathrm{Corr}_S^0(X,X)$ and an integer $n \in \mathbb{Z}$ corresponding to Tate twists. The category $\mathcal{M}_\sim/S$ is a $\mathbb{Q}$-linear $\otimes$-category, with the tensor structure being given by fibre product over S.

We are mostly concerned with the case when $\sim$ is taken to be rational equivalence $\sim_{\mathrm{rat}}$, in which case we denote $\mathcal{M}_\sim/S$ by CHM/S, or homological equivalence $\sim_{\mathrm{hom}}$ with respect to singular cohomology (or equivalently any choice of ℓ-adic cohomology), in which case we denote the resulting category by HomM/S. These categories are referred to as *relative Chow motives over S* and *relative homological* *motives over S* respectively[1]. Write $H^i_B(Z(\mathbb{C}),\mathbb{Q})$ for the singular cohomology of a variety Z/k. Since points are homologically equivalent and the cycle class map commutes with pushforwards and pullbacks, homological equivalence is coarser than rational equivalence. As a result,

$$A^d_{\sim_{\mathrm{hom}}}(Z) = \mathrm{im}(A^d_{\sim_{\mathrm{rat}}}(Z) \to H^{2d}_B(Z(\mathbb{C}),\mathbb{Q})(d))$$

and we obtain a forgetful map

$$\mathrm{CHM}/S \to \mathrm{HomM}/S,$$

which is full.

If $\mathrm{SmProjVar}/S$ denotes the category of (not necessarily irreducible) smooth projective varieties over S, then there is a functor $h\colon (\mathrm{SmProjVar}/S)^{\mathrm{op}} \to \mathrm{CHM}/S$ which assigns to a variety X/S its motive $(X,\Delta_X,0)$ where Δ_X is the diagonal cycle of $X\times_S X$. By composition, the same is also true of homological motives.

For any adequate equivalence relation, the construction of $\mathcal{M}_\sim/S$ is compatible with change of S, i.e. given $f\colon S'\to S$, we obtain pullback functors $f^*\colon \mathcal{M}_\sim/S \to \mathcal{M}_\sim/S'$ by base changing triples in the obvious way.

Remark 4.4.3. This construction has been extended to the case when $S\to k$ is not necessarily smooth but remains quasi-projective by work of Corti–Hanamura [CH00]. These constructions test the boundaries of what might be considered to be a "pure case". On the other hand, it has recently been shown that Corti–Hanamura's construction is equivalent to the heart of a certain "weight-structure" within Cisinski–Déglise's category of Voevodsky style motives over S [Fan16].

Definition 4.4.4. Let $F/\mathbb{Q}$ be a number field. We define the category CHM_F/S of *relative Chow motives over S with coefficients in F* to be $((\mathrm{CHM}/S)_F)^\natural$. We define HomM_F/S analogously.

Definition 4.4.5. Let AbVar/S denote the category of abelian varieties over S. We denote by $\mathrm{CHM}^{\mathrm{ab}}_F/S, \mathrm{HomM}^{\mathrm{ab}}_F/S$ the smallest rigid linear symmetric tensor subcategories which contain motives of abelian varieties and are closed under taking subobjects and Tate twists.

[1]It may be better to refer to HomM/S as "naive homological motives". This is because, unlike in the case of $S=k$, our homological motives admit non-trivial maps between objects which should be considered to live in different cohomological degrees. As a result, they do not coincide with any reasonable notion of "relative numerical motives".

Theorem 4.4.6. *There is a section $\mathcal{I}$ of the projection $\mathcal{N} \colon \mathrm{CHM}_F^{\mathrm{ab}}/S \to \mathrm{HomM}_F^{\mathrm{ab}}/S$ which is a linear symmetric tensor functor, commutes with Tate twists and is such that*

$$h|_{\mathrm{AbVar}^{\mathrm{op}}} = \mathcal{I} \circ \mathcal{N} \circ h|_{\mathrm{AbVar}^{\mathrm{op}}}.$$

Proof. This follows from work of O'Sullivan [O'S11, pf. of Thm. 6.1.1]. More precisely, O'Sullivan checks that there is a section of the projection map from $\mathrm{CHM}^{\mathrm{ab}}/S$ onto its quotient by its maximal proper tensor ideal (a relative analogue of numerical motives). But homological and numerical equivalence coincide for abelian varieties [Lie68, Thm. 4], so the maximal proper tensor ideal of $\mathrm{HomM}^{\mathrm{ab}}/S$ is trivial and $\mathrm{HomM}^{\mathrm{ab}}/S$ must be the quotient described above. The same reasoning applies for motives with coefficients. □

Remark 4.4.7. Morphisms in the image of $\mathcal{I}$ are *symmetrically distinguished* in the sense of [O'S11, Def. 6.2.1]. O'Sullivan checks that the pullback of a symmetrically distinguished cycle is symmetrically distinguished [O'S11, Thm. iii) p2]. From this, it is easy to see check that, given a morphism $f \colon S' \to S$, there is a natural isomorphism $f^* \circ \mathcal{I} \implies \mathcal{I} \circ f^*$ since both compositions yield a symmetrically distinguished Chow cycle lying over a numerical cycle, but there is only one such cycle (cf. [O'S11, Thm. 6.2.5]).

Theorem 4.4.8 ([DM91, Thm. 3.1])**.** *Let A/S be an abelian variety of dimension n, then within CHM_F/S there is a decomposition*

$$h(A) = \bigoplus_{i=0}^{2n} h^i(A),$$

such that, if $[n] \colon A \to A$ denotes multiplication by n, then

$$h([n]) = \bigoplus_{i=0}^{2n} n^i \cdot \mathrm{id}_{h^i(A)}.$$

The analogous statement for homological motives also holds but is automatic. The second condition ensures that, the decomposition is compatible with change of A and S, and any of the standard realisations. Another consequence is:

Theorem 4.4.9 (Künneth Formula)**.** *The decomposition $h(A) = \bigoplus_i h^i(A)$ respects* *the Künneth formula, i.e.*

$$h^k(A \times A') = \bigoplus_{i+j=k} h^i(A) \otimes h^j(A').$$

Theorem 4.4.10 ([Kin98, Prop. 2.2.1])**.** *Given an abelian variety A/S, the map*

$$\mathrm{End}(A)^{\mathrm{op}} \otimes F \to \mathrm{End}_{\mathrm{CHM}_{F/S}}(h^1(A))$$

is an isomorphism.

4.5 Realisations

We now outline how to construct realisations for relative motives (see also [CH00] where they define realisations for more general S).

Notation 4.5.1. Let $F/\mathbb{Q}$ is be a finite field extension and fix a smooth quasi-projective variety $S \xrightarrow{t} k$ over a number field as before. Let $\mathrm{Sh}_F/S(\mathbb{C})$ denote the category of sheaves of F-vector spaces on $S(\mathbb{C})$.

Definition 4.5.2. A *(pure) variation of Hodge structure of weight* n on $S(\mathbb{C})$ consists of a pair $(V, \mathcal{F}^\bullet)$ with V a finite rank local system of $\mathbb{Q}$-vector spaces on $S(\mathbb{C})$ and a descending filtration $\mathcal{F}^\bullet$ of the associated locally free sheaf $V \otimes_{\mathbb{Q}} \mathcal{O}_{S(\mathbb{C})}$ by holomorphic sub-bundles such that

i) for every $s \in S(\mathbb{C})$, the filtration on the fibre V_s given by $\mathcal{F}^\bullet$ defines a $\mathbb{Q}$-Hodge structure of weight n in the sense of Definition 4.1.4,

ii) (Griffiths transversality) if $\nabla\colon V \otimes \mathcal{O}_{S(\mathbb{C})} \to (V \otimes \mathcal{O}_{S(\mathbb{C})}) \otimes_{\mathcal{O}_{S(\mathbb{C})}} \Omega^1_S$ denotes the flat connection on $V \otimes \mathcal{O}_{S(\mathbb{C})}$ corresponding to the locally constant sheaf V, then
$$\nabla(\mathcal{F}^i(V \otimes \mathcal{O}_{S(\mathbb{C})})) \subseteq \mathcal{F}^{i-1}(V \otimes \mathcal{O}_{S(\mathbb{C})}) \otimes_{\mathcal{O}_{S(\mathbb{C})}} \Omega^1_{S(\mathbb{C})}.$$

Since the filtration is by sub-bundles, the Hodge structures on each of the fibres all have the same Hodge numbers. In this way it makes sense to speak of the Hodge type of a variation of Hodge structures.

A morphism of variations of Hodge structure is then a morphism of local systems for which the induced maps on the fibres are morphisms of Hodge structures. We denote the category of variations of Hodge structure on $S(\mathbb{C})$ by $\mathrm{VHS}/S(\mathbb{C})$ and define $\mathrm{VHS}_F/S(\mathbb{C})$ as in Section 4.3. We adopt the same notation for Tate twists as in Definition 4.1.4.

Notation 4.5.3. Given a variety X/k, let $F_{X(\mathbb{C})}$ denote the constant sheaf on $X(\mathbb{C})$ with coefficient group F.

Variations of Hodge structure are defined so that we have:

Theorem 4.5.4 (Griffiths, see [Voi07, Sec. 10.2])**.** *Let $X \xrightarrow{p} S$ be a smooth projective variety over S. Then $R^i p_* F_{X(\mathbb{C})}$ has a canonical pure variation of Hodge structure of weight i.*

Note that in the relative setting, singular cohomology generalises to a locally constant sheaf whilst the analogue of de Rham cohomology is a locally free sheaf.

We wish to define a corresponding realisation functor for relative motives. This *Hodge realisation* will be a functor $H_B^\bullet \mathrm{HomM}_F/S \to \mathrm{VHS}_F/S(\mathbb{C})$ such that the following diagram commutes:

$$\begin{tikzcd}
(\mathrm{SmProjVar}/S)^{\mathrm{op}} \arrow[r, "h"] \arrow[dr] & \mathrm{HomM}_F/S \arrow[d, "H_B^\bullet"] \\
& \mathrm{VHS}_F/S(\mathbb{C})
\end{tikzcd}$$

Suppose we have smooth projective maps $(X \xrightarrow{p} S), (Y \xrightarrow{q} S)$. For simplicity, assume that S, X, Y are all connected. Let d_S, d_X denote the dimensions of S, X respectively. Consider the diagram:

$$\begin{tikzcd}
& X \times_S Y \arrow[dl, "q'"'] \arrow[dr, "p'"] & \\
X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& S \arrow[d, "t"] & \\
& k &
\end{tikzcd}$$

If we stipulate that $H_B^\bullet(h(X)(i)) = \bigoplus_j R^j p_* \mathbb{Q}(i)$, then in view of the universal property of Construction 4.3.7, we need only define compatible maps

$$\mathrm{Corr}_S^i(X,Y) \to \bigoplus_j \mathrm{Hom}_{\mathrm{VHS}_F/S}(R^j p_* F_{X(\mathbb{C})}, R^{j+2i} q_* F_{Y(\mathbb{C})}(i)).$$

As a starting point, recall that for homological equivalence, we have an *absolute cycle class* map

$$\begin{aligned}\mathrm{Corr}^i_S(X,Y) = A^{d_X-d_S+i}_{\sim_{\mathrm{hom}}}(X\times_S Y)\\ \to H_B^{2d_X-2d_S+2i}((X\times_S Y)(\mathbb{C}),F)(d_X-d_S+i)\end{aligned}$$

taking values in singular cohomology.

Lemma 4.5.5. *In the above situation, there is a canonical map*

$$\begin{aligned}&H_B^{2d_X-2d_S+2i}((X\times_S Y)(\mathbb{C}),F)(d_X-d_S+i) \\ &\qquad\to \bigoplus_j \mathrm{Hom}_{\mathrm{Sh}_F/S(\mathbb{C})}(R^jp_*F_{X(\mathbb{C})},R^{j+2i}q_*F_{Y(\mathbb{C})}).\end{aligned} \tag{4.5}$$

Composing with the cycle class map, we obtain a map

$$\mathrm{Corr}^i_S(X,Y)\to \bigoplus_j \mathrm{Hom}_{\mathrm{VHS}_F/S(\mathbb{C})}(R^jp_*F_{X(\mathbb{C})},R^{j+2i}q_*F_{Y(\mathbb{C})}(2i)). \tag{4.6}$$

Proof. By the projection formula we have

$$Rp_*F_{X(\mathbb{C})}\otimes Rq_*F_{Y(\mathbb{C})}\cong Rq_*(q^*Rp_*F_{X(\mathbb{C})}\otimes F_{Y(\mathbb{C})})$$

which, by proper base change, is isomorphic to

$$\cong Rq_*Rp'_*F_{(X\times_S Y)(\mathbb{C})}.$$

Taking cohomology we find that

$$\begin{aligned}&R^{2d_X-2d_S+2i}(q\circ p')_*F_{(X\times_S Y)(\mathbb{C})}(d_X-d_S+i)\\ &\qquad\cong H^{2d_X-2d_S+2i}(Rp_*F_{X(\mathbb{C})}(d_X-d_S)\otimes Rq_*F_{Y(\mathbb{C})}(i))\end{aligned}$$

By the Künneth formula

$$\begin{aligned}&H^{2d_X-2d_S+2i}(Rp_*F_{X(\mathbb{C})}(d_X-d_S)\otimes Rq_*F_{Y(\mathbb{C})}(i))\\ &\qquad\cong \bigoplus_j R^{2d_X-2d_S-j}p_*F_{X(\mathbb{C})}(d_X-d_S)\otimes R^{j+2i}q_*F_{Y(\mathbb{C})}(i).\end{aligned}$$

Verdier duality for p states that

$$Rp_* R\underline{\mathrm{Hom}}_{D^b_c(\mathrm{Sh}_F/X(\mathbb{C}))}(F_{X(\mathbb{C})}, p^! F_{S(\mathbb{C})}) = R\underline{\mathrm{Hom}}_{D^b_c(\mathrm{Sh}_F/S(\mathbb{C}))}(Rp_* F_{X(\mathbb{C})}, F_{S(\mathbb{C})}), \tag{4.7}$$

where $D^b_c(\mathrm{Sh}_F/Z(\mathbb{C}))$ denotes the bounded derived category of constructible sheaves on $Z(\mathbb{C})$. We always have that $p^! D_S = D_X$ where D_Z denotes the dualising complex on $Z(\mathbb{C})$. Since X and S are smooth, $D_X = F_{X(\mathbb{C})}(d_S)[2d_X]$ and $D_S = F_{S(\mathbb{C})}(d_S)[2d_S]$. As a result, $p^! F_{S(\mathbb{C})}[2d_X - 2d_S]$. Since p is proper, $p_!$ coincides with p_*. Thus (4.7) becomes

$$Rp_* F_{X(\mathbb{C})}(d_X - d_S)[2d_X - 2d_S] = R\underline{\mathrm{Hom}}_{D^b_c(\mathrm{Sh}_F/S(\mathbb{C}))}(Rp_* F_{X(\mathbb{C})}, F_{S(\mathbb{C})}). \tag{4.8}$$

Since $Rp_*\mathbb{Q}$ is quasi-isomorphic to a complex of locally constant sheaves with zero transition maps (a consequence of the decomposition theorem of [Del68] together with Theorem 4.5.4), $R\underline{\mathrm{Hom}}_{D^b_c(\mathrm{Sh}_F/S(\mathbb{C}))}(Rp_* F_{X(\mathbb{C})}, F_{S(\mathbb{C})})$ is simply the complex whose $-j^{\text{th}}$ term is given by $\mathrm{Hom}_{\mathrm{Sh}_F/S(\mathbb{C})}((Rp_* F_{X(\mathbb{C})})_j, F_{S(\mathbb{C})})$. In particular, taking cohomology of (4.8) gives

$$R^j p_* F_{X(\mathbb{C})}(d_X - d_S) \cong (R^{2d_X - 2d_S - j} p_* F_{X(\mathbb{C})})^\vee.$$

As a result,

$$\bigoplus_j R^{2d_X - 2d_S - j} p_* F_{X(\mathbb{C})}(d_X - d_S) \otimes R^{j+2i} q_* F_{Y(\mathbb{C})}(i) \\ \cong \bigoplus_j \underline{\mathrm{Hom}}_{\mathrm{Sh}_F/S(\mathbb{C})}(R^j p_* F_{X(\mathbb{C})}, R^{j+2i} q_* F_{Y(\mathbb{C})}(i)).$$

Putting this together we have a canonical isomorphism

$$R^{2d_X - 2d_S + 2i}(q \circ p')_* F_{(X \times_S Y)(\mathbb{C})}(d_X - d_S + i) \\ \xrightarrow{\sim} \bigoplus_j \underline{\mathrm{Hom}}_{\mathrm{Sh}_F/S(\mathbb{C})}(R^j p_* F_{X(\mathbb{C})}, R^{j+2i} q_* F_{Y(\mathbb{C})}(i)).$$

We now obtain (4.5) by applying t_* and precomposing with the degeneracy map

$$H_B^{2d_X - 2d_S + 2i}((X \times_S Y)(\mathbb{C}), F)(d_X - d_S + i) \\ := R^{2d_X - 2d_S + 2i}(t \circ q \circ p')_* F_{(X \times_S Y)(\mathbb{C})}(d_X - d_S + i) \\ \to t_* R^{2d_X - 2d_S + 2i}(q \circ p')_* F_{(X \times_S Y)(\mathbb{C})}(d_X - d_S + i).$$

For (4.6), we must prove that correspondences yield morphisms of Hodge structures. This follows by considering $\bigoplus_j \mathrm{Hom}_{\mathrm{Sh}_F/S(\mathbb{C})}(R^j p_* F_{X(\mathbb{C})}, R^{j+2i} q_* F_{Y(\mathbb{C})})$ as a Hodge structure. The morphisms of sheaves which are also morphisms of Hodge structures are precisely those lying within the Hodge type $(0,0)$-subspace, but the image of the absolute cycle class map lies in the $(0,0)$-subspace and all operations (e.g. the Künneth formula) respect the Hodge structure fibrewise. $\square$

These maps are compatible with composition and send the class of Δ_X to the identity map.

Notation 4.5.6. Given $X \xrightarrow{p} S$, for simplicity, we write $H^i_B(X)$ for $R^i p_* F_{X(\mathbb{C})}$. Note that for an abelian scheme $H^i_B(X(\mathbb{C})) = H^\bullet_B(h^i(X))$, by Theorem 4.4.8.

Since $\mathrm{VHS}_F/S(\mathbb{C})$ is pseudo-abelian, applying the universal property of Construction 4.3.7 we obtain:

Corollary 4.5.7. *For any S/k a smooth quasi-projective variety. There are relative Hodge realisation functors*

$$H^\bullet_B \colon \mathrm{HomM}_F/S \to \mathrm{VHS}_F/S(\mathbb{C}),$$

which send $h(X \xrightarrow{p} S)(i)$ to $\bigoplus_j R^j p_* F_{X(\mathbb{C})}(i)$. These are natural in S.

Remark 4.5.8. The functor $H^\bullet_B$ is not faithful in general. To see this, let S be any quasi-projective variety with non-trivial cycles in some codimension $c \neq 0$ and let $X = Y = S$. Then $H^\bullet_B$ maps

$$A^i_{\sim_{\mathrm{hom}}}(S) \longrightarrow \bigoplus_j \underline{\mathrm{Hom}}_{\mathrm{Sh}_F/S(\mathbb{C})}(R^j p_* F_{X(\mathbb{C})}, R^{j+2i} q_* F_{Y(\mathbb{C})}(i)),$$

but the latter is zero whenever $i \neq 0$. In contrast, Corti–Hanamura have defined a faithful realisation functor, $\mathrm{HomM}_F/S \to D^b_c(\mathrm{Sh}_F/S(\mathbb{C}))$, taking values in the derived category of constructible sheaves, which maps $h((X \xrightarrow{p} S))(i)$ to $Rp_*\mathbb{Q}(i)[2i]$ see [CH00, Thm. 2.19].

On the other hand, if $(X \xrightarrow{p} S)$ and $(Y \xrightarrow{q} S)$ are abelian varieties then $H^\bullet_B$ is injective on $\mathrm{Hom}_{\mathrm{HomM}_F/S}(h^i(X), h^i(Y))$ for any i. To see this, we must use that the complex in $D^b_c(\mathrm{Sh}_F/S(\mathbb{C}))$ corresponding to $h^i(X)$ is $R^i p_* F_{X(\mathbb{C})}[-i]$ (this follows from Theorem 4.4.8) and so is concentrated in degree i and the same is true for Y. As a result, taking cohomology is injective on morphisms $R^i p_* F_{X(\mathbb{C})}[-i] \to R^i q_* F_{Y(\mathbb{C})}[-i]$ in $D^b_c(\mathrm{Sh}_F/S(\mathbb{C}))$. The desired injectivity now follows by observing that our construction is obtained from the (faithful) realisation functor of Corti–Hanamura by taking cohomology.

Remark 4.5.9. In Corollary 4.5.7, by naturality in S we mean that given $f\colon S' \to S$, there is a natural isomorphism $\xi\colon f^* \circ H_B^\bullet \implies H_B^\bullet \circ f^*$. For $(X \xrightarrow{p} S)$ this is given by the proper base change map $f^* R^i p_* F_{X(\mathbb{C})} \to R^i p_{S',*} f^* F_{X(\mathbb{C})}$.

For use in Section 4.11, we record that the above can be repeated in the étale case (where the results of [CH00] also hold).

Notation 4.5.10. Let ℓ be any prime and λ a prime of F dividing ℓ. Write F_λ for the completion of F at λ. Let λ be a uniformiser of F_λ. Given a scheme X, we write $\acute{\mathrm{E}}\mathrm{t}_\lambda/S$ for the category of lisse λ-adic sheaves on X and $F_{\lambda,X}$ for the constant λ-adic sheaf on a scheme X with coefficient group F_λ.

Lemma 4.5.11. *There are relative étale realisation functors*

$$H_\lambda^\bullet\colon \mathrm{HomM}_F/S \to \acute{\mathrm{E}}\mathrm{t}_\lambda/S,$$

which send $X \xrightarrow{p} S$ to $\bigoplus_i R^i p_* F_{\lambda,X}$. These are natural in S.

As before, these are faithful on maps $h^i(X) \to h^i(Y)$, when X, Y are abelian varieties. Though not necessary for our purposes, these constructions admit comparison isomorphisms between $H_\lambda^\bullet(-)$ and $H_B^\bullet(-)$.

4.6 The canonical construction

Notation 4.6.1. For an algebraic group $G/\mathbb{Q}$ and a field F of characteristic zero, as in Section 4.3, $\mathrm{Rep}_F(G)$ denotes the category of representations of G_F over F. Via Lemmas 4.3.12, 4.3.14, we shall usually consider an object $V \in \mathrm{Rep}_F(G)$ as a representation V of G over $\mathbb{Q}$ together with a map $F \hookrightarrow \mathrm{End}_G(V)$. We also set $\mathrm{Rep}(G) := \mathrm{Rep}_{\mathbb{Q}}(G)$.

Notation 4.6.2. Throughout $(G, \mathfrak{X})$ will denote a Shimura datum (which we sometimes equate with (G, h) for $h \in \mathfrak{X}$). We shall always assume that our Shimura data are such that the identity connected component of the centre of G is an almost–direct product of a $\mathbb{Q}$-split torus and an $\mathbb{R}$-anisotropic torus[2]. This en-

[2]Since any torus is an almost–direct product of its maximal split subtorus and maximal anisotropic subtorus, this is equivalent to stating that the maximal $\mathbb{Q}$-anisotropic subtorus of $Z(G)$ remains anisotropic over $\mathbb{R}$. As such, it is also equivalent to "SV5" as stated in [Mil17, Sec. 5] and implies "SV4".

sures that all real cocharacters of the centre are in fact defined over $\mathbb{Q}$. By definition, for any $h \in \mathfrak{X}$, representation $\mathrm{Ad}_G \circ h$ of $\mathbb{S}$ on $\mathrm{Lie}(G)$ has Hodge type contained in $\{(-1,1),(0,0),(1,-1)\}$. As a result, under the *weight homomorphism* $h \circ w \colon \mathbb{G}_{m,\mathbb{R}} \overset{(-)^{-1}}{\to} \mathbb{S}_{\mathbb{R}} \to G_{\mathbb{R}}$, $\mathbb{G}_{m,\mathbb{R}}$ acts trivially on $\mathrm{Lie}(G)$ and so $h \circ w$ has image in $Z(G)_{\mathbb{R}}$ and as a result is independent of the choice of h. Therefore, as a real cocharacter of the centre, $h \circ w$ must be rational.

Upon fixing a choice of neat open compact $K \leq G(\mathbb{A}_f)$, we denote by S the canonical model $\mathrm{Sh}_K(G,\mathfrak{X})$ over the reflex field of the corresponding Shimura variety. We follow a similar convention for $(G',\mathfrak{X}')$ with $K' \leq G'(\mathbb{A}_f)$ etc. If $f\colon (G',\mathfrak{X}') \to (G,\mathfrak{X})$ is a morphism of Shimura data for which $f(K') \leq K$, then we also denote by f the induced map $f\colon S' \to S$.

Construction 4.6.3 (cf. [Mil17, Prop. 5.9])**.** Given an element $(\rho\colon G_F \to \mathrm{GL}(V)) \in \mathrm{Rep}_F(G)$, we may define a variation of Hodge structure on $S(\mathbb{C})$ as follows: consider V as $\mathbb{Q}$-representation of G together with an action of F. Then the underlying local system corresponds to the cover

$$G(\mathbb{Q})\backslash(\mathfrak{X} \times (G(\mathbb{A}_f)/K) \times V) \to G(\mathbb{Q})\backslash(\mathfrak{X} \times (G(\mathbb{A}_f)/K)),$$

where $g \in G(\mathbb{Q})$ acts by $(h_x,t,v) \mapsto (gh_xg^{-1}, gt, \rho(g)v)$. This indeed forms a local system as our conditions on the centre Z_G ensure that $Z_G(\mathbb{Q})$ is discrete in $Z_G(\mathbb{A}_f)$. The stalk at a point (h_x,t) is identified with corresponding fibre $\{(h_x,t,v) \mid v \in V\} \cong V$ and as such may be given the $\mathbb{Q}$-Hodge structure defined by the map $\rho \circ h_x\colon \mathbb{S} \to G_{\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$. This is independent of the choice of representative (h_x,t), inherits an F-structure and, as the weight homomorphism is rational, the Hodge structures at all points have the same weight. As a result, we have canonically defined a Hodge structure on each stalk.

It remains to show this "family" of $\mathbb{Q}$-Hodge structures defines a variation of Hodge structure. This requires checking that the Hodge structures vary analytically [Pin90, Prop. 1.7] and Griffiths transversality [Pin90, Prop. 1.10]. For example, in the case of $V = \mathrm{Ad}_G$, the adjoint representation, transversality is automatic as the definition of Shimura data ensures that the Hodge filtration on $V = (\mathrm{Ad}_G)_{S(\mathbb{C})} \otimes \mathcal{O}_{S(\mathbb{C})}$ is given by

$$0 = \mathcal{F}^1 V \subseteq \mathcal{F}^0 V \subseteq \mathcal{F}^{-1} V = V,$$

which only has a single step which is not trivial or the whole space. Transversality for arbitrary representations is then shown by reduction to this case.

This extends to a functor $\mu_G^{\mathcal{H}} \colon \mathrm{Rep}_F(G) \to \mathrm{VHS}_F/S(\mathbb{C})$ referred to as the *canonical construction* (where $\mathcal{H}$ stands for Hodge).

Example 4.6.4. If we did not assume the condition on the centre of G, then

$$G(\mathbb{Q})\backslash(\mathfrak{X} \times (G(\mathbb{A}_f)/K) \times V) \to G(\mathbb{Q})\backslash(\mathfrak{X} \times (G(\mathbb{A}_f)/K))$$

need not form a local system. To see this, let $L/\mathbb{Q}$ be a real quadratic extension and $G = \mathrm{Res}_{L/\mathbb{Q}} \mathrm{GL}_2$. There is a Shimura datum $(G, \mathfrak{X})$ for G constructed analogously to Example 4.2.2. The centre Z_G of G is $\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_{m,L}$, which is splits over $\mathbb{R}$ but contains a non-zero anisotropic subtorus over $\mathbb{Q}$ (namely the kernel of the norm map). As a result, $Z_G(\mathbb{Q})$ is not discrete in $Z_G(\mathbb{A}_f)$. To see this note that $\mathcal{O}_L^\times \subseteq \prod_p G(\mathbb{Z}_p)$ is of rank one as an abelian group and so, as any open subset of $W \subseteq \prod_p G(\mathbb{Z}_p)$ has finite index, $W \cap Z_G(\mathbb{Q})$ always contains infinitely many points.

Now let $K \leq G(\mathbb{A}_f)$ be a neat open compact subgroup and V be the standard representation of G, which is a two dimensional L-vector space. Consider the action of $G(\mathbb{Q})$ on

$$\mathfrak{X} \times (G(\mathbb{A}_f)/K) \times V \to \mathfrak{X} \times (G(\mathbb{A}_f)/K). \tag{4.9}$$

The stabiliser of an element $(h_x, t) \in \mathfrak{X} \times G(\mathbb{A}_f)/K$ is given by

$$\mathrm{Stab}_{G(\mathbb{Q})}(h_x) \cap \mathrm{Stab}_{G(\mathbb{Q})}(t) \supseteq Z_G(\mathbb{Q}) \cap (G(\mathbb{Q}) \cap K) := U.$$

By the above discussion, U must be of rank one as an abelian group. So, as U acts faithfully on V, after quotienting (4.9) by the action of $G(\mathbb{Q})$, the fibres will not be vector spaces and we certainly do not obtain a local system.

Construction 4.6.5. Let $V \in \mathrm{Rep}_F(G)$ and f be as above. There is a canonical isomorphism of local systems $\kappa_V \colon f^*\mu_G^{\mathcal{H}}(V) \to \mu_{G'}^{\mathcal{H}}(f^*V)$ and this is also a morphism of variations of Hodge structure as it respects the Hodge structure on each fibre. The collection $\kappa := (\kappa_V)_V$ then defines a natural isomorphism:

$$\begin{tikzcd}
\mathrm{Rep}_F(G) \arrow[r, "\mu_G^{\mathcal{H}}"] \arrow[d, "f^*"'] & \mathrm{VHS}_F/S(\mathbb{C}) \arrow[d, "f^*"] \arrow[dl, phantom, "\swarrow\!\!\!\!\swarrow"] \\
\mathrm{Rep}_F(G') \arrow[r, "\mu_{G'}^{\mathcal{H}}"'] & \mathrm{VHS}_F/S'(\mathbb{C})
\end{tikzcd}$$

It is natural to ask if there is a lift of the canonical construction to the category of Chow motives with coefficients in F, i.e. a functor $\mathrm{Rep}_F(G) \to \mathrm{CHM}_F/S$

that together with $H_B^\bullet$ factorises $\mu_G^{\mathcal{H}}$. By considering mixed Shimura varieties, we are able to define such a lift on a certain subcategory.

4.7 Mixed Shimura varieties

Mixed Shimura data, as defined by Pink [Pin90], are an extension of the traditional definition of Shimura data. A *mixed Shimura datum* consists of a pair $(P, \tilde{\mathfrak{X}})$ with $P/\mathbb{Q}$ an algebraic group and a subspace $\tilde{\mathfrak{X}} \subseteq \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ satisfying various requirements (see [Pin90, Sec. 2.1] for the precise conditions[3]). In the case that P is reductive, i.e. that P has trivial unipotent radical, we recover the classical definition of Shimura data, which we shall refer to as the *pure* case.

For any neat open compact $K \leq P(\mathbb{A}_f)$ there is an associated mixed Shimura variety $\mathrm{Sh}_K(P, \tilde{\mathfrak{X}})$, which is algebraic over its reflex field. A morphism of mixed Shimura data $f\colon (P', \tilde{\mathfrak{X}}') \to (P, \tilde{\mathfrak{X}})$ is a map $P' \to P$ for which $f(\tilde{\mathfrak{X}}') \subseteq \tilde{\mathfrak{X}}$. Pairs of neat open compact subgroups $K \leq P(\mathbb{A}_f)$ and $K' \leq P'(\mathbb{A}_f)$ with $f(K') \leq K$ give rise to algebraic maps $\mathrm{Sh}_{K'}(P', \tilde{\mathfrak{X}}') \to \mathrm{Sh}_K(P, \tilde{\mathfrak{X}})$.

Any mixed Shimura datum $(P, \tilde{\mathfrak{X}})$ admits a map to the pure Shimura datum $(G, \mathfrak{X})$ where G is the quotient of P by its unipotent radical $R_u(P)$ and $\mathfrak{X}$ is given by postcomposing elements of $\tilde{\mathfrak{X}}$ with $\pi\colon P \to G$.

We shall always assume that our mixed Shimura varieties satisfy the stronger conditions that: the centre of $G = P/R_u(P)$ is an almost–direct product of a $\mathbb{Q}$-split torus and a torus which is $\mathbb{R}$-anisotropic, and so the weight cocharacter $\pi \circ h \circ w\colon \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is rational for $h \in \tilde{\mathfrak{X}}$. These ensure that there is a canonical construction for mixed Shimura varieties associating variations of mixed Hodge structure on $\mathrm{Sh}_K(P, \tilde{\mathfrak{X}})$ to representations of $\mathrm{Rep}(P)$ (see [Pin90, Sec. 1.18]).

Universal abelian varieties can be seen as instances of mixed Shimura varieties (see Example 4.7.6). In this section, we shall observe that the theory of mixed Shimura varieties automates the creation of certain abelian varieties over pure Shimura varieties in a functorial way.

Definition 4.7.1. Let $(G, \mathfrak{X})$ be a (pure) Shimura datum and $V \in \mathrm{Rep}_F(G)$. We consider V as a $\mathbb{Q}$-representation together with an F-structure $F \hookrightarrow \mathrm{End}_G(V)$. For any choice of $h_x \in \mathfrak{X}$, $V \otimes_{\mathbb{Q}} \mathbb{C}$ decomposes as a direct sum of one dimen-

[3]We do not require the additional generality of allowing $\mathfrak{X}$ to be a finite cover of a subspace of $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$

sional $\mathbb{C}$-subrepesentations upon each of which $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts as multiplication by $z^{p_i}\bar{z}^{q_i}$ for some p_i, q_i. We say that V has *Hodge type* given by set $\{(p_1, q_1), (p_2, q_2), ..., (p_n, q_n)\}$ of (p_i, q_i) occurring in the above decomposition. Since different choices of h_x define isomorphic $\mathbb{R}$-Hodge structures, this is independent of the choice of h_x.

Recall, it is a condition of a pure Shimura datum that the adjoint representation Ad_G has Hodge type contained in $\{(-1,1),(0,0),(1,-1)\}$. Similarly, it is a condition of a mixed Shimura datum that the adjoint representation is admits a filtration whose graded pieces carry pure Hodge structures of Hodge type contained in either $\{\{(-1,1),(0,0),(1,-1)\}\}$, $\{(-1,0),(0,-1)\}$ or $\{(-1,-1)\}$.

The Hodge type of a representation $V \in \mathrm{Rep}(G)$ coincides with the Hodge type of $\mu_G^{\mathcal{H}}(V)$ as a variation of Hodge structure on $S(\mathbb{C})$.

Notation 4.7.2. Let $\mathrm{Rep}_F(G)^{\mathrm{AV}}$ denote the full subcategory of $\mathrm{Rep}_F(G)$ whose objects have Hodge type contained in $\{(-1,0),(0,-1)\}$.

Given $V \in \mathrm{Rep}_F(G)^{\mathrm{AV}}$, considering V as a representation over $\mathbb{Q}$, we may form the semi-direct product $V \rtimes G$ as an algebraic group over $\mathbb{Q}$. Let $p\colon V \rtimes G \to G$ denote the projection map and $\tilde{\mathfrak{X}}$ consist of the elements $t \in \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, (V \rtimes G)_{\mathbb{C}})$ for which $p \circ t \in \mathfrak{X}_{\mathbb{C}}$.

Lemma 4.7.3. *Let $(G, \mathfrak{X})$ be a (pure) Shimura datum $V \in \mathrm{Rep}_F(G)^{\mathrm{AV}}$, then $(V \rtimes G, \tilde{\mathfrak{X}})$ is a mixed Shimura datum.*

Proof. The unipotent radical of $V \rtimes G$ is V. If, in the notation of [Pin90, Sec. 2.1], we set $U = V$, then it is easy to check the conditions directly. Alternatively, use that $(V \rtimes G, \tilde{\mathfrak{X}})$ is an instance of a unipotent extension in the sense of [Pin90, Prop. 2.17]. Note that we are assuming $(G, \mathfrak{X})$ has rational weight and the centre is an almost–direct product of a $\mathbb{Q}$-split and $\mathbb{R}$-anisotropic torus. The datum $(V \rtimes G, \tilde{\mathfrak{X}})$ then also satisfies the corresponding strengthened condition of a mixed Shimura variety. $\square$

Mixed Shimura data of the form $(V \rtimes G, \tilde{\mathfrak{X}})$ are the only non-pure data we shall need to consider.

Lemma 4.7.4. *Let $(G, \mathfrak{X})$ be a Shimura datum, $K \leq G(\mathbb{A}_f)$ a neat open compact subgroup and $V \in \mathrm{Rep}(G)$. Then for any choice of K-stable full rank $\hat{\mathbb{Z}}$-lattice $L \leq V(\mathbb{A}_f)$, $L \rtimes K$ is a neat open compact subgroup of $V \rtimes G$.*

Proof. Only neatness is not completely immediate. To see this, consider the representation U of $V \rtimes G$ which has underlying vector space $V \times \mathbb{A}^1_{\mathbb{Q}}$ and on which (v, g) acts as $(w, \lambda) \mapsto (g \cdot w + \lambda v, \lambda)$. If W is any faithful representation of G, then $U \oplus W$ is a faithful representation of $V \rtimes G$. Moreover, the eigenvalues of $(v, g) \in (V \rtimes G)(\mathbb{A}_f)$ acting on U coincide with those of $(0, g)$ acting on U and so the eigenvalues of (v, g) acting on $U \oplus W$ coincide with those of $(0, g)$. As a result, (v, g) is neat if and only if g is and $L \rtimes K$ is neat if and only if K is. $\square$

Lemma 4.7.5. *For any Shimura datum $(G, \mathfrak{X})$ and V, K, L as above, the map $\mathrm{Sh}_{L\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}) \to \mathrm{Sh}_K(G, \mathfrak{X})$ has the structure of an abelian variety.*

Proof. This is [Pin90, 3.22 a)] (the zero section is given by the Levi section $\iota \colon G \to V \rtimes G$). $\square$

Moreover, this is functorial in the sense that given a homomorphism of representations $f \colon V \to V'$ with $L \leq V(\mathbb{A}_f)$ and $L' \leq V'(\mathbb{A}_f)$ and $f(L) \leq L'$, then the induced map of mixed Shimura varieties $\mathrm{Sh}_{L\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}) \to \mathrm{Sh}_{L\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}')$ respects the group structure.

Example 4.7.6. If a Shimura datum $(G, \mathfrak{X})$ has a PEL-datum (see Definition 4.9.2) with standard representation V, then for any neat open compact K and K-stable $\hat{\mathbb{Z}}$-lattice L of $V(\mathbb{A}_f)$ (we shall always take our lattices to be of full rank), $\mathrm{Sh}_{L\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}) \to \mathrm{Sh}_K(G, \mathfrak{X})$ is isogeneous to the universal abelian variety defined by the PEL-datum.

Lemma 4.7.7. *i) Let (G, h) be a Shimura datum and K a neat open compact subgroup. Given $V, W \in \mathrm{Rep}_F(G)^{\mathrm{AV}}$ and K-stable $\hat{\mathbb{Z}}$-lattices L_V, L_W of $V(\mathbb{A}_f)$, $W(\mathbb{A}_f)$, then as abelian varieties over S, there is a canonical isomorphism*

$$\begin{aligned}\mathrm{Sh}_{(L_V\oplus L_W)\rtimes K}((V \oplus W) \rtimes G, &\tilde{\mathfrak{X}}_{V\oplus W}) \\ &\cong \mathrm{Sh}_{L_V\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}_V) \times_S \mathrm{Sh}_{L_W\rtimes K}(W \rtimes G, \tilde{\mathfrak{X}}_W),\end{aligned}$$

where $\tilde{\mathfrak{X}}_V, \tilde{\mathfrak{X}}_V, \tilde{\mathfrak{X}}_{V\oplus W}$ are as in Notation 4.7.2.

ii) Given a morphism of pure Shimura data $f \colon (G', h') \to (G, h)$, neat open compact subgroups $K' \leq G'(\mathbb{A}_f), K \leq G(\mathbb{A}_f)$ with $f(K') \leq K$, and $V \in \mathrm{Rep}_F(G)^{\mathrm{AV}}$ together with a K-stable $\hat{\mathbb{Z}}$-lattice L, then there is a canonical isomorphism of abelian S'-schemes

$$\mathrm{Sh}_{L\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}})_{\mathrm{Sh}_{K'}(G',\mathfrak{X}')} \cong \mathrm{Sh}_{f^*L\rtimes K}(f^*V \rtimes G, \tilde{\mathfrak{X}}'),$$

where f^*L is the lattice L considered as a K'-stable $\hat{\mathbb{Z}}$-lattice.

Proof. Both statements follow immediately from the characterisation of fibre products for mixed Shimura data given in [Pin90, Sec. 3.10]. $\square$

Construction 4.7.8. We now define a functor $\mu_G^{\mathrm{mot}}\colon \mathrm{Rep}_F(G)^{\mathrm{AV}} \to \mathrm{CHM}_F/S$ as follows: given $V \in \mathrm{Rep}_F(G)^{\mathrm{AV}}$, let L be a full rank $\hat{\mathbb{Z}}$-lattice of $V(\mathbb{A}_f)$ which is stable under K. We then set $\mu_G^{\mathrm{mot}}(V) = h^1(\mathrm{Sh}_{L\rtimes K}(V \rtimes K, \tilde{X}))^\vee$ as a motive with rational coefficients which equip this with an F-structure $F \hookrightarrow \mathrm{End}_{\mathrm{CHM}/S}(\mu_G^{\mathrm{mot}}(V))$ in the following way. Let

$$T := \{\alpha \in \mathrm{End}_G(V) \mid \alpha(L) \subseteq L\}.$$

For any $\alpha \in \mathrm{End}_G(V)$, $\alpha(L)$ is a $\hat{\mathbb{Z}}$-lattice and so there exists an $n \in \mathbb{N}$ such that $n \cdot \alpha(L) \leq L$. In other words, $T \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{End}_G(V)$. Thus, we may act on $h^1(\mathrm{Sh}_{L\rtimes K}(V \rtimes K, \tilde{\mathfrak{X}}))^\vee$ by $F = (T \cap F) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the first factor acting via functoriality of mixed Shimura varieties and the second by $\mathbb{Q}$-linearity of CHM/S. This uses that the actions of $\mathbb{Z}$ as subring of $T \cap F$ (i.e. addition via the group law as an abelian variety) and as a subring of $\mathbb{Q}$ coincide, which follows from Theorem 4.4.8. In contrast, this would not be true of $h(S_{K,V})^\vee$ and this does not define an element of CHM_F/S.

Given a morphism $f\colon V \to V'$, let $n \in \mathbb{N}$ be large enough to ensure that $f(nL) \leq L'$. We obtain maps

$$(\pi_n^\vee)_*\colon h^1(\mathrm{Sh}_{L\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}))^\vee \to h^1(\mathrm{Sh}_{nL\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}))^\vee,$$

$$f_*\colon h^1(\mathrm{Sh}_{nL\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}))^\vee \to h^1(\mathrm{Sh}_{L'\rtimes K}(V' \rtimes G, \tilde{\mathfrak{X}}'))^\vee,$$

where the first map is obtained by applying $h^1(-)^\vee$ to the dual of the map of abelian varieties $\pi_n\colon\ \mathrm{Sh}_{nL\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}) \to \mathrm{Sh}_{L\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}})$ which is given by functoriality of mixed Shimura varieties, whilst the second is $h^1(-)^\vee$ of the map of mixed Shimura varieties induced by f. We then set $\mu_G^{\mathrm{mot}}(f)$ to be $1/n$ times the composite $f_* \circ (\pi_m^\vee)_*$. By construction the morphisms $\mu_G^{\mathrm{mot}}(f)$ will respect the F-action.

Proposition 4.7.9. *Given a choice of $\hat{\mathbb{Z}}$-lattice for each $V \in \mathrm{Rep}_F(G)^{\mathrm{AV}}$ as above, then the corresponding μ_G^{mot} is a well-defined functor $\mathrm{Rep}_F(G)^{\mathrm{AV}} \to \mathrm{CHM}_F/S$. The functor μ_G^{mot} is independent of the choice of lattice for each V up to canonical natural isomorphism.*

Proof. We first remark that $\mu_G^{\mathrm{mot}}(f)$ is independent of the choice of n. This follows as the constructions for n and for nm differ by $1/m \cdot (\pi_m^\vee)_* \circ \pi_{m,*} = 1/m \cdot [m]_*$, but, for an abelian variety A/S, $[m]$ acts on $h^1(A)^\vee$ by multiplication by m (Theorem 4.4.8).

That μ_G^{mot} respects composition follows from the commutativity of the following diagram, for any $f\colon V \to V'$, $m \in \mathbb{N}$ and n s.t. $f(nL) \leq L'$

$$\begin{tikzcd}
\mathrm{Sh}_{nL \rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}) \arrow[r, "\pi_m^\vee"] \arrow[d, "f"'] & \mathrm{Sh}_{mnL \rtimes K}(V \rtimes K, \tilde{\mathfrak{X}}) \arrow[d, "f"] \\
\mathrm{Sh}_{L' \rtimes K}(V' \rtimes G, \tilde{\mathfrak{X}}') \arrow[r, "\pi_m^\vee"'] & \mathrm{Sh}_{mL' \rtimes K}(V' \rtimes G, \tilde{\mathfrak{X}}')
\end{tikzcd}$$

and thus it is clear that μ_G^{mot} defines a functor.

Given choices L_1, L_2 for each V and corresponding functors $\mu_{G,1}^{\mathrm{mot}}, \mu_{G,2}^{\mathrm{mot}}$, define a natural transformation $\psi\colon \mu_{G,1}^{\mathrm{mot}} \to \mu_{G,2}^{\mathrm{mot}}$ by defining ψ_V to be $1/n$ times the map

$$h^1(\mathrm{Sh}_{L_1 \rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}))^\vee \overset{(\pi_n^\vee)_*}{\to} h^1(\mathrm{Sh}_{nL_1 \rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}))^\vee \overset{\mathrm{id}_*}{\to} h^1(\mathrm{Sh}_{L_2 \rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}))^\vee \tag{4.10}$$

for any n such that $nL_1 \leq L_2$. That this defines a natural transformation again follows from the commutativity of the above square. Moreover, for every V, as an isogeny (4.10) is invertible after applying $h^1(-)^\vee$, we find that ψ defines a natural isomorphism. □

Remark 4.7.10. If $f\colon V \to W$ is a non-zero homomorphism of representations of G over $\mathbb{Q}$ and we fix a neat open compact subgroup K of G and K-stable $\hat{\mathbb{Z}}$-lattices $L_V \leq V$, $L_W \leq W$ such that $f(L_V) \leq L_W$, then $\mathrm{Sh}_{L_V \rtimes K}(V \rtimes G, \tilde{\mathfrak{X}}_V) \to \mathrm{Sh}_{L_W \rtimes K}(W \rtimes G, \tilde{\mathfrak{X}}_W)$ is non-zero as a morphism of abelian varieties (for example, using the explicit description of the points over $\mathbb{C}$). Together with Theorem 4.4.10 this demonstrates that μ_G^{mot} is faithful.

Notation 4.7.11. Given $V \in \mathrm{Rep}_F(G)^{\mathrm{AV}}$, we shall denote the mixed Shimura variety $\mathrm{Sh}_{L \rtimes K}(V \rtimes G, \tilde{\mathfrak{X}})$ simply by $S_{K,V}$. We use $p\colon S_{K,V} \to S$ and $\iota\colon S \to S_{K,V}$ to denote the maps induced by the projection and Levi section as well as the induced maps on their analytifications. We continue accordingly for (G', h').

Lemma 4.7.12. *Given a morphism of Shimura data $f\colon (G', \mathfrak{X}') \to (G, \mathfrak{X})$, a neat open compact $K \leq G(\mathbb{A}_f)$, $K' \leq G'(\mathbb{A}_f)$ with $f(K') \leq K$, and a choice of stable $\hat{\mathbb{Z}}$-lattices for all elements of $\mathrm{Rep}_F(G), \mathrm{Rep}_F(G')$, then the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Rep}_F(G)^{\mathrm{AV}} \arrow[r, "\mu_G^{\mathrm{mot}}"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\swarrow"] & \mathrm{CHM}_F/S \arrow[d, "f^*"] \\
\mathrm{Rep}_F(G')^{\mathrm{AV}} \arrow[r, "\mu_{G'}^{\mathrm{mot}}"'] & \mathrm{CHM}_F/S'
\end{tikzcd}$$

up to a natural isomorphism $\psi\colon f^* \circ \mu_G^{\mathrm{mot}} \implies \mu_{G'}^{\mathrm{mot}} \circ f^*$.

Proof. From Lemma 4.7.7 *ii)* and that the canonical projectors defining h^i commute with pullback, we obtain isomorphisms

$$(h^1(S_{K,V})^\vee)_{S'} \cong h^1(S'_{K',f^*V})^\vee.$$

The natural isomorphism is then given by taking these maps and possibly composing the maps defined in the proof of Proposition 4.7.9 if the lattice chosen for f^*V is not f^*L. $\square$

4.8 Direct images for mixed Shimura varieties

In this section we check that μ_G^{mot} lifts the canonical construction and is compatible with base change.

Lemma 4.8.1. *Given a Shimura datum* $(G, \mathfrak{X})$ *and* $V \in \mathrm{Rep}_F(G)^{\mathrm{AV}}$*, then there is a canonical identification of* $\mu_G^{\mathcal{H}}(V)$ *and the dual of* $H_B^1((S_{K,V})(\mathbb{C})) = R^1p_*F_{(S_{K,V})(\mathbb{C})}$*, where* $p\colon (S_{K,V})(\mathbb{C}) \to S(\mathbb{C})$ *denotes the usual projection.*

Proof. The canonical construction can be extended to mixed Shimura varieties as we now recall. Let $(P, \tilde{\mathfrak{X}})$ be a mixed Shimura datum and $Q \leq P(\mathbb{A}_f)$ a neat open compact subgroup. A representation $W \in \mathrm{Rep}_F(P)$, which we consider as a $\mathbb{Q}$-representation $\rho\colon P \to \mathrm{GL}(W)$ together with an F-structure, defines a local system

$$\mu_P^{\mathcal{H}}(W) := P(\mathbb{Q})\backslash(\tilde{\mathfrak{X}} \times (P(\mathbb{A}_f)/Q) \times W)$$

on $\mathrm{Sh}_Q(P, \tilde{\mathfrak{X}})(\mathbb{C}) = P(\mathbb{Q})\backslash(\tilde{\mathfrak{X}} \times (P(\mathbb{A}_f)/Q))$. Similarly to Construction 4.6.3, each fibre $\{(x, k, v) \mid v \in W\} \cong W$ has a well defined mixed Hodge structure given by $\rho \circ h_x$ and $\mu_P^{\mathcal{H}}(W)$ has the structure of a graded-polarisable variation of Hodge structure (i.e. the graded pieces are polarised in the sense of [PS08, Def. 2.9]). This extends to a $\otimes$-functor

$$\mu_P^{\mathcal{H}}\colon \operatorname{Rep}_F(P) \to \mathrm{VHS}_F/\operatorname{Sh}_Q(P, \tilde{\mathfrak{X}})(\mathbb{C}).$$

This is functorial in the sense that, given $f\colon (P', \mathfrak{X}') \to (P, \mathfrak{X})$ and $Q \leq P(\mathbb{A}_f), Q' \leq P'(\mathbb{A}_f)$ with $f(Q') \leq Q$, then there is a canonical isomorphism

$$f^*\mu_P^{\mathcal{H}}(W) = \mu_{P'}^{\mathcal{H}}(f^*W).$$

For the purposes of the lemma, the key fact is that pushforwards of sheaves arising via the canonical construction correspond to group cohomology. More specifically, in the notation of the lemma, the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Rep}(V \rtimes G) \arrow[r, "\mu_{V\rtimes G}^{\mathcal{H}}"] \arrow[d, "H^i(V,-)"'] & \mathrm{VHS}/(S_{K,V})(\mathbb{C}) \arrow[d, "R^i p_*"] \\
\operatorname{Rep}(G) \arrow[r, "\mu_G^{\mathcal{H}}"'] & \mathrm{VHS}/S(\mathbb{C})
\end{tikzcd}$$

where the left vertical map is group cohomology (see [Wil97, Thm. II.2.3, Prop. I.1.6c)]). In the case of the one dimensional trivial representation, this yields identifications

$$\mu_G^{\mathcal{H}}(H^1(V, F)) = R^1p_*F_{(S_{K,V})(\mathbb{C})}.$$

But, $H^1(V, F) = V^\vee$ as desired. $\square$

Notation 4.8.2. Write φ_V for the isomorphism $\varphi_V\colon H_B^1((S_{K,V})(\mathbb{C}))^\vee \to \mu_G^{\mathcal{H}}(V)$ and $\varphi = (\varphi_V)_V$ for the collection as V ranges over $V \in \operatorname{Rep}_F(G)^{\mathrm{AV}}$.

Lemma 4.8.3. *i) Let (G, h) be a Shimura datum and $\alpha\colon V_1 \to V_2$ a morphism in $\operatorname{Rep}_F(G)^{\mathrm{AV}}$. Fix a neat open compact subgroup $K \leq G(\mathbb{A}_f)$ and let α also denote the map $(S_{K,V_1})(\mathbb{C}) \to (S_{K,V_2})(\mathbb{C})$. Then the following diagram commutes:*

$$\begin{tikzcd}
H_B^1((S_{K,V_1})(\mathbb{C}))^\vee \arrow[r, "\varphi_{V_1}"] \arrow[d, "(\alpha^*)^\vee"'] & \mu_G^{\mathcal{H}}(V_1) \arrow[d, "\mu_G^{\mathcal{H}}(\alpha)"] \\
H_B^1((S_{K,V_2})(\mathbb{C}))^\vee \arrow[r, "\varphi_{V_2}"'] & \mu_G^{\mathcal{H}}(V_2)
\end{tikzcd}$$

ii) Let $f\colon (G', h') \to (G, h)$ be a morphism of Shimura data and $K \leq G(\mathbb{A}_f), K' \leq G'(\mathbb{A}_f)$ neat open compact subgroups with $f(K') \leq K$. For any $V \in \operatorname{Rep}_F(G)^{\mathrm{AV}}$, the following diagram commutes:

$$\begin{tikzcd}
f^*H^1_B((S_{K,V})(\mathbb{C}))^\vee \arrow[r, "f^*(\varphi_V)"] \arrow[d, "H^1_B(\psi_V)"'] & f^*\mu^{\mathcal{H}}_G(V) \arrow[d, "\kappa_V"] \\
H^1_B((S'_{K',f^*V})(\mathbb{C}))^\vee \arrow[r, "\varphi_{f^*V}"'] & \mu^{\mathcal{H}}_{G'}(f^*V)
\end{tikzcd}$$

Proof. We prove the first case, the other is similar. The strategy is to reduce to a group theoretic context via a Tannakian argument. Fix a connected component S^0 of $S(\mathbb{C})$ and let S^0_{K,V_i} denote the connected component $p_i^{-1}(S^0)$. In [Wil97, Thm. II.2.2] it is checked that the canonical construction produces variations of Hodge structure which are admissible in the sense of [Kas86]. Since the V_i are unipotent, objects in the image of $\mu^{\mathcal{H}}_{V_i \rtimes G}$ (in the notation used in the proof of Lemma 4.8.1) admit a filtration by objects pulled back from S^0. Let VHS'/S^0 denote the category of admissible variations of Hodge structure on S^0 and p_i-$\mathrm{UVar}/S^0_{K,V_i}$ denote the full subcategory of $\mathrm{VHS}'/S^0_{K,V_i}$ whose objects admit a filtration whose graded objects are pulled back from elements of VHS'/S^0. The functors $\mu^{\mathcal{H}}_G, \mu^{\mathcal{H}}_{V_i \rtimes G}$ take values in these categories.

Fix $y \in S^0$ and for $i = 1, 2$ set $x_i = \iota_i(y)$, where ι_i denotes the canonical Levi section. For $i = 1, 2$, let P_{i,x_i} denote the Tannaka dual of p_i-$\mathrm{UVar}/S^0_{K,V_i}$ and G_y the Tannaka dual of VHS'/S^0 all with the obvious fibre functors. The map $P_{i,x_i} \to G_y$ induced by p_i^* is surjective (e.g. [DM82, Prop. 2.21a)]). Lastly, set $V_{i,x_i} = \ker(P_{i,x_i} \to G_y)$.

Consider the diagram:

$$\begin{tikzcd}
p_1\text{-}\mathrm{UVar}/S^0_{K,V_1} \arrow[dr, "R^j p_{1,*}"'] & & p_2\text{-}\mathrm{UVar}/S^0_{K,V_2} \arrow[ll, "\alpha^*"'] \arrow[dl, "R^j p_{2,*}"] \\
& \mathrm{VHS}'/S^0 &
\end{tikzcd}$$

This does not commute, but there is an obvious natural transformation $R^j p_{2,*} \implies R^j p_{1,*}\alpha^*$. The calculation of higher direct images in p_i-$\mathrm{UVar}/S_{K,V_i}$ coincides with the usual higher direct image as elements of $\mathrm{VHS}_F/S^0_{K,V_i}$ (cf. [Wil97, Sec. I.4]). The maps $R^j p_{i,*}$ are not $\otimes$-functors, but we claim that when viewed in the Tannakian setting, the above triangle becomes:

$$\begin{tikzcd}
\mathrm{Rep}(P_{1,x_1}) \arrow[dr, "H^j(V_{1,x_1},-)"'] & & \mathrm{Rep}(P_{2,x_2}) \arrow[ll, "\alpha^*"'] \arrow[dl, "H^j(V_{2,x_2},-)"] \\
& \mathrm{Rep}(G_y) &
\end{tikzcd}$$

and the natural transformation becomes the usual map

$$H^j(V_{2,x_2},-) \implies H^j(V_{1,x_1},\alpha^*(-)).$$

To see this, note that p_i^* corresponds to inflation from G_y and has right adjoint $p_{i,*}$, whilst $(-)^{V_{i,x_i}}$ is right adjoint to inflation.

Since the canonical construction is a $\otimes$-functor, after taking duals we obtain a diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & V_{i,x_i} \arrow[r] \arrow[d, dashed] & P_{i,x_i} \arrow[r] \arrow[d, "t_i"] & G_y \arrow[r] \arrow[d, "r"] & 1 \\
0 \arrow[r] & V_i \arrow[r] & P_i \arrow[r] & G \arrow[r] & 1
\end{tikzcd} \tag{4.11}$$

where t_i is the dual of $\mu^{\mathcal{H}}_{V_i \rtimes G}$ and r the dual of $\mu^{\mathcal{H}}_G$. Moreover, the left vertical map $V_{i,x_i} \to V_i$ is an isomorphism [Wil97, p. 96] (this would not be true without restricting to admissible variations of Hodge structure). This shows that the following square commutes:

$$\begin{tikzcd}
\mathrm{Rep}(P_i) \arrow[r, "t_i^*"] \arrow[d, "H^1(V_i,-)"'] & \mathrm{Rep}(P_{i,x}) \arrow[d, "H^1(V_{i,x_i},-)"] \\
\mathrm{Rep}(G) \arrow[r, "r^*"'] & \mathrm{Rep}(G_y)
\end{tikzcd} \tag{4.12}$$

as in the proof of Lemma 4.8.1. In the case of the trivial representation $\mathbb{Q}$, this yields maps $r^*H^1(V_i,\mathbb{Q}) \to H^1(V_{i,x_i},\mathbb{Q})$ which are dual to φ_{V_i}. Since the diagrams of (4.11) are compatible with α^*, the squares of (4.12) form a prism:

$$\begin{tikzcd}
& \mathrm{Rep}(P_2) \arrow[rr] \arrow[dl, "\alpha^*"'] \arrow[ddl] & & \mathrm{Rep}(P_{2,x_2}) \arrow[dl, "\alpha^*"'] \arrow[ddl] \\
\mathrm{Rep}(P_1) \arrow[rr] \arrow[d] & & \mathrm{Rep}(P_{1,x}) \arrow[d] & \\
\mathrm{Rep}(G) \arrow[rr] & & \mathrm{Rep}(G_y) &
\end{tikzcd}$$

A purely group theoretic argument now checks that, consequently, there is a commutative square:

$$\begin{tikzcd}
H^1(V_{1,x_1}, \mathbb{Q}) & r^*H^1(V_1, \mathbb{Q}) \arrow[l] \\
H^1(V_{2,x_2}, \mathbb{Q}) \arrow[u, "\alpha^*"] & r^*H^1(V_2, \mathbb{Q}) \arrow[l] \arrow[u, "r^*\alpha^*"']
\end{tikzcd}$$

Taking Tannaka and linear duals we now obtain the square in *i)*. $\square$

We are now able to prove Theorem 4.2.5 of the introduction.

Theorem 4.8.4. *Let (G, h) be an arbitrary Shimura datum and $K \leq G(\mathbb{A}_f)$ neat open compact. Denote by S the Shimura variety $\mathrm{Sh}_K(G, h)$. Then the following diagram commutes,*

$$\begin{tikzcd}
\mathrm{Rep}_F(G)^{\mathrm{AV}} \arrow[rr, "\mu_G^{\mathrm{mot}}"] \arrow[dr, "\mu_G^{\mathcal{H}}"'] & \arrow[d, phantom, "\Leftarrow"] & \mathrm{CHM}_F/S \arrow[dl, "H_B^{\bullet}"] \\
& \mathrm{VHS}_F/S(\mathbb{C}) &
\end{tikzcd}$$

up to natural isomorphism given by $\varphi\colon H_B^* \circ \mu_G^{\mathrm{mot}} \implies \mu_G^{\mathcal{H}}$ (where φ is as in Notation 4.8.2). Moreover, under pullback by $f\colon (G', \mathfrak{X}') \to (G, \mathfrak{X})$, the triangles for $(G, \mathfrak{X}), (G', \mathfrak{X}')$ form a commutative prism:

$$\begin{tikzcd}
\mathrm{Rep}_F(G)^{\mathrm{AV}} \arrow[rr, "\mu_G^{\mathrm{mot}}"] \arrow[dr, "\mu_G^{\mathcal{H}}"'] \arrow[dd, "f^*"'] & & \mathrm{CHM}_F/S \arrow[dl, "H_B^{\bullet}"] \arrow[dd, "f^*"] \\
& \mathrm{VHS}_F/S(\mathbb{C}) \arrow[dd, "f^*"] & \\
\mathrm{Rep}_F(G')^{\mathrm{AV}} \arrow[rr, "\mu_{G'}^{\mathrm{mot}}"] \arrow[dr, "\mu_{G'}^{\mathcal{H}}"'] & & \mathrm{CHM}_F/S' \arrow[dl, "H_B^{\bullet}"] \\
& \mathrm{VHS}_F/S'(\mathbb{C}) &
\end{tikzcd}$$

for which each face has a given natural transformation, all of which are compatible.

Proof. That φ_V defines a natural isomorphism for the first triangle is Lemma 4.8.3 *i)*. The commutativity of the other individual faces in the prism is given by the natural isomorphisms: ψ of Lemma 4.7.12 for the rear face, κ of Construction 4.6.5 for the front left face, and ξ of Remark 4.5.9 for the front right.

Due to O'Sullivan's Theorem 4.4.6 (cf. Remark 4.4.7), we need only prove the compatibility statement for homological motives. As a result we reduce to showing that the two natural isomorphisms $H_B^\bullet \circ f^* \circ \mu_G^{\mathrm{mot}} \implies H_B^\bullet \circ \mu_{G'}^{\mathrm{mot}} \circ f^*$ (which are functors from $\mathrm{Rep}_F(G) \to \mathrm{VHS}_F/S'(\mathbb{C})$) defined by

$$f^* H_B^1((S_{K,V})(\mathbb{C}))^\vee \overset{H_B^1(\psi_V)}{\to} H_B^1((S'_{K',f^*V})(\mathbb{C}))^\vee,$$
$$f^* H_B^1((S_{K,V})(\mathbb{C}))^\vee \overset{f^*\varphi_V}{\to} \mu_G^{\mathcal{H}}(V)_{S'(\mathbb{C})} \overset{\kappa_V^{-1}}{\to} \mu_{G'}^{\mathcal{H}}(f^*V) \overset{\varphi_{f^*V}^{-1}}{\to} H_B^1((S'_{K',f^*V})(\mathbb{C}))^\vee,$$

coincide, here κ is as defined in Construction 4.6.5 and ψ is as defined in Lemma 4.7.12. This follows from Lemma 4.8.3 *ii)*. □

4.9 Ancona's construction

In the case of PEL-type Shimura data Ancona has recently described a lift of $\mu_G^{\mathcal{H}}$ defined on all of $\mathrm{Rep}_F(G)$ [Anc15]. But, as defined, this construction depends on the choice of PEL-datum and it is not immediately clear that it is well behaved with respect to pullbacks.

In this section we briefly recall Ancona's construction, but in the language of mixed Shimura varieties.

Notation 4.9.1. Given an algebra $B/\mathbb{Q}$, we write B_F for $B \otimes_{\mathbb{Q}} F$. Similarly if W is a B-module, then W_F denotes $W \otimes_{\mathbb{Q}} F$.

Definition 4.9.2. A *PEL-datum* is a tuple $(B, *, V, \langle\ ,\ \rangle, h)$ consisting of: a semi-simple $\mathbb{Q}$-algebra B with a positive (anti-)involution $*$ on B, that is an anti-commutative involution such that $\mathrm{Tr}_{B_{\mathbb{R}}/\mathbb{R}}(bb^*) > 0$ for all $0 \neq b \in B_{\mathbb{R}}$. Together with a finite dimensional B-module V equipped with an alternating non-degenerate $\mathbb{Q}$-valued pairing $\langle\ ,\ \rangle$ on V such that, for $b \in B, u, v \in V$

$$\langle bu, v\rangle = \langle u, b^*v\rangle,$$

and finally a choice of $\mathbb{R}$-algebra homomorphism $h\colon \mathbb{C} \to \mathrm{End}_{B_{\mathbb{R}}}(V_{\mathbb{R}})$ such that

$$\langle h(z)u, v\rangle = \langle u, h(\bar{z})v\rangle \quad \forall z \in \mathbb{C},\ u, v \in V$$

$$\langle u, h(i)u\rangle \text{ is positive definite,}$$

(the first condition ensures that $\langle u, h(i)v\rangle$ is symmetric).

Let G be the algebraic group whose R-points, for any $\mathbb{Q}$-algebra R, are defined by

$$G(R) = \left\{ g \in \mathrm{Aut}_{B_R}(V_R) \;\middle|\; \begin{array}{l} \exists\, \mu(g) \in R^{\times} \text{ such that } \langle gu, gv \rangle = \mu(g)\langle u, v \rangle \\ \text{for all } u, v \in V \otimes R \end{array} \right\}.$$

For $z \in \mathbb{C}^{\times}$, we automatically have that $h(z) \in G(\mathbb{R})$. We also denote by h the induced map $\mathbb{S} \to G_{\mathbb{R}}$. Then (G, h) satifies (1.5.1), (1.5.2) and (1.5.3) of [Del71] and so defines a Shimura datum (see [Kot92, Lem. 4.1]). A Shimura datum (G, h) which arises in this way is said to be of *PEL-type* and the corresponding $(B, *, V, \langle\ ,\ \rangle, h)$ is said to be a *PEL-datum* for (G, h).

Lemma 4.9.3. *For any (G, h) of PEL-type, the centre of G is an almost–direct product of a $\mathbb{Q}$-split torus and an $\mathbb{R}$-anisotropic torus.*

Proof. We want to show that the largest anisotropic subtorus of $Z(G)$ remains anisotropic over $\mathbb{R}$. The largest anisotropic subtorus of $Z(G)$ is contained in the kernel of the multiplier character $\mu\colon G \to \mathbb{G}_m$ and so must be contained in $Z(G_1)$. We claim that $(Z(G_1)^{\circ})_{\mathbb{R}}$ is always anisotropic.

Any semisimple $\mathbb{R}$-algebra with positive involution is a product of simple factors of one of the following three types: $(M_n(\mathbb{R}), A \mapsto A^t)$, $(M_n(\mathbb{C}), A \mapsto \bar{A}^t)$ or $(M_n(\mathbb{H}), A \mapsto \bar{A}^t)$ where $\mathbb{H}$ denotes the quaternions and where $\bar{A}$ denotes coefficientwise complex conjugation in the $M_n(\mathbb{C})$-case and the involution $a + bi + cj + dij \mapsto a - bi - cj - dij$ in the $M_n(\mathbb{H})$-case (for example [Kot92, p. 386]).

In particular, all symplectic $B_{\mathbb{R}}$-modules split as an orthogonal direct sum of submodules only acted on non-trivially by a single simple factor of one of the above types and G_1 splits accordingly. As such, it suffices assume that $B_{\mathbb{R}}$ is simple of each of the three above types. Moreover, we are able to reduce to the case of $B_{\mathbb{R}}$ isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ by an easy Morita equivalence argument.

We shall make repeated use of the following result of Kottwitz. Given any semisimple $\mathbb{R}$-algebra $(B, *)$ and two triples $(V, \langle\ ,\ \rangle, h)$ and $(V', \langle\ ,\ \rangle', h')$, that together with $(B, *)$ satisfy the conditions of Definition 4.9.2 with $\mathbb{R}$ in place of $\mathbb{Q}$, then if V and V' are isomorphic as $B \otimes_{\mathbb{R}} \mathbb{C}$-modules, with $\mathbb{C}$ acting via h and h' respectively, then $(V, \langle\ ,\ \rangle)$ and $(V'\langle\ ,\ \rangle')$ are isomorphic as symplectic $(B, *)$-modules [Kot92, Lemma 4.2].

First assume that $(B_{\mathbb{R}}, *) = (\mathbb{R}, * = \mathrm{id})$. Then

$$\left(W = \mathbb{R}^{\oplus 2}, \langle\ ,\ \rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, h(i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)$$

is a triple as above with corresponding $B_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$-module $\mathbb{C}$. As a result, as a symplectic $(B_{\mathbb{R}}, *)$-module $V_{\mathbb{R}}$ must split as an orthogonal direct sum of terms isomorphic to W. By definition, $G_1(\mathbb{R})$ for $W^{\oplus n}$ is Sp_{2n}. In particular, it has finite centre so that $(Z(G)^{\circ})_{\mathbb{R}}$ is anisotropic.

Now assume that $(B_{\mathbb{R}}, *) = (\mathbb{C}, * = z \mapsto \bar{z})$. In this case, $B_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ has two irreducible modules. The two triples given by $(\mathbb{C}, \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(xi\bar{y}), h(i) = i)$ with $z \in \mathbb{C}$ acting by multiplication by z and $\bar{z}$ respectively correspond to these two irreducibles. Both have the same underlying symplectic $(B_{\mathbb{R}}, *)$-module so $V_{\mathbb{R}}$ must be isomorphic to $\mathbb{C}^n$. For this symplectic module, $G_1(\mathbb{R})$ consists of elements of $\mathrm{GL}_n(\mathbb{C})$ which also lie in $\mathrm{Sp}_{2n}(\mathbb{R})$. This is precisely the unitary group $U_n(\mathbb{R})$. Now $Z(U_n(\mathbb{R})) = U_1(\mathbb{R})$, which is indeed anisotropic.

Finally, for the quaternion case we shall assume that $(B_{\mathbb{R}}, *) = (\mathbb{H}^{\mathrm{op}}, *)$ (with $\mathbb{H}^{\mathrm{op}}$ an expositional choice). In this case, $B_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$ has a unique irreducible module which is of $\mathbb{R}$-dimension 4. This is realised by the triple $(\mathbb{H}, \mathrm{tr}_{\mathbb{H}/\mathbb{R}}(xj\tilde{y}), h(i) = j)$ where $\mathbb{H}^{\mathrm{op}}$ acts by right multiplication and $y \mapsto \tilde{y}$ is the involution given by $y = a + bi + cj + dij \mapsto a + bi - cj + dij$. In this case, $\mathrm{End}_{\mathbb{H}^{\mathrm{op}}}(\mathbb{H}) \cong \mathbb{H}$ with $\mathbb{H}$ acting by left multiplication. On $\mathrm{End}_{\mathbb{H}^{\mathrm{op}}}(\mathbb{H})$, taking adjoints with respect to $\mathrm{tr}(xj\tilde{y})$ coincides with the map $y \mapsto \tilde{y}$. The embedding $\mathbb{H} \hookrightarrow M_2(\mathbb{C})$ which sends $i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ defines a splitting of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$. The involution of $M_2(\mathbb{C})$ induced by $y \mapsto \tilde{y}$ is then matrix transposition. As a result, $G_1(\mathbb{C}) = \{c \in \mathrm{Aut}_{\mathbb{H}^{\mathrm{op}}}(\mathbb{H}) \mid cc^* = \mathrm{id}\} = O_2(\mathbb{C})$ is the orthogonal group. More generally, for $\mathbb{H}^{\oplus n}$ we then have $G_1(\mathbb{C}) = O_{2n}(\mathbb{C})$, which does indeed have finite centre. $\square$

If we fix a PEL-datum for (G, h), then we say that $V \in \mathrm{Rep}(G)$ is the *standard representation* of G. Shimura data with a fixed choice of PEL-datum have an explicit moduli interpretation (see [Mil17, Sec. 8]).

Example 4.9.4. We give an example of two distinct PEL-data for the same Shimura datum. First consider the PEL-datum $(\mathbb{Q}, *, \mathbb{Q}^{\oplus 2}, \langle\ ,\ \rangle, h)$, where $* = \mathrm{id}$, $\langle\ ,\ \rangle$ is the alternating pairing represented by $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and h sends $a + bi \mapsto$

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. The corresponding Shimura datum is then the usual datum $(\mathrm{GL}_2, \mathcal{H})$ defined in Example 4.2.2.

There is also a PEL-datum $(M_2(\mathbb{Q}), * = (-)^t, \mathbb{Q}^{\oplus 4}, \langle\ ,\ \rangle)$, where the involution is transposition, $M_2(\mathbb{Q})$ acts diagonally on $\mathbb{Q}^{\oplus 4} = \mathbb{Q}^{\oplus 2} \oplus \mathbb{Q}^{\oplus 2}$ acting on each factor in the standard way, the pairing is represented by $\left(\begin{array}{c|c} 0 & I_2 \\ \hline -I_2 & 0 \end{array}\right)$, and h is given by $a + bi \mapsto \left(\begin{array}{c|c} aI_2 & -bI_2 \\ \hline bI_2 & aI_2 \end{array}\right)$. Then G is isomorphic to GL_2, which is embedded within $\mathrm{GL}_4(\mathbb{Q})$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{array}{c|c} aI_2 & bI_2 \\ \hline cI_2 & dI_2 \end{array}\right)$, so that the associated Shimura datum is again $(\mathrm{GL}_2, \mathcal{H})$. This is an example of the Morita equivalence used in the proof of Lemma 4.9.3.

Remark 4.9.5. Suppose $(B, *, V, \langle\ ,\ \rangle, h)$ is a PEL-datum. Then h is uniquely determined, up to G conjugacy, by $(B, *, V, \langle\ ,\ \rangle)$ [Kot92, Lem. 4.3]. If we assume that B is simple and linear or symplectic in the sense of [Mil17, Sec. 8], then any 4-tuple $(B, *, V, \langle\ ,\ \rangle)$ satisfying the relevant parts of Definition 4.9.2 admits an h (which is necessarily unique up to conjugacy) [Mil17, Prop. 8.12].

Proposition 4.9.6. *Given a Shimura datum (G, h) with a choice of PEL-datum $(B, *, V, \langle\ ,\ \rangle, h)$, then for all fields $F/\mathbb{Q}$, all objects of $\mathrm{Rep}_F(G)$ are, up to isomorphism, direct summands of some space of the form $\bigoplus_{i=1}^k V_F^{\otimes a_k} \otimes V_F^{\otimes b_k}$.*

Proof. As V is a faithful G-representation, this follows from the proof of [DM82, Prop. 2.20]. $\square$

Theorem 4.9.7 ([Anc15, Thm. 6.1])**.** *Given a Shimura datum (G, h) with a PEL-datum $(B, *, V, \langle\ ,\ \rangle, h)$, let K be a neat open compact subgroup of $G(\mathbb{A}_f)$ and L a $\hat{\mathbb{Z}}$-lattice of V_F (considered as a representation over $\mathbb{Q}$). Then for any $n \in \mathbb{N}$, there is a canonical inclusion of rings $a\colon \mathrm{End}_{\mathrm{Rep}_F(G)}(V_F^{\otimes n}) \hookrightarrow \mathrm{End}_{\mathrm{HomM}_F/S}(h^1(S_{V_F,K})^{\vee \otimes n})$ such that the diagram*

$$\begin{tikzcd}
\mathrm{End}_{\mathrm{Rep}_F(G)}(V_F^{\otimes n}) \arrow[rr, "a"] \arrow[dr, "\mu_G^{\mathcal{H}}"'] & & \mathrm{End}_{\mathrm{HomM}/S}(h^1(S_{V_F,K})^{\vee \otimes n}) \arrow[dl, "H_B^i"] \\
& \mathrm{End}_{\mathrm{VHS}/S(\mathbb{C})}(\mu_G^{\mathcal{H}}(V_F)^{\otimes n}) &
\end{tikzcd}$$

commutes. Here, we have used the isomorphism $\varphi_{V_F}\colon H_B^1((S_{V_F,K})(\mathbb{C}))^\vee \to \mu_G^{\mathcal{H}}(V_F)$ of Lemma 4.8.1 to identify $\mathrm{End}(\mu_G^{\mathcal{H}}(V_F)^{\otimes n})$ and $\mathrm{End}(H^1((S_{V_F,K})(\mathbb{C}))^{\otimes n})$.

Remark 4.9.8. Ancona's strategy is to lift endomorphisms of V_F itself (in our presentation, this is via functoriality of mixed Shimura varieties) and permutations of $V_F^{\otimes n}$ in the obvious way and then additionally lift cycles arising from the polarisation via the Poincaré-Lefschetz isomorphisms (which have been described motivically). Ancona then shows that endomorphisms of the above kinds generate all of $\mathrm{End}_{\mathrm{Rep}_F(G)}(V_F^{\otimes n})$ in the case of PEL-type Shimura varieties. This is not true for arbitrary Shimura varieties, and to obtain such a result more generally would require identifying more algebraic cycles.

Construction 4.9.9. There is a $\otimes$-functor $\mathrm{Anc}_G \colon \mathrm{Rep}_F(G) \to \mathrm{HomM}_F/S$ defined as follows: set $\mathrm{Anc}_G(V_F^{\otimes n}) = h^1(S_{V_F,K})^{\vee\otimes n}$ and let $\mathrm{Anc}_G(\alpha)$ for $\alpha \in \mathrm{End}(V_F^{\otimes n})$ be defined via the map of Theorem 4.9.7. By Hom-tensor adjunction, Theorem 4.9.7 also defines a motivic lift of the map $\mathbb{1} \to V \otimes V^\vee$. More generally, to define the image of elements of $\mathrm{Hom}(V_F^{\otimes a} \otimes V_F^{\vee\otimes b}, V_F^{\otimes c} \otimes V_F^{\vee\otimes d})$ it suffices to fix the image of $\mathrm{Hom}(V_F^{\otimes(a+d)}, V_F^{\otimes(b+c)})$, but for weight reasons this is zero unless $a - b = c - d$, in which case it is covered by Theorem 4.9.7.

This also allows us to define, for any choice of idempotent e the image of a direct summand $e \cdot (\bigoplus V_F^{\otimes a_n} \otimes V_F^{\vee\otimes b_n})$. Since every element of $W \in \mathrm{Rep}_F(G)$ is of this form by Proposition 4.9.6, if we pick a fixed isomorphism $\theta_W \colon W \xrightarrow{\sim} e_W \cdot (\bigoplus V_F^{\otimes a_{W,n}} \otimes V_F^{\vee\otimes b_{W,n}})$ for each W, then we can compatibly extend Anc_G to all of $\mathrm{Rep}_F(G)$. Finally, by composition with the section of Theorem 4.4.6, we obtain a functor $\mathrm{Rep}_F(G) \to \mathrm{CHM}_F/S$, which we also denote Anc_G.

Lemma 4.9.10. *The construction of* Anc_G *is, up to natural isomorphism, independent of all choices made.*

Proof. Fix $W \in \mathrm{Rep}_F(G)$ and two summands isomorphic to W of a tensor space, $e \cdot \bigoplus V_F^{a_k} \otimes V_F^{\vee\otimes b_k}$, $e' \cdot \bigoplus V_F^{\otimes a'_k} \otimes V_F^{\vee\otimes b'_k}$. We must provide an isomorphism

$$e \cdot \bigoplus h^1(S_{V_F,K})^{\vee a_k} \otimes h^1(S_{V_F,K})^{\otimes b_k} \to e' \cdot \bigoplus h^1(S_{V_F,K})^{\vee a'_k} \otimes h^1(S_{V_F,K})^{\otimes b'_k}.$$

Given the compatibility of the Künneth formula with mixed Shimura varieties, we may assume that W is irreducible and there is a corresponding isomorphism $e \cdot (V_F^{\otimes a} \otimes V_F^{\vee\otimes b}) \to e \cdot (V_F^{\otimes a'} \otimes V_F^{\vee\otimes b'})$.

As before, it suffices to assume that $b = b' = 0$. For weight reasons, we must then have that $a = a'$. Finally, since Lemma 4.9.7 lifts all elements of $\mathrm{End}_{\mathrm{Rep}_F(G)}(V_F^{\otimes a})$, we obtain a motivic lift of the isomorphism between the two tensor space representatives of W. This construction is natural, and so gives the desired natural isomorphism. □

Remark 4.9.11. Let $(G, \mathfrak{X})$ be a Shimura datum with a chosen PEL-datum for which all objects of $\mathrm{Rep}(G)^{\mathrm{AV}}$ are direct summands of $V^{\oplus n}$ for varying n. Then the argument given above can be adapted to show that Anc_G extends μ_G^{mot} up to natural isomorphism. If the PEL-datum is of "symplectic type", then this always holds (see Section 4.12). This can also be checked to hold much more generally.

4.10 Compatibility with base change

In this section we give conditions to ensure Ancona's construction is compatible with base change analogously to Theorem 4.8.4.

Lemma 4.10.1 ([Anc15, Thm. 8.6])**.** *Let (G, h) be a Shimura datum of PEL-type with a fixed PEL-datum $(B, *, V, \langle\ ,\ \rangle, h)$. Fix also a choice of neat open compact subgroup $K \leq G(\mathbb{A}_f)$ and denote by S the Shimura variety $\mathrm{Sh}_K(G, h)$. Then the following diagram commutes,*

$$\begin{tikzcd} \mathrm{Rep}_F(G) \arrow[rr, "\mathrm{Anc}_G"] \arrow[dr, "\mu_G^{\mathcal{H}}"'] & \arrow[d, phantom, "\Leftarrow"] & \mathrm{CHM}_F/S \arrow[dl, "H_B^\bullet"] \\ & \mathrm{VHS}_F/S(\mathbb{C}) & \end{tikzcd}$$

up to canonical natural isomorphism.

Proof. We describe the natural isomorphism. In the notation of Construction 4.9.9, write $\eta_{G,V}$ for

$$\mu_G^{\mathcal{H}}(\theta_W^{-1}) \circ (e_W \cdot \bigoplus (\varphi_{V_F}^{\otimes a_{W,n}} \otimes \varphi_{V_F}^{\vee, \otimes b_{W,n}})),$$

where φ_{V_F} is as defined in Notation 4.8.2. That $\eta_G := (\eta_{G,V})_V$ defines a natural isomorphism now follows from Lemma 4.8.3 *i)*. □

Unfortunately, it is not formal to show that $\mathrm{Anc}_{(-)}$ commutes with morphisms of Shimura varieties (i.e. there is a commutative prism analogous to that of Theorem 4.8.4). For example, consider the identity map $S \to S$, but where each S has a distinct choice of PEL-datum with standard representation V_1, V_2 respectively. We know that $f^*V_2 \cong e \cdot (\bigoplus V_1^{\otimes a_n} \otimes V_2^{\vee \otimes b_n})$ for some e (Theorem 4.9.6). We desire an isomorphism

$$f^* \operatorname{Anc}_G(V_2) = h^1(S_{K,V_2})^\vee \xrightarrow{\sim} e \cdot \left(\bigoplus h^1(S_{K,V_1})^{\vee \otimes a_n} \otimes h^1(S_{K,V_1}^{\otimes b_n}) \right),$$

but this need not be obtainable using only the functoriality of mixed Shimura varieties. However, in the following restricted setting we shall only ever need to construct maps which do arise from functoriality of mixed Shimura varieties (and tensor products and direct sums thereof).

Definition 4.10.2. Let $f\colon (G', h') \to (G, h)$ be a morphism of PEL-type Shimura data each with a choice of PEL-datum whose standard representations are denoted V', V. If

$$(\star) \qquad f^*V \cong e \cdot V'^{\oplus n} \text{ for some } n \in \mathbb{N} \text{ and idempotent } e \in \operatorname{End}_{\operatorname{Rep}(G')}(V'^{\oplus n}),$$

then we say that f is an *admissible* morphism of Shimura varieties with PEL-data.

Note that if f is admissible, then $f^*V_F \cong e_F \cdot V_F'^{\oplus n}$ for any F. Admissibility implies that there is exists a map $(S_{K,V})_{S'} \to S'_{K',V'}$ as abelian varieties over S'.

Example 4.10.3. In Example 4.9.4, we described two PEL-data for $(\mathrm{GL}_2, \mathcal{H})$, one with standard representation $V = \mathbb{Q}^{\oplus 2}$ and the other with standard representation $W = \mathbb{Q}^{\oplus 2} \oplus \mathbb{Q}^{\oplus 2}$. The identity map $(\mathrm{GL}_2, \mathcal{H}) \to (\mathrm{GL}_2, \mathcal{H})$ is admissible for each of the two ways of assigning each $(\mathrm{GL}_2, \mathcal{H})$ a distinct choice of the two PEL-data. Indeed, $\mathrm{id}^* V \cong (i_1 \circ \pi_1) \cdot W$ and $\mathrm{id}^* W \cong V^{\oplus 2}$.

More generally, for any PEL-datum $(B, *, V, \langle\ ,\ \rangle, h)$ with associated Shimura datum (G, h), then, for $n > 0$ $(M_n(B), *, V^{\oplus n}, \langle\ ,\ \rangle^{\oplus n}, h^{\oplus n})$ is also a PEL-datum for (G, h), as can be seen by Morita equivalence. Then the identity map is admissible for any choice of one of these data for the source and target (G, h).

Example 4.10.4. Given a PEL-datum $(B, *, V, \langle\ ,\ \rangle, h)$ and $B' \subseteq B$ a $\mathbb{Q}$-subalgebra, then $(B', *, V, \langle\ ,\ \rangle, h)$ is also a PEL-datum. If $(G, h), (G', h)$ denote the respective Shimura data, then the induced map $(G', h) \hookrightarrow (G, h)$ with the above choices is an admissible morphism.

The admissibility of a given morphism of Shimura data with a chosen PEL-datum is easy to verify in practice and seems to hold for many examples which arise in applications. We do not impose any direct condition on the morphism respecting the chosen PEL-data. In Section 4.12, we check that any morphism of a Siegel Shimura datum to an arbitrary Shimura datum is admissible. We have been unable to find any examples of morphisms of Shimura data with chosen PEL-data which are not admissible.

We now assume $f\colon (G', h') \to (G, h)$ is admissible and fix one such isomorphism as in $(\star)$.

Construction 4.10.5. We now have canonical isomorphisms:

$$\begin{aligned} f^* \operatorname{Anc}_G(V) &= h^1(S_{K,V})^{\vee}_{S'} \\ &= h^1((S_{K,V})_{S'})^{\vee} \end{aligned}$$

as the canonical projectors h^i commute with pullbacks [DM91, Thm. 3.1],

$$\begin{aligned} &\overset{\text{Lem. 4.7.7}}{=} h^1(S'_{K',f^*V})^{\vee} \\ &\overset{(\star)}{=} h^1(S'_{K',e\cdot V'^{\oplus n}})^{\vee} \\ &= e\cdot (h^1(S'_{K',V'})^{\oplus n})^{\vee} \\ &= \operatorname{Anc}_{G'} f^*V. \end{aligned}$$

by Lemma 4.7.7 i) and the Künneth formula 4.4.9. Write λ_V for this composite. For V_F, the base change to $\operatorname{Rep}_F(G)$, there is an analogous λ_{V_F}.

Notation 4.10.6. As functors on $\operatorname{Rep}_F(G)$, we extend this to a putative natural isomorphism $\lambda\colon f^* \circ \operatorname{Anc}_G \implies \operatorname{Anc}_{G'} \circ f^*$ as follows: Let $W \in \operatorname{Rep}_F(G)$. Since the construction of $\operatorname{Anc}_{G'}$ is independent of the choice of the $\theta'_{W'}$ (Lemma 4.9.10), we are free to assume that, for $W \in \operatorname{Rep}_F(G)$ with $\theta_W\colon W \xrightarrow{\sim} e_W \cdot (\bigoplus V_F^{\otimes a_n} \otimes V_F^{\vee\otimes b_n})$, then θ'_{f^*W} is obtained from $f^*\theta_W$ by taking the tensor products and direct sums of (the base change of) the isomorphism of $(\star)$. In other words,

$$f^* \operatorname{Anc}_G(W) = e_W \cdot (\bigoplus h^1(S_{K,V_F})^{\vee\otimes a_k}_{S'} \otimes h^1(S_{K,V_F})^{\otimes b_k}_{S'})$$

whilst

$$\operatorname{Anc}_{G'} f^*(W) = e_W \cdot (\bigoplus (e\cdot \bigoplus h^1(S'_{K',V'_F}))^{\vee\otimes a_k} \otimes (e\cdot \bigoplus h^1(S'_{K',V'_F}))^{\otimes b_k})).$$

There is now an obvious choice for λ_W given by taking sums and products of λ_{V_F} and its dual.

Lemma 4.10.7. *Given $f\colon (G', h') \to (G, h)$ an admissible morphism of PEL-type Shimura varieties with fixed PEL data, then the following diagram commutes:*

$$
\begin{tikzcd}
\mathrm{Rep}_F(G) \arrow[r, "\mathrm{Anc}_G"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\swarrow"] & \mathrm{HomM}_F/S \arrow[d, "f^*"] \\
\mathrm{Rep}_F(G') \arrow[r, "\mathrm{Anc}_{G'}"'] & \mathrm{HomM}_F/S'
\end{tikzcd}
$$

up to natural isomorphism given by $\lambda\colon f^* \circ \mathrm{Anc}_{G'} \implies \mathrm{Anc}_{G'} \circ f^*$.

Proof. Since the functor $H_B^\bullet(-)$ is injective on $\mathrm{Hom}_{\mathrm{HomM}_F/S'}(h^i(A_1), h^i(A_2))$ for A_1, A_2 abelian varieties over S' (see Remark 4.5.8), it is enough to check that $H_B^\bullet(\lambda)\colon H_B^\bullet \circ f^* \circ \mathrm{Anc}_G \implies H_B^\bullet \circ \mathrm{Anc}_{G'} \circ f^*$ is a natural isomorphism. But we already have a natural isomorphism $H_B^\bullet \circ f^* \circ \mathrm{Anc}_G \implies H_B^\bullet \circ \mathrm{Anc}_{G'} \circ f^*$, given by composing the given natural isomorphisms of the other faces of the prism. To show that $H_B^\bullet(\lambda)$ is a natural isomorphism, it suffices to check that it coincides with the one already constructed. We need only check this for V_F itself, i.e. that

$$H_B^\bullet(\lambda_{V_F}) = \eta^{-1}_{S', f^*V_F} \circ \kappa_{V_F} \circ f^*(\eta_{S,V_F}) \circ \xi^{-1}_{h^1(S_{K,V_F})}.$$

Here η_{S,V_F} is as defined in the proof of Lemma 4.10.1, κ is as defined in Construction 4.6.5 and ξ is as defined in Remark 4.5.9.

Applying $\xi_{h^1(S_{K,V_F})}$ to both sides, this means checking the equality of:

$$
\begin{aligned}
&f^*H_B^1((S_{K,V_F})(\mathbb{C})) \to H^1((S_{K',f^*V_F})(\mathbb{C})) \overset{(\star)}{\to} e \cdot H_B^1((S'_{K',V'_F})(\mathbb{C})),\\
&f^*H_B^1((S_{K,V_F})(\mathbb{C})) \overset{f^*\varphi_{V_F}}{\to} f^*\mu_G^{\mathcal{H}}(V_F) \overset{\kappa_V}{\to} \mu_{G'}^{\mathcal{H}}(f^*V_F) \overset{\varphi^{-1}_{f^*V_F}}{\to} H_B^1((S_{K,f_F^V})(\mathbb{C}))\\
&\qquad\qquad\qquad \overset{(\star)}{\to} e \cdot \bigoplus H_B^1((S'_{K',V'_F})(\mathbb{C})),
\end{aligned}
$$

where, in the second line the composite of the last two maps is η^{-1}_{S',f^*V_F}, as defined in Lemma 4.10.1. The equality now follows from the commutativity of:

$$
\begin{tikzcd}
f^*H_B^1((S_{K,V_F})(\mathbb{C})) \arrow[r, "f^*(\varphi_{V_F})"] \arrow[d, "H^1(\lambda_{V_F})"'] & f^*\mu_G^{\mathcal{H}}(V_F) \arrow[d, "\kappa_{V_F}"] \\
H_B^1((S'_{K',f^*V_F})(\mathbb{C})) \arrow[r, "\varphi_{f^*V_F}"'] & \mu_{G'}^{\mathcal{H}}(f^*V_F)
\end{tikzcd}
$$

as shown in Lemma 4.8.3 *ii)*. □

Note that the statement of Lemma 4.10.7 is independent of the choice of realisation.

Corollary 4.10.8. *Given an admissible morphism of Shimura data $f\colon (G',h') \to (G,h)$, then the analogous prism for* Anc_G *to that of Theorem 4.8.4 commutes in the sense that all the faces commute up to the given natural isomorphisms and all natural isomorphisms are compatible.*

Proof. As before we need only consider the analogue for homological motives. First consider the analogous prism for homological motives. From Lemmas 4.10.1, 4.10.7 and the results of Section 4.4, 4.6 we have commutativity of the individual faces. It remains to check that, as natural isomorphisms $H_B^\bullet \circ f^* \circ \mathrm{Anc}_G \implies H_B^\bullet \circ \mathrm{Anc}_{G'} \circ f^*$,

$$\begin{aligned}&H_B^\bullet(\lambda),\\ &\eta_{G'}^{-1} \circ \kappa^{-1} \circ f^*(\eta_G),\end{aligned}$$

agree. This can be seen from the proof of Lemma 4.10.7 and Lemma 4.8.3. □

4.11 Étale canonical construction

Canonical constructions arise more generally than just the Hodge realisation, and both μ_G^{mot} and Ancona's construction should also be lifts of any such construction. We sketch this for the étale realisation following [Wil97, Sec. II.4]. We use the notation for the étale realisation described in Lemma 4.5.11.

Notation 4.11.1. Let $(G,\mathfrak{X})$ be a Shimura datum and K be a neat open compact subgroup of $G(\mathbb{A}_f)$. We consider the associated Shimura variety $S := \mathrm{Sh}_K(G,\mathfrak{X})$ as defined over its reflex field $E/\mathbb{Q}$ via the theory of canonical models (which is independent of K). Let $V \in \mathrm{Rep}_F(G)$ and L be a K-stable full rank $\hat{\mathbb{Z}}$-sublattice of $V(\mathbb{A}_f)$. If $\mathrm{Sh}_{L\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}})$ is the mixed Shimura variety defined in Lemma 4.7.3, then the projection and section maps force $\mathrm{Sh}_{L\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}})$ to also have reflex field E [Pin90, Sec. 11.2(b)]. We denote the canonical model of $\mathrm{Sh}_{L\rtimes K}(V \rtimes G, \tilde{\mathfrak{X}})$ by $S_{K,V}$. The projection and Levi section then define regular maps $p\colon S_{K,V} \to S$, $\iota\colon S \to S_{K,V}$.

Construction 4.11.2. Let $(G,\mathfrak{X})$ be a Shimura datum and $K \le G(\mathbb{A}_f)$ neat open compact. If $K' \le K$ is an open normal subgroup, then there is a right action of K/K' on $\mathrm{Sh}_{K'}(G,\mathfrak{X})$. Since we are assuming that the centre of G is an almost–direct product of a $\mathbb{Q}$-split and $\mathbb{R}$-anisotropic torus, the action of K/K' is free on $\mathbb{C}$-points and

$$\mathrm{Sh}_{K'}(G,\mathfrak{X}) \longrightarrow \mathrm{Sh}_K(G,\mathfrak{X})$$

is an étale cover of smooth algebraic varieties with Galois group K/K' (see [Pin92, Prop. 3.3.3. and (3.4.1)]).

Taking the inverse limit over $K' \leq K$ we obtain a pro-Galois covering of $\mathrm{Sh}_K(G, \mathfrak{X})$ with Galois group K. Let ℓ be a prime and λ a prime of F lying over ℓ. Write F_λ for the completion of F at λ as before. Then any F_λ-linear continuous representation of K will define a lisse λ-adic sheaf on $\mathrm{Sh}_K(G, \mathfrak{X})$.

Given $(G_F \xrightarrow{\rho} \mathrm{GL}(V)) \in \mathrm{Rep}_F(G)$, we obtain such a representation via

$$K \hookrightarrow G(\mathbb{A}_f) \twoheadrightarrow G(\mathbb{Q}_\ell) \hookrightarrow G(F_\lambda) = G_F(F_\lambda) \xrightarrow{\rho(F_\lambda)} \mathrm{GL}(V)(F_\lambda).$$

This defines a functor

$$\mu_G^{\text{ét}} \colon \mathrm{Rep}_F(G) \to \text{Ét}_{F_\lambda}/S,$$

which we refer to as the *étale canonical construction.*

Lemma 4.11.3. *Given a Shimura datum $(G, \mathfrak{X})$ and $V \in \mathrm{Rep}_F(G)^{\mathrm{AV}}$, then there is a canonical identification $\varphi_{V,\lambda} \colon H^1_\lambda(S_{K,V})^\vee \xrightarrow{\sim} \mu_G^{\acute{e}t}(V)$.*

Proof. The étale canonical construction extends verbatim to mixed Shimura varieties. As in the Hodge case, the diagram

$$\begin{array}{ccc} \mathrm{Rep}_F(V \rtimes G) & \xrightarrow{\mu^{\text{ét}}_{V \rtimes G}} & \text{Ét}_{F_\lambda}/S_{K,V} \\ {\scriptstyle H^i(V,-)}\downarrow & & \downarrow{\scriptstyle R^i p_*} \\ \mathrm{Rep}_F(G) & \xrightarrow[\mu^{\text{ét}}_G]{} & \text{Ét}_{F_\lambda}/S \end{array}$$

commutes [Wil97, Thm. II.4.7, Thm. I.4.3]. The dual of the desired isomorphism is given by commutativity in the case of the trivial representation F. $\square$

Lemma 4.11.4. *i) Let $(G, \mathfrak{X})$ be a Shimura datum and $\alpha \colon V_1 \to V_2$ a morphism in $\mathrm{Rep}_F(G)^{\mathrm{AV}}$. Fix a neat open compact subgroup $K \leq G(\mathbb{A}_f)$ and let α also denote the map $S_{K,V_1} \to S_{K,V_2}$. Then the following diagram commutes:*

$$\begin{array}{ccc} H^1_\lambda(S_{K,V_1})^\vee & \xrightarrow{\varphi_{V_1,\lambda}} & \mu_G^{\text{ét}}(V_1) \\ {\scriptstyle (\alpha^*)^\vee}\downarrow & & \downarrow{\scriptstyle \mu_G^{\mathcal{H}}(\alpha)} \\ H^1_\lambda(S_{K,V_2})^\vee & \xrightarrow[\varphi_{V_2,\lambda}]{} & \mu_G^{\text{ét}}(V_2) \end{array}$$

ii) Let $f\colon (G',\mathfrak{X}') \to (G,\mathfrak{X})$ be a morphism of Shimura data and $K \le G(\mathbb{A}_f), K' \le G'(\mathbb{A}_f)$ neat open compact subgroups for which $f(K') \le K$. Write E' for the reflex field of $(G',\mathfrak{X}')$ (so $E' \supseteq E$). For any $V \in \mathrm{Rep}_F(G)^{\mathrm{AV}}$, the following diagram commutes:

$$\begin{tikzcd} f^*(H^1_\lambda((S_{K,V}))^\vee)_{E'} \arrow[r, "f^*(\varphi_{V,\lambda})"] \arrow[d] & f^*\mu^{\text{ét}}_G(V)_{E'} \arrow[d] \\ H^1_\lambda(S'_{K',f^*V})^\vee \arrow[r, "\varphi_{f^*V,\lambda}"'] & \mu^{\text{ét}}_{G'}(f^*V) \end{tikzcd}$$

Here, on the top row, f^* denotes pullback via the map $S_{K',f^*V} \to (S_{K,V})_{E'}$ and $(-)_{E'}$ pullback via $(S_{K,V})_{E'} \to S_{K,V}$.

Proof. As for Lemma 4.8.3, but using [Wil97, Cor. I.3.2 i)]. $\square$

We now obtain results analogous to Theorem 4.8.4 and Lemmas 4.10.1, 4.10.7, whose proofs are almost identical.

Lemma 4.11.5. *Let $(G,\mathfrak{X})$ be an arbitrary Shimura datum and $K \le G(\mathbb{A}_f)$ neat open compact. Denote by S the Shimura variety $\mathrm{Sh}_K(G,\mathfrak{X})$. Then the following diagram commutes,*

$$\begin{tikzcd} \mathrm{Rep}_F(G)^{\mathrm{AV}} \arrow[rr, "\mu^{\mathrm{mot}}_G"] \arrow[dr, "\mu^{\text{ét}}_G"'] & \arrow[d, phantom, "\Leftarrow"] & \mathrm{CHM}_F/S \arrow[dl, "H^\bullet_\lambda"] \\ & \text{Ét}_{F_\lambda}/S & \end{tikzcd}$$

up to natural isomorphism given by $\varphi\colon H^\bullet_\lambda \circ \mu^{\mathrm{mot}}_G \implies \mu^{\mathcal{H}}_G$. Moreover, under pullback by $f\colon (G',\mathfrak{X}') \to (G,\mathfrak{X})$, the triangles for $(G,\mathfrak{X}),(G',\mathfrak{X}')$ form a commutative prism for which the given natural transformations on each face are compatible.

Lemma 4.11.6. *i) Let (G,h) be a Shimura datum of PEL type with a fixed PEL datum $(B,*,V,\langle\ ,\ \rangle,h)$. Fix also a choice of neat open compact subgroup $K \le G(\mathbb{A}_f)$ and denote by S the Shimura variety $\mathrm{Sh}_K(G,h)$. Then the following diagram commutes,*

$$\begin{tikzcd} \mathrm{Rep}_F(G) \arrow[rr, "\mathrm{Anc}_G"] \arrow[dr, "\mu^{\text{ét}}_G"'] & \arrow[d, phantom, "\Leftarrow"] & \mathrm{CHM}_F/S \arrow[dl, "H^\bullet_\lambda"] \\ & \text{Ét}_{F_\lambda}/S & \end{tikzcd}$$

up to canonical natural isomorphism.

ii) Given a morphism of Shimura data $f\colon (G', h') \to (G, h)$, each of PEL-type with a fixed datum, which is admissible in the sense of Definition 4.10.2, then the triangles for (G, h) and for (G', h') together with base change form a commutative prism as in Theorem 4.10.8. Each face has a prescribed natural isomorphism which altogether are compatible.

4.12 Symplectic case

Example 4.12.1. Consider the PEL-datum $(\mathbb{Q}, *, V = \mathbb{Q}^{\oplus 2}, \langle\ ,\ \rangle, h)$ of Example 4.9.4 defining the Shimura datum $(\mathrm{GL}_2, \mathcal{H})$. In this case, the objects of $\mathrm{Rep}(G)^{\mathrm{AV}}$ are all isomorphic to $V^{\oplus n}$ for some n (this can be read of from the classification of [FH91, Sec. 15.5] for example). As a result, any morphism of Shimura data $f\colon (\mathrm{GL}_2, \mathcal{H}) \to (G, h)$, with (G, h) also having a fixed choice of PEL-datum, must be admissible in the sense of Definition 4.10.2.

In fact, this holds more generally.

Definition 4.12.2. We say that a PEL-datum $(B, *, V, \langle\ ,\ \rangle, h)$ is of *symplectic type* if $(B_{\mathbb{R}}, *)$ decomposes as a product of algebras with positive involution isomorphic to $(M_n(\mathbb{R}), * = (-)^t)$.

In this case $G_1(\mathbb{R})$ splits as a direct product of terms isomorphic to $\mathrm{Sp}_{2k}(\mathbb{R})$ (cf. the proof of Lemma 4.9.3).

Lemma 4.12.3. *Let $(G', \mathfrak{X}')$ be a Shimura datum with a choice of PEL-datum $(B', *', V', \langle\ ,\ \rangle', h')$ of symplectic type. Then for any Shimura datum (G, h) with a choice of PEL-datum, any map $f\colon (G', h') \to (G, h)$ is admissible, i.e. satisfies $(\star)$ of Definition 4.10.2.*

Proof. We shall show that every object of $\mathrm{Rep}(G')^{\mathrm{AV}}$ is a direct summand of $V'^{\oplus n}$ for some n. For this, it suffices to show the analogous statement after base change to $\mathbb{R}$. Let W be an $\mathbb{R}$-representation of $G'_{\mathbb{R}}$ of Hodge type $\{(-1, 0), (0, -1)\}$. First consider the restriction of W to $G'_{1,\mathbb{R}} \cong \prod_i \mathrm{Sp}_{2g_i}$. Then W splits as a direct sum of irreducibles on which $G'_{1,\mathbb{R}}$ acts via projection to the i^{th} factor for some i.

Let $U_1 \subset \mathbb{S}$ denote the kernel of the norm map. We may assume that, under $\mathrm{pr}_i \circ h'$, the image of U_1 within Sp_{2g_i} is as given in the usual Siegel datum (cf. Example 4.2.2). For any representation T of a Sp_{2g_i}, we say W has U_1-weights within some set of integers $\{a_1, ..., a_n\}$ if upon restriction to $\mathrm{pr}_i \circ h'(U_1)$, $W \otimes_{\mathbb{Q}} \mathbb{C}$ decomposes as a sum of one dimensional representations on which (with respect to a fixed isomorphism) $U_{1,\mathbb{C}} \cong \mathbb{G}_m$ acts by multiplication by a_i for some i.

Examining the classification of irreducible Sp_{2g}-modules (e.g. as given [FH91, Sec. 17.2]) it is easy to see that the only irreducible representation of Sp_{2g_i} with U_1-weights $\{1, -1\}$ is the standard representation of Sp_{2g_i}. As a result, all irreducible representations of $G'_{1,\mathbb{R}}$ are direct summands of the standard representation.

Since the action of scalar matrices on W is determined by its $\mathbb{G}_m$-weight (by which we mean weight in the traditional sense), the map $\mathrm{Rep}(G'_{\mathbb{R}})^{\mathrm{AV}} \to \mathrm{Rep}(G'_{1,\mathbb{R}})$ is faithful. In particular, there is at most one representation, up to isomorphism, of $G'_{\mathbb{R}}$ restricting to any representation of $G'_{1,\mathbb{R}}$. Since all irreducible representations of $G'_{1,\mathbb{R}}$ are summands of the standard representation and the standard representation of $G'_{\mathbb{R}}$ is one representation restricting to the standard representation of $G'_{1,\mathbb{R}}$, we must have the all irreducible objects of $\mathrm{Rep}(G'_{\mathbb{R}})^{\mathrm{AV}}$ are direct summands of $V'_{\mathbb{R}}$. $\square$

Remark 4.12.4. The proof also shows that, for any Shimura datum with a chosen PEL-datum of symplectic type, Anc_G extends μ_G^{mot} up to natural isomorphism (see Remark 4.9.11).

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