DYNKIN GAMES WITH POISSON RANDOM INTERVENTION TIMES*

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Abstract. This paper introduces a new class of Dynkin games, where the two players are allowed to make their stopping decisions at a sequence of exogenous Poisson arrival times. The value function and the associated optimal stopping strategy are characterized by the solution of a backward stochastic differential equation. The paper further provides a replication strategy for the game, and applies the model to study the optimal conversion and calling strategies of convertible bonds, and their asymptotics when the Poisson intensity goes to infinity.

Key words. constrained Dynkin game, penalized BSDE, optimal stopping strategy, replication strategy, convertible bond.

AMS subject classifications. 60G40, 91A05, 91G80, 93E20.

1. Introduction. Dynkin games are the games on stopping times, where two players determine their optimal stopping times as their strategies. The game was first introduced by Dynkin [14], and later generalized by Neveu [28] in 1970s. In this game, two players observe two stochastic processes, say L and U, and their aims are to maximize/minimize the expected value of the payoff

$$R(\sigma,\tau)=L_\tau \mathbb{1}_{\{\tau\le\sigma\}}+U_\sigma \mathbb{1}_{\{\sigma<\tau\}}$$

over stopping times τ and σ, respectively. In a discrete-time setting, under the assumption that $U\ge L$, Neveu proved the existence of the game value and its associated optimal strategy.

Since then, there has been a considerable development of Dynkin games. The corresponding continuous time models were developed, among others, by Bismut [6], Alario-Nazaret et al [1], Lepeltier and Maingueneau [21] and Morimoto [27]. In order to relax the condition $U\ge L$, Yasuda [36] proposed to extend the class of strategies to randomized stopping times, and proved that the game value exists under merely an integrability condition. Rosemberg et al [30], Touzi and Vielle [34] and Laraki and Solan [19] further extended his work in this direction. If the two players in the game are with asymmetric payoffs, then it gives arise to a nonzero-sum Dynkin game. See, for example, Hamadene and Zhang [16] and more recently De Angelis et al [12] with more references therein. A robust version of Dynkin games can be found in Bayraktar and Yao [3] if the players are ambiguous about their probability model.

The setups in all the aforementioned works are either in continuous time where stopping times take any value in a certain time interval, or in discrete time where stopping times only take values in a pre-specified time grid. In this paper, we consider a hybrid of continuous and discrete times, and introduce a new type of Dynkin games, where both players are allowed to stop at a sequence of random times generated by an exogenous Poisson process serving as a signal process. We call such a Dynkin game a *constrained Dynkin game*.

The underlying Poisson process can be regarded as an exogenous constraint on the players' abilities to stop, so it may represent the liquidity effect, i.e. the Poisson

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process indicates the times at which the underlying stochastic processes are available to stop. Moreover, the Poisson process can also be seen as an information constraint. The players are allowed to make their stopping decisions at all times, but they are only able to observe the underlying stochastic processes at Poisson times.

Our first main result is Theorem 2.3, which characterizes the value of the constrained Dynkin game and its associated optimal stopping strategy in terms of the solution of a penalized backward stochastic differential equation (BSDE). The latter is widely used to approximate the solution of a reflected BSDE with double obstacles and the corresponding continuous time Dynkin game. The main idea to solve the constrained Dynkin game is to introduce a family of auxiliary games (see (3.9)-(3.10)), for which standard dynamic programming principle holds. Furthermore, following from the convergence of penalized BSDE to reflected BSDE (see, for example, [11] and [15]) and the penalized BSDE characterization (2.6) of the constrained Dynkin game, we also make a connection with standard Dynkin games in continuous time. That is, the value of the constrained Dynkin game will converge to the value of its continuous time counterpart when the Poisson intensity goes to infinity.

Our second main result is about replication of the constrained Dynkin game (see Theorem 5.1). This has an application to the hedging problems in finance. In the existing literature of financial applications of optimal stopping with Poisson times, the vast majority of papers focus on the risk-neutral valuation without even mentioning the issue of hedging (see [13] and [20] among others). This somewhat lacks a foundation since, as is well known, the major argument supporting the risk-neutral valuation is the existence of hedging strategies. We address this issue by constructing a replication strategy for the constrained Dynkin game (which in particular covers the optimal stopping case). For such a replication problem, a new element is the jump risk stemming from the Poisson process. To hedge this jump risk, we introduce a pricing process generated by the jump times of the Poisson process. We then construct the replication strategies recursively for a sequence of constrained Dynkin games starting from different Poisson arrival times, and for each game, the replication strategy is constructed via two linear BSDEs. The first BSDE is used to replicate the payoff of the game before the next jump time, and the second equation is used to replicate the payoff after this jump time.

With the above replication strategies behind the risk-neutral valuation, we then apply the constrained Dynkin game to study convertible bonds. In a convertible bond, the bondholder decides whether to keep the bond to collect coupons or to convert it to the firm's stocks. She will choose a conversion strategy to maximize the bond value. On the other hand, the issuing firm has the right to call the bond, and presumably acts to maximize the equity value of the firm by minimizing the bond value. This creates a two-person, zero-sum Dynkin game.

Traditionally, convertible bond models often assume that both the bond holder and the firm are allowed to stopped at any stopping time adapted to the firm's fundamental (such as its stock prices). In reality, there may exist some liquidation constraint as an external shock, and both players only make their decisions when such a shock arrives. We model such a liquidation shock as the arrival times of an exogenous Poisson process, and thus the convertible bond model falls into the framework of constrained Dynkin games. A similar idea has first appeared in the modeling of debt run problems (see [23]), which can be formulated as optimal stopping problems with Poisson arrival times.

Furthermore, in a Markovian setting, we derive explicitly the optimal stopping strategies for both the bondholder and the firm. We show that if the initial stock price is not too high (otherwise the game will stop at the first Poisson arrival time), the optimal stopping rules of the two players depend on the relationship between the coupon rate c, dividend rate q, interest rate r and surrender price K. For the firm, its optimal stopping strategy is to either call the bond back as soon as possible (if $c \geq rK$) or postpone the calling time of the bond as late as possible (if $c < rK$). In contrast, the investor's optimal stopping strategy depends on the relationship between c and qK. If $c > qK$, the investor will delay her conversion time as late as possible; if $c \leq qK$, her conversion strategy is determined by an optimal conversion boundary, the latter of which is obtained by solving a free boundary problem.

Turning to the literature, the optimal stopping problem with constraints on the stopping times was introduced by Dupuis and Wang [13], when they used it to model perpetual American options exercised at exogenous Poisson arrival times. See also Lempa [20] and Menaldi and Robin [25] for further extensions of this type of optimal stopping problems. On the other hand, Liang [22] made a connection between such kind of optimal stopping problems with penalized BSDE. The corresponding optimal switching (impulse control) problems were studied by Liang and Wei [24] and more recently by Menaldi and Robin [26] with more general signal times and state spaces.

The study of convertible bonds dated back to Brennan and Schwartz [7] and Ingersoll [17]. However, it was Sirbu et al [31] who first analyzed the optimal strategy of perpetual convertible bonds (see also Sirbu and Shreve [32] for the finite horizon counterpart). They reduced the problem from a Dynkin game to an optimal stopping problem, and discussed when call precedes conversion and vice versa. Several more realistic features of convertible bonds have been taken into account since then. For example, Bielecki et al [4] considered the problem of the decomposition of a convertible bond into bond component and option component. Crepey and Rahal [10] studied the convertible bond with call protection, which is typically path dependent. Chen et al [9] considered the tax benefit and bankruptcy cost for convertible bonds. For a complete literature review, we refer to the aforementioned papers with references therein.

The paper is organized as follows. Section 2 contains the problem formulation and main result, with its proof provided in section 3. In section 4, we establish a connection with standard Dynkin games. Section 5 is about replication of the constrained Dynkin game. In section 6, we apply the constrained Dynkin game to study the convertible bonds in a Markovian setting, and derive the explicit optimal stopping strategies and the corresponding free boundaries under various situations. Section 7 carries out an asymptotic analysis of the game values and the free boundaries when the Poisson intensity goes to infinity.

2. Constrained Dynkin games. Let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with $\mathbb{F}$ being the minimal augmented filtration of W. Let $(T_i)_{i\geq 0}$ be the arrival times of an independent Poisson process with intensity λ and minimal augmented filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$. Denote the smallest filtration generated by $\mathbb{F}$ and $\mathbb{H}$ as $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. Without loss of generality, we also assume that $T_0 = 0$ and $T_\infty = \infty$.

Let T be a finite $\mathbb{F}$-stopping time representing the terminal time of the game, and ξ be an $\mathcal{F}_T$-measurable random variable representing the corresponding payoff. Define a random variable $M : \Omega \to \mathbb{N}$ such that T_M is the next Poisson arrival time after T, i.e. $M(\omega) = \sum_{i\geq 1} i \mathbb{1}_{\{T_{i-1}(\omega)\leq T(\omega)<T_i(\omega)\}}$.

For any integer $i \geq 0$, define the control set

$$\mathcal{R}_{T_i}(\lambda) = \{\mathbb{G}\text{-stopping time } \tau \text{ for } \tau(\omega) = T_N(\omega) \text{ where } i \leq N \leq M(\omega)\}.$$

The subscript T_i in $\mathcal{R}_{T_i}(\lambda)$ represents the smallest stopping time that is allowed to choose, and λ represents the intensity of the underlying Poisson process.

Consider the following *constrained Dynkin game*, where two players choose their respective stopping times $\sigma, \tau \in \mathcal{R}_{T_1}(\lambda)$ in order to minimize/maximize the expected value of the discounted payoff

$$R(\sigma,\tau) = \int_0^{\sigma\wedge\tau\wedge T} e^{-rs} f_s\, ds + e^{-rT}\xi \mathbb{1}_{\{\sigma\wedge\tau\geq T\}} + e^{-r\tau} L_\tau \mathbb{1}_{\{\tau<T,\tau\leq\sigma\}} + e^{-r\sigma} U_\sigma \mathbb{1}_{\{\sigma<T,\sigma<\tau\}}, \tag{2.1}$$

where $r > 0$ is the discount rate, and f, as a real-valued $\mathbb{F}$-progressively measurable process, is the running payoff. The terminal payoff is U if σ happens firstly, L if τ happens firstly or σ and τ happen simultaneously, and ξ otherwise, where L and U are two real-valued $\mathbb{F}$-progressively measurable processes.

Let us define the upper and lower values of the constrained Dynkin game

$$\overline{v}^\lambda = \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \sup_{\tau\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[R(\sigma,\tau)\right], \tag{2.2}$$

$$\underline{v}^\lambda = \sup_{\tau\in\mathcal{R}_{T_1}(\lambda)} \inf_{\sigma\in\mathcal{R}_{T_1}(\lambda)} \mathbb{E}\left[R(\sigma,\tau)\right]. \tag{2.3}$$

The game (2.2)-(2.3) is said to have value v^λ if $v^\lambda = \overline{v}^\lambda = \underline{v}^\lambda$. It is standard to show that if there exists a saddle point $(\sigma^*,\tau^*) \in \mathcal{R}_{T_1}(\lambda)\times\mathcal{R}_{T_1}(\lambda)$ such that $\mathbb{E}\left[R(\sigma^*,\tau)\right] \leq \mathbb{E}\left[R(\sigma^*,\tau^*)\right] \leq \mathbb{E}\left[R(\sigma,\tau^*)\right]$ for every $(\sigma,\tau) \in \mathcal{R}_{T_1}(\lambda)\times\mathcal{R}_{T_1}(\lambda)$, then the value of this game exists and equals to $v^\lambda = \mathbb{E}\left[R(\sigma^*,\tau^*)\right]$.

There are two new features of the above constrained Dynkin game. First, there is a control constraint in the sense that only stopping at Poisson arrival times is allowed. Second, the players are not allowed to stop at the initial starting time. Instead, they are only allowed to stop from the first Poisson time onwards.

We also consider an auxiliary game related to the above constrained Dyknin game by replacing the control set in (2.2)-(2.3) with $\mathcal{R}_{T_0}(\lambda)$, so the players are also allowed to stop at the initial starting time. That is

$$\overline{\hat{v}}^\lambda = \inf_{\sigma\in\mathcal{R}_{T_0}(\lambda)} \sup_{\tau\in\mathcal{R}_{T_0}(\lambda)} \mathbb{E}\left[R(\sigma,\tau)\right], \tag{2.4}$$

$$\underline{\hat{v}}^\lambda = \sup_{\tau\in\mathcal{R}_{T_0}(\lambda)} \inf_{\sigma\in\mathcal{R}_{T_0}(\lambda)} \mathbb{E}\left[R(\sigma,\tau)\right]. \tag{2.5}$$

Note that the difference between (2.4)-(2.5) and (2.2)-(2.3) is that the former is allowed to stop at the initial starting time $T_0 = 0$, while the latter not. In other words, the players in (2.4)-(2.5) first make their stopping decisions and then move forward, while in (2.2)-(2.3) they first move forward and then make their decisions. We shall show that if the game (2.2)-(2.3) has value v^λ, then the value of (2.4)-(2.5) also exists and is given by $\hat{v}^\lambda = \min\{U_0, \max\{v^\lambda, L_0\}\}$, so the key is to solve the game (2.2)-(2.3).

2.1. Main result. To solve the above constrained Dynkin games, we introduce the following BSDE defined on a random horizon $[0, T]$:

$$V^{\lambda}_{t\wedge T} = \xi + \int_{t\wedge T}^{T} \left[f_s + \lambda \left(L_s - V^{\lambda}_s \right)^+ - \lambda \left(V^{\lambda}_s - U_s \right)^+ - r V^{\lambda}_s \right] ds - \int_{t\wedge T}^{T} Z^{\lambda}_s \, dW_s \tag{2.6}$$

for $t \geq 0$. Note that the above BSDE (2.6) is often used to construct the solution of a reflected BSDE with two reflecting barriers L and U (cf. (4.3)). Intuitively, when V^{λ} falls below L (or goes above U), there will be a penalty $\lambda(L - V^{\lambda})$ (or $\lambda(V^{\lambda} - U)$) incurred, so BSDE (2.6) is also refereed to as the penalized equation.

Assumption 2.1. *For $t \in [0, T]$, $L_t \leq U_t$, a.s. Moreover, (i) when T is an unbounded stopping time, the running payoff f and the terminal payoffs L, U and ξ are all bounded; (ii) when T is a bounded stopping time, f, L, U and ξ are square-integrable, i.e. $\mathbb{E}[\sup_{0\leq t\leq T} |X_t|^2] < \infty$ for $X = f, L, U$ and ξ.*

The assumption $L \leq U$ is crucial to the existence of the game value. On the other hand, the conditions (i) and (ii) are to guarantee the existence and uniqueness of the solution to BSDE (2.6), which will in turn be used to construct the game value and its associated optimal stopping strategy.

Proposition 2.2. *Suppose that Assumption 2.1 holds. Then, there exists a unique solution (V, Z) to BSDE (2.6). Moreover, (i) when T is unbounded, V is a bounded and continuous $\mathbb{F}$-adapted process, and $Z \in \mathcal{M}^2_{loc}(0, T; \mathbb{R}^d)$, where the latter denotes the space of all $\mathbb{F}$-progressively measurable processes Z such that*

$$||Z||^2_{loc} := \mathbb{E}\left[\int_0^{t\wedge T} |Z_s|^2 \, ds \right] < \infty \quad \text{for } t \geq 0;$$

(ii) when T is bounded, then V is a continuous square-integrable $\mathbb{F}$-adapted process, and $Z \in \mathcal{M}^2(0, T; \mathbb{R}^d)$.

The proof essentially follows from Theorem 4.1 in [29] (for bounded T) and Section 5 in [8] (for unbounded T), so we omit its proof and refer to [29] and [8] for the details. We are now in a position to state the main result of this paper.

Theorem 2.3. *Suppose that Assumption 2.1 holds. Let $(V^{\lambda}, Z^{\lambda})$ be the unique solution to BSDE (2.6). Then, the value of the constrained Dynkin game (2.2)-(2.3) exists and is given by $v^{\lambda} = \overline{v}^{\lambda} = \underline{v}^{\lambda} = V^{\lambda}_0$. The corresponding optimal stopping strategy is given by*

$$\begin{cases} \sigma^*_{T_1} = \inf\{T_N \geq T_1 : V^{\lambda}_{T_N} \geq U_{T_N}\} \wedge T_M; \\ \tau^*_{T_1} = \inf\{T_N \geq T_1 : V^{\lambda}_{T_N} \leq L_{T_N}\} \wedge T_M. \end{cases} \tag{2.7}$$

Moreover, the value of the Dynkin game (2.4)-(2.5) also exists and is given by $\hat{v}^{\lambda} = \min\{U_0, \max\{v^{\lambda}, L_0\}\}$, with the associated optimal stopping strategy $\sigma^*_{T_0}$ and $\tau^*_{T_0}$.

2.2. Examples. Theorem 2.3 solves a wide class of problems in a unified manner, covering from Markovian to non-Markovian situations and from finite to infinite horizons. In the one-dimensional homogenous Markovian setting, there usually exists a threshold strategy. For this, we will discuss a specific convertible-bond example in section 6. In the rest of the section, we list several path-dependent examples, which are difficult to deal with under Markovian framework (at least it needs a case-by-case study) but covered by Theorem 2.3.

(i) *Path-dependent payoffs L and U.* Let T be fixed so it is a constant stopping time and S be a one-dimensional positive diffusion process adapted to $\mathbb{F}$. For $\delta > 0$, consider an Israeli option written on S with maturity T, where the holder may exercise to get a normal claim but the writer is punished by an amount δS for annulling the contract early (see [18]). The payoffs L and U may take the form $L_t = \max\{m, S_t^*\}$ and $U_t = \max\{m, S_t^*\} + \delta S_t$ for $m > S_0$ and $S_t^* = \sup_{0\le u\le t} S_u$. This is so called Israeli Russian option. For $L_t = \int_0^t S_u du$ and $U_t = \int_0^t S_u du + \delta S_t$, it is called Israeli integral option (see [2]). Under mild integrability assumption on S as in Assumption 2.1, Theorem 2.3 shows that the values of both Israeli options exist and the associated optimal strategies can be characterized via the solution to (2.6).

(ii) *Path-dependent stopping time T.* Stopping times are widely used in insurance as indicators of a variety of risks. Let S be a one-dimensional positive diffusion process adapted to $\mathbb{F}$. We may consider the following stopping times as the terminal time of the game: drawdown stopping time $T = \inf\{t \ge 0 : S_t^* - S_t \ge m\}$ for $m \ge 0$; occupation stopping time $T = \inf\{t \ge m : \int_0^t 1_{\{S_u\in A\}} du \ge m\}$ for $A \subset \mathbb{R}_+$. Note that unlike the standard first-passage-time (see θ^λ in section 6), both types of path-dependent stopping times need tailor-made analysis under Markovian framework, but can be covered by Theorem 2.3 in a unified manner.

3. Proof of Theorem 2.3. We first give an equivalent formulation of the constrained Dynkin game (2.2)-(2.3). Given the arrival time T_i, define pre-T_i σ-field

$$\mathcal{G}_{T_i} = \left\{ A \in \bigvee_{s\ge 0} \mathcal{G}_s : A \cap \{T_i \le s\} \in \mathcal{G}_s \text{ for } s \ge 0 \right\}$$

and $\tilde{\mathbb{G}} = (\mathcal{G}_{T_i})_{i\ge 0}$. It is obvious that the upper and lower values of the constrained Dynkin game can be rewritten as

$$\overline{v}^\lambda = \inf_{N^\sigma\in\mathcal{N}_1(\lambda)} \sup_{N^\tau\in\mathcal{N}_1(\lambda)} \mathbb{E}\left[R(T_{N^\sigma}, T_{N^\tau})\right], \tag{3.1}$$

$$\underline{v}^\lambda = \sup_{N^\tau\in\mathcal{N}_1(\lambda)} \inf_{N^\sigma\in\mathcal{N}_1(\lambda)} \mathbb{E}\left[R(T_{N^\sigma}, T_{N^\tau})\right], \tag{3.2}$$

where

$$\mathcal{N}_n(\lambda) = \left\{ \tilde{\mathbb{G}}\text{-stopping time } N \text{ for } n \le N(\omega) \le M(\omega) \right\}.$$

The subscript n in $\mathcal{N}_n(\lambda)$ represents the smallest stopping time that is allowed to choose, and λ represents the intensity of the underlying filtration $\tilde{\mathbb{G}}$. Both players are allowed to stop at a sequence of integers $n, n+1, \cdots, M$.

We also observe that a pair of processes (V^λ, Z^λ) solve (2.6), if and only if the corresponding discounted processes $(Q_t^\lambda, \tilde{Z}_t^\lambda) = (e^{-rt}V_t^\lambda, e^{-rt}Z_t^\lambda)$, for $t \in [0,T]$, solve the following BSDE

$$Q_{t\wedge T}^\lambda = \tilde{\xi} + \int_{t\wedge T}^T \left[\tilde{f}_s + \lambda\left(\tilde{L}_s - Q_s^\lambda\right)^+ - \lambda\left(Q_s^\lambda - \tilde{U}_s\right)^+ \right] ds - \int_{t\wedge T}^T \tilde{Z}_s^\lambda\, dW_s, \tag{3.3}$$

where $\tilde{\xi} = e^{-rT}\xi$ and $\tilde{\phi}_s = e^{-rs}\phi_s$ for $\phi = f, L, U$.

Thus, to prove Theorem 2.3, it is equivalent to show that $Q_0^\lambda = \overline{q}^\lambda = \underline{q}^\lambda$, where

$$\overline{q}^\lambda := \inf_{N^\sigma\in\mathcal{N}_1(\lambda)} \sup_{N^\tau\in\mathcal{N}_1(\lambda)} \mathbb{E}\left[\tilde{R}\left(T_{N^\sigma}, T_{N^\tau}\right)\right], \tag{3.4}$$

$$\underline{q}^{\lambda} := \sup_{N^{\tau}\in\mathcal{N}_1(\lambda)} \inf_{N^{\sigma}\in\mathcal{N}_1(\lambda)} \mathbb{E}\left[\tilde{R}\left(T_{N^{\sigma}}, T_{N^{\tau}}\right)\right], \tag{3.5}$$

with

$$\tilde{R}(\sigma,\tau) = \int_0^{\sigma\wedge\tau\wedge T} \tilde{f}_s\,ds + \tilde{\xi}\mathbb{1}_{\{\sigma\wedge\tau\geq T\}} + \tilde{L}_{\tau}\mathbb{1}_{\{\tau<T,\tau\leq\sigma\}} + \tilde{U}_{\sigma}\mathbb{1}_{\{\sigma<T,\sigma<\tau\}},$$

and the optimal stopping strategy is given by

$$\begin{cases} N_1^{\sigma,*} = \inf\{N\geq 1 : Q^{\lambda}_{T_N} \geq \tilde{U}_{T_N}\}\wedge M, \\ N_1^{\tau,*} = \inf\{N\geq 1 : Q^{\lambda}_{T_N} \leq \tilde{L}_{T_N}\}\wedge M. \end{cases} \tag{3.6}$$

To prove the above assertions, we start with the following lemma.

LEMMA 3.1. *Suppose that Assumption 2.1 holds. Then, for any $1\leq n\leq M$, the solution of BSDE (3.3) at time T_{n-1} is the unique solution of the recursive equation*

(3.7)

$$\begin{aligned} Q^{\lambda}_{T_{n-1}} = \mathbb{E}\Bigg[&\int_{T_{n-1}}^{T_n\wedge T} \tilde{f}_s\,ds + \tilde{\xi}\mathbb{1}_{\{T_n>T\}} \\ &+ \left(\mathbb{1}_{\{Q^{\lambda}_{T_n}\geq\tilde{U}_{T_n}\}}\tilde{U}_{T_n} + \mathbb{1}_{\{Q^{\lambda}_{T_n}\leq\tilde{L}_{T_n}\}}\tilde{L}_{T_n} + \mathbb{1}_{\{\tilde{L}_{T_n}<Q^{\lambda}_{T_n}<\tilde{U}_{T_n}\}}Q^{\lambda}_{T_n}\right)\mathbb{1}_{\{T_n\leq T\}}\Bigg|\mathcal{G}_{T_{n-1}}\Bigg]. \end{aligned}$$

Proof. Applying Itô's formula to $\alpha_t Q^{\lambda}_t$, where $\alpha_t = e^{-\lambda t}$, we obtain, for $t\in[0,T]$,

$$\alpha_t Q^{\lambda}_t = \alpha_T Q^{\lambda}_T + \int_t^T \alpha_s\left[\tilde{f}_s + \lambda F_s(Q^{\lambda}_s)\right]ds - \int_t^T \alpha_s\tilde{Z}^{\lambda}_s\,dW_s,$$

where $F_s(Q^{\lambda}_s) := Q^{\lambda}_s + (\tilde{L}_s - Q^{\lambda}_s)^+ - (Q^{\lambda}_s - \tilde{U}_s)^+$. Consequently,

$$\begin{aligned} &Q^{\lambda}_{T_{n-1}} \\ &= \frac{\alpha_T}{\alpha_{T_{n-1}}}\tilde{\xi} + \int_{T_{n-1}}^T \frac{\alpha_s}{\alpha_{T_{n-1}}}\left[\tilde{f}_s + \lambda F_s(Q^{\lambda}_s)\right]ds - \int_{T_{n-1}}^T \frac{\alpha_s}{\alpha_{T_{n-1}}}\tilde{Z}^{\lambda}_s\,dW_s \\ &= \mathbb{E}\left[e^{-\lambda(T-T_{n-1})}\tilde{\xi} + \int_{T_{n-1}}^T e^{-\lambda(s-T_{n-1})}\left[\tilde{f}_s + \lambda F_s(Q^{\lambda}_s)\right]ds\middle|\mathcal{G}_{T_{n-1}}\right]. \end{aligned}$$

On the other hand, we use the conditional density $\lambda e^{-\lambda(x-T_{n-1})}\,dx$ of T_n to calculate the right-hand side of (3.7):

$$\begin{aligned} &\mathbb{E}\left[\int_{T_{n-1}}^{T_n\wedge T}\tilde{f}_s\,ds\middle|\mathcal{G}_{T_{n-1}}\right] \\ &= \mathbb{E}\left[e^{-\lambda(T-T_{n-1})}\int_{T_{n-1}}^T \tilde{f}_s\,ds + \int_{T_{n-1}}^T \lambda e^{-\lambda(x-T_{n-1})}\int_{T_{n-1}}^x \tilde{f}_s\,ds\,dx\middle|\mathcal{G}_{T_{n-1}}\right] \\ &= \mathbb{E}\left[e^{-\lambda(T-T_{n-1})}\int_{T_{n-1}}^T \tilde{f}_s\,ds + \int_{T_{n-1}}^T \tilde{f}_s\int_s^T \lambda e^{-\lambda(x-T_{n-1})}\,dx\,ds\middle|\mathcal{G}_{T_{n-1}}\right] \\ &= \mathbb{E}\left[\int_{T_{n-1}}^T e^{-\lambda(s-T_{n-1})}\tilde{f}_s\,ds\middle|\mathcal{G}_{T_{n-1}}\right], \end{aligned}$$

where we used integration by parts in the second equality. Similarly, we have

$$\mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_n>T\}}\middle|\mathcal{G}_{T_{n-1}}\right]=\mathbb{E}\left[e^{-\lambda(T-T_{n-1})}\tilde{\xi}\middle|\mathcal{G}_{T_{n-1}}\right],$$

and

$$\begin{aligned}&\mathbb{E}\left[\left(\mathbb{1}_{\{Q^\lambda_{T_n}\geq\tilde{U}_{T_n}\}}\tilde{U}_{T_n}+\mathbb{1}_{\{Q^\lambda_{T_n}\leq\tilde{L}_{T_n}\}}\tilde{L}_{T_n}+\mathbb{1}_{\{\tilde{L}_{T_n}<Q^\lambda_{T_n}<\tilde{U}_{T_n}\}}Q^\lambda_{T_n}\right)\mathbb{1}_{\{T_n\leq T\}}\middle|\mathcal{G}_{T_{n-1}}\right]\\ &=\mathbb{E}\left[\int_{T_{n-1}}^{T}\lambda e^{-\lambda(s-T_{n-1})}\left(\mathbb{1}_{\{Q^\lambda_s\geq\tilde{U}_s\}}\tilde{U}_s+\mathbb{1}_{\{Q^\lambda_s\leq\tilde{L}_s\}}\tilde{L}_s+\mathbb{1}_{\{\tilde{L}_s<Q^\lambda_s<\tilde{U}_s\}}Q^\lambda_s\right)ds\middle|\mathcal{G}_{T_{n-1}}\right].\end{aligned}$$

It follows that (3.7) holds. Since the recursive equation (3.7) obviously admits a unique solution, $Q^\lambda_{T_{n-1}}$ is then the unique solution of (3.7) for $1\leq n\leq M$. □

As a direct consequence of Lemma 3.1, if we define $\hat{Q}^\lambda=\min\{\tilde{U},\max\{Q^\lambda,\tilde{L}\}\}$, then by the assumption $L\leq U$ (so $\tilde{L}\leq\tilde{U}$),

$$\hat{Q}^\lambda=\mathbb{1}_{\{Q^\lambda\geq\tilde{U}\}}\tilde{U}+\mathbb{1}_{\{Q^\lambda\leq\tilde{L}\}}\tilde{L}+\mathbb{1}_{\{\tilde{L}<Q^\lambda<\tilde{U}\}}Q^\lambda,$$

and thus, $\hat{Q}^\lambda$ satisfies the following recursive equation: For $1\leq n\leq M$,

(3.8)

$$\begin{aligned}&\hat{Q}^\lambda_{T_{n-1}}\\ &=\min\left\{\tilde{U}_{T_{n-1}},\max\left\{\mathbb{E}\left[\int_{T_{n-1}}^{T_n\wedge T}\tilde{f}_s\,ds+\tilde{\xi}\mathbb{1}_{\{T_n>T\}}+\hat{Q}^\lambda_{T_n}\mathbb{1}_{\{T_n\leq T\}}\middle|\mathcal{G}_{T_{n-1}}\right],\tilde{L}_{T_{n-1}}\right\}\right\},\end{aligned}$$

which also admits a unique solution since we can calculate its solution backwards in a recursive way. We show that $\hat{Q}^\lambda_{T_{n-1}}$ is the value of another constrained Dynkin game. Introduce the upper and lower values of an auxiliary constrained Dynkin game as

$$\bar{\tilde{q}}^\lambda_{T_{n-1}}=\operatorname*{ess\,inf}_{N^\sigma\in\mathcal{N}_{n-1}(\lambda)}\operatorname*{ess\,sup}_{N^\tau\in\mathcal{N}_{n-1}(\lambda)}\mathbb{E}\left[\tilde{R}_{n-1}(T_{N^\sigma},T_{N^\tau})|\mathcal{G}_{T_{n-1}}\right],\tag{3.9}$$

$$\underline{\hat{q}}^\lambda_{T_{n-1}}=\operatorname*{ess\,sup}_{N^\tau\in\mathcal{N}_{n-1}(\lambda)}\operatorname*{ess\,inf}_{N^\sigma\in\mathcal{N}_{n-1}(\lambda)}\mathbb{E}\left[\tilde{R}_{n-1}(T_{N^\sigma},T_{N^\tau})|\mathcal{G}_{T_{n-1}}\right],\tag{3.10}$$

where

$$\tilde{R}_{n-1}(\sigma,\tau)=\int_{T_{n-1}\wedge T}^{\sigma\wedge\tau\wedge T}\tilde{f}_s\,ds+\tilde{\xi}\mathbb{1}_{\{\sigma\wedge\tau\geq T\}}+\tilde{L}_\tau\mathbb{1}_{\{\tau<T,\tau\leq\sigma\}}+\tilde{U}_\sigma\mathbb{1}_{\{\sigma<T,\sigma<\tau\}}$$

with $\tilde{R}_0(\sigma,\tau)=\tilde{R}(\sigma,\tau)$, and

$$\mathcal{N}_{n-1}(\lambda)=\left\{\tilde{\mathbb{G}}\text{-stopping time } N \text{ for } n-1\leq N(\omega)\leq M(\omega)\right\}.$$

Note that, when $n=1$, (3.9)-(3.10) corresponds to the auxiliary Dynkin game (2.4)-(2.5), which will be used to solve the original constrained Dynkin game. The difference between the auxiliary game and the original game is that the players first make their stopping decisions and then move forward in the auxiliary game, while in original game they first move forward and then make their decisions.

LEMMA 3.2. *Suppose that Assumption 2.1 holds. Then, for any $1 \leq n \leq M$, the value of the auxiliary constrained Dynkin game (3.9)-(3.10) exists. Its value, denoted by $\hat{q}^{\lambda}_{T_{n-1}}$, satisfies the recursive equation (3.8), namely,*

$$\begin{aligned}
&\hat{q}^{\lambda}_{T_{n-1}} \\
&= \min\left\{\tilde{U}_{T_{n-1}}, \max\left\{\mathbb{E}\left[\int_{T_{n-1}}^{T_n\wedge T} \tilde{f}_s\,ds + \tilde{\xi}\mathbb{1}_{\{T_n>T\}} + \hat{q}^{\lambda}_{T_n}\mathbb{1}_{\{T_n\leq T\}}\middle|\mathcal{G}_{T_{n-1}}\right], \tilde{L}_{T_{n-1}}\right\}\right\}.
\end{aligned}$$

Hence, $\hat{q}^{\lambda}_{T_{n-1}} = \hat{Q}^{\lambda}_{T_{n-1}}$ a.s. The optimal stopping strategy of (3.9)-(3.10) is given by

$$\left\{\begin{array}{l} \hat{N}^{\sigma,*}_{n-1} = \inf\{N \geq n-1 : \hat{q}^{\lambda}_{T_N} = \tilde{U}_{T_N}\} \wedge M; \\ \hat{N}^{\tau,*}_{n-1} = \inf\{N \geq n-1 : \hat{q}^{\lambda}_{T_N} = \tilde{L}_{T_N}\} \wedge M. \end{array}\right. \tag{3.11}$$

Proof. Without loss of generality, we may assume $\tilde{f}_s = 0$.
Step 1. Since $T_{M-1} \leq T < T_M$, the upper value of the auxiliary game (3.9) is equivalent to

$$\begin{aligned}
\bar{\hat{q}}^{\lambda}_{T_{n-1}} = \operatorname*{ess\,inf}_{N^\sigma\in\mathcal{N}_{n-1}(\lambda)} \operatorname*{ess\,sup}_{N^\tau\in\mathcal{N}_{n-1}(\lambda)} \mathbb{E}\Big[&\tilde{\xi}\mathbb{1}_{\{N^\sigma=N^\tau=M\}} + \tilde{L}_{T_{N^\tau}}\mathbb{1}_{\{n-1\leq N^\tau\leq M-1, N^\tau\leq N^\sigma\}} \\
&+ \tilde{U}_{T_{N^\sigma}}\mathbb{1}_{\{n-1\leq N^\sigma\leq M-1, N^\sigma<N^\tau\}}|\mathcal{G}_{T_{n-1}}\Big].
\end{aligned}$$

We claim that

$$\bar{\hat{q}}^{\lambda}_{T_{M-1}} = \min\left\{\tilde{U}_{T_{M-1}}, \max\left\{\mathbb{E}\left[\tilde{\xi}|\mathcal{G}_{T_{M-1}}\right], \tilde{L}_{T_{M-1}}\right\}\right\}, \tag{3.12}$$

and, for $n-1 \leq i \leq M-2$,

$$\bar{\hat{q}}^{\lambda}_{T_i} = \min\left\{\tilde{U}_{T_i}, \max\left\{\mathbb{E}\left[\bar{\hat{q}}^{\lambda}_{T_{i+1}}|\mathcal{G}_{T_i}\right], \tilde{L}_{T_i}\right\}\right\}. \tag{3.13}$$

If (3.12)-(3.13) hold, then

$$\begin{aligned}
\bar{\hat{q}}^{\lambda}_{T_{n-1}} &= \min\left\{\tilde{U}_{T_{n-1}}, \max\left\{\mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{n=M\}} + \bar{\hat{q}}^{\lambda}_{T_n}\mathbb{1}_{\{n\leq M-1\}}|\mathcal{G}_{T_{n-1}}\right], \tilde{L}_{T_{n-1}}\right\}\right\} \\
&= \min\left\{\tilde{U}_{T_{n-1}}, \max\left\{\mathbb{E}\left[\tilde{\xi}\mathbb{1}_{\{T_n>T\}} + \bar{\hat{q}}^{\lambda}_{T_n}\mathbb{1}_{\{T_n\leq T\}}|\mathcal{G}_{T_{n-1}}\right], \tilde{L}_{T_{n-1}}\right\}\right\},
\end{aligned}$$

which is the recursive equation (3.8).

Similarly, we obtain that $\underline{\hat{q}}^{\lambda}_{T_{n-1}}$ also satisfies the recursive equation (3.8). Since (3.8) admits a unique solution, it is clear that $\bar{\hat{q}}^{\lambda}_{T_{n-1}} = \underline{\hat{q}}^{\lambda}_{T_{n-1}} = \hat{q}^{\lambda}_{T_{n-1}} = \hat{Q}^{\lambda}_{T_{n-1}}$ a.s.
Step 2. Next, we show (3.12)-(3.13). Indeed, for $i = M-1$,

$$\begin{aligned}
\bar{\hat{q}}^{\lambda}_{T_{M-1}} = &\operatorname*{ess\,inf}_{N^\sigma\in\mathcal{N}_{M-1}(\lambda)} \operatorname*{ess\,sup}_{N^\tau\in\mathcal{N}_{M-1}(\lambda)} \mathbb{E}\Big[\tilde{\xi}\mathbb{1}_{\{N^\sigma=N^\tau=M\}} + \tilde{L}_{T_{M-1}}\mathbb{1}_{\{M-1=N^\tau\leq N^\sigma\}} \\
&+ \tilde{U}_{T_{M-1}}\mathbb{1}_{\{M-1=N^\sigma<N^\tau\}}|\mathcal{G}_{T_{M-1}}\Big] \\
= &\min_{N^\sigma\in\mathcal{N}_{M-1}(\lambda)} \max_{N^\tau\in\mathcal{N}_{M-1}(\lambda)} \Big\{\mathbb{E}[\tilde{\xi}|\mathcal{G}_{T_{M-1}}]\mathbb{1}_{\{N^\sigma=N^\tau=M\}} + \tilde{L}_{T_{M-1}}\mathbb{1}_{\{M-1=N^\tau\leq N^\sigma\}} \\
&+ \tilde{U}_{T_{M-1}}\mathbb{1}_{\{M-1=N^\sigma<N^\tau\}}\Big\} \\
= &\min\left\{\tilde{U}_{T_{M-1}}, \max\left\{\mathbb{E}[\tilde{\xi}|\mathcal{G}_{T_{M-1}}], \tilde{L}_{T_{M-1}}\right\}\right\}.
\end{aligned}$$

In general, for $n-1 \le i \le M-2$, we have

$$\bar{\hat{q}}^{\lambda}_{T_i} = \operatorname*{ess\,inf}_{N^\sigma \in \mathcal{N}_i(\lambda)} \operatorname*{ess\,sup}_{N^\tau \in \mathcal{N}_i(\lambda)} \mathbb{E}\Big[\tilde{\xi}\mathbb{1}_{\{N^\sigma = N^\tau = M\}} + \tilde{L}_{T_{N^\tau}}\mathbb{1}_{\{i \le N^\tau \le M-1, N^\tau \le N^\sigma\}} + \tilde{U}_{T_{N^\sigma}}\mathbb{1}_{\{i \le N^\sigma \le M-1, N^\sigma < N^\tau\}} | \mathcal{G}_{T_i}\Big].$$

Taking conditional expectation on $\mathcal{G}_{T_{i+1}}$ further yields

$$\begin{aligned}\bar{\hat{q}}^{\lambda}_{T_i} &= \operatorname*{ess\,inf}_{N^\sigma \in \mathcal{N}_i(\lambda)} \operatorname*{ess\,sup}_{N^\tau \in \mathcal{N}_i(\lambda)} \mathbb{E}\Big[\tilde{L}_{T_i}\mathbb{1}_{\{i = N^\tau \le N^\sigma\}} + \tilde{U}_{T_i}\mathbb{1}_{\{i = N^\sigma < N^\tau\}} \\ &\quad + \mathbb{E}\Big[\tilde{\xi}\mathbb{1}_{\{N^\sigma = N^\tau = M\}} + \tilde{L}_{T_{N^\tau}}\mathbb{1}_{\{i+1 \le N^\tau \le M-1, N^\tau \le N^\sigma\}} \\ &\quad + \tilde{U}_{T_{N^\sigma}}\mathbb{1}_{\{i+1 \le N^\sigma \le M-1, N^\sigma < N^\tau\}} | \mathcal{G}_{T_{i+1}}\Big] | \mathcal{G}_{T_i}\Big] \\ &= \min\left\{\tilde{U}_{T_i}, \max\left\{\mathbb{E}\Big[\bar{\hat{q}}^{\lambda}_{T_{i+1}} | \mathcal{G}_{T_i}\Big], \tilde{L}_{T_i}\right\}\right\},\end{aligned}$$

where the second equality holds since the operations $\operatorname{ess\,inf}_{N^\sigma \in \mathcal{N}_{i+1}(\lambda)} \operatorname{ess\,sup}_{N^\tau \in \mathcal{N}_{i+1}(\lambda)}$ and $\mathbb{E}[\cdot|\mathcal{G}_{T_i}]$ are interchangeable, which will be proved in the next step.

Step 3. In this step, we show the operations $\operatorname{ess\,inf}_{N^\sigma \in \mathcal{N}_{i+1}(\lambda)} \operatorname{ess\,sup}_{N^\tau \in \mathcal{N}_{i+1}(\lambda)}$ and $\mathbb{E}[\cdot|\mathcal{G}_{T_i}]$ are interchangeable, i.e. (3.16) below holds. To this end, for fixed i and $N^\sigma \in \mathcal{N}_i(\lambda)$, we note that the family

$$\left(\mathbb{E}\Big[\tilde{R}_i(T_{N^\sigma}, T_{N^\tau})|\mathcal{G}_{T_i}\Big], N^\tau \in \mathcal{N}_i(\lambda)\right) \tag{3.14}$$

is an increasing directed set. Indeed, if we choose arbitrary $N^\tau_1, N^\tau_2 \in \mathcal{N}_i(\lambda)$ and let $X_j = \mathbb{E}\Big[\tilde{R}_i(T_{N^\sigma}, T_{N^\tau_j})|\mathcal{G}_{T_i}\Big]$, for $j = 1, 2$. Then, defining the stopping time N^τ as $N^\tau = N^\tau_1\mathbb{1}_{\{X_1 \ge X_2\}} + N^\tau_2\mathbb{1}_{\{X_1 < X_2\}}$, we have $N^\tau \in \mathcal{N}_i(\lambda)$ and $\mathbb{E}\Big[\tilde{R}_i(T_{N^\sigma}, T_{N^\tau})|\mathcal{G}_{T_i}\Big] \ge \max\{X_1, X_2\}$.

Similarly, we also have, for fixed i, the family

$$\left(\operatorname*{ess\,sup}_{N^\tau \in \mathcal{N}_i(\lambda)} \mathbb{E}\Big[\tilde{R}_i(T_{N^\sigma}, T_{N^\tau})|\mathcal{G}_{T_i}\Big], N^\sigma \in \mathcal{N}_i(\lambda)\right) \tag{3.15}$$

is a decreasing directed set. Under Assumption 2.1, it is obvious that both (3.14) and (3.15) are uniformly integrable. Therefore, by Proposition VI-1-1 of Neveu [28], we obtain

$$\begin{aligned}\mathbb{E}\Big[\bar{\hat{q}}^{\lambda}_{T_{i+1}}\Big|\mathcal{G}_{T_i}\Big] &= \mathbb{E}\left[\operatorname*{ess\,inf}_{N^\sigma \in \mathcal{N}_{i+1}(\lambda)} \operatorname*{ess\,sup}_{N^\tau \in \mathcal{N}_{i+1}(\lambda)} \mathbb{E}\Big[\tilde{R}_{i+1}(T_{N^\sigma}, T_{N^\tau})|\mathcal{G}_{T_{i+1}}\Big]\middle|\mathcal{G}_{T_i}\right] \\ &= \operatorname*{ess\,inf}_{N^\sigma \in \mathcal{N}_{i+1}(\lambda)} \mathbb{E}\left[\operatorname*{ess\,sup}_{N^\tau \in \mathcal{N}_{i+1}(\lambda)} \mathbb{E}\Big[\tilde{R}_{i+1}(T_{N^\sigma}, T_{N^\tau})|\mathcal{G}_{T_{i+1}}\Big]\middle|\mathcal{G}_{T_i}\right] \\ &= \operatorname*{ess\,inf}_{N^\sigma \in \mathcal{N}_{i+1}(\lambda)} \operatorname*{ess\,sup}_{N^\tau \in \mathcal{N}_{i+1}(\lambda)} \mathbb{E}\Big[\tilde{R}_{i+1}(T_{N^\sigma}, T_{N^\tau})|\mathcal{G}_{T_i}\Big].\end{aligned} \tag{3.16}$$

Step 4. It remains to prove that $\left(\hat{N}^{\sigma,*}_{n-1}, \hat{N}^{\tau,*}_{n-1}\right)$ in (3.11) are indeed the optimal stopping times for the auxiliary Dynkin game (3.9)-(3.10), i.e. for every $(N^\sigma, N^\tau) \in$

$\mathcal{N}_{n-1}(\lambda) \times \mathcal{N}_{n-1}(\lambda)$,

$$\mathbb{E}\left[\tilde{R}_{n-1}\left(T_{\hat{N}^{\sigma,*}_{n-1}}, T_{N^\tau}\right)|\mathcal{G}_{T_{n-1}}\right] \le \mathbb{E}\left[\tilde{R}_{n-1}\left(T_{\hat{N}^{\sigma,*}_{n-1}}, T_{\hat{N}^{\tau,*}_{n-1}}\right)|\mathcal{G}_{T_{n-1}}\right] \le \mathbb{E}\left[\tilde{R}_{n-1}\left(T_{N^\sigma}, T_{\hat{N}^{\tau,*}_{n-1}}\right)|\mathcal{G}_{T_{n-1}}\right].$$

To this end, it suffices to prove that

(i) $\left(\hat{q}^\lambda_{T_{m\wedge \hat{N}^{\sigma,*}_{n-1}\wedge \hat{N}^{\tau,*}_{n-1}}}\right)_{m\ge n-1}$ is a $\tilde{\mathbb{G}}$-martingale;

(ii) $\left(\hat{q}^\lambda_{T_{m\wedge \hat{N}^{\sigma,*}_{n-1}\wedge N^\tau}}\right)_{m\ge n-1}$ is a $\tilde{\mathbb{G}}$-supermartingale for any $N^\tau \in \mathcal{N}_{n-1}(\lambda)$;

(iii) $\left(\hat{q}^\lambda_{T_{m\wedge N^\sigma\wedge \hat{N}^{\tau,*}_{n-1}}}\right)_{m\ge n-1}$ is a $\tilde{\mathbb{G}}$-submartingale for any $N^\sigma \in \mathcal{N}_{n-1}(\lambda)$.

Indeed, we have

$$\begin{aligned}
&\mathbb{E}\left[\hat{q}^\lambda_{T_{(m+1)\wedge \hat{N}^{\sigma,*}_{n-1}\wedge \hat{N}^{\tau,*}_{n-1}}}\middle|\mathcal{G}_{T_m}\right]\\
&=\mathbb{E}\left[\left(\sum_{j=n-1}^m \mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}\wedge \hat{N}^{\tau,*}_{n-1}=j\}} + \mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}\wedge \hat{N}^{\tau,*}_{n-1}\ge m+1\}}\right)\hat{q}^\lambda_{T_{(m+1)\wedge \hat{N}^{\sigma,*}_{n-1}\wedge \hat{N}^{\tau,*}_{n-1}}}\middle|\mathcal{G}_{T_m}\right]\\
&=\sum_{j=n-1}^m \mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}\wedge \hat{N}^{\tau,*}_{n-1}=j\}}\hat{q}^\lambda_{T_j} + \mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}\wedge \hat{N}^{\tau,*}_{n-1}\ge m+1\}}\mathbb{E}\left[\hat{q}^\lambda_{T_{m+1}}\middle|\mathcal{G}_{T_m}\right]\\
&=\sum_{j=n-1}^m \mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}\wedge \hat{N}^{\tau,*}_{n-1}=j\}}\hat{q}^\lambda_{T_j} + \mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}\wedge \hat{N}^{\tau,*}_{n-1}\ge m+1\}}\hat{q}^\lambda_{T_m} = \hat{q}^\lambda_{T_{m\wedge \hat{N}^{\sigma,*}_{n-1}\wedge \hat{N}^{\tau,*}_{n-1}}},
\end{aligned}$$

where the second last equality follows from the definition of $\left(\hat{N}^{\sigma,*}_{n-1}, \hat{N}^{\tau,*}_{n-1}\right)$ in (3.11), so the martingale property (i) has been proved.

To prove the supermartingale property (ii), we note that

$$\begin{aligned}
&\mathbb{E}\left[\hat{q}^\lambda_{T_{(m+1)\wedge \hat{N}^{\sigma,*}_{n-1}\wedge N^\tau}}\middle|\mathcal{G}_{T_m}\right]\\
&=\mathbb{E}\left[\hat{q}^\lambda_{T_{(m+1)\wedge \hat{N}^{\sigma,*}_{n-1}}}\mathbb{1}_{\{N^\tau\ge m+1\}} + \hat{q}^\lambda_{T_{\hat{N}^{\sigma,*}_{n-1}\wedge N^\tau}}\mathbb{1}_{\{N^\tau\le m\}}\middle|\mathcal{G}_{T_m}\right]\\
&=\mathbb{E}\left[\left(\sum_{j=n-1}^m \mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}=j\}} + \mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}\ge m+1\}}\right)\hat{q}^\lambda_{T_{(m+1)\wedge \hat{N}^{\sigma,*}_{n-1}}}\mathbb{1}_{\{N^\tau\ge m+1\}}\right.\\
&\qquad\left. + \hat{q}^\lambda_{T_{\hat{N}^{\sigma,*}_{n-1}\wedge N^\tau}}\mathbb{1}_{\{N^\tau\le m\}}\middle|\mathcal{G}_{T_m}\right]\\
&=\left(\sum_{j=n-1}^m \mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}=j\}}\hat{q}^\lambda_{T_j} + \mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}\ge m+1\}}\mathbb{E}\left[\hat{q}^\lambda_{T_{m+1}}\middle|\mathcal{G}_{T_m}\right]\right)\mathbb{1}_{\{N^\tau\ge m+1\}}\\
&\qquad + \hat{q}^\lambda_{T_{\hat{N}^{\sigma,*}_{n-1}\wedge N^\tau}}\mathbb{1}_{\{N^\tau\le m\}}.
\end{aligned}$$

Using the definition of $\hat{N}^{\sigma,*}_{n-1}$ in (3.11), we further have

$$\mathbb{E}\left[\hat{q}^\lambda_{T_{m+1}}\middle|\mathcal{G}_{T_m}\right] \le \max\left\{\mathbb{E}\left[\hat{q}^\lambda_{T_{m+1}}\middle|\mathcal{G}_{T_m}\right], \tilde{L}_{T_m}\right\} = \hat{q}^\lambda_{T_m} \text{ on } \{\hat{N}^{\sigma,*}_{n-1}\ge m+1\}.$$

In turn,

$$\begin{aligned}
&\mathbb{E}\left[\hat{q}^{\lambda}_{T_{(m+1)\wedge \hat{N}^{\sigma,*}_{n-1}\wedge N^{\tau}}}\middle|\mathcal{G}_{T_m}\right]\\
&\leq \left(\sum_{j=n-1}^{m}\mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}=j\}}\hat{q}^{\lambda}_{T_j}+\mathbb{1}_{\{\hat{N}^{\sigma,*}_{n-1}\geq m+1\}}\hat{q}^{\lambda}_{T_m}\right)\mathbb{1}_{\{N^{\tau}\geq m+1\}}+\hat{q}^{\lambda}_{T_{\hat{N}^{\sigma,*}_{n-1}\wedge N^{\tau}}}\mathbb{1}_{\{N^{\tau}\leq m\}}\\
&=\hat{q}^{\lambda}_{T_{m\wedge \hat{N}^{\sigma,*}_{n-1}}}\mathbb{1}_{\{N^{\tau}\geq m+1\}}+\hat{q}^{\lambda}_{T_{\hat{N}^{\sigma,*}_{n-1}\wedge N^{\tau}}}\mathbb{1}_{\{N^{\tau}\leq m\}}=\hat{q}^{\lambda}_{T_{m\wedge \hat{N}^{\sigma,*}_{n-1}\wedge N^{\tau}}},
\end{aligned}$$

which proves the supermartingale property (ii). Likewise, the submartingale property (iii) can be proved in a similar way, and the proof of the lemma is completed. □

We are now in a position to prove Theorem 2.3. By Lemmas 3.1 and 3.2, we have

$$\begin{aligned}
Q^{\lambda}_0 &= \mathbb{E}\left[\int_0^{T_1\wedge T}\tilde{f}_s\,ds+\tilde{\xi}\mathbb{1}_{\{T_1>T\}}+\hat{Q}^{\lambda}_{T_1}\mathbb{1}_{\{T_1\leq T\}}\right]\\
&= \mathbb{E}\left[\int_0^{T_1\wedge T}\tilde{f}_s\,ds+\tilde{\xi}\mathbb{1}_{\{T_1>T\}}+\hat{q}^{\lambda}_{T_1}\mathbb{1}_{\{T_1\leq T\}}\right]\\
&\geq \mathbb{E}\bigg[\int_0^{T_1\wedge T}\tilde{f}_s\,ds+\tilde{\xi}\mathbb{1}_{\{T_1>T\}}+\mathbb{E}\left[\tilde{R}_1(T_{\hat{N}^{\sigma,*}_1},T_{N^{\tau}})|\mathcal{G}_{T_1}\right]\mathbb{1}_{\{T_1\leq T\}}\bigg] \qquad (3.17)
\end{aligned}$$

for any $N^{\tau}\in\mathcal{N}_1(\lambda)$, where last inequality follows from the supermartingale property (ii). Moreover, recall that

$$\begin{aligned}
&\mathbb{E}\left[\tilde{R}_1(T_{\hat{N}^{\sigma,*}_1},T_{N^{\tau}})|\mathcal{G}_{T_1}\right]\\
&=\mathbb{E}\Bigg[\int_{T_1\wedge T}^{T_{\hat{N}^{\sigma,*}_1}\wedge T_{N^{\tau}}\wedge T}\tilde{f}_s\,ds+\tilde{\xi}\mathbb{1}_{\left\{T_{\hat{N}^{\sigma,*}_1}\wedge T_{N^{\tau}}\geq T\right\}}\\
&\quad+\tilde{L}_{T_{N^{\tau}}}\mathbb{1}_{\left\{T_{N^{\tau}}<T,T_{N^{\tau}}\leq T_{\hat{N}^{\sigma,*}_1}\right\}}+\tilde{U}_{T_{\hat{N}^{\sigma,*}_1}}\mathbb{1}_{\left\{T_{\hat{N}^{\sigma,*}_1}<T,T_{\hat{N}^{\sigma,*}_1}<T_{N^{\tau}}\right\}}|\mathcal{G}_{T_1}\Bigg].
\end{aligned}$$

Plugging the above expression into (3.17) further yields

$$\begin{aligned}
Q^{\lambda}_0 &\geq \mathbb{E}\Bigg[\int_0^{T_{\hat{N}^{\sigma,*}_1}\wedge T_{N^{\tau}}\wedge T}\tilde{f}_s\,ds+\tilde{\xi}\mathbb{1}_{\left\{T_{\hat{N}^{\sigma,*}_1}\wedge T_{N^{\tau}}\geq T\right\}}+\tilde{L}_{T_{N^{\tau}}}\mathbb{1}_{\left\{T_{N^{\tau}}<T,T_{N^{\tau}}\leq T_{\hat{N}^{\sigma,*}_1}\right\}}\\
&\quad+\tilde{U}_{T_{\hat{N}^{\sigma,*}_1}}\mathbb{1}_{\left\{T_{\hat{N}^{\sigma,*}_1}<T,T_{\hat{N}^{\sigma,*}_1}<T_{N^{\tau}}\right\}}\Bigg]=\mathbb{E}\left[\tilde{R}(T_{\hat{N}^{\sigma,*}_1},T_{N^{\tau}})\right],
\end{aligned}$$

for any $\tilde{\mathbb{G}}$-stopping time $N^{\tau}\in\mathcal{N}_1(\lambda)$. Taking the supremum over $N^{\tau}\in\mathcal{N}_1(\lambda)$, we obtain

$$Q^{\lambda}_0\geq \sup_{N^{\tau}\in\mathcal{N}_1(\lambda)}\mathbb{E}\left[\tilde{R}(T_{\hat{N}^{\sigma,*}_1},T_{N^{\tau}})\right]\geq \inf_{N^{\sigma}\in\mathcal{N}_1(\lambda)}\sup_{N^{\tau}\in\mathcal{N}_1(\lambda)}\mathbb{E}\left[\tilde{R}(T_{N^{\sigma}},T_{N^{\tau}})\right]=\overline{q}^{\lambda}.$$

Similarly, we also have $Q^{\lambda}_0\leq \underline{q}^{\lambda}$. It then follows from $\overline{q}^{\lambda}\geq\underline{q}^{\lambda}$ that $Q^{\lambda}_0=\underline{q}^{\lambda}=\overline{q}^{\lambda}$.

Finally, we verify that $Q^{\lambda}_0=\mathbb{E}[\tilde{R}(T_{\hat{N}^{\sigma,*}_1},T_{\hat{N}^{\tau,*}_1})]$, so $(\hat{N}^{\sigma,*}_1,\hat{N}^{\tau,*}_1)$ are the optimal stopping strategy. Indeed, with $N^{\sigma}=\hat{N}^{\sigma,*}_1$ and $N^{\tau}=\hat{N}^{\tau,*}_1$, (3.17) becomes an equality due to the martingale property (i), i.e.

$$\begin{aligned}Q_0^\lambda &= \mathbb{E}\left[\int_0^{T_1\wedge T}\tilde{f}_s\,ds+\tilde{\xi}\mathbb{1}_{\{T_1>T\}}+\hat{q}_{T_1}^\lambda\mathbb{1}_{\{T_1\leq T\}}\right]\\ &= \mathbb{E}\left[\int_0^{T_1\wedge T}\tilde{f}_s\,ds+\tilde{\xi}\mathbb{1}_{\{T_1>T\}}+\mathbb{E}\left[\tilde{R}_1(T_{\hat{N}_1^{\sigma,*}},T_{\hat{N}_1^{\tau,*}})|\mathcal{G}_{T_1}\right]\mathbb{1}_{\{T_1\leq T\}}\right]\\ &= \mathbb{E}\left[\tilde{R}(T_{\hat{N}_1^{\sigma,*}},T_{\hat{N}_1^{\tau,*}})\right].\end{aligned}$$

We conclude the proof by proving that the optimal stopping times $\left(\hat{N}_1^{\sigma,*},\hat{N}_1^{\tau,*}\right)$ are actually $\left(N_1^{\sigma,*},N_1^{\tau,*}\right)$ in (3.6). Indeed,

$$\begin{aligned}\hat{N}_1^{\sigma,*} &= \inf\{N\geq 1:\hat{q}_{T_N}^\lambda=\tilde{U}_{T_N}\}\wedge M\\ &= \inf\{N\geq 1:\hat{Q}_{T_N}^\lambda=\tilde{U}_{T_N}\}\wedge M\\ &= \inf\{N\geq 1:Q_{T_N}^\lambda\geq\tilde{U}_{T_N}\}\wedge M=N_1^{\sigma,*},\end{aligned}$$

and, similarly, $\hat{N}_1^{\tau,*}=N_1^{\tau,*}$.

4. Connection with standard Dynkin games. We show that, when $\lambda\to\infty$, the value v^λ of the constrained Dynkin game converges to the value of a standard Dynkin game. The setup is the same as in section 2 except that the control set is replaced with $\mathcal{R}_t$, which is defined as

$$\mathcal{R}_t=\{\mathbb{F}\text{-stopping time }\tau\text{ for }t\leq\tau(\omega)\leq T\}.$$

Define the corresponding upper and lower values of the standard Dynkin game as

$$\overline{v}=\inf_{\sigma\in\mathcal{R}_0}\sup_{\tau\in\mathcal{R}_0}\mathbb{E}\left[R(\sigma,\tau)\right],\tag{4.1}$$

$$\underline{v}=\sup_{\tau\in\mathcal{R}_0}\inf_{\sigma\in\mathcal{R}_0}\mathbb{E}\left[R(\sigma,\tau)\right].\tag{4.2}$$

This game is said to have value v if $v=\overline{v}=\underline{v}$, and $(\sigma^*,\tau^*)\in\mathcal{R}_0\times\mathcal{R}_0$ is called a saddle point of the game if $\mathbb{E}\left[R(\sigma^*,\tau)\right]\leq\mathbb{E}\left[R(\sigma^*,\tau^*)\right]\leq\mathbb{E}\left[R(\sigma,\tau^*)\right]$ for every $(\sigma,\tau)\in\mathcal{R}_0\times\mathcal{R}_0$.

Proposition 4.1. *Suppose that Assumption 2.1 holds and, moreover, both L and U are continuous and satisfy $L_T\leq\xi\leq U_T$. Then, the value v of the Dynkin game (4.1)-(4.2) exists and, moreover, $\lim_{\lambda\uparrow\infty}v^\lambda=v$.*

Proof. To solve the Dynkin game (4.1)-(4.2), we introduce the following reflected BSDE defined on a random horizon $[0,T]$:

$$V_{t\wedge T}=\xi+\int_{t\wedge T}^T(f_s-rV_s)ds+\int_{t\wedge T}^T dK_s^+-\int_{t\wedge T}^T dK_s^--\int_{t\wedge T}^T Z_s\,dW_s\tag{4.3}$$

for $t\geq 0$, under the constraints (i) $L_t\leq V_t\leq U_t$, for $0\leq t\leq T$; (ii) $\int_0^T(V_t-L_t)\,dK_t^+=\int_0^T(U_t-V_t)\,dK_t^-=0$. By a solution to the reflected BSDE (4.3), we mean a triplet of $\mathbb{F}$-progressively measurable processes (V,Z,K), where $K:=K^+-K^-$ with K^+ and K^- being increasing processes starting from $K_0^+=K_0^-=0$.

It follows from Hamadene et al [15] that (4.3) is well-posed and admits a unique solution. Using arguments similar to the ones in Cvitanic and Karatzas [11], it is standard to show that the value of the Dynkin game (4.1)-(4.2) exists and is given by the solution of the reflected BSDE (4.3), i.e. $v = \overline{v} = \underline{v} = V_0$.

To prove the second assertion, we note that BSDE (2.6) can be regarded as a sequence of penalized BSDEs for (4.3), where the local time processes K^+ and K^- are approximated by

$$K_t^{\lambda,+} := \int_0^t \lambda \left(L_s - V_s^\lambda\right)^+ ds; \quad K_t^{\lambda,-} := \int_0^t \lambda \left(V_s^\lambda - U_s\right)^+ ds,$$

with $K^\lambda := K^{\lambda,+} - K^{\lambda,-}$. Since $\lim_{\lambda\uparrow\infty} \mathbb{E}[\sup_{t\in[0,T]} |V_t^\lambda - V_t|^2] = 0$ (see, for example, [15] and [11]), the second assertion follows immediately. □

5. Replication of constrained Dynkin games. In this section, we discuss about replication of the constrained Dynkin game. This provides a foundation for the risk-neutral valuation of convertible bonds introduced in the next section.

Let $\bar{N}_t := \sum_{n\geq 1} \mathbb{1}_{\{T_n \leq t\}} - \lambda t$, $t \geq 0$, be the compensated Poisson martingale on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{G}, \mathbb{P})$. Suppose there exist $(d+2)$ underlying assets, whose pricing processes follow

$$dS_t^i = S_t^i(r - q^i)dt + S_t^i \sigma^i dW_t,\ 1 \leq i \leq d; \tag{5.1}$$

$$dP_t = P_{t-} r dt + P_{t-} \bar{\sigma} d\bar{N}_t; \tag{5.2}$$

$$dB_t = B_t r dt, \tag{5.3}$$

where $r > 0$ is the risk-free interest rate, $\bar{\sigma} > 0$ represents the volatility of P, and q^i and $\sigma^i := (\sigma^{ij})_{1\leq j\leq d}$ represent, respectively, the dividend and volatility of S^i. Assume that the volatility matrix $\sigma := (\sigma^{ij})_{1\leq i,j\leq d}$ is invertible. The risky assets $(S^i)_{1\leq i\leq d}$ are the underlying assets used to hedge the Brownian noise of the game. The risky asset P is used to hedge the jump risk of the Poisson process. In practice, it could be the cash flow of a credit default swap delivering payoffs at jump times $(T_n)_{n\geq 1}$ (see, for example, [5] for the single jump case). Finally, B represents the risk-free bank account. Therefore, $\mathbb{P}$ could be interpreted as a risk-neutral probability measure.

From section 2.3 (Lemmas 3.1 and 3.2 in particular), we know that the solution V^λ of BSDE (2.6) provides the values of the constrained Dynkin game (2.2)-(2.3) starting at different Poisson arrival times T_{n-1}, for $1 \leq n \leq M$, and they satisfy the recursive equation

$$\begin{aligned} e^{-rT_{n-1}} V_{T_{n-1}}^\lambda = \mathbb{E}\Bigg[&\int_{T_{n-1}}^{T_n\wedge T} e^{-rs} f_s ds + e^{-rT} \xi \mathbb{1}_{\{T_n > T\}} \\ &+ e^{-rT_n} \min\{U_{T_n}, \max\{V_{T_n}^\lambda, L_{T_n}\}\} \mathbb{1}_{\{T_n \leq T\}} | \mathcal{G}_{T_{n-1}} \Bigg]. \end{aligned} \tag{5.4}$$

Thus, the discounted payoff of the game starting at T_{n-1} is

$$\begin{aligned} &\left(\int_{T_{n-1}}^T e^{-rs} f_s\, ds + e^{-rT} \xi \right) \mathbb{1}_{\{T_n > T\}} \\ &+ \left(\int_{T_{n-1}}^{T_n} e^{-rs} f_s\, ds + e^{-rT_n} \min\{U_{T_n}, \max\{V_{T_n}^\lambda, L_{T_n}\}\} \right) \mathbb{1}_{\{T_n \leq T\}}, \end{aligned} \tag{5.5}$$

with $V^{\lambda}_{T_n}$ being the value of the game starting at T_n. Compared to the original payoff (2.1), the above payoff (5.5) (with $n=1$) only involves the first Poisson arrival time T_1, and the optimality of stopping strategies is encoded in $V^{\lambda}_{T_1}$. Thus, the replication of the constrained Dynkin game (2.2)-(2.3) naturally depends on the replication of the same game but starting at Poisson arrival time T_1, the later of which in turn depends on the replication of the game starting from T_2 and so on and so forth. In particular, the discounted payoff of the game starting at T_{M-1} is $\int_{T_{M-1}}^{T} e^{-rs} f_s ds + e^{-rT}\xi$, since $T_{M-1} \leq T < T_M$ by the definition of the random variable M.

For $1 \leq n \leq M$, consider the constrained Dynkin game starting at Poisson arrival time T_{n-1}. We aim to construct a replication portfolio $(\pi^{S,n}_t, \pi^{P,n}_t, \pi^{B,n}_t)$, $t \in [T_{n-1}, T]$, to replicate the discounted payoff (5.5), where $\pi^{S,n} = (\pi^{S^i,n})_{1\leq i\leq d}$ represents the amount of the money invested in $(S^i)_{1\leq i\leq d}$, and $\pi^{P,n}$ and $\pi^{B,n}$ represent the amount of the money invested in P and B, respectively. Let X^n_t be the corresponding wealth of each player at time t. Then, $X^n_t = \sum_{i=1}^d \pi^{S^i,n}_t + \pi^{P,n}_t + \pi^{B,n}_t$, and the self-financing condition implies that

$$
\begin{aligned}
X^n_t &= X^n_{T_{n-1}} + \int_{T_{n-1}}^{t} \left(\sum_{i=1}^{d} \frac{\pi^{S^i,n}_s}{S^i_s} dS^i_s + \frac{\pi^{P,n}_s}{P_{s-}} dP_s + \frac{\pi^{B,n}_s}{B_s} dB_s + \sum_{i=1}^{d} q^i \pi^{S^i,n}_s ds \right) \\
&= X^n_{T_{n-1}} + \int_{T_{n-1}}^{t} \left(rX^n_s ds + \pi^{S,n}_s \sigma dW_s + \pi^{P,n}_s \bar{\sigma} d\bar{N}_s \right), \qquad (5.6)
\end{aligned}
$$

for $t \in [T_{n-1}, T]$. The problem is to find a replication portfolio $(\pi^{S,n}, \pi^{P,n}, \pi^{B,n})$ such that the discounted wealth $e^{-rT} X^n_T$ replicates the discounted payoff (5.5), and to prove that $X^n_{T_{n-1}} = V^{\lambda}_{T_{n-1}}$, i.e. the constrained Dynkin game starting from T_{n-1} is replicable and its value is indeed given by $V^{\lambda}_{T_{n-1}}$.

THEOREM 5.1. *Let $(Y^{\xi,\theta}, Z^{\xi,\theta})$ be the unique solution of the linear BSDE defined on the random horizon $[\theta, T]$ with a parameter $\theta \in [0,T]$, i.e.*

$$
Y^{\xi,\theta}_{t\wedge T} = \left(\int_{\theta}^{T} e^{r(T-s)} f_s ds + \xi \right) - \int_{t\wedge T}^{T} rY^{\xi,\theta}_s ds - \int_{t\wedge T}^{T} Z^{\xi,\theta}_s dW_s, \qquad (5.7)
$$

for $t \geq \theta$. Then, for the constrained Dynkin game starting at T_{M-1}, its replication wealth and the corresponding replication portfolio are given by

$$
\left\{
\begin{array}{l}
X^M_t = Y^{\xi, T_{M-1}}_t; \\
(\pi^{S,M}_t, \pi^{P,M}_t, \pi^{B,M}_t) = (Z^{\xi,T_{M-1}}_t \sigma^{-1}, 0, X^M_t - \pi^{S,M}_t), \ t \in [T_{M-1}, T],
\end{array}
\right. \qquad (5.8)
$$

where σ^{-1} is the inverse of the volatility matrix $(\sigma^{ij})_{1\leq i,j\leq d}$. Moreover, the value of the game is given by $X^M_{T_{M-1}} = V^{\lambda}_{T_{M-1}}$.

In general, let (Y^{θ}, Z^{θ}) be the unique solution of the linear BSDE defined on $[\theta, T]$ with a parameter $\theta \in [0,T]$, i.e.

$$
Y^{\theta}_{t\wedge T} = \left(\int_{\theta}^{T} e^{r(T-s)} f_s\, ds + \xi \right) - \int_{t\wedge T}^{T} \left[rY^{\theta}_s - \lambda (Y^{\theta,s}_s - Y^{\theta}_s) \right] ds - \int_{t\wedge T}^{T} Z^{\theta}_s dW_s, \qquad (5.9)
$$

for $t \geq \theta$, and $(Y^{\theta,\bar{\theta}}, Z^{\theta,\bar{\theta}})$ be the unique solution of the linear BSDE defined on $[\bar{\theta}, T]$

with parameters $\theta, \bar{\theta}$ satisfying $0 \leq \theta < \bar{\theta} \leq T$, i.e.

$$Y_{t\wedge T}^{\theta,\bar{\theta}} = \left(\int_\theta^{\bar{\theta}} e^{r(T-s)} f_s ds + e^{r(T-\bar{\theta})} \min\{U_{\bar{\theta}}, \max\{V_{\bar{\theta}}^\lambda, L_{\bar{\theta}}\}\} \right) - \int_{t\wedge T}^T rY_s^{\theta,\bar{\theta}} ds - \int_{t\wedge T}^T Z_s^{\theta,\bar{\theta}} dW_s, \tag{5.10}$$

for $t \geq \bar{\theta}$, where V^λ is the unique solution to BSDE (2.6). Then, for the constrained Dynkin game starting at T_{n-1} for $1 \leq n \leq M-1$, its replication wealth and the corresponding replication portfolio are given by

$$\begin{cases} X_t^n &= Y_t^{T_{n-1}} \mathbb{1}_{\{t<T_n\}} + Y_t^{T_{n-1},T_n} \mathbb{1}_{\{t\geq T_n\}}; \\ \pi_t^{S,n} &= \left(Z_t^{T_{n-1}} \mathbb{1}_{\{t\leq T_n\}} + Z_t^{T_{n-1},T_n} \mathbb{1}_{\{t>T_n\}} \right) \sigma^{-1}; \\ \pi_t^{P,n} &= \left(Y_t^{T_{n-1},t} - Y_t^{T_{n-1}} \right) \mathbb{1}_{\{t\leq T_n\}} \bar{\sigma}^{-1}; \\ \pi_t^{B,n} &= X_t - \pi_t^{S,n} - \pi_t^{P,n},\ t \in [T_{n-1}, T]. \end{cases} \tag{5.11}$$

Moreover, the value of the game is given by $X_{T_{n-1}}^n = V_{T_{n-1}}^\lambda$.

Proof. We first replicate the constrained Dynkin game starting at T_{M-1}. It is clear that $(Y^{\xi,T_{M-1}}, Z^{\xi,T_{M-1}}\sigma^{-1}, 0)$ satisfies the wealth equation (5.6) and, moreover, by applying Itô's formula to $e^{-rt}Y_t^{\xi,\theta}$, we further have

$$e^{-r(t\wedge T)} Y_{t\wedge T}^{\xi,\theta} = \left(\int_\theta^T e^{-rs} f_s ds + e^{-rT}\xi \right) - \int_{t\wedge T}^T e^{-rs} Z_s^{\xi,\theta} dW_s.$$

Thus, $e^{-rT}Y_T^{\xi,T_{M-1}}$ replicates the discounted payoff (5.5) with $n = M$ and, moreover,

$$X_{T_{M-1}}^M = Y_{T_{M-1}}^{\xi,T_{M-1}} = \mathbb{E}\left[\int_{T_{M-1}}^T e^{-r(s-T_{M-1})} f_s ds + e^{-r(T-T_{M-1})}\xi \,\middle|\, \mathcal{G}_{T_{M-1}} \right] = V_{T_{M-1}}^\lambda.$$

In general, we prove the assertion for $1 \leq n \leq M-1$ by induction. Suppose the assertion holds for the game starting at T_n and $X_{T_n}^{n+1} = V_{T_n}^\lambda$. Then, for the game starting at T_{n-1}, by the construction of X^n in (5.11) and the terminal data for BSDEs (5.9) and (5.10), we have

$$\begin{aligned} e^{-rT} X_T^n &= e^{-rT} \left(Y_T^{T_{n-1}} \mathbb{1}_{\{T<T_n\}} + Y_T^{T_{n-1},T_n} \mathbb{1}_{\{T\geq T_n\}} \right) \\ &= \left(\int_{T_{n-1}}^T e^{-rs} f_s\, ds + e^{-rT}\xi \right) \mathbb{1}_{\{T_n>T\}} \\ &\quad + \left(\int_{T_{n-1}}^{T_n} e^{-rs} f_s\, ds + e^{-rT_n} \min\{U_{T_n}, \max\{V_{T_n}^\lambda, L_{T_n}\}\} \right) \mathbb{1}_{\{T_n\leq T\}}. \end{aligned} \tag{5.12}$$

Therefore, $e^{-rT}X_T^n$ replicates the discounted payoff (5.5).

Next, we show that $(X^n, \pi^{S,n}, \pi^{P,n})$ given in (5.11) indeed satisfies the wealth equation (5.6). To this end, note that

$$X_t^n = X_{t\wedge T_n-}^n + (X_{t\wedge T_n}^n - X_{t\wedge T_n-}^n) + (X_t^n - X_{t\wedge T_n}^n),$$

for $t \in [T_{n-1}, T]$. Since $X^n_{t\wedge T_n -} = Y^{T_{n-1}}_{t\wedge T_n -}$ by the definition of X^n, we have

$$\begin{aligned} X^n_{t\wedge T_n -} &= Y^{T_{n-1}}_{T_{n-1}} + \int_{T_{n-1}}^{t\wedge T_n -} \left[rY^{T_{n-1}}_s - \lambda(Y^{T_{n-1},s}_s - Y^{T_{n-1}}_s)\right] ds + \int_{T_{n-1}}^{t\wedge T_n -} Z^{T_{n-1}}_s dW_s \\ &= X^n_{T_{n-1}} + \int_{T_{n-1}}^{t\wedge T_n -} rY^{T_{n-1}}_s ds + \int_{T_{n-1}}^{t\wedge T_n -} Z^{T_{n-1}}_s dW_s \\ &\quad - \int_{T_{n-1}}^{t} (Y^{T_{n-1},s}_s - Y^{T_{n-1}}_s)\mathbb{1}_{\{s\le T_n\}}\lambda ds. \end{aligned}$$

Furthermore, at the Poisson arrival time T_n, X^n has a jump with size

$$X^n_{t\wedge T_n} - X^n_{t\wedge T_n -} = \left(Y^{T_{n-1},T_n}_{T_n} - Y^{T_{n-1}}_{T_n -}\right)\mathbb{1}_{\{T_n\le t\}} = \int_{T_{n-1}}^{t} (Y^{T_{n-1},s}_s - Y^{T_{n-1}}_s)\mathbb{1}_{\{s\le T_n\}} dN_s.$$

On the other hand, since $X^n_t = Y^{T_{n-1},T_n}_t$ on the event $\{t \ge T_n\}$, we have

$$X^n_t - X^n_{t\wedge T_n} = \int_{t\wedge T_n}^{t} rY^{T_{n-1},T_n}_s ds + \int_{t\wedge T_n}^{t} Z^{T_{n-1},T_n}_s dW_s.$$

In turn, we deduce, using the constructions of X^n, $\pi^{S,n}$ and $\pi^{P,n}$ in (5.11), that

$$X^n_t = X^n_{T_{n-1}} + \int_{T_{n-1}}^{t} rX^n_s ds + \int_{T_{n-1}}^{t} \pi^{S,n}_s \sigma dW_s + \int_{T_{n-1}}^{t} \pi^{P,n}_s \bar{\sigma} d\bar{N}_s.$$

Finally, applying Itô's formula to $e^{-rt}X^n_t$ and using (5.12), we obtain that

$$\begin{aligned} e^{-rT_{n-1}} X^n_{T_{n-1}} &= \left(\int_{T_{n-1}}^{T} e^{-rs} f_s\, ds + e^{-rT}\xi\right)\mathbb{1}_{\{T_n > T\}} \\ &\quad + \left(\int_{T_{n-1}}^{T_n} e^{-rs} f_s\, ds + e^{-rT_n}\min\{U_{T_n}, \max\{V^{\lambda}_{T_n}, L_{T_n}\}\}\right)\mathbb{1}_{\{T_n\le T\}} \\ &\quad - \int_{T_{n-1}}^{T} e^{-rs}\pi^{S,n}_s \sigma dW_s - \int_{T_{n-1}}^{T} e^{-rs}\pi^{P,n}_s \bar{\sigma} d\bar{N}_s. \end{aligned}$$

In turn, taking conditional expectation with respect to $\mathcal{G}_{T_{n-1}}$ and using (5.4), we conclude that $X^n_{T_{n-1}} = V^{\lambda}_{T_{n-1}}$. □

6. Application to convertible bonds with random intervention times. In this section, using the constrained Dynkin game introduced in section 2, we study convertible bonds for which both players are only allowed to stop at a sequence of random intervention times.

Traditionally, convertible bond models often assume that both the bond holder and the issuing firm are allowed to stopped at any stopping time adapted to the firm's fundamental (such as its stock prices). In reality, there may exist some liquidation constraint as an external shock, and both players only make their decisions when such a shock arrives. We model such a liquidation shock as the arrival times of an exogenous Poisson process. A similar idea has first appeared in the modeling of debt run problems (see [23]), which can be formulated as optimal stopping problems with Poisson arrival times.

Assumption 6.1. *Let $d = 1$. The firm's stock price S^s, under the risk-neutral probability measure $\mathbb{P}$, follows*

$$S_t^s = s + \int_0^t (r-q) S_u^s \, du + \int_0^t \sigma S_u^s \, dW_u, \tag{6.1}$$

with $S_0^s = s > 0$, where the constants r, q, σ represent the risk-free interest rate, the dividend rate and the volatility of the stock, satisfying the parameter assumption $r > q$[1].

The firm issues convertible bonds as perpetuities with a constant coupon rate c. Consider an investor purchasing a share of this convertible bond at initial time $t = 0$. By holding the convertible bond, the investor will continuously receive the coupon rate c from the firm until the contract is terminated. The investor has the right to convert her bond to the firm's stocks, while the firm has the right to call the bond and force the bondholder to surrender her bond to the firm at a sequence of Poisson arrival times $(T_n)_{n \geq 1}$ with a constant intensity $\lambda > 0$. Hence, there are two situations that the contract maybe terminated:

(i) if the firm calls the bond at some $\mathbb{G}$-stopping time σ firstly, the bondholder will receive a pre-specified surrender price K at time σ;

(ii) if the investor chooses to convert her bond at some $\mathbb{G}$-stopping time τ firstly or both players choose to stop the contract simultaneously, the bondholder will obtain γS_τ at time τ from converting her bond with a pre-specified conversion rate $\gamma \in (0,1)$.

In summary, the investor will obtain the following discounted payoff at initial time $t = 0$:

$$P(s;\sigma,\tau) = \int_0^{\sigma\wedge\tau} e^{-ru} c \, du + e^{-r\tau} \gamma S_\tau^s \mathbb{1}_{\{\tau \leq \sigma\}} + e^{-r\sigma} K \mathbb{1}_{\{\sigma < \tau\}}, \tag{6.2}$$

with $\sigma, \tau \in \tilde{\mathcal{R}}_{T_1}(\lambda)$, where

$$\tilde{\mathcal{R}}_{T_i}(\lambda) = \{\mathbb{G}\text{-stopping time } \tau \text{ for } \tau(\omega) = T_N(\omega) \text{ where } N \geq i\}.$$

The investor will choose $\tau \in \tilde{\mathcal{R}}_{T_1}(\lambda)$ to maximize the bond value, while the firm will choose $\sigma \in \tilde{\mathcal{R}}_{T_1}(\lambda)$ to maximize the equity value of the firm by minimizing the bond value. This leads to a constrained Dynkin game as introduced in section 2. The upper value and lower value of this *constrained convertible bond* are

$$\overline{v}^\lambda(s) = \inf_{\sigma \in \tilde{\mathcal{R}}_{T_1}(\lambda)} \sup_{\tau \in \tilde{\mathcal{R}}_{T_1}(\lambda)} \mathbb{E}\left[P(s;\sigma,\tau)\right], \tag{6.3}$$

$$\underline{v}^\lambda(s) = \sup_{\tau \in \tilde{\mathcal{R}}_{T_1}(\lambda)} \inf_{\sigma \in \tilde{\mathcal{R}}_{T_1}(\lambda)} \mathbb{E}\left[P(s;\sigma,\tau)\right]. \tag{6.4}$$

Note that the constrained Dynkin game in section 2 does not exactly cover the above constrained convertible bond, since the model in section 2 has a random terminal time T, while the convertible bond is perpetual. However, in the following proposition, we shall show that when

$$s \geq \bar{s}^\lambda := \frac{q+\lambda}{r+\lambda} \frac{K}{\gamma},$$

[1] The case $r \leq q$ can be treated in a similar way.

the optimal stopping strategy is trivial. In this region, it is always optimal for both the investor and the firm to stop at the first Poisson arrival time. Intuitively, when the stock price is high, the stock is attractive enough to lead both the investor to convert her bond to stocks and the firm to prevent the investor from converting by calling the bond as early as possible.

PROPOSITION 6.2. *Suppose that Assumption 6.1 holds. Then, the value of the constrained convertible bond, denoted as $v^{\lambda}(s)$, exists and satisfies $L^{\lambda}(s) \leq v^{\lambda}(s) \leq U^{\lambda}$ for $s \in (0,\infty)$, where*

$$L^{\lambda}(s) := \frac{c}{r+\lambda} + \frac{\lambda}{q+\lambda}\gamma s; \quad U^{\lambda} := \frac{c+\lambda K}{r+\lambda}.$$

Moreover, in the domain $s \in \left[\bar{s}^{\lambda},\infty\right)$, it holds that $v^{\lambda}(s) = L^{\lambda}(s)$, and the optimal stopping strategy is $\tau^{*,\lambda} = \sigma^{*,\lambda} = T_1$.

Proof. Choosing $\tau \equiv T_1$ in (6.4) yields a lower bound of the convertible bond price:

$$\begin{aligned}
\underline{v}^{\lambda}(s) &= \sup_{\tau\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \inf_{\sigma\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \mathbb{E}\left[\int_0^{\sigma\wedge\tau} e^{-ru}c\,du + e^{-r\tau}\gamma S^s_{\tau}\mathbb{1}_{\{\tau\leq\sigma\}} + e^{-r\sigma}K\mathbb{1}_{\{\sigma<\tau\}}\right] \\
&\geq \inf_{\sigma\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \mathbb{E}\left[\int_0^{T_1} e^{-ru}c\,du + e^{-rT_1}\gamma S^s_{T_1}\right] \\
&= \mathbb{E}\left[\int_0^{\infty} \lambda e^{-\lambda m}\left(\int_0^{m} e^{-ru}c\,du + e^{-rm}\gamma S^s_m\right) dm\right] \\
&= \int_0^{\infty} \lambda e^{-\lambda m}\int_0^{m} e^{-ru}c\,du\,dm + \lambda\gamma\mathbb{E}\left[\int_0^{\infty} e^{-(r+\lambda)m}S^s_m\,dm\right] \\
&= \frac{c}{r+\lambda} + \frac{\lambda}{q+\lambda}\gamma s = L^{\lambda}(s),
\end{aligned}$$

where we used the integration by parts in the last equality.

On the other hand, by choosing $\sigma \equiv T_1$ in (6.3), we get an upper bound of the convertible bond price:

$$\begin{aligned}
\bar{v}^{\lambda}(s) &= \inf_{\sigma\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \sup_{\tau\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \mathbb{E}\left[\int_0^{\sigma\wedge\tau} e^{-ru}c\,du + e^{-r\tau}\gamma S^s_{\tau}\mathbb{1}_{\{\tau\leq\sigma\}} + e^{-r\sigma}K\mathbb{1}_{\{\sigma<\tau\}}\right] \\
&\leq \sup_{\tau\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \mathbb{E}\left[\int_0^{T_1} e^{-ru}c\,du + e^{-rT_1}\gamma S^s_{T_1}\mathbb{1}_{\{\tau=T_1\}} + e^{-rT_1}K\mathbb{1}_{\{\tau>T_1\}}\right] \\
&= \frac{c}{r+\lambda} + \max\left\{\frac{\lambda}{q+\lambda}\gamma s, \frac{\lambda K}{r+\lambda}\right\} = \max\{L^{\lambda}(s), U^{\lambda}\}.
\end{aligned}$$

In the domain $s \in \left[\bar{s}^{\lambda},\infty\right)$, we always have $L^{\lambda}(s) \geq U^{\lambda}$ so $\bar{v}^{\lambda}(s) \leq L^{\lambda}(s) \leq \underline{v}^{\lambda}(s)$. Thus, the value of the convertible bond exists, and $v^{\lambda}(s) = \bar{v}^{\lambda}(s) = \underline{v}^{\lambda}(s) = L^{\lambda}(s)$, with the optimal stopping strategy $\tau^{*,\lambda} = \sigma^{*,\lambda} = T_1$.

In the domain $s \in \left(0,\bar{s}^{\lambda}\right)$, we have $L^{\lambda}(s) < U^{\lambda}$. Introduce an $\mathbb{F}$-stopping time

$$\theta^{\lambda} := \inf\{u \geq 0 : S^s_u \geq \bar{s}^{\lambda}\}.$$

Then, it follows from the dynamic programming principle that

$$\overline{v}^{\lambda}(s) = \inf_{\sigma\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \sup_{\tau\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \mathbb{E}\left[\int_0^{\sigma\wedge\tau\wedge\theta^{\lambda}} e^{-ru}c\,du + e^{-r\theta^{\lambda}} v^{\lambda}\left(S^s_{\theta^{\lambda}}\right)\mathbb{1}_{\{\sigma\wedge\tau\geq\theta^{\lambda}\}}\right.$$
$$\left. + \left(e^{-r\tau}\gamma S^s_{\tau}\mathbb{1}_{\{\tau\leq\sigma\}} + e^{-r\sigma}K\mathbb{1}_{\{\sigma<\tau\}}\right)\mathbb{1}_{\{\sigma\wedge\tau<\theta^{\lambda}\}}\right].$$

By the definition of the stopping time θ^{λ}, $v^{\lambda}\left(S^s_{\theta^{\lambda}}\right) = v^{\lambda}\left(\bar{s}^{\lambda}\right) = L^{\lambda}(\bar{s}^{\lambda}) = U^{\lambda}$. Thus, in the domain $s\in(0,\bar{s}^{\lambda})$, (6.3)-(6.4) are equivalent to

$$\overline{v}^{\lambda}(s) = \inf_{\sigma\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \sup_{\tau\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \mathbb{E}\left[\tilde{P}(s;\sigma,\tau)\right], \tag{6.5}$$

$$\underline{v}^{\lambda}(s) = \sup_{\tau\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \inf_{\sigma\in\tilde{\mathcal{R}}_{T_1}(\lambda)} \mathbb{E}\left[\tilde{P}(s;\sigma,\tau)\right], \tag{6.6}$$

where the payoff $\tilde{P}(s;\sigma,\tau)$ is

$$\int_0^{\sigma\wedge\tau\wedge\theta^{\lambda}} e^{-ru}c\,du + e^{-r\theta^{\lambda}}U^{\lambda}\mathbb{1}_{\{\sigma\wedge\tau\geq\theta^{\lambda}\}} + e^{-r\tau}\gamma S^s_{\tau}\mathbb{1}_{\{\tau<\theta^{\lambda},\tau\leq\sigma\}} + e^{-r\sigma}K\mathbb{1}_{\{\sigma<\theta^{\lambda},\sigma<\tau\}}.$$

Note that if we introduce the $\mathbb{G}$-stopping time

$$T_M := \inf\{T_N \geq \theta^{\lambda} : N\geq 1\}, \tag{6.7}$$

since the payoff function $\tilde{P}(s;\sigma,\tau)$ does not change after T_M, we may replace the control set $\tilde{R}_{T_1}(\lambda)$ in (6.5)-(6.6) with $\mathcal{R}_{T_1}(\lambda)$, the latter of which consists of $\mathbb{G}$-stopping times $T_1, T_2, \cdots, T_M$.

Now, we apply Theorem 2.3 with $T=\theta^{\lambda}$, $L_t = \gamma S^s_t$, $U_t = K$, $f_t = c$ and $\xi = U^{\lambda}$ to (6.5)-(6.6), and obtain the existence of the value of the convertible bond in the domain $s\in(0,\bar{s}^{\lambda})$. □

Thanks to the above proposition, *we focus our analysis to the domain* $s\in\left(0,\bar{s}^{\lambda}\right)$ *in the rest of this section.* We characterize the value of the convertible bond and the corresponding optimal stopping strategy via the solution of ODEs and the associated free boundaries, respectively.

Proposition 6.3. *Suppose that Assumption 6.1 holds. Define the infinitesimal generator* $\mathcal{L}_0 = \frac{1}{2}\sigma^2 s^2\partial^2_{ss} + (r-q)s\partial_s - r$. *For* $s\in(0,\bar{s}^{\lambda})$, *the value of the convertible bond* $v^{\lambda}(s)$ *is the unique solution to the following ODEs:*

(i) If $c > qK$, *then* $v^{\lambda}(s) > \gamma s$, *and*

$$-\mathcal{L}_0 v^{\lambda} = c - \lambda(v^{\lambda} - K)^{+} \tag{6.8}$$

with the boundary condition $v^{\lambda}(\bar{s}^{\lambda}) = U^{\lambda}$;

(ii) If $c < rK$, *then* $v^{\lambda}(s) < K$, *and*

$$-\mathcal{L}_0 v^{\lambda} = c + \lambda(\gamma s - v^{\lambda})^{+} \tag{6.9}$$

with the boundary condition $v^{\lambda}(\bar{s}^{\lambda}) = U^{\lambda}$.

Proof. It is immediate from Theorem 2.3 and (6.5)-(6.6) that the convertible bond value is $v^{\lambda}(s) = V^{\lambda,s}_0$, for $s\in(0,\bar{s}^{\lambda})$, where $V^{\lambda,s}$ is the first component of the solution to the penalized BSDE

(6.10)

$$V^{\lambda,s}_{t\wedge\theta^{\lambda}} = U^{\lambda} + \int_{t\wedge\theta^{\lambda}}^{\theta^{\lambda}} \left[c + \lambda\left(\gamma S^s_u - V^{\lambda,s}_u\right)^{+} - \lambda\left(V^{\lambda,s}_u - K\right)^{+} - rV^{\lambda,s}_u\right] du - \int_{t\wedge\theta^{\lambda}}^{\theta^{\lambda}} Z^{\lambda,s}_u\,dW_u.$$

Moreover, the optimal stopping strategy is

$$\text{(6.11)} \qquad \begin{cases} \sigma^{*,\lambda} = \inf\{T_N \geq T_1 : V^{\lambda,s}_{T_N} \geq K\} \wedge T_M; \\ \tau^{*,\lambda} = \inf\{T_N \geq T_1 : V^{\lambda,s}_{T_N} \leq \gamma S^s_{T_N}\} \wedge T_M, \end{cases}$$

with T_M given in (6.7).

On the other hand, by the Markov property of the stock price S, $V^{\lambda,s}_t = v^\lambda(S^s_t)$. In turn, Itô's formula further implies that

(6.12)

$$v^\lambda(S^s_{\theta^\lambda}) - v^\lambda(S^s_{t\wedge\theta^\lambda}) = \int_{t\wedge\theta^\lambda}^{\theta^\lambda} \left[\mathcal{L}_0 v^\lambda(S^s_u) + r v^\lambda(S^s_u)\right] du + \int_{t\wedge\theta^\lambda}^{\theta^\lambda} \sigma s \partial_s v^\lambda(S^s_u) dW_u.$$

It then follows from (6.10) and (6.12) that $v^\lambda(s)$, for $s \in (0, \bar{s}^\lambda)$, solves the ODE

$$\text{(6.13)} \qquad -\mathcal{L}_0 v^\lambda = c + \lambda(\gamma s - v^\lambda)^+ - \lambda(v^\lambda - K)^+,$$

with the boundary condition $v^\lambda(\bar{s}^\lambda) = U^\lambda$. Note that if $c < rK$, Proposition 6.2 yields

$$v^\lambda(s) \leq U^\lambda = \frac{c + \lambda K}{r + \lambda} < \frac{rK + \lambda K}{r + \lambda} = K,$$

and if $c > qK$, it follows that

$$v^\lambda(s) \geq L^\lambda(s) = \frac{c}{r+\lambda} + \frac{\lambda}{q+\lambda}\gamma s > \frac{qK}{r+\lambda} + \frac{\lambda}{q+\lambda}\gamma s > \gamma s.$$

The ODEs (6.8)-(6.9) then follow immediately. □

The rest of this section is devoted to the characterization of the optimal stopping strategy of the constrained convertible bond via its associated free boundaries.

6.1. The Case I: $qK < c < rK$**.** From Proposition 6.3, when $qK < c < rK$, we always have $\gamma s < v^\lambda(s) < K$. Thus, following from (6.11), the optimal stopping strategy is

$$\tau^{*,\lambda} = \sigma^{*,\lambda} = T_M.$$

Intuitively, when the coupon rate c satisfies $c < rK$, i.e. $\frac{c}{r} < K$, the firm shall never spend K to call the bond back, since it only needs to pay the coupon rate c as a perpetual bond, whose value is $\frac{c}{r}$. Thus, the firm shall never call until T_M.

When the coupon rate c satisfies $c > qK$, i.e. $c > qK > q\frac{r+\lambda}{q+\lambda}\gamma s > q\gamma s$, the investor shall never convert her bond into stocks, since the stock dividends she will receive by holding γ shares of the stock are no more than what she would otherwise receive from the bond coupons. Thus, the investor shall never convert until T_M.

In Figure 7.1, the bold horizontal line $\bar{s}^\lambda$ represents the conversion and calling boundary. We simulate three Poisson times $T_1 = 0.3$, $T_2 = 0.5$, $T_3 = 0.8$, and two stock price paths. The investor (and the firm) will convert (and call) the bond at T_1 for the stock path 1. They will continue at T_1 and T_2, and terminate the contract at T_3 for the stock path 2.

We further calculate the convertible bond value by solving the corresponding ODE explicitly. Note that in such a situation, $v^\lambda = v^{1,\lambda}$ solves

$$\text{(6.14)} \qquad \begin{cases} -\mathcal{L}_0 v^{1,\lambda} - c &= 0, \text{ for } 0 < s < \bar{s}^\lambda; \\ v^{1,\lambda}(0+) &= \dfrac{c}{r}; \\ v^{1,\lambda}(\bar{s}^\lambda) &= U^\lambda. \end{cases}$$

We put the perpetual bond value $\frac{c}{r}$ at the boundary $v^{1,\lambda}(0+) := \lim_{s\downarrow 0} v^{1,\lambda}(s)$, because in such a situation, there is no motivation for the firm to call or for the investor to convert the bond.

The general solution of (6.14) has the form $v^{1,\lambda}(s) = A_+ s^{\alpha^+} + A_- s^{\alpha^-} + \frac{c}{r}$, for $0 < s < \bar{s}^\lambda$, where

$$\alpha^\pm = \frac{-(r - q - \frac{\sigma^2}{2}) \pm \sqrt{(r - q - \frac{\sigma^2}{2})^2 + 2r\sigma^2}}{\sigma^2}. \tag{6.15}$$

Since $\alpha^- < 0$, we obtain $A_- = 0$ by the boundary condition at $v^{1,\lambda}(0+)$. Using the other boundary condition, we further obtain

$$v^{1,\lambda}(s) = A^{1,\lambda} s^\alpha + \frac{c}{r}, \tag{6.16}$$

where $\alpha = \alpha^+$ and $A^{1,\lambda} = \frac{\lambda}{r+\lambda} \frac{rK-c}{r} \left(\bar{s}^\lambda\right)^{-\alpha}$.

In Figure 2, we further plot the value function $v^{1,\lambda}(s)$, which always stays between $[L^\lambda(s), U^\lambda]$ for $s \in (0, \bar{s}^\lambda)$. Since $L^\lambda > \gamma s$ and $U^\lambda < K$, the value function also stays between $(\gamma s, K)$, which means it is never optimal for the firm or the investor to stop in the region $s \in (0, \bar{s}^\lambda)$.

6.2. The Case II: $c \geq rK$**.** It is obvious that $c > qK$ if $c \geq rK$. Thus, from Proposition 6.3, we always have $v^\lambda(s) > \gamma s$, and following from (6.11), the optimal conversion strategy for the investor is

$$\tau^{*,\lambda} = T_M,$$

i.e. it is never optimal for the investor to convert until T_M. Instead, the investor's optimal strategy is to keep the convertible bond to receive its coupons (up to T_M).

On the other hand, following from (6.8), $v^\lambda = v^{2,\lambda}$ solves

$$\begin{cases} -\mathcal{L}_0 v^{2,\lambda} - c + \lambda(v^{2,\lambda} - K)^+ = 0, & \text{for } 0 < s < \bar{s}^\lambda; \\ v^{2,\lambda}(0+) = U^\lambda; \\ v^{2,\lambda}(\bar{s}^\lambda) = U^\lambda. \end{cases} \tag{6.17}$$

We put U^λ at the boundary $v^{2,\lambda}(0+) := \lim_{s\downarrow 0} v^{2,\lambda}(s)$. In this situation, since the coupon rate c is too large, the firm would prefer to convert as soon as possible to stop paying the bond coupons. It is clear that $v^{2,\lambda}(s) = U^\lambda \geq K$. In turn, by (6.11), it is optimal for the firm to call as soon as possible, i.e. at the first Poisson arrival time

$$\sigma^{*,\lambda} = T_1.$$

In Figure 3, the bold horizontal line $\bar{s}^\lambda$ represents the conversion boundary for the investor. Once again, we simulate three Poisson times $T_1 = 0.25$, $T_2 = 0.5$, $T_3 = 0.8$, and two stock price paths. For the stock price path 1, the firm will call the bond at T_1 firstly, and for the stock price path 2, both the firm and the investor will terminate the contract at T_1.

Figure 4 further plots the value function $v^{2,\lambda}$, which is a constant U^λ for $s \in (0, \bar{s}^\lambda)$. Since the value function always stays above K, and therefore also above γs, it is never optimal for the investor to convert in the region $(0, \bar{s}^\lambda)$.

6.3. The Case III: $c \le qK$**.** It is obvious that $c < rK$ if $c \le qK$. Thus, from Proposition 6.3, we always have $v^{\lambda}(s) < K$, and following from (6.11), the optimal calling time for the firm is

$$\sigma^{*,\lambda} = T_M,$$

i.e. it is never optimal for the firm to call until T_M. Furthermore, following from (6.9), $v^{\lambda} = v^{3,\lambda}$ solves

$$\left\{\begin{array}{rcl} -\mathcal{L}_0 v^{3,\lambda} - c - \lambda(\gamma s - v^{3,\lambda})^+ &=& 0, \text{ for } 0 < s < \bar{s}^{\lambda}; \\ v^{3,\lambda}(0+) &=& \dfrac{c}{r}; \\ v^{3,\lambda}(\bar{s}^{\lambda}) &=& U^{\lambda}. \end{array}\right. \tag{6.18}$$

Next, we solve (6.18) explicitly. Since $c \le qK$, the intersection point of the lower bound $L^{\lambda}(s)$ of the convertible bond value and the investor's payoff function γs is no greater than $\bar{s}^{\lambda}$ (so γs is no less than $L^{\lambda}(s)$ between this intersection point and $\bar{s}^{\lambda}$). Thus, it may happen that, in the region $s \in (0, \bar{s}^{\lambda})$, the investor converts the bond earlier than T_M. Since $v^{3,\lambda}(s) > \gamma s$ when $s \downarrow 0$, and $v^{3,\lambda}(s) \le \gamma s$ for $s = \bar{s}^{\lambda}$, we define

$$x^{*,\lambda} = \inf\left\{s \in (0, \bar{s}^{\lambda}] : v^{3,\lambda}(s) \le \gamma s\right\}. \tag{6.19}$$

By definition it is obvious $v^{3,\lambda} > \gamma s$ for $s \in (0, x^{*,\lambda})$, and by the continuity of $v^{3,\lambda}(\cdot)$, $v^{3,\lambda}(x^{*,\lambda}) = \gamma x^{*,\lambda}$. Let us at the moment assume that $v^{3,\lambda} \le \gamma s$ for $s \in (x^{*,\lambda}, \bar{s}^{\lambda}]$. Later, we will verify this condition. If this condition holds, (6.18) is equivalent to the following free boundary problem

$$-\mathcal{L}_0 v^{3,\lambda} - c = 0, \text{ for } 0 < s < x^{*,\lambda}; \tag{6.20}$$

$$-\mathcal{L}_0 v^{3,\lambda} - c + \lambda(v^{3,\lambda} - \gamma s) = 0, \text{ for } x^{*,\lambda} < s < \bar{s}^{\lambda}; \tag{6.21}$$

$$v^{3,\lambda}(0+) = \frac{c}{r}; \tag{6.22}$$

$$v^{3,\lambda}(\bar{s}^{\lambda}) = U^{\lambda}; \tag{6.23}$$

$$v^{3,\lambda}(x^{*,\lambda}-) = \gamma x^{*,\lambda}; \tag{6.24}$$

$$v^{3,\lambda}(x^{*,\lambda}+) = \gamma x^{*,\lambda}; \tag{6.25}$$

$$\left(v^{3,\lambda}\right)'(x^{*,\lambda}-) = \left(v^{3,\lambda}\right)'(x^{*,\lambda}+). \tag{6.26}$$

We first observe that, with the boundary condition (6.22), ODEs (6.20)-(6.21) imply

$$v^{3,\lambda}(s) = \left\{\begin{array}{ll} A^{3,\lambda} s^{\alpha} + \frac{c}{r}, & \text{if } s \in (0, x^{*,\lambda}); \\ B_+ s^{\beta^+} + B_- s^{\beta^-} + \frac{c}{r+\lambda} + \frac{\lambda}{q+\lambda}\gamma s, & \text{if } s \in (x^{*,\lambda}, \bar{s}^{\lambda}), \end{array}\right. \tag{6.27}$$

where $\alpha = \alpha^+$ is given in (6.15),

$$\beta^{\pm} = \frac{-(r - q - \frac{\sigma^2}{2}) \pm \sqrt{(r - q - \frac{\sigma^2}{2})^2 + 2(r+\lambda)\sigma^2}}{\sigma^2}, \tag{6.28}$$

and four unknowns $(A^{3,\lambda}, B_+, B_-, x^{*,\lambda})$ are to be determined. Using the continuity across $x^{*,\lambda}$, i.e. (6.24)-(6.25), the smooth pasting across $x^{*,\lambda}$, i.e. (6.26), and the boundary condition at $s = \bar{s}^{\lambda}$, i.e. (6.23), we obtain that $x^{*,\lambda} \in \left(0, \bar{s}^{\lambda}\right]$ is the (unique) solution to the following algebraic equation

$$C_1 x^{\beta^+ - \beta^- + 1} + C_2 x^{\beta^+ - \beta^-} + C_3 x + C_4 = 0, \tag{6.29}$$

with

$$
(6.30)\qquad \begin{cases} C_1 = \left(\alpha - \frac{\lambda}{q+\lambda} - \frac{q}{q+\lambda}\beta^+\right)\gamma; \\ C_2 = -\left(\alpha\frac{c}{r} - \frac{c}{r+\lambda}\beta^+\right); \\ C_3 = -\left(\alpha - \frac{\lambda}{q+\lambda} - \frac{q}{q+\lambda}\beta^-\right)(\bar{s}^\lambda)^{\beta^+-\beta^-}\gamma; \\ C_4 = \left(\alpha\frac{c}{r} - \frac{c}{r+\lambda}\beta^-\right)(\bar{s}^\lambda)^{\beta^+-\beta^-}, \end{cases}
$$

and the coefficients are determined by

$$
(6.31)\qquad \begin{cases} A^{3,\lambda} &= \left(x^{*,\lambda}\right)^{-\alpha}\left(\gamma x^{*,\lambda} - \frac{c}{r}\right); \\ B_+ &= \frac{\frac{q}{q+\lambda}\gamma x^{*,\lambda} - \frac{c}{r+\lambda}}{(x^{*,\lambda})^{\beta^+} - (\bar{s}^\lambda)^{\beta^+-\beta^-}(x^{*,\lambda})^{\beta^-}}; \\ B_- &= \frac{\frac{q}{q+\lambda}\gamma x^{*,\lambda} - \frac{c}{r+\lambda}}{(x^{*,\lambda})^{\beta^-} - (\bar{s}^\lambda)^{\beta^--\beta^+}(x^{*,\lambda})^{\beta^+}}. \end{cases}
$$

It remains to verify the condition $v^{3,\lambda} \leq \gamma s$ for $s \in (x^{*,\lambda}, \bar{s}^\lambda]$. Indeed, since $A^{3,\lambda} > 0$, $\alpha > 1$, $B_+ < 0$, $\beta^+ > 1$ and $B_- > 0$, $\beta^- < 0$, it is clear that $v^{3,\lambda}$ is convex in the interval $(0, x^{*,\lambda})$ and concave in the interval $(x^{*,\lambda}, \bar{s}^\lambda]$. Moreover, $\left(v^{3,\lambda}\right)'(x^{*,\lambda}) < \gamma$. This verifies the condition.

The optimal conversion time for the investor is therefore given as

$$
\tau^{*,\lambda} = \inf\{T_N : S^s_{T_N} \geq x^{*,\lambda}\} \wedge T_M.
$$

In Figure 5, the top bold horizontal line $\bar{s}^\lambda$ represents the calling boundary for the firm, and the bottom bold horizontal line $x^{*,\lambda}$ represents the conversion boundary for the investor. Once again, we simulate three Poisson times $T_1 = 0.3$, $T_2 = 0.5$, $T_3 = 0.8$, and two stock price paths. For the stock price path 1, both the investor and the firm will terminate the contract at T_1; and for the stock path 2, the investor will continue at T_1 and convert at T_2, while the firm will not call the bond back at neither T_1 nor T_2.

In Figure 6, we further plot the value function $v^{3,\lambda}$, which crosses the payoff function γs in the region $(0, \bar{s}^\lambda]$, so the crossing point $x^{*,\lambda}$ is the optimal conversion boundary for the investor. Furthermore, the value function $v^{3,\lambda}$ is strictly dominated by K for $s \in (0, \bar{s}^\lambda)$, so the firm never calls the bond back in this region.

7. Asymptotics as $\lambda \to \infty$. We study the asymptotic behavior of the convertible bond price and its associated free boundaries when the Poisson intensity $\lambda \to \infty$. Intuitively, they will converge to their continuous time counterparts. We prove this intuition in this section.

7.1. Review of standard convertible bonds. The setting is the same as in section 6 except that both the investor and the firm choose their respective optimal stopping strategies as $\mathbb{F}$-stopping times taking values in $[0, \infty]$. Then, the upper and lower values of the standard convertible bond are given by

$$
(7.1)\qquad \overline{v} = \inf_{\sigma \in \tilde{\mathcal{R}}_0} \sup_{\tau \in \tilde{\mathcal{R}}_0} \mathbb{E}\left[P(s;\sigma,\tau)\right],
$$

$$
(7.2)\qquad \underline{v} = \sup_{\tau \in \tilde{\mathcal{R}}_0} \inf_{\sigma \in \tilde{\mathcal{R}}_0} \mathbb{E}\left[P(s;\sigma,\tau)\right],
$$

and the control set $\tilde{\mathcal{R}}_0$ is defined as

$$\tilde{\mathcal{R}}_0 = \{\mathbb{F}\text{-stopping time } \tau \text{ for } \tau \geq 0\}.$$

We say this game has value v if $v = \overline{v} = \underline{v}$, and has a saddle point $(\sigma^*, \tau^*) \in \tilde{\mathcal{R}}_0 \times \tilde{\mathcal{R}}_0$ if $\mathbb{E}\left[P(s;\sigma^*,\tau)\right] \leq \mathbb{E}\left[P(s;\sigma^*,\tau^*)\right] \leq \mathbb{E}\left[P(s;\sigma,\tau^*)\right]$ for every $(\sigma,\tau) \in \tilde{\mathcal{R}}_0 \times \tilde{\mathcal{R}}_0$.

The proof of the following result follows along the similar arguments in [35] and is thus omitted. We refer to [35] for its further details.

PROPOSITION 7.1. *Suppose that Assumption 6.1 holds. Let $\bar{s} := \frac{K}{\gamma}$, and define an $\mathbb{F}$-stopping time $\theta = \inf\{u \geq 0 : S_u^s \geq \bar{s}\}$. Then, the value of the standard convertible bond $v(s)$ is given as follows:*

(i) The Case I: $qK < c < rK$,

$$v^1(s) = \begin{cases} A^1 s^\alpha + \frac{c}{r}, & \text{if } s \in (0,\bar{s}); \\ \gamma s, & \text{if } s \in [\bar{s},\infty), \end{cases} \tag{7.3}$$

with $\alpha = \alpha^+$ as in (6.15) and $A^1 = \frac{rK-c}{r}(\bar{s})^{-\alpha}$. The optimal stopping strategy is given by

$$\sigma^* = \tau^* = \theta. \tag{7.4}$$

(ii) The Case II: $c \geq rK$,

$$v^2(s) = \begin{cases} K, & \text{if } s \in (0,\bar{s}); \\ \gamma s, & \text{if } s \in [\bar{s},\infty). \end{cases} \tag{7.5}$$

The optimal stopping strategy is given by

$$\sigma^* = 0; \quad \tau^* = \theta. \tag{7.6}$$

(iii) The Case III: $c \leq qK$,

$$v^3(s) = \begin{cases} A^3 s^\alpha + \frac{c}{r}, & \text{if } s \in (0,x^3); \\ \gamma s, & \text{if } s \in [x^3,\infty), \end{cases} \tag{7.7}$$

with $\alpha = \alpha^+$ and $A^3 = \left(\gamma x^3 - \frac{c}{r}\right)(x^3)^{-\alpha}$. The optimal stopping strategy is given by

$$\sigma^* = \theta, \quad \tau^* = \inf\{t \geq 0 : S_t^s \geq x^3\}, \tag{7.8}$$

where

$$x^3 = \begin{cases} x^* := \frac{\alpha}{\alpha-1}\frac{c}{\gamma r}, & \text{if } c \leq \frac{\alpha-1}{\alpha} rK; \\ \bar{s}, & \text{if } c > \frac{\alpha-1}{\alpha} rK. \end{cases}$$

7.2. Asymptotics. We conclude the paper by studying, when $\lambda \to \infty$, (i) the convergence of the constrained convertible bond price v^λ to its continuous-time counterpart v; (ii) the convergence of the optimal conversion/calling boundaries for the constrained convertible bond to its continuous-time counterparts.

It is easy to check that $\bar{s}^\lambda \to \bar{s}$, $A^{1,\lambda} \to A^1$, $L^\lambda(s) \to \gamma s$ and $U^\lambda \to K$ with the convergence rate $1/\lambda$ by using their explicit forms. As a consequence, we have

$$v^{1,\lambda}(s) \to v^1(s); \quad v^{2,\lambda}(s) \to v^2(s),$$

with the convergence rate $1/\lambda$. Hence, we only need to establish the convergence results for Case III when $c \leq qK$. To this end, we first establish the monotonic property of $x^{*,\lambda}$, as defined in (6.19), with respect to λ in the following lemma.

PROPOSITION 7.2. *Suppose that Assumption 6.1 holds and that $c \leq qK$. Then, $x^{*,\lambda}$ is non-decreasing with respect to λ.*

Proof. By the definition of $x^{*,\lambda}$ in (6.19), it is sufficient to prove $v^{3,\lambda}$ is non-decreasing in λ. Recall that $v^{3,\lambda}$ is the solution to the ODE (6.18) in the domain $s \in (0, \bar{s}^\lambda)$, and $v^{3,\lambda} = L^\lambda$ in the domain $s \in [\bar{s}^\lambda, \infty)$.

Let us suppose $\lambda_1 < \lambda_2$ and it is easy to check that $\bar{s}^{\lambda_1} < \bar{s}^{\lambda_2}$. For $s \geq \bar{s}^{\lambda_1}$, we have $v^{3,\lambda_1} = L^{\lambda_1}$. Then,

$$\begin{aligned} v^{3,\lambda_1}(s) - v^{3,\lambda_2}(s) &\leq L^{\lambda_1}(s) - L^{\lambda_2}(s) \\ &= \frac{c(\lambda_2 - \lambda_1)}{(r+\lambda_1)(r+\lambda_2)} - \frac{q(\lambda_2 - \lambda_1)}{(q+\lambda_1)(q+\lambda_2)}\gamma s \\ &\leq \frac{(q-r)qK(\lambda_2 - \lambda_1)}{(r+\lambda_1)(q+\lambda_2)(r+\lambda_2)} < 0. \end{aligned}$$

On the other hand, for $s < \bar{s}^{\lambda_1}$, note that $v^{3,\lambda_1}(0+) = v^{3,\lambda_2}(0+) = \frac{c}{r}$ and $v^{3,\lambda_1}(\bar{s}^{\lambda_1}) < v^{3,\lambda_2}(\bar{s}^{\lambda_1})$. Define the set $\mathcal{N} = \left\{s \in \left(0, \bar{s}^{\lambda_1}\right) : v^{3,\lambda_1}(s) > v^{3,\lambda_2}(s)\right\}$, and suppose that $\mathcal{N} \neq \emptyset$. Then on $\mathcal{N}$, we have

$$\begin{cases} -\mathcal{L}_0 v^{3,\lambda_1} = c + \lambda_1(\gamma s - v^{3,\lambda_1})^+; \\ -\mathcal{L}_0 v^{3,\lambda_2} = c + \lambda_2(\gamma s - v^{3,\lambda_2})^+, \end{cases}$$

which implies

$$\begin{aligned} -\mathcal{L}_0(v^{3,\lambda_1} - v^{3,\lambda_2}) &= \lambda_1(\gamma s - v^{3,\lambda_1})^+ - \lambda_2(\gamma s - v^{3,\lambda_2})^+ \\ &\leq \lambda_2\left[(\gamma s - v^{3,\lambda_1})^+ - (\gamma s - v^{3,\lambda_2})^+\right] \leq 0. \end{aligned}$$

Hence, we have $v^{3,\lambda_1} \leq v^{3,\lambda_2}$ on $\mathcal{N}$, which is in contradiction with the definition of $\mathcal{N}$. ◻

Since $x^{*,\lambda}$ is bounded by $\bar{s}^\lambda (\leq \bar{s})$, Proposition 7.2 then implies that $\lim_{\lambda\to\infty} x^{*,\lambda}$ exists, denoted by x^∞. Moreover, by Proposition 7.1, we have $x^\infty \leq x^*$ if $c \leq \frac{\alpha-1}{\alpha} rK$, and $x^\infty \leq \bar{s}$ if $c > \frac{\alpha-1}{\alpha} rK$.

On the other hand, by (6.29), $x^{*,\lambda}$ is the solution to the following allergic equation

$$\left[\left(\frac{x}{\bar{s}^\lambda}\right)^{\beta^+-\beta^-} - 1\right]\left[\left(\alpha - \frac{\lambda}{q+\lambda}\right)\gamma x - \alpha\frac{c}{r} - \beta^+\left(\frac{q}{q+\lambda}\gamma x - \frac{c}{r+\lambda}\right)\right] = (\beta^+ - \beta^-)\left(\frac{q}{q+\lambda}\gamma x - \frac{c}{r+\lambda}\right). \tag{7.9}$$

Sending $\lambda \to \infty$ in (7.9), since the right hand side of (7.9) has the limit 0, we obtain

$$\lim_{\lambda\to\infty} \underbrace{\left[\left(\frac{x^{*,\lambda}}{\bar{s}^\lambda}\right)^{\beta^+-\beta^-} - 1\right]}_{I^\lambda} \underbrace{\left[\left(\alpha - \frac{\lambda}{q+\lambda}\right)\gamma x^{*,\lambda} - \alpha\frac{c}{r} - \beta^+\left(\frac{q}{q+\lambda}\gamma x^{*,\lambda} - \frac{c}{r+\lambda}\right)\right]}_{II^\lambda} = 0.$$

This implies at least one of I^λ and II^λ has the limit 0.

If $c < \frac{\alpha-1}{\alpha} rK$, we have $\lim_{\lambda\to\infty} I^\lambda = -1$, since

$$\lim_{\lambda\to\infty} \frac{x^{*,\lambda}}{\bar{s}^\lambda} = \frac{x^\infty}{\bar{s}} \leq \frac{x^*}{\bar{s}} = \frac{\alpha}{\alpha-1}\frac{c}{rK} < 1.$$

This implies $\lim_{\lambda\to\infty} II^\lambda = 0$, i.e. $x^\infty = x^*$.

If $c > \frac{\alpha-1}{\alpha} rK$, we have

$$\lim_{\lambda\to\infty} II^\lambda = (\alpha-1)\gamma x^\infty - \alpha\frac{c}{r} < (\alpha-1)(\gamma x^\infty - K) \le 0,$$

which implies $\lim_{\lambda\to\infty} I^\lambda = 0$, i.e. $x^\infty = \bar{s}$.

If $c = \frac{\alpha-1}{\alpha} rK$, it is easy to check that $x^\infty = x^* = \bar{s}$.

Hence, we have established the convergence of $x^{*,\lambda} \to x^3$ as $\lambda \to \infty$. As a consequence, it also follows that $v^{3,\lambda}(s) \to v^3(s)$. However, due to the lack of explicit solutions for Case III, it is unclear what is the corresponding convergence rate.

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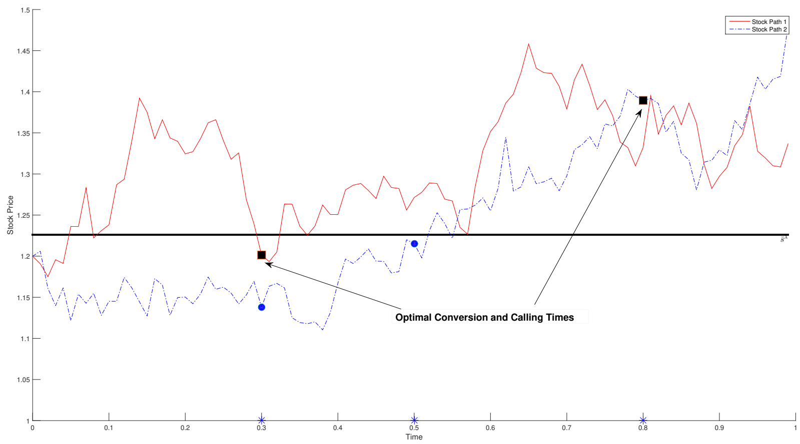

FIG. 7.1. *Scenario Simulation for Case I. The figure shows two simulated stock price paths in the case of $qK < c < rK$. Parameter values are $K = 1$, r=0.05, q=0.03, σ=0.2, γ=1 and λ=1. The initial stock price is set to s=1.2. The bold horizontal line describes the conversion and calling boundary $\bar{s}^{\lambda}$. Given the Poisson times T_1=0.3, T_2=0.5 and T_3=0.8, the investor will convert and the firm will call the bond both at T_1 (for path 1) and T_3 (for path 2).*

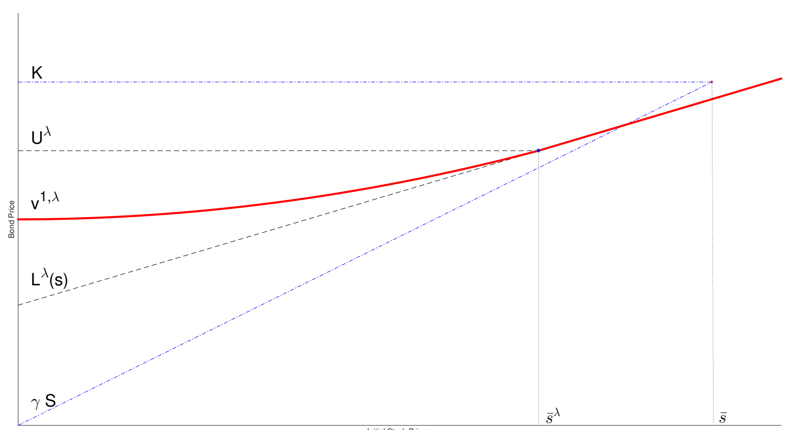

FIG. 7.2. *The value function $v^{1,\lambda}$ for Case I.*

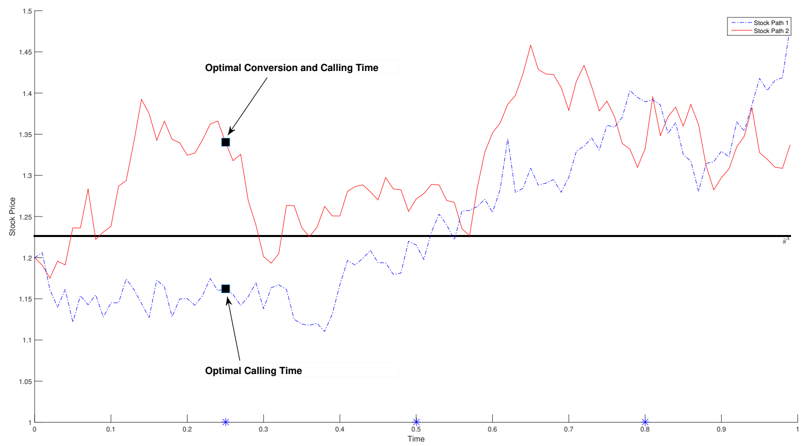

FIG. 7.3. *Scenario Simulation for Case II. The figure shows two simulated stock price paths in the case of $c \geq rK$. The parameters are the same as those in Figure 7.1.The bold horizontal line describes the conversion boundary $\bar{s}^{\lambda}$. Given the Poisson times T_1=0.25, T_2=0.5 and T_3=0.8, the firm will call the bond at T_1 (marked square) for the stock price path 1; and both the firm and the investor will terminate the contract at T_1 (marked square) for the stock price path 2.*

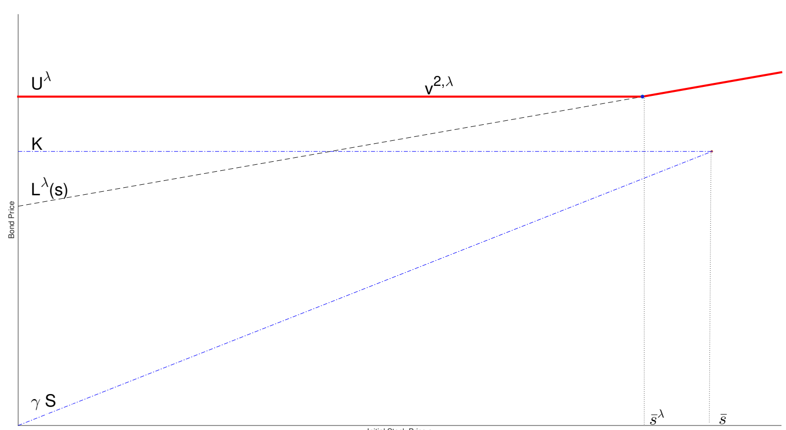

FIG. 7.4. *The value function $v^{2,\lambda}$ for Case II.*

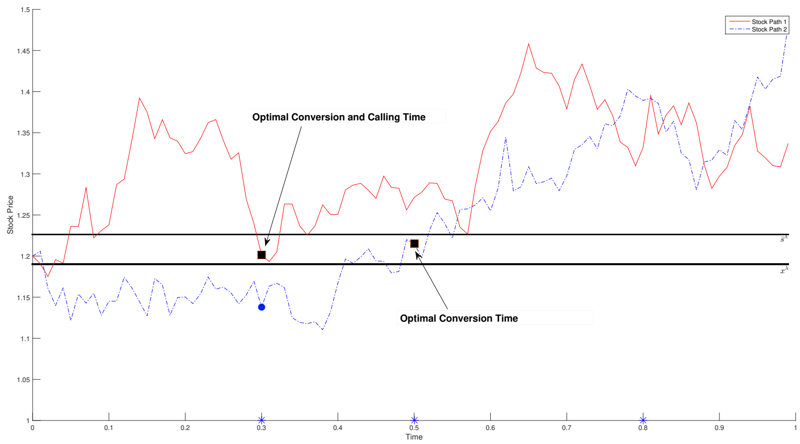

FIG. 7.5. *Scenario Simulation for Case III. The figure shows two simulated stock price paths in the case of $c \leq qK$. The parameters are the same as those in Figure 7.1. The top bold horizontal line is the calling boundary $\bar{s}^{\lambda}$, and the bottom bold horizontal line is the conversion boundary $x^{*,\lambda}$. Given the Poisson times T_1=0.3, T_2=0.5 and T_3=0.8, both the investor and the firm will terminate the contract at T_1 (marked square) for the stock price path 1; and the invertor will convert her bond T_2 (marked square) for the stock price path 2.*

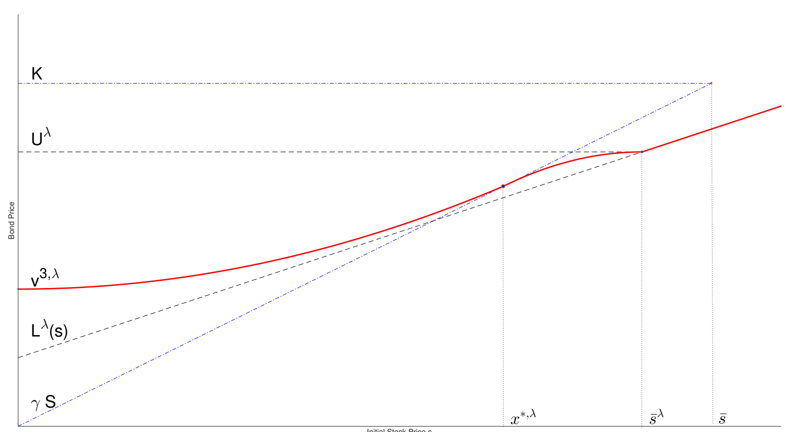

FIG. 7.6. *The value function $v^{3,\lambda}$ for Case III.*