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Axiomatising the Bayesian Paradigm in Parallel Small Worlds

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There is currently much interest in scenario-focused decision analysis (SFDA), a methodology which provides, among other things, supporting analyses in circumstances in which there are deep uncertainties about the future, i.e. when experts and decision makers (DMs) cannot come to any agreement on some of the probabilities to use in a Bayesian model. This lack of agreement can mean that sensitivity and robustness analyses show that virtually any strategy may be optimal under the beliefs of one or more participants. Scenario-focused analyses fix the deep uncertainties at interesting values in different scenarios and conduct a (Bayesian) decision analysis within each. The results can be informative to the DMs, helping them understand different possible futures and their reactions to them. However, theoretical axiomatisations of subjective expected utility (SEU), the core of decision analysis, do not immediately extend to the context of SFDA. The purpose of this paper is to provide an axiomatisation of SEU that supports SFDA. Scenarios have much in common with Savage's concept of a *small worlds*. We discuss the parallels and then explore two difficulties in extending his and other writers' axiomatisations. The development of SEU offered here overcomes these difficulties. Throughout attention is given to the implications of the theoretical development for the practice of decision analysis.

Key words: deep uncertainty, small world, reference experiment, scenario-focused decision analysis (SFDA), subjective expected utility (SEU)

1. Introduction

“With some inaccuracy, descriptions of uncertain consequences can be classified into two major categories, those which use exclusively the language of probability distributions and those which call for some other principle, either to replace or to supplement.”

K.J. Arrow, 1951

Many Bayesians would argue that no further tools are needed to model uncertainty than subjective probability; some might concede that there is value in a careful and judicious use of sensitivity and robustness analyses along with thoughtful discussion to clarify conceptual vagueness: see French (1995, 2003). However, many non-Bayesian writers have pointed to circumstances in which some uncertainties about future events are too *deep* to agree on probabilities; some effectively deny the conceptual existence of probabilities for such events. Knight (1921) argued for a distinction between circumstances of *risk* in which probabilities could be agreed and those of *uncertainty* in which they could not. Discussions of such issues may be found in the literature centred around decision tables and criteria such as minimax, maximim regret and Hurwicz- α (Luce and Raiffa 1957, French 1986). During the last century, such circumstances were described as exhibiting *strict* uncertainty; nowadays, the term *deep* uncertainty is more common. For more recent perspectives on deep uncertainty, see Lempert *et al* (2003), Spiegelhalter and Riesch (2011), Cox (2012), Durbach and Stewart (2012) and Marchau *et al* (2019).

I shall *not* adopt Knight's position. Conceptually, I adopt a fully subjective Bayesian stance and believe all uncertainties can *in principle* be expressed probabilistically. However, *in practice* it may not be possible in the time available before the results of any analysis are required. Warren Walker, who has led thinking in this area for the past two decades, defines deep uncertainty as "the condition in which analysts do not know or the parties to a decision cannot agree upon (1) the appropriate models to describe interactions among a system's variables (2) the probability distributions to represent uncertainty about key parameter's in the model, and/or (3) how to value the desirability of alternative outcomes" (Walker *et al* 2013). Such lack of agreement may be resolved by further research studies and discussion, but that takes time. This definition of deep uncertainty gives a modern interpretation of Knight's perspective; and, indeed, of Arrow's as embodied in the opening quote. It recognises that our knowledge evolves over time, but perhaps not quickly enough for the decisions that are needed today. For further discussion, see French (2013, 2015), in which I discuss the many of the issues that motivated this work and my perspectives on uncertainty and its modelling in risk and decision analysis.

In such circumstances, scenario planning has often helped (Schoemaker 1995). Recently, various authors have discussed how more quantitative forms of analysis may be combined with scenario planning (Stewart *et al* 2013, Wright and Goodwin 1999), leading to *scenario-focused decision analysis* (SFDA). The aim, in part, has been to address deep uncertainties through 'fixing' these in particular scenarios, so that conventional analyses may be conducted *within* each. By comparing the results *across* scenarios some understanding may be garnered of the robustness of strategies. French (2013, 2015) provides

some further motivation for the merging of scenario thinking with more conventional decision analyses. There have been several studies which use multiple scenarios to conduct parallel quantitative multi-criteria decision analyses (Montibeller *et al* 2006, Ram *et al* 2011, Schroeder and Lambert 2011, Comes *et al* 2013). In all cases, the actual analyses within scenarios were deterministic using multi-attribute value theory. French *et al* (2011) provides an unpublished study in which decision tree analysis is used in each scenario to evaluate the sustainability of nuclear energy in the UK; we discuss parts of this study below. Williamson and Goldstein (2012) indicate that their emulation methods for large complex decision trees can be integrated with scenario planning. As I write this, the UK is mired in *Brexit* planning and negotiations, in preparation for leaving the European Union. Deliberations abound with half-defined, contradictory and often counterfactual assumptions. Many uncertainties face us. Many stakeholders hold quite different values. SFDA has the potential to clarify issues and differences, enlightening the debate.

Intuitively, the methodology of SFDA seems sensible and straightforward. There are significant questions relating to cognitive aspects: e.g. how to elicit subjective probabilities and utilities within the context of a scenario and how to draw the fullest possible information from the results without naively making implicit, but unfounded assumptions about the relative likelihood of scenarios. But mathematically the process looks sound. On deeper consideration (French 2015), however, there are some fundamental issues that require discussion and attention. As Savage (1972) noted, any Bayesian model describes a *small world*. There have been many axiomatisations of the Bayesian model, but these generally have taken the concept of a small world from Savage without great discussion (Laskey and Lehner 1994). Small worlds and scenarios have many similarities: the former provide a

setting for a statistical or decision analysis; the latter a backdrop for qualitative strategic debate in scenario planning. Nonetheless, there are differences. In scenario planning, several scenarios are used. These might or might not overlap. Bayesian analyses are usually set in a single small world. Savage did consider the use of several small worlds, but as a nested sequence leading up to the ‘full’ *grand world*, which he essentially took as being conceptually as close to reality as one can get with a model.

Two issues arise because of the differences between small worlds and scenarios.

1. To measure subjective probabilities and utilities, we need something like a metre rule with gradations that are, at least conceptually, as fine as we need them to be. In practice, we might use a probability wheel or similar, and there are many examples of using such elicitation tools in introductory decision analysis texts (French *et al* 2009, Reilly and Clemen 2013). This is conceptualised as a *reference* or *auxiliary experiment*, which essentially provides a uniform probability distribution against which uncertainties relating to the events and consequences of interest may be compared and measured (French *et al* 2009, pp72-73). In Savage’s and other axiomatisations, this reference experiment is common to all small worlds in the nested sequence, ensuring that probability and utility are measured on common consistent scales. In SFDA the scenarios are not nested.
2. In standard Bayesian subjective expected utility (SEU) theory, reality is taken to act as an attractor so that, with sufficient data, behaviours in the small worlds converge to models of real behaviours. Some scenarios, however, may describe worlds quite different, possibly dramatically different from the future that actually unfolds. Indeed they may be conceptually or practically impossible on current knowledge, i.e. counterfactual.

Analyses within such scenarios cannot describe nor converge to Savage's grand world, yet may be very informative to the decision makers' (DMS') deliberations. For example, in an environmental impact study Rothlisberger *et al* (2012) needed to assess what the likelihoods of different ecologies of an infested lake would have been, if no infestation had taken place.

Thus there is a need to consider the validity and good sense of the Bayesian approach with SFDA. To address these issues, I argue, that we need to separate the reference experiment from the field or σ -field of events that describe possible outcomes in the small world. If our modelling requires only a finite set of events and finite additivity, then the former is all that is needed. But if we need a larger set of events, the full power of continuous random variables and integration to evaluate expectations, then we need the real world events to form a σ -field (French and Rios Insua 2000). We return to this point in Section 4. This separation of the real world from the reference experiment leads to more complex and less mathematically attractive axiomatisations of Bayesian theory, but ones that are both more fit for purpose and also align better with the constructive nature of Bayesian analysis¹.

The focus of this paper is on providing a theoretical justification of SFDA and considering the implications of this for the meaningfulness of some potential comparisons between scenarios. We do not consider in any depth cognitive issues relating to elicitation and the transparency of the process. Nor do we consider any implications of the development for Bayesian statistics, which, of course, relies on the same SEU model.

In the next section, we consider an example of SFDA to motivate the developments that follow. In particular, we emphasise the need to develop SEU theory to fit with scenarios

that not only may differ substantially in context, but also in terms of the beliefs and values that the DM wishes to explore. Our discussion in Section 3 turns to the small worlds which model the scenarios, offering the quantitative context in which Bayesian SEU analyses can take place. As noted in point 1, if analyses in different scenarios, i.e. *parallel* small worlds, are to be commensurate to some level, judgements need to be assessed against the *same* reference experiment. Separating the reference experiment from the small worlds of concern is the tool that allows us to develop SEU theory for SFDA. We axiomatise SEU theory across parallel small worlds with a common reference experiment in Section 4, providing consistent scales of probability across them all, thus meeting point 1 above. Outline proofs are provided in an Appendix, though very similar proofs have been published elsewhere, e.g. French and Rios Insua (2000). The constructive nature of the proofs illustrates many of the more discursive points made in the paper. In Section 5 we extend the example of Section 2 and use this to give substance to the discussion of the implications of the theoretical development for the practice of SFDA. We also note that in addition to the theoretical foundation provided, further work is needed on many cognitive issues. Finally, Section 7 offers some brief concluding remarks.

2. Nuclear sustainability: an example of SFDA

Many developments of decision analysis have found early application in support of decision making in the nuclear domain (Keeney and Nair 1977). Post-Chernobyl there have been many applications in emergency planning and remedial clean-up (Papamichail and French 2013). The disposal of nuclear waste is another area which has seen much application (Keeney and von Winterfeldt 1994, Morton *et al* 2009)

Some time back — before the Fukushima Disaster — I was involved in a multi-disciplinary project to consider the sustainability of nuclear power generation with the UK energy portfolio (French *et al* 2011). Sustainability was to be interpreted in a very broad fashion including economic, environmental, security and safety criteria and was to be considered from the differing perspectives of a wide variety of stakeholders. Furthermore, our time horizon was essentially 60+ years because of the long build-operate-decommission life cycle of a nuclear reactor. Over such periods there were many uncertainties to consider and some were deep. For instance, the economic viability of nuclear and renewable generation related to whether some form of energy storage could be developed to match the slowly adjustable output of nuclear plants and the vagaries of most renewables smoothly to relatively fast changing energy demand. The development of large scale energy storage and the date by which it might come on stream were deeply uncertain and subject to much disagreement. We could examine, however, ‘interesting’ scenarios in which economic storage became available at different dates. The viability of any energy portfolio would also be determined by government policy. Again ‘interesting’ scenarios could be established in which different policies were assumed. Public and hence political acceptability of nuclear power could be affected by a future nuclear accident, and indeed the Fukushima Disaster has led to gross changes in commitments to nuclear energy in many countries. Again many deep uncertainties about such accidents could be explored through different scenarios.

These motivations for exploring differing scenarios relate to deep uncertainties, the belief/probability side of the Bayesian paradigm. But there is also a preference/utility side, and scenarios can elucidate some of the issues there. Debates about nuclear power have been riven by many opposing stakeholder perspectives. If decision analyses are to support

discussion between different stakeholder groups at least some of the analyses need to reflect each group's values. The idea is to explore what the future would look like if each group's values dominated in the decisions and behaviours that shape it. Conducting a decision analysis against such scenarios enables each group to see not only how the various energy policies might perform in their own ideal world, but also how they might perform in other stakeholders' ideal worlds.

Sometimes an event can catalyse radical changes in a decision maker's (DM's) beliefs and preferences (French *et al* 1997). If an event changes the world irrevocably, she² may need to rethink her perspectives, beliefs and values. Moreover, if she recognises the possibility of such an event, then she may be able to anticipate how her judgements might change. In French (2013), I argue that it is only when we have experienced a phenomenon many times that we understand our beliefs and preferences about it. An unanticipated novel event can catalyse a step change in scientific knowledge, even a paradigm shift. Similarly, when we encounter some entirely novel circumstances, we need to evolve our preferences to those circumstances in order to determine what we really want. The near disaster at Three Mile Island (TMI) in 1979 led to new understandings of how hydrogen bubbles might form in a reactor vessel, and hence had significant implications for reactor design. On the value side, TMI catalysed both individuals and society to re-evaluate the costs and benefits of nuclear power. Because such events are so novel, so complex, one cannot build 'rational' models anticipating how beliefs and preferences will evolve. There is no simple *a priori* application of Bayes Theorem to anticipate a paradigm shift in unanticipated circumstances, though with hindsight one might tell a simplistic story along those lines. Nor can one apply models of preference evolution. It has become common over the past 25 years to use qualitative

scenario planning to stimulate decision makers to imagine dramatically different futures and their responses to these. Working *within* each scenario it is possible in many cases for the DM's to anticipate how their beliefs and preferences might evolve.

This example illustrates three important features that need be allowed for in developing an axiomatisation of the SEU model that justifies its use in SFDA:

- deep uncertainties may be fixed in a scenario allowing probability models to be constructed which reflect current knowledge as it would play out in those circumstances;
- a particular stakeholder's values may be explored in a scenario, allowing the DM to explore differing stakeholder perspectives across scenarios;
- gross changes in the DM's beliefs and preferences caused by a dramatic – paradigmatic or catastrophic – change in the her world and environment may be anticipated and explored.

3. Small worlds and Bayesian SEU theory

Most SEU models are built upon a mathematical structure with three components, $\{\Theta, C, A\}$, where:

$$\Theta = \{\theta \mid \theta \text{ is a state of the world}\} \text{ is the state space;} \quad (1)$$

$$C = \{c \mid c \text{ is a consequence}\} \text{ is the consequence space;} \quad (2)$$

$$A = \{a \mid a \text{ is an act which the DM can choose}\} \text{ is the action space.} \quad (3)$$

A state of the world is a possible description of the present and future with all uncertainties resolved, Θ being taken to span all possibilities. However great her uncertainty, the DM is certain that one of the descriptions in Θ is true. The set of consequences C contains all

possible outcomes that may arise from her acts $a \in A$. Each act relates outcomes to each possible state of the world so we might also define A as:

$$A = \{a \mid a: \theta \rightarrow C\}. \quad (4)$$

For the present we shall define A no further and take it to contain all acts that the DM wishes to consider.

The triple $\{\Theta, C, A\}$ is effectively what Savage (1972) dubbed a *small world*. The interpretations of $\{\Theta, C, A\}$ make a host of implausible assumptions about our knowledge of reality and, indeed, ourselves: e.g. that we can anticipate the future well enough that Θ spans *all* possibilities. Our purpose in this paper is to develop an axiomatisation that allows *some* weakening of such assumptions in the direction of greater realism. But first we step back from detail to consider wider issues of the interpretation of a small world, something that concerned Savage deeply and which he did not resolve entirely to his own satisfaction. Savage's approach (§5.5, Savage (1972)) was to consider a *nested* sequence of small worlds increasing in complexity and detail up to a *grand world*, which was essentially reality or as close as he could get to it. He discussed the consistency needed between assumptions about behaviours in one small world and the next in order that each might become a better approximation to the grand world, noting, for instance, that a smaller world state needs to be an event in a larger world that contains it. Savage recognised that the detail and complexity of the grand world would be so great that the nested sequence would never reach it. He found it “difficult to say with any completeness” the form that the process of constructing the sequence would take. Phillips (1984) in his theory of requisite modelling elaborated on the the process somewhat. The key point in Savage's conception was that, as

one proceeded up the sequence through cycles of model evaluation and criticism towards the grand world, each small world would get closer in structure to reality and that, with sufficient data, behaviours predicted by Bayesian models built in the small worlds would converge in some very informal sense to behaviours in reality. That conception is central to many Bayesian discussions of scientific inference; see, e.g., Box (1980). In short, there is an assumption that as observations accumulate, reality attracts good Bayesian models. At least, that assumption underpins much of Bayesian *statistics*: whether it underpins Bayesian *decision analysis* we leave to the discussion in Section 5.

There are many axiomatic developments of the SEU model. French and Rios Insua (2000) provide a survey of several, classifying them broadly into two categories:

- **Revealed preference approaches:** these essentially define and axiomatise properties that might be expected of a *rational* DM's preferences over the action space. With the addition of some structural axioms, it can then be shown that her preferences over A may be modelled by SEU based on a subjective probability distribution over Θ and a utility function over C . Note that the DM's preferences over A are the primitive judgements and from these her beliefs about the relative likelihood of different states in Θ and her preferences over C are deduced, i.e. *revealed*. There are many such approaches, but perhaps the best known and most cited is that of Savage (1972).
- **Constructive approaches:** these define and axiomatise properties that might be expected of a *rational* DM's beliefs about the relative likelihood of states in Θ and her preferences between possible consequences in C and from these *construct* her preferences between actions in A , showing that these may be modelled by expected utilities. DeGroot (1970) provides an archetypal constructive development of the SEU model.

I have always found constructive approaches more persuasive. While, to quote Locke, to others “men’s actions are the best interpreters of their thoughts”, to guide my own thinking, inference and decision making I want a form of analysis that begins with my uncertainties about the world and my preferences between possible future consequences and moves me towards an understanding of the relative advantages of different actions and, indeed, a ranking of them that can guide my choice. As an analyst, I have found that my clients wish for the same. Moreover, the mathematical proofs that constructive axiomatisations lead to the SEU model also mirror the elicitation processes commonly used by analysts to help DMS construct a consistent model of their beliefs and preferences (French *et al* 2009, Keeney and Raiffa 1976, Slovic 1995). Constructive approaches provide transparent justifications of the arguments implicit in an analysis. Revealed preferences lack that transparency. In mathematical terms, of course, they are equivalent since they axiomatise and justify identical results. Nonetheless, the majority of discussion in this paper concerns a constructive development that justifies the use of SEU in SFDA.

I also need a constructive approach because much as I would like to be an instinctive Bayesian, I doubt very much that I am. Too many behavioural studies have shown that humans are not as rational in their inference and decision making as they would wish (Kahneman and Tversky 1974, French *et al* 2009); I am unlikely to be an exception. Nor are my clients. Having described this fallability somewhat pejoratively as being subject to *heuristics and biases*, Kahneman (2011) and others now refer to two style of thinking. In broad terms – these are not offered as precise definitions:

- **System 1 thinking** is somewhat superficial, takes place on the fringes or outside of consciousness and is subject to heuristics and biases;

- **System 2 thinking** is more conscious, more analytic, more careful and usually supported by models articulated within explicit arguments to justify the final inference or decision.

The precise definitions of System 1 and System 2 Thinking are moot within the behavioural sciences, as is the terminology. Shleifer (2012), Evans and Stanovich (2013) and Evans (2012) discuss the controversy. Whatever the terminology, decision analysis has long recognised that instinctive responses and behaviours might not reflect the rationality and consistency that careful, conscious thought might wish. I shall follow Kahneman (2011) in using the non-pejorative System 1 vs System 2 Thinking terminology. Moreover, I also recognise that there may not be a true dichotomy here, but increasing depth between subconscious informal thought and conscious explicit formal thought. For behavioural scientists System 2 Thinking need only be conscious, explicit and analytic; there is no presumption that it is rational in any particular sense. However, Bayesian decision analysis seeks to help the DM adopt Bayesian ways of thought, guiding her away from the foibles of System 1 Thinking.

An analyst has to communicate with and support clients who intuitively think, respond and understand using System 1 styles of thought; yet he must develop an analysis using fully explicit System 2 styles of thinking to support and help build their individual understanding and group consensus (Edwards *et al* 2007, French 2003, 2013, 2015, French *et al* 2009). Even when I am conducting an analysis for myself, there is a need to recognise that my thought processes will inevitably move between Systems 1 and 2 styles of thinking as I move from framing the issues through analysis to inferring, deciding and implementing. From my perspective, the Bayesian SEU model fits in this process by providing a model

of rational behaviour rather than a model of how I actually am. In French (1986), I suggested that the processes of statistical and decision analysis might be thought of as the creation of a model decision maker (MDM) in the small world in which the analysis is to be constructed³ and exploring how she made inferences and decisions in circumstances that modelled the real DM's. The MDM's beliefs and preferences should as close to the real DM's as possible, but constrained to be fully rational in the sense of fitting an axiomatic SEU theory. Shafer (1986) terms this process *argument by analogy*. By observing this inference or decision process model, the real DM gains insight to guide her behaviour in the real world. The Bayesian model is no more than a close metaphor for the cognitive process that the DM should like to follow. For further discussion, see French (2015).

Axiomatisations of SEU are an application of *measurement theory*, which is the body of knowledge that clarifies how behaviours in a quantitative model reflect qualitative behaviours in some system (Krantz *et al* 1971, Suppes *et al* 1989, Luce *et al* 1990, Roberts 1979). Among the many insights the theory brings, it emphasises that there must be some properties in the qualitative system that are reflected the characteristics of the number system used in the quantitative model. Generally this insight leads us into the theory of *scale types*. Specifically, since probability is central to SEU theory both in representing uncertainty and in the construction of a utility scale to represent preference, the axiomatisation must include an assumption that introduces a 0–1 scale of probability. In Savage's Theory, this is essentially provided by his sixth axiom. DeGroot (1970) explicitly assumes the existence of a uniform probability distribution in his fifth axiom. These axioms effectively introduce a randomising device such as a probability wheel, more formally called a *reference* or *auxiliary experiment*, against which the DM may measure

off her uncertainties. However, these theories do so by including the events generated by the reference experiment *within* the $(\sigma-)$ field of events that describes the small world of concern. In terms of measurement, this is equivalent to constructing a theory of length in a world in which metre rules are helpfully lying alongside every object that one might conceivably wish to measure. In practice, of course, one places a rule alongside an object when its length is needed. So too in the practice of decision analysis, the analyst introduces a probability wheel at those points when he needs to interact with the DM to elicit her uncertainties and preferences.

In a series of writings, several of us have introduced axiomatisations of subjective probability and SEU theory that keep the reference experiment separate from the field of events of real concern to the DM within the small world that she is considering (French 1986, 1982, French and Rios Insua 2000, French *et al* 2009, Xie 1995, Xie and French 1997). The theoretical development of this paper builds on those extending the development from that needed for a single or nested sequence of small worlds to several parallel ones. We refer to the earlier work extensively for further motivating discussion. For definiteness, the axioms and notation below build on those in French and Rios Insua (2000) to develop an axiomatisation relevant to parallel small worlds.

We note that the assumptions made in an axiom may have one or more of the following qualities:

- **Rationality**, A_R . Such axioms encode behaviours or judgements that should be considered rational. For instance, transitivity, which is required by PSW1 below, is commonly accepted as a good principle of rationality. These axioms go to the heart of System 2 thinking, describing the aspects of the MDM's behaviour that make her an ideal to whom the real decision maker is willing to look for guidance.

- **Measurement device**, A_{Me} . Such axioms introduce the measurement device that is to be used to quantify the qualitative behaviours and characterise their properties, e.g. Axiom PSW2 below introduces the reference experiment.
- **Context structure**, A_C . Such axioms encode features that must be present in the small world and the MDM's judgements for any analysis to take place. For instance, Axiom PSW3 below includes an assumption that there is some event in the small world that is more likely than an impossibility: strictly, it assumes that certainty is more likely than impossibility.
- **Mathematical structure**, A_{Ma} . Such axioms introduce properties that nice mathematical models need, e.g. countable additivity. It would be conceptually possible to develop statistical, decision and risk analysis on the basis of finite additivity only; the number of entities in the universe is finite after all. But countable additivity gives the mathematical model much more power to explore and understand behaviour.

Categorising axioms according to whether they make rationality, context, measurement or mathematical assumptions emphasises different aspects of building a model in analysis to guide DMs towards rational System 2 thinking. Recognising and understanding that can help in explaining and motivating the analysis to the DMs and in what ways the model can be useful. Of course, few axioms can be categorised simply in just one of these ways. Mathematical elegance and the drive to succinctness means that inevitably several such qualities may be present in each axiom. We indicate for each axiom below the qualities that pertain within it. The reason for doing this is that to address the two concerns raised in Section 1 we should look at the measurement and contextual aspects of the axioms, respectively.

4. Axiomatising SEU theory across parallel small worlds with a common reference experiment

The first step that we take is to widen our notation to cover parallel small worlds, $\{\Theta_w, C_w, A_w\}, w = 1, 2, \dots, W$. We assume that each of the small worlds represents a set of circumstances or futures that a DM would find informative to consider. Each might represent, for example, some realisation of a deep uncertainty, some different sets of knowledge or cultural values. Our aim is to help her by constructing, i.e. axiomatising, in each small world a *rational* MDM who faces a decision which captures the essence of the decision facing her. For each small world we assume that the MDM begins with a qualitative feeling of relative likelihood between events defined on Θ_w , which we denote by the binary relation \succeq_{Θ_w} , and with a qualitative feeling of preference between consequences in C_w which we denote by the binary relation \succeq_{C_w} . It will help in later sections to acknowledge that the MDM's beliefs and preferences are part of the conceptual small world that the real DM is considering and from here on denote each small world, $w = 1, 2, \dots, W$, by $\{\Theta_w, C_w, A_w, \succeq_{\Theta_w}, \succeq_{C_w}\}$.

We follow common practice and introduce $\succ_{\Theta_w}, \sim_{\Theta_w}, \succ_{C_w}$ and \sim_{C_w} with the following meanings:

\succeq_{Θ_w} – at least as likely as;

\succ_{Θ_w} – strictly more likely than, defined by $E \succ_{\Theta_w} F \iff (E \succeq_{\Theta_w} F \text{ and } F \not\succeq_{\Theta_w} E)$;

\sim_{Θ_w} – equally likely as, defined by $E \sim_{\Theta_w} F \iff (E \succeq_{\Theta_w} F \text{ and } F \succeq_{\Theta_w} E)$;

\succeq_{C_w} – at least as good as;

\succ_{C_w} – strictly better than, defined by $c_1 \succ_{C_w} c_2 \iff (c_1 \succeq_{C_w} c_2 \text{ and } c_2 \not\succeq_{C_w} c_1)$;

\sim_{C_w} – indifferent to, defined by $c_1 \sim_{C_w} c_2 \iff (c_1 \succeq_{C_w} c_2 \text{ and } c_2 \succeq_{C_w} c_1)$.

N.B. these interpretations are well defined when \succeq_{Θ_w} and \succeq_{C_w} are weak orders (French 1986), as assumed in Axioms PSW1 and PSW9 below.

In developing SEU for parallel small worlds, we shall need to be clear on the interpretation of the elements of $\{\Theta_w, C_w, A_w, \succeq_{\Theta_w}, \succeq_{C_w}\}$. Firstly, the Θ_w need not be related to each other in any simple fashion; neither need the A_w nor the C_w . It may be that one starts with some encompassing small world $\{\Theta, C, A, \succeq_{\Theta}, \succeq_C\}$, then constructs each of the $\{\Theta_w, C_w, A_w, \succeq_{\Theta_w}, \succeq_{C_w}\}$, perhaps by assuming that some event on Θ happens or that some $a \in A$ becomes unavailable, thus forming the new small world by conditioning. However, the spirit of scenario-planning encourages us to consider a much wider range of small worlds, which may be quite unrelated, almost paradigmatically different. Similarly, the \succeq_{Θ_w} and \succeq_{C_w} need not be related in any direct sense, perhaps because the MDM in each small world may represent the beliefs and preferences of different stakeholders or because the small world relates to the aftermath of a dramatic event that causes the DM to reflect on and change her beliefs and values⁴.

For her beliefs and preferences to be assessed consistently across the parallel small worlds, the MDM must relate all her judgements to the *same* reference experiment. This will ensure that her probability assessments represent the same levels of uncertainty within all scenarios. Without further assumptions, numerical utilities will not be comparable across scenarios; utilities are only unique up to an affine transformation. Later we will discuss further assumptions that may give some comparability between the utility functions.

We begin with the axioms that ensure that the MDM's assessment of relative likelihoods over the Θ_w can be modelled by probabilities. For each $w = 1, 2, \dots, W$, we let \mathcal{Q}_w be a field or σ -field of events defined on Θ_w . To assess the probabilities of events in \mathcal{Q}_w we assume

that the MDM compares each with events defined on some fair – or uniform – randomising device. We represent events on this randomising device by intervals in $[0, 1]$. For instance if the randomising device is a probability wheel, the event that the pointer stops spinning in a given sector would be identified with an interval of length equal to the proportion of the full circle represented by that sector. We let \mathcal{B} be the σ -field of open and closed intervals on $[0, 1]$. To be absolutely clear: we used the *same* \mathcal{B} in comparisons with $\mathcal{Q}_w, \forall w$.

First we need to assume that she is prepared to compare events in \mathcal{Q}_w with events in \mathcal{B} , but we do not want to do this in such a way that the minimal σ -field of all events in both \mathcal{Q}_w and \mathcal{B} is necessarily formed. At no point in a decision analysis is it necessary to consider complex events and strategies involving unions and intersections of events drawn from both; only simple comparisons are needed. Thus we extend \succeq_{Θ_w} to $\mathcal{Q}_w \cup \mathcal{B}$, but note this is the *set* union of \mathcal{Q}_w and \mathcal{B} not the minimal σ -field containing both. Our first axiom demands that \succeq_{Θ} is a weak order, i.e. complete and transitive.

Axiom PSW1 [Qualities: A_R, A_{Me}, A_C]

$\forall w = 1, 2, \dots, W, \succeq_{\Theta_w}$ is a weak order on $\mathcal{Q}_w \cup \mathcal{B}$.

Next we need assume that her judgements of the relative likelihood of events in \mathcal{B} correspond to a uniform distribution in the obvious way. Note that this axiom implies that \succeq_{Θ_w} , for $w = 1, 2, \dots, W$, are identical on \mathcal{B} .

Axiom PSW2 [Qualities: A_{Me}]

Let I, J be intervals, open or closed, in \mathcal{B} with lengths ℓ_I and ℓ_J .

Then, $\forall w = 1, 2, \dots, W, I \succeq_{\Theta_w} J \iff \ell_I \geq \ell_J$.

In most axiomatisations of subjective probability the next assumption would be to assume that \succeq_{Θ_w} is a qualitative probability over a σ -field containing both \mathcal{Q}_w and \mathcal{B} (French and

Rios Insua 2000), but we want to avoid going that far. We need it to be a qualitative probability on each of \mathcal{Q}_w and \mathcal{B} . This is achieved by Axioms PSW3 and PSW4. However, elicitation of her probabilities over Θ_w only requires that the MDM can compare events in \mathcal{Q}_w directly with events in \mathcal{B} , and that she does so consistently over set unions within each of \mathcal{Q}_w and \mathcal{B} . This requirement is encoded by Axiom PSW5.

Axiom PSW3 [Qualities: A_R, A_C]

$\forall w = 1, 2, \dots, W$, \succeq_{Θ_w} restricted to \mathcal{Q}_w is a qualitative probability.

Thus in addition to it being a weak order on \mathcal{Q}_w :

(a) $\forall R \in \mathcal{Q}_w, R \succeq_{\Theta_w} \emptyset$. Moreover $\Theta_w \succ_{\Theta_w} \emptyset$.

(b) $\forall R, S, T \in \mathcal{Q}_w$ with $R \cap T = \emptyset = S \cap T$: $R \succeq_{\Theta_w} S \iff R \cup T \succeq_{\Theta_w} S \cup T$.

Axiom PSW4 [Qualities: A_R, A_{Me}]

\succeq_{Θ_w} restricted to \mathcal{B} is a qualitative probability.

Thus in addition to it being a weak order on \mathcal{B} :

(a) $\forall E \in \mathcal{B}, E \succeq_{\Theta_w} \emptyset$. Moreover $[0, 1] \succ_{\Theta_w} \emptyset$.

(b) $\forall E, F, G \in \mathcal{B}$ with $E \cap G = \emptyset = F \cap G$: $E \succeq_{\Theta_w} F \iff E \cup G \succeq_{\Theta_w} F \cup G$.

Axiom PSW5 [Qualities: A_R, A_{Me}]

$\forall w = 1, 2, \dots, W$, \succeq_{Θ_w} on $\mathcal{Q}_w \cup \mathcal{B}$ obeys:

(a) $\Theta_w \sim_{\Theta_w} [0, 1]$.

(b) $\forall R, S \in \mathcal{Q}_w$ and $\forall E, F \in \mathcal{B}$ with $R \cap S = \emptyset = E \cap F$:

$(R \succeq_{\Theta_w} E, S \succeq_{\Theta_w} F) \Rightarrow R \cup S \succeq_{\Theta_w} E \cup F$.

$(E \succeq_{\Theta_w} R, F \succeq_{\Theta_w} S) \Rightarrow E \cup F \succeq_{\Theta_w} R \cup S$.

The next axiom effectively assumes that there is a π_R , $\forall R \in \mathcal{Q}_w$, such that the MDM holds $R \sim_{\Theta_w} [0, \pi_R]$. It is encoded in terms of closed sets, however, to capture the iterative, bounding process of elicitation commonly used to assess π_R , which becomes, of course, her subjective probability of R .

Axiom PSW6 [Qualities: A_{Me} , A_{Ma}]

$\forall w = 1, 2, \dots, W$, $\forall R \in \mathcal{Q}_w$, both:

$$\{\pi \in [0, 1] \mid [0, \pi] \succeq_{\Theta_w} R\} \text{ and } \{\pi \in [0, 1] \mid R \succeq_{\Theta_w} [0, \pi]\}$$

are closed.

Axioms PSW1 to PSW6 are enough to ensure that the resulting probabilities are finitely additive. Axiom PSW7 provides the necessary extension to ensure countable additivity.

Axiom PSW7 [Qualities: A_{Ma}]

$\forall w = 1, 2, \dots, W$, if $R_1 \supseteq R_2 \supseteq R_3 \supseteq \dots$ is a decreasing sequence of events in \mathcal{Q}_w

such that $R_i \succeq_{\Theta_w} [0, \pi]$, $\forall i$ for some fixed $\pi \in [0, 1]$, $\bigcap_{i=1}^{\infty} R_i \succeq_{\Theta_w} [0, \pi]$.

Central to the Bayesian approach is the use of Bayes Theorem to prescribe how the MDM should update her beliefs in the light of observed data. To do this there is a need to be able to construct and manipulate conditional probabilities coherently. For any $R, S, T \in \mathcal{Q}_w$ for which the MDM holds $T \succ_{\Theta_w} \emptyset$, we define $(R \mid T) \succeq_{\Theta_w} (S \mid T)$ if she holds R to be at least as likely as S when she knows that T has occurred.

Axiom PSW8 [Qualities: A_R]

$\forall w = 1, 2, \dots, W$, $\forall R, S, T \in \mathcal{Q}_w$ with $T \succ_{\Theta_w} \emptyset$:

$$(R \mid T) \succeq_{\Theta_w} (S \mid T) \iff R \cap T \succeq_{\Theta_w} S \cap T.$$

Axioms PSW1—PSW8 allow that a subjective probability representation of \succeq_{Θ_w} may be constructed on each small world $\{\Theta_w, C_w, A_w, \succeq_{\Theta_w}, \succeq_{C_w}\}$, $w = 1, 2, \dots, W$.

Lemma 1

$\forall w = 1, 2, \dots, W$:

(a) if \mathcal{Q}_w is a field, then under Axioms PSW1—PSW6 there is a unique, finitely additive probability distribution P_{Θ_w} on Θ_w agreeing with \succeq_{Θ_w} in the sense that $\forall R, S \in \mathcal{Q}_w$,

$$R \succeq_{\Theta_w} S \iff P_{\Theta_w}(R) \geq P_{\Theta_w}(S);$$

(b) if \mathcal{Q}_w is a σ -field, then under Axioms PSW1—PSW7 there is a unique, countably additive probability distribution P_{Θ_w} on Θ_w agreeing with \succeq_{Θ_w} in the sense that $\forall R, S \in \mathcal{Q}_w$,

$$R \succeq_{\Theta_w} S \iff P_{\Theta_w}(R) \geq P_{\Theta_w}(S);$$

(c) if, in addition, Axiom PSW8 holds, then $\forall R, S, T \in \mathcal{Q}_w$ for which $T \succ_{\Theta_w} \emptyset$,

$$(R | T) \succeq_{\Theta_w} (S | T) \iff P_{\Theta_w}(R | T) \geq P_{\Theta_w}(S | T).$$

PROOF: See Appendix.

We can now move onto to the axioms that underpin the utility part of SEU theory. Initially, we state a set of axioms that allow the utility function to be constructed for the case that C_w is finite; or, if C_w is conceptually infinite, then the set $\{c \in C_w \mid c = a(\theta) \text{ for some } a \in A_w \text{ and } \theta \in \Theta_w\}$ is finite. First we need assume that the MDM is prepared to compare any two consequences in C_w and do so in a transitive way.

Axiom PSW9 [Qualities: A_R, A_C]

$\forall w = 1, 2, \dots, W$, \succeq_{C_w} is a weak order on C_w .

The elicitation process, i.e. the measurement process, used to assess utilities very often asks for comparisons between possible consequences and hypothetical bets based on the randomising device that involving a finite number of consequences. For this reason we introduce \mathcal{P}_{S_w} , the set of simple distributions over C_w , i.e. distributions that give positive

probability to a finite number of elements of C_w . We identify each simple distribution $p \in \mathcal{P}_{S_w}$ with a bet that gives prizes (consequences) $c \in C_w$ according to the probabilities p . We shall assume that the MDM is prepared to make comparisons between any $c \in C_w$ and any $p \in \mathcal{P}_{S_w}$, extending \succeq_{C_w} to \succeq_{S_w} as:

Axiom PSW10 [Qualities: A_R, A_{Me}]

$\forall w = 1, 2, \dots, W$, the relation \succeq_{S_w} extends \succeq_{C_w} such that:

- (a) \succeq_{S_w} is a weak order on $C \cup \mathcal{P}_{S_w}$;
- (b) \succeq_{S_w} restricted to C_w is identical to \succeq_{C_w} ;
- (c) For each $c \in C_w$, let $\Psi_c \in \mathcal{P}_{S_w}$ be the degenerate distribution which gives probability 1 to c . Then $c_1 \succeq_{C_w} c_2 \iff \Psi_{c_1} \succeq_{S_w} \Psi_{c_2}$.

Denote by $c_1 \alpha c_2$ with $c_1, c_2 \in C_w$ and $0 \leq \alpha \leq 1$ the simple distribution which gives probability α to c_1 and probability $(1 - \alpha)$ to c_2 . By definition, $c_1 \alpha c_2 \in \mathcal{P}_{S_w}$. The next axiom assumes that for any pair of such bets with the same two consequences the MDM prefers the bet with the higher probability of gaining the better prize.

Axiom PSW11 [Qualities: A_R, A_{Me}]

$\forall w = 1, 2, \dots, W$, $\forall c_1, c_2 \in C_w$ with $c_1 \succ_{C_w} c_2$, $c_1 \alpha c_2 \succ_{S_w} c_1 \alpha' c_2 \iff \alpha > \alpha'$

We also make the assumption that C_w is bounded⁵ in the sense that there is a best and worst consequence in C_w .

Axiom PSW12 [Qualities: A_C]

$\forall w = 1, 2, \dots, W$, $\exists c_w^*, c_{w*} \in C_w$ such that $c_w^* \succ_{S_w} c_{w*}$ and $\forall c \in C_w$,

$$c_w^* \succeq_{S_w} c \succeq_{S_w} c_{w*}.$$

In the same way that Axiom PSW6 essentially assumed that the MDM could identify a simple event in the reference experiment which she perceived as equally likely to each event in \mathcal{Q}_w so Axiom PSW13 essentially assumes that for each $c \in C_w$ she can identify a simple gamble between c_w^* and c_{w*} based on the reference experiment that she holds to be of equal value to it: $\exists \alpha \in [0, 1]$ such that $c \sim_{C_w} c_{w*} \alpha c_w^*$. Again, however, this is stated more generally in terms of closed sets to capture the commonly used process that is used in elicitation to bracket the utility of c .

Axiom PSW13 [Qualities: A_{Me} , A_{Ma}]

$\forall w = 1, 2, \dots, W$, $\forall c_1, c_2, c_3 \in C_w$ such that $c_1 \succeq_{C_w} c_2 \succeq_{C_w} c_3$, with at least one strict preference:

$$\{\alpha \in [0, 1] \mid c_1 \alpha c_3 \succeq_{S_w} c_2\} \text{ and } \{\alpha \in [0, 1] \mid c_2 \succeq_{S_w} c_1 \alpha c_3\}$$

are both closed.

The next two axioms assume that for any simple distribution over C_w the MDM can construct a gamble of the form $c_1 \alpha c_2$ which she considers of equal value. Let $\langle \pi_1, b_1; \pi_2, b_2; \dots; \pi_r, b_r \rangle \in \mathcal{P}_{S_w}$ be the simple distribution which leads to $b_i \in C \cup \mathcal{P}_{S_w}$ with probability π_i , $\sum_i \pi_i = 1$. Note that the outcomes involved may differ from distribution to distribution, but for convenience of notation we assume that b_1, b_2, \dots, b_r are common in any comparisons of different distributions: i.e. that outcomes with probability zero are introduced to align the two distributions in the obvious manner. Note also that by allowing that an outcome may lie in \mathcal{P}_{S_w} , we are introducing compound gambles into the model, i.e. gambles in which one or more prizes might be entries into further gambles in \mathcal{P}_{S_w} .

Axiom PSW14, known as *Substitutability*, assumes that the MDM is indifferent between two distributions if one outcome is substituted by another for which she is indifferent.

Axiom PSW14 [Qualities: A_R]

$\forall w = 1, 2, \dots, W$, let $b, b' \in C_w \cup \mathcal{P}_{S_w}$, $b \sim_{S_w} b'$. Let $p = \langle \dots; \pi, b; \dots \rangle \in \mathcal{P}_{S_w}$ lead to b with probability π , and let $p' = \langle \dots; \pi, b'; \dots \rangle \in \mathcal{P}_{S_w}$ differ from p only in that b' has been substituted for b . Then:

$$\langle \dots; \pi, b; \dots \rangle \sim_{S_w} \langle \dots; \pi, b'; \dots \rangle.$$

Axiom PSW15 is a requirement that the MDM's preferences over \mathcal{P}_{S_w} are independent of the number of randomisations involved in generating the ultimate probabilities of outcomes in C_w . It is known as the *Reduction of Compound Lotteries*.

Axiom PSW15 [Qualities: A_R]

$\forall w = 1, 2, \dots, W$, consider the mixture of simple distributions $\langle \pi_1, p_1; \pi_2, p_2; \dots; \pi_s, p_s \rangle$, where $p_i = \langle \pi_{i1}, c_1; \pi_{i2}, c_2; \dots; \pi_{ir}, c_r \rangle \in \mathcal{P}_{S_w}$. Let $\rho_j = \sum_{i=1}^s \pi_i \pi_{ij}$ for $j = 1, 2, \dots, r$. Then:

$$\langle \pi_1, p_1; \pi_2, p_2; \dots; \pi_s, p_s \rangle \sim_{S_w} \langle \rho_1, c_1; \rho_2, c_2; \dots; \rho_r, c_r \rangle.$$

Axioms PSW1—PSW15 provide the basis for the construction of the SEU representation of the MDM's preferences over \mathcal{P}_{S_w} .

Lemma 2

In the presence of Axioms PSW1—PSW8, Axioms PSW9—PSW15 imply that $\forall w = 1, 2, \dots, W$, $\exists u_w : C \rightarrow \mathfrak{R}$, an agreeing utility function on C_w such that $\forall c, c' \in C_w$:

$$c \succeq_{C_w} c' \iff u_w(c) \geq u_w(c').$$

Moreover, $\forall p, q \in \mathcal{P}_{S_w}$:

$$p \succeq_{S_w} q \iff E(u_w(c) | p) \geq E(u_w(c) | q),$$

where the expectations are taken with respect to the distributions p and q respectively. In addition, $u_w(\cdot)$ is unique up to a positive affine transformation.

PROOF: See Appendix.

Remember that, for each small world, distributions in \mathcal{P}_{S_w} only give positive probability to a finite set of consequences in C_w . Thus the expected utilities in the second part of Lemma 2 are finite sums. Shortly we will discuss the more general case of continuous distributions and integration. First though we remain with the finite case. Note that \mathcal{P}_{S_w} is not A_w . To justify using SEU theory as a basis for prescriptive analysis, we need to ensure that the MDM perceives the possible actions in A_w as equivalent to actions in \mathcal{P}_{S_w} , at least in so far as helping her construct her preferences over A_w . Define for $a \in A_w$:

$$P_{C_w}(c' | a) = \sum_{\{\theta \in \Theta_w | c' = a(\theta)\}} P_{\Theta_w}(\theta). \quad (5)$$

which gives the probability that action $a \in A_w$ leads to the consequence $c' \in C_w$. Then for each $a \in A_w$ define $p_a \in \mathcal{P}_{S_w}$ as $\langle P_{C_w}(c_1 | a), c_1; P_{C_w}(c_2 | a), c_2; \dots; P_{C_w}(c_r | a), c_r \rangle$, where $\{c_1, c_2, \dots, c_r\}$ contains all consequences for which a gives positive probability. We assume that the MDM perceives that a and p_a have the same value to her; or rather that she prefers a to b if and only if she prefers p_a to p_b . We introduce the obvious notation \succeq_{A_w} for her preferences on A_w .

Axiom PSW16 [Qualities: A_R]

$\forall w = 1, 2, \dots, W$, suppose that $a, b \in A_w$ are such that both lead only to a finite number of consequences and let $\{c_1, c_2, \dots, c_r\} \subset C_w$ contain these,

i.e. $P_{C_w}(\{c_1, c_2, \dots, c_r\} | a) = 1 = P_{C_w}(\{c_1, c_2, \dots, c_r\} | b)$. Then:

$$a \succeq_{A_w} b \iff p_a \succeq_{S_w} p_b$$

Axiom PSW16 justifies the use of SEU to construct the MDM's preference ranking over A_w when the actions can lead to at most a finite number of consequences, *viz.*

$$a \succeq_{A_w} b \iff E(u_w(c) | p_a) \geq E(u_w(c) | p_b) \quad (6)$$

We interpret $E(u_w(c) | p_a)$ as the *subjective expected utility* of $a \in A_w$ and usually denote it as simply $E(u_w(c) | a)$ or, even more simply, $E(u_w | a)$.

The construction implicit in these axioms defines $u_w(\cdot)$ uniquely with $u_w(c^*) = 1$ and $u_w(c_*) = 0$. However any affine transformation of $u_w(\cdot)$ will also represent the same preferences since $\forall \alpha > 0$ and $\forall -\infty < \beta < \infty$:

$$E(u_w(c) | a) \geq E(u_w(c) | b) \iff E(\alpha u_w(c) + \beta | a) \geq E(\alpha u_w(c) + \beta | b) \quad (7)$$

Moreover, if $u_w(\cdot)$ and $u'_w(\cdot)$ represent the same preferences then $u'_w(\cdot) = \alpha u_w(\cdot) + \beta$ for some $\alpha > 0$ and some $-\infty < \beta < \infty$ (DeGroot 1970, French and Rios Insua 2000).

Pulling the threads together gives:

Theorem 1

$\forall w = 1, 2, \dots, W$

(a) *Axioms PSW1—PSW6, PSW9—PSW16 are sufficient to justify the SEU representation with finitely additive probabilities of the MDM's preferences over A_w , viz.:*

$$a \succeq_{A_w} a' \iff E(u_w | a) \geq E(u_w | a').$$

where:

$$E(u_w | a) = \sum_{\theta \in \Theta_w} u_w(a(\theta)) P_{\Theta_w}(\theta),$$

$u_w(\cdot)$ is unique up to a positive affine transformation, and $P_{\Theta_w}(\cdot)$ is finitely additive.

(b) *If Axiom PSW7 holds, $P_{\Theta_w}(\cdot)$ is countably additive.*

(c) *If Axiom PSW8 holds, the use of Bayes Theorem to model the updating of belief in the light of data is justified.*

PROOF: See Appendix.

For many practical decision analyses the assumption of finiteness is not a constraint. The theory is sufficient to justify Bayesian analysis on most decision trees and influence diagrams. However, for much of statistics and for more complex decision analyses, the theory needs to be extended to cases in which one or more of the spaces are infinite. It seems to me that there are two ways forward here. Either way requires us to remember that none of the small worlds $\{\Theta_w, C_w, A_w, \succeq_{\Theta_w}, \succeq_{C_w}\}$ are the real world; each is only a putative model.

The first – my preferred route, at least conceptually – is to axiomatise a finite SEU model with finite additivity; in doing so, we minimise the influence of axioms with A_{Ma} qualities. Then we approximate the analysis in this finite model by using continuous probability distribution and utility functions, and use integration to approximate the finite sums that form the expected utilities. After all in practice in any analysis, we elicit:

- a finite number of properties of any probability distribution and then fit a convenient, tractable, often standard distributional form; and
- the utilities of a finite number of consequences, perhaps properties such as risk attitude, and then sketch in or adopt a convenient, tractable utility function.

Moreover, we use computational methods such as MCMC in our analyses which approximate the ‘correct’ analytic forms. So why pretend that we are approximating a fully axiomatically justified countably additive, infinite small world? Why not simply admit that the axiomatised SEU model is based on a finite small world with finitely additive probabilities?

The alternative route is, of course, to extend the axiomatisation to justify an infinite SEU model in which probabilities are countably additive and that sufficient measurability conditions hold for all necessary expectations to exist. This is the route that most of the Bayesian literature has (implicitly) followed. So we now indicate how this can be done. Firstly, to avoid restating all the axioms, we redefine \mathcal{P}_{S_w} as the set of *all* distributions over C_w . Since C_w is no longer finite, we specify that all distributions in \mathcal{P}_{S_w} are taken to be over the *same* (σ -)field \mathcal{D}_w . \succeq_{S_w} is extended similarly to be a relation on the new $C \cup \mathcal{P}_{S_w}$. However, we need to modify the statements, though not the intent, of Axioms PSW14 and PSW15, since they explicitly refer to specific finite distributions over C_w .

Axiom PSW14' [Qualities: A_R]

$\forall w = 1, 2, \dots, W$, let $b, b' \in C \cup \mathcal{P}_{S_w}$, $b \sim_{S_w} b'$. Let $p \in \mathcal{P}_{S_w}$ be a distribution which leads to b with probability π , and let $p' \in \mathcal{P}_{S_w}$ differ from p only in that b' has been substituted for b , i.e. p' gives b' with probability π and all other consequences with the same probabilities that b does.

Then $p \sim_{S_w} p'$.

Axiom PSW15' [Qualities: A_R]

$\forall w = 1, 2, \dots, W$, consider the mixture of distributions $\pi \circ p$ in which $p \in \mathcal{P}_{S_w}$ is selected to give an outcome $b \in C_w \cup \mathcal{P}_{S_w}$ according to the probability distribution $\pi \in \mathcal{P}_{S_w}$.

Let $\rho \in \mathcal{P}_{S_w}$ be the distribution with probability density defined by $\rho(c) = \int_{\mathcal{P}_{S_w}} p(c) d\pi(p)$.

Then $\rho \sim_{S_w} \pi \circ p$.

Finally, we need to ensure that \mathcal{Q}_w and \mathcal{D}_w are compatible and that $\forall a \in A_w$, the utility function $u_w(a(\cdot))$ is measurable with respect to \mathcal{Q}_w so that the expectation:

$$E(u_w | a) = \int_{\Theta_w} u_w(a(\theta)) dP_{\Theta_w}(\theta) \quad (8)$$

is well defined. Thus the axioms which ensure this relate much more to the mathematical structure of the small world than the rationality that we require of the MDM. Since we are still assuming through Axiom PSW12 that the consequence space is bounded, the following axiom is sufficient to ensure that $u_w(\cdot)$ is measurable⁶.

Axiom PSW17 [Qualities: A_{Ma}]

$\forall w = 1, 2, \dots, W,$

(a) $\forall Q \in \mathcal{Q}_w$ and $\forall a \in A_w, \{a(\theta) \in C_w \mid \theta \in Q\} \in \mathcal{D}_w$

(b) $\forall x \in [0, 1], \{c \mid x \geq u_w(c)\} \in \mathcal{D}_w$; *i.e.* $u_w(\cdot)$ is measurable with respect to \mathcal{D}_w .

These axioms are sufficient to justify the SEU representation in each small world.

Theorem 2

$\forall w = 1, 2, \dots, W,$ *Axioms PSW1–PSW13, PSW14', PSW15', PSW16 and PSW17 are sufficient to justify the SEU representation of the MDM's preferences over A_w , viz.:*

$$a \succeq_{A_w} a' \iff E(u_w \mid a) \geq E(u_w \mid a').$$

where:

$$E(u_w \mid a) = \int_{\theta \in \Theta_w} u_w(a(\theta)) dP_{\Theta_w}(\theta),$$

$u_w(\cdot)$ is unique up to a positive affine transformation, $P_{\Theta_w}(\cdot)$ is countably additive, and the use of Bayes Theorem to model the updating of belief in the light of data is justified.

PROOF: See Appendix.

Two points related to this axiomatisation are worthy of emphasis. First, while this development lacks the mathematical beauty and sophistication of Savage's or DeGroot's, it does justify the processes used in the practice of statistical and decision analysis. Particularly

in the finite case, the axioms directly justify the constructions that we use with our clients to build subjective probabilities and utilities. In French (1986) and French *et al* (2009), we demonstrate these in a much less mathematical way, showing where the assumptions are brought to bear to justify the elicitation and analysis in a simple example. Second, the axiomatisation refers to the MDM's judgements \succeq_{Θ_w} and \succeq_{C_w} . In assuming that the MDM can make consistent judgements of finer and finer discrimination as required by Axioms PSW6 and PSW13, we are assuming that she is not susceptible to System 1 thinking nor of bounded cognitive capacity. She is there to provide an idealised metaphor of a decision maker to help guide the real DM towards a more rational decision than she would otherwise make. Of course, the actual DM may find the MDM's entirely explicit System 2 analysis unconvincing and follow her own judgement, which may be based on System 1 Thinking; and her choice may in some instances be none the worse for that (Goldstein and Gigerenzer 2002). However, the point is that this axiomatisation has at its heart a recognition of the practical process of statistical and decision analysis in which System 2 analysis is used to challenge the DM's System 1 Thinking to help build her understanding of the issues.

Rather obviously, this axiomatisation of SEU for several small worlds contains and hence justifies SEU in a single small world.

It is important to be clear on what comparisons may be represented through the subjective probabilities, utility values and expected utilities that have been developed:

- *Within* each small world $\{\Theta_w, C_w, A_w, \succeq_{\Theta_w}, \succeq_{C_w}\}$, $w = 1, 2, \dots, W$:

— P_w represents the MDM's beliefs over Θ_w , i.e.

$$\forall E, F \in \mathcal{Q}_w, E \succeq_{\Theta_w} F \iff P_{\Theta_w}(E) \geq P_{\Theta_w}(F).$$

In particular, certainties are represented by $P_{\Theta_w}(\Theta_w) = 1$, since $\Theta_w \sim_{\Theta} [0, 1]$.

— $u_w(\cdot)$ represents the MDM's preferences over C_w , i.e.

$$\forall c, c' \in C_w, c \succeq_{C_w} c' \iff u_w(c) \geq u_w(c').$$

— Expected utilities represent the MDM's preferences over A_w , i.e.

$$\forall a, a' \in A_w, a \succeq_{A_w} a' \iff E(u_w | a) \geq E(u_w | a').$$

- It should be remembered that the MDM may represent quite different belief and preferences within each small world. This may be because the MDM has different knowledge and evidence in different scenarios or because she represents different DMS or stakeholders. We have not introduced into the notation a conditioning history \mathcal{H} such as used by Jeffrey (1961) and other writers; but if we had, it would represent not just the MDM's entire experience, but also her culture, world view, etc. – and it would condition her utilities as well as her probabilities. This implies that we allow that \succeq_{Θ_w} might be conditioned on counterfactual information. Note also that in allowing that the MDM's beliefs and preferences may be quite different in each small world, we are breaking away from any idea that the small worlds need be nested in a single grand world. The framework of multiple small worlds allows us to justify not only scenario-focused decision analyses, but also analyses such as in Rothlisberger *et al* (2012) in which one considers worlds in which some past event did *not* happen.
- *Between* small worlds the MDM beliefs exhibit some comparability in the sense that she believes an event in one small world to be equally likely to an event in another if $E \in \mathcal{Q}_{w_1}$, $F \in \mathcal{Q}_{w_2}$ and $G \in \mathcal{B}$ are such that $E \sim_{\Theta_{w_1}} G$ and $F \sim_{\Theta_{w_2}} G$. Similarly, by comparisons through the common reference experiment she can assess whether one event in one small world is more likely than another event in another small world or not. But again remember that her probabilities will almost certainly be conditioned on different evidence and experience in the different small worlds.

- Without further assumptions, however, preferences are not comparable *across* small worlds, because each $u_w(\cdot)$ is only unique up to different affine transformations. This, of course, has the implication that any idea of using maximin-like ideas to select strategies that are least bad across the small worlds would lay the analysis open to spurious choices of scale.

5. Nuclear sustainability: further discussion

To give some substance to the discussion, we return to the nuclear sustainability example of Section 2. French *et al* (2011) describe a realistic, but hypothetical SFDA of this problem. Because the impact of nuclear generation depends on other generation in the energy portfolio, strategies included planned changes in the use of gas, coal and renewables. Thus a strategy was a specification of energy policy as foreseen for the period 2010-2060. Thus the analysis took different energy portfolios and projected various costs and impacts arising between 2010 and 2060. Key uncertainties included:

- whether a hydrogen economy would develop within a useful timeframe;
- whether carbon capture and storage would become economic in the near future;
- the completion of a connector to the European Supergrid;
- the future regulation of nuclear power;
- the possibility of a major nuclear accident in the world changing public acceptance of nuclear power significantly;
- the development of the UK economy.

A wide range of scenarios were considered. Table 1 summarises the results for six, three relating to different economic contexts and three to different regulation regimes. In the

latter case, we looked to cultural theory as a guide to different risk attitudes (Douglas 1992) in building each of the three scenarios. [N.b. ‘current’ relates to the late 2000s.]

- **Base Case:** current planning and regulatory frameworks; economic growth in line with rest of world.
- **Hierarchist:** stringent regulation, but faster planning and low probability of abandoning nuclear in event of accident.
- **Entrepreneurial:** lower regulation, faster planning cycles, less surplus capacity, more likely to abandon nuclear in event of accident.
- **Egalitarian:** stringent and slower regulation and planning, extremely likely to abandon nuclear in the event of an accident.
- **UK Economic Decline:** UK performs poorly relative to the rest of the world; other factors as in the Base Case.
- **UK Economic Growth:** UK performs well relative to the rest of the world; other factors as in the Base Case.

Four energy policies are presented in Table 1. N.b. coal needed to be phased out in all scenarios because of international commitments.

- **Status quo:** the UK’s current planning for building new nuclear plants and managing the rest of the energy portfolio.
- **No new nuclear:** abandon plans for new nuclear generation and increase gas and renewables.
- **Reduce gas:** reduce generation by gas, increase nuclear and renewables.
- **Go it alone:** abandon planned link with European Supergrid and increase renewables.

Energy Policy		Base Case		Hierachist		Entrepreneurial		Egalitarian		UK Economic Decline		UK Economic Growth	
		Industry	Consumer	Industry	Consumer	Industry	Consumer	Industry	Consumer	Industry	Consumer	Industry	Consumer
Status quo	Rank	3	3	3	4	2	2	4	4	4	4	2	3
	Exp	1526	270	1419	209	2058	186	1427	241	1010	191	2309	245
	Loss												
No new nuclear	Rank	1	1	1	1	4	4	1	1	1	1	1	1
	Exp	1445	256	1395	203	2147	195	1363	231	948	179	2269	240
	Loss												
Reduce gas	Rank	4	4	4	2	3	3	2	2	2	2	3	4
	Exp	1572	271	1431	208	2141	192	1378	235	982	185	2326	246
	Loss												
Go it alone	Rank	2	2	2	3	1	1	3	3	3	3	4	2
	Exp	1524	269	1416	208	2054	186	1407	239	1006	190	2279	243
	Loss												

Table 1: *Possible output from SFDA of nuclear sustainability*

Because we were dealing mainly with costs, we worked in expected loss, the negative of expected utility. These were discounted over time and considered for two stakeholder groups: the power industry and consumers. Note again that these were hypothetical studies; probability and value judgements were made by the project team in line with each scenario and stakeholder group.

For each energy policy under each scenario, Table 1 provides the expected losses to the industry and consumer groups. The ranking of the policies within each scenario is also given. Note that the industry and consumer losses are measured on very different scales because they are impacted by different costs.

Unfortunately, the project funding completed before we had the opportunity to explore our different scenario-focused decision analyses with senior members of the nuclear community and government agencies who were shaping the UK energy policy at the time. So

neither could we tailor the value judgements to reflect the preferences of actual stakeholders nor were questions about the transparency and usefulness of the information addressed. Nevertheless, this example analysis is sufficient to give body to our discussion. So to return to the general remarks made at the end of the previous section.

Probabilities are comparable across scenarios, but utilities are not. This clearly has implications for the interpretation of the results of scenario-focused analyses. They provide the DM with guidance on the ranking of actions within each scenario, but there is no guidance on how she should compare actions across scenarios. Thus in Table 1, it is appropriate to compare the expected losses down the columns, but not across the rows⁷; rankings are provided within columns.

The analysis provides no guidance on how to rank the actions overall. The motivation of SFDA is that, by exploring the pros and cons of different actions in each scenario, the DM will see the issues before her more clearly and be able to make a better decision. Here an energy policy with no new nuclear build ranks highest for both consumers and industry in all scenarios except the Entrepreneurial. She might note that *reduced* regulation as in the Entrepreneurial scenario is extremely unlikely, and so feel that abandoning nuclear energy is the best overall. However, she should remember that the six scenarios are a very partial, possibly unbalanced sampling of all possible futures.

6. Further discussion of SFDA

Stewart *et al* (2013) explore whether some form of weighted aggregation might be made across the scenarios, noting that the weights can represent neither uncertainties nor preferences. Weights cannot represent uncertainties because the scenarios neither span all

possible futures nor are disjoint. Weights cannot represent preferences, at least in the sense that preferences are modelled by utilities, because the DM cannot choose to resolve uncertainties as ranking scenarios would imply. Stewart *et al* suggest that some notion of the relative importance of the scenarios to the DM might provide a meaningful interpretation of weights that would allow aggregation across scenarios. But the meaning of relative importance is a nebulous one, presumably relating to the motivation for selecting the scenarios that are included in the analysis. The selection of scenarios is currently being researched in several contexts⁸. Moreover, such an approach would require expected utilities in the scenarios assessed on the same scale, a condition that we have noted is not generally true.

It is possible to imagine circumstances in which a common utility scale across scenarios might be constructed. Suppose that there are two consequences which are common to all scenarios. Suppose that each is such that the MDM is indifferent to receiving it whether it is received in one scenario or another and that she consistently prefers one consequence to the other. Then such a pair of consequences would allow the utility functions to be aligned through affine transformations. In some contexts, e.g. if the consequences are essentially monetary, this it might be the case. In the example, the consequences to nuclear industry are mainly monetary so their expected losses might be comparable across scenarios. But this is less likely to be the case for consumers for whom, e.g., unreliable supply would have lifestyle consequences.

An alternative approach requiring complex elicitation might be to use a swing-weighting approach (French *et al* 2009, Keeney and Raiffa 1976) to assess the relative importance of scenarios and bring the utility functions to the same scale simultaneously. Note that any perception of value needs to be made by the *same* DM, otherwise there will be need to

justify interpersonal comparison of preferences (French 1986, French and Argyris 2018). Thus if scenarios are used to represent different stakeholder perceptions, the MDM would be a *supra-decision-maker* (Keeney and Raiffa 1976).

Notwithstanding these suggestions, we should remember that SFDA has been developed to deal with circumstances in which there are deep uncertainties, which bring a potential for catastrophic or paradigmatic changes in beliefs or preferences, or significantly different stakeholder perspectives (French *et al* 1997). Thus it is unlikely that the analyses will be fully commensurate across scenarios.

Introducing Bayesian modelling into the scenarios brings with it some conceptual difficulties. There will be many scenarios which describes worlds very far from reality and Savage's grand world. Thus Savage's and other earlier axiomatisations of the Bayesian SEU model may not cohere with scenario-focused thinking. In fact there are echoes of the motivation behind the development of subjective probabilities conditioned on counterfactuals and causal decision theory (Gibbard and Harper 1988, Meek and Glymour 1994, Pearl 2009), but note that the 'what if' or 'counterfactual' conditioning implicit in scenario-focused thinking is much more informal than those much more theoretical developments. Indeed, they and some related developments in the literature on causality make strong implicit assumptions that the small world relates closely to the real world.

The axiom system presented in section 4 provides a justification for using the Bayesian model in SFDA. In each scenario, i.e. each small world, the MDM is constructed to represent an 'interesting' set of beliefs and preferences given the history and culture of and the knowledge available to the decision maker in that hypothetical context. Note again that the real DM might wish to think herself (altruistically) into the position of other stakeholders,

so we refer to ‘interesting’ judgements rather than simply her own. If she is convinced by the rationality that is built into the Bayesian SEU model, she will need to accept those axioms with the $A_{rationality}$ quality. Moreover, the need to elicit and manipulate subjective probabilities and utilities means she must also accept those axioms with A_{Me} and A_{Ma} qualities. Thus the acceptability of the axiom system as a justification of SFDA depends primarily on the acceptability of those axioms with the A_C quality: *viz.* Axioms PSW1, PSW3, PSW9 and PSW12. The context issues in these relate to:

- The elements in the each small world $\Theta_w, C_w, A_w, w = 1, 2, \dots, W$ are sufficiently well defined and understood for the MDM — and therefore also the real DM — to express her beliefs and preferences over them as weak orders. We are assuming that it is not necessary for the states, consequences and actions to be representations of the *real* world for the DM to be able to make the required judgements: she can think hypothetically and possibly counterfactually.
- The rather obvious demand that the ‘certain’ event is strictly more likely than the null event and the more constraining condition that there exists c_w^* and c_{w*} which bound the set of consequences.

These are hardly more demanding conceptually than when there is a single small world. Thus the key questions in justifying SFDA in any particular case relate to whether the real DM can think herself well enough into each scenario for \succeq_{Θ_w} and \succeq_{S_w} to be elicited. That is more a question for empirical behavioural studies than theoretical discussion of the acceptability of the axioms.

There are many further issues to be addressed and resolved before SFDA is fully established and justified as a sound method of supporting decision-making. These relate particularly to the judgements required of the DM during elicitation and the transparency and

acceptability of the process of SFDA to her. We will not explore these in any depth here, but note the following need for research.

- Cognitive studies are needed into the elicitation of subjective probabilities and utilities within scenarios. How should questions be asked of the DM that satisfactorily condition her responses on each scenario? This may be particularly difficult when a scenario assumes counterfactuals.
- How should the results of the analysis be presented to the DM? Simply providing an SEU ranking within each scenario as offered in Table 1 is clearly not sufficient. Away from scenarios, current decision analysis uses sensitivity analysis and other tools to build qualitative understanding and articulate discussion (French 2003). How should these processes be extended to SFDA in a meaningful and transparent fashion?
- Indeed, will DMs find the results of an SFDA transparent and useful? Comparing the two case studies in Montibeller *et al* (2006) suggests that this is not a foregone conclusion.
- We have not discussed the development of appropriate scenarios within SFDA. There is an extensive literature on constructing scenarios for qualitative scenario analysis (Schoemaker 1995, Mahmoud *et al* 2009); but SFDA may require a modified approach.

7. Conclusion

In this paper I have explored the theoretical underpinning of SFDA, arguing that current axiomatisations of SEU require extending to deal with two issues:

1. the need to introduce the *same* reference experiment into all scenarios to ensure consistency between the analyses within scenarios;
2. the lack of nesting of scenarios, the possibility that some may contain counterfactual assumptions, and that some may not be an approximation to a true grand world.

Section 4 shows that, at the cost of some loss of mathematical elegance, axiomatisations of the SEU model can be developed which identify the reference experiment quite separately from the underlying small world(s) and that this can be done in a constructive fashion which mimics and thus directly justifies the processes of decision analysis. Moreover, the approach extends the SEU model naturally to the case of SFDA. While the development has paid considerable attention to the practice of decision analysis, there are many cognitive issues that need addressing by behavioural studies. Some of these are listed at the end of the previous section.

Appendix. Outline Proofs

Lemma 1

$\forall w = 1, 2, \dots, W$:

(a) if \mathcal{Q}_w is a field, then under Axioms PSW1—PSW6 there is a unique, finitely additive probability distribution P_{Θ_w} on Θ_w agreeing with \succeq_{Θ_w} in the sense that $\forall R, S \in \mathcal{Q}_w$,

$$R \succeq_{\Theta_w} S \iff P_{\Theta_w}(R) \geq P_{\Theta_w}(S);$$

(b) if \mathcal{Q}_w is a σ -field, then under Axioms PSW1—PSW7 there is a unique, countably additive probability distribution P_{Θ_w} on Θ_w agreeing with \succeq_{Θ_w} in the sense that $\forall R, S \in \mathcal{Q}_w$,

$$R \succeq_{\Theta_w} S \iff P_{\Theta_w}(R) \geq P_{\Theta_w}(S);$$

(c) if, in addition, Axiom PSW8 holds, then $\forall R, S, T \in \mathcal{Q}_w$ for which $T \succ_{\Theta_w} \emptyset$,

$$(R|T) \succeq_{\Theta_w} (S|T) \iff P_{\Theta_w}(R|T) \geq P_{\Theta_w}(S|T).$$

PROOF (outline):

(a) For any $w = 1, 3, \dots, W$: consider $R \in \mathcal{Q}_w$. Then, by PSW1, PSW3, PSW4 and PSW5(a):

$$1 \in \{\pi \in [0, 1] \mid [0, \pi] \succeq_{\Theta_w} R\} = H, \text{ say}; \quad (9)$$

$$0 \in \{\pi \in [0, 1] \mid R \succeq_{\Theta_w} [0, \pi]\} = K, \text{ say}. \quad (10)$$

PSW1 ensures that \succeq_{Θ_w} is complete, so $H \cup K = [0, 1]$. Using the connectedness of $[0, 1]$ and the closure of H and K given by PSW6 gives $H \cap K \neq \emptyset$. Hence $\exists \pi_R \in [0, 1]$ such that $R \sim_{\Theta_w} [0, \pi_R]$. By PSW1 and PSW2, π_R is unique.

Define $P_{\Theta_w}(R) = \pi_R$, $\forall R \in \mathcal{Q}_w$. Clearly $P_{\Theta_w}(R) \geq 0$. By PSW5(a), $P_{\Theta_w}(\Theta_w) = 1$.

Next let $R, S \in \mathcal{Q}_w$ be such that $R \subset S$. Then $S = R \cup (R^c \cap S)$ and $R \cap (R^c \cap S) = \emptyset$. We have $(R^c \cap S) \succeq_{\Theta_w} \emptyset$. So, by PSW5:

$$S = R \cup (R^c \cap S) \succeq_{\Theta_w} R \cup \emptyset = R. \quad (11)$$

So $R \subset S \implies S \succeq_{\Theta_w} R$, which implies $P_{\Theta_w}(S) \geq P_{\Theta_w}(R)$.

Next we show that $P_{\Theta_w}(\cdot)$ is additive. Let $R, S \in \mathcal{Q}_w$ be such that $R \cap S = \emptyset$. Then $R \sim_{\Theta_w} [0, P_{\Theta_w}(R)]$ and $R \cup S \sim_{\Theta_w} [0, P_{\Theta_w}(R \cup S)]$. Additivity will follow if we can show that $S \sim_{\Theta_w} (P_{\Theta_w}(R), P_{\Theta_w}(R \cup S)]$. If $S \succ_{\Theta_w} (P_{\Theta_w}(R), P_{\Theta_w}(R \cup S)]$, then using the implication of PSW5 for strictly more likely, \succ_{Θ_w} :

$$R \cup S \succ_{\Theta_w} [0, P_{\Theta_w}(R)] \cup (P_{\Theta_w}(R), P_{\Theta_w}(R \cup S)] = [0, P_{\Theta_w}(R \cup S)] \quad (12)$$

which is untrue. A similar contradiction follows if $S \prec_{\Theta_w} (P_{\Theta_w}(R), P_{\Theta_w}(R \cup S))$. So $S \sim_{\Theta_w} (P_{\Theta_w}(R), P_{\Theta_w}(R \cup S))$; and $[0, P_{\Theta_w}(S)] \sim_{\Theta_w} (P_{\Theta_w}(R), P_{\Theta_w}(R \cup S))$. Hence $P_{\Theta_w}(R \cup S) = P_{\Theta_w}(R) + P_{\Theta_w}(S)$.

Simple induction gives for $R_i \in \mathcal{Q}_w$, $i = 1, 2, \dots, n$ with $R_i \cap R_j = \emptyset$ for $i \neq j$:

$$P_{\Theta_w}\left(\bigcup_{i=1}^n R_i\right) = \sum_{i=1}^n P_{\Theta_w}(R_i) \quad (13)$$

Thus $P_{\Theta_w}(\cdot)$ is a finitely additive probability distribution on Θ_w .

Let $R, S \in \mathcal{Q}_w$ be such that $R \succeq_{\Theta_w} S$. Then:

$$[0, P_{\Theta_w}(R)] \sim_{\Theta_w} R \succeq_{\Theta_w} S \sim_{\Theta_w} [0, P_{\Theta_w}(S)] \quad (14)$$

So, by PSW2 $P_{\Theta_w}(R) \geq P_{\Theta_w}(S)$. Hence $P_{\Theta_w}(\cdot)$ agrees with \succeq_{Θ_w} on Θ_w .

(b) Suppose now that \mathcal{Q}_w and \mathcal{B} are σ -fields. Let $R_i \in \mathcal{Q}_w$ for $i = 1, 2, 3, \dots$ be such that $R_1 \supseteq R_2 \supseteq R_3 \supseteq \dots$ and $\bigcap_{i=1}^{\infty} R_i = \emptyset$. Since \succeq_{Θ_w} is a qualitative probability and since $P_{\Theta_w}(\cdot)$ agrees with it, $P_{\Theta_w}(R_1) \geq P_{\Theta_w}(R_2) \geq P_{\Theta_w}(R_3) \geq \dots \geq 0$. It follows that $\lim_{i \rightarrow \infty} P_{\Theta_w}$ exists. Suppose that this limit is strictly positive, $\lambda > 0$ say. Then $\forall i$, $P_{\Theta_w}(R_i) > \lambda$. So $\forall i$, $R_i \succeq_{\Theta_w} [0, \lambda] \succ_{\Theta_w} \emptyset$. Hence, by PSW7, $\bigcap_{i=1}^{\infty} R_i \succ_{\Theta_w} \emptyset$, which is a contradiction. So $\lim_{i \rightarrow \infty} P_{\Theta_w} = 0$, which implies that $P_{\Theta_w}(\cdot)$ is countably additive.

(c) The condition $T \succ_{\Theta_w} \emptyset$ in PSW8 together with the results in part (a) ensure that $P_{\Theta_w}(T) > 0$. So $P_{\Theta_w}(R | T)$ and $P_{\Theta_w}(S | T)$ exist. Moreover,

$$P_{\Theta_w}(R | T) \geq P_{\Theta_w}(S | T) \iff P_{\Theta_w}(R \cap T) \geq P_{\Theta_w}(S \cap T). \quad (15)$$

Hence the result follows immediately from PSW8 and the the fact that $P_{\Theta_w}(\cdot)$ agrees with \succeq_{Θ_w} .

Lemma 2 *In the presence of Axioms PSW1—PSW8, Axioms PSW9—PSW15 imply that $\forall w = 1, 2, \dots, W$, $\exists u_w : C \rightarrow \mathfrak{R}$, an agreeing utility function on C_w such that $\forall c, c' \in C_w$:*

$$c \succeq_{C_w} c' \iff u_w(c) \geq u_w(c').$$

Moreover, $\forall p, q \in \mathcal{P}_{S_w}$:

$$p \succeq_{S_w} q \iff E(u_w(c) | p) \geq E(u_w(c) | q),$$

where the expectations are taken with respect to the distributions p and q respectively. In addition, $u_w(\cdot)$ is unique up to a positive affine transformation.

PROOF (outline):

In the following we shall adopt the convention that for any $w = 1, 2, \dots, W$, all $p \in \mathcal{P}_{S_w}$ may be structured so that $p = \langle \pi_1, c_1; \pi_2, c_2; \dots; \pi_r, c_r \rangle$ with $c_1 = c_w^* \succeq_{S_w} c_2 \succeq_{S_w} c_3 \succeq_{S_w} \dots \succeq_{S_w} c_r = c_{w^*}$. In other words, the consequences of any simple distribution are ordered from best to worst. Moreover, we assume that c_w^* and c_{w^*} are included in the consequences, though, of course, that may be with zero probabilities. Again, we will assume that the π_i are zero as necessary to allow a common indexing of consequences when comparing different distributions. Since we shall only compare a finite number of distributions, this will not affect the overall finite number of consequences in our argument.

For each $w = 1, 2, \dots, W$, we argue from PSW9 – PSW13, analogously to the proof of part (a) of Lemma 3, that $\forall c \in C_w$, there exists a unique α such that $c \sim_{S_w} c_w^* \alpha c_{w^*}$. Let $p = \langle \pi_1, c_1; \pi_2, c_2; \dots; \pi_r, c_r \rangle \in \mathcal{P}_{S_w}$. Let $c_i \sim_{S_w} c_w^* \alpha_i c_{w^*}$, for $i = 1, 2, \dots, r$. Note that by PSW10(c), $c_w^* \sim_{S_w} c_w^* 1 c_{w^*}$ and $c_{w^*} \sim_{S_w} c_w^* 0 c_{w^*}$. Next for $i = 1, 2, \dots, r$, use axioms PSW14 and PSW15 to substitute $c_w^* \alpha_i c_{w^*}$ for c_i , in turn, to deduce:

$$p = \langle \pi_1, c_1; \pi_2, c_2; \dots; \pi_r, c_r \rangle \sim_{S_w} c_w^* \left(\sum_{i=1}^r \pi_i \alpha_i \right) c_{w^*}. \quad (16)$$

Similarly, given $q = \langle \tau_1, c_1; \tau_2, c_2; \dots; \tau_r, c_r \rangle$, it follows that $q \sim_{S_w} c_w^* \left(\sum_{i=1}^r \tau_i \alpha_i \right) c_{w^*}$. Hence by PSW11:

$$p \succeq_{S_w} q \iff \left(\sum_{i=1}^r \pi_i \alpha_i \right) \geq \left(\sum_{i=1}^r \tau_i \alpha_i \right). \quad (17)$$

Setting $u(c_i) = \alpha_i$ gives the SEU representation.

Although this constructive proof defines $u(c)$ uniquely for each $c \in C_w$, the linearity of the expectation operator means that it is only unique up to an affine transformation.

Theorem 1

$\forall w = 1, 2, \dots, W$

(a) *Axioms PSW1–PSW6, PSW9–PSW16 are sufficient to justify the SEU representation with finitely additive probabilities of the MDM's preferences over A_w , viz.:*

$$a \succeq_{A_w} a' \iff E(u_w | a) \geq E(u_w | a').$$

where:

$$E(u_w | a) = \sum_{\theta \in \Theta_w} u_w(a(\theta)) P_{\Theta_w}(\theta),$$

$u_w(\cdot)$ is unique up to a positive affine transformation, and $P_{\Theta_w}(\cdot)$ is finitely additive.

(b) If Axiom PSW7 holds, $P_{\Theta_w}(\cdot)$ is countably additive.

(c) If Axiom PSW8 holds, the use of Bayes Theorem to model the updating of belief in the light of data is justified.

PROOF (outline):

Noting from (5) that:

$$E(u_w | a) = \sum_{\theta \in \Theta_w} u_w(a(\theta))P_{\Theta_w}(\theta) = \sum_{c \in C_w} u_w(c) \sum_{\{\theta | c=a(\theta)\}} P_{\Theta}(\theta) \quad (18)$$

Lemmas 3 and 4 together with Axiom PSW16 immediately yield part (a).

Parts (b) and (c) are restatements of parts (b) and (c) of Lemma 3.

Theorem 2

$\forall w = 1, 2, \dots, W$, Axioms PSW1–PSW13, PSW14', PSW15', PSW16 and PSW17 are sufficient to justify the SEU representation of the MDM's preferences over A_w , viz.:

$$a \succeq_{A_w} a' \iff E(u_w | a) \geq E(u_w | a').$$

where:

$$E(u_w | a) = \sum_{\theta \in \Theta_w} u_w(a(\theta))P_{\Theta_w}(\theta),$$

$u_w(\cdot)$ is unique up to a positive affine transformation, $P_{\Theta_w}(\cdot)$ is countably additive, and the use of Bayes Theorem to model the updating of belief in the light of data is justified.

PROOF (outline):

As before but using Axioms PSW14', PSW15', for each $w = 1, 2, \dots, W$, define $u_w(c) \in [0, 1]$ by $c \sim_{S_w} c_w^*(u_w(c))c_{w*}$. Note that this means that $u_w(c)$, $c \in C_w$ corresponds to a probability distribution which gives c_w^* with probability $u_w(c)$ and c_{w*} with probability $(1 - u_w(c))$ and zero probability to the rest of C . By PSW17, $u_w(c)$ is a measurable function with respect to the σ -field \mathcal{D}_w and, for any $a \in A_w$, $P_{\Theta_w}(a(\cdot))$ is a probability distribution over C_w also compatible with its σ -field \mathcal{D}_w . So $E(u_w | a) = \int_{\Theta_w} u_w(a(\theta))dP_{\Theta_w}(\theta)$ exists.

The argument now proceeds as before, from $a \in A_w$ construct $p_a \in \mathcal{P}_{S_w}$ such that $c \in C$ is received with probability $\int_{\{\theta \in \Theta_w | c=a(\theta)\}} dP_{\Theta_w}(\theta)$. Next $u_w \circ p_a$ gives c_w^* with probability $E(u_w | a)$, c_{w^*} with probability $1 - E(u_w | a)$, while the rest of C has zero probability. So, by PSW15', $p_a \succeq_{S_w} p_b$ iff and only if $E(u_w | a) \geq E(u_w | b)$. A simple application of PSW16 now gives the main result. The uniqueness of $u_w(\cdot)$ up to a positive affine transformation and the countable additivity of P_{Θ_w} follow from the earlier results.

Endnotes

1. Note that here and elsewhere where I refer to the *constructive* nature of Bayesian decision analysis and the related nature of the textitconstructive mathematical proofs that I use, I am not referring specifically to any psychological or educational theory of *Constructivism*, though there are clear parallels.

2. We shall adopt the convention of referring to the DM as feminine and the analyst as male, thus creating greater clarity in the presentation. Note also that there is a subtlety here. We have moved from discussing DMs in the plural to the singular. The Bayesian model is a model of individual rationality. Extending it to a mathematical structure that applies to groups is fraught with difficulty. Rather one implements it in processes which help groups move toward consensus: see, e.g, French *et al* (2009).

3. Elsewhere, I have used *model decision problem* (MDP) in the same sense as Savage's small world.

4. Although we allow that each small world may represent the perspectives of different stakeholders, we imagine that the analysis is conducted for *one* real DM. The Bayesian SEU model is an individualistic one. Developing the processes which support groups of decision makers is covered elsewhere (French *et al* 2009). We assume that the DM constructs the small worlds altruistically to represent other stakeholder perspectives where necessary.

5. Unbounded theories of utility may be developed along the lines in DeGroot (1970), although we shall not do so here. In the vast majority of practical cases, the context bounds the problem in the sense here.

6. For the unbounded case, an axiom with the structure of Assumption U_3 in DeGroot (1970) may be used.

7. We might also note that it is not appropriate to compare expected losses between industry and consumers; but that arises because of issues with interpersonal comparisons of preference not the development within this paper (French and Argyris 2018).

8. see, e.g, <https://portal.iket.kit.edu/CONFIDENCE/index.php>

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Biography

Simon French began working on Bayesian Statistics and Decision Analysis in the 1970s. Over the years he has worked on both theory and applications in a wide variety of contexts, particularly in the nuclear industry, publishing many books and papers. His interest in dealing with deep uncertainties has grown recently as he has looked at applications in crisis management and long term environmental issues. In 2017 he was awarded the INFORMS Frank P Ramsey Medal for distinguished contributions to decision analysis.