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# GEOMETRICALLY CONVERGENT SIMULATION OF THE EXTREMA OF LÉVY PROCESSES 

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#### Abstract

We develop a novel approximate simulation algorithm for the joint law of the position, the running supremum and the time of the supremum of a general Lévy process at an arbitrary finite time. We identify the law of the error in simple terms. We prove that the error decays geometrically in $L^{p}$ (for any $p \geq 1$ ) as a function of the computational cost, in contrast with the polynomial decay for the approximations available in the literature. We establish a central limit theorem and construct non-asymptotic and asymptotic confidence intervals for the corresponding Monte Carlo estimator. We prove that the multilevel Monte Carlo estimator has optimal computational complexity (i.e. of order $\epsilon^{-2}$ if the mean squared error is at most $\epsilon^{2}$ ) for locally Lipschitz and barrier-type functions of the triplet and develop an unbiased version of the estimator. We illustrate the performance of the algorithm with numerical examples.


## 1. Introduction

Consider a Lévy processes $X=\left(X_{t}\right)_{t \geq 0}$ over the time interval $[0, T]$ for a given constant time $T>$ 0 . The triplet $\chi=\left(X_{T}, \bar{X}_{T}, \tau_{T}\right)$, consisting of the position $X_{T}$, the supremum $\bar{X}_{T}$ of $X$ over the interval $[0, T]$ and the first time $\tau_{T}$ at which $X$ attains its supremum, plays a key role in numerous areas of applied probability (e.g. ruin probabilities in insurance mathematics [KKM04], barrier and lookback options and technical trading in mathematical finance [BL02, Mor02, MP12], buffer size in queuing theory [Asm03, MP15] and the prediction of the ultimate supremum and its time in optimal stopping [BDP11, BvS14], to name a few). However, the information about the law of $\bar{X}_{T}$ (let alone of $\chi$ ) is very difficult to extract from the characteristics of the Lévy process $X$ [Cha13]. Moreover, the known properties of the law of $\bar{X}_{T}$ are typically not explicit in the characteristics [CM16], making its exact simulation challenging (e.g. the first exact simulation algorithm for the supremum of a stable process was developed recently [GCMUB19]).

The central importance of $\chi$ in applied probability, combined with its intractability when $X$ is not compound Poisson with drift, has lead to an abundance of works on its approximation over the last quarter of the century [AGP95, BGK97, BGK99, DL11a, DL11b, Che11, DH11, Der11, KKPvS11, FCKSS14, GX17, Iva18, BI20]. These methodologies naturally yield Monte Carlo (MC) and Multilevel Monte Carlo (MLMC) algorithms for $\chi$. Without exception, the errors of these algorithms achieve polynomial decay in the computational cost. The following natural question arises: does there exist an algorithm whose error decays geometrically in the cost? The simple and general algorithm SB-Alg below answers this question affirmatively. Subsection 1.1 gives an intuitive introduction to the algorithm and describes its main properites, while Subsection 1.2 compares SB-Alg with the existing literature, cited

[^0]at the beginning of this paragraph. (See also the YouTube presentation [GCMUB21] for an overview of the paper.)
1.1. Contribution. The present paper has two main contributions: (I) a novel stick-breaking approximation (SBA) for $\chi$, given in Equation (1.2) and sampled by the algorithm SB-Alg below, and an explicit characterisation of the law of its error (see Theorem 1 below); (II) an analysis of the SBA as a Monte Carlo algorithm for functions of interest in applied probability. Contribution (II), described in Section 2 below, includes the geometric decay of the strong error and the central limit theorem for the MC estimator based on SB-Alg for various classes of functions of $\chi$ arising in applications (e.g. locally Lipschitz and barrier-type). Moreover, Section 2 develops the MLMC and unbiased extensions of the SBA, both of which have optimal computational complexity. In the present subsection we describe Contribution (I).

The SBA is based on the stick-breaking representation of the triplet $\chi$, derived from the description of the law of the concave majorant of a Lévy process given in [PUB12]. More precisely, the stick-breaking representation of $\chi$ states that the following a.s. equality holds

$$
\begin{equation*}
\chi=\left(X_{T}, \bar{X}_{T}, \tau_{T}\right)=\sum_{k=1}^{\infty}\left(Y_{L_{k-1}}-Y_{L_{k}},\left(Y_{L_{k-1}}-Y_{L_{k}}\right)^{+}, \ell_{k} \cdot \mathbb{1}_{\left\{Y_{L_{k-1}}-Y_{L_{k}}>0\right\}}\right), \tag{1.1}
\end{equation*}
$$

where the Lévy process $Y$ has the same law as $X$ and is independent of the stick-breaking process, $\ell=\left(\ell_{n}\right)_{n \in \mathbb{N}}$ on $[0, T]$, based on the uniform law $\mathrm{U}(0,1)$, i.e., $L_{0}=T, \ell_{n}=V_{n} L_{n-1}$ and $L_{n}=L_{n-1}-\ell_{n}$ for $n \in \mathbb{N}$ where $\left(V_{n}\right)_{n \in \mathbb{N}}$ is a $\mathrm{U}(0,1)$-iid sequence, see Figure 1.1. (In (1.1) and throughout the paper we denote $x^{+}=\max \{x, 0\}$ for any $x \in \mathbb{R}$.) The coupling ( $X, \ell, Y$ ) satisfying the almost sure equality in (1.1) is constructed in Subsection 4.1 below. Note that, in particular, it satisfies $Y_{T}=X_{T}$ a.s.


Figure 1.1. The figure illustrates the first $n=4$ sticks of a stick-breaking process. The increments of $Y$ in (1.1) are taken over the intervals $\left[L_{k}, L_{k-1}\right]$ of length $\ell_{k}$. Crucially, the time $L_{n}$ featuring in the vector $\left(Y_{L_{n}}, \bar{Y}_{L_{n}}, \bar{\tau}_{L_{n}}(Y)\right)$ in (1.4) of Theorem 1 is exponentially small in $n$ and independent of $Y$.

Given the representation in (1.1), the SBA is defined as follows:

$$
\begin{align*}
\chi_{n}= & \sum_{k=1}^{n}\left(Y_{L_{k-1}}-Y_{L_{k}},\left(Y_{L_{k-1}}-Y_{L_{k}}\right)^{+}, \ell_{k} \cdot \mathbb{1}_{\left\{Y_{L_{k-1}}-Y_{L_{k}}>0\right\}}\right)  \tag{1.2}\\
& +\left(Y_{L_{n}}, Y_{L_{n}}^{+}, L_{n} \cdot \mathbb{1}_{\left\{Y_{L_{n}}>0\right\}}\right) .
\end{align*}
$$

Since the residual sum $\sum_{k=n+1}^{\infty}\left(Y_{L_{k-1}}-Y_{L_{k}}\right)$ equals $Y_{L_{n}}$ for any $n \in \mathbb{N}$, the first component of $\chi_{n}$ coincides with that of $\chi$, while, as we shall see in Theorem 1 below, $Y_{L_{n}}^{+}$and $L_{n} \cdot \mathbb{1}_{\left\{Y_{L_{n}}>0\right\}}$ reduce the errors of the corresponding partial sums in (1.2). The coupling ( $X, \ell, Y$ ) makes it possible to compare $\chi$ and $\chi_{n}$ on the same probability space and analyse the strong error $\chi-\chi_{n}$.

Denote the distribution of $X_{t}$ by $F(t, x)=\mathbb{P}\left(X_{t} \leq x\right)$, where $x \in \mathbb{R}$ and $t>0$. Then an algorithm that simulates exactly from the law of the SBA $\chi_{n}$ is given as follows:

```
SB-Alg
Require: \(n \in \mathbb{N}\), fixed time horizon \(T>0\)
    Set \(\Lambda_{0}=T, \mathcal{X}_{0}=(0,0,0)\)
    for \(k=1, \ldots, n\) do
        Sample \(v_{k} \sim \mathrm{U}(0,1)\) and put \(\lambda_{k}=v_{k} \Lambda_{k-1}\) and \(\Lambda_{k}=\Lambda_{k-1}-\lambda_{k}\)
        Sample \(\xi_{k} \sim F\left(\lambda_{k}, \cdot\right)\) and put \(\mathcal{X}_{k}=\mathcal{X}_{k-1}+\left(\xi_{k}, \xi_{k}^{+}, \lambda_{k} \cdot \mathbb{1}_{\left\{\xi_{k}>0\right\}}\right)\)
    end for
    Sample \(\varsigma_{n} \sim F\left(\Lambda_{n}, \cdot\right)\) and return \(\mathcal{X}_{n}+\left(\varsigma_{n}, \varsigma_{n}^{+}, \Lambda_{n} \cdot \mathbb{1}_{\left\{\varsigma_{n}>0\right\}}\right)\)
```

SB-Alg clearly outputs a random vector with the same law as $\chi_{n}$ in (1.2), using a total of $n+1$ sampling steps. Theorem 1 and Section 2 below show that $\chi_{n}$ in (1.2) is an increasingly accurate approximation of $\chi$ as $n$ grows. Intuitively this is because, by (4.1), the sum in the definition of $\chi_{n}$ consists of the first $n$ terms taken in a size-biased order (see Subsection 4.1 below) making the remainder very small. It will become clear from Theorem 1 that the last step in SB-Alg reduces the error further. The computational cost of the algorithm is proportional to $n$ if we can sample any increment of $X$ in constant time. We stress that SB-Alg is not a version of the random walk approximation (see Equation (2.2) below) on a randomised grid as it does not require the computation of either max or arg max of a discretisation of $X$. Instead, the approximation for the supremum and its time are obtained by summing non-negative numbers, making SB-Alg numerically very stable. The convergence analysis of SB-Alg relies on the following result, which describes explicitly the law of its error.

The following notation, needed to state Theorem 1, will be used throughout the paper: for a rightcontinuous function $f:[0, \infty) \rightarrow \mathbb{R}$ with left-hand limits, we denote by $\bar{f}_{t}=\sup \left\{f_{s}: s \in[0, t]\right\}$ its supremum over the interval $[0, t]$ and by $\tau_{t}(f)=\inf \left\{s \in[0, t]: \bar{f}_{s}=\bar{f}_{t}\right\}$ the first time the supremum $\bar{f}_{t}$ is attained.

Theorem 1. Assume the Lévy process $X$ is not compound Poisson with drift and let $(X, \ell, Y)$ be the coupling constructed in Subsection 4.1 below, satisfying (1.1). For any $n \in \mathbb{N}$, define the vector of errors of the SBA by

$$
\begin{align*}
\chi-\chi_{n} & =\left(0, \Delta_{n}^{\mathrm{SB}}, \delta_{n}^{\mathrm{SB}}\right)=\left(0, \Delta_{n}-Y_{L_{n}}^{+}, \delta_{n}-L_{n} \cdot \mathbb{1}_{\left\{Y_{L_{n}}>0\right\}}\right), \quad \text { where } \\
\Delta_{n} & =\bar{X}_{T}-\sum_{k=1}^{n}\left(Y_{L_{k-1}}-Y_{L_{k}}\right)^{+} \quad \text { and } \quad \delta_{n}=\tau_{T}-\sum_{k=1}^{n} \ell_{k} \cdot \mathbb{1}_{\left\{Y_{L_{k-1}}-Y_{L_{k}}>0\right\}} . \tag{1.3}
\end{align*}
$$

Then, conditionally on $L_{n}$,

$$
\begin{align*}
& \left(Y_{L_{n}}, \Delta_{n}, \delta_{n}\right) \stackrel{d}{=}\left(Y_{L_{n}}, \bar{Y}_{L_{n}}, \tau_{L_{n}}(Y)\right), \quad \text { and hence } \\
& \quad\left(\Delta_{n}^{\mathrm{SB}}, \delta_{n}^{\mathrm{SB}}\right) \stackrel{d}{=}\left(\bar{Y}_{L_{n}}-Y_{L_{n}}^{+}, \tau_{L_{n}}(Y)-L_{n} \cdot \mathbb{1}_{\left\{Y_{L_{n}}>0\right\}}\right) . \tag{1.4}
\end{align*}
$$

Moreover, the inequalities $0 \leq \Delta_{n+1}^{\mathrm{SB}} \leq \Delta_{n}^{\mathrm{SB}} \leq \Delta_{n}, 0 \leq \delta_{n} \leq L_{n}$ and $\left|\delta_{n}^{\mathrm{SB}}\right| \leq L_{n}$ hold a.s.
Non-asymptotic (i.e. for fixed $n$ ) explicit descriptions of the law of the error, such as (1.4) in Theorem 1, are not common among the simulation algorithms for the supremum and related functionals of the path. Since $L_{n}$ and $Y$ are independent, the representation in (1.4) is easy to work with and provides a cornerstone for the results of Section 2. Note that, by Theorem 1, the sequences $\left(\Delta_{n}^{\mathrm{SB}}\right)_{n \in \mathbb{N}}$, $\left(\Delta_{n}\right)_{n \in \mathbb{N}}$ and $\left(\delta_{n}\right)_{n \in \mathbb{N}}$ are nonincreasing almost surely and converge to 0 . Furthermore, the following
observations based on Theorem 1 motivate the final step in SB-Alg (i.e. the inclusion of the last summand in the definition in (1.2)): (I) the tail of the error $\Delta_{n}^{\mathrm{SB}}$ may be strictly lighter than that of $\Delta_{n}$ (as $\bar{X}_{t}-X_{t}^{+}=\min \left\{\bar{X}_{t}, \bar{X}_{t}-X_{t}\right\}$ and $\bar{X}_{t}-X_{t} \stackrel{d}{=} \sup _{s \in[0, t]}\left(-X_{s}\right)$ for all $t>0$ [Ber96, Prop. VI.3]); (II) for a large class of Lévy processes, $\delta_{n}^{\mathrm{SB}}$ is asymptotically centred at 0 , i.e. $\mathbb{E}\left[\delta_{n}^{\mathrm{SB}} / L_{n}\right] \rightarrow 0$ as $n \rightarrow \infty$, while $\mathbb{E}\left[\delta_{n} / L_{n}\right]$ converges to a strictly positive constant (see Proposition 4 below for details). Theorem 1 is proved in Subsection 4.2.

Since $\mathbb{E} L_{n}=T 2^{-n}$ and $L_{n}$ is independent of $Y$, the convergence of SB-Alg is geometric (see also Section 2). Indeed, the error $\left(\Delta_{n}^{\mathrm{SB}}, \delta_{n}^{\mathrm{SB}}\right)$, satisfies the following weak limit.

Corollary 1. If weak convergence $X_{t} / a(t) \xrightarrow{d} Z_{1}$ (as $t \searrow 0$ ) holds for some (necessarily) $\alpha$-stable process $Z$ and a function $a$, which is necessarily $1 / \alpha$-regularly varying at zero, then

$$
\begin{equation*}
\left(\frac{Y_{L_{n}}}{a\left(L_{n}\right)}, \frac{\Delta_{n}}{a\left(L_{n}\right)}, \frac{\Delta_{n}^{\mathrm{SB}}}{a\left(L_{n}\right)}, \frac{\delta_{n}}{L_{n}}, \frac{\delta_{n}^{\mathrm{SB}}}{L_{n}}\right) \xrightarrow{d}\left(Z_{1}, \bar{Z}_{1}, \bar{Z}_{1}-Z_{1}^{+}, \tau_{1}(Z), \tau_{1}(Z)-\mathbb{1}_{\left\{Z_{1}>0\right\}}\right) \text { as } n \rightarrow \infty . \tag{1.5}
\end{equation*}
$$

The assumption in Corollary 1 essentially amounts to both tails of the Lévy measure of $X$ being regularly varying at zero with index $-1 / \alpha$ (see [Iva18, Thm 2]). This is a rather weak requirement, typically satisfied by Lévy based models in applied probability, which allows an arbitrary modification of the Lévy measure away from zero (see discussion in [Iva18, Sec. 4]). Moreover, the index $\alpha$ is given by (4.21) and the function $a(t)$ is typically of the form $a(t) \sim C_{0} t^{1 / \alpha}$ for some constant $C_{0}>0$. The scaling in the limit (1.5) is stochastic; however, since $\mathbb{E} L_{n}=T 2^{-n}$, the rate of decay of the error is clearly geometric. Corollary 1 is proved in Subsection 4.2 by applying Theorem 1 to the small-time weak limit of $X$.
1.2. Connections with existing literature. In the present subsection we discuss briefly the literature on the approximations of $\chi$ and compare it with SB-Alg.

The random walk approximation (RWA) (defined in (2.2) below) is based on $\left(X_{k T / n}\right)_{k \in\{1, \ldots, n\}}$, the skeleton of the Lévy process $X$. It is a widely used method for approximating $\chi$ with computational cost proportional to the discretisation parameter $n$. In the case of Brownian motion, the asymptotic law of the error was studied in [AGP95]. The papers [BGK97, BGK99] (resp. [DL11a, DL11b]) identified the dominant error term of the RWA for barrier and lookback options under the exponential Lévy models when $X$ is a Brownian motion with drift (resp. jump diffusion). Based on Spitzer's identity, [Che11] developed bounds on the decay of the error in $L^{1}$ for general Lévy processes, extending the results of [DL11a]. Ideas from [Iva18] were employed in [BI20] to obtain sharper bounds on the convergence of the error of the RWA in $L^{p}$ for general Lévy processes and any $p>0$. Such results are useful for the analysis of MC and MLMC schemes based on the RWA, see [GX17] for the case of certain parametric Lévy models. We will describe in more detail these contributions in Section 2 as we contrast them with the analogous results for SB-Alg.

Exploiting the the Wiener-Hopf factorisation, [KKPvS11] introduced the Wiener-Hopf approximation (WHA) of $\left(X_{T}, \bar{X}_{T}\right)$. This approximation is given by $\left(X_{G_{n}}, \bar{X}_{G_{n}}\right)$, where $G_{n}$ is the sum of $n$ independent exponential random variables with mean $T / n$, so that $\mathbb{E} G_{n}=T$ with variance $T^{2} / n$. Implementing the WHA requires the ability to sample the supremum at an independent exponential time, which is only done approximately for a specific parametric class of Lévy processes with exponential moments and arbitrary path variation $[\mathrm{KKPvS11}]$. The computational cost of the WHA is proportional to $n$. The decay of the bias and the MLMC version of the WHA were later studied in [FCKSS14]. As observed
in [GX17, Sec. 1], the WHA currently cannot be directly applied to various parametric models used in practice which possess increments that can be simulated exactly (e.g. the variance gamma process).

The jump-adapted Gaussian approximation (JAGA) was introduced in [Der11, DH11] to approximate Lipschitz functions in the supremum norm of Lévy-driven stochastic differential equations with Lipschitz coefficients. The algorithm is based on an approximation of the skeleton $\left\{X_{t_{k}}\right\}_{k=1}^{n}$ where the time grid includes the times of the jumps of $X$ whose magnitude is larger than some cutoff level $\kappa$ and the smalljump component of $X$ is approximated by an additional Brownian motion. Typically, the cost and bias of the JAGA are proportional to $n+\kappa^{-\beta}$ and $\left(n^{-1 / 2}+n^{1 / 4} \kappa\right) \sqrt{\log n}$, respectively, where $\beta$ is the Blumenthal-Getoor index, see (4.5). The complexity of the MLMC version of the JAGA for Lipschitz functions of $\left(X_{T}, \bar{X}_{T}\right)$ is compared with that of SB-Alg in Subsection 2.4 below.

In contrast with Theorem 1 for the SBA, the laws of the errors of all the other algorithms discussed in the present subsection are intractable. The error of the SBA $\chi_{n}$ in (1.2) decays geometrically in $L^{p}$ (see Theorem 2 below) as opposed to the polynomial decay for the other algorithms (see Subsection 2.1.1 below). The error in $L^{p}$ of the SBA applied to locally Lipschitz and barrier-type functions arising in applications also decays geometrically (see Propositions $2 \& 3$ below). To the best of our knowledge, such errors have not been analysed for algorithms other than the RWA, which has polynomial decay (see Subsection 2.2.1 for details). The rate of the decay of the bias is directly linked to the computational complexity of MC and MLMC estimates. Indeed, if the mean squared error is to be at most $\epsilon>0$, the MC algorithm based on the SBA has (near optimal) complexity of order $\mathcal{O}\left(\epsilon^{-2} \log \epsilon\right)$ (see Appendix A. 1 below for the definition of $\mathcal{O}$ ). The MLMC scheme based on SB-Alg has (optimal) complexity of order $\mathcal{O}\left(\epsilon^{-2}\right)$, which is in general neither the case for the RWA [GX17] nor the WHA [FCKSS14] (see details in Subsection 2.4.1).
1.3. Organisation. The remainder of this paper is organised as follows. We develop the theory for the SBA as a Monte Carlo algorithm in Section 2. Each result is compared with its analogue (if it exists) for the algorithms discussed in Subsection 1.2 above. In Section 3 we provide numerical examples illustrating the performance of SB-Alg. The proofs of the results in Sections 1 and 2 are presented in Section 4.

## 2. SBA Monte Carlo: theory and applications

The present section describes the geometric convergence of SB-Alg and analyses the Monte Carlo estimation of the functions of interest in applied probability. In Subsection 2.1 we establish the geometric decay of the error in $L^{p}$. In Subsection 2.2 we show that the error in $L^{p}$ (and hence the bias) of SB-Alg applied to the aforementioned functions also decays geometrically. In Subsection 2.3 we study the error of the MC estimator based on SB-Alg for the expected value of those functions via a central limit theorem and provide the corresponding asymptotic and non-asymptotic confidence intervals. Subsection 2.4 gives the computational complexity of the MC and MLMC estimators based on SB-Alg. Subsection 2.5 describes the unbiased estimator.
2.1. Geometric decay in $L^{p}$ of the error of the $\mathbf{S B A}$. In the present subsection we study the decay in $L^{p}$ of the error $\left(\Delta_{n}^{\mathrm{SB}}, \delta_{n}^{\mathrm{SB}}\right)$ of the SBA $\chi_{n}$ given in (1.3). Let $\left(\sigma^{2}, \nu, b\right)$ be the generating triplet of $X$ associated with the cutoff function $x \mapsto \mathbb{1}_{\{|x|<1\}}$ (see [Sat13, Ch. 2, Def. 8.2]). The existence of the moments of $X_{T}$ and $\bar{X}_{T}$, necessary for the following result, can be characterised [Sat13, Thm 25.3] in
terms of the integrals

$$
\begin{equation*}
I_{+}^{p}=\int_{[1, \infty)} x^{p} \nu(d x), \quad I_{-}^{p}=\int_{(-\infty,-1]}|x|^{p} \nu(d x), \quad p \geq 0 . \tag{2.1}
\end{equation*}
$$

Throughout we use the standard $\mathcal{O}$ notation, see Appendix A. 1 below for definition.
Theorem 2. Under the assumptions of Theorem 1, the following holds for any $p \geq 1$.
(a) The inequality $\max \left\{\mathbb{E}\left[\left|\delta_{n}^{S B}\right|^{p}\right], \mathbb{E}\left[\delta_{n}^{p}\right]\right\} \leq T^{p}(1+p)^{-n}$ holds for any $n \in \mathbb{N}$.
(b) If $\min \left\{I_{+}^{p}, I_{-}^{p}\right\}<\infty\left(\right.$ resp. $\left.I_{+}^{p}<\infty\right)$, then $\mathbb{E}\left[\left(\Delta_{n}^{\mathrm{SB}}\right)^{p}\right]$ (resp. $\left.\mathbb{E}\left[\Delta_{n}^{p}\right]\right)$ is bounded above by $\mathcal{O}\left(\eta_{p}^{-n}\right)$ as $n \rightarrow \infty$, where $\eta_{p}$ lies in the interval $[3 / 2,2]$ for any Lévy process $X$. Both $\eta_{p}$, defined in (4.22), and the constants in $\mathcal{O}\left(\eta_{p}^{-n}\right)$ are explicit in the characteristics $\left(\sigma^{2}, \nu, b\right)$ of $X$ (see (4.24)).

By Theorem 1, the error $\Delta_{n}^{\mathrm{SB}}$ is bounded above by the supremum of the Lévy process over the stochastic interval $\left[0, L_{n}\right]$ with average length equal to $\mathbb{E} L_{n}=T 2^{-n}$. The key step in the proof of Theorem 2, given in Lemma 2 below, consists of controlling the expectation of the supremum of $X$ over short time intervals (see Subsection 4.3 below for details).

Since $\eta_{2}=2$ (see definition in (4.22)), an application of Theorem 2(b) for $p \in\{1,2\}$ yields $\mathbb{E} \Delta_{n}^{\mathrm{SB}}=$ $\mathcal{O}\left((3 / 2)^{-n}\right)$ and $\mathbb{E}\left[\left(\Delta_{n}^{\mathrm{SB}}\right)^{2}\right]=\mathcal{O}\left(2^{-n}\right)$. These two moments are used in the analysis of the MLMC estimator based on SB-Alg (see Subsection 2.4 below). A further application of Theorem 2 yields a geometric bound on the $L^{p}$-Wasserstein distance $\mathcal{W}_{p}\left(\mathcal{L}(\chi), \mathcal{L}\left(\chi_{n}\right)\right)$ between the laws $\mathcal{L}(\chi)$ and $\mathcal{L}\left(\chi_{n}\right)$ of the corresponding random vectors (see (4.25) below for the definition of the Wasserstein distance and Subsection 4.3 for the proof of Corollary 2).

Corollary 2. Assume $\min \left\{I_{+}^{p}, I_{-}^{p}\right\}<\infty$ for some $p \geq 1$. Under the assumptions of Theorem 1 we have $\mathcal{W}_{p}\left(\mathcal{L}(\chi), \mathcal{L}\left(\chi_{n}\right)\right)=\mathcal{O}\left(\eta_{p}^{-n / p}\right)$ as $n \rightarrow \infty$. As in Theorem 2(b) above, $\eta_{p}$ lies in the interval $[3 / 2,2]$ and the constant in $\mathcal{O}\left(\eta_{p}^{-n / p}\right)$, given in Equation (4.26), is explicit.
2.1.1. Comparison. The algorithm based on the RWA with time-step $T / n$ outputs

$$
\begin{equation*}
\left(X_{T}, \max _{k \in\{1, \ldots, n\}} X_{k T / n}, \frac{T}{n} \arg \max _{k \in\{1, \ldots, n\}} X_{k T / n}\right) . \tag{2.2}
\end{equation*}
$$

The $L^{1}$ bounds on the error $\Delta_{n}^{\mathrm{RW}}=\bar{X}_{T}-\max _{k \in\{1, \ldots, n\}} X_{k T / n}$ have a long history. Using the weak limit of the error of the RWA, the $L^{1}$ bound $\mathbb{E} \Delta_{n}^{\mathrm{RW}}=\mathcal{O}\left(n^{-1 / 2}\right)$ is established for the Brownian motion with drift in [AGP95, BGK99]. The same bound holds when the jumps of $X$ have finite activity (i.e. $\nu(\mathbb{R})<\infty$ and $\sigma \neq 0$ ) [DL11a]. The approach of [DL11a], based on Spitzer's identity, was extended in [Che11, Thm 5.2.1] to the case without a Brownian component. If $X$ has paths of finite variation, these bounds were further improved via a different methodology in [BI20]. In particular, by [BI20, Thm 4.1], we have: $\mathbb{E} \Delta_{n}^{\mathrm{RW}}=\mathcal{O}\left(n^{-1 / 2}\right)$ if $X$ has a Brownian component (i.e. $\sigma \neq 0$ ), $\mathbb{E} \Delta_{n}^{\mathrm{RW}}=\mathcal{O}\left(n^{-1}\right)$ if $X$ has paths of finite variation (i.e. $\int_{(-1,1)}|x| \nu(d x)<\infty$ and $\sigma=0$ ) and $\mathbb{E} \Delta_{n}^{\mathrm{RW}}=\mathcal{O}\left(n^{\delta-1 / \beta}\right)$ otherwise, for any small $\delta>0$ and $\beta \in[1,2]$ defined in (4.5) below.

Bounds for $\mathbb{E}\left[\left(\Delta_{n}^{\mathrm{RW}}\right)^{p}\right], p>0$, analysed in [DL11a, BI20], are as follows. By [BI20, Thm 4.1], for $\alpha \in[0,2]$ given in (4.21) below, the decay is $\mathcal{O}\left(n^{-1}\right)$ for $p>\alpha$ and $\mathcal{O}\left(n^{\delta-p / \alpha}\right)$ for $0<p \leq \alpha$ and any small $\delta>0$ (we may take $\delta=0$ if either $\alpha=1$ and $X$ is of finite variation or $\alpha=2$ ). If $X$ is spectrally negative (i.e. $\nu((0, \infty))=0$ ) and has jumps of finite variation (i.e. $\left.\int_{(-1,0)}|x| \nu(d x)<\infty\right)$, then for $p>1$ the decay is of order $\mathcal{O}\left(n^{-p}\right)\left(\right.$ resp. $\left.\mathcal{O}\left(n^{-p / 2} \log (n)^{p}\right)\right)$ if $\sigma=0($ resp. $\sigma \neq 0)$ [DL11a, Lem. 6.5].

Interestingly, as noted in [BI20, Rem. 4.4], if $X$ has jumps of both signs, then for any $p>0$, the error of the RWA satisfies $\lim \inf _{n \rightarrow \infty} n \mathbb{E}\left[\left(\Delta_{n}^{\mathrm{RW}}\right)^{p}\right]>0$. Put differently, the error cannot be of order $o\left(n^{-1}\right)$ (see Appendix A. 1 below for the definition of $o$ ).

Intuitively, the error committed by the RWA is due to the skeleton missing the fluctuations of the process over the interval of length $1 / n$ where the process attained its supremum. Since these fluctuations can be substantial in the presence of high jump activity and heavy tails, the decay of the resulting error is polynomial in $n$. In contrast, the error of the SBA is by Theorem $2(\mathrm{~b})$ bounded by $\mathcal{O}\left(\eta_{p}^{-n}\right)$ with $\eta_{p} \in[3 / 2,2]$, as it commits the same error as the RWA but over the interval $\left[0, L_{n}\right]$ with average length of $T / 2^{n}$. Numerical results show that the biases of the RWA and the SBA over $2^{n}$ and $n$ steps, respectively, are comparable (Figure 3.1 below).

Recall that the WHA, applicable to a specific parametric class of Lévy processes [KKPvS11], is given by $\left(X_{G_{n}}, \bar{X}_{G_{n}}\right)$, where $G_{n}$ is an independent gamma random variable with mean $\mathbb{E} G_{n}=T$ and variance $T^{2} / n$. Since $\bar{X}_{s+t}-\bar{X}_{s}$ is stochastically dominated by $\bar{X}_{t}$ and $X_{t+s}-X_{s} \stackrel{d}{=} X_{t}$, the $L^{p}$ norm of the error is linked to both, the small time behaviour of $t \mapsto\left(X_{t}, \bar{X}_{t}\right)$ and the deviations of $G_{n}$ from $T$. Therefore, the moments of the errors depend on those of $\left|G_{n}-T\right|$ and satisfy $\mathbb{E}\left[\left|X_{T}-X_{G_{n}}\right|^{p}\right]=\mathcal{O}\left(n^{-1 / q}\right)$ and $\mathbb{E}\left[\left|\bar{X}_{T}-\bar{X}_{G_{n}}\right|^{p}\right]=\mathcal{O}\left(n^{-1 / q}\right)$ for $p \in\{1,2\}$, where $q=4$ if $p=1$ and $X$ is of infinite variation and $q=2$ otherwise [FCKSS14, Prop. 4.5]. These bounds are based on a martingale decomposition of the Lévy process $X$ (see [FCKSS14, Lem. 4.4]), while the analogous results in our paper use the Lévy-Itô decomposition, see Lemma 2 below.

Intuitively, the error in the WHA is due to the censored fluctuations of $X$ over a stochastic interval of length $\left|G_{n}-T\right|$. This is analogous to the error of the SBA over a stochastic interval of length $L_{n}$. However, since $\mathbb{E}\left[\left|G_{n}-T\right|\right]$ is asymptotically equal to $T \sqrt{2 /(n \pi)}$ (by the central limit theorem and [Bil99, Thm 5.4]) and $\mathbb{E}\left[L_{n}\right]=T 2^{-n}$, the speed of convergence is polynomial in the WHA and geometric in the SBA.

The first two moments of the error of the JAGA with cost $n$ were analysed in [DH11, Der11], resulting in the bound $\mathcal{O}\left(n^{-\min \left\{1,1 / \beta_{+}\right\}}+n^{1 / 4-1 / \beta_{+}} \sqrt{\log n}\right)$ if $X$ has no Brownian component (i.e. $\sigma=0$ ) and $\mathcal{O}\left(n^{1 / 4-\min \left\{3 / 4,1 / \beta_{+}\right\}} \sqrt{\log n}\right)$ otherwise, where $\beta_{+}$, given in (4.6), is slightly larger than the BlumenthalGetoor index $\beta \in[0,2]$ in (4.5). Intuitively, this error is the result of missing the fluctuations of $X$ between consecutive points on the random grid and the error incurred from approximating the smalljump component with an additional Brownian motion.
2.2. SBA for certain functions of $\chi$ : geometric decay of the strong error. Throughout the paper we consider a measurable function $g: \mathbb{R} \times \mathbb{R}_{+} \times[0, T] \rightarrow \mathbb{R}$ satisfying $\mathbb{E}|g(\chi)|<\infty$, where $\mathbb{R}_{+}=[0, \infty)$. We focus our attention on the classes of functions that arise in application areas such as financial mathematics [Sch03, CT04], risk theory [SC10, AA10] and insurance [CMDS ${ }^{+}$13]. More specifically, we study the following three classes of functions: (I) Lipschitz in Proposition 1, (II) locally Lipschitz in Proposition 2 and (III) barrier-type in Proposition 3. These results are a consequence of the representation of the law of the error in Theorem 1, bounds from Theorem 2 and a tail estimate (without integrability assumptions) for the error $\Delta_{n}$ in Lemma 4.

Lipschitz functions arise in applications, for example, in the pricing of hindsight [BGK99, SS03, DL11a, GX17] and perpetual American [Mor02] puts under exponential Lévy models. Indeed, for fixed $S_{0}, K_{0}>0$, these two examples require computing the expectations of $\left(K_{0}-S_{0} e^{X_{T}-\bar{X}_{T}}\right)^{+}$and $e^{X_{T}-\bar{X}_{T}}$,
both of which are bounded and Lipschitz in $\left(X_{T}, \bar{X}_{T}\right)$ since $\bar{X}_{T} \geq X_{T}$. The next result, proved in Subsection 4.4 below, shows that the convergence of SB-Alg is also geometric for these functions.

Proposition 1. Assume $\left|g(x, y, t)-g\left(x, y^{\prime}, t^{\prime}\right)\right| \leq K\left(\left|y-y^{\prime}\right|+\left|t-t^{\prime}\right|\right)$ for some $K>0$ and all $x \in \mathbb{R}$, $y, y^{\prime} \in \mathbb{R}_{+}, t, t^{\prime} \in[0, T]$. Suppose $p \geq 1$ satisfies $\min \left\{\|g\|_{\infty}, I_{+}^{p}, I_{-}^{p}\right\}<\infty$, where $\|g\|_{\infty}=\sup \{|g(x, y, t)|:$ $\left.(x, y, t) \in \mathbb{R} \times \mathbb{R}_{+} \times[0, T]\right\}$, and let $\eta_{p} \in[3 / 2,2]$ be as in (4.22). Then, under the assumptions of Theorem 1, we have

$$
\mathbb{E}\left[\left|g(\chi)-g\left(\chi_{n}\right)\right|^{p}\right]=\mathcal{O}\left(\eta_{p}^{-n}\right) \quad \text { as } n \rightarrow \infty .
$$

Moreover, the constant in $\mathcal{O}\left(\eta_{p}^{-n}\right)$, given in Equation (4.29) below, is explicit in $K,\|g\|_{\infty}$ and the characteristics $\left(\sigma^{2}, \nu, b\right)$ of the Lévy process $X$.

The pricing of lookback puts, hindsight calls [BGK99, DL11a, GX17] and perpetual American calls [Mor02] involve expectations of continuous functions of $\chi$, such as $\left(S_{0} e^{\bar{X}_{T}}-K_{0}\right)^{+}$and $e^{\bar{X}_{T}}$, which are only locally Lipschitz. By Proposition 2, under appropriate assumptions on large positive jumps, the error of SB-Alg decays geometrically for such functions.

Proposition 2. Assume that $\left|g(x, y, t)-g\left(x, y^{\prime}, t^{\prime}\right)\right| \leq K\left(\left|y-y^{\prime}\right|+\left|t-t^{\prime}\right|\right) e^{\lambda \max \left\{y, y^{\prime}\right\}}$ for some $K, \lambda>0$ and all $\left(x, y, y^{\prime}, t, t^{\prime}\right) \in \mathbb{R} \times \mathbb{R}_{+}^{2} \times[0, T]^{2}$. Let $p \geq 1$ and $q>1$ satisfy $\int_{[1, \infty)} e^{\lambda p q x} \nu(d x)<\infty$ and let $\eta_{p q^{\prime}} \in[3 / 2,2]$ be as in (4.22), where $q^{\prime}=(1-1 / q)^{-1}$. Then, under the assumptions of Theorem 1 ,

$$
\mathbb{E}\left[\left|g(\chi)-g\left(\chi_{n}\right)\right|^{p}\right]=\mathcal{O}\left(\eta_{p q^{\prime}}^{-n / q^{\prime}}\right) \quad \text { as } n \rightarrow \infty
$$

Moreover, the constant in $\mathcal{O}\left(\eta_{p q^{\prime}}^{-n / q^{\prime}}\right)$, given in Equation (4.32) below, is explicit in $p, q, K, \lambda$ and the characteristics $\left(\sigma^{2}, \nu, b\right)$ of the Lévy process $X$.

In order to obtain the smallest value $\eta_{p q^{\prime}}^{-1 / q^{\prime}}$ in Proposition 2, one needs to take the largest possible $q$ allowed by the assumptions (see Remark 5 below for details). Hence, the rate of decay is determined by the exponential moments of the Lévy measure $\left.\nu\right|_{[1, \infty)}$. In the context of financial mathematics, it is natural to assume that the returns in the exponential Lévy model have finite variance, i.e. $\mathbb{E} e^{2 X_{t}}<\infty$. This is equivalent to $\int_{[1, \infty)} e^{2 x} \nu(d x)<\infty$ [Sat13, Thm 25.3], implying for example $q=2$ (for $\lambda=1$ and $p=1$ ) with the bound $\mathcal{O}\left(2^{-n / 2}\right)$. The proof of Proposition 2 is in Subsection 4.4. A numerical example is in Subsection 3.1.

Barrier-type functions of $\chi$, which are discontinuous in the trajectory of the Lévy process, arise in the pricing of contingent convertibles [CMDS $\left.{ }^{+} 13\right]$, the evaluation of ruin probabilities [KKM04] and as payoffs of barrier options [BGK97, BGK99, SS03]. By Theorem 1, the error $\Delta_{n}^{\mathrm{SB}}$ in (1.3) of the second coordinate $\bar{X}_{T}-\Delta_{n}^{\mathrm{SB}}$ of the SBA $\chi_{n}$ satisfies $0 \leq \Delta_{n}^{\mathrm{SB}} \searrow 0$ a.s. as $n \rightarrow \infty$. Hence, the limit $\mathbb{P}\left(\bar{X}_{T}-\Delta_{n}^{\mathrm{SB}} \leq x\right) \searrow \mathbb{P}\left(\bar{X}_{T} \leq x\right)$ as $n \rightarrow \infty$ holds for any fixed $x>0$. The rate of convergence in this limit is both crucial for the control of the bias of barrier-type functions and intimately linked to the quality of the right-continuity of the distribution function $x \mapsto \mathbb{P}\left(\bar{X}_{T} \leq x\right)$ of $\bar{X}_{T}$. We will thus need the following assumption.

Assumption 1. Given $M, K, \gamma>0$, the inequality $\mathbb{P}\left(\bar{X}_{T} \leq M+x\right)-\mathbb{P}\left(\bar{X}_{T} \leq M\right) \leq K x^{\gamma}$ holds for all $x \geq 0$.

Proposition 3. Define $g(\chi)=h\left(X_{T}\right) \mathbb{1}_{\left\{\bar{X}_{T} \leq M\right\}}$, where $h: \mathbb{R} \rightarrow \mathbb{R}$ is bounded and measurable and $M>0$. Let Assumption 1 hold for $M$ and some $K, \gamma>0$. Fix any $p, q \geq 1$ and let $\eta_{q} \in[3 / 2,2]$ be as in (4.22). Then, under the assumptions of Theorem 1, we have

$$
\mathbb{E}\left[\left|g(\chi)-g\left(\chi_{n}\right)\right|^{p}\right]=\mathcal{O}\left(\eta_{q}^{-n \gamma /(\gamma+q)}\right), \quad \text { as } n \rightarrow \infty
$$

Moreover, the constant in $\mathcal{O}\left(\eta_{q}^{-n \gamma /(\gamma+q)}\right)$, given in Equation (4.33) below, is explicit in $K, \gamma, p, q,\|h\|_{\infty}$ and the characteristics $\left(\sigma^{2}, \nu, b\right)$ of the Lévy process $X$.

The proof of Proposition 3 is in Subsection 4.4 below. Minimising $\eta_{q}^{-\gamma /(\gamma+q)}$ as a function of $q$ is not trivial (see Remark 6 below for the optimal choice of $q$ ). In the special case when $\gamma=1$ (i.e. the distribution function of $\bar{X}_{T}$ is Lipschitz from the right at $M$ ) we have: (a) if $X$ has paths of finite variation, then $\eta_{1}=2$ and the optimal choice $q=1$ gives the bound $\mathcal{O}\left(2^{-n / 2}\right) ;(\mathrm{b})$ if $\sigma \neq 0$, then the optimal choice $q=2$ yields the bound $\mathcal{O}\left(2^{-n / 3}\right)$.

The rate of decay in Proposition 3 is essentially controlled by the rate of convergence in the Kolmogorov distance of $\bar{X}_{T}-\Delta_{n}^{\mathrm{SB}}$ to $\bar{X}_{T}$. In general, as mentioned above, $\bar{X}_{T}-\Delta_{n}^{\mathrm{SB}}$ is known to converge to $\bar{X}_{T}$ weakly. As the Kolmogorov distance does not metrise the topology of weak convergence (cf. [Pet95, Ex. 1.8.32, p.43]), we require an additional assumption, such as 1 , to obtain a rate in Proposition 3.

Assumption 1 holds for a wide class of Lévy processes. By the Lebesgue differentiation theorem [Coh13, Thm 6.3.3], the function $x \mapsto \mathbb{P}\left(\bar{X}_{T} \leq x\right)$ is differentiable a.e. and Assumption 1 holds for $\gamma=1$ and Lebesgue almost every $M$. If the density of $\bar{X}_{T}$ exists and is bounded around $M$, then $x \mapsto \mathbb{P}\left(\bar{X}_{T} \leq x\right)$ is locally Lipschitz at $M$, again satisfying Assumption 1 with $\gamma=1$. This is the case if the density of $\bar{X}_{T}$ is continuous at $M$, which holds for stable processes or if $\sigma \neq 0$ [CM16], and, more generally, if $X$ converges weakly under the zooming-in procedure and $\alpha>1$ in (4.21), see [BI20, Lem. 5.7]. Moreover, by [CM16, Prop. 2] and [Ber96, Sec. VI.4, Thm 19], the density of $\bar{X}_{T}$ is continuous at $M$ if the ascending ladder height process of $X$ has positive drift (e.g. if $X$ is spectrally negative of infinite variation) or if $X$ is in a certain class of subordinated Brownian motions [KMR13, Prop. 4.5]. However, the continuity of the density of $\bar{X}_{T}$ is known to fail if $X$ is of bounded variation with no negative jumps and has a Lévy measure with atoms [KKR12, Lem. 2.4]. Furthermore, for any $\gamma \in(0,1)$, the function $x \mapsto \mathbb{P}\left(\bar{X}_{T} \leq x\right)$ may be continuous at $M$ but not locally $\gamma$-Hölder continuous (see example in Appendix B below) even if the Lévy measure has no atoms, demonstrating again the necessity of an condition such as Assumption 1 in Proposition 3.

We stress that, even if the density is locally bounded at $M$, it appears to be very difficult to give bounds (based on the Lévy characteristics) on the value it takes at $M$. This means that, unlike in the case of a (locally)-Lipschitz function $g(\chi)$, in the context of barrier options we cannot provide non-asymptotic confidence intervals based on Proposition 3, cf. Subsection 2.3 below.
2.2.1. Comparison. The results in [DH11, Der11, DL11a, FCKSS14, BI20], discussed in Subsection 2.1.1 above, yield bounds in $L^{p}$ on the error of a Lipschitz function of $\left(X_{T}, \bar{X}_{T}\right)$. The orders of decay are the same as those reported in Subsection 2.1.1 above for the respective approximations. The error of the time of the supremum $\tau_{T}$, geometrically convergent for the SBA by Theorem 2(a) and Proposition 1, appears not to have been studied for the other algorithms.

In the case of locally Lipschitz functions, only the decay of the error in $L^{1}$ for the RWA seems to have been analysed. Define for any $q>0$ the integral

$$
\begin{equation*}
E_{+}^{q}=\int_{[1, \infty)} e^{q x} \nu(d x) \tag{2.3}
\end{equation*}
$$

If $X$ has finite activity (i.e. $\nu(\mathbb{R})<\infty$ ), then the bias equals $\mathcal{O}\left(n^{-1 / 2}\right)$ if $\sigma \neq 0$ and $E_{+}^{q}<\infty$ for some $q>2$ [DL11a, Prop. 5.1] and $o\left(n^{-(q-1) / q}\right)$ if $\sigma=0$ and $E_{+}^{q}<\infty$ for some $q>1$ [DL11a, Rem. 5.3]. In the case $\sigma=0$ and $\nu(\mathbb{R})=\infty$, for any $q>1$ satisfying $E_{+}^{q}<\infty$ and any arbitrarily small $\delta>0$, the bias decays as follows: $\mathcal{O}\left((n / \log (n))^{\delta-(q-1) / q}\right)$ if the process is of finite variation (i.e. $\int_{(-1,1)}|x| \nu(d x)<$ $\infty), \mathcal{O}\left(n^{\delta-(q-1) / q}\right)$ if $\int_{(-1,1)}|x| \log |x| \nu(d x)<\infty$ and $\mathcal{O}\left(n^{\delta-(q-1) /(2 q)}\right)$ otherwise [DL11a, Thm 6.2]. If the Lévy process $X$ is spectrally negative with jumps of finite variation (i.e. $\nu((0, \infty))=0$ and $\left.\int_{(-1,0)}|x| \nu(d x)<\infty\right)$ and if $E_{+}^{q}<\infty$ for some $q>1$, the error decays as $\mathcal{O}\left(n^{-1}\right)\left(\right.$ resp. $\left.\mathcal{O}\left(n^{-1 / 2} \log (n)\right)\right)$ if $\sigma=0$ (resp. $\sigma \neq 0$ ) [DL11a, Prop. 6.4].

The discontinuous payoffs under variance gamma (VG), normal inverse Gaussian (NIG) and spectrally negative $\alpha$-stable (with $\alpha>1$ ) processes are considered in [GX17]. Under the assumption that the density of the supremum is bounded around the barrier level in all three models, the errors in $L^{p}$ of the RWA decay as $\mathcal{O}\left(n^{\delta-1}\right), \mathcal{O}\left(n^{\delta-1 / 2}\right)$ and $\mathcal{O}\left(n^{\delta-1 / \alpha}\right)$ for arbitrarily small $\delta>0$, respectively [GX17, Prop. 5.5]. In the case $\nu(\mathbb{R})<\infty$ and $\sigma \neq 0$, the error decays as $\mathcal{O}(1 / \sqrt{n})$, see [DL11b, Prop. 2.2 \& Rem. 2.3]. This result was first established in [BGK97] for the Brownian motion with drift.

As noted in [BI20, Sec. 5.3], if $X$ has a jointly continuous density $(t, x) \mapsto \frac{\partial}{\partial x} \mathbb{P}\left(X_{t} \leq x\right)$ bounded for $(t, x)$ away from the origin $(0,0)$ (e.g. if Orey's condition holds for $\gamma>1$ [Sat13, Prop. 28.3] or $\sigma>0$, see also the paragraphs following Proposition 3 ), $\nu(\mathbb{R})=\infty$ and $\alpha \geq 1$ (defined in (4.21)), then the error in $L^{p}$ of the RWA for a barrier option decays as $\mathcal{O}\left(n^{\delta-1 / \alpha}\right)$ for any small $\delta>0$. Moreover, by [BI20, Lem. 5.8], $\lim \inf _{n \rightarrow \infty} n \mathbb{P}\left(\bar{X}_{T}>x \geq \max _{k \in\{1, \ldots, n\}} X_{k T / n}\right)>0$ if $X$ has jumps of both signs. Put differently, the error in $L^{p}$ of the RWA for a general barrier option cannot be of order $o\left(n^{-1}\right)$. As far as we know, such results for the WHA [KKPvS11] are currently unavailable.
2.3. The central limit theorem (CLT) and the confidence intervals (CIs). Let $\left(\chi_{n}^{i}\right)_{i \in\{1, \ldots, N\}}$ be the output produced by $N \in \mathbb{N}$ independent runs of SB-Alg using $n$ steps. The Monte Carlo estimator $\sum_{i=1}^{N} g\left(\chi_{n}^{i}\right) / N$ of $\mathbb{E} g(\chi)$, where $g: \mathbb{R} \times \mathbb{R}_{+} \times[0, T] \rightarrow \mathbb{R}$ is a measurable function of interest in applied probability (e.g. in one of the classes from Subsection 2.2 above), has an error

$$
\begin{equation*}
\Delta_{n, N}^{g}=\frac{1}{N} \sum_{i=1}^{N} g\left(\chi_{n}^{i}\right)-\mathbb{E} g(\chi) \tag{2.4}
\end{equation*}
$$

Our aim is to understand the rate of convergence of the error in (2.4) as the number of samples $N$ tends to infinity.

Theorem 3 (CLT). Assume $\mathbb{P}\left[\chi \in D_{g}\right]=0$, where $D_{g}$ is the discontinuity set of $g$, and
(a) there is a measurable function $G: \mathbb{R} \times \mathbb{R}_{+} \times[0, T] \rightarrow \mathbb{R}_{+}$such that:
(i) $|g(x, y, t)| \leq G(x, y, t)$ for all $(x, y, t) \in \mathbb{R} \times \mathbb{R}_{+} \times[0, T]$,
(ii) for all $x \in \mathbb{R},(y, t) \mapsto G(x, y, t)$ is nondecreasing in both coordinates,
(iii) $\mathbb{E}\left[G\left(X_{T}, \bar{X}_{T}, T\right)^{2}\right]<\infty$,
(b) $\mathbb{E} g(\chi)=\mathbb{E} g\left(\chi_{n}\right)+o\left(\eta_{g}^{-n}\right)$ for some $\eta_{g}>1$.

Denote $\mathbb{V}[g(\chi)]=\mathbb{E}\left[(g(\chi)-\mathbb{E}[g(\chi)])^{2}\right]$ and set $n_{N}=\left\lceil\log N / \log \left(\eta_{g}^{2}\right)\right\rceil$ for every $N \in \mathbb{N}$, where we denote $\lceil x\rceil=\inf \{n \in \mathbb{N}: n \geq x\}$ for $x \in \mathbb{R}$. Then the following weak convergence holds

$$
\begin{equation*}
\sqrt{N} \Delta_{n_{N}, N}^{g} \xrightarrow{d} N(0, \mathbb{V}[g(\chi)]), \quad \text { as } N \rightarrow \infty \tag{2.5}
\end{equation*}
$$

Theorem 3 is not an iid CLT since the bias of the MC estimator forces the increase in the number of steps taken by SB-Alg as the number of samples $N \rightarrow \infty$. Its proof (see Subsection 4.5 below) establishes Lindeberg's condition and then applies the CLT for triangular arrays. The condition $\mathbb{P}\left[\chi \in D_{g}\right]=0$ is satisfied if e.g. the Lebesgue measure of $D_{g}$ is zero and 0 is regular for $X$ for both half-lines [Cha13, Thm 3]. This assumption is important as it allows us to construct asymptotic confidence intervals for barrier options using the limit in (2.5). Assumption (a) ensures the convergence of $\mathbb{V}\left[g\left(\chi_{n}\right)\right]$ to $\mathbb{V}[g(\chi)]$ and might seem restrictive at first sight. However, the function $G$ is very easy to identify (see Remark 7 below) in the contexts of Propositions 1, 2 and 3, where Assumption (b) also clearly holds.

Since $\left|\Delta_{n, N}^{g}\right| \leq\left|\mathbb{E} g(\chi)-\mathbb{E} g\left(\chi_{n}\right)\right|+\left|\Delta_{n, N}^{g}-\mathbb{E} \Delta_{n, N}^{g}\right|$, we may construct a confidence interval for the MC estimator $\sum_{i=1}^{N} g\left(\chi_{n}^{i}\right) / N$ at level $1-\epsilon \in(0,1)$ using the implication:

$$
\left.\begin{array}{l}
\left|\mathbb{E} g(\chi)-\mathbb{E} g\left(\chi_{n}\right)\right|<r_{1},  \tag{2.6}\\
\mathbb{P}\left(\left|\Delta_{n, N}^{g}-\mathbb{E} \Delta_{n, N}^{g}\right|<r_{2}\right) \geq 1-\epsilon,
\end{array}\right\} \Longrightarrow \mathbb{P}\left(\left|\Delta_{n, N}^{g}\right|<r_{1}+r_{2}\right) \geq 1-\epsilon
$$

In (2.6), $r_{1}$ may be chosen as a function of the number $n$ of steps in SB-Alg in various ways depending on the properties of $g$ (see Propositions 1 and 2 of Subsection 2.2). Note that this requires the explicit dependence of the constant on the model characteristics.

Having fixed $n$, pick $r_{2}$ in (2.6) as a function of $\epsilon$ either via concentration inequalities (not relying on Theorem 3) or the CLT in Theorem 3:
(i) Non-asymptotic CI: by Chebyshev's inequality $\mathbb{P}\left(\left|\Delta_{n, N}^{g}-\mathbb{E} \Delta_{n, N}^{g}\right|>r\right) \leq \mathbb{V}\left[g\left(\chi_{n}\right)\right] /\left(r^{2} N\right)$, we only need to bound the variance $\mathbb{V}\left[g\left(\chi_{n}\right)\right]$ (e.g. by the function $G$ in Remark 7). See e.g. [Che08, Thm 1] for a sharper choice of $r_{2}$.
(ii) Asymptotic CI: since $\Delta_{n, N}^{g}-\mathbb{E} \Delta_{n, N}^{g}=(1 / N) \sum_{i=1}^{N} g\left(\chi_{n}^{i}\right)-\mathbb{E} g\left(\chi_{n}\right)$, we may use the CLT for fixed $n$ in Remark 8 below (as in (i) we bound $\mathbb{V}\left[g\left(\chi_{n}\right)\right]$ by elementary methods).

In the case we do not have access to the constants in the bound on the bias in (2.6) in terms of the model parameters (e.g. barrier options in Proposition 3), we apply the CLT result in Theorem 3 to the estimator $\Delta_{n_{N}, N}^{g}$ directly, to obtain an asymptotic CI. See Subsection 3.2 below for the numerical examples of asymptotic and non-asymptotic CIs.
2.4. Computational complexity of SB-Alg and the multilevel Monte Carlo. Assume that the expected computational cost of drawing a sample from the distribution $F(t, \cdot)$ in $\mathrm{SB}-\mathrm{Alg}$ is bounded above by a constant that does not depend on $t \in[0, T]$. Then the expected computational cost of a single draw from the law of $\chi_{n}$ via SB-Alg is bounded by $\mathcal{O}(n)$. The CLT in Theorem 3 (applicable to (locally) Lipschitz and barrier-type functions, cf. Subsection 2.3 above) implies that the $L^{2}$-norm of the error in (2.4) of the MC estimator can be made smaller than $\epsilon$, i.e. $\mathbb{E}\left[\left(\Delta_{n, N}^{g}\right)^{2}\right] \leq \epsilon^{2}$, at a computational cost of $\mathcal{O}\left(\epsilon^{-2} \log \epsilon\right)$ as $\epsilon \rightarrow 0$. The cost of the Monte Carlo estimator based on SB-Alg is thus only a log-factor away from the optimal Monte Carlo cost of $\mathcal{O}\left(\epsilon^{-2}\right)$, arising when one has access to exact simulation with finite expected running time.

The main aim of MLMC, introduced in [Hei01, Gil08], is to reduce the computational cost of an MC algorithm for a given level of accuracy. We will apply a general MLMC result [CGST11, Thm 1], stated
in our setting for ease of reference as Theorem 4 in Appendix A. 2 below. Let $P=g(\chi)$ and $P_{n}=g\left(\chi_{n}\right)$, $n \in \mathbb{N}$, for any function $g$ that satisfies the assumptions of Theorem 3 (see also Remark 7 below). Note that the expected computational cost of a single draw in Theorem 4 is allowed to grow geometrically in $n$. Since in the context of the present section sampling $P_{n}$ has a cost of $\mathcal{O}(n)$, we may choose an arbitrarily small rate $q_{3}>0$ in Theorem 4 .

A key component of any MLMC scheme is the coupling ( $P_{n}, P_{n+1}$ ). In the case of SB-Alg (and the notation therein), this consists of using the same sequence of sticks $\left(\lambda_{k}\right)_{k \in\{1, \ldots, n\}}$ and increments $\left(\xi_{k}\right)_{k \in\{1, \ldots, n\}}$ in the consecutive levels and setting $\varsigma_{n}=\xi_{n+1}+\varsigma_{n+1}$, cf. the coupling of Subsection 4.1. Since

$$
\begin{equation*}
\mathbb{V}\left[P_{n+1}-P_{n}\right] \leq \mathbb{E}\left[\left(P_{n+1}-P_{n}\right)^{2}\right] \leq 2\left(\mathbb{E}\left[\left(P_{n+1}-P\right)^{2}\right]+\mathbb{E}\left[\left(P-P_{n}\right)^{2}\right]\right) \tag{2.7}
\end{equation*}
$$

Assumption (b) in Theorem 4 follows easily from the bound $\mathbb{E}\left[\left(P-P_{n}\right)^{2}\right]=\mathcal{O}\left(2^{-n q_{2}}\right)$ for all functions $g$ of interest (see Propositions 1, 2 and 3 above for the corresponding $q_{2}>0$ ). These observations imply that the computational complexity of the MLMC estimator in (A.1) is bounded above by $\mathcal{O}\left(\epsilon^{-2}\right)$ (take $q_{3}=q_{2} / 2$ for all choices of $g$ in the propositions above). The implementation of the MLMC estimator based on SB-Alg for a barrier-type function $g$ under the NIG model numerically confirms this bound, see Subsection 3.3 below.
2.4.1. Comparison. The computational complexity of MC and MLMC procedures based on the SB-Alg is given by $\mathcal{O}\left(\epsilon^{-2}|\log \epsilon|\right)$ and $\mathcal{O}\left(\epsilon^{-2}\right)$, respectively, for a function $g(\chi)$, which is Lipschitz, locally Lipschitz or barrier-type. This makes SB-Alg robust, as its performance does not depend on the structure of the problem. In particular, minor changes in model parameters will not result in major differences in the computational complexity. We compare this to the extant MC and MLMC algorithms in the literature.

Lipschitz function $g$. We first review the results for Lipschitz functions of $\left(X_{T}, \bar{X}_{T}\right)$. For the RWA, $\alpha$ as in (4.21) below and a small $\delta>0(\delta=0$ if $\alpha \in\{1,2\})$, [BI20, Thm 4.1] implies that the cost of an MC estimator is $\mathcal{O}\left(\epsilon^{-2-\max \{1, \alpha+\delta\}}\right)$. In particular, if $\sigma \neq 0$, the complexity of the RWA is $\mathcal{O}\left(\epsilon^{-4}\right)$ (see also [DL11a, Che11, GX17]). Their MLMC counterparts, derived following the procedure of [GX17], together with the bounds in [BI20, Thm 4.1] and (2.7), have a complexity of $\mathcal{O}\left(\epsilon^{-2} \log ^{2}(\epsilon)\right)$. Moreover, if the process is spectrally negative without a Brownian component and either an infinite variation stable process [GX17, Prop. 5.5] or of finite variation [DL11a, Lem. 6.5], then the MLMC estimator for a Lipschitz function of ( $X_{T}, \bar{X}_{T}$ ) has optimal cost $\mathcal{O}\left(\epsilon^{-2}\right)$. For the WHA (see Subsection 1.2 above), the MC (resp. MLMC) estimator for a Lipschitz function of $\left(X_{T}, \bar{X}_{T}\right)$ has a complexity of $\mathcal{O}\left(\epsilon^{-4}\right)$ (resp. $\mathcal{O}\left(\epsilon^{-3}\right)$ ) if the process is of finite variation and of $\mathcal{O}\left(\epsilon^{-6}\right)\left(\right.$ resp. $\left.\mathcal{O}\left(\epsilon^{-4}\right)\right)$ otherwise [FCKSS14, Thm 4.6]. For the JAGA, the complexity of the MC estimator is $\mathcal{O}\left(\epsilon^{-2} \max \left\{\epsilon^{-\max \left\{1, \beta_{+}\right\}}, \epsilon^{-4 \beta_{+} /\left(4-\beta_{+}\right)} \log (1 / \epsilon)^{2 \beta_{+} /\left(4-\beta_{+}\right)}\right\}\right)$ if $\sigma=0$ and $\mathcal{O}\left(\epsilon^{-2-\max \left\{2,4 \beta_{+} /\left(4-\beta_{+}\right)\right\}}\right)$otherwise (see (4.6) for the definition of $\left.\beta_{+} \in(0,2]\right)$. The complexity of the MLMC estimator is $\mathcal{O}\left(\epsilon^{-2} \log (1 / \epsilon)^{\left.3 \cdot \mathbb{I}_{\{\sigma \neq 0\}}\right)}\right.$ if $\beta_{+}<1, \mathcal{O}\left(\epsilon^{-2} \log (1 / \epsilon)^{\left.2+\mathbb{I}_{\{\sigma \neq 0\}}\right)}\right.$ if $\beta_{+}=1$, $\mathcal{O}\left(\epsilon^{-2-4\left(1-1 / \beta_{+}\right)} \log (1 / \epsilon)^{2-2 / \beta_{+}}\right)$if $\beta_{+} \in(1,4 / 3]$ and $\sigma \neq 0$, and $\mathcal{O}\left(\epsilon^{-2-8\left(\beta_{+}-1\right) /\left(4-\beta_{+}\right)} \log (1 / \epsilon)^{4\left(\beta_{+}-1\right) /\left(4-\beta_{+}\right)}\right)$ otherwise. In the worst case $\beta_{+}=2$, the MLMC estimator based on the JAGA has a complexity of $\mathcal{O}\left(\epsilon^{-6}\right)$.

Locally Lipschitz function $g$. In the case of locally Lipschitz functions, only the MC analysis of the RWA appears to be available in the literature. The error in this case is at best $\mathcal{O}\left(\epsilon^{-3}\right)$, attained only when the Lévy process is spectrally negative, with jumps of finite variation and no Brownian
component (i.e. $\nu\left(\mathbb{R}_{+}\right)=0, \int_{(-1,0)}|x| \nu(d x)<\infty$ and $\sigma=0$ ) and the inequality $E_{+}^{q}<\infty$ holds for some $q>1$ [DL11a, Prop. 6.4] (recall the definition of $E_{+}^{q}$ in (2.3) above). If $X$ has a Brownian component (i.e. $\sigma \neq 0$ ), then the cost is either $\mathcal{O}\left(\epsilon^{-4}\right)$ if $\nu(\mathbb{R})<\infty$ and $E_{+}^{q}<\infty$ for some $q>2$ [DL11a, Prop. 6.4] or $\mathcal{O}\left(\epsilon^{-4} \log ^{2}(\epsilon)\right)$ if $X$ is spectrally negative with jumps of finite variation and $E_{+}^{q}<\infty$ for some $q>1$ [DL11a, Prop. 5.1]. If $\sigma=0$ and $X$ has infinite activity, then for any arbitrarily small $\delta>0$, the condition $E_{+}^{q}<\infty$ (for some $q>1$ ) implies an MC complexity of $\mathcal{O}\left(\epsilon^{-2-2 q /(q-1)-\delta}\right)$. In the last case, the decay may be improved to $\mathcal{O}\left(\epsilon^{-2-q /(q-1)-\delta}|\log (\epsilon)|\right)\left(\right.$ resp. $\left.\mathcal{O}\left(\epsilon^{-2-q /(q-1)-\delta}\right)\right)$ if $\int_{(-1,1)}|x| \nu(d x)<\infty$ (resp. $\left.\int_{(-1,1)}|x| \log |x| \nu(d x)<\infty\right)$ [DL11a, Thm 6.2].

Barrier-type function $g$. To the best of our knowledge, there are no non-parametric MLMC results in the literature for barrier options under the RWA. Recently the MLMC for the RWA under VG, NIG and spectrally negative $\alpha$-stable (with $\alpha>1$ ) processes has been shown in [GX17] to have the computational cost of $\mathcal{O}\left(\epsilon^{-2-\delta}\right), \mathcal{O}\left(\epsilon^{-3-\delta}\right)$ and $\mathcal{O}\left(\epsilon^{-1-\alpha-\delta}\right)$ for small $\delta>0$, respectively. We are not aware of any results for WHA, introduced in [KKPvS11], for barrier options.
2.5. Unbiased estimators. Randomising the number of levels and samples at each level in the MLMC estimator from the previous section yields an unbiased estimator (A.3) below, see e.g. [RG15, Vih18]. There are numerous ways of implementing such a debiasing technique, typically based on a random variable $R$ on the integers satisfying $\mathbb{P}[R=n]>0$ for all $n \in \mathbb{N}$, with the tail of the law of $R$ in some way linked to the asymptotic decay of the level variances in the MLMC. While other estimators from [Vih18] could be considered, here we focus on the single term estimator (STE) and the independent sum estimator (ISE). For these two estimators, a sequence $\left(R_{j}\right)_{j \in\{1, \ldots, N\}}$ of independent random variables specifies the number of samples $N_{k}$ at level $k \in \mathbb{N}$ as follows: $N_{k}=\sum_{j=1}^{N} \mathbb{1}_{\left\{R_{j}=k\right\}}$ for STE and $N_{k}=\sum_{j=1}^{N} \mathbb{1}_{\left\{R_{j} \geq k\right\}}$ for ISE. For both estimators, we use the uniform stratified sampling of the sequence $\left(R_{i}\right)_{i \in\{1, \ldots, N\}}$ : each $R_{j}$ is drawn independently and distributed as $R$ conditioned to be between its $(j-1) / N$ and $j / N$ quantiles.

The probabilities $(\mathbb{P}[R=n])_{n \in \mathbb{N}}$ that maximise the asymptotic inverse relative efficiencies (see Appendix A. 3 below for definition) for the STE and ISE, denoted by by $\left(p_{n}^{\mathrm{ST}}\right)_{n \in \mathbb{N}}$ and $\left(p_{n}^{\mathrm{IS}}\right)_{n \in \mathbb{N}}$, respectively, are in general given by the formulae in (A.4). In the case of the unbiased estimator for $\mathbb{E} P$, where $P=g(\chi)$, the optimal probabilities take the form:

- (Lipschitz) If $g$ is as in Proposition 1, we set

$$
p_{n}^{\mathrm{ST}}=\frac{2^{-n / 2} / \sqrt{n}}{\sum_{k=1}^{\infty} 2^{-k / 2} / \sqrt{k}}, \quad p_{n}^{\mathrm{IS}}=\frac{2^{-(n-1) / 2}}{\sqrt{n}}-\frac{2^{-n / 2}}{\sqrt{n+1}}
$$

- (Locally Lipschitz) If $g, q$ and $q^{\prime}=(1-1 / q)^{-1}$ are as in Proposition 2, we set

$$
p_{n}^{\mathrm{ST}}=\frac{2^{-n /\left(2 q^{\prime}\right)} / \sqrt{n}}{\sum_{k=1}^{\infty} 2^{-k /\left(2 q^{\prime}\right)} / \sqrt{k}}, \quad p_{n}^{\mathrm{IS}}=\frac{2^{-(n-1) /\left(2 q^{\prime}\right)}}{\sqrt{n}}-\frac{2^{-n /\left(2 q^{\prime}\right)}}{\sqrt{n+1}} .
$$

- (Barrier-type) If $g, \gamma$ and $q$ are as in Proposition 3, we set

$$
p_{n}^{\mathrm{ST}}=\frac{\eta_{q}^{-n \gamma /(2 \gamma+2 q)} / \sqrt{n}}{\sum_{k=1}^{\infty} \eta_{q}^{-k \gamma /(2 \gamma+2 q)} / \sqrt{k}}, \quad p_{n}^{\mathrm{IS}}=\frac{\eta_{q}^{-(n-1) \gamma /(2 \gamma+2 q)}}{\sqrt{n}}-\frac{\eta_{q}^{-n \gamma /(2 \gamma+2 q)}}{\sqrt{n+1}} .
$$

It is interesting to note that the choices in the Lipschitz (resp. locally Lipschitz) case is independent of the structure of the Lévy process $X$ (resp. dependent only through its exponential moments). This
invariance reinforces the idea that $\mathrm{SB}-\mathrm{Alg}$ is robust. It is a consequence of the fact that $\eta_{p}$ (defined in (4.22)) equals 2 for $p \geq 2$.

## 3. Numerical examples

The implementation of SB-Alg above can be found in the repository [GCMUB18] together with a simple algorithm for the simulation of the increments of the VG, NIG and weakly stable processes. This implementation was used in Sections 3.1 below.
3.1. Numerical comparison: SBA and RWA. Let $X=\left(X_{t}\right)_{t \geq 0}$ be given by $X_{t}=B_{Z_{t}}+b t$, where $Z$ is a subordinator with Lévy measure $\nu_{Z}(d x)=\mathbb{1}_{\{x>0\}} \gamma x^{-\alpha-1} e^{-\lambda x} d x(\alpha \in[0,1), \gamma, \lambda>0)$ and drift $\sigma_{Z} \geq 0, B$ is a standard Brownian motion and $b \in \mathbb{R}$. The Lévy measure of $X$ by [Sat13, Thm 30.1] equals $\nu(d x) / d x=\frac{\gamma}{\sqrt{2 \pi}}|x|^{-2 \alpha-1} \int_{0}^{\infty} s^{-\alpha-3 / 2} e^{-\lambda s x^{2}-s^{-1} / 2} d s$, implying that the Blumenthal-Getoor index of $X$ is $\beta=2 \alpha \in[0,2)$, and its Brownian component equals $\sigma^{2}=\sigma_{Z}^{2}$. Moreover, the increment $X_{t}$ can be simulated in constant expected computational time for any $t>0$.

We consider the estimator $\sum_{i=1}^{N} g\left(\chi_{n}^{i}\right) / N$, where $\left(\chi_{n}^{i}\right)_{i \in\{1, \ldots, N\}}$ are $N$ iid samples produced by running the SB-Alg over $n$ steps. We compare the results with the output of the RWA in (2.2), based on a time step of size $T / 2^{n}$ and the same number $N$ of iid samples. The function $g(\chi)$ corresponds to either a lookback put or an up-and-out call under the exponential Lévy model $S=S_{0} \exp (X)$. Figure 3.1 shows that the accuracy of the two algorithms is comparable as suggested by Propositions 2 and 3 above (note $E_{ \pm}^{q}<\infty$ if and only if $q^{2}<2 \lambda$, since $\left.\mathbb{E}\left[e^{q X_{t}}\right]=e^{b t} \mathbb{E}\left[e^{q^{2} Z_{t} / 2}\right]\right)$.


Figure 3.1. We take $\alpha=0.75, \gamma=0.1, \lambda=4, \sigma_{Z}=0.05, b=-0.05$ and $S_{0}=2$, $K_{0}=3, M=5, T=1$ and $N=10^{7}$. The value $\mathbb{E} g(\chi)$ is obtained by running SB-Alg for $n=100$ steps and using $N=10^{8}$ samples. The RWA is approximately $\left(2^{n} / n\right)$-times slower than the SBA for the same amount of bias, making it infeasible for $n>15$ as at least $60000<2^{n}$ steps are needed in the time interval $[0,1]$.
3.2. Asymptotic and non-asymptotic CIs. Let $X$ be a Normal Inverse Gaussian process (NIG) with parameters $(b, \kappa, \sigma, \theta)$, i.e. a Lévy process with characteristic function $\mathbb{E}\left[e^{i u X_{t}}\right]=\exp (t(b+1 / \kappa)-$ $\left.(t / \kappa) \sqrt{1-2 i u \theta \kappa+\kappa \sigma^{2} u^{2}}\right)$, whose Lévy measure is given by

$$
\frac{\nu(d x)}{d x}=\frac{C}{|x|} e^{A x} K_{1}(B|x|), \quad \text { with } \quad A=\frac{\theta}{\sigma^{2}}, \quad B=\frac{\sqrt{\theta^{2}+\sigma^{2} / \kappa}}{\sigma^{2}}, \quad C=\frac{\sqrt{\theta^{2}+2 \sigma^{2} / \kappa}}{2 \pi \sigma \kappa^{3 / 2}},
$$

where $K_{1}$ is the modified Bessel function of the second kind, which satisfies

$$
K_{1}(x)=\frac{1}{x}+\mathcal{O}(1), \text { as } x \rightarrow 0, \quad K_{1}(x)=e^{-x} \sqrt{\frac{\pi}{2|x|}}(1+\mathcal{O}(1 /|x|)), \text { as } x \rightarrow \infty .
$$

We simulate the increments of the NIG process by [CT04, Alg. 6.12]. Figure 3.2 presents confidence intervals at level $1-\epsilon=99 \%$ for the prices of hindsight put and barrier up-and-out call under the NIG model $S=S_{0} \exp (X)$.

The non-asymptotic CI for the hindsight put is constructed via Chebyshev's inequality as discussed in Subsection 2.3 above. In particular, note that the payoff of the hindsight put $g:(x, y, t) \mapsto\left(K_{0}-S_{0} e^{y}\right)^{+}$ is non-increasing in $y$ and does not depend on $x$ and $t$. Since $\bar{X}_{T}$ dominates the second coordinate $\bar{X}_{T}-\Delta_{n}^{\mathrm{SB}}$ of the SBA $\chi_{n}$ in (1.2), we apply $\mathbb{E} g\left(\chi_{n}\right) \geq \mathbb{E} g(\chi)$ and find

$$
\left.\begin{array}{l}
0 \leq \mathbb{E} g\left(\chi_{n}\right)-\mathbb{E} g(\chi)<r_{1}, \\
\mathbb{P}\left(\left|\Delta_{n, N}^{g}-\mathbb{E} \Delta_{n, N}^{g}\right|<r_{2}\right) \geq 1-\epsilon
\end{array}\right\} \Longrightarrow \mathbb{P}\left(-r_{1}-r_{2}<\Delta_{n, N}^{g}<r_{2}\right) \geq 1-\epsilon,
$$

where $\Delta_{n, N}^{g}$ is defined in (2.4), reducing the upper bound of the CI to the error $r_{2}$, which depends on the bound on $g$ and the number of samples $N$ but not on $n$.

As explained in Section 2.3 above, if explicit constants in the bounds on the bias are not available in terms of the model parameters, as is the case with an up-and-out call option (see Proposition 3 above and remarks following it), we resort to the CLT in Theorem 3 above. The plot on the right in Figure 3.2 depicts the asymptotic CI for an up-and-out call as a function of $\log _{2} N$, where $N$ is the number of samples used to estimate $\mathbb{E} g(\chi)$ and the asymptotic variance in (2.5) of Theorem 3 is estimated using the sample.

Non-asymptotic CI for $g(\chi)=\left(K_{0}-\bar{S}_{T}\right)^{+} \quad$ Asymptotic CI for $g(\chi)=\left(S_{T}-K_{0}\right)^{+} \cdot \mathbb{1}_{\left\{\bar{S}_{T} \leq M\right\}}$



Figure 3.2. The pictures show the point estimation and CIs for the hinsight put (left) and the up-and-out call (right) under the NIG model. NIG parameters: $\sigma=1, \theta=0.1$, $\kappa=0.1$ and $b=-0.05$. Option parameters: $S_{0}=2, K_{0}=3, M=8$ and $T=1$. The number of samples in the plot on the left equals $N=10^{7}$. The confidence level of $1-\epsilon=99 \%$ applies to both plots.
3.3. MLMC for a barrier payoff under NIG. We apply the MLMC algorithm for the SBA to the up-and-out call option in [GX17, Sec. 6.3] (with payoff $g(\chi)=\left(S_{T}-K_{0}\right)^{+} \cdot \mathbb{1}_{\left\{\bar{S}_{T} \leq M\right\}}$, where $\left.S_{T}=S_{0} \exp \left(X_{T}\right)\right)$ under the NIG model. The top left (resp. right) plot in Figure 3.3 graphs the
estimated and theoretically predicted mean (resp. variance) of the difference of two consecutive levels (as a function of $n$ ).

It is common practice in MLMC to estimate the bias and level variances (rather than use the theoretical bounds such as those in Theorem 4) first and then compute the numbers of samples $\left(N_{k}\right)_{k \in\{1, \ldots, n\}}$ at each level by solving a simple optimisation problem. This often improves the overall performance of the algorithm but requires an initial computational investment. The fact that $\left(N_{k}\right)_{k \in\{1, \ldots, n\}}$ are based on estimates gives rise to some oscillation in their behaviour and, consequently, in that of the computational cost. However, as expected from (A.2), the bottom left plot in Figure 3.3 shows that $\left(N_{k}\right)_{k \in\{1, \ldots, n\}}$ constitute approximately straight lines for various levels of accuracy. The bottom right plot in Figure 3.3 shows that the computational complexity is approximately constant, as expected from the analysis in Section 2.4 above. Moreover, the difference in the complexity between the MC and MLMC is numerically seen to be small. This is not surprising since, as explained in Section 2.4 above, the two differ by a log-factor. The analogous figure for the MLMC based on the RWA for the identical model parameters and option is given in [GX17, Fig. 7].


Figure 3.3. The pictures show the level bias decay, level variance decay, samples per level and complexities of MC and MLMC implementations for the up-and-out call $g(\chi)=e^{-r T}\left(S_{T}-K\right)^{+} \mathbb{1}_{\left\{\bar{S}_{T}<M\right\}}$ and the NIG process. NIG parameters: $\sigma=0.1836$, $\theta=-0.1313, \kappa=1.2819$ and $b=0.1571$ (see [GX17, Sec. 3] and the reference therein). Option parameters: $S_{0}=100, K_{0}=100, M=115, T=1$ and $r=0.05$. The bounds in the top two graphs are based on Proposition 3 (with $\gamma=q=1$ ) and synchronous coupling. See Subsection 2.4 for the computational complexity of MC and MLMC in the bottom right.

The computational complexity of MLMC in Figure 3.3 is greater than that of the MC (for $\epsilon>1 / 8000$ ) due to the size of the leading constant. Overall, the performance of both MC and MLMC in this examples is good, with the actual decay rates of the bias and level variances being better than the theoretical bounds by a factor of 2 .

## 4. Proofs and technical results

Let $X=\left(X_{t}\right)_{t \geq 0}$ be a Lévy process, which we assume not to be compound Poisson with drift. By Doeblin's diffuseness lemma [Kal02, Lem. 13.22], this is equivalent to the following requirement, which we assume throughout the remainder of the paper.

Assumption 2. $\mathbb{P}\left(X_{t}=x\right)=0$ for all $x \in \mathbb{R}$ and for some (and hence all) $t>0$.
4.1. The concave majorant of $X$ and its coupling with $(\ell, Y)$. Given a countable set $\mathcal{S}$ and a function $\phi: \mathcal{S} \rightarrow(0, \infty)$ such that $\sum_{s \in \mathcal{S}} \phi(s)<\infty$, size-biased sampling of $\mathcal{S}$ based on the function $\phi$ produces a random enumeration $\left(s_{n}\right)_{n \in \mathbb{N}}$ of $\mathcal{S}$ using the following sequential construction: let $Z_{0}=\emptyset$ and assume we have already sampled the points in $Z_{n-1}=\left\{s_{1}, \ldots, s_{n-1}\right\}$ for some $n \in \mathbb{N}$; then, conditional on $Z_{n-1}$, the random element $s_{n}$ in $\mathcal{S} \backslash Z_{n-1}$ follows the law $\mathbb{P}\left(s_{n}=s \mid Z_{n-1}\right)=\phi(s) / \sum_{s^{\prime} \in \mathcal{S} \backslash Z_{n-1}} \phi\left(s^{\prime}\right)$, $s \in \mathcal{S} \backslash Z_{n-1}$.

The concave majorant of a path of $\left(X_{t}\right)_{t \in[0, T]}$ is the point-wise smallest concave function $C:[0, T] \rightarrow$ $\mathbb{R}$ satisfying $C_{t} \geq X_{t}$ for all $t \in[0, T]$. Since $X$ is not compound Poisson with drift, it is possible to obtain a complete description of the law of $C$ (see [PUB12] for details), which we now recall. Note that $t \mapsto C_{t}$ is a piecewise linear function comprising of infinitely many line segments known as faces. Each face has a positive length and a height, which is a real number. If the faces are ordered chronologically (i.e. as they arise with increasing $t$ ), the concavity of $C$ implies that the sequence of the corresponding slopes is strictly decreasing (see Figure 4.1(a) below). The lengths of the faces constitute a countable set of positive numbers with a finite sum clearly equal to $T$.

We may thus order randomly the faces of $C$ using size-biased sampling on lengths, see Figure 4.1 below. This random ordering almost surely differs from the chronological one, with longer faces much more likely to appear near the beginning of the sequence. For any $n \in \mathbb{N}=\{1,2, \ldots\}$, let $g_{n}$ (resp. $d_{n}$ ) be the left (resp. right) end point of the $n$-th face of $C$ in the size-biased enumeration. The size-biased sequence of lengths and heights of the faces of $C$ satisfies the following equality in law [PUB12, Thm 1]:

$$
\begin{equation*}
\left(\left(d_{n}-g_{n}, C_{d_{n}}-C_{g_{n}}\right)\right)_{n \in \mathbb{N}} \stackrel{d}{=}\left(\left(\ell_{n}, Y_{L_{n-1}}-Y_{L_{n}}\right)\right)_{n \in \mathbb{N}} \tag{4.1}
\end{equation*}
$$

where $Y$ is a copy of $X$, independent of the stick-breaking process $\ell=\left(\ell_{n}\right)_{n \in \mathbb{N}}$ on $[0, T]$ based on the uniform law $\mathrm{U}(0,1)$. We stress that the equality in law (4.1) holds in the sense of random processes indexed by $\mathbb{N}$. Surprisingly, by (4.1), the law of the sequence of lengths $\left(d_{n}-g_{n}\right)_{n \in \mathbb{N}}$ does not depend on $X$. This fact is the basis for a coupling of $(\ell, Y)$ and $X$ such that (4.1) holds a.s.

This coupling, constructed below, is crucial for the analysis of the error in the SB-Alg above and will be used throughout the paper. Indeed, under such a coupling, (1.1) holds a.s. since the location (resp. time) of the supremum of $X$ over $[0, T]$ equals the sum of all the heights (resp. lengths) of the faces of $C$ with positive slope. (Note that the function $t \mapsto C_{t}$ is concave and thus of finite variation, making the sequence of heights $\left(C_{d_{n}}-C_{g_{n}}\right)_{n \in \mathbb{N}}=\left(Y_{L_{n-1}}-Y_{L_{n}}\right)_{n \in \mathbb{N}}$ absolutely summable.) In particular, it implies $Y_{T}=X_{T}$ a.s.

Consider the countable set of faces of the concave majorant $C$ of $X$. Each face consists of a pair $(x, y)$, where $x>0$ is the length and $y \in \mathbb{R}$ is the height of the face. Since the lengths of the faces are positive and summable with sum $T$, it is possible to perform size-biased sampling of the faces based on the function $\phi:(x, y) \mapsto x$, which then yields the random enumeration $\left(\left(d_{n}-g_{n}, C_{d_{n}}-C_{g_{n}}\right)\right)_{n \in \mathbb{N}}$ of the faces of $C$. This enumeration, by [PUB12, Thm 1], satisfies the distributional equality (4.1). Furthermore, in this case, the size-biased sampling has a geometric interpretation as illustrated by Figure 4.1 below, wherein $\left(g_{n}\right)_{n \in \mathbb{N}}$ and $\left(d_{n}\right)_{n \in \mathbb{N}}$ are the left and right endpoints of the $n$-th face, respectively. Note that Assumption 2 and (4.1) imply that there is no face of $C$ that is horizontal. Hence the time at which the supremum is attained is a.s. unique.


Figure 4.1. Selecting the first three faces of the concave majorant: the total length of the thick blue segment(s) on the abscissa equal the stick sizes $T, T-\left(d_{1}-g_{1}\right)$ and $T-\left(d_{1}-g_{1}\right)-\left(d_{2}-g_{2}\right)$, respectively. The independent random variables $U_{1}, U_{2}, U_{3}$ are uniform on the sets $[0, T],[0, T] \backslash\left(g_{1}, d_{1}\right),[0, T] \backslash \bigcup_{i=1}^{2}\left(g_{i}, d_{i}\right)$, respectively. Note that the residual length of unsampled faces after $n$ samples is $L_{n}$.

We now explain how to couple ( $\ell, Y$ ) with $X$ in such a way that (4.1) (and hence (1.1)) holds a.s. Start by recalling from (4.1) that $\left(\left(d_{n}-g_{n}, C_{d_{n}}-C_{g_{n}}\right)\right)_{n \in \mathbb{N}} \stackrel{d}{=}\left(\left(\ell_{n}^{\prime}, Y_{L_{n-1}^{\prime}}^{\prime}-Y_{L_{n}^{\prime}}^{\prime}\right)\right)_{n \in \mathbb{N}}$, where $Y^{\prime}$ is a copy of $X$, independent of the stick-breaking process $\ell^{\prime}=\left(\ell_{n}^{\prime}\right)_{n \in \mathbb{N}}$ on $[0, T]$ based on the uniform law $\mathrm{U}(0,1)$ and $L_{n-1}^{\prime}=\sum_{k=n}^{\infty} \ell_{k}^{\prime}$. Now recall that the Skorokhod space $\mathcal{D}[0, T]$ of right-continuous functions on $[0, T]$ with left-hand limits (see [Bi199, p. 109]) is a Polish space [Bil99, p. 112] and thus a Borel space [Kal02, Thm A1.2]. By possibly extending the original probability space, [Kal02, Thm 6.10] asserts the existence of a random element $Y$ in $\mathcal{D}[0, T]$ such that

$$
\begin{equation*}
\left(\left(d_{n}-g_{n}\right)_{n \in \mathbb{N}},\left(C_{d_{n}}-C_{g_{n}}\right)_{n \in \mathbb{N}}, Y\right) \stackrel{d}{=}\left(\left(\ell_{n}^{\prime}\right)_{n \in \mathbb{N}},\left(Y_{L_{n-1}^{\prime}}^{\prime}-Y_{L_{n}^{\prime}}^{\prime}\right)_{n \in \mathbb{N}}, Y^{\prime}\right) \tag{4.2}
\end{equation*}
$$

Consequently, the process $Y$ has the same law as $Y^{\prime} \stackrel{d}{=} X$. If we define the sequence $\ell=\left(\ell_{n}\right)_{n \in \mathbb{N}}$ through $\ell_{n}=d_{n}-g_{n}$ and $L_{n-1}=\sum_{k=n}^{\infty} \ell_{k}$ for each $n \in \mathbb{N}$, then (4.2) implies that $Y$ is independent of $\ell$. Again, by (4.2), the increment of $Y$ over the interval $\left[L_{n}, L_{n-1}\right]$ is equal to $Y_{L_{n-1}}-Y_{L_{n}}=C_{d_{n}}-C_{g_{n}}$ a.s. Thus, this coupling between $(\ell, Y)$ and $X$ is the desired one, as (4.1) holds a.s.

The coupling $(X, \ell, Y)$ can also be obtained without the abstract result [Kal02, Thm 6.10] using the ' 3214 ' transformation from [PUB12], which is explicit in the trajectory of $X$. Since the details of the coupling are not important in this paper, we use the abstract result for brevity.
4.2. The law of the error and the proof of Theorem 1. In the present subsection we will prove Theorem 1. We also state and prove Proposition 4, which explains why the error $\delta_{n}^{\mathrm{SB}}$ of the $\mathrm{SBA} \chi_{n}$ is typically smaller than $\delta_{n}$.

Proof of Theorem 1. By the coupling from Subsection 4.1, the equality in (1.1) holds a.s., i.e. we have $\chi=\left(X_{T}, \bar{X}_{T}, \tau_{T}\right)=\sum_{k=1}^{\infty}\left(Y_{L_{k-1}}-Y_{L_{k}},\left(Y_{L_{k-1}}-Y_{L_{k}}\right)^{+}, \ell_{k} \cdot \mathbb{1}_{\left\{Y_{L_{k-1}}-Y_{L_{k}}>0\right\}}\right)$. Hence, from the definition in (1.3), we clearly obtain

$$
\left(Y_{L_{n}}, \Delta_{n}, \delta_{n}\right)=\sum_{k=n+1}^{\infty}\left(Y_{L_{k-1}}-Y_{L_{k}},\left(Y_{L_{k-1}}-Y_{L_{k}}\right)^{+}, \ell_{k} \cdot \mathbb{1}_{\left\{Y_{L_{k-1}}-Y_{L_{k}}>0\right\}}\right)
$$

In particular, we have $\delta_{n} \leq \sum_{k=n+1}^{\infty} \ell_{k}=L_{n}$ and thus $\left|\delta_{n}^{\mathrm{SB}}\right| \leq L_{n}$.
We now apply (4.1) to conclude that the tail sum in the display above has the required law. Note first that, given $L_{n},\left(\ell_{n+k}\right)_{k \in \mathbb{N}}$ is a stick-breaking process on the interval $\left[0, L_{n}\right]$. Thus, since $Y$ and $\ell$ are independent, the law of the sequence $\left(\left(\ell_{n+k}, Y_{L_{k+n-1}}-Y_{L_{k+n}}\right)\right)_{k \in \mathbb{N}}$, given $L_{n}$, is the same law as that of the right-hand side of (4.1) applied to the interval [0, $L_{n}$ ]. Put differently, by (4.1), this sequence has the same law as the sequence of the faces of the concave majorant of the Lévy process $Y$ over the interval $\left[0, L_{n}\right]$ in size-biased order. Hence, identity (1.1) applied to the interval $\left[0, L_{n}\right]$ (instead of $[0, T])$, together with the independence of $Y$ and $\ell$, yields the first equality in law in (1.4):

$$
\left(Y_{L_{n}}, \bar{Y}_{L_{n}}, \tau_{L_{n}}(Y)\right) \stackrel{d}{=} \sum_{k=n+1}^{\infty}\left(Y_{L_{k-1}}-Y_{L_{k}},\left(Y_{L_{k-1}}-Y_{L_{k}}\right)^{+}, \ell_{k} \cdot \mathbb{1}_{\left\{Y_{L_{k-1}}-Y_{L_{k}}>0\right\}}\right)
$$

The second distributional identity in (1.4) follows from the definition of $\left(\Delta_{n}^{\mathrm{SB}}, \delta_{n}^{\mathrm{SB}}\right)$ as a measurable transformation of $\left(Y_{L_{n}}, \Delta_{n}, \delta_{n}\right)$.

For any $n \in \mathbb{N}$, the second identity in (1.4) implies $0 \leq \Delta_{n}^{\mathrm{SB}}$. The definition of $\Delta_{n}$ in (1.3) and the inequality $Y_{L_{n}}^{+} \leq\left(Y_{L_{n}}-Y_{L_{n+1}}\right)^{+}+Y_{L_{n+1}}^{+}$yield the following:

$$
\Delta_{n+1}^{\mathrm{SB}}=\Delta_{n+1}-Y_{L_{n+1}}^{+}=\Delta_{n}-\left(Y_{L_{n}}-Y_{L_{n+1}}\right)^{+}-Y_{L_{n+1}}^{+} \leq \Delta_{n}-Y_{L_{n}}^{+}=\Delta_{n}^{\mathrm{SB}} \leq \Delta_{n}
$$

This concludes the proof of the theorem.
Proposition 4. Let $X$ satisfy Assumption 2. Then the following statements hold.
(a) For any $t>0$, we have $\mathbb{E} \tau_{t}(X)=\int_{0}^{t} \mathbb{P}\left(X_{s}>0\right) d s$.
(b) If $t^{-1} \int_{0}^{t} \mathbb{P}\left(X_{s}>0\right) d s-\mathbb{P}\left(X_{t}>0\right) \rightarrow 0$ as $t \searrow 0$, then $\mathbb{E}\left[\delta_{n}^{\mathrm{SB}} / L_{n}\right] \rightarrow 0$ as $n \rightarrow \infty$.
(c) If $\mathbb{P}\left(X_{t}>0\right) \rightarrow \rho_{0} \in[0,1]$ as $t \searrow 0$, then (b) holds and $\mathbb{E}\left[\delta_{n} / L_{n}\right] \rightarrow \rho_{0}$ as $n \rightarrow \infty$.
(d) If $\mathbb{P}\left(X_{t}>0\right)=\rho_{0} \in[0,1]$ for all $t \in(0, T]$, then $\mathbb{E}\left[\delta_{n}^{S B} \mid L_{n}\right]=\mathbb{E}\left[\delta_{n} \mid L_{n}\right]-L_{n} \rho_{0}=0$ a.s.

Remark 1. (i) Note that $\tau_{T} \in\left[\tau_{T}-\delta_{n}, \tau_{T}-\delta_{n}+L_{n}\right]$ and, given $L_{n}$, SBA $\chi_{n}$ chooses randomly the endpoints of the interval via a Bernoulli random variable with mean $\mathbb{P}\left(Y_{L_{n}}>0 \mid L_{n}\right)$.
(ii) The assumption in (d) holds if e.g. $X$ is a subordinated stable or a symmetric Lévy process. Moreover, it implies that the third coordinate in $\chi_{n}$ is unbiased, since the expectation of its error vanishes: $\mathbb{E}\left[\delta_{n}^{\mathrm{SB}}\right]=0$. In contrast we have $\mathbb{E}\left[\delta_{n}\right]=\rho_{0} T / 2^{n}$.
(iii) The bias of the third coordinate of $\chi_{n}$, conditional on $L_{n}=t$, equals $\int_{0}^{t} \mathbb{P}\left(X_{s}>0\right) d s-t \mathbb{P}\left(X_{t}>\right.$ 0 ) by (4.3) below. This quantity is generally well behaved as $t \rightarrow 0$. More specifically, we have $t^{-1} \int_{0}^{t} \mathbb{P}\left(X_{s}>0\right) d s-\mathbb{P}\left(X_{t}>0\right) \rightarrow 0$ as $t \searrow 0$ (thus satisfying the assumption in (b)) if $t \mapsto \mathbb{P}\left(X_{t}>0\right)$ is slowly varying at 0 [BGT89, Prop. 1.5.8].
(iv) Note that the assumption in (c) implies that of (b). This assumption, known as Spitzer's condition [Ber96, Thm VI.3.14], is satisfied if for example $X$ converges weakly under the zooming-in procedure [BI20, Sec. 2.2].

Proof. Denote $\rho(t)=\mathbb{P}\left(X_{t}>0\right)$ for all $t>0$.
(a) Apply (1.4) to the interval $[0, t]$ with $n=1$, to get $\tau_{t}(X) \stackrel{d}{=} U t \mathbb{1}_{\left\{X_{t U}>0\right\}}+\tau_{t(1-U)}(Y)$, where $U \sim \mathrm{U}(0,1)$ is independent of $Y$, which itself is a copy of $X$. Hence,

$$
\mathbb{E} \tau_{t}(X)=t^{-1} \int_{0}^{t}\left(s \mathbb{E} \mathbb{1}_{\left\{X_{s}>0\right\}}+\mathbb{E} \tau_{t-s}(Y)\right) d s=t^{-1} \int_{0}^{t}\left(s \rho(s)+\mathbb{E} \tau_{s}(X)\right) d s
$$

where $\rho(s)=\mathbb{P}\left(X_{s}>0\right)$. Since $t \mapsto \tau_{t}$ is right-continuous and nondecreasing, so is $t \mapsto \mathbb{E} \tau_{t}$. The integral equation in the display above, the continuity of $\rho(t)$ for $t>0$ and a bootstrap argument imply that $t \mapsto \mathbb{E} \tau_{t}(X)$ is absolutely continuous with a derivative, say $h$. Put differently, we have $\mathbb{E} \tau_{t}(X)=\int_{0}^{t} h(s) d s$ for all $t>0$. Multiplying the equality in the display by $t$ and applying integration by parts yields $\int_{0}^{t} s h(s) d s=\int_{0}^{t} s \rho(s) d s$ for all $t>0$. Hence the integrands must agree a.e. with respect to the Lebesgue measure. In particular, $\mathbb{E} \tau_{t}=\int_{0}^{t} h(s) d s=\int_{0}^{t} \rho(s) d s$ as desired.
(b) By Theorem 1, conditional on $L_{n}$, we have $\delta_{n}^{S B} \stackrel{d}{=} \tau_{L_{n}}(Y)-L_{n} \cdot \mathbb{1}_{\left\{Y_{L_{n}}>0\right\}}$. Hence, by (a),

$$
\begin{equation*}
\mathbb{E}\left[\delta_{n}^{\mathrm{SB}} \mid L_{n}\right]=\int_{0}^{L_{n}} \rho(s) d s-L_{n} \rho\left(L_{n}\right) \tag{4.3}
\end{equation*}
$$

Since $L_{n} \searrow 0$ as $n \rightarrow \infty$, the assumption in (b) and (4.3) imply that $\mathbb{E}\left[\delta_{n}^{S B} \mid L_{n}\right] / L_{n} \rightarrow 0$ a.s. as $n \rightarrow \infty$. Jensen's inequality applied to $x \mapsto|x|$ and the inequality $\left|\delta_{n}^{\mathrm{SB}} / L_{n}\right| \leq 1$ from Theorem 1 imply that $\left|\mathbb{E}\left[\delta_{n}^{\mathrm{SB}} \mid L_{n}\right] / L_{n}\right| \leq \mathbb{E}\left[\left|\delta_{n}^{\mathrm{SB}}\right| / L_{n} \mid L_{n}\right] \leq 1$. Hence, the dominated convergence theorem [Kal02, Thm 1.21] gives $\mathbb{E}\left[\delta_{n}^{\mathrm{SB}} / L_{n}\right]=\mathbb{E}\left[\mathbb{E}\left[\delta_{n}^{\mathrm{SB}} \mid L_{n}\right] / L_{n}\right] \rightarrow 0$ as $n \rightarrow \infty$.
(c) Since the assumption implies that of (b), the conclusion of (b) holds. Moreover, by (b),

$$
\lim _{n \rightarrow \infty} \mathbb{E}\left[\delta_{n} / L_{n} \mid L_{n}\right]=\lim _{n \rightarrow \infty} \mathbb{E}\left[\delta_{n}^{\mathrm{SB}} / L_{n}+\mathbb{1}_{\left\{Y_{L_{n}}>0\right\}} \mid L_{n}\right]=\lim _{n \rightarrow \infty} \rho\left(L_{n}\right)=\rho_{0} \quad \text { a.s. }
$$

Hence the dominated convergence theorem, applied as in the proof of (b), gives the result.
(d) Since $\rho(t)=\rho_{0}$ for all $t \in[0, T]$, the right-hand side in (4.3) equals 0 a.s., as claimed. Similarly, we have $\mathbb{E}\left[\delta_{n} \mid L_{n}\right]=\mathbb{E}\left[\delta_{n}^{\mathrm{SB}}+L_{n} \cdot \mathbb{1}_{\left\{Y_{L_{n}}>0\right\}} \mid L_{n}\right]=L_{n} \rho_{0}$ a.s.

Proof of Corollary 1. We assumethe existence of a function $a$ on the positive reals, such that $\left(X_{t \delta} / a(\delta)\right)_{t \geq 0}$ converges weakly to some process $\left(Z_{t}\right)_{t \geq 0}$ as $\delta \searrow 0$ in the sense of finite-dimensional distributions. It is known that the limiting process is then self-similar [BGT89, Thm 8.5.2] and thus $\alpha$-stable and the function $a$ is regularly varying with index $1 / \alpha \in[2, \infty)$. Moreover, the convergence extends to the Skorokhod space $\mathcal{D}[0, \infty)$ [JS03, Cor. VII.3.6]. (For a detailed description of $a$ and the limit criteria see [Iva18, Thm 2].)

Note that $Z^{\delta}=\left(Y_{t \delta} / a(\delta)\right)_{t \in[0,1]}$ converges to $Z=\left(Z_{t}\right)_{t \in[0,1]}$ in $\mathcal{D}[0,1]$ and that $\tau_{1}\left(Z^{\delta}\right)=\tau_{\delta}(Z) / \delta$. It is well known that the supremum mapping $x \mapsto \sup _{t \in[0,1]} x_{t}$ and the projection $x \mapsto x_{1}$ are continuous a.s. with respect to the law of $Y$. Next, since the time of the maximum of a stable process $\left(Z_{t} \vee Z_{t-}\right)_{t \in[0,1]}$ is a.s. unique, then $\tau_{1}$ is a.s. continuous with respect to the law of $Z$ (see e.g. [Kal02, Lem. 14.12]). Thus, as $\delta \searrow 0$, this yields

$$
\chi^{\delta}=\left(Y_{\delta} / a(\delta), \bar{Y}_{\delta} / a(\delta), \tau_{\delta}(Y) / \delta\right)=\left(Z_{1}^{\delta}, \bar{Z}_{1}^{\delta}, \tau_{1}\left(Z^{\delta}\right)\right) \xrightarrow{d}\left(Z_{1}, \bar{Z}_{1}, \tau_{1}(Z)\right)=\chi^{0} .
$$

By the equality in law given in (1.4), we obtain

$$
\begin{equation*}
\left(Y_{L_{n}} / a\left(L_{n}\right), \Delta_{n} / a\left(L_{n}\right), \delta_{n} / L_{n}\right) \stackrel{d}{=}\left(Y_{L_{n}} / a\left(L_{n}\right), \bar{Y}_{L_{n}} / a\left(L_{n}\right), \tau_{L_{n}}(Y) / L_{n}\right) \tag{4.4}
\end{equation*}
$$

Hence, the result will follow if we prove that $\chi^{L_{n}} \xrightarrow{d} \chi^{0}$. Recall that the weak convergence is equivalent to $\mathbb{E} f\left(\chi^{\delta}\right) \rightarrow \mathbb{E} f\left(\chi^{0}\right)$ as $\delta \searrow 0$ for every bounded and continuous $f$. Since $\ell$ and $Y$ are independent and $L_{n} \rightarrow 0$ a.s., conditional on the sequence $\left(L_{n}\right)_{n \in \mathbb{N}}$ we get $\mathbb{E}\left[f\left(\chi^{L_{n}}\right) \mid L_{n}\right] \rightarrow \mathbb{E} f\left(\chi^{0}\right)$. The sequence of random variables $\left(\mathbb{E}\left[f\left(\chi^{L_{n}}\right) \mid L_{n}\right]\right)_{n \in \mathbb{N}}$ is bounded (since $f$ is) and converges to $\mathbb{E} f\left(\chi^{0}\right)$ a.s. Hence, by the dominated convergence theorem, it converges in $L^{1}$, implying $\chi^{L_{n}} \xrightarrow{d} \chi^{0}$. Hence, the weak limit holds for the left-hand side of (4.4), which yields Corollary 1.
4.3. Convergence in $L^{p}$ and the proof of Theorem 2. Recall that $\left(\sigma^{2}, \nu, b\right)$ is the generating triplet of $X$ associated with the cutoff function $x \mapsto \mathbb{1}_{\{|x|<1\}}$ (see [Sat13, Ch. 2, Def. 8.2]). The moments of the Lévy measure $\nu$ at infinity are linked with the moments of $X_{t}^{+}$and $\bar{X}_{t}$ for any $t>0$ as follows. By dominating $X$ path-wise with a Lévy process $Z$ equal to $X$ with its jumps in $(-\infty,-1]$ removed and applying [Sat13, Thm 25.3] to $Z$, we find that, for any $p>0$, the conditions $I_{+}^{p}<\infty$ and $E_{+}^{p}<\infty$ (see (2.1) and (2.3) for definition) imply $\mathbb{E}\left[\left(X_{t}^{+}\right)^{p}\right]<\infty$ and $\mathbb{E} \exp \left(p X_{t}^{+}\right)<\infty$, respectively, for all $t>0$. Similarly, by applying [Sat13, Thm 25.18] to $Z$ we obtain that $I_{+}^{p}<\infty$ and $E_{+}^{p}<\infty$ imply $\mathbb{E}\left[\bar{X}_{t}^{p}\right]<\infty$ and $\mathbb{E} \exp \left(p \bar{X}_{t}\right)<\infty$, respectively.

Let $\beta$ be the Blumenthal-Getoor index [BG61], defined as

$$
\begin{equation*}
\beta=\inf \left\{p>0: I_{0}^{p}<\infty\right\}, \quad \text { where } \quad I_{0}^{p}=\int_{(-1,1)}|x|^{p} \nu(d x), \quad \text { for any } p \geq 0 \tag{4.5}
\end{equation*}
$$

and note that $\beta \in[0,2]$ since $I_{0}^{2}<\infty$. Moreover, $I_{0}^{1}<\infty$ if and only if the jumps of $X$ have finite variation, in which case we may define the natural drift $b_{0}=b-\int_{(-1,1)} x \nu(d x)$. Note that $I_{0}^{p}<\infty$ for any $p>\beta$ but $I_{0}^{\beta}$ can be either finite or infinite. If $I_{0}^{\beta}=\infty$ we must have $\beta<2$ and can thus pick $\delta \in(0,2-\beta)$, satisfying $\beta+\delta<1$ whenever $\beta<1$, and define

$$
\begin{equation*}
\beta_{+}=\beta+\delta \cdot \mathbb{1}_{\left\{I_{0}^{\beta}=\infty\right\}} \in[\beta, 2] \tag{4.6}
\end{equation*}
$$

Note that $\beta_{+}$is either equal to $\beta$ or arbitrarily close to it. In either case we have $I_{0}^{\beta_{+}}<\infty$.
The main aim of the present subsection is to prove Theorem 2 and Propositions $1,2 \& 3$. With this in mind, we first establish three lemmas and a corollary.

Lemma 1. The Lévy measure $\nu$ of $X$ satisfies the following for all $\kappa \in(0,1]$ :

$$
\begin{equation*}
\bar{\nu}(\kappa)=\nu(\mathbb{R} \backslash(-\kappa, \kappa)) \leq \kappa^{-\beta_{+}} I_{0}^{\beta_{+}}+\bar{\nu}(1), \quad \bar{\sigma}^{2}(\kappa)=\int_{(-\kappa, \kappa)} x^{2} \nu(d x) \leq \kappa^{2-\beta_{+}} I_{0}^{\beta_{+}} \tag{4.7}
\end{equation*}
$$

Moreover the following inequalities hold:

$$
\begin{align*}
\int_{(-1,-\kappa] \cup[\kappa, 1)}|x|^{p} \nu(d x) & \leq \kappa^{-\left(\beta_{+}-p\right)^{+}} I_{0}^{\beta_{+}}, \quad \text { for } p \in \mathbb{R}  \tag{4.8}\\
\int_{(-\kappa, \kappa)}|x|^{p} \nu(d x) & \leq \kappa^{p-\beta_{+}} I_{0}^{\beta_{+}}, \quad \text { for } p \geq \beta_{+} \tag{4.9}
\end{align*}
$$

Proof. Multiplying the integrands by (I) $(|x| / \kappa)^{\beta_{+}}$, (II) $(\kappa /|x|)^{2-\beta_{+}}$, (III) $(|x| / \kappa)^{\beta_{+}-p}$ if $p \leq \beta_{+}$or $|x|^{\beta_{+}-p}$ otherwise and (IV) $(\kappa /|x|)^{p-\beta_{+}}$, respectively, and extending the integration set to $(-1,1)$ yields the bounds.

Recall the definition in (2.1) of $I_{+}^{p}$ and $I_{-}^{p}$ for $p \geq 0$. Denote $\lceil x\rceil=\inf \{m \in \mathbb{Z}: m \geq x\}$ for any $x \in \mathbb{R}$. Recall that the Stirling numbers of the second kind $\left\{\begin{array}{c}m \\ k\end{array}\right\}$ arise in the formula for the moments of a Poisson random variable $H$ with mean $\mu \geq 0$ : for any $m \in \mathbb{N}$ we have

$$
\mathbb{E}\left[H^{m}\right]=\sum_{k=1}^{m}\left\{\begin{array}{c}
m  \tag{4.10}\\
k
\end{array}\right\} \mu^{k}, \quad \text { where } \quad\left\{\begin{array}{c}
m \\
k
\end{array}\right\}=\frac{1}{k!} \sum_{i=0}^{k}(-1)^{i}\binom{k}{i}(k-i)^{m} .
$$

In particular, we have $\left\{\begin{array}{c}m \\ 0\end{array}\right\}=0$ for all $m \in \mathbb{N}$. Throughout, we will use the following inequality

$$
\begin{equation*}
\left(\sum_{k=1}^{m} x_{i}\right)^{p} \leq m^{(p-1)^{+}} \sum_{k=1}^{m} x_{i}^{p}, \quad \text { where } m \in \mathbb{N}, x_{1}, \ldots, x_{m} \geq 0 \text { and } p \geq 0 \tag{4.11}
\end{equation*}
$$

This inequality follows easily from the concavity of $x \mapsto x^{p}$ when $p<1$ and Jensen's inequality when $p \geq 1$.

Lemma 2. For all $t \in[0, T]$ and $p>0$, the condition $I_{+}^{p}<\infty$ implies

$$
\begin{equation*}
\mathbb{E}\left[\bar{X}_{t}^{p}\right] \leq m_{X}^{p}(t)=4^{(p-1)^{+}}\left(C_{p, 1} t^{p / \beta_{+}}+C_{p, 2} t^{p / 2}+C_{p, 3} t^{p}+C_{p, 4} t^{\min \left\{1, p / \beta_{+}\right\}}\right) \tag{4.12}
\end{equation*}
$$

where the constants $\left\{C_{p, i}\right\}_{i=1}^{4}$ are given by

$$
\begin{align*}
C_{p, 1} & =2^{(p-1)^{+}} T^{p-p / \beta_{+}}\left(I_{0}^{\beta_{+}}\right)^{p}+T^{-p / \beta_{+}}\left(2^{p} T^{p / 2}\left(I_{0}^{\beta_{+}}\right)^{p / 2} \cdot \mathbb{1}_{\{p \leq 2\}}\right. \\
& \left.+2\left(p^{2} /(p-1)\right)^{p} \exp \left(T I_{0}^{\beta_{+}}-p\right) \cdot \mathbb{1}_{\{p>2\}}\right), \\
C_{p, 2} & =|\sigma|^{p} \Gamma\left(\frac{p+1}{2}\right) \frac{2^{p / 2}}{\sqrt{\pi}}, \quad C_{p, 3}=2^{(p-1)^{+}}\left(b^{+} \cdot \mathbb{1}_{\left\{I_{0}^{1}=\infty\right\}}+b_{0}^{+} \cdot \mathbb{1}_{\left\{I_{0}^{1}<\infty\right\}}\right)^{p},  \tag{4.13}\\
C_{p, 4} & =T^{\left(1-p / \beta_{+}\right)^{+}}\left(I_{+}^{p}+I^{\prime}\right) \sum_{k=1}^{\lceil p\rceil}\left\{\begin{array}{c}
\lceil p\rceil \\
k
\end{array}\right\} T^{k-1}\left(I^{\prime}+\nu([1, \infty))\right)^{k-1},
\end{align*}
$$

where $I^{\prime}=\int_{(0,1)} x^{\beta_{+}} \nu(d x)$ and $\Gamma(\cdot)$ is the Gamma function. Moreover, if $I_{+}^{1}<\infty$, then

$$
\mathbb{E}\left[\bar{X}_{t}\right] \leq|\sigma| \sqrt{\frac{2}{\pi}} \sqrt{t}+ \begin{cases}\left(b^{+}+I_{+}^{1}\right) t+2 \sqrt{I_{0}^{2}} \sqrt{t}, & \beta_{+}=2  \tag{4.14}\\ \left(b^{+}+I_{+}^{1}\right) t+2 T^{-1 / \beta_{+}}\left(\sqrt{T I_{0}^{\beta_{+}}}+T I_{0}^{\beta_{+}}\right) t^{1 / \beta_{+}}, & \beta_{+} \in(1,2) \\ \left(b_{0}^{+}+\int_{(0, \infty)} x \nu(d x)\right) t, & \beta_{+} \leq 1\end{cases}
$$

Remark 2. (i) The formula in (4.14) essentially follows from [Che11, Lem. 5.2.2 \& Eq. (5.2)] for $\beta_{+} \in$ $\left(1,2\right.$ ] and from [DL11a, Prop. 3.4] for $\beta_{+} \leq 1$. A new proof of (4.14) given below is based on the methodology used to establish a more general inequality in (4.12). Moreover, the dominant powers of $t$ in both bounds (4.12) and (4.14) coincide in the case $p=1$ with slightly better constants in (4.14). The estimate in (4.12) works for all $p>0$ and is for the reasons of clarity applied in the proofs that follow even in the case $p=1$.
(ii) Note that $C_{p, 2}=0$ if $\sigma=0$ and, if $X$ is spectrally negative, we have $C_{p, 4}=0$.
(iii) The constants in (4.13) are well defined even if the assumption $I_{+}^{p}<\infty$ fails. The inequality in (4.12) holds trivially in this case since $C_{p, 4}=\infty$.

Recall that the Lévy-Itô decomposition [Sat13, Thms $19.2 \& 19.3]$ of the Lévy process $X$ with generating triplet $\left(\sigma^{2}, \nu, b\right)$ at a level $\kappa \in(0,1]$ is given by $X_{t}=b_{\kappa} t+\sigma B_{t}+J_{t}^{1, \kappa}+J_{t}^{2, \kappa}$ for all $t \geq 0$, where $b_{\kappa}=b-\int_{(-1,1) \backslash(-\kappa, \kappa)} x \nu(d x)$ and $J^{1, \kappa}=\left(J_{t}^{1, \kappa}\right)_{t \geq 0}$ (resp. $\left.J^{2, \kappa}=\left(J_{t}^{2, \kappa}\right)_{t \geq 0}\right)$ is Lévy with triplet
$\left(0,\left.\nu\right|_{(-\kappa, \kappa)}, 0\right)$ (resp. $\left(0,\left.\nu\right|_{\mathbb{R} \backslash(-\kappa, \kappa)}, b-b_{\kappa}\right)$ - recall that we are using the cutoff function $\left.x \mapsto \mathbb{1}_{\{|x| \leq 1\}}\right)$ and $B=\left(B_{t}\right)_{t \geq 0}$ is a standard Brownian motion. Moreover, the processes $B, J^{1, \kappa}, J^{2, \kappa}$ are independent, $J^{1, \kappa}$ is an $L^{2}$-bounded martingale with the magnitude of jumps at most $\kappa$ and $J^{2, \kappa}$ is a compound Poisson process with intensity $\bar{\nu}(\kappa)$ (see (4.7) above) and no drift.

Proof. By the discussion above we have $\bar{X}_{t} \leq b_{\kappa}^{+} t+|\sigma| \bar{B}_{t}+\bar{J}_{t}^{1, \kappa}+\bar{J}_{t}^{2, \kappa}$. Then (4.11) implies

$$
\begin{equation*}
\mathbb{E}\left[\bar{X}_{t}^{p}\right] \leq 4^{(p-1)^{+}}\left(\left(b_{\kappa}^{+}\right)^{p} t^{p}+|\sigma|^{p} \mathbb{E}\left[\bar{B}_{t}^{p}\right]+\mathbb{E}\left[\left(\bar{J}_{t}^{1, \kappa}\right)^{p}\right]+\mathbb{E}\left[\left(\bar{J}_{t}^{2, \kappa}\right)^{p}\right]\right), \tag{4.15}
\end{equation*}
$$

where $\bar{B}_{t} \stackrel{d}{=}\left|B_{t}\right|$ and so $\mathbb{E}\left[\bar{B}_{t}\right]=t^{p / 2} \Gamma\left(\frac{p+1}{2}\right) 2^{p / 2} / \sqrt{\pi}\left[\right.$ Kal02, Prop. 13.13], which yields $C_{p, 2}$ in all cases. By Lemma 1 we have

$$
b_{\kappa}^{+} \leq\left\{\begin{array}{lll}
b_{0}^{+}+\int_{(-\kappa \kappa)}|x| \nu(d x) \leq b_{0}^{+}+\kappa^{1-\beta_{+}} I_{0}^{\beta_{+}}, & I_{0}^{1}<\infty & \text { (i.e. } \left.\beta_{+} \leq 1\right) \\
b^{+}+\kappa^{1-\beta_{+}} I_{0}^{\beta_{+}}, & I_{0}^{1}=\infty & \text { (i.e. } \left.\beta_{+}>1\right)
\end{array}\right.
$$

Hence, by (4.11), we obtain

$$
\begin{align*}
\left(b_{\kappa}^{+}\right)^{p} & \leq\left(\kappa^{1-\beta_{+}} I_{0}^{\beta_{+}}+\mathbb{1}_{\left\{I_{0}^{1}=\infty\right\}} b^{+}+\mathbb{1}_{\left\{I_{0}^{1}<\infty\right\}} b_{0}^{+}\right)^{p} \\
& \leq 2^{(p-1)^{+}}\left(\kappa^{p-p \beta_{+}}\left(I_{0}^{\beta_{+}}\right)^{p}+\mathbb{1}_{\left\{I_{0}^{1}=\infty\right\}}\left(b^{+}\right)^{p}+\mathbb{1}_{\left\{I_{0}^{1}<\infty\right\}}\left(b_{0}^{+}\right)^{p}\right) . \tag{4.16}
\end{align*}
$$

$\bar{J}_{t}^{2, \kappa}$ is dominated by the sum of the positive jumps of $J^{2, \kappa}$ over the interval $[0, t]$, which has the same law as $\sum_{k=1}^{N_{t}} R_{k}$ for iid random variables $\left(R_{k}\right)_{k \in \mathbb{N}}$ with law $\left.\nu\right|_{[\kappa, \infty)} / \nu([\kappa, \infty))$ and an independent Poisson random variable $N_{t}$ with mean $t \nu([\kappa, \infty))$. Note that since $N_{t}$ is a nonnegative integer, then $N_{t}^{(p-1)^{+}+1} \leq N_{t}^{\lceil p\rceil}$. Hence, the independence between $\left(R_{k}\right)_{k \in \mathbb{N}}$ and $N_{t}$, the inequality $\left(\sum_{k=1}^{N_{t}} R_{k}\right)^{p} \leq$ $N_{t}^{(p-1)^{+}} \sum_{k=1}^{N_{t}} R_{k}^{p}$ (which follows from (4.11)) and (4.10) yield

$$
\begin{aligned}
\mathbb{E}\left[\left(\bar{J}_{t}^{2, \kappa}\right)^{p}\right] & \leq \mathbb{E}\left[\left(\sum_{k=1}^{N_{t}} R_{k}\right)^{p}\right] \leq \mathbb{E}\left[N_{t}^{(p-1)^{+}} \sum_{k=1}^{N_{t}} R_{k}^{p}\right]=\mathbb{E}\left[R_{1}^{p}\right] \mathbb{E}\left[N_{t}^{(p-1)^{+}+1}\right] \leq \mathbb{E}\left[R_{1}^{p}\right] \mathbb{E}\left[N_{t}^{\lceil p\rceil}\right] \\
& =\left(\int_{[\kappa, \infty)} x^{p} \frac{\nu(d x)}{\nu([\kappa, \infty))}\right)\left(\sum_{k=1}^{[p\rceil}\left\{\begin{array}{c}
\lceil p\rceil \\
k
\end{array}\right\}\left(t \nu([\kappa, \infty))^{k}\right) .\right.
\end{aligned}
$$

Denote $I^{\prime}=\int_{(0,1)} x^{\beta_{+}} \nu(d x)$. The first inequality in (4.7) and the bound in (4.8) of Lemma 1 applied to $\left.\nu\right|_{(0, \infty)}$ and the facts $\kappa \leq 1$ and $t \leq T$ yield

$$
\begin{align*}
\mathbb{E}\left[\left(\bar{J}_{t}^{2, \kappa}\right)^{p}\right] & \leq t\left(I_{+}^{p}+\int_{[\kappa, 1)} x^{p} \nu(d x)\right) \sum_{k=1}^{\lceil p\rceil}\left\{\begin{array}{c}
\lceil p\rceil \\
k
\end{array}\right\}\left(t \kappa^{-\beta_{+}} I^{\prime}+t \nu([1, \infty))\right)^{k-1} \\
& \leq t \kappa^{-\left(\beta_{+}-p\right)^{+}}\left(I_{+}^{p}+I^{\prime}\right) \sum_{k=1}^{\lceil p\rceil}\left\{\begin{array}{c}
\lceil p\rceil \\
k
\end{array}\right\}\left(t \kappa^{-\beta_{+}} I^{\prime}+T \nu([1, \infty))\right)^{k-1} . \tag{4.17}
\end{align*}
$$

Assume $p \leq 2$. Jensen's inequality applied to the function $x \mapsto x^{2 / p}$ and Doob's martingale inequality [Kal02, Prop. 7.6] applied to $J^{1, \kappa}$ yield

$$
\begin{equation*}
\mathbb{E}\left[\left(\bar{J}_{t}^{1, \kappa}\right)^{p}\right] \leq \mathbb{E}\left[\left(\bar{J}_{t}^{1, \kappa}\right)^{2}\right]^{p / 2} \leq 2^{p} \mathbb{E}\left[\left(J_{t}^{1, \kappa}\right)^{2}\right]^{p / 2}=2^{p}(\bar{\sigma}(\kappa))^{p} t^{p / 2} \tag{4.18}
\end{equation*}
$$

where $\bar{\sigma}(\kappa)$ denotes the positive square root of $\bar{\sigma}^{2}(\kappa)$. Hence (4.15) for $p=1$, the first inequality in (4.17) and the estimate in (4.18) give

$$
\begin{equation*}
\mathbb{E} \bar{X}_{t} \leq\left(b_{\kappa}^{+}+\int_{[\kappa, 1)} x \nu(d x)+I_{+}^{1}\right) t+\left(|\sigma| \sqrt{\frac{2}{\pi}}+2 \bar{\sigma}(\kappa)\right) \sqrt{t} \tag{4.19}
\end{equation*}
$$

If $\beta_{+}=2$, then taking $\kappa=1$ in (4.19) yields the first formula in (4.14). If $\beta_{+} \leq 1$ then $I_{0}^{1}<\infty$. Letting $\kappa \rightarrow 0$ in (4.19) we obtain the third formula in (4.14). Set $\kappa=(t / T)^{1 / \beta_{+}}$and apply Lemma 1 to get $t \bar{\sigma}^{2}(\kappa) \leq t^{2 / \beta_{+}} T^{1-2 / \beta_{+}} I_{0}^{\beta_{+}}$. Hence $t \int_{[\kappa, 1)} x \nu(d x) \leq t^{1 / \beta_{+}} T^{1-1 / \beta_{+}} I_{0}^{\beta_{+}}$, and (4.16) \& (4.19) yield the second formula in (4.14), completing the proof of (4.14). To prove (4.12) for general $p \in(0,2]$, we again set $\kappa=(t / T)^{1 / \beta_{+}}$and use the inequalities $t \leq T$ and (4.16)-(4.18) as before. More specifically, (I) (4.16), (II) (4.17) and (III) (4.16) \& (4.18) establish the values of (I) $C_{p, 3}$, (II) $C_{p, 4}$ and (III) $C_{p, 1}$, respectively. This concludes the proof for the case $p \leq 2$.

Assume $p>2$. The only bound from the case $p \leq 2$ above that does not apply in this case is the one on $\mathbb{E}\left[\left(\bar{J}_{t}^{1, \kappa}\right)^{p}\right]$. Doob's martingale inequality and the bound $|x|^{p} \leq(p / e)^{p} e^{|x|}$ for all $x \in \mathbb{R}$ yield

$$
\mathbb{E}\left[\left(\bar{J}_{t}^{1, \kappa}\right)^{p}\right] \leq\left(\frac{p}{p-1}\right)^{p} \mathbb{E}\left[\left|J_{t}^{1, \kappa}\right|^{p}\right]=\left(\frac{\kappa p}{p-1}\right)^{p} \mathbb{E}\left[\left(\kappa^{-1}\left|J_{t}^{1, \kappa}\right|\right)^{p}\right] \leq\left(\frac{\kappa p^{2} / e}{p-1}\right)^{p} \mathbb{E}\left[e^{\kappa^{-1}\left|J_{t}^{1, \kappa}\right|}\right]
$$

Note $\mathbb{E}\left[e^{\kappa^{-1}\left|J_{t}^{1, \kappa}\right|}\right] \leq \mathbb{E}\left[e^{\kappa^{-1} J_{t}^{1, \kappa}}+e^{-\kappa^{-1} J_{t}^{1, \kappa}}\right]=e^{t \psi_{\kappa}\left(\kappa^{-1}\right)}+e^{t \psi_{\kappa}\left(-\kappa^{-1}\right)}$, where $\psi_{\kappa}$ is the Lévy-Khintchine exponent of $J_{1}^{1, \kappa}$, i.e. $\psi_{\kappa}(u)=\int_{(-\kappa, \kappa)}\left(e^{u x}-1-u x\right) \nu(d x)$ for $u \in \mathbb{R}$. Using the elementary bound $e^{x}-1-x \leq x^{2}$ for all $|x| \leq 1$ and (4.7), we find that $\psi_{\kappa}(u) \leq u^{2} \bar{\sigma}^{2}(\kappa) \leq u^{2} \kappa^{2-\beta_{+}} I_{0}^{\beta_{+}}$for $|u| \leq \kappa^{-1}$. By setting $\kappa=(t / T)^{1 / \beta_{+}}$, we obtain

$$
\begin{equation*}
\mathbb{E}\left[\left(\bar{J}_{t}^{1, \kappa}\right)^{p}\right] \leq 2\left(\frac{\kappa p^{2} / e}{p-1}\right)^{p} e^{t \kappa^{-\beta_{+}} I_{0}^{\beta_{+}}}=2 t^{p / \beta_{+}} T^{-p / \beta_{+}}\left(\frac{p^{2}}{p-1}\right)^{p} e^{T I_{0}^{\beta_{+}-p}} \tag{4.20}
\end{equation*}
$$

As before we obtain (4.12) as follows: (I) (4.16), (II) (4.17) and (III) (4.16) \& (4.20) establish the values of (I) $C_{p, 3}$, (II) $C_{p, 4}$ and (III) $C_{p, 1}$, respectively, which completes the proof.

Recall that $\beta$, $I_{0}^{1}$ and $\beta_{+}$are defined in (4.5) and (4.6) above. To describe the dominant power (as $t \downarrow 0)$ in the preceding results, define $\alpha \in[\beta, 2]$ and $\alpha_{+} \in\left[\beta_{+}, 2\right]$ by

$$
\alpha=2 \cdot \mathbb{1}_{\{\sigma \neq 0\}}+\mathbb{1}_{\{\sigma=0\}}\left\{\begin{array}{ll}
1, & I_{0}^{1}<\infty \text { and } b_{0} \neq 0  \tag{4.21}\\
\beta, & \text { otherwise },
\end{array} \quad \text { and } \quad \alpha_{+}=\alpha+\left(\beta_{+}-\beta\right) \cdot \mathbb{1}_{\{\alpha=\beta\}} .\right.
$$

Note that the index $\alpha$ agrees with the one in [BI20, Eq. (2.5)] and $\alpha_{+}>0$ since, by Assumption 2, $X$ is not compound Poisson with drift. Define

$$
\begin{equation*}
\eta_{p}=1+\mathbb{1}_{\{p>\alpha\}}+\frac{p}{\alpha_{+}} \cdot \mathbb{1}_{\{p \leq \alpha\}} \in(1,2], \quad \text { for any } \quad p>0 \tag{4.22}
\end{equation*}
$$

and note that $\eta_{p} \geq 3 / 2$ for $p \geq 1$.
Remark 3. (i) In Theorem 2 and Propositions 1,2 and 3 we assumed that $p \geq 1$ for reasons of clarity. This is not a necessary assumption and the proofs can be made to work with minor modifications for any $p>0$. However, since $\eta_{p} \rightarrow 1$ as $p \rightarrow 0$, the convergence may become arbitrarily slow as $p \rightarrow 0$ (to be expected since $x^{p} \rightarrow 1$ as $p \rightarrow 0$ for any $x>0$ ).
(ii) The constants $C_{p, 2}$ and $C_{p, 3}$ in Lemma 2 above satisfy the following: (a) if $\alpha<2$, then $\sigma=0$ and hence $C_{p, 2}=0$; (b) if $\alpha<1$, then $I_{0}^{1}<\infty$ and $b_{0}=0$ and hence $C_{p, 3}=0$.

Corollary 3. Pick $p>0$, let $\left\{C_{p, i}\right\}_{i=1}^{4}$ be as in Lemma 2 and define the constants $C_{p}(X)$ and $C_{p}^{*}(X)$ as follows:

$$
\begin{align*}
& \frac{C_{p}(X)}{4^{(p-1)^{+}}}= \begin{cases}C_{p, 1} T^{\frac{p}{\beta_{+}}-\frac{p}{\alpha_{+}}}+C_{p, 2}+C_{p, 3} T^{p-\frac{p}{\alpha_{+}}}+C_{p, 4} T^{\min \left\{1, \frac{p}{\beta_{+}}\right\}-\frac{p}{\alpha_{+}}}, & p \leq \alpha, \\
C_{p, 1} T^{\frac{p}{\beta_{+}}-1}+C_{p, 2} T^{\frac{p}{2}-1}+C_{p, 3} T^{p-1}+C_{p, 4}, & p>\alpha,\end{cases}  \tag{4.23}\\
& C_{p}^{*}(X)=C_{p}(X) \cdot \mathbb{1}_{\left\{I_{+}^{p}<\infty\right\}}+C_{p}(-X) \cdot \mathbb{1}_{\left\{I_{+}^{p}=\infty\right\}} .
\end{align*}
$$

Then, if $I_{+}^{p}<\infty\left(\right.$ resp. $\left.\min \left\{I_{+}^{p}, I_{-}^{p}\right\}<\infty\right)$, the inequality

$$
\mathbb{E}\left[\bar{X}_{t}^{p}\right] \leq C_{p}(X) t^{\eta_{p}-1} \quad\left(\text { resp. } \mathbb{E}\left[\left(\bar{X}_{t}-X_{t}^{+}\right)^{p}\right] \leq C_{p}^{*}(X) t^{\eta_{p}-1}\right)
$$

holds for all $t \in[0, T]$.
Proof. Since $\bar{X}_{t}-X_{t}^{+}=\min \left\{\bar{X}_{t}, \bar{X}_{t}-X_{t}\right\}$ is stochastically dominated by both $\bar{X}_{t}$ and $\overline{(-X)}$, then it suffices to prove the result for $\bar{X}_{t}$. (It is critical here, as seen in the definition of $C_{p}^{*}(X)$ in (4.23), that the definition of $\alpha$ is the same for $X$ and $-X$.) Since $t^{q+r} \leq t^{q} T^{r}$ for $t \in[0, T]$ and $r \geq 0$, then it suffices to show that the exponent of $t$ in each term of (4.12) is at least $\eta_{p}-1$. By Remark 3(ii), this is trivially the case when $p \leq \alpha \leq \alpha_{+} \leq 2$. Recall that $\alpha_{+}$is arbitrarily close (or equal) to $\alpha$. Hence, in the case $p>\alpha$, we may assume that $p>\alpha_{+} \geq \beta_{+}$and use Remark 3(ii) to obtain the result and conclude the proof.
Remark 4. If $X$ is spectrally negative (i.e. $\quad \nu\left(\mathbb{R}_{+}\right)=0$ ), then $C_{p, 4}=0$ and therefore $\mathbb{E}\left[\bar{X}_{t}^{p}\right]=$ $\mathcal{O}\left(t^{p / \max \left\{1, \alpha_{+}\right\}}\right)$as $t \searrow 0$, implying the rate in [DL11a, Lem. 6.5], which is the best in the literature to date for the spectrally negative case. In certain specific cases, Lemma 2 implies a rate better than the one stated in Corollary 3. For example, if $\beta<1$ (thus $\beta_{+}<1$ ), $\sigma=0, I_{+}^{p}<\infty$ and the natural drift satisfies $b_{0}<0$ (thus $\alpha=1$ ), then by Lemma 2 we have $\mathbb{E}\left[\bar{X}_{t}^{p}\right]=\mathcal{O}\left(t^{p / \beta+}\right)$ if $p \leq \beta$, which is sharper than the bound $\mathbb{E}\left[\bar{X}_{t}^{p}\right]=\mathcal{O}\left(t^{p}\right)$ implied by Corollary 3. Analogous improvements can be stated for $\bar{X}_{t}-X_{t}^{+}$, when either $\left(I_{+}^{p}<\infty \& b_{0}<0\right)$ or $\left(I_{-}^{p}<\infty \& b_{0}>0\right)$. For the sake of presentation, throughout the paper we work with bounds in Corollary 3.
Lemma 3. Let $X$ be Lévy process satisfying 2 and let $\Delta_{n}$ and $\Delta_{n}^{\mathrm{SB}}$ be as in Theorem 1. If $\mathbb{E}\left[\bar{X}_{t}^{p}\right] \leq C t^{q}$ (resp. $\left.\mathbb{E}\left[\left(\bar{X}_{t}-X_{t}^{+}\right)^{p}\right] \leq C t^{q}\right)$ for some $C, q, p>0$ and all $t \in[0, T]$, then

$$
\mathbb{E}\left[\Delta_{n}^{p}\right] \leq C T^{q}(1+q)^{-n} \quad\left(\text { resp. } \mathbb{E}\left[\left(\Delta_{n}^{\mathrm{SB}}\right)^{p}\right] \leq C T^{q}(1+q)^{-n}\right) \text { for all } n \in \mathbb{N} .
$$

Proof. By assumption and (1.4) in Theorem 1, we have $\mathbb{E}\left[\Delta_{n}^{p} \mid L_{n}\right]=\mathbb{E}\left[\bar{Y}_{L_{n}}^{p} \mid L_{n}\right] \leq C L_{n}^{q}$ and thus $\mathbb{E}\left[\Delta_{n}^{p}\right] \leq \mathbb{E}\left[C L_{n}^{q}\right]=C T^{q}(1+q)^{-n}$. The result for $\Delta_{n}^{\mathrm{SB}}$ is analogously proven.
Proof of Theorem 2. (a) By Theorem 1, the errors $\delta_{n}$ and $\left|\delta_{n}^{\mathrm{SB}}\right|$ are both bounded by $L_{n}$. Since $\mathbb{E}\left[L_{n}^{p}\right]=$ $T^{p}(1+p)^{n}$, the claim follows.
(b) By Corollary 3, we may apply Lemma 3 to obtain part (b) of the theorem. Indeed,

$$
\begin{equation*}
\mathbb{E}\left[\Delta_{n}^{p}\right] \leq C_{p}(X) T^{\eta_{p}-1} \eta_{p}^{-n} \quad\left(\text { resp. } \mathbb{E}\left[\left(\Delta_{n}^{\mathrm{SB}}\right)^{p}\right] \leq C_{p}^{*}(X) T^{\eta_{p}-1} \eta_{p}^{-n}\right) \tag{4.24}
\end{equation*}
$$

where $C_{p}(X)$ (resp. $\left.C_{p}^{*}(X)\right)$ is as in (4.23) in Corollary 3.
For $p \geq 1$, let $\|\cdot\|_{p}$ denote the $p$-norm on $\mathbb{R}^{d}$. The $L^{p}$-Wasserstein distance between distributions $\mu_{x}$ and $\mu_{y}$ on $\mathbb{R}^{d}$ is defined as

$$
\begin{equation*}
\mathcal{W}_{p}\left(\mu_{x}, \mu_{y}\right)=\inf _{X \sim \mu_{x}, Y \sim \mu_{y}} \mathbb{E}\left[\|\mathcal{X}-\mathcal{Y}\|_{p}^{p}\right]^{1 / p} \tag{4.25}
\end{equation*}
$$

where the infimum is taken over all couplings of $(\mathcal{X}, \mathcal{Y})$, such that $\mathcal{X}$ and $\mathcal{Y}$ follow the laws $\mu_{x}$ and $\mu_{y}$, respectively.

Proof of Corollary 2. Recall that the coupling of $\left(\chi, \chi_{n}\right)$ in Subsection 4.1 yields $\chi-\chi_{n}=\left(0, \Delta_{n}^{\mathrm{SB}}, \delta_{n}^{\mathrm{SB}}\right)$ (cf. Theorem 1 above). By Theorem 2(a), Equation (4.24) and the inequality $1+p \geq 2 \geq \eta_{p}$ (since $p \geq 1$ ), we have

$$
\mathbb{E}\left[\left\|\chi-\chi_{n}\right\|_{p}^{p}\right]=\mathbb{E}\left[\left|\Delta_{n}^{\mathrm{SB}}\right|^{p}+\left|\delta_{n}^{\mathrm{SB}}\right|^{p}\right] \leq C_{p}^{*}(X) T^{\eta_{p}-1} \eta_{p}^{-n}+T^{p}(1+p)^{-n} \leq\left(C_{p}^{*}(X) T^{\eta_{p}-1}+T^{p}\right) \eta_{p}^{-n}
$$

Since for any coupling of $\left(\chi, \chi_{n}\right)$ we have $\mathcal{W}_{p}\left(\mathcal{L}(\chi), \mathcal{L}\left(\chi_{n}\right)\right) \leq \mathbb{E}\left[\left\|\chi-\chi_{n}\right\|_{p}^{p}\right]^{1 / p}$, the $L^{p}$-Wasserstein distance is bounded by $C^{\prime} \eta_{p}^{-n / p}$, where the constant takes the form

$$
\begin{equation*}
C^{\prime}=\left(C_{p}^{*}(X) T^{\eta_{p}-1}+T^{p}\right)^{1 / p} \tag{4.26}
\end{equation*}
$$

concluding the proof.
4.4. Proofs of Propositions 1, 2 and 3. The following result about the tail probabilities of $\Delta_{n}$ (defined in Theorem 1) is key in the proofs below.

Lemma 4. Let $X$ be a Lévy process satisfying 2. Fix $p>0$ and $T>0$. Let $C_{p}(Z)$ be the constant in (4.23) of Corollary 3 for the Lévy process $Z=X-J^{2,1}$, where $J^{2,1}$ is the compound Poisson process in the Lévy-Itô decomposition of $X$ (see the paragraph preceding the proof of Lemma 2). Using the notation $\bar{\nu}(1)=\nu(\mathbb{R} \backslash(-1,1))$, for any $r, p>0$, we have

$$
\begin{align*}
\mathbb{P}\left(\Delta_{n} \geq r\right) & \leq \bar{\nu}(1) T 2^{-n}+r^{-p} C_{p}(Z) T^{\eta_{p}-1} \eta_{p}^{-n}  \tag{4.27}\\
\mathbb{E}\left[\min \left\{\Delta_{n}, r\right\}^{p}\right] & \leq r^{p} \bar{\nu}(1) T 2^{-n}+C_{p}(Z) T^{\eta_{p}-1} \eta_{p}^{-n} \tag{4.28}
\end{align*}
$$

Proof. Since $\mathbb{P}\left(\Delta_{n} \geq r\right)=\mathbb{P}\left(\min \left\{\Delta_{n}, r\right\}^{p} \geq r^{p}\right) \leq \mathbb{E}\left[\min \left\{\Delta_{n}, r\right\}^{p}\right] / r^{p}$ by Markov's inequality, we only need to prove (4.28).

Let $Y$ be as in Theorem 1. Pick any $t>0$. Let $A$ be the event on which $J^{2,1}$ does not have a jump on the interval $[0, t]$. Then $\mathbb{P}(A)=e^{-\bar{\nu}(1) t} \leq 1-\bar{\nu}(1) t$, or equivalently $P\left(A^{c}\right) \leq \bar{\nu}(1) t$. By Corollary 3 applied to $Z$ we have $\mathbb{E}\left[\bar{Z}_{t}^{p}\right] \leq C_{p}(Z) t^{\eta_{p}-1}$. Since $\bar{X}_{t}=\bar{Z}_{t}$ a.s. on the event $A$ we get $\min \left\{\bar{X}_{t}, r\right\}^{p} \leq r^{p} \cdot \mathbb{1}_{A^{c}}+\bar{Z}_{t}^{p} \cdot \mathbb{1}_{A} \leq r^{p} \cdot \mathbb{1}_{A^{c}}+\bar{Z}_{t}^{p}$, implying

$$
\mathbb{E}\left[\min \left\{\bar{X}_{t}, r\right\}^{p}\right] \leq r^{p} \bar{\nu}(1) t+C_{p}(Z) t^{\eta_{p}-1}
$$

This inequality, Theorem $1, \mathbb{E}\left[L_{n}\right]=T 2^{-n}$ and the equality in law $X \stackrel{d}{=} Y$ imply (4.28): $\mathbb{E}\left[\min \left\{\Delta_{n}, r\right\}^{p}\right]=$ $\mathbb{E}\left[\mathbb{E}\left[\min \left\{\bar{Y}_{L_{n}}, r\right\}^{p} \mid L_{n}\right]\right] \leq \mathbb{E}\left[r^{p} \bar{\nu}(1) L_{n}+C_{p}(Z) L_{n}^{\eta_{p}-1}\right]$.

Proof of Proposition 1. Assume first $\|g\|_{\infty}<\infty$. Since $\min \{a+b, c\} \leq \min \{a, c\}+b$ for all $a, b, c \geq 0$, we have

$$
\left|g(x, y, t)-g\left(x, y^{\prime}, t^{\prime}\right)\right| \leq \min \left\{K\left|y-y^{\prime}\right|,\|2 g\|_{\infty}\right\}+K\left|t-t^{\prime}\right|
$$

Recall that the output of SB-Alg is a copy of $\chi_{n}$. Since, by Theorem 1, we a.s. have $0 \leq \Delta_{n}^{\mathrm{SB}} \leq \Delta_{n}$ and $\left|\delta_{n}^{\mathrm{SB}}\right| \leq L_{n}$, by (4.11) and (4.28) we obtain

$$
\begin{aligned}
\mathbb{E}\left[\left|g(\chi)-g\left(\chi_{n}\right)\right|^{p}\right] & \leq 2^{(p-1)^{+}}\left(\mathbb{E}\left[K^{p} \min \left\{\Delta_{n},\|2 g\|_{\infty} / K\right\}^{p}\right]+K^{p} \mathbb{E}\left[L_{n}^{p}\right]\right) \\
& \leq 2^{(p-1)^{+}}\left[\|2 g\|_{\infty}^{p} \bar{\nu}(1) T 2^{-n}+K^{p}\left(C_{p}(Z) T^{\eta_{p}-1} \eta_{p}^{-n}+T^{p}(1+p)^{-n}\right)\right]
\end{aligned}
$$

where $Z=X-J^{2,1}$. Now assume that $\min \left\{I_{+}^{p}, I_{-}^{p}\right\}<\infty$. Then, again by Theorems $1 \& 2$ and Equation (4.24), we obtain

$$
\begin{aligned}
\mathbb{E}\left[\left|g(\chi)-g\left(\chi_{n}\right)\right|^{p}\right] & \leq 2^{(p-1)^{+}} K^{p}\left(\mathbb{E}\left[\Delta_{n}^{p}\right]+\mathbb{E}\left[L_{n}^{p}\right]\right) \\
& \leq 2^{(p-1)^{+}} K^{p}\left(C_{p}^{*}(X) T^{\eta_{p}-1} \eta_{p}^{-n}+T^{p}(1+p)^{-n}\right) .
\end{aligned}
$$

Since $\eta_{p} \leq 2 \leq 1+p$ for $p \geq 1$, this yields the result: $\mathbb{E}\left[\left|g(\chi)-g\left(\chi_{n}\right)\right|\right] \leq C^{\prime} \eta_{p}^{-n}$ for

$$
C^{\prime}=2^{(p-1)^{+}} \begin{cases}\|2 g\|_{\infty}^{p} \bar{\nu}(1) T+K^{p}\left(C_{p}(Z) T^{\eta_{p}-1}+T^{p}\right), & \|g\|_{\infty}<\infty  \tag{4.29}\\ K^{p}\left(C_{p}^{*}(X) T^{\eta_{p}-1}+T^{p}\right), & \|g\|_{\infty}=\infty\end{cases}
$$

The proof is thus complete.
Proof of Proposition 2. Recall that the second component of $\chi_{n}$ (resp. $\chi$ ) equals $\bar{X}_{T}-\Delta_{n}^{\mathrm{SB}}$ (resp. $\bar{X}_{T}$ ). Recall from Theorem 1 that $\left|\delta_{n}^{\mathrm{SB}}\right| \leq L_{n}$. Since $0 \leq \Delta_{n}^{\mathrm{SB}} \leq \Delta_{n}$, the locally Lipschitz property of $g$ implies:

$$
\left|g(\chi)-g\left(\chi_{n}\right)\right| \leq K\left(\Delta_{n}+L_{n}\right) e^{\lambda \bar{X}_{T}}
$$

From the definition of $q^{\prime}$ we get $1 / q^{\prime}+1 / q=1$. Thus Hölder's inequality gives:

$$
\begin{equation*}
\mathbb{E}\left[\left|g(\chi)-g\left(\chi_{n}\right)\right|^{p}\right] \leq K^{p} \mathbb{E}\left[\left(\Delta_{n}+L_{n}\right)^{p q^{\prime}}\right]^{\frac{1}{q^{\prime}}} \mathbb{E}\left[e^{\lambda p q \bar{X}_{T}}\right]^{\frac{1}{q}} \tag{4.30}
\end{equation*}
$$

where the second expectation on the right-hand side of (4.30) is finite by assumption $E_{+}^{\lambda p q}<\infty$ and the argument in the first paragraph of Subsection 4.3 above.

We now estimate both expectations on the right-hand side of (4.30). Note that $I_{+}^{r}<\infty$ for all $r>0$ as $E_{+}^{\lambda p q}<\infty$. By (4.11), we have $\mathbb{E}\left[\left(\Delta_{n}+L_{n}\right)^{p q^{\prime}}\right] \leq 2^{\left(p q^{\prime}-1\right)^{+}} \mathbb{E}\left[\Delta_{n}^{p q^{\prime}}+L_{n}^{p q^{\prime}}\right]$. Hence Theorem 2, (4.24) and the inequality $(x+y)^{1 / q^{\prime}} \leq x^{1 / q^{\prime}}+y^{1 / q^{\prime}}$ for $x, y \geq 0$ imply

$$
\begin{aligned}
\mathbb{E}\left[\left(\Delta_{n}+L_{n}\right)^{p q^{\prime}}\right]^{1 / q^{\prime}} & \leq 2^{\left(p-1 / q^{\prime}\right)^{+}}\left(C_{p q^{\prime}}(X) T^{\eta_{p q^{\prime}}-1} \eta_{p q^{\prime}}^{-n}+T^{p q^{\prime}}\left(1+p q^{\prime}\right)^{-n}\right)^{1 / q^{\prime}} \\
& \leq 2^{\left(p-1 / q^{\prime}\right)^{+}}\left(C_{p q^{\prime}}(X)^{1 / q^{\prime}} T^{\left(\eta_{p q^{\prime}}-1\right) / q^{\prime}} \eta_{p q^{\prime}}^{-n / q^{\prime}}+T^{p}\left(1+p q^{\prime}\right)^{-n / q^{\prime}}\right)
\end{aligned}
$$

It remains to obtain an explicit bound for the expectation $\mathbb{E}\left[\exp \left(\lambda p q \bar{X}_{T}\right)\right]$. By removing all jumps smaller than -1 from $X$, we obtain a Lévy process $Z$ with triplet $\left(\sigma^{2},\left.\nu\right|_{[-1, \infty)}, b\right)$ that dominates $X$ path-wise. Set $Z_{t}^{*}=\sup _{s \in[0, t]}\left|Z_{s}\right|$ and note $Z_{T}^{*} \geq \bar{Z}_{T} \geq \bar{X}_{T}$. Define the function $h: x \mapsto e^{\lambda p q x}-1$ on $\mathbb{R}$. Then, for any $c>0$, by Fubini's theorem we have

$$
\begin{aligned}
\mathbb{E}\left[h\left(Z_{T}^{*}-c\right)\right] & \leq \mathbb{E}\left[h\left(Z_{T}^{*}-c\right) \mathbb{1}_{\left\{Z_{T}^{*}>c\right\}}\right]=\int_{c}^{\infty} \mathbb{P}\left(Z_{T}^{*}>z\right) h^{\prime}(z-c) d z \\
& =\int_{0}^{\infty} \mathbb{P}\left(Z_{T}^{*}>z+c\right) h^{\prime}(z) d z \leq \int_{0}^{\infty} \frac{\mathbb{P}\left(\left|Z_{T}\right|>z\right)}{\mathbb{P}\left[Z_{T}^{*} \leq c / 2\right]} h^{\prime}(z) d z=\frac{\mathbb{E}\left[h\left(\left|Z_{T}\right|\right)\right]}{\mathbb{P}\left[Z_{T}^{*} \leq c / 2\right]},
\end{aligned}
$$

where the second inequality holds by [Sat13, p. 167, Eq. (25.15)]. Hence, we get

$$
\begin{equation*}
\mathbb{E}\left[e^{\lambda p q \bar{X}_{T}}\right] \leq \mathbb{E}\left[e^{\lambda p q Z_{T}^{*}}\right]=e^{\lambda p q c} \mathbb{E}\left[1+h\left(Z_{T}^{*}-c\right)\right] \leq e^{\lambda p q c}\left(1+\frac{\mathbb{E}\left[e^{\lambda p q\left|Z_{T}\right|}\right]-1}{\mathbb{P}\left[Z_{T}^{*} \leq c / 2\right]}\right) \tag{4.31}
\end{equation*}
$$

Using the Lévy-Khintchine formula [Sat13, Thm 25.17] for the Lévy process $Z$ we get

$$
\mathbb{E}\left[e^{\lambda p q\left|Z_{T}\right|}\right] \leq \mathbb{E}\left[e^{\lambda p q Z_{T}}\right]+\mathbb{E}\left[e^{-\lambda p q Z_{T}}\right]=e^{T \Psi \Psi_{Z}(\lambda p q)}+e^{T \Psi Z(-\lambda p q)},
$$

where $\Psi_{Z}(u)=b u+\sigma^{2} u^{2} / 2+\int_{[-1, \infty)}\left(e^{u x}-1-u x \mathbb{1}_{\{x<1\}}\right) \nu(d x)$ for $u \in(-\infty, \lambda p q]$. Markov's inequality implies $\mathbb{P}\left[Z_{T}^{*} \leq c / 2\right] \geq 1-(2 / c) \mathbb{E}\left[Z_{T}^{*}\right]$. Moreover, by Lemma 2, we have

$$
\mathbb{E}\left[Z_{T}^{*}\right] \leq \mathbb{E}\left[\bar{Z}_{T}-\inf _{s \in[0, T]} Z_{s}\right] \leq m_{\{Z\}}^{\mathbb{1}}(T)+m_{\{-Z\}}^{\mathbb{I}}(T) .
$$

Hence, from (4.31), for any $c>\left(m_{\{Z\}}^{\mathbb{1}}(T)+m_{\{-Z\}}^{\mathbb{1}}(T)\right) / 2$ we get

$$
\mathbb{E}\left[e^{\lambda p q \bar{X}_{T}}\right] \leq e^{\lambda p q c}\left(1+\frac{e^{T \Psi_{Z}(\lambda p q)}+e^{T \Psi_{Z}(-\lambda p q)}-1}{1-\frac{2}{c}\left(m_{\{Z\}}^{\mathbb{1}}(T)+m_{\{-Z\}}^{\mathbb{1}}(T)\right)}\right) .
$$

Therefore, using (4.30) and the inequalities $\eta_{p q^{\prime}} \leq 2 \leq 1+p q^{\prime}$ (as $p q^{\prime} \geq 1$ ), we obtain the bound $\mathbb{E}\left[\left|g(\chi)-g\left(\chi_{n}\right)\right|^{p}\right] \leq C^{\prime} \eta_{p q^{\prime}}^{-n / q^{\prime}}$, where

$$
\begin{equation*}
C^{\prime}=\frac{C_{p q^{\prime}}(X)^{1 / q^{\prime}} T^{\left(\eta_{p q^{\prime}}-1\right) / q^{\prime}}+T^{p}}{2^{-\left(p-1 / q^{\prime}\right)^{+}} K^{-p} e^{-\lambda p c}}\left(1+\frac{e^{T \Psi_{Z}(\lambda p q)}+e^{T \Psi_{Z}(-\lambda p q)}-1}{1-\frac{2}{c}\left(m_{\{Z\}}^{\mathbb{1}}(T)+m_{\{-Z\}}^{\mathbb{1}}(T)\right)}\right)^{1 / q} \tag{4.32}
\end{equation*}
$$

the constant $C_{p q^{\prime}}(X)$ is defined in (4.23) and $m_{\{Z\}}^{\mathbb{1}}(T)$ and $m_{\{-Z\}}^{\mathbb{1}}(T)$ are given in Lemma 2.
Remark 5. The rate $\eta_{p q^{\prime}}^{-1 / q^{\prime}}$ in the bound of Proposition 2 is smallest (as a function of $q$ ) for the largest $q$ satisfying the exponential moment condition in Proposition 2. Indeed, let $r=p q^{\prime}$ and note that, since $p$ is fixed, minimising $\eta_{p q^{\prime}}^{-1 / q^{\prime}}$ in $q$ is equivalent to maximising $\eta_{r}^{1 / r}$ in $r$. By (4.22), the function $r \mapsto \eta_{r}^{1 / r}$ is decreasing and hence takes its maximal value at the smallest possible $r$ (i.e. largest possible $q$ ).
Proof of Proposition 3. Recall from Theorem 1 that $0 \leq \Delta_{n}^{\mathrm{SB}} \leq \Delta_{n}$. Let $\epsilon_{n}=\eta_{q}^{-n /(\gamma+q)}$ and note

$$
\begin{aligned}
\mathbb{E}\left[\frac{\left|h\left(X_{t}\right)\right|^{p}}{\|h\|_{\infty}^{p}}\left|\mathbb{1}_{\left\{\bar{X}_{T}-\Delta_{n}^{\mathrm{SB}} \leq x\right\}}-\mathbb{1}_{\left\{\bar{X}_{T} \leq x\right\}}\right|^{p}\right] \leq & \mathbb{P}\left(\bar{X}_{T}-\Delta_{n}^{\mathrm{SB}} \leq x<\bar{X}_{T}\right) \\
\leq & \mathbb{P}\left(\bar{X}_{T}-\Delta_{n} \leq x<\bar{X}_{T}\right) \\
= & \mathbb{P}\left(\bar{X}_{T}-\Delta_{n} \leq x<\bar{X}_{T}-\epsilon_{n}\right) \\
& +\mathbb{P}\left(\bar{X}_{T}-\Delta_{n} \leq x<\bar{X}_{T} \leq x+\epsilon_{n}\right) \\
\leq & \mathbb{P}\left(\epsilon_{n}<\Delta_{n}\right)+\mathbb{P}\left(x<\bar{X}_{T} \leq x+\epsilon_{n}\right) .
\end{aligned}
$$

By (4.27) in Lemma 4 we have

$$
\mathbb{P}\left(\epsilon_{n}<\Delta_{n}\right) \leq \bar{\nu}(1) T 2^{-n}+\epsilon_{n}^{-q} C_{q}(Z) T^{\eta_{q}-1} \eta_{q}^{-n}=\bar{\nu}(1) T 2^{-n}+C_{q}(Z) T^{\eta_{q}-1} \eta_{q}^{-n \gamma /(\gamma+q)}
$$

The assumed Hölder continuity of the distribution function of $\bar{X}_{T}$ in Assumption 1 implies that $\mathbb{P}(x<$ $\left.\bar{X}_{T} \leq x+\epsilon_{n}\right) \leq K \epsilon_{n}^{\gamma}$. Given the formula for $C_{q}(Z)$ in (4.23), the constant

$$
\begin{equation*}
C^{\prime}=\|h\|_{\infty}^{p}\left(\bar{\nu}(1) T+C_{q}(Z) T^{\eta_{q}-1}+K\right), \tag{4.33}
\end{equation*}
$$

is explicit and satisfies $\mathbb{E}\left[\left|g(\chi)-g\left(\chi_{n}\right)\right|^{p}\right] \leq C^{\prime} \eta_{q}^{-n \gamma /(\gamma+q)}$.
Remark 6. Minimising the rate $\eta_{q}^{-\gamma /(\gamma+q)}$ as a function of $q$ in Proposition 3 is somewhat involved. On the interval $\left(\alpha_{+}, \infty\right)$, the rate $q \mapsto \eta_{q}^{-\gamma /(\gamma+q)}=2^{-\gamma /(\gamma+q)}$ is strictly increasing, so the optimal $q$ always lies in $\left(0, \alpha_{+}\right]$. On the interval $\left(0, \alpha_{+}\right]$the problem is equivalent to maximising the map $r \mapsto e^{f(r)}=\eta_{q}^{\gamma /(\gamma+q)}$ on the interval $(0,1]$, where $r=\frac{q}{\alpha_{+}} \in(0,1]$ and $f: x \mapsto \log (1+x) /\left(1+\frac{\alpha_{+}}{\gamma} x\right)$. Since

$$
\frac{\gamma}{\alpha_{+}}\left(1+\frac{\alpha_{+}}{\gamma} x\right)^{2} \frac{d}{d x} f(x)=\frac{\frac{\gamma}{\alpha_{+}}-1}{1+x}-(\log (1+x)-1)
$$

the critical point of $f$, obtained by solving for $s=\log (1+x)-1$ in $s e^{s}=e^{-1}\left(\frac{\gamma}{\alpha_{+}}-1\right)$, is given by $r_{0}=e^{W\left(e^{-1}\left(\gamma / \alpha_{+}-1\right)\right)+1}-1$, where $W$ is the Lambert $W$ function, defined as the inverse of $x \mapsto x e^{x}$. Since $f$ is increasing on $\left[0, r_{0}\right]$ and decreasing on $\left(r_{0}, \infty\right)$, then $r=\min \left\{r_{0}, 1\right\}$ maximises $\left.f\right|_{(0,1]}$, implying that the optimal $q$ equals

$$
q=\alpha_{+} \min \left\{1, e^{W\left(e^{-1}\left(\gamma / \alpha_{+}-1\right)\right)+1}-1\right\}
$$

In particular, the choice $q=\alpha_{+}$is optimal if and only if $\gamma / \alpha_{+} \geq 2 \log (2)-1=0.38629 \ldots$, and leads to the bound $\mathcal{O}\left(2^{-n /\left(1+\alpha_{+} / \gamma\right)}\right)$. Hence, if $\gamma=1$, the best bound in Proposition 3 is $\mathcal{O}\left(2^{-n /\left(1+\alpha_{+}\right)}\right)$.

### 4.5. The proof of the central limit theorem.

Proof of Theorem 3. Recall $n_{N}=\left\lceil\log N / \log \left(\eta_{g}^{2}\right)\right\rceil$ and note that $1 \geq \sqrt{N} \eta_{g}^{-n_{N}} \geq \eta_{g}^{-1}$. Hence Assumption (b) yields

$$
\begin{equation*}
\sqrt{N} \mathbb{E} \Delta_{n_{N}, N}^{g} \rightarrow 0 \quad \text { as } N \rightarrow \infty \tag{4.34}
\end{equation*}
$$

The coupling in Subsection 4.1, used in Theorem 1, implies that for all $n \in \mathbb{N}$ the following relations between $\chi$ and the SBA $\chi_{n}$ in (1.2) hold a.s.: $Y_{T}=X_{T}, \bar{X}_{T}-\Delta_{n}^{\mathrm{SB}} \leq \bar{X}_{T}$ and $\tau_{T}-\delta_{n}^{\mathrm{SB}} \leq T$. Hence parts (i) and (ii) of Assumption (a) imply that $g\left(\chi_{n}\right)$ and $g\left(\chi_{n}\right)^{2}$ are dominated by $\zeta=G\left(X_{T}, \bar{X}_{T}, T\right)$ and $\zeta^{2}$, respectively. Since $\zeta$ and $\zeta^{2}$ are integrable by assumption, the dominated convergence theorem yields

$$
\begin{equation*}
\mathbb{V}\left[g\left(\chi_{n}\right)\right]=\mathbb{E}\left[g\left(\chi_{n}\right)^{2}\right]-\left[\mathbb{E} g\left(\chi_{n}\right)\right]^{2} \rightarrow \mathbb{E}\left[g(\chi)^{2}\right]-[\mathbb{E} g(\chi)]^{2}=\mathbb{V}[g(\chi)] \quad \text { as } n \rightarrow \infty \tag{4.35}
\end{equation*}
$$

Recall that $\left(\chi_{n}^{i}\right)_{i \in\{1, \ldots, N\}}$ is the output produced by $N$ independent runs of SB-Alg using $n$ steps. Define the normalised centred random variables

$$
\zeta_{i, N}=\left(g\left(\chi_{n_{N}}^{i}\right)-\mathbb{E} g\left(\chi_{n_{N}}^{i}\right)\right) / \sqrt{N \mathbb{V}[g(\chi)]}, \quad \text { where } i \in\{1, \ldots, N\}
$$

Hence (4.35) implies $\sum_{i=1}^{N} \mathbb{E} \zeta_{i, N}^{2}=\mathbb{V}[g(\chi)]^{-1}(1 / N) \sum_{i=1}^{N} \mathbb{V}\left[g\left(\chi_{n_{N}}^{i}\right)\right] \rightarrow 1$ as $N \rightarrow \infty$. Moreover, we have

$$
\sum_{i=1}^{N} \zeta_{i, N}=\sqrt{N / \mathbb{V}[g(\chi)]} \Delta_{n_{N}, N}^{g}+o(1) \quad \text { as } N \rightarrow \infty
$$

where $o(1)$ is a deterministic sequence, proportional to the one in (4.34). Hence, (2.5) holds if and only if $\sum_{i=1}^{N} \zeta_{i, N} \xrightarrow{d} N(0,1)$ as $N \rightarrow \infty$.

To conclude the proof, we shall use Lindeberg's CLT [Kal02, Thm 5.12], for which it remains to prove that Lindeberg's condition holds, i.e. $\sum_{i=1}^{N} \mathbb{E}\left[\zeta_{i, N}^{2} \mathbb{1}_{\left\{\zeta_{i, N}>r\right\}}\right] \rightarrow 0$ as $N \rightarrow \infty$ for all $r>0$. By the coupling from the second paragraph of this proof, we find $\left|g\left(\chi_{n}^{i}\right)\right| \leq\left|\zeta_{i}\right|$ for all $i \in\{1, \ldots, N\}$ and $n \in \mathbb{N}$, where $\left(\zeta_{i}\right)_{i \in\{1, \ldots, N\}}$ are iid with the law equal to $G\left(X_{T}, \bar{X}_{T}, T\right)$. Crucially, $\zeta_{i}$ does not depend on the number of steps $n_{N}$ in the SB-Alg. Moreover, note that iid random variables $\xi_{i}=\left(\left|\zeta_{i}\right|+\mathbb{E}\left|\zeta_{i}\right|\right)$ satisfy $\mathbb{E} \xi_{i}^{2}<\infty$ and $\left|\zeta_{i, N}\right| \leq \xi_{i} / \sqrt{N \mathbb{V}[g(\chi)]}$ for any $i \in\{1, \ldots, N\}$. Hence we find

$$
\mathbb{V}[g(\chi)] \sum_{i=1}^{N} \mathbb{E}\left[\zeta_{i, N}^{2} \mathbb{1}_{\left\{\zeta_{i, N}>r\right\}}\right] \leq \sum_{i=1}^{N} \frac{1}{N} \mathbb{E}\left[\xi_{i}^{2} \mathbb{1}_{\left\{\xi_{i}>r N \mathbb{V}(g(\chi))\right\}}\right]=\mathbb{E}\left[\xi_{1}^{2} \mathbb{1}_{\left\{\xi_{1}>r N \mathbb{V}(g(\chi))\right\}}\right] \rightarrow 0
$$

as $N \rightarrow \infty$, implying Lindeberg's condition and our theorem.
Remark 7. Identifying the appropriate $G$ in Theorem 3 is usually simple. For instance, the following choices of $G$ can be made in the contexts of interest.
(a) Let $g$ be Lipschitz (as in Proposition 1). Then we can take
(i) $G(x, y, t)=\|g\|_{\infty}$, if $\|g\|_{\infty}<\infty$;
(ii) $G(x, y, t)=|g(x, y, t)|+2 K(y+t)$, if $I_{+}^{2}<\infty$.
(b) Let $g$ be locally Lipschitz with the Lipschitz constant exponentially increasing at rate $\lambda>0$ (as in Proposition 2). Then we can take
(i) $G(x, y, t)=K e^{\lambda y}$, if $g(x, y, t) \leq K e^{\lambda y}$ and $E_{+}^{2 \lambda}<\infty$ (lookback and hindsight options fall in this category);
(ii) $G(x, y, t)=|g(x, y, t)|+2 K(y+t) e^{\lambda y}$ if $E_{+}^{2 \lambda q}<\infty$ for some $q>1$.
(c) If $g$ is a barrier option (as in Proposition 3), then take $G(x, y, t)=\|g\|_{\infty}$.

Remark 8. If we are prepared to centre, it is possible to apply the standard iid CLT to the estimator based on SB-Alg. Indeed, for fixed $n$, assuming $\mathbb{V}\left[P_{n}\right]<\infty$ where $P_{n}=g\left(\chi_{n}\right)$, the classical CLT yields

$$
\frac{1}{\sqrt{N \mathbb{V}\left[P_{n}\right]}} \sum_{i=1}^{N}\left(P_{n}^{i}-\mathbb{E} P_{n}\right) \xrightarrow{d} N(0,1) \quad \text { as } N \rightarrow \infty .
$$

In contrast, the gist of Theorem 3 is that one need not centre the sample with a function of $n$, which itself depends on the sample.

## Appendix A. MLMC and the debiasing

A.1. $\mathcal{O}$ and $o$. The following standard notation is used throughout the paper: for functions $f, g: \mathbb{N} \rightarrow$ $(0, \infty)$ we write $f(n)=\mathcal{O}(g(n))$ (resp. $f(n)=o(g(n)))$ as $n \rightarrow \infty$ if $\lim _{\sup }^{n \rightarrow \infty}$ $f(n) / g(n)$ is finite (resp. 0). Put differently, $f(n)=\mathcal{O}(g(n))$ is equivalent to $f(n)$ being bounded above by $C_{0} g(n)$ for some constant $C_{0}>0$ and all $n \in \mathbb{N}$. In particular, $f(n)=\mathcal{O}(g(n))$ does not imply that $f$ and $g$ decay at the same rate. We also write $f(\epsilon)=\mathcal{O}(g(\epsilon))$ (resp. $f(\epsilon)=o(g(\epsilon)))$ as $\epsilon \downarrow 0$, for functions $f, g:(0, \infty) \rightarrow(0, \infty)$ if $\limsup _{\epsilon \downarrow 0} f(\epsilon) / g(\epsilon)$ is finite (resp. 0 ).
A.2. ML. We start by recalling a version of [CGST11, Thm 1].

Theorem 4. Consider a family of square integrable random variables $P, P_{1}, P_{2}, \ldots$ and $P_{0}=0$. Let $\left\{D_{k}^{i}\right\}_{k, i \in \mathbb{N}}$ be independent with $D_{k}^{i} \stackrel{d}{=} P_{k}-P_{k-1}$ for all $k, i \in \mathbb{N}$. Assume that for some $q_{1} \geq\left(q_{2} \wedge q_{3}\right) / 2>0$ and all $n \in \mathbb{N}$ we have
(a) $\left|\mathbb{E} P-\mathbb{E} P_{n}\right| \leq c_{1} 2^{-n q_{1}}$,
(b) $\mathbb{V}\left[P_{n+1}-P_{n}\right] \leq c_{2} 2^{-n q_{2}}$,
(c) the expected computational cost $\mathcal{C}(n)$ of constructing a single sample of $\left(P_{n}, P_{n-1}\right)$ is bounded by $c_{3} 2^{n q_{3}}$,
where $c_{1}, c_{2}, c_{3}$ are positive constants. Then for every $\epsilon>0$ there exist $n, N_{1}, \ldots, N_{n} \in \mathbb{N}$ (see Remark 9(i) below for explicit formulae) such that the estimator

$$
\begin{equation*}
\hat{P}=\sum_{k=1}^{n} \frac{1}{N_{k}} \sum_{i=1}^{N_{k}} D_{k}^{i} \quad \text { is } L^{2} \text {-accurate at level } \epsilon, \quad \mathbb{E}\left[(\hat{P}-\mathbb{E} P)^{2}\right]<\epsilon^{2}, \tag{A.1}
\end{equation*}
$$

and the computational complexity is of order

$$
\mathcal{C}_{M L}(\epsilon)= \begin{cases}\mathcal{O}\left(\epsilon^{-2}\right) & \text { if } q_{2}>q_{3} \\ \mathcal{O}\left(\epsilon^{-2} \log ^{2} \epsilon\right) & \text { if } q_{2}=q_{3} \\ \mathcal{O}\left(\epsilon^{-2-\left(q_{3}-q_{2}\right) / q_{1}}\right) & \text { if } q_{2}<q_{3}\end{cases}
$$

Remark 9. (i) In [CGST11], the number of levels equals $n=\left\lceil\log _{2}\left(\sqrt{2} c_{1} \epsilon^{-1}\right) / q_{1}\right\rceil$ and the number of samples at level for $k \in\{1, \ldots, n\}$ is

$$
N_{k}= \begin{cases}\left\lceil 2 c_{2} \epsilon^{-2} 2^{-\left(q_{2}+q_{3}\right) k / 2} /\left(1-2^{-\left(q_{2}-q_{3}\right) / 2}\right)\right\rceil & \text { if } q_{2}>q_{3}  \tag{A.2}\\ \left\lceil 2 c_{2} \epsilon^{-2} n 2^{-q_{3} k}\right\rceil & \text { if } q_{2}=q_{3} \\ \left\lceil 2 c_{2} \epsilon^{-2} 2^{n\left(q_{3}-q_{2}\right) / 2-\left(q_{2}+q_{3}\right) k / 2} /\left(1-2^{-\left(q_{3}-q_{2}\right) / 2}\right)\right\rceil & \text { if } q_{2}<q_{3}\end{cases}
$$

Clearly, the number of levels $n$ is obtained from the bound on the bias in Assumption (a), while the number of samples (A.2) at levels $k \in\{1, \ldots, n\}$ are obtained from a simple constrained optimisation using the bounds on the variances and computational costs. In practice, if one has no access to the constants involved in the bounds in Assumptions (a)-(c), one estimates them via Monte Carlo simulation for small $n$. In the setting of this paper this is the case for barrier options, see Proposition 3 and the paragraphs succeeding it.
(ii) The coupling $\left(P_{n}, P_{n-1}\right)$ that can be simulated, implicit in Assumptions (b) and (c) of Theorem 4, constitutes a crucial extension of any MC algorithm necessary for an MLMC estimator to be define. It is clear from (b) that a trivial independent coupling is undesirable in this context. In fact, typically, the optimal coupling (the one where $\mathbb{V}\left[P_{n+1}-P_{n}\right]$ equals the $L^{2}$-Wasserstein distance between the laws of $P_{n}-\mathbb{E} P_{n}$ and $P_{n+1}-\mathbb{E} P_{n+1}$, cf. (4.25) above) is very expensive (resp. impossible) to simulate, making the bound in (c) very large (resp. infeasible). Hence a "compromise" coupling is needed. This is, however, not the case for the problems analysed in this paper as the cost scales only linearly in $n$. In contrast, Assumption (a) requires no specific coupling since $\left|\mathbb{E} P_{n}-\mathbb{E} P\right|$ only compares $P$ and $P_{n}$ through their means. Thus, $q_{1}$ may be computed using the optimal coupling, even if unavailable for simulation.
A.3. The debiasing. A certain random selection of the variables $\left\{D_{n}^{k}\right\}_{n, k \in \mathbb{N}}$ in Theorem 4 leads to an unbiased estimator for $\mathbb{E} P$ (see [McL11, RG15]). More precisely, following [Vih18, Thm 7], define the estimator

$$
\begin{equation*}
\hat{P}=\sum_{k=1}^{\infty} \frac{1}{\mathbb{E} N_{k}} \sum_{n=1}^{N_{k}} D_{k}^{n} \tag{A.3}
\end{equation*}
$$

where the sequence of nonnegative random integers $\left(N_{k}\right)_{k \in \mathbb{N}}$, independent of $\left\{D_{n}^{k}\right\}_{n, k \in \mathbb{N}}$, satisfies $\mathbb{E} N_{k}>$ 0 for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} N_{k}<\infty$, i.e. $N_{k}=0$ for all sufficiently large indices. The sequence $\left(N_{k}\right)_{k \in \mathbb{N}}$ can be constructed as a deterministic functional of a finite sample of positive integers $\left(R_{j}\right)_{j=1}^{N}$ as follows: (a) single term estimator (STE): $N_{k}=\sum_{j=1}^{N} \mathbb{1}_{\left\{R_{j}=k\right\}}$; and (b) independent sum estimator (ISE): $N_{k}=\sum_{j=1}^{N} \mathbb{1}_{\left\{R_{j} \geq k\right\}}$ (see [Vih18, Thms $\left.3 \& 5\right]$ ). For instance, one may take $\left(R_{n}\right)_{n=1}^{N}$ to be iid with common distribution $p_{n}=\mathbb{P}[R=n]>0, n \in \mathbb{N}$. The computational complexities of STE and ISE are linked with the optimal choice for the law of $R$ [Vih18, Sec. 6]. One of the choices analysed in [Vih18] is that of the Uniform Stratified Estimator (USE), described in Theorem 5 below. Let $F_{R}$ :
$x \mapsto \sum_{n=1}^{\lfloor x\rfloor} p_{n}, x>0$, be the distribution function of $R$ (where we denote $\lfloor x\rfloor=\sup \{k \in \mathbb{Z}: k \leq x\}$ ), let $F_{R}^{-1}: u \mapsto \inf \left\{k \in \mathbb{N}: F_{R}(k) \geq u\right\}, u \in(0,1)$, be its generalised inverse. Put $\bar{p}_{n}=1-F_{R}(n-1)$ for $n \in \mathbb{N}$ and recall $\mathcal{C}(n)$ defined in Theorem 4 above.

Theorem 5 ([Vih18, Thm 19]). For some fixed $N \in \mathbb{N}$ let $\left(U_{k}\right)_{k \in\{1, \ldots, N\}}$ be independent with $U_{k} \sim$ $\mathrm{U}\left(\frac{k-1}{N}, \frac{k}{N}\right)$ and put $R_{k}=F_{R}^{-1}\left(U_{k}\right)$ for $k \in\{1, \ldots, N\}$.
(a) Assume $\sum_{n=1}^{\infty} \mathbb{E}\left[\left(P_{n}-P_{n-1}\right)^{2}\right] / p_{n}<\infty$ and define $N_{j}=\sum_{k=1}^{N} \mathbb{1}_{\left\{R_{k}=j\right\}}$ whose mean is $\mathbb{E} N_{j}=N p_{j}$. Then $\hat{P}_{\mathrm{ST}, N}$ in (A.3) is the uniform stratified STE satisfying $\mathbb{E} \hat{P}_{\mathrm{ST}, N}=\mathbb{E} P$ and $\lim _{N \rightarrow \infty} N \mathbb{V}\left[\hat{P}_{\mathrm{ST}, N}\right]=$ $\sum_{n=1}^{\infty} \mathbb{V}\left[P_{n}-P_{n-1}\right] / p_{n}$ with cost $N \sum_{n=1}^{\infty} p_{n} \mathcal{C}(n)$.
(b) Assume $\sum_{n=1}^{\infty} \mathbb{E}\left[\left(P-P_{n-1}\right)^{2}\right] / \bar{p}_{n}<\infty$ and define $N_{j}=\sum_{k=1}^{N} \mathbb{1}_{\left\{R_{k} \geq j\right\}}$ whose mean is $\mathbb{E} N_{j}=N \bar{p}_{j}$. Then $\hat{P}_{\mathrm{IS}, N}$ in (A.3) is the uniform stratified ISE satisfying $\mathbb{E} \hat{P}_{\mathrm{IS}, N}=\mathbb{E} P$ and $\lim _{N \rightarrow \infty} N \mathbb{V}\left[\hat{P}_{\mathrm{IS}, N}\right]=$ $\sum_{n=1}^{\infty}\left(\mathbb{V}\left[P-P_{n-1}\right]-\mathbb{V}\left[P-P_{n}\right]\right) / \bar{p}_{n}$ with cost $N \sum_{n=1}^{\infty} \bar{p}_{n} \mathcal{C}(n)$.

Remark 10. The asymptotic inverse relative efficiencies (see [Vih18, Sec. 6, p. 12] for definition) of STE and ISE, denoted by $I R E_{\mathrm{ST}}$ and $I R E_{\mathrm{IS}}$, respectively, are given by

$$
\begin{aligned}
& I R E_{\mathrm{ST}}=\left(\sum_{n=1}^{\infty} \frac{\mathbb{V}\left[P_{n}-P_{n-1}\right]}{p_{n}}\right)\left(\sum_{n=1}^{\infty} p_{n} \mathcal{C}(n)\right) \geq \sum_{n=1}^{\infty} \sqrt{V_{\mathrm{ST}}(n) \mathcal{C}(n)} \\
& \operatorname{IRE} E_{\mathrm{IS}}=\left(\sum_{n=1}^{\infty} \frac{\mathbb{V}\left[P-P_{n-1}\right]-\mathbb{V}\left[P-P_{n}\right]}{\bar{p}_{n}}\right)\left(\sum_{n=1}^{\infty} \bar{p}_{n} \mathcal{C}(n)\right) \geq \sum_{n=1}^{\infty} \sqrt{V_{\mathrm{IS}}(n) \mathcal{C}(n)},
\end{aligned}
$$

where $V_{\mathrm{ST}}(n)=\mathbb{V}\left[P_{n}-P_{n-1}\right], V_{\mathrm{IS}}(n)=\mathbb{V}\left[P-P_{n-1}\right]-\mathbb{V}\left[P-P_{n}\right]$. The lower bounds follow from the Cauchy-Schwarz inequality, do not depend on the choice of the law $\left(p_{n}\right)_{n \in \mathbb{N}}$ and are attained by taking

$$
\begin{equation*}
p_{n}^{\mathrm{ST}}=\frac{\sqrt{V_{\mathrm{ST}}(n) / \mathcal{C}(n)}}{\sum_{k=1}^{\infty} \sqrt{V_{\mathrm{ST}}(k) / \mathcal{C}(k)}} \quad \text { and } \quad \bar{p}_{n}^{\mathrm{IS}}=\frac{\sqrt{V_{\mathrm{IS}}(n) / \mathcal{C}(n)}}{\sum_{k=1}^{\infty} \sqrt{V_{\mathrm{IS}}(k) / \mathcal{C}(k)}} \tag{A.4}
\end{equation*}
$$

Hence these choices are clearly optimal.

## Appendix B. Regularity of the density of the supremum $\bar{X}_{T}$

In this appendix we discuss the necessity of the Assumption 1 in Proposition 3.
Example 1. For any $\gamma \in(0,1)$ there exists a Lévy process $X$ with an absolutely continuous Lévy measure $\nu$ such that $\lim \inf _{u \downarrow 0} u^{\alpha-2} \bar{\sigma}^{2}(u)>0$ holds for some $\alpha \in(0,1)$ and Assumption 1 fails for $\gamma$ at countably many $M>0$.

Recall $\bar{\sigma}^{2}(\kappa)=\int_{(-\kappa, \kappa)} x^{2} \nu(d x)$ for $\kappa \in(0,1)$ and note that $X$ in Example 1 has smooth transition densities by [Sat13, Prop. 28.3].

Proof. The essence of the proof is to construct any such $M$ as a singularity of the density of $\nu$. For simplicity and to make things explicit, we shall prove it for a single and fixed $M>0$. To that end, let $S$ be an $\alpha$-stable process with positivity parameter $\rho=\mathbb{P}\left(S_{1}>0\right) \in(0,1)$ satisfying $\alpha \rho+\alpha+\rho<\gamma$. Let $Z$ be an independent Lévy process with finite Lévy measure $\nu_{Z}$ given by $\nu_{Z}((-\infty, x] \backslash\{0\})=$ $\min \left\{1,(\max \{x, M\}-M)^{\rho}\right\}$ and put $X=S+Z$. Hereafter consider only small enough $\epsilon>0$, namely, $\epsilon<\min \left\{(T / 2)^{1 / \alpha}, \min \{M, 1\} / 2\right\}$. Our goal is to bound from below the probability $\mathbb{P}\left(\bar{X}_{T} \in[M, M+3 \epsilon)\right)$. To do this, we consider the event where $Z$ jumps exactly once, $S$ is small, $\bar{S} \leq M$ at the time of that jump and $S$ does not increase too much after the jump.

Since the density of $S_{1}$ is positive, continuous and bounded, it follows from the scaling property that there is some constant $K_{1}>0$ (not depending on $\epsilon$ ) such that for all $t \leq \epsilon^{\alpha}$,

$$
\mathbb{P}\left(S_{t} \in[0, \epsilon), \bar{S}_{t} \leq M\right)=\mathbb{P}\left(S_{1} \in\left[0, t^{-1 / \alpha} \epsilon\right), \bar{S}_{1} \leq t^{-1 / \alpha} M\right) \geq K_{1}
$$

From [Bin73, Thm 4A], we also know that $\mathbb{P}\left(\bar{S}_{t} \leq \epsilon\right) \geq K_{2} \epsilon^{\alpha \rho}$ for some constant $K_{2}>0$ and all $t>T-\epsilon^{\alpha} / 2$. Now, $Z_{T} \in[M, M+\epsilon)$ has probability $e^{-T} T \epsilon^{\rho}$ since it can only happen if $Z$ had a single jump on $[0, T]$, whose time $U$ is then conditionally distributed $\mathrm{U}(0, T)$. For fixed $t \in(0, T)$, the Markov property gives

$$
\mathbb{P}\left[\sup _{s \in[0, T-t]} S_{s+t}-S_{t} \in A,\left(S_{t}, \bar{S}_{t}\right) \in B \times C\right]=\mathbb{P}\left[\bar{S}_{T-t} \in A\right] \mathbb{P}\left[\left(S_{t}, \bar{S}_{t}\right) \in B \times C\right],
$$

for all measurable $A, B, C \subset \mathbb{R}$. Hence, multiplying by the density of $U$ at $t$, integrating and using the independence of $(U, Z)$ and $S$, we obtain

$$
\begin{aligned}
& \mathbb{P}\left(\bar{X}_{T} \in[M, M+3 \epsilon)\right) \geq \mathbb{P}\left(Z_{T} \in[M, M+\epsilon), S_{U} \in[0, \epsilon), \bar{S}_{U} \leq M, \bar{X}_{T} \in[M, M+3 \epsilon)\right) \\
& \geq e^{-T} T \epsilon^{\rho} \int_{0}^{T} \mathbb{P}\left(\sup _{s \in[0, T-t]} S_{s+t}-S_{t} \leq \epsilon, S_{t} \in[0, \epsilon), \bar{S}_{t} \leq M \mid Z_{T} \in[M, M+\epsilon), U=t\right) \frac{d t}{T} \\
& \geq e^{-T} \epsilon^{\rho} \int_{0}^{\epsilon^{\alpha}} \mathbb{P}\left(\bar{S}_{T-t} \leq \epsilon\right) \mathbb{P}\left(S_{t} \in[0, \epsilon), \bar{S}_{t} \leq M\right) d t \geq e^{-T} K_{1} K_{2} \epsilon^{\alpha \rho+\alpha+\rho} .
\end{aligned}
$$

This implies that $x \mapsto \mathbb{P}\left(\bar{X}_{T} \leq x\right)$ is not locally $\gamma$-Hölder continuous at $M$.

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